Analysis and design of MPC frameworks for dynamic operation of nonlinear constrained systems

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Notation

In the following, we list the main symbols and acronyms used in this thesis. Additional notation is defined in the corresponding sections.

Abbreviations and acronyms

MPC	model predictive control
w.l.o.g.	without loss of generality
w.r.t.	with respect to
s.t.	such that
SISO	single-input single-output
MIMO	multi-input multi-output
QP	quadratic program
LP	linear program
SDP	semidefinite program
LMI	linear matrix inequality
LQR	linear quadratic regulator
LTI	linear time-invariant
LTV	linear time-varying
LDI	linear difference/differential inclusion
LPV	linear parameter varying
CLF	control Lyapunov function
i-ISS	incremental input to state stability
i-OSS	incremental output to state stability
i-IOSS	incremental input-output to state stability
FBI	Francis-Byrnes-Isidori
BINF	Byrnes-Isidori normal form

Real numbers, complex numbers, and sets

$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$)	set of non-negative (positive) real numbers
$\mathbb{C}$	set of complex numbers
$\lvert z \rvert$	absolute value of $z \in \mathbb{C}$
$\mathbf{i}$	imaginary unit, i.e., $\mathbf{i}^2 = -1$
$\mathbb{I}$	set of integers
$\mathbb{I}_{\geq a}$	set of integers greater than or equal to $a \in \mathbb{R}$
$\lfloor a \rfloor$	largest integer smaller than or equal to $a \in \mathbb{R}$
$\lceil a \rceil$	smallest integer greater than or equal to $a \in \mathbb{R}$
$[a, b]$; $[a, b)$; $(a, b]$	intervals $\{x \in \mathbb{R} \mid a \leq x \leq b\}$, $\{x \in \mathbb{R} \mid a \leq x < b\}$, $\{x \in \mathbb{R} \mid a < x \leq b\}$ for $a, b \in \mathbb{R}$
$\mathbb{I}_{[a,b]}$	set of integers in the interval $[a, b]$
$\mathrm{mod}(a, b)$	modulo operator for $a, b \in \mathbb{I}_{\geq 0}$, with $\mathrm{mod}(a, b) = a$ for $a \in \mathbb{I}_{[0,b-1]}$ and $\mathrm{mod}(a, a) = 0$
$\mathrm{int}(\mathbb{X})$	interior of a set $\mathbb{X} \subset \mathbb{R}^n$
$\mathbb{B}_\epsilon(x)$	ball of radius ϵ centered around $x \in \mathbb{R}^n$, i.e., $\mathbb{B}_\epsilon(x) = \{z \in \mathbb{R}^n \mid \lVert x - z \rVert \leq \epsilon\}$
$\mathrm{vert}(\Theta)$	vertices $\theta_i \in \mathbb{R}^n$ of a polytopic set $\Theta \subset \mathbb{R}^n$

Comparison functions [146]

$\mathcal{K}$	A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function, i.e., $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing, and $\alpha(0) = 0$.
$\mathcal{K}_\infty$	A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}_\infty$ function, i.e., $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and α is radially unbounded, i.e., $\lim_{r\to\infty} \alpha(r) = \infty$.
$\mathcal{L}$	A function $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{L}$ function, i.e., $\delta \in \mathcal{L}$, if it is continuous, decreasing, and $\lim_{r\to\infty} \delta(r) = 0$.

Vectors, matrices, and norms

I_n	identity matrix with dimension $n \times n$
$0_{n\times m}$	$n \times m$ matrix of zeros
$A^\top$	transpose of matrix $A \in \mathbb{R}^{n\times m}$
$\|x\|$	euclidean norm of $x \in \mathbb{R}^n$
$\|x\|_Q^2$	weighted euclidean norm $x^\top Qx$ for a positive definite matrix $Q = Q^\top \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$
$\|x(\cdot)\|_Q^2$	For a sequence $x(\cdot) \in \mathbb{X}^T$, $T \in \mathbb{I}_{\geq 1}$, $\mathbb{X} \subseteq \mathbb{R}^n$ and a positive definite matrix $Q = Q^\top \in \mathbb{R}^{n\times n}$ we denote $\sum_{k=0}^{T-1} \|x(k)\|_Q^2 =: \|x(\cdot)\|_Q^2$.
$\lambda_{\max}(Q)$ ($\lambda_{\min}(Q)$)	maximum (minimum) eigenvalue of a symmetric matrix $Q = Q^\top$
$\lambda_{\max}(P/Q)$ ($\lambda_{\min}(P/Q)$)	maximum (minimum) generalized eigenvalue of symmetric matrices $P = P^\top$, $Q = Q^\top$, i.e., largest (smallest) constant $\lambda \in \mathbb{R}$ where $P - \lambda Q$ is singular
$\mathrm{diag}(X_1, \ldots, X_r)$	block-diagonal matrix with main diagonal blocks $X_1, \ldots, X_r$, $r \in \mathbb{I}_{\geq 1}$
$A \succ 0$ ($A \succeq 0$)	matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite), i.e., $A = A^\top$ and $x^\top Ax > 0$ ($x^\top Ax \geq 0$) for all $x \in \mathbb{R}^n$ with $x \neq 0$
$A \prec 0$ ($A \preceq 0$)	matrix $A \in \mathbb{R}^{n\times n}$ is negative definite (negative semidefinite), i.e., the matrix $-A$ is positive definite (positive semidefinite)
(x, y)	stacked vector $[x^\top, y^\top]^\top \in \mathbb{R}^{n+m}$ for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

Derivatives

$\frac{\partial F}{\partial x}\big\|_{(\bar{x},\bar{y})}$	For a continuously differentiable function $F(x, y)$, $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$, $\frac{\partial F}{\partial x}\big\|_{(\bar{x},\bar{y})} \in \mathbb{R}^{m\times n_1}$ denotes the matrix of partial derivatives of F w.r.t. x, evaluated at the point $(\bar{x}, \bar{y})$ (Jacobi matrix).
$\frac{\partial^2 F}{\partial x^2}\big\|_{(\bar{x},\bar{y})}$	For a twice continuously differentiable function $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $\frac{\partial^2 F}{\partial x^2}\big\|_{(\bar{x},\bar{y})} \in \mathbb{R}^{n_1\times n_1}$ denotes the matrix of second order partial derivatives of F w.r.t. x, evaluated at the point $(\bar{x}, \bar{y})$ (Hessian matrix).

Abstract

In this thesis, we develop and analyse model predictive control (MPC) schemes that are suitable for dynamic operation and nonlinear constrained systems. While most theoretical contributions for MPC consider the problem of stabilizing a given steady-state, in this thesis we are particularly interested in the challenges related to dynamic operation, e.g., trajectory tracking, online changing operation conditions, or output regulation, which are encountered in a wide range of practical control problems. In this thesis, we pursue two complementary MPC approaches for dynamic operation. First, we propose novel MPC design procedures, which are tailored to the challenges of dynamic operation. Second, we consider rather intuitive MPC formulations which provide advantages in terms of implementability and derive desirable guarantees for the closed loop. Altogether, both approaches lead to the development of novel MPC design procedures for dynamic operation and corresponding theoretical analysis yielding desirable closed-loop properties, which are the contributions of this thesis.

Design procedures

To be more precise, this thesis contains three novel MPC *design procedures* that are suitable for dynamic operation (see Chapter 3). Firstly, we present a *reference generic* offline design procedure for terminal ingredients (terminal cost/set), which is applicable for the stabilization of dynamic reference trajectories and nonlinear systems. This design procedure yields parametrized terminal ingredients that do *not* need to be recomputed if the desired mode of operation (trajectory/setpoint) changes online. Secondly, we use these terminal ingredients in combination with additional optimization variables in the form of *artificial reference trajectories* to develop a tracking MPC framework that is suitable for online changing dynamic target signals. In particular, this framework ensures recursive feasibility and constraint satisfaction irrespective of online changes in the user defined desired target signal and yields exponential stability of the optimal mode of operation. Furthermore, an extension is developed that allows for a partial time scale separation between trajectory tracking and trajectory planning. Thirdly, we propose an *economic* MPC framework for dynamic operation of nonlinear constrained systems with a general (*economic*) cost function. The resulting approach yields desirable closed-loop (economic) performance guarantees and can operate under online changing cost functions. The applicability and practical benefits of the proposed MPC designs are demonstrated with different nonlinear examples from literature.

Analysis methods

Regarding the *analysis* of MPC formulations without any tailored design procedure (see Chapter 4), we have the following three contributions. Firstly, we analyse the closed loop resulting from a trajectory *tracking* MPC implementation without terminal ingredients. Assuming that the system is *incrementally stabilizable*, we can ensure stability of any reachable dynamic reference trajectory using a sufficiently long prediction horizon. We also extend the existing theoretical results for MPC without terminal ingredients, deriving improved performance bounds. Secondly, we extend the setup to the *output regulation* problem, where an exosystem generates an output reference that needs to be tracked. For *minimum-phase* systems, we show that simply minimizing a quadratic output stage cost can successfully solve the output regulation problem and stabilize the regulator manifold. We also provide a design using an incremental input regularization that is suitable for *non-minimum-phase* systems. Thirdly, we investigate the case of *unreachable* dynamic reference trajectories. Using tools from economic MPC, we show that the MPC scheme (approximately) "finds" the optimal *reachable* trajectory in closed loop. All the results are illustrated through numerical examples.

The main goal of this thesis is to demonstrate that MPC is a control method that is suitable for dynamic operation of nonlinear constrained systems. To this end, we consider trajectory tracking, output regulation and general economic problems, introduce corresponding MPC designs and derive rigorous theoretical properties for the resulting closed loop such as stability, performance and constraint satisfaction.

Deutsche Kurzfassung

Analyse und Entwurf prädiktiver Regelungsmethoden zum dynamischen Betrieb nichtlinearer Systeme

Die prädiktive Regelung (Englisch: *model predictive control*, MPC) ist eine moderne, optimierungsbasierte Regelungsmethode, die für allgemeine nichtlineare Systeme geeignet ist und allgemeine Gütekriterien und Beschränkungen berücksichtigen kann. Aufgrund dieser Flexibilität ist MPC sowohl in industriellen Anwendungen als auch in der theoretischen Forschung weit verbreitet. Die vorliegende Arbeit befasst sich mit der Entwicklung und Analyse von MPC-Algorithmen zum dynamischen Systembetrieb nichtlinearer beschränkter Systeme. Wir befassen uns im Speziellen mit den Herausforderungen, die mit der Anwendung von MPC-Algorithmen zum dynamischen Betrieb nichtlinearer Systeme verbunden sind. Ein solcher dynamischer Systembetrieb, z. B. Stabilisierung einer dynamischen Referenztrajektorie, Ausgangsregelung (Englisch: *output regulation*) oder allgemeine Änderungen des Systembetrieb in Echtzeit, ist in einer Vielzahl von Anwendungen notwendig. Wir beschäftigen uns mit zwei sich ergänzenden MPC-Methoden zum dynamischen Systembetrieb. Einerseits entwickeln wir neue MPC-Algorithmen und Entwurfsverfahren, die an den dynamischen Systembetrieb angepasst sind. Andererseits betrachten wir einfache, intuitive MPC-Formulierungen ohne aufwendige Entwurfsverfahren und analysieren die Eigenschaften des geschlossenen Regelkreises. Der Beitrag der vorliegenden Arbeit ist die Entwicklung neuer MPC Entwurfsverfahren zum dynamischen Systembetrieb und eine entsprechende theoretische Analyse, welche die gewünschten Eigenschaften für den geschlossenen Regelkreis garantiert.

Entwurfsverfahren

Die vorliegende Arbeit beinhaltet drei neue MPC-Entwurfsverfahren zum dynamischen Systembetrieb (s. Kapitel 3): Zunächst stellen wir ein *referenz-generisches* Entwurfsverfahren vor zur Berechnung von parametrisierten Endkosten/Endbeschränkungen für nichtlineare Systeme, mit der die Stabilisierung dynamischer Referenztrajektorien erreicht wird. Die Besonderheit liegt darin, dass die entsprechenden Berechnungen *nicht* erneut ausgeführt werden müssen, wenn sich die gewünschte Systembetriebsart (Referenztrajektorie/Sollwert) während des Systembetriebes ändert. Zweitens werden *künstliche Referenztrajektorien* verwendet, um eine MPC-Methode zu entwickeln, welche für Änderungen der dynamischen Soll-Signalen in Echtzeit geeignet ist. Im Speziellen garantiert diese MPC-Methode rekursive Lösbarkeit und die Einhaltung von Beschränkungen unabhängig von den Soll-Signalen. Desweiteren sorgt die Methode dafür, dass die optimale Systembetriebsart exponentiell stabilisiert wird. Zusätzlich wird eine Methode entwickelt, um die Trajektorienplanung und deren Stabilisierung teilweise zu entkoppeln. Drittens entwickeln wir eine *ökonomische* MPC-Methode zur optimalen dynamischen Regelung von nichtlinearen beschränkten Systemen mit allgemeinem (*ökonomischem*) Gütekriterium. Der resultierende geschlossene Regelkreis erfüllt eine wünschenswerte (*ökonomische*) Regelgüte und ermöglicht einen dynamischen Systembetrieb trotz Änderungen in der Kostenfunktion in Echtzeit. Die Anwendbarkeit und die Vorteile der entwickelten MPC-Entwurfsverfahren werden anhand verschiedener nichtlinearer Beispiele aus der Literatur demonstriert.

Analysemethoden

Die vorliegende Arbeit beinhaltet die folgenden drei Beiträge zur *Analyse* von MPC-Algorithmen ohne aufwendige Entwurfsverfahren (s. Kapitel 4). Zuerst analysieren wir die Eigenschaften eines MPC-Algorithmus zur Stabilisierung von Referenztrajektorien ohne jegliches Offline-Design. Wir garantieren, dass jede dynamisch erreichbare Referenztrajektorie stabilisiert wird, falls das System *inkrementell stabilisierbar* ist und ein ausreichend langer Prädiktionshorizont verwendet wird. Zusätzlich werden die existierenden theoretischen Ergebnisse für MPC-Algorithmen ohne aufwendige Entwurfsverfahren erweitert. Als zweites betrachten wir das Ausgangsregelungsproblem (Englisch: *output regulation*), in dem ein Exo-System ein Ausgangssignal generiert, welches vom System stabilisiert werden soll. Wir zeigen, dass ein einfacher MPC-Algorithmus, der nur eine quadratische Kost auf den Ausgangs minimiert, das Ausgangsregelungsproblem erfolgreich lösen kann, falls das System *minimalphasig* ist. Für *nicht-minimalphasige* Systeme entwickeln wir ein MPC-Entwurfsverfahren mit inkrementeller Regularisierung des Steuereingangs. Zuletzt betrachten wir den Fall, in dem die gegebene dynamische Referenztrajektorie *nicht* erreichbar ist. Durch die Nutzung von *ökonomischen* MPC-Analysemethoden können wir zeigen, dass der MPC-Algorithmus die optimale *erreichbare* Referenztrajektorie "findet". Die theoretischen Ergebnisse werden durch verschiedene numerische Beispiele illustriert.

Das Ziel dieser Arbeit ist es zu zeigen, dass die prädiktive Regelung (MPC) eine regelungstechnische Methode ist, welche für den dynamischen Betrieb nichtlinearer beschränkter Systeme geeignet ist. Zu diesem Zweck betrachten wir die Stabilisierung dynamischer Referenztrajektorien, das Ausgangsregelungsproblem und die optimale *ökonomische* Regelung, entwerfen entsprechende MPC-Algorithmen und zeigen Eigenschaften des geschlossenen Regelkreises, wie z. B. Stabilität, Regelgüte und Einhaltung von Beschränkungen.

Chapter 1

Introduction

1.1 Motivation

Model predictive control (MPC) [126, 236] is a modern optimization-based control strategy, which is based on the repeated solution of finite horizon (open-loop) optimal control problems. This method generates feedback in closed-loop operation by implementing only the initial part of the optimized input trajectory and repeating the online optimization in the next sampling time in a receding horizon fashion. Since MPC only requires the solution to *finite-horizon open-loop* optimal control problems, it is less restricted by the curse of dimensionality associated with optimal control and dynamic programming [35]. The core advantages of MPC are: (i) the direct inclusion of hard state and input constraints; (ii) the applicability to general nonlinear systems; (iii) the consideration of general performance criteria. Due to these properties, MPC is a control method that is widely used in practice and actively researched in academia. There exist well-established theoretical results in MPC to stabilize a given steady-state [193], which is the standard setting considered in literature. In addition, much active research is devoted to the numerical challenges in MPC or robustness issues due to model mismatch, compare for example [79] and [33, 153].

However, in control applications, the challenges and control goals often go beyond the stabilization of a pre-determined steady-state. In this thesis, we are particularly interested in addressing the challenges associated with *dynamic operation*, which has received significantly less attention in the corresponding MPC literature. By *dynamic operation*, we encompass the following three theoretical challenges encountered in practice:

(i) Stationary operation is not desired:

There are numerous control applications in which the system should not be operated at steady-state and instead a trajectory/path should be followed or periodic operation is desired. In motion control problems, as for example encountered in robotics or autonomous driving, the system needs to move along a specified trajectory or path. Also, in many control problems, the system needs to be operated in a time-varying or periodic fashion due to external time-varying variables influencing the dynamics, constraints or cost functions. Examples for this problem setup include heating, ventilation, and air conditioning (HVAC) systems [235, 255] or water distribution networks [165, 282], which are all affected by the day/night cycle in temperature, price or demand. Lastly, there also exist time-invariant control problems that require dynamic/periodic operation in order to improve the performance compared to steady-state operation. Examples for such problems are power generation using kites [80], where dynamic movement utilizes the wind energy to produce electricity, and maximizing the yield of a continuous stirred-tank reactor (CSTR) [24].

(ii) Desired mode of operation changes online:
Changes in the desired mode of operation are frequently encountered in control applications. In robotics or autonomous driving, the desired reference/path is often generated online by a separate external unit (e.g., using artificial intelligence and visual feedback) and thus can change unpredictably, compare, e.g., [JK30]. In economically-oriented problems like HVAC [235, 255], kites [80] or power networks [JK27], the optimal mode of operation is dependent on external quantities, such as the weather forecast or the supply/demand, both of which tend to fluctuate in an unpredictable fashion, compare [102]. All of these control problems require a controller to change the mode of operation online based on external variables.

(iii) Desired mode of operation cannot be directly specified in terms of a given state and input setpoint/trajectory:
In many control applications, the control goal can *not* be easily specified in terms of a reachable state and input setpoint/trajectory. In motion control problems (e.g., robotics or autonomous driving), the desired control goal is typically specified in terms of a path/trajectory in the Cartesian space, which corresponds to some output of the system, compare, e.g., [94, JK30]. In addition, since this Cartesian reference is often provided by an external unit, it is often not physically realizable (due to the dynamics or constraints of the systems), compare, e.g., [151, JK30]. Furthermore, in process control, HVAC [235, 255], power networks [JK27] or water distribution networks [165, 282], the control goal can often be more naturally

expressed in terms of *economic criteria*, e.g., the production yield or energy consumption. In all of these control problems, a desired state trajectory/setpoint is at most indirectly specified, while the true control goal is more naturally expressed in terms of output variables or general economic cost functions [96].

Hence, in many practical control problems, the desired mode of operation is dynamic, subject to online changes, and not explicitly specified in terms of suitable state and input pairs. The goal of this thesis is to develop frameworks to design and analyse MPC schemes that guarantee closed-loop properties despite the presented challenges encountered in *dynamic operation* of nonlinear constrained systems. In the following two sections, we discuss the related literature and summarize the main contributions of this thesis.

1.2 Related work

In this section, we provide a brief overview over the research in MPC, with a specific focus on stability theory and existing approaches related to the challenges inherent to *dynamic operation*. A more detailed discussion regarding the contribution of this thesis in comparison to the related work can be found in the corresponding sections. The basic principle of MPC is as follows: At each time t, we measure the state $x(t)$ and determine an optimal input trajectory $u^*(\cdot|t)$ over a finite horizon N by solving an optimal control problem. In particular, we predict the corresponding future state trajectory $x^*(\cdot|t)$ and minimize a cost function $\mathcal{J}_N$ over the finite horizon window N. Then, we apply the first part of the resulting optimal input trajectory over the sampling interval and repeat this procedure at the next sampling time $t+1$ with a shifted horizon. A more in-depth introduction into MPC theory can be found in [126, 236], compare also the overview articles [190, 229].

Stability in MPC

Most standard MPC schemes consider a positive definite (typically quadratic) stage cost ℓ to stabilize a desired steady-state. Although such formulations are often successfully applied in practice, without additional modifications or assumptions, the closed loop resulting from the application of MPC is in general *not* stabilizing and may also not necessarily be (inherently) robust, compare [212, 231] and [118]. There is a long history and development of MPC design procedures, especially related to terminal ingredients (terminal cost and terminal set) that can be added to the optimal control problem to ensure stability of a desired steady-state. In particular, first stability results for nonlinear systems have been obtained using zero-terminal constraints [192], which have subsequently been relaxed to dual mode formulations [196]. These design procedures have been generalized to terminal costs V_f and terminal sets $\mathbb{X}_f$ [107], infinite-horizon or truncated predictions as a terminal cost [74, 175], quadratic terminal costs and ellipsoidal terminal sets [55], culminating in general conditions on terminal costs and sets, compare [193] for an overview. Results with implicitly-enforced terminal set constraints can be found in [162]. Under appropriate stabilizability conditions and with a sufficiently long prediction horizon N, stability in MPC can also be guaranteed without any terminal ingredients (terminal cost and terminal set) [7, 37, 86, 120, 123, 127, 237, 267], which is often termed *unconstrained* MPC (due to the lack of a terminal set constraint). There also exist design procedures that utilize relaxed conditions on the terminal ingredients in combination with a sufficiently long prediction horizon N, compare [119], [145] and [238, 239]. A discussion on the drawbacks and merits of using terminal ingredients in MPC can be found in [188], and this issue will also be briefly elaborated on in Sections 2.3 and 5.2. Lastly, there also exist alternative stabilizing design procedures using Lyapunov, passivity or contraction constraints [195, 216, 230] and various a posteriori analysis techniques [152, 168, 227].

In summary, although there exists a long history of research in stability properties of MPC, the analysis of MPC without terminal ingredients is typically limited to positive definite stage costs with a global reachability assumption (with the notable exceptions [119] and [37]) and the resulting bounds are sometimes conservative. In Section 4.1, we extend/improve the results in [37, 119, 175] by providing improved bounds for MPC without terminal ingredients based on: a) *local* stabilizability conditions; b) *detectable/observable* positive semidefinite stage costs; and c) *extended prediction horizons* with a (locally) stabilizing controller.

Trajectory tracking with MPC

In trajectory tracking, the problem of stabilizing a given steady-state is replaced by the stabilization of a time-varying (typically state and input) trajectory. The theoretical results for MPC with and without terminal ingredients for the stabilization of steady-states directly extend to this case, if suitably adjusted terminal ingredients can be constructed and modified reachability conditions hold, compare [236, Thm. 2.39] and [126, Ass. 6.30]. An important generalization of the trajectory tracking problem is path following, where only an (output) path is specified and the exact timing and velocity in which this path should be tracked is an additional degree of freedom. The theoretical results from trajectory tracking can be extended to this path-following setting by using an extended state and input space to formulate the problem, compare [91, 94]. Suitable design procedures for the terminal ingredients for asymptotically constant reference trajectories and periodic reference trajectories can be found in [91, 93] and [23], respectively. Results without terminal ingredients can be found in [194] and [95]. However, these results are not directly applicable to *output*[1] reference trajectories/paths, since MPC theory without terminal constraints typically relies on a positive definite stage cost, with the main exception being [119], compare also [126, Sec. 10.3].

In summary, the existing design procedures for terminal ingredients are limited to special classes of trajectories and the consideration of output trajectories in MPC without terminal ingredients is an open problem. In Section 3.1, we provide a novel design procedure for the terminal ingredients that is applicable to general time-varying reference trajectories and avoids repeated design procedures in case of changing reference trajectories. Furthermore, in Sections 4.1 and 4.2, we provide a framework to study the stabilization of *input-output* and *output* trajectories using MPC without terminal constraints, without requiring the construction of a corresponding state trajectory.

Changing mode of operation

Online changes in the steady-state to be stabilized are common in practice and hence there has been much research on MPC formulations for setpoint tracking.

Terminal ingredients

The standard design procedure for terminal ingredients [55, 193] is typically based on the linearization around the desired steady-state and hence requires an additional design step each time the desired setpoint changes. For linear systems, a polytopic terminal set can be constructed offline using the concept of *maximal output admissible sets* [114] in combination with an augmented state space model (cf. [58, 163]). Given a stabilizing feedback for different setpoints, an infinite-horizon terminal cost for tracking has been proposed in [176], similar to [74]. A fixed quadratic terminal cost for different setpoints has been suggested in [105] using the concept of pseudo linearization, which, however, turns out to be difficult to apply in practice. In [274, 275], the nonlinear system

[1] In the continuous-time setting, an output stage cost ℓ for *flat* outputs is 0-detectable and hence the integral cost is positive definite [95].

is locally described as an LTV/LDI system and thus a quadratic terminal cost V_f can be computed, which is valid for a local set of steady-states. By switching between different local LTV/LDI models from a discrete set and their corresponding quadratic terminal costs V_f, a piece-wise quadratic terminal cost V_f for the (typically one-dimensional) steady-state manifold can be computed.

In summary, the existing design methods for terminal ingredients in case of online changing operation are limited to setpoint tracking and both the parametrization and scalability issues may limit the practical application. In Section 3.1.3, we provide a design procedure for terminal ingredients using an LPV description, which results in a continuously differentiable terminal cost, is not restricted to low dimensional steady-state manifolds, and is even applicable to dynamic trajectories.

Recursive feasibility - Artificial references

In addition to the challenges related to the computation of suitable terminal ingredients, online changes in the reference setpoint to be stabilized can cause feasibility issues, even in the linear case. In [32], a predictive command governor is used to determine a (parametrized) virtual reference sequence to ensure constraint satisfaction of a given primal controller. In [58], the dual mode MPC formulation (cf. [196]) is extended to tracking by using invariant sets [114] and including a *feasibility recovery mode*, which adjusts the reference to ensure feasibility, analogous to a reference governor [113]. An extension to the nonlinear case can be found in [57], compare also [56]. Similarly, in [189] the reference setpoint is updated in an external loop based on a feasibility condition. For the special case of integrating processes, a joint MPC formulation that computes the stationary output/reference and the control input is proposed in [52]. A general formulation for setpoint tracking based on artificial references is presented in [161, 163]. The basic idea is to include an artificial reference setpoint r as an additional decision variable in the MPC and then use a stabilizing MPC formulation to track the artificial setpoint r in combination with an offset cost of the artificial reference point r relative to the desired reference. This tracking MPC formulation has been subsequently extended to zone tracking [100], modified terminal constraints [256], periodic/dynamic trajectories [149, 165, 166], robust formulations [159, 223, 296], nonlinear dynamics [164], non-convex steady-state manifolds [63, 64], positive semidefinite stage costs ℓ for input-output (data-driven) models [JK4], distributed formulations [2, 61, 104, 135], and (local) optimality guarantees [103]. Similar formulations are also considered in offset-free tracking MPC [201, 218], compare also [174]. Bounds on the tracking error under bounded variation of the reference are derived in [88] by modifying the design in [163], compare also [77, 199] for error bounds using a robust MPC design.

Overall, the consideration of dynamic/periodic operation has received little attention (with the notable expectations [149, 165, 166]) and the computational complexity associated with artificial periodic/dynamic trajectories may limit the practical applicability. In Section 3.2, we unify and extend the existing theoretical results using artificial *periodic* reference trajectories for *nonlinear* constrained systems. In addition, we overcome scalability issues that are inherent in this MPC formulation with artificial periodic reference trajectories by providing a partial time scale separation.

Economic costs in MPC

Traditionally, economic objectives are considered using a two-layer control architecture, where the economically optimal setpoint/trajectory is determined in the upper layer using a so-called *real-time optimization* (RTO) and the lower layer consists of a controller that stabilizes/tracks this setpoint/trajectory. Directly minimizing the economic cost in an economic MPC formulation [96] can improve (transient) performance compared to simply stabilizing the optimal steady-state, compare, e.g., [234]. Although economic MPC is a relatively recent research field, there exist well-established performance guarantees for economic MPC schemes with terminal ingredients [16, 19, 78, 167], including performance guarantees w.r.t. periodic operation [295]. Tracking formulations that are locally equivalent to an economic MPC formulation are presented in [76, 293, 294]. Performance bounds for economic MPC without terminal ingredients can be found in [122, 128, 129, 130, 132, 210], compare also [291] and [8] for modifications to reduce the length of the necessary prediction horizon N (in case the system is optimally operated at steady-state). These results are based on a *dissipativity* condition and *turnpike* arguments, compare [JK1, JK2, 70, 92, 98, JK25, 203, 207].

In Section 4.3, we study unreachable reference trajectories in the context of economic MPC without terminal ingredients and extend the analysis in [122, 132] to time-varying problems and relax the standard (global) controllability condition.

Changing cost functions in economic MPC

In order to cope with online changes in the economic cost function and hence the economically optimal mode of operation, additional modifications are needed. In case that only a finite set of transitions is possible, connecting orbits between the modes of operation can be precomputed and recursive feasibility and performance can be guaranteed using a dwell-time assumption [20]. At the expense of transient performance, a tracking MPC formulation based on artificial setpoints/trajectories can be used to stabilize the (economically) optimal setpoint/trajectory [165], compare also [133, 139]. Purely economic MPC formulations based on artificial reference setpoints can be found in [87, 102, 206, 208], compare also [68, 82, 101] for robust modifications. Similar economic formulations are also used in [150] for asymptotic consensus of distributed agents with conflicting economic objectives. To address dynamic/periodic operation, economic MPC schemes with periodicity constraints have been proposed in [138, 282], but performance guarantees w.r.t. an optimal periodic orbit can only be established under very restrictive conditions. For linear systems, performance guarantees w.r.t. periodic operation are established in [43] based on strong duality, additional constraint tightening, and a sufficiently large positive definite offset cost.

In summary, the literature does *not* address MPC formulations that are: suitable for nonlinear systems and online changing modes of operation; consider a purely economic objective; and provide economic performance guarantees compared to optimal dynamic/periodic operation. In Section 3.3, we present an economic MPC formulation for nonlinear systems using artificial periodic reference trajectories and provide closed-loop performance guarantees relative to periodic optimal operation.

1.3 Contribution and outline of this thesis

In the following, we detail the outline of this thesis and clarify the contributions.

Chapter 2: Preliminaries - Stability in MPC

In this chapter, we provide a basic introduction to MPC. In particular, we briefly summarize the basic stability results for MPC with and without terminal ingredients, which forms the basis of this thesis. In addition, we provide a discussion regarding the respective merits of these two complementary MPC designs.

Chapter 3: Novel design procedures for MPC schemes with dynamic operation

In this chapter, we present a framework to design MPC schemes for dynamic operation of nonlinear constrained systems. In particular, we tackle the various challenges associated with dynamic operation by providing MPC formulations using artificial periodic reference trajectories for tracking of output trajectories and minimization of general economic cost functions, respectively.

In Section 3.1, we consider the problem of designing a trajectory tracking MPC with terminal ingredients for reachable dynamic state and input trajectories. In order to avoid repeated cumbersome design procedures due to changes in the reference trajectory, we propose a *reference generic* offline computation of the terminal ingredients for a set of reachable reference trajectories. The proposed approach considers the linearization of the nonlinear system around all possible points in the constraint set and describes the linearized system as a quasi-LPV system, where the reference r is the external parameter. Based on this description and a suitable parametrization of the terminal ingredients, we derive a design procedure in terms of LMIs. The proposed offline design needs to be executed only *once*, resulting in parametrized terminal ingredients that can also be directly used if the reference trajectory changes unpredictably.

In Section 3.2, we employ an MPC approach based on artificial reference trajectories in order to address the issues of recursive feasibility under online changing references and unreachable output references. We extend and unify the existing theoretical results for tracking MPC with artificial periodic references to consider: nonlinear systems, periodic (possibly unreachable) output target signals, general terminal ingredients (both terminal equality constraints and terminal cost/set), and ensure exponential stability of the optimal reachable periodic trajectory. In MPC approaches with artificial references, a reference constraint is typically specified offline, which determines the region of operation and limits the size of the terminal set and may reduce the performance. We present a method to automate this trade-off between region of operation and performance by introducing the terminal set size as an additional optimization variable in the MPC. For problems with a large period length, the joint optimization of the artificial reference trajectory and stabilizing MPC may not be real-time implementable for computational reasons. To circumvent this problem, we present a *partially decoupled* formulation, where a standard tracking MPC and a periodic trajectory planning problem are solved in parallel on different time scales, while retaining the feasibility and convergence properties of the original formulation.

In Section 3.3, we study the more general problem of *economic* optimal control. In particular, we consider the case where the economic stage cost depends on external variables that can unpredictably change online. To ensure recursive feasibility despite these online changes, we again consider an MPC formulation using artificial periodic reference trajectories. Instead of considering a tracking formulation, we propose a fully economic formulation to improve the performance. We demonstrate that naive extensions of existing economic MPC formulations to the periodic setup do *not* necessarily yield the desired performance guarantees. Instead, we suitably modify the MPC formulation and guarantee that (on average) the closed loop economic performance is no worse than the performance at a locally optimal periodic orbit. We also extend the design procedures for terminal ingredients to economic costs and dynamic/periodic reference trajectories.

In Section 3.4, we demonstrate the applicability of the proposed MPC design methods and show performance improvement compared to state of the art approaches. We first compare the performance of the different MPC formulations using a setpoint tracking example with a CSTR. We demonstrate the additional performance improvements of using suitable terminal ingredients, online optimized terminal set size, and direct *economic* formulations. Then, we showcase the applicability to periodic output regulation (including unreachable target signals) with a nonlinear ball and plate system. Finally, we demonstrate economic performance improvement using dynamic operation with an HVAC and a CSTR example.

In summary, the main contributions of this chapter are the following:

- We develop a *reference generic* offline computation for the terminal ingredients.
- We extend and unify existing trajectory tracking MPC formulations by using an artificial periodic reference trajectory providing a novel proof, introducing online optimization of the terminal set size, and presenting a partially decoupled reference planner to ensure applicability to long period lengths.
- We propose a novel economic MPC formulation using artificial periodic reference trajectories and establish corresponding performance guarantees.
- We demonstrate the performance benefits of the proposed designs using numerical examples from literature.

The results of Chapter 3 have been previously presented in [JK15, JK16, JK22, JK26].

Chapter 4: Analysis of MPC schemes for dynamic operation without offline design

In this chapter, we present a framework to analyse trajectory tracking MPC formulations without terminal ingredients. Instead of providing an intricate design procedure, we consider simple and intuitive MPC formulations without any offline design or terminal ingredients, and provide intrinsic system theoretic properties that guarantee desired closed-loop performance for trajectory tracking and output regulation, also in the case of unreachable reference trajectories.

In Section 4.1, we study a simple trajectory tracking MPC for reachable reference trajectories, which requires no offline design. Given a (local) incremental stabilizability condition, we ensure exponential stability with a user-specified region of attraction, given a sufficiently large prediction horizon N. The presented results also improve existing results for MPC without terminal constraints to yield less conservative performance bounds based on a *local* reachability assumption and sublevel set arguments. The corresponding theoretical derivations are also extended to positive semidefinite (input-output) stage costs using a *detectability* condition (i-IOSS) and improved bounds are derived using a stronger *observability* condition (ν-step i-OSS). Furthermore, guaranteed stability with *significantly shorter prediction horizons* is enabled by using an *extended prediction horizon* with some known (locally) stabilizing control law.

In Section 4.2, we extend the problem setup to output regulation. In the output regulation problem, only an *output* reference trajectory is specified, which is generated by an exosystem. For minimum-phase systems, we show that simply minimizing quadratic output stage cost can guarantee exponential stability of the regulator manifold. This result is particularly appealing, since, in contrast to classical control results, we do *not* need to solve the Francis-Byrnes-Isidori (FBI) equations. We additionally provide theoretical results for non-minimum-phase systems by using an (incremental) input regularization.

In Section 4.3, we study the properties of the trajectory tracking MPC without terminal ingredients in case the reference trajectory is not reachable. Using tools from economic MPC and suitable uniqueness conditions, we ensure that the MPC scheme (practically) stabilizes the reachable trajectory which has the minimal distance to the desired (unreachable) trajectory. We also extend the existing theory for *economic* MPC without terminal constraints to consider only *local* stabilizability/controllability conditions using sublevel set arguments.

In Section 4.4, we consider the special case of linear system. We show that the posed conditions coincide with classical conditions considered in the output regulation literature and prove that the considered dissipation-based characterization of the non-resonance condition is equivalent to the classical rank-based condition. In addition, we provide less conservative bounds, especially for the MPC formulation with extended prediction horizon.

In Section 4.5, we provide numerical examples to illustrate the presented theoretical results. We consider a simple linear example to compare the provided performance bounds. We demonstrate the applicability to trajectory tracking with a nonlinear CSTR example. Then, we show the practicality of the output regulation MPC formulation using offset-free tracking of a nonlinear cement milling circuit under noisy error feedback. Finally, we study the case of unreachable reference trajectories using the example of an asynchronous motor.

In summary, the main contributions of this chapter are the following:

- We analyse a simple trajectory tracking MPC formulation and establish exponential stability for reachable reference trajectories using an incremental stabilizability assumption.
- We extend this theory to output regulation and provide sufficient conditions to ensure stability of the regulator manifold.
- We studied the case of unreachable reference trajectories and guarantee (practical) stability of the optimal reachable trajectory.
- We extend the existing theory for MPC without terminal ingredients in multiple directions.
- We illustrate the applicability of the theoretical results using numerical examples.

The results of Chapter 4 have been previously presented in [JK10, JK19, JK21, JK23, JK24].

Chapter 5: Conclusions

In this chapter, we summarize the contributions of this thesis, contrast the two complementary MPC frameworks, and provide some directions for future research.

Appendices

Appendices A, B and C contain supplementary material regarding: suboptimality estimates in MPC with terminal ingredients; extending the reference generic design of the terminal ingredients to more general tracking stage costs; and incremental stability.

Chapter 2

Preliminaries - Stability in MPC

In this chapter, we recapitulate the basic stability results for MPC with and without terminal ingredients and briefly discuss their respective merits.

2.1 Stabilizing MPC with terminal ingredients

In the following, we briefly summarize the standard theoretical results for stabilizing MPC with terminal ingredients, analogous to the general results that can, for example, be found in [236] or [126, Chap. 5].

We consider a nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)), \quad x(0) = x_0,$$

with the state $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$, the control input $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, the dynamics $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, the initial condition $x_0 \in \mathbb{X}$, and the time step $t \in \mathbb{I}_{\geq 0}$. We consider general pointwise-in-time constraints

$$(x(t), u(t)) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}, \quad t \in \mathbb{I}_{\geq 0}. \tag{2.1}$$

The control goal is to stabilize a feasible setpoint $(x_s, u_s) \in \mathbb{Z}$, $f(x_s, u_s) = x_s$, while satisfying the constraints (2.1).

At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t)$, the MPC control law is determined based on the following optimization problem:

Problem 2.1.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_N(x(\cdot|t), u(\cdot|t)) \tag{2.2a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{2.2b}$$
$$x(0|t) = x(t), \tag{2.2c}$$
$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{2.2d}$$
$$x(N|t) \in \mathbb{X}_{\mathrm{f}}, \tag{2.2e}$$

where

$$\mathcal{J}_N(x(\cdot|t), u(\cdot|t)) = \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_{\mathrm{f}}(x(N|t)). \tag{2.2f}$$

In this problem, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the stage cost, $V_{\mathrm{f}} : \mathbb{X}_{\mathrm{f}} \to \mathbb{R}$ is the terminal cost, $\mathbb{X}_{\mathrm{f}} \subseteq \mathbb{X}$ is the terminal set, $N \in \mathbb{I}_{\geq 1}$ is the prediction horizon, $u(\cdot|t) \in \mathbb{U}^N$ is the predicted input trajectory and $x(\cdot|t) \in \mathbb{X}^{N+1}$ is the corresponding predicted state trajectory. We assume that ℓ, V_{f} and f are continuous, and the sets $\mathbb{Z}$ and $\mathbb{X}_{\mathrm{f}}$ are compact, which guarantees that Problem 2.1 has a minimizer, assuming a feasible solution exists, compare [236, Prop. 2.4]. We denote the[1] minimizer of Problem 2.1 by $u^*(\cdot|t)$ and the corresponding state trajectory by $x^*(\cdot|t)$. The value function is defined as $V_N(x(t)) := \mathcal{J}_N(x^*(\cdot|t), u^*(\cdot|t))$. The receding horizon control law is defined by the following algorithm.

Algorithm 2.2. *(Stabilizing MPC Algorithm with terminal ingredients)*
Offline: Specify the constraint set $\mathbb{Z}$, the stage cost ℓ, and the prediction horizon N. Design a suitable terminal cost V_{f} and a terminal set $\mathbb{X}_{\mathrm{f}}$.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, solve Problem 2.1, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t). \tag{2.3}$$

In the following, we present standard conditions that can be used to ensure asymptotic stability of the steady-state x_{s}.

Assumption 2.3. *(Stabilizing stage cost) There exist functions $\underline{\alpha}_\ell$, $\overline{\alpha}_\ell$, $\alpha_{\mathrm{f}} \in \mathcal{K}_\infty$ such that $\underline{\alpha}_\ell(\|x - x_{\mathrm{s}}\|) \leq \ell_{\min}(x) \leq \overline{\alpha}_\ell(\|x - x_{\mathrm{s}}\|)$ for all $(x, u) \in \mathbb{Z}$, $V_{\mathrm{f}}(x) \leq \alpha_{\mathrm{f}}(\|x - x_{\mathrm{s}}\|)$ for all $x \in \mathbb{X}_{\mathrm{f}}$, and $\ell(x_{\mathrm{s}}, u_{\mathrm{s}}) = 0$, $V_{\mathrm{f}}(x_{\mathrm{s}}) = 0$, with $\ell_{\min}(x) := \inf_{u \in \mathbb{U}} \ell(x, u)$.*

[1] In case the minimizer is not unique, one minimizer can be selected.

Assumption 2.4. *(Terminal ingredients) There exists a terminal control law* $k_{\mathrm{f}} : \mathbb{X}_{\mathrm{f}} \to \mathbb{U}$ *such that the following conditions hold for all* $x \in \mathbb{X}_{\mathrm{f}}$*:*

(i) $(x, k_{\mathrm{f}}(x)) \in \mathbb{Z}$,

(ii) $f(x, k_{\mathrm{f}}(x)) \in \mathbb{X}_{\mathrm{f}}$,

(iii) $V_{\mathrm{f}}(f(x, k_{\mathrm{f}}(x))) - V_{\mathrm{f}}(x) \leq -\ell(x, k_{\mathrm{f}}(x)) + \ell(x_{\mathrm{s}}, u_{\mathrm{s}})$.

Assumption 2.5. *(Weak controllability) There exists a function* $\alpha_{\mathrm{V}} \in \mathcal{K}_\infty$ *such that for any state* $x(t) \in \mathbb{X}$ *such that Problem 2.1 is feasible, the following bound holds:* $V_N(x(t)) \leq \alpha_{\mathrm{V}}(\|x(t) - x_{\mathrm{s}}\|)$.

Conditions (i)–(ii) of Assumption 2.4 ensure that the terminal set $\mathbb{X}_{\mathrm{f}}$ is positively invariant under the terminal control law k_{f} and the resulting state and input satisfy the constraints. Assumption 2.3 ensures that the stage cost ℓ is positive definite w.r.t. the steady-state x_{s} and thus Assumption 2.4 (iii) ensures that V_{f} is a (local) control Lyapunov function (CLF). Assumption 2.5 is a technical condition that follows directly from Assumptions 2.3–2.4 if $x_{\mathrm{s}} \in \mathrm{int}(\mathbb{X}_{\mathrm{f}})$.

The following theorem summarizes the resulting closed-loop properties.

Theorem 2.6. *Let Assumptions 2.3–2.5 hold. If the initial condition* x_0 *is such that Problem 2.1 is feasible at* $t = 0$*, then the closed-loop system* (2.3) *resulting from Algorithm 2.2 satisfies the constraints* (2.1)*, Problem 2.1 is feasible for all* $t \in \mathbb{I}_{\geq 0}$*, and* x_{s} *is asymptotically stable. Furthermore, the following performance bound holds for the closed loop:*

$$\mathcal{J}_\infty^{\mathrm{cl}}(x_0) := \sum_{t=0}^{\infty} \ell(x(t), u(t)) \leq V_N(x_0).$$

This result with a corresponding proof can, for example, be found in [236, Thm. 2.19] and similarly in [126, Thm. 5.5]. Similar continuous-time results can be found in [55, 107]. We point out that this stability result does not necessarily require that the global minimizer of Problem 2.1 is found at each time $t \in \mathbb{I}_{\geq 0}$. Instead, the same guarantees hold for suboptimal solutions with a suitable warm-start, compare [253].

The simplest design to satisfy Assumption 2.4 is a terminal equality constraint ($\mathbb{X}_{\mathrm{f}} = \{x_{\mathrm{s}}\}$, $k_{\mathrm{f}} = u_{\mathrm{s}}$, $V_{\mathrm{f}} = 0$, TEC) [192], which directly satisfies Assumption 2.4 and requires some technical (local) controllability property to ensure satisfaction of Assumption 2.5. A less conservative design procedure has been derived in [55] for the continuous-time case, which is briefly recapitulated analogous to [236, Sec. 2.5.5].

Suppose that f is twice-continuously differentiable and the stage cost is quadratic, i.e., $\ell(x,u) = \|x - x_s\|_Q^2 + \|u - u_s\|_R^2$. Assume further that $(x_s, u_s) \in \text{int}(\mathbb{Z})$ and the linearized dynamics around (x_s, u_s) are given by $x(t+1) = Ax(t) + Bu(t)$, with (A, B) stabilizable. Then, we can compute the discrete-time linear quadratic regulator (LQR) for the system (A, B) and cost function $(Q + \epsilon I_n, R)$ with some $\epsilon > 0$, resulting in a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a stabilizing feedback matrix $K \in \mathbb{R}^{m \times n}$. There exists a small enough constant $\alpha > 0$ such that $V_f(x) = \|x - x_s\|_P^2$, $k_f(x) = u_s + K(x - x_s)$, $\mathbb{X}_f = \{x \in \mathbb{X} | \ V_f(x) \leq \alpha\}$ satisfy Assumption 2.4. Furthermore, Assumption 2.5 directly follows by using $x_s \in \text{int}(\mathbb{X}_f)$ and the fact that V_f is a local upper bound to the value function V_N. This design procedure uses the fact that due to the factor $\epsilon > 0$, the linearized dynamics strictly satisfy condition (iii) in Assumption 2.4 and the difference between the nonlinear system and the linearization can be bounded locally.

2.2 Stabilizing MPC without terminal ingredients

In the following, we briefly summarize the standard theoretical results for stabilizing MPC without terminal ingredients, analogous to the general results that can, for example, be found in [126, Chap. 6].

The setup and MPC algorithm are exactly as in Section 2.2, only the optimization problem is modified such that no terminal constraints or terminal cost are considered.

Problem 2.7.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_N(x(\cdot|t), u(\cdot|t)) \tag{2.4a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{2.4b}$$

$$x(0|t) = x(t), \tag{2.4c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{2.4d}$$

with

$$\mathcal{J}_N(x(\cdot|t), u(\cdot|t)) = \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)). \tag{2.4e}$$

The value function is defined as $V_N(x(t)) := \mathcal{J}_N(x^*(\cdot|t), u^*(\cdot|t))$, where $u^*(\cdot|t)$ is the minimizer and $x^*(\cdot|t)$ is the corresponding state trajectory. The following algorithm specifies the receding horizon control law.

Algorithm 2.8. *(Stabilizing MPC Algorithm without terminal ingredients)*
Offline: Specify the constraint set $\mathbb{Z}$, the stage cost ℓ, and the prediction horizon N.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, solve Problem 2.7, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t). \tag{2.5}$$

Instead of appropriately designing terminal ingredients, MPC schemes without terminal ingredients rely on suitable stabilizability conditions.

Assumption 2.9. *(Exponential cost controllability) There exist a constant $\gamma \geq 1$ such that for any $x \in \mathbb{X}$ and any prediction horizon $N \in \mathbb{I}_{\geq 1}$, Problem* (2.7) *is feasible and the value function satisfies $V_N(x) \leq \gamma \ell(x,u)$, $\forall (x,u) \in \mathbb{Z}$.*

Assumption 2.9 ensures that the constraint set is control invariant and that for any feasible state, there exists an open-loop input trajectory which asymptotically steers the system to the desired steady-state x_s.

The following theorem summarizes the corresponding closed-loop properties.

Theorem 2.10. *Let Assumptions 2.3 and 2.9 hold. Then, there exists a constant $N_0 > 0$ such that for any prediction horizon $N > N_0$, the closed-loop system* (2.5) *resulting from Algorithm 2.8 satisfies the constraints* (2.1) *and x_s is asymptotically stable. Furthermore, there exists a constant $\alpha_N \in (0,1]$ such that the following suboptimality estimate holds for the closed loop:*

$$\mathcal{J}^{\mathrm{cl}}_\infty(x_0) := \sum_{t=0}^{\infty} \ell(x(t), u(t)) \leq \frac{V_N(x_0)}{\alpha_N} \leq \frac{V_\infty(x_0)}{\alpha_N}.$$

This result and a corresponding proof can be found in [126, Thm. 6.20], compare also [123]. Similar continuous-time results can be found in [237].

2.3 Summary and discussion

In this chapter, we presented basic stability results for MPC schemes with and without terminal ingredients. In both cases, the stage cost ℓ is assumed to be positive definite w.r.t. the desired steady-state x_s. In MPC formulations with terminal ingredients, suitable offline design methods are required to satisfy Assumption 2.4, while MPC formulations without terminal ingredients instead rely on stabilizability conditions (Ass. 2.9) and a sufficiently long prediction horizon N. In this thesis, we consider MPC schemes with and without terminal ingredients, which are treated separately in Chapter 3 and Chapter 4, respectively. Although the corresponding MPC algorithms are quite similar, the methods used in the analysis and design are quite distinct. Since these two approaches are at the core of this thesis, we will summarizes some of the main difference and respective advantages in the following. A detailed discussion on this issue can also be found in [188] and [126, Sec. 7.4].

The first and probably most crucial difference is the fact that MPC schemes with terminal ingredients require an offline design step to determine suitable terminal ingredients. This step can often be time consuming for challenging applications or may result in conservatively small terminal sets $\mathbb{X}_f$. A second drawback of MPC schemes with terminal ingredients is the fact that the region of attraction can be severely reduced due to the terminal set constraint (2.2e). These two factors are some of the main reasons why MPC without terminal ingredients is often seen as an attractive alternative for nonlinear systems. The drawbacks of MPC without terminal ingredients are often more indirect. Given the *exponential cost controllability* condition (Ass. 2.9), it is possible to compute a sufficiently large prediction horizon N_0 such that closed-loop properties can be guaranteed. However, these estimates can often be very conservative and a short prediction horizon may result in a destabilizing MPC, compare for example [231]. This is in contrast to MPC schemes with terminal ingredients, which guarantee all desired closed-loop properties even for arbitrary short prediction horizons N, assuming initial feasibility of Problem 2.1.

We point out that a local version of the *exponential cost controllability* condition (Ass. 2.9) is strongly related to existence of a suitable local CLF satisfying the conditions in Assumption 2.4, compare [252]. However, an important difference is that the terminal cost V_f (Ass. 2.4) needs to have a simple analytical expression, while MPC schemes without terminal ingredients only require a constant $\gamma \geq 1$ over approximating the value function, which can be significantly easier to obtain in practice.

Another point of difference is the resulting closed-loop performance bound. In particular, MPC without terminal ingredients provides suboptimality estimates $\alpha_N \in (0, 1]$, that relate the closed-loop performance with the infinite-horizon optimal performance. In order to obtain similar guarantees for MPC with terminal constraints, the difference between the finite-horizon cost $V_N(x)$ and the infinite horizon cost $V_\infty(x)$ needs to be bounded, as as exemplified in [126, Thm. 5.22] and [131, Thm. 6.2/6.4] (assuming that the terminal cost locally approximates the infinite-horizon cost well). Given the simplified setting typically considered in MPC without terminal constraints, suboptimality estimates $\alpha_N \in (0,1]$ can also be directly computed for MPC with terminal ingredients. A corresponding proof can be found in Appendix A (compare also Prop. 4.37), which, to the best knowledge of the author, also constitutes a novel result.

The differences between MPC formulation with and without terminal ingredients will be revisited in Section 5.2, with a particular focus on the novel MPC formulations developed in this thesis for dynamic operation.

Chapter 3

Novel design procedures for MPC schemes with dynamic operation

In this chapter, we present a framework to design MPC schemes with terminal ingredients for dynamic operation. In particular, the proposed design methods and MPC formulations address the challenges associated with dynamic operation identified in Section 1.1: (i) non-stationary operation, (ii) online changes in the mode of operation, (iii) optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories.

In Section 3.1, *non-stationary operation* is considered with a trajectory tracking MPC for reachable dynamic trajectories. Furthermore, a novel *reference generic* offline design for the terminal ingredients is derived, which is mainly relevant for *online changes in the mode of operation*. In Section 3.2, *online changes in the mode of operation* are considered by using an MPC formulation with *artificial reference* trajectories. This formulation is subsequently further improved by introducing online optimized terminal ingredients. In addition, a *partially decoupled* tracking and trajectory planning MPC formulation is developed to allow for a partial time scale separation. In Section 3.3, we further extend this formulation to account for general *economic* control goals, thus considering the case that the *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories*. In Section 3.4, we demonstrate the practicality of the different proposed methods using numerous numerical examples. The results presented in this chapter are based on Köhler et al. [JK15, JK16, JK22, JK26].

3.1 Trajectory tracking MPC and reference generic offline computations

In this section, we consider the problem of tracking a given state and input trajectory using an MPC scheme with terminal ingredients. As shown in Section 2.1, terminal ingredients (terminal cost, terminal set, and terminal control law) can be useful to ensure desired closed-loop properties (recursive feasibility, stability and constraint satisfaction) for MPC schemes. The main contribution of this section is the development of a novel *reference generic* offline design procedure for the corresponding terminal ingredients (Sec. 3.1.3). This section is based on and taken in parts literally from [JK15][1] and [JK16][2].

3.1.1 Trajectory tracking MPC

In the following, we present a trajectory tracking MPC formulation using terminal ingredients and establish exponential stability of reachable reference trajectories. We consider the following nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)), \quad x(0) = x_0,$$

with the state $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$, the control input $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, the dynamics $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, the initial condition $x_0 \in \mathbb{X}$, and the time step $t \in \mathbb{I}_{\geq 0}$. We impose pointwise-in-time constraints on the state and input

$$(x(t), u(t)) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}, \quad t \in \mathbb{I}_{\geq 0}. \tag{3.1}$$

We assume that $\mathbb{Z}$ is compact and f is continuous. We consider the problem of stabilizing a state and input reference trajectory $r(t) := (x_\mathrm{r}(t), u_\mathrm{r}(t)) \in \mathbb{X} \times \mathbb{U} \subseteq \mathbb{R}^{n+m}$.

Assumption 3.1. *(Reachable reference trajectory) There exists a set* $\mathbb{Z}_\mathrm{r} \subseteq \mathrm{int}(\mathbb{Z})$ *such that the reference trajectory satisfies*

$$r(t) \in \mathbb{Z}_\mathrm{r}, \quad x_\mathrm{r}(t+1) = f(x_\mathrm{r}(t), u_\mathrm{r}(t)), \quad \forall t \in \mathbb{I}_{\geq 0}. \tag{3.2}$$

[1] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear model predictive control framework using reference generic terminal ingredients." In: *IEEE Trans. Automat. Control* 65.8 (2020). extended version: arXiv:1909.12765, pp. 3576–3583©2019 IEEE.

[2] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear tracking model predictive control scheme for unreachable dynamic target signals." In: *Automatica* 118 (2020). extended version: arXiv:1911.03304, p. 109030©2020 Elsevier Ltd.

This assumption characterizes the fact that the reference trajectory r is reachable, i.e., follows the dynamics f and lies (strictly) in the constraint set $\mathbb{Z}$. We use the following set to characterize the conditions (3.2):

$$(r,r^+)\in\mathcal{R}:=\{(r,r^+)=((x_\mathrm{r},u_\mathrm{r}),(x_\mathrm{r}^+,u_\mathrm{r}^+))\in\mathbb{Z}_\mathrm{r}\times\mathbb{Z}_\mathrm{r}\mid x_\mathrm{r}^+=f(x_\mathrm{r},u_\mathrm{r})\}.$$

The case of unreachable reference trajectories will be considered in Section 3.2 using artificial reference trajectories.

Remark 3.2. *(Reference constraint set) The set $\mathcal{R}$ is later used to characterize all feasible reference trajectories for which the terminal ingredients should guarantee certain properties. This set can be modified to incorporate additional incremental input constraints $\|u_\mathrm{r}(t+1)-u_\mathrm{r}(t)\|_\infty\leq\epsilon$ with some $\epsilon>0$, which can reduce the conservatism of the design of the terminal ingredients. Additional restrictions to the considered set of reference trajectories that could be considered include asymptotically constant trajectories [91, 93] or periodic trajectories [23]. Setpoints are included as a special case by choosing $\mathbb{Z}_\mathrm{r}$ based on the steady-state manifold and considering $\mathcal{R}=\{(r,r^+)\in\mathbb{Z}_\mathrm{r}\times\mathbb{Z}_\mathrm{r}\mid r=r^+\}$, compare also Remark 3.20.*

Define the tracking error $e_\mathrm{r}(t):=x(t)-x_\mathrm{r}(t)$. The control goal is to achieve (uniform) stability of the tracking error $e_\mathrm{r}=0$, while ensuring satisfaction of the constraints (3.1). For simplicity of exposition, we consider a quadratic reference tracking stage cost

$$\ell(x,u,r)=\|x-x_\mathrm{r}\|_Q^2+\|u-u_\mathrm{r}\|_R^2, \tag{3.3}$$

with positive definite weighting matrices $Q\in\mathbb{R}^{n\times n}$, $R\in\mathbb{R}^{m\times m}$.

We assume that the future reference trajectory is exactly known and denote the reference trajectory r over the prediction horizon $N\in\mathbb{I}_{\geq 1}$ by $r(\cdot|t)\subseteq\mathbb{Z}_\mathrm{r}^{N+1}$, with $r(k|t):=r(t+k)$ for all $t\in\mathbb{I}_{\geq 0}$, $k\in\mathbb{I}_{[0,N]}$. We consider a terminal set $\mathbb{X}_\mathrm{f}\subseteq\mathbb{X}\times\mathbb{Z}$ and a continuous terminal cost $V_\mathrm{f}:\mathbb{X}\times\mathbb{Z}\to\mathbb{R}_{\geq 0}$. At each time $t\in\mathbb{I}_{\geq 0}$, given the current state $x(t)$ and the reference trajectory $r(\cdot|t)\in\mathbb{Z}_\mathrm{r}^{N+1}$, the MPC control law is determined based on the following optimization problem:

Problem 3.3.

$$\underset{u(\cdot|t)}{\text{minimize}}\quad \mathcal{J}_N(x(\cdot|t),u(\cdot|t),r(\cdot|t)) \tag{3.4a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.4b}$$

$$x(0|t) = x(t), \tag{3.4c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.4d}$$

$$(x(N|t), r(N|t)) \in \mathbb{X}_\mathrm{f}, \tag{3.4e}$$

where

$$\mathcal{J}_N(x(\cdot|t), u(\cdot|t), r(\cdot|t)) := \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t), r(k|t)) + V_\mathrm{f}(x(N|t), r(N|t)). \tag{3.4f}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, and the value function $V_N(x(t), r(\cdot|t)) := \mathcal{J}_N(x^*(\cdot|t), u^*(\cdot|t), r^*(\cdot|t))$. The following algorithm summarizes the closed-loop operation.

Algorithm 3.4. *(Trajectory tracking MPC Algorithm)*
Offline: Specify the constraint set $\mathbb{Z}$, the stage cost ℓ (Q, R), the prediction horizon N, and design suitable terminal ingredients (V_f, $\mathbb{X}_\mathrm{f}$).
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, obtain the reference trajectory $r(\cdot|t)$, solve Problem 3.3, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t), \quad t \in \mathbb{I}_{\geq 0}. \tag{3.5}$$

We consider the following conditions for the terminal ingredients.

Assumption 3.5. *(Terminal ingredients) There exists a terminal control law $k_\mathrm{f} : \mathbb{X} \times \mathbb{Z} \to \mathbb{U}$ such that the following properties hold for any $(r, r^+) \in \mathcal{R}$ and any $(x, r) \in \mathbb{X}_\mathrm{f}$:*

$$V_\mathrm{f}(x^+, r^+) \leq V_\mathrm{f}(x, r) - \ell(x, k_\mathrm{f}, r), \tag{3.6a}$$

$$(x, k_\mathrm{f}(x, r)) \in \mathbb{Z}, \tag{3.6b}$$

$$(x^+, r^+) \in \mathbb{X}_\mathrm{f}, \tag{3.6c}$$

with $x^+ = f(x, k_\mathrm{f}(x, r))$.

Assumption 3.6. *(Local upper bound - value function) There exist constants* $c_{\mathrm{u}} \geq 1$, $\epsilon > 0$ *such that for any reference trajectory* $r(\cdot|t)$ *satisfying the conditions in Assumption 3.1, any* $x(t) \in \mathbb{X}$ *with* $\|x(t) - x_{\mathrm{r}}(t)\|_Q \leq \epsilon$*, Problem 3.3 is feasible and the value function satisfies*

$$V_N(x(t), r(\cdot|t)) \leq c_{\mathrm{u}} \|x(t) - x_{\mathrm{r}}(t)\|_Q^2. \tag{3.7}$$

Assumption 3.5 summarizes standard conditions on the terminal ingredients, which correspond to the conditions in Assumption 2.4 for $\mathcal{R} = \{(x_{\mathrm{s}}, u_{\mathrm{s}}), (x_{\mathrm{s}}, u_{\mathrm{s}})\}$. Assumption 3.6 ensures that Problem 3.3 is locally feasible and the value function admits a local quadratic upper bound. The design of suitable terminal ingredients satisfying Assumptions 3.5–3.6 is discussed in detail in the remainder of this section.

Remark 3.7. *(More general terminal conditions) The conditions in Assumption 3.5 could be relaxed in two directions at the expense of a more involved offline design and cumbersome notation. Firstly, the terminal cost* V_{f}*, the terminal control law* k_{f} *and the terminal set* $\mathbb{X}_{\mathrm{f}}$ *are parametrized by one point* $r \in \mathbb{Z}_{\mathrm{r}}$ *of the reference trajectory. Instead, the terminal ingredients could depend on the full future reference trajectory* $r(\cdot|t)$*, as for example considered in [91, 93] and [23] for asymptotically constant and periodic reference trajectories, respectively. Secondly, Assumption 3.1 ensures that the reference* $r(t)$ *is contained within a control invariant subset* $\mathbb{Z}_\infty \subseteq \mathbb{Z}_{\mathrm{r}}$*. Thus, Assumption 3.5 could be relaxed such that conditions* (3.6) *only need to be satisfied for points* $r \in \mathbb{Z}_\infty$*. The exact characterization of the set* $\mathbb{Z}_\infty$ *is, however, challenging and thus we consider the stricter conditions formulated in Assumption 3.5. Furthermore, in Assumption 3.6 the quadratic bounds could also be generalized using* $\mathcal{K}_\infty$*-functions (cf. Ass. 2.3).*

The following theorem summarizes the theoretical properties of the closed-loop system (3.5).

Theorem 3.8. *Let Assumptions 3.1, 3.5, and 3.6 hold. If the initial condition* x_0 *is such that Problem 3.3 is feasible at* $t = 0$*, then the closed-loop system* (3.5) *resulting from Algorithm 3.4 satisfies the constraints* (3.1)*, Problem 3.3 is feasible for all* $t \in \mathbb{I}_{\geq 0}$*, and* $e_{\mathrm{r}} = 0$ *is (uniformly) exponentially stable. Furthermore, the following performance bound holds for the closed loop:*

$$\mathcal{J}_\infty^{\mathrm{cl}}(x_0) := \sum_{t=0}^{\infty} \ell(x(t), u(t), r(t)) \leq V_N(x_0, r(\cdot|0)).$$

Proof. This theorem is a straightforward extension of standard MPC results in [236], compare also [93]. Given the optimal solution $u^*(\cdot|t)$, the candidate sequence

$$u(k|t+1) = \begin{cases} u^*(k+1|t) & k \in \mathbb{I}_{[0,N-2]} \\ k_\mathrm{f}(x^*(N|t), r(N|t)) & k = N-1 \end{cases},$$

is a feasible solution of Problem 3.3 and implies

$$V_N(x(t+1), r(\cdot|t+1)) \leq V_N(x(t), r(\cdot|t)) - \ell(x(t), u(t), r(t)). \tag{3.8}$$

Compactness of $\mathbb{Z}$ in combination with the local quadratic upper bound (3.7) implies

$$\|x(t) - x_\mathrm{r}(t)\|_Q^2 \leq V_N(x(t), r(\cdot|t)) \leq c_\mathrm{v}\|x(t) - x_\mathrm{r}(t)\|_Q^2,$$

for some $c_\mathrm{v} \geq c_\mathrm{u} \geq 1$, for all $x(t)$ such that Problem 3.3 is feasible, compare [236, Prop. 2.16]. Uniform exponential stability follows from standard Lyapunov arguments using the time-varying Lyapunov function V_N, compare [126, Thm. 2.22], [236, Thm. 2.32]. The performance bound follows by summing up Inequality (3.8) and using $V_N \geq 0$ with ℓ, $V_\mathrm{f} \geq 0$. ∎

This theorem ensures that the trajectory tracking MPC enjoys all the standard desirable properties. We point out that the result in Theorem 3.8 does not require local feasibility of Problem 3.3, and thus Assumption 3.6 could be relaxed to a weak-controllability condition, similar to Assumption 2.5. However, the considered stronger property also holds for the standard design procedures discussed in the remainder of this section and will be vital for the MPC formulations using artificial reference trajectories in Section 3.2.

We point out that Problem 3.3 can be modified using the constraint tightening in [JK17, JK29] to ensure robust reference tracking, compare [JK15, App. B] for the technical details. As a complementary result to Theorem 3.8, in Section 4.1 trajectory tracking without terminal ingredients is considered, which is particularly interesting in case Assumption 3.1 does not hold, compare Section 4.3.

3.1.2 Terminal ingredients

In the following, we discuss terminal ingredients satisfying Assumptions 3.5–3.6. In particular, we discuss terminal equality constraints and terminal sets based of a given local CLF.

First, we consider a simple terminal equality constraint (TEC), which is also called a zero-terminal constraint (ZTC) [192] in case $(x_\mathrm{r}, u_\mathrm{r}) = 0$.

Definition 3.9. *(Local incremental finite-time controllability) A system is said to be locally incrementally uniformly finite-time controllable on a set $\tilde{\mathbb{Z}} \subseteq \mathbb{X} \times \mathbb{U}$, if there exist constants $\nu \in \mathbb{I}_{\geq 1}$, $c_\mathrm{ctr} \geq 1$, $\epsilon_\mathrm{ctr} > 0$ such that for any trajectory $z(k+1) = f(z(k), v(k))$, $(z(k), v(k)) \in \tilde{\mathbb{Z}}$, $k \in \mathbb{I}_{\geq 0}$, $z(0) = z_0 \in \mathbb{X}$ and any initial condition $x_0 \in \mathbb{X}$ satisfying $\|x_0 - z_0\| \leq \epsilon_\mathrm{ctr}$, there exists an input sequence $u(\cdot) \in \mathbb{U}^\nu$ satisfying*

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0, \quad x(\nu) = z(\nu), \tag{3.9a}$$

$$\|x(k) - z(k)\|^2 + \|u(k) - v(k)\|^2 \leq c_\mathrm{ctr} \|x_0 - z_0\|^2, \quad k \in \mathbb{I}_{[0,\nu-1]}. \tag{3.9b}$$

This controllability condition is for example satisfied if the dynamics f is continuously differentiable and the linearization along any feasible trajectory is ν-step (uniformly) controllable. The following proposition shows that a simple terminal equality constraint satisfies all the desired properties, if this controllability condition holds.

Proposition 3.10. *Suppose that the system is locally incrementally uniformly finite-time controllable on the set $\mathbb{Z}_\mathrm{r}$ (Def. 3.9). Then, for any prediction horizon $N \geq \nu$, Assumptions 3.5–3.6 hold with $\mathbb{X}_\mathrm{f} = \{(x, r) \in \mathbb{X} \times \mathbb{Z}_\mathrm{r} \mid x = x_\mathrm{r}\}$, $k_\mathrm{f}(x, r) = u_\mathrm{r}$, $V_\mathrm{f}(x, r) = 0$.*

Proof. Satisfaction of Assumption 3.5 follows directly from the definition of the terminal equality constraint and the reachable reference trajectory (cf. Ass. 3.1). Consider Problem 3.3 at time t with a state $x(t) \in \mathbb{X}$ satisfying $\|x(t) - x_\mathrm{r}(t)\|_Q \leq \epsilon$, with a constant $\epsilon \in (0, \epsilon_\mathrm{ctr}\sqrt{\lambda_{\min}(Q)}]$ and a prediction horizon $N \in \mathbb{I}_{\geq \nu}$. Using controllability from Definition 3.9, there exists an input trajectory $u(\cdot|t) \in \mathbb{U}^\nu$ that drives the state $x(t)$ to $x_\mathrm{r}(\nu|t)$ in ν steps. We obtain a feasible candidate input sequence $u(\cdot|t) \in \mathbb{U}^N$ for Problem 3.3 by appending $u(k|t) = u_\mathrm{r}(k|t)$, $k \in \mathbb{I}_{[\nu,N-1]}$ $x(k|t) = x_\mathrm{r}(k|t)$, $k \in \mathbb{I}_{[\nu,N]}$. Inequality (3.9b) ensures

$$\begin{aligned} &\|x(k|t) - x_\mathrm{r}(k|t)\|_Q^2 + \|u(k|t) - u_\mathrm{r}(k|t)\|_R^2 \\ \leq & c_\mathrm{ctr} \frac{\max\{\lambda_{\max}(Q), \lambda_{\max}(R)\}}{\lambda_{\min}(Q)} \|x(t) - x_\mathrm{r}(t)\|_Q^2 \\ \leq & c_\mathrm{ctr} \frac{\max\{\lambda_{\max}(Q), \lambda_{\max}(R)\}}{\lambda_{\min}(Q)} \epsilon^2, \quad k \in \mathbb{I}_{[0,\nu-1]}. \end{aligned} \tag{3.10}$$

Given that $r(k|t) \in \mathbb{Z}_\mathrm{r} \subseteq \mathrm{int}(\mathbb{Z})$, there exists a small enough (uniform) constant $\epsilon \in (0, \epsilon_\mathrm{ctr}\sqrt{\lambda_{\min}(Q)}]$ such that Inequality (3.10) ensures $(x(k|t), u(k|t)) \in \mathbb{Z}$, $k \in \mathbb{I}_{[0,N-1]}$. Thus, the considered candidate input sequence $u(\cdot|t) \in \mathbb{U}^N$ satisfies the constraints in Problem 3.3 and thus Problem 3.3 is locally feasible. Finally, the local quadratic upper bound (3.7) follows directly from Inequality (3.10) with $c_{\mathrm{u}} := \nu \cdot c_{\mathrm{ctr}} \dfrac{\max\{\lambda_{\max}(Q), \lambda_{\max}(R)\}}{\lambda_{\min}(Q)} \geq 1$. ∎

While a terminal equality constraint is easy to design and implement, it can also be very conservative, yielding a small region of attraction and potentially unnecessarily aggressive closed-loop behaviour.

The standard alternative to using a terminal equality constraint involves the design of a local CLF.

Proposition 3.11. *Suppose there exists a constant $\alpha_1 > 0$ such that the terminal cost $V_{\mathrm{f}} : \mathbb{X} \times \mathbb{Z} \to \mathbb{R}_{\geq 0}$ and the terminal control law $k_{\mathrm{f}} : \mathbb{X} \times \mathbb{Z} \to \mathbb{U}$ satisfy Inequality (3.6a) for all $(x, r) \in \mathbb{X} \times \mathbb{Z}_{\mathrm{r}}$ satisfying $V_{\mathrm{f}}(x, r) \leq \alpha_1$. Assume further that V_{f} admits a quadratic upper bound, i.e., there exists a constant $c_{\mathrm{u}} \geq 1$ such that*

$$V_{\mathrm{f}}(x, r) \leq c_{\mathrm{u}} \|x - x_{\mathrm{r}}\|_Q^2, \quad \forall (x, r) \in \mathbb{X} \times \mathbb{Z}_{\mathrm{r}}. \tag{3.11}$$

Then, there exists a constant $\alpha \in (0, \alpha_1]$ such that Assumptions 3.5–3.6 hold with $\mathbb{X}_{\mathrm{f}} = \{(x, r) \in \mathbb{X} \times \mathbb{Z}_{\mathrm{r}} \mid V_{\mathrm{f}}(x, r) \leq \alpha\}$.

Proof. To show satisfaction of Assumption 3.5, we need to show that the positive invariance property (3.6c) and constraint satisfaction (3.6b) hold. First, note that Inequality (3.6a) and $\ell \geq 0$ ensure that the terminal set $\mathbb{X}_{\mathrm{f}}$ is positively invariant for $\alpha \leq \alpha_1$. Denote $\Delta x = x - x_{\mathrm{r}}$, $\Delta u = k_{\mathrm{f}}(x, r) - u_{\mathrm{r}}$. Inequality (3.6a) in combination with $V_{\mathrm{f}} \geq 0$ ensures that

$$\|\Delta x\|_Q^2 + \|\Delta u\|_R^2 = \ell(x, k_{\mathrm{f}}(x, r), r) \leq V_{\mathrm{f}}(x, r) \leq \alpha_1, \quad \forall (x, r) \in \mathbb{X}_{\mathrm{f}}.$$

Given that $\mathbb{Z}_{\mathrm{r}} \subseteq \mathrm{int}(\mathbb{Z})$, there exists a small enough constant $\epsilon_2 > 0$ such that $\mathbb{B}_{\epsilon_2}(r) \subseteq \mathbb{Z}$ for all $r \in \mathbb{Z}_{\mathrm{r}}$. Thus, Condition (3.6b) holds for $\alpha \leq \alpha_2$ with

$$\alpha_2 := \epsilon_2^2 \min\{\lambda_{\min}(Q), \lambda_{\min}(R)\} > 0. \tag{3.12}$$

Hence, choosing $\alpha := \min\{\alpha_1, \alpha_2\} > 0$ ensures satisfaction of Assumption 3.5. Assumption 3.5 ensures that for all $(x, r) \in \mathbb{X}_{\mathrm{f}}$, the terminal control law k_{f} is a feasible solution of Problem 3.3, which implies $V_N(x(t), r(\cdot|t)) \leq V_{\mathrm{f}}(x(t), r(t)) \leq c_{\mathrm{u}} \|x(t) - x_{\mathrm{r}}(t)\|_Q^2$, compare [236, Sec. 2.4]. Assumption 3.6 follows by noting that $\|x - x_{\mathrm{r}}\|_Q^2 \leq \epsilon^2 := \dfrac{\alpha}{c_{\mathrm{u}}}$ implies $(x, r) \in \mathbb{X}_{\mathrm{f}}$. ∎

Such an MPC design with a local CLF is also called *quasi-infinite horizon* (QINF) MPC, since the terminal cost is a local over-approximation of the infinite-horizon tail-cost, compare [55]. A joint offline design procedure for V_{f} and k_{f} based on a local Taylor approximation is discussed in the next section. In addition, a construction of V_{f} in terms of a finite-tail sequence with a stabilizing feedback k_{f} can be found in Proposition 4.34, similar to [175].

Discussion

One of the main benefits of using a terminal cost/set (QINF, Prop. 3.11) is that the desired properties also hold for an arbitrarily small prediction horizon N (assuming Problem 3.3 is initially feasible). Furthermore, the values of c_{u} and ϵ can be computed explicitly and are typically significantly less conservative than the ones associated with terminal equality constraints (TEC, Prop 3.10). This has a significant impact on the closed-loop performance, which is quantitatively investigated with a numerical example in Section 3.4.1. The main advantage of using a terminal equality constraint (Prop. 3.10) is the fact that no offline design is required.

3.1.3 Reference generic offline computations

In Proposition 3.11, we provided a design for the terminal ingredients, given a known local CLF V_{f}. In the following, we first review state of the art approaches to compute such a local CLF and terminal ingredients offline, especially for dynamic trajectory tracking. Then, we extend the existing methods by presenting a *reference generic* offline computation, which addresses the challenge *desired mode of operation changes online* (cf. Sec. 1.1 (ii)). In particular, in Lemma 3.12 we provide sufficient conditions for continuously parametrized terminal ingredients based on the Jacobian. Then, in Lemma 3.13 and Proposition 3.15 tractable semidefinite programs (SDPs) are derived to compute the corresponding parametrized terminal ingredients. The overall offline procedure is summarized in Algorithm 3.22. The extension of this offline design procedure to more general (non-quadratic) tracking stage costs and economic stage costs can be found in Appendix B and Section 3.3.5, respectively. Details regarding the continuous-time case can be found in [JK15, App. C].

Existing design procedure for terminal ingredients

For linear stabilizable systems, a terminal set and terminal cost can be computed based on the LQR and the maximal output admissible set [114]. For the purposes of stabilizing a given setpoint, a suitable design procedure for nonlinear systems with a stabilizable linearization has been provided in [55, 236], which is also discussed in Section 2.1.

In Section 1.1, we identified *non-stationary operation* as the first challenge regarding dynamic operation. In the offline design of terminal ingredients, this requires an extension of the design in [55, 236] to dynamic trajectories. For the case of dynamic trajectory tracking, the nonlinear system can be locally approximated as a linear time-varying (LTV) system. Using such a description, in [91, 93] a time-varying terminal cost $V_f(x,t)$ and terminal control law $k_f(x,t)$ are computed offline for asymptotically constant trajectories using the Riccati differential equation. Similarly, in [23] periodic trajectories are considered and periodically time-varying terminal ingredients $V_f(x,t)$, $k_f(x,t)$ are computed using linear matrix inequalities (LMIs). Although there already exist design methods for trajectory tracking, these methods are limited to special classes of trajectories (periodic or asymptotically constant) and the complexity of the offline design scales with the length of the reference trajectory $r(\cdot)$.

The second challenge for dynamic operation is that the *desired mode of operation changes online* (cf. Sec. 1.1, (ii)). Regarding the design of terminal ingredients, online changing reference trajectories or setpoints typically necessitate repeated re-design of the terminal ingredients. Thus, offline procedures independent of the reference are desired, which we refer to as *reference generic* offline computations. For the special case of setpoint tracking, this issue has received significant attention in literature. In [105], the issue of finding a setpoint independent quadratic terminal cost has been investigated based on the concept of pseudo linearization. While in principle very appealing, the computation of such a pseudo linearization for general nonlinear systems seems unpractical. In [176], a locally stabilizing controller is assumed and the terminal cost and constraints are defined implicitly based on the infinite horizon tail cost. The main drawback of this method is the implicit description of the terminal cost, which can significantly increase the online computational demand. Terminal ingredients based on finite tail sequences can be found in [175] and will also be discussed in Section 4.1.5. In [164], the feasible setpoints are partitioned into disjoint sets and for each such set a linear stabilizing controller and quadratic terminal cost are designed using the methods in [274, 275] based on a local LTV system description. This method is mainly limited to systems with a one dimensional steady-state manifold, due to the otherwise complex and difficult partitioning. In addition, the piece-wise definition can also lead to numerical difficulties, especially in combination with artificial reference trajectories (Sec. 3.2). Thus, there exist *reference generic* design methods for setpoint tracking, but the parametrization and scalability issues may limit the practical application. Furthermore, to the best knowledge of the author, similar methods for dynamic trajectories are not addressed in literature.

The last challenge identified in Section 1.1 is that the *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories* (cf. Sec. 1.1, (iii)). This corresponds to the consideration of a general stage cost ℓ, which is neither quadratic in (x, u) nor positive (semi-)definite. Such more general stage costs are studied in economic MPC [96]. A corresponding design for terminal ingredients for fixed steady-states can be found in [16]. In addition to the usual stabilizing quadratic component, the *economic* terminal cost $V_f(x)$ also contains a linear gradient correcting term. In [208, Rk. 8], for changing setpoints r, the computation of a continuously parametrized terminal cost $V_f(x, r)$ was proposed using the pole placement formula. However, to the best knowledge of the author, this approach cannot be directly translated into a simple optimization problem and hence has never been implemented in a numerical example. A (possibly conservative) positive definite terminal cost V_f based on a CLF has been proposed in [10, 11]. For the special case of output-tracking/path-following stage costs ℓ and asymptotically constant reference trajectories a design has been proposed in [91, 94].

Overall, design procedures for economic terminal costs V_f for dynamic trajectories and *reference generic* offline computations have not been addressed in literature.

We present a *reference generic* offline computation, which is applicable to general dynamic trajectories (not just asymptotically constant or periodic) and does not necessitate repeated offline computations under online changing operation conditions. The main idea is to parametrize the Jacobian of the nonlinear dynamics as a quasi-linear parameter-varying (LPV) system and then use gain-scheduling methods to compute parametrized terminal ingredients using LMIs [40]. The extension of this design procedure to more general output tracking stage costs and economic stage costs can be found in Appendix B and Section 3.3.5, respectively.

Sufficient conditions based on the Jacobian

Suppose the dynamics f are continuously differentiable and denote the Jacobian of f evaluated around an arbitrary point $(x_\mathrm{r}, u_\mathrm{r}) = r \in \mathbb{Z}_\mathrm{r}$ by

$$A(r) = \left[\frac{\partial f}{\partial x}\right]\Bigg|_{(x_\mathrm{r},u_\mathrm{r})}, \quad B(r) = \left[\frac{\partial f}{\partial u}\right]\Bigg|_{(x_\mathrm{r},u_\mathrm{r})}. \tag{3.13}$$

Lemma 3.12. *Suppose that f is twice continuously differentiable. Assume that there exist a continuously parametrized matrix $K : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{m\times n}$ and a continuously parametrized positive definite matrix $P : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{n\times n}$ such that for any $(r, r^+) \in \mathcal{R}$, the following matrix inequality is satisfied*

$$(A(r) + B(r)K(r))^\top P(r^+)(A(r) + B(r)K(r)) \preceq P(r) - (Q + K(r)^\top RK(r)) - \epsilon I_n, \tag{3.14}$$

with some $\epsilon > 0$. Then, there exists a constant $\alpha > 0$ such that $V_\mathrm{f}(x, r) = \|x - x_\mathrm{r}\|^2_{P(r)}$, $k_\mathrm{f}(x, r) = u_\mathrm{r} + K(r) \cdot (x - x_\mathrm{r})$, and $\mathbb{X}_\mathrm{f} = \{(x, r) \in \mathbb{X} \times \mathbb{Z}_\mathrm{r} \mid V_\mathrm{f}(x, r) \leq \alpha\}$ satisfy Assumptions 3.5–3.6.

Proof. The proof is very much in line with the result for setpoints in [55, 236]. Given Proposition 3.11, we only need to show that V_f locally satisfies (3.6a). The local quadratic upper bound (3.11) holds with $c_\mathrm{u} := \max_{r\in\mathbb{Z}_\mathrm{r}} \lambda_{\max}(P(r))$ using the fact that P is continuous and $\mathbb{Z}_\mathrm{r}$ is compact. In the following, we show that there exists a small enough constant $\alpha_1 > 0$ such that Inequality (3.6a) holds for all $V_\mathrm{f}(x, r) \leq \alpha_1$. Denote $\Delta x := x - x_\mathrm{r}$ and $\Delta u := k_\mathrm{f}(x, r) - u_\mathrm{r} = K(r)\Delta x$. Using a first order Taylor approximation at $r = (x_\mathrm{r}, u_\mathrm{r})$, we get

$$f(x, k_\mathrm{f}(x, r)) = f(x_\mathrm{r}, u_\mathrm{r}) + A(r)\Delta x + B(r)\Delta u + \Phi(\Delta x, r),$$

with the remainder term Φ. The terminal cost satisfies

$$\begin{aligned} V_\mathrm{f}(x^+, r^+) &= \|f(x, u) - f(x_\mathrm{r}, u_\mathrm{r})\|^2_{P(r^+)} = \|(A(r) + B(r)K(r))\Delta x + \Phi(\Delta x, r)\|^2_{P(r^+)} \\ \leq& \|(A(r) + B(r)K(r))\Delta x\|^2_{P(r^+)} + \|\Phi(\Delta x, r)\|^2_{P(r^+)} \\ &+ 2\|\Phi(\Delta x, r)\|_{P(r^+)}\|(A(r) + B(r)K(r))\Delta x\|_{P(r^+)} \\ \overset{(3.3),(3.14)}{\leq}& V_\mathrm{f}(x, r) - \epsilon\|\Delta x\|^2 - \ell(x, k_\mathrm{f}(x, r), r) + \|\Phi(\Delta x, r)\|^2_{P(r^+)} \\ &+ 2\|\Phi(\Delta x, r)\|_{P(r^+)}\|(A(r) + B(r)K(r))\Delta x\|_{P(r^+)}. \end{aligned} \tag{3.15}$$

Using the continuity of $P(r),\ K(r)$ and the compactness of the constraint set $\mathbb{Z}_\mathrm{r}$, there exist finite non-negative constants

$$k_\mathrm{u} := \max_{r\in\mathbb{Z}_\mathrm{r}} \|K(r)\| \geq 0, \quad c_\mathrm{l} := \min_{r\in\mathbb{Z}_\mathrm{r}} \lambda_{\min}(P(r)) \geq \lambda_{\min}(Q) > 0, \tag{3.16}$$
$$c_{\mathrm{u},2} := \max_{r\in\mathbb{Z}_\mathrm{r}} \lambda_{\max}(P(r) - (\epsilon I + Q + K(r)^\top R K(r))) \geq 0.$$

Suppose that for all $V_\mathrm{f}(x,r) \leq \alpha_1$, the remainder term Φ is locally Lipschitz[3] continuous in the first argument with a constant L_Φ satisfying

$$\|\Phi(\Delta x, r)\| \leq L_\Phi \|\Delta x\|, \quad L_\Phi := \sqrt{\frac{c_{\mathrm{u},2}+\epsilon}{c_\mathrm{u}}} - \sqrt{\frac{c_{\mathrm{u},2}}{c_\mathrm{u}}}. \tag{3.17}$$

Then, for all $V_\mathrm{f}(x,r) \leq \alpha_1$, we have

$$\begin{aligned} &\|\Phi(\Delta x, r)\|^2_{P(r^+)} + 2\|\Phi(\Delta x, r)\|_{P(r^+)} \|(A(r)+B(r)K(r))\Delta x\|_{P(r^+)} \\ \overset{(3.14),(3.16),(3.17)}{\leq} &\left(L_\Phi^2 c_\mathrm{u} + 2L_\Phi \sqrt{c_\mathrm{u}}\sqrt{c_{\mathrm{u},2}}\right)\|\Delta x\|^2 \\ = &\left(c_\mathrm{u}\left(L_\Phi + \sqrt{\frac{c_{\mathrm{u},2}}{c_\mathrm{u}}}\right)^2 - c_{\mathrm{u},2}\right)\|\Delta x\|^2 \overset{(3.17)}{\leq} \epsilon\|\Delta x\|^2, \end{aligned}$$

which, in combination with Inequality (3.15), implies Inequality (3.6a). Twice continuous differentiability of f in combination with compactness of $\mathbb{Z}$, $\mathbb{Z}_\mathrm{r}$ implies that there exists some constant $T > 0$ with

$$\|\Phi(\Delta x, r)\| \leq T\left(\|\Delta x\|^2 + \|\Delta u\|^2\right) \overset{(3.16)}{\leq} T(1+k_\mathrm{u}^2)\|\Delta x\|^2, \quad \forall r \in \mathbb{Z}_\mathrm{r}.$$

Thus, for all $V_\mathrm{f}(x,r) \leq \alpha_1$, we have $\|\Delta x\| \leq \sqrt{\frac{\alpha}{c_\mathrm{l}}}$ and the Lipschitz bound (3.17) holds by choosing

$$\alpha_1 := c_\mathrm{l}\left(\frac{L_\Phi}{T(1+k_\mathrm{u}^2)}\right)^2 > 0. \tag{3.18}$$

■

As a summary, given matrices $P,\ K$ satisfying (3.14), we can compute a local Lip-

[3] In line with existing procedures [55], we first derive a sufficient local Lipschitz bound L_Φ and then obtain a local region α_1 (3.18). Alternatively, it is possible to directly use the quadratic bound $\|\Phi(\Delta x, r)\| \leq c\|\Delta x\|^2$ and work with higher order terms to obtain α_1, compare [JK24, Prop. 1].

schitz bound (3.17), which in turn implies a maximal terminal set size α_1. Then, in Proposition 3.11 the constraint sets $\mathbb{Z}$ and $\mathbb{Z}_\mathrm{r}$ in combination with K, P imply an upper bound α_2 to ensure constraint satisfaction. Thus, Assumptions 3.5–3.6 hold for any $\alpha \leq \min\{\alpha_1, \alpha_2\}$. This result is an extension of the design in [55, 236] to arbitrary dynamic references. We point out that any terminal cost V_f satisfying Assumptions 3.5–3.6 also constitutes a (local) incremental CLF (cf. App. C) for a class of system trajectories, compare also the discussion in Section 4.1.4.

Design procedure based on a quasi-LPV presentation

Lemma 3.12 states that matrices satisfying Inequality (3.14) can directly be used to construct terminal ingredients satisfying Assumptions 3.5–3.6 with a suitable terminal set size α. In the following, we formulate computationally tractable optimization problems to compute parametrized matrices that satisfy the conditions in Lemma 3.12. The following lemma transforms the conditions in Inequality (3.14) to equivalent conditions that are linear in the arguments.

Lemma 3.13. *Suppose there exist continuously parametrized matrices $X : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{n\times n}$, $Y : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{m\times n}$, and $X_{\min} \in \mathbb{R}^{n\times n}$ that satisfy the following constraints*

$$\min_{X,\,Y,\,X_{\min}} \; -\log\det X_{\min} \tag{3.19a}$$

$$\text{s.t.} \begin{pmatrix} X(r) & (A(r)X(r)+B(r)Y(r))^\top & (Q+\epsilon I_n)^{1/2}X(r) & (R^{1/2}Y(r))^\top \\ * & X(r^+) & 0 & 0 \\ * & * & I_n & 0 \\ * & * & * & I_m \end{pmatrix} \succeq 0, \tag{3.19b}$$

$$X_{\min} \preceq X(r), \tag{3.19c}$$

$$\forall (r, r^+) \in \mathcal{R}. \tag{3.19d}$$

Then, $P = X^{-1}$, $K = YP$ satisfy (3.14) *for all $(r, r^+) \in \mathcal{R}$.*

Proof. The proof uses standard LMI techniques, compare [40]. Defining $X = P^{-1}$ and multiplying (3.14) from left and right with X yields

$$X(r) - \begin{pmatrix} A(r)X(r)+B(r)Y(r) \\ (Q+\epsilon I_n)^{1/2}X(r) \\ R^{1/2}Y(r) \end{pmatrix}^\top \begin{pmatrix} P(r^+) & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{pmatrix} \begin{pmatrix} A(r)X(r)+B(r)Y(r) \\ (Q+\epsilon I_n)^{1/2}X(r) \\ R^{1/2}Y(r) \end{pmatrix} \succeq 0.$$

Using the Schur complement, this reduces to (3.19b), which is linear in the matrices X, Y. ■

The optimization problem (3.19) is convex, linear in X, Y and minimizes the worst-case terminal cost ($P(r) \preceq X_{\min}^{-1}$ $\forall r \in \mathbb{Z}_{\mathrm{r}}$). So far, the result is only conceptual, since the optimization problem (3.19) is an infinite programming problem (infinite dimensional optimization variables with infinite dimensional constraints). In particular, we need a finite parametrization of X, Y and the infinite constraints need to be converted into a finite set of sufficient constraints.

We approach this problem from the perspective of quasi-LPV systems and gain scheduling [243]. First, we write the Jacobian (3.13) as

$$A(r) = A_0 + \sum_{j=1}^{p} \theta_j(r) A_j, \; B(r) = B_0 + \sum_{j=1}^{p} \theta_j(r) B_j, \tag{3.20}$$

with some nonlinear continuously differentiable parameters $\theta = (\theta_1, \ldots, \theta_p) : \mathbb{Z}_{\mathrm{r}} \to \mathbb{R}^p$ and constant matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $i \in \mathbb{I}_{[0,p]}$. This can always be achieved with $p \leq n(n+m)$. We impose the same structure on the optimization variables with

$$X(r) = X_0 + \sum_{j=1}^{p} \theta_j(r) X_j, \; Y(r) = Y_0 + \sum_{j=1}^{p} \theta_j(r) Y_j, \tag{3.21}$$

and constant matrices $X_i \in \mathbb{R}^{n \times n}$, $Y_i \in \mathbb{R}^{m \times n}$, $i \in \mathbb{I}_{[0,p]}$. Using the parametrization (3.20)-(3.21), the optimization problem (3.19) contains only a finite number of optimization variables, but still needs to be verified for all $(r, r^+) \in \mathcal{R}$.

Remark 3.14. *(Input affine systems) For input affine systems of the form $f(x,u) = f_{\mathrm{x}}(x) + Bu$, the Jacobian* (3.20) *and correspondingly the parameters θ_i only depend on x_{r}. Thus, the resulting terminal ingredients are solely parametrized by the state x_{r}.*

Finite dimensional SDP

In order to convert the infinite dimensional constraints in (3.19) to a set of sufficient LMIs, we match the constraint sets $\mathcal{R}$ on the reference r to polytopic constraints on the parameters θ. We need a polytopic set $\overline{\Theta} \subseteq \mathbb{R}^p \times \mathbb{R}^p$ that satisfies

$$(\theta(r), \theta(r^+)) \in \overline{\Theta}, \quad \forall (r, r^+) \in \mathcal{R}. \tag{3.22}$$

A particularly simple structure that satisfies this condition is the following joint polytopic constraint set

$$\overline{\Theta} := \{(\theta,\theta^+) \in \Theta \times \Theta \mid \theta^+ \in \{\theta\} \oplus \Omega\}, \tag{3.23}$$

with hyperboxes $\Theta = \{\theta \in \mathbb{R}^p \mid \theta_i \in [\underline{\theta}_i, \overline{\theta}_i],\ i \in \mathbb{I}_{[1,p]}\}$, $\Omega = \{\Delta\theta \in \mathbb{R}^p \mid \Delta\theta_i \in [\underline{v}_i, \overline{v}_i],\ i \in \mathbb{I}_{[1,p]}\}$. The constants $\underline{\theta}_i$, $\overline{\theta}_i$, $\underline{v}_i$, $\overline{v}_i$ need to satisfy $\theta(r) \in \Theta$ for all $r \in \mathbb{Z}_{\mathrm{r}}$ and $\theta(r^+) \in \theta(r) \oplus \Omega$ for all $(r, r^+) \in \mathcal{R}$. The following proposition provides a finite dimensional SDP to compute a terminal cost using the 6^p vertices of the set $\overline{\Theta}$ (3.23).

Proposition 3.15. *Let Condition* (3.22) *hold with* $\overline{\Theta}$ *according to* (3.23)*. Suppose that there exist matrices* X_i, Y_i, $i \in \mathbb{I}_{[0,p]}$, Λ_i, $i \in \mathbb{I}_{[1,p]}$, $X_{\min}$ *that satisfy the following constraints*

$$\min_{X_i,\, Y_i,\, \Lambda_i,\, X_{\min}} \quad -\log\det X_{\min} \tag{3.24a}$$

$$\text{s.t.} \begin{pmatrix} X(\theta) & X(\theta)A(\theta)^\top + Y(\theta)^\top B(\theta)^\top & (Q+\epsilon)^{1/2}X(\theta) & (R^{1/2}Y(\theta))^\top \\ * & X(\theta^+) & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix} \succeq \begin{pmatrix} \sum_{i=1}^p \theta_i^2 \Lambda_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.24b}$$

$$X_{\min} \preceq X(\theta), \tag{3.24c}$$

$$\forall (\theta,\theta^+) \in \mathrm{vert}(\overline{\Theta}), \tag{3.24d}$$

$$\begin{pmatrix} 0 & (A_iX_i + B_iY_i)^\top \\ (A_iX_i + B_iY_i) & 0 \end{pmatrix} - \Lambda_i \succeq 0, \quad \Lambda_i \succeq 0, \quad i \in \mathbb{I}_{[1,p]}. \tag{3.24e}$$

Then, the matrices $P = X^{-1}$, $K = YP$, *satisfy Inequality* (3.14) *for all* $(r, r^+) \in \mathcal{R}$, *with* X, Y *according to* (3.21).

Proof. Due to Lemma 3.13, it suffices to show that $X(r)$, $Y(r)$ satisfy the constraints in (3.19). Condition (3.22) and $\Lambda_i \succeq 0$ imply that any solution that satisfies the constraints (3.24b) for all $(\theta,\theta^+) \in \overline{\Theta}$, also satisfies the constraints (3.19b) for all $(r, r^+) \in \mathcal{R}$. It remains to show that it suffices to check the inequality on the vertices of the constraint set $\overline{\Theta}$. This last result is a consequence of multi-convexity [21, Cor. 3.2], compare also [112]. In particular, if a function f is multi-concave along the edges of the constraint set $\overline{\Theta}$, then it attains its minimum at a vertex of $\overline{\Theta}$ and thus it suffices to verify (3.24b) over the vertices of $\overline{\Theta}$. The edges of $\overline{\Theta}$ (3.23) are characterized by $\{\theta_i,\ \theta_i^+,\ \theta_i^+ - \theta_i\}$,

$i \in \mathbb{I}_{[1,p]}$. A function is multi-concave if the second derivative w.r.t. these directions is negative semidefinite, compare [21, Cor. 3.4]. Similar to [21, Cor. 3.5], the additional constraint (3.24e) ensures that the function is multi-concave. Thus, it suffices to verify Inequality (3.24b) on the vertices of the constraint set $\overline{\Theta}$. ■

Remark 3.16. *(Simplified formulation) The result in Proposition 3.15 remains valid, if the set* $\overline{\Theta}$ *in* (3.23) *is replaced by the set* $\overline{\Theta} = \{(\theta, \theta^+) \mid \theta \in \Theta,\ \theta^+ - \theta \in \Omega\}$. *This set has only* 4^p *vertices and the induced conservatism of this over-approximation is negligible if* Ω *is small compared to* Θ. *The computational complexity can be further reduced by considering (block-)diagonal multipliers* $\Lambda_i = \lambda_i I$.

As discussed in Remark 3.2, we can include additional constraints on the considered set of references $\mathcal{R}$, which makes the offline computation less conservative. The advantages and applicability of the proposed design procedure are demonstrated in the numerical examples in Section 3.4, compare also the numerical examples in [JK15].

Remark 3.17. *(LPV methods) The main result is that we can formulate the offline design procedure similar to the gain-scheduling synthesis of (quasi)-LPV systems and thus can draw on a well established field to formulate offline LMI procedures, compare [243]. We point out that the usage of LPV methods and gain scheduling for nonlinear systems has a long history (cf., e.g., connection to incremental stability [110] and application to tracking MPC [56]) and continues to be an active field, compare computational MPC approaches in [116], incremental system properties [148, 270, 280, 281] and the recent overview in [202]. If the parameters* θ_i *are chosen based on a vertex representation* ($\theta_i \geq 0, \sum_{i=1}^{p} \theta_i = 1$)*, the multi-convexity condition* (3.24e) *can alternatively be replaced by positivity conditions of the polynomials, compare for example [200]. In [184], a convexification with an additional matrix is considered. In [15], a piece-wise parameter-dependent Lyapunov function is computed by partitioning the set* Θ *into smaller hyperboxes and computing constant matrices* P, K *for each interval. In case* $\Omega = \mathbb{R}^p$*, necessary and sufficient conditions for poly-quadratic stability can be found in [69] based on [75]. Parameter dependent Lyapunov functions for bounded rate of variant using a linear fractional transformation (LFT) and full block multipliers can be found in [276, 285]. If we restrict the parametrization to constant matrices* P*, we can use the synthesis procedures in [249] based on a LFT and full block multipliers, compare also [248, 250, 268].*

Remark 3.18. *(Sum-of-squares) An alternative solution to this problem is sum-of-squares (SOS) optimization [220], which is also frequently used to compute Lyapunov or storage functions, compare, e.g., [JK1, 62, 85, 225]. Assuming* A, B *are polynomial, consider matrices* X, Y *polynomial in* r *(with a specified order* d*) and ensure that the matrix in* (3.19b) *or* (3.14) *is positive definite on $\mathbb{Z}_r$. A similar approach is suggested in [178] to compute a control contraction metric (CCM) for continuous-time systems, which is a strongly related problem. SOS optimization is also used for the computation of invariant sets in setpoint tracking and reference governors in [62], which is closely related to the tracking formulation considered in Section 3.2. We do not pursue SOS approaches here since many systems require a high order polynomial to approximate the nonlinear dynamics and the computational complexity grows exponentially in n^d, thus potentially prohibiting the practical application. In addition, considering the constraint $r \in \mathbb{Z}_r$ is crucial in the discrete-time case and requires additional generalized S-procedure variables (cf., e.g., [JK1, Sec. 4]). The connection between CCM and LPV gain-scheduling design is discussed in [280], compare also Appendix C.*

Remark 3.19. *(Gridding) A common heuristic to ensure that parameter dependent LMIs such as* (3.19) *hold for all $(r, r^+) \in \mathcal{R}$ is to consider the constraints on sufficiently many sample points in the constraint set, compare [21, Sec. 4.2]. Due to continuity, the constraint is typically satisfied on the full constraint set, if it (strictly) holds on a sufficiently fine grid. If this method is applied, it is crucial to a posteriori verify satisfaction of Condition* (3.6a)*, e.g., by using a finer grid. For the simple structure of $\mathcal{R}$ in Assumption 3.1, this gridding can be achieved by gridding $r \in \mathbb{Z}_r$, computing $x_r^+ = f(x_r, u_r)$, and considering all u_r^+ such that $(x_r^+, u_r^+) \in \mathbb{Z}_r$. This approach does not introduce additional conservatism, but is computationally challenging for high dimensional systems. If some parameters (e.g., u_r) enter the LMIs affinely and are subject to polytopic constraints, it suffices to consider the vertices of the corresponding constraint set.*

Remark 3.20. *(Special case - setpoint tracking) Setpoint tracking is included in the previous derivation as a special case with $\mathcal{R} = \{(r, r^+) \in \mathbb{Z}_r \times \mathbb{Z}_r \mid r^+ = r\}$ and the steady-state manifold $\mathbb{Z}_r$. In this case, the SDP in Proposition 3.15 significantly simplifies with $\Omega = \{0\}$, $\overline{\Theta} = \{(\theta, \theta^+) \in \Theta \times \Theta \mid \theta^+ = \theta\}$, resulting in only 2^p vertices. Furthermore, it suffices to consider the steady-state manifold $\mathbb{Z}_r$ to determine the parameters $\theta(r)$ that characterize the Jacobian $A(r)$, $B(r)$, compare, e.g., [62]. Thus, the dimension p is typically significantly smaller in the setpoint tracking case, which also reduces the computational complexity of SOS or heuristic gridding.*

Remark 3.21. *(General stage cost ℓ) For simplicity of exposition, we consider positive definite quadratic stage costs $\ell(x, u, r)$* (3.3) *in this section. The presented reference generic offline design procedure can be readily extended to more general (twice continuously differentiable) tracking stage costs ℓ, such as output tracking stage costs $\ell(x, u, r) = \|h(x, u) - h(x_r, u_r)\|^2_{S(r)}$ for some smooth output $h : \mathbb{Z} \to \mathbb{Y}$, compare Appendix B. The non-trivial extension to economic (indefinite) stage costs ℓ_{eco} is detailed in Section 3.3.5. Analogous derivations for the continuous-time case can be found in [JK15, App. C].*

Terminal set size α

The terminal set size α derived in Lemma 3.12 and Proposition 3.11 can be quite conservative which reduce the region of attraction and closed-loop performance of the MPC formulation. In the following, we outline how a non-conservative value α can be computed (given P and K) in order to reduce the conservatism.

For simplicity, we consider a polytopic constraint set $\mathbb{Z} = \{r = (x,u) \in \mathbb{R}^{n+m} | L_j \cdot r \leq l_j, j \in \mathbb{I}_{[1,n_z]}\}$. The largest constant α_2 such that $\alpha \leq \alpha_2$ implies constraint satisfaction (3.6b) can be computed with

$$\begin{aligned} \alpha_2 := \max_{\alpha} \ & \alpha \\ \text{s.t. } & \|P(r)^{-1/2} \begin{pmatrix} I_n & K^\top(r) \end{pmatrix} L_j^\top\|^2 \alpha \leq (l_j - L_j r)^2, \\ & \forall r \in \mathbb{Z}_{\mathrm{r}}, \quad j \in \mathbb{I}_{[1,n_z]}. \end{aligned} \tag{3.25}$$

This problem can be efficiently solved by gridding the constraint set $\mathbb{Z}_{\mathrm{r}}$, solving the resulting linear program (LP) for each point r and taking the minimum. In the special case that P, K are constant, this reduces to one small scale LP. Similar procedures can be applied for nonlinear Lipschitz continuous constraints, compare Section 3.2.2.

Determining a non-conservative constant α_1, related to the local CLF V_{f} can be significantly more difficult. Conceptually, a corresponding value can be computed with the following non-convex optimization problem:

$$\begin{aligned} \alpha_1 := \max_{\alpha} & \ \alpha \\ \text{s.t. } & (3.6a) \text{ holds } \forall (r, r^+) \in \mathcal{R}, \ \forall x : V_{\mathrm{f}}(x,r) \leq \alpha. \end{aligned} \tag{3.26}$$

In the special case of stabilizing a fixed steady-state ($\mathcal{R} = \{0,0\}$), this reduces to the procedure suggested in [55, Rk. 3.1]. Convex solvers, such as sequential quadratic programming (SQP), cannot be employed to determine α_1 using (3.26), since the problem is highly non-convex and local minima would result in terminal ingredients that do not satisfy Assumption 3.5. Alternatively, heuristic sampling approaches can be used to directly check the constraints in (3.26) (cf. [JK15, Alg. 1]). The conservatism of using Lipschitz bounds can also be reduced by considering Hölder continuity and computing the corresponding bounds using sampling[4] [117, 232]. The overall offline procedure to compute the terminal ingredients (Ass. 3.5) is summarized in the following algorithm.

[4]For general nonlinear dynamics, even the computation of the Lipschitz constant L_Φ is challenging and may require global solvers or heuristic sampling.

Algorithm 3.22. *Reference generic offline computation - Terminal ingredients*

1: *Define* θ *parametrizing the Jacobian* (3.20).

2: *Compute* P, K *using LMIs:*
Determine hyperboxes Θ, Ω *according to* (3.22), (3.23).
Solve SDP (3.24)
(Alternatives: Gridding, SOS,...)

3: *Compute the size of the terminal set* $\alpha := \min\{\alpha_1, \alpha_2\}$*:*
a) compute α_2 *using* (3.25) *or* (3.12)*,*
b) compute α_1 *using* (3.26) *or* (3.18).

Remark 3.23. *(Linear difference inclusion, LDI) The applicability of the proposed approach strongly hinges on sufficiently smooth dynamics, since we use arguments based on the Jacobian* (3.13) *and local Taylor approximations (Lemma 3.12), and thus cannot be applied if* f *is not continuously differentiable. Even if* f *is twice continuously differentiable, the terminal set size* α_1 *may be very small. Both of these issues can be addressed by replacing the Jacobian* (3.13) *with a local LDI of the following form:*
For any $r \in \mathbb{Z}_r$, *any* $(x, u) \in \mathbb{B}_\epsilon(r)$, *there exists some* $(A, B) \in \Theta_{\mathrm{LDI}}(r)$ *such that*

$$f(x, u) - f(x_r, u_r) = A(x - x_r) + B(u - u_r).$$

Compared to the first-order Taylor approximation, this characterization has no remainder term Φ, *but for any reference point* $r \in \mathbb{Z}_r$ *a set of matrices* A, B *needs to be considered (typically a convex hull). For* r *fixed, this is conceptually similar to [274, 275] where the nonlinear system is locally characterized as an LTV system. Using such a characterization, the complexity of the resulting SDP increases, since for any point* r, *multiple matrices* A, B *need to be considered. An advantage is that the value* α_1, *which is typically difficult to compute, can be set a priori through the constant* ϵ. *A detailed numerical investigation regarding the applicability of this approach is, however, still missing. In the special case of constant matrices* P, K, *the SDP significantly simplifies, analogous to [289, Lemma 2]. Similar LDI descriptions are also used in [286] to determine suitable time-varying terminal ingredients online.*

Remark 3.24. *(Application to robust MPC) In this thesis, we focus on nominal system properties for dynamic operation and neglect stability and feasibility issues due to model mismatch. However, we wish to point out that Algorithm 3.22 can analogously be used to compute an incremental Lyapunov function* $V_\delta(x, x_r)$ *(cf. [18]). Such incremental Lyapunov functions or similarly contraction metrics (cf. [169, 178]) have recently been increasingly employed to derive corresponding (tube-based) robust MPC approaches for nonlinear systems, compare [29, 39, JK8,*

JK13, JK17, JK18, JK29, JK30, 251, 257, 258, 273]. A suitable adaptation of the LMIs to compute incremental Lyapunov functions tailored to such a robust MPC approach can be found in [215]. A more detailed exposition regarding the connection of the proposed offline design to incremental stability, contraction metrics and nonlinear robust MPC approaches can be found in Appendix C.

Summary

In this section, we studied trajectory tracking for reachable reference trajectories using an MPC scheme with terminal ingredients. The main contribution of this section was the development of a *reference generic offline* design procedure for the terminal ingredients based on a quasi-LPV parametrization of the linearized dynamics. The presented offline procedure (Alg. 3.22) is more involved than the offline computation for one specific setpoint [55, 236] or trajectory [23, 91]. However, the main advantage of the proposed procedure is the fact that *no* repeated offline computations are necessary to account for changing operation conditions. This feature is particularly relevant for MPC designs based on artificial reference trajectories, which are studied in the next section. The applicability of this design procedure to nonlinear systems is demonstrated with numerical examples in Section 3.4, compare also the numerical examples in [JK15].

3.2 Tracking MPC formulations using artificial reference trajectories

In Section 3.1, we presented a trajectory tracking MPC formulation for reachable dynamic reference trajectories. In this section, we extend the problem to *output* target signals, which may be *unreachable* and subject to unpredictable *changes online*. To cope with this problem, we consider an MPC formulation using an artificial periodic reference trajectory and present a theoretical analysis that generalizes and unifies existing results to *nonlinear systems*, *periodic unreachable output trajectories* and *generalized conditions on the terminal ingredients* (Sec. 3.2.1). Furthermore, we improve this MPC formulation by including an online optimization of the terminal set size and the reference constraint set, which significantly improves the performance by using an additional scalar optimization variable (Sec. 3.2.2). In addition, we show how stabilization and dynamic trajectory planning can be formulated as partially decoupled optimization problems, which reduces the computational demand by introducing a partial time scale separation while ensuring recursive feasibility and convergence (Sec. 3.2.3). This section is based on and taken in parts literally from [JK16][5].

3.2.1 Nonlinear tracking MPC for dynamic target signals

In the following, we generalize the trajectory tracking problem considered in Section 3.1, by considering a predicted exogenous target signal y_e for some plant output y. Compared to the reference tracking problem (Sec. 3.1): a) the desired reference/target is specified with some output target y_e instead of a state and input reference trajectory $(x_\mathrm{r}, u_\mathrm{r})$; b) the desired target y_e is not necessarily reachable (Ass. 3.1 does not hold) and thus only the distance should be minimized; c) the target signal y_e may change unpredictably during online operation. Thus, by considering unpredictably changing unreachable output target signals y_e this section mainly focuses on the challenge that the *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories* (cf. Sec. 1.1, (iii)) and *online changes in the mode of operation* (cf. Sec. 1.1, (ii)). To handle these challenges, we use an MPC formulation with artificial reference trajectories. To allow for tractable formulations, we limit the class of *non-stationary operation* (cf. Sec. 1.1, (i)) to T-periodic trajectories instead of the general time-varying trajectories considered in Section 3.1.

Related work

MPC formulations using artificial reference setpoints/trajectories are a promising tool to handle *unreachable* target signals, compare [103, 161, 163]. By using terminal constraints for the artificial reference, these MPC formulations provide a large region of attraction and ensure recursive feasibility independent of the (typically exogenous) target signal y_e. This MPC formulation was originally proposed in [163] for *linear* systems and piece-wise constant *state* references, using an artificial setpoint and the maximal admissible invariant set for tracking as the terminal constraint. In [166], this approach is extended to *periodic output* target signals, using artificial periodic trajectories, an additional strict convexity assumption, and a terminal equality constraint, compare also [165]. In [164], the setpoint tracking formulation in [163] is extended to *nonlinear* systems and piece-wise constant output target values y_e. We generalize and unify the methodologies from [163, 164, 166] to design nonlinear MPC schemes that exponentially stabilize the optimal *reachable* periodic trajectory given a possibly *unreachable* periodic output target signal,

[5] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear tracking model predictive control scheme for unreachable dynamic target signals." In: *Automatica* 118 (2020). extended version: arXiv:1911.03304, p. 109030©2020 Elsevier Ltd.

using general conditions on the terminal ingredients and suitable convexity conditions on the set of feasible periodic output trajectories.

MPC formulation

We consider a predicted exogenous target signal $y_{\mathrm{e}}(\cdot|t) \in \mathbb{Y}^T$ of length $T \in \mathbb{I}_{\geq 1}$ (ideally representing a T-periodic signal, cf. Ass. 3.26), a nonlinear continuous output function $h : \mathbb{Z} \to \mathbb{Y}$ and some compact output space $\mathbb{Y} \subseteq \mathbb{R}^{n_{\mathrm{y}}}$. We want to minimize the weighted distance between this target signal and the output, i.e., minimize $\sum_{t=0}^{\infty} \|h(x(t), u(t)) - y_{\mathrm{e}}(t|t)\|_S^2$, with some positive definite weighting matrix $S \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{y}}}$. At time $t \in \mathbb{I}_{\geq 0}$, given the target signal $y_{\mathrm{e}}(\cdot|t) \in \mathbb{Y}^T$, an optimal periodic orbit can be determined based on the following periodic optimal control problem.

Problem 3.25.

$$\underset{r(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_{T,\mathrm{e}}(r(\cdot|t), y_{\mathrm{e}}(\cdot|t)) \tag{3.27a}$$

subject to

$$(r(j|t), r(\mathrm{mod}(j+1, T)|t)) \in \mathcal{R}, \quad j \in \mathbb{I}_{[0,T-1]}, \tag{3.27b}$$

with

$$\mathcal{J}_{T,\mathrm{e}}(r(\cdot|t), y_{\mathrm{e}}(\cdot|t)) := \sum_{j=0}^{T-1} \| \underbrace{h(x_{\mathrm{r}}(j|t), u_{\mathrm{r}}(j|t))}_{=:y_{\mathrm{r}}(j|t)} - y_{\mathrm{e}}(j|t)\|_S^2 =: \|y_{\mathrm{r}}(\cdot|t) - y_{\mathrm{e}}(\cdot|t)\|_S^2, \tag{3.27c}$$

where mod denotes the modulo operator. The solution to this optimization problem is the[6] optimal reachable T-periodic reference trajectory $r_T^*(\cdot|t) = (x_T^*(\cdot|t), u_T^*(\cdot|t))$ and the minimum is denoted by $V_{T,\mathrm{e}}(y_{\mathrm{e}}(\cdot|t)) := \mathcal{J}_{T,\mathrm{e}}(r_T^*(\cdot|t), y_{\mathrm{e}}(\cdot|t))$. The corresponding output reference is denoted by $y_T^*(\cdot|t) \in \mathbb{Y}^T$, with $y_T^*(k|t) = h(r_T^*(k|t))$, $t \in \mathbb{I}_{\geq 0}$, $k \in \mathbb{I}_{[0,T-1]}$. For the stability analysis, we assume that the target signal y_{e} is consistent and T-periodic.

Assumption 3.26. *(Consistently periodic target signal) For any $t \in \mathbb{I}_{\geq 0}$, the target signal y_{e} satisfies*

$$y_{\mathrm{e}}(\mathrm{mod}(k+1, T)|t) = y_{\mathrm{e}}(k|t+1), \qquad \forall k \in \mathbb{I}_{[0,T-1]}.$$

[6] Uniqueness of the minimizer will be ensured in the later derivation using suitable convexity conditions.

This assumption characterizes the fact that the prediction of y_e is exact. Assuming the target signal is T-periodic (Ass. 3.26), the optimal reachable T-periodic reference r_T^* is also consistent, i.e., $r_T^*(\mathrm{mod}(k+1,T)|t) = r_T^*(k|t+1)\ \forall t \in \mathbb{I}_{\geq 0}$, $k \in \mathbb{I}_{[0,T-1]}$.

At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t)$ and the target signal $y_\mathrm{e}(\cdot|t) \in \mathbb{Y}^T$, the MPC control law is determined by the following optimization problem:

Problem 3.27.

$$\underset{u(\cdot|t),\, r(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_N(x(\cdot|t), u(\cdot|t), r(\cdot|t)) + \mathcal{J}_{T,\mathrm{e}}(r(\cdot|t), y_\mathrm{e}(\cdot|t)) \tag{3.28a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.28b}$$
$$x(0|t) = x(t), \tag{3.28c}$$
$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.28d}$$
$$(x(N|t), r(N|t)) \in \mathbb{X}_\mathrm{f}, \tag{3.28e}$$
$$(r(j|t), r(j+1|t)) \in \mathcal{R}, \quad j \in \mathbb{I}_{[0,T-1]}, \tag{3.28f}$$
$$r(l+T|t) = r(l|t), \quad l \in \mathbb{I}_{[0,\max\{0,N-T\}]}. \tag{3.28g}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, the artificial reference trajectory $r^*(\cdot|t) = (x_\mathrm{r}^*(\cdot|t), u_\mathrm{r}^*(\cdot|t))$, and the value function

$$W_{N,T}(x(t), y_\mathrm{e}(\cdot|t)) := \mathcal{J}_N(x^*(\cdot|t), u^*(\cdot|t), r^*(\cdot|t)) + \mathcal{J}_{T,\mathrm{e}}(r^*(\cdot|t), y_\mathrm{e}(\cdot|t)).$$

Compared to the trajectory tracking MPC formulation in Section 3.1, in the considered MPC formulation the reference trajectory r is a decision variable and correspondingly the constraints on the reference trajectory $\mathbb{Z}_\mathrm{r}$, $\mathcal{R}$ (cf. Ass. 3.1) can be adjusted, compare also Section 3.2.2. The rationale behind this optimization problem is to penalize the (standard) tracking cost $\mathcal{J}_N$ w.r.t. some artificial periodic reference r together with the distance of the output of this artificial reference to the target signal using $\mathcal{J}_{T,\mathrm{e}}$. As we will see later in the theoretical analysis (Thm. 3.31) and the numerical examples (Sec. 3.4.1–3.4.2), this formulation ensures that the closed loop smoothly tracks the optimal reachable periodic trajectory x_T^*. The following algorithm summarizes the closed-loop operation.

Algorithm 3.28. *(Tracking MPC Algorithm with artificial reference trajectory)*
Offline: Specify the constraint sets $\mathbb{Z}$, $\mathbb{Z}_{\mathrm{r}}$, the weighting matrices (Q, R, S), the prediction horizon N, the period length T, and design suitable terminal ingredients V_{f}, $\mathbb{X}_{\mathrm{f}}$ (cf. Sec. 3.1).
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, obtain the target signal $y_{\mathrm{e}}(\cdot|t)$, solve Problem 3.27, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t), \quad t \in \mathbb{I}_{\geq 0}. \tag{3.29}$$

Theoretical analysis

In the following, we derive the theoretical properties of the closed-loop system based on Problem 3.27. The theoretical analysis mainly requires two assumptions: a) suitable conditions on the terminal ingredients (Ass. 3.5–3.6) and b) a convexity and uniqueness condition for the set of periodic output trajectories $y_{\mathrm{r}}(\cdot|t) \in \mathbb{Y}^T$.

Assumption 3.29. *(Convexity and uniqueness) There exists a (unique) Lipschitz continuous function $g : \mathbb{Y}^T \to \mathbb{Z}_{\mathrm{r}}^T$ such that for any trajectory $r(\cdot|t)$ satisfying the constraints in Problem 3.25, it holds $(g_{\mathrm{x}}(y_{\mathrm{r}}(\cdot|t)), g_{\mathrm{u}}(y_{\mathrm{r}}(\cdot|t))) := g(y_{\mathrm{r}}(\cdot|t)) = r(\cdot|t) = (x_{\mathrm{r}}(\cdot|t), u_{\mathrm{r}}(\cdot|t))$. Furthermore, the set of feasible solutions of Problem 3.25 is convex in $y_{\mathrm{r}}(\cdot|t)$, i.e., given two feasible solutions $r_1, r_2 \in \mathbb{Z}_{\mathrm{r}}^T$ with corresponding outputs $y_{\mathrm{r},1}, y_{\mathrm{r},2} \in \mathbb{Y}^T$, the reference $r = g(y_{\mathrm{r}})$ is a feasible solution of Problem 3.25 with $y_{\mathrm{r}} = \beta y_{\mathrm{r},1} + (1-\beta) y_{\mathrm{r},2}$, $\beta \in [0, 1]$.*

Similar convexity conditions are also used in [166, Ass. 2] and [164, Ass. 1-2] for the linear periodic problem and the nonlinear setpoint tracking problem, respectively. This assumption implies that Problem 3.25 is a strictly convex optimization problem and the minimizer r_T^* is unique. Thus, for any $y_{\mathrm{r}} \neq y_T^*$ it is possible to incrementally change y_{r} such that it remains feasible and the cost $\mathcal{J}_{T,\mathrm{e}}$ decreases. Furthermore, due to convexity the directional derivative of $\mathcal{J}_{T,\mathrm{e}}$ at y_T^* in any feasible direction is non-negative, i.e., for any reference $r(\cdot|t)$ satisfying the constraints in Problem 3.25, the corresponding output $y_{\mathrm{r}}(\cdot|t)$ satisfies

$$\left.\frac{\partial \mathcal{J}_{T,\mathrm{e}}}{\partial y_{\mathrm{r}}(\cdot|t)}\right|_{(y_T^*(\cdot|t), y_{\mathrm{e}}(\cdot|t))} (y_{\mathrm{r}}(\cdot|t) - y_T^*(\cdot|t)) \geq 0, \tag{3.30}$$

which can be equivalently written as

$$\mathcal{J}_{T,\mathrm{e}}(r(\cdot|t), y_{\mathrm{e}}(\cdot|t)) \geq V_{T,\mathrm{e}}(y_{\mathrm{e}}(\cdot|t)) + \|y_T^*(\cdot|t) - y_{\mathrm{r}}(\cdot|t)\|_S^2. \tag{3.31}$$

Remark 3.30. *(Convexity and uniqueness) Similar to [164, Rk. 1], existence of g can be ensured based on the implicit function theorem, if a rank condition on the linearization of a suitably defined T-step system is satisfied and f, h are continuously differentiable, compare also the nonresonance condition in Section 4.2. In case such a unique map g does not exist, the issue of stabilizing a unique state trajectory can be resolved by fixing a unique map g in Problem 3.27, compare [135] and the MPC formulation in [164, Eq. (9)].*
Even in case the set of reachable periodic trajectories or the steady-state manifold $\mathbb{Z}_{\mathrm{r}}$ ($T = 1$) are non-convex, the convexity condition on the output trajectory (Ass. 3.29) may still hold, compare [63] and the numerical example in Section 3.4.1. Furthermore, in case the set of feasible steady-state outputs $\mathcal{Y}_{\mathrm{r}}$ is a non-convex set in normal form (e.g., star-shaped), satisfaction of the convexity condition can be explicitly enforced by using a homeomorphic transformation of the output $\tilde{y} = \phi(h(x,u))$, compare [64]. If the convexity condition in Assumption 3.29 is not satisfied, the MPC scheme will not necessarily stabilize the optimal reachable trajectory x_T^*, but could instead stabilize a suboptimal periodic trajectory. The main alternative to using a tracking MPC scheme with simultaneous optimization of the artificial trajectory (Problem 3.27) would be to directly solve Problem 3.25 and then apply a tracking MPC for this reachable reference trajectory r_T^*, compare Section 3.1. If Problem 3.25 is solved with a standard convex solver, the solver may end in the same local minimum as the closed loop based on Algorithm 3.28. Thus, even if the convexity condition is not satisfied, the MPC formulation in Problem 3.27 is still an appropriate choice.

The following theorem establishes *exponential* stability of the optimal *reachable* trajectory x_T^* given suitable terminal ingredients (Ass. 3.5–3.6) and the convexity condition on the set of feasible periodic output trajectories (Ass. 3.29). This result generalizes and unifies the results in [164, 166] by considering *nonlinear* dynamics, *periodic* reference trajectories, establishing *exponential* stability, and unifying the consideration of different terminal ingredients (Ass. 3.5–3.6, Prop. 3.10–3.11).

Theorem 3.31. *Let Assumptions 3.5 and 3.6 hold. If the initial condition x_0 is such that Problem 3.27 is feasible at $t = 0$, then the closed-loop system* (3.29) *resulting from Algorithm 3.28 satisfies the constraints* (3.1) *and Problem 3.27 is feasible for all $t \in \mathbb{I}_{\geq 0}$, independently of the target signal y_{e}. If additionally Assumptions 3.26 and 3.29 hold, then the optimal reachable trajectory x_T^* is (uniformly) exponentially stable for the resulting closed-loop system* (3.29).

Proof. **Part I:** Recursive Feasibility: It suffices to note that feasibility of Problem 3.27 does not depend on the target signal y_{e}. Correspondingly, the candidate input sequence $u(\cdot|t+1)$ in Theorem 3.8 with the shifted reference $r(k|t+1) = r^*(\mathrm{mod}(k+1,T)|t)$,

$k \in \mathbb{I}_{[0,T-1]}$ is a feasible solution of Problem 3.27.
Part II: Stability: Consider a periodic target signal (Ass. 3.26), which is denoted by $y_{\mathrm{e}}(t+k) := y_{\mathrm{e}}(k|t)$. Thus, the minimizer of Problem 3.25 is a periodic trajectory, i.e., $x_T^*(k+t) := x_T^*(k|t)$, $t \in \mathbb{I}_{\geq 0}$, $k \in \mathbb{I}_{[0,T-1]}$, and we write $\mathcal{J}_{T,\mathrm{e}}(r(\cdot),t) := \mathcal{J}_{T,\mathrm{e}}(r(\cdot),y_{\mathrm{e}}(\cdot|t))$, $V_{T,\mathrm{e}} := V_{T,\mathrm{e}}(y_{\mathrm{e}}(\cdot|t))$, with $\mathcal{J}_{T,\mathrm{e}}$ (periodically) time-varying in the second argument and $V_{T,\mathrm{e}}$ constant in time. Define the candidate Lyapunov function $W(x(t),t) := W_{N,T}(x(t),y_{\mathrm{e}}(\cdot|t)) - V_{T,\mathrm{e}}$ and the tracking error $e_T(t) := x(t) - x_T^*(t)$. In the following, we show that there exist constants $\alpha_{\mathrm{W}}, c_{\mathrm{v}} > 0$ such that

$$W(x(t+1),t+1) \leq W(x(t),t) - \|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2, \tag{3.32a}$$
$$\alpha_{\mathrm{W}}\|e_T(t)\|_Q^2 \leq W(x(t),t) \leq c_{\mathrm{v}}\|e_T(t)\|_Q^2, \tag{3.32b}$$

holds for all $x(t)$ such that Problem 3.27 is feasible. The shifted reference $r(\cdot|k+1)$ in Part I satisfies $\mathcal{J}_{T,\mathrm{e}}(r(\cdot|t+1),t+1) = \mathcal{J}_{T,\mathrm{e}}(r^*(\cdot|t),t)$. Thus, feasibility in combination with Inequality (3.8) implies $W(x(t+1),t+1) - W(x(t),t) \leq -\ell(x(t),u(t),r^*(0|t))$, which implies (3.32a). Lipschitz continuity of g implies

$$\begin{aligned}\|x_{\mathrm{r}}^*(0|t) - x_T^*(0|t)\|_Q &\leq \|x_{\mathrm{r}}^*(\cdot|t) - x_T^*(\cdot|t)\|_Q \\ =&\|g_{\mathrm{x}}(y_{\mathrm{r}}^*(\cdot|t)) - g_{\mathrm{x}}(y_T^*(\cdot|t))\|_Q \leq L_{\mathrm{g}}\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S,\end{aligned} \tag{3.33}$$

with some constant $L_{\mathrm{g}} > 0$. Thus, strong convexity (cf. (3.31)) implies

$$\begin{aligned}\mathcal{J}_{T,\mathrm{e}}(r^*(\cdot|t),t) - V_{T,\mathrm{e}} &\geq \|y_T^*(\cdot|t) - y_{\mathrm{r}}^*(\cdot|t)\|_S^2 \\ \geq& 1/L_{\mathrm{g}}^2\|x_T^*(\cdot|t) - x_{\mathrm{r}}^*(\cdot|t)\|_Q^2 \geq 1/L_{\mathrm{g}}^2\|x_T^*(0|t) - x_{\mathrm{r}}^*(0|t)\|_Q^2.\end{aligned}$$

Correspondingly, using the fact that $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$ for all $a, b \in \mathbb{R}$ yields the lower bound

$$\begin{aligned}W(x(t),t) &\geq \|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 + \mathcal{J}_{T,\mathrm{e}}(r^*(\cdot|t),t) - V_{T,\mathrm{e}} \\ \geq& \|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 + 1/L_{\mathrm{g}}^2\|x_T^*(0|t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \geq \alpha_W\|e_T(t)\|_Q^2,\end{aligned}$$

with $\alpha_{\mathrm{W}} := \frac{1}{2}\min\{1, 1/L_{\mathrm{g}}^2\} > 0$. In case $\|e_T(t)\|_Q^2 \leq \epsilon^2$, Assumption 3.6 ensures that $r(\cdot|t) = r_T^*(\cdot|t)$ is a feasible solution of Problem 3.27, which implies $W(x(t),t) \leq c_{\mathrm{u}}\|e_T(t)\|_Q^2$. As in Theorem 3.8, compact constraints together with this local upper bound imply the upper bound in (3.32b) with some constant $c_{\mathrm{v}} \geq c_{\mathrm{u}}$, compare [236, Prop. 2.16]. Inequalities (3.32) imply (uniform) stability of x_T^* for the closed-loop system,

but not necessarily asymptotic or exponential stability.
Part III: Exponential stability - case distinction:
Case 1: Consider

$$\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \geq \gamma \|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2, \tag{3.34}$$

with a later specified constant $\gamma > 0$. Then, Inequalities (3.32a) and (3.33) imply

$$\begin{aligned}
&W(x(t+1), t+1) - W(x(t), t) \\
\overset{(3.32a)}{\leq} & - \|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \overset{(3.34)}{\leq} -\frac{1}{2}\left(\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 + \gamma\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2\right) \\
\overset{(3.33)}{\leq} & -\frac{1}{2}\left(\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 + \frac{\gamma}{L_{\mathrm{g}}^2}\|x_{\mathrm{r}}^*(0|t) - x_T^*(0|t)\|_Q^2\right) \\
\leq & -\frac{1}{4}\min\left\{1, \frac{\gamma}{L_{\mathrm{g}}^2}\right\}\|x(t) - x_T^*(t)\|_Q^2.
\end{aligned}$$

Case 2: Assume

$$\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \leq \gamma \|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2. \tag{3.35}$$

Compactness of $\mathbb{Y}$ implies that there exists a constant $\overline{y} > 0$ such that $\|y_{\mathrm{r}} - \tilde{y}_{\mathrm{r}}\|_S^2 \leq \overline{y}^2$, for any trajectories y_{r}, $\tilde{y}_{\mathrm{r}} \in \mathbb{Y}^T$. This implies

$$\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \overset{(3.35)}{\leq} \gamma\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2 \leq \gamma\overline{y}^2.$$

For $\gamma \leq \gamma_1 := \epsilon^2/\overline{y}^2$, we have $\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \leq \epsilon^2$. Thus, Assumption 3.6 implies

$$\begin{aligned}
&\|x^*(1|t) - x_{\mathrm{r}}^*(1|t)\|_Q^2 \leq \mathcal{J}_N(x^*(\cdot|t), u^*(\cdot|t), r^*(\cdot|t))) \\
\overset{(3.7)}{\leq} & c_{\mathrm{u}}\|x(t) - x_{\mathrm{r}}^*(0|t)\|_Q^2 \overset{(3.35)}{\leq} \gamma c_{\mathrm{u}}\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2 \leq \gamma c_{\mathrm{u}}\overline{y}^2.
\end{aligned} \tag{3.36}$$

For $\gamma \leq \gamma_2 := \epsilon^2/(4c_{\mathrm{u}}\overline{y}^2) \leq \gamma_1$ this implies $\|x(t+1) - x_{\mathrm{r}}^*(1|t)\|_Q \leq \epsilon/2$. Consider $y_{\mathrm{r}}(j|t+1) = h(r(j|t+1))$, $j \in \mathbb{I}_{[0,T-1]}$, where $r(\cdot|t+1)$ is the candidate reference from Part I of the proof. At time $t+1$, define an auxiliary reference

$$\hat{y}_{\mathrm{r}} := \beta y_{\mathrm{r}}(\cdot|t+1) + (1-\beta)y_T^*(\cdot|t+1),\ \beta \in [0,1], \tag{3.37}$$

with the corresponding state and input trajectory $g(\hat{y}_{\mathrm{r}}) = \hat{r} = (\hat{x}_{\mathrm{r}}, \hat{u}_{\mathrm{r}})$. Convexity (cf.

Ass. 3.29) ensures that the auxiliary reference $\hat{r}$ is a feasible solution of Problem 3.25 at $t+1$. The definition (3.37) implies

$$\hat{y}_{\mathrm{r}} - y_{\mathrm{r}}(\cdot|t+1) = (1-\beta)(y_T^*(\cdot|t+1) - y_{\mathrm{r}}(\cdot|t+1)). \tag{3.38}$$

The cost $\mathcal{J}_{T,\mathrm{e}}$ satisfies

$$\begin{aligned}
&\mathcal{J}_{T,\mathrm{e}}(\hat{r}, t+1) - \mathcal{J}_{T,\mathrm{e}}(r^*_{\cdot|t}, t) = (\hat{y}_{\mathrm{r}} - y_{\mathrm{r}}(\cdot|t+1))^\top S(\hat{y}_{\mathrm{r}} + y_{\mathrm{r}}(\cdot|t+1) - 2y_{\mathrm{e}}(\cdot|t+1)) \\
\overset{(3.37)}{=}&(1-\beta)(y_T^*(\cdot|t+1) - y_{\mathrm{r}}(\cdot|t+1))^\top S \\
&((1+\beta)y_{\mathrm{r}}(\cdot|t+1) + (1-\beta)y_T^*(\cdot|t+1) - 2y_{\mathrm{e}}(\cdot|t+1)) \\
=& - (1-\beta^2)\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2 \\
&+ (1-\beta)\left.\frac{\partial \mathcal{J}_{T,\mathrm{e}}}{\partial y_{\mathrm{r}}(\cdot|t+1)}\right|_{(y_T^*(\cdot|t+1), y_{\mathrm{e}}(\cdot|t+1))} (y_T^*(\cdot|t+1) - y_{\mathrm{r}}(\cdot|t+1)) \\
\overset{(3.30)}{\leq}& - (1-\beta^2)\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2.
\end{aligned} \tag{3.39}$$

Lipschitz continuity (cf. (3.33)) implies

$$\begin{aligned}
&\|x(t+1) - \hat{x}_{\mathrm{r}}(0)\|_Q \leq \|x(t+1) - x_{\mathrm{r}}^*(1|t)\|_Q + \|x_{\mathrm{r}}^*(1|t) - \hat{x}_{\mathrm{r}}(0)\|_Q \\
\leq & \frac{\epsilon}{2} + L_{\mathrm{g}}\|y_{\mathrm{r}}(\cdot|t+1) - \hat{y}_{\mathrm{r}}\|_S \overset{(3.38)}{=} \frac{\epsilon}{2} + L_{\mathrm{g}}(1-\beta)\|y_{\mathrm{r}}(\cdot|t+1) - y_T^*(\cdot|t+1)\|_S.
\end{aligned}$$

For $\beta \in [\beta_1, 1]$ with $\beta_1 := 1 - \epsilon/(2L_{\mathrm{g}}\overline{y})$, this implies $\|x(t+1) - \hat{x}_{\mathrm{r}}(0)\|_Q \leq \epsilon$. Thus, Assumption 3.6 ensures that there exists some state and input sequence $(\hat{x}, \hat{u})$ such that $(\hat{x}, \hat{u}, \hat{r})$ is a feasible solution of Problem 3.27 at time $t+1$ and the tracking cost satisfies

$$\begin{aligned}
\mathcal{J}_N(\hat{x}, \hat{u}, \hat{r}) &\overset{(3.7)}{\leq} c_{\mathrm{u}}\|x(t+1) - \hat{x}_{\mathrm{r}}(0)\|_Q^2 \\
&\leq 2c_{\mathrm{u}}(\|x(t+1) - x_{\mathrm{r}}^*(1|t)\|_Q^2 + \|x_{\mathrm{r}}^*(1|t) - \hat{x}_{\mathrm{r}}(0)\|_Q^2) \\
&\leq 2c_{\mathrm{u}}\|x(t+1) - x_{\mathrm{r}}^*(1|t)\|_Q^2 + 2c_{\mathrm{u}}L_{\mathrm{g}}^2\|y_{\mathrm{r}}(\cdot|t+1) - \hat{y}_{\mathrm{r}}\|_S^2 \\
&\overset{(3.38)}{\leq} 2c_{\mathrm{u}}\|x(t+1) - x_{\mathrm{r}}^*(1|t)\|_Q^2 + 2c_{\mathrm{u}}L_{\mathrm{g}}^2(1-\beta)^2\|y_{\mathrm{r}}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2,
\end{aligned} \tag{3.40}$$

where the second to last inequality follows from Lipschitz continuity, compare (3.33).

Correspondingly, we have

$$\begin{aligned}
& W(x(t+1),t+1) - W(x(t),t) \\
\leq & \mathcal{J}_N(\hat{x},\hat{u},\hat{r}) + \mathcal{J}_{T,\mathrm{e}}(\hat{r},t+1) - \mathcal{J}_{T,\mathrm{e}}(r^*(\cdot|t),t) - \|x(t) - x_\mathrm{r}^*(0|t)\|_Q^2 \\
\overset{(3.39),(3.40)}{\leq} & 2c_\mathrm{u}\|x(t+1) - x_\mathrm{r}^*(1|t)\|_Q^2 - \|x(t) - x_\mathrm{r}^*(0|t)\|_Q^2 \\
& - \underbrace{\left((1-\beta^2) - 2c_\mathrm{u}L_\mathrm{g}^2(1-\beta)^2\right)}_{=:c_2(\beta)} \|y_\mathrm{r}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2 \\
\overset{(3.33),(3.36)}{\leq} & (2c_\mathrm{u}^2\gamma - c_2(\beta)/2)\|y_\mathrm{r}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2 \\
& - c_2(\beta)/(2L_\mathrm{g}^2)\|x_\mathrm{r}^*(0|t) - x_T^*(0|t)\|_Q^2 - \|x(t) - x_\mathrm{r}^*(0|t)\|_Q^2 \\
\leq & (2c_\mathrm{u}^2\gamma - c_2(\beta)/2)\|y_\mathrm{r}^*(\cdot|t) - y_T^*(\cdot|t)\|_S^2 \\
& - \min\left\{\frac{1}{2}, c_2(\beta)/(4L_\mathrm{g}^2)\right\} \|x(t) - x_T^*(0|t)\|_Q^2.
\end{aligned}$$

Let $\beta = \beta_2 := \arg\max_{\beta\in[\beta_1,1]} c_2(\beta)$, with $c_2(\beta_2) > 0$. For $\gamma \leq \gamma_3 := c_2(\beta_2)/(4c_\mathrm{u}^2)$, this implies

$$W(x(t+1),t+1) - W(x(t),t) \leq -\min\left\{\frac{1}{2}, c_2(\beta_2)/(4L_\mathrm{g}^2)\right\} \|x(t) - x_T^*(t)\|_Q^2.$$

Combine: Combining these two cases yields

$$\begin{aligned}
W(x(t+1),t) \leq & W(x(t),t) - \gamma_T\|x(t) - x_T^*(t)\|_Q^2, \\
\gamma_T := & \min\left\{\frac{c_2(\beta_2)}{4L_\mathrm{g}^2}, \frac{1}{4}, \frac{\gamma}{4L_\mathrm{g}^2}\right\}, \ \gamma := \min\{\gamma_1,\gamma_2,\gamma_3\}.
\end{aligned} \tag{3.41}$$

Uniform exponential stability follows using Inequalities (3.32b), (3.41) and standard Lyapunov arguments. ∎

Discussion

Theorem 3.31 ensures exponential stability of the optimal reachable trajectory x_T^* by showing quadratic lower and upper bounds and an exponential decay of the Lyapunov function $W := W_{N,T} - V_{T,\mathrm{e}}$. The exponential decay of W is shown by utilizing two distinct candidate solutions, namely $(\hat{x},\hat{u},\hat{r})$ and the standard candidate solution from Theorem 3.8. In particular, we distinguish whether the tracking error $\|x_\mathrm{r}^*(0|t) - x(t)\|_Q^2$ is large/small (γ) compared to the output tracking cost $\mathcal{J}_{T,\mathrm{e}} - V_{T,\mathrm{e}}$. If the reference tracking error is large, then the standard candidate solution (cf. Thm. 3.8) ensures a sufficient exponential decrease in the Lyapunov function W. On the other hand, if the reference tracking error is small enough ($\gamma\overline{y}^2$), then the convexity condition (Ass. 3.29) ensures that the artificial reference r can be incrementally ($\beta < 1$) moved towards the optimal reachable reference r_T^*, which decreases the output tracking cost $\mathcal{J}_{T,\mathrm{e}}$. The local quadratic bound (3.7) (Ass. 3.6) on the value function V_N ensures that the optimization problem is feasible with the incrementally moved reference $\hat{r}$ and that the increase in the tracking cost $\mathcal{J}_N$ is quadratically bounded. Finally, there exists a sufficiently small change ($\beta_2 < 1$) such that this auxiliary candidate solution $(\hat{x}, \hat{u}, \hat{r})$ ensures an exponential decay in W.

Remark 3.32. *(Model uncertainty and offset-free tracking) Similar to the derivations in [164, 166], Theorem 3.31 assumes no model mismatch, which is rarely the case in practical applications. To ensure robust recursive feasibility despite disturbances, Problem 3.27 needs to be adjusted using constraint tightening techniques from robust MPC (cf. [159]). A corresponding formulation for nonlinear robust setpoint tracking MPC ($T = 1$) can be found in the recent paper [JK30], which combines the formulation presented in Section 3.2.2 with the nonlinear robust MPC formulation in [JK29]. An additional modification to explicitly include the model mismatch in tracking MPC formulations with artificial setpoints has been proposed in [247] (for linear systems), which allows for stronger robust convergence guarantees.*
In addition to possible feasibility issues, model mismatch typically also implies non-zero offset, even in case of constant references. For the special case of setpoint tracking ($T = 1$), this issue is typically resolved using offset-free MPC formulations (cf. [201, 218] and references therein), which rely on a disturbance estimator for constant offsets or velocity formulations (cf. [36, 59, 159, 176, 213]). In order to transfer this concept to T-periodic trajectories, the dimension of the disturbance model must be increased (cf. [173]) or the velocity formulation needs to be adjusted for periodic signals (cf. Sec. 4.2.3). An alternative approach to ensure offset-free tracking is to use a parameter estimation scheme with an adaptive MPC formulation, under appropriate assumptions on the model mismatch, compare, e.g., [45, 219] or [82]. Extending the proposed approach to ensure offset-free tracking for dynamic/periodic trajectories despite deterministic model mismatch requires further research.

Remark 3.33. *(Extensions and open issues) The proposed approach can be extended to stabilize the economically optimal reachable reference trajectory r, as an extension to the linear approach in [165], assuming that Problem 3.25 remains (strictly/strongly) convex (Ass. 3.29). In Section 3.3, we extend the proposed formulation to use a purely economic cost function, similar to [87, 206, 208]. Recently, in [JK4] for the special case of linear systems, the analysis* *in Theorem 3.31 was extended to stage costs ℓ with Q only positive semidefinite using suitable observability conditions, which is especially relevant for input-output and data-driven models, compare [JK3, JK4, JK5, JK6, 51, 65, 66, 180, 181, 182, 183]. The consideration of nonperiodic dynamic target signals in this framework is still an open topic.*

Remark 3.34. *(Reference governor) Setpoint stabilization is a special case in Theorem 3.31 with $T = 1$ and $\mathbb{Z}_{\mathrm{r}}$ corresponding to the feasible steady-state manifold (cf. Rk. 3.20). In this case, the considered MPC formulation is strongly related to the reference governor problem [113], compare also earlier MPC formulations [58] using a feasibility recovery mode. In particular, for a horizon of $N = 0$, we can define the MPC control law as $u(t) = k_{\mathrm{f}}(x(t), r^*(0|t))$, in which case Problem 3.27 only adjusts the artificial reference r to ensure constraint satisfaction, while a locally stabilizing controller k_{f} is applied in closed loop, analogous to a reference governor. Due to this similarity, the reference generic offline computations (Sec. 3.1.3) can equally be used to compute the necessary ingredients for a reference governor, compare also [62].*

Remark 3.35. *(Stability properties) For the special case of setpoint tracking ($T = 1$), an analogous MPC formulation has been considered in [164]. In the following, we wish to highlight some of the main differences. The considered theoretical derivation is based on the rather general conditions on the terminal ingredients (Ass. 3.5–3.6), which allow a unified analysis of terminal equality constraints and differently parametrized terminal cost/set formulation, including the continuously parametrized terminal ingredients from Section 3.1.3. Furthermore, Theorem 3.31 ensures exponential stability with a convergence rate $\frac{c_{\mathrm{v}}-\gamma_T}{c_{\mathrm{v}}} < 1$ and and the Lyapunov function W, while the proof in [164] uses an argument of contradiction[7], which does not guarantee any specific rate of convergence. For the linear case, a similar difference can be found in literature, with a proof of contradiction in [163] and stronger Lyapunov decrease properties in [166] and [2, Lemma V.4].*

3.2.2 Online optimized terminal set size and reference constraint set

In the following, we generalize the parametrization of the terminal set $\mathbb{X}_{\mathrm{f}}$ and the reference constraint set $\mathbb{Z}_{\mathrm{r}}$ in order to further improve performance. In particular, for the design in Proposition 3.11 (analogous to the design in [55]), the size of the terminal set $\mathbb{X}_{\mathrm{f}}$ depends on the constant $\alpha > 0$, which is the minimum of two values: α_1 and α_2. The first (α_1) needs to be such that Inequality (3.6a) holds, compare Lemma 3.12 and the optimization problem (3.26). The second (α_2) ensures constraint satisfaction (3.6b) and

[7] In [164], it is shown that $x(t) = x_{\mathrm{r}}^*(t)$ implies $x(t) = x_T^*$. Combining this with the fact that $\lim_{t\to\infty} \|x(t) - x_{\mathrm{r}}^*(t)\| = 0$, the authors in [164] assert $\lim_{t\to\infty} \|x(t) - x_T^*\| = 0$.

depends on the difference between $\mathbb{Z}$ and $\mathbb{Z}_\mathrm{r}$, compare the proof of Proposition 3.11 and the optimization problem (3.25). Thus, by specifying a reference constraint set $\mathbb{Z}_\mathrm{r}$ offline, we trade off achievable terminal set size (and hence convergence speed of the closed loop) against operation close to the boundary of the constraint set $\mathbb{Z}$. In the following, we show how α can be optimized online, instead of using a preassigned reference constraint set $\mathbb{Z}_\mathrm{r}$.

Related work

For linear systems with polytopic constraints, the maximal invariant set for tracking directly yields a polytopic terminal set $\mathbb{X}_\mathrm{f} \subseteq \mathbb{X} \times \mathbb{Z}_\mathrm{r}$ [163]. Due to scalability issues with this approach, different modifications have been considered in the literature. In [296], an ellipsoidal invariant set for tracking is considered, resulting in a quadratic expression for $\alpha(r)$. In [256], the terminal set is a polytope centered around the artificial reference r and scaled with an additional optimization variable α. In [164], for nonlinear systems, the reference constraint set $\mathbb{Z}_\mathrm{r}$ is partitioned and for each partition i a constant size α_i is used. More recently, in [62] an explicit polynomial expression for $\alpha(r)$ is computed offline using SOS optimization. In case that a fixed dynamic trajectory r is considered, a time-varying function $\alpha(t)$ can be computed offline [91, 93].

We extend the previous approaches to nonlinear dynamics, nonlinear (Lipschitz continuous) constraints, and dynamic reference trajectories. Instead of finding an explicit nonlinear function $\alpha(r(\cdot|t))$ offline (as done in [62, 164, 296]), we include α as a scalar optimization variable in the MPC optimization problem. Thus, the proposed modification does not require any additional complex offline design and the online computational complexity is only marginally increased.

Proposed formulation

In order to automate the trade-off regarding α, we need a condition, which is simple to evaluate and guarantees satisfaction of Condition (3.6b) for a specific $r \in \mathbb{Z}$. To this end, we assume Lipschitz continuity of the constraints.

Assumption 3.36. *(Lipschitz continuous constraints) There exist Lipschitz continuous functions $g_i : \mathbb{R}^{n+m} \to \mathbb{R}$ with Lipschitz constants $L_i \geq 0$, $i \in \mathbb{I}_{[1,n_\mathrm{z}]}$ such that*

$$\left\{(x,u) \in \mathbb{R}^{n+m} \mid g_i(x,u) \leq 0,\ i \in \mathbb{I}_{[1,n_\mathrm{z}]}\right\} \subseteq \mathbb{Z}.$$

Lemma 3.37. *Let Assumption 3.36 hold. Suppose that* $\ell(x, k_\mathrm{f}(x,r)) \leq V_\mathrm{f}(x,r)$, $\forall (x,r) \in \mathbb{X} \times \mathbb{Z}$. *Then, there exist continuous functions* $c_i : \mathbb{Z} \to \mathbb{R}_{\geq 0}$, $i \in \mathbb{I}_{[1,n_\mathrm{z}]}$ *such that for any* $r = (x_\mathrm{r}, u_\mathrm{r}) \in \mathbb{Z}$, $x \in \mathbb{X}$, *the following inequalities hold*

$$g_i(x, k_\mathrm{f}(x,r)) \leq g_i(x_\mathrm{r}, u_\mathrm{r}) + c_i(r)\sqrt{V_\mathrm{f}(x,r)},\ i \in \mathbb{I}_{[1,n_\mathrm{z}]}.$$

Proof. For all $i \in \mathbb{I}_{[1,n_\mathrm{z}]}$, a constant function c_i can be directly obtained using

$$\begin{aligned} & g_i(x, k_\mathrm{f}(x,r)) - g_i(x_\mathrm{r}, u_\mathrm{r}) \leq L_i \|(x, k_\mathrm{f}(x,r)) - (x_\mathrm{r}, u_\mathrm{r})\| \\ \leq & L_i \sqrt{\frac{\ell(x, k_\mathrm{f}(x,r), r)}{\min\{\lambda_{\min}(Q), \lambda_{\min}(R)\}}} \overset{(3.6a)}{\leq} \underbrace{\frac{L_i}{\sqrt{\min\{\lambda_{\min}(Q), \lambda_{\min}(R)\}}}}_{=:c_i} \sqrt{V_\mathrm{f}(x,r)}. \qquad \blacksquare \end{aligned}$$

Note that $V_\mathrm{f}(x,r) \geq \ell(x, k_\mathrm{f}(x,r))$ follows directly if Inequality (3.6a) holds for all $r \in \mathbb{Z}$ (not just $r \in \mathbb{Z}_\mathrm{r}$). While this proof only provides a constructive formula for constant functions c_i, for many important special cases less conservative functions c_i depending on r can be constructed. In particular, in case of polytopic constraints, i.e., $g_i(r) = L_i r - l_i$, and V_f, k_f parametrized according to Lemma 3.12, simple functions are given by the following formula

$$c_i(r) := \|P^{-1/2}(r) \begin{pmatrix} I_n & K^\top(r) \end{pmatrix} L_i^\top\|, \quad i \in \mathbb{I}_{[1,n_\mathrm{z}]}, \tag{3.42}$$

which is based on the support function [60, Equation (10)]. In case P, K are computed using the parametrization $P = X^{-1}$, $K = YP$ from Lemma 3.13 or Proposition 3.15 with X, Y continuously differentiable, then the functions c_i in Equation (3.42) are continuous and can be efficiently implemented in CasADi [17] with $P^{-1/2}$ as the symbolic Cholesky decomposition of X. Given the functions c_i, some arbitrary lower bound $\alpha_{\min} > 0$, and the terminal cost V_f, the proposed MPC formulation is given by the following optimization problem.

Problem 3.38.

$$\underset{u(\cdot|t),\, r(\cdot|t),\, \alpha_\mathrm{s}(t)}{\text{minimize}} \quad \mathcal{J}_N(x(\cdot|t), u(\cdot|t), r(\cdot|t)) + \mathcal{J}_{T,\mathrm{e}}(r(\cdot|t), y_\mathrm{e}(\cdot|t)) \tag{3.43a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.43b}$$

$$x(0|t) = x(t), \tag{3.43c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.43d}$$

$$V_{\mathrm{f}}(x(N|t), r(N|t)) \leq \alpha_{\mathrm{s}}^2(t), \tag{3.43e}$$

$$\sqrt{\alpha_{\min}} \leq \alpha_{\mathrm{s}}(t) \leq \sqrt{\alpha_1}, \tag{3.43f}$$

$$x_{\mathrm{r}}(j+1|t) = f(x_{\mathrm{r}}(j|t), u_{\mathrm{r}}(j|t)), \quad j \in \mathbb{I}_{[0,T-1]}, \tag{3.43g}$$

$$g_i(r(j|t)) + \alpha_{\mathrm{s}}(t) \cdot c_i(r(j|t)) \leq 0, \quad i \in \mathbb{I}_{[1,n_z]}, \quad j \in \mathbb{I}_{[0,T-1]}, \tag{3.43h}$$

$$r(l+T|t) = r(l|t), \quad l \in \mathbb{I}_{[0,\max\{0,N-T\}]}. \tag{3.43i}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, the artificial reference trajectory $r^*(\cdot|t) = (x_{\mathrm{r}}^*(\cdot|t), u_{\mathrm{r}}^*(\cdot|t))$, an online optimized terminal set size $\alpha_{\mathrm{s}}^*(t)$, and the value function $W_{N,T}(x(t), y_{\mathrm{e}}(\cdot|t))$. Compared to Problem 3.27, the reference constraints (3.28f) are replaced by (3.43g)–(3.43h), the terminal set constraint (3.28e) is replaced by (3.43e) and we have one additional scalar optimization variable α_{s} subject to (3.43f). For a fixed value $\alpha = \alpha_{\mathrm{s}}^2$, Problem 3.38 corresponds to Problem 3.27 in case the terminal set $\mathbb{X}_{\mathrm{f}}$ is structured as in Proposition 3.11 and the reference constraint set is given by $\mathbb{Z}_{\mathrm{r}} = \left\{ r \in \mathbb{R}^{n+m} \mid g_i(r) + \alpha_{\mathrm{s}} c_i(r) \leq 0,\ i \in \mathbb{I}_{[1,n_z]} \right\}$. The following algorithm summarizes the closed-loop operation.

Algorithm 3.39. *(Tracking MPC Algorithm with artificial reference trajectory and online optimized terminal set size)*
Offline: Specify the constraint set $\mathbb{Z}$, the weighting matrices (Q, R, S), the prediction horizon N, the period length T, and the minimal size $\alpha_{\min} > 0$. Design suitable terminal ingredients V_{f}, k_{f} (cf. Sec. 3.1) and construct the functions c_i (cf. Lemma 3.37 or (3.42)).
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, obtain the target signal $y_{\mathrm{e}}(\cdot|t)$, solve Problem 3.38, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t), \quad t \in \mathbb{I}_{\geq 0}. \tag{3.44}$$

Theoretical analysis

In order to analyse the resulting closed loop, we define the following reference constraint set

$$\tilde{\mathbb{Z}}_{\mathrm{r}} = \left\{ r \in \mathbb{Z} \mid g_i(r) + \sqrt{\alpha_{\min}} c_i(r) \leq 0,\ i \in \mathbb{I}_{[1,n_z]} \right\},$$

which ensures that any solution of Problem 3.38 satisfies $r^*(j|t) \in \tilde{\mathbb{Z}}_{\mathrm{r}}$, $j \in \mathbb{I}_{[0,T-1]}$ since $\alpha_{\mathrm{s}} \geq \sqrt{\alpha_{\min}}$ (cf. (3.43f)). The following proposition confirms that the proposed enhancement preserves the theoretical properties in Theorem 3.31.

Proposition 3.40. *Consider $\mathbb{Z}_{\mathrm{r}} = \tilde{\mathbb{Z}}_{\mathrm{r}}$. Let Assumptions 3.26, 3.29, and 3.36 hold. Suppose that V_{f}, k_{f}, α_1 satisfy the conditions in Proposition 3.11 and that $V_{\mathrm{f}}(x,r) \geq \ell(x, k_{\mathrm{f}}(x,r))$ for all $(x,r) \in \mathbb{X} \times \mathbb{Z}$. If the initial condition x_0 is such that Problem 3.38 is feasible at $t = 0$, then the closed-loop system* (3.44) *resulting from Algorithm 3.39 satisfies the constraints* (3.1)*, Problem 3.38 is feasible for all $t \in \mathbb{I}_{\geq 0}$, and the optimal reachable trajectory x_T^* is (uniformly) exponentially stable.*

Proof. First, note that every feasible solution of Problem 3.27 is also a feasible solution of Problem 3.38 with $\alpha_{\mathrm{s}} = \sqrt{\alpha_{\min}}$, if the terminal set $\mathbb{X}_{\mathrm{f}}$ is chosen based on Proposition 3.11 with $\alpha = \alpha_{\min}$. Recursive feasibility follows with the same candidate input using $\alpha_{\mathrm{s}}(t+1) = \alpha_{\mathrm{s}}^*(t)$. Parts II and III of Theorem 3.31 remain true since Inequality (3.7) holds for all references $r(\cdot|t) \in \tilde{\mathbb{Z}}_{\mathrm{r}}$ with $\epsilon = \sqrt{\alpha_{\min}/c_{\mathrm{u}}} > 0$ (cf. Prop. 3.11). Satisfaction of Assumption 3.29 with $\tilde{\mathbb{Z}}_{\mathrm{r}}$ ensures that the reference $\hat{y}_{\mathrm{r}}$ (3.37) satisfies the constraints in Problem 3.38 with $\alpha_{\mathrm{s}} = \sqrt{\alpha_{\min}}$. ■

In summary, the optimization over α_{s} provides an additional degree of freedom which can significantly enlarge the terminal set, lead to faster convergence and results in a larger region of attraction. The performance benefits and applicability of this online optimization of α will be demonstrated with numerical examples in Sections 3.4.1 and 3.4.2.

Remark 3.41. *(Connection to (tube-based) robust MPC approaches) The tightened constraints on the reference r* (3.43h) *are analogous to the tightened constraints in recent robust MPC schemes, where α_{s} characterizes a tube in terms of a sublevel set of an incremental Lyapunov function [29, JK13, JK17, JK18, JK29, JK30]. Due to this correspondence, we conjecture that Lipschitz continuity of the constraints (Ass. 3.36) can be relaxed to general continuity conditions (cf. [JK29, App. B]) and even non-smooth collision avoidance constraints can be considered (cf. [JK36]), at the expense of additional computational complexity. Furthermore, for linear* *systems with polytopic constraints (g_i linear), a polytopic[8] function $\tilde{V}_{\mathrm{f}}$ and constants c_i can be computed (cf. [JK11] and [JK29, Sec. IV.B]). In this case, Problem 3.38 reduces to a linearly constrained quadratic program (QP). For this special case and $T = 1$, the constraints* (3.43h) *correspond to [256, Eq. (18f)].*

Remark 3.42. *(Relaxed reference constraints using contraction rate) It is possible to further relax the tightened reference constraints* (3.43h)*, by taking into account the fact that the terminal cost is contractive with some constant $\rho \in [0, 1)$, i.e.,*

$$V_{\mathrm{f}}(f(x, k_{\mathrm{f}}(x, r)), r^+) \leq \rho^2 V_{\mathrm{f}}(x, r), \quad \forall\, (r, r^+) \in \mathcal{R}, \quad \forall x : V_{\mathrm{f}}(x, r) \leq \alpha_1, \tag{3.45}$$

compare [JK15, Prop. 1]. In particular, by redefining $\alpha_{\mathrm{s}} = \sqrt{\alpha} - \sqrt{\alpha_{\min}}$ we can replace the constraints (3.43e), (3.43f), (3.43h) *with the following less conservative constraints*

$$\begin{aligned}
&V_{\mathrm{f}}(x(N|t), r(N|t)) \leq (\alpha_{\mathrm{s}}(t) + \sqrt{\alpha_{\min}})^2, \\
&g_i(r(j|t) + (\alpha_{\mathrm{s}}(t)\rho^{\mathrm{mod}(j+T-N,T)} + \sqrt{\alpha_{\min}})c_i(r(j|t)) \leq 0, \quad i \in \mathbb{I}_{[1,n_z]}, \quad j \in \mathbb{I}_{[0,T-1]}, \\
&\alpha_{\mathrm{s}}(t) \in [0, \sqrt{\alpha_1} - \sqrt{\alpha}_{\min}],
\end{aligned}$$

where mod *denotes the modulo operator. The result in Proposition 3.40 remains valid with the candidate solution $\alpha_{\mathrm{s}}(t+1) = \rho\alpha_{\mathrm{s}}^*(t)$, which satisfies $(\alpha_{\mathrm{s}}(t+1) + \sqrt{\alpha_{\min}})^2 = \alpha(t+1) \geq$* $\max\{\rho^2\alpha^*(t), \alpha_{\min}\}$*, with $\alpha^*(t) = (\alpha_{\mathrm{s}}^*(t) + \sqrt{\alpha}_{\min})^2$. These relaxed constraints are especially useful in transient operation with active constraints on the reference r and $\rho^T \ll 1$.*

3.2.3 Partially decoupled reference updates

In the following, we demonstrate that the joint stabilization and trajectory planning MPC formulation (Problem 3.27/3.38) can be formulated as two partially decoupled optimization problems. With this partially decoupled formulation, we can significantly reduce the online computational demand by introducing a partial time scale separation.

Motivation

The main premise of the proposed approach using artificial reference trajectories (Sec. 3.2) is that the operating conditions change on a time scale similar to the system dynamics, which in turn necessitates frequent online updates of the reference trajectory r. The most

[8] In this case the terminal set $\mathbb{X}_{\mathrm{f}}$ and the constants c_i are computed based on the polytopic Lyapunov function $\tilde{V}_{\mathrm{f}}$, while a different quadratic terminal cost V_{f} is used in the cost function $\mathcal{J}_N$.

challenging problems are those, where the operating conditions change at a time scale similar to the system dynamics, while the target signal and hence the optimal system operation is determined based on long term considerations that involve a significantly larger time scale, i.e., the period length T is very large. An example of such a multi time scale problem would be the power grid, compare, e.g., [154], where real time decisions are made every 5 minutes, while the planning horizon is 7 days yielding $T \geq 2 \cdot 10^3$. For such problems, it is vital that the reference r is updated frequently, while at the same time it may be computationally too expensive to solve the joint planning and regulation problem (Problem 3.27/3.38) in each time step t.

Related work

The issue of time scale separations in predictive control is a long standing problem with a divers set of methodologies and heuristics considered in literature. For example, in process control, planning and control are often performed on different time scales with different models [171]. One classical approach to improve scalability of MPC to longer prediction horizons N is to parametrize the predicted input sequence using move blocking [115]. Another standard method to limit the computational demand is to optimize less frequently by considering a longer sampling period (or equivalently a multi-step implementation [125]). The lack of fast feedback can be compensated by using an additional lower-level feedback [134], as is common practice in process control [71, 229]. Since for many nonlinear systems it may be difficult to design a simple stabilizing feedback, a tracking MPC analogous to Section 3.1 can be used in the lower-level, as suggested in [191]. The decoupled formulation in [191] is a special case of the MPC formulation derived in the following, considering terminal equality constraints ($\alpha = 0$) and no updates in the reference trajectory ($M = \infty$).

The methodology closest to to the proposed approach is contract-based MPC [90, 172], which can be used to decompose large problems in a hierarchical setting [26] by using (tube-based) robust MPC methods to capture uncertainty in the planning of other subsystems. Similarly, in a long horizon planning problem a simplified model can be used to capture the long term effects with a reduced computational complexity [27, 44].

Basic idea

The basic idea of the proposed approach is illustrated in Figure 3.1. At each time $t \in \mathbb{I}_{\geq 0}$, a standard tracking MPC (cf. Sec. 3.1) computes the control action $u(t)$ based on the measured plant state $x(t)$ and a fixed reference trajectory $r(\cdot|t)$. The reference

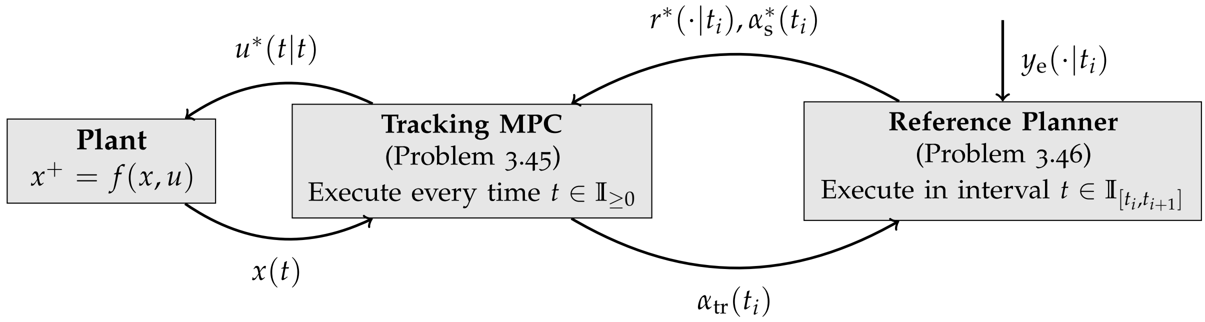

Figure 3.1. Schematic overview of the communication and parallel computation in the partially decoupled planning and tracking MPC approach. The tracking MPC is executed in every time step $t \in \mathbb{I}_{\geq 0}$, while the communication to the reference planner is limited to $t_i = i \cdot M$, $M \in \mathbb{I}_{\geq 1}$. Thus, the reference planning optimization problem (Problem 3.46) can be solved in the time interval $[t_i, t_{i+1}]$.

planner operates in parallel to compute the new reference trajectory $r(\cdot|t_{i+1})$ with the corresponding terminal set size $\alpha_s(t_{i+1})$ based on the target signal $y_e(\cdot|t_i)$. In particular, the reference planner operates with a different sampling period $t_i = i \cdot M$ and sends the updated reference every $M \in \mathbb{I}_{\geq 1}$ time steps to the tracking MPC. Thus, the computationally more demanding reference planning problem can be solved in the time interval $[t_i, t_{i+1}]$. The planner accounts for the most recent state of the plant $x(t_i)$ indirectly based on $\alpha_{tr}(t_{i+1})$, which corresponds to a feasible terminal set size in the tracking MPC for the reference trajectory $r(\cdot|t_i)$ (prior to the update). Recursive feasibility is ensured by imposing additional constraints on the terminal state of the tracking MPC and the corresponding point on the reference trajectory. This partial coupling is visualized in Figure 3.2. In particular, consistency is ensured by enforcing: a) At time t_{i+1} there exists a state trajectory such that $x(N|t_{i+1})$ is in a terminal set around $r^*(N+M|t_i)$ with the reduced size $\alpha_{tr}(t_{i+1})$; b) The new computed reference trajectory $r(\cdot|t_{i+1})$ with the corresponding terminal set size $\alpha_s(t_{i+1})$ is such that the following implication holds

$$V_f(x, r^*(N+M|t_i)) \leq \alpha_{tr}(t_{i+1}) \quad \Rightarrow \quad V_f(x, r(N|t_{i+1})) \leq \alpha_s^2(t_{i+1}). \tag{3.46}$$

In Figure 3.2, we see that although this consistency constraint limits the update of reference trajectory, it still allows for a large degree of freedom since the additional constraint only restricts the point $r(N|t_{i+1})$.

The consistency constraint can be seen as an analogue to contract-based design methods. From a communication and computation point of view, the decomposition

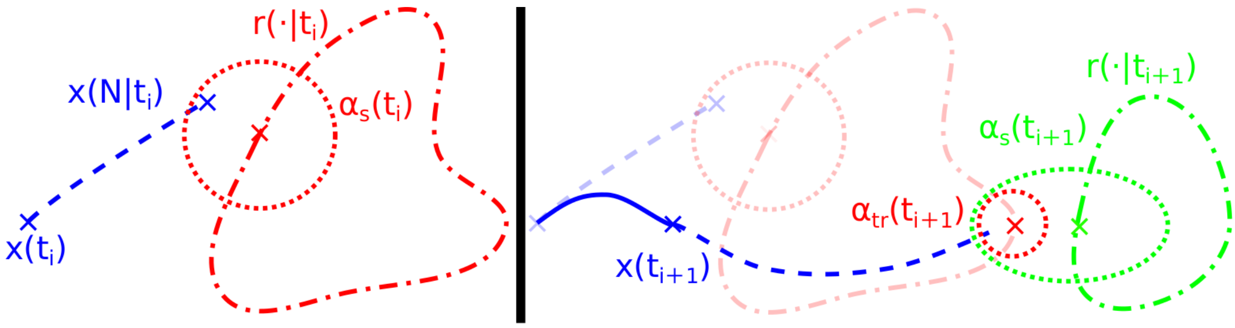

Figure 3.2. Illustration of the needed coupling constraint between the tracking MPC and the reference planner: Predicted state trajectory $x(\cdot|t)$, $t \in \{t_i, t_{i+1}\}$ (blue, dashed), closed-loop state trajectory (blue, solid), artificial reference trajectory $r(\cdot|t_i)$ (red, dashed-dotted) with terminal set based on $\alpha_s(t_i)$ and $\alpha_{tr}(t_{i+1})$ (red, dotted), new artificial reference trajectory $r(\cdot|t_{i+1})$ (green, dashed-dotted) with terminal set based on $\alpha_s(t_{i+1})$ (green, dotted) at time t_i (left) and time t_{i+1} (right).

is similar to cascade and hierarchical approaches. The proposed approach strongly differs from existing methods by providing a formulation tailored to the tracking MPC with artificial reference trajectories, guaranteeing feasibility for nonlinear systems, and allowing for a flexible trade-off between computational complexity and fast/frequent reference updates using a factor $M \in \mathbb{I}_{\geq 1}$.

In the following, we detail the different components. First, a continuity condition on the terminal cost V_f will be exploited to derive simple sufficient conditions for the set inclusion (3.46). Next, the tracking MPC with a contractive terminal set constraint is presented, which allows for a priori bounds on $\alpha_{tr}(t_{i+1})$. Then, the reference planner with the coupling constraint is presented, the overall algorithm is summarized, and finally Proposition 3.48 provides feasibility and convergence guarantees.

Continuity of the terminal cost

To obtain a simple to evaluate sufficient condition for the set inclusion (3.46), we consider the following continuity assumption for the terminal cost V_f.

Assumption 3.43. *(Continuity of the terminal cost) There exists a function $\gamma_f \in \mathcal{K}_\infty$ such that for any $(x_r, u_r) = r \in \mathbb{Z}$, $(\tilde{x}_r, \tilde{u}_r) = \tilde{r} \in \mathbb{Z}$, $x \in \mathbb{X}$:*

$$\sqrt{V_f(x, \tilde{r})} \leq \sqrt{V_f(x, r)}(1 + \gamma_f(\|r - \tilde{r}\|)) + \sqrt{V_f(x_r, \tilde{r})}. \tag{3.47}$$

The following proposition shows that the terminal cost based on the reference generic offline computation (cf. Sec. 3.1.3) satisfies Assumption 3.43.

Proposition 3.44. *Suppose* $V_{\mathrm{f}}(x,r) = \|x - x_{\mathrm{r}}\|_{P(r)}^2$ *,* $P = X^{-1}$ *with* X *according to* (3.21) *and* $\theta : \mathbb{Z} \to \mathbb{R}^p$ *continuously differentiable. Then, Assumption 3.43 holds with a linear function* γ_{f}*.*

Proof. The fact that θ is continuously differentiable directly implies that X is continuously differentiable w.r.t. r. Thus, also the matrix $P = X^{-1}$ is continuously differentiable in r for any $r \in \mathbb{Z}$, using the fact that X is positive definite with uniform lower and upper bounds. This property in combination with compact constraints and uniform bounds on P ensures that there exists a Lipschitz constant $L_{\mathrm{P}} > 0$ such that

$$P(\tilde{r}) - P(r) \preceq L_{\mathrm{P}} \cdot P(r)\|r - \tilde{r}\|, \quad \forall r, \tilde{r} \in \mathbb{Z}.$$

This implies

$$\begin{aligned} \|x\|_{P(\tilde{r})} \leq & \sqrt{\|x\|_{P(r)}^2 + L_{\mathrm{P}}\|x\|_{P(r)}^2\|r - \tilde{r}\|} = \|x\|_{P(r)}\sqrt{1 + L_{\mathrm{P}}\|r - \tilde{r}\|} \\ \leq & \|x\|_{P(r)}(1 + L_{\mathrm{P}}\|r - \tilde{r}\|), \end{aligned} \tag{3.48}$$

for any $x \in \mathbb{X}$. Thus, Inequality (3.47) follows from

$$\begin{aligned} \sqrt{V_{\mathrm{f}}(x,\tilde{r})} = & \|x - \tilde{x}_{\mathrm{r}}\|_{P(\tilde{r})} \leq \|x - x_{\mathrm{r}}\|_{P(\tilde{r})} + \|x_{\mathrm{r}} - \tilde{x}_{\mathrm{r}}\|_{P(\tilde{r})} \\ \overset{(3.48)}{\leq} & \sqrt{V_{\mathrm{f}}(x,r)}(1 + L_{\mathrm{P}}\|r - \tilde{r}\|) + \|x_{\mathrm{r}} - \tilde{x}_{\mathrm{r}}\|_{P(\tilde{r})}, \end{aligned}$$

with $\gamma_{\mathrm{f}}(c) := L_{\mathrm{P}} \cdot c$. ∎

This result essentially follows from the continuously differentiable parametrization and the triangular inequality. In the special case of constant matrices P, Condition (3.47) is satisfied with $\gamma_{\mathrm{f}} = 0$. In addition to the continuity property (3.47), the following design exploits the fact that the terminal cost is contractive with some factor $\rho \in [0,1)$, compare (3.45).

Contractive tracking MPC

Suppose that at time $t_i = i \cdot M$, $i \in \mathbb{I}_{\geq 0}$, we have trajectories $x(\cdot|t_i)$, $u(\cdot|t_i)$, $r^*(\cdot|t_i)$, $\alpha_{\mathrm{s}}^*(t_i)$, that satisfy the constraints in Problem 3.38. Given the reference trajectory $r^*(\cdot|t_i)$, the tracking MPC considers the shifted reference trajectory $r(j|t) = r^*(\mathrm{mod}(j + t - t_i, T)|t_i)$, $j \in \mathbb{I}_{[0,T-1]}$, $t \in \mathbb{I}_{[t_i,t_i+M-1]}$ (until the reference trajectory is updated at t_{i+1}). The terminal set size is updated as follows

$$\alpha_{\mathrm{tr}}(t) = \rho^{2(t-t_i)} \max\{\alpha_{\min}, V_{\mathrm{f}}(x(N|t_i), r(N|t_i))\}, \quad t \in \mathbb{I}_{[t_i, t_i+M-1]}, \tag{3.49}$$

with the contraction rate $\rho \in [0,1)$ satisfying Condition (3.45). At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t)$, the reference trajectory $r(\cdot|t) \in \tilde{\mathbb{Z}}_{\mathrm{r}}^{N+1}$ and $\alpha_{\mathrm{tr}}(t)$, the reference tracking MPC is given by the following optimization problem:

Problem 3.45.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_N(x(\cdot|t), u(\cdot|t), r(\cdot|t)) \tag{3.50a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.50b}$$

$$x(0|t) = x(t), \tag{3.50c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.50d}$$

$$V_{\mathrm{f}}(x(N|t), r(N|t)) \leq \alpha_{\mathrm{tr}}(t). \tag{3.50e}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$ and the corresponding state trajectory $x^*(\cdot|t)$. Problem 3.45 is similar to Problem 3.3 in Section 3.1.1, but with a contractive terminal set constraint (3.50e). Note that the contractive terminal constraint (3.50e) with α_{tr} according to (3.49) is similar to the contractive constraint used in [216].

Reference planner

In parallel, the reference is updated based on a reference planning problem. At each time $t_i = i \cdot M$, $i \in \mathbb{I}_{\geq 0}$, given $r^*(\cdot|t_i)$, $\alpha_{\mathrm{tr}}(t_i)$ and the target signal $y_{\mathrm{e}}(\cdot|t_i)$, the reference trajectory is updated based on the following optimization problem:

Problem 3.46.

$$\underset{r(\cdot|t_{i+1}),\ \alpha_{\mathrm{s}}(t_{i+1})}{\text{minimize}} \quad \mathcal{J}_{T,\mathrm{e}}(r(\cdot|t_{i+1}), \overline{y}_{\mathrm{e}}(\cdot|t_{i+1})) \tag{3.51a}$$

subject to

$$\rho^M \sqrt{\alpha_{\mathrm{tr}}(t_i)}(1+\gamma_{\mathrm{f}}(\|r^*(N+M|t_i)-r(N|t_{i+1})\|))$$
$$+\sqrt{V_{\mathrm{f}}(x_{\mathrm{r}}^*(N+M|t_i),r(N|t_{i+1}))} \leq \alpha_{\mathrm{s}}(t_{i+1}), \tag{3.51b}$$
$$\sqrt{\alpha_{\min}} \leq \alpha_{\mathrm{s}}(t_{i+1}) \leq \sqrt{\alpha_1}, \tag{3.51c}$$
$$x_{\mathrm{r}}(j+1|t_{i+1}) = f(x_{\mathrm{r}}(j|t_{i+1}),u_{\mathrm{r}}(j|t_{i+1})), \quad j \in \mathbb{I}_{[0,T-1]}, \tag{3.51d}$$
$$g_k(r(j|t_{i+1})) + \alpha_{\mathrm{s}}(t_{i+1}) \cdot c_k(r(j|t_{i+1})) \leq 0, \quad k \in \mathbb{I}_{[1,n_z]}, \quad j \in \mathbb{I}_{[0,T-1]}, \tag{3.51e}$$
$$r(l+T|t_{i+1}) = r(l|t_{i+1}), \quad l \in \mathbb{I}_{[0,\max\{0,N+M-T\}]}, \tag{3.51f}$$
$$\overline{y}_{\mathrm{e}}(j|t_{i+1}) = y_{\mathrm{e}}(\mathrm{mod}(j+M,T)|t_i), \quad j \in \mathbb{I}_{[0,T-1]}. \tag{3.51g}$$

The solution to this optimization problem is an optimal reference trajectory $r^*(\cdot|t_{i+1})$ and the corresponding terminal set size $\alpha_{\mathrm{s}}^*(t_{i+1})$. The constraint (3.51b) is a simple sufficient condition for the consistency constraint (3.46) exploiting the continuity bound in Assumption 3.43. Note that the constraints on the reference r can be further relaxed using the formula in Remark 3.42. Since we start to solve Problem 3.46 at time t_i, the target signal $y_{\mathrm{e}}(\cdot|t_{i+1})$ is not yet available and instead in (3.51g) the currently available target signal $y_{\mathrm{e}}(\cdot|t_i)$ is shifted by M time steps (assuming it is T-periodic). We point out that we use the information available at time t_i, in order to compute the reference trajectory at time t_{i+1}.

Overall algorithm

The overall procedure is summarized in Algorithm 3.47.

Algorithm 3.47. *(Tracking MPC Algorithm with partially decoupled reference updates)*
Offline: Specify the constraint set $\mathbb{Z}$, the weighting matrices (Q, R, S), the prediction horizon N, the period length T, the reference update frequency M, and the minimal size $\alpha_{\min} > 0$. Design suitable terminal ingredients V_{f}, k_{f} (cf. Sec. 3.1). Construct the functions c_i (cf. Lemma 3.37 or (3.42)*) and γ_{f} (cf. Ass. 3.43 or Prop. 3.44).*
Online: Execute at each time step $t_i = i \cdot M$, $i \in \mathbb{I}_{\geq 0}$:
Obtain the reference $r^*(\cdot|t_i)$ from the reference planner (Problem 3.46).
Get $x(N|t_i)$ from the candidate solution of the Tracking MPC (Problem 3.45).
Compute $\alpha_{\mathrm{tr}}(t_i)$ using (3.49)*.*
Execute in parallel:

Tracking MPC
for $t \in \mathbb{I}_{[t_i, t_i+M-1]}$ ***do***
Update $r(\cdot|t)$, $\alpha_{\mathrm{tr}}(t)$ (3.49).
Solve Problem 3.45.
Apply the control input $u(t) = u^*(0|t)$.
end for

Reference planner
Obtain the target signal $y_{\mathrm{e}}(\cdot|t_i)$.
Solve Problem 3.46.
Get $r^*(\cdot|t_{i+1})$, $\alpha_{\mathrm{s}}^*(t_{i+1})$.

Problem 3.45 represents a standard tracking MPC (Sec. 3.1.1) that is executed in each time step t with a fixed (periodic) reference trajectory r and a shrinking terminal set α_{tr}. On the other hand, Problem 3.46 can be solved in the interval $[t_i, t_{i+M}]$, thus allowing to solve larger planning problems ($T \gg 1$) by updating the reference r less frequently ($M \in \mathbb{I}_{\geq 1}$). Condition (3.51b) constrains how the updated reference r may deviate from the previous solution, which partially couples the planning (Problem 3.46) and regulation problem (Problem 3.45). Compared to a joint optimization (cf. Problems 3.27/3.38), the practical convergence under changing operation conditions may be slower, since the reference is updated less frequently and the constraint (3.51b) may limit the rate of change, compare the numerical example in Section 3.4.2. However, the partially decoupled updates in Algorithm 3.47 can significantly reduce the computational demand, especially in case of long planning horizons T.

Theoretical analysis

The following proposition shows that the proposed partially decoupled reference updates ensure feasibility and convergence.

Proposition 3.48. *Let the conditions in Proposition 3.40 and Assumption 3.43 hold. Suppose further that Algorithm 3.47 is initialized at* $t = t_0 = 0$ *with* $r^*(\cdot|t_0)$, $\alpha_{\mathrm{s}}^*(t_0)$ *satisfying the constraints* (3.51c)–(3.51f) *in Problem 3.46,* $\alpha_{\mathrm{tr}}(t_0) \leq \alpha_{\mathrm{s}}^*(t_0)$ *and* $x(t_0)$ *are such that Problem 3.45 is feasible. Then, the closed-loop system resulting from Algorithm 3.47 satisfies the constraints* (3.1)*, Problem 3.45 is feasible for all* $t \in \mathbb{I}_{\geq 0}$ *and Problem 3.46 is feasible for all* $t_i = i \cdot M$, $i \in \mathbb{I}_{\geq 0}$. *Assume further*[9] *that there exists a constant* $\underline{c} > 0$ *such that for every constraint* $k \in \mathbb{I}_{[1,n_z]}$*, we have either* $\inf_{r\in\mathbb{Z}} c_k(r) \geq \underline{c}$ *or* $\sup_{r\in\mathbb{Z}} c_k(r) = 0$*. Then, the resulting reference* $r^*(\cdot|t_i)$ *converges to the optimal reachable trajectory* x_T^* *in finite time and the state* $x(t)$ *converges (uniformly) exponentially to* x_T^*.

Proof. **Part I:** Recursive feasibility: First, for $t \in \mathbb{I}_{[t_i, t_i+M-1]}$ the reference $r(\cdot|t)$ satisfies the tightened constraints (3.51e) with $\alpha_{\mathrm{s}}^*(t_i)$. Thus, feasibility of Problem 3.45 at time t_i

[9] This condition excludes the special case where $c_k(r) = 0$ for some (but not all) $r \in \mathbb{Z}$ and can always be ensured by adding a small positive constant to c_k.

implies recursive feasibility of Problem 3.45 at time t with the updated terminal set size $\alpha_{\mathrm{tr}}(t) \leq \alpha_{\mathrm{s}}^*(t_i)$ according to Equation (3.49), using the standard MPC candidate solution from Theorem 3.31/Proposition 3.40 and the contractivity (3.45). Correspondingly, at time t_{i+1}, the candidate solution satisfies

$$V_{\mathrm{f}}(x(N|t_{i+1}), r^*(N+M|t_i)) \leq \rho^{2M}\alpha_{\mathrm{tr}}(t_i). \tag{3.52}$$

The constraint (3.51b) ensures

$$\begin{aligned}
&\sqrt{V_{\mathrm{f}}(x(N|t_{i+1}), r^*(N|t_{i+1}))} \\
\overset{(3.47),(3.52)}{\leq} &\rho^M\sqrt{\alpha_{\mathrm{tr}}(t_i)}(1+\gamma_{\mathrm{f}}(\|r^*(N+M|t_i) - r^*(N|t_{i+1)}\|)) \\
&+\sqrt{V_{\mathrm{f}}(x_{\mathrm{r}}^*(N+M|t_i), r^*(N|t_{i+1}))} \\
\overset{(3.51b)}{\leq} &\alpha_{\mathrm{s}}^*(t_{i+1}),
\end{aligned}$$

which in combination with the update (3.49) and the constraint (3.51c) implies $\alpha_{\mathrm{tr}}(t_{i+1}) \leq \alpha_{\mathrm{s}}^*(t_{i+1})$. At time t_{i+1} a feasible solution of Problem 3.46 is given by the previous reference r shifted by M steps, i.e., $r(j|t_{i+1}) = r^*(\mathrm{mod}(j+M,T)|t_i)$, $j \in \mathbb{I}_{[0,T-1]}$, with the candidate terminal set size

$$\alpha_{\mathrm{s}}(t_{i+1}) = \max\left\{\rho^M\sqrt{\alpha_{\mathrm{tr}}(t_i)} + 0.5\left(1-\rho^M\right)\sqrt{\alpha_{\min}}, \sqrt{\alpha_{\min}}\right\},$$

and Condition (3.51b) is strictly satisfied using the fact that $V_{\mathrm{f}}(x_{\mathrm{r}}^*(N+M|t_i), r(N|t_{i+1})) = 0$ by definition.

Part II: Convergence: Consider the auxiliary candidate reference $\hat{r}$ based on $\hat{y}_{\mathrm{r}}$ from the definition (3.37) with some $\beta_{t_i} \in [0,1)$. Suppose β_{t_i} is chosen such that $\|\hat{r} - r(\cdot|t_{i+1})\| \leq \epsilon$, with some constant $\epsilon > 0$. There exists a constant $\epsilon_1 > 0$ such that for $\epsilon \leq \epsilon_1$ this auxiliary reference $\hat{r}$ satisfies the constraint (3.51b) with

$$\begin{aligned}
&\rho^M\sqrt{\alpha_{\mathrm{tr}}(t_i)}(1+\gamma_{\mathrm{f}}(\|r(N|t_{i+1}) - \hat{r}(N)\|)) + \sqrt{V_{\mathrm{f}}(x_{\mathrm{r}}(N|t_{i+1}), \hat{r}(N))} \\
\leq&\rho^M\sqrt{\alpha_{\mathrm{tr}}(t_i)} + \sqrt{\alpha_1}\rho^M\gamma_{\mathrm{f}}(\epsilon_1) + \epsilon_1\sqrt{c_u} := \rho^M\sqrt{\alpha_{\mathrm{tr}}(t_i)} + 0.5\left(1-\rho^M\right)\sqrt{\alpha_{\min}} \\
\leq&\alpha_{\mathrm{s}}(t_{i+1}).
\end{aligned}$$

We show satisfaction of (3.51e) for the auxiliary reference $\hat{r}$ with a case distinction.

Case 1: Suppose that $\alpha_{\mathrm{s}}(t_{i+1}) = \sqrt{\alpha_{\min}}$. In this case, Condition (3.51e) is equivalent

to $\hat{r} \in \tilde{\mathbb{Z}}_{\mathrm{r}} = \mathbb{Z}_{\mathrm{r}}$, which is guaranteed by the convexity condition (cf. Ass. 3.29) as in Proposition 3.40.
Case 2: $\alpha_{\mathrm{s}}(t_{i+1}) = \rho^M \sqrt{\alpha_{\mathrm{tr}}(t_i)} + 0.5\left(1-\rho^M\right)\sqrt{\alpha_{\min}}$. Given c_k continuous (cf. Ass. 3.43) and $\mathbb{Z}$ compact, there exists a function $\delta \in \mathcal{K}_\infty$ such that for any $r, \tilde{r} \in \mathbb{Z}$: $c_k(r) - c_k(\tilde{r}) \leq \delta(\|r-\tilde{r}\|)$, $k \in \mathbb{I}_{[1,n_z]}$. For constraints $k \in \mathbb{I}_{[1,n_z]}$ with $c_k = 0$, feasibility of Condition (3.51e) is independent of α_{s} and thus follows from convexity (Ass. 3.29). For the other constraints $k \in \mathbb{I}_{[1,n_z]}$ satisfaction of Condition (3.51e) at t_{i+1} follows from feasibility of (3.51e) at t_i together with the definition of the candidate reference $r(j|t_{i+1}) = r^*(\mathrm{mod}(j+M,T)|t_i)$, $\alpha_{\mathrm{tr}}(t_i) \leq \alpha_{\mathrm{s}}^*(t_i)$ and $c_k(r) \geq \underline{c}$:

$$\begin{aligned}
& g_k(\hat{r}(j)) + \alpha_{\mathrm{s}}(t_{i+1}) c_k(\hat{r}(j)) \\
\overset{(3.51\mathrm{e})}{\leq}\; & g_k(\hat{r}(j)) - g_k(r(j|t_{i+1})) + \alpha_{\mathrm{s}}(t_{i+1}) c_k(\hat{r}(j)) - \sqrt{\alpha_{\mathrm{tr}}(t_i)} c_k(r(j|t_{i+1})) \\
\overset{\text{Ass. 3.36}}{\leq}\; & L_k \|r(j|t_{i+1}) - \hat{r}(j)\| + \left(\alpha_{\mathrm{s}}(t_{i+1}) - \sqrt{\alpha_{\mathrm{tr}}(t_i)}\right) c_k(r(j|t_{i+1})) \\
& + \alpha_{\mathrm{s}}(t_{i+1}) \delta(\|r(j|t_{i+1}) - \hat{r}(j)\|) \\
\leq\; & L_k \epsilon + \sqrt{\alpha_1}\delta(\epsilon) - \left(1-\rho^M\right)\left(\sqrt{\alpha_{\mathrm{tr}}(t_i)} - 0.5\sqrt{\alpha_{\min}}\right) c_k(r(j|t_{i+1})) \\
\overset{(3.49)}{\leq}\; & L_k \epsilon + \sqrt{\alpha_1}\delta(\epsilon) - \left(1-\rho^M\right) \cdot 0.5\sqrt{\alpha_{\min}} \cdot \underline{c} \leq 0,
\end{aligned}$$

where the last inequality holds for $\epsilon \leq \epsilon_2$ with $\epsilon_2 > 0$ sufficiently small. The reference satisfies

$$\|r(\cdot|t_{i+1}) - \hat{r}\| \leq L_{\mathrm{g}} \|\hat{y}_{\mathrm{r}} - y_{\mathrm{r}}(\cdot|t_i)\|_S \overset{(3.38)}{=} (1-\beta_t) L_{\mathrm{g}} \|y_T^*(\cdot|t_{i+1}) - y_{\mathrm{r}}(\cdot|t_{i+1})\|_S,$$

with the Lipschitz constant $L_g > 0$ (cf. Ass. 3.29). Thus, choosing

$$\beta_{t_i} = \max\left\{1 - \frac{\epsilon}{L_{\mathrm{g}} \|y_{\mathrm{r}}(\cdot|t_{i+1}) - y_T^*(\cdot|t_{i+1})\|_S}, 0\right\} < 1,$$

with $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$, the candidate reference $\hat{r}$ is feasible. In case $\|y_{\mathrm{r}}(\cdot|t_{i+1}) - y_T^*(\cdot|t)\|_S \geq \epsilon/L_{\mathrm{g}}$, this implies

$$\mathcal{J}_{T,\mathrm{e}}(\hat{r}, t_{i+1}) - \mathcal{J}_{T,\mathrm{e}}(r^*(\cdot|t_i), t_i) \overset{(3.39)}{\leq} -(1-\beta_t)(\epsilon/L_{\mathrm{g}})^2 \leq -\epsilon^3/(L_{\mathrm{g}}^3 \overline{y}),$$

with $\overline{y}$ as defined in the proof of Theorem 3.31. In case $\|y_{\mathrm{r}}(\cdot|t_{i+1}) - y_T^*(\cdot|t)\|_S \leq \epsilon/L_{\mathrm{g}}$, the candidate reference converges to the optimal reference trajectory in one step, i.e., $\beta_t = 0$, $\mathcal{J}_{T,\mathrm{e}}(\hat{r}, t_{i+1}) = V_{T,\mathrm{e}}$. This decrease in combination with $\mathcal{J}_{T,\mathrm{e}} - V_T \leq \overline{y}^2$ ensures convergence of $\mathcal{J}_{T,\mathrm{e}}$ to V_T and thus r to r_T^* in at most $T_{\max} := M\lceil(\overline{y}L_{\mathrm{g}}/\epsilon)^3 + 1\rceil$ time steps. Exponential convergence of x to x_T^* follows from exponential stability (Thm. 3.8) and finite-time convergence of the reference r. ∎

The result in Proposition 3.48 essentially builds on three properties. First, the constraints on the reference planner (Problem 3.46) and the tracking MPC (Problem 3.45) with the contracting terminal set size (3.49) are such that the proposed algorithm is recursively feasible (cf. Fig. 3.2 for an illustration). Second, in each time step t_i, it is possible to incrementally reduce the tracking cost $\mathcal{J}_{T,\mathrm{e}}$, which in combination with convexity (Ass. 3.29) and compact constraints implies that the reference planner converges to the optimal reference r_T^* in finite time. Third, once the reference planner converged in finite time, the tracking MPC (Problem 3.45) ensures exponential stability of the reference $r = r_T^*$ (cf. Thm. 3.8) and thus exponential convergence of $x(t)$ to $x_T^*(t)$. We point out that we only showed convergence for this partially coupled approach, as opposed to uniform stability[10] in Theorem 3.31/Proposition 3.40.

Remark 3.49. *(Extensions) It is possible to adjust Algorithm 3.47 such that the reference planner does not require explicit information from the tracking MPC, by replacing the update* $\alpha_{\mathrm{tr}}(t)$ *in* (3.49) *and only using the fact that the terminal set is* ρ*-contractive. Although this may simplify the computation, the closed-loop convergence of the tracking MPC is typically significantly faster, which is why the update* (3.49) *can speed up the convergence rate of the reference planner. It is possible to implement Algorithm 3.47 in an asynchronous fashion with* M *changing online by adjusting the constraint* (3.51b) *to hold for any* $M \in [M_{\min}, M_{\max}] \subset \mathbb{I}_{\geq 1}$. *Thus, the reference planner needs to solve Problem 3.46 until* $t_i + M_{\max}$*, but the reference can also be updated earlier starting at* $t_i + M_{\min}$*. The computational complexity of the reference planner (Problem 3.46) can be further reduced using a simplifying parametrization for the set of artificial periodic reference trajectories* $r(\cdot|t)$*. For example, the reference input* u_{r} *could be parametrized to reduce the number of decision variables, e.g., using move-blocking [115]. In this case, the constraints in the optimization problem would be time-varying and we may experience some performance degradation due to the more restricted class of reference trajectories* $r(\cdot|t)$*, i.e., in general* $\lim_{i\to\infty} \mathcal{J}_{T,\mathrm{e}}(r(\cdot|t_i), y_{\mathrm{e}}(\cdot|t_i)) > V_{T,e}$.

[10] We conjecture that stability can be preserved by adding a suitable regularization penalizing $\|r(k|t_{i+1}) - r^*(k+M|t_i)\|$, $k \in \mathbb{I}_{[0,N]}$ in the cost $\mathcal{J}_{T,\mathrm{e}}$ to compensate the potential cost increase in the tracking MPC cost $\mathcal{J}_N$, compare also the discussion in Remark 3.83.

Summary

In this section, we studied tracking MPC formulations based on *artificial reference* trajectories. The proposed design is applicable to *output* target signals, which may be *unreachable* and subject to unpredictable *changes online*, and provides a large region of attraction. The provided theoretical analysis ensures exponential stability of the optimal reachable periodic trajectory and unifies/generalizes previous theoretical results (cf. [164, 166]) by considering *nonlinear* dynamics and *periodic operation* (Sec. 3.2.1). We further extended this formulation by introducing an additional degree of freedom in the parametrization of the terminal set and the reference constraint set (Sec. 3.2.2). This modification automates a trade-off typically faced in the offline design procedure which results in improved performance and an increased the region of operation. In addition, we introduced a partially decoupled MPC formulation that allows for a partial time scale separation between trajectory tracking and trajectory planning (Sec. 3.2.3). This formulation is particularly relevant to ensure real-time implementability of the proposed MPC formulation in case of long planning horizons T. In the next section, we further extended these tracking MPC formulation with artificial reference trajectories to directly consider general *economic* stage costs.

3.3 Economic MPC with artificial reference trajectories

In Section 3.2, we presented a tracking MPC formulation for (possibly unreachable) output target signals using artificial reference trajectories. In this section, we extend the problem to general time-varying *economic* stage costs ℓ_{eco}, possibly subject to unpredictable changes online. In particular, we consider an MPC formulation with an artificial periodic trajectory and a purely economic cost formulation (no tracking stage cost ℓ). We demonstrate by means of a simple academic example that a naive extension of existing approaches (cf. [87, 206, 208]) to the periodic economic problem, does *not* necessarily yield the desired closed-loop performance guarantees (Sec. 3.3.1). We present a revised economic MPC formulation, imposing additional constraints on the artificial periodic trajectory (Sec. 3.3.2) and derive performance guarantees relative to the limiting artificial reference trajectory (Sec. 3.3.3). Finally, in Section 3.3.4 improved a priori performance bounds are derived using a self-tuning weight (cf. [206, 208]). The design of the terminal ingredients for economic dynamic problems are detailed in Section 3.3.5. Some variants and extensions to the proposed economic MPC framework are presented in Section 3.3.6.

This section is based on and taken in parts literally from [JK26][11].

3.3.1 Dynamic operation and pitfalls in economic MPC

In the following, we generalize the tracking problem considered in Section 3.2, by considering an *economic* (not necessarily positive definite) time-varying stage cost $\ell_{\mathrm{eco}}(x, u, t, y_{\mathrm{e}})$, which may depend on some exogenous parameters y_{e}. In addition, we demonstrate that a naive extension of existing generalized terminal setpoint constraints to periodic reference trajectories r, does *not* necessarily imply desirable economic performance guarantees.

As in Section 3.2, we allow for *online changes in the mode of operation* (cf. Sec. 1.1, (ii)) by using T-periodic artificial reference trajectories. Compared to the tracking problem (Sec. 3.2), the control goal is directly specified with the economic stage cost ℓ_{eco} and the optimal mode of operation is not necessarily T-periodic. Instead of guaranteeing stability of some (optimal) T-periodic trajectory, we guarantee that the closed-loop performance is at least as good (on average) as a (local) optimal T-periodic trajectory. Thus, by considering an *economic* MPC formulation this section mainly addresses the additional challenge when the *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories* (cf. Sec. 1.1, (iii)).

Motivation

Periodic and time-varying operation is natural in many control problems. For example, water distribution networks [165, 282], electrical networks [224] or building and HVAC systems [235, 255] are inherently time-varying/periodic control problems due to changes in the cost or dynamics related to the day-night cycle. Even in the time-invariant problem of maximizing the production in (nonlinear) CSTRs, periodic/dynamic operation can be economically beneficial, compare [24]. Periodic operations also naturally arise in periodic/cyclic scheduling [241] and power generation using kites [80]. Furthermore, in most of these problems, the control goal can be more naturally expressed with an economic objective, e.g., maximize the production yield in process control or minimize the energy consumption in HVAC. Thus, we present a economic MPC framework with performance guarantees for (periodic) time-varying problems with an economic stage cost ℓ_{eco}, possibly dependent on external variables y_{e}.

[11] J. Köhler, M. A. Müller, and F. Allgöwer. "Periodic optimal control of nonlinear constrained systems using economic model predictive control." In: *J. Proc. Contr.* 92 (2020). extended version: arXiv:2005.05245, pp. 185–201©2020 Elsevier Ltd.

Related work

Economic MPC [96] is a variant of MPC that directly aims at improving a user-specified economic stage cost ℓ_{eco}, which can improve (transient) performance compared to simply stabilizing the optimal mode of operation (cf. [234]). In case the system is optimally operated at steady-state, economic MPC schemes with suitable terminal ingredients directly enjoy relative performance guarantees [16, 19, 78, 167]. Similarly, in [10, 241, 295] performance guarantees relative to a periodic trajectory are obtained by using terminal constraints for an *a priori known optimal periodic trajectory*. Analogous results for more general sets than periodic orbits can be found in [81, 186]. All of these approaches do *not* incorporate online changes in the optimal system operation.

In case of optimal operation at steady-state, results for economic MPC without any terminal ingredients (cf. Sec. 4.3 for a thorough discussion) can be found in [122, 132], with extensions to the periodic and general time-varying setup in [210] and [128, 129, 130], respectively. Due to the absence of terminal ingredients these approaches can directly handle online changes in the cost function and require no a priori specification of the optimal mode of operation, but the performance guarantees require difficult to verify a priori assumptions (strict dissipativity, overtaking optimality) and possibly a very long prediction horizon N.

If there exists only a finite set of possible modes of operation, feasible transition trajectories can be computed offline to avoid feasibility issues [20].

To operate under online changing conditions, the economically optimal periodic trajectory can be stabilized using a tracking MPC with artificial reference trajectories (cf. Sec. 3.2) in combination with an economic cost $\mathcal{J}_{T,\text{eco}}$ for the artificial reference (assuming convexity, cf. Rk. 3.33), compare [165] for corresponding results for linear systems. Similar stability results for linear systems can be found in [43] based on strong duality. For nonlinear systems, stability of the optimal periodic trajectory is established in [133] by regularizing the non-periodicity in the input and economic cost, instead of directly using a tracking stage cost. However, the system may not be optimally operated periodically and the usage of a tracking/regularization cost can limit the economic performance.

Purely economic MPC formulations based on artificial setpoints have been presented in [87, 206, 208], compare also [102]. These approaches can operate reliably under online changing operation conditions and provide performance guarantees relative to the optimal steady-state. Economic MPC schemes with periodicity constraints have been proposed in [138, 282], but performance guarantees w.r.t. an optimal periodic orbit can only be established under very restrictive conditions (cf. [138, Lemma 3, Thm. 4]) and may not necessarily hold for linear convex problems (cf. [282, Example 6]).

In summary, the related literature does *not* address MPC formulations that are: suitable for *nonlinear* systems and *online changing* modes of operation; consider a purely *economic* objective; and provide economic performance guarantees compared to optimal *dynamic/periodic operation*. We extend the economic MPC formulations in [87, 206, 208] to the periodic case and suitably modify the optimization problem to guarantee closed-loop performance bounds. As a special case, we obtain a modified version of the MPC approaches based on periodicity constraints [138, 282] with performance guarantees, if we consider a prediction horizon of $N = 0$.

Setup

Compared to the setup in Sections 3.1–3.2, we consider an economic stage cost ℓ_{eco} and a time-varying periodic problem, with a known period length $T \in \mathbb{I}_{\geq 1}$. For many systems (e.g., HVAC or water distribution networks) this periodicity is inherent to the problem setup (in the dynamics or cost function). In time-invariant problems (e.g., CSTRs), this period length T is a user-specified decision variable that influences the possible performance improvement compared to steady-state operation (cf. [133]). Both cases are illustrated in the numerical examples in Sections 3.4.3–3.4.4.

We consider a nonlinear time-varying discrete-time system

$$x(t+1) = f(x(t), u(t), t), \quad x(0) = x_0,$$

with the state $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$, the control input $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, the dynamics $f : \mathbb{X} \times \mathbb{U} \times \mathbb{I}_{\geq 0} \to \mathbb{X}$, the initial condition $x_0 \in \mathbb{X}$, and the time step $t \in \mathbb{I}_{\geq 0}$. We impose time-varying constraints on the state and input

$$(x(t), u(t), t) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U} \times \mathbb{I}_{\geq 0}, \quad t \in \mathbb{I}_{\geq 0}. \tag{3.53}$$

For the artificial reference trajectory, we consider a reference constraint set $\mathbb{Z}_{\mathrm{r}}$, which satisfies the following condition: There exists a constant $\epsilon > 0$ such that for all $(r, t) \in \mathbb{Z}_{\mathrm{r}}$, $\tilde{r} \in \mathbb{B}_\epsilon(r)$, we have $(\tilde{r}, t) \in \mathbb{Z}$ (analogous to Assumption 3.1). The economic performance measure is given by a general (non-convex) time-varying function $\ell_{\mathrm{eco}} : \mathbb{X} \times \mathbb{U} \times \mathbb{I}_{\geq 0} \times \mathbb{Y} \to \mathbb{R}$, which may depend on external parameters $y_{\mathrm{e}} \in \mathbb{Y}$, with $\mathbb{Y} \subset \mathbb{R}^p$ compact. This function is called the economic stage cost ℓ_{eco} and the main control goal is the minimization of the closed-loop economic stage cost ℓ_{eco} while satisfying constraints.

We assume that the dynamics f, the economic stage cost ℓ_{eco}, and the constraint sets $\mathbb{Z}$, $\mathbb{Z}_{\text{r}}$ are periodically time-varying with the (known) period length T, i.e.,

$$f(x,u,t) = f(x,u,t+T), \quad \forall (x,u,t) \in \mathbb{X} \times \mathbb{U} \times \mathbb{I}_{\geq 0},$$
$$\ell_{\text{eco}}(x,u,t,y_{\text{e}}) = \ell_{\text{eco}}(x,u,t+T,y_{\text{e}}), \quad \forall (x,u,t,y_{\text{e}}) \in \mathbb{X} \times \mathbb{U} \times \mathbb{I}_{\geq 0} \times \mathbb{Y},$$
$$(r,t) \in \mathbb{Z} \Leftrightarrow (r,t+T) \in \mathbb{Z}, \quad (r,t) \in \mathbb{Z}_{\text{r}} \Leftrightarrow (r,t+T) \in \mathbb{Z}_{\text{r}}, \quad \forall t \in \mathbb{I}_{\geq 0}.$$

The set of feasible T-periodic artificial reference trajectories $r \in (\mathbb{X} \times \mathbb{U})^T$ at time t is denoted by

$$(r,t) \in \mathbb{Z}_T := \Big\{ (r(\cdot),t) = (x_{\text{r}}(\cdot), u_{\text{r}}(\cdot)),t) \in (\mathbb{X} \times \mathbb{U})^T \times \mathbb{I}_{\geq 0} \mid (r(k),t+k) \in \mathbb{Z}_{\text{r}},$$
$$x_{\text{r}}(\text{mod}(k+1,T)) = f(x_{\text{r}}(k), u_{\text{r}}(k), t+k), \quad k \in \mathbb{I}_{[0,T-1]} \Big\}$$

and is assumed to be non-empty. We define a periodic shift operation $\mathcal{R}_T$, which satisfies $\mathcal{R}_T^T r = r$ and $(r,t) \in \mathbb{Z}_T \Rightarrow (\mathcal{R}_T^k r, t+k) \in \mathbb{Z}_T$, $\forall t \in \mathbb{I}_{\geq 0}$, $k \in \mathbb{I}_{\geq 0}$.

In the following, we often denote $\ell_{\text{eco}}(r,t,y_{\text{e}}) = \ell_{\text{eco}}(x_{\text{r}}, u_{\text{r}}, t, y_{\text{e}})$ for $r = (x_{\text{r}}, u_{\text{r}})$ with some abuse of notation.

For simplicity, we assume that ℓ_{eco} and f are continuous and the sets $\mathbb{Z}, \mathbb{Z}_{\text{r}}$ are compact for a fixed $t \in \mathbb{I}_{\geq 0}$, i.e., $\|r\| \leq C$, $\forall (t,r) \in \mathbb{Z}$, which implies that ℓ_{eco} is bounded for all states and inputs satisfying the constraints (3.53).

At each time step $t \in \mathbb{I}_{\geq 0}$, a predicted sequence of parameters $y_{\text{e}}(\cdot|t) \in \mathbb{Y}^T$ is available as an external signal with $y_{\text{e}}(t) = y_{\text{e}}(0|t)$, similar to the target signal in Section 3.2. For the performance analysis, we assume that the parameter predictions y_{e} are consistent and T-periodic.

Assumption 3.50. *(Consistently periodic parameter predictions) For any $t \in \mathbb{I}_{\geq 0}$, the parameter predictions y_{e} satisfies*

$$y_{\text{e}}(\text{mod}(k+1,T)|t) = y_{\text{e}}(k|t+1), \qquad \forall k \in \mathbb{I}_{[0,T-1]}.$$

Remark 3.51. *(Price signal) The parameters y_{e} might incorporate online changing prices or general changes in the desired production/operation. In case some of the constraints in $\mathbb{Z}$ are relaxed to soft constraints using penalty terms in the stage cost ℓ_{eco}, the external parameters y_{e} can also model online changes in these soft constraints.*

At time $t \in \mathbb{I}_{\geq 0}$, given the parameter prediction $y_{\text{e}}(\cdot|t) \in \mathbb{Y}^T$, an optimal T-periodic orbit can be determined based on the following periodic optimal control problem

Problem 3.52.

$$\underset{r(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_{T,\text{eco}}(r(\cdot|t), t, y_{\text{e}}(\cdot|t)) \tag{3.54a}$$

subject to

$$(r(\cdot|t), t) \in \mathbb{Z}_T, \tag{3.54b}$$

with

$$\mathcal{J}_{T,\text{eco}}(r(\cdot|t), t, y_{\text{e}}(\cdot|t)) := \sum_{j=0}^{T-1} \ell_{\text{eco}}(r(j|t), t+j, y_{\text{e}}(j|t)). \tag{3.54c}$$

The solution to this optimization problem is an optimal T-periodic orbit $r_T^*(\cdot|t)$, $r_T^*(j|t) = (x_T^*(j|t), u_T^*(j|t))$, $j \in \mathbb{I}_{[0,T-1]}$. If the external parameters are consistently T-periodic (Ass. 3.50), then an optimal periodic trajectory at the next time step is given by shifting a previous optimal trajectory, i.e., $r_T^*(\cdot|t+1) = \mathcal{R}_T r_T^*(\cdot|t)$. Given some initialization at $t = 0$, the closed-loop average economic cost is defined as

$$\overline{\mathcal{J}}^{\text{cl}}_{\text{eco},\infty} := \limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell_{\text{eco}}(x(k), u(k), k, y_{\text{e}}(k)). \tag{3.55}$$

If the optimal periodic trajectory is known in advance and the parameter predictions are consistent (Ass. 3.50), standard economic MPC formulations with terminal ingredients (cf. [10, 241, 295]) guarantee that the average closed-loop performance is no worse than the average performance at the optimal periodic orbit, i.e.,

$$\overline{\mathcal{J}}^{\text{cl}}_{\text{eco},\infty} \leq \frac{\mathcal{J}_{T,\text{eco}}(r_T^*(\cdot|0), 0, y_{\text{e}}(\cdot|0))}{T}. \tag{3.56}$$

The control goal is to minimize this closed-loop average cost (3.55) (ideally satisfying (3.56)) and satisfy the constraints (3.53). In addition, the controller is expected to retain feasibility under online changes in the parameters y_{e} and adapt the mode of operation accordingly.

Pitfalls - Generalized periodic constraints

In the following, we show that a naive extension of existing generalized terminal setpoint constraints in [87, 206, 208] to periodic reference trajectories r, does *not* necessarily imply the desirable economic performance guarantees and thus requires further modifications which we will introduce in Section 3.3.2.

Consider the scalar time-invariant system with $f(x, u) = u$, $x \in \{0, 1, 2\}$, which is depicted as a graph in Figure 3.3 with some arbitrary small positive constant $\epsilon > 0$, similar to [210, Example 4]. The optimal periodic orbit is $x_T^* = (1, 2)$ (and phase shifts thereof) with cost $\mathcal{J}_{T,\text{eco}}(r_T^*) = \epsilon$ and $T = 2$.

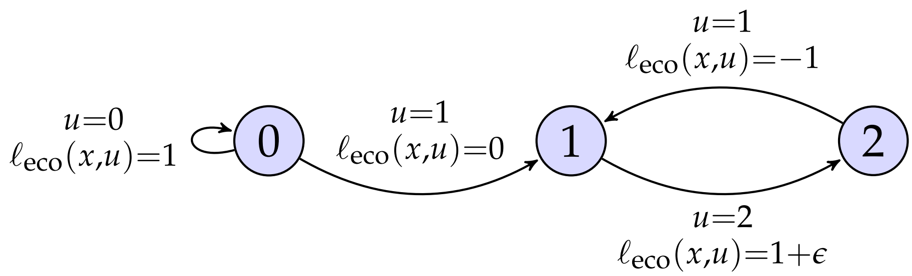

Figure 3.3. Academic counter example - Illustration of feasible transitions©2020 Elsevier Ltd.

The following economic MPC scheme with an artificial periodic trajectory can be viewed as a generalization of the methods in [87, 206, 208] based on artificial setpoints:

$$\min_{u(\cdot|t),\ r(\cdot|t)} \sum_{k=0}^{N-1} \ell_{\text{eco}}(x(k|t), u(k|t)) + \mathcal{J}_{T,\text{eco}}(r(\cdot|t)) \tag{3.57a}$$

$$\text{s.t. } x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.57b}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.57c}$$

$$r(\cdot|t) \in \mathbb{Z}_T, \quad x(N|t) = x_{\text{r}}(0|t), \quad x(0|t) = x(t). \tag{3.57d}$$

This optimization problem computes an open-loop trajectory $x^*(\cdot|t)$ starting at $x(t)$ that ends on some periodic trajectory $r^*(\cdot|t) \in \mathbb{Z}_T$. In closed-loop operation, the optimization problem (3.57) is solved in each time step $t \in \mathbb{I}_{\geq 0}$ and the first part of the optimized input trajectory is applied to the system, i.e., $u(t) = u^*(0|t)$, $x(t+1) = x^*(1|t)$, $t \in \mathbb{I}_{\geq 0}$. Although the economic MPC schemes in [87, 206, 208] often have additional modifications (e.g., terminal cost, self-tuning weights, additional constraints on r), the problem we discuss in the following remains the same.

Consider the initial condition $x_0 = 0$ and a prediction horizon of $N = 2$. The artificial reference is the optimal periodic orbit $x_{\text{r}}(k|t) \in \{1, 2\}$, $k \in \mathbb{I}_{[1,2]}$. The only feasible trajectories that satisfy $x(N|t) = x(2|t) \in \{1, 2\}$ are $u(\cdot|t) = (0, 1)$ and $u(\cdot|t) = (1, 2)$, and the corresponding open-loop cost is $1+0$ and $0+1+\epsilon$, respectively. Thus, the optimal solution to the optimization problem (3.57) satisfies $x^*(1|t)=0=x(t)$, $t\in\mathbb{I}_{\geq 0}$. Correspondingly, the closed-loop system based on the optimization problem (3.57) stays at $x(t)=0$, yielding the closed-loop economic cost $\ell_{\text{eco}}(x(t),u(t))=1$, $\forall t\in\mathbb{I}_{\geq 0}$ and does not achieve the same performance as the artificial periodic reference r^*. This issue can persist, even if we choose an arbitrarily large (even) prediction horizon N. In particular, with the MPC formulation in (3.57), we can only ensure

$$\overline{\mathcal{J}}^{\text{cl}}_{\text{eco},\infty}\leq \max_{k\in\mathbb{I}_{[0,T-1]}}\ell_{\text{eco}}(r_T^*(k))=1+\epsilon.$$

This is in contrast to existing results for the steady-state case ($T=1$) in [87, 206, 208], which ensure the superior bound (3.56). The same problem appears in economic MPC schemes without terminal constraints for periodic problems, compare [210, Examples 4 and 18]. One way to alleviate this problem is to apply the first T components of the open-loop input sequence $u^*(\cdot|t)$ (cf. multi-step MPC in [210, 272]), which transforms the problem to a higher dimensional steady-state problem (cf. T-step system in [210, 211, 272]). Since we wish to consider problems with possibly large period lengths T, this solution seems, however, inadequate. If we would use an economic MPC scheme based on periodicity constraints [138, 282], the closed-loop system would also stay at $x(t)=0$ for all $t\in\mathbb{I}_{\geq 0}$, since there exists only one feasible periodic orbit starting at $x_0=0$. The theoretical results in [138] do not apply, since the one-step controllability condition [138, Lemma 4] is not satisfied.

To summarize, as also discussed in [241], the existing approaches with online optimized periodic reference trajectories r *do not come with any closed-loop performance guarantees similar to* (3.56). In the following, we show that the performance guarantees (3.56) can be recovered, by suitably adjusting the economic MPC formulation.

3.3.2 Proposed economic MPC formulation

In the following, we detail the proposed economic MPC scheme and discuss the relation to other existing methods. The main idea is to directly minimize the predicted economic stage cost ℓ_{eco} with some continuous (economic) terminal cost $V_{\text{f,eco}}:\mathbb{X}\times\mathbb{Z}_T\times\mathbb{Y}^T\to\mathbb{R}$ and a terminal set $\mathbb{X}_{\text{f}}\subseteq\mathbb{X}\times\mathbb{Z}_T$ around the artificial reference trajectory r. In addition, we use an updating scheme to ensure that the optimized artificial reference trajectory r converges to the best possible periodic orbit r_T^*. At each time $t\in\mathbb{I}_{\geq 0}$, given the current state $x(t)$, the predicted parameters $y_{\text{e}}(\cdot|t)\in\mathbb{Y}^T$, memory states $\kappa(\cdot|t)\in\mathbb{R}^T$, a self-tuning weight $\beta(t) \geq 0$, and a constant $c_\kappa > 0$, the MPC control law is defined by the following optimization problem:

Problem 3.53.

$$\begin{aligned}\underset{u(\cdot|t),\ r(\cdot|t)}{\text{minimize}} \quad & \sum_{k=0}^{N-1} \ell_{\text{eco}}(x(k|t), u(k|t), t+k, y_{\text{e}}(\text{mod}(k,T)|t)) \\ & + V_{\text{f,eco}}(x(N|t), r(\cdot|t), t+N, \overline{y}_{\text{e}}(\cdot|t)) \\ & + \beta(t) \cdot \mathcal{J}_{T,\text{eco}}(r(\cdot|t), t+N, \overline{y}_{\text{e}}(\cdot|t))\end{aligned} \tag{3.58a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t), t+k), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.58b}$$

$$x(0|t) = x(t), \tag{3.58c}$$

$$(x(k|t), u(k|t), t+k) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{3.58d}$$

$$(x(N|t), r(\cdot|t), t+N) \in \mathbb{X}_{\text{f}}, \tag{3.58e}$$

$$(r(\cdot|t), t+N) \in \mathbb{Z}_T, \tag{3.58f}$$

$$\overline{y}_{\text{e}}(\cdot|t) = \mathcal{R}_T^N y_{\text{e}}(\cdot|t), \tag{3.58g}$$

$$\Delta\overline{\kappa}(t) = \sum_{j=0}^{T-1} \left[\ell_{\text{eco}}(r(j|t), t+N+j, \overline{y}_{\text{e}}(j|t)) - \kappa(j|t)\right], \tag{3.58h}$$

$$\ell_{\text{eco}}(r(j|t), t+N+j, \overline{y}_{\text{e}}(j|t)) \leq \kappa(j|t) - c_\kappa \Delta\overline{\kappa}(t), \quad j \in \mathbb{I}_{[0,T-1]}. \tag{3.58i}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, the artificial reference trajectory $r^*(\cdot|t) = (x_{\text{r}}^*(\cdot|t), u_{\text{r}}^*(\cdot|t))$, and the value function

$$\begin{aligned}W(t) := & \sum_{k=0}^{N-1} \ell_{\text{eco}}(x^*(k|t), u^*(k|t), t+k, y_{\text{e}}(\text{mod}(k,T)|t)) \\ & + V_{\text{f,eco}}(x^*(N|t), r^*(\cdot|t), t+N, \overline{y}_{\text{e}}(\cdot|t)) + \beta(t) \cdot \mathcal{J}_{T,\text{eco}}(r^*(\cdot|t), t+N, \overline{y}_{\text{e}}(\cdot|t)).\end{aligned}$$

The self-tuning weight β and the memory state κ are updated as follows

$$\beta(t+1) = \mathcal{B}(\beta(\cdot), \kappa(\cdot), x(\cdot)), \tag{3.59a}$$

$$\kappa(j|t+1) = \ell_{\text{eco}}(r^*(\text{mod}(j+1,T)|t), t+N+1+j, \overline{y}_{\text{e}}(j|t+1)), \quad j \in \mathbb{I}_{[0,T-1]}. \tag{3.59b}$$

The economic cost of the artificial reference trajectory r^* is saved in the memory states

κ (3.59b) using the more recent parameter predictions y_e available at time $t+1$. The tuning weight $\beta(t)$ can be determined by some general (causal) update rule $\mathcal{B}$ [206, 208], or simply chosen by a user as constant (cf. Prop. 3.84, [87]) or a time-varying signal (cf. [206, Update rule 1]). The input trajectory minimizes the predicted economic cost (3.58a) with a terminal cost $V_{\mathrm{f,eco}}$, to be specified later (Ass. 3.55). The economic cost of the artificial periodic reference trajectory r is weighted with a self-tuning (time-varying) weight $\beta(t)$ (cf. (3.58a)), similar to [87, 206, 208]. The resulting state and input trajectory satisfy the dynamics (3.58b)–(3.58c) and the posed state and input constraints (3.58d). In addition, the terminal state of the predicted state sequence satisfies a terminal set constraint $\mathbb{X}_\mathrm{f}$ (cf. Ass. 3.55 below) involving the artificial reference trajectory (3.58e). The artificial reference is a feasible periodic orbit (3.58f). The variable $\overline{y}_\mathrm{e}$ (cf. (3.58g)) corresponds to the parameter trajectory $y_\mathrm{e}(\cdot|t)$, periodically shifted by the prediction horizon N, to be in phase with the artificial reference trajectory r. The reason why the economic terminal cost $V_{\mathrm{f,eco}}$ may depend on the full parameter prediction $\overline{y}_\mathrm{e}(\cdot|t) \in \mathbb{Y}^T$, instead of simply considering $\overline{y}_\mathrm{e}(0|t)$ becomes evident in the design of the terminal cost (cf. Sec. 3.3.5).

Conditions (3.58h)–(3.58i) pose additional constraints on the improvement of the economic cost of the artificial reference r compared to $\kappa(j|t)$, similar to [87, 206, 208]. In particular, if $\Delta\kappa$ is negative (the cost $\mathcal{J}_{T,\mathrm{eco}}$ of the reference improves), then $\ell_{\mathrm{eco}}(r(j|t), t+N+j, \overline{y}_\mathrm{e}(j|t))$ can be larger than $\kappa(j|t)$. Hence, the constraint (3.58i) is less restrictive than $\ell_{\mathrm{eco}}(r(j|t), t+N+j, \overline{y}_\mathrm{e}(j|t)) \leq \kappa(j|t)$. The memory states κ in combination with the self-tuning weight β and the constant c_κ are crucial to establish the desired performance guarantees and are discussed in more detail in the following theoretical analysis, compare also the alternative MPC formulation in Section 3.3.6.

Compared to the tracking MPC formulation in Section 3.2, we directly minimize the economic cost ℓ_{eco} over the prediction horizon N and require additional constraints (3.58h)–(3.58i) and update steps (3.59). Compared to the MPC formulations in [87, 206, 208] ($T=1$), a T-dimensional memory state is considered and the revised constraints (3.58h)–(3.58i) are used.

The constant $c_\kappa > 0$ and the terminal ingredients $V_{\mathrm{f,eco}}$, $\mathbb{X}_\mathrm{f}$ are designed offline, which is detailed in Section 3.3.5. The memory states $\kappa(\cdot|0)$ can be initialized arbitrarily large such that the constraint (3.58i) is inactive at $t=0$. The tuning variable β can be initialized with any positive scalar, most naturally $\beta(0)=1$. The overall procedure is summarized by the following algorithm.

Algorithm 3.54. *(Economic MPC Algorithm with artificial reference trajectory)*

Offline: Specify the constraint sets $\mathbb{Z}$, $\mathbb{Z}_{\mathrm{r}}$, the economic cost ℓ_{eco}, the prediction horizon N, and the period length T. Design suitable terminal ingredients $V_{\mathrm{f,eco}}$, $\mathbb{X}_{\mathrm{f}}$ (cf. Sec. 3.3.5). Specify the update rule $\mathcal{B}$ and the constant c_κ. Initialize the self-tuning weight $\beta(0)$ and the memory states $\kappa(\cdot|0)$.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, obtain the predicted parameters $y_{\mathrm{e}}(\cdot|t)$, solve Problem 3.53, and apply the control input $u(t) := u^*(0|t)$. Update the self-tuning weight β (3.59a) *and the memory state κ* (3.59b).

Existing schemes = special cases

In the following, we discuss in detail how various existing methods for economic MPC are contained in this formulation as special cases.

The proposed formulation can best be viewed as an extension to the MPC formulations in [87, 206, 208], which consider an artificial reference setpoint ($T = 1$). In particular, if we assume a time-invariant problem setup and choose $T = 1$, we get the optimization problem and closed-loop operation in [87, 206, 208]. For $c_\kappa \geq 0$, the constraints (3.58h)–(3.58i) are equivalent to $\ell_{\mathrm{eco}}(r(t)) \leq \kappa(t) = \ell_{\mathrm{eco}}(r^*(t-1))$ which is used in [87, 206, 208] to ensure that the cost of the artificial reference r is non-increasing. Although one can directly see that [87, 206, 208] is a special case of the posed formulation, it is not obvious from the onset that the extension of [87, 206, 208] to periodic problems should be given by Problem 3.53. A more intuitive extension might be to use the constraint $\mathcal{J}_{T,\mathrm{eco}}(r(\cdot|t)) \leq \kappa(t) = \mathcal{J}_{T,\mathrm{eco}}(r^*(\cdot|t-1))$ (as an alternative to (3.58h)–(3.58i)). The possibly suboptimal performance of such an approach has, however, been illustrated in the numerical example in Section 3.3.1. In Section 3.3.6 we show that we can guarantee the same properties with this more intuitive constraint, if we instead suitably reformulate the cost function. Another possible formulation for periodic orbits would be the constraint $\ell_{\mathrm{eco}}(r(j|t)) \leq \kappa(j|t)$ (choosing $c_\kappa = 0$ in (3.58i)). This modification is sufficient to avoid the pitfall in Section 3.3.1, if the artificial reference is initialized as an optimal periodic orbit r_T^*. However, this more restrictive constraint can potentially prevent the artificial reference trajectory r to converge to the optimal periodic orbit r_T^*.

If we consider a prediction horizon of $N = 0$ and a terminal equality constraint $\mathbb{X}_{\mathrm{f}} = \{x_{\mathrm{r}}(0) = x\}$, then the proposed formula yields a modified version of the MPC scheme using periodicity constraints [138, 282]. The only difference would be the additional performance constraints on the periodic orbit (3.58h)–(3.58i). Crucially, if we choose a suitable terminal cost and terminal set (cf. Ass. 3.59), then we can establish closed-loop performance guarantees (Thm. 3.65), which are in general not valid for MPC schemes using periodicity constraints [138, 282]. In particular, the terminal ingredients (Ass. 3.59) relax the one-step controllability condition [138, Lemma 4] to a stabilizability condition. In Lemma 3.78, we discuss how to retain the performance guarantees with a terminal equality constraint.

The tracking MPC scheme [165] can be viewed as a modified version, which uses a tracking stage cost ℓ in Problem 3.53, similar to the MPC formulations in Section 3.2, and hence does not require the additional tuning variable β or memory states κ.

The standard economic MPC formulations for periodic orbits [10, 295] and steady-states [16, 19] are contained as a special case, if we fix the artificial reference trajectory $r = r_T^*$.

3.3.3 Relative performance guarantees

In Proposition 3.56, we show that the proposed formulation is recursively feasible, using standard conditions on the terminal ingredients (Ass. 3.55). If the external parameters are consistently T-periodic (Ass. 3.50), Proposition 3.58 shows that the average closed-loop performance is no worse than the performance of the limiting artificial references.

Terminal ingredients: The following assumption captures the (standard) conditions for the terminal ingredients.

Assumption 3.55. *(Economic terminal ingredients) There exists a terminal control law* $k_\mathrm{f} : \mathbb{X} \times \mathbb{Z}_T \to \mathbb{U}$ *such that the following properties hold for any time* $t \in \mathbb{I}_{\geq 0}$*, parameter prediction* $y_\mathrm{e} \in \mathbb{Y}^T$*, periodic reference* $(r, t) \in \mathbb{Z}_T$*, and any state* $(x, r, t) \in \mathbb{X}_\mathrm{f}$*:*

$$(x^+, r^+, t+1) \in \mathbb{X}_\mathrm{f}, \tag{3.60a}$$

$$(x, u, t) \in \mathbb{Z}, \tag{3.60b}$$

$$V_{\mathrm{f,eco}}(x^+, r^+, t+1, y_\mathrm{e}^+) - V_{\mathrm{f,eco}}(x, r, t, y_\mathrm{e}) \leq \ell_\mathrm{eco}(r(0), t, y_\mathrm{e}(0)) - \ell_\mathrm{eco}(x, u, t, y_\mathrm{e}(0)), \tag{3.60c}$$

with $x^+ = f(x, u, t)$*,* $r^+ = \mathcal{R}_T r$*,* $y_\mathrm{e}^+ = \mathcal{R}_T y_\mathrm{e}$*,* $u = k_\mathrm{f}(x, r, t)$*.*

Compared to Assumption 3.5, we consider a time-varying setup and an economic stage cost ($\ell_\mathrm{eco}(r, t, y_\mathrm{e}) \neq 0$). Due to the definition of the reference constraint set $\mathbb{Z}_T$, Assumption 3.55 can be satisfied with a simple terminal equality constraint (TEC):

$$\mathbb{X}_\mathrm{f} = \{(x, r, t) \in \mathbb{X} \times \mathbb{Z}_T \mid x = x_\mathrm{r}(0)\},\ k_\mathrm{f}(x, r, t) = u_\mathrm{r}(0),\ V_{\mathrm{f,eco}}(x, r, t, y_\mathrm{e}) = 0, \tag{3.61}$$

similar to Proposition 3.10. However, for the improved performance guarantees discussed in Section 3.3.4 below we will require stronger conditions for the terminal set (cf. Ass. 3.59 and Sec. 3.3.6), compare also [208] and Lemma 3.78.

Recursive feasibility: The following proposition shows that feasibility of the proposed economic MPC algorithm is guaranteed, independent of the exogenous parameters y_e.

Proposition 3.56. *Let Assumption 3.55 hold. If Problem 3.53 is feasible at $t = 0$, then the closed-loop system resulting from Algorithm 3.54 satisfies the constraints* (3.53) *and Problem 3.53 is feasible for all $t \in \mathbb{I}_{\geq 0}$, independently of the parameter prediction $y_\mathrm{e}(\cdot|t)$.*

Proof. This result is a straightforward extension of existing results for MPC with artificial reference trajectories [87, 102, 138, 163, 164, 165, 166, 206, 208, 282], compare also Theorem 3.31. Given the feasible reference trajectory $(r^*(\cdot|t), t+N) \in \mathbb{Z}_T$ at time t, the shifted reference trajectory $r(\cdot|t+1) = \mathcal{R}_T r^*(\cdot|t)$ satisfies (3.58f). This reference trajectory satisfies the constraints (3.58i) with equality, since $\Delta\overline{\kappa}(t+1) = 0$. A corresponding candidate input sequence is given by

$$u(k|t+1) = \begin{cases} u^*(k+1|t) & k \in \mathbb{I}_{[0,N-2]} \\ k_\mathrm{f}(x^*(N|t), r^*(\cdot|t), t+N) & k = N-1 \end{cases}.$$

The resulting state and input sequences satisfy the constraints (3.58d) and the terminal constraint (3.58e) due to Assumption 3.55. ∎

Self-tuning weight: Define the change in the weight β as $\gamma(t) := \beta(t+1) - \beta(t)$, $t \in \mathbb{I}_{\geq 0}$. The following assumption characterizes the properties the update scheme $\mathcal{B}$ (3.59a) should have such that relative performance guarantees hold despite online changing values of β.

Assumption 3.57. *[206, Ass. 1] (Self-tuning weight - bounded rate of change γ) There exists a constant $c_\gamma \in \mathbb{R}$ such that the sequence $\beta(\cdot)$ satisfies $\limsup_{t\to\infty} \gamma(t) \leq 0$ and $\gamma(t) \leq c_\gamma$, $\beta(t) \geq 0$ for all $t \in \mathbb{I}_{\geq 0}$.*

A more nuanced discussion and an alternative condition on $\mathcal{B}$ resulting in slightly weaker performance guarantees can be found in [206]. Define the cost of the optimal artificial reference trajectory as

$$\overline{\kappa}(t+1) := \sum_{j=0}^{T-1} \kappa(j|t+1) = \mathcal{J}_{T,\mathrm{eco}}(r^*(\cdot|t), t+N, \overline{y}_\mathrm{e}(\cdot|t+1)), \quad t \in \mathbb{I}_{\geq 0}. \tag{3.62}$$

If the external parameters $y_e(\cdot|t)$ are consistently T-periodic (Ass. 3.50), then the constraint (3.58i), and the update (3.59b) with $c_\kappa \geq 0$ ensure that $\Delta\overline{\kappa}(t) := \overline{\kappa}(t+1) - \overline{\kappa}(t) \leq 0$ and thus $\overline{\kappa}$ is non-increasing. Boundedness of ℓ_{eco} implies boundedness of $\overline{\kappa}$. Thus $\overline{\kappa}(t)$ converges to some finite limit $\overline{\kappa}_\infty := \lim_{t\to\infty} \overline{\kappa}(t)$.

Average performance bounds: The following proposition establishes that the closed-loop performance is no worse than $\overline{\kappa}_\infty$, i.e., the performance of the limiting artificial trajectories r.

Proposition 3.58. *Let Assumptions 3.50, 3.55, and 3.57 hold. If Problem 3.53 is feasible at $t = 0$, then the closed-loop system resulting from Algorithm 3.54 satisfies the following performance bound*

$$\limsup_{K\to\infty} \frac{\sum_{t=0}^{TK-1} \ell_{\text{eco}}(x(t), u(t), t, y_e(t))}{TK} \leq \frac{\overline{\kappa}_\infty}{T}. \tag{3.63}$$

Proof. Proposition 3.56 provides a feasible candidate solution $u(\cdot|t+1)$, $r(\cdot|t+1)$ to Problem 3.53 at time $t+1$, given feasibility at time $t \in \mathbb{I}_{\geq 0}$. Hence, we can use the cost of the candidate solution to upper bound the value function $W(t+1)$. The terminal cost (Ass. 3.55) in combination with consistent parameters $y_e(\cdot|t)$ (Ass. 3.50) and T-periodicity ensure

$$\begin{aligned} &W(t+1) - W(t) + \ell_{\text{eco}}(x(t), u(t), t, y_e(t)) \\ \leq &\ell_{\text{eco}}(r^*(0|t), t+N, \overline{y}_e(0|t)) + \gamma(t)\overline{\kappa}(t+1) \overset{(3.59\text{b})}{=} \kappa(T-1|t+1) + \gamma(t)\overline{\kappa}(t+1), \end{aligned}$$

similar to [206, Thm. 1], [208, Thm. 1]. The definition of κ in (3.59b), consistent parameters $y_e(\cdot|t)$ (Ass. 3.50) and the constraints (3.58h)–(3.58i) ensure

$$\begin{aligned} \kappa(j|t+1) &\overset{(3.59\text{b})}{=} \ell_{\text{eco}}(r^*(j+1|t), t+N+j+1, y_e(j|t+1)) \\ &\overset{(3.58\text{i})}{\leq} \kappa(j+1|t) - c_\kappa \Delta\overline{\kappa}(t), \quad j \in \mathbb{I}_{[0,T-1]}, \end{aligned} \tag{3.64}$$

with $\kappa(T|t) := \kappa(0|t)$. Using Inequality (3.64) recursively implies

$$\kappa(T-1|t+k+1) \leq \kappa(k|t) - c_\kappa \sum_{j=0}^{k} \Delta\overline{\kappa}(t+j), \quad k \in \mathbb{I}_{[0,T-1]}. \tag{3.65}$$

Using the definition of $\overline{\kappa}$ in (3.62), we can bound the T-step sum as

$$\begin{aligned}&\sum_{k=0}^{T-1}\kappa(T-1|t+1+k) \overset{(3.65)}{\leq} \sum_{k=0}^{T-1}\left(\kappa(k|t) - c_\kappa\sum_{j=0}^{k}\Delta\overline{\kappa}(t+j)\right)\\ \leq &\sum_{k=0}^{T-1}\kappa(k|t) - c_\kappa T\sum_{k=0}^{T-1}\Delta\overline{\kappa}(t+k)\\ \overset{(3.62)}{=}&\overline{\kappa}(t) - c_\kappa T\sum_{k=0}^{T-1}\Delta\overline{\kappa}(t+k) = \overline{\kappa}(t) + c_\kappa T(\overline{\kappa}(t) - \overline{\kappa}(t+T)).\end{aligned}$$

Thus, the closed-loop transient cost over one period T satisfies

$$\begin{aligned}&W(t+T) - W(t) + \sum_{k=t}^{t+T-1}\ell_{\text{eco}}(x(k), u(k), k, y_{\text{eco}}(k)) \qquad (3.66)\\ \leq&\overline{\kappa}(t) + c_\kappa T(\overline{\kappa}(t) - \overline{\kappa}(t+T)) + \sum_{k=0}^{T-1}\gamma(t+k)\overline{\kappa}(t+1+k).\end{aligned}$$

Abbreviate $\ell_{\text{eco}}(t) := \ell_{\text{eco}}(x(t), u(t), t, y_{\text{e}}(t))$ and define $\overline{\kappa}_{\text{e}}(t) := \overline{\kappa}(t) - \overline{\kappa}_\infty$. Then, Inequality (3.66) evaluated over a time interval $K \cdot T$, $K \in \mathbb{I}_{\geq 1}$ starting at $t = 0$ can be rewritten as

$$\begin{aligned}&W(K\cdot T) - W(0) \qquad (3.67)\\ \leq&K\overline{\kappa}_\infty + c_\kappa T(\overline{\kappa}(0) - \overline{\kappa}(TK)) + \sum_{k=0}^{K-1}\overline{\kappa}_{\text{e}}(k\cdot T) + \sum_{t=0}^{KT-1}[\gamma(t)\overline{\kappa}_\infty + \gamma(t)\overline{\kappa}_{\text{e}}(t+1) - \ell_{\text{eco}}(t)].\end{aligned}$$

The remainder of the proof is analogous to [206, Thm. 1]. Boundedness of ℓ_{eco}, $V_{\text{f,eco}}$ and $\beta(t) \geq 0$ ensures that $W(TK)$ is lower bounded and thus

$$0 \leq \liminf_{K\to\infty}\frac{W(TK) - W(0)}{K}.$$

Taking averages on both sides of Inequality (3.67) yields

$$
\begin{aligned}
0 \leq & \liminf_{K\to\infty} \frac{W(TK)-W(0)}{K} \\
\overset{(3.67)}{\leq} & \overline{\kappa}_\infty + \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1}\overline{\kappa}_\mathrm{e}(k\cdot T) + \lim_{K\to\infty}\frac{c_\kappa T}{K}(\overline{\kappa}(0)-\overline{\kappa}(TK)) \\
& + \limsup_{K\to\infty}\frac{1}{K}\left[\sum_{t=0}^{KT-1}\gamma(t)\overline{\kappa}_\infty + \gamma(t)\overline{\kappa}_\mathrm{e}(t+1)\right] - \limsup_{K\to\infty}\frac{1}{K}\sum_{t=0}^{KT-1}\ell_\mathrm{eco}(t) \\
\leq & \overline{\kappa}_\infty - \limsup_{K\to\infty}\frac{1}{K}\sum_{t=0}^{TK-1}\ell_\mathrm{eco}(t),
\end{aligned}
$$

which implies the desired performance bound (3.63). The first inequality uses the fact that

$$
\liminf_n (a_n - b_n) \leq \liminf_n -b_n + \limsup_n a_n = \limsup_n a_n - \limsup_n b_n.
$$

The second inequality follows from

$$
\begin{aligned}
& \gamma(t)\leq c_\gamma, \quad \overline{\kappa}_\mathrm{e}(t)\in[0,\infty), \quad t\in\mathbb{I}_{\geq 0}, \\
& \lim_{t\to\infty}\overline{\kappa}_\mathrm{e}(t)=0, \quad \limsup_{t\to\infty}\gamma(t)\leq 0, \quad \lim_{t\to\infty}\Delta\overline{\kappa}(t)=0. \qquad \blacksquare
\end{aligned}
$$

Proposition 3.58 extends the performance bounds in [206, Thm. 1] from the steady-state case ($T=1$) to periodic problems ($T>1$). In particular, we showed that the constraints (3.58h)–(3.58i) with c_κ allow us to relate the difference in the value function W over T-steps with $\overline{\kappa}$, the cost of the artificial reference trajectory.

3.3.4 Improved a priori performance bounds

In the following, we provide sufficient conditions to ensure that the cost of the artificial periodic orbit converges to a local minimum.

Terminal ingredients: The following assumption is a stronger version of Assumption 3.55, which is used to derive the improved performance guarantees.

Assumption 3.59. *(Contractive terminal set) Consider $\mathbb{X}_\mathrm{f}$, $V_{\mathrm{f,eco}}$, k_f satisfing Assumption 3.55. There exist a function $V_\delta : \mathbb{X}\times\mathbb{Z}_T \to \mathbb{R}_{\geq 0}$ and functions $\alpha_{\delta,1}, \alpha_{\delta,2}, \alpha_{\delta,3}, \alpha_{\delta,4} \in \mathcal{K}_\infty$ such that for any time $t\in\mathbb{I}_{\geq 0}$, any periodic references $(r,t)\in\mathbb{Z}_T$, $(\tilde{r},t)\in\mathbb{Z}_T$ and any $(x,r,t)\in\mathbb{X}_\mathrm{f}$,* *the following inequalities hold*

$$V_\delta(x^+, r^+, t+1) - V_\delta(x, r, t) \leq -\alpha_{\delta,1}(\|x - x_\mathrm{r}(0)\|), \tag{3.68a}$$
$$\alpha_{\delta,2}(\|x - x_\mathrm{r}(0)\|) \leq V_\delta(x, r, t) \leq \alpha_{\delta,3}(\|x - x_\mathrm{r}(0)\|), \tag{3.68b}$$
$$|V_\delta(x, r, t) - V_\delta(x, \tilde{r}, t)| \leq \alpha_{\delta,4}(\|r - \tilde{r}\|), \tag{3.68c}$$

with $x^+ = f(x, u, t)$, $u = k_\mathrm{f}(x, r, t)$, $r^+ = \mathcal{R}_T r$. *Furthermore, there exist functions* $\alpha : \mathbb{Z}_T \to \mathbb{R}_{\geq 0}$, $\alpha_{\delta,5} \in \mathcal{K}_\infty$ *and constants* $\overline{\alpha} \geq \underline{\alpha} > 0$ *such that the terminal set is given by* $\mathbb{X}_\mathrm{f} = \{(x, r, t) \in \mathbb{X} \times \mathbb{Z}_T \mid V_\delta(x, r, t) \leq \alpha(r, t)\}$ *and the following conditions hold*

$$|\alpha(r, t) - \alpha(\tilde{r}, t)| \leq \alpha_{\delta,5}(\|r - \tilde{r}\|),\ \alpha(r, t) = \alpha(r^+, t+1) \in [\underline{\alpha}, \overline{\alpha}]. \tag{3.69}$$

These conditions ensure that the terminal set $\mathbb{X}_\mathrm{f}$ has a non-empty interior (in x for r, t fixed) and thus explicitly exclude terminal equality constraints. In particular, V_δ needs to be a (continuous) incremental CLF with feedback k_f, which can be designed analogous to the tracking terminal cost V_f in Section 3.1.3 with $\alpha_{\delta,1}, \alpha_{\delta,2}, \alpha_{\delta,3}$ quadratic (cf. Lemma 3.12). For this design choice Condition (3.68c) holds with a quadratic function $\alpha_{\delta,4}$, compare Proposition 3.44. Furthermore, the fact that α may depend on (r, t) allows the usage of the online optimized terminal set size from Section 3.2.2, with the continuous function

$$\alpha(r, t) := \min\left\{ \min_{i \in \mathbb{I}_{[1,n_\mathrm{z}]}, j \in \mathbb{I}_{[0,T-1]}} \frac{-g_i(r(j), t)}{c_i(r(j), t)}, \sqrt{\alpha_1} \right\}, \tag{3.70}$$

if $\mathbb{Z}_T, g_i, c_i, \alpha_1$ are chosen appropriately. The computation of a suitable economic terminal cost $V_\mathrm{f,eco}$ is detailed in Section 3.3.5.

The following lemma shows that the reference r can be incrementally changed in closed-loop operation without losing recursive feasibility.

Lemma 3.60. *Let Assumptions 3.55 and 3.59 hold. Then, there exists a constant* $\epsilon > 0$ *such that at each time* $t \in \mathbb{I}_{\geq}$, *for any* $(r, t) \in \mathbb{Z}_T$, $(x, r, t) \in \mathbb{X}_\mathrm{f}$, *for all* $(\tilde{r}, t+1) \in \mathbb{Z}_T$ *satisfying* $\tilde{r} \in \mathbb{B}_\epsilon(\mathcal{R}_T r)$, *it holds that*

$$(x^+, \tilde{r}, t+1) \in \mathbb{X}_\mathrm{f}, \quad x^+ = f(x, k_\mathrm{f}(x, r, t), t).$$

Proof. First, note that Assumption 3.59 ensures that the positive invariance condition (3.60a) is strictly satisfied, i.e,

$$
\begin{aligned}
&V_\delta(x^+, \mathcal{R}_T r, t+1) \overset{(3.68a)}{\leq} V_\delta(x, r, t) - \alpha_{\delta,1}(\|x - x_\mathrm{r}(0)\|) \\
\overset{(3.68b)}{\leq}\ & V_\delta(x, r, t) - \alpha_{\delta,1}(\alpha_{\delta,3}^{-1}(V_\delta(x, r, t))) \leq \sup_{c \in [0, \alpha(r,t)]} c - \alpha_{\delta,1}(\alpha_{\delta,3}^{-1}(c)) \\
\overset{(3.69)}{\leq}\ & \alpha(r, t) - \underbrace{\min\{\alpha_{\delta,1}(\alpha_{\delta,3}^{-1}(\underline{\alpha}/2)), \underline{\alpha}/2\}}_{=:\Delta\alpha > 0},
\end{aligned}
\tag{3.71}
$$

where the last step follows using the case distinction $c \leq \underline{\alpha}/2$ and $c \geq \underline{\alpha}/2$ and the fact that $\alpha(r, t) \geq \underline{\alpha}$. Given $\|\mathcal{R}_T r - \tilde{r}\| \leq \epsilon$, we have

$$
\begin{aligned}
&V_\delta(x^+, \tilde{r}, t+1) \overset{(3.68c)}{\leq} V_\delta(x^+, \mathcal{R}_T r, t+1) + \alpha_{\delta,4}(\epsilon) \\
\overset{(3.71)}{\leq}\ & \alpha(r, t) - \Delta\alpha + \alpha_{\delta,4}(\epsilon) \overset{(3.69)}{\leq} \alpha(\tilde{r}, t) + \alpha_{\delta,5}(\epsilon) + \alpha_{\delta,4}(\epsilon) - \Delta\alpha = \alpha(\tilde{r}, t),
\end{aligned}
$$

with $\epsilon := (\alpha_{\delta,4} + \alpha_{\delta,5})^{-1}(\Delta\alpha)$. ∎

This result is an extension of [208, Lemma 1]. For comparison, in the tracking MPC scheme in Section 3.2 we only required that the reference can be incrementally moved if $x \in \mathbb{B}_\epsilon(x_\mathrm{r})$ using Assumption 3.6.

Self-tuning weight: Given a state x at time $t \in \mathbb{I}_{\geq 0}$, the set of periodic reference trajectories r with a terminal set $\mathbb{X}_\mathrm{f}$ that can be reached within the prediction horizon N is defined as

$$
\begin{aligned}
\mathcal{R}_N(x, t) := \Big\{ & r \in (\mathbb{X} \times \mathbb{U})^T \mid (r, t+N) \in \mathbb{Z}_T,\ \exists u \in \mathbb{U}^N \text{ s.t.} \\
& x(t) = x,\ (x(N+t), r, t+N) \in \mathbb{X}_\mathrm{f}, \\
& x(k+1) = f(x(k), u(k), k),\ (x(k), u(k), k) \in \mathbb{Z},\ k \in \mathbb{I}_{[t, t+N-1]} \Big\}.
\end{aligned}
$$

Given additionally $\overline{y}_\mathrm{e}(\cdot|t) \in \mathbb{Y}^T$, $\kappa(\cdot|t) \in \mathbb{R}^T$, we define the set of reference trajectories that also satisfy the constraints (3.58h)–(3.58i) as

$$
\overline{\mathcal{R}}_N(x, t, \overline{y}_\mathrm{e}(\cdot|t), \kappa(\cdot|t)) := \{ r \in \mathcal{R}_N(x, t) \mid \text{s.t. } r \text{ satisfies (3.58h)–(3.58i)} \}.
$$

Given a point $x \in \mathbb{X}$ at time $t \in \mathbb{I}_{\geq 0}$ with some fixed $\overline{y}_\mathrm{e}(\cdot|t) \in \mathbb{Y}^T$, $\kappa(\cdot|t) \in \mathbb{R}^T$, the cost of the best reachable periodic orbit is given as

$$\mathcal{J}_{T,\text{eco,min}}(x,t,\overline{y}_{\text{e}}(\cdot|t),\kappa(\cdot|t)) := \min_{r(\cdot)\in\overline{\mathcal{R}}_N(x,t,\overline{y}_{\text{e}}(\cdot|t),\kappa(\cdot|t))} \mathcal{J}_{T,\text{eco}}(r(\cdot),t+N,\overline{y}_{\text{e}}(\cdot|t)). \quad (3.72)$$

Assumption 3.61. *(Self-tuning weight - increasing weight) The update rule $\mathcal{B}$ is such that for any y_{e} satisfying Assumption 3.50 and for all sequences $x(\cdot)$, $\kappa(\cdot)$, it holds that*

$$\overline{\kappa}_\infty - \liminf_{t\to\infty} \mathcal{J}_{T,\text{eco,min}}(x(t),t,\overline{y}_{\text{e}}(\cdot|t),\kappa(\cdot|t)) > 0 \quad \Rightarrow \quad \liminf_{t\to\infty} \beta(t) = \infty.$$

The main idea is that in closed-loop operation the self-tuning weight β increases if necessary and thus ensures that the artificial trajectory converges to the optimal mode of operation, compare [206, 208]. A detailed discussion on update schemes $\mathcal{B}$ satisfying Assumptions 3.57 and 3.61 is given in [206].

Periodic continuity: As discussed in Sections 3.3.1–3.3.2, the constraints (3.58h)–(3.58i) are crucial for the desired properties. However, these constraints limit how the shape of the artificial reference trajectory may change. In particular, for $c_\kappa = 0$ this constraint ensures that the reference can only be updated if the economic cost on all points $r(j|t)$, $j \in \mathbb{I}_{[0,T-1]}$ of the reference trajectory r does not increase. For c_κ arbitrarily large, the constraint (3.58i) becomes inactive, if the overall cost of the artificial trajectory decreases ($\Delta\kappa < 0$). However, both for numerical and technical reasons we consider the smooth constraints (3.58i) with a finite value c_κ. Thus, we require the following technical continuity assumption on the periodic optimal control problem (Problem 3.52).

Assumption 3.62. *(Continuity of periodic optimal control problem) There exists a constant $c_\kappa > 0$ such that at any time step $t \in \mathbb{I}_{\geq 0}$, for any parameters $\overline{y}_{\text{e}} \in \mathbb{Y}^T$, for any periodic trajectory $(r,t) \in \mathbb{Z}_T$, which is not a local minimum of Problem 3.52, and any $\epsilon > 0$, there exists a reference trajectory $(\tilde{r},t) \in \mathbb{Z}_T$ with $\|r-\tilde{r}\| \leq \epsilon$, $\mathcal{J}_{T,\text{eco}}(\tilde{r},t,\overline{y}_{\text{e}}) < \mathcal{J}_{T,\text{eco}}(r,t,\overline{y}_{\text{e}})$, that satisfies*

$$c_\kappa \geq \frac{\ell_{\text{eco}}(\tilde{r}(j),t+j,\overline{y}_{\text{e}}(j)) - \ell_{\text{eco}}(r(j),t+j,\overline{y}_{\text{e}}(j))}{\mathcal{J}_{T,\text{eco}}(r,t,\overline{y}_{\text{e}}) - \mathcal{J}_{T,\text{eco}}(\tilde{r},t,\overline{y}_{\text{e}})}, \quad j \in \mathbb{I}_{[0,T-1]}. \quad (3.73)$$

This assumption ensures that it is possible to incrementally change the overall cost $\mathcal{J}_{T,\text{eco}}$, with incremental changes in the reference r and a change in the local cost ℓ_{eco} proportional to the overall improvement in $\mathcal{J}_{T,\text{eco}}$. If we expand the fraction by $\Delta r = r - \tilde{r}$ and take the limit $\Delta r \to 0$, we can see that this condition is similar to a continuity assumption on the fraction of the gradients of ℓ_{eco} and $\mathcal{J}_{T,\text{eco}}$. A modified MPC formulation, which does not require Assumption 3.62, is presented in Section 3.3.6.

The following lemma shows that the continuity condition (Ass. 3.62) in combination with the contractive terminal set (Ass. 3.59) ensure convergence to local minima.

Lemma 3.63. *Let Assumptions 3.50, 3.55, 3.59 and 3.62 hold. Suppose that Problem 3.53 is feasible at time $t \in \mathbb{I}_{\geq 0}$ and the optimal artificial reference trajectory $r^*(\cdot|t)$ is not a local minimizer of Problem 3.52. Then, there exists a reference $\tilde{r} \in \overline{\mathcal{R}}_N(x(t+1), t+1, \overline{y}_e(\cdot|t+1), \kappa(\cdot|t+1))$, which is a feasible candidate solution of Problem 3.53 at time $t+1$, and satisfies*

$$\mathcal{J}_{T,\text{eco}}(\tilde{r}, t+N+1, \overline{y}_e(\cdot|t+1)) < \mathcal{J}_{T,\text{eco}}(\mathcal{R}_T r^*(\cdot|t), t+N+1, \overline{y}_e(\cdot|t+1)). \qquad (3.74)$$

Proof. Given that $\mathcal{R}_T r_T^*(\cdot|t)$ is not a local minimizer, Assumption 3.62 ensures that there exists a feasible periodic reference trajectory $\tilde{r}$, that improves the reference cost $\mathcal{J}_{T,\text{eco}}$ (3.74) and satisfies $\|r - \tilde{r}\| \leq \epsilon$ and (3.73). Satisfaction of the posed constraints (3.58h)–(3.58i) follows from (3.73), by noting that

$$\Delta\overline{\kappa}(t) = \mathcal{J}_{T,\text{eco}}(\tilde{r}, t+N+1, \overline{y}_e(\cdot|t+1)) - \mathcal{J}_{T,\text{eco}}(\mathcal{R}_T r^*(\cdot|t), t+N+1, \overline{y}_e(\cdot|t+1)).$$

With ϵ according to Lemma 3.60, the candidate input $u(\cdot|t+1)$ from Proposition 3.58 satisfies the terminal set constraint (3.58e) with the incrementally changed reference $\tilde{r}$ and is thus a feasible solution of Problem 3.53. ∎

A priori performance bounds: The following proposition establishes a priori performance bounds on the artificial reference trajectory.

Proposition 3.64. *Let Assumptions 3.50, 3.55, 3.59 and 3.62 hold. Assume that Problem 3.53 is feasible at $t = 0$. If the update rule $\mathcal{B}$ satisfies Assumption 3.61, then $\overline{\kappa}_\infty$ is a local minimum of Problem 3.52 for the closed-loop system resulting from Algorithm 3.54.*

Proof. This result is a direct extension of the self-tuning economic MPC results in [208, Thm. 2/3, Cor. 1]. Using a proof of contradiction one can show that Assumption 3.62 implies $\overline{\kappa}_\infty = \lim_{t\to\infty} \mathcal{J}_{T,\text{eco,min}}(x(t), t, \overline{y}_e(\cdot|t), \kappa(\cdot|t))$, compare [208, Thm. 2], [206, Thm. 2]. Suppose there exists a limiting artificial reference r, which is not a local minimizer of Problem 3.52. Lemma 3.63 ensures that there exists a feasible reference $\tilde{r}$ with an improved cost, which implies $\mathcal{J}_{T,\text{eco,min}} < \overline{\kappa}_\infty$ and thus contradicts the assumption. ∎

The following theorem summarizes the theoretical properties of the proposed MPC scheme.

Theorem 3.65. *Let Assumptions 3.55 hold and assume that Problem 3.53 is feasible at $t = 0$. Then, Problem 3.53 is recursively feasible for all $t \in \mathbb{I}_{\geq 0}$ for the closed-loop system resulting* *from Algorithm 3.54. Assume further that Assumptions 3.50, 3.57, 3.59, 3.61 and 3.62 hold. Then,* $\overline{\kappa}_\infty$ *is a local minimum of Problem 3.52 and the following performance bound holds*

$$\limsup_{K\to\infty} \frac{\sum_{t=0}^{TK-1} \ell_{\text{eco}}(x(t), u(t), t, y_{\text{e}}(t))}{K} \leq \overline{\kappa}_\infty.$$

Proof. The results follow directly from Propositions 3.56, 3.58, and 3.64. ■

Corollary 3.66. *Let Assumptions 3.50, 3.55, 3.59, and 3.62 hold. Assume that Problem 3.53 is feasible at* $t = 0$*. If the update rule* $\mathcal{B}$ *is chosen as update scheme* 2 *or* 6 *in [206], then the closed-loop average economic performance resulting from Algorithm 3.54 is no worse than the performance at a locally optimal periodic orbit (local minimum of Problem 3.52).*

Proof. This results follows directly from Theorem 3.65. It suffices to note that the update schemes 2 and 6 in [206] satisfy Assumptions 3.57 and 3.61, compare [206, Lemmas 1 and 4]. ■

This result extends the performance guarantees in [208] to economic MPC schemes with artificial *periodic* trajectories ($T > 1$) and thus provides performance guarantees relative to (locally) optimal periodic operation. The proposed economic MPC formulation (Alg. 3.54) ensures the desired closed-loop performance, if

(a) The parameter predictions are consistent (Ass. 3.50),

(b) Suitable terminal ingredients are employed (Ass. 3.55 and 3.59),

(c) An update rule $\mathcal{B}$ is used (Ass. 3.57 and 3.61),

(d) The periodic continuity condition holds (Ass. 3.62).

Condition (a) is intuitively needed to yield performance guarantees using MPC. Explicit design procedures satisfying Condition (b) are detailed in Section 3.3.5. Relaxations of Conditions (c) and (d) can be found in Section 3.3.6.

Remark 3.67. *(Continuous-time formulation) For simplicity, we have presented the proposed MPC framework in a discrete-time setting. However, the approach can be directly applied to continuous-time problems by defining the discrete-time stage cost* ℓ_{eco} *and dynamics* f *implicitly as the integration of some continuous-time dynamics* f_{c} *and the average continuous-time cost* $\ell_{\text{eco,c}}$ *over some sampling period* T_{s}*. One advantage of considering a continuous-time formulation is that the design of terminal ingredients satisfying Assumption 3.5 (cf. Sec. 3.3.5) simplifies, compare [JK15, App. C]. Furthermore, in a continuous-time setting it is possible to use a variable* *sampling time* $T_\mathrm{s} \in [\underline{T}_\mathrm{s}, \overline{T}_\mathrm{s}] \subseteq \mathbb{R}_{>0}$*, by considering the decision variable* $u = (u_\mathrm{c}, T_\mathrm{s})$*, where* u_c *denotes the (typically piece-wise constant) control input. As a result, in a time-invariant setting the fixed constant* T *does not directly impose a time length on the set of periodic orbits* $\mathbb{Z}_T$*, but only a finite parametrization. The constants* $\underline{T}_\mathrm{s}$*,* $\overline{T}_\mathrm{s}$ *need to be chosen such that the (possibly explicit) discretization scheme is stable and the MPC can react fast enough. This modification is equally applicable to the tracking MPC formulation in Section 3.2, assuming that a continuous-time output target* y_e *is specified. The advantages of such a continuous-time MPC formulation are also explored in a numerical example in [JK26, App. A]. We point out that the benefits of using such a variable continuous-time period length have also been recently investigated in [133] using a direct multiple shooting method.*

Remark 3.68. *(MPC formulations using periodicity constraints) For a horizon* $N = 0$*, we can define the MPC control law as* $u(t) = k_\mathrm{f}(x(t), r^*(\cdot|t), t)$*. In this case, Problem 3.53 only determines a periodic optimal reference trajectory* $r^*(\cdot|t)$*, which is restricted to be close to the current state* $x(t)$ *due to the terminal set constraint. If the terminal set constraint* $\mathbb{X}_\mathrm{f}$ *were chosen as a terminal equality constraint* (3.61)*, Problem 3.53 would compute a periodic trajectory, starting at the current state* $x(t)$*, which was also proposed in the MPC formulations in [138, 282]. Such an economic MPC formulation with a periodicity constraint directly guarantees the relative performance bounds in Proposition 3.58. However, in contrast to Proposition 3.64, in general such a formulation does not ensure convergence to a local minimum. In particular, in [138, Lemma 3, Thm. 4], convergence could only be guaranteed using a restrictive one-step controllability condition. Furthermore, even in the linear convex case, such a formulation may fail to convergence, compare [282, Example 6]. By using the contractive terminal set* $\mathbb{X}_\mathrm{f}$ *from Assumption 3.59 with* $N = 0$*, the proposed formulation can relax the periodicity constraint employed in [138, 282] and thus provide stronger performance guarantees.*

Remark 3.69. *(Model uncertainty in economic MPC) The consideration of model uncertainty in the considered economic MPC framework is interesting for multiple reasons. Recursive feasibility and constraint satisfaction can be handled analogous to the tracking MPC in Sections 3.1–3.2, compare Remarks 3.24, 3.32 and 3.41. The economic stage cost* ℓ_eco *can be adjusted to account for the uncertainty in the predictions, which allows for robust performance guarantees, compare [30, 31, 82, JK35, 271, 272]. Optimal operation with uncertain deterministic model-mismatch/offsets requires additional care and has been considered in literature for the special case of* $T = 1$*. In [82], a set bounding deterministic disturbances is updated online and used in the cost function, extending [206, 208] to a robust setting and reducing conservatism online (for linear systems). Instead of only estimating a disturbance/offset as done in offset-free tracking [201, 218], the gradient of the economic stage cost* ℓ_eco *at steady-state can be additionally estimated using* *modifier adaptation. This gradient can be directly used in the economic MPC scheme to ensure convergence to the (economically) optimal steady-state, compare [9, 99, 217, 269].*

3.3.5 Terminal costs for economic periodic operation

In the following, we provide design procedures to compute a suitable economic terminal cost $V_{\mathrm{f,eco}}$ satisfying Assumption 3.59. We extend the approach in [16] to dynamic/periodic trajectories, by considering the linearized dynamics and local quadratic approximations of the economic stage cost ℓ_{eco} (using the first and second derivative). The online evaluation of this terminal cost $V_{\mathrm{f,eco}}$ involves an additional adjoint p_{eco} (similar to the local gradient correction employed in [291]), which needs to be recomputed online for any periodic reference trajectory r. In order to reduce the computational complexity, we also present a simpler, more conservative, design procedure with a positive definite terminal cost $V_{\mathrm{f,eco}}$ similar to the design in [10, 11]. In addition, we also show how Algorithm 3.54 can be adjusted to retain the a priori performance bounds (Thm. 3.65) with a simple terminal equality constraint.

Linear-quadratic local auxiliary stage cost

The following lemma extends the results in [16, Lemma 22-23] to compute an auxiliary stage cost $\ell_{\mathrm{eco,q}}$ which locally upper bounds the stage cost ℓ_{eco}.

Lemma 3.70. *Suppose that V_δ, k_{f}, α satisfy the conditions (3.68)–(3.69) in Assumption 3.59. Suppose further that the sublevel sets of V_δ are convex in x, the controller k_{f} is twice continuously differentiable in x, continuous in r, and satisfies $k_{\mathrm{f}}(x_{\mathrm{r}}, r, t) = u_{\mathrm{r}}(0)$. In addition, assume that the stage cost ℓ_{eco} and the dynamics f are twice continuously differentiable w.r.t. $\xi = (x, u) \in \mathbb{R}^{n+m}$. Then, the function $\overline{\ell}_{\mathrm{eco}}(x, r, t, y_{\mathrm{e}}) := \ell_{\mathrm{eco}}(x, k_{\mathrm{f}}(x, r, t), t, y_{\mathrm{e}}) - \ell_{\mathrm{eco}}(r(0), t, y_{\mathrm{e}})$ is twice continuously differentiable w.r.t. x. Furthermore, for any $\epsilon > 0$, there exists a constant $\alpha_1 > 0$ and a positive semidefinite matrix $S : \mathbb{Z}_{\mathrm{r}} \times \mathbb{Y} \to \mathbb{R}^{(n+m)\times(n+m)}$ such that the following conditions hold for any $(r, t) \in \mathbb{Z}_T$, $y_{\mathrm{e}} \in \mathbb{Y}$ and any $x \in \mathbb{X}$ with $V_\delta(x, r, t) \leq \alpha_1$:*

$$S(r(0), t, y_{\mathrm{e}}) \succeq \ell_{\mathrm{eco},\xi\xi}(r(0), t, y_{\mathrm{e}}), \tag{3.75a}$$

$$\ell_{\mathrm{eco,q}}(x, r, t, y_{\mathrm{e}}) \geq \overline{\ell}_{\mathrm{eco}}(x, r, t, y_{\mathrm{e}}) + \frac{\epsilon}{2}\|x - x_{\mathrm{r}}(0)\|^2, \tag{3.75b}$$

with

$$\ell_{\text{eco,q}}(x,r,t,y_{\text{e}}) := \|x - x_{\text{r}}(0)\|^2_{Q_{\text{eco}}(r,t,y_{\text{e}})} + \overline{\ell}_{\text{eco},x}(x_{\text{r}}(0),r,t,y_{\text{e}}) \cdot (x - x_{\text{r}}(0)), \tag{3.75c}$$

$$\begin{aligned} Q_{\text{eco}}(r,t,y_{\text{e}}) := &\begin{pmatrix} I_n \\ k_{\text{f},x}(x_{\text{r}}(0),r,t) \end{pmatrix}^\top S(r(0),t,y_{\text{e}}) \begin{pmatrix} I_n \\ k_{\text{f},x}(x_{\text{r}}(0),r,t) \end{pmatrix} \\ &+ 2\epsilon I_n + \sum_{j=1}^{m} \ell_{\text{eco},u_j}(r(0),t,y_{\text{e}}) k_{\text{f},j,xx}(x_{\text{r}}(0),r,t), \end{aligned} \tag{3.75d}$$

where $\ell_{\text{eco},\xi\xi} : \mathbb{Z}_{\text{r}} \times \mathbb{Y} \to \mathbb{R}^{(n+m)\times(n+m)}$ denotes the Hessian of ℓ_{eco} w.r.t. $\xi = (x,u) \in \mathbb{R}^{n+m}$, $\ell_{\text{eco},u_j} : \mathbb{Z}_{\text{r}} \times \mathbb{Y} \to \mathbb{R}$ the Jacobian of ℓ_{eco} w.r.t. the j-th component of u, $k_{\text{f},x} : \mathbb{X} \times \mathbb{Z}_T \to \mathbb{R}^{m\times n}$ the Jacobian of k_{f} w.r.t. x, $\overline{\ell}_{\text{eco},x} : \mathbb{X} \times \mathbb{Z}_T \times \mathbb{Y} \to \mathbb{R}^{1\times n}$ the Jacobian of $\overline{\ell}_{\text{eco}}$ w.r.t. x, and $k_{\text{f},j,xx} : \mathbb{X} \times \mathbb{Z}_T \to \mathbb{R}^{n\times n}$ the Hessian of the j-th component of k_{f} w.r.t. x, $j \in \mathbb{I}_{[1,m]}$.

Proof. The derivative of $\overline{\ell}_{\text{eco}}$ w.r.t. x is the total derivative of ℓ_{eco} w.r.t. x, for $u = k_{\text{f}}$. Hence, the Jacobian and Hessian of $\overline{\ell}_{\text{eco}}$ are given by

$$\overline{\ell}_{\text{eco},x} = \ell_{\text{eco},\xi} \begin{pmatrix} I_n \\ k_{\text{f},x} \end{pmatrix}, \quad \overline{\ell}_{\text{eco},xx} = \begin{pmatrix} I_n & k_{\text{f},x}^\top \end{pmatrix} \ell_{\text{eco},\xi\xi} \begin{pmatrix} I_n & k_{\text{f},x}^\top \end{pmatrix}^\top + \sum_{j=1}^{m} \ell_{\text{eco},u_j} k_{\text{f},j,xx},$$

where $\ell_{\text{eco},\xi} : \mathbb{Z}_{\text{r}} \times \mathbb{Y} \to \mathbb{R}^{1\times(n+m)}$ denotes the Jacobian of ℓ_{eco} w.r.t. $\xi = (x,u)$. Twice continuous differentiability of ℓ_{eco} and compact constraints imply that there exists a finite constant

$$c := \sup_{(r,t)\in\mathbb{Z}_{\text{r}}, y_{\text{e}}\in\mathbb{Y}} \lambda_{\max}(\ell_{\text{eco},\xi\xi}(r,t,y_{\text{e}})).$$

Thus, the matrix $S := (\max\{c,0\}) I_{n+m}$ is positive semidefinite and satisfies $S \succeq \ell_{\text{eco},\xi\xi}$. The construction in (3.75d), the definition of the Hessian $\overline{\ell}_{\text{eco},xx}$ and $S \succeq \ell_{\text{eco},\xi\xi}$ directly imply $Q_{\text{eco}}(r,t,y_{\text{e}}) \succeq \overline{\ell}_{\text{eco},xx}(x_{\text{r}}(0),r,t,y_{\text{e}}) + 2\epsilon I_n$. Similar to [16, Lemma 22], there exists a small enough constant $\alpha_1 > 0$ (uniform in r,t,y_{e}) such that $Q_{\text{eco}}(r,t,y_{\text{e}}) \succeq \overline{\ell}_{\text{eco},xx}(x,r,t,y_{\text{e}}) + \epsilon I_n$, $\forall (r,t) \in \mathbb{Z}_T$, $y_{\text{e}} \in \mathbb{Y}$, $x \in \mathbb{X} : V_\delta(x,r,t) \leq \alpha_1$. Abbreviate $\Delta x = x - x_{\text{r}}(0)$, which implies $\ell_{\text{eco,q}} = \|\Delta x\|^2_{Q_{\text{eco}}} + \overline{\ell}_{\text{eco},x} \cdot \Delta x$. Convexity of the sublevel sets of V_δ implies that $V_\delta(x_{\text{r}}(0) + s\Delta x, r, t) \leq \alpha_1$ for all $s \in [0,1]$ and any $V_\delta(x_{\text{r}}(0) + \Delta x, r, t) \leq \alpha_1$. Hence, we can use the mean value theorem for vector functions [236, Prop. A.11 (b)], similar to [16, Lemma 23], to obtain

$$\begin{aligned}
&\ell_{\mathrm{eco,q}}(x,r,t,y_\mathrm{e}) - \overline{\ell}_{\mathrm{eco}}(x,r,t,y_\mathrm{e}) \\
=& \int_0^1 (1-s)\Delta x^\top \left(Q_{\mathrm{eco}}(r,t,y_\mathrm{e}) - \overline{\ell}_{\mathrm{eco},xx}(x_\mathrm{r}(0)+s\cdot\Delta x,r,t,y_\mathrm{e})\right)\Delta x ds \\
\geq& \int_0^1 (1-s)\epsilon\|\Delta x\|^2 ds = \epsilon/2\|\Delta x\|^2. \qquad\blacksquare
\end{aligned}$$

Basically, $\ell_{\mathrm{eco,q}}$ is a local linear-quadratic over-approximation of the stage cost ℓ_{eco}. Hence, we will formulate a sufficient condition for Inequality (3.60c) using the auxiliary stage cost $\ell_{\mathrm{eco,q}}$. We point out that Lemma 3.70 does not impose any definiteness conditions on the Hessian of the stage cost ℓ_{eco}, but instead upper bounds the Hessian using the positive semidefinite matrix S.

Sufficient conditions based on the linearization

Denote the Jacobian of f evaluated around an arbitrary point $(x_\mathrm{r},u_\mathrm{r},t)=(r,t)\in\mathbb{Z}_\mathrm{r}$ by

$$A(r,t) := \left[\frac{\partial f}{\partial x}\right]\Bigg|_{(x_\mathrm{r},u_\mathrm{r},t)}, \quad B(r,t) := \left[\frac{\partial f}{\partial u}\right]\Bigg|_{(x_\mathrm{r},u_\mathrm{r},t)}.$$

Given some periodic trajectory $(r(\cdot|t),t)\in\mathbb{Z}_T$, the Jacobian w.r.t. x of the system dynamics f in closed loop with the terminal control law k_f is given by

$$A_{\mathrm{cl}}(r(\cdot|t),t) := A(r(0|t),t) + B(r(0|t),t)k_{\mathrm{f},x}(r(\cdot|t),t).$$

In the following, we introduce a periodic adjoint trajectory $p_{\mathrm{eco}}(j|t)\in\mathbb{R}^n$, $j\in\mathbb{I}_{[0,T-1]}$, which can be computed online based on the following set of $n\cdot T$ linear (in p_{eco}) equality constraints

$$\begin{aligned}
&A_{\mathrm{cl}}^\top(\mathcal{R}_T^j r(\cdot|t),t+j)p_{\mathrm{eco}}(j+1|t) \\
=&p_{\mathrm{eco}}(j|t) - \overline{\ell}_{\mathrm{eco},x}^\top(x_\mathrm{r}(j|t),\mathcal{R}_T^j r(\cdot|t),t+j,y_\mathrm{e}(j|t)),\ j\in\mathbb{I}_{[0,T-1]},
\end{aligned} \tag{3.76}$$

with $p_{\mathrm{eco}}(N|t) := p_{\mathrm{eco}}(0|t)$. In the setpoint case ($T=1$), this reduces to $p_{\mathrm{eco}}^\top(A_{\mathrm{cl}}-I_n) = -\overline{\ell}_{\mathrm{eco},x}^\top$, similar to [16, 208]. Similar to the adjoints used in [291], this vector p_{eco} corrects the effect of $\overline{\ell}_{\mathrm{eco},x}$, the gradient of the stage cost.

The following proposition shows that such an online computed adjoint vector p_{eco} in combination with an offline computed matrix valued function P_{eco} provides a suitable terminal cost for dynamic operation with economic cost.

Proposition 3.71. *Suppose the conditions in Lemma 3.70 hold. Assume that there exists a continuously parametrized positive definite matrix* $P_{\text{eco}} : \mathbb{Z}_T \to \mathbb{R}^{n\times n}$ *such that for all* $(r,t) \in \mathbb{Z}_T$, $y_{\text{e}} \in \mathbb{Y}$, *the following matrix inequality is satisfied*

$$A_{\text{cl}}^\top(r,t)P_{\text{eco}}(\mathcal{R}_T r, t+1)A_{\text{cl}}(r,t) - P_{\text{eco}}(r,t) \preceq -Q_{\text{eco}}(r,t,y_{\text{e}}) - \tilde{\epsilon} I_n, \tag{3.77}$$

with some $\tilde{\epsilon} > 0$ *and* $P_{\text{eco}}(r,t+T) = P_{\text{eco}}(r,t)$. *Then, for any periodic reference* $(r(\cdot|t),t) \in \mathbb{Z}_T$, *parameters* $y_{\text{e}}(\cdot|t) \in \mathbb{Y}^T$, *the conditions* (3.76) *have a unique solution* $p_{\text{eco}}(\cdot|t)$, *which is denoted by the continuous function* $p_{\text{eco}} : \mathbb{Z}_T \times \mathbb{Y}^T \to \mathbb{R}^n$ *with* $p_{\text{eco}}(r(\cdot|t),t,y_{\text{e}}(\cdot|t)) := p_{\text{eco}}(0|t)$. *There exists a constant* $\alpha_1 > 0$ *such that the terminal cost*

$$V_{\text{f,eco}}(x,r,t,y_{\text{e}}) := \|x_{\text{r}}(0) - x\|^2_{P_{\text{eco}}(r,t)} + p_{\text{eco}}^\top(r,t,y_{\text{e}}) \cdot (x - x_{\text{r}}(0)), \tag{3.78}$$

satisfies Condition (3.60c) *with* $\mathbb{X}_{\text{f}} = \{(x,r,t) \in \mathbb{X} \times \mathbb{Z}_T \mid V_\delta(x,r,t) \leq \alpha_1\}$.

Proof. **Part I:** Condition (3.77) ensures that the linearized (time-varying) dynamics along the periodic trajectory r are (uniformly) exponentially stable, which implies

$$\det(I_n - \Pi_{j=0}^{T-1} A_{\text{cl}}^\top(\mathcal{R}_T^j r, t+j)) > 0. \tag{3.79}$$

Thus, the constraints (3.76) have a unique solution p_{eco} for any $(r,t) \in \mathbb{Z}_T$, $y_{\text{e}} \in \mathbb{Y}^T$, compare (3.83). Hence, the map $p_{\text{eco}} : \mathbb{Z}_T \times \mathbb{Y}^T \to \mathbb{R}^n$ is well defined and continuous in r, y_{e} due to the uniform bound (3.79) and A_{cl}, $\overline{\ell}_{\text{eco},x}$ continuous in r and y_{e}.
Part II: Denote $\Delta x = x - x_{\text{r}}(0)$. The first order Taylor approximation at $x = x_{\text{r}}(0)$ yields

$$\Delta x^+ = f(x,k_{\text{f}}(x,r,t),t) - f(x_{\text{r}}(0),u_{\text{r}}(0),t) = A_{\text{cl}}(r,t)\Delta x + \Phi(\Delta x,r,t),$$

with the remainder term Φ. Twice continuous differentiability of f and compact constraints imply that the remainder term is uniformly Lipschitz continuous in the terminal set, i.e., $\|\Phi(\Delta x,r,t)\| \leq L_\Phi \|\Delta\|$ for all $(r,t) \in \mathbb{Z}_T$, with a constant L_Φ arbitrary small for α_1 arbitrary small. Using this bound in combination with Condition (3.77) implies that there exists a sufficiently small constant $\alpha_1 > 0$ such that the nonlinear system (locally) satisfies

$$\|\Delta x^+\|^2_{P_{\text{eco}}(\mathcal{R}_T r,t+1)} - \|\Delta x\|^2_{P_{\text{eco}}(r,t)}, \leq -\|\Delta x\|^2_{Q_{\text{eco}}(r,t,y_{\text{e}})}, \tag{3.80}$$

for all $(x,r,t) \in \mathbb{X}_{\text{f}}$, $y_{\text{e}} \in \mathbb{Y}$, compare the proof of Lemma 3.12 for details. Given that T is finite, $\overline{\ell}_{\text{eco},x}$ is uniformly bounded and Condition (3.79) holds, the function p_{eco} admits a uniform norm bound $\overline{p}_{\text{eco}} := \max_{(r,t)\in\mathbb{Z}_T, y_\text{e}\in\mathbb{Y}^T} \|p_{\text{eco}}(r,t,y_\text{e})\|$. Using the definition of p_{eco}, we get

$$\begin{aligned} & p_{\text{eco}}^\top(\mathcal{R}_T r, t+1, \mathcal{R}_T y_\text{e})\Delta x^+ \\ \leq & p_{\text{eco}}^\top(\mathcal{R}_T r, t+1, \mathcal{R}_T y_\text{e}) A_{\text{cl}}(r,t)\Delta x + \overline{p}_{\text{eco}}\|\Phi(\Delta x, r, t)\| \\ \overset{(3.76)}{=} & p_{\text{eco}}^\top(r,t,y_\text{e})\Delta x - \overline{\ell}_{\text{eco},x}(x_\text{r}(0), r, t, y_\text{e}(0))\Delta x + \overline{p}_{\text{eco}}\|\Phi(\Delta x, r, t)\| \\ \leq & p_{\text{eco}}^\top(r,t,y_\text{e})\Delta x - \overline{\ell}_{\text{eco},x}(x_\text{r}(0), r, t, y_\text{e}(0))\Delta x + \epsilon/2\|\Delta x\|^2, \end{aligned} \tag{3.81}$$

where the last inequality holds for a sufficiently small constant $\alpha_1 > 0$, given the properties of the remainder term Φ. By combining Inequalities (3.80)–(3.81) and using the auxiliary stage cost $\ell_{\text{eco,q}}$ from Lemma 3.70, the terminal cost (3.78) satisfies

$$\begin{aligned} & V_{\text{f,eco}}(x^+, \mathcal{R}_T r, t+1, \mathcal{R}_T y_\text{e}) - V_{\text{f,eco}}(x,r,t,y_\text{e}) \\ \leq & -\|\Delta x\|^2_{Q_{\text{eco}}(r,t,y_\text{e}(0))} - \overline{\ell}_{\text{eco},x}(x_\text{r}(0), r, t, y_\text{e}(0))\Delta x + \frac{\epsilon}{2}\|\Delta x\|^2 \overset{(3.75\text{b})}{\leq} -\overline{\ell}_{\text{eco}}(x,r,t,y_\text{e}(0)), \end{aligned}$$

for all $(x,r,t) \in \mathbb{X}_\text{f}$, $y_\text{e} \in \mathbb{Y}^T$ and hence Condition (3.60c) holds. ■

The following result combines the economic terminal cost $V_{\text{f,eco}}$ from Proposition 3.71 with the (standard) terminal set design in Proposition 3.11, to provide a complete design procedure for terminal ingredients satisfying Assumptions 3.55 and 3.59.

Corollary 3.72. *Suppose that the stage cost ℓ_{eco} and the dynamics f are twice continuously differentiable w.r.t. (x,u). Assume that there exist a continuously parametrized matrix $k_{\text{f},x}$: $\mathbb{Z}_T \to \mathbb{R}^{m\times n}$ and continuously parametrized positive definite matrix $P_{\text{eco}} : \mathbb{Z}_T \to \mathbb{R}^{n\times n}$, with $P_{\text{eco}}(r,t+T) = P_{\text{eco}}(r,t)$, $k_{\text{f},x}(r,t+T) = k_{\text{f},x}(r,t)$ such that for all $(r,t) \in \mathbb{Z}_T$, $y_\text{e} \in \mathbb{Y}$, Inequality (3.77) holds with some $\tilde{\epsilon} > 0$. Then, there exists a function $\alpha : \mathbb{Z}_T \to \mathbb{R}_{\geq 0}$ such that*

$$k_\text{f}(x,r,t) = u_\text{r}(0) + k_{\text{f},x}(r,t)\cdot(x - x_\text{r}(0)), \tag{3.82a}$$

$$\mathbb{X}_\text{f} = \{(x,r,t) \in \mathbb{X}\times\mathbb{Z}_T \mid \|x - x_\text{r}(0)\|^2_{P_{\text{eco}}(r,t)} \leq \alpha(r,t)\}, \tag{3.82b}$$

and $V_{\text{f,eco}}$ according to (3.78) satisfy Assumptions 3.55 and 3.59.

Proof. Given that P_{eco} and $Q_{\text{eco}} + \tilde{\epsilon} I_n$ are positive definite with uniform lower and upper bounds, the conditions (3.68a)–(3.68b) in Assumption 3.59 are satisfied with the incremental Lyapunov function $V_\delta(x,r,t) = \|x - x_\text{r}(0)\|^2_{P_{\text{eco}}(r,t)}$ and quadratic functions $\alpha_{\delta,1}, \alpha_{\delta,2}, \alpha_{\delta,3} \in \mathcal{K}_\infty$. Convexity of the terminal set $\mathbb{X}_\text{f}$ w.r.t. x (compare conditions

Lemma 3.70) follows from V_δ quadratic in x and P_{eco} positive definite. Condition (3.68c) follows from V_δ quadratic and the assumed continuity of P_{eco} w.r.t. r, similar to Proposition 3.44. Conditions (3.60a), (3.60c), (3.68a)–(3.68c) hold for any $\alpha \leq \alpha_1$, using Proposition 3.71. Furthermore, given that references r inside $\mathbb{Z}_{\text{r}}$ strictly satisfy the constraints in $\mathbb{Z}$ and $k_{\text{f},x}$ is bounded, there exists a small enough constant $\alpha_2 > 0$ such that conditions (3.60b), (3.69) hold for any $\alpha \leq \alpha_2$. Hence, choosing the constant terminal set size $\alpha(r,t) := \min\{\alpha_1, \alpha_2\}$ satisfies all the conditions (Condition (3.69) is trivially satisfied in this case). ∎

With this result, we can directly specify a procedure to compute suitable terminal ingredients. First, symbolic expressions for the Jacobians A, B, $\ell_{\text{eco},\xi}$ and the Hessian $\ell_{\text{eco},\xi\xi}$ are computed. Then, a positive semidefinite matrix S is computed, which satisfies Condition (3.75a). This can either be achieved with a constant matrix S (cf. proof of Lemma 3.70) or by computing a suitably parametrized matrix S using LMIs.

Given S, we have to compute a (continuously) parametrized matrix P_{eco} such that Condition (3.77) holds. Suppose we want to compute a feedback of the form (3.82a) ($k_{\text{f},xx} = 0$). In this case, Condition (3.77) with Q_{eco} according to Condition (3.75d) is equivalent to Inequality (B.2) in Appendix B with the following (output) tracking stage cost

$$\tilde{\ell} = \|(C + Dk_{\text{f},x})\Delta x\|_S^2 + (2\epsilon + \tilde{\epsilon})\|\Delta x\|^2,$$
$$C = \begin{pmatrix} I_n \\ 0_{m\times n} \end{pmatrix} \in \mathbb{R}^{(n+m)\times n},\ D = \begin{pmatrix} 0_{n\times m} \\ I_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times m}.$$

Hence, we can use Lemma B.3/Proposition B.4 in Appendix B to compute suitable matrices $k_{\text{f},x}$ and P_{eco}, using LMIs and a quasi-LPV parametrization, analogous to the design in Section 3.1.3.

Given that p_{eco} needs to satisfy Condition (3.76) with *equality*, an offline parametrization for p_{eco} seems intractable (with the exception of linear systems, compare Section 3.3.6). Hence, we can simply add[12] the constraint (3.76) to Problem 3.53 and compute $p_{\text{eco}}(\cdot|t)$ online. Finally, regarding the terminal set size α, we first compute the constant $\alpha_1 > 0$ such that Condition (3.60c) holds for all $V_\delta(x,r,t) \leq \alpha_1$, e.g., using the optimization problem (3.26). There are two options to compute a terminal set size α that also ensures constraint satisfaction (3.60b). The fact that the set $\mathbb{Z}_{\text{r}}$ is smaller than the set $\mathbb{Z}$ can be used to compute a constant $\alpha \in (0, \alpha_1]$, similar to the optimization problem

[12] Due to the prediction horizon N the time index t changes to $t+N$ in (3.76) and $y_{\text{e}}(\cdot|t)$ is replaced by $\overline{y}_{\text{e}}(\cdot|t) = \mathcal{R}_T^N y_{\text{e}}(\cdot|t)$.

in (3.25). However, such a constant α depends on the choice of $\mathbb{Z}_\mathrm{r}$ and thus can yield arbitrary small values α (and thus slow convergence of r, compare Lemma 3.60), or requires restrictive constraints on the set of periodic trajectories $\mathbb{Z}_T$. This problem can be alleviated by using a reference dependent terminal set size $\alpha(r,t) \in [\underline{\alpha}, \overline{\alpha}]$ (cf. (3.70)), which can be implemented using an additional scalar optimization variable α in Problem 3.53, as done in Section 3.2.2. The overall design procedure is summarized in Algorithm 3.73.

Algorithm 3.73. *Offline computation - Terminal ingredients for Economic MPC*

1: *Compute the symbolic Jacobians, A, B, $\ell_{\mathrm{eco},\xi}$ and the Hessian $\ell_{\mathrm{eco},\xi\xi}$.*
2: *Determine the matrix $S \succeq 0$ such that $S \succeq \ell_{\mathrm{eco},\xi\xi}$* (3.75a).
3: *Compute the matrices P_eco and $k_{\mathrm{f},x}$ such that Inequality* (3.77) *holds (cf. SDP App. B).*
4: *Compute the maximal terminal set size $\alpha_1 > 0$ (cf. Eq.* (3.26)*).*
5: *Derive $\alpha(r,t) \in (0, \alpha_1]$ for constraint satisfaction:*
   a) Compute the constant $\alpha_2 > 0$ (cf. Eq. (3.25)*) and set $\alpha = \min\{\alpha_1, \alpha_2\}$.*
   b) Use $\alpha(r,t)$ with an additional scalar optimization variable (cf. Sec. 3.2.2).

The proposed procedure is a combination and extension of the reference generic offline computations in Section 3.1.3, the terminal cost for economic MPC [16] and the online computation of p_eco using Equation (3.76). Regarding the online operation, we simply include the constraints (3.76) to compute $p_\mathrm{eco}(\cdot|t)$ and possibly constraints to compute $\alpha(r,t)$ online (cf. Sec. 3.2.2) in Problem 3.53. This procedure significantly simplifies in the special case of linear systems with linear/quadratic stage costs ℓ_eco, which is discussed in Section 3.3.6. Furthermore, in the special case of artificial setpoints ($T=1$), we recover the schemes in [87, 206, 208] and Algorithm 3.73 provides a corresponding procedure to derive suitable terminal ingredients.

Remark 3.74. *(Parametrization using y_e) The matrix P_eco can also be parametrized by y_e, to yield a less conservative terminal cost $V_{\mathrm{f,eco}}$. However, the incremental Lyapunov function V_δ used for the terminal set $\mathbb{X}_\mathrm{f}$ (Ass. 3.59) may not depend on y_e to ensure recursive feasibility independent of online changes in the parameters y_e. Thus, the choice of $\mathbb{X}_\mathrm{f}$ in Corollary 3.72 is only valid for P_eco independent of y_e.*

Remark 3.75. *(Computation of p_eco) As already discussed, the vector p_eco needs to be computed online using Equations* (3.76)*, which adds $n \cdot T$ optimization variables and $n \cdot T$ equality constraints (linear in p_eco) to Problem 3.53. Abbreviate $A(j|t) = A(r(j|t), t+j)$, $B(j|t) = B(r(j|t), t+j)$, $\ell_{\mathrm{eco},\xi}(j|t) = \ell_{\mathrm{eco},\xi}(r(j|t), t+j, y_\mathrm{e}(j|t))$ and suppose the feedback $k_{\mathrm{f},x}$ is parametrized in the form $k_{\mathrm{f},x}(j|t) = Y(j|t)X^{-1}(j|t)$ with matrices X, Y (cf. Sec. 3.1.3 and*

App. B). Then, multiplying the constraints (3.76) *by* X *from the left yields the following equivalent constraint*

$$\begin{aligned}&(A(j|t)X(j|t)+B(j|t)Y(j|t))^\top p_{\text{eco}}(j+1|t)\\ =&X(j|t)p_{\text{eco}}(j|t)-\begin{pmatrix}X(j|t) & Y^\top(j|t)\end{pmatrix}\ell^\top_{\text{eco},\xi}(j|t),\end{aligned}$$

where we use $k_{\text{f},x}X=Y$ *and the formula for* $\overline{\ell}_{\text{eco},x}$ *from the proof of Lemma 3.70. The resulting constraint can be implemented directly in terms of* X, Y.

The constraints (3.76) *can also be compressed and replaced by directly using an explicit expression for* $p_{\text{eco}}(0|t)$. *Denote* $\overline{\ell}_{\text{eco},x}(j|t)=\overline{\ell}_{\text{eco},x}(x_\text{r}(j|t),r(\cdot|t),t+j,y_\text{e}(j|t))$ *and* $\overline{A}_{\text{cl}}(k|t)=\Pi_{j=0}^{k-1}A_{\text{cl}}(\mathcal{R}_T^j r(\cdot|t),t+j)$. *Then,* $p_{\text{eco}}(0|t)$ *satisfying Condition* (3.76) *can be equivalently computed using*

$$p_{\text{eco}}(r(\cdot|t),t,y_\text{e}(\cdot|t)):=\left(I_n-\overline{A}_{\text{cl}}^\top(T|t)\right)^{-1}\sum_{j=0}^{T-1}\overline{A}_{\text{cl}}^\top(j|t)\overline{\ell}^\top_{\text{eco},x}(j|t). \tag{3.83}$$

Furthermore, if the period length T *is very large, an approximate solution can be obtained by assuming* $\overline{A}_{\text{cl}}(j|t)\approx 0$ *for* $j\in\mathbb{I}_{[T_\text{c},T]}$ *with some* $T_\text{c}<T$, *which results in* $p_{\text{eco}}(0|t)\approx\sum_{j=0}^{T_\text{c}-1}\overline{A}_{\text{cl}}^\top(j|t)\overline{\ell}^\top_{\text{eco},x}(j|t)$. *The fact that we need to take the full reference trajectory* r *into account to compute the correct gradient correction* p_{eco} *indicates that the computation of an (indefinite) economic terminal cost for nonperiodic time-varying trajectories may be non-trivial.*

Positive definite terminal cost $V_{\text{f,eco}}$

The terminal cost $V_{\text{f,eco}}$ designed in Corollary 3.72 is not positive definite with respect to the reference trajectory, which provides a better performance, analogous to the terminal cost in [16]. However, the computational complexity of Problem 3.53 may increase due to long expression of p_{eco} in Equation (3.83). In the following, we provide an alternative design, resulting in a positive definite terminal cost $V_{\text{f,eco}}$, which may be more conservative but easier to implement. The design procedure is inspired by the positive definite economic terminal cost in [11, Prop. 2] and [10, Prop. 27].

Proposition 3.76. *Suppose that the conditions in Corollary 3.72 hold. Then, there exists a constant* $c>0$ *such that* $\mathbb{X}_\text{f}$, k_f *according to* (3.82) *and the following terminal cost satisfy Assumptions 3.55 and 3.59:*

$$V_{\text{f,eco}}(x,r,t):=\|x-x_\text{r}(0)\|^2_{P_{\text{eco}}(r,t)}+c\|x-x_\text{r}(0)\|_{P_{\text{eco}}(r,t)}.$$

Proof. Denote $\Delta x = x - x_{\mathrm{r}}(0)$, $u = k_{\mathrm{f}}(x,r,t)$, $x^+ = f(x,u,t)$, $r^+ = \mathcal{R}_T r$, $\Delta x^+ = x^+ - x_{\mathrm{r}}^+(0)$. Given that $V_\delta(x,r,t) = \|\Delta x\|^2_{P_{\mathrm{eco}}(r,t)}$ satisfies Conditions (3.68a)–(3.68b) with $\alpha_{\delta,1}, \alpha_{\delta,2}, \alpha_{\delta,3}$ quadratic, there exists a constant $\rho \in [0,1)$ such that $V_\delta(x^+,r^+,t+1) \leq \rho^2 V_\delta(x,r,t)$ for any $(x,r,t) \in \mathbb{X}_{\mathrm{f}}$. Furthermore, we have

$$\overline{\ell}_{\mathrm{eco}}(x,r,t,y_{\mathrm{e}}) \overset{(3.75b)}{\leq} \ell_{\mathrm{eco,q}}(x,r,t,y_{\mathrm{e}}) \leq \|\Delta x\|^2_{Q_{\mathrm{eco}}(r,t,y_{\mathrm{e}})} + a_1 \|\Delta x\|_{P_{\mathrm{eco}}(r,t)}, \tag{3.84}$$

with some constant $a_1 > 0$ using uniform bounds on the gradient $\overline{\ell}_{\mathrm{eco},x}$ and the eigenvalues of P_{eco} (continuous and compact constraints). Inequality (3.60c) follows by choosing $c := a_1/(1-\rho) > 0$ with

$$\begin{aligned} &V_{\mathrm{f,eco}}(x^+,r^+,t+1) = \|\Delta x^+\|^2_{P_{\mathrm{eco}}(r^+,t+1)} + c\|\Delta x^+\|_{P_{\mathrm{eco}}(r^+,t+1)} \\ \leq &\|\Delta x\|^2_{P_{\mathrm{eco}}(r,t)} - \|\Delta x\|^2_{Q_{\mathrm{eco}}(r,t,y_{\mathrm{e}})} + c\rho\|\Delta x\|_{P_{\mathrm{eco}}(r,t)} \\ = &V_{\mathrm{f,eco}}(x,r,t) - \|\Delta x\|^2_{Q_{\mathrm{eco}}(r,t,y_{\mathrm{e}})} - a_1\|\Delta x\|_{P_{\mathrm{eco}}(r,t)} \overset{(3.84)}{\leq} V_{\mathrm{f,eco}}(x,r,t) - \overline{\ell}_{\mathrm{eco}}(x,r,t,y_{\mathrm{e}}). \blacksquare \end{aligned}$$

Similar design procedures can be used with polynomial bounds on the incremental Lyapunov function V_δ and a polynomial continuity bound on ℓ_{eco}, compare [JK26, Prop. 5].

Terminal equality constraints

In the following, we discuss how to replace the general terminal set (Assumption 3.59) with a simple terminal equality constraint (TEC, cf. (3.61)). In principle, Conditions (3.68a)–(3.68c) are quite general and not restrictive, but the explicit knowledge of V_δ (which characterizes the terminal set $\mathbb{X}_{\mathrm{f}}$) and the design of the economic terminal cost $V_{\mathrm{f,eco}}$ can pose challenges. A simple alternative is to consider a TEC, which requires no complex design procedure. The following analysis is similar to [87] and [164], which also considered TEC in the steady-state case. We consider the following finite-time local incremental controllability condition, similar to Definition 3.9.

Assumption 3.77. *(Local incremental finite-time controllability) There exist constants $\nu \in \mathbb{I}_{\geq 1}$, $\epsilon > 0$ such that for any references $(r,t) \in \mathbb{Z}_T$, $(\tilde{r},t) \in \mathbb{Z}_T$ with $\|r - \tilde{r}\| \leq \epsilon$, there exists an input sequence $u \in \mathbb{U}^\nu$ such that*

$$\begin{aligned} x(0) = &x_{\mathrm{r}}(0), \; x(\nu) = \tilde{x}_{\mathrm{r}}(\nu), \\ x(k+1) = &f(x(k),u(k),t+k), \; (x(k),u(k),t+k) \in \mathbb{Z}, \; k \in \mathbb{I}_{[0,\nu-1]}. \end{aligned}$$

This condition is for example satisfied with $\nu \leq n$ if the linearization along any feasible periodic trajectory is (uniformly) controllable [236, Ass. 2.37], [165, Ass. 2], compare also [87, Ass. 7] [210, Ass. 10]. Typically, an additional continuity bound on ℓ_{eco} is imposed (cf. [164, Ass. 4] or (3.9b)), which is, however, not necessary in the considered setup with the bounded stage cost ℓ_{eco} and the self-tuning weight β. The following result is an adaptation of Lemmas 3.60 and 3.63 to TEC using Assumption 3.77 and a multi-step implementation.

Lemma 3.78. *Let Assumptions 3.50, 3.62 and 3.77 hold. Consider a terminal equality constraint* (3.61) *and a prediction horizon $N \in \mathbb{I}_{\geq \nu}$. Suppose that Problem 3.53 is feasible at time $t \in \mathbb{I}_{\geq 0}$ and the optimal artificial reference trajectory $r^*(\cdot|t)$ is not a local minimizer of Problem 3.52. Suppose further that Algorithm 3.54 is replaced by the following ν-step implementation*

$$u(t+k) = u^*(k|t), \quad k \in \mathbb{I}_{[0,\nu-1]}, \quad x(t+\nu) = x^*(\nu|t). \tag{3.85}$$

Then, there exists a reference $\tilde{r} \in \overline{\mathcal{R}}_N(x(t+\nu), t+\nu, \overline{y}_{\text{e}}(\cdot|t+\nu), \kappa(\cdot|t+\nu))$, which is a feasible solution of Problem 3.53 at time $t+\nu$ and satisfies

$$\mathcal{J}_{T,\text{eco}}(\tilde{r}, t+N+\nu, \overline{y}_{\text{e}}(\cdot|t+\nu)) < \mathcal{J}_{T,\text{eco}}(\mathcal{R}_T^{\nu} r^*(\cdot|t), t+N+\nu, \overline{y}_{\text{e}}(\cdot|t+\nu)).$$

Proof. Given that $(\mathcal{R}_T^{\nu} r^*(\cdot|t), t+N+\nu) \in \mathbb{Z}_T$ is not a local minimizer, Assumption 3.62 ensures that there exists a reference $(\tilde{r}, t+N+\nu) \in \mathbb{Z}_T$, that satisfies the posed constraints (3.58h)–(3.58i), improves the reference cost $\mathcal{J}_{T,\text{eco}}$ and satisfies $\|\tilde{r} - \mathcal{R}_T^{\nu} r^*(\cdot|t)\| \leq \epsilon$. Due to the multi-step implementation (3.85), the sequence $u(k|t+\nu) = u^*(k+\nu|t)$, $k \in \mathbb{I}_{[0,N-\nu-1]}$ yields a state trajectory $x(\cdot|t+\nu)$ that satisfies $x(N-\nu|t+\nu) = x^*(N|t) = x_{\text{r}}^*(0|t)$. Correspondingly, we can append the input sequence $u(\cdot|t+\nu)$ with the candidate solution $u \in \mathbb{U}^{\nu}$ from Assumption 3.77, which satisfies the constraints (3.58d)–(3.58f). ∎

Compared to the results in Lemmas 3.60 and 3.63 based on a contractive terminal set, the resulting properties with terminal equality constraints are only valid if we apply the first ν parts of the computed input sequence. For comparison, in the MPC formulation in Section 3.2 with the positive definite tracking stage cost ℓ, such a multi-step implementation is not needed, since the closed-loop system eventually converges to a neighbourhood of the artificial reference trajectory r, compare also [164, Thm. 2], [165, Thm. 3] [166, Thm. 2]. The main benefit of the terminal equality constraint implementation is the simple design. Although we need to use a multi-step implementation with ν steps, we would like to point out that ν is independent of T, and hence this method does not suffer from the same limitations as the approaches based on T-step systems, such as [210, 272]. On the other hand, an implementation with a suitable terminal cost (Ass. 3.59) can use a shorter prediction horizon N, requires no multi-step implementation and typically yields better closed-loop performance, compare, e.g., the numerical example in [JK26, App. A].

Remark 3.79. *(Alternatives to multi-step) In [87], a similar economic MPC scheme for setpoints* $(T = 1)$ *has been considered with terminal equality constraints. However, instead of a ν-step MPC implementation* (3.85)*, in [87, Algorithm 3] it was suggested to augment the MPC with an algorithm that decides at each time $t \in \mathbb{I}_{\geq 0}$ if the previous candidate solution or the standard MPC feedback is applied. In particular, if the cost of the artificial reference r does not improve by a minimal amount $\tilde{\epsilon}$, the previous candidate solution is applied. Given Assumption 3.77, after at most ν steps, it is possible to incrementally move the reference trajectory and thus improve the cost. Hence, by augmenting the MPC with such an algorithm, it may not be necessary to apply the first ν steps of the computed input trajectory, which can speed up convergence.*

For the linear case, a different alternative to a multi-step implementation can be found in [43]. In particular, by tightening the state and input constraints $\mathbb{Z}$ along the prediction horizon N in an increasing fashion, the candidate solution employed in Proposition 3.58 strictly satisfies the posed state and input constraints. Thus, if the reference trajectory is incrementally moved the input sequence can be suitably adjusted using controllability such that both the posed state and input constraints and the terminal equality constraint hold.

Remark 3.80. *(Extended horizon terminal cost) Another method to obtain a terminal cost in MPC is to use an extended prediction horizon $M \in \mathbb{I}_{\geq 0}$ (cf. [175] and Sec. 4.1.5), which has been extended to the economic MPC setting in [167]. In particular, a terminal cost based on some stabilizing feedback k_{f} could be chosen as* $V_{\mathrm{f,eco}}(x, r, t, y_{\mathrm{e}}) = \sum_{k=0}^{M} \ell_{\mathrm{eco}}(x_{\mathrm{f}}(k), u_{\mathrm{f}}(k), t + k, y_{\mathrm{e}}(k))$*, where $x_{\mathrm{f}}(k)$ corresponds to the closed-loop system with* $u_{\mathrm{f}}(k) = k_{\mathrm{f}}(x_{\mathrm{f}}(k), \mathcal{R}_T^k r, t + k)$ *starting at $x_{\mathrm{f}}(0) = x$. The advantage of such a truncated series terminal cost is the relative easy implementation, which only requires a stabilizing feedback k_{f} in the offline design and does not pose any additional differentiability assumptions. Furthermore, the resulting terminal cost is neither quadratic nor positive definite and thus for M large can improve performance. However, in the economic case for any finite $M \in \mathbb{I}_{\geq 0}$, such a terminal cost does not satisfy Assumption 3.55 and thus deteriorates the resulting performance bounds, compare [167, Thm. 5].*

3.3.6 Variations and extensions

In the following, we elaborate on some extensions and variants of the proposed economic MPC framework. Overall, the proposed economic MPC framework provides desired performance guarantees, if the constant c_κ, the self-tuning weight $\beta(t)$ and the terminal ingredients $V_{\mathrm{f,eco}}$, $\mathbb{X}_{\mathrm{f}}$ are chosen properly (Ass. 3.55, 3.57, 3.59, 3.61, and 3.62). The design of the terminal ingredients is elaborated in Section 3.3.5. In the following, we show how the problem can be reformulated to get rid of the constant c_κ and the continuity condition in Assumption 3.62, while retaining the performance guarantees from Theorem 3.65. Furthermore, we provide performance bounds for constant weights β (not satisfying Assumption 3.61) in Proposition 3.84. Finally, we also discuss the computational complexity for the special case of convex problems.

Modified reference cost

The proposed economic MPC formulation in Problem 3.53 uses standard conditions for the terminal ingredients (Ass. 3.55) and contains many economic MPC formulations as special cases, compare [10, 16, 19, 87, 138, 206, 208, 282, 295]. However, the formulation also requires the additional constraints (3.58h)–(3.58i), based on the continuity condition (Ass. 3.62), which is non-standard. In the following, we briefly show an alternative solution to this problem, based on a modified cost for the artificial reference trajectory r. The following result is based on [JK25, Prop. 1], which in turn is motivated by the analysis of non-monotonic Lyapunov functions [6].

Lemma 3.81. *Let Assumption 3.55 hold. Then, for any $y_{\mathrm{e}} \in \mathbb{Y}^T$, $(r,t) \in \mathbb{Z}_T$, $(x,r,t) \in \mathbb{X}_{\mathrm{f}}$, the modified terminal cost*

$$\tilde{V}_{\mathrm{f,eco}}(x,r,t,y_{\mathrm{e}}) := V_{\mathrm{f,eco}}(x,r,t,y_{\mathrm{e}}) + \sum_{k=0}^{T-2} \frac{T-1-k}{T} \ell_{\mathrm{eco}}(r(k),t+k,y_{\mathrm{e}}(k)) \tag{3.86}$$

satisfies

$$\tilde{V}_{\mathrm{f,eco}}(x^+,\mathcal{R}_T r,t+1,\mathcal{R}_T y_{\mathrm{e}}) - \tilde{V}_{\mathrm{f,eco}}(x,r,t,y_{\mathrm{e}}) \leq -\ell_{\mathrm{eco}}(x,u,t,y_{\mathrm{e}}(0)) + \mathcal{J}_{T,\mathrm{eco}}(r,t,y_{\mathrm{e}})/T,$$

with $x^+ = f(x,u,t)$, $u = k_{\mathrm{f}}(x,r,t)$.

Proof. Abbreviate $\ell_{\mathrm{eco}}(k) = \ell_{\mathrm{eco}}(r(k),t+k,y_{\mathrm{e}}(k))$, $\tilde{V}_{\mathrm{f,eco}} = \tilde{V}_{\mathrm{f,eco}}(x,r,t,y_{\mathrm{e}})$, $\tilde{V}^+_{\mathrm{f,eco}} =$

$\tilde{V}_{\mathrm{f,eco}}(x^+, \mathcal{R}_T r, t+1, \mathcal{R}_T y_{\mathrm{e}})$. The modified terminal cost satisfies

$$\begin{aligned} T(\tilde{V}^+_{\mathrm{f,eco}} - \tilde{V}_{\mathrm{f,eco}}) &= T(V^+_{\mathrm{f,eco}} - V_{\mathrm{f,eco}}) + \sum_{k=0}^{T-2}(T-1-k)(\ell_{\mathrm{eco}}(k+1) - \ell_{\mathrm{eco}}(k)) \\ &\overset{(3.60c)}{\leq} - T\ell_{\mathrm{eco}}(x,u,t,y_{\mathrm{e}}(0)) + (T+1-T)\ell_{\mathrm{eco}}(0) + \ell_{\mathrm{eco}}(T-1) + \sum_{k=1}^{T-2}\ell_{\mathrm{eco}}(k) \\ &= - T\ell_{\mathrm{eco}}(x,u,t,y_{\mathrm{e}}(0)) + \mathcal{J}_{T,\mathrm{eco}}(r,t,y_{\mathrm{e}}). \end{aligned}$$ ∎

We point out that the modification of the cost in Lemma 3.81 is applicable to terminal equality constraints (cf. Lemma 3.78) and terminal costs/sets (Ass. 3.59). The following proposition shows that this modified reference cost can ensure the same performance bounds as the economic MPC formulation in Problem 3.53, without using the continuity condition (Ass. 3.62).

Proposition 3.82. *Let Assumptions 3.50, 3.55, and 3.57 hold. Consider Problem 3.53 with* $V_{\mathrm{f,eco}}$ *replaced by* $\tilde{V}_{\mathrm{f,eco}}$ (3.86) *and the constraints* (3.58h)–(3.58i) *replaced by*

$$\mathcal{J}_{T,\mathrm{eco}}(r(\cdot|t), t+N, \overline{y}_{\mathrm{e}}(\cdot|t)) \leq \overline{\kappa}(t) := \sum_{j=0}^{T-1}\kappa(j|t). \tag{3.87}$$

If the modified Problem 3.53 is feasible at $t = 0$*, then the closed-loop system resulting from Algorithm 3.54 satisfies the performance bound* (3.63).

Proof. Similar to Proposition 3.58, the candidate solution from Proposition 3.56 with the modified terminal cost implies

$$\begin{aligned} &W(t+1) - W(t) + \ell_{\mathrm{eco}}(x(t),u(t),t,y_{\mathrm{e}}(t)) \\ \leq &\mathcal{J}_{T,\mathrm{eco}}(r^*(\cdot|t), t+N, \overline{y}_{\mathrm{e}}(\cdot|t+1))/T + \gamma(t)\overline{\kappa}(t+1) \overset{(3.62)}{=} \overline{\kappa}(t+1)/T + \gamma(t)\overline{\kappa}(t+1). \end{aligned}$$

Correspondingly, the T-step bound (3.66) holds with $c_\kappa = 0$, since $\overline{\kappa}(t+1) \leq \overline{\kappa}(t)$. The remainder of the proof follows from the arguments in Proposition 3.58, similar to [206, Thm. 1], [208, Thm. 1]. ∎

The properties in Proposition 3.64 and Theorem 3.65 hold equally with the modified terminal cost $\tilde{V}_{\mathrm{f,eco}}$ and a contractive terminal set (Ass. 3.59), with the simpler constraint (3.87) (without requiring Assumption 3.62). The main advantage of using this modified terminal cost $\tilde{V}_{\mathrm{f,eco}}$ is that the technical continuity condition Assumption 3.62 is not required. Furthermore, the number of constraints in Problem 3.53 is smaller and the set of feasible artificial reference trajectories is larger. This modified reference cost yields an objective function, which, to the best knowledge of the author, differs from any existing MPC formulation (for $T > 1$). The practical effect on the closed-loop performance is studied in the numerical example in Section 3.4.4. For the academic example considered in Section 3.3.1 with a terminal equality constraint and $T = 2$, the modified cost is given by $\tilde{V}_{\mathrm{f,eco}}(r) = \frac{1}{2}\ell_{\mathrm{eco}}(r(0))$. With this modified cost the closed loop also "does the right thing", i.e., converges to the optimal T-periodic orbit $\{1, 2\}$.

Overall, compared to the MPC formulation initially presented in Section 3.3.2 this modified MPC formulation is superior in terms of theoretical properties (Ass. 3.62 not required) and practical implementation (fewer nonlinear constraints in Problem 3.53). The main benefit of first presenting Problem 3.53 is the fact that this formulation is a clear generalization of existing economic MPC formulations [10, 16, 19, 87, 138, 206, 208, 282, 295] which are contained as a special case (which is less obvious with the modified cost $\tilde{V}_{\mathrm{f,eco}}$), compare the discussion in Section 3.3.2.

Remark 3.83. *(Partially decoupled reference updates) Similar to the MPC formulation in Section 3.2.3, the computational complexity of Problem 3.53 can be reduced by using two partially decoupled optimization problems to compute the input $u \in \mathbb{U}^N$ and the reference trajectory $(r, t) \in \mathbb{Z}_T$, respectively. In addition to the computational savings, such a formulation has additional distinct advantages in the economic setting. First, since the reference planner only minimizes $\mathcal{J}_{T,\mathrm{eco}}$, no self-tuning weight is needed. Second, in the tracking case in Section 3.2.3 we could only guarantee convergence but not stability. We conjecture that stronger transient performance guarantees in the economic setting can be recovered, if the formulation is slightly adjusted. In particular, considering the modified terminal cost $\tilde{V}_{\mathrm{f,eco}}$ (cf. Prop. 3.82), the primal economic MPC formulation directly ensures that the difference in the value function is bounded by: the closed-loop stage cost ℓ_{eco}, the average cost at the periodic reference $\mathcal{J}_{T,\mathrm{eco}}/T$ and an additional term bounding the deterioration in $\tilde{V}_{\mathrm{f,eco}}$ due to the change in the reference trajectory. Thus, if the reference planner also considers a bound on this performance deterioration, we conjecture that the partially decoupled formulation enjoys transient performance guarantees analogous to Theorem 3.65. Note that for the tracking MPC formulation in Section 3.2.3, a similar bound on the increased cost can be considered in the reference planner. However, in the tracking case the reference trajectory changes the full tracking cost $\mathcal{J}_N$, not just some terminal cost. Thus, bounding the cost deterioration in $\mathcal{J}_N$ using conservative bounds may significantly slow down convergence.*

Constant weights β

A large self-tuning weight $\beta(t)$ can deteriorate the transient performance, but is useful to ensure convergence of the artificial reference to a local minimizer (cf. Prop. 3.64). In the following, we show that similar performance bounds hold when choosing a large constant weight β. In particular, in [87] a competing approach to [206, 208] has been considered with a constant weight β. Instead of changing the weight β online to achieve (locally) *optimal* performance, a fixed weight β is considered and a suboptimality bound on the performance is established. The following proposition shows that the same result applies here, as an alternative to Proposition 3.64, similar to [87, Prop. 2].

Proposition 3.84. *Let Assumptions 3.50, 3.55 and 3.62 hold and assume that β is constant, i.e., $\beta(t) = \beta$, $t \in \mathbb{I}_{\geq 0}$. Assume that Problem 3.53 is feasible at some time $t \in \mathbb{I}_{\geq 0}$. Then, there exists a function $\underline{\beta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $\epsilon > 0$, $\beta \geq \underline{\beta}(\epsilon)$ implies $\overline{\kappa}(t+1) \leq \mathcal{J}_{T,\text{eco,min}}(x(t), t, \overline{y}_\text{e}(\cdot|t), \kappa(\cdot|t)) + \epsilon$.*

Proof. Denote the minimizer and the minimum of the optimization problem (3.72) at time $t \in \mathbb{I}_{\geq 0}$ by $r^*_{\min}(\cdot|t)$ and $\mathcal{J}_{T,\text{eco,min}}(t)$, respectively. By definition, there exists a feasible input sequence $\tilde{u}$ with corresponding state sequence $\tilde{x}$ such that $r^*_{\min}(\cdot|t)$ satisfies the constraints in Problem 3.53. Due to optimality, we have

$$\begin{aligned} W(t) \leq & \sum_{k=0}^{N-1} \ell_\text{eco}(\tilde{x}(k), \tilde{u}(k), t+k, y_\text{e}(\text{mod}(k,T)|t)) \\ & + V_\text{f,eco}(\tilde{x}(N), r^*_{\min}(\cdot|t), t+N, \overline{y}_\text{e}(\cdot|t)) + \beta \mathcal{J}_{T,\text{eco}\min}(t). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \beta(\mathcal{J}_{T,\text{eco}}(r^*(\cdot|t), t+N, \overline{y}_\text{e}(\cdot|t)) - \mathcal{J}_{T,\text{eco,min}}(t)) \\ \leq & \sum_{k=0}^{N-1} \ell_\text{eco}(\tilde{x}(k), \tilde{u}(k), t+k, y_\text{e}(\text{mod}(k,T)|t)) - \ell_\text{eco}(x^*(k|t), u^*(k|t), t+k, y_\text{e}(\text{mod}(k,T)|t)) \\ & + V_\text{f,eco}(\tilde{x}(N), r^*_{\min}(\cdot|t), t+N, \overline{y}_\text{e}(\cdot|t)) - V_\text{f,eco}(x^*(N|t), r^*(\cdot|t), t+N, \overline{y}_\text{e}(\cdot|t)) \leq \eta, \end{aligned}$$

with some (uniform) finite constant $\eta > 0$. The last inequality follows from boundedness of ℓ_eco, $V_\text{f,eco}$ (continuous functions and compact constraints) and N finite. This inequality directly implies

$$\overline{\kappa}(t+1) = \mathcal{J}_{T,\text{eco}}(r^*(\cdot|t), t+N, \overline{y}_\text{e}(\cdot|t)) \leq \mathcal{J}_{T,\text{eco,min}}(t) + \epsilon,$$

for $\beta \geq \underline{\beta}(\epsilon) := \eta/\epsilon$. ■

Thus, for a large enough weight β, the cost of the artificial periodic orbit r is arbitrarily (ϵ) close to the cost of the optimal reachable periodic orbit (3.72). Combining this result with Lemma 3.63 and the stronger terminal ingredients (Ass. 3.59), $\mathcal{J}_{T,\text{eco,min}}$ is a local minimum of Problem 3.52. Correspondingly, it is possible to derive performance bounds similar to Theorem 3.65 with an additional suboptimality term ϵ, compare [87, Thm. 2]. For the special case of $T = 1$ (artificial setpoint), more details on the effect of β on the closed loop can be found in [206, 208] and [87].

Convex problems

In the following, we discuss the special case, when the periodic optimal control problem (Problem 3.52) is convex. Suppose that the dynamics f are affine, i.e., $f(x,u,t) = A(t)x + B(t)u + c(t)$, and the constraint sets $\mathbb{Z}$ and $\mathbb{Z}_\text{r}$ are polytopes, which implies that $\mathbb{Z}_T$ is a convex polytope. For ℓ_eco convex, this implies that the periodic optimal control problem (Problem 3.52) is convex and Corollary 3.66 guarantees that the closed-loop performance is no worse than operation at an optimal T-periodic orbit. In the following, we discuss how the design procedure for the terminal ingredients and the online optimization simplifies for the considered special case. Since we have a linear (time-varying) system, we consider a linear time-varying feedback $k_{f,x} = K(t)$ and a time-varying matrix $S(t, y_\text{e})$ satisfying Condition (3.75a). Thus, the matrix $Q_\text{eco}(t, y_\text{e})$ in Lemma 3.70 is independent of r and we can consider a time-varying matrix $P_\text{eco}(t)$ to satisfy Condition (3.77) in Proposition 3.71. Matrices $P_\text{eco}(t)$, $K(t)$ satisfying Condition (3.77) can be computed by solving T coupled LMIs similar to [23]. Alternatively, the computation of $K(t)$, $P_\text{eco}(t)$ can be achieved using the discrete-time LQR for a suitably defined T-step system with $\tilde{x} \in \mathbb{X}$ and $\tilde{u} \in \mathbb{U}^T$.

For the terminal set $\mathbb{X}_\text{f}$, we can either use an ellipsoidal set $\mathbb{X}_\text{f} = \{(x,r,t) \mid \|x - x_\text{r}(0)\|^2_{P_\text{eco}(t)} \leq \alpha\}$ or a polytopic (periodically time-varying) invariant set $\mathbb{X}_\text{f}$[13].

In case ℓ_eco is quadratic in (x,u), Lemma 3.70 and Proposition 3.71 contain no nonlinear terms that need to be locally over-approximated and hence we can set $\epsilon = \tilde{\epsilon} = 0$ and α_1 arbitrary large. Furthermore, if ℓ_eco is convex and quadratic in (x,u) and $V_\text{f,eco}$, $\mathbb{X}_\text{f}$ is chosen according to Corollary 3.72, Problem 3.53 contains a quadratic cost function and convex linear and quadratic constraints.[14]

[13]The optional consideration of an online optimized terminal set size $\alpha(r,t)$ can be expressed using linear constraints, for both cases.

[14]In addition to the possibly ellipsoidal terminal set, the constraints (3.58i) are quadratic, leading to a

In the special case that ℓ_{eco} is linear in (x, u), the vector p_{eco} can be explicitly computed (independent of the online optimized reference r) for given parameters y_{e} using Equation (3.83). Furthermore, since $S = 0$ the terminal cost is linear ($P_{\mathrm{eco}} = 0$). Alternatively, with a polytopic incremental Lyapunov function V_δ, a polytopic terminal cost $V_{\mathrm{f,eco}}$ analogous to Proposition 3.76 can be used in Problem 3.53, which can be implemented using linear constraints. Thus, if a polyhedral terminal set is chosen, Problem 3.53 with a linear cost ℓ_{eco} and a linear or polytopic terminal cost $V_{\mathrm{f,eco}}$ only requires the solution to a linear program (LP), which can be done efficiently. In case that some of the input variables u are also subject to integer constraints (cf. for example periodic scheduling problems with discrete decisions [241] and the HVAC numerical example in Section 3.4.3), the problem can be formulated as a mixed-integer linear program (MILP).

Summary

In this section, we studied an economic MPC formulation with *artificial periodic reference* trajectories for time-varying *economic* stage costs ℓ_{eco}, possibly subject to unpredictable changes online. We showed that a naive extension of existing economic MPC approaches (cf. [87, 206, 208]) to artificial periodic reference trajectories does *not* necessarily yield the desired closed-loop performance guarantees (Sec. 3.3.1). We provided performance guarantees relative to the (limiting) artificial reference trajectory (Sec. 3.3.3) by either imposing additional constraints on the artificial reference (Sec. 3.3.2) or using a novel cost function inspired by non-monotonic Lyapunov functions (Sec. 3.3.6). Furthermore, by using a self-tuning weight $\beta(t)$ (Sec. 3.3.3) or a large constant weight β (Sec. 3.3.6) for the artificial reference trajectory we ensured that the artificial reference (approximately) converges to a local minimum. As a result, we proved an average performance bound relative to a locally optimal periodic orbit. We showed that in the special case of $N = 0$ (no prediction horizon) the proposed approach reduces to a modified version of the MPC formulations based on periodicity constraints in [138, 282] but with stronger performance guarantees (Rk. 3.68).

In addition, we presented a design procedure for the terminal ingredients that is applicable to economic costs and artificial dynamic trajectories (Sec. 3.3.5). In particular, we combined the reference generic offline design (Sec. 3.1.3) with the standard design for economic terminal costs [16], extended to periodic artificial reference trajectories using a periodic adjoint trajectory p_{eco} in the MPC formulation. In the special case of

non negligible increase in the online computation. Given that p_{eco} in Equation (3.83) is linear in r, the terminal cost $V_{\mathrm{f,eco}}$ is quadratic in the decision variables. However, convexity of the overall cost function is not obvious and may require β large enough.

linear system dynamics f with a linear economic stage cost ℓ_{eco} the adjoint trajectory p_{eco} can be computed offline and the overall economic MPC problem reduces to an LP (Sec. 3.3.6).

In the next section, the performance benefits of the presented MPC design methods in this chapter are demonstrated using numerical examples.

3.4 Numerical examples

In the previous sections, we developed a *reference-generic* offline design of terminal ingredients for nonlinear trajectory tracking MPC (Sec. 3.1.3), a tracking MPC formulation with *artificial reference* trajectories (Sec. 3.2), *optimized terminal sets* (Sec. 3.2.2), *partially decoupled* tracking and trajectory planning MPC formulation (Sec. 3.2.3), and an *economic* MPC formulation with artificial reference trajectories (Sec. 3.3). In the following, we provide numerical examples to demonstrate the practical applicability of these MPC design procedures and quantify the performance benefits of various design options. In Section 3.4.1, we consider a setpoint tracking problem ($T = 1$) for a nonlinear CSTR and demonstrate the performance benefits of using using suitable terminal ingredients (Sec. 3.1.3), online optimized terminal sets (Sec. 3.2.2), and economic MPC formulations (Sec. 3.3). In Section 3.4.2, we consider tracking of periodic target signals (Sec. 3.2) for a nonlinear ball and plate system and investigate the practicality of the partially decoupled reference updates (Sec. 3.2.3). In Section 3.4.3, we consider economic optimal operation (Sec. 3.3) with a simple building temperature (HVAC) example and a time-varying periodic setup. In Section 3.4.4, we consider the (classical) time-invariant problem of maximizing the yield of a nonlinear CSTR and demonstrate performance benefits of dynamic operation using the economic MPC formulation from Section 3.3. Additional examples showing the applicability of the reference generic offline computation (Sec. 3.1.3) to nonlinear dynamic problems can be found in [JK13, JK15, JK29, JK36]. For the following examples, the offline and online computation is done in Matlab using SeDuMi-1.3 [261] and CasADi [17], respectively. This section is based on and taken in parts literally from [JK16][15] and [JK26][16].

[15] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear tracking model predictive control scheme for unreachable dynamic target signals." In: *Automatica* 118 (2020). extended version: arXiv:1911.03304, p. 109030©2020 Elsevier Ltd.

[16] J. Köhler, M. A. Müller, and F. Allgöwer. "Periodic optimal control of nonlinear constrained systems using economic model predictive control." In: *J. Proc. Contr.* 92 (2020). extended version: arXiv:2005.05245, pp. 185–201©2020 Elsevier Ltd.

3.4.1 Setpoint tracking - performance comparison

The following example demonstrates the performance benefits of the parametrized terminal ingredients (Sec. 3.1.3), the online optimized terminal set (Sec. 3.2.2), and the economic MPC formulation (Sec. 3.3) at the example of setpoint tracking ($T = 1$). We consider the following CSTR model

$$\dot{x} = \begin{pmatrix} \frac{1}{\theta_\mathrm{f}}(1 - x_1) - kx_1 e^{-\frac{M}{x_2}} \\ \frac{1}{\theta_\mathrm{f}}(x_\mathrm{f} - x_2) + kx_1 e^{-\frac{M}{x_2}} - \alpha_\mathrm{f} u(x_2 - x_\mathrm{c}) \end{pmatrix}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad u \in \mathbb{R},$$

with the concentration x_1, the temperature x_2, and the coolant flow rate u, taken from [191]. The parameters are $\theta_\mathrm{f} = 20$, $k = 300$, $M = 5$, $x_\mathrm{f} = 0.3947$, $x_\mathrm{c} = 0.3816$, $\alpha_\mathrm{f} = 0.117$. The discrete-time model is defined with an Euler discretization and the sampling time $T_\mathrm{s} = 0.1$s. We consider the setpoint tracking problem considered in Section 3.2 with $T = 1$ and the output $y = h(x, u) = x_2$.

The constraint set is given by $\mathbb{Z} = [0, 1]^2 \times [0, 2]$, the weights for the quadratic stage cost (cf. (3.3)) are $Q = I_2$ and $R = 0.01$, the output weighting (cf. (3.27c)) is $S = 10^3$, and the feasible steady-state manifold for the reference r is $\mathbb{Z}_\mathrm{r} = \{(x, u) \mid f(x, u) = x,\ x_2 \in [0.43, 0.86]\}$. We point out that the convexity and uniqueness conditions in Assumption 3.29 hold, even though the set $\mathbb{Z}_\mathrm{r}$ is clearly not convex, compare Figure 3.4.

We compute the terminal ingredients using Algorithm 3.22 and Lemma 3.13 with a quasi-LPV parametrization using $\theta : \mathbb{Z} \to \mathbb{R}^4$, gridding[17] (cf. Rk. 3.19) the one-dimensional steady-state manifold $\mathbb{Z}_\mathrm{r}$ with 100 points, and setting $\epsilon = 1$. The overall offline computations are accomplished in less than 10 seconds.

Regarding the terminal set size α, we compare the fixed value $\alpha = 1.3 \cdot 10^{-2}$ based on $\mathbb{Z}_\mathrm{r}$ (cf. (3.25)) with the online optimized, reference dependent value $\alpha(r) \in [\alpha_{\min}, \alpha_1] = [10^{-1}, 10^{-4}]$ (cf. Sec. 3.2.2). The functions c_i are chosen according to (3.42). The resulting size of the terminal set for different setpoints $r \in \tilde{\mathbb{Z}}_\mathrm{r}$ can be seen in Figure 3.4. The online optimization of $\alpha(r)$ (blue), allows us to consider points $r \notin \mathbb{Z}_\mathrm{r}$ and thus yields a significantly larger operating area. For points $r \in \mathbb{Z}_\mathrm{r}$, the constant terminal set size α (red) is considerably smaller and thus conservative. In [164], it has been suggested to partition $\mathbb{Z}_\mathrm{r}$ and compute different constants α_i for each partitioning. The

[17] The convex formulation from Proposition 3.15 is equally applicable. However, gridding results in less conservative terminal ingredients and for the one-dimensional steady-state manifold the computational complexity of gridding can be neglected. A quantitative comparison of gridding and Proposition 3.15 for a different CSTR model including dynamic trajectories ($T > 1$) and a Runge-Kutta discretization (resulting in non-trivial parameters θ_i) can be found in [JK15, Sec. V.A].

terminal set with 3 equally spaced partitions of $\mathbb{Z}_r$ is also displayed in Figure 3.4. The terminal set based on partitioning is always an inner approximation to the continuously parametrized terminal sets with $\alpha(r) \geq \alpha_i$ for all $r \in \mathbb{Z}_r$. In addition, the continuously parametrized terminal ingredients are well suited for standard solvers with automatic differentiation, contrary to the piecewise constant definitions used in [164]. In [62], it was suggested to compute an explicit polynomial expression for $\alpha(r)$ offline using SOS. For the present example, an explicit polynomial map may require a high order polynomial to explicitly capture the shape $\alpha(r)$ in Figure 3.4. Thus, the approach in [62] may experience scalability issues for $T > 1$ or multi-dimensional steady-state manifolds.

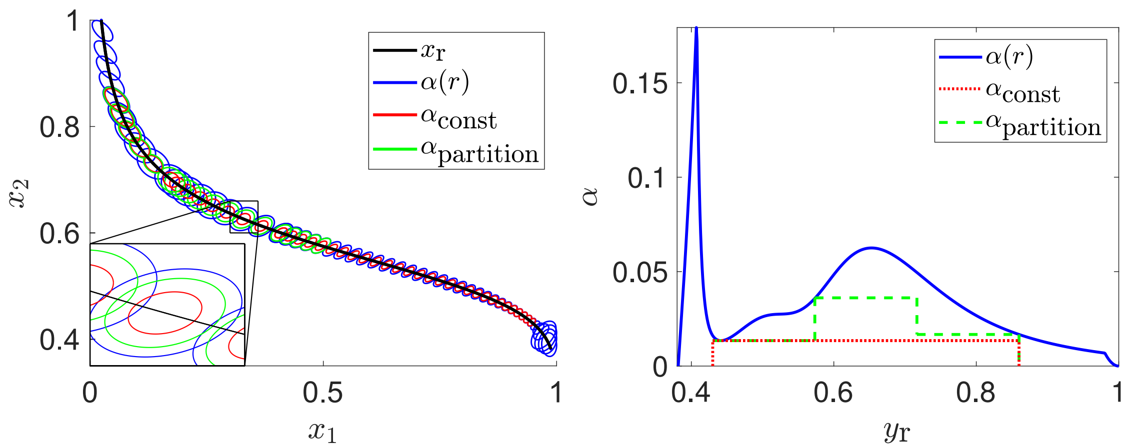

Figure 3.4. Left: Temperature vs. concentration: Setpoints x_r (black) with corresponding terminal set $\mathbb{X}_f$ for setpoint dependent $\alpha(r)$ (blue ellipses), constant $\alpha = 0.013$ (red ellipses) and piece-wise constant $\alpha_{partition}$ (green ellipses). Right: Terminal set size α over setpoints y_r, for online optimized $\alpha(r)$ (solid, blue), constant α (dotted, red) and piece-wise constant $\alpha_{partition}$ (dashed, green).©2020 Elsevier Ltd.

Starting at $x_0 = (0.9492, 0.43)$, the output target is $y_e = 0.6519$, as in the numerical example in [191]. We implemented the setpoint tracking MPC from Section 3.2.1 ($T = 1$) using a terminal equality constraint (Prop. 3.10, TEC) and the parametrized terminal ingredients (Lemma 3.12, QINF-α). Furthermore, we implemented the approach using an online optimized terminal set size (Prop. 3.40, QINF-$\alpha(r)$). In addition, we implemented the economic formulation (Sec. 3.3) by replacing the tracking stage cost $\ell(x, u, r)$ (3.3) with $\ell_{eco}(x, u) = \|y - y_e\|^2$. This approach was also implemented with a terminal equality constraint (TEC) and the parametrized terminal ingredients (QINF-$\alpha/\alpha(r)$, Cor. 3.72). We also considered a reference governor[18] (Gov), which corresponds to the

[18] The local controller $k_f(x, r)$ (Ass. 3.5) is applied. The reference r is updated by increments $y_r^+ = y_r, +0.003$, if $(x, r^+) \in \mathbb{X}_f$ with the constant terminal set size $\alpha = 0.013$.

candidate solution ($N = 0$) in the stability proof (cf. Rk. 3.34).

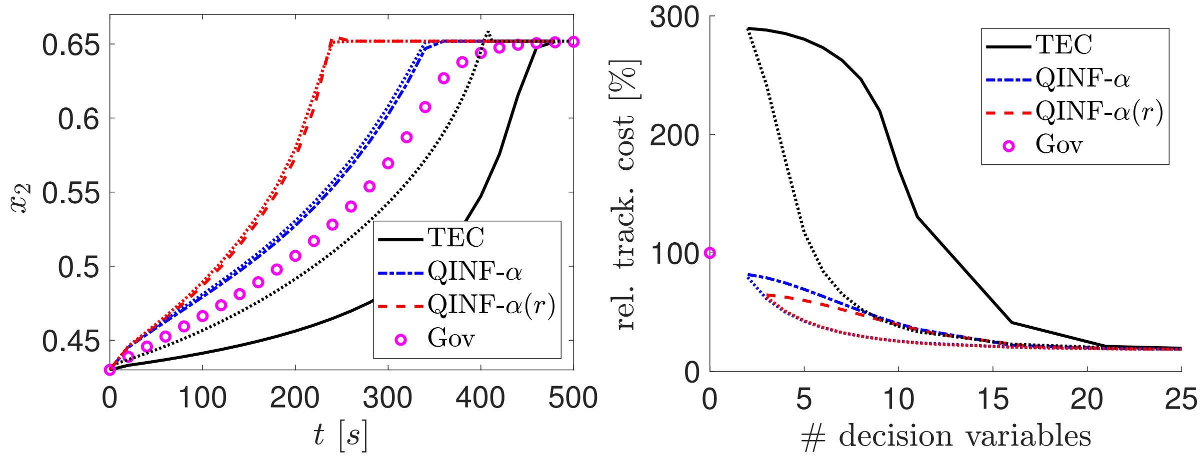

Figure 3.5. Left: Exemplary closed loop (temperature vs. time): Setpoint tracking MPC (Sec. 3.2) with terminal equality constraint and (TEC, black, solid, $N = 9$), terminal cost/set with fixed α (red, dashed-dotted, QINF-α, $N = 1$) and online optimized $\alpha(r)$ (red, dashed, QINF-$\alpha(r)$, $N = 1$); Economic formulations ℓ_{eco} (Sec. 3.3) with terminal equality constraint (TEC, black, dotted, $N = 4$), terminal cost/set with fixed α (red, dotted, QINF-α, $N = 1$) and online optimized $\alpha(r)$ (red, dotted, QINF-$\alpha(r)$, $N = 1$); and reference governor (magenta, circles, Gov). Right: Tracking cost $\sum_{t=0}^{5000} \|y(t) - y_{\text{e}}(t)\|^2$ relative to reference governor v.s. number of decision variables (condensed[19] formulation) in Problems 3.27/3.38/3.53.©2020 Elsevier Ltd.

Figure 3.5 shows a quantitative performance comparison and exemplary closed-loop trajectories for the different MPC formulation. The different prediction horizons N have been chosen such that a similar performance is achieved, which again highlights the fact that the proposed design procedures (QINF-$\alpha(r)$) can improve performance while simultaneously reducing the computational complexity. For $N = 1$, Problem 3.27/3.38/3.53 only requires 2/3 scalar optimization variables. Comparing the performance of the tracking MPC (Sec. 3.2) with suitably designed terminal ingredients (QINF) to a simple terminal equality constraint (TEC) formulations, we see a significant reduction of the tracking error even for $N = 1$. We see a similar performance improvement when the terminal set $\mathbb{X}_{\text{f}}$ with a fixed constant α (Sec. 3.2.1) is replaced by an online optimized size $\alpha(r)$ (QINF-$\alpha(r)$). Furthermore, we see that a suitably designed reference governor can compete with a badly designed tracking MPC (TEC), for short horizons N. More-

[19]The equality constraints to compute the predicted state trajectory $x(\cdot|t)$ and the adjoints $p_{\text{eco}}(\cdot|t)$ are assumed to be condensed and thus the state trajectory $x(\cdot|t) \in \mathbb{R}^{n(N+1)}$ and the adjoints $p_{\text{eco}}(\cdot|t) \in \mathbb{R}^{nT}$ are not treated as decision variables. Similarly, x_{r} and u_{r} are treated as an explicit function of $y_{\text{r}} \in \mathbb{R}$ and are thus captured with a scalar decision variable y_{r}.

over, the benefits of optimizing the terminal set size $\alpha(r)$ online are clearly visible. In addition, it seems that the economic formulation ℓ_{eco} (Sec. 3.3) consistently achieves a better performance in the considered tracking problem. The considered example clearly demonstrates that a) the inclusion of suitable terminal ingredients is a major factor to ensure desired closed-loop performance (as articulated in [188]) and b) the outlined procedure to compute terminal ingredients, including online optimization of the terminal set size (Sec. 3.2.2), are well suited to improve the performance in nonlinear tracking MPC.

Remark 3.85. *(Alternative MPC formulations) The closed-loop performance of the terminal equality constraint MPC (TEC) is very sensitive to the offset weighting, e.g., for $S = 10^2$ the convergence rate decreases by one order of magnitude, while the MPC formulation with terminal cost/set (QINF) is almost unaffected. In [164, Sec. III.B], it was suggested to implicitly enforce the terminal constraint by scaling the terminal cost V_{f} with some sufficiently large scaling factor ω. For the considered example this corresponds to $\omega \approx 4 \cdot 10^3$. Such a large terminal cost results in numerical difficulties due to the ill-conditioning and the corresponding optimal solution is virtually indistinguishable from the terminal equality constraint MPC (since the high terminal cost implicitly enforces a terminal set constraint on sublevel sets). An alternative solution to this problem would be to directly implement a stabilizing MPC without any terminal ingredients or artificial steady-state, compare Section 2.2 and Chapter 4. Such an MPC formulation is quite sensitive to the stage cost Q, R and the considered prediction horizon N. For $N \in \mathbb{I}_{[1,30]}$, this MPC formulation simply gets stuck at a steady-state close to the initial state, while for $N = 40$, the MPC formulation yields fast convergence similar to the proposed tracking MPC formulation (Sec. 3.2) with $N = 15$ (QINF-α) or $N = 20$ (TEC). If we use a stabilizing MPC with terminal ingredients and a fixed artificial reference r, a prediction horizon of $N \geq 500$ is required to ensure initial feasibility, yielding a performance comparable to the proposed QINF-$\alpha(r)$ with $N = 30$. Thus, the artificial steady-state r significantly reduces the computational demand and results in a smooth closed-loop operation.*

3.4.2 Periodic trajectory tracking - partially decoupled reference updates

The following example shows the applicability of the tracking MPC formulation in Section 3.2 for periodic output target signals and nonlinear systems. In addition, we demonstrate the practicality of the partially decoupled approach from Section 3.2.3.

We consider a nonlinear ball and plate system, taken from [223] with

$$\ddot{z}_1 = \frac{5}{7}\left(z_1\dot{\beta}_1^2 + \dot{\beta}_1 z_2 \dot{\beta}_2 + g\sin(\beta_1)\right), \quad \ddot{z}_2 = \frac{5}{7}\left(z_2\dot{\beta}_2^2 + \dot{\beta}_2 z_1 \dot{\beta}_1 + g\sin(\beta_2)\right),$$
$$x = (z_1, z_2, \dot{z}_1, \dot{z}_2, \beta_1, \beta_2, \dot{\beta}_1, \dot{\beta}_2) \in \mathbb{R}^8, \quad u = (\ddot{\beta}_1, \ddot{\beta}_2) \in \mathbb{R}^2, \quad y = (z_1, z_2) \in \mathbb{R}^2,$$

with the position z_i and the angle β_i, $i \in \mathbb{I}_{[1,2]}$. We use an Euler discretization of this model with step size $T_\mathrm{s} = 0.1$s to get a nonlinear discrete-time system. The constraint set is $\mathbb{Z} = [-0.06, 0.06]^2 \times [-0.2 \times 0.2]^2 \times [-\pi/3, \pi/3]^2 \times [-1, 1]^2 \times [-2, 2]^2 \subseteq \mathbb{R}^{10}$, the stage cost weights are $Q = I_8$ and $R = 0.1 \cdot I_2$, and the output weighting is $S = I_2$. The Jacobian matrices $A(r)$, $B(r)$ are parametrized with $\theta \in \mathbb{R}^9$ (cf. Sec. 3.1.3). Constant matrices P, K satisfying the conditions in Lemma 3.12 with $\mathbb{Z}_\mathrm{r} = \mathbb{Z}$ are computed using Proposition 3.15. The offline computation considers $2^9 = 512$ vertices and is accomplished in 40 seconds. In [JK22], parametrized matrices P, K are computed for the same example, which allows for a larger constraint set (e.g., $|\dot{\beta}_i| \leq 2$). However, the matrices are only parametrized using the 3 most relevant parameters θ_i and the offline computation considers $2^9 \cdot 2^3 = 4096$ vertices, which increases the overall offline computation time to approximately one hour.

In [166, 223], a linearized version of this model has been considered to study periodic reference tracking. Given the theoretical results in Section 3.2 (and the reference generic terminal ingredients from Section 3.1.3), we extend these results to the nonlinear model. The initial condition and the target signal y_e are chosen similar to [166, 223] and the period length is $T = 16$. The target signal y_e is first an (unreachable) rectangular signal and suddenly changes to a circle (cf. Fig. 3.6). We implement the proposed approach with $N = 1$ and online optimized terminal set size (Sec. 3.2.2). In addition, we also implement the approach with the partially decoupled reference update (Sec. 3.2.3) with $N = 1$, $M = 2$. The resulting closed-loop trajectories can be seen in Figure 3.6. Initially, a large terminal set size α is optimal as it allows the controller to quickly move the reference but restricts the reference to have a large distance to the constraints. Then, the reference moves continuously to the optimal reachable trajectory x_T^* and the terminal set size α decreases to $\alpha_{\min} = 10^{-8}$. As a result, the closed-loop trajectory shows initially fast convergence and then smoothly converges to the optimal trajectory x_T^*. The same effect can again be observed when the target signal y_e suddenly changes at $t = 5.5s$. The approach using partially decoupled reference updates (Sec. 3.2.3) has a slower convergence rate. On the other hand, this approach only uses $N \cdot m = 2$ optimization variables to determine the control input u and the reference update, which requires

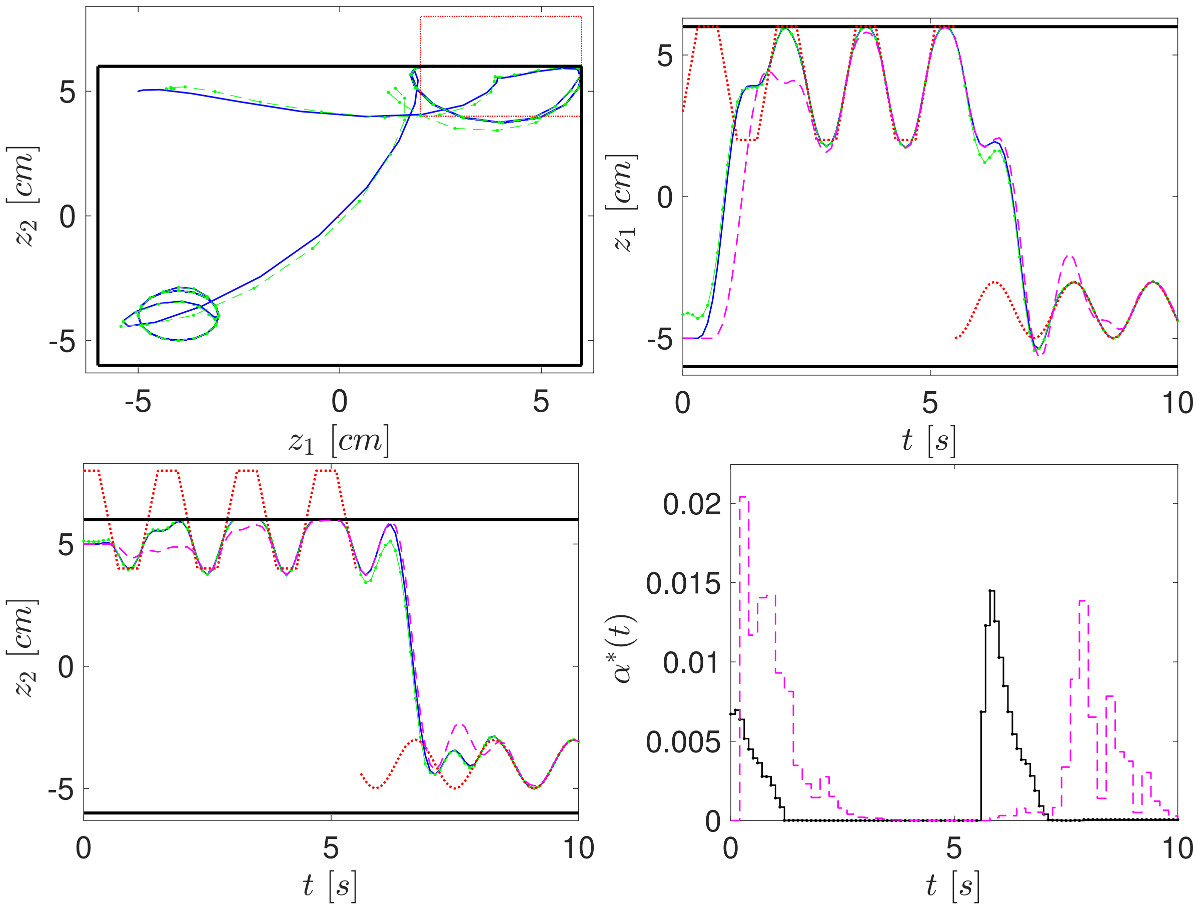

Figure 3.6. Trajectories of z_1, z_2 for the closed-loop system x (blue, solid), the artificial reference r (dash-dot, green), the target signal y_e (dotted, red) and the state constraints (black, thick solid line). The closed-loop trajectories z_1, z_2 for the closed-loop system based on partially decoupled updates (Sec. 3.2.3) (dashed, magenta) is also displayed. Bottom right: Closed-loop evolution of the online optimized terminal set size α for the joint optimization (Sec. 3.2.2) (black, solid) and based on the partially decoupled updates (Sec. 3.2.3) (dashed, magenta).©2020 Elsevier Ltd.

$n + mT + 1 = 41$ optimization variables, can be solved in intervals of $M \cdot T_s = 200$ms, thus greatly reducing the online computational demand. For comparison, the joint optimization in Problem 3.38 requires $m(N + T) + n + 1 = 43$ decision variables and needs to be solved every $T_s = 100$ms. The tracking error $\sum_{t=0}^{\infty} \|y(t) - y_e(t)\|_S^2$ with different values $M \in \mathbb{I}_{[1,5]}$ relative to the joint optimization in Problem 3.38 is displayed in the following table.

M	1	2	3	4	5
relative cost	111%	116%	155%	217%	254%

Overall, this example demonstrates that the MPC formulation from Section 3.2 is suitable for nonlinear dynamic tracking problems and that we can flexibly trade-off computationally demand vs. convergence speed with $M \in \mathbb{I}_{\geq 1}$ using the partially decoupled formulation from Section 3.2.3.

3.4.3 Economic optimal operation - periodic problem

In the following, we show the applicability of the proposed economic MPC framework (Sec. 3.3) to periodic problems subject to online changing performance measures. We consider a simple building temperature evolution example from [242, Sec. IV.A] governed by

$$m\dot{x}(t) = -k(x(t) - T_{\text{amb}}(t)) + q_{\text{amb}}(t) - u(t),$$

with air temperature x, cooling rate u, ambient temperature T_{amb}, rate of direct heat by the ambient q_{amb} and model constants $m, k > 0$. The cooling rate u is generated using $N_{\text{chiller}} = 2$ chillers and is subject to the following (time-invariant) disjoint constraint set $u \in \mathbb{U} = \{0\} \times [0.75, 1] \times [1.5, 2]$, which is implemented using an additional discrete decision variable $v \in \{0, 1, 2\}$, corresponding to the number of active chillers. The state is subject to periodically time-varying comfort bounds $T_{\min}(t) \leq x \leq T_{\max}(t)$. The corresponding discrete-time system is given by

$$x(t + 1) = Ax(t) + Bu(t) + e(t), \; (x(t), u(t), t) \in \mathbb{Z}.$$

The objective is to minimize the electricity cost[20] given by the economic stage cost $\ell_{\text{eco}} = y_e \cdot u$, with the external price profile y_e. This example has a linear cost, affine

[20]The proposed framework can also consider peak-demand prices using the formulation in [242] or unpredictably changing constraint sets (reflecting comfort levels set by a user) using soft constraints (cf. Rk. 3.51).

dynamics (cf. convex setup in Sec. 3.3.6) and a time-varying setup with period length $T = 24$.

In [242], for fixed periodic price signals y_e (Ass. 3.50), it was shown that periodic economic MPC formulations [10, 241, 295] outperform tracking MPC formulations, such as [165] and the MPC formulations in Sections 3.1 and 3.2. We consider the more challenging problem, where the price profile y_e changes each day. Furthermore, we assume that only the price profile for the current day is available as a forecast, which is modelled using the external predictions y_e that change every 24 hours, i.e., $y_\mathrm{e}(\cdot|t+1) = \mathcal{R}_T y_\mathrm{e}(\cdot|t)$ for all $t \in \mathbb{I}_{\geq 0}$: $\mathrm{mod}(t, 24) \neq 0$. The considered price profile y_e is taken from the real data considered in [235] over the span of one week.

We implemented the economic MPC formulation from Section 3.3 using $N = 2$, $\beta = 10$ with the modified cost $\tilde{V}_{\mathrm{f,eco}}$ from Proposition 3.82 and a terminal equality constraint (TEC), where Lemma 3.78 holds with $\nu = 1$ for the considered scalar stable system. As a result, Problem 3.53 is a small scale mixed-integer linear program (MILP), which is solved to optimality using *intlinprog* from Matlab. The resulting closed loop can be seen in Figure 3.7. The closed loop yields a periodic like operation for each day, with small changes between each day based on the different price profile y_e. The adjustment of the closed-loop response based on the price y_e can be directly seen with the applied input u, which is always at a maximum when the electricity price y_e is low. We also compared the proposed MPC framework to the optimal[21] operation in Figure 3.7, assuming full knowledge of the future price profile $y_\mathrm{e}(t)$ for all coming days. The proposed framework results in state and input trajectories very similar to the optimal operation with a minimal increase of 0.1% in the overall electricity price ℓ_{eco}.

Overall, this example shows the applicability of the proposed economic MPC scheme (Sec. 3.3) for convex time-varying problems. In particular, we showed reliable operation under unpredictable online changing condition with close to optimal (hindsight) performance.

3.4.4 Economic performance improvement using dynamic operation

In the following, we consider the classical problem of increasing the yield of a CSTR with dynamic operation, compare [24]. In this example, we compare the performance of the proposed economic MPC approach (Sec. 3.3) with periodicity constraint MPC [138, 282] and tracking MPC formulations [165] (cf. Sec. 3.2). We first demonstrate average performance improvement of the economic MPC framework (Sec. 3.3) compared to fixed

[21] To allow for a consistent comparison, the same initial and final state is considered.

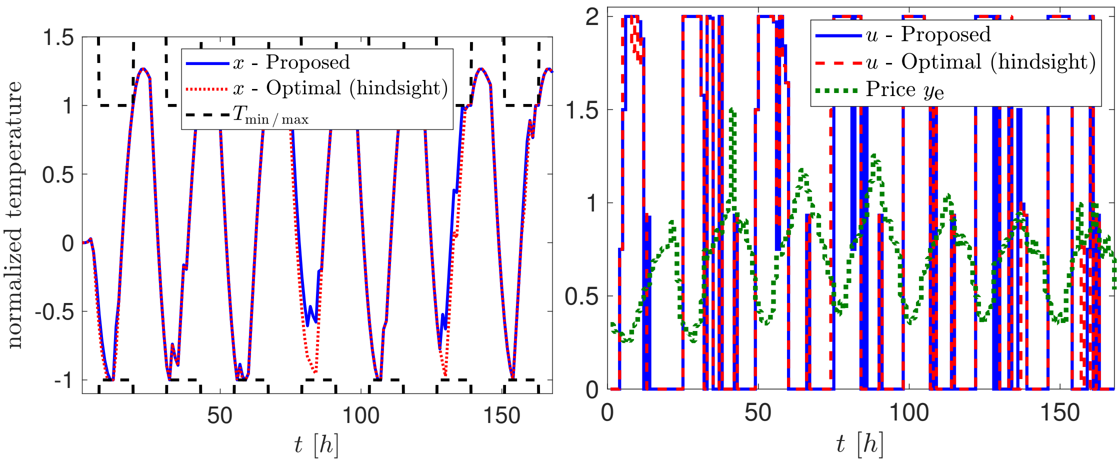

Figure 3.7. Transient performance under online changing price signals y_e for temperature control problem. Left: Closed-loop temperature x of the proposed approach (blue, solid) and the optimal operation (red, dotted), with time-varying constraints $T_{\min/\max}$ (black, dashed). Right: Closed-loop applied cooling rate u for the proposed approach (blue, solid) and optimal operation (red, dotted); and the price signal y_e (green, dotted).©2020 Elsevier Ltd.

periodic operation ($T > 1$) or steady-state operation ($T = 1$). Then, we show reliable economic performance under online changing dynamic operation due to changing cost functions.

We consider a continuous-time model of a CSTR

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 - x_1 - 10^4 x_1^2 \exp\left(\frac{-1}{x_3}\right) - 400 x_1 \exp\left(\frac{-0.55}{x_3}\right) \\ 10^4 x_1^2 \exp\left(\frac{-1}{x_3}\right) - x_2 \\ u - x_3 \end{pmatrix},$$

where $u \in \mathbb{R}$ is related to the heat flux and $x = (x_1,\ x_2,\ x_3) \in \mathbb{R}^3$ correspond to the concentration of the reaction, the desired product and the temperature, compare [24, 209], [96, Sec. 3.4]. The constraint sets are

$$\mathbb{Z}_r = [0.05, 0.4] \times [0.05, 0.2]^2 \times [0.059, 0.439], \quad \mathbb{Z} = [0.03, 1]^3 \times [0.049, 0.449].$$

We consider the economic stage cost $\ell_{eco}(x, u, y_e) = -x_2 + y_e \cdot (u - u_s)^2$ with $u_s = 0.1491$ and $y_e \in \mathbb{Y} = [0, 1]$. If the external parameter is $y_e = 0$, the stage cost ℓ_{eco} tries to maximize the production of the desired product x_2. The online tunable part in the cost function is a regularization of the input u relative to the optimal steady-state input u_s.

For $y_e = 1$, the system is optimally operated at a steady-state (x_s, u_s), while for $y_e = 0$ dynamic operation can significantly outperform steady-state operation, compare [96, Sec. 3.4]. Hence, treating y_e as an external variable allows a user to smoothly transition between steady-state and dynamic operation. The discrete-time model is defined with a fourth order Runge-Kutta discretization and a sampling time of $T_s = 0.05$.[22] For the following simulations, the initial condition is always chosen as the optimal steady-state x_s and $\beta(t) = 10$ for all $t \in \mathbb{I}_{\geq 0}$.

Average performance improvement

We first consider the problem of maximizing the concentration x_2 ($y_e = 0$) to show average performance improvements. In the absence of transient changes (y_e constant), the average performance of periodicity constraint MPC [138, 282] and tracking MPC formulations (cf. [165], Sec. 3.2) are equivalent for a fixed T (assuming that convergence is achieved). Similarly, the proposed framework yields the same asymptotic performance as the economic MPC schemes in [10, 241, 295], assuming that r converges. We implemented the proposed approach with $T \in \{1, 10, 20\}$, $N \in \mathbb{I}_{[1,50]}$ and tested different proposed designs regarding the terminal ingredients (V_f, $\mathbb{X}_f$, Cor. 3.72, Prop. 3.76, Lemma 3.78) and the cost function ($\tilde{V}_f$, Lemma 3.81, Prop. 3.82). The detailed numerical results for all the considered implementations can be found in [JK26, App. A]. In the following, we only consider the approach utilizing the positive definite terminal cost V_f from Proposition 3.76 in combination with the modified cost $\tilde{V}_f$ from Lemma 3.81, which seems most suitable for practical applications (in terms of computational complexity and performance). Figure 3.8 exemplarily shows the performance of this approach with $T \in \{1, 10, 20\}$ for increasing N in comparison to the average cost at the optimal periodic orbit of length $T = \{10, 20\}$ and the optimal steady-state ($T = 1$). We note that, neglecting small initial deviations[23], the proposed EMPC outperforms optimal periodic operation with the same period length T, even though a constant value β is used (Ass. 3.61 does not hold and Prop. 3.84 only guarantees ϵ suboptimality). The performance increases (for both purely periodic operation and the economic MPC) if we increase T or N. This implies that the proposed economic MPC framework (Sec. 3.3) utilizing periodic orbits ($T > 1$) and additional predictions ($N \geq 1$) with a purely

[22]In [96, 209] a sampling time of $T_s = 0.1$ is used. However, with the considered fourth order explicit Runge-Kutta discretization, a sampling time of $T_s = 0.1$ does not preserve stability of the continuous-time system. In addition, we consider $x_i \geq 0.03$ instead of $x_i \geq 0$, to avoid discretization errors for $x_i \approx 0$.

[23]The average performance is computed in the interval $t \in \mathbb{I}_{[1000,2000]}$ starting with initial condition $x = x_s$. Thus, for very short horizons N, $\overline{\kappa}$ has not yet converged.

economic formulation can outperform periodicity constrained formulations [138, 282] ($N = 0$), steady-state formulations [87, 102, 206, 208] ($T = 1$) and periodic tracking formulations (cf. [164, 165, 166], Sec. 3.2, with ℓ positive definite), even if the operating conditions do not change online (Ass. 3.50 holds).

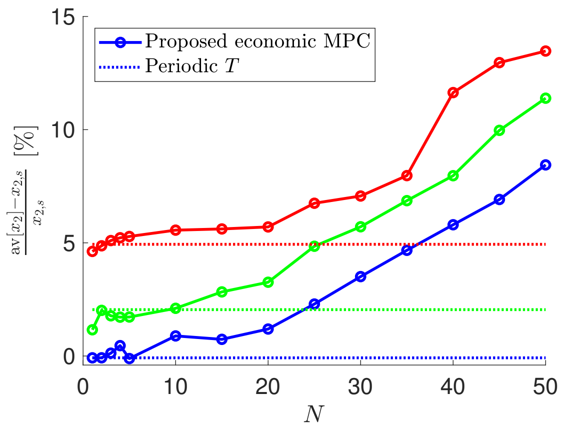

Figure 3.8. Average performance improvement due to dynamic operation relative to the optimal steady-state x_s - CSTR. Periodic operation (dotted) vs. proposed economic MPC scheme (solid with circles) for $T = 1$ (blue), $T = 10$ (green) and $T = 20$ (red).©2020 Elsevier Ltd.

Transient performance under online changing conditions

In the following, we study the performance of the proposed scheme under online changing conditions, i.e., y_e unpredictably time-varying. This corresponds to a scenario where a user can freely tune the desired mode of operation online. The resulting closed loop for the proposed economic MPC scheme with $N = 10$, $T = 20$ can be seen in Figure 3.9. As $y_e \to 0$ (e.g., $t = 15$ or $t = 246$), the system operates dynamically to increase production x_2 and once the weight y_e on the input deviation increases the system quickly minimizes the control effort and smoothly converges to the new optimal mode of operation, i.e., the steady-state x_s (e.g., $t \in \mathbb{I}_{[185,246]}$, $t \in \mathbb{I}_{[400,470]}$). In this scenario, the MPC scheme on average increases production by 2.8% compared to steady-state operation, while a 5% increase was achieved for $y_e \equiv 0$ (cf. Fig. 3.8). For comparison, the performance of the economic MPC (Sec. 3.3), the tracking MPC (cf. [165], Sec. 3.2), both with $T = 20$, $N \in \mathbb{I}_{[0,50]}$, and the periodicity constraint MPC [138, 282] with $T \in \mathbb{I}_{[0,90]}$ can be seen in Figure 3.9. The number of decision variables in a condensed formulation are $n + m \cdot (T + N)$ for the economic MPC (Sec. 3.3) and the tracking MPC (cf. [165], Sec. 3.2), and $m \cdot T$ for the periodicity constraint MPC [138, 282] ($N = 0$). Thus, the x-axis ($N + T$) in Figure 3.9 is a measure for the computational complexity. First, note that we can improve the performance of the economic MPC by increasing N. Similarly, the performance of the periodicity constraint MPC [138, 282] improves for a larger period length T, but at a smaller pace. Thus, given the same number of decision variables, the proposed economic MPC formulation can achieve a better performance. For small values of N the performance of the tracking MPC is similar to the economic MPC. However, in contrast to the economic MPC formulation, the economic performance of the tracking MPC does not improve significantly with a large horizon N (since the region of attraction is not the limiting factor). Additional numerical results, comparing the performance to economic MPC without terminal constraints [122, 128, 129, 130, 132, 210] and investigating the effect of the various degrees of freedoms in the formulation on closed-loop performance (terminal ingredients (Sec. 3.3.5), alternative cost formulations (Sec. 3.3.6), continuous-time formulations (Rk. 3.67)) can be found in [JK26, App. A].

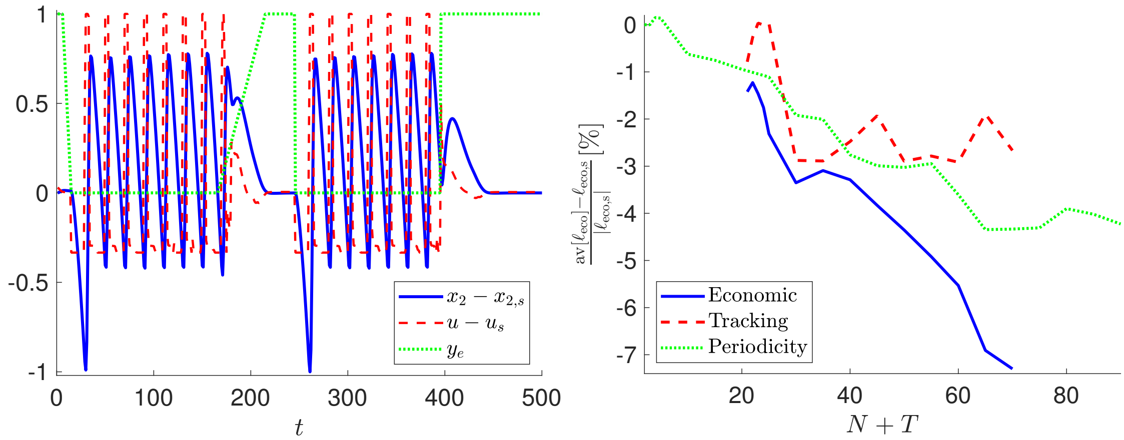

Figure 3.9. Dynamic operation under online changing conditions - CSTR. Left: Exemplary closed-loop trajectories of economic MPC (Sec. 3.3) for $N = 10$, $T = 20$. Deviation in production $x_2 - x_{2,\mathrm{s}}$ (blue, solid), deviation in heat flux $u - u_\mathrm{s}$ (red, dashed), price signal y_e (green, dotted). All signals are normalized to $|x| \leq 1$. Right: Transient performance $\ell_{\mathrm{eco}}(x, u, y_\mathrm{e}) = -x_2 + y_\mathrm{e} \cdot (u - u_\mathrm{s})^2$ relative to steady-state operation $\ell_{\mathrm{eco,s}} = -x_{2,\mathrm{s}}$. Economic MPC approach from Section 3.3 (blue, solid), tracking MPC (cf. [165], Sec. 3.2) (red, dashed) with $T = 20$ and $N \in \mathbb{I}_{[0,50]}$ and periodicity constraint MPC [138, 282] (green, dotted) with $T \in \mathbb{I}_{[0,90]}$.©2020 Elsevier Ltd.

To summarize, in the considered example we have shown the applicability of the proposed economic MPC with artificial periodic reference trajectories (Sec. 3.3) to nonlinear economic control problems. In particular, the proposed approach: (i) improves performance compared to (fixed) steady-state or periodic operation, (ii) reliably operates under online changing conditions, (iii) in general achieves better performance than periodicity constraint formulations [138, 282] or tracking formulations (cf. [165], Sec. 3.2).

3.5 Summary

In this chapter, we presented different MPC designs for dynamic operation of nonlinear constrained systems. In particular, we investigated tracking of unreachable target signals (Sec. 3.2) and periodic/dynamic economic operation (Sec. 3.3). In both cases, we used artificial periodic reference trajectories to provide a large region of attraction and guarantee feasibility independent of online changes in the control goal (target signal y_{e}/economic cost ℓ_{eco}). In case of consistent periodic target signals/cost functions, we derived desired closed-loop performance guarantees: stability of the optimal reachable trajectory/average performance no worse than performance at a (local) optimal periodic orbit. In addition, we investigated various extensions to enhance performance, such as offline computation of parametrized terminal ingredients V_{f}, k_{f} (Sec. 3.1.3, Sec. 3.3.5), reference dependent terminal sets size α (Sec. 3.2.2) and partially decoupled reference updates (Sec. 3.2.3). We demonstrated the efficacy of the proposed MPC designs using nonlinear constrained examples from literature (Sec. 3.4). In the next chapter, we investigate the complementary problem of achieving similar closed-loop guarantees with simpler MPC formulations that require no offline design and use no terminal ingredients.

Chapter 4

Analysis of MPC schemes for dynamic operation without offline design

In this chapter, we present a framework to analyse the closed-loop properties of MPC formulations without terminal ingredients for dynamic operation. The following results can be viewed as complementary to Chapter 3, focusing on simpler MPC formulation without terminal ingredients or artificial reference trajectories. In particular, we address the challenges associated with dynamic operation identified in Section 1.1: (i) *non-stationary operation* and (iii) *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories*. The considered MPC formulations naturally cope with challenge (ii) *online changes in the mode of operation*, since neither terminal ingredients nor offline design procedures are necessary.

In Section 4.1, tracking for reachable dynamic state and input trajectories is considered. Incremental system properties and a lower bound on the prediction horizon N are derived, which guarantee exponential stability of the closed loop. Also, closed-loop stability with positive semidefinite input-output stage costs is analysed using an additional detectability/observability condition. Furthermore, improved lower bounds on the prediction horizon are derived using an extended prediction horizon. In Section 4.2, the more general output regulation setup is considered, with the main additional challenge that the optimal mode of operation is only indirectly specified in terms of minimizing a quadratic output stage cost. In the analysis, we uncover strong connections to the classical conditions considered the output regulation literature, including feasibility of the regulator equations, detectability conditions, the minimum-phase property, and the nonresonance condition. The results are intriguing, since we guarantee stability of the regulator manifold, without explicitly solving the regulator equations. In Section 4.3, we investigate unreachable reference trajectories and derive sufficient conditions to ensure practical stability of the (unknown) optimal reachable trajectory using tools from economic MPC. In Section 4.4, the theoretical results are revisited for the special case of linear system dynamics. In Section 4.5, we illustrate the theoretical results and demonstrate improved bounds using numerical examples. The results presented in this chapter are based on Köhler et al. [JK10, JK19, JK21, JK23, JK24].

This chapter also contains various results on MPC without terminal constraints, including improved performance bounds, that may be of independent interest[1].

4.1 Trajectory tracking MPC without terminal ingredients

In Section 3.1, we studied trajectory tracking of reachable reference trajectories by designing an MPC scheme with suitable terminal ingredients. In this section, we consider the complementary problem of analysing an MPC scheme without terminal ingredients (similar to Section 2.2) and derive system theoretic conditions that guarantee stability of the reference trajectory. The main contributions of this section are: a) improved performance and stability bounds for MPC without terminal ingredients based on a *local* cost controllability condition and positive definite stage costs or alternatively a *detectability/observability* condition on the stage cost; b) sufficient conditions in terms of incremental system properties for these abstract conditions in the context of trajectory tracking; and c) a closed-loop analysis in case an *extended* prediction horizon is used. We first present the setup and the considered MPC formulation (Sec. 4.1.1). Next, we prove exponential stability based on a local cost controllability condition for positive definite sage costs (Sec. 4.1.2) and for positive semidefinite stage costs using additional detectability/observability conditions (Sec. 4.1.3). Then, we show that the considered assumptions can be reduced to incremental system properties (Sec. 4.1.4) and demonstrate that improved bounds can be derived using an extended prediction horizon (Sec. 4.1.5).

[1] Theorem 4.5 improves the results in [37] based on *local* cost controllability conditions, which is further improved in Remark 4.32 using an LP analysis. Theorem 4.12 extends the results in [119] for detectable stage costs ℓ by deriving performance bounds and considering local stabilizability conditions. Proposition 4.14 further improves these bounds using an observability condition. Theorem 4.37 derives significantly shorter prediction horizons by using an extended prediction. Theorem 4.50 shows that stability can be ensured, even if the stage cost is not (directly) detectable. Theorem 4.80 relaxes the controllability assumptions in [122] using sublevel set arguments for the rotated value function.

This section is based on and taken in parts literally from [JK24][2], [JK19][3], and [JK10][4].

4.1.1 Trajectory tracking MPC

In the following, we present the considered trajectory tracking MPC formulation without terminal ingredients. The setup is analogous to Section 3.1 and is detailed again for completeness. We consider a nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)), \quad x(0) = x_0,$$

with the state $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$, the control input $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, the dynamics $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, the initial condition $x_0 \in \mathbb{X}$, and the time step $t \in \mathbb{I}_{\geq 0}$. We impose pointwise-in-time constraints on the state and input

$$(x(t), u(t)) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}, \quad t \in \mathbb{I}_{\geq 0}, \tag{4.1}$$

and assume that $\mathbb{Z}$ is compact and f is continuous. We consider the problem of stabilizing a state and input reference trajectory $r(t) := (x_{\mathrm{r}}(t), u_{\mathrm{r}}(t)) \in \mathbb{X} \times \mathbb{U} \subseteq \mathbb{R}^{n+m}$ and assume that the future reference trajectory is exactly known. Denote the tracking error by $e_{\mathrm{r}}(t) := x(t) - x_{\mathrm{r}}(t)$. The control goal is to achieve constraint satisfaction (4.1) and (uniform) asymptotic stability of the tracking error $e_{\mathrm{r}} = 0$ for a (preferably large) set of initial conditions, called the region of attraction. To this end, we consider a continuous (tracking) stage cost $\ell : \mathbb{Z} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$. At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t)$, the MPC control law is determined based on the following time-varying optimization problem:

Problem 4.1.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_N(x(\cdot|t), u(\cdot|t), t) \tag{4.2a}$$

[2] J. Köhler, M. A. Müller, and F. Allgöwer. "Nonlinear reference tracking: An economic model predictive control perspective." In: *IEEE Trans. Automat. Control* 64.1 (2019), pp. 254–269©2018 IEEE.

[3] J. Köhler, M. A. Müller, and F. Allgöwer. "Constrained nonlinear output regulation using Model Predictive Control." In: *IEEE Trans. Automat. Control* (2021). extended version: arXiv:2005.12413©2021 IEEE.

[4] J. Köhler and F. Allgöwer. "Stability and performance in MPC using a finite-tail cost." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control.* 2021, pp. 166–171©2021 the authors.

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.2b}$$

$$x(0|t) = x(t), \tag{4.2c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.2d}$$

where

$$\mathcal{J}_N(x(\cdot|t), u(\cdot|t), t) := \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t), t+k). \tag{4.2e}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, and the value function $V_N(x(t), t) := \mathcal{J}_N(x^*(\cdot|t), u^*(\cdot|t), t)$. To simplify the theoretical exposition regarding feasibility, we define $V_N(x(t), t) = \infty$ in case Problem 4.1 does not admit a feasible solution. The following algorithm summarizes the closed-loop operation.

Algorithm 4.2. *(Trajectory tracking MPC Algorithm)*
Offline: Specify the constraint set $\mathbb{Z}$, the stage cost ℓ, and the prediction horizon N.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, solve Problem 4.1, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t), \quad t \in \mathbb{I}_{\geq 0}. \tag{4.3}$$

Compared to the MPC formulations in Chapter 3, we consider a simple reference tracking MPC formulation without terminal ingredients or artificial reference trajectories. Thus, the considered MPC formulation requires *no* complex design procedures or additional optimization variables. Furthermore, the absence of hard terminal constraints avoids potential feasibility issues due to changing reference trajectories or disturbances. Due to this fact, Algorithm 4.2 can be directly applied if the reference trajectory changes online. In contrast to the tracking MPC formulation in Section 3.2 based on artificial reference trajectories, we also do not need to restrict the analysis to periodic reference trajectories. These properties make the considered MPC scheme particularly suitable for practical implementations.

4.1.2 Theoretical analysis

In the following, we derive sufficient conditions for exponential stability of the tracking error $e_\mathrm{r} = 0$. There exists a large body of literature that study the stability and performance of similar MPC schemes without terminal ingredients, often also called *unconstrained* MPC (due to the lack of terminal constraints), compare [7, 37, 86, 119, 120, 123, 126, 127, 237, 267].

As is standard in the corresponding literature [37, 120, 123, 127, 237, 267], we assume that the stage cost is positive definite, analogous to Assumption 2.3, which is characterized using the function $\ell_{\min}(x,t) := \inf_{u\in\mathbb{U}} \ell(x,u,t)$.

Assumption 4.3. *(Tracking stage cost) There exist functions $\underline{\alpha}_\ell, \overline{\alpha}_\ell \in \mathcal{K}_\infty$ such that $\underline{\alpha}_\ell(\|x - x_\mathrm{r}(t)\|) \leq \ell_{\min}(x,t) \leq \overline{\alpha}_\ell(\|x - x_\mathrm{r}(t)\|)$ for all $(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$ and $\ell(x_\mathrm{r}(t), u_\mathrm{r}(t), t) = 0$, for all $t \in \mathbb{I}_{\geq 0}$.*

We point out that this assumption includes the case where the stage cost is independent of u and u_r, in which case the input reference u_r does not need to be known (cf. Ass. 4.16).

The standard analysis for MPC without terminal ingredients relies on an asymptotic/exponential *cost controllability* [120, 123, 127, 237, 267], which provides a suitable bound for the infinite-horizon cost for all $x \in \mathbb{X}$. However, this condition can be rather restrictive in case of unstable system dynamics or hard state constraints. In [37], this analysis has been extended to only require a *local*[5] bound on the cost function, which is significantly less restrictive, compare also [86, 252]. Thus, in accordance with the existing literature, we consider a local exponential cost controllability assumption.

Assumption 4.4. *(Local exponential cost controllability) There exist constants $\gamma > 1$, $\epsilon > 0$ such that for all $N \in \mathbb{I}_{\geq 1}$, for all $(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$ satisfying $\ell_{\min}(x,t) \leq \epsilon$, Problem 4.1 is feasible and the value function satisfies*

$$V_N(x,t) \leq \gamma \cdot \ell_{\min}(x,t). \tag{4.4}$$

We note that for ℓ quadratic this assumption is similar to Assumption 3.6. Sufficient conditions for Assumption 4.4 will be discussed in detail in Section 4.1.4.

[5] We point out that local cost controllability conditions have also been used earlier in [239, Ass. 4], [238, Ass. 1]. However, the corresponding guarantees require an explicit terminal constraint to be included in Problem 4.1, unlike the method in [37] based on sublevel set arguments, which are also adopted in this chapter.

The following theorem drives a lower bound $N_{\overline{V}}$ for prediction horizon N such that asymptotic stability of the reference trajectory under the closed-loop system is ensured.

Theorem 4.5. *Let Assumptions 4.3–4.4 hold. Then, for any $\overline{V} > 0$, there exists a constant $N_{\overline{V}} > 0$ such that for all $N > N_{\overline{V}}$ and any initial condition $(x_0, 0) \in \mathbb{X}_{\overline{V}} := \{(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0} \mid V_N(x,t) \leq \overline{V}\}$, the closed-loop system* (4.3) *resulting from Algorithm 4.2 satisfies the constraints* (4.1)*, Problem 4.1 is feasible for all $t \in \mathbb{I}_{\geq 0}$, and $e_{\mathrm{r}} = 0$ is (uniformly) asymptotically stable. Furthermore, there exists a constant $\alpha_M \in (0,1]$ such that the following performance bound holds for the closed loop:*

$$\mathcal{J}_{\infty}^{\mathrm{cl}}(x_0) := \sum_{t=0}^{\infty} \ell(x(t), u(t), t) \leq V_N(x_0, 0)/\alpha_M \leq V_\infty(x_0, 0)/\alpha_M. \tag{4.5}$$

Proof. The proof is split into three parts. Part I and II show that the value function V_N satisfies the following bounds at time $t \in \mathbb{I}_{\geq 0}$, assuming $(x(t), t) \in \mathbb{X}_{\overline{V}}$:

$$\ell_{\min}(x(t), t) \leq V_N(x(t), t) \leq \gamma_{\overline{V}} \ell_{\min}(x, t), \tag{4.6a}$$

$$V_N(x(t+1), t+1) - V_N(x(t), t) \leq -\alpha_M \ell(x(t), u(t), t), \tag{4.6b}$$

with later specified constants $\gamma_{\overline{V}} \geq \alpha_M > 0$. Part III establishes that $(x(t), t) \in \mathbb{X}_{\overline{V}}$ holds recursively for all $t \in \mathbb{I}_{\geq 0}$, derives the performance bound (4.5) and establishes uniform asymptotic stability. Abbreviate $\ell(k|t) := \ell(x^*(k|t), u^*(k|t), t+k)$, $e_{\mathrm{r}}^*(k|t) := x^*(k|t) - x_{\mathrm{r}}(t+k)$ and $V(k|t) := V_{N-k}(x^*(k|t), t+k)$, $k \in \mathbb{I}_{[0,N-1]}$.

Part I: The lower bound in Inequality (4.6a) follows directly from $\ell \geq 0$ and

$$V_N(x(t), t) \geq \ell(x(t), u(t), t) \geq \ell_{\min}(x(t), t).$$

Define $\gamma_{\overline{V}} := \max\left\{\gamma, \frac{\overline{V}}{\epsilon}\right\}$. Assumption 4.4 in combination with $V_N(x(t), t) \leq \overline{V}$ implies the upper bound in Inequality (4.6a). This can be shown with a case distinction based on whether $\ell_{\min}(x(t), t) \leq \epsilon$, similar to [37]. If $\ell_{\min}(x(t), t) \leq \epsilon$, Assumption 4.4 directly guarantees the upper bound since $\gamma_{\overline{V}} \geq \gamma$ by definition. On the other hand, if $\ell_{\min}(x(t), t) > \epsilon$, we have $V_N(x(t), t) \leq \overline{V} \leq \overline{V}/\epsilon \cdot \ell_{\min}(x(t), t) \leq \gamma_{\overline{V}} \ell_{\min}(x(t), t)$.

Part II: In the following, we show that Condition (4.6b) holds with some $\alpha_M \in (0,1]$. The basic idea of the following proof is to decompose the prediction horizon N in two parts: $\mathbb{I}_{[0,k_x-1]}$ and $\mathbb{I}_{[k_x,N-1]}$, with some later specified $k_x \in \mathbb{I}_{[0,N-1]}$. For the interval $\mathbb{I}_{[k_x,N-1]}$, we can then exploit a local upper bound on the value function (cf. Ass. 4.4) to use standard arguments (cf. [37, 123]) and thus derive Inequality (4.6b) with $\alpha_M > 0$ for

N large enough.
For any $k' \in \mathbb{I}_{[0,N-1]}$, the principle of optimality ensures

$$V_N(x(t),t) = \sum_{k=0}^{N-1} \ell(k|t) = \sum_{k=0}^{k'-1} \ell(k|t) + V(k'|t). \tag{4.7}$$

We denote the smallest element $k \in \mathbb{I}_{[0,N-1]}$, which satisfies $V(k|t) \leq \gamma\epsilon$ by k_x, assuming it exists. In the following, we prove $k_x \in \mathbb{I}_{[0,N_0]}$ with $N_0 := \lceil \gamma_{\overline{V}} - \gamma \rceil = \left\lceil \max\left\{0, \frac{\overline{V}-\gamma\epsilon}{\epsilon}\right\}\right\rceil \in \mathbb{I}_{\geq 0}$. For contradiction, assume that $k_x > N_0$. Then, by definition (cf. (4.7), $\ell \geq 0$) we have $V(k|t) > \gamma\epsilon$ for all $k \in \mathbb{I}_{[0,N_0]}$. Using the same arguments as in Part I with $\overline{V}$ replaced by $\gamma\epsilon$ ensures that $V(k|t) > \gamma\epsilon$ implies $\ell(k|t) > \epsilon$. Thus, Condition (4.7) ensures

$$V(N_0|t) < V(0|t) - \epsilon N_0 \leq \overline{V} - \epsilon N_0 \leq \overline{V} - \epsilon(\gamma_{\overline{V}} - \gamma) \leq \gamma\epsilon,$$

which contradicts $k_x > N_0$. Hence, we have $k_x \in \mathbb{I}_{[0,N_0]}$ and $V(k|t) \leq V(k_x|t) \leq \gamma\epsilon$ for all $k \in \mathbb{I}_{[k_x,N-1]}$. Furthermore, $V(k|t) \leq \gamma\epsilon$, $k \in \mathbb{I}_{[k_x,N-1]}$ implies $V(k|t) \leq \gamma\ell(k|t)$, $k \in \mathbb{I}_{[k_x,N-1]}$ using Assumption 4.4 and a case distinction whether $\ell(k|t) \leq \epsilon$, analogous to the arguments in Part I. In addition, it holds

$$V(k_x|t) \leq \gamma \min\{\ell(k_x|t), \epsilon\} \leq \gamma \min\{\ell(0|t), \epsilon\}, \tag{4.8}$$

where the second inequality follows from the definition of k_x, i.e., $\ell(0|t) \leq \epsilon$ implies $k_x = 0$. We can use the bounds in [123, Variant 2], [267] for the remaining horizon of length $N - k_x \geq N - N_0 =: M$ to show

$$\ell(N-1|t) \leq \left(\frac{\gamma-1}{\gamma}\right)^{N-k_x-1} V(k_x|t) \overset{(4.8)}{\leq} \gamma \left(\frac{\gamma-1}{\gamma}\right)^{M-1} \min\{\ell(0|t), \epsilon\}. \tag{4.9}$$

For $M \geq M_0 := \frac{2\log\gamma - \log(\gamma-1)}{\log\gamma - \log(\gamma-1)} = \frac{\log\gamma}{\log\gamma - \log(\gamma-1)} + 1$, Inequality (4.9) ensures

$\ell(N-1|t) \leq \epsilon$. Thus, we can again invoke Assumption 4.4, which yields

$$\begin{aligned}
&V_N(x(t+1),t+1) \leq \sum_{k=1}^{N-1} \ell(k|t) - \ell(N-1|t) + V_2(x^*(N-1|t),t+N-1) \\
&\stackrel{(4.4)}{\leq} V_N(x(t),t) - \ell(0|t) + (\gamma-1)\ell(N-1|t) \stackrel{(4.9)}{\leq} V_N(x(t),t) - \ell(0|t) + \frac{(\gamma-1)^M}{\gamma^{M-2}}\ell(0|t) \\
&\leq V_N(x(t),t) - \underbrace{\left(1 - \frac{(\gamma-1)^M}{\gamma^{M-2}}\right)}_{:=\alpha_M} \ell(0|t). \qquad (4.10)
\end{aligned}$$

For $M > M_1 := \frac{2\log\gamma}{\log\gamma - \log(\gamma-1)}$, this ensures $\alpha_M > 0$. Thus, for

$$N > N_{\overline{V}} := \underbrace{\frac{2\log\gamma + \max\{0, -\log(\gamma-1)\}}{\log\gamma - \log(\gamma-1)}}_{=:\underline{M}} + N_0. \qquad (4.11)$$

we have $M > \underline{M} := \max\{M_0, M_1\}$ and all the previous bounds hold.
Part III: Condition (4.6b) with $\ell \geq 0$ and $\alpha_M > 0$ ensures that V_N is non-increasing and thus $(x(t),t) \in \mathbb{X}_{\overline{V}}$ holds for all $t \in \mathbb{I}_{\geq 0}$. Hence, the results in Part I and II hold for all $t \in \mathbb{I}_{\geq 0}$. Inequalities (4.6) in combination with the lower/upper bounds on $\ell_{\min}$ (Ass. 4.3) ensure that $V_N(x,t)$ is a (uniform) time-varying Lyapunov function, which implies uniform asymptotic stability of $e_r = 0$, compare [126, Thm. 2.22], [236, Thm. 2.32]. The performance bound (4.5) follows directly from Inequality (4.6b) with $V_\infty(x,t) \geq V_N(x,t) \geq 0$. ∎

We have shown that for any prediction horizon $N > N_{\overline{V}}$ and any initial condition x_0 with $V_N(x_0,0) \leq \overline{V}$, the closed-loop system asymptotically stabilizes the reference trajectory. The basic idea of this proof can be summarized as follows. Given a specified region of attraction $(x(t),t) \in \mathbb{X}_{\overline{V}}$, we have $V(k|t) \leq \gamma\epsilon$ for all $k \in \mathbb{I}_{[N_0,N-1]}$, which implies $V(k|t) \leq \gamma\ell(k|t)$ (cf. Ass. 4.4). Thus, standard arguments from MPC without terminal ingredients (cf. [123]) can be used for the remaining horizon $M = N - N_0$ to bound the cost of the appended piece. A simpler version of this proof can be found in [JK24, Thm. 2], which results in the more conservative bounds $N_0 = \gamma_{\overline{V}} - 1 \geq \gamma_{\overline{V}} - \gamma$. For $N_0 = 0$, the formulas and bounds for α_M, $\underline{M}$ in (4.10), (4.11) are equivalent to [123, Variant 2], [267], where Assumption 4.4 is assumed globally. In Section 4.1.5, we show how the local bound $\underline{M}$ can be further reduced using analysis methods from the literature (cf. LP analysis in [120, 127]) or employing finite-tail costs from [175]. We point out that the proof of $V(k_x|t) \leq \gamma\epsilon$ is similar to the proof in [162] to demonstrate implicit satisfaction of the terminal set constraint based on sublevel set arguments. Note that Problem 4.1 can be modified using constraint tightening to ensure robust reference tracking under bounded disturbance, compare [JK17] for the technical details.

Remark 4.6. *(Comparison - performance bounds) The resulting theoretical guarantees are closely related to [37], where setpoint stabilization without terminal constraints is studied based on a local cost controllability assumption (cf. Ass. 4.4). The analysis presented in [37] directly uses the regional upper bound $\gamma_{\overline{V}}$ and proceeds as in [123, Variant 2], [267], without splitting the horizon in two components N_0, M. This approach results in $N_{\overline{V}} = \dfrac{2\log\gamma_{\overline{V}}}{\log\gamma_{\overline{V}} - \log(\gamma_{\overline{V}} - 1)}$ (assuming w.l.o.g. $\gamma_{\overline{V}} \geq 2$). For $\gamma = \gamma_{\overline{V}}$, this bound is equivalent to the derived bound in Equation* (4.11)*. However, for $\gamma_{\overline{V}} \gg \gamma$, the derived bound can be significantly less conservative. In the limit, for $\overline{V} \to \infty$, the bound in [37, Rk. 5] approaches $N \geq 2\frac{\overline{V}}{\epsilon}\log\overline{V}$, while the derived bound* (4.11) *approaches $N \geq \frac{\overline{V}}{\epsilon}$. The proposed proof explicitly exploits the fact that we have additional (tighter) bounds for the points $k \in \mathbb{I}_{[N_0,N-1]}$, which are close enough to the reference trajectory r. This proof highlights some intrinsic properties and difficulties in establishing regional results based on local properties. If we wish to increase the region of attraction $\mathbb{X}_{\overline{V}}$, we can simply increase the prediction horizon N by $N_0 = \frac{\overline{V}-\gamma\epsilon}{\epsilon}$ compared to the local bound $M \geq \underline{M}$. This leaves the suboptimality index α_M in the performance bound* (4.5) *untouched. Such simple connections are rather hard to conclude using the method in [37].*

Remark 4.7. *(Tightness) The resulting bound on the region of attraction $\mathbb{X}_{\overline{V}}$ has a certain tightness property in the following sense. Consider a system satisfying the local cost controllability condition in Assumption 4.4 and we wish to estimate the largest constant $\overline{V}$, resulting in closed-loop stability for $(x_0, 0) \in \mathbb{X}_{\overline{V}}$. One can show that any lower bound on the prediction horizon N, solely based on this information, has to satisfy $N \geq \overline{V}/\epsilon \geq N_0$ in order to ensure stability. In other words, one can construct examples that satisfy Assumption 4.4 and there exist initial conditions $V_N(x_0, 0) \leq \overline{V}$ with $N\epsilon < \overline{V}$ that cannot be stabilized.[6] Thus, alternative estimates on the sufficient prediction horizon $N_{\overline{V}}$, which are only based on Assumption 4.4, can never be smaller than $\frac{\overline{V}}{\epsilon}$.*

[6] We define the region of attraction based on sublevel sets V_N instead of V_∞. If the guaranteed region of attraction is defined as a sublevel set of V_∞, we can also construct examples that are not closed-loop stable using a prediction horizon $N < \overline{V}/\epsilon$.

4.1.3 Positive semidefinite cost and detectability/observability

In the following, we extend the theoretical analysis in Theorem 4.5 to stage costs ℓ that are only positive semidefinite, i.e., do not satisfy Assumption 4.3. This extension is interesting and relevant for multiple reasons. A standard example for positive semidefinite stage costs are input-output stage costs ℓ, which appear naturally in case only some output reference y_r is specified, compare, e.g., output path following in [94]. Furthermore, in case an input-output model resulting from system identification is used (cf., e.g., input-output LPV systems [1, 51] and non-parametric input-output models [180, 181, 182, 183]) the consideration of positive semidefinite input-output stage costs is natural. In addition, recent data-driven MPC approaches [JK3, JK4, JK5, JK6, 65, 66] use an implicit (linear) model of the system in terms of input-output data and thus cannot directly use a stage cost ℓ satisfying Assumption 4.3. Furthermore, the consideration of input-output stage costs is a pre-requisite for the theoretical analysis of the output regulation MPC considered in Section 4.2. Overall, input-output stage costs ℓ and input-output models are of practical and theoretical relevance, but the existing theoretical results for MPC either consider terminal ingredients or assume that the stage cost ℓ is positive definite (with the exception of [119]).

Theorem 4.12 extends the theoretical analysis in Theorem 4.5 based on a *detectability* condition for the considered stage cost ℓ. Proposition 4.14 improves the quantitative performance bounds by using a suitable *observability* condition for the stage cost ℓ. The theoretical analysis in this section is an extension of the results in [119], using a *local* bound on the cost function, deriving a performance bound, and improving the overall bounds on the prediction horizon.

Theoretical analysis

Problem 4.1 and Algorithm 4.2 remain unchanged and only Assumption 4.3 and thus also Assumption 4.4 are modified. In particular, without Assumption 4.3, $\ell_{\min}(x,t)$ being small does not necessarily imply that e_r is small and thus Assumption 4.4 is unrealistic. Instead, the role of $\ell_{\min}$ is replaced by a continuous function $\sigma : \mathbb{X} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the *state measure* [119], which assume to be bounded in terms of the tracking error $e_\mathrm{r} = x - x_\mathrm{r}(t)$.

Assumption 4.8. *(State measure) There exist functions* $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ *such that* $\underline{\alpha}(\|x - x_\mathrm{r}(t)\|) \leq \sigma(x,t) \leq \overline{\alpha}(\|x - x_\mathrm{r}(t)\|)$ *for all* $(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$.

The following conditions replace Assumptions 4.3–4.4, similar to [119, SA3/4].

Assumption 4.9. *(Local exponential cost controllability) There exist constants $\gamma_s, \epsilon > 0$ such that for all $N \in \mathbb{I}_{\geq 1}$, for all $(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$ satisfying $\sigma(x,t) \leq \epsilon$, Problem 4.1 is feasible and the value function satisfies*

$$V_N(x,t) \leq \gamma_s \sigma(x,t). \tag{4.12}$$

This condition is analogous to Assumption 4.4, with the difference that $\ell_{\min}$ is replaced by σ. Compared to [119, SA4], Assumption 4.9 only requires a *local* bound. This is similar to how the local bounds in Assumption 4.4 and in [37] are a less restrictive version of the global bound assumed in [120, 123, 127, 237, 267].

Assumption 4.10. *(Detectable stage cost ℓ) There exists a function $W : \mathbb{X} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and constants $\gamma_o, \epsilon_o > 0$ such that*

$$W(x,t) \leq \gamma_o \sigma(x,t), \tag{4.13a}$$

$$W(f(x,u),t+1) - W(x,t) \leq -\epsilon_o \sigma(x,t) + \ell(x,u,t), \tag{4.13b}$$

for any $(x,u,t) \in \mathbb{Z} \times \mathbb{I}_{\geq 0}$.

Remark 4.11. *(Connection to dissipativity) Assumption 4.10 is a special case of the strict dissipativity condition typically used in economic MPC [96, 203], with ℓ playing the role of the supply rate. The main difference is that W satisfies the upper bound* (4.13a)*, while in economic MPC only boundedness (from below) of the storage W is assumed, compare [136]. This small technical difference is the main reason that the analysis of economic MPC schemes without terminal ingredients is more involved and the resulting performance bounds are significantly more conservative than results for stabilizing MPC without terminal ingredients, compare [122, 132] and Section 4.3. Note that Assumption 4.10 is trivially satisfied with $W = 0$ if $\ell(x,u) \geq \sigma(x)$, which is the case we studied in Theorem 4.5.*

The following theorem combines the ideas of [119, Thms. 1–2] to address positive semidefinite stage costs ℓ (Ass. 4.10) with the methods in Theorem 4.5 to consider a less restrictive *local* bound for the cost function (Ass. 4.9).

Theorem 4.12. *Let Assumptions 4.9 and 4.10 hold. Then, for any $\overline{Y} > 0$, there exists a constant $N_{\overline{Y}} > 0$ such that for all $N > N_{\overline{Y}}$ and any initial condition $(x_0, 0) \in \mathbb{X}_{\overline{Y}} := \{(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0} \mid V_N(x,t) + W(x,t) \leq \overline{Y}\}$, the closed-loop system* (4.3) *resulting from Algorithm 4.2 satisfies the constraints* (4.1)*, and Problem 4.1 is feasible for all $t \in \mathbb{I}_{\geq 0}$. Furthermore, there exists a constant $\alpha_M \in (0,1]$ such that the following performance bound holds for the closed* *loop:*

$$\mathcal{J}_{\infty}^{\mathrm{cl}}(x_0) := \sum_{t=0}^{\infty} \ell(x(t), u(t), t) \leq \frac{1}{\alpha_M} V_{\infty}(x_0, 0) + \frac{1-\alpha_M}{\alpha_M} \gamma_{\mathrm{o}} \sigma(x_0, 0). \tag{4.14}$$

If additionally Assumption 4.8 holds, then $e_{\mathrm{r}} = 0$ is (uniformly) asymptotically stable.

Proof. The proof is split into three parts. Part I and II show that the function $Y_N := V_N + W$ satisfies the following bounds at time $t \in \mathbb{I}_{\geq 0}$, assuming $(x(t), t) \in \mathbb{X}_{\overline{Y}}$:

$$\epsilon_{\mathrm{o}} \sigma(x(t), t) \leq Y_N(x(t), t) \leq \gamma_{\overline{Y}} \sigma(x(t), t), \tag{4.15a}$$

$$Y_N(x(t+1), t+1) - Y_N(x(t), t) \leq -\alpha_M \epsilon_{\mathrm{o}} \sigma(x(t), t), \tag{4.15b}$$

with later specified constants $\gamma_{\overline{Y}} > 0$, $\alpha_M \in (0, 1]$. Part III establishes that $(x(t), t) \in \mathbb{X}_{\overline{Y}}$ holds recursively for all $t \in \mathbb{I}_{\geq 0}$, derives the performance bound (4.14) and establishes uniform asymptotic stability. Abbreviate $\ell(k|t) := \ell(x^*(k|t), u^*(k|t), t+k)$, $\sigma(k|t) = \sigma(x^*(k|t), t+k)$, $Y(k|t) = V_{N-k}(x^*(k|t), t+k) + W(x^*(k|t), t+k)$, $k \in \mathbb{I}_{[0,N-1]}$ and $Y(N|t) = W(x^*(N|t), t+N)$.

Part I: The lower bound in Inequality (4.15a) follows with $\ell, W \geq 0$, and

$$Y_N(x(t), t) \geq \ell(0|t) + W(x(t), t) \overset{(4.13b)}{\geq} \epsilon_{\mathrm{o}} \sigma(0|t) + W(x^*(1|t), t+1) \geq \epsilon_{\mathrm{o}} \sigma(0|t).$$

In case $\sigma(x(t), t) \leq \epsilon$, we directly obtain the bound $Y_N(x(t), t) \leq (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}}) \sigma(x(t), t)$ using Conditions (4.12) and (4.13a). The upper bound in Inequality (4.15a) holds with $\gamma_{\overline{Y}} := \max\left\{\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}}, \frac{\overline{Y}}{\epsilon}\right\}$ using this inequality, $(x(t), t) \in \mathbb{X}_{\overline{Y}}$ and a case distinction whether or not $\sigma(x(t), t) \leq \epsilon$, analogous to Theorem 4.5.

Part II: The detectability condition (Ass. 4.10) implies for any $k_1, k_2 \in \mathbb{I}_{[0,N]}$, $k_2 \geq k_1$:

$$\begin{aligned} Y(k_2|t) - Y(k_1|t) =& W(x^*(k_2|t), t+k_2) - W(x(k_1|t), t+k_1) - \sum_{j=k_1}^{k_2-1} \ell(j|t) \\ \overset{(4.13b)}{\leq}& -\epsilon_{\mathrm{o}} \sum_{j=k_1}^{k_2-1} \sigma(j|t). \end{aligned} \tag{4.16}$$

Denote the smallest element $k \in \mathbb{I}_{[0,N-1]}$, which satisfies $Y(k|t) \leq (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\epsilon$ by k_x, assuming it exists. In the following, we prove $k_x \in \mathbb{I}_{[0,N_0]}$ with

$$N_0 := \left\lceil \frac{\gamma_{\overline{Y}} - (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})}{\epsilon_{\mathrm{o}}} \right\rceil = \left\lceil \max\left\{ \frac{\overline{Y} - (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\epsilon}{\epsilon \cdot \epsilon_0}, 0 \right\} \right\rceil. \tag{4.17}$$

For contradiction, suppose $k_x > N_0$. Then, Inequality (4.16) with $\sigma \geq 0$ implies $Y(k|t) \geq Y(N_0|t) > (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\epsilon$ for all $k \in \mathbb{I}_{[0,N_0]}$. Using the same arguments as in Part I, this implies $\sigma(k|t) > \epsilon$, $k \in \mathbb{I}_{[0,N_0]}$ (cf. proof Thm. 4.5). In this case, Inequality (4.16) with $k_1 = 0$, $k_2 = N_0$ implies

$$Y(N_0|t) < Y(0|t) - N_0\epsilon \cdot \epsilon_0 \leq \overline{Y} - N_0\epsilon \cdot \epsilon_0 \overset{(4.17)}{\leq} (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\epsilon,$$

which contradicts $k_x > N_0$. Thus, we have $k_x \in \mathbb{I}_{[0,N_0]}$. Note that $Y(k_x|t) \leq (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\epsilon$ implies $Y(k_x|t) \leq (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\sigma(k_x|t)$ using the case distinction from Part I with $\overline{Y} = (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})\epsilon$. Hence, it holds

$$Y(k_x|t) \leq (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}}) \min\{\sigma(k_x|t), \epsilon\} \leq (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}}) \min\{\sigma(0|t), \epsilon\}, \tag{4.18}$$

where the second inequality follows from the definition of k_x, i.e., $\sigma(0|t) \leq \epsilon$ implies $k_x = 0$. Considering Inequality (4.16) with $k_1 = k_x$ and $k_2 = N$ implies that there exists a $k' \in \mathbb{I}_{[k_x,N-1]}$ such that

$$\sigma(k'|t) \leq \frac{Y(k_x|t) - Y(N|t)}{\epsilon_{\mathrm{o}}(N - k_x)} \overset{(4.18)}{\leq} \frac{(\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}}) \min\{\sigma(0|t), \epsilon\}}{\epsilon_{\mathrm{o}}(N - N_0)}. \tag{4.19}$$

Define $M := N - N_0$. For $M > M_0 := (\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})/\epsilon_{\mathrm{o}}$, we have $k' \neq 0$ and $\sigma(k'|t) \leq \epsilon$. Thus, Assumption 4.9 ensures that starting at $(x^*(k'|t), t + k')$, there exists a feasible state and input trajectory satisfying the bound (4.12), which implies

$$\begin{aligned} V_N(x(t+1), t+1) + \ell(0|t) &\leq \sum_{k=0}^{k'-1} \ell(k|t) + V_{N-k'+1}(x^*(k'|t), t + k') \qquad (4.20) \\ &\overset{(4.12)}{\leq} V_N(x(t), t) + \gamma_{\mathrm{s}}\sigma(k'|t) \overset{(4.19)}{\leq} V_N(x(t), t) + \frac{\gamma_{\mathrm{s}}(\gamma_{\mathrm{s}} + \gamma_{\mathrm{o}})}{\epsilon_{\mathrm{o}} M} \sigma(0|t). \end{aligned}$$

Combining Conditions (4.20) and (4.13b), the function Y_N satisfies Inequality (4.15b) with

$$\alpha_M := 1 - \frac{\gamma_s(\gamma_s + \gamma_o)}{\epsilon_o^2 M}. \tag{4.21}$$

Furthermore, $\alpha_M \in (0,1]$ for $M > M_1 := \gamma_s(\gamma_s+\gamma_o)/\epsilon_o^2$. All the arguments hold with $N > N_{\overline{Y}} := N_0 + \underline{M}$ with $\underline{M} := \max\{M_0, M_1\} = \frac{\gamma_s+\gamma_o}{\epsilon_o}\max\left\{1, \frac{\gamma_s}{\epsilon_o}\right\}$.
Part III: Inequality (4.15b) ensures that Y_N is non-increasing and thus $(x(t),t) \in \mathbb{X}_{\overline{Y}}$ holds for for all $t \in \mathbb{I}_{\geq 0}$. Analogous to Theorem 4.5, Inequalities (4.15) in combination with Assumption 4.8 ensure uniform uniform asymptotic stability of $e_r = 0$. Combining Conditions (4.15b), (4.20) and the fact that $\epsilon_o(1-\alpha_M) = \frac{\gamma_s(\gamma_s+\gamma_o)}{\epsilon_o M}$, we have

$$\begin{aligned}&\alpha_M(V_N(x(t+1),t+1) - V_N(x(t),t)) + (1-\alpha_M)(Y_N(x(t+1),t+1) - Y_N(x(t),t))\\ \leq&\alpha_M(1-\alpha_M)\epsilon_o\sigma(0|t) - \alpha_M\ell(0|t) - (1-\alpha_M)\alpha_M\epsilon_o\sigma(0|t) = -\alpha_M\ell(0|t).\end{aligned}$$

Using this inequality in a telescopic sum and $Y_N = V_N + W$, we get

$$\alpha_M\mathcal{J}_\infty^{\mathrm{cl}}(x_0) \leq \alpha_M V_N(x_0,0) + (1-\alpha_M)Y_N(x_0,0) = V_N(x_0,0) + (1-\alpha_M)W(x_0,0).$$

The performance bound (4.14) follows with $V_\infty(x,t) \geq V_N(x,t)$ and Inequality (4.13a). ∎

Compared to [119, Thm. 1], Theorem 4.12 considers a less restrictive *local* bound (Ass. 4.9) to show stability. In addition, we provide a performance bound (4.14) similar to the suboptimality estimates usually obtained in MPC without terminal constraints (with ℓ positive definite), compare, e.g., [123, 126, 237] and Theorem 4.5. In particular, the performance bound (4.14) and the definition of α_M (4.21) imply that as $N \to \infty$, we recover $\mathcal{J}_\infty^{\mathrm{cl}}(x_0) = V_\infty(x_0,0)$, similar to standard results with ℓ positive definite [123, 126, 237]. We note that for MPC formulations with terminal ingredients (Sec. 3.1), the consideration of positive semidefinite stage costs ℓ is rather straightforward using the Lyapunov function $Y_N = V_N + W$, compare, e.g., [JK3, Thm. 2], [236, Thm. 2.24].

Improved performance bounds using observability

In the following, we improve the performance bounds in Theorem 4.12 by using an additional *observability* condition for the stage cost ℓ. In particular, the horizon $N_{\overline{Y}}$ in Theorem 4.12 scales with $\gamma_s(\gamma_s+\gamma_o)$, which is comparable to the bound in [123, Variant 1] and [119]. However, if ℓ is positive definite (Ass. 4.3) the derivations in [37, 267], [123, Variant 2] and Theorem 4.5 provide bounds where $N_{\overline{Y}}$ scales with $2\gamma \log \gamma$, which can be significantly less conservative. In the following, we show how we can obtain similar bounds for $N_{\overline{Y}}$, α_M using an additional/stronger finite-time *observability* condition.

Assumption 4.13. *(Observable stage cost ℓ) There exist constants $\nu \in \mathbb{I}_{\geq 1}$ and $c_{\mathrm{o}} > 0$ such that for any trajectory $(x(t), u(t)) \in \mathbb{Z}$, $x(t+1) = f(x(t), u(t))$, $t \in \mathbb{I}_{\geq 0}$, the following bound holds*

$$\sigma(x(t+\nu), t+\nu) \leq c_{\mathrm{o}} \sum_{k=t}^{t+\nu-1} \ell(x(k), u(k), k), \ \forall t \in \mathbb{I}_{\geq 0}. \tag{4.22}$$

The following proposition shows that we can improve the performance bounds in Theorem 4.12 with this observability condition.

Proposition 4.14. *Let Assumptions 4.9, 4.10, and 4.13 hold. Then, the results in Theorem 4.12 hold with $N_{\overline{Y}}$, α_M replaced by*

$$N_{\overline{Y},\nu} := \nu \left(\left\lceil \frac{\max\{\log(\gamma_{\mathrm{s}} c_{\mathrm{o}}), \log(\gamma_{\mathrm{s}}^2 c_{\mathrm{o}}/\epsilon_{\mathrm{o}})\}}{\log(\gamma_{\mathrm{s}} c_{\mathrm{o}}+1) - \log(\gamma_{\mathrm{s}} c_{\mathrm{o}})} \right\rceil + N_{0,\nu} + 1 \right), \tag{4.23a}$$

$$\alpha_{M,\nu} := 1 - \frac{\gamma_{\mathrm{s}}^2 c_{\mathrm{o}}}{\epsilon_{\mathrm{o}}} \left(\frac{\gamma_{\mathrm{s}} c_{\mathrm{o}}}{\gamma_{\mathrm{s}} c_{\mathrm{o}}+1} \right)^{M_\nu}, \tag{4.23b}$$

with $M_\nu := \left\lfloor \frac{N-\nu(N_{0,\nu}+1)}{\nu} \right\rfloor$ and $N_{0,\nu} := \left\lceil \max\left\{ c_{\mathrm{o}} \frac{\overline{Y}-\gamma_{\mathrm{s}}\epsilon}{\epsilon}, 0 \right\} \right\rceil$.

Proof. Abbreviate $\ell(k|t) := \ell(x^*(k|t), u^*(k|t), t+k)$, $V(k|t) := V_{N-k}(x^*(k|t), t+k)$, $k \in \mathbb{I}_{[0,N-1]}$, $\sigma(k|t) = \sigma(x^*(k|t), t+k)$, $k \in \mathbb{I}_{[0,N]}$. The following proof is similar to Theorem 4.5 and uses arguments from [123, Variant 2], [37, 267] over ν-steps by exploiting Assumption 4.13. We define

$$\gamma_{\overline{Y},\mathrm{s}} := \max\left\{ \gamma_{\mathrm{s}}, \frac{\overline{Y}}{\epsilon} \right\}, \quad N_{0,\nu} := \lceil c_{\mathrm{o}}(\gamma_{\overline{Y},s} - \gamma_{\mathrm{s}}) \rceil = \left\lceil \max\left\{ c_{\mathrm{o}} \frac{\overline{Y}-\gamma_{\mathrm{s}}\epsilon}{\epsilon}, 0 \right\} \right\rceil \in \mathbb{I}_{\geq 0}. \tag{4.24}$$

We denote the smallest element $k \in \mathbb{I}_{[0,N-1]}$, which satisfies $V(k|t) \leq \gamma_{\mathrm{s}}\epsilon$ by k_x, assuming it exists. In the following, we prove $k_x \in \mathbb{I}_{[\nu N_{0,\nu}, N-1]}$. For contradiction, assume $k_x > \nu N_{0,\nu}$. Then, analogous to the proof in Theorem 4.5, we have $V(k|t) > \gamma_{\mathrm{s}}\epsilon$, $k \in \mathbb{I}_{[0,\nu N_{0,\nu}]}$, which implies $\sigma(k|t) > \epsilon$ for all $k \in \mathbb{I}_{[0,\nu N_{0,\nu}]}$. Thus, Assumption 4.13

implies

$$\begin{aligned} V(0|t) - V(\nu N_{0,\nu}|t) = & \sum_{k=0}^{\nu N_{0,\nu}-1} \ell(k|t) = \sum_{k=0}^{N_{0,\nu}-1} \sum_{j=0}^{\nu-1} \ell(\nu \cdot k + j|t) \\ & \geq \frac{1}{c_{\mathrm{o}}} \sum_{k=0}^{N_{0,\nu}-1} \sigma(\nu \cdot (k+1)|t) > \frac{\epsilon N_{0,\nu}}{c_{\mathrm{o}}} \overset{(4.24)}{\geq} \overline{Y} - \gamma_{\mathrm{s}}\epsilon. \end{aligned}$$

Since $V(0|t) \leq Y_N(x(t),t) \leq \overline{Y}$ by definition, this contradicts $V(\nu N_{0,\nu}) > \gamma_{\mathrm{s}}\epsilon$ and thus shows $k_x \in \mathbb{I}_{[0,\nu N_{0,\nu}]}$. Furthermore, we have $V(k|t) \leq \gamma_{\mathrm{s}} \min\{\sigma(k|t), \epsilon\}$ for all $k \in \mathbb{I}_{[k_x, N-1]}$, analogous to the proof in Theorem 4.5. In addition, it holds

$$V(\nu N_{0,\nu}|t) \leq V(k_x|t) \leq \gamma_{\mathrm{s}} \min\{\sigma(k_x|t), \epsilon\} \leq \gamma_{\mathrm{s}} \min\{\sigma(0|t), \epsilon\}, \tag{4.25}$$

where the last inequality follows from the definition of k_x, i.e., $\sigma(0|t) \leq \epsilon$ implies $k_x = 0$. Using observability (Ass. 4.13), we obtain for any $p \in \mathbb{I}_{[\nu(N_{0,\nu}+1), N-1]}$:

$$\sum_{k=p}^{N-1} \ell(k|t) = V(p|t) \leq \gamma_{\mathrm{s}}\sigma(p|t) \overset{(4.22)}{\leq} \gamma_{\mathrm{s}} c_{\mathrm{o}} \sum_{k=p-\nu}^{p-1} \ell(k|t). \tag{4.26}$$

Furthermore, we get

$$\begin{aligned} \sum_{k=p-\nu}^{N-1} \ell(k|t) = & \sum_{k=p-\nu}^{p-1} \ell(k|t) + \sum_{k=p}^{N-1} \ell(k|t) \\ \overset{(4.26)}{\geq} & \frac{1}{\gamma_{\mathrm{s}} c_{\mathrm{o}}} \sum_{k=p}^{N-1} \ell(k|t) + \sum_{k=p}^{N-1} \ell(k|t) = \frac{\gamma_{\mathrm{s}} c_{\mathrm{o}} + 1}{\gamma_{\mathrm{s}} c_{\mathrm{o}}} \sum_{k=p}^{N-1} \ell(k|t) \end{aligned} \tag{4.27}$$

for any $p \in \mathbb{I}_{[\nu(N_{0,\nu}+1), N-1]}$. Define $M_\nu := \left\lfloor \frac{N-\nu(N_{0,\nu}+1)}{\nu} \right\rfloor \in \mathbb{I}_{\geq 0}$. Applying Inequality (4.27) recursively (similar to [123, Variant 2], [37, 267]), we obtain

$$\begin{aligned} \frac{1}{c_{\mathrm{o}}} \sigma(N|t) \overset{(4.22)}{\leq} & \sum_{k=N-\nu}^{N-1} \ell(k|t) \leq \sum_{k=\nu(M_\nu + N_{0,\nu})}^{N-1} \ell(k|t) \\ \overset{(4.27)}{\leq} & \left(\frac{\gamma_{\mathrm{s}} c_{\mathrm{o}}}{\gamma_{\mathrm{s}} c_{\mathrm{o}} + 1} \right)^{M_\nu} \sum_{k=\nu N_{0,\nu}}^{N-1} \ell(k|t) \leq \left(\frac{\gamma_{\mathrm{s}} c_{\mathrm{o}}}{\gamma_{\mathrm{s}} c_{\mathrm{o}} + 1} \right)^{M_\nu} V(\nu N_{0,\nu}|t). \end{aligned} \tag{4.28}$$

For $M_\nu \geq M_{\nu,0} := \dfrac{\log(\gamma_{\mathrm{s}} c_{\mathrm{o}})}{\log(\gamma_{\mathrm{s}} c_{\mathrm{o}} + 1) - \log(\gamma_{\mathrm{s}} c_{\mathrm{o}})}$, Inequality (4.28) implies $\sigma(N|t) \leq \epsilon$. We can use Assumption 4.9 to obtain

$$V_N(x(t+1),t+1) - V_N(x(t),t) + \ell(0|t) \overset{(4.12)}{\leq} \gamma_s \sigma(N|t)$$
$$\overset{(4.28)}{\leq} \gamma_s c_o \left(\frac{\gamma_s c_o}{\gamma_s c_o + 1}\right)^{M_\nu} V(\nu N_{0,\nu}|t) \overset{(4.25)}{\leq} \gamma_s^2 c_o \left(\frac{\gamma_s c_o}{\gamma_s c_o + 1}\right)^{M_\nu} \sigma(0|t). \tag{4.29}$$

Analogous to the derivation in Theorem 4.12, combining Inequalities (4.13b) and (4.29), we obtain

$$Y_N(x(t+1),t+1) - Y_N(x(t),t) \leq -\alpha_{M,\nu} \cdot \epsilon_o \cdot \sigma(x(t),t)$$

with $\alpha_{M,\nu}$ according to Equation (4.23b). For $M_\nu > M_{\nu,2} := \dfrac{\log(\gamma_s^2 c_o/\epsilon_o)}{\log(\gamma_s c_o + 1) - \log(\gamma_s c_o)}$, we have $\alpha_{M,\nu} > 0$. Finally, defining $\underline{M}_\nu := \max\{M_{\nu,0}, M_{\nu,1}\}$, the derivations hold for $M > \underline{M}_\nu$ with $N > N_{\overline{Y},\nu} = \nu(\lceil \underline{M}_\nu \rceil + N_{0,\nu} + 1)$, which corresponds to the expression in Equation (4.23a). The remainder of the proof is analogous to Theorem 4.12. ∎

This proposition showed that even for positive semidefinite stage costs ℓ, we can derive performance bounds that scale similarly to [37, 267], [123, Variant 2] and Theorem 4.5, assuming that the observability condition (Ass. 4.13) holds. Note that the suboptimality $1 - \alpha_{M,\nu}$ in Equation (4.23b) decays exponentially in M_ν, compared to $1 - \alpha_M$ in Equation (4.21), which decays with $1/M$. Furthermore, the horizon $N_{\overline{Y},\nu}$ only grows linearly with $\overline{Y}$, similar to Theorem 4.5. A simpler proof with more conservative bounds w.r.t. $\overline{Y}$, directly using the bounds in [37, 267], [123, Variant 2] over ν steps, can be found in [JK19, App. A].

4.1.4 Conditions for cost controllability, detectability and observability using incremental system properties

In the following, we provide intuitive conditions in terms of intrinsic (incremental) system properties, that ensure satisfaction of the local cost controllability condition (Ass. 4.4/4.9) and the detectability/observability condition of the stage cost ℓ (Ass. 4.10/4.13). We first consider the setting in Section 4.1.2 with a quadratic positive definite stage cost ℓ (Ass. 4.16) and provide sufficient conditions for the local cost controllability (Ass. 4.4) using an incremental stabilizability property (Prop. 4.19). We also discuss the connection between Assumption 4.4 and the design of terminal ingredients (Prop. 4.20). Then, we consider the more general setup in Section 4.1.3 for the special case of quadratic stage costs ℓ that only penalize inputs and outputs (Ass. 4.22). In Proposition 4.23, we show that the corresponding local cost controllability (Ass. 4.9) holds if the system is incrementally stabilizable and the output is Lipschitz continuous. Sufficient conditions for the detectability/observability condition of the stage cost ℓ (Ass. 4.10/4.13) are provided in Propositions 4.25/4.27 using incremental detectability (i-IOSS) and finite-step observability (ν-step i-OSS).

Throughout this section, we consider reachable reference trajectories.

Assumption 4.15. *(Reachable reference trajectory) There exists a set* $\mathbb{Z}_{\mathrm{r}} \subseteq \mathrm{int}(\mathbb{Z})$ *such that the reference trajectory satisfies*

$$r(t) \in \mathbb{Z}_{\mathrm{r}}, \quad x_{\mathrm{r}}(t+1) = f(x_{\mathrm{r}}(t), u_{\mathrm{r}}(t)), \quad \forall t \in \mathbb{I}_{\geq 0}.$$

This assumption characterizes the fact that the reference trajectory r is reachable, i.e., follows the dynamics f and lies (strictly) in the constraint set $\mathbb{Z}$, analogous to Assumption 3.1. The case of unreachable reference trajectories cannot be treated with the arguments in Theorem 4.5/4.12 and will be considered in Section 4.3 using arguments from economic MPC.

Positive definite stage cost - local cost controllability

In order to draw on connections to exponential stability and the terminal ingredients in Section 3.1, we consider a quadratic stage cost.

Assumption 4.16. *(Quadratic tracking stage cost) The stage cost* ℓ *is given by*

$$\ell(x,u,t) = \|x - x_{\mathrm{r}}(t)\|_Q^2 + \|u - u_{\mathrm{r}}(t)\|_R^2, \quad t \in \mathbb{I}_{\geq 0}, \tag{4.30}$$

with a positive definite weighting matrix $Q \in \mathbb{R}^{n \times n}$ *and a positive semidefinite matrix* $R \in \mathbb{R}^{m \times m}$.

Note that Assumption 4.16 implies satisfaction of Assumption 4.3 with $\ell_{\min}(x,t) = \|x - x_{\mathrm{r}}(t)\|_Q^2$, $\underline{\alpha}_\ell(c) = c^2 \cdot \lambda_{\min}(Q)$, and $\overline{\alpha}_\ell(c) = c^2 \cdot \lambda_{\max}(Q)$.

Remark 4.17. *(Non-quadratic stage cost) While the theoretical analysis in Theorem 4.5 applies to a rather general class of stage costs, verifying satisfaction of Assumption 4.4 for non-quadratic stage costs and choosing a suitable non-quadratic stage cost can be significantly more difficult. However, e.g., Brockett's nonholonomic integrator or unicycle models cannot be stabilized by the considered MPC formulation (Problem 4.1) if a quadratic sage cost is chosen, compare [212]. For* *such systems, also the design procedures for the terminal ingredients from Section 3.1.3 are not applicable and the terminal cost V_f may need to be non-smooth, compare [42]. Nonetheless, it may still be possible to verify Assumption 4.4 and thus derive closed-loop guarantees by considering a suitable non-quadratic stage cost ℓ and constructing an explicit open-loop input that yields the desired bound, compare [119, Examples 1-2], [284].*

In the following, we show that Assumption 4.4 follows naturally from an exponential stabilizability property. Since we are interested in the stability of an arbitrary (reachable) reference trajectory, we frame this assumption in the context of incremental stability [18, 29, 264, 265], which is related to contractivity and universal stabilizability [178].

Definition 4.18. *(Local uniform incremental exponential stabilizability) A system is said to be locally incrementally uniformly exponentially stabilizable on a set $\tilde{\mathbb{Z}} \subseteq \mathbb{X} \times \mathbb{U}$, if there exist constants $\rho \in [0,1)$, ϵ_0, c_1, $c_2 > 0$ such that for any trajectory $(z(k), v(k)) \in \tilde{\mathbb{Z}}$, $z(k+1) = f(z(k), v(k))$, $k \in \mathbb{I}_{\geq 0}$, for any initial condition $x(0) \in \mathbb{X}$ satisfying $\|x(0) - z(0)\|^2 \leq \epsilon_0$, there exists an input sequence $u(\cdot) \in \mathbb{U}$ such that*

$$\|x(k) - z(k)\| \leq c_1 \rho^k \|x(0) - z(0)\|, \; \|u(k) - v(k)\| \leq c_2 \|x(k) - z(k)\|, \; k \in \mathbb{I}_{\geq 0}, \quad (4.31)$$

with $x(k+1) = f(x(k), u(k))$, $k \in \mathbb{I}_{\geq 0}$.

This definition ensures that any (reachable) reference trajectory is locally exponentially stabilizable with some linearly bounded control input. For brevity, in this thesis, we often simply refer to this system property as *incrementally stabilizable*, since we only consider local, uniform, and exponential bounds. In case this condition holds with $u(\cdot) = v(\cdot)$ ($c_2 = 0$) and $\tilde{\mathbb{Z}} = \mathbb{X} \times \mathbb{U}$, then the system is incrementally stable [18, 29, 67, 265], which is closely related to contractive systems [169] and convergent dynamics [221], compare [264]. These incremental system properties, including their numerical verification, are discussed in detail in Appendix C.

Proposition 4.19. *Let Assumptions 4.15–4.16 hold. Suppose the system is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}_\mathrm{r}$ (Def. 4.18). Then, Assumption 4.4 holds.*

Proof. Without loss of generality, suppose $t = 0$ and $\ell_{\min}(x(0), 0) = \|e_\mathrm{r}(0)\|_Q^2 \leq \epsilon$, with some later specified constant $\epsilon \in (0, \lambda_{\min}(Q) \cdot \epsilon_0]$. Consider $(z(k), v(k)) = (x_\mathrm{r}(k), u_\mathrm{r}(k))$, $k \in \mathbb{I}_{\geq 0}$, which satisfies the conditions in Definition 4.18 due to Assumption 4.15.
Part I: First, we show that the input sequence $u(\cdot|0) = u(\cdot)$ from Definition 4.18 is a feasible solution of Problem 4.1. Note that Inequalities (4.31) ensure that the

corresponding state and input trajectories satisy

$$\begin{aligned}&\|x(k|0)-x_{\mathrm{r}}(k)\|^2+\|u(k|0)-u_{\mathrm{r}}(k)\|^2 \overset{(4.31)}{\leq} c_1^2(1+c_2^2)\rho^{2k}\|e_{\mathrm{r}}(0)\|^2\\ \overset{(4.30)}{\leq}& \frac{c_1^2(1+c_2^2)\rho^{2k}}{\lambda_{\min}(Q)}\ell_{\min}(x(0),0)\leq \frac{c_1^2(1+c_2^2)}{\lambda_{\min}(Q)}\epsilon, \quad k\in\mathbb{I}_{[0,N-1]}. \end{aligned} \tag{4.32}$$

Given that $(x_{\mathrm{r}}(k),u_{\mathrm{r}}(k))\in \mathrm{int}(\mathbb{Z})$, there exists a sufficiently small (uniform) constant $\epsilon\in(0,\lambda_{\min}(Q)\cdot\epsilon_0]$ such that Inequality (4.32) guarantees satisfaction of the constraints (4.2d), i.e., $(x(k|0),u(k|0))\in\mathbb{Z}$, $k\in\mathbb{I}_{[0,N-1]}$.
Part II: Given that the considered input trajectory $u(\cdot|0)$ satisfies the constraints in Problem 4.1, we have $V_N(x(0),0)\leq \mathcal{J}_N(x(\cdot|0),u(\cdot|0),0)$. Analogous to Inequality (4.32), we directly get

$$\begin{aligned}\ell(x(k|0),u(k|0),k)=&\|x(k|0)-x_{\mathrm{r}}(k)\|_Q^2+\|u(k|0)-u_{\mathrm{r}}(k)\|_R^2\\ \overset{(4.31)}{\leq}& \frac{c_1^2(\lambda_{\max}(Q)+\lambda_{\max}(R)c_2^2)}{\lambda_{\min}(Q)}\rho^{2k}\|e_{\mathrm{r}}(0)\|_Q^2.\end{aligned}$$

Using the geometric series $\sum_{k=0}^{N-1}\rho^{2k}=\frac{1-\rho^{2N}}{1-\rho^2}\leq\frac{1}{1-\rho^2}$, the bound (4.4) holds with

$$\gamma:=\frac{C}{1-\rho^2},\quad C:=\frac{c_1^2(\lambda_{\max}(Q)+\lambda_{\max}(R)c_2^2)}{\lambda_{\min}(Q)}.$$ ■

This proposition shows that we can locally bound the value function V_N, if the reference trajectory is reachable and the system is incrementally stabilizable (Def. 4.18). To ensure satisfaction of Assumption 4.4 for a specific reference trajectory r, it suffices if this specific reference trajectory can be exponentially stabilized. Assuming this property for any reachable trajectory on a constraint set $\mathbb{Z}_{\mathrm{r}}$ enables us to provide guarantees for generic classes of reference trajectories. Similar results for setpoint stabilization can be found in [123], [126, Sec. 6.2].

The following proposition clarifies how the terminal ingredients in Section 3.1 are related to the cost controllability condition (Ass. 4.4).

Proposition 4.20. *Let Assumptions 4.15–4.16 hold. Suppose there exist functions V_{f} and k_{f}, and a set $\mathbb{X}_{\mathrm{f}}$ satisfying Assumptions 3.5–3.6. Then, Assumption 4.4 holds with $\gamma=c_{\mathrm{u}}$ and c_{u} according to Inequality* (3.7).

Proof. First, we note that Assumptions 3.5–3.6 ensure that for any $\|e_{\mathrm{r}}(t)\|_Q^2\leq\epsilon^2$, Problem 3.3 is feasible and the value function corresponding to Problem 3.3 is bounded by $c_\mathrm{u}\|e_\mathrm{r}(t)\|_Q^2$. Since any feasible solution of Problem 3.3 is a feasible solution of Problem 4.1, also Problem 4.1 is feasible for all $\|e_\mathrm{r}(t)\|_Q^2 \leq \epsilon^2$. Second, since the terminal cost V_f is non-negative, for the same input trajectory $u(\cdot|t)$, the cost function in Problem 4.1 is never larger than the cost function in Problem 3.3. Thus, the value function in Problem 4.1 is never larger than the value function in Problem 3.3 and hence Inequality (3.7) ensures that Assumption 4.4 holds with $\gamma = c_\mathrm{u}$. ∎

This result is analogous to the setpoint stabilization result in [252]. Combining Proposition 4.20 and Theorem 4.5, we arrive at the following statement: Whenever it is possible to compute a terminal cost for reference tracking (with a quadratic upper bound), e.g., using the method in Section 3.1 or [23, 91], then there exists a sufficiently large prediction horizon N such that the MPC scheme without terminal ingredients (Alg. 4.2) solves the trajectory tracking problem. Thus, the MPC formulation without terminal ingredients can be applied whenever the tracking MPC with terminal ingredients (Sec. 3.1) ensures (exponential) stability. The main difference is the fact that Problem 4.1 does not require an explicit offline design to compute V_f, $\mathbb{X}_\mathrm{f}$, but may require a large prediction horizon N to satisfy the conditions in Theorem 4.5.

Remark 4.21. *(Necessity of condition) We wish to point out that the incremental stabilizability condition (Def. 4.18) is not just a sufficient condition to ensure exponential stability of $e_\mathrm{r} = 0$, but also a necessary condition in a certain sense (compare also the necessary conditions for universal trajectory tracking in [287]). Suppose that the considered MPC scheme guarantees exponential stability for all reference trajectories satisfying Assumption 4.15 with some non-vanishing region of attraction (non-empty interior) and some prediction horizon N. Then, clearly, the control law defining the MPC policy satisfies the conditions in Definition 4.18 (the input bound requires $R \succ 0$). In particular, the properties in Definition 4.18 are essentially equivalent to the properties guaranteed in Theorem 4.5 (neglecting the extended region of attraction and the performance bound).*

Input-output stage cost - detectability/observability

For the case of positive semidefinite stage cost (cf. Sec. 4.1.3), we consider the important special case of quadratic input-output stage cost.

Assumption 4.22. *(Quadratic input-output tracking stage cost) The stage cost ℓ is given by*

$$\ell(x,u,t) = \|h(x,u) - y_\mathrm{r}(t)\|_Q^2 + \|u - u_\mathrm{r}(t)\|_R^2, \quad t \in \mathbb{I}_{\geq 0}, \tag{4.33}$$

with positive definite weighting matrices $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, a Lipschitz continuous output map $h : \mathbb{Z} \to \mathbb{Y} \subseteq \mathbb{R}^p$, and the output reference $y_\mathrm{r}(t) := h(x_\mathrm{r}(t), u_\mathrm{r}(t))$.

We point out that we require R positive definite and hence both y_r and u_r are required for the implementation. The relaxation of this condition to allow for $R = 0$ is one of the main technical challenges addressed in Section 4.2. Lipschitz continuity of h is needed to derive a quadratic bound on the value function using the geometric series (Prop. 4.23) and to show observability of the stage cost ℓ (Prop. 4.27).

The following proposition shows that incremental stabilizability implies the local cost controllability (Ass. 4.9), similar to Proposition 4.19.

Proposition 4.23. *Let Assumptions 4.15 and 4.22 hold. Suppose the system is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}_\mathrm{r}$ (Def. 4.18). Then, Assumptions 4.8–4.9 hold with $\sigma(x(t), t) = \|x - x_\mathrm{r}(t)\|^2$.*

Proof. Assumption 4.8 holds directly with $\underline{\alpha}(c) = \overline{\alpha}(c) = c^2$. Consider w.l.o.g. $t = 0$. Local feasibility of the input sequence $u(\cdot)$ from Definition 4.18 for Problem 4.1 follows analogous to Proposition 4.19 with a suitably adjusted $\epsilon \in (0, \epsilon_0]$, by replacing $\ell_{\min}/\lambda_{\min}(Q)$ with σ. Analogous to Inequality (4.32), h Lipschitz continuous with Lipschitz constant L_h implies

$$\begin{aligned}
&\ell(x(k|0), u(k|0), k) = \|h(x(k|0), u(k|0)) - h(x_\mathrm{r}(k), u_\mathrm{r}(k))\|_Q^2 + \|u(k|0) - u_\mathrm{r}(k)\|_R^2 \\
\leq &L_\mathrm{h}^2 \lambda_{\max}(Q) \|x(k|0) - x_\mathrm{r}(k)\|^2 + (\lambda_{\max}(Q) L_\mathrm{h}^2 + \lambda_{\max}(R)) \|u(k|0) - u_\mathrm{r}(k))\|^2 \\
\overset{(4.31)}{\leq} &(L_\mathrm{h}^2 \lambda_{\max}(Q)(1 + c_2^2) + c_2^2 \lambda_{\max}(R)) c_1^2 \rho^{2k} \|e_\mathrm{r}(0)\|_Q^2, \quad k \in \mathbb{I}_{[0,N-1]}.
\end{aligned}$$

Using the geometric series, the bound (4.12) holds with

$$\gamma_\mathrm{s} := \frac{C_\mathrm{s}}{1 - \rho^2}, \quad C_\mathrm{s} := c_1^2 (L_\mathrm{h}^2 \lambda_{\max}(Q)(1 + c_2^2) + c_2^2 \lambda_{\max}(R)). \qquad \blacksquare$$

Analogous to Proposition 4.19, if there exist terminal ingredients $V_\mathrm{f}, k_\mathrm{f}, \mathbb{X}_\mathrm{f}$ satisfying the standard conditions (Ass. 3.5–3.6) with the input-output stage cost (Ass. 4.22) and $V_\mathrm{f}(x, t) \leq c_\mathrm{u} \sigma(x, t)$, then Assumption 4.9 also holds with $\gamma_\mathrm{s} = c_\mathrm{u}$.

Assumption 4.10 with the input-output stage cost ℓ (Ass. 4.22) requires that for $(u, h(x, u)) \equiv (u_\mathrm{r}, y_\mathrm{r})$, the state x (exponentially) converges to x_r, which corresponds to a detectability condition on the output h. One standard characterization of detectability for nonlinear systems is incremental input-output to state stability (i-IOSS) with a corresponding i-IOSS Lyapunov function [14, 50, 147, 205].

Assumption 4.24. *(exponential i-IOSS) There exists an i-IOSS Lyapunov function* $V_{\mathrm{o}} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geq 0}$ *and constants* $\underline{c}_{\mathrm{o}}, \overline{c}_{\mathrm{o}}, c_{\mathrm{o},1}, c_{\mathrm{o},2} > 0$, $\rho_{\mathrm{o}} \in [0,1)$ *such that for all* $(x,u) \in \mathbb{Z}$, $(z,v) \in \mathbb{Z}$:

$$\underline{c}_{\mathrm{o}}\|x-z\|^2 \leq V_{\mathrm{o}}(x,z) \leq \overline{c}_{\mathrm{o}}\|x-z\|^2, \tag{4.34a}$$

$$V_{\mathrm{o}}(f(x,u),f(z,v)) - \rho_{\mathrm{o}}V_{\mathrm{o}}(x,z) \leq c_{\mathrm{o},1}\|u-v\|^2 + c_{\mathrm{o},2}\|h(x,u)-h(z,v)\|^2. \tag{4.34b}$$

Proposition 4.25. *Let Assumptions 4.15, 4.22, and 4.24 hold. Then, Assumptions 4.8 and 4.10 hold with* $\sigma(x(t),t) = \|x - x_{\mathrm{r}}(t)\|^2$.

Proof. Consider some $t \in \mathbb{I}_{\geq 0}$, $(z,v) = (x_{\mathrm{r}}(t), u_{\mathrm{r}}(t))$, and $W(x,t) = c \cdot V_{\mathrm{o}}(x, x_{\mathrm{r}}(t))$ with $c := \min\left\{\frac{\lambda_{\min}(R)}{c_{\mathrm{o},1}}, \frac{\lambda_{\min}(Q)}{c_{\mathrm{o},2}}\right\} > 0$. The upper bound (4.13a) follows directly from Inequality (4.34a) with $\gamma_{\mathrm{o}} := c \cdot \overline{c}_{\mathrm{o}}$ and $\sigma(x,t) = \|x - x_{\mathrm{r}}(t)\|^2$. Inequality (4.13b) holds with $\epsilon_{\mathrm{o}} := c \cdot (1-\rho_{\mathrm{o}}) \cdot \underline{c}_{\mathrm{o}} > 0$ using

$$\begin{aligned}
&W(f(x,u),t+1) - W(x,t) \\
\overset{(4.34b)}{\leq}\; & c\left(c_{\mathrm{o},1}\|u-u_{\mathrm{r}}(t)\|^2 + c_{\mathrm{o},2}\|h(x,u)-h(x_{\mathrm{r}}(t),u_{\mathrm{r}}(t))\|^2\right) - c(1-\rho_{\mathrm{o}})V_{\mathrm{o}}(x,x_{\mathrm{r}}(t)) \\
\overset{(4.34a)}{\leq}\; & \frac{c \cdot c_{\mathrm{o},1}}{\lambda_{\min}(R)}\|u-u_{\mathrm{r}}(t)\|_R^2 + \frac{c \cdot c_{\mathrm{o},2}}{\lambda_{\min}(Q)}\|h(x,u)-y_{\mathrm{r}}(t)\|_Q^2 - c(1-\rho_{\mathrm{o}})\underline{c}_{\mathrm{o}}\|x-x_{\mathrm{r}}(t)\|^2 \\
\overset{(4.33)}{\leq}\; & -\epsilon_{\mathrm{o}}\sigma(x,t) + \ell(x,u,t).
\end{aligned}$$

∎

A corresponding i-IOSS Lyapunov function V_{o} can be computed using results for *differential detectability* [245] or more generally results from *incremental dissipativity* [270].

Regarding Assumption 4.13, we consider a finite step incremental output to state stability (i-OSS) condition, which is similar to standard observability characterizations used in the literature for optimization-based observer design [236, Def. 4.28], [12, Def. 1].

Assumption 4.26. *(ν-step i-OSS) There exist constants* $\nu \in \mathbb{I}_{\geq 1}$, $c_{\mathrm{obs}} > 0$ *such that for any trajectory satisfying* $x(k+1) = f(x(k),u(k))$, $z(k+1) = f(z(k),u(k))$, $(x(k),u(k)) \in \mathbb{Z}$, $(z(k),u(k)) \in \mathbb{Z}$, $k \in \mathbb{I}_{\geq 0}$, *the following inequality holds*

$$\|x(t)-z(t)\|^2 \leq c_{\mathrm{obs}} \sum_{j=t}^{t+\nu-1} \|h(x(j),u(j)) - h(z(j),u(j))\|^2, \quad \forall t \in \mathbb{I}_{\geq 0}. \tag{4.35}$$

Assumption 4.26 implies that two trajectories subject to the same input u generate the same output y over ν steps only if they had the same initial condition.

Proposition 4.27. *Let Assumptions 4.15, 4.22 and 4.26 hold and suppose that f is Lipschitz continuous. Then, Assumptions 4.8 and 4.13 hold with $\sigma(x(t),t) = \|x - x_{\mathrm{r}}(t)\|^2$.*

Proof. Suppose w.l.o.g. $t = 0$. Consider $\tilde{x}(0) = x(0)$, $z(k) = x_{\mathrm{r}}(k)$, $v(k) = u_{\mathrm{r}}(k)$, and $\tilde{x}(k+1) = f(\tilde{x}(k), v(k))$, $x(k+1) = f(x(k), u(k))$, $k \in \mathbb{I}_{[0,\nu-1]}$. Assumption 4.26 yields

$$\sigma(x(0),0) = \|x(0) - z(0)\|^2 \overset{(4.35)}{\leq} c_{\mathrm{obs}} \sum_{j=0}^{\nu-1} \|h(\tilde{x}(j), v(j)) - y_{\mathrm{r}}(j)\|^2. \tag{4.36}$$

The assumed Lipschitz continuity of f implies

$$\|x(k) - \tilde{x}(k)\|^2 \leq c_1 \sum_{j=0}^{\nu-1} \|u(j) - v(j)\|^2, \quad k \in \mathbb{I}_{[1,\nu]}, \tag{4.37}$$

with some constant $c_1 > 0$. Similarly, Lipschitz continuity of f implies for any $k \in \mathbb{I}_{[1,\nu]}$:

$$\begin{aligned} c_2\sigma(x(k),k) =& c_2\|x(k) - z(k)\|^2 \leq 2c_2\|\tilde{x}(k) - z(k)\|^2 + 2c_2\|x(k) - \tilde{x}(k)\|^2 \\ \overset{(4.37)}{\leq}& 2c_2 L_{\mathrm{f}}^{2k}\|x(0) - z(0)\|^2 + 2c_2c_1 \sum_{j=0}^{\nu-1} \|u(j) - v(j)\|^2 \\ \leq& \sigma(x(0),0) + \sum_{j=0}^{\nu-1} \|u(j) - v(j)\|^2, \end{aligned} \tag{4.38}$$

with the Lipschitz constant $L_{\mathrm{f}} \geq 0$ and $c_2 = \dfrac{1}{2\max\{1, L_{\mathrm{f}}^{2\nu}, c_1\}} > 0$, where we used $\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$ in the first inequality. Using further Lipschitz continuity of h and $\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$, we have

$$\begin{aligned} &\|h(\tilde{x}(j), v(j)) - y_{\mathrm{r}}(j)\|^2 \\ \leq& 2\|h(x(j), u(j)) - y_{\mathrm{r}}(j)\|^2 + 2L_{\mathrm{h}}^2(\|\tilde{x}(j) - x(j)\|^2 + \|u(j) - v(j)\|^2), \quad j \in \mathbb{I}_{[0,\nu-1]}. \end{aligned} \tag{4.39}$$

Combining all these inequalities, we arrive at

$$
\begin{aligned}
c_2\sigma(x(\nu),\nu) &\overset{(4.38)}{\leq} \sigma(x(0),0) + \sum_{j=0}^{\nu-1} \|u(j)-v(j)\|^2 \\
&\overset{(4.36)}{\leq} \sum_{j=0}^{\nu-1} c_{\text{obs}} \|h(\tilde{x}(j),v(j)) - y_{\text{r}}(j)\|^2 + \|u(j)-v(j)\|^2 \\
&\overset{(4.39)}{\leq} \sum_{j=0}^{\nu-1} 2c_{\text{obs}}(\|h(x(j),u(j)) - y_{\text{r}}(j)\|^2 + L_{\text{h}}^2(\|\tilde{x}(j)-x(j)\|^2 + \|u(j)-v(j)\|^2)) \\
&\quad + \|u(j)-v(j)\|^2 \\
&\overset{(4.37)}{\leq} \sum_{j=0}^{\nu-1} 2c_{\text{obs}}\|h(x(j),u(j)) - y_{\text{r}}(j)\|^2 + (1+2c_{\text{obs}}L_{\text{h}}^2(1+\nu c_1))\|u(j)-v(j)\|^2 \\
&\overset{(4.33)}{\leq} \underbrace{\max\left\{\frac{2c_{\text{obs}}}{\lambda_{\min}(Q)}, \frac{1+2c_{\text{obs}}L_{\text{h}}^2(1+\nu c_1)}{\lambda_{\min}(R)}\right\}}_{=:c_{\text{obs}}} \sum_{j=0}^{\nu-1} \ell(x(j),u(j),j). \qquad \blacksquare
\end{aligned}
$$

Assumption 4.13 can be viewed as a final-state observability property (cf. [236, Def. 4.29]) and correspondingly Proposition 4.27 is analogous to [236, Prop. 4.31], which showed that observability in combination with (uniform) continuity implies final-state observability.

Note that in case a multi-step implementation is used (cf. [125]), Assumption 4.13 directly ensures that we have a positive definite stage cost over ν steps, which is in accordance with the classical notion of observability. We point out that in the continuous-time case, the integral of observable input-output stage costs is positive definite, which in turn simplifies the closed-loop analysis.

Remark 4.28. *(Input-output model - extended state) An important class of systems are input-output models, such as the Nonlinear AutoregRessive model with eXogeneous inputs (NARX), with the (typically non-minimal) state $x(t) = (y(t-\nu),\dots,y(t-1),u(t-\nu),\dots,u(t-1)) \in \mathbb{Y}^\nu \times \mathbb{U}^\nu$ compare, e.g., [JK5, JK6, 83, 180, 181, 182, 183]. Such systems naturally satisfy Assumptions 4.10 and 4.13 for quadratic input-output stage costs (Ass. 4.22). To show this, consider w.l.o.g. $r(t)=0$. With $\ell(x,u) = \|y\|^2 + \|u\|^2$ and $\sigma(x) = \|x\|^2$, Inequality* (4.22) *in Assumption 4.13 holds with equality for $c_{\text{o}} = 1$. Furthermore, for such systems Assumption 4.10 holds with*

$$W(x(t)) = \frac{1}{\nu}\sum_{k=1}^{\nu}(\nu+1-k)\cdot(\|y(t-k)\|^2 + \|u(t-k)\|^2) =: \|x(t)\|_{P_{\text{o}}}^2,$$

$\gamma_o = 1$, and $\epsilon_o = 1/\nu$. The proof of this statement is analogous to [JK33, Lemma 4]. In particular, Condition (4.13b) *holds with equality. Note that by using an IOSS Lyapunov function W, the analysis in [181, 182] could also be carried out with the natural Lyapunov function $Y = V_N + W$ instead of considering $\sum_{j=0}^{n} V_N(x(k-j))$ as a Lyapunov function.*

Summary

To summarize the results in this section so far: Suppose the system is incrementally exponentially stabilizable (Def. 4.18) and a positive definite quadratic stage cost ℓ (Ass. 4.16) is used in Problem 4.1. By combining Proposition 4.19 with the analysis in Theorem 4.5, we arrive at the following result: For any reachable reference trajectory r (Ass. 4.15) and any desired region of attraction (characterized with $\overline{V}$), there exists a large enough prediction horizon N such that the reference tracking error e_{r} is exponentially[7] stable for all initial conditions in the region of attraction. An explicit bound $N_{\overline{V}}$ can be computed based on system constants and the desired region of attraction. If the system is incrementally stabilizable on some subset $\mathbb{Z}_{\mathrm{stab}} \subseteq \mathbb{Z}_{\mathrm{r}}$, we can obtain similar guarantees if $r(t) \in \mathbb{Z}_{\mathrm{stab}}$, $t \in \mathbb{I}_{\geq 0}$. Overall, this result allows us to give guarantees on a whole set of reference trajectories, without requiring an analysis or design step for a specific reference trajectory r. Furthermore, if it is possible to compute suitable terminal ingredients, then the conditions hold and an explicit bound can directly be computed (Prop. 4.19). We obtain similar guarantees in the case of quadratic input-output stage costs (Ass. 4.22) given an additional detectability/observability condition (Ass. 4.24/4.26). However, the quantitative bounds on the horizon $N_{\overline{Y}}$ in Theorem 4.12/Proposition 4.14 may be more conservative.

4.1.5 Shorter horizons and improved bounds

The previous results provide insightful connections between the region of attraction $\mathbb{X}_{\overline{V}}, \mathbb{X}_{\overline{Y}}$, the suboptimality index α_M, system constants $\rho, \nu, c_o, \epsilon_o$, and a sufficiently large prediction horizon $N_{\overline{V}}, N_{\overline{Y}}$. However, for some applications, the resulting guarantees may be too conservative and the horizon $N_{\overline{V}}, N_{\overline{Y}}$ may be too large to implement the MPC scheme in real-time due to computational restrictions. Thus, in the following we discuss different methods to achieve the same guarantees with a shorter horizon N. To simplify the subsequent discussion, we focus on the setting with a positive definite stage cost ℓ (Ass. 4.3,4.16, Thm. 4.5). The following improvements are focused on the local

[7]With $\ell_{\min}$ quadratic, the bounds (4.6) ensure uniform exponential stability of $e_{\mathrm{r}} = 0$.

prediction horizon bounds $M > \underline{M}$ (cf. (4.11)), which has also received more attention in literature.

First, using results from literature (cf. [120, 123, 126, 127, 267]), we discuss the effect of multi-step implementations (Rk. 4.29), horizon-dependent bounds γ_N for the (local) cost controllability (Rk. 4.30), and terminal weights (Rk. 4.31). Furthermore, we improve the performance bounds derived in Theorem 4.5 using a linear programming analysis similar to [120, 127] (Rk. 4.32). Then, in Theorem 4.37, we merge and extend results on finite-tail sequences [175], approximate/relaxed terminal costs [238, 239, 267] and implicit terminal constraints [162]. In particular, we modify/simplify Problem 4.1 and derive guarantees without directly resorting to the standard cost controllability condition (Ass. 4.4).

Improved bounds - results from literature

Remark 4.29. *(Multi-step implementation) The performance bounds in Theorem 4.5 can be improved using a multi-step implementation, i.e., applying the first $N_c \in \mathbb{I}_{\geq 1}$ steps (N_c is also called control horizon) of the optimal open-loop input sequence $u^*(\cdot|t)$ to the system and only re-optimizing at time $t + N_c$. Corresponding bounds for the suboptimality estimate α_N can be found in [127, Thm. 5.4], which can be significantly less conservative. In particular, for $N_c = \lfloor N/2 \rfloor$ the lower bounds on the stabilizing horizon N scale linearly with γ. The fact that Problem 4.1 only needs to be solved every N_c steps also reduces the computational demand and thus may allow for a larger prediction horizon N. However, this is also a practical drawback since the lack of frequent re-optimization may deteriorate robustness properties. This drawback can be partially compensated by solving Problem 4.1 in each time step t, but with a time-varying/shrinking prediction horizon $N(t) = N - \mathrm{mod}(t, N_c)$, compare [125]. Alternatively, given a fixed desired suboptimality estimate α_N, we can opportunistically apply a more recent updated solution, whenever the guarantees (relaxed dynamic programming inequality) remain valid, compare [283].*

Remark 4.30. *(Horizon dependent cost controllability bounds) The constant $\gamma > 1$ in Assumption 4.4 can be replaced by horizon dependent constants $\gamma_N > 1$, $N \in \mathbb{I}_{\geq 2}$. Similar to Proposition 4.19, if the system is incrementally stabilizable (Def. 4.18), these constants can be computed as $\gamma_k = \frac{1-\rho^{2k}}{1-\rho^2} \cdot C$, using the geometric series. In the proof of Theorem 4.5,* *Inequality* (4.9)*) can then be replaced by*

$$\ell(N-1|t) \leq \left(\prod_{k=2}^{M-M_0} \frac{\gamma_k - 1}{\gamma_k}\right) V(N_0 + M_0|t) \leq \left(\frac{\gamma_M - 1}{\gamma_M}\right)^{M_0} \left(\prod_{k=2}^{M-M_0} \frac{\gamma_k - 1}{\gamma_k}\right) V(k_x|t),$$

with $N_0 = \lceil \gamma_{\overline{V}} - \gamma_M \rceil$, $M_0 = \dfrac{\log(\gamma_M)}{\log(\gamma_M) - \log(\gamma_M - 1)} + 1$, *and* $\ell(k|t) \leq \epsilon$ *for* $k \geq N_0 + M_0$. *The resulting suboptimality index is then given by*

$$\alpha_M := 1 - (\gamma_M - 1)(\gamma_2 - 1) \left(\frac{\gamma_M - 1}{\gamma_M}\right)^{M_0 - 1} \left(\prod_{k=2}^{M-M_0} \frac{\gamma_k - 1}{\gamma_k}\right).$$

Thus, similar to [126, Prop. 6.19], [123, Extension (a)], [267, Thm. 1], we can directly improve the performance bounds in Theorem 4.5 by using horizon dependent bounds $\gamma_k \leq \gamma$. *Compared to the formula in [126, Prop. 6.19] based on global cost controllability* ($N_0 = 0$)*, this bound requires a larger horizon* $N = N_0 + M$ *and the first terms in the product are* $\gamma_M \geq \gamma_k$, $k \in \mathbb{I}_{[M-M_0+1, M-1]}$.

Remark 4.31. *(Terminal weight) It is possible to ensure stability with a significantly smaller prediction horizon* N *(compared to Theorem 4.5), if the cost function* $\mathcal{J}_N$ *in Problem 4.1 is adjusted by multiplying the stage cost* ℓ *at* $k = N - 1$ *with some weight* $\omega > 1$, *compare [127], [126, Sec. 10.2], [123, Extension (d)]. Similar to Proposition 4.19, if the system is incrementally stabilizable (Def. 4.18), the local cost controllability (Ass. 4.4) holds with* $\gamma_k = \dfrac{1 - \rho^{2(k-1)}}{1 - \rho^2} C + \rho^{2(k-1)} C \cdot \omega$, *using the geometric series. Thus, given some* $\nu \in \mathbb{I}_{\geq 1}$ *with* $C\rho^{2\nu} < 1$, *we can choose* $\omega := \dfrac{1 - \rho^{2\nu}}{(1 - \rho^2)(1 - \rho^{2\nu} C)} \geq 1$ *to ensure* $\omega \geq \gamma_{\nu+1}$. *Then, [127, Thm. 5.1] ensures closed-loop stability for* $N > \nu$ *(assuming Assumption 4.4 holds globally) if the first* $N_c = \nu$ *inputs are applied in a multi-step implementation (cf. Rk. 4.31). Thus, by combining a large terminal weight, horizon dependent bounds, and a multi-step implementation, we can ensure stability with a horizon* $N > \dfrac{\log(C)}{\log(1/\rho^2)} + 1$, *which is significantly shorter than the bound* $\underline{M}$ *in Theorem 4.5 and independent of* γ. *In addition, a large terminal weight may increase the region of attraction* $\mathbb{X}_{\overline{V}}$ *characterized by* $\overline{V}$, *compare [162]. However, a large terminal weight* ω *can also deteriorate closed-loop performance.*

Remark 4.32. *(Linear programming analysis) In case of global cost controllability, better (tight) estimates for the suboptimality index* α_M *have been obtained using the LP analysis in [126, Prop. 6.18], [120, 127]. In particular, the bound* M_0 *based on Inequality* (4.9) *to ensure* $\ell(N-1) \leq \ell(k_x|t)$ *cannot be improved and essentially corresponds to the bound derived* *in [162]. However, for γ large the bound M_1 in the proof of Theorem 4.5 to ensure $\alpha_M > 0$ introduces additional conservatism.*

In the following, we show how an LP analysis can be used to improved the bonds for $\underline{M}$, α_M in Theorem 4.5. In order to provide a more general analysis, we assume that $\ell_{\min}(x(t),t) \leq V_k(x(t),t) \leq \gamma_k \ell_{\min}(x(t),t)$, for all $k \in \mathbb{I}_{\geq 2}$, and all $\ell_{\min}(x(t),t) \leq \epsilon$, with constants $\epsilon > 0$, $\gamma_k \geq 1$, and γ_k non-decreasing in k (cf. Rk. 4.30). We seek the worst-case (in terms of α_M) $\ell(k|t)$, $k \in \mathbb{I}_{[0,N-1]}$, $V_N(x(t+1),t+1)$ that could occur in closed-loop operation, which is represented by the scalar optimization variables $\ell_k \geq 0$, $k \in \mathbb{I}_{[0,N-1]}$, $\mu \geq 0$. Correspondingly, the best estimate α_M satisfying Inequality (4.6b) corresponds to minimum of $(\sum_{k=0}^{N-1} \ell_k - \mu)/\ell_0$, for all feasible ℓ_k, μ. Analogous to Theorem 4.5, we know that $V(k|t) \leq \gamma_M \ell_k$ for $k \geq N_0 := \left\lceil \max\left\{ \frac{\overline{V} - \gamma_M \epsilon}{\epsilon}, 0 \right\} \right\rceil$. Thus, we know that $\sum_{j=k}^{N-1} \ell_j \leq \gamma_M \min\{\ell_k, \epsilon\}$, for all $k \geq N_0$. Furthermore, analogous to Inequality (4.9), this bound ensures

$$\sum_{j=k}^{N-1} \ell_j \leq \left(\frac{\gamma_M - 1}{\gamma_M} \right)^{k-N_0} \underbrace{V(N_0|t)}_{\leq \gamma_M \epsilon} \leq \epsilon, \quad k \in \mathbb{I}_{\geq N_0 + M_0}, \tag{4.40}$$

where the last inequality follows from $k \geq N_0 + M_0$, with $M_0 := \left\lceil \frac{\log(\gamma_M)}{\log(\gamma_M) - \log(\gamma_M - 1)} \right\rceil$, compare Remark 4.30. The bound (4.40) ensures $\ell_k \leq \epsilon$, for $k \geq N_0 + M_0$. Thus, the value function at the next time step can be upper bounded as $\mu \leq \sum_{j=1}^{k-1} \ell_j + \gamma_{N-k+1} \ell_k$, $k \geq N_0 + M_0$ using the local cost controllability (assuming $M_0 \geq 1$ w.l.o.g.). Furthermore, we have $\sum_{j=k}^{N-1} \ell_j \leq \gamma_{N-k} \ell_k$, $k \geq N_0 + M_0$. Given the derived bounds, the best estimate α_M guaranteed to satisfy $V_N(x(t+1),t+1) \leq V_N(x(t),t) - \alpha_M \ell(x(t),u(t),t)$ can be computed based on the following

LP, similar to [120, Lemma 4.6]:

$$\alpha_M := \min_{\ell_k,\, \mu} \sum_{k=0}^{N-1} \ell_k - \mu, \tag{4.41a}$$

$$\text{s.t. } \ell_k \geq 0, \quad k \in \mathbb{I}_{[0,N-1]}, \quad \mu \geq 0, \tag{4.41b}$$

$$\sum_{j=k}^{N-1} \ell_j \leq \gamma_M \ell_k, \quad k \in \mathbb{I}_{[N_0, N_0+M_0-1]}, \tag{4.41c}$$

$$\sum_{j=k}^{N-1} \ell_j \leq \gamma_{N-k} \ell_k, \quad k \in \mathbb{I}_{[N_0+M_0, N-2]}, \tag{4.41d}$$

$$\mu \leq \sum_{j=1}^{k-1} \ell_j + \gamma_{N-k+1} \ell_k, \quad k \in \mathbb{I}_{[N_0+M_0, N-1]} \tag{4.41e}$$

$$\ell_{N_0} \leq \ell_0, \tag{4.41f}$$

$$\ell_0 = 1. \tag{4.41g}$$

The objective (4.41a) *corresponds to* α_M*, since* $\ell_0 = 1$ *(w.l.o.g.), compare Condition* (4.41g)*. Condition* (4.41f) *corresponds to Inequality* (4.8) *with* $k_x = N_0$ *(w.l.o.g.). Compared to the LP in [120, 127], Condition* (4.41c) *considers* γ_M *instead of* γ_{N-k}*, and the bounds* (4.41c)*,* (4.41d)–(4.41e) *are only imposed for* $k \geq N_0$ *and* $k \geq N_0 + M_0$*, respectively. In case the cost controllability is assumed globally, we have* $N_0 = M_0 = 0$ *and obtain the LP in [120, 127] as a special case.*

In the following, we reformulate the LP (4.41) *in order to derive a more compact formula for* α_M *and thus allow for a clearer comparison to the bounds in the literature [120, 127]. The values* ℓ_j*,* $j \in \mathbb{I}_{[0,N_0-1]}$ *are irrelevant and can simply be compensated by redefining* $\tilde{\mu} = \mu - \sum_{k=0}^{N_0-1} \ell_k$*. Using the same arguments as in [127, Prop. 5.2], [126, Sec. 6.8], the constraint* (4.41e) *is active for* $j = N-2$ *(assuming* $N \geq N_0 + M_0$*), i.e.,* $\mu = \sum_{k=1}^{N-2} \ell_k + \gamma_2 \ell_{N-1}$*. In addition, we know that* $\ell_{N_0} = \ell_0 = 1$*. Thus, by considering* $M = N - N_0$ *and* $\tilde{\ell}_k = \ell_{k+N_0}$*,* $\tilde{\mu} =$

$\sum_{k=0}^{M-2} \tilde{\ell}_k - 1 + \gamma_2 \tilde{\ell}_{M-1}$, *we arrive at the following equivalent LP*

$$\alpha_M := \min_{\tilde{\ell}_k} 1 - (\gamma_2 - 1)\tilde{\ell}_{M-1} \tag{4.42a}$$

$$\text{s.t. } \tilde{\ell}_k \geq 0, \quad k \in \mathbb{I}_{[0,M-1]}, \tag{4.42b}$$

$$\sum_{j=k}^{M-1} \tilde{\ell}_j \leq \gamma_M \tilde{\ell}_k, \quad k \in \mathbb{I}_{[0,M_0-1]}, \tag{4.42c}$$

$$\sum_{j=k}^{M-1} \tilde{\ell}_j \leq \gamma_{M-k} \tilde{\ell}_k, \quad k \in \mathbb{I}_{[M_0,M-2]}, \tag{4.42d}$$

$$\sum_{j=k}^{M-2} \tilde{\ell}_j + \gamma_2 \tilde{\ell}_{M-1} \leq \gamma_{M-k+1} \tilde{\ell}_k, \quad k \in \mathbb{I}_{[M_0,M-2]} \tag{4.42e}$$

$$\tilde{\ell}_0 = 1. \tag{4.42f}$$

Note that Inequality (4.42e) *corresponds to Inequality* (4.41e) *using the explicit expression for* $\tilde{\mu}$. *The LP* (4.42) *is equivalent to the LP in [120, 127] with a shorter horizon* $M = N - N_0$ *and with Inequalities* (4.42d)–(4.42e) *only for* $k \geq M_0$. *Suppose that* $\gamma_{k_1+k_2} \leq \gamma_{k_1} + \gamma_{k_2}$ *holds, i.e., the bound is "submultiplicative" (cf. [126]). Then, for* $k \in \mathbb{I}_{[M_0,M-2]}$, *the constraints* (4.42d) *are implied by the constraints* (4.42e) *and can be dropped (cf. [126, Sec. 6.8]). Analogous to [127, Thm. 5.4.], the optimal solution to the LP* (4.42) *satisfies the remaining constraints with equality and can thus be recursively derived, resulting in* $\tilde{\ell}_k^* = \left(\prod_{j=1}^{M-k-2} \frac{\gamma_{2+j}}{\gamma_{2+j}-1}\right) \frac{\gamma_2}{\gamma_{M-k+1}-1} \tilde{\ell}_{M-1}^*$, $k \in \mathbb{I}_{[M_0,M-2]}$. *The solution* $\tilde{\ell}_k$, $k \in \mathbb{I}_{[0,M_0]}$ *can also be defined recursively based on the constraints* (4.42c), *resulting in* $\tilde{\ell}_{M_0-1}^* = \left(\frac{\gamma_M - 1}{\gamma_M}\right)^{M_0-1}$. *Finally, using these equations and Condition* (4.42c) *with* $k = M_0 - 1$, *the resulting suboptimality index is given by*

$$\alpha_M = 1 - \frac{(\gamma_2 - 1)(\gamma_M - 1)\left(\frac{\gamma_M - 1}{\gamma_M}\right)^{M_0-1}}{1 + \sum_{k=M_0}^{M-2} \left(\prod_{j=1}^{M-k-2} \frac{\gamma_{2+j}}{\gamma_{2+j}-1}\right) \frac{\gamma_2}{\gamma_{M-k+1}-1}}.$$

Considering $\gamma_k = \gamma$, *this formula simplifies to*

$$\alpha_M = 1 - \frac{(\gamma - 1)^M}{\gamma^{M_0-1}(\gamma^{M-M_0} - (\gamma - 1)^{M-M_0})}, \tag{4.43}$$

which corresponds to the formula in [127, Thm. 5.4] for $M_0 = 1$ (we assumed $M_0 \geq 1$). The overall resulting lower bound for M to ensure $\alpha_M > 0$ with Equation (4.43) is significantly less conservative than the bound in Theorem 4.5, but is more conservative than the bound in [127, Thm. 5.4] (based on global cost controllability) since we can only impose the constraint (4.42e) for $k \geq M_0$. We point out that alternatively it is possible to directly utilize the bounds in the literature based on global cost controllability (cf. [120, 123, 126, 127, 267]) to compute α_M, $\underline{M}$ by redefining $N_0 := \left\lceil \frac{\gamma_M(\overline{V}-\epsilon)}{\epsilon} \right\rceil$, which, however, may result in a significantly smaller region of attraction $\mathbb{X}_{\overline{V}}$.

Relaxed terminal costs - extended prediction horizon

Most of the literature on MPC without terminal ingredients (cf. [37, 86, 120, 123, 126, 127, 237, 267]) is based on the cost controllability condition (cf. Ass. 4.4) and thus for γ large the resulting guarantees typically become conservative. In the following, we overcome this issue by explicitly using a known continuous control law $\kappa : \mathbb{X} \times \mathbb{I}_{\geq 0} \to \mathbb{U}$ in the MPC formulation.

Given some initial state x at time t, we define the closed-loop state and input response under the feedback κ by the continuous functions $\phi_{\mathrm{x}} : \mathbb{I}_{\geq 0} \times \mathbb{X} \times \mathbb{I}_{\geq 0} \to \mathbb{X}$ and $\phi_{\mathrm{u}} : \mathbb{I}_{\geq 0} \times \mathbb{X} \times \mathbb{I}_{\geq 0} \to \mathbb{U}$:

$$\begin{aligned}
\phi_{\mathrm{x}}(0,x,t) &:= x, \\
\phi_{\mathrm{u}}(k,x,t) &:= \kappa(\phi_{\mathrm{x}}(k,x,t), t+k), \quad k \in \mathbb{I}_{\geq 0}, \\
\phi_{\mathrm{x}}(k+1,x,t) &:= f(\phi_{\mathrm{x}}(k,x,t), \phi_{\mathrm{u}}(k,x,t)), \quad k \in \mathbb{I}_{\geq 0}.
\end{aligned}$$

The following assumptions captures the desired properties under the stabilizing feedback κ.

Assumption 4.33. *(Known locally stabilizing controller) There exist constants $\rho \in [0,1)$, $C \geq 1$, $\epsilon > 0$ such that for all $(x,t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$, satisfying $\ell_{\min}(x,t) \leq \epsilon$, we have*

$$\ell(\phi_{\mathrm{x}}(k,x,t), \phi_{\mathrm{u}}(k,x,t), t+k) \leq C\rho^{2k}\ell_{\min}(x,t), \quad k \in \mathbb{I}_{\geq 0}, \tag{4.44a}$$

$$(\phi_{\mathrm{x}}(k,x,t), \phi_{\mathrm{u}}(k,x,t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{\geq 0}. \tag{4.44b}$$

Analogous to Proposition 4.19, Assumption 4.33 implies satisfaction of the cost controllability (Ass. 4.4) with $\gamma = C/(1-\rho^2)$.

Given these (local) stability properties of the feedback κ, we can construct a local CLF based on a finite-tail sequence. Given a horizon length $M \in \mathbb{I}_{\geq 1}$, we define the finite-tail cost $V_{\mathrm{f},M}:\mathbb{X}\times\mathbb{I}_{\geq 0}\rightarrow\mathbb{R}_{\geq 0}$ as

$$V_{\mathrm{f},M}(x,t):= \tag{4.45}$$
$$\begin{cases}\sum_{k=0}^{M-1}\ell(\phi_{\mathrm{x}}(k,x,t),\phi_{\mathrm{u}}(k,x,t),t+k) & \text{if } (\phi_{\mathrm{x}}(k,x,t),\phi_{\mathrm{u}}(k,x,t))\in\mathbb{Z},\ k\in\mathbb{I}_{[0,M-1]}\\ \infty & \text{otherwise}\end{cases}$$

The following proposition shows that $V_{\mathrm{f},M}$ is a relaxed terminal cost.

Proposition 4.34. *Let Assumptions 4.3 and 4.33 hold. Then, for any $M\in\mathbb{I}_{\geq 1}$, there exist constants $\underline{c}_M$, $\overline{c}_M\geq 1$, $\alpha_M\leq 1$ such that for all $(x,t)\in\mathbb{X}\times\mathbb{I}_{\geq 0}$ satisfying $\ell_{\min}(x,t)\leq\epsilon$, we have*

$$\underline{c}_M\ell_{\min}(x,t)\leq V_{\mathrm{f},M}(x,t)\leq\overline{c}_M\ell_{\min}(x,t), \tag{4.46a}$$
$$V_{\mathrm{f},M}(f(x,\kappa(x,t)),t+1)\leq V_{\mathrm{f},M}(x,t)-\alpha_M\ell(x,\kappa(x,t),t), \tag{4.46b}$$
$$V_{\mathrm{f},\infty}(x,t)\leq V_{\mathrm{f},M}(x,t)+\frac{1-\alpha_M}{1-\rho^2}\ell_{\min}(x,t). \tag{4.46c}$$

Furthermore, there exists a constant $\underline{M}>0$ such that for any $M>\underline{M}$, we have $\alpha_M>0$. In addition, for any $M\in\mathbb{I}_{\geq 1}$, there exists a constant $c_{M,M+1}\geq 0$ such that for all $(x,t)\in\mathbb{X}\times\mathbb{I}_{\geq 0}$ with $V_{\mathrm{f},M}(x)\leq\epsilon$, we have

$$V_{\mathrm{f},M+1}(x,t)-V_{\mathrm{f},M}(x,t)\leq c_{M,M+1}V_{\mathrm{f},M}(x,t). \tag{4.46d}$$

Proof. Denote $\ell_k=\ell(\phi_{\mathrm{x}}(k,x,t),\phi_{\mathrm{u}}(k,x,t),t+k)$, $k\in\mathbb{I}_{\geq 0}$. Inequality (4.46a) follows directly from Inequalities (4.44) and $\ell_{\min}(x,t)\leq\epsilon$ with $\underline{c}_M:=1$, $\overline{c}_M:=C\dfrac{1-\rho^{2M}}{1-\rho^2}$. Furthermore, for $\ell_{\min}(x,t)\leq\epsilon$, satisfaction of Inequality (4.46b) follows with

$$\begin{aligned}&V_{\mathrm{f},M}(f(x,\kappa(x,t)),t+1)-V_{\mathrm{f},M}(x,t)=-\ell_0+\ell_M\\ &\overset{(4.44a)}{\leq}-\ell_0+C\rho^{2M}\ell_{\min}(x,t)\leq-\underbrace{(1-C\rho^{2M})}_{=:\alpha_M}\ell_0.\end{aligned}$$

For $M > \underline{M} := \frac{\log(C)}{\log(1/\rho^2)}$, we have $\alpha_M > 0$. Inequality (4.46c) follows similarly with

$$\lim_{M'\to\infty} V_{\mathrm{f},M'}(x,t) - V_{\mathrm{f},M}(x,t) = \sum_{k=M}^{\infty} \ell_k$$
$$\overset{(4.44a)}{\leq} C\ell_{\min}(x,t) \sum_{k=M}^{\infty} \rho^2 = C\frac{\rho^{2M}}{1-\rho^2}\ell_{\min}(x,t) = \frac{1-\alpha_M}{1-\rho^2}\ell_{\min}(x,t).$$

Note that given $\ell_0 \leq V_{\mathrm{f},M}(x,t)$, Inequality (4.46d) directly follows from Inequality (4.46c) for $c_{M,M+1} = 1-\alpha_M = C\rho^{2M}$. In the following, we use an LP analysis similar to [120, 127] to prove Inequality (4.46d) with $c_{M,M+1} = C\rho^{2M}\frac{1-\rho^2}{1-\rho^{2M}} \leq 1-\alpha_M$. Assumption 4.33 and $\ell_j \leq V_{\mathrm{f},M}(x,t) \leq \epsilon$, $j \in \mathbb{I}_{[0,M-1]}$ ensure

$$\ell_k \leq C\rho^{2(k-j)}\ell_j,\ j \in \mathbb{I}_{[0,M-1]},\ k \in \mathbb{I}_{[j+1,M]}.$$

Furthermore, Condition (4.46d) is equivalent to $\ell_M \leq c_{M,M+1}\sum_{k=0}^{M-1}\ell_k$. We normalize the stage cost using $\tilde{\ell}_k := \ell_k/V_{\mathrm{f},M}(x,t)$, $k \in \mathbb{I}_{[0,M]}$. A valid constant $c_{M,M+1}$ can be computed using the following LP:

$$c_{M,M+1} := \max_{\tilde{\ell}_k} \tilde{\ell}_M \tag{4.47a}$$
$$\text{s.t.} \sum_{k=0}^{M-1} \tilde{\ell}_k = 1, \tag{4.47b}$$
$$\tilde{\ell}_M \leq C\rho^{2(M-k)}\tilde{\ell}_k,\ k \in \mathbb{I}_{[0,M-1]}. \tag{4.47c}$$

Using a standard argument of contradiction, the constraints (4.47c) are all active and thus the analytical solution is given by $c_{M,M+1} := C\rho^{2M}\frac{1-\rho^2}{1-\rho^{2M}} = C^2\rho^{2M}/\bar{c}_M$. ∎

This proposition ensures that for $M > \underline{M}$, $V_{\mathrm{f},M}$ is a local CLF, which is analogous to the result in [175]. Inequality (4.46b) is similar to the standard inequality required for the terminal cost V_f (cf. Ass. 3.5), with an additional relaxation factor $\alpha_M \in (0,1]$. A similar relaxed terminal cost has been considered in [119, Ass. 5] with $\alpha_M = 0$. In [239, Ass. 2, Prop. 2], [267, A3] a relaxed condition similar to (4.46b) is considered, which does not necessarily require $V_{\mathrm{f},M}$ to be a CLF. The continuous-time analogue to the finite-tail cost $V_{\mathrm{f},M}$ would be an integral-based terminal cost, which was suggested in [238], however, without a fixed input $u = \kappa$. Note that $V_\mathrm{f} = V_{\mathrm{f},M}/\alpha_M$ satisfies the conditions in Proposition 3.11 with $\alpha_1 = \epsilon\underline{c}_M/\alpha_M > 0$ and can thus also be used to derive valid terminal ingredients satisfying Assumptions 3.5–3.6.

Given the relaxed terminal cost, we can specify a modified MPC optimization problem. At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t)$, the MPC control law is determined based on the following optimization problem:

Problem 4.35.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_{N,M}(x(\cdot|t), u(\cdot|t), t) \tag{4.48a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.48b}$$

$$x(0|t) = x(t), \tag{4.48c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.48d}$$

where

$$\mathcal{J}_{N,M}(x(\cdot|t), u(\cdot|t), t) := \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t), t+k) + V_{\mathrm{f},M}(x(N|t), t+N). \tag{4.48e}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, and the value function $V_{N,M}(x(t),t) = \mathcal{J}_{N,M}(x^*(\cdot|t), u^*(\cdot|t), t)$. To simplify the theoretical exposition regarding feasibility, we define $V_{N,M}(x(t),t) = \infty$, in case Problem 4.35 does not admit a feasible solution. The following algorithm summarizes the closed-loop operation.

Algorithm 4.36. *(Trajectory tracking MPC Algorithm - finite-tail terminal cost)*
Offline: Specify the constraint set $\mathbb{Z}$, the stage cost ℓ, the prediction horizon $N \in \mathbb{I}_{\geq 1}$, the extended horizon $M \in \mathbb{I}_{\geq 1}$, and the control law κ.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t)$, solve Problem 4.35, and apply the control input $u(t) := u^*(0|t)$.

Problem 4.35 can be seen as a modified version of Problem 3.3, where the terminal cost V_{f} is replaced by an approximate terminal cost $V_{\mathrm{f},M}$ and the terminal constraint is omitted (as, e.g., done in [162]). Alternatively, we can also see Problem 4.35 as a special case of Problem 4.1 by taking into account the formula we used for $V_{\mathrm{f},M}$ in Equation (4.45). In particular, if we consider the prediction horizon $\tilde{N} = N + M$ in Problem 4.1 and constrain the last M inputs to be $u(k|t) = \kappa(x(k|t), t+k)$, $k \in \mathbb{I}_{[N,N+M-1]}$, we end up with Problem 4.35. Note that Problem 4.35 optimizes over an input sequence $u(\cdot|t)$ of length N, but predicts the system response over an extended prediction horizon $N+M$, as also done in [83, 164, 167, 175].

The following theorem establishes the closed-loop properties.

Theorem 4.37. *Let Assumptions 4.3 and 4.33 hold. Then, for any $\overline{V}>0$, $M\in\mathbb{I}_{\geq 1}$, there exists a constant $N_{\overline{V},M}>0$ such that for all $N>N_{\overline{V},M}$ and any initial condition $(x_0,0)\in\mathbb{X}_{\overline{V}}:=\{(x,t)\in\mathbb{X}\times\mathbb{I}_{\geq 0}\mid V_{N,M}(x,t)\leq\overline{V}\}$, the closed-loop system resulting from Algorithm 4.36 satisfies the constraints* (4.1)*, Problem 4.35 is feasible for all $t\in\mathbb{I}_{\geq 0}$, and $e_{\mathrm{r}}=0$ is (uniformly) asymptotically stable. Furthermore, there exist constants $\alpha_{N,M}\in(0,1]$, $\epsilon_{N,M}\in(0,1]$ such that the following performance bound holds for the closed loop:*

$$\mathcal{J}_{\infty}^{\mathrm{cl}}(x_0):=\sum_{k=0}^{\infty}\ell(x(k),u(k),k)\leq\frac{1}{\epsilon_{N,M}}V_{N,M}(x_0,0)\leq\frac{1}{\alpha_{N,M}}V_{\infty}(x_0,0). \tag{4.49}$$

Proof. The proof is split into three parts, analogous to Theorem 4.5. Part I and II show that the value function $V_{N,M}$ satisfies the following bounds at time $t\in\mathbb{I}_{\geq 0}$, assuming $(x(t),t)\in\mathbb{X}_{\overline{V}}$:

$$\ell_{\min}(x(t),t)\leq V_{N,M}(x(t),t)\leq\gamma_{\overline{V}}\ell_{\min}(x,t), \tag{4.50a}$$

$$V_{N,M}(x(t+1),t+1)-V_{N,M}(x(t),t)\leq-\epsilon_{N,M}\ell(x(t),u(t),t), \tag{4.50b}$$

with later specified constants $\gamma_{\overline{V}}\geq\epsilon_{N,M}>0$. Part III establishes that $(x(t),t)\in\mathbb{X}_{\overline{V}}$ holds recursively for all $t\in\mathbb{I}_{\geq 0}$, derives the performance bound (4.49) and establishes uniform asymptotic stability. Abbreviate $\ell(k|t):=\ell(x^*(k|t),u^*(k|t),t+k)$ and $V(k|t):=V_{N-k,M}(x^*(k|t),t+k)$, $k\in\mathbb{I}_{[0,N]}$ with $u^*(N|t)=\kappa(x^*(N|t),t+N)$.

Part I: The lower bound in Inequality (4.50a) follows directly from $\ell\geq 0$, analogous to Theorem 4.5. Note that for $\ell_{\min}(x(t),t)\leq\epsilon$, Assumption 4.33 ensures that $u(k|t)=\kappa(x(k|t),t+k)$, $k\in\mathbb{I}_{[0,N-1]}$ is a feasible solution to Problem 4.35, yielding $V_{N,M}(x(t),t)\leq\gamma\ell_{\min}(x(t),t)$ with $\gamma:=C/(1-\rho^2)$. Define $\gamma_{\overline{V}}:=\max\left\{\gamma,\frac{\overline{V}}{\epsilon}\right\}$. The upper bound in Inequality (4.50a) follows from this local upper bound and $V_{N,M}(x(t),t)\leq\overline{V}$, analogous to Theorem 4.5 .

Part II: In the following, we show that Condition (4.50b) holds with some $\epsilon_{N,M}\in(0,1]$. For any $k'\in\mathbb{I}_{[0,N]}$, the principle of optimality ensures

$$V_{N,M}(x(t),t)=\sum_{k=0}^{N-1}\ell(k|t)+V_{\mathrm{f},M}(x^*(N|t),t+N)=\sum_{k=0}^{k'-1}\ell(k|t)+V(k'|t).$$

Define $N_0 := \lceil \gamma_{\overline{V}} - \gamma \rceil = \left\lceil \max\left\{0, \frac{\overline{V} - \gamma\epsilon}{\epsilon}\right\}\right\rceil \in \mathbb{I}_{\geq 0}$. Analogous to Theorem 4.5, one can show $k_x \in \mathbb{I}_{[0,N_0]}$ and $V(k|t) \leq \gamma\epsilon$ for all $k \in \mathbb{I}_{[k_x,N]}$, where k_x is the smallest element $k \in \mathbb{I}_{[0,N]}$, which satisfies $V(k|t) \leq \gamma\epsilon$. Correspondingly, this implies $V(k|t) \leq \gamma\ell(k|t)$, $k \in \mathbb{I}_{[k_x,N-1]}$ using a case distinction whether $\ell(k|t) \leq \epsilon$. Furthermore, it holds that

$$V(k_x|t) \leq \gamma \min\{\ell(k_x|t), \epsilon\} \leq \gamma \min\{\ell(0|t), \epsilon\}, \tag{4.51}$$

where the second inequality follows from the definition of k_x, i.e., $\ell(0|t) \leq \epsilon$ implies $k_x = 0$. Define $\rho_\gamma^2 := \frac{\gamma - 1}{\gamma} \in [0, 1)$. Given that $V(k|t) \leq \gamma\ell(k|t)$, $k \in \mathbb{I}_{[k_x,N]}$, we can use the bounds in [123, Variant 2], [267] for the remaining horizon of length $N - k_x \geq N - N_0$ to show

$$V_{\mathrm{f},M}(x^*(N|t), t+N) = V(N|t) \leq \rho_\gamma^{2(N-k_x)} V(k_x|t) \overset{(4.51)}{\leq} \rho_\gamma^{2(N-N_0)} \gamma \min\{\ell(0|t), \epsilon\}. \tag{4.52}$$

Note that for $N \geq N_1 := N_0 + \frac{\log(\gamma)}{\log(1/\rho_\gamma^2)}$, we have $\ell(N|t) \leq V(N|t) \leq \epsilon$ and thus we can use the bounds from Proposition 4.34. This yields

$$\begin{aligned}
&V_{N,M}(x(t+1), t+1) \leq \sum_{k=1}^{N-1} \ell(k|t) + V_{1,M}(x^*(N|t), t+N) \\
\leq &\sum_{k=1}^{N-1} \ell(k|t) + V_{\mathrm{f},M+1}(x^*(N|t), t+N) \\
= &V_{N,M}(x(t), t) - \ell(0|t) + V_{\mathrm{f},M+1}(x^*(N|t), t+N) - V_{\mathrm{f},M}(x^*(N|t), t+N) \\
\overset{(4.46d)}{\leq} &V_{N,M}(x(t), t) - \ell(0|t) + c_{M,M+1} V_{\mathrm{f},M}(x^*(N|t), t+N) \\
\overset{(4.52)}{\leq} &V_{N,M}(x(t), t) - \ell(0|t) + c_{M,M+1} \rho_\gamma^{2(N-N_0)} \gamma \ell(0|t) \\
\overset{\text{Prop. 4.34}}{=} &V_{N,M}(x(t), t) - \underbrace{\left(1 - C^2 \rho^{2M} \rho_\gamma^{2(N-N_0)} \frac{\gamma}{\overline{c}_M}\right)}_{=:\epsilon_{N,M}} \ell(0|t).
\end{aligned}$$

For $N > N_2 := N_0 + \frac{\log(\rho^{2M} C^2 \gamma / \overline{c}_M)}{\log(1/\rho_\gamma^2)}$, this ensures $\epsilon_{N,M} > 0$. Thus, given any $M \in \mathbb{I}_{\geq 1}$, for

$$N > N_M := \max\{N_1, N_2\} = N_0 + \frac{\max\left\{\log(\rho^{2M} C^2 \gamma / \bar{c}_M), \log(\gamma)\right\}}{\log(1/\rho_\gamma^2)}$$
$$= N_0 + \frac{\log(\gamma) + \max\{\log(c_{M,M+1}), 0\}}{\log(\gamma) - \log(\gamma - 1)}, \tag{4.53}$$

all the previous bounds hold.

Part III: Condition (4.50b) with $\ell \geq 0$ and $\epsilon_{N,M} > 0$, ensure that $V_{N,M}$ is non-increasing and thus $(x(t), t) \in \mathbb{X}_{\overline{V}}$ holds for all $t \in \mathbb{I}_{\geq 0}$. Hence, the results in Part I and II hold for all $t \in \mathbb{I}_{\geq 0}$. Inequalities (4.50) and Assumption 4.3 ensure uniform asymptotic stability of $e_\mathrm{r} = 0$. Regarding the performance bound (4.49), the first inequality directly follows from Inequality (4.50b). Given an initial condition $(x_0, 0) \in \mathbb{X}_{\overline{V}}$, define an infinite horizon optimal trajectory $x_\infty(k)$, $u_\infty(k)$, $k \in \mathbb{I}_{\geq 0}$ with $V_\infty(x_0, 0) = \sum_{k=0}^{\infty} \ell(x_\infty(k), u_\infty(k), k)$. Note that $V_\infty(x_0, 0) \geq V_{N,M}(x_0, 0)$ would directly imply (4.49) with $\alpha_{N,M} = \epsilon_{N,M}$ and thus we can w.l.o.g. consider $V_\infty(x_0, 0) \leq V_{N,M}(x_0, 0) \leq \overline{V}$. Using Assumption 4.33, $\ell_{\min}(x_\infty(k), k) \leq \epsilon$ implies $V_\infty(x_\infty(k), k) \leq \gamma \ell_{\min}(x_\infty(k), k)$. Thus, we can use the same steps from Part II to show that the infinite-horizon optimal trajectory satisfies

$$V_\infty(x_\infty(N), N) \leq \rho_\gamma^{2(N-N_0)} \min\{\gamma\epsilon, V_\infty(x_0, 0)\}.$$

We have $\ell_{\min}(x_\infty(N), N) \leq V_\infty(x_\infty(N), N) \leq \epsilon$ using $N \geq N_1$ and thus Inequality (4.46a) yields $V_{\mathrm{f},M}(x_\infty(N), N) \leq \bar{c}_M \ell_{\min}(x_\infty(N), N)$. The initial part of the infinite-horizon optiml trajectory is a feasible candidate solution to Problem 4.35, implying

$$V_{N,M}(x_0, 0) \leq V_\infty(x_0, 0) + V_{\mathrm{f},M}(x_\infty(N), N)$$
$$\leq \left(1 + \bar{c}_M \rho_\gamma^{2(N-N_0)}\right) V_\infty(x_0, 0).$$

Inequality (4.49) follows with $\alpha_{N,M} := \dfrac{\epsilon_{N,M}}{1 + \bar{c}_M \rho_\gamma^{2(N-N_0)}} \in (0, \epsilon_{N,M}]$. ∎

Remark 4.38. *(Comparison to state of the art) This result is interesting for multiple reasons. First, it generalizes and unifies the "standard" results for MPC without terminal ingredients (Thm. 4.5, [37, 120, 123, 126, 127, 267], $M = 0$) and MPC with terminal ingredients (Sec. 3.1, [55, 74, 162, 236], $\alpha_M = 1$), by considering an "approximate" terminal cost (cf. Prop 4.34). Compared to the standard MPC arguments with terminal ingredients, weaker conditions are imposed on the terminal cost and no explicit terminal set constraint is used. The price we have to pay for this relaxation is that a horizon $N > N_M$ is required. However, compared to the standard* *bounds in MPC without terminal ingredients (cf. [37, 120, 123, 126, 127, 267], $M = 0$), the bounds derived in Theorem 4.37 can be significantly less conservative, as explained in the following. To simplify the following discussion, suppose that Assumption 4.33 holds globally (as assumed in most of the literature) and approximate $\gamma \approx \bar{c}_M$ (which holds for M large). Then, Theorem 4.37 ensures stability, if $\rho^{2M}\rho_\gamma^{2N}C^2 < 1$. For comparison, the standard bounds (cf. [123, Variant 2],[37, 131, 267]) (essentially) require $\gamma^2\rho_\gamma^{2N} < 1$. Neglecting[8] the difference between C and γ, for $M = 0$ we recover the existing bounds as a special case. More importantly, the provided bound may be significantly less conservative, if ρ is significantly smaller than ρ_γ. Note that for ρ_γ based on Proposition 4.34 it holds:*

$$\gamma = \frac{C}{1-\rho^2}, \quad 1-\rho_\gamma^2 = \frac{1}{\gamma} = \frac{1-\rho^2}{C} \geq 1-\rho^2.$$

Hence, for C large, the bounds based on the cost controllability can become very conservative. In Section 4.5, the different bounds are quantitatively compared with numerical examples. Using the bounds γ_k from Remark 4.30, we can replace $\rho_\gamma^{2(N-N_0)}$ by $\prod_{k=N_0+1}^{N} \frac{\gamma_{k,M}-1}{\gamma_{k,M}}$, with $V_{N,M}(x,t) \leq \gamma_{N,M}\ell_{\min}(x,t)$, $\gamma_{k,M} = C\frac{1-\rho^{2(k+M)}}{1-\rho^2}$, resulting in less conservative bounds. We conjecture that the bounds in Theorem 4.37 can be further improved using an LP analysis, analogous to [123, Variant 3], [120, 127] (cf. Rk. 4.32).

Although the underlying MPC formulation including $V_{\mathrm{f},M}$ is almost equivalent to the MPC formulation in [175], the proof deviates significantly. In particular, in [175, Eq. (13)] it is simply assumed that the following inequality holds for any feasible solution

$$\ell(\phi_{\mathrm{x}}(M,x^*(N|t),t+N),\phi_{\mathrm{u}}(M,x^*(N|t),t),t+N) < \ell(x(t),u^*(0|t),t).$$

In our analysis, on the contrary, we provide explicit verifiable assumptions in terms of lower bounds on the prediction horizon N_M that ensure the desired closed-loop properties.

The difference in the suboptimality estimate $\alpha_{N,M}$ and $\epsilon_{N,M}$ illustrates the different role of the prediction horizon N and the extended prediction horizon M regarding stability (4.50) and performance (4.49), with $\lim_{N+M\to\infty}\epsilon_{N,M} = 1$, $\lim_{N\to\infty}\alpha_{N,M} = 1$. In particular, in case $M = 0$, we have $\alpha_{N,M} = \epsilon_{N,M}$ and the same constant is used for the relaxed dynamic programming inequality (4.50b) and the performance bound (4.49). However, in the considered case we can improve the stability properties by increasing the extended prediction horizon $M \in \mathbb{I}_{\geq 1}$. Although an increased extended horizon M

[8] In case the bounds with γ_k are used (cf. Rk. 4.30), this difference becomes less relevant.

also improves the suboptimality estimate $\alpha_{N,M}$, we can only approach infinite horizon performance ($\alpha_{N,M} = 1$) if the prediction horizon N approaches infinity. This is rather natural, since we did not assume that the local feedback κ is optimal in any sense.

Summary

In this section, we analysed the closed-loop properties of trajectory tracking MPC without terminal ingredients. Given a local cost controllability condition, we provided performance bounds in dependence of the considered region of attraction $\mathbb{X}_V$ and the prediction horizon N (Sec. 4.1.2). In addition, we extended this analysis to positive semidefinite stage costs using a detectability or observability condition (Sec. 4.1.3). Furthermore, we provided significantly less conservative bounds by employing an LP analysis or using an extended prediction horizon (Sec. 4.1.5). Finally, we showed that for reachable reference trajectories and quadratic stage costs, the considered assumptions can be reduced to incremental system properties, such as incremental stabilizability, i-IOSS, and finite-step i-OSS (Sec. 4.1.4). In the next section, we study the more general output regulation problem, where the desired state and input reference trajectory x_r, u_r is not known.

4.2 Output regulation MPC

In Section 4.1, we analysed the closed-loop properties of a trajectory tracking MPC scheme without any terminal ingredients. In this section, we generalize the setup to the *output regulation* problem (Sec. 4.2.1). The proposed output regulation MPC simply minimizes a quadratic output stage cost. Hence, contrary to classical design approaches, the application of the proposed output regulation MPC does *not* require the solution to the regulator (Francis-Byrnes-Isidori, FBI) equations or any other offline design procedure. We show that for minimum-phase systems such a simple design ensures exponential stability of the regulator manifold, if a sufficiently large prediction horizon N is used (Sec. 4.2.2). We also provide a stability proof in case of unstable zero dynamics using an incremental input regularization and an additional nonresonance condition (Sec. 4.2.3). This section is based on and taken in parts literally from [JK19][9].

[9] J. Köhler, M. A. Müller, and F. Allgöwer. "Constrained nonlinear output regulation using Model Predictive Control." In: *IEEE Trans. Automat. Control* (2021). extended version: arXiv:2005.12413©2021 IEEE.

4.2.1 Output regulation problem

In the following, we generalize the reference trajectory tracking problem considered in Section 4.1 to the output regulation problem. Compared to the reference tracking problem (Sec. 4.1) we have the following differences: a) the plant is augmented with an exosystem that generates disturbances and the output reference; b) the optimal mode of operation is the regulator manifold, which is only implicitly characterized by an output y. In particular, the lack of input regularization poses a non-trivial problem and constitutes the key technical challenge addressed in this section. By considering the output regulation problem this section mainly addresses the additional challenge when the *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories* (cf. Sec. 1.1, (iii)).

Related work

Output regulation is one of the fundamental problems in control theory, combining dynamic trajectory tracking, disturbance rejection and output-feedback in a common framework [48, 53, 144, 222], compare also the (robust) servomechanism problem [72]. The classical solution is to solve the regulator/FBI equations [53, 144]. This reduces the problem to the stabilization of a dynamic state and input trajectory, which can, e.g., be studied using the notion of convergent dynamics [222]. Alternatively, the plant can be augmented using the *internal model principle* [109]. This approach directly lends itself to the error feedback case and can also be applied to nonlinear systems using an *immersion property* and an analytical description of the zero dynamics (cf. [143]), compare [47, 185, 228]. Hence, the classical solutions to the nonlinear output regulation problem require a non-trivial offline design procedure (e.g., solving a partial differential equation [144]), which is a bottleneck for practical implementation. In the following, we present an MPC approach that solves the output regulation problem and does *not* require any offline design such as, e.g., solving the regulator equations.

The special case of constant exogenous signals is often studied in MPC under the rubric of offset-free tracking or setpoint tracking. Existing solutions compute the optimal steady-state offline/online [164, 174], use velocity formulations [36, 176] or deploy disturbance observers [201, 213], to reduce the problem to the stabilization of a given steady-state. In case of exogenous signals with a known period length T, the output regulation problem can be solved by computing the optimal T-periodic trajectory offline [89] or online (cf. Sec. 3.2). In [5], output regulation is studied using a local (polynomial) approximation to the regulator equations and the dynamic programming equations, but no closed-loop guarantees are obtained. In summary, the existing MPC approaches reduce the output regulation problem to the stabilization of a given state and input trajectory by explicitly computing the solution to the regulator equations online or offline, and thus reduce the problem to trajectory tracking MPC (cf. Sec. 3.1/4.1). In the proposed MPC formulation, the analysis is based on detectability notions similar to Assumption 4.10 and Theorem 4.12, and hence we require neither a positive definite stage cost nor terminal ingredients. Thus, the implementation does *not* require a computation of the solution to regulator equations.

Setup

We consider the following nonlinear discrete-time system

$$x_{\mathrm{p}}(t+1) = f_{\mathrm{p}}(x_{\mathrm{p}}(t), w(t), u(t)), \quad x_{\mathrm{p}}(0) = x_{\mathrm{p},0}, \tag{4.54a}$$

$$w(t+1) = s(w(t)), \quad w(0) = w_0, \tag{4.54b}$$

$$y(t) = h(x_{\mathrm{p}}(t), w(t), u(t)), \tag{4.54c}$$

with the plant state $x_{\mathrm{p}}(t) \in \mathbb{X}_{\mathrm{p}} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$, the state of the exosystem $w(t) \in \mathbb{W} \subseteq \mathbb{R}^q$, the control input $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, the output $y(t) \in \mathbb{Y} \subseteq \mathbb{R}^p$, the time step $t \in \mathbb{I}_{\geq 0}$, the initial conditions $x_{\mathrm{p},0} \in \mathbb{X}_{\mathrm{p}}$, $w_0 \in \mathbb{W}$, the plant dynamics $f_{\mathrm{p}} : \mathbb{X}_{\mathrm{p}} \times \mathbb{W} \times \mathbb{U} \to \mathbb{X}_{\mathrm{p}}$, the dynamic of the exosystem $s : \mathbb{W} \to \mathbb{W}$ and the output equation $h : \mathbb{X}_{\mathrm{p}} \times \mathbb{W} \times \mathbb{U} \to \mathbb{Y}$. The exogenous signal w affects both the plant dynamics (4.54a) and the output (4.54c), and thus can model deterministic dynamic disturbances and references. Compared to the setup in Section 4.1, the overall system is comprised of a plant and an autonomous exosystem that jointly generate some output y. We impose general point-wise in time constraints of the form

$$(x_{\mathrm{p}}(t), w(t), u(t)) \in \mathbb{Z} \subseteq \mathbb{X}_{\mathrm{p}} \times \mathbb{W} \times \mathbb{U}, \quad t \in \mathbb{I}_{\geq 0}. \tag{4.55}$$

The control goal is to achieve output nulling ($\lim_{t\to\infty} \|y(t)\| = 0$), while satisfying the constraints (4.55). The classical solution to the output regulation problem is to compute functions $\pi_{\mathrm{x}} : \mathbb{W} \to \mathbb{X}_{\mathrm{p}}$, $\pi_{\mathrm{u}} : \mathbb{W} \to \mathbb{U}$, which satisfy

$$\pi_{\mathrm{x}}(s(w)) = f_{\mathrm{p}}(\pi_{\mathrm{x}}(w), w, \pi_{\mathrm{u}}(w)), \quad \forall w \in \mathbb{W}, \tag{4.56a}$$

$$0 = h(\pi_{\mathrm{x}}(w), w, \pi_{\mathrm{u}}(w)), \quad \forall w \in \mathbb{W}. \tag{4.56b}$$

Equations (4.56) are called the discrete-time regulator equations or FBI equations. In [53, Thm. 2], it was shown that the regulator equations are locally solvable, if the zero dynamics of the plant are hyperbolic and the exosystem is neutrally stable.[10] Given a solution to the regulator equations (4.56), output regulation can be reduced to the problem of stabilizing the state and input reference trajectory $(\pi_x(w(t)), \pi_u(w(t)))$, $t \in \mathbb{I}_{\geq 0}$.

Assumption 4.39. *(Regulator equations) The regulator equations* (4.56) *admit a solution* π_x, π_u. *Furthermore, there exists a constant* $\epsilon > 0$ *such that for any* $w \in \mathbb{W}$ *and any* $x_p \in \mathbb{B}_\epsilon(\pi_x(w))$, $u \in \mathbb{B}_\epsilon(\pi_u(w))$, *we have* $(x_p, w, u) \in \mathbb{Z}$.

Uniqueness of both, π_x and π_u, will be ensured based on an additional minimum-phase or nonresonance condition posed latter.

Assumption 4.39 ensures that the nonlinear constrained output regulation problem is locally solvable, analogous to Assumptions 3.1/4.15 in the trajectory tracking case. Correspondingly, if the functions π_x, π_u are known we can directly solve the output regulation problem with the trajectory tracking MPC schemes from Sections 3.1/4.1. Also, in case we know some feedback κ that can (locally) stabilize the trajectory $(\pi_x(w(t)), \pi_u(w(t)))$, the output regulation problem can be solved for initial conditions close to the regulator manifold (cf. [JK19, Prop. 1]). Similarly, in [222], π_u is used as a feedforward input and the dynamics are assumed to be convergent, which is closely related to incrementally stable dynamics (cf. [264]). We point out that such classical solutions only provide a *local* solution to the *constrained* output regulation problem and require knowledge of π_x, π_u, the solution to the regulator equations (4.56). Both of these restrictions will be relaxed in the proposed MPC approach.

Remark 4.40. *(Classical design procedures) Although we view the design using* π_x, π_u *and a stabilizing feedback as the classical solution to the output regulation problem (cf. [48, 53, 144, 222]), especially in the area of error feedback much progress has taken place. In particular, the immersion property, which requires that the dynamic output feedback is able to generate the feedforward input* $\pi_u(w)$*, is highly relevant. In [47, 185, 228], an internal model property is used to augment the model before designing a controller. The construction of this internal model uses a function* τ*, which is constructed using an analytical description of the zero dynamics (cf. [47, Lemma 7.1]) and hence requires an analytical expression of the model in the Byrnes-Isidori normal form (BINF). Connections between the proposed MPC design and classical tools such as*

[10] The zero dynamics is hyperbolic, if the eigenvalues of the Jacobian linearization do not lie on the unit circle. The exosystem is neutrally stable, if the equilibrium $w = 0$ is stable in the sense of Lyapunov and the eigenvalues of its Jacobian linearization lie on the unit circle.

the regulator equations (4.56)*, zero dynamics and the BINF will appear throughout this section. However, one crucial difference will be that these concepts are only used in the analysis, while the implementation requires no complex offline procedures, which is one of the main benefits of the considered MPC framework.*

4.2.2 Output regulation MPC for minimum-phase systems

In the following, we present the proposed output regulation MPC scheme. We show that simply minimizing a quadratic cost on the predicted output y over a sufficiently long prediction horizon N solves the constrained nonlinear output regulation problem, if the system is minimum-phase, i.e., has stable zero dynamics.

We first present the output regulation MPC and introduce preliminaries regarding relative degree and the zero dynamics. As one of the main technical contributions, we show that the minimum-phase property implies a cost detectability (cf. Ass. 4.10) for a look-ahead stage cost $\ell_{\mathrm{y},d}$. Then, we show the desired closed-loop properties by extending the proof of Theorem 4.12.

Output regulation MPC

In the following, we present the output regulation MPC. We denote the overall state by $x(t) := (x_{\mathrm{p}}(t), w(t)) \in \mathbb{X} := \mathbb{X}_{\mathrm{p}} \times \mathbb{W} \subseteq \mathbb{R}^n$ and the overall dynamics by $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ with $f(x,u) := (f_{\mathrm{p}}(x_{\mathrm{p}}, w, u), s(w))$. Given a solution to the regulator equations, we can define the regulator manifold as $\mathcal{A} = \{x = (x_{\mathrm{p}}, w) \in \mathbb{X} \mid x_{\mathrm{p}} = \pi_{\mathrm{x}}(w)\}$. Correspondingly, the output regulation problem is equivalent to the stabilization of the (unknown) set $\mathcal{A}$, where the output y is zero. Note that due to the autonomous exosystem (4.54b), the overall system cannot be stabilizable and thus only stability of a subset can be achieved. In order to drive the system to the regulator manifold, we consider the following stage cost

$$\ell_{\mathrm{y}}(x,u) := \|h(x_{\mathrm{p}}, w, u)\|_Q^2, \tag{4.57}$$

which penalizes the output y with some positive definite matrix $Q = Q^\top \in \mathbb{R}^{p \times p}$.

At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t) = (x_{\mathrm{p}}(t), w(t))$, the MPC control law is determined based on the following optimization problem:

Problem 4.41.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_{N,\mathrm{y}}(x(\cdot|t), u(\cdot|t)) \tag{4.58a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.58b}$$

$$x(0|t) = x(t), \tag{4.58c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.58d}$$

where

$$\mathcal{J}_{N,\mathrm{y}}(x(\cdot|t), u(\cdot|t)) := \sum_{k=0}^{N-1} \ell_{\mathrm{y}}(x(k|t), u(k|t)). \tag{4.58e}$$

For simplicity, we assume that f_{p}, s, and h are continuous and the constraint set $\mathbb{Z}$ is compact. The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state trajectory $x^*(\cdot|t)$, and the value function $V_{N,\mathrm{y}}(x(t)) := \mathcal{J}_{N,\mathrm{y}}(x^*(\cdot|t), u^*(\cdot|t))$. To simplify the theoretical exposition, we define $V_{N,\mathrm{y}}(x(t)) = \infty$, in case Problem 4.41 does not admit a feasible solution. The following algorithm summarizes the closed-loop operation.

Algorithm 4.42. *(Output regulation MPC Algorithm)*
Offline: Specify the constraint set $\mathbb{Z}$, the weighting matrix Q, and the prediction horizon N.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t) = (x_{\mathrm{p}}(t), w(t))$, solve Problem 4.41, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u^*(0|t)) = x^*(1|t), \quad t \in \mathbb{I}_{\geq 0}. \tag{4.59}$$

Remark 4.43. *(Error feedback and robustness) In order to solve Problem 4.41, we need to be able to predict both the plant state x_{p} and the exosystem state w. Thus, we assume that $x = (x_{\mathrm{p}}, w)$ can be measured online and an accurate prediction model is available. The output regulation problem is classically posed without state measurements and solved using a dynamic error feedback, compare [144], [142, Ch. 8] and [47, 228]. In this thesis, we restrict ourselves to the nominal case of exact state measurements, but the proposed output regulation MPC can be naturally extended to the error feedback case using an observer and tools from output-feedback*

MPC [106]. A corresponding theoretical analysis can be found in [JK19, App. B], where we show finite-gain $\mathcal{L}_2$-stability in the presence of noisy output measurements, given some simplifying assumptions (mainly no state constraints).

Compared to the trajectory tracking problem analysed in Theorem 4.12 with the quadratic input-output stage cost (cf. Ass. 4.22), we face two additional challenges in the output regulation setting. First, instead of studying the stability of a time-varying reference trajectory $r(t)$, we study the stability of the regulator manifold $\mathcal{A}$. Second, the stage cost ℓ_{y} does not contain any input regularization.

The extension to set stability is rather straightforward and only requires a modification of Assumption 4.8. In general, the results on MPC without terminal constraints can be directly extended in this direction, compare [119, 155]. On the other hand, the lack of input regularization poses a non-trivial problem and constitutes the main technical challenges addressed in the analysis of the output regulation MPC. In particular, if we would include an input regularization in the output stage cost ℓ_{y} (4.57), then the stability analysis from Theorem 4.12 could be directly applied to the output regulation setting (cf. [JK19, Cor. 1]). However, in order to implement an input regularization, i.e., penalize $\|u - \pi_u(w)\|^2$, we would require knowledge of $\pi_{\mathrm{u}}(w)$ and hence would need to solve the regulator equations (4.56).

Cost controllability

First, we require a bound on the value function (cf. Ass. 4.4/4.9), which will be derived based on a stabilizability condition similar to Definition 4.18.

Definition 4.44. *(Local incremental uniform exponential stabilizability of the plant) The plant is said to be locally incrementally uniformly exponentially stabilizable on a set $\tilde{\mathbb{Z}} \subseteq \mathbb{X} \times \mathbb{U}$, if there exist constants $\rho \in [0,1)$, ϵ_0, c_1, $c_2 > 0$ such that for any trajectory $(z_{\mathrm{p}}(k), w(k), v(k)) \in \tilde{\mathbb{Z}}$, $z_{\mathrm{p}}(k+1) = f_{\mathrm{p}}(z_{\mathrm{p}}(k), w(k), v(k))$, $w(k+1) = s(w(k))$, $k \in \mathbb{I}_{\geq 0}$ and for any initial condition $x_{\mathrm{p}}(0) \in \mathbb{X}_{\mathrm{p}}$ satisfying $\|x_{\mathrm{p}}(0) - z_{\mathrm{p}}(0)\|^2 \leq \epsilon_0$, there exists an input sequence $u(\cdot) \in \mathbb{U}$ such that*

$$\|x_{\mathrm{p}}(k) - z_{\mathrm{p}}(k)\| \leq c_1 \rho^k \|x_{\mathrm{p}}(0) - z_{\mathrm{p}}(0)\|, \ \|u(k) - v(k)\| \leq c_2 \|x_{\mathrm{p}}(k) - z_{\mathrm{p}}(k)\|, \quad (4.60)$$

with $x_{\mathrm{p}}(k+1) = f_{\mathrm{p}}(x_{\mathrm{p}}(k), w(k), u(k))$, $k \in \mathbb{I}_{\geq 0}$.

Note that compared to the trajectory tracking problem in Section 4.1, we only assume stabilizability of the plant dynamics x_{p} and not the overall state x, since the exosystem (4.54b) is autonomous and typically Lyapunov stable. For brevity, in this thesis, we often simply refer to this system property as *incremental stabilizability*, since we only consider local, uniform, and exponential bounds.

Analogous to Propositions 4.19/4.23, this stabilizability property can be used to derive a local bound on the value function, assuming the regulator equations admit a solution (Ass. 4.39).

Proposition 4.45. *Let Assumption 4.39 hold. Suppose the plant is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.44) and that h is Lipschitz continuous. Then, there exist constants $\gamma_{\mathrm{s}}, \epsilon > 0$ such that for all $N \in \mathbb{I}_{\geq 1}$ and all $x \in \mathbb{X}$ satisfying $\sigma(x) := \|x_{\mathrm{p}} - \pi_{\mathrm{x}}(w)\|^2 \leq \epsilon$, Problem 4.41 is feasible and the value function satisfies $V_{N,\mathrm{y}}(x) \leq \gamma_{\mathrm{s}}\sigma(x)$.*

Proof. Considering $(z_{\mathrm{p}}(k), v_{\mathrm{p}}(k)) = (\pi_{\mathrm{x}}(w(k|t)), \pi_{\mathrm{u}}(w(k))) \in \mathrm{int}(\mathbb{Z}), k \in \mathbb{I}_{[0,N-1]}$ (Ass. 4.39), there exists a constant $\epsilon \in (0, \epsilon_0]$ such that $u(\cdot)$ from Definition 4.44 is a feasible solution of Problem 4.41 if $\sigma(x(t)) = \|x_{\mathrm{p}}(t) - \pi_{\mathrm{x}}(w(t))\|^2 = \|x_{\mathrm{p}}(t) - z_{\mathrm{p}}(t)\|^2 \leq \epsilon$ (cf. [JK19, Prop. 1]), analogous to Propositions 4.19/4.23. Similar to Propositions 4.19/4.23, Inequalities (4.60) imply that the corresponding state and input trajectory satisfy

$$\|x_{\mathrm{p}}(k|t) - \pi_{\mathrm{x}}(w(k|t)\|^2 + \|u(k|t) - \pi_{\mathrm{u}}(w(k|t))\|^2 \leq c_1^2(1 + c_2^2)\rho^{2k}\sigma(x(t)), \tag{4.61}$$

with $c_1, c_2 > 0$, $\rho \in [0, 1)$ from Definition 4.44. Lipschitz continuity of h with Lipschitz constant $L_{\mathrm{h}} \geq 0$ implies

$$\begin{aligned} V_{N,\mathrm{y}}(x(t)) &\leq \sum_{k=0}^{N-1} \|h(x_{\mathrm{p}}(k|t), w(k|t), u(k|t))\|_Q^2 \\ &\overset{(4.56\mathrm{b}),(4.61)}{\leq} c_1^2(1 + c_2^2)L_{\mathrm{h}}^2\lambda_{\max}(Q)\sigma(x(t)) \sum_{k=0}^{N-1} \rho^{2k} \leq \underbrace{\frac{c_1^2(1 + c_2^2)L_{\mathrm{h}}^2\lambda_{\max}(Q)}{1 - \rho^2}}_{=:\gamma_{\mathrm{s}}} \sigma(x(t)). \blacksquare \end{aligned}$$

The stabilizability condition (Def. 4.44) could be relaxed to only hold for $(z_{\mathrm{p}}, v) = (\pi_{\mathrm{x}}(w), \pi_{\mathrm{u}}(w))$, which is less restrictive (compare convergent dynamics in [222]). However, the benefit of considering incremental stabilizability is the fact that it can be verified without solving the regulator equations (4.56), which is one of the main motivations of the proposed output regulation MPC.

Relative degree - Byrnes-Isidori normal form

In the following, we consider a single-input-single-output (SISO) system without a direct feed through term, i.e., $m = p = 1$ and $h(x_{\mathrm{p}}, w, u) = h(x)$. We assume that the system has no direct feed through in order to consider the same setup as in the relevant literature, compare [197]. The case of square multi-input-multi-output (MIMO) systems is discussed in Remark 4.52 below. For ease of notation, Assumptions 4.46–4.48 below regarding the relative degree and the zero dynamics will be posed globally.

We consider the case, where the system has a well defined relative degree $d \in \mathbb{I}_{\geq 0}$, which is characterized using the Byrnes-Isidori normal form (BINF), similar to [197, Prop. 2.1].

Assumption 4.46. *(Byrnes-Isidori normal form) There exist a constant $d \in \mathbb{I}_{[0,n_\mathrm{p}-1]}$ and functions $\Phi_k : \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \to \mathbb{R}$, $k \in \mathbb{I}_{[1,d+1]}$, $\Phi_\eta : \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \to \mathbb{R}^{n_\mathrm{p}-d-1}$, $\tilde{\Phi} : \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \to \mathbb{R}^{n_\mathrm{p}}$, $F_{d+1} : \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \times \mathbb{U} \to \mathbb{R}$, and $F_\eta : \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \times \mathbb{U} \to \mathbb{R}^{n_\mathrm{p}-d-1}$ such that the following conditions hold for all $(x_\mathrm{p}, w, u) \in \mathbb{X} \times \mathbb{U}$:*

$$\Phi_1(x_\mathrm{p}, w) := h(x_\mathrm{p}, w), \tag{4.62a}$$

$$\Phi_{k+1}(x_\mathrm{p}, w) := \Phi_k(f_\mathrm{p}(x_\mathrm{p}, w, u), s(w)), \quad k \in \mathbb{I}_{[1,d]}, \tag{4.62b}$$

$$\tilde{\Phi}(\Phi(x_\mathrm{p}, w), w) = x_\mathrm{p}, \tag{4.62c}$$

$$F_{d+1}(\Phi(x_\mathrm{p}, w), w, u) := \Phi_{d+1}(f_\mathrm{p}(x_\mathrm{p}, w, u), s(w)), \tag{4.62d}$$

$$F_\eta(\Phi(x_\mathrm{p}, w), w, u) := \Phi_\eta(f_\mathrm{p}(x_\mathrm{p}, w, u), s(w)), \tag{4.62e}$$

with $\Phi := [\Phi_1, \ldots, \Phi_{d+1}, \Phi_\eta^\top]^\top : \mathbb{X} \to \mathbb{R}^{n_\mathrm{p}}$. Furthermore, the functions Φ, $\tilde{\Phi}$, F_{d+1}, and F_η are Lipschitz continuous.

Given the transformation Φ and its inverse $\tilde{\Phi}$ (cf. (4.62c)), the plant dynamics can be equivalently represented with the state $\zeta = [z_1, \ldots, z_{d+1}, \eta^\top]^\top = \Phi(x_\mathrm{p}, w)$. Conditions (4.62a)–(4.62b) ensure that the states z_k, $k \in \mathbb{I}_{[1,d+1]}$ correspond to a discrete-time integrator chain of the output h and Equations (4.62d)–(4.62e) define the remaining dynamics. The corresponding equivalent dynamics in BINF are given by

$$z_k(t+1) = z_{k+1}(t), \quad k \in \mathbb{I}_{[1,d]}, \tag{4.63a}$$

$$z_{d+1}(t+1) = F_{d+1}(\zeta(t), w(t), u(t)), \tag{4.63b}$$

$$\eta(t+1) = F_\eta(\zeta(t), w(t), u(t)), \tag{4.63c}$$

$$y(t) = z_1(t), \quad t \in \mathbb{I}_{\geq 0}. \tag{4.63d}$$

In the following, we denote $\Phi_\mathrm{z} := [\Phi_1, \ldots, \Phi_{d+1}]^\top$. Lipschitz continuity of $\tilde{\Phi}$ and Φ ensures that stability of the original plant x_p can be equivalently studied based on the transformed state ζ. With the representation (4.63), we directly have $[y(t), \ldots, y(t+$

$d)]^\top = \Phi_\mathrm{z}(x(t))$, i.e., the input $u(t)$ cannot influence the output $y(t+k)$, $k \in \mathbb{I}_{[0,d]}$. If the relative degree is well-defined, then $\frac{\partial F_{d+1}}{\partial u} \neq 0$, i.e., at time $t \in \mathbb{I}_{\geq 0}$ the input $u(t)$ can influence the output $y(t+d+1)$, which will be ensured through Assumption 4.47 below. We point out that in Assumption 4.46 (and in Ass. 4.48 below) we only consider the BINF for the plant state x_p, but not the exosystem state w. In particular, the zero dynamics of x also contain the dynamics in w, which are in general not contractive.

Assumption 4.47. *(Well-defined zero dynamics) There exist a control law* $\tilde{\alpha} : \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \to \mathbb{U}$ *and constants* $c_{\mathrm{h}_1}, c_{\mathrm{h}_2} > 0$ *such that*

$$c_{\mathrm{h}_1}|\Delta u| \leq |F_{d+1}(\zeta, w, \tilde{\alpha}(\zeta, w) + \Delta u)| \leq c_{\mathrm{h}_2}|\Delta u|, \tag{4.64}$$

for all $(\zeta, w, \tilde{\alpha}(\zeta, w) + \Delta u) \in \mathbb{R}^{n_\mathrm{p}} \times \mathbb{W} \times \mathbb{U}$.

Consider the set $L_\mathrm{D} := \{x \in \mathbb{X} \mid \Phi_\mathrm{z}(x) = 0\}$, as in [198]. If $x(t) \in L_\mathrm{D}$, then $y(t+k) = 0$ for $k \in \mathbb{I}_{[0,d]}$. Condition (4.64) ensures that there exists a *unique* feedback law $\alpha(x) := \tilde{\alpha}(\Phi(x), w)$ such that the manifold L_D is positively invariant, which ensures that the system has well-defined zero dynamics, compare [47, Sec. V]. The requirement of a *unique* control law is relevant for well-posedness of the zero dynamics and also the reason we restrict ourselves to SISO (or square MIMO) systems.

Minimum-phase systems and detectability

The following assumption ensures that the system is minimum-phase, i.e., the zero dynamics are asymptotically stable, using an ISS Lyapunov function.

Assumption 4.48. *(Minimum-phase) There exist constants* $\underline{c}_\eta, \overline{c}_\eta > 0$, $\rho_\eta \in [0,1)$, *and an ISS Lyapunov function* $V_\eta : \mathbb{R}^{n_\mathrm{p}-d-1} \times \mathbb{W} \to \mathbb{R}_{\geq 0}$ *such that for all* $(w, z, \eta, u) \in \mathbb{W} \times \mathbb{R}^{d+1} \times \mathbb{R}^{n_\mathrm{p}-d-1} \times \mathbb{U}$, *we have*

$$\underline{c}_\eta \|\eta - \tilde{\eta}_w\|^2 \leq V_\eta(\eta, w) \leq \overline{c}_\eta \|\eta - \tilde{\eta}_w\|^2, \tag{4.65a}$$

$$V_\eta(F_\eta(\zeta, w, u), w^+) \leq \rho_\eta V_\eta(\eta, w) + \|z\|^2 + (u - \tilde{\alpha}(\zeta, w))^2, \tag{4.65b}$$

with $\tilde{\eta}_w = \Phi_\eta(\pi_\mathrm{x}(w), w)$, $w^+ = s(w)$, $\zeta = (z, \eta)$.

Given a system with exosystem state w and consistently zero output ($z \equiv 0$, $u \equiv \tilde{\alpha}$, cf. Ass. 4.46–4.47), Assumption 4.48 implies that the state η exponentially converges to $\tilde{\eta}_w$, which corresponds to the "stationary" value of η for $(x_\mathrm{p}, u) = (\pi_\mathrm{x}(w), \pi_\mathrm{u}(w))$. Furthermore, the dynamics of η with $z = 0, u = \tilde{\alpha}$ are a diffeomorphic copy of the plant dynamics f_p on L_D. Thus, Assumption 4.48 characterizes the stability of the zero dynamics of the plant, i.e., the minimum-phase property. We point out that in [158], the *strong* minimum-phase property has been characterized using the notion of *output-input stability*, which is similar to the considered ISS characterization, but does not require the BINF (Ass. 4.46), compare [158, Example 2]. Furthermore, in [84] the minimum-phase property is characterized using a more general dissipation inequality.

The following proposition shows that the minimum-phase property guarantees the stage cost detectability condition (Ass. 4.10) with a look-ahead stage cost $\ell_{\text{y},d}$ as an extension to Proposition 4.25.

Proposition 4.49. *Let Assumptions 4.39, 4.46, 4.47, and 4.48 hold. Then, there exists a function* $W : \mathbb{X} \to \mathbb{R}_{\geq 0}$ *and constants* $\epsilon_\text{o}, \gamma_\text{o} > 0$ *such that for any* $(x, u) \in \mathbb{Z}$

$$W(x) \leq \gamma_\text{o}\sigma(x), \tag{4.66a}$$

$$W(f(x,u)) - W(x) \leq -\epsilon_\text{o}\sigma(x) + \ell_{\text{y},d}(x,u), \tag{4.66b}$$

with $\ell_{\text{y},d}(x,u) := h(x)^2 + F_{d+1}(\Phi(x), w, u)^2$ *and* $\sigma(x) = \|x_\text{p} - \pi_\text{x}(w)\|^2$.

Proof. Assumptions 4.47–4.48 directly imply

$$\begin{aligned}
&V_\eta(F_\eta(\zeta, w, u), s(w)) - V_\eta(\eta, w) \\
\overset{(4.65)}{\leq} & -(1-\rho_\eta)\cdot \underline{c}_\eta \|\eta - \Phi_\eta(\pi_\text{x}(w), w)\|^2 + (u - \tilde{\alpha}(\zeta, w))^2 + \|z\|^2 \\
\overset{(4.64)}{\leq} & -(1-\rho_\eta)\cdot \underline{c}_\eta \|\eta - \Phi_\eta(\pi_\text{x}(w), w)\|^2 + \frac{F_{d+1}(\zeta, w, u)^2}{c_{\text{h}_1}} + \|z\|^2.
\end{aligned} \tag{4.67}$$

Note that the dynamics in $z = \Phi_\text{z}(x) = [z_1, \ldots, z_{d+1}]^\top \in \mathbb{R}^{d+1}$ (4.63a)–(4.63b) correspond to a finite impulse response (FIR) filter with input F_{d+1} and output y, which is hence detectable. For linear systems, detectability is equivalent to the existence of a quadratic IOSS Lyapunov function (cf. Ass. 4.24), compare [50]. Thus, there exists a positive definite matrix P_z and some constants $\tilde{c}_{\text{o},1}, \tilde{c}_{\text{o},2} > 0$ satisfying

$$\|z(t+1)\|_{P_\text{z}}^2 - \|z(t)\|_{P_\text{z}}^2 \leq -\|z(t)\|^2 + \tilde{c}_{\text{o},1} F_{d+1}(\zeta(t), w(t), u(t))^2 + \tilde{c}_{\text{o},2} y(t)^2, \tag{4.68}$$

for all $t \in \mathbb{I}_{\geq 0}$. The function $\tilde{W}(\zeta, w) := c_1 V_\eta(\eta, w) + c_2 \|z\|_{P_\text{z}}^2$ with $c_2 := \frac{1}{\max\{\tilde{c}_{\text{o},2}, 2\tilde{c}_{\text{o},1}\}} > 0$,

$c_1 := \frac{\min\{c_{\text{h}_1}, c_2\}}{2} > 0$ satisfies

$$\begin{aligned}
&\tilde{W}(\zeta(t+1), w(t+1)) - \tilde{W}(\zeta(t), w(t)) \\
\overset{(4.67)\text{–}(4.68)}{\leq} & - \tilde{\epsilon}_{\text{o}}(\|z(t)\|^2 + \|\eta(t) - \Phi_\eta(\pi_{\text{x}}(w(t)), w(t))\|^2) + F_{d+1}(\zeta(t), w(t), u(t))^2 + y(t)^2 \\
\leq & \ell_{\text{y},d}(x(t), u(t)) - \epsilon_{\text{o}} \underbrace{\|x_{\text{p}}(t) - \pi_{\text{x}}(w(t))\|^2}_{=\sigma(x)},
\end{aligned}$$

with $\tilde{\epsilon}_{\text{o}} := \min\left\{\frac{c_2}{2}, c_1(1-\rho_\eta)\underline{c}_\eta\right\} > 0$ and $\epsilon_{\text{o}} = \tilde{\epsilon}_{\text{o}}/L_{\tilde{\Phi}}^2$, where $L_{\tilde{\Phi}} > 0$ is the Lipschitz constant of $\tilde{\Phi}$ from Assumption 4.46. The function $W(x) := \tilde{W}(\Phi(x_{\text{p}}, w), w)$ also satisfies the upper bound (4.66a) using

$$\begin{aligned}
W(x) =& c_1 V_\eta(\Phi_\eta(x_{\text{p}}, w), w) + c_2 \|\Phi_{\text{z}}(x_{\text{p}}, w)\|_{P_{\text{z}}}^2 \\
\leq & c_1 \bar{c}_\eta \|\Phi_\eta(x_{\text{p}}, w) - \Phi_\eta(\pi_{\text{x}}(w), w)\|^2 + c_2 \lambda_{\max}(P_{\text{z}}) \|\Phi_{\text{z}}(x_{\text{p}}, w) - \Phi_{\text{z}}(\pi_{\text{x}}(w), w)\|^2 \\
\leq & \max\{c_1 \bar{c}_\eta, c_2 \lambda_{\max}(P_{\text{z}})\} L_\Phi^2 \underbrace{\|x_{\text{p}} - \pi_{\text{x}}(w)\|^2}_{=\sigma(x)},
\end{aligned}$$

and thus satisfies Inequalities (4.66). ∎

Due to the well-defined zero dynamics (Ass. 4.47), minimizing F_η in the look-ahead stage cost $\ell_{\text{y},d}$ corresponds to an input regularization with respect to the input $u = \alpha(x)$. Based on this fact, the result in Proposition 4.49 can be intuitively interpreted in the form of a detectability notion. In particular, detectability ensures that for $(u, y) \equiv 0$, the plant state x^{p} is asymptotically stable. The minimum-phase property implies that the state $x_{\text{p}} - \pi_{\text{x}}(w)$ is asymptotically stable on the set L_{D}, which corresponds to the zero dynamics with $y \equiv 0$, $u \equiv \alpha(x)$. Hence, the minimum-phase property is similar to detectability for a shifted input $\tilde{u} = u - \alpha(x)$ and replaces the i-IOSS condition (Ass. 4.24) used in Proposition 4.25 and Theorem 4.12 in the later analysis. We point out that this (implicit) input regularization w.r.t. $u = \alpha(x)$ is different compared to a standard input regularization w.r.t. $\pi_{\text{u}}(w)$, since in general $\alpha(x_{\text{p}}, w) \neq \pi_{\text{u}}(w)$, except for $x_{\text{p}} = \pi_{\text{x}}(w)$.

The look-ahead stage cost satisfies $\ell_{\text{y},d}(x(t), u(t)) = y(t)^2 + y(t+d+1)^2$ and hence it is possible to directly implement an MPC scheme with this look-ahead stage cost $\ell_{\text{y},d}$ without explicitly using the BINF from Assumption 4.46 and guarantee stability analogous to Theorem 4.12. The same stage cost has also been suggested in [5, Eq. (44)] to study infinite horizon optimal regulation and approximations thereof. Even though this stage cost can be implemented, in Theorem 4.50 we show that we obtain the same properties using the output stage cost ℓ_{y} (4.57), albeit with a potentially larger prediction horizon N. The benefit of directly using the output stage cost ℓ_{y} are a more intuitive MPC formulation and the fact that the relative degree d is not used in the implementation, which is particularly important for the MIMO case (cf. Rk. 4.52).

Closed-loop stability

In the following, we show that, given the minimum-phase property (Ass. 4.48), the proposed output regulation MPC (Problem 4.41) ensures stability of the regulator manifold.

Theorem 4.50. *Let Assumptions 4.39, 4.46, 4.47, and 4.48 hold. Suppose the plant is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.44). Assume further that π_{x} (Ass. 4.39) and h are Lipschitz continuous. Then, for any constant $\overline{Y} > 0$, there exists a constant $N_{\overline{Y}} > 0$ such that for all $N > N_{\overline{Y}}$ and any initial condition $x_0 \in \mathbb{X}_{\overline{Y}} := \{x \in \mathbb{X} \mid 2V_{N,\mathrm{y}}(x) - \|\Phi_{\mathrm{z}}(x)\|^2 + W(x) \leq \overline{Y}\}$, the closed-loop system* (4.59) *resulting from Algorithm 4.42 satisfies the constraints* (4.55)*, Problem 4.41 is feasible for all $t \in \mathbb{I}_{\geq 0}$, and the regulator manifold $\mathcal{A}$ is exponentially stable.*

Proof. The proof is structured as follows: We first show that the minimizer for the cost function $\mathcal{J}_{N,\mathrm{y}}$ in Problem 4.41 coincides with the minimizer for a finite-horizon cost based on the look-ahead stage cost $\ell_{\mathrm{y},d}$ with an additional positive semidefinite terminal cost. Then, we exploit the fact that $\ell_{\mathrm{y},d}$ satisfies the detectability condition (Prop. 4.49) and extend the proof of Theorem 4.12. Finally, we establish exponential stability of the regulator manifold by providing corresponding lower and upper bounds for σ.
Part I: The output stage cost $\ell_{\mathrm{y}}(x)$ and the look-ahead stage cost$\ell_{\mathrm{y},d}(x,u)$ are such that for any trajectory satisfying the dynamics (4.54), we have $\ell_{\mathrm{y},d}(x(t),u(t)) = \ell_{\mathrm{y}}(x(t)) + \ell_{\mathrm{y}}(x(t+d+1))$, $t \in \mathbb{I}_{\geq 0}$, compare the BINF (4.63). Define the open-loop cost by $\mathcal{J}_{\tilde{N},\mathrm{y},d}(x(\cdot|t),u(\cdot|t)) := \sum_{k=0}^{\tilde{N}-1} \ell_{\mathrm{y},d}(x(k|t),u(k|t))$, $\tilde{N} \in \mathbb{I}_{\geq 1}$. For any $N > d+1$, we have

$$2\mathcal{J}_{N,\mathrm{y}}(x(\cdot|t),u(\cdot|t)) = \|\Phi_{\mathrm{z}}(x(t))\|^2 + \mathcal{J}_{N-d-1,\mathrm{y},d}(x(\cdot|t),u(\cdot|t)) + \|\Phi_{\mathrm{z}}(x(N-d-1|t))\|^2,$$

where we use the fact that $\mathcal{J}_{d+1,\mathrm{y}}(x(\cdot|t),u(\cdot|t)) = \|\Phi_{\mathrm{z}}(x(t))\|^2$, compare the BINF (4.63). Hence, minimizing the cost $\mathcal{J}_{N,\mathrm{y}}$ yields the same minimizers as minimizing the look-ahead stage cost $\ell_{\mathrm{y},d}$ over a shorter prediction horizon $N_d := N-d-1$ and adding a positive semidefinite terminal cost.

Part II: In the following, we analyse the closed loop using the shifted value function

$$\begin{aligned}\tilde{V}_{N,\mathrm{y}}(x(t)) :=& 2V_{N,\mathrm{y}}(x(t)) - \|\Phi_{\mathrm{z}}(x(t))\|^2 \\ =& \mathcal{J}_{N_d,\mathrm{y},d}(x^*(\cdot|t), u^*(\cdot|t)) + \|\Phi_{\mathrm{z}}(x^*(N_d|t))\|^2.\end{aligned}$$

Proposition 4.45 ensures that $\tilde{V}_{N,\mathrm{y}}(x) \leq 2V_{N,\mathrm{y}}(x) \leq \tilde{\gamma}_{\mathrm{s}}\sigma(x)$, $\tilde{\gamma}_{\mathrm{s}} := 2\gamma_{\mathrm{s}}$, for $\sigma(x) \leq \epsilon$. Furthermore, due to Proposition 4.49, we have $W(f(x,u)) - W(x) \leq -\epsilon_o\sigma(x) + \ell_{\mathrm{y},d}(x,u)$. Consider the Lyapunov candidate function $Y_{N,\mathrm{y}}(x) := W(x) + \tilde{V}_{N,\mathrm{y}}(x)$, which implies $\mathbb{X}_{\overline{Y}} = \{x \in \mathbb{X} \mid Y_{N,\mathrm{y}}(x) \leq \overline{Y}\}$. Analogous to the proof in Theorem 4.12, we have

$$\epsilon_{\mathrm{o}}\sigma(x) \leq Y_{N,\mathrm{y}}(x) \leq \tilde{\gamma}_{\overline{Y}}\sigma(x), \quad \tilde{\gamma}_{\overline{Y}} := \max\{\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}}, \overline{Y}/\epsilon\}. \tag{4.69}$$

Abbreviate $\sigma(k|t) = \sigma(x^*(k|t))$, $Y(k|t) = Y_{N-k,\mathrm{y}}(x^*(k|t))$, $k \in \mathbb{I}_{[0,N_d-1]}$ and $Y(N_d|t) = \Phi_{\mathrm{z}}(x^*(N_d|t)) + W(x^*(N_d|t))$. The following steps are analogous to Theorem 4.12. Proposition 4.49 implies that for any $k_1, k_2 \in \mathbb{I}_{[0,N_d]}$, $k_2 \geq k_1$:

$$Y(k_2|t) - Y(k_1|t) \leq -\epsilon_{\mathrm{o}} \sum_{j=k_1}^{k_2-1} \sigma(j|t). \tag{4.70}$$

There exists a point $k_x \in \mathbb{I}_{[0,N_0]}$ with $N_0 := \left\lceil \frac{\gamma_{\overline{Y}} - (\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})}{\epsilon_{\mathrm{o}}} \right\rceil$ such that $Y(k_x|t) \leq (\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})\min\{\epsilon, \sigma(0|t)\}$. Inequality (4.70) with $k_1 = k_x$ and $k_2 = N_d$ implies that there exists a $k' \in \mathbb{I}_{[k_x, N_d-1]}$ such that

$$\sigma(k'|t) \leq \frac{Y(k_x|t) - Y(N_d|t)}{\epsilon_{\mathrm{o}}(N_d - k_x)} \leq \frac{(\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})\min\{\sigma(0|t), \epsilon\}}{\epsilon_{\mathrm{o}}(N_d - N_0)}.$$

Define $M_d := N_d - N_0$. For $M_d > M_1 := (\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})/\epsilon_{\mathrm{o}}$, we have $k' \neq 0$ and $\sigma(k'|t) \leq \epsilon$. Analogous to the derivation of Inequality (4.20), we can use Proposition 4.45 to arrive at

$$\tilde{V}_{N,\mathrm{y}}(x(t+1)) + \ell_{\mathrm{y},d}(x(t), u(t)) \leq \tilde{V}_{N,\mathrm{y}}(x(t)) + \frac{\tilde{\gamma}_{\mathrm{s}}(\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})}{\epsilon_{\mathrm{o}} M_d}\sigma(x(t)).$$

Combining this inequality with Inequality (4.66b), we have

$$Y_{N,\mathrm{y}}(x(t+1)) - Y_{N,\mathrm{y}}(x(t)) \leq -\alpha_M \epsilon_{\mathrm{o}} \cdot \sigma(x(t)), \quad \alpha_M := 1 - \frac{\tilde{\gamma}_{\mathrm{s}}(\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})}{\epsilon_{\mathrm{o}} M_d}, \tag{4.71}$$

with $\alpha_M \in (0, 1]$ for $M > M_2 := \tilde{\gamma}_{\mathrm{s}}(\tilde{\gamma}_{\mathrm{s}} + \gamma_{\mathrm{o}})/\epsilon_{\mathrm{o}}^2$. All the arguments hold with $N >$

$N_{\overline{Y}} := N_0 + d + 1 + \underline{M}$ with $\underline{M} := \max\{M_1, M_2\}$ and $x(t) \in \mathbb{X}_{\overline{Y}}$ holds recursively for all $t \in \mathbb{R}_{\geq 0}$.

Part III: Define the point-to-set distance $\|x\|_{\mathcal{A}} := \inf_{s\in\mathcal{A}} \|x - s\|$. In the following, we show that there exists a constant $c_\pi > 0$ such that $c_\pi \sigma(x) \leq \|x\|_{\mathcal{A}}^2 \leq \sigma(x)$, which in combination with Inequalities (4.69) and (4.71) ensures exponential stability of $\mathcal{A}$ using standard Lyapunov arguments. For given $(x_\mathrm{p}, w) \in \mathbb{X}$, denote some minimizer by $\tilde{w} := \arg\min_{\tilde{w}\in\mathbb{W}} \|(\pi_\mathrm{x}(\tilde{w}), \tilde{w}) - (x_\mathrm{p}, w)\|$. Given the assumed Lipschitz continuity of π_x with some Lipschitz constant $L_\pi \geq 0$, we have

$$\begin{aligned}\sigma(x) =&\|x_\mathrm{p} - \pi_\mathrm{x}(w)\|^2\\ \leq&2(\|x_\mathrm{p} - \pi_\mathrm{x}(\tilde{w})\|^2 + \|\pi_\mathrm{x}(\tilde{w}) - \pi_\mathrm{x}(w)\|^2)\\ \leq&2\max\{L_\pi^2, 1\}\|(x_\mathrm{p}, w) - (\pi_\mathrm{x}(\tilde{w}), \tilde{w})\|^2 =: 1/c_\pi \|x\|_{\mathcal{A}}^2,\end{aligned}$$

where the first inequality uses $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$ for any $a, b \in \mathbb{R}^{n_\mathrm{p}}$. Furthermore,

$$\|x\|_{\mathcal{A}} = \|(\pi_\mathrm{x}(\tilde{w}), \tilde{w}) - (x_\mathrm{p}, w)\| \leq \|x_\mathrm{p} - \pi_\mathrm{x}(w)\| = \sqrt{\sigma(x)},$$

which finishes the proof. ∎

Discussion

This result implies that the proposed output regulation MPC scheme (Problem 4.41) solves the nonlinear constrained regulation problem if:

(a) The regulator problem is (strictly) feasible (Ass. 4.39),

(b) The plant is incrementally stabilizable (Def. 4.44) and minimum-phase (Ass. 4.46–4.48),

(c) A sufficiently large prediction horizon $N > N_{\overline{Y}}$ is used.

Condition (a) ensures that the output regulation is well posed, Condition (b) restricts the applicability to certain system classes and Condition (c) can always be ensured at the expense of additional computational complexity. We emphasize that in order to apply the proposed MPC scheme, we do not need to solve the regulator equations (4.56). This is only possible, since we do not use a positive definite stage cost ℓ or terminal ingredients, both of which would drastically simplify the theoretical analysis but would necessitate knowledge of $\pi_{\mathrm{x}}(w)$. Thus, compared to classical solutions (cf. [53, 144, 222]) the proposed MPC scheme has the following advantages:

- Explicit solution to the regulator equations (4.56) is not required,
- No explicit stabilizing controller κ is needed for the implementation,
- The MPC scheme enjoys a larger region of attraction.

Compared to the MPC schemes in [89], Section 3.2, and [164, 176], we do not pose any periodicity conditions on w or restrict ourselves to constant values w. The restriction to minimum-phase systems will be relaxed in Section 4.2.3 using an incremental input regularization.

Remark 4.51. *(Instability for non-minimum-phase systems) The output regulation MPC based on Problem 4.41 is in general not stabilizing for non-minimum-phase systems. Consider the following academic linear system $x^+ = 0.5x + u$, $y = x - u$ with no constraints, i.e., $\mathbb{Z} = \mathbb{X} \times \mathbb{U} = \mathbb{R}^2$. This system is stable and has a direct feed through. The solution of Problem 4.41 with the output stage cost $\ell_{\mathrm{y}} = y^2 = (x-u)^2$ satisfies $u^*(0|t) = x(t)$, $x(t+1) = 1.5x(t)$, $y(t) = 0$ and $V_{N,\mathrm{y}}(x(t)) = 0$ for any horizon $N \in \mathbb{I}_{\geq 1}$ and all $t \in \mathbb{I}_{\geq 0}$. Thus, the MPC scheme minimizes the output $\|y\|$, but the resulting state and input trajectory is unstable. This problem is inherently related to the singular cost ℓ_{y} and the existence of unstable zero dynamics. A similar phenomenon appears in high-gain controllers which force the output y to zero in a short time and thus often fail to stabilize systems with unstable zero dynamics (non-minimum-phase), compare [73, Sec. 3.4]. If the system has an arbitrarily small (non-zero) initial condition x_0 and is subject to compact input constraints $u(t) \in \mathbb{U}$, the closed-loop output satisfies $y(t) = 0$, $t \in \mathbb{I}_{[0,K]}$ with some finite constant K. Then, once the state is sufficiently large, the input $u = x$ does not satisfy the input constraint and thus the output y becomes non-zero. This demonstrates that for general non-minimum-phase systems, an input regularization as used in Section 4.1 is vital. Thus, in the next section an incremental input regularization will be used to ensure stability for non-minimum-phase systems.*

Remark 4.52. *(MIMO systems) The results in this section can be naturally extended to square ($m = p$) MIMO systems with $\ell_{\mathrm{y}}(x) := \|y\|_Q^2 := \sum_{i=1}^{p} y_i^2 q_i$, $q_i > 0$, albeit with a more involved notation. In this case the BINF (4.63) (cf. Ass. 4.46) contains integrator states $z_{i,k}$, $i \in \mathbb{I}_{[1,p]}$, $k \in \mathbb{I}_{[1,d_i+1]}$ and nonlinear maps F_{i,d_i+1} for each output component y_i, with different relative degrees $d_i \in \mathbb{I}_{\geq 0}$. Assumptions 4.47–4.48 remain unchanged with $F_d := (F_{i,d_i+1}, \ldots, F_{p,d_p+1})$. Proposition 4.49 remains true with the look-ahead stage cost $\ell_{\mathrm{y},d}(x(t), u(t)) = \ell_{\mathrm{y}}(x(t)) + \sum_{i=1}^{p} q_i y_i^2(t + d_i + 1)$. In Theorem 4.50, we consider $\mathcal{J}_{N-\overline{d}-1,\mathrm{y}}$ with $\overline{d} = \max_i d_i$ and obtain the* *different non-negative "terminal cost"* $\|\Phi_z(x(N-\overline{d}-1|t))\|_{Q_d}^2 + \sum_{i=1}^{p}\sum_{k=N-(\overline{d}-d_i)}^{N-1} q_i y_i^2(k|t)$ *with* $Q_d = \mathrm{diag}(q_i) \in \mathbb{R}^{\sum_{i=1}^{p}(1+d_i)\times\sum_{i=1}^{p}(1+d_i)}$. *The remainder of the proof remains unchanged.*

Remark 4.53. *(Classical design) The application of the proposed output regulation MPC with the stage cost* ℓ_{y} *does* not *require the solution to the regulator equations* $\pi_{\mathrm{x}}, \pi_{\mathrm{u}}$*, the transformation of the system to the BINF, knowledge of the relative degree* d*, or even a stabilizing controller* κ*. The analysis uses the fact that the nonlinear functions* Φ_{z}*,* π_{x}*,* π_{u}*,* $V_\eta, \dots$ *exist, but the exact formulas for these terms are not required for the actual implementation. This fact and the constraint handling capabilities of the MPC are the main benefits of the proposed MPC framework, compare also the discussion in Remark 4.40.*

We point out that the zero dynamics are also vital in the classical output regulation literature [143] and while there exist results for non-minimum-phase systems, "most methods [...] only address systems in normal form with a (globally) stable zero dynamics" [228]. In Section 4.2.3, we will provide a slightly modified MPC design using an incremental input regularization that also ensures stability in the presence of unstable zero dynamics.

Remark 4.54. *(Implicit terminal cost - extremely short prediction horizons) The analysis contains a terminal cost* $\mathcal{J}_{d+1,\mathrm{y}}(x(N|t))$*, which is locally equivalent to the value function* $V_{N,\mathrm{y}}$ *with the unique optimal input* $u = \alpha(x)$ *(cf. Ass- 4.46–4.47). Thus, in the absence of constraints, a horizon* $N > d+1$ *is sufficient to ensure stability for such minimum-phase systems, which can be significantly less conservative than the usual bounds obtained in MPC without terminal constraints [123, 237, 267], compare also the bounds in Section 4.1.5. We conjecture similar (short horizon) guarantees can be derived in the presence of state and input constraints, which is subject of current research. Due to this property we also conjecture that the incremental stabilizability condition (Def. 4.44) can be replaced by an additional continuity bound of the feedback* α *in Assumption 4.47. However, Definition 4.44 allows us to treat the output regulation MPC for minimum-phase systems similarly to the trajectory tracking MPC (Sec. 4.1) and the output regulation MPC for non-minimum-phase systems (Sec. 4.2.3).*

Remark 4.55. *(Detectability conditions do not hold with* ℓ_{y}*) Although Theorem 4.50 ensures stability and utilizes a proof similar to Theorem 4.12, this is only possible by utilizing the look-ahead stage cost* $\ell_{\mathrm{y},d}$ *in the analysis. The detectability condition (cf. Ass. 4.10, Inequalities* (4.66)*) is in general not valid with the output stage cost* ℓ_{y}*. Consider the trivial SISO FIR filter* $y(t) = u(t-2)$*, with* $x(t) = (u(t-1), u(t-2))$*,* $\sigma(x) = \|x\|^2$*, which clearly satisfies the conditions in Theorem 4.50. Inequality* (4.66b) *for* $x = (0,0)$ *implies that* $W((u,0)) = 0$ *for all* $u \in \mathbb{R}$*. Now consider* $x = (x_0, 0)$ *with* $x_0 \neq 0$ *and* $u \in \mathbb{R}$*: Inequality* (4.66b) *implies* $W((u,x_0)) \leq W((x_0,0)) - \epsilon_o x_0^2 + 0 < 0$*, which contradicts the assumption that* W *is non-*

negative. Thus, this system does not satisfy Assumption 4.10/Inequalities (4.66) *with the output stage cost* ℓ_{y}.

Remark 4.56. *(Input affine system) In the special case of input affine systems* $f(x,u) = f_0(x) + g_0(x)u$, *we have* $\alpha(x) = -\frac{f_0^{d+2}(x)}{f_0^{d+1}\circ g_0(x)}$, *where we abbreviate* $f_0^{k+1}(x) := f_0^k \circ f_0$, $k \in \mathbb{I}_{[0,d+1]}$. *Thus, the well-defined relative degree (Ass. 4.47) reduces to* f_0 *Lipschitz continuous and* $f_0^{d+1} \circ g_0$ *non-singular with a uniform lower and upper bound on* $|f_0^{d+1}(x) \circ g_0(x)|$ *for all* $x \in \mathbb{X}$.

Remark 4.57. *(Flat systems and observability) In the special case that* y *is a flat output we have no zero dynamics, i.e.,* $n_{\mathrm{p}} = d+1$ *in Assumption 4.46. Thus, the system* (4.63) *reduces to an FIR filter with the input* F_{d+1}. *Hence, similar to Propositions 4.27 and 4.49, one can show that the look-ahead stage cost* $\ell_{\mathrm{y},d}$ *satisfies the stronger observability condition (Ass. 4.13) with* $c_{\mathrm{o}} = 1$, $\nu = d+1 = n_{\mathrm{p}}$ *(cf. Rk. 4.28). Correspondingly, less conservative bounds on the prediction horizon* $N_{\overline{Y}}$ *can be derived following the arguments in Proposition 4.14. We point out that for continuous-time flat systems, the integral of the stage cost* ℓ_{y} *is positive definite, which in turn simplifies the closed-loop analysis (cf. [95]).*

4.2.3 Output regulation MPC for non-minimum-phase systems

The theoretical analysis in Section 4.2.2 is only applicable to minimum-phase systems and, as demonstrated with a simple example in Remark 4.51, the lack of input regularization can lead to instability for non-minimum-phase systems. The trajectory tracking MPC formulation in Section 4.1 avoids these problems by using an input regularization, which would require knowledge of the optimal feedforward input $\pi_{\mathrm{u}}(w)$ for the considered output regulation problem. In the following, we show how these restrictions can be relaxed for periodic exogenous signals by using an incremental input regularization.

Incremental input formulation for periodic exogenous signals

The main idea is to reformulate the problem such that the optimal feedforward input vanishes by using an incremental input regularization in the MPC formulation. To allow for this reformulation, we focus on periodic exogenous signals.

Assumption 4.58. *(Periodic exogenous signals) There exists a known period length* $T \in \mathbb{I}_{\geq 1}$ *such that* $w(t+T) = w(t)$ *for all* $t \in \mathbb{I}_{\geq 0}$ *with* w *evolving according to* (4.54b).

In the classical output regulation literature (cf. [48, 53, 144, 222]), the exosystem is assumed to be neutrally/Poisson stable, which in the linear case reduces to constant or harmonic/periodic exogenous signals w and hence Assumption 4.58 holds with T being the least common multiple of the different period lengths.

Define a memory state for the past applied inputs as $\xi(t) := (u(t-1), \dots, u(t-T)) \in \mathbb{U}^T$ with some initial condition $\xi(0) = \xi_0 \in \mathbb{U}^T$. Define the change in the periodicity of the control input as $\Delta u(t) := u(t) - u(t-T)$. We consider the following stage cost

$$\ell_{\mathrm{y},\Delta}(x, u, \Delta u) := \|h(x, w, u)\|_Q^2 + \|\Delta u\|_R^2, \tag{4.72}$$

with positive definite matrices $Q \in \mathbb{R}^{p\times p}$, $R \in \mathbb{R}^{m\times m}$. Compared to the output stage cost in Equation (4.57) this cost contains an additional incremental input regularization.

At each time $t \in \mathbb{I}_{\geq 0}$, given the current state $x(t) = (x_\mathrm{p}(t), w(t))$ and the past control inputs $\xi(t)$, the output regulation MPC control law is determined based on the following optimization problem:

Problem 4.59.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \mathcal{J}_{N,\mathrm{y},\Delta}(x(\cdot|t), u(\cdot|t), \Delta u(\cdot|t)) \tag{4.73a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.73b}$$

$$\Delta u(k|t) = u(k|t) - u(t+k-T), \quad k \in \mathbb{I}_{[0,\max\{N,T\}-1]}, \tag{4.73c}$$

$$\Delta u(k|t) = u(k|t) - u(k-T|t), \quad k \in \mathbb{I}_{[T,N-1]}, \tag{4.73d}$$

$$x(0|t) = x(t), \tag{4.73e}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.73f}$$

where

$$\mathcal{J}_{N,\mathrm{y},\Delta}(x(\cdot|t), u(\cdot|t), \Delta u(\cdot|t)) := \sum_{k=0}^{N-1} \ell_{\mathrm{y},\Delta}(x(k|t), u(k|t), \Delta u(k|t)). \tag{4.73g}$$

The solution to this optimization problem is an optimal input trajectory $u^*(\cdot|t)$, the corresponding state and incremental input trajectory $x^*(\cdot|t)$, $\Delta u^*(\cdot|t)$, and the value function $V_{N,\mathrm{y},\Delta}(x(t), \xi(t)) := \mathcal{J}_{N,\mathrm{y},\Delta}(x^*(\cdot|t), u^*(\cdot|t), \Delta u^*(\cdot|t))$. To simplify the theoretical exposition, we define $V_{N,\mathrm{y},\Delta}(x(t), \xi(t)) = \infty$ in case Problem 4.41 does not admit a feasible solution. The following algorithm summarizes the closed-loop operation.

Algorithm 4.60. *(Output regulation MPC Algorithm for non-minimum-phase systems)*
Offline: Specify the constraint set $\mathbb{Z}$, the weighting matrices Q, R, the prediction horizon N, and the period length T.
Online: At each time step $t \in \mathbb{I}_{\geq 0}$, measure the current state $x(t) = (x_{\mathrm{p}}(t), w(t))$ and the past control inputs $\xi(t)$, solve Problem 4.59, and apply the control input $u(t) := u^*(0|t)$.

The resulting closed-loop system is given by

$$x(t+1) = f(x(t), u(t)) = x^*(1|t), \quad u(t) = u^*(0|t), \quad t \in \mathbb{I}_{\geq 0}. \tag{4.74}$$

The difference to Problem 4.41 in Section 4.2.2 is the usage of an incremental input regularization $\|\Delta u\|_R^2$ that penalizes nonperiodic input signals u. Although the optimal feedforward solution $(\pi_{\mathrm{x}}(w), \pi_{\mathrm{u}}(w))$ is unknown, we know that w and hence $\pi_{\mathrm{u}}(w)$ is T-periodic (Ass. 4.58). Thus, intuitively speaking, we know that the optimal solution should drive $(y, \Delta u)$ to the origin using the considered stage cost $\ell_{\mathrm{y},\Delta}$. We point out that a stage cost penalizing nonperiodic trajectories has also been recently considered in [133] for periodic optimal control.

Note that the computational complexity of Problem 4.59 is almost equivalent to Problem 4.41. In particular, the computational complexity of Problem 4.41 only depends on the prediction horizon N and not on the period length T, and the prediction horizon N is not necessarily larger than T. Hence, in contrast to the approaches in Sections 3.2–3.3, the MPC formulation does not scale with the period length T and large values of T are not a problem.

Theoretical analysis

The basic idea of the following theoretical analysis is that for the augmented plant state $x_{\mathrm{p,a}} = (x_{\mathrm{p}}, \xi) \in \mathbb{X}_{\mathrm{p}} \times \mathbb{U} =: \mathbb{X}_{\mathrm{p,a}}$ and the augmented state $x_{\mathrm{a}} := (x_{\mathrm{p,a}}, w) \in \mathbb{X}_{\mathrm{p,a}} \times \mathbb{W} =: \mathbb{X}_{\mathrm{a}}$ we can treat Δu as a control input and thus the stage cost (4.72) is similar to the input-output stage cost (4.30) in Section 4.1. Hence, by showing stabilizability and detecability conditions similar to Assumptions 4.9–4.10 for the augmented system we can use the arguments in Theorem 4.12 to conclude stability of the regulator manifold.

Augmented system

Define the block cyclic permutation matrix $E_0 \in \mathbb{R}^{mT\times mT}$ and the selection matrices E_1, $E_2 \in \mathbb{R}^{mT\times m}$ as

$$E_0 := \begin{pmatrix} 0_{m\times(T-1)m} & I_m \\ I_{(T-1)m} & 0_{(T-1)m\times m} \end{pmatrix}, \quad E_1 := \begin{pmatrix} I_m \\ 0_{(T-1)m\times m} \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0_{(T-1)m\times m} \\ I_m \end{pmatrix}.$$

The dynamics of the memory state ξ and the control input u can be compactly expressed as

$$u(t) = E_2^\top \xi(t) + \Delta u(t), \tag{4.75a}$$
$$\xi(t+1) = E_0\xi(t) + E_1\Delta u(t) = (E_0 - E_1E_2^\top)\xi(t) + E_1u(t). \tag{4.75b}$$

The matrix E_0 satisfies $\prod_{k=0}^{T-1} E_0 = E_0^T = I_{mT}$, the eigenvalues of E_0 are $\lambda_k = e^{2\pi\mathrm{i}k/T}$, $k \in \mathbb{I}_{[0,T-1]}$, all with a geometric and algebraic multiplicity of m (due to the block structure) and the eigenvalues of $E_0 - E_1E_2^\top$ are $\lambda_k = 0$, $k \in \mathbb{I}_{[0,mT-1]}$. We note that $E_1^\top E_0 = E_2^\top$.

Feasibility of the regulator equations (Ass. 4.39) allows us to naturally construct an augmented regulator manifold $\mathcal{A}_\mathrm{a} := \{x_\mathrm{a} = (x_\mathrm{p,a}, w) \in \mathbb{X}_\mathrm{a} \mid x_\mathrm{p,a} = \pi_\mathrm{x,a}(w)\}$ with $\pi_\mathrm{x,a} := (\pi_\mathrm{x}, \pi_\mathrm{u} \circ s^{T-1}, \ldots, \pi_\mathrm{u}) : \mathbb{W} \to \mathbb{X}_\mathrm{p,a}$, $\pi_\mathrm{u,a}(w) := 0$. The corresponding state measure for the augmented system is given by

$$\sigma_\mathrm{a}(x_\mathrm{a}) := \|(x_\mathrm{p}, \xi) - \pi_\mathrm{x,a}(w)\|^2 \geq \|x_\mathrm{p} - \pi_\mathrm{x}(w)\|^2,$$

which satisfies $\sigma_\mathrm{a}(x_\mathrm{a}(t)) = 0$ if and only if $(x_\mathrm{p}(t), \xi(t)) = (\pi_\mathrm{x}(w(t)), \pi_\mathrm{u}(w(t-1)), \ldots, \pi_\mathrm{u}(w(t-T)))$ using periodicity (Ass. 4.58). The main benefit of analysing the augmented system is the fact that $\pi_\mathrm{u,a}(w) = 0$, which allows for the implementation of the input regularization.

Stabilizability

The following proposition shows cost controllability similar to Proposition 4.45.

Proposition 4.61. *Let Assumptions 4.39 and 4.58 hold. Suppose the plant is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.44) and that h is Lipschitz continuous. Then, there exist constants $\gamma_\mathrm{s,a}, \epsilon > 0$ such that for all $N \in \mathbb{I}_{\geq 1}$ and all $x_\mathrm{a} \in \mathbb{X}_\mathrm{a}$ satisfying $\sigma_\mathrm{a}(x_\mathrm{a}) \leq \epsilon$, Problem 4.59 is feasible and the value function satisfies $V_{N,\mathrm{y},\Delta}(x_\mathrm{a}) \leq \gamma_\mathrm{s,a}\sigma(x_\mathrm{a})$.*

Proof. The set of feasible input sequences $u(\cdot|t) \in \mathbb{U}^N$ for Problem 4.41 and Problem 4.59 are equivalent. Hence, for $\sigma_{\mathrm{a}}(x_{\mathrm{a}}) \leq \epsilon$, with $\epsilon \in (0, \epsilon_0]$ from Proposition 4.45 the input sequence from Proposition 4.45 is a feasible solution to Problem 4.59. Denote $u(k-T|t) = u(t+k-T)$, $w(k-T|t) = w(k|t)$ for $k \in \mathbb{I}_{[0,T-1]}$. The corresponding input sequence satisfies the following bound

$$\begin{aligned}
&\|\Delta u(k|t)\|_R^2 = \|u(k|t) - u(k-T|t)\|_R^2 \\
\overset{\text{Ass. 4.58}}{=}&\|u(k|t) - \pi_{\mathrm{u}}(w(k|t)) + \pi_{\mathrm{u}}(w(k-T|t)) - u(k-T|t)\|_R^2 \\
\leq& 2\|u(k|t) - \pi_{\mathrm{u}}(w(k|t))\|_R^2 + 2\|\pi_{\mathrm{u}}(w(k-T|t)) - u(k-T|t)\|_R^2, \quad k \in \mathbb{I}_{[0,N-1]}.
\end{aligned}$$

Furthermore, analogous to Proposition 4.45 we have

$$\|u(k|t) - \pi_{\mathrm{u}}(w(k|t))\|_R^2 \leq c_1^2 c_2^2 \rho^{2k} \lambda_{\max}(R) \|x_{\mathrm{p}}(t) - \pi_{\mathrm{x}}(w(t))\|^2,\ k \in \mathbb{I}_{[0,N-1]}, \tag{4.76}$$

with $c_1, c_2 > 0$, $\rho \in [0,1)$ from Definition 4.44. In addition, the definition of the augmented state measure σ_{a} directly implies

$$\sum_{k=0}^{T-1} \|u(k-T|t) - \pi_{\mathrm{u}}(w(k-T|t))\|_R^2 \leq \lambda_{\max}(R) \sigma_{\mathrm{a}}(x_{\mathrm{a}}(t)). \tag{4.77}$$

Combining these bounds, we get

$$\begin{aligned}
&\sum_{k=0}^{N-1} \|\Delta u(k|t)\|_R^2 \\
\leq& \sum_{k=0}^{N-1} 2\|u(k|t) - \pi_{\mathrm{u}}(w(k|t))\|_R^2 + 2\|u(k-T|t) - \pi_{\mathrm{u}}(w(k-T|t))\|_R^2 \\
\leq& \sum_{k=0}^{N-1} 4\|u(k|t) - \pi_{\mathrm{u}}(w(k|t))\|_R^2 + 2\sum_{k=0}^{T-1} \|u(k-T|t) - \pi_{\mathrm{u}}(w(k-T|t))\|_R^2 \\
\overset{(4.76),(4.77)}{\leq}& \sum_{k=0}^{N-1} 4c_1^2 c_2^2 \rho^{2k} \lambda_{\max}(R) \|x_{\mathrm{p}}(t) - \pi_{\mathrm{x}}(w(t))\|^2 + 2\lambda_{\max}(R)\sigma_{\mathrm{a}}(x_{\mathrm{a}}(t)) \\
\leq& \underbrace{\left(\frac{4c_1^2 c_2^2}{1-\rho^2} + 2\right) \lambda_{\max}(R)}_{=:\gamma_{\mathrm{s,u}}} \sigma_{\mathrm{a}}(x_{\mathrm{a}}(t)).
\end{aligned}$$

The desired bound follows by combining this bound on the input regularization with the bound on the output stage cost in Proposition 4.45 with $\gamma_{\mathrm{s,a}} := \gamma_{\mathrm{s}} + \gamma_{\mathrm{s,u}}$. ∎

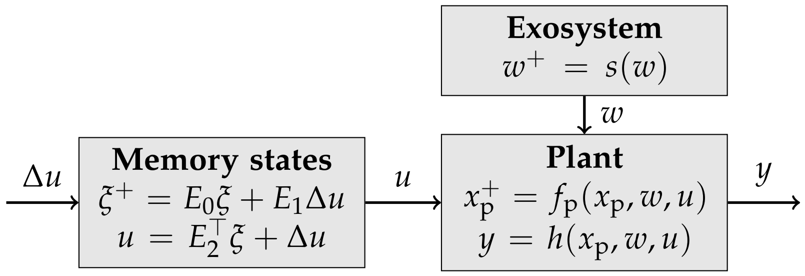

Figure 4.1. Illustration of the augmented system as a series connection of two detectable systems.

An alternative proof based on a joint incremental Lyapunov function can be found in [JK19, Prop. 5].

Detectability and the nonresonance condition

In order to use the stability analysis in Theorem 4.12, we require a cost detectability condition similar to Assumption 4.10. Analogous to Proposition 4.25, the detectability condition holds if the augmented plant is i-IOSS (cf. Assumption 4.24). The augmented plant $x_\mathrm{p,a}$ is a series connection of the two detectable systems: x_p and ξ with the overall input Δu and overall output $y = h$, obtained by connecting the input/output u, compare Figure 4.1.

In order to prove detectability of a series connection both systems need to be detectable and an additional *nonresonance condition* is required.

Assumption 4.62. *(exponential i-IOSS plant) There exists an i-IOSS Lyapunov function* $W : \mathbb{X}_\mathrm{p} \times \mathbb{X}_\mathrm{p} \times \mathbb{W} \to \mathbb{R}_{\geq 0}$ *and constants* $\underline{c}_\mathrm{o}$, $\overline{c}_\mathrm{o}$, $c_{\mathrm{o},1}$, $c_{\mathrm{o},2} > 0$, $\rho_\mathrm{o} \in [0, 1)$ *such that for all* $(x_\mathrm{p}, w, u) \in \mathbb{Z}$, $(z_\mathrm{p}, w, v) \in \mathbb{Z}$*:*

$$\underline{c}_\mathrm{o}\|x_\mathrm{p} - z_\mathrm{p}\|^2 \leq V_\mathrm{o}(x_\mathrm{p}, z_\mathrm{p}, w) \leq \overline{c}_\mathrm{o}\|x_\mathrm{p} - z_\mathrm{p}\|^2, \tag{4.78a}$$

$$\begin{aligned} &V_\mathrm{o}(f_\mathrm{p}(x_\mathrm{p}, w, u), f_\mathrm{p}(z_\mathrm{p}, w, v), s(w)) - \rho_\mathrm{o} V_\mathrm{o}(x_\mathrm{p}, z_\mathrm{p}, w) \\ \leq & c_{\mathrm{o},1}\|u - v\|^2 + c_{\mathrm{o},2}\|h(x_\mathrm{p}, w, u) - h(z_\mathrm{p}, w, v)\|^2. \end{aligned} \tag{4.78b}$$

Abbreviate the augmented plant dynamics by $f_\mathrm{p,a}(x_\mathrm{p,a}, w, u) := (f_\mathrm{p}(x_\mathrm{p}, w), (E_0 - E_1 E_2^\top)\xi + E_1 u)$.

Assumption 4.63. *(Nonresonance condition) There exists an incremental storage function* $V_\mathrm{R} : \mathbb{X}_\mathrm{p,a} \times \mathbb{X}_\mathrm{p,a} \times \mathbb{W} \to \mathbb{R}_{\geq 0}$ *and constants* $c_\mathrm{R,u}$, $c_\mathrm{R} > 0$ *such that for all* $(x_\mathrm{p}, w, u) \in \mathbb{Z}$,

$(z_\mathrm{p}, w, v) \in \mathbb{Z}$, $\xi \in \mathbb{U}^T$, $\Xi \in \mathbb{U}^T$:

$$V_\mathrm{R}(x_\mathrm{p,a}, z_\mathrm{p,a}, w) \leq c_\mathrm{R,u}\|x_\mathrm{p,a} - z_\mathrm{p,a}\|^2, \tag{4.79a}$$

$$\begin{aligned} &V_\mathrm{R}(f_\mathrm{p,a}(x_\mathrm{p,a}, w, u), f_\mathrm{p,a}(z_\mathrm{p,a}, w, v), s(w)) - V_\mathrm{R}(x_\mathrm{p,a}, z_\mathrm{p,a}, w) \\ \leq &c_\mathrm{R}\left(\|\Delta u - \Delta v\|^2 + \|h(x_\mathrm{p}, w, u) - h(z_\mathrm{p}, w, v)\|^2\right) - \|u - v\|^2, \end{aligned} \tag{4.79b}$$

with $x_\mathrm{p,a} = (x_\mathrm{p}, \xi)$, $z_\mathrm{p,a} = (z_\mathrm{p}, \Xi)$, $\Delta u = u - E_2^\top \xi$, *and* $\Delta v = v - E_2^\top \Xi$.

Given that condition (4.79) corresponds to an incremental dissipativity condition, it can be verified using the results in [270] based on differential dissipativity. Loosely speaking, conditions (4.79) imply that if two systems have a similar initial condition, produce a similar output and are driven by a similar incremental input Δu, Δv, then the input u, v applied to the plant has to be similar. In particular, if both systems are driven by a periodic input u, v and generate the same output trajectory y, then the two periodic input trajectories u, v must be equivalent. Thus, this condition excludes the possibility of two distinct periodic inputs u, v resulting in the same output y. This condition seems to be a relaxed version of *input detectability/observability*, as for $\Delta u = \Delta v = 0$ (periodic inputs) it essentially requires that $y \equiv 0$ implies $u \equiv 0$ (assuming zero initial conditions), similar to [137, Def. 3]. In the linear case, this is equivalent to assuming that the poles generating a T-periodic input signal u with Equations (4.75) (assuming $\Delta u = 0$) are not cancelled by zeros of the plant, which corresponds to the well established *nonresonance condition*, compare Section 4.4 for a detailed proof. We point out that in [185] a different nonlinear extension of the *nonresonance condition* has been proposed, which is characterized using a rank condition on the lie derivatives as opposed to the proposed dissipativity characterization. Although both characterizations correspond to the classical *nonresonance condition* in the linear case, the considered formulation using dissipation inequalities with a storage function allows us to directly construct an i-IOSS Lyapunov function to establish detectability of the augmented plant, as shown in the following proposition. Define the augmented stage cost $\ell_\mathrm{a}(x_\mathrm{a}, u) := \|h(x, u, w)\|_Q^2 + \|u - E_2^\top \xi\|_R^2$, which satifies $\ell_\mathrm{a}(x_\mathrm{a}(t), u(t)) = \|y(t)\|_Q^2 + \|\Delta u(t)\|_R^2$ and is thus equivalent to $\ell_{\mathrm{y},\Delta}$ from Equation (4.72). Abbreviate the augmented state dynamics by $f_\mathrm{a} : \mathbb{X}_\mathrm{a} \times \mathbb{U} \to \mathbb{X}_\mathrm{a}$ with $f_\mathrm{a}(x_\mathrm{p,a}, w, u) := (f_\mathrm{p,a}(x_\mathrm{p,a}, w, u), s(w))$ and the constraint set of the augmented state by $\mathbb{Z}_\mathrm{a} := \{(x_\mathrm{p}, \xi, w, u) \in \mathbb{X}_\mathrm{a} \times \mathbb{U} \mid (x_\mathrm{p}, w, u) \in \mathbb{Z}\}$.

Proposition 4.64. *Let Assumptions 4.39, 4.58, 4.62 and 4.63 hold. Then, there exists a function*

$W_{\mathrm{a}}:\mathbb{X}_{\mathrm{a}}\rightarrow\mathbb{R}_{\geq 0}$ and constants $\gamma_{\mathrm{a,o}},\epsilon_{\mathrm{a,o}}>0$ such that

$$\begin{aligned} W_{\mathrm{a}}(x_{\mathrm{a}}) \leq& \gamma_{\mathrm{a,o}}\sigma_{\mathrm{a}}(x_{\mathrm{a}}), && (4.80a)\\ W_{\mathrm{a}}(f_{\mathrm{a}}(x_{\mathrm{a}},u))-W_{\mathrm{a}}(x_{\mathrm{a}}) \leq& -\epsilon_{\mathrm{a,o}}\sigma_{\mathrm{a}}(x_{\mathrm{a}})+\ell_{\mathrm{a}}(x_{\mathrm{a}},u), && (4.80b) \end{aligned}$$

for any $(x_{\mathrm{a}},u)\in\mathbb{Z}_{\mathrm{a}}\subseteq\mathbb{X}_{\mathrm{a}}\times\mathbb{U}$.

Proof. Consider the feasible trajectory $z_{\mathrm{p,a}}=(z_{\mathrm{p}},\Xi)=\pi_{\mathrm{x,a}}(w)$, $\Delta v=\pi_{\mathrm{u,a}}(w)=0$, $v=E_2^\top\Xi$, $z_{\mathrm{p,a}}^+=(z_{\mathrm{p}}^+,\Xi^+)=\pi_{\mathrm{x,a}}(w^+)$, $w^+=s(w)$ (cf. Assumption 4.39). First note that the linear dynamics of ξ with the input Δu and the output $u=E_2^\top\xi+\Delta u$ are observable. Thus, there exists a quadratic i-IOSS Lyapunov function $V_\xi(\xi,\Xi)=\|\xi-\Xi\|^2_{P_{\xi,\mathrm{o}}}$ with a positive definite matrix P_ξ (cf. [50]) satisfying

$$V_\xi(\xi^+,\Xi^+)\leq\rho_\xi V_\xi(\xi,\Xi)+\|\Delta u-\Delta v\|^2+\|u-v\|^2, \tag{4.81}$$

with $\rho_\xi\in[0,1)$ and $\xi^+=E_0\xi+E_1\Delta u$, $\Delta u=u-E_2^\top\xi$. Consider the storage function

$$W_{\mathrm{a}}(x_{\mathrm{p,a}},w):=c_3\left(V_{\mathrm{o}}(x_{\mathrm{p}},z_{\mathrm{p}},w)+V_\xi(\xi,\Xi)+c_2V_{\mathrm{R}}(x_{\mathrm{p,a}},z_{\mathrm{p,a}},w)\right),$$

with $c_2:=c_{\mathrm{o},1}+1$, V_{o}, V_{R} from Assumptions 4.62 and 4.63, and a later specified constant $c_3>0$. The upper bound (4.80a) holds with $\gamma_{\mathrm{a,o}}:=c_3(\max\{\bar{c}_{\mathrm{o}},\lambda_{\max}(P_{\xi,\mathrm{o}})\}+c_2c_{\mathrm{R,u}})$. Inequality (4.80b) holds with

$$\begin{aligned} &(W_{\mathrm{a}}(f_{\mathrm{a}}(x_{\mathrm{a}},u))-W_{\mathrm{a}}(x_{\mathrm{a}}))/c_3\\ \overset{(4.78b),(4.79b),(4.81)}{\leq}&-(1-\rho_{\mathrm{o}})V_{\mathrm{o}}(x_{\mathrm{p}},z_{\mathrm{p}},w)+c_{\mathrm{o},1}\|u-v\|^2+c_{\mathrm{o},2}\|h(x_{\mathrm{p}},w,u)-h(z_{\mathrm{p}},v,w)\|^2\\ &-(1-\rho_\xi)V_\xi(\xi,\Xi)+\|\Delta u-\Delta v\|^2+\|u-v\|^2\\ &+c_2\left(c_{\mathrm{R}}\|\Delta u-\Delta v\|^2+c_{\mathrm{R}}\|h(x_{\mathrm{p}},w,u)-h(z_{\mathrm{p}},v,w)\|^2-\|u-v\|^2\right)\\ =&-(1-\rho_{\mathrm{o}})V_{\mathrm{o}}(x_{\mathrm{p}},z_{\mathrm{p}},w)-(1-\rho_\xi)V_\xi(\xi,\Xi)+(1+c_2c_{\mathrm{R}})\|\Delta u-\Delta v\|^2\\ &+\underbrace{(c_{\mathrm{o},1}+1-c_2)}_{=0}\|u-v\|^2+(c_{\mathrm{o},2}+c_2c_{\mathrm{R}})\|h(x_{\mathrm{p}},w,u)-\underbrace{h(z_{\mathrm{p}},v,w)}_{\overset{(4.56b)}{=}0}\|^2\\ \overset{(4.78a)}{\leq}&-(1-\rho_{\mathrm{o}})\underline{c}_{\mathrm{o}}\|x_{\mathrm{p}}-z_{\mathrm{p}}\|^2-(1-\rho_\xi)\lambda_{\min}(P_\xi)\|\xi-\Xi\|^2\\ &+\frac{(1+c_2c_{\mathrm{R}})}{\lambda_{\min}(R)}\|\Delta u-\Delta v\|^2_R+\frac{c_{\mathrm{o},2}+c_2c_{\mathrm{R}}}{\lambda_{\min}(Q)}\|h(x_{\mathrm{p}},w,u)\|^2_Q\\ \leq&-\epsilon_{\mathrm{a}}/c_3\cdot\sigma_{\mathrm{a}}(x_{\mathrm{a}})+\ell_{\mathrm{a}}(x_{\mathrm{a}},u)/c_3, \end{aligned}$$

with $c_3 := \min\left\{\frac{\lambda_{\min}(R)}{1+c_2c_R}, \frac{\lambda_{\min}(Q)}{c_{o,2}+c_2c_R}\right\} > 0$, $\sigma_a(x_a) = \|x_{p,a} - z_{p,a}\|^2 = \|x_p - z_p\|^2 + \|\xi - \Xi\|^2$, $\epsilon_a := \min\{(1-\rho_o)\underline{c}_o, (1-\rho_\xi)\lambda_{\min}(P_{\xi,o})\} \cdot c_3 > 0$. ■

Final result

With Propositions 4.61 and 4.64 and Theorem 4.12, we can summarize the theoretical properties of the incremental input regularized output regulation MPC scheme (Problem 4.59).

Theorem 4.65. *Let Assumptions 4.39, 4.58, 4.62 and 4.63 hold. Suppose further that the plant is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.44) and that π_x, π_u, s, and h are Lipschitz continuous. Then, for any $\overline{Y} > 0$, there exists a constant $N_{\overline{Y}} > 0$ such that for all $N > N_{\overline{Y}}$ and any initial condition $(x_0, \xi_0) =: x_{a,0} \in \mathbb{X}_{\overline{Y},a} := \{x_a \in \mathbb{X}_a \mid V_{N,y,\Delta}(x_a) + W_a(x_a) \leq \overline{Y}\}$, the closed loop system* (4.74) *resulting from Algorithm 4.60 satisfies the constraints* (4.55)*, Problem 4.59 is feasible for all $t \in \mathbb{I}_{\geq 0}$, and the (augmented) regulator manifold $\mathcal{A}_a$ is exponentially stable.*

Proof. Propositions 4.61 and 4.64 derive inequalities on the value function $V_{N,y,\Delta}$, the storage W_a and the state measure σ_a, which are equivalent to the conditions in Assumptions 4.9 and 4.10. Thus, we can directly use the arguments in Theorem 4.12 with the Lyapunov function $Y_{N,a} := V_{N,y,\Delta} + W_a$, resulting in inequalities analogous to Inequalities (4.69), (4.71) in Theorem 4.50. Exponential stability of the regulator manifold follows from $\pi_{x,a}$ Lipschitz continuous (cf. proof Thm. 4.50, Part III). ■

Discussion

This result implies that the output regulation MPC scheme (Problem 4.59) solves the nonlinear constrained regulation problem if:

(a) The regulator problem is (strictly) feasible (Ass. 4.56),

(b) The plant is incrementally stabilizable (Def. 4.44) and detectable (Ass. 4.62),

(c) The exogenous signal is periodic (Ass. 4.58) and a technical nonresonance condition holds (Ass. 4.63),

(d) A sufficiently large prediction horizon $N > N_{\overline{Y}}$ is used.

Compared to Section 4.2.3, the possibly restrictive minimum-phase property is relaxed to a detectability condition. However, we require periodicity of the exogenous signal w with a known period length T and a technical nonresonance condition. Overall, the rigorous theoretical guarantees in combination with the fact that no complex design procedure is required for the implementation makes the proposed output regulation MPC framework suitable for practical application.

Remark 4.66. *(Existing MPC solutions for periodic problems) For the special case of periodic signals w, there also exist competing approaches to solve the regulator problem. Given w_0 and the period length T, the T-periodic trajectory $(\pi_{\mathrm{x}}(w(t)), \pi_{\mathrm{u}}(w(t)))$, $t \in \mathbb{I}_{[0,T-1]}$ can be obtained by solving one (potentially large) nonlinear program (NLP), as suggested in [89]. Then, the output regulation problem reduces to the problem of stabilizing a given state and input trajectory, for which MPC approaches with and without terminal ingredients exist, compare Sections 3.1 and 4.1, respectively. If we consider online changing operating conditions or the error feedback setting (Rk. 4.43), the estimates for w may change online and thus the large scale NLP would have to be solved repeatedly during online operation. The problem of recomputing a periodic reference trajectory online can be integrated in the MPC formulation using artificial reference trajectories, compare Section 3.2.*

The main advantage of the proposed MPC approach is its simplicity. No offline design for the terminal ingredients is required. No periodic trajectory $(\pi_{\mathrm{x}}(w(t)), \pi_{\mathrm{u}}(w(t)))$, $t \in \mathbb{I}_{\geq 0}$ needs to be computed offline/online. The overall algorithm, design, and online optimization problem is simple. One of the main drawbacks is that, depending on system dynamics, a large prediction horizon N may be required resulting in a potentially larger computational complexity compared to direct trajectory stabilization with given maps $\pi_{\mathrm{x}}, \pi_{\mathrm{u}}$ satisfying the regulator equations (4.56) *(cf. Sec. 3.1/4.1). The advantages of avoiding offline computations are especially relevant if the MPC formulation is combined with some model/parameter update scheme to address system uncertainty (cf. adaptive MPC formulations in [45, 82, JK11, JK13, 219, JK37]).*

Remark 4.67. *(Offset-free setpoint tracking - incremental input penalty) The problem of offset-free setpoint tracking is a special case of the output regulation problem with $s(w) = w$ and $T = 1$. In this case, Δu penalizes the change in the control input u, which is quite common in the MPC literature, especially in case of offset-free setpoint tracking, compare [36, 176, 213] and [174, Cor. 4]. Thus, the proposed formulation is rather intuitive and similar to existing standard approaches for setpoint tracking MPC. For comparison, in [176] a linear dynamic controller is used to characterize the terminal cost and set and in [164] artificial setpoints are used to track changing setpoints. The issue of estimating the disturbances has been treated in [201, 213] for linear and nonlinear systems with disturbance observers and can also be treated* *in the proposed framework, compare Remark 4.43. Compared to many of the existing approaches, the lack of any complex offline design is one of the main benefits of the proposed MPC framework, as it requires no terminal ingredients, artificial setpoints, or a solution to the regulator equations.*

Remark 4.68. *(Nonresonance condition = tracking condition) In the case of nonlinear setpoint tracking MPC, it is often assumed that there exists a unique (Lipschitz continuous) map from any output y to a corresponding steady-state and input (x_{p}, u), compare [164, Ass. 1] or [176, Ass. 1]. Given f_{p}, h continuously differentiable, this condition is equivalent to a rank condition on the Jacobian (cf. [164, Rk. 1], [236, Lemma 1.8]), which is equivalent to the nonresonance condition for constant exogenous signals, compare [185]. We point out that the rank-based and dissipation-based nonresonance characterizations are equivalent in the linear case (cf. Prop. 4.86 in Sec. 4.4). Thus, the tracking condition [164, Ass. 1] is strongly related (if not equivalent) to the dissipation-based characterization in Assumption 4.63 for $T = 1$. Furthermore, a similar characterization to [164, Ass. 1] can be used for periodic trajectories (cf. Ass. 3.29 in Sec. 3.2), which seems to be an alternative characterization for the property in Assumption 4.63.*

Summary

In this section, we studied the closed-loop properties of a simple tracking MPC scheme without terminal ingredients for the output regulation problem. In particular, the proposed output regulation MPC simply minimizes a quadratic output stage cost and does *not* require any complex offline design procedures (e.g., solving the regulator/FBI equations). We proved that this simple design ensures exponential stability of the regulator manifold for a sufficiently long prediction horizon N, if the plant is *minimum-phase* (stable zero dynamics) and incrementally exponentially stabilizable. We also provided a modified MPC formulation that uses an incremental input regularization, assuming that the exogenous signals are *periodic*. For this modified MPC formulation, we could relax the minimum-phase assumption to a *detectability* condition (i-IOSS) in combination with a technical *nonresonance* assumption. Overall, the proposed output regulation MPC formulations are particularly appealing since no complex design procedures are required and the closed loop converges to the (in general unknown) regulator manifold. In the next section, we study the more general case when the underlying tracking problem is not feasible (Ass. 4.15/4.39 does *not* hold) by considering *unreachable reference trajectories*.

4.3 Unreachable reference trajectories

In Sections 4.1 and 4.2, we studied the closed-loop properties of simple tracking MPC schemes without terminal ingredients, assuming that the underlying tracking problem is feasible (Ass. 4.15/4.39 holds). In this section, we study the closed-loop properties in case this assumption is not satisfied, i.e., if the reference trajectory is not reachable (Sec. 4.3.1). In particular, we leverage tools used in economic MPC without terminal constraints, such as dissipativity, to derive a region of attraction and guarantee (practical) stability of the *unknown* optimal reachable trajectory (Sec. 4.3.2). This section is based on and taken in parts literally from [JK24][11].

4.3.1 Unreachable reference trajectories in MPC

In the following, we generalize the trajectory tracking problem considered in Section 4.1 by allowing for unreachable reference trajectory (Ass. 4.15 does not hold) in the theoretical analysis. Thus, by considering an *unreachable* reference trajectories this section mainly addresses the additional challenge when the *optimal mode of operation is not directly specified in terms of given state and input setpoints/trajectories* (cf. Sec. 1.1, (iii)). The setup and the MPC algorithm are equivalent to Section 4.1, only Assumption 4.15 is dropped and thus a different theoretical analysis is required. We present these results only for the trajectory tracking case (Sec. 4.1) to avoid a cumbersome notation, but they can be naturally extended to the output regulation setting (Sec. 4.2), compare Remark 4.84.

Motivation

The motivation to consider unreachable reference trajectories in this chapter is equivalent to the motivation in Section 3.2. In particular, the reference trajectory is often generated by an external unit. Furthermore, even if the reference trajectory is generated based on a model, often a different (typically coarser) model is used in the reference planning. Thus, the considered reference trajectory often does not satisfy the dynamics and hence we cannot guarantee stability of this reference trajectory. Since the provided reference trajectory does not constitute a viable mode of operation, the optimal mode of operation is unclear a priori. Thus, we investigate sufficient conditions to ensure that the closed-loop "finds" the optimal mode of operation.

[11] J. Köhler, M. A. Müller, and F. Allgöwer. "Nonlinear reference tracking: An economic model predictive control perspective." In: *IEEE Trans. Automat. Control* 64.1 (2019), pp. 254–269©2018 IEEE.

Related work

For unreachable reference trajectories, the trajectory tracking MPC (Problem 4.1) corresponds to an economic MPC formulation (cf. [96]), since the (point-wise) minimizer of the stage cost does in general *not* coincide with the optimal mode of operation. Similar issues also appear in the case of steady-state operation (cf. [234]), although in this case it is easier to determine the optimal steady-state and thus use terminal ingredients (Chap. 3). We point out that Problem 4.1 and the optimal mode of operation are time-varying. In case the system is optimally operated at a steady-state, the optimal mode of operation can be equivalently characterized using *dissipativity* [19, 207], which implies the *turnpike* property [70, 92, 98]. Based on these properties, closed-loop performance and (practical) stability can be ensured if a sufficiently long prediction horizon is used [122, 132], compare also modified economic MPC formulations in [8, 291] and the robust extension in [JK35]. Extensions of this dissipativity characterization for more general optimal modes of operation (periodic orbits, sets,...) and corresponding closed-loop properties can be found in [81, JK25, 186, 203, 210, 295]. The following derivation extends the analysis in [122, 132] to time-varying problems and relaxes the standard controllability conditions by employing sublevel set arguments. More recently, in [128, 129, 130], for general time-varying economic MPC problems, dissipativity, turnpike and closed-loop performance/stability results have been derived.

Alternatively, the problem of unreachable setpoints/trajectories could also be tackled using artificial reference trajectories (Sec. 3.2) or treating unpredictable changes in the reference as a disturbance [77, 199].

4.3.2 Theoretical analysis

We consider the trajectory tracking MPC from Section 4.1 with a quadratic tracking stage cost (Ass. 4.16). We first define the optimal mode of operation and introduce suitable continuity and dissipativity conditions. Then, we establish suitable turnpike properties in Lemmas 4.77 and 4.78. Finally, Theorems 4.79 and 4.80 show that the closed loop practically tracks the optimal (unknown) mode of operation if a sufficiently long prediction horizon N is used.

Strict dissipativity and rotated cost

In the following, the reference trajectory can be seen as an arbitrary bounded time-varying signal, which is not necessarily related to the system dynamics. For the theoretical analysis, we limit ourselves to $T-$periodic reference trajectories, i.e., there exists $T \in \mathbb{I}_{\geq 1}$: $r(t+T) = r(t)$, $\forall t \in \mathbb{I}_{\geq 0}$, to avoid some technical challenges associated with general time-varying problems. In view of the more recent results in [128, 129, 130] for general time-varying problems, the following results can be extended to nonperiodic trajectories.

Given the periodic reference trajectory $r = (x_{\mathrm{r}}, u_{\mathrm{r}})$, we can define the best reachable $T-$periodic trajectory with the following periodic optimal control problem (analogous to Problem 3.25/3.52).

Problem 4.69.

$$\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad \mathcal{J}_T(x(\cdot), u(\cdot), 0) \tag{4.82a}$$

subject to

$$x(k+1) = f(x(k), u(k)),\ (x(k), u(k)) \in \mathbb{Z},\ k \in \mathbb{I}_{[0,T-1]},\ x(T) = x(0), \tag{4.82b}$$

with $\mathcal{J}_T$ *according to Equation* (4.2e) *and* ℓ *from Equation* (4.30).

The solution to this optimization problem is the[12] optimal reachable T-periodic state and input trajectory $x_T(\cdot) \in \mathbb{X}^T$, $u_T(\cdot) \in \mathbb{U}^T$ and the minimum is denoted by $V_{T,\min} := \mathcal{J}_T(x_T(\cdot), u_T(\cdot), 0)$. Contrary to existing results for optimal periodic operation (cf. [166, 210, 295] or Sec. 3.2/3.3), the implementation of the considered MPC scheme (Alg. 4.2) does *not* require knowledge of the period length T or the optimal reachable trajectory x_T. Instead, this is only used as an analysis tool. For the implementation of the MPC algorithm we only require the next N steps of the reference trajectory $r = (x_{\mathrm{r}}, u_{\mathrm{r}})$.

Clearly, if the goal is to stay close to the reference trajectory $(x_{\mathrm{r}}, u_{\mathrm{r}})$, a desirable property of the MPC would be to track the *unknown* reference x_T. To establish such a property, we require (strict) constraint satisfaction of x_T, similar to Assumption 4.15.

Assumption 4.70. *(Strict feasibility of the optimal reachable trajectory) The optimal reachable trajectory from Problem 4.69 satisfies* $(x_T(t), u_T(t)) \in \mathrm{int}(\mathbb{Z})$, $t \in \mathbb{I}_{\geq 0}$.

This condition is intuitively needed to ensure that the MPC can steer the system to the optimal reachable trajectory x_T, without imposing additional controllability conditions. This requirement may limit the applicability of the following theory to scenarios, where

[12] Uniqueness of the minimizer will be ensured in the later derivation using a suitable dissipativity condition.

the original reference trajectory r is mainly not reachable due to the dynamics. We can relax this requirement to allow for active input constraints, if the stabilizability condition (Def. 4.18) is strengthened to stability ($c_2 = 0$).

Denote the stage cost of the optimal reachable trajectory by $\ell_T(k) = \ell(x_T(k), u_T(k), k)$, $k \in \mathbb{I}_{[0,T-1]}$. The stage cost difference $\ell_T(k) - \ell(x,u,k)$ can be positive or negative, depending on $(x,u,k) \in \mathbb{Z} \times \mathbb{I}_{\geq 0}$. Moreover, even if the system is initialized at the optimal $T-$periodic solution, i.e., $x_0 = x_T(0)$, the solution of Problem 4.1 does in general not follow this optimal trajectory due to the finite time horizon N. Thus, we consider the notion of practical asymptotic stability.

Lemma 4.71. *([132, Thm. 2.4], [126, Thm. 2.23])*
Let $V : \mathbb{R}^n \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a time-varying practical Lyapunov function, with

$$\begin{aligned} \alpha_1(\|e(t)\|) \leq V(e(t),t) \leq& \alpha_2(\|e(t)\|), \\ V(e(t+1),t+1) - V(e(t),t) \leq& -\alpha_3(\|e(t)\|) + \theta, \end{aligned}$$

for all $V(e(t),t) \leq \overline{V}$, $\overline{V} > \eta := \alpha_2(\alpha_3^{-1}(\theta)) + \theta$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\theta > 0$. Then, for all initial conditions $V(e(0),0) \leq \overline{V}$, the origin $e = 0$ is uniformly practically asymptotically stable and the system uniformly converges to the set $\{(e,t) \in \mathbb{R}^n \times \mathbb{I}_{\geq 0} | V(e,t) \leq \eta\}$.

Denote the tracking error with respect to the optimal reachable trajectory by $e_T(t) := x(t) - x_T(t)$. We study the reference tracking problem in the economic MPC framework using strict dissipativity.

Assumption 4.72. *(Strict dissipativity) There exists a bounded time-varying periodic storage function $\lambda : \mathbb{X} \times \mathbb{I}_{\geq 0} \to \mathbb{R}$, $\lambda(x,k) = \lambda(x, k+T)$ such that for all $(x,u,k) \in \mathbb{Z} \times \mathbb{I}_{\geq 0}$, we have*

$$L(x,u,k) := \ell(x,u,k) - \ell_T(k) + \lambda(x,k) - \lambda(f(x,u),k+1) \geq \alpha_\ell(\|x - x_T(k)\|), \quad (4.83)$$

with $\alpha_\ell \in \mathcal{K}_\infty$. Furthermore, there exists a function $\gamma_\lambda \in \mathcal{K}_\infty$ such that for all $(x,k) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$, the storage functions is bounded by

$$|\lambda(x,k)| \leq \gamma_\lambda(\|x - x_T(k)\|). \quad (4.84)$$

Condition (4.83) corresponds to a strict dissipativity characterization with the supply rate $s(x,u,k) = \ell(x,u,k) - \ell_T(k)$. This condition ensures that the rotated stage cost $L : \mathbb{Z} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite w.r.t. the optimal reachable trajectory x_T.

Inequality (4.84) is a (local) continuity assumption on the storage function λ, which, in combination with $\mathbb{Z}_{\mathbb{X}} := \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} : (x, u) \in \mathbb{Z}\}$ compact ensures boundedness of λ on $\mathbb{Z}_{\mathbb{X}}$. In [210, 295], periodic optimal control for time-invariant problems is considered and the corresponding rotated stage cost is positive definite with respect to a periodic orbit instead of a specific point on the orbit, compare also [JK25] for a discussion on the different notions of periodic dissipativity. In the considered trajectory tracking problem the phase is uniquely specified by the time-varying stage cost ℓ, compare also [128, 129, 130] regarding dissipativity in time-varying (nonperiodic) economic MPC problems.

For general nonlinear dynamical systems with an arbitrary dynamical reference trajectory $(x_\mathrm{r}, u_\mathrm{r})$, computing suitable storage functions λ is a challenging task, compare, e.g., [JK1, 225] for SOS approaches. For our purposes, it suffices to show the existence of such storage functions, which is discussed in the following lemma.

Lemma 4.73. *Let Assumption 4.70 hold. Assume that the system is locally (uniformly) controllable around x_T [260, Def. 3.6.4] and that the system is uniformly suboptimally operated off the trajectory x_T according to [211, Def. 12][13]. Then, Assumption 4.72 is satisfied.*

Proof. **Part I:** Proving existence of a bounded storage λ is an extension of the results in [207], with the main difference in [207, Thm. 4] and [204, Thm. 4.12]. The local controllability assumption in combination with the definition of uniform suboptimal operation, enables us to construct a periodic trajectory with lower cost to prove dissipativity by contradiction. The rest of the proof does not change compared to the steady-state case and guarantees that there exists a bounded storage λ on $\mathbb{Z}_{\mathbb{X}}$. For $x \notin \mathbb{Z}_{\mathbb{X}}$ we can define $\lambda(x) := c \in \mathbb{R}$ with a sufficiently small constant c such that Inequality (4.83) holds (cf. proof [JK25, Prop. 2]).

Part II: In order to show local continuity (4.84), we use the assumed local uniform controllability in combination with continuity of the stage cost ℓ, analogous to the continuity result in [226]. In particular, w.l.o.g. suppose $\lambda(x_T(k), k) = 0$ and define the supply rate $\tilde{s}(x, u, k) := \ell(x, u, k) - \ell_T(k) - \alpha_\ell(\|x - x_T(k)\|)$. Then, given an arbitrary point $x(k)$ with $\|e_T(k)\|$ sufficiently small, we can steer the system to $x_T(k + \nu)$ in $\nu \in \mathbb{I}_{\geq 1}$ steps with a uniformly bounded supply $\tilde{s}$ by combining Assumption 4.70 with the assumed

[13] The original proof [204, Thm. 4.12] only showed dissipativity on the control invariant set $\mathbb{X}_\infty$, but can be extended to hold on $\mathbb{Z}_{\mathbb{X}}$, by adjusting the definition of uniform suboptimal operation to account for trajectories $x \notin \mathbb{X}_\infty$ (cf. [JK25, Sec. III.E]).

uniform controllability. Using a telescopic sum for Inequality (4.83), this implies

$$\underbrace{\lambda(x_T(k+\nu),k+\nu)}_{=0} - \lambda(x(k),k) \leq \sum_{j=k}^{k+\nu-1} \tilde{s}(x(j),u(j),j) \leq \alpha_\lambda(\|e_T(k)\|),\ \alpha_\lambda \in \mathcal{K}_\infty,$$

which proves the local lower bound in Inequality (4.84). The local upper bound follows with the same reasoning by steering the system to some x, starting at $x_T(k)$. Local validity of Condition (4.84) and boundedness of λ directly implies satisfaction of Condition (4.84) for all $x \in \mathbb{X}$. ■

To study stability properties, we further define the rotated value function, which plays the role of a positive definite Lyapunov function with respect to the optimal reachable trajectory x_T. The rotated open-loop cost $\tilde{\mathcal{J}}_N : \mathbb{X}^N \times \mathbb{U}^N \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$\begin{aligned} &\tilde{\mathcal{J}}_N(x(\cdot|t), u(\cdot|t), t) := \sum_{k=0}^{N-1} L(x(k|t), u(k|t), t+k) \\ &\overset{(4.83)}{=} \mathcal{J}_N(x(\cdot|t), u(\cdot|t), t) + \lambda(x(t), t) - \lambda(x(N|t), t+N) - \underbrace{\sum_{k=0}^{N-1} \ell_T(t+k)}_{=:c_N(t)}. \end{aligned} \tag{4.85}$$

The rotated MPC problem is given by the following optimization problem.

Problem 4.74.

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \tilde{\mathcal{J}}_N(x(\cdot|t), u(\cdot|t), t) \tag{4.86a}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{[0,N-1]}, \tag{4.86b}$$

$$x(0|t) = x(t), \tag{4.86c}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N-1]}. \tag{4.86d}$$

The solution to this optimization problem is an optimal (rotated) input trajectory $\tilde{u}^*(\cdot|t)$, the corresponding state trajectory $\tilde{x}^*(\cdot|t)$, and the rotated value function $\tilde{V}_N(x(t),t) := \tilde{\mathcal{J}}_N(\tilde{x}^*(\cdot|t), \tilde{u}^*(\cdot|t), t)$. To simplify the theoretical exposition regarding feasibility, we define $\tilde{V}_N(x(t),t) = \infty$, in case Problem 4.74 does not admit a feasible solution. Note that unlike the case with terminal constraints [19, 295], this rotated cost does in general not have the same minimizer as the original cost ($u^* \neq \tilde{u}^*$), which makes the stability proof more involved, compare [122, 132, 210].

Stabilizability and turnpike property

In the following, we derive a turnpike property with respect to the optimal reference x_T based on local stabilizability and strict dissipativity. The following proposition bounds the open-loop cost, by combing the ideas of Proposition 4.19 with the strict dissipativity in Assumption 4.72.

Proposition 4.75. *Let Assumptions 4.16, 4.70 and 4.72 hold. Suppose the system is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.18). Then, there exist constants c_T, γ, $\tilde{c} > 0$ and a function $\alpha_{\mathrm{u}} \in \mathcal{K}_\infty$ such that for all $N \in \mathbb{I}_{\geq 1}$, for all $(x(t), t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$ satisfying $\|e_T(t)\| \leq c_T$, the following bounds hold*

$$V_N(x(t), t) \leq \gamma \|e_T(t)\|_Q^2 + c_N(t) + \tilde{c}\|e_T(t)\|, \tag{4.87a}$$

$$\tilde{V}_N(x(t), t) \leq \alpha_{\mathrm{u}}(\|e_T(t)\|). \tag{4.87b}$$

Proof. Consider an input sequence $u(\cdot|t) \in \mathbb{U}^N$ stabilizing the optimal reachable trajectory (x_T, u_T) (cf. Definition 4.18). Feasibility of this trajectory for Problems 4.1/4.74 follows from $(x_T(t), u_T(t)) \in \operatorname{int}(\mathbb{Z})$ and c_T small enough, compare the proof of Proposition 4.19. Denote the corresponding state trajectory by $x(\cdot|t)$ with $e_T(k|t) := x(k|t) - x_T(t+k)$, $k \in \mathbb{I}_{[0,N-1]}$. Based on ℓ quadratic, $\mathbb{Z}$ compact, and the bounds in Inequalities (4.31), the corresponding stage cost satisfies

$$\begin{aligned}\ell(x(k|t), u(k|t), t+k) \leq& \ell_T(t+k) + \tilde{c}_1\|e_T(k|t)\|^2 + \tilde{c}_2\|e_T(k|t)\| \\ \overset{(4.31)}{\leq}& \ell_T(t+k) + \tilde{c}_3\rho^{2k}\|e_T(t)\|^2 + \tilde{c}_4\rho^k\|e_T(t)\|,\ k \in \mathbb{I}_{[0,N-1]},\end{aligned}$$

with some positive constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4 > 0$. Inequality (4.87a) follows by summing up this inequality and using the geometric series with $\rho \in [0, 1)$, compare [JK24, Prop. 3] for details. The rotated cost satisfies

$$\begin{aligned}\tilde{\mathcal{J}}_N(x(\cdot|t), u(\cdot|t), t) \overset{(4.85)}{=}& \mathcal{J}_N(x(\cdot|t), u(\cdot|t), t) - c_N(t) + \lambda(x(t), t) - \lambda(x(N|t), t+N) \\ \leq& \gamma\|e_T(t)\|_Q^2 + \tilde{c}\|e_T(t)\| + \gamma_\lambda(\|e_T(t)\|) + \gamma_\lambda(\underbrace{\|e_T(N|t)\|}_{\leq c_1\|e_T(t)\|}).\end{aligned}$$

Correspondingly we get the bound

$$\alpha_{\mathrm{u}}(r) := \gamma\lambda_{\max}(Q)\cdot r^2 + \tilde{c}\cdot r + \gamma_\lambda(r) + \gamma_\lambda(c_1\cdot r), \tag{4.88}$$

with $\alpha_{\mathrm{u}} \in \mathcal{K}_\infty$. In conclusion, we have constructed a feasible input sequence $u(\cdot|t)$ that exponentially stabilizes the optimal reference x_T, resulting in a local upper bound on the value function V_N and the rotated value function $\tilde{V}_N$. ■

In addition to the upper bound on the value function and the rotated value function, we also need an additional local continuity bound on the value function.

Assumption 4.76. *(Local continuity value function) There exists a function $\alpha_{\mathrm{V}} \in \mathcal{K}_\infty$ such that for all $(x_1, x_2, t) \in \mathbb{X}\times\mathbb{X}\times\mathbb{I}_{\geq 0}$ satisfying $x_1 \in \mathbb{B}_{c_T}(x_T(t))$, $x_2 \in \mathbb{B}_{c_T}(x_T(t))$ with c_T from Proposition 4.75, the value function from Problem 4.1 satisfies*

$$|V_N(x_1,t) - V_N(x_2,t)| \leq \alpha_{\mathrm{V}}(\|x_T(t) - x_1\| + \|x_T(t) - x_2\|). \tag{4.89}$$

For general nonlinear systems, this local continuity condition can be ensured if the system is locally finite-time controllable (cf. Def. 3.9, Ass. 3.77), compare [210, Thm. 16] and [122, Thm. 6.4].

Given the bounds in Proposition 4.75, strict dissipativity (Ass. 4.72) and continuity of the value function (Ass. 4.76), we can establish a turnpike property and local bounds on the corresponding finite horizon cost.

Lemma 4.77. *Let Assumptions 4.16, 4.70, 4.72 and 4.76 hold. Suppose the system is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.18). Then, there exist functions σ, $\tilde{\sigma} \in \mathcal{L}$ such that the following properties hold for all $N \in \mathbb{I}_{\geq 1}$, all $M \in \mathbb{I}_{[1,N]}$, and all $\|e_T(t)\| \leq c_T$ with c_T according to Proposition 4.75:*
The optimal solution of Problem 4.1 contains at least M points $k \in \mathbb{I}_{[0,N-1]}$ that satisfy

$$\|x^*(k|t) - x_T(t+k)\| \leq \sigma(N-M+1). \tag{4.90}$$

The optimal solutions of Problems 4.1 and 4.74 contain at least M points $k_x \in \mathbb{I}_{[0,N-1]}$ that simultaneously satisfy

$$\|x^*(k_x|t) - x_T(t+k_x)\| \leq \tilde{\sigma}(N-M),\ \|\tilde{x}^*(k_x|t) - x_T(t+k_x)\| \leq \tilde{\sigma}(N-M). \tag{4.91}$$

Furthermore, for $N \geq M + \tilde{\sigma}^{-1}(c_T)$ the corresponding open-loop costs satisfy

$$\tilde{\mathcal{J}}_{k_x}(x^*(\cdot|t), u^*(\cdot|t), t) \leq \tilde{\mathcal{J}}_{k_x}(\tilde{x}^*(\cdot|t), \tilde{u}^*(\cdot|t), t) + 2\gamma_\lambda(\tilde{\sigma}(N-M)) + \alpha_{\mathrm{V}}(2\tilde{\sigma}(N-M)). \tag{4.92}$$

Proof. The proof is composed of four parts. In Part I, the rotated cost of the optimal solution of Problem 4.1 is bounded. In Part II, the turnpike of the optimal solution (4.90) is established. In Part III, the combined turnpike property (4.91) is derived. In Part IV, Inequality (4.92) is shown. Denote $e^*(k|t) = x^*(k|t) - x_T(t+k)$, $\tilde{e}^*(k|t) = \tilde{x}^*(k|t) - x_T(t+k)$, $k \in \mathbb{I}_{[0,N]}$.

Part I: First, define the set $\tilde{\mathbb{Z}}_{\mathbb{X}} := \{\tilde{x} \in \mathbb{X} \mid \exists (x,u) \in \mathbb{Z} : f(x,u) = \tilde{x}\}$, with[14] $x^*(N|t) \in \tilde{\mathbb{Z}}_{\mathbb{X}}$ and $\tilde{\mathbb{Z}}_{\mathbb{X}}$ compact since f continuous and $\mathbb{Z}$ compact. We bound the rotated cost of the optimal solution of Problem 4.1 using Proposition 4.75 and Assumption 4.72:

$$\begin{aligned}
&\tilde{\mathcal{J}}_N(x^*(\cdot|t), u^*(\cdot|t), t) \\
\overset{(4.85)}{=} & V_N(x(t), t) - c_N(t) + \lambda(x(t), t) - \lambda(x^*(N|t), t+N) \\
\overset{(4.84),(4.87a)}{\leq} & \gamma\|e_T(t)\|_Q^2 + \tilde{c}\|e_T(t)\| + \gamma_\lambda(\|e_T(t)\|) + \gamma_\lambda(\|e_T^*(N|t)\|) \\
\overset{(4.88)}{\leq} & \alpha_{\mathrm{u}}(\|e_T(t)\|) + C \leq \alpha_{\mathrm{u}}(c_T) + C,
\end{aligned}$$

with

$$C := \max_{k \in \mathbb{I}_{[0,T-1]}} \max_{x \in \tilde{\mathbb{Z}}_{\mathbb{X}}} \gamma_\lambda(\|x - x_T(k)\|). \tag{4.93}$$

Similarly, the rotated value function satisfies

$$\tilde{V}_N(x(t), t) \leq \tilde{\mathcal{J}}_N(x^*(\cdot|t), u^*(\cdot|t), t) \leq \alpha_{\mathrm{u}}(c_T) + C.$$

Part II: Define

$$\sigma(N-M+1) := \alpha_\ell^{-1}\left(\frac{\alpha_{\mathrm{u}}(c_T) + C}{N-M+1}\right), \quad \sigma \in \mathcal{L}.$$

Suppose there exist $N-M+1$ instances $k \in \mathbb{I}_{[0,N-1]}$ with $\|e_T^*(k|t)\| > \sigma(N-M+1)$.

[14] This set is only required since, in contrast to the MPC formulation in [122], Problem 4.1 does not include the additional constraint $x^*(N|t) \in \mathbb{Z}_{\mathbb{X}}$.

Using Assumption 4.72, the rotated cost satisfies

$$\begin{aligned}\tilde{\mathcal{J}}_N(x^*(\cdot|t),u^*(\cdot|t),t) &= \sum_{k=0}^{N-1} L(x^*(k|t),u^*(k|t),t+k)\\ &> (N-M+1)\alpha_\ell(\sigma(N-M+1)) = \alpha_\mathrm{u}(c_T)+C.\end{aligned}$$

This is a contradiction to the derived bound for $\tilde{\mathcal{J}}_N$. Thus, at least M instances $k \in \mathbb{I}_{[0,N-1]}$ satisfy $\|e_T^*(k|t)\| \leq \sigma(N-M+1)$.

Part III: Similar to Part II, given any $N_0 \in \mathbb{I}_{\geq 1}$. There exist at most N_0 instances $k \in \mathbb{I}_{[0,N-1]}$ that satisfy $\|e_T^*(k|t)\| > \sigma(N_0+1)$ or $\|\tilde{e}_T^*(k|t)\| > \sigma(N_0+1)$ respectively. Using arguments similar to [122, Sec. 7, Eq. (21)], there exist at least $M = N - 2N_0$ points k_x, that simultaneously satisfy

$$\|e_T^*(k_x|t)\| \leq \tilde{\sigma}(N-M), \quad \|\tilde{e}_T^*(k_x|t)\| \leq \tilde{\sigma}(N-M),$$

with

$$\tilde{\sigma}(N-M) := \alpha_\ell^{-1}\left(2\frac{\alpha_\mathrm{u}(c_T)+C}{N-M}\right), \quad \tilde{\sigma} \in \mathcal{L}.$$

Part IV: Given the derivation in Part III, Assumptions 4.72 and 4.76, and $\tilde{\sigma}(N-M) \leq c_T$, we have

$$\begin{aligned}&\tilde{\mathcal{J}}_{k_x}(x^*(\cdot|t),u^*(\cdot|t),t)\\ \overset{(4.85)}{=}& V_N(x(t),t)+\lambda(x(t),t)-\lambda(x^*(k_x|t),t+k_x)-c_{k_x}(t)\\ &-V_{N-k_x}(x^*(k_x|t),t+k_x)\\ \leq& \mathcal{J}_{k_x}(\tilde{x}^*(\cdot|t),\tilde{u}^*(\cdot|t),t)+\lambda(x(t),t)-\lambda(x^*(k_x|t),t+k_x)-c_{k_x}(t)\\ &+V_{N-k_x}(\tilde{x}^*(k_x|t),t+k_x)-V_{N-k_x}(x^*(k_x|t),t+k_x)\\ \overset{(4.85)}{=}& \tilde{\mathcal{J}}_{k_x}(\tilde{x}^*(\cdot|t),\tilde{u}^*(\cdot|t),t)+\lambda(\tilde{x}^*(k_x|t),t+k_x)-\lambda(x^*(k_x|t),t+k_x)\\ &+V_{N-k_x}(\tilde{x}^*(k_x|t),t+k_x)-V_{N-k_x}(x^*(k_x|t),t+k_x)\\ \overset{(4.84),(4.89),(4.91)}{\leq}& \tilde{\mathcal{J}}_{k_x}(\tilde{x}^*(\cdot|t),\tilde{u}^*(\cdot|t),t)+2\gamma_\lambda(\tilde{\sigma}(N-M))+\alpha_\mathrm{V}(2\tilde{\sigma}(N-M)).\end{aligned}$$ ■

The following lemma shows that similar bounds hold on sublevel sets of the rotated value function $\tilde{V}_N$, which is necessary to derive a suitable region of attraction.

Lemma 4.78. *Let Assumptions 4.70, 4.72, and 4.76 hold. Then, for any $\overline{V} > 0$, there exists* *a function $\tilde{\sigma}_{\overline{V}} \in \mathcal{L}$ such that for any $(x(t), t) \in \mathbb{X} \times \mathbb{I}_{\geq 0}$, $N \in \mathbb{I}_{\geq 1}$ with $\tilde{V}_N(x(t), t) \leq \overline{V}$ and any $M \in \mathbb{I}_{[1,N]}$, the optimal solutions of Problems 4.1 and 4.74 contain at least M points $k_x \in \mathbb{I}_{[0,N-1]}$, that simultaneously satisfy*

$$\|e_T^*(k_x|t)\| \leq \tilde{\sigma}_{\overline{V}}(N-M), \quad \|\tilde{e}_T^*(k_x|t)\| \leq \tilde{\sigma}_{\overline{V}}(N-M).$$

Furthermore, for $N \geq M + \tilde{\sigma}_{\overline{V}}^{-1}(c_T)$ the corresponding open-loop costs satisfy

$$\begin{aligned} &\tilde{\mathcal{J}}_{k_x}(x^*(\cdot|t), u^*(\cdot|t), t) - \tilde{\mathcal{J}}_{k_x}(\tilde{x}^*(\cdot|t), \tilde{u}^*(\cdot|t), t) \\ \leq &2\gamma_\lambda(\tilde{\sigma}_{\overline{V}}(N-M)) + \alpha_{\mathrm{V}}(2\tilde{\sigma}_{\overline{V}}(N-M)). \end{aligned} \tag{4.94}$$

Proof. The rotated Lyapunov function (Problem 4.74) satisfies

$$\tilde{V}_N(x(t), t) = \mathcal{J}_N(\tilde{x}^*(\cdot|t), \tilde{u}^*(\cdot|t), t) - c_N(t) + \lambda(x(t), t) - \lambda(\tilde{x}^*(N|t), t+N).$$

Correspondingly we have

$$\begin{aligned} &\tilde{\mathcal{J}}_N(x^*(\cdot|t), u^*(\cdot|t), t) = V_N(x(t), t) - c_N(t) + \lambda(x(t), t) - \lambda(x^*(N|t), t+N) \\ \leq &\mathcal{J}_N(\tilde{x}^*(\cdot|t), \tilde{u}^*(\cdot|t), t) - c_N(t) + \lambda(x(t), t) - \lambda(x^*(N|t), t+N) \\ = &\tilde{V}_N(x(t), t) + \lambda(\tilde{x}^*(N|t), t+N) - \lambda(x^*(N|t), t+N) \\ \leq &\overline{V} + 2C, \end{aligned}$$

with C according to Equation (4.93). The rest of the proof is analogous to Lemma 4.77 with

$$\tilde{\sigma}_{\overline{V}}(N-M) := \alpha_\ell^{-1}\left(2\frac{\overline{V} + 2C}{N-M}\right). \qquad \blacksquare$$

We have established turnpike properties of the optimal solutions $x^*(\cdot|t)$, $\tilde{x}^*(\cdot|t)$ on any sublevel sets of $\tilde{V}_N$. Note that the bounds are quantitatively significantly more conservative than the local bounds in Lemma 4.77.

Closed-loop properties

Given these preliminaries, we can study the closed-loop properties of the MPC scheme for unreachable reference trajectories. To study practical tracking, we introduce the set $\mathbb{S}_{c_T} := \{(x, t) \in \mathbb{X} \times \mathbb{I}_{\geq 0} | \tilde{V}_N(x, t) \leq \alpha_\ell(c_T)\}$ with c_T according to Proposition 4.75 and α_ℓ according to Assumption 4.72. The following theorem establishes local practical tracking of the optimal reachable trajectory x_T.

Theorem 4.79. *Let Assumptions 4.16, 4.70, 4.72 and 4.76 hold. Suppose the system is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.18). Then, there exists a constant $\tilde{N}_1 > 0$ such that for all $N \geq \tilde{N}_1$ and any initial condition $(x_0, 0) \in \mathbb{S}_{c_T}$, the closed-loop system* (4.3) *resulting from Algorithm 4.2 satisfies the constraints* (4.1)*, Problem 4.1 is feasible for all $t \in \mathbb{I}_{\geq 0}$, and $e_T = 0$ is uniformly practically asymptotically stable.*

Proof. The proof is split into three parts. Part I and II show that the rotated value function $\tilde{V}_N$ satisfies the following bounds at time $t \in \mathbb{I}_{\geq 0}$, assuming $(x(t), t) \in \mathbb{S}_{c_T}$:

$$\alpha_\ell(\|e_T(t)\|) \leq \tilde{V}_N(x(t), t) \leq \alpha_{\mathrm{u}}(\|e_T(t)\|), \tag{4.95a}$$

$$\tilde{V}_N(x(t+1), t+1) - \tilde{V}_N(x(t), t) \leq -\alpha_\ell(\|e_T(t)\|) + \tilde{\theta}(N-2), \tag{4.95b}$$

with some $\tilde{\theta} \in \mathcal{L}$. Part III establishes that $(x(t), t) \in \mathbb{S}_{c_T}$ holds recursively for all $t \in \mathbb{I}_{\geq 0}$ and establishes uniform practical asymptotic stability. Abbreviate $e_T^*(k|t) := x^*(k|t) - x_T(t+k)$ and $\tilde{e}_T^*(k|t) := \tilde{x}^*(k|t) - x_T(t+k)$, $k \in \mathbb{I}_{[0,N-1]}$.

Part I: Boundedness of $\tilde{V}_N$: The lower bound is based on the strict dissipativity property in Assumption 4.72, yielding

$$\tilde{V}_N(x(t), t) \geq L(x(t), \tilde{u}^*(0|t), t) \geq \alpha_\ell(\|e_T(t)\|).$$

This property in combination with $\tilde{V}_N(x(t), t) \leq \alpha_\ell(c_T)$ implies $\|e_T(t)\| \leq c_T$. The upper bound then directly follows from Proposition 4.75.

Part II: Decrease of $\tilde{V}_N$: With Lemma 4.77, Inequalities (4.91) and $M = 2$, there exists a $k_x \in \mathbb{I}_{[1,N-1]}$ such that

$$\|e_T^*(k_x|t)\| \leq \tilde{\sigma}(N-2), \quad \|\tilde{e}_T^*(k_x|t)\| \leq \tilde{\sigma}(N-2).$$

For $N \geq \tilde{N}_0 := 2 + \tilde{\sigma}^{-1}(c_T)$, we can apply the results of Proposition 4.75 and Lemma 4.77.

Using the dynamic programming principle, we have

$$\begin{aligned}
&\tilde{V}_N(x(t+1),t+1)\\
\leq &\sum_{k=1}^{k_x-1} L(x^*(k+1|t),u^*(k+1|t),t+k)+\tilde{V}_{N-k_x+1}(x^*(k_x|t),t+k_x)\\
\overset{(4.87b),(4.92)}{\leq} &\tilde{\mathcal{J}}_{k_x}(\tilde{x}^*(\cdot|t),\tilde{u}^*(\cdot|t),t)-L(x(t),u(t),t)\\
&+\alpha_{\mathrm{u}}(\tilde{\sigma}(N-2))+2\lambda_\gamma(\tilde{\sigma}(N-2))+\alpha_{\mathrm{V}}(2\tilde{\sigma}(N-2))\\
\overset{(4.83),(4.91)}{\leq} &\tilde{V}_N(x(t),t)-\alpha_\ell(\|e_T(t)\|)+\tilde{\theta}(N-2)),
\end{aligned}$$

with $\tilde{\theta}:=\alpha_{\mathrm{u}}(\tilde{\sigma})+2\gamma_\lambda(\tilde{\sigma})+\alpha_{\mathrm{V}}(2\tilde{\sigma})\in\mathcal{L}$.

Part III: Practical asymptotic stability: In order to apply this argument recursively, we have to ensure that $\tilde{V}_N(x(t),t)\leq\alpha_\ell(c_T)$ holds recursively. Using Inequalities (4.95) and Lemma 4.71, we can recursively establish the following bound for all $t\in\mathbb{I}_{\geq 0}$:

$$\tilde{V}_N(x(t),t)\leq\alpha_{\mathrm{u}}(\alpha_\ell^{-1}(\tilde{\theta}(N-2)))+\tilde{\theta}(N-2)=:\alpha_\theta(N-2)\leq\alpha_\ell(c_T),$$

for all

$$N\geq\tilde{N}_1:=2+\alpha_\theta^{-1}(\alpha_\ell(c_T)). \tag{4.96}$$

Thus, for all $N\geq\tilde{N}_1\geq\tilde{N}_0$ the set $\mathbb{S}_{c_T}$ is positively invariant. Furthermore, the origin $e_T=0$ is uniformly practically asymptotically stable with the practical Lyapunov function $\tilde{V}_N$ (cf. Lemma 4.71). ■

The following theorem shows that we can increase the region of attraction by increasing the prediction horizon N.

Theorem 4.80. *Let Assumptions 4.16, 4.70, 4.72, and 4.76 hold. Suppose the system is locally incrementally uniformly exponentially stabilizable on the set $\mathbb{Z}$ (Def. 4.18). Then, for any $\overline{V}>0$, there exists a constant $\tilde{N}_{\overline{V}}>0$ such that for all $N>\tilde{N}_{\overline{V}}$ and any initial condition $(x_0,0)\in\mathbb{X}_{\overline{V}}:=\{(x,t)\in\mathbb{X}\times\mathbb{I}_{\geq 0}\mid\tilde{V}_N(x,t)\leq\overline{V}\}$, the closed-loop system* (4.3) *resulting from Algorithm 4.2 satisfies the constraints* (4.1)*, Problem 4.1 is feasible for all $t\in\mathbb{I}_{\geq 0}$, and $e_T=0$ is uniformly practically asymptotically stable.*

Proof. Part I and II show that the rotated value function $\tilde{V}_N$ satisfies the following bounds at time $t \in \mathbb{I}_{\geq 0}$, assuming $(x(t), t) \in \mathbb{X}_{\overline{V}}$:

$$\begin{aligned}\alpha_\ell(\|e_T(t)\|) \leq \tilde{V}_N(x(t), t) \leq& \alpha_{\mathrm{u},\overline{V}}(\|e_T(t)\|),\\ \tilde{V}_N(x(t+1), t+1) - \tilde{V}_N(x(t), t) \leq& -\tilde{\alpha}_{N,\overline{V}}(\|e_T(t)\|) + \tilde{\theta}(N-2),\end{aligned}$$

with some $\alpha_{\mathrm{u},\overline{V}}, \tilde{\alpha}_{N,\overline{V}} \in \mathcal{K}_\infty$. Part III establishes that $(x(t), t) \in \mathbb{X}_{\overline{V}}$ holds recursively for all $t \in \mathbb{I}_{\geq 0}$ and shows uniform practical asymptotic stability. Abbreviate $e_T^*(k|t) := x^*(k|t) - x_T(t+k)$ and $\tilde{e}_T^*(k|t) := \tilde{x}^*(k|t) - x_T(t+k)$, $k \in \mathbb{I}_{[0,N-1]}$.

Part I: Boundedness of $\tilde{V}_N$: Using the bound $\overline{V}$ and Proposition 4.75, the function

$$\alpha_{\mathrm{u},\overline{V}}(r) := \begin{cases} \max\{\alpha_\mathrm{u}(r), \frac{r}{c_T}\overline{V}\} & \text{if } r \leq c_T \\ \max\{\alpha_\mathrm{u}(c_T), \overline{V}\} + \epsilon(r - c_T) & \text{else} \end{cases},$$

satisfies $\alpha_{u,\overline{V}} \in \mathcal{K}_\infty$ and $\tilde{V}_N(x(t), t) \leq \alpha_{\mathrm{u},\overline{V}}(\|e_T(t)\|)$ for arbitrary small $\epsilon > 0$. The lower bound is analogous to Theorem 4.79.

Part II: Decrease of $\tilde{V}_N$: The following derivation is split in two Parts (a), (b) depending on whether $\|e_T(t)\| \leq \tilde{c}_T := \alpha_\mathrm{u}^{-1}(\alpha_\ell(c_T)) \leq c_T$ or not, which are unified in (c).

(a) Assume $\|e_T(t)\| \leq \tilde{c}_T$, which implies $(x(t), t) \in \mathbb{S}_{c_T}$ by Proposition 4.75. Then we can use the derivations from Theorem 4.79 to conclude

$$\tilde{V}_N(x(t+1), t+1) \leq \tilde{V}_N(x(t), t) - \alpha_\ell(\|e_T(t)\|) + \tilde{\theta}(N-2).$$

For $N \geq \tilde{N}_1$ we have positive invariance of $\mathbb{S}_{c_T}$.

(b) Assume $\|e_T(t)\| > \tilde{c}_T$. Lemma 4.78 with $M = 1$ ensures that there exists a $k_x \in \mathbb{I}_{[0,N-1]}$ such that

$$\|e_T^*(k_x|t)\| \leq \tilde{\sigma}_{\overline{V}}(N-1), \quad \|\tilde{e}_T^*(k_x|t)\| \leq \tilde{\sigma}_{\overline{V}}(N-1).$$

For $N \geq 1 + \tilde{\sigma}_{\overline{V}}(\tilde{c}_T)$, we have $\|e_T^*(k_x|t)\| \leq \tilde{c}_T$ and thus $k_x \geq 1$. Using Proposition 4.75, we get

$$\tilde{V}_{N-k_x+1}(x^*(k_x|t), t+k_x) \leq \alpha_\mathrm{u}(\|e_T^*(k_x|t)\|) \leq \alpha_\mathrm{u}(\tilde{\sigma}_{\overline{V}}(N-1)).$$

This yields

$$\begin{aligned}
&\tilde{V}_N(x(t+1),t+1)\\
\leq &\sum_{k=1}^{k_x-1} L(x^*(k+1|t),u^*(k+1|t),t+k)+\tilde{V}_{N-k_x+1}(x^*(k_x|t),t+k_x)\\
\overset{(4.94)}{\leq} &\tilde{\mathcal{J}}_{k_x}(\tilde{x}^*(\cdot|t),\tilde{u}^*(\cdot|t),t)-L(x(t),u(t),t)\\
&+\alpha_{\mathrm{u}}(\tilde{\sigma}_{\overline{V}}(N-1))+2\gamma_\lambda(\tilde{\sigma}_{\overline{V}}(N-1))+\alpha_{\mathrm{V}}(2\tilde{\sigma}_{\overline{V}}(N-1))\\
\leq &\tilde{V}_N(x(t),t)-\alpha_\ell(\|e_T(t)\|)+\tilde{\theta}_{\overline{V}}(N-1),
\end{aligned}$$

with $\tilde{\theta}_{\overline{V}} := \alpha_{\mathrm{u}}(\tilde{\sigma}_{\overline{V}})+2\gamma_\lambda(\tilde{\sigma}_{\overline{V}})+\alpha_{\mathrm{V}}(2\tilde{\sigma}_{\overline{V}}) \in \mathcal{L}$. For

$$N > \tilde{N}_2 := \tilde{\theta}_{\overline{V}}^{-1}(\alpha_\ell(\tilde{c}_T))+1, \tag{4.97}$$

we have $\tilde{\theta}_{\overline{V}}(N-1) < \alpha_\ell(\tilde{c}_T)$ and thus

$$\tilde{V}_N(x(t+1),t+1)-\tilde{V}_N(x(t),t) \leq -\tilde{\alpha}_{2,N},$$

with $\tilde{\alpha}_{2,N} := \alpha_\ell(c_T)-\tilde{\theta}_{\overline{V}}(N-1) > 0$.
(c) Given (a) and (b) we can unify the results as follows. Let $N > \tilde{N}_{\overline{V}} := \max\{\tilde{N}_1,\tilde{N}_2\}$ with $\tilde{N}_1$, $\tilde{N}_2$ according to Equations (4.96) and (4.97). The function

$$\tilde{\alpha}_{N,\overline{V}}(r) := \begin{cases} \min\left\{\alpha_\ell(r),\tilde{\alpha}_{3,N}\dfrac{r}{\tilde{c}_T}\right\} & \text{if} \quad r \leq \tilde{c}_T \\ \min\{\alpha_\ell(\tilde{c}_T),\tilde{\alpha}_{3,N}\}+\dfrac{r-\tilde{c}_T}{r_{\max}-\tilde{c}_T}\epsilon_2 & \text{else} \end{cases}$$

with $r_{\max} = \max_{x_1,x_2\in\mathbb{Z}_{\mathbb{X}}}\|x_1-x_2\|$, $\epsilon_2 = \frac{1}{2}(\tilde{\theta}(N-2)+\tilde{\alpha}_{2,N})$ and $\tilde{\alpha}_{3,N} = \tilde{\alpha}_{2,N}+\tilde{\theta}(N-2)-\epsilon_2 > 0$ satisfies $\tilde{\alpha}_{N,\overline{V}} \in \mathcal{K}_\infty$ and

$$\tilde{V}_N(x(t+1),t+1)-\tilde{V}_N(x(t),t) \leq -\tilde{\alpha}_{N,\overline{V}}(\|e_T(t)\|)+\tilde{\theta}(N-2).$$

Part III: Recursive feasibility and practical asymptotic stability: Based on the decrease condition in Part II b) we have $\tilde{V}_N(x(t),t) \leq \overline{V}$ recursively satisfied and the derivations in Part I and II hold for all $t \in \mathbb{I}_{\geq 0}$. Furthermore, $(x(t),t)$ converges to the set $\mathbb{S}_{c_T}$ in at most $\tilde{k} := \left\lceil \frac{\overline{V}-\alpha_\ell(c_T)}{\tilde{\alpha}_{2,N}} \right\rceil$ steps. From the derivations in Part II (a) and Theorem 4.79 we further know that this set is positively invariant for the closed-loop dynamics. Thus, the origin $e_T = 0$ is uniformly practically asymptotically stable.

We have shown, for any prediction horizon $N > \tilde{N}_{\overline{V}}$ and any initial condition with $\tilde{V}_N(x(0),0) \leq \overline{V}$, the closed-loop uniformly converges to the set $\mathbb{S}_{c_T}$, which is centered around the optimal trajectory x_T. ∎

Discussion

We point out that in the economic MPC analysis in Theorem 4.80 the method to extend the region of attraction differs strongly from the analysis in Theorems 4.5 for reachable reference trajectories. Note that Theorem 4.80 also implies the following closed-loop asymptotic average performance bound:

$$\limsup_{K\to\infty} \frac{1}{K}\sum_{t=0}^{K-1} \ell(x(t),u(t),t) = \limsup_{K\to\infty} \frac{1}{K}\sum_{t=0}^{K-1} L(x(t),u(t),t) \leq \frac{V_{T,\min}}{T} + \tilde{\theta}(N-2).$$

Overall, the resulting closed-loop system approximately achieves the same performance as the *unknown* optimal reachable trajectory x_T.

Remark 4.81. *(Interpretation) By combining Theorem 4.80 with Lemma 4.73, we have a set of suitable assumptions that ensure practical tracking for general reference trajectories. Given an in general unreachable (periodic) reference trajectory $(x_\mathrm{r}, u_\mathrm{r})$, there exists a trajectory (x_T, u_T) (Problem 4.69), which is reachable and as close as possible to the given reference trajectory. Assume that the system is uniformly suboptimally operated off x_T, which means that any trajectory with a close to optimal cost has a turnpike property with respect to x_T. Such an assumption is not very restrictive due to the positive definite cost Q, assuming the minimizer x_T is unique. Then, Lemma 4.73 ensures strict dissipativity (Ass. 4.72) if the system is locally controllable around x_T. Finally, Theorem 4.80 ensures practical tracking of x_T, provided the prediction horizon N is chosen large enough. The main requirements are local stabilizability/controllability (Def. 4.18), that x_T lies strictly in the constraint set (Ass. 4.70) and is unique. To summarize, the closed-loop system "does the right thing" in that it "finds" the best trajectory x_T, which is as close as possible to the unreachable reference, given that a large enough prediction horizon N is used.*

Remark 4.82. *(Related MPC results) Compared to the results in [210], the period length T is not needed for the online implementation. This is mainly due to the different setup and the corresponding difference in the dissipativity assumption (time-varying vs. orbit).*

Compared to the result in [122, Thm. 4.2], where economic MPC in case of steady-state optimality is considered, the presented theorem relaxes the controllability assumption. A direct extension of the results in [122] to arbitrary reference trajectories would require a finite-time *controllability assumption on the full state space $\mathbb{X}$ (similar assumptions are used in [126, Sec. 8], [210], [98]). In contrast, our result only requires local stabilizability and utilizes level sets similar to [37]. On level sets we automatically have a finite-time controllability property with respect to the set $\mathbb{S}_{c_T}$, as derived in Lemma 4.78 and exploited in Theorem 4.80. We note that the more recent results in [129] also implicitly require a global stabilizability/controllability condition (cf. [129, Ass. 1]). Furthermore, the proof in [129, Thm. 4] also uses a case distinction analogous to Theorem 4.80 to ensure positive invariance of the sublevel set of the rotated Lyapunov function $\tilde{V}_N$.*

In contrast to competing results for unreachable reference trajectories based on artificial reference trajectories (cf. Sec. 3.2, [89, 166]), the presented approach can practically track the optimal reference without explicitly computing it.

Compared to reachable reference trajectories (Sec. 4.1), a drawback is that the bounds on the prediction horizon N depend on quantities that are defined by the optimal reachable trajectory (c_T,C), for which hard bounds are difficult to obtain in practice (similar problems also appear in [122], where economic MPC in case of steady-state optimality is considered). Hence, in economic MPC without terminal ingredients the bounds on the prediction horizon N are typically more of a conceptual nature.

Remark 4.83. *(Nonperiodic problems) In case we have a nonperiodic reference trajectory r, closed-loop stability and performance bounds for the MPC scheme over any finite interval $K \in \mathbb{I}_{\geq 1}$ can still be ensured with the presented theorems. In particular, consider a T-periodic reference $(\tilde{x}_{\mathrm{r}}, \tilde{u}_{\mathrm{r}})$, with $(\tilde{x}_{\mathrm{r}}(t), \tilde{u}_{\mathrm{r}}(t)) = (x_{\mathrm{r}}(t), u_{\mathrm{r}}(t))$, for all $t \in \mathbb{I}_{[0,N+K]}$ and the period length $T \geq N + K$. Clearly, over the finite interval K the closed-loop system for both reference trajectories with the same initial condition cannot be distinguished. Thus, the theorems regarding periodic reference trajectories can be used to describe the performance over any finite-time interval. This is especially relevant in case the reference trajectory becomes reachable after some finite-time interval, see the numerical example in Section 4.5. An extension of the presented analysis to general nonperiodic reference trajectories could be based on the concept of overtaking optimality [128], compare the more recent results in [129, 130].*

Remark 4.84. *(Unreachable trajectories in output regulation) The considered arguments regarding unreachable reference trajectories can also be used for the output regulation problem (Sec. 4.2) in case the regulator equations do not admit a solution (Ass. 4.39 does not hold). This is especially relevant if the system is under-actuated ($p > m$) or opposing goals (minimum energy and exact tracking) are combined (cf. [34]). In this case, the optimal trajectory x_T is replaced by some maps $\tilde{\pi}_{\mathrm{x}} : \mathbb{W} \to \mathbb{X}_{\mathrm{p}}$, $\tilde{\pi}_{\mathrm{u}} : \mathbb{W} \to \mathbb{U}$, that also satisfy the dynamics (cf.* (4.56a)*), but do not always yield zero output (Condition* (4.56b) *does not hold). For the special case of linear* *dynamics and no exosystem, in [34] modified regulator equations are derived to compute the solution with minimal error. Note that the "optimal" mapping yielding the smallest output is in general a function of the initial condition w_0. In particular, for a given initial condition w_0 the output regulation problem can be equivalently written as an output reference tracking problem (cf. Sec. 4.1) with time-varying dynamics. Correspondingly, in this case the role of the distance $\|e_T\|^2$ is replaced by the "state measure" $\sigma(x) = \|x_{\mathrm{p}} - \tilde{\pi}_{\mathrm{x}}(w)\|^2$. Note that for w periodic (Ass. 4.58), we have a time-invariant problem and the optimal mode of operation is periodic, similar to [210]. However, a crucial difference is the fact that w is autonomous and that hence the optimal (reachable) mode of operation may depend on the initial condition w_0, compare also [81]. Note that in [JK25, Cor. 4] also economic MPC without terminal ingredients for time-invariant problems and optimal periodic operation is analysed. However, the corresponding conditions are rarely applicable in practice.*

Summary

In this section, we studied the closed-loop properties of a tracking MPC scheme without terminal ingredients in case the reference trajectory is *unreachable*. We provided sufficient conditions in terms of uniqueness of the optimal mode of operation and stabilizability/controllability conditions that guarantee desirable properties of the closed loop. In particular, we characterized a sufficiently long prediction horizon N and a region of attraction that ensure that the closed loop (practically) tracks the (unknown) optimal mode of operation. In the next section, we study the special case of *linear* system dynamics and show how the assumptions imposed in this chapter simplify.

4.4 Linear systems

In this section, we show how the derivations and conditions in Sections 4.1–4.3 simplify in case of linear system dynamics. We first show how the conditions in Section 4.2 reduce to classical conditions considered in the output regulation literature (Sec. 4.4.1). Then, we provide simpler conditions for the tracking MPC in case of unreachable reference trajectories (Sec. 4.4.2). Finally, we show less conservative bounds for MPC without terminal ingredients (Sec. 4.4.3). This section is based on and taken in parts literally from [JK24][15] and [JK19][16].

[15] J. Köhler, M. A. Müller, and F. Allgöwer. "Nonlinear reference tracking: An economic model predictive control perspective." In: *IEEE Trans. Automat. Control* 64.1 (2019), pp. 254–269©2018 IEEE.

[16] J. Köhler, M. A. Müller, and F. Allgöwer. "Constrained nonlinear output regulation using Model Predictive Control." In: *IEEE Trans. Automat. Control* (2021). extended version: arXiv:2005.12413©2021 IEEE.

4.4.1 Output regulation

In the following, we revisit the output regulation problem from Section 4.2 for the special case of linear systems and show how the corresponding system theoretic conditions simplify. Furthermore, in Proposition 4.86 we show the equivalence of the (classical) rank/eigenvalue-based characterization of nonresonance and the proposed dissipation-based condition (Ass. 4.63).

Setup

In the special case of linear systems, the system (4.54) reduces to

$$\begin{aligned} x_{\mathrm{p}}^+ &= f_{\mathrm{p}}(x_{\mathrm{p}}, w, u) := Ax_{\mathrm{p}} + Bu + P_{\mathrm{x}}w, \\ w^+ &= s(w) := Sw, \\ y &= h(x_{\mathrm{p}}, w, u) = Cx_{\mathrm{p}} + Du - P_{\mathrm{y}}w. \end{aligned}$$

Stabilizability/Detectability

The incremental stabilizability condition (Def. 4.44) reduces to stabilizability of (A, B) and a stabilizing input is given by the linear control law $u(k) = K(x(k) - z(k)) + v(k)$, with $(A + BK)$ Schur. Detectability of (A, C) is equivalent to i-IOSS (Ass. 4.62) with a quadratic i-IOSS Lyapunov function $V_{\mathrm{o}}(x_{\mathrm{p}}, z_{\mathrm{p}}, w) = \|x_{\mathrm{p}} - z_{\mathrm{p}}\|_{P_{\mathrm{o}}}^2$, compare [50]. The finite step i-OSS condition (Ass. 4.26) reduces to observability of (A, C) with the lag $\nu \in \mathbb{I}_{[1,n_{\mathrm{p}}]}$. The regulator equations (4.56) (Ass. 4.39) reduce to

$$\Pi S = A\Pi + B\Gamma + P_{\mathrm{x}}, \quad 0 = C\Pi + D\Gamma - P_{\mathrm{y}}, \tag{4.98}$$

with $\pi_{\mathrm{x}}(w) = \Pi w$, $\pi_{\mathrm{u}}(w) = \Gamma w$, and matrices $\Pi \in \mathbb{R}^{n_{\mathrm{p}} \times q}$, $\Gamma \in \mathbb{R}^{m \times q}$. Note that satisfaction of Assumption 4.10 for $R \succ 0$ and (A, C) detectable has also been shown in [136, Cor. 2], analogous to Proposition 4.25. In case of polytopic constraints $\mathbb{Z}$, quadratic cost functions V_{f}, ℓ, and a linear feedback κ, the MPC optimization problems in Sections 4.1-4.3 reduce to standard quadratic programs (QPs).

Nonresonance condition

Consider the case where the matrix S has only eigenvalues λ of the form $\lambda = e^{2\pi \mathrm{i} k/T}$ with some period length $T \in \mathbb{I}_{\geq 1}$ (Ass. 4.58), which encompasses constant and sinusoidal exogenous signals w. Correspondingly, all eigenvalues of S are on the unit circle, i.e.,

$|\lambda| = 1$, as is standard in literature [53, (A1)], [144, H1]. For simplicity, we consider square systems, i.e., $m = p$. To characterize the transmission zeros of a linear transfer matrix, we use Rosenbrock's system matrix

$$G(\lambda) := \begin{pmatrix} A - \lambda I_{n_\mathrm{p}} & B \\ C & D \end{pmatrix}.$$

In particular, $\lambda \in \mathbb{C}$ is a zero of the transfer matrix if the matrix $G(\lambda)$ does not have full rank, compare [73]. The classical nonresonance condition (cf. [141, Lemma 4.1]) reduces to $\mathrm{rank}(G(\lambda_k)) = n_\mathrm{p} + m$ for all λ_k which are eigenvalues of S, i.e., the transmission zeros of the plant do not coincide with the poles of the exosystem. Feasibility of the regulator equations (4.98) can be ensured if this nonresonance condition holds, compare [141, Lemma 4.1]. Furthermore, the matrices Π, Γ are unique since $m = p$. Hence, in case $\mathbb{Z} = \mathbb{R}^{n_\mathrm{p}+q+m}$, Assumption 4.39 holds if $\mathrm{rank}(G(\lambda)) = n_\mathrm{p} + m$ for all λ_k which are eigenvalues of S.

The augmented plant dynamics with input Δu and output y (cf. Fig. 4.1) are characterized by the matrices

$$A_\mathrm{a} := \begin{pmatrix} A & BE_2^\top \\ 0 & E_0 \end{pmatrix}, \quad C_\mathrm{a} := \begin{pmatrix} C & DE_2^\top \end{pmatrix}, \quad B_\mathrm{a} = \begin{pmatrix} B \\ E_1 \end{pmatrix}, \quad D_\mathrm{a} = D.$$

The following proposition shows that if the above *nonresonance condition* holds for all T-periodic exosystems, then the augmented plant is detectable, i.e., the result in Proposition 4.64 remains valid.

Proposition 4.85. *Consider a square linear system with (A, C) detectable. Suppose that $\mathrm{rank}(G(\lambda_k)) = n_\mathrm{p} + m$ for all $k \in \mathbb{I}_{[0,T-1]}$ with $\lambda_k = e^{2\pi \mathrm{i} k/T}$. Then, $(A_\mathrm{a}, C_\mathrm{a})$ is detectable.*

Proof. Detectability of $(A_\mathrm{a}, C_\mathrm{a})$ is equivalent to

$$\mathrm{rank} \begin{pmatrix} A - \lambda I_{n_\mathrm{p}} & BE_2^\top \\ 0 & E_0 - \lambda I_{mT} \\ C & DE_2^\top \end{pmatrix} = n + Tm, \tag{4.99}$$

for all $\lambda \in \mathbb{C}$, which satisfy $|\lambda| \geq 1$. First, consider an eigenvalue $\lambda_k = e^{2\pi \mathrm{i} k/T}$, in which case $\mathrm{rank}(E_0 - \lambda_k I_{mT}) = m(T-1)$. W.l.o.g. consider $m = 1$. The eigenvector $\xi_k = [e^{-2\pi \mathrm{i} k/T}, \ldots, e^{-2\pi \mathrm{i} kT/T}]^\top$ satisfies $(E_0 - \lambda_k I)\xi_k = 0$. Furthermore, we have $E_2^\top \xi_k = e^{-2\pi \mathrm{i} kT/T} = 1$. Thus, Condition (4.99) is equivalent to $\mathrm{rank}(G(\lambda_k)) = n_\mathrm{p} + m$. For

$\lambda_k \neq e^{2\pi \mathrm{i}k/T}$, the rank condition reduces to detectability of (A, C). Thus, $(A_\mathrm{a}, C_\mathrm{a})$ is detectable. ∎

This result considers $\lambda_k = e^{2\pi \mathrm{i}k/T}$ for $k \in \mathbb{I}_{[0,T-1]}$ instead of only the eigenvalues of S (cf. [141, Lemma 4.1]). This is due to the fact that the incremental input regularization in Section 4.2.3 simply penalizes the non-periodicity of the control input. In case of redundant inputs $m > p$, we may be able to achieve the same output trajectory y with different input trajectories u, in which case the augmented plant (cf. Fig. 4.1) is not detectable. Thus, we only consider square linear systems.

The following proposition shows that the dissipation-based characterization in Assumption 4.63 is equivalent to the rank condition in Proposition 4.85 for linear systems.

Proposition 4.86. *Consider a square linear system with (A, C) detectable. Then, Assumption 4.63 holds if and only if* $\mathrm{rank}(G(\lambda_k)) = n_\mathrm{p} + m$ *for all $\lambda_k = e^{2\pi \mathrm{i}k/T}$, $k \in \mathbb{I}_{[0,T-1]}$.*

Proof. **Part I:** Suppose Assumption 4.63 holds, but there exists some $\lambda_k = e^{2\pi \mathrm{i}k/T}$ such that $\mathrm{rank}(G(\lambda_k)) < n_\mathrm{p} + m$, i.e., there exists some (complex) vector $(x_\mathrm{p}, u) \neq 0$ such that $Ax_\mathrm{p} + Bu = \lambda_k x_\mathrm{p}$, $Cx_\mathrm{p} + Du = 0$. This implies the existence of a $T-$periodic state and input trajectory (x_p, u), which satisfies $Cx_\mathrm{p}(t) + Du(t) = 0$, $t \in \mathbb{I}_{\geq 0}$. The periodicity of this trajectory implies that the incremental input satisfies $\Delta u = 0$. Using linearity, we consider $(z_\mathrm{p}, v, \Delta v) = 0$ in Assumption 4.63 without loss of generality. Plugging the trajectories in Condition (4.79b), using a telescopic sum and $V_\mathrm{R} \geq 0$, we arrive at

$$\sum_{t=0}^{k-1} \|u(t)\|^2 \leq V_\mathrm{R}(x_\mathrm{p,a}(0), z_\mathrm{p,a}(0), w(0)), \quad \forall k \in \mathbb{I}_{\geq 1}. \tag{4.100}$$

Since u periodic and the sum in Inequality (4.100) takes a finite value for any $k \in \mathbb{I}_{\geq 1}$, this immediately implies $u \equiv 0$. Finally, since $y \equiv 0$ and (A, C) detectable, $x_\mathrm{p} \equiv 0$. Thus, the only periodic solution that satisfies $y \equiv 0$ is the trivial solution $(x_\mathrm{p}, u) \equiv 0$ and thus $\mathrm{rank}(G(\lambda_k)) = n_\mathrm{p} + m$.

Part II: Suppose $\mathrm{rank}(G(\lambda_k)) = n_\mathrm{p} + m$ and (A, C) is detectable. Then, Proposition 4.85 ensures that $(A_\mathrm{a}, C_\mathrm{a})$ is detectable. Thus, (cf. [50]) there exists a positive definite matrix $P_\mathrm{o,a} \in \mathbb{R}^{(n_\mathrm{p}+Tm)\times(n_\mathrm{p}+Tm)}$ satisfying

$$\|A_\mathrm{a} x_\mathrm{p,a} + B_\mathrm{a}\Delta u\|^2_{P_\mathrm{o,a}} - \|x_\mathrm{p,a}\|^2_{P_\mathrm{o,a}} \leq -\epsilon\|x_\mathrm{p,a}\|^2 + \|\Delta u\|^2 + \|Cx_\mathrm{p,a} + D\Delta u\|^2 \tag{4.101}$$

for any $x_\mathrm{p,a} \in \mathbb{R}^{n_\mathrm{p}+Tm}$, $u_\mathrm{a} \in \mathbb{R}^m$ with some $\epsilon > 0$. Consider $V_\mathrm{R}(x_\mathrm{p,a}, z_\mathrm{p,a}, w) = c \cdot \|x_\mathrm{p,a} - z_\mathrm{p,a}\|^2_{P_\mathrm{o,a}}$ with $c := 2/\epsilon > 0$. Inequality (4.79a) holds with $c_\mathrm{R,u} := c \cdot \lambda_\mathrm{max}(P_\mathrm{o,a})$. The definition of the memory state ξ implies

$$\|u\|^2 \overset{(4.75)}{=} \|E_2^\top \xi + \Delta u\|^2 \leq 2\|\Delta u\|^2 + 2\underbrace{\|E_2 E_2^\top\|}_{=1}\|\xi\|^2. \quad (4.102)$$

W.l.o.g. consider $(z_{\mathrm{p,a}}, v, \Delta v, w) = 0$ in Assumption 4.63 using linearity. Condition (4.79b) holds with $c_{\mathrm{R}} := c + 2 > 0$ using

$$\begin{aligned} & V_{\mathrm{R}}(A_{\mathrm{a}} x_{\mathrm{p,a}} + B_{\mathrm{a}} \Delta u, 0, 0) - V_{\mathrm{R}}(x_{\mathrm{p,a}}, 0, 0) \\ \overset{(4.101)}{\leq} & -c \cdot \epsilon \|x_{\mathrm{p,a}}\|^2 + c\|\Delta u\|^2 + c\|C x_{\mathrm{p}} + D u\|^2 \\ \leq & -2\|\xi\|^2 + (c_{\mathrm{R}} - 2)\|\Delta u\|^2 + c_{\mathrm{R}}\|h(x_{\mathrm{p}}, w, u)\|^2 \\ \overset{(4.102)}{\leq} & -\|u\|^2 + c_{\mathrm{R}}\left(\|\Delta u\|^2 + \|h(x_{\mathrm{p}}, w, u)\|^2\right). \end{aligned}$$

■

We point out again that the rank condition $\mathrm{rank}(G(\lambda_k)) = n_{\mathrm{p}} + m$ is similar to the eigenvalue/rank conditions used for input observability/detectability in [137, Thms. 2–3], which is a closely related problem. A crucial relaxation in the conditions in Proposition 4.86 is that only periodic inputs need to be observable/detectable and thus the rank condition only needs to hold for the values λ_k corresponding to the period length T, as opposed to all λ_k (outside the unit disc).

Minimum-phase - stable zeros

In the following, we revisit the conditions used in Section 4.2.2. We consider a SISO system as in Section 4.2.2. The relative degree $d \in \mathbb{I}_{\geq 0}$ in Assumption 4.46 corresponds to $CA^kB = 0$, $k \in \mathbb{I}_{[0,d]}$ and $CA^{d+1}B \neq 0$, and the maps $\Phi, \tilde{\Phi}$ are linear. Furthermore, the zero dynamics are always well-defined (Ass. 4.47), using $\alpha(x) = K_\alpha x$ with $K_\alpha = -\frac{CA^{d+2}}{CA^{d+1}B}$. If we consider the closed-loop system with $u = K_\alpha x + \tilde{u}$ we have $\eta^+ = A_\eta \eta + A_{\eta,z} z + B_{\eta,u}\tilde{u} + B_{\eta,w} w$ and the zero dynamics are stable if A_η is Schur. In this case, the dynamics in η are obviously also ISS w.r.t. $z, \tilde{u}$ with a quadratic function $V_\eta = \|\eta - \tilde{\eta}_w\|^2_{P_\eta}$ and thus Assumption 4.48 holds. The eigenvalues of A_η characterizing the zero dynamics correspond to the zeros of the transfer function (assuming (A, B, C, D) corresponds to a minimal realization), compare [73, 143].

Remark 4.87. *(Singular LQR) In the absence of constraints and without exosystem, Problem 4.41 with linear dynamics corresponds to the singular ($R = 0$) LQR with a finite horizon. Hence, Theorem 4.50 ensures stability for a receding horizon implementation of finite-horizon singular*

LQR, if (A, B) is stabilizable and the zero dynamics are stable.

Remark 4.88. *(Dissipativity) As discussed in Remark 4.11, the detectability condition in Assumption 4.10 is a special case of strict dissipativity [136], which is often considered in economic MPC. There exist recent papers analysing dissipativity for linear quadratic problems, e.g., also for (A, C) not detectable [124, Thm. 6.1]. However, the connection between zero dynamics and detectability (cf. Prop. 4.49), and hence (strict) dissipativity, seems largely unexplored in the recent economic MPC literature. This may be particularly relevant, since Remark 4.55 shows that even for trivial systems the detectability condition (Ass. 4.10) may not hold, while closed-loop stability can still be guaranteed.*

Summary

Suppose we have a linear square system that satisfies

- (A, B) stabilizable, (A, C) detectable,
- Eigenvalues of S satisfy $\lambda_k = e^{2\pi \mathbf{i} k/T}$, $T \in \mathbb{I}_{\geq 1}$,
- Nonresonance condition: $\text{rank}(G(\lambda_k)) = n_\text{p} + m$, $\forall k \in \mathbb{I}_{[0,T-1]}$, $\lambda_k = e^{2\pi \mathbf{i} k/T}$,
- No constraints[17]: $\mathbb{Z} = \mathbb{R}^{n_\text{p}+q+m}$.

Then, the conditions in Theorem 4.65 hold and the output regulation MPC with incremental input regularization (Alg. 4.59) solves the output regulation problem for N sufficiently large. Furthermore, in case $\text{rank}(G(\lambda)) = n_\text{p} + m$ for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| \geq 1$ (stable zeros, Ass. 4.48), also the MPC scheme without input regularization (Alg. 4.42) solves the output regulation problem for N sufficiently large. In case the joint system (x_p, w) is detectable, we can also design an error feedback MPC that ensures finite-gain $\mathcal{L}_2$-stability for noisy output measurements (cf. [JK19, App. B]).

In the linear case, we clearly see that the considered conditions align with the typical assumptions employed to solve the output regulation problem, compare the necessary and sufficient conditions in [72, Thm. 2]. We emphasize again that one of the main benefits of the output regulation MPC (in addition to the constraint handling capabilities) is the fact that $\pi_\text{x}, \pi_\text{u}$ are not used in the implementation, since the computation thereof can be a practical bottleneck for nonlinear systems.

[17] The result equally holds for general nonlinear constraint sets $\mathbb{Z}$ if we can ensure that $(\pi_\text{x}(w), \pi_\text{u}(w), w)$ is in the interior of $\mathbb{Z}$, compare Assumption 4.39.

4.4.2 Unreachable reference trajectory - strong duality

In the following, we show how the conditions regarding unreachable reference trajectories (Sec. 4.3) simplify in the linear setting.

Lemma 4.89. *Let Assumptions 4.16 and 4.70 hold. Assume that $\mathbb{Z}$ is a convex set, f is linear and R is positive definite. Then, Assumption 4.72 is satisfied with linear periodic storage functions $\lambda(x(k),k) = \lambda_k^\top e_T(k)$, $\gamma_\lambda(r) = r\max_k \|\lambda_k\|$. The rotated stage cost is given by*

$$L(x,u,k) = \|x - x_T(k)\|_Q^2 + \|u - u_T(k)\|_R^2.$$

Proof. Assumption 4.70 ensures that the constraints $\mathbb{Z}$ are not active in Problem 4.69. The first part of this lemma is a standard result in convex optimization (strong duality) with the dual variables λ [41, 292]. This result requires a strictly convex cost (R, Q positive definite) and convex constraints (linear system, $\mathbb{Z}$ convex) with Slater's condition. The special structure of the rotated cost is due to the inactive constraints, see also [70, Prop. 4.3]. ∎

Thus, for linear systems with a quadratic stage cost we have linear storage functions and a quadratic rotated stage cost L. This results can also be generalized for quadratic input-output stage costs (Ass. 4.33), with $y = h(x,u) = Cx + Du$ and (A,C) detectable. Ignoring the time-varying aspect, we have a linear-quadratic stage cost ℓ and can thus use the results in [124] to construct a linear-quadratic storage function λ, where the quadratic part serves the same role as an i-IOSS Lyapunov function (cf. Prop. 4.25). Notably, due to the compact constraints $\mathbb{Z}$, detectability of (A,C) can be relaxed to having no unobservable eigenvalue λ on the unit circle to derive *pre-dissipativity* using an indefinite quadratic term in the storage function, compare [124] for details.

Based on Lemma 4.89, we can also simplify the conditions in Theorem 4.79/4.80 to require ℓ quadratic (Ass. 4.16), $f(x,u) = Ax + Bu$, (A,B) controllable[18], $\mathbb{Z}$ convex, and $(x_T, u_T) \in \text{int}(\mathbb{Z})$ (Ass. 4.70). Furthermore, since L is quadratic the bounds on the rotated Lyapunov function in Inequalities (4.95) simplify to

$$\begin{aligned} \|e_T(t)\|_Q^2 \leq \tilde{V}_N(x(t),t) \leq& \gamma_{\overline{V}}\|e_T(t)\|_Q^2, \\ \tilde{V}_N(x(t+1),t+1) - \tilde{V}_N(x(t),t) \leq& -\|e_T(t)|_Q^2 + \tilde{\theta}_{\overline{V}}(N-1), \end{aligned}$$

[18] Except for V_N locally continuous (Ass. 4.76) we only need (A,B) stabilizable. In case $\mathbb{Z} = \mathbb{R}^{n+m}$, the value function is linear-quadratic and thus naturally locally continuous. For $\mathbb{Z}$ polytopic and any $N \in \mathbb{I}_{\geq 1}$ the value function is piece-wise quadratic and hence locally continuous (although the author is not aware of a proof regarding a uniform continuity bound for all $N \in \mathbb{I}_{\geq 1}$). Note that the rotated value function $\tilde{V}_N$ is locally quadratic and hence continuous since $(x_T, u_T) \in \text{int}(\mathbb{Z})$ and L quadratic.

with some $\tilde{\theta}_{\overline{V}} \in \mathcal{L}$ and $\gamma_{\overline{V}} > 0$ analogous to the case of reachable reference trajectories (cf. Thm. 4.5). The formulas in Section 4.3 can be correspondingly simplified but remain conservative with $C = \max_k \max_{x \in \tilde{\mathbb{Z}}_{\mathbb{X}}} \lambda_k^\top (x - x_T(k))$ potentially very large.

In the linear setting the assumptions are simpler and easier to verify. Conceptually similar results were obtained in [132, Thm. 3.11] for linear quadratic economic MPC, assuming no constraints, i.e., $\mathbb{Z} = \mathbb{R}^{n+m}$.

4.4.3 Stability in linear MPC

In the following, we show how some of the bounds in Section 4.1 simplify in case of linear system dynamics $f(x,u) = Ax + Bu$. W.l.o.g. consider $r = 0$. The incremental stabilizability condition (Def. 4.18) reduces to stabilizability of (A,B) and a stabilizing input is given by the linear control law $u(k) = K(x(k) - z(k)) + v(k)$, with $(A + BK)$ Schur. For ℓ quadratic and positive definite (Ass. 4.16) we can compute the LQR resulting in a feedback matrix $K \in \mathbb{R}^{m \times n}$ and a positive definite matrix $P \in \mathbb{R}^{n \times n}$. The smallest cost controllability constant γ (Ass. 4.4) can be computed with $\gamma = \lambda_{\max}(P/Q)$, where $\lambda_{\max}(P/Q)$ denotes the maximal generalized eigenvalue. Similarly, the horizon dependent constants γ_k (Rk. 4.30) can be computed based on the finite horizon LQR.

Detectable stage cost

Detectability of (A,C) is equivalent to i-IOSS (Ass. 4.24) with a quadratic i-IOSS Lyapunov function $V_{\mathrm{o}}(x,z) = \|x - z\|_{P_{\mathrm{o}}}^2$, compare [50]. Computing a storage W satisfying Assumptions 4.10 with the largest possible $\epsilon_{\mathrm{o}} > 0$ can be posed as an equivalent LMI with $W = \|x\|_{P_{\mathrm{o}}}^2$. Thus, in the linear setting, there exist explicit formulas for the Lyapunov functions $\|x\|_P^2$, $\|x\|_{P_{\mathrm{o}}}^2$ associated with the cost controllability and the detectability, respectively. This can be used to improve the bounds in Theorem 4.12/Proposition 4.23 by choosing $\sigma(x) \neq \|x\|^2$. For example, by choosing $\sigma(x) = \|x\|_P^2$, we get $\gamma_{\mathrm{s}} = 1$, $\gamma_{\mathrm{o}} = \lambda_{\max}(P_{\mathrm{o}}/P)$, which may result in less conservative bounds $\underline{M}$.

Global cost controllability

For linear systems with no state constraints ($\mathbb{Z} = \mathbb{R}^n \times \mathbb{U}$) and A Schur stable, in addition to the local cost controllability (Ass. 4.4), we have $V_N(x) \leq \gamma_{\mathrm{gl}} \ell_{\min}(x)$ for all $x \in \mathbb{R}^n$ with some constant $\gamma_{\mathrm{gl}} \geq \gamma$, which can be computed based on the Lyapunov equation $A^\top PA - P + Q = 0$. Thus, in this case we can also show *global* exponential stability with a sufficiently large horizon $N > N_{\mathrm{gl}}$ (cf. [119, Cor. 4]). In addition, the bounds regarding the region of attraction (cf. proof Thm. 4.5) can be improved with $V(N_0|t) \leq \left(\frac{\gamma_{\text{gl}}-1}{\gamma_{\text{gl}}}\right)^{N_0} \overline{V} \leq \gamma\epsilon$ for $N_0 \geq \frac{\log(\overline{V})-\log(\gamma\epsilon)}{\log(\gamma_{\text{gl}})-\log(\gamma_{\text{gl}}-1)}$. The corresponding bound results in $N_{\overline{V}} = N_0 + \underline{M}$, $\underline{M} := \frac{\log(\gamma_{\text{gl}}\gamma)-N_0(\log(\gamma_{\text{gl}})-\log(\gamma_{\text{gl}}-1))}{\log(\gamma)-\log(\gamma-1)}$, which reduces to the standard bound in [123, Variant 2], [267], Theorem 4.5 in case $\gamma = \gamma_{\text{gl}}$.

Extended prediction horizon

Regarding the MPC with extended horizon (Sec. 4.1.5), we can improve the theoretical analysis in multiple directions in the linear setting.

A natural choice for the feedback κ (Ass. 4.33) is the LQR feedback $u = Kx$. Non-conservative bounds for the constants $\underline{c}_M, \overline{c}_M, \alpha_M$ (Prop. 4.34) can directly be computed using the solution of the finite-horizon discrete-time Lyapunov equation $P_M = \sum_{k=0}^{M-1} A_K^k Q_K A_K^k$ with $A_K = A + BK$, $Q_K = Q + K^\top RK$. Correspondingly, the constants can be computed as a (generalized) eigenvalue, e.g., $\alpha_M = \lambda_{\min}((A_K^\top P_M A_K - P_M)/Q_K)$.

In addition, Assumption 4.33 can be naturally strengthened in case of linear systems with polytopic constraints. In particular, Conditions (4.44) hold for all $x \in \mathbb{X}_0$, where the polytopic set $\mathbb{X}_0$ is the maximal positive invariant set (cf. [114]). Furthermore, since this set is finitely determined, there exists a constant $\underline{M} \in \mathbb{I}_{\geq 1}$ such that $V_{\text{f},M}(x) < \infty$, $M \geq \underline{M}$, implies $x \in \mathbb{X}_0$, compare also [83, Thm. 5]. Hence, we can use the bounds (4.44) without requiring $V_{\text{f},M} \leq \epsilon$ and thus avoid the additional conservatism in the bound N_1 in Theorem 4.37. Notably, this property with $M \geq \underline{M}$ ensures persistent feasibility for any $N \in \mathbb{I}_{\geq 1}$, compare [83].

In case the stabilizing feedback κ corresponds to the LQR, $\mathbb{Z}$ is polytopic, and $M \geq \underline{M}$, we can also strengthen the guarantees regarding the suboptimality index $\alpha_{N,M} \leq \epsilon_{N,M}$ in Theorem 4.37. In particular, we have $V_{\text{f,M}}(x) \leq V_{\infty,0}(x)$ for any $x \in \mathbb{X}_0$, i.e., the finite-tail cost is a lower bound to the infinite horizon cost. Thus, the performance bound (4.49) simplifies to $\alpha_{N,M} := \epsilon_{N,M}$ and hence we can achieve infinite horizon optimal performance with a finite value N, if $M \to \infty$. We note that this result is not surprising, since $\lim_{M\to\infty} V_{\text{f,M}}$ corresponds to a standard LQR terminal cost and if the standard terminal set constraint $\mathbb{X}_{\text{f}}$ is inactive (which is the case for a sufficiently large horizon N), then infinite horizon optimal performance is achieved.

In the following, we briefly discuss that for linear systems improved bounds, similar to the ones from Theorem 4.37, also hold for a standard MPC without terminal constraints (Problem 4.1), i.e., without explicitly using an extended prediction horizon M with some stabilizing feedback κ. The main advantage when considering linear systems is that locally, if the constraints are not active, non-conservative characterizations of the optimal solution (e.g., a relaxed dynamic programming inequality) can be directly derived based on the finite-horizon LQR. Hence, we can replace Assumption 4.33 by similar stability properties for the finite-horizon LQR with a time-varying control law κ. Then, we can use the fact that for N large enough, the principle of optimality guarantees that in the last M steps the open-loop optimal trajectory coincides with the finite-horizon LQR. The main benefit of this alternative analysis is the fact that we can directly exploit the typically stronger guarantees of the finite-horizon LQR in the MPC stability analysis.

Summary

In this section, we revisited the results in Sections 4.1–4.3 for the special case of linear system dynamics. We showed that the conditions considered in the output regulation MPC (Sec. 4.2) reduce to classical conditions from the output regulation literature. Furthermore, we discussed that the strict dissipativity condition used in the analysis of unreachable reference trajectories (Sec. 4.3) is naturally satisfied in the linear setting due to strong duality. Finally, we also showed how the performance bounds for MPC without terminal ingredients (Sec. 4.1) simplify and improve for linear systems. In the next section, we demonstrate the practicality of the theoretical results derived in this chapter using numerical examples.

4.5 Numerical examples

In the previous sections, we studied MPC without terminal constraints for tracking of reachable and unreachable reference trajectories (Sec. 4.1/4.3), output regulation MPC (Sec. 4.2), and derived lower bounds on the prediction horizon N to guarantee stability/performance. In the following, we provide numerical examples that demonstrate the applicability of the considered MPC formulation for nonlinear dynamic problems and numerically investigate the conservatism of the derived bounds. In Section 4.5.1, we consider a linear academic example and compare the different bounds on the prediction horizon N. In Section 4.5.2, we illustrate the applicability of the trajectory tracking MPC for reachable reference trajectories and nonlinear systems (Sec. 4.1), and discuss potential sources of conservatism in the derived bounds. Then, Section 4.5.3 shows the applicability of the output regulation MPC (Sec. 4.2) at the example of nonlinear *offset-free* tracking MPC with error feedback (Rk. 4.43). Finally, Section 4.5.4 demonstrates the applicability of the tracking MPC in the case of unreachable dynamic reference trajectories (Sec. 4.3) with a simple linear example. For the following examples, the offline and online computation is done in Matlab using SeDuMi-1.3 [261] and CasADi [17], respectively. This section is based on and taken in parts literally from [JK24][19] and [JK19][20].

4.5.1 Performance bounds - MPC without terminal ingredients

In the following, we consider a simple system to compare the bounds from Theorems 4.5, 4.12, 4.37, Proposition 4.14, and Remark 4.30. In order to demonstrate the necessity of a sufficiently long prediction horizon N we study the classical four tank system from [231]. To enable a simple analytical comparison, we first examine a linear system without constraints. The corresponding linear discrete-time model is characterized by

$$A = \begin{pmatrix} 0.932 & 0 & 0.041 & 0 \\ 0 & 0.918 & 0 & 0.033 \\ 0 & 0 & 0.924 & 0 \\ 0 & 0 & 0 & 0.937 \end{pmatrix}, \; B = \begin{pmatrix} 0.017 & 0.001 \\ 0.001 & 0.023 \\ 0 & 0.061 \\ 0.072 & 0 \end{pmatrix}, \; C = \begin{pmatrix} I_2 & 0_2 \end{pmatrix}, \; D = 0_2,$$

compare [JK3, JK4, JK5]. Note that the system is square, linear, stable, and non-minimum-phase. Without loss of generality, we consider the problem of stabilizing the origin (in the linear setting without constraints, any reachable reference trajectory r simply shifts the state and input). We use the quadratic stage cost $\ell = \|x\|_Q^2 + \|u\|_R^2$, $R = 10^{-4} I_m$, $Q = C^\top C + 10^{-2} I_n$, which primarily penalizes the output $y = Cx$, similar to [JK3, JK4, 231]. For any horizon $N \in \mathbb{I}_{[2,16]}$ the MPC formulation without terminal ingredients (Alg. 4.2) results in an unstable closed-loop system. Furthermore, for horizons $N \leq 28$ the value function V_N is *not* a valid Lyapunov function.

Stability based on cost controllability

For the system under consideration, the cost controllability (Ass 4.4) is satisfied for $\gamma \approx 73$. Consequently, Theorem 4.5 ensures stability with $N \geq 622$. Alternatively, we can ensure stability for $N \geq 313$ based on the LP analysis in [120, 127] (cf. Rk. 4.32). Using the improved bounds in Remark 4.30 based on γ_k we obtain stability with $N \geq 170$. Using the bounds γ_k in the LP analysis, we can improve this bound further and guarantee stability for $N \geq 162$ (cf. [127, Thm. 5.4], Rk. 4.32).

[19] J. Köhler, M. A. Müller, and F. Allgöwer. “Nonlinear reference tracking: An economic model predictive control perspective.” In: *IEEE Trans. Automat. Control* 64.1 (2019), pp. 254–269©2018 IEEE.

[20] J. Köhler, M. A. Müller, and F. Allgöwer. “Constrained nonlinear output regulation using Model Predictive Control.” In: *IEEE Trans. Automat. Control* (2021). extended version: arXiv:2005.12413©2021 IEEE.

These rather large and conservative bounds are mainly a consequence of the high value of γ, which is due to the relatively large penalty on the output y in the stage cost ℓ. Hence, in case the stage cost is fixed (not a tuning variable), analysis based on cost controllability with $\ell_{\min}$ as done in most of the literature (cf. [120, 123, 127, 237, 267]) can sometimes be quite conservative.

Extended prediction horizon

The conservatism induced by a large constant γ can be partially compensated by employing the extended prediction horizon from Theorem 4.37. In Figure 4.2, we can see the suboptimality index $\alpha_{N,M} = \epsilon_{N,M}$ (cf. Sec. 4.4.3) depending on the prediction horizon N for different values of $M \in \{5, 10, 20\}$. In addition, we show the performance bounds from Remark 4.30 with γ_k. For simplicity, we did not use the formula involving γ_k (cf. Rk. 4.30) for the extended prediction horizon, since the difference is rather small for M large. We can clearly see that the minimum prediction horizon to guarantee performance and stability can be significantly relaxed by using an extended prediction horizon M. For example, for $M = 20$, we can guarantee stability for any horizon $N \geq 0$. However, for small values of M the bounds in Remark 4.30 involving γ_k can also be less conservative. Numerically we find that for $M = 5$ the closed loop with Algorithm 4.36 is stable for $N > 2$ and the value function $V_{N,M}$ is a Lyapunov function for $N > 14$, which are significantly shorter horizons compared to $N > 17$ and $N > 28$ for Algorithm 4.2. Overall, using an extended prediction horizon allows for significantly shorter prediction horizons compared to the bounds directly based on cost controllability. Nevertheless, depending on the choice of ℓ, the bounds can still be conservative since they also use $\ell_{\min}$ and γ.

Detectable/Observable stage cost

Although the stage cost is positive definite, we can also use the more general derivations in Theorem 4.12 and Proposition 4.14 based on detectability/observability of the stage cost (Ass. 4.10/4.13). For simplicity, we consider the state measure $\sigma(x) = \|x\|_P^2$ with P based on the LQR. Correspondingly, Assumption 4.9 holds with $\gamma_s = 1$ and Assumption 4.10 holds with $W = 0$, $\gamma_o = 0$ and $\epsilon_o = \lambda_{\min}(Q/P) = 0.0137$. Thus, Theorem 4.12 ensures stability for $N > 5.3 \cdot 10^3$ which is quite conservative.

We can additionally make use of the fact the stage cost is observable, i.e., satisfies Assumption 4.13 with $\nu = 2$ and $c_o \approx 40.65$, which can be computed as a generalized eigenvalue. For these constants Proposition 4.14 guarantees stability for $N > 660$.

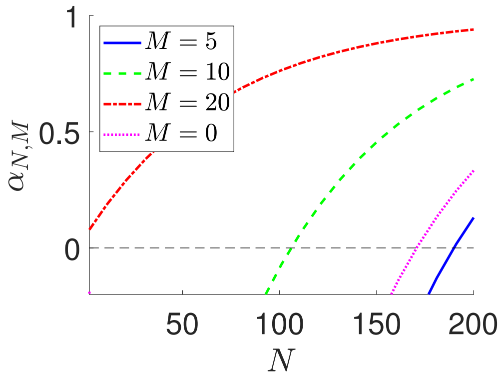

Figure 4.2. Guaranteed suboptimality index $\alpha_{N,M}$ with extended prediction horizon for $M \in \{5, 10, 20\}$ (blue, solid; green, dashed; red, dash-dot), $N \in \mathbb{I}_{[0,200]}$ and suboptimality index α_N without extended prediction horizon ($M = 0$, magenta, dotted) based on γ_k (Rk. 4.30), $N \in \mathbb{I}_{[2,200]}$.

Note that this bound is only slightly more conservative than the bounds based on Theorems 4.5, which exploit the positive definiteness of ℓ.

If the stage cost is slightly changed to $Q = C^\top C + 10^{-6} I_4$ (closer to an I/O cost), then we get $\gamma > 6 \cdot 10^3$ and the bound in Theorem 4.5 deteriorates with $N_{\overline{V}} > 10^5$, while the bound in Proposition 4.14 yields $N_{\overline{Y},\nu} \approx 10^4$. Thus, although the standard bounds based on cost controllability (cf. Thm. 4.5, [120, 123, 127, 237, 267]) are typically less conservative, the bounds based on detectability/observability (cf. Thm. 4.12/Prop. 4.14) can be superior for positive definite stage costs ℓ in cases where the stage cost is mainly composed of an input-output stage costs with only a small positive definite component.

Cost tuning

Based on this example, the guarantees and properties of MPC without terminal ingredients may seem rather conservative. This is, however, mainly due to the choice of the stage cost ℓ, which is chosen to highlight the general need for a sufficiently large prediction horizon N also for simple linear stable systems without constraints. If we instead choose $Q = I_n$, we get $\gamma \approx 7.8$, Theorem 4.5 ensures stability for any horizon $N \geq 30$, and Theorem 4.37 with $M = 1$ ensures stability for any horizon $N \geq 1$. Besides, with the bounds γ_k (cf. Rk. 4.30) we can even guarantee stability for any $N \in \mathbb{I}_{\geq 2}$. Thus, the resulting guarantees are highly dependent on the choice of the stage cost ℓ.

Discussion

We considered a simple stable unconstrained linear system from the literature [231] to study general properties of MPC without terminal ingredients. Depending on the prediction horizon N, the closed loop may be unstable and we can use the results in Theorem 4.5 (analogous to the literature [120, 123, 127, 237, 267]) to derive a lower bound on the prediction horizon N to guarantee stability. In addition, the bounds can be improved using the constants $\gamma_k \leq \gamma$ (cf. Rk. 4.30) and the LP analysis (cf. [120, 127], Rk. 4.32). Significantly less conservative bounds can be obtained if an extended horizon M (cf. Thm. 4.37) is used. Furthermore, we showed that even in the case of positive definite stage costs ℓ, the bounds derived from the observability of the stage cost ℓ (cf. Prop. 4.14) can be less conservative compared to the standard bounds based on cost controllability (cf. [120, 123, 127, 237, 267], Thm. 4.5), since additional information is used in Assumption 4.13. Finally, we note that the cost controllability condition (Ass. 4.4) often gives an easy insight how the stage cost could be modified in order to guarantee stability with shorter horizons N. The same conclusions hold for nonlinear constrained systems with the main difference that verifying the corresponding conditions (e.g., Ass. 4.4) is more challenging and the resulting bounds also depend on the considered region of attraction $\mathbb{X}_{\overline{V}}$.

4.5.2 Reference tracking and region of attraction

The following example illustrates the theoretical results in Section 4.1 for trajectory tracking MPC with reachable references and nonlinear constrained systems. We consider a discrete-time version of a continuous stirred tank reactor (CSTR) taken from [191]:

$$f(x,u) = \begin{pmatrix} x_1 + \frac{h}{\theta}(1-x_1) - hkx_1 e^{-\frac{M}{x_2}} \\ x_2 + \frac{h}{\theta}(x_f - x_2) + hkx_1 e^{-\frac{M}{x_2}} - h\alpha u(x_2 - x_c) \end{pmatrix},$$

with the temperature x_1, the concentration x_2, the coolant flow rate u and the sampling time $T_\mathrm{s} = 0.1\mathrm{s}$ (cf. the example in Sec. 3.4.1). The constraint set is given by $\mathbb{Z} = [0,1] \times [0.5,1] \times [0,2]$ and we have a quadratic stage cost (Ass. 4.16) with $Q = I_2$, $R = 10^{-4}$. The reference constraint set is given by $\mathbb{Z}_\mathrm{r} = [0.05, 0.4] \times [0.6, 0.9] \times [0.5, 1.5]$. The local cost controllability condition (Ass. 4.4) is validated for generic references $r \in \mathbb{Z}_\mathrm{r}$ by computing a parametrized terminal cost V_f and terminal control law k_f (cf.

Alg. 3.22)[21] resulting in $\gamma = 62$ (cf. Prop. 4.20).

We consider a reachable (Ass. 4.15) periodic reference trajectory that connects two equilibria as shown in Figure 4.3. A numerical investigation shows that the same controller k_f implies the tighter bound $\gamma = 21.9$, $\epsilon = 10^{-5}$ along this specific reference trajectory. We note that the restriction to *local* cost controllability (Ass. 4.4) is crucial since for $\epsilon = 10^{-4}$ the same local control law is not feasible and thus validating a cost controllability constant γ is challenging. We point out that the improved bounds based on γ_k (Rk. 4.30) are more difficult to apply for this problem since computing γ for general nonlinear problems is already time consuming. According to Theorem 4.5, a prediction horizon of $N > N_{\overline{V}} = \underline{M} \approx 132$ ensures local ($\overline{V} = \gamma\epsilon$) exponential stability. The reference trajectory and the local sets $\|e_\mathrm{r}(t)\|_Q^2 \leq \epsilon$ are shown in Figure 4.3. This small region of attraction is mainly due to the active input constraint.

If a prediction horizon of $N = 200$ is used, Theorem 4.5 provides the region of attraction $\overline{V} = (\gamma + N - \underline{M})\epsilon \approx 9 \cdot 10^{-4}$. This guaranteed region of attraction with one exemplary open-loop solution is depicted in Figure 4.3. Note that the guaranteed region of attraction could be significantly increased by generalizing the trajectory tracking MPC formulation to path following [3, 91, 94]. Theorem 4.5 ensures that after $N_0 \approx 68$ steps the optimal open-loop solution enters the local region of attraction, while in Figure 4.3 we have $V(k_x|t) \leq \gamma\epsilon$ for $k_x \geq 7$ and $\ell(k_x|t) \leq \epsilon$ for $k_x \geq 9$, which highlights the potential conservatism of this result.

Discussion

Remark 4.90. *(Comparison - performance bounds N) The proposed reference tracking MPC ensures stability with a prediction horizon of* $N = 200$ *and requires no complex design procedure. As discussed in Remark 4.6, the bounds in [37] are based on the same local cost controllability condition and hence could also be used to derive a lower bound on the prediction horizon. In order to guarantee exponential stability for* $\overline{V} = 9 \cdot 10^{-4} = 90\epsilon$*, these derivations require a prediction horizon of* $N > 805$*. Thus, the proof technique presented in Theorem 4.5 is significantly less conservative compared to the method in [37]. If we improve the bound* $\underline{M}$ *using the LP analysis in Remark 4.32, we ensure local stability with* $N \geq 81$ *and can guarantee the same region of attraction* $\mathbb{X}_{\overline{V}}$ *with* $N \geq 150$*.*

In the closed-loop simulations, we find that the MPC with prediction horizon $N = 10$ *ensures exponential stability for all initial conditions considered in Theorem 4.5, confirming again that*

[21] Lemma 3.13 is used with $\mathbb{Z}_\mathrm{r}$ gridded using 30^3 points. The conditions in Proposition 3.11 hold with $\alpha_1 = 0.1$, which is validated with a finer grid consisting of size $5 \cdot 10^{-3}$ in $\mathbb{Z}_\mathrm{r}$, each with 10^3 random points Δx, compare [JK15, Alg. 1].

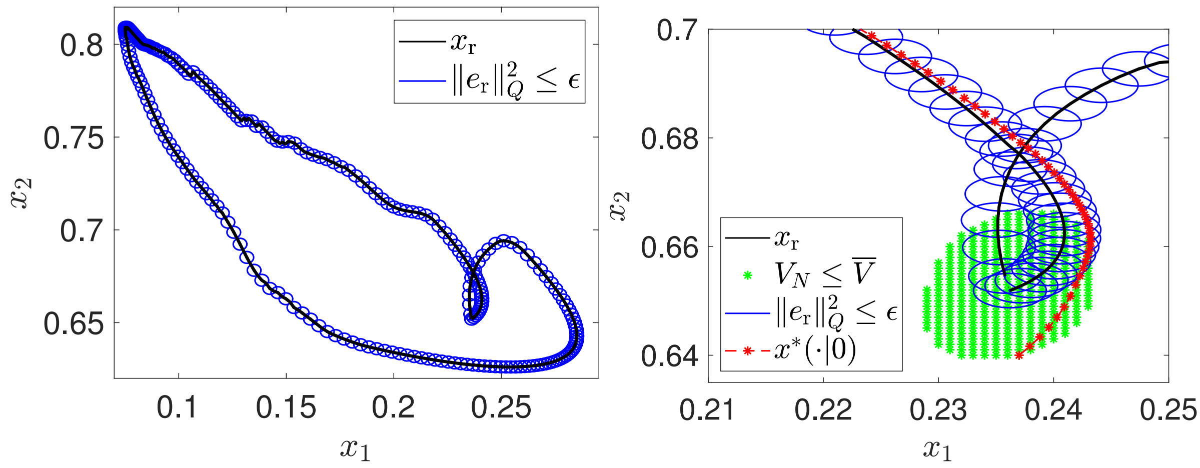

Figure 4.3. Concentration vs. temperature. Left: periodic reference trajectory x_r (solid) with local region of attraction $\|e_r(t)\|_Q^2 \le \epsilon$ (ellipses). Right: Increased region of attraction $V_N(x_0, 0) \le \overline{V}$ (dots) and exemplary open-loop solution $x^*(\cdot|0)$ (dashed).©2018 IEEE

the improved theoretical bounds on N in Theorem 4.5 can still be quite conservative.

If we use the approach in Theorem 4.37 with $M = 1$, *we obtain* $c_{M,M+1} \approx 1$ *and thus local stability can be ensured for* $N_M = \frac{\log(\gamma)}{\log(\gamma)-\log(\gamma-1)} = 66$, *improving the bounds in Theorem 4.5 by a factor* 2. *If we further increase* M *to* 10 *we get* $c_{M,M+1} = 0.075$ *and thus* $N_2 = \frac{\log(\gamma c_{M,M+1})}{\log(\gamma)-\log(\gamma-1)} \approx 10.8$. *This is a significant decrease in the overall prediction horizon* $N + M$ *and thus the computational complexity to ensure local stability.*

Remark 4.91. *(Alternative MPC solutions) To apply the methods in [23, 91], one has to construct terminal sets and costs along the reference trajectory. Those approaches require offline computations for an explicit reference trajectory. As a consequence, online changes in the reference trajectory are hard to deal with. On the other hand, the guarantees for the tracking MPC without terminal ingredients (Sec. 4.1) are valid for all feasible reference trajectories r (Ass. 4.15), assuming that the cost controllability (Ass. 4.4) is validated for all feasible reference trajectories. It is also possible to use parametrized terminal ingredients based on the design in Section 3.1. However, the resulting terminal cost can be conservative (*$\gamma = 21$ *vs.* $c_u = 62$*) and the offline design can be time consuming.*

We considered a nonlinear constrained trajectory tracking problem and showed the applicability of the theoretical results in Section 4.1. We demonstrated that the results in Theorem 4.5 significantly improve the bounds in [37] ($N \ge 200$ vs. $N \ge 805$). In addition, the LP analysis in Remark 4.32 further improved the bounds in Theorem 4.5 ($N \ge 81$ vs. $N \ge 132$). Based on the extended horizon MPC formulation (cf. Thm. 4.37), the lower bounds on the prediction horizon could again be significantly reduced (e.g., $M = 1$, $N_M \geq 66$ vs. $N \geq 132$). Thus, the benefits of the derived theoretical results (Thm. 4.5, Rk. 4.32, Thm. 4.37) in terms of less conservative performance bounds for nonlinear MPC without terminal ingredients has been demonstrated. However, this numerical example also demonstrates potential sources of conservatism regarding the theoretical guarantees.

4.5.3 Output regulation and and offset-free tracking

In the following example, we demonstrate the applicability of the output regulation MPC (Sec. 4.2) to nonlinear offset-free tracking MPC using both, the pure output cost formulation (Sec. 4.2.2) and the incremental input formulation (Sec. 4.2.3). We consider the following nonlinear model of a cement milling circuit taken from [174]:

$$\begin{aligned}
0.3\dot{x}_1 &= -x_1 + (1-\alpha(x_2,u_2))\phi(x_2),\\
\dot{x}_2 &= -\phi(x_2) + u_1 + x_3,\\
0.01\dot{x}_3 &= -x_3 + \alpha(x_2,u_2)\phi(x_2),\\
\phi(x_2) &= \max\{0, -0.1116\cdot x_2^2 + 16.50x_2\},\\
\alpha(x_2,u_2) &= \frac{\phi^{0.8}(x_2)\cdot u_2^4}{3.56\cdot 10^{10} + \phi^{0.8}(x_2)\cdot u_2^4},
\end{aligned}$$

with $x \in \mathbb{R}^3$, $u \in \mathbb{R}^2$. The discrete-time model is computed using the 4th order Runge Kutta method and a sampling time of one minute[22]. The system is subject to compact input constraints and no state constraints with $\mathbb{Z} = \mathbb{X} \times \mathbb{U} = \mathbb{R}^n \times [80,150] \times [165,180]$. The output is given by $y = (x_1, x_3) - (w_1, w_2)$, where w corresponds to the constant output reference $w = (110, 425)$.

In the following, we briefly show that the considered assumptions hold on the subset $x_2 \in [45,55]$, $w \in \mathbb{W} = [100,120] \times [410,430]$, which provides a sufficiently large region of attraction. First, the unique solution to the regulator equations (Ass. 4.39) can be

[22]The time unit in the model is hours.

analytically computed as

$$\pi_{\mathrm{x}}(w) = \left[w_1,\ 73.9 - \sqrt{5.5 \cdot 10^3 - 8.9(w_1 + w_2)},\ w_2\right]^\top,$$
$$\pi_{\mathrm{u}}(w) = \left[w_1,\ 434 \left(\frac{w_2}{w_1}\right)^{0.25} (w_1 + w_2)^{-0.2}\right]^\top,$$

which is also Lipschitz continuous on the considered region. The plant is open-loop incrementally stable and hence trivially satisfies the incremental stabilizability condition (cf. Definition 4.44). In particular, we computing (via gridding) a constant contraction metric [169, 178], which corresponds to a quadratic incremental Lyapunov function V_δ (cf. App. C). Correspondingly, the system also trivially satisfies the detectability condition (Ass. 4.62) with $V_{\mathrm{o}} = V_\delta$. Furthermore, one can show that the system is flat and contains no zero-dynamics. Hence, the conditions regarding the minimum-phase property (Ass. 4.46–4.48) are trivially satisfied. Similarly, the nonresonance condition (Ass. 4.63) follows due to the absence of zero-dynamics (cf. the discussion in Section 4.2.3) and a corresponding quadratic incremental storage function V_{R} can be computed similar to [245, 270]. Hence, we have shown that all the considered assumptions hold. However, due to the complexity of the system the resulting bounds on the sufficiently long prediction horizon N from the derived theorems are too conservative to be applied. Thus, we simply implement the two MPC schemes (Sec. 4.2.2/Sec. 4.2.3) with $N = 6$, $Q = I_2$ and $R = 0/R = 10^{-2} \cdot I_2$.

The resulting closed loop for $x_{\mathrm{p}}(0) = (120, 55, 450)$ can be seen in Figure 4.4. Both MPC formulations smoothly track the output reference, while satisfying the active input constraints. If we compare the MPC formulation with and without input regularization (Sec. 4.2.2/Sec. 4.2.3) , the resulting closed-loop state trajectories are almost indistinguishable, while the absence of input regularization leads to more aggressive control inputs.

Noisy error feedback and inherent robustness

Now we consider more the realistic scenario of noisy error feedback as discussed in Remark 4.43, i.e., only noisy output measurements $\tilde{y} = y + \eta$ are available, with η uniformly distributed in $[-1, 1]^2$. As in [174], we design an extended Kalman filter (EKF) as an observer and implement the output regulation MPC in a certainty equivalent fashion, compare [JK19, App. B] for details. The EKF uses an initial variance of

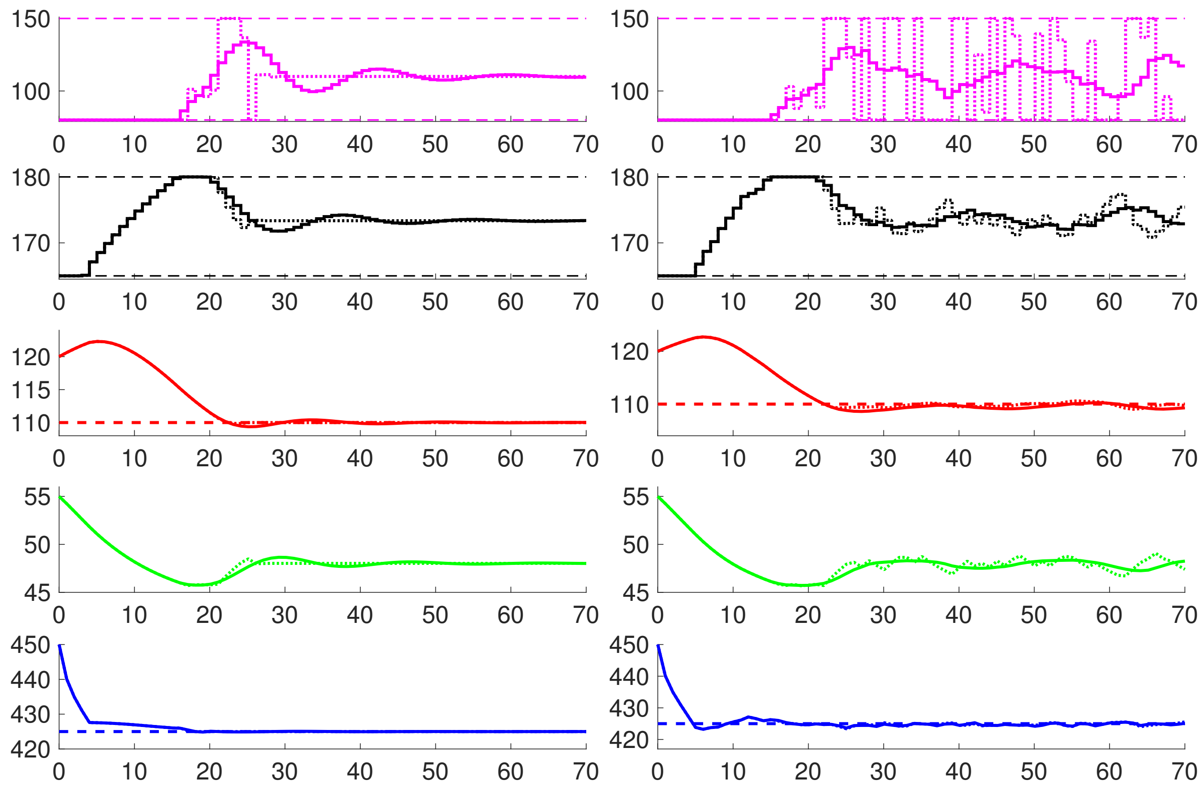

Figure 4.4. Offset-free tracking with state feedback (left) and noisy error feedback (right): With incremental input regularization (solid) and without input regularization (dotted). From top to bottom: u_1 (magenta), u_2 (black), x_1 (red), x_2 (green), x_3 (blue). Output reference $(w_1, w_2) = (110, 425)$ and input constraints are dashed. Time in minutes.©2021 IEEE.

$\Sigma = 100 \cdot I_n$ and unit variance for noise and disturbances in the design. The initial state estimate is given by $\hat{x}(0) = (\hat{x}_\mathrm{p}(0), \hat{w}(0)) = (100, 50, 400, 100, 400)$. The resulting closed loop can be seen in Figure 4.4. We can see that for both MPC formulations the control performance is rather insensitive to the noise and estimation error. The resulting closed-loop state trajectories for the two MPC formulations are almost indistinguishable, while the absence of input regularization leads to more aggressive control inputs, especially in u_1.

Discussion

The main benefit of the proposed approach is its simplicity. For the implementation, we only require a prediction model, input and output weights Q, R, and a sufficiently long prediction horizon N needs to be chosen. In case of error feedback, we additionally need to design a stable observer, e.g., here an EKF. Most importantly, the proposed design does not require any complex offline computations. This is in contrast to most approaches for output regulation (cf. [53, 89, 176, 222]), that typically first need to compute a solution to the regulator equations (4.56), which is in general non-trivial. Furthermore, compared to classical approaches to output regulation (cf., e.g., [53, 222]), the proposed approach offers a large region of attraction despite the presence of hard input constraints.

We note that at least in the nominal state feedback case, the tracking MPC formulations in Section 3.2 are also applicable to this problem and can guarantee stability with a short prediction horizon N, even in case of unreachable trajectories (Ass. 4.39 does not hold). Compared to such tracking MPC formulations with artificial references, the proposed approach has the following advantages:

(a) No complex offline design for terminal ingredients,

(b) No feasibilities issues and strong stability properties in the noisy error feedback case due to the absence of terminal constraints,

(c) A larger region of attraction,

(d) No additional decision variables to compute the optimal mode of operation $x = \pi_\mathrm{x}(w)$ online, especially in case $T \neq 1$.

Thus, the proposed output regulation MPC has practical benefits compared to classical output regulation methods (cf. [53, 222]) and competing tracking MPC formulations (cf. Sec. 3.2).

4.5.4 Unreachable reference trajectories and linear systems

In the following, we study a linear system with an unreachable reference trajectory to illustrate the theoretical results in Section 4.3. We consider an asynchronous motor with the parameters[23] from [54], and the sampling time of $T_s = 0.1\text{ms}$. This model corresponds to a linear system subject to ellipsoidal constraints

$$x^+ = Ax + Bu, \quad x = \left(i_{\text{s},\alpha}, i_{\text{s},\beta}, \Psi_{r,\alpha}, \Psi_{r,\beta}\right), \quad u = \left(u_{\text{s},\alpha},\ u_{\text{s},\beta}\right),$$
$$\mathbb{Z} = \left\{(x,u) \mid i_{\text{s},\alpha}^2 + i_{\text{s},\beta}^2 \leq \bar{i}^2,\ \Psi_{\text{r},\alpha}^2 + \Psi_{\text{r},\beta}^2 \leq \overline{\Psi}_{\text{r}},\ u_{\text{s},\alpha}^2 + u_{\text{s},\beta}^2 \leq \bar{u}^2\right\},$$

with the stator current $i_{\text{s},\alpha\beta}$, the rotor flux $\Psi_{r,\alpha\beta}$, and the input voltage u.

Periodic operation

Typically, induction machines are controlled in the rotated dq-frame, in which the system is described by nonlinear differential equations. In the $\alpha\beta$-frame, we have a linear system and stationary operation is described by a periodic trajectory. Thus, we can transform a nonlinear setpoint stabilization problem into a linear reference tracking problem.

For the reference tracking MPC we use the stage cost $Q = I_4$, $R = I_2$, which implies $\gamma = 2.5$ using Proposition 4.20 and the LQR (cf. Sec. 4.4.3). Based on Theorem 4.5, the prediction horizon $N > N_{\overline{V}} = \underline{M} \approx 3.7$ is sufficient for local ($\overline{V} = \gamma\epsilon$) stability of the periodic operation. Note that using the formula from Remark 4.30 with $\gamma_2 = 2.22$, we can guarantee (local) stability for any horizon $N \in \mathbb{I}_{\geq 1}$.

An exemplary plot of two periodic trajectories with the local region of attraction ($V_\infty(x,t) \leq \gamma\epsilon$) projected on the stator current $i_{\text{s},\alpha\beta}$ can be seen in Figure 4.5. We want to focus on the problem of transitioning from one periodic operation to another.

Changing mode of operation and unreachable reference trajectory

In order to transition from one reachable periodic trajectory to another, one can generate a reference trajectory connecting the two periodic trajectories. In practice, computing a reachable reference trajectory can be a difficult task[24]. Thus, as a reference trajectory

[23] To simplify the following derivations, the stator flux Ψ_s is scaled by a factor of 10^3 and the input voltage is scaled by a factor of 10. The angular velocity ω is constant. We consider the constraints $\bar{i} = 1$, $\overline{\Psi}_\text{r} = 284$, and $\bar{u} = 0.02$.

[24] It is possible to compute a connecting reference trajectory in the dq-frame based on a flatness property of the asynchronous motor, compare, e.g., nonlinear constrained trajectory optimization based on

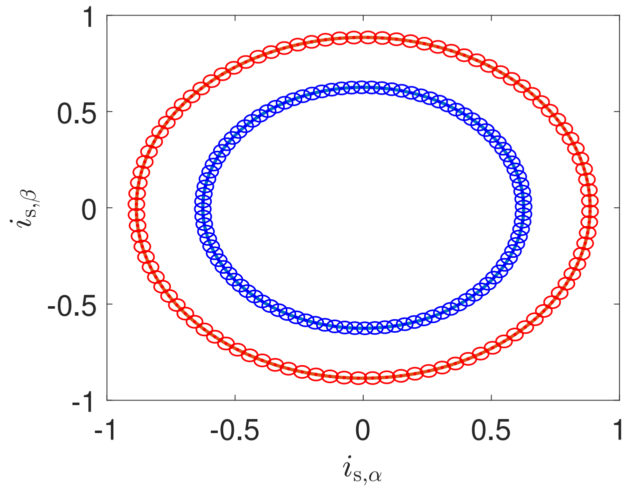

Figure 4.5. Phase plot stator current $i_{s,\alpha\beta}$: two periodic reference trajectories (solid) with (projected) local region of attraction $V_\infty(x,t) \leq \gamma\epsilon$ (ellipses). Each corresponds to a specific operation point in the dq-frame.©2018 IEEE

x_r we linearly interpolate between the two trajectories over $T = 1800$ steps creating an almost reachable reference trajectory. Correspondingly, there exists an optimal reachable[25] reference trajectory x_T (Problem 4.69). Theorem 4.79 can be used[26] to ensure local practical stability of the optimal reachable trajectory x_T if a sufficiently large prediction horizon N is used. In Figure 4.6, we can see the (partially) unreachable reference trajectory x_r, the optimal reachable trajectory x_T and the local region of attraction $\mathbb{S}_{c_T}$. In addition, an exemplary closed-loop simulation with $N = 10$ can be seen, which practically tracks the optimal reference trajectory x_T. An alternative, naive, method would be to directly change the reference trajectory at some point t, for which reliable theoretical guarantees are hard to obtain and practical simulations also lead to unsatisfactory results[27].

flatness in [97].

[25] Problem 4.69 is originally based on periodic reference trajectories. As discussed in Remark 4.83, nonperiodic trajectories can be treated as periodic trajectories over any finite-time span.

[26] Theorem 4.79 with the simplifications in Section 4.4.2 requires a prediction horizon of $N \geq \tilde{N}_1 \approx 5 \cdot 10^6$, which is quite conservative. This conservatism is due to the conservative bound C used in the turnpike property (Lemma 4.77).

[27] Theorem 4.5 can be used to analyse the region of attraction. Due to the large distance between the trajectories and the presence of hard state and input constraints, we have $V_N(x_0, 0) \approx 10^{10}\epsilon$ and thus we would require a prediction horizon $N \approx 10^{10}$, which is unrealistic. Also numerically we find that even with relatively large horizons ($N = 400$), there exists no point $k_x \in \mathbb{I}_{[0,N-1]}$ with $\ell(k_x|t) \leq \epsilon$. If we simply implement this method with a prediction horizon of $N = 10$, the MPC has in general undesirable non-smooth operation accompanied by numerical difficulties stemming from active nonlinear state and input constraints, and we experience repeated feasibility issues and constraint violations.

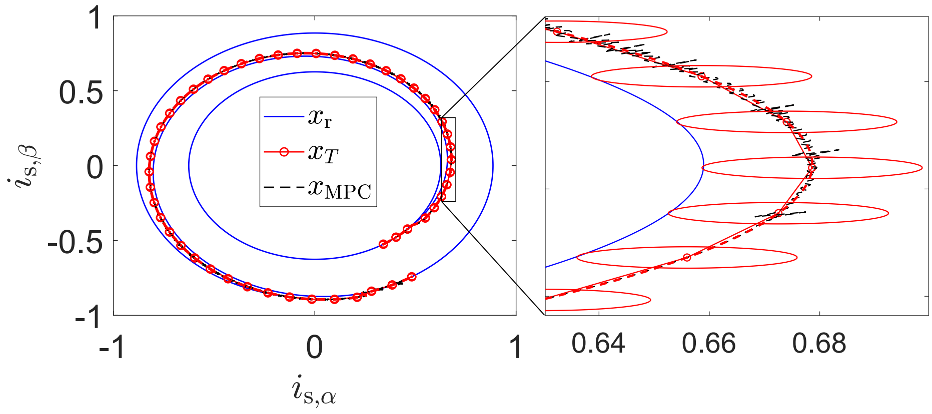

Figure 4.6. Phase plot stator current $i_{s,\alpha\beta}$: unreachable reference trajectory x_r (blue, solid) connecting two periodic reference trajectories, optimal reachable trajectory x_T (red, dashed-circle) with (projected) local region of attraction $\mathbb{S}_{c_T}$ (red ellipses) and closed-loop MPC trajectory x_{MPC} (black, dashed).©2018 IEEE

Discussion

We have demonstrated that the simple reference tracking MPC formulation (Alg. 4.2) ensures practical tracking of the *unknown* optimal reachable trajectory x_T. Thus, we have provided a method to enable operation changes without complex design procedures. Note that most other reference tracking approaches [23, 91] are unsuited to track such an unreachable, nonperiodic reference trajectory. Furthermore, MPC formulations based on periodic artificial reference trajectories (cf. Sec. 3.2) would experience computational challenges due to the long period length, with a possible exception being the partially decoupled approach from Section 3.2.3.

4.6 Summary

In this chapter, we analysed the closed-loop properties of simple MPC formulations for dynamic operation of nonlinear constrained systems, complementary to the MPC design methods proposed in Chapter 3. In particular, we analysed tracking MPC formulations without terminal ingredients for reachable and unreachable reference trajectories (Sec. 4.1/Sec. 4.3) and output regulation (Sec. 4.2). In all the considered cases, we ensured desired closed-loop properties (constraint satisfaction, stability, performance), given suitable system properties (stabilizability, detectability, nonresonance, etc.) and lower bounds on the prediction horizon N. In particular, we provided corresponding formulas relating a desired guaranteed performance $\alpha_M, \alpha_{M,\nu}, \alpha_{N,M}, \tilde{\theta}(N)$ (w.r.t. the infinite horizon optimal solution) and region of attraction $\mathbb{X}_{\overline{V}}$, $\mathbb{X}_{\overline{Y}}$ to a sufficiently large prediction horizon $N_{\overline{V}}$, $N_{\overline{Y}}$, $N_{\overline{Y},\nu}$. In addition, we provided various improvements over existing theoretical bounds for MPC without terminal ingredients, such as considering *local* stabilizability conditions (Sec. 4.1.2, Sec. 4.3.2), analysis based on stage cost observability (Sec. 4.1.3), and analysis based on an extended horizon M (Sec. 4.1.5), compare footnote 1. We demonstrated the practicality of the theoretical analysis using numerical examples (Sec. 4.5).

Chapter 5

Conclusions

In the following, we summarize the main results of this thesis, discuss the respective advantages of the two complementary MPC frameworks proposed in Chapters 3 and 4, and mention possible directions for future research.

5.1 Summary

This thesis addressed the problem of dynamic operation of nonlinear constrained systems. We presented two complementary MPC frameworks, each with corresponding design methods and theoretical analysis tools, in order to address the challenges associated with dynamic operation. In the first framework (Chap. 3), we focused on novel design procedures and MPC formulations, while in the second framework (Chap. 4), we considered simple MPC formulations (without terminal ingredients) and derived system theoretic conditions on the control problem that ensure the desired closed-loop properties.

Design procedures

In Chapter 3, we investigated constructive design procedures to enhance existing MPC formulations to ensure applicability and high performance for dynamic problems. We first considered the problem of tracking a reachable reference trajectory and discussed some of the practical limitations of the existing design methods. In order to overcome these limitations, we presented a *reference generic* offline design procedure (Sec. 3.1), resulting in parametrized terminal ingredients, which are valid for any reachable reference trajectory. The main idea was to recast the offline design problem equivalent to a gain-scheduling problem for quasi-LPV systems by suitably parametrizing the Jacobian of the dynamics and the terminal ingredients. This allowed us to use the rich literature for LPV systems and derive an SDP, which only needs to be solved *once*, irrespective of changes in the reference trajectory.

As a second contribution in Chapter 3, we considered unreachable output target signals, which may be subject to unpredictable changes online (Sec. 3.2). In order to address these challenges, we extended the previous trajectory tracking MPC formulation using *artificial reference* trajectories and restricted the MPC formulation to periodic trajectories. This MPC formulation can directly incorporate the parametrized terminal ingredients and thus avoid conservative terminal equality constraints. Given relatively general conditions on the terminal ingredients and a convexity condition, we proved exponential stability of the optimal reachable trajectory. We extended this MPC formulation using an online optimized terminal set size and reference constraint set to improve performance. In order to address the computational complexity associated with long artificial periodic reference trajectories, we introduced a *partially decoupled* MPC formulation, which provides a partial time scale separation between trajectory planning and tracking.

As a third contribution in Chapter 3, we studied a more general dynamic control problem with an economic objective (Sec. 3.3). We combined the MPC formulation using artificial periodic reference trajectories with a purely economic objective and avoided the use of any positive definite tracking cost. Thus, the considered formulation is purely economically oriented and can yield a better transient performance than tracking formulations. In order to arrive at the desired performance guarantees, we required stronger assumptions on the terminal ingredients, a self-tuning weight for the artificial reference and an additional technical modification to avoid potential pitfalls in the periodic economic setting.

Since Chapter 3 involves different design procedures with potential performance benefits, we included many nonlinear numerical examples to study the practicality and performance benefits of the proposed formulations (Sec. 3.4). In particular, we demonstrated significant performance improvements when using parametrized terminal ingredients and an online optimized terminal set size (Sec. 3.4.1). We show that the partially decoupled reference updates allow for a tunable reduction in the computational complexity with small performance degradation (Sec. 3.4.2). We establish substantial economic performance benefits for the proposed economic formulation (Sec. 3.4.3–3.4.4).

In conclusion, the results presented in Chapter 3 provide a collection of design methods and ensure highly performant dynamic operation using MPC.

Analysis methods

In Chapter 4, we studied the closed-loop properties of simple and intuitive MPC formulations for dynamic operation. In particular, we considered MPC formulations without terminal ingredients, which require no offline design and are hence easy to implement. We first considered the problem of tracking a reachable reference trajectory using an MPC without terminal ingredients (Sec. 4.1). If the system is locally incrementally stabilizable, then the problem satisfies a local cost controllability condition for any reachable reference trajectory and thus exponential stability can be guaranteed using a sufficiently large prediction horizon N. We extended this analysis to positive semidefinite input-output stage costs ℓ using a suitable detectability/observability condition and a modified proof. In order to allow for shorter prediction horizons, we also studied the closed-loop properties with an extended prediction horizon based on a known control law κ.

As a second contribution, we considered the problem of *output regulation*, where an output reference is generated by an exosystem (Sec. 4.2). The considered MPC formulation does *not* require the solution of the regulator/FBI equations and instead implicitly tracks the output-zeroing regulator manifold. Our analysis revealed that an additional *minimum-phase* property is crucial to provide closed-loop guarantees. Since this simple formulation may result in stability issues in case of unstable zero dynamics, we also proposed an input regularization for periodic problems to overcome this limitation to minimum-phase systems. With this modified formulation, we derived the desired guarantees under a suitable *detectability* condition (i-IOSS) and a technical *nonresonance* condition.

Finally, as a third contribution, we studied the properties of the previous trajectory tracking MPC formulation in case the reference trajectory is not reachable (Sec. 4.3). We derived sufficient conditions such that the simple tracking MPC formulation (practically) stabilizes the best reachable trajectory if a sufficiently long prediction horizon N is employed. This result was derived using tools from economic MPC, such as uniqueness conditions in terms of dissipation inequalities.

In the special case of linear system dynamics, the conditions for the theoretical results in Chapter 4 reduce to standard system properties, such as detectability, stabilizability and conditions on the zeros of the transfer matrix (Sec. 4.4). We illustrated the theoretical results and their practical applicability with numerical examples (Sec. 4.5). In addition, we quantitatively compared the derived stability and performance bounds to state of the art results.

In conclusion, Chapter 4 provides a variety of analysis methods that guarantee stability and a desired performance with simple MPC formulations without any complex design procedures merely on the basis of system theoretic properties (e.g., incremental stabilizability, detectability, minimum-phase). This chapter also contains various improved bounds for MPC without terminal constraints based on *local* stabilizability conditions, positive *semidefinite* stage costs and extended prediction horizons that may be of independent interest.

5.2 Discussion

In the following, we elaborate on the complementary nature of the theoretical results in Chapters 3 and 4, and more generally on MPC formulations with and without terminal ingredients for nonlinear dynamic problems.

In both chapters, we first considered the problem of tracking a reachable reference trajectory (cf. Sec. 3.1/4.1). In both MPC approaches, we require the same system property, namely (local) incremental stabilizability (cf. Def. 4.18, Prop. 4.20, Rk. 4.21). The main difference is the fact that the design in Section 3.1 requires an explicit computation of a local CLF V_{f}, while the MPC approach in Section 4.1 only requires a scalar bound $\gamma > 0$ on V_{f} (cf. Prop. 4.20), which is often easier to compute (cf. App. C). However, this simpler design may come at the price of performance degradation or the requirement to use a long prediction horizon N, compare the numerical example in [JK15]. This is analogous to the standard trade-off faced when choosing an MPC scheme with or without terminal ingredients for setpoint stabilization, compare [188]. We point out that Section 4.1.5 offers an intermediate solution which only requires the offline design of a stabilizing feedback κ and ensures stability with a shorter prediction horizon N. An important difference compared to the standard problem of stabilizing a given steady-state is that both the design of the terminal ingredients (Sec. 3.1) and the verification of the cost controllability condition (cf. Sec. 4.1.4, App. C) are more challenging in the considered case of dynamic operation.

Although the setups considered in Sections 3.2 and 4.2 are not equivalent, both setups include tracking of periodic output signals as an important special case. In particular, the uniqueness condition used in Section 3.2 (cf. Ass. 3.29) is essentially equivalent to the nonresonance condition (Ass. 4.63) used in Section 4.2.3, compare Remark 4.68. A first distinction can be made regarding the class of reference trajectories or target signals. The approach in Section 3.2 naturally accommodates unreachable target signals and sudden changes in the state of the exosystem. Although unpredictable changes in the reference can generally lead to feasibility issues for the MPC formulation in Section 4.2, for many practical problems (no state constraints, incrementally stable) we can retain the theoretical properties due to the absence of terminal constraints (cf. [JK19, App. B]). In case of unreachable reference trajectories (Ass. 4.15/4.39 does not hold), we can still provide guarantees without artificial references (Sec. 4.3), but this result requires dissipation conditions, which are difficult to verify, and potentially a significantly longer prediction horizon N. On the other hand, for minimum-phase systems, the approach presented in Section 4.2 is not limited to periodic trajectories and can be directly applied to general time-varying reference trajectories. Furthermore, even if the underlying problem is periodic, the complexity of the MPC formulation in Section 4.2 does not directly depend on the period length T, while the approach in Section 3.2 explicitly computes a T-periodic reference trajectory. This complexity limitation was also the main motivation to provide the partially decoupled formulation (Sec. 3.2.2), which can reduce this problem.

The last and most general control problem considered in this thesis concerns optimal dynamic operation with an economic stage cost ℓ_{eco}. In Section 3.3, we presented an economic MPC formulation with guaranteed performance, which requires no prior assumptions on the optimal mode of operation. The main limitation of this approach is the fixed period length T used for the artificial reference trajectory. Although we did not study the general economic case in Chapter 4, the analysis in Section 4.3 regarding unreachable reference trajectories can be extended to general time-varying economic problems, compare [128, 129, 130]. The main benefit of such an economic MPC formulation without terminal ingredients is that the approach can be directly applied to general time-varying problems and is not limited to performance guarantees w.r.t. periodic operation. However, the corresponding performance guarantees may require a very long prediction horizon N and use a turnpike property w.r.t. a specific trajectory (cf. overtaking optimality in [128]). This turnpike property holds for example in case of linear systems with strictly convex cost (cf. Sec. 4.4.2 and [130]), but does not necessarily hold for general nonlinear time-invariant problems [210].

Overall, the results of Chapters 3 and 4 consider similar problems (trajectory tracking, output regulation, economic operation) and the resulting MPC formulations have different advantages and disadvantages. The MPC formulations from Chapter 3 typically require an explicit offline design procedure and are often restricted to periodic problems, with potential scalability issues for very long period lengths T. However, neither short prediction horizons N nor unreachable reference targets pose a problem for the MPC formulations in Chapter 3. On the other hand, the MPC formulations from Chapter 4 often require a significantly longer prediction horizon N and some of the conditions may be difficult to verify a priori, but the approaches are directly applicable to general time-varying problems and can "find" the optimal mode of operation in closed loop. This shows the complementary nature of the two MPC frameworks presented in this thesis, each with their own respective advantages.

5.3 Outlook

In this thesis, we provided analysis results and design procedures for dynamic operation with MPC, based on which additional future research topics can be pursued.

In this thesis, we neglected issues regarding external disturbances and uncertain predictions. To account for such a model mismatch, robust MPC methods are required to ensure recursive feasibility, compare, e.g., [33, 118, JK29, 160] and references therein. First steps in this direction can be found in [JK15, App. B] and [JK30], where the tracking MPC formulations in Sections 3.1 and 3.2 have been augmented with the (tube-based) robust MPC approach from [JK17, JK29], which is based on incremental stabilizability (Def. 4.18). Additionally, an analysis for this robust MPC formulation without terminal ingredients can be found in [JK17], which is applicable to the trajectory tracking tracking MPC in Section 4.1 in case of positive definite stage costs (Sec. 4.1.1). Initial robustness results for the output regulation problem (Sec. 4.2) have been derived in [JK19, App. B], but this issue requires further research. In particular, in the output regulation setting, the overall system is not stabilizable and disturbances in the exosystem result in deviations in the reference trajectory which complicates the closed-loop stability analysis. Hence, in addition to the classical robustness issues considered in MPC, uncertainty in the reference trajectory, as for example treated in [77, 88, 199], is largely an open problem. For the economic MPC formulation in Section 3.3, the consideration of uncertain predictions also requires additional attention. Specifically, in the economic setting robust performance guarantees may require the usage of a cost function that takes uncertainty into account [30, 31, 82, JK35, 271, 272]. Furthermore, in various applications such as HVAC, additional uncertainty in forecasts and changing prices need to be taken into account to guarantee closed-loop performance.

The results presented in this thesis have been derived based on the initial results in Sections 3.1 and 4.1 for trajectory tracking. A natural extension of this work would be to consider more general path-following MPC formulations [91, 94] (cf. also the recent works [28, 244]) which introduce an additional degree of freedom and can overcome intrinsic performance limitations of tracking formulations [3, 4]. In this direction, the reference generic offline computations (Sec. 3.1) could be extended to the path following problem. Furthermore, although there exist some results for path-following MPC without terminal ingredients [95, 194], a more general theoretical framework could be derived in combination with the output regulation MPC from Section 4.2. In addition, adopting a continuous-time parametrization for the periodic artificial trajectories used in Sections 3.2–3.3 could significantly relax the periodicity condition, compare Remark 3.67 and [133].

The offline computation of the parametrized terminal ingredients (Sec. 3.1) could be extended in multiple directions. Improving the scalability of the SDP (Lemma 3.13, Prop. 3.15), e.g., by using distributed formulations similar to [60, 279], would be valuable. A more detailed investigation regarding the alternative LDI-based formulation (cf. Rk. 3.23) would be of interest to make the approach applicable to a broader class of nonlinear systems and easier to automate by avoiding difficulties in computing α_1 (cf. (3.26) and Prop. 3.11). Quantitative/numerical comparisons regarding the performance/conservatism of the quadratically parametrized terminal cost V_{f} (Sec. 3.1.3) and the finite-tail cost $V_{\mathrm{f},M}$ (Sec. 4.1.5) would also be of practical relevance. Furthermore, exploring the applicability of the proposed offline procedure to other control problems, such as computing invariant sets for reference governors (cf. Rk. 3.34), verifying detectability for moving horizon estimation (MHE) [14, 147, 205], and developing robust MPC formulations (cf. App. C), is of interest.

Further investigations regarding the performance in case of changing reference signals for the tracking MPC in Section 3.2 would be interesting (cf. [88]), in particular deriving regret like bounds similar to [157, 214]. In addition, the issue of decoupling the computational complexity from the period length T has been considered in Section 3.2.3, but this problem deserves further research. Alternative solutions based on simplified parametrizations of the artificial reference trajectory may prove useful, e.g., using move-blocking [49, 115] or models of different granularity [27, 44]. In addition, the corresponding theoretical properties in the economic case (Sec. 3.3) may require further attention (cf. Rk. 3.83). Due to the similarities with contract-based MPC design (cf. [26, 90, 172]), it may be possible to extend the approach in Section 3.2.3 to derive hierarchical and distributed MPC formulations with more flexible asynchronous communication protocols.

Regarding the results in Chapter 4, the resulting bounds for a sufficiently long prediction horizon N are sometimes too conservative to be applied in practice and thus less conservative bounds are desired. A promising avenue for less conservative bounds is the extended prediction horizon MPC (Sec. 4.1.5). We conjecture the bounds for the extended prediction horizon can be improved using an LP analysis analogous to [120, 127] (cf. Rk. 4.32). Furthermore, an analysis of the extended prediction horizon MPC (Sec. 4.1.5) for detectable/observable stage costs (Sec. 4.1.3) may be particularly interesting for data-driven input-output models [1, JK3, JK4, JK5, JK6, JK7, 51, 65, 66, 180, 181, 182, 183], since the design of standard terminal ingredients is often non-trivial.

A further investigation into proofs for MPC without terminal ingredients that utilize system properties like minimum-phase or do not use the value function as a Lyapunov function (cf. [119], Sec. 4.1.2) may be promising. For example, a completely different analysis regarding stability with short horizons may be possible with the output stage cost ℓ_{y} for minimum-phase systems (cf. Rk. 4.54). In addition, it may be possible to obtain practical heuristic tuning methods for stage costs $\ell = \|y\|^2 + \lambda\|u\|^2$. In particular, for λ small, we expect guaranteed stability for minimum-phase systems and thus automatic tuning methods like λ-tracking (cf. [46, 140]) may be applicable to MPC. We point out that we can also define an appropriate output $y = Cx + Du$ to guarantee the minimum-phase property (cf. zero assignment problem [25, Sec. 4.5.1]) and thus obtain a systematically motivated way to design the stage cost.

Regarding the output regulation problem (Sec. 4.2), it would be interesting to further generalize some of the assumptions to derive necessary and sufficient conditions (e.g., using converse Lyapunov theorems for i-IOSS). In particular, considering a more general characterization of the minimum-phase property (independent of the BINF) would be interesting, e.g., based on output-input stability [158] or dissipation inequalities similar to [84]. Furthermore, for the error-feedback case (Rk. 4.43) the consideration of more general nonlinear bounds in the proof would be interesting.

The results in Section 4.3 are largely an extension of existing approaches for economic MPC without terminal constraints, which require more research to address some of the practical issues. In particular, the resulting guarantees are often only of conceptual nature and the exact bounds on the prediction horizon are too conservative to be applied in practice (since the bound on the storage function λ can be very large, even for linear problems). This limitation can be reduced by modifying the MPC optimization problem (cf. [291] or [8]). Extending such approaches to dynamic problems is both theoretically and practically an interesting open problem.

In conclusion, the results derived in this thesis open up various research directions regarding novel MPC design and analysis methods, which are relevant for dynamic operation of nonlinear constrained systems using model predictive control.

Appendix A

Suboptimality estimates for MPC with terminal ingredients

In the following, we briefly derive suboptimality estimates $\alpha_N \in (0,1]$ w.r.t. infinite-horizon performance for MPC with terminal ingredients. We point out that similar suboptimality estimates can be found in [131, Thm. 6.2/6.4] and [126, Thm. 5.22], considering an additional non-vanishing constant $\eta > 0$ bounding the difference between the terminal cost V_f and the infinite-horizon optimal cost V_∞. Furthermore, in [121, Thm. 6.6], for N large enough, a suboptimality bound based on relatively general conditions is derived. The following (simple) exposition shows that for any horizon $N \in \mathbb{I}_{>0}$, for conditions comparable to the literature for MPC without terminal ingredients (cf. [120, 123, 127]), MPC with terminal ingredients also guarantees a suboptimality index $\alpha_N > 0$ w.r.t. the infinite horizon optimal performance.

We consider the setup from Section 2.1, with the following simplifying conditions.

Assumption A.1. *Assumptions 2.3–2.5 hold with* $\alpha_\mathrm{f} = \alpha_\mathrm{V} = \gamma\alpha_\ell$ *with a constant* $\gamma \geq 1$. *Furthermore,* $\mathbb{X}_\mathrm{f} = \mathbb{X}$.

We point out that assuming $\mathbb{X}_\mathrm{f} = \mathbb{X}$ is comparable to considering the asymptotic controllability condition (Assumption 2.9) on the full state space. Both conditions can be relaxed using sublevel set arguments, compare, e.g., [37] and Section 4.1. The linear bound γ is naturally satisfied if the stage cost ℓ and the terminal cost V_f are quadratic.

The following theorem provides suboptimality estimates for MPC with terminal ingredients.

Theorem A.2. *Let Assumption A.1 hold. Then, for any* $N \in \mathbb{I}_{>0}$, *there exists a constant* $\alpha_N \in (0,1]$ *such that for any* $x_0 \in \mathbb{X}$, *the closed-loop system* (2.3) *resulting from Algorithm 2.2* *satisfies the following performance bound:*

$$\mathcal{J}_{\infty}^{\mathrm{cl}}(x_0) := \sum_{t=0}^{\infty} \ell(x(t), u(t)) \leq V_N(x_0) \leq \frac{V_\infty(x_0)}{\alpha_N}.$$

Furthermore, $\lim_{N\to\infty} \alpha_N = 1$.

Proof. Given an initial condition $x_0 \in \mathbb{X}$, denote an infinite horizon optimal trajectory as $x_\infty(k)$, $u_\infty(k)$, $k \in \mathbb{I}_{\geq 0}$ with $V_\infty(x_0) = \sum_{k=0}^{\infty} \ell(x_\infty(k), u_\infty(k))$. Since $\mathbb{X}_\mathrm{f} = \mathbb{X}$, the initial part of this trajectory is a feasible candidate solution to Problem 2.1, implying

$$V_N(x_0) \leq V_\infty(x_0) + V_\mathrm{f}(x_\infty(N)). \tag{A.1}$$

Assumption A.1 ensures that $V_\infty(x) \leq V_\mathrm{f}(x) \leq \gamma \ell_{\min}(x)$ for all $x \in \mathbb{X}$. Using standard arguments from MPC without terminal ingredients (cf. Sec. 4.1.2, [123, Variant 2]), this implies that the optimal solution satisfies

$$V_\mathrm{f}(x_\infty(N)) \leq \gamma V_\infty(x_\infty(N)) \leq \rho_\gamma^{2N} \gamma V_\infty(x_0), \tag{A.2}$$

with $\rho_\gamma^2 := \dfrac{\gamma - 1}{\gamma} \in [0, 1)$. Combining (A.1)–(A.2), we arrive at

$$V_N(x_0) \leq (1 + \rho_\gamma^{2N} \gamma) V_\infty(x_0) =: \frac{1}{\alpha_N} V_\infty(x_0).$$

Note that for any $N \in \mathbb{I}_{>0}$, we have $\alpha_N > 0$ and furthermore, $\lim_{N\to\infty} \alpha_N = 1$. ■

Similar derivations for a more general setup can be found in Theorem 4.37.

Appendix B

Terminal ingredients - extensions

In the following, we discuss how the derivation in Section 3.1 can be extended to deal with an output tracking stage cost ℓ. This section is based on and taken in parts literally from [JK15, App. D][1].

As an alternative to the quadratic stage cost in (3.3), consider the following output reference tracking stage cost

$$\ell(x, u, r) = \|h(x, u) - h(x_\mathrm{r}, u_\mathrm{r})\|^2_{S(r)}, \tag{B.1}$$

with a nonlinear twice continuously differentiable output function $h : \mathbb{Z} \to \mathbb{Y}$, $\mathbb{Y} \subseteq \mathbb{R}^{n_\mathrm{y}}$ and a continuously parametrized positive definite weighting matrix $S : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{y}}$. Such a stage cost can be used for output regulation, output trajectory tracking, output path following or manifold stabilization, compare [91, 94] and Sections 4.1–4.2. In addition, the design of terminal ingredients for economic stage costs (cf. Sec. 3.3.5) also builds on the following derivation.

We denote the Jacobian of the output h around an arbitrary point $r \in \mathbb{Z}_\mathrm{r}$ by

$$C(r) = \left[\frac{\partial h}{\partial x}\right]\bigg|_{(x_\mathrm{r}, u_\mathrm{r})}, \quad D(r) = \left[\frac{\partial h}{\partial u}\right]\bigg|_{(x_\mathrm{r}, u_\mathrm{r})}.$$

The following lemma establishes sufficient conditions for Assumptions 3.5–3.6 with the stage cost (B.1) based on the Jacobian, similar to Lemma 3.12.

Lemma B.1. *Suppose that f, h are twice continuously differentiable. Assume that there exists a continuously parametrized matrix $K : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{m \times n}$ and a continuously parametrized positive definite matrix $P : \mathbb{Z}_\mathrm{r} \to \mathbb{R}^{n \times n}$ such that for any $(r, r^+) \in \mathcal{R}$, the following matrix inequality*

[1] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear model predictive control framework using reference generic terminal ingredients." In: *IEEE Trans. Automat. Control* 65.8 (2020). extended version: arXiv:1909.12765, pp. 3576–3583©2019 IEEE.

is satisfied

$$\begin{aligned}&(A(r)+B(r)K(r))^\top P(r^+)(A(r)+B(r)K(r))-P(r)\\ \preceq &-(C(r)+D(r)K(r))^\top S(r)(C(r)+D(r)K(r))-\tilde{\epsilon}I_n\end{aligned} \tag{B.2}$$

with some constant $\tilde{\epsilon}>0$*. Then, there exists a sufficiently small constant* $\alpha>0$ *such that* $V_{\mathrm{f}}(x,r)=\|x-x_{\mathrm{r}}\|_{P(r)}^2$, $k_{\mathrm{f}}(x,r)=u_{\mathrm{r}}+K(r)\cdot(x-x_{\mathrm{r}})$, $\mathbb{X}_{\mathrm{f}}=\{(x,r)\in\mathbb{X}\times\mathbb{Z}_{\mathrm{r}}\mid V_{\mathrm{f}}(x,r)\leq\alpha\}$ *satisfy Assumptions 3.5–3.6.*

Proof. A first order Taylor approximation at $r=(x_{\mathrm{r}},u_{\mathrm{r}})$ yields

$$h(x,k_{\mathrm{f}}(x,r))-h(x_{\mathrm{r}},u_{\mathrm{r}})=(C(r)+D(r)K(r))\Delta x+\tilde{\Phi}(\Delta x,r),$$

with the remainder term $\tilde{\Phi}$ and $\Delta x=x-x_{\mathrm{r}}$. The stage cost satisfies

$$\begin{aligned}\ell(x,k_{\mathrm{f}}(x,r),r)\geq&\|(C(r)+D(r)K(r))\Delta x\|_{S(r)}^2+\|\tilde{\Phi}(\Delta x,r)\|_{S(r)}^2\\ &-2\|\tilde{\Phi}(\Delta x,r)\|_{S(r)}\|(C(r)+D(r)K(r))\Delta x\|_{S(r)}.\end{aligned} \tag{B.3}$$

Given continuity and compactness, there exists a constant

$$c_{\mathrm{y}}:=\max_{r\in\mathbb{Z}_{\mathrm{r}}}\|C(r)+D(r)K(r)\|_{S(r)}, \tag{B.4}$$

and we consider w.l.o.g.[2] $c_{\mathrm{y}}^2\geq\tilde{\epsilon}/2$. For a sufficiently small $\alpha>0$, the remainder term $\tilde{\Phi}$ satisfies the following (local) Lipschitz bound for all $(x,r)\in\mathbb{X}_{\mathrm{f}}$:

$$\|\tilde{\Phi}(\Delta x,r)\|_{S(r)}=:\tilde{L}_{x,r}\|\Delta x\|, \tag{B.5}$$

$$\tilde{L}_{x,r}\leq\tilde{L}_{\Phi}:=c_{\mathrm{y}}-\sqrt{c_{\mathrm{y}}^2-\tilde{\epsilon}/2}. \tag{B.6}$$

[2] In case $\tilde{\epsilon}/2\geq c_{\mathrm{y}}^2$, the following derivation yields the same bound using $L_{x,r}(L_{x,r}-2c_{\mathrm{y}})\leq-c_{\mathrm{y}}^2\leq-\tilde{\epsilon}/2$.

This implies

$$\begin{aligned}
&\ell(x,k_{\mathrm{f}}(x,r))\\
\overset{(B.3),(B.5)}{\geq} &\|(C(r)+D(r)K(r))\Delta x\|_{S(r)}^2+\tilde{L}_{r,x}^2\|\Delta x\|^2-2\tilde{L}_{r,x}\|\Delta x\|^2\|C(r)+D(r)K(r)\|_{S(r)}\\
\overset{(B.4)}{\geq} &\|(C(r)+D(r)K(r))\Delta x\|_{S(r)}^2+\tilde{L}_{r,x}(\tilde{L}_{r,x}-2c_{\mathrm{y}})\|\Delta x\|^2\\
\overset{(B.6)}{\geq} &\|(C(r)+D(r)K(r))\Delta x\|_{S(r)}^2+\tilde{L}_{\Phi}(\tilde{L}_{\Phi}-2c_{\mathrm{y}})\|\Delta x\|^2\\
\overset{(B.6)}{=} &\|(C(r)+D(r)K(r))\Delta x\|_{S(r)}^2-\tilde{\epsilon}/2\|\Delta x\|^2.
\end{aligned}$$

The second to last step follows by using the fact that the function $L(L-2c_{\mathrm{y}})$ attains it minimum for $L\in[0,\tilde{L}_{\Phi}]$ at $L=\tilde{L}_{\Phi}<c_{\mathrm{y}}$. Combining the derived bound on $\ell(x,k_{\mathrm{f}}(x,r))$ with (B.2) ensures that the terminal cost V_{f} satisfies Inequality (3.15) in Lemma 3.12 with the modified stage cost and with $\epsilon=\tilde{\epsilon}/2$. The remainder of the proof is analogous to Lemma 3.12. ■

Remark B.2. *(Special case: quadratic cost) For the linear output $h(x,u)=(Q^{1/2}x,R^{1/2}u)^\top\in\mathbb{R}^{n+m}$ and $S=I_{n+m}$, we recover the conditions in Lemma 3.12 with the quadratic stage cost* (3.3).

Lemma B.3. *Suppose that there exist continuously parametrized matrices $X:\mathbb{Z}_{\mathrm{r}}\to\mathbb{R}^{n\times n}$, $Y:\mathbb{Z}_{\mathrm{r}}\to\mathbb{R}^{m\times n}$ and $X_{\min}\in\mathbb{R}^{n\times n}$ that satisfy the following constraints*

$$\min_{X(r),Y(r),X_{\min}} -\log\det X_{\min} \tag{B.7a}$$

$$\text{s.t.}\begin{pmatrix} X(r) & (A(r)X(r)+B(r)Y(r))^\top & (C(r)X(r)+D(r)Y(r))^\top & \sqrt{\tilde{\epsilon}}X(r)\\ * & X(r^+) & 0 & 0\\ * & * & S^{-1}(r) & 0\\ * & * & * & I_n\end{pmatrix}\succeq 0, \tag{B.7b}$$

$$X_{\min}\preceq X(r),\quad \forall (r,r^+)\in\mathcal{R}. \tag{B.7c}$$

Then, $P=X^{-1}$, $K=YP$ satisfy (B.2) *for all $(r,r^+)\in\mathcal{R}$.*

Proof. The proof is similar to Lemma 3.13, compare also [40]. Define $X(r)=P^{-1}(r)$ and

$Y(r) = K(r)X(r)$. Multiplying (B.2) from left and right with $X(r)$ yields

$$\begin{aligned}&(A(r)X(r)+B(r)Y(r))^\top P(r^+)(A(r)X(r)+B(r)Y(r)) - X(r) + \tilde{\epsilon}X(r)I_nX(r)\\&+(C(r)X(r)+D(r)Y(r))^\top S(r)(C(r)X(r)+D(r)Y(r)) \preceq 0.\end{aligned}$$

This can be equivalently written as

$$X(r) - \begin{pmatrix} A(r)X(r)+B(r)Y(r) \\ C(r)X(r)+D(r)Y(r) \\ \sqrt{\tilde{\epsilon}}X(r)\end{pmatrix}^\top \begin{pmatrix} P(r^+) & 0 & 0 \\ 0 & S(r) & 0 \\ 0 & 0 & I_n\end{pmatrix} \begin{pmatrix} A(r)X(r)+B(r)Y(r) \\ C(r)X(r)+D(r)Y(r) \\ \sqrt{\tilde{\epsilon}}X(r)\end{pmatrix} \succeq 0.$$

Using the Schur complement, this reduces to (B.7), which is linear in X, Y. ∎

Similar to Proposition 3.15, we can obtain a tractable SDP, if in addition to the parametrizations (3.20)–(3.21), the parameters θ_i are chosen such that

$$S^{-1}(r) = S_0 + \sum_{i=1}^{p} \theta_i(r) S_i.$$

Proposition B.4. *Let Condition* (3.22) *hold with* $\overline{\Theta}$ *according to* (3.23)*. Suppose that there exist matrices* X_i*,* Y_i*,* $i \in \mathbb{I}_{[0,p]}$*,* Λ_i*,* $i \in \mathbb{I}_{[1,p]}$*,* $X_{\min}$ *that satisfy the following constraints*

$$\min_{X_i,\, Y_i,\, \Lambda_i,\, X_{\min}} \; -\log\det X_{\min} \tag{B.8a}$$

$$\text{s.t.} \begin{pmatrix} X(\theta) & (A(\theta)X(\theta)+B(\theta)Y(\theta))^\top & (C(\theta)X(\theta)+D(\theta)Y(\theta))^\top & \sqrt{\tilde{\epsilon}}X(\theta) \\ * & X(\theta^+) & 0 & 0 \\ * & * & S^{-1}(\theta) & 0 \\ * & * & * & I_n \end{pmatrix} \succeq \begin{pmatrix} \sum_{i=1}^p \theta_i^2 \Lambda_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{B.8b}$$

$$X_{\min} \preceq X(\theta), \tag{B.8c}$$

$$\forall (\theta, \theta^+) \in \text{vert}(\overline{\Theta}), \tag{B.8d}$$

$$\begin{pmatrix} 0 & (A_iX_i+B_iY_i)^\top & (C_iX_i+D_iY_i)^\top \\ (A_iX_i+B_iY_i) & 0 & 0 \\ (C_iX_i+D_iY_i) & 0 & 0 \end{pmatrix} \preceq \Lambda_i, \quad \Lambda_i \succeq 0, \quad i \in \mathbb{I}_{[1,p]}. \tag{B.8e}$$

Then, $P = X^{-1}$ *and* $K = YP$ *satisfy* (B.2).

Proof. The proof is analogous to Proposition 3.15 based on Lemma B.3. The constraint (B.8e) ensures multi-convexity. ■

Remark B.5. *(Stability properties) Depending on the output* h *and the reference* r*, there may exist multiple solutions that achieve exact output tracking. Thus, we can in general not expect stability of the reference* r*, but instead convergence to a corresponding set or manifold, compare [91]. Stability can be ensured using additional assumptions regarding detectability of the output* h *involving incremental input-output to state stability (i-IOSS) arguments, compare, e.g., [JK3, Thm. 2], [236, Thm. 2.24] and also the results in Sections 4.1–4.2.*

Appendix C

Incremental system properties

The goal of this section is to concisely summarize some of the main theoretical results in the literature regarding incremental system properties, especially in connection with their application in MPC theory. The content of this section is largely a summary of the relevant theoretical results in [18, 169, 178, 264] and a discussion regarding the relation to conditions and design procedures in this thesis. First, we define incremental stability, incremental Lyapunov functions, and contraction metrics. Then, we discuss control contraction metrics, universal stabilizability and its relation to incremental stabilizability (Def. 4.18). Furthermore, we contrast the SDP design for (local) incremental Lyapunov functions (Sec. 3.1.3) with approaches from the literature on incremental stability and contraction metrics. Finally, we discuss the applicability of incremental system properties in the context of MPC.

Incremental stability

In the following, we define some standard concepts regarding incremental stability. We consider a nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)), \quad x(0) = x_0, \tag{C.1}$$

with the state $x(t) \in \mathbb{X} = \mathbb{R}^n$, the control input $u(t) \in \mathbb{U} = \mathbb{R}^m$, the dynamics $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, the initial condition $x_0 \in \mathbb{X}$, and the time step $t \in \mathbb{I}_{\geq 0}$. Given an initial condition $x \in \mathbb{X}$, and an input sequence $u(\cdot)$, we denote the system response with input u at time t by $\phi(t, x, u(\cdot))$. We consider the following definition regarding incremental stability, analogous to [18].

Definition C.1. *(Incremental stability) System* (C.1) *is incrementally (globally) exponentially stable, if there exist constants* $C \geq 1$, $\rho \in [0, 1)$ *such that for all* $u(\cdot) \in \mathbb{U}$, *all* $\xi, \eta \in \mathbb{X}$ *and all*

$t \in \mathbb{I}_{\geq 0}$, *the following holds*

$$\|\phi(t,\xi,u(\cdot)) - \phi(t,\eta,u(\cdot))\| \leq C\rho^k \|\xi - \eta\|.$$

This definition implies that if the system is driven by some fixed input trajectory $u(\cdot)$, then the state converges to a fixed trajectory (assuming existence), independent of the initial condition. This system property is related to the notion of *convergent dynamics* [221]. Incremental stability can equivalently be characterized using an incremental Lyapunov function $V_\delta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geq 0}$ satisfying

$$c_\mathrm{l}\|x_1 - x_2\|^2 \leq V_\delta(x_1,x_2) \leq c_\mathrm{u}\|x_1 - x_2\|^2, \tag{C.2a}$$

$$V_\delta(f(x_1,u), f(x_2,u)) \leq \rho^2 V_\delta(x_1,x_2), \tag{C.2b}$$

for any $(x_1, x_2, u) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U}$ with constants $c_\mathrm{l}, c_\mathrm{u} > 0$, $\rho \in [0,1)$, compare [18, Thm. 1], [264, Thm. 9]. We note that in [264] a time-varying system f with no input u is considered and the resulting incremental Lyapunov function is time-varying. However, in the continuous-time converse Lyapunov proof in [18] external signals are explicitly considered and a time-invariant function V_δ is obtained. In the considered setting of exponential stability, there exists a finite constant $T \in \mathbb{I}_{\geq 1}$ such that

$$V_\delta(x_1,x_2) := \sup_{u(\cdot)\in\mathbb{U}^T} \sum_{t=0}^{T-1} \|\phi(t,x_1,u(\cdot)) - \phi(t,x_2,u(\cdot))\|^2,$$

is a valid incremental Lyapunov function (cf. proof. Prop. 4.34). Incremental stability can also be studied using the Jacobian and contraction metrics/differential Lyapunov functions [108, 169]. Suppose f is continuously differentiable and denote the Jacobian of f by

$$A(x,u) = \left[\frac{\partial f}{\partial x}\right]\Bigg|_{(x,u)}, \quad B(x,u) = \left[\frac{\partial f}{\partial u}\right]\Bigg|_{(x,u)}.$$

The differential dynamics of system (C.1) are given by

$$\delta_\mathrm{x}(t+1) = A(x,u)\delta_\mathrm{x}(t) + B(x,u)\delta_\mathrm{u}(t), \tag{C.3}$$

where $\delta_\mathrm{x}(t) \in \mathbb{R}^n$ and $\delta_\mathrm{u}(t) \in \mathbb{R}^m$ are called the virtual displacement and correspond to an infinitesimal displacement w.r.t. x, u. Uniform exponential stability of the differential dynamics can be studied using a contraction metric [169] or equivalently a differential

Lyapunov function [108].

Definition C.2. *(Contraction metric) The system* (C.1) *is contracting with the contraction metric* $M : \mathbb{X} \to \mathbb{R}^{n\times n}$, *if there exist constants* $c_1, c_2 > 0$, $\rho \in [0, 1)$ *such that for all* $(x, u) \in \mathbb{X} \times \mathbb{U}$, *we have*

$$c_1 I_n \preceq M(x) \preceq c_2 I_n \tag{C.4a}$$
$$A^\top(x,u) M(f(x,u)) A(x,u) \preceq \rho^2 M(x). \tag{C.4b}$$

Note that the conditions (C.4) correspond to Lyapunov inequalities for uniform exponential stability of the differential dynamics (C.3) with the differential Lyapunov function $V_{\mathrm{x}}(x, \delta_{\mathrm{x}}) = \|\delta_{\mathrm{x}}\|^2_{M(x)}$ and $\delta_{\mathrm{u}} = 0$, compare [108]. Condition (C.4b) is often equivalently expressed in terms of the generalized Jacobian $F(x, u) = \Theta(f(x, u))A(x, u)\Theta^{-1}(x)$, with $M = \Theta^\top\Theta$ [169, 264].

In the considered case of exponential stability and continuously differentiable dynamics, incremental stability (Def. C.1) is equivalent to the existence of a contraction metric (Def. C.2) (cf. [111, 264]) and V_δ is also called a contraction analysis Lyapunov function [264]. In particular, given any two states $x_1, x_2 \in \mathbb{X}$, denote the set of continuously differentiable curves $\gamma : [0, 1] \to \mathbb{X}$ with $\gamma(0) = x_1$, $\gamma(1) = x_2$ by $\Gamma(x_1, x_2)$. Then, the Riemannian energy function

$$V_\delta(x_1, x_2) = \inf_{\gamma \in \Gamma(x_1,x_2)} \left(\int_0^1 \left.\frac{\partial \gamma}{\partial s}\right|_s^\top M(\gamma(s)) \left.\frac{\partial \gamma}{\partial s}\right|_s \right) ds, \tag{C.5}$$

is a valid incremental Lyapunov function, i.e., Inequalities (C.2) hold. The minimizing curve γ^* is called the geodesic [178]. Under the conditions of the Hopf-Rinow theorem, this geodesic exists and the Riemannian distance satisfies $d(x_1, x_2) = \sqrt{V_\delta(x_1, x_2)}$ (cf. [178]). In case of a constant metric $M \in \mathbb{R}^{n\times n}$, the Riemannian energy is given by the quadratic function $V_\delta(x_1, x_2) = \|x_1 - x_2\|^2_M$ and the geodesic γ^* is a straight line, compare also the results in [67, 221] regarding convergent dynamics and incremental quadratic stability. We point out that earlier results regarding the equivalence of differential stability and incremental stability for continuous-time systems can be found in [111, Thm. 3.4], based on an operator view point using the Gâteaux derivative, which results in an LTV system equivalent to (C.3). Note that Inequalities (C.4) are similar to the matrix inequalities considered in Lemma 3.12 and correspondingly the approaches discussed in Section 3.1.3 can be used to compute a contraction metric (SOS, LPV, gridding,...).

Incremental stabilizability

In the following, we consider the problem of designing a smooth feedback[1] $\kappa : \mathbb{X} \to \mathbb{U}$ such that the closed-loop with $u = \kappa(x) + v$ is contracting [178]. Denote the Jacobian by $K(x) := \left[\frac{\partial \kappa}{\partial x}\right]\Big|_x$. The contraction condition (C.4) for the closed loop changes to

$$(A(x,u) + B(x,u)K(x))^\top M(f(x,u))(A(x,u) + B(x,u)K(x)) \preceq \rho^2 M(x). \tag{C.6}$$

If this condition holds, the metric M is called a *control contraction metric* (CCM) and the system is said to be *universally exponentially stabilizable* [178]. Furthermore, in [178], for continuous-time systems, Condition (C.6) is translated into a necessary and sufficient condition for the contraction metric M, independent of the feedback K. Using the dual metric $W = M^{-1}$ and $Y = KW$, the joint design of the feedback K and metric W is translated into a convex infinite dimensional problem, which can be solved using SOS optimization and polynomial parametrization. In order to stabilize a given trajectory $(x_\mathrm{r}, u_\mathrm{r})$, the feedback can be computed with a corresponding path integral of $\delta_\mathrm{u} = K(x)\delta_\mathrm{x}$ along the geodesic $\gamma^* \in \Gamma(x, x_\mathrm{r})$ (cf. [178, Eq. (6)]). In [156], an efficient evaluation of this control input based on the pseudospectral method was proposed. In [277], a continuous-time dynamic realization was presented based on the gradient flow, which converges to the geodesic and thus to the stabilizing control law.

In the following, we discuss the relation of *universal stabilizability* and the offline design in [178] to the considered local incremental stabilizability (Def. 4.18) and the SDP design in Sections 3.1.3. At first glance, the two system properties seem almost equivalent: For any reachable trajectory (z, v) we can construct a control law/input sequence that exponentially stabilizes the trajectory. The main difference are the relaxations in Definition 4.18, to only consider trajectories in some constraint set $\tilde{\mathbb{Z}}$ and to only consider local initial conditions. Thus, we can always ensure that the system is incrementally stabilizable (Def. 4.18) by constructing a CCM and explicit bounds $c_1, c_2 > 0$, $\rho \in [0, 1)$ can be computed based on M, K. However, for many physical systems global incremental stabilizability may be too restrictive. In particular, the model (especially discrete-time

[1]In general, universal stabilizability only requires the existence of a tracking feedback κ_tr to drive x to z, which allows for $\kappa_\mathrm{tr}(x, z, v) \neq \kappa(x) - \kappa(z) + v$. In this case, the matrix K may also depend on u. The corresponding feedback is given by a path integral, which has a unique solution if f and K are affine in u, even though $\delta_\mathrm{u}(x, u, \delta_\mathrm{x})$ is not completely integrable [178] (assuming an additional technical conditions holds [177]). Existence of a feedback $K(x)$ can be ensured by posing slightly stronger conditions on the contraction metric $M(x)$ [178, Conditions C1/C2]. In case these conditions do not hold, we can consider the augment state (x, u) and treat the change in the control input as the new input, as suggested in [257].

polynomial models) may only be a reasonable approximation of the true system on some subset $\tilde{\mathbb{Z}} \subseteq \mathbb{X} \times \mathbb{U}$. In case the considered subset $\tilde{\mathbb{Z}}$ is positively invariant and connected, converse Lyapunov results and incremental Lyapunov functions based on contraction metrics can still be applied (cf. [13] and [108]), compare also [111] for convex subsets. However, such a set may be rather conservative, while the considered notion (Def. 4.18) allows us to consider stabilizability of trajectories that satisfy some general a priori known constraints $\tilde{\mathbb{Z}}$, which is especially relevant in the constrained control case considered in the present thesis. The fact that we only assume *local* stabilizability is a direct consequence of the fact that we consider $(z, v) \in \tilde{\mathbb{Z}} \neq \mathbb{R}^n \times \mathbb{R}^m$, with $\tilde{\mathbb{Z}}$ neither convex nor positive invariant, which is not treated in the literature [18, 169, 178, 264]. In particular, in case $\tilde{\mathbb{Z}} = \mathbb{R}^n \times \mathbb{R}^m$ local incremental stability directly implies global incremental stability [18, Prop. 3.4]. Thus, although the formulation in Definition 4.18 may seem heuristic compared to the "standard" global conditions (cf. [18, 169, 178, 264]), the corresponding relaxations in the conditions are crucial to allow the applicability for many nonlinear constrained systems (cf. the following discussion regarding offline design methods). Note that even though these technical issues may prevent us from guaranteeing stabilizability on the full set $\tilde{\mathbb{Z}}$, using additional continuity properties, Lemma 3.12 ensures that $V_\delta(x, z) = \|x - z\|^2_{M(x)}$ is a valid (local) incremental CLF (cf. Prop. 4.19).

Design procedures

In the following, we focus on the different methods in the literature to compute incremental Lyapunov functions/contraction metrics and contrast them with the SDP design in Section 3.1.3. For continuous-time polynomial systems, the conditions (C.4)/(C.6) can be translated into an SOS problem [178]. However, for discrete-time polynomial systems global exponential stability is very restrictive and the restriction $(z, v) \in \tilde{\mathbb{Z}}$ requires additional S-procedure variables, which increases the computational complexity. In earlier work, the connection between gain scheduling and incremental stability is shown, allowing for a simple LMI design in case $M(x) = M$ [110, Thm. 7.3], compare also [254] where a general LPV synthesis based on a LFT was suggested. The design procedure in Section 3.1.3 can be seen as an extension of these ideas by parametrizing the differential/incremental Lyapunov function and using the LMI techniques in [21]. We point out that the LPV design methods require bounds on the parameters θ, which are easy to obtain in case a compact constraint set $\tilde{\mathbb{Z}}$ is considered. This highlights the benefits of considering the relaxed (local, constraint) stabilizability condition (Def. 4.18) instead of the standard global incremental stability/stabilizability [18, 169, 178, 264]. More recently, the connection between LPV/gain-scheduling design, CCMs, and incremental stability has received a lot of attention [148, 280, 281]. These recent contributions can take restrictions of the targeted reference behaviour $\mathcal{B}$ into account using *virtual CCMs* (VCCMs) [281] and also address dynamic output feedback with a corresponding (non-trivial) controller realization [148]. The corresponding synthesis problem (cf. [281, Prop. 8]) are state dependent matrix inequalities similar to Lemma 3.13. We point out that these methods have also been extended to incremental dissipativity for quadratic storage and supply functions [270] and robust analysis based on differential IQCs [278].

In addition, we point out that for special classes of systems (feedback linearizable, strict-feedback form) an incrementally stabilizing feedback can be directly constructed, without using differential dynamics, compare [170] and [290]. Sample based methods to verify/compute incremental Lyapunov functions and contraction metrics can be found in [187] and [262, 266]. Figure C.1 summarizes the different approaches to compute incremental Lyapunov functions V_δ with a corresponding stabilizing feedback κ.

Incremental system properties in MPC

In the following, we discuss how incremental Lyapunov functions or more generally incremental storage functions can be used in MPC.

Parametrized terminal cost and cost controllability

In Section 3.1, we showed that a parametrized local CLF $V_\mathrm{f}(x,r)$ for arbitrary dynamic reference trajectories can be efficiently used in trajectory tracking MPC formulations (Sec. 3.1/3.2). Since the local CLF V_f is equivalent to a (local) incremental Lyapunov function, any other approach to compute incremental Lyapunov functions could in principle be used to replace the LPV parametrization and the SDP based design in Section 3.1.3. However, a major advantage of the (simple) approach in Section 3.1.3 is the fact that the corresponding terminal cost/incremental Lyapunov function V_f is easy to evaluate online. In particular, with $V_\mathrm{f}(x,r) = (x - x_\mathrm{x})^\top (X_0 + \sum_j \theta_j(r) X_j)^{-1}(x - x_\mathrm{r})$ (cf. Lemma 3.12/3.13, (3.21)), evaluating V_f mainly requires the evaluation of p scalar functions θ_j and one $n \times n$ matrix inverse $P = X^{-1}$, which allows for an efficient implementation using, e.g., CasADi [17]. On the other hand, the incremental Lyapunov functions based on CCMs seem unsuitable for such MPC formulations due to the rather implicit description in equation (C.5) (with the notable exception of constant metrics M).

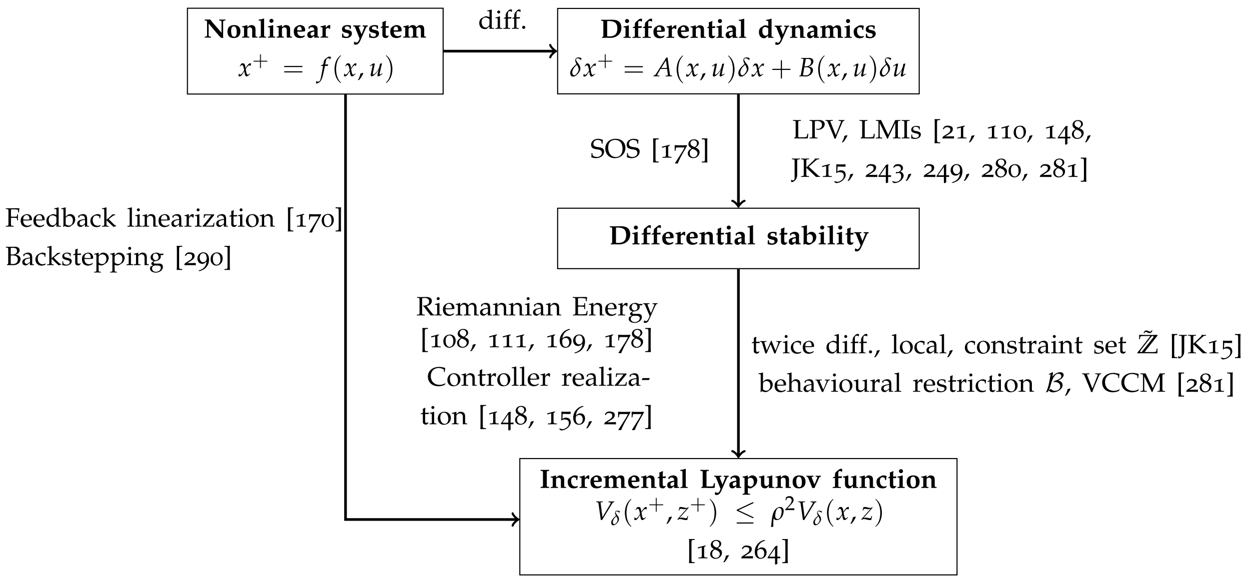

Figure C.1. Overview: From nonlinear systems to incremental stability. For special system classes an incrementally stabilizing feedback can be directly designed (cf. [170, 290]). Otherwise, assuming continuously differentiable dynamics f, the differential dynamics can be analysed and differential stability can be enforced/analysed using SOS optimization, LPV gain scheduling or other LMI methods (cf. [21, 110, 148, JK15, 243, 249, 280, 281]). A corresponding incremental Lyapunov function V_δ (cf. [18, 264]) is then given by the Riemannian Integral (cf. [108, 111, 169, 178]) . The corresponding control input is based on a path integral, which can be computed online (cf. [156]) or a suitable dynamic controller realization can be designed (cf. [148, 277]). Additional restrictions on the class of trajectories/references can be considered using VCCMs [281]. Alternatively, by relaxing the requirement to *local* stability, [JK15] (cf. Sec. 3.1.3) allows for a simpler parametrization of the incremental Lyapunov function and the consideration of general constraints $\tilde{\mathbb{Z}}$.

Thus, the difference in the parametrization of the corresponding incremental Lyapunov functions is crucial for applications that require an explicit description of V_δ.

Note that although the incremental Lyapunov function V_δ based on contraction metrics (C.5) is in general too complex to be directly used in an MPC scheme, it is possible to compute an upper bound of the form $V_\delta(x_1,x_2) \leq \|x_1 - x_2\|^2_{\overline{M}}$, with $\overline{M} \succeq M(x)$, $\forall x \in \mathbb{X}$ (cf. [258, Lemma IV.3]). Such upper bounds on the incremental Lyapunov function can be used to derive a cost controllability constant γ (cf. Ass. 4.4, Prop. 4.19), which is used in the theoretical analysis in Chapter 4.

Uncertainty propagation in robust MPC

Consider a perturbed system $x(t+1) = f(x(t), u(t), w(t))$, where $w(t) \in \mathbb{W}$ is some unknown but bounded disturbance/uncertainty. The concepts related to incremental stability are well suited to bound the deviation of a perturbed trajectory from some nominal planned trajectory (e.g., using the incremental $\mathcal{L}_2$-gain). Such bounds are crucial in robust MPC, where a possible deviation from the nominally predicted trajectory needs to be taken into account to ensure constraint satisfaction despite disturbances and model mismatch. In particular, we can consider the stronger notion of incremental input to state stability (i-ISS) [29, 265, 290]:

$$V_\delta(f(x,v,w_1), f(z,v,w_2)) \leq \rho^2 V_\delta(x,z) + \alpha_w(\|w_1 - w_2\|), \quad \alpha_w \in \mathcal{K}, \tag{C.7}$$

which imposes some additional continuity conditions on the dynamics f and the incremental Lyapunov function V_δ (C.2). Given some compact bound $w \in \mathbb{W}$, we can compute a robust positive invariant (RPI) set $\Omega = \{(x,z) \mid V_\delta(x,z) \leq \overline{w}\}$, $\overline{w} := \alpha_w(\max_{w \in \mathbb{W}} \|w\|)/(1-\rho^2)$ for the joint dynamics of the perturbed state x and a nominal state z [29, 257, 258], i.e., $(f(x,u,w), f(z,u,0)) \in \Omega$ for all $(x,z) \in \Omega$. Then, we can directly employ a tube-based nonlinear robust MPC scheme to ensure robust constraint satisfaction, compare [29, 257, 258]. In case V_δ and thus Ω is based on contraction metrics, the set Ω has a highly nonlinear expression (cf. (C.5)), which significantly increases the online computational complexity (with the notable exception of constant metrics M). However, we can compute an ellipsoidal set that over-approximates the RPI set $\Omega \subseteq \{(x,z) \mid \|x-z\|^2_{\underline{M}} \leq \overline{w}\}$, with $M(x) \succeq \underline{M}$ (cf. [258, Lemma IV.3]), which can be used to efficiently implement the constraint tightening. In addition, we can also exploit the fact that the Riemannian distance $d(x_1,x_2) = \sqrt{V_\delta(x_1,x_2)}$ satisfies the triangular inequality to simplify the robust MPC design, compare [215, Ass. 1], [38, Prop. 5.3]. Furthermore, by dropping the standard optimization over the nominal initial state $z(0|t)$ (cf. [29, 257, 258]), the robust MPC reduce to a robust trajectory optimization (independent of x) with reduced computational complexity. Alternatively, by using the simpler (and more conservative) parametrization in Section 3.1.3 with $V_\delta(x,z) = \|x-z\|^2_{M(x)}$, the initial state constraint can be reduced to a quadratic constraint in z.

Alternatively, we can directly use the bound (C.7) to compute an over-approximation of the k-step reachable set as

$$\mathcal{R}_k \subseteq \{z \mid V_\delta(x^*(k|t),z) \leq \epsilon_k\} \subseteq \{z \mid \|x^*(k|t)-z\|^2_{\underline{M}} \leq \epsilon_k\}, \quad k \in \mathbb{I}_{[0,N]},$$

with $\epsilon_k > 0$ based on a geometric series involving $\rho \in [0,1)$, compare [JK17, JK29]. Thus, robust closed-loop properties can be ensured by tightening the constraints sets along the prediction horizon based on this over-approximation. This approach is quite easy to apply, since it only requires a modification of the open-loop constraints based on bounds of the incremental Lyapunov function V_δ. The approach can also be applied if the system is incrementally *stabilizable* by additionally tightening the input constraints to ensure that we can stabilize the previous optimal solution [JK17, JK29]. Thus, this approach requires neither an explicit description of the incremental Lyapunov function V_δ for of the incrementally stabilizing feedback κ, but only suitable bounds which can be directly obtained in case CCMs are used. We point out that the SDP design procedure for V_δ in Section 3.1.3 can be adjusted to directly optimize the parameters used in the robust MPC (e.g., ρ), compare [215]. This robust MPC design based on *incremental stabilizability* has also been extended to deal with state and input dependent uncertainties [JK29], output feedback [JK18], robust adaptive MPC [JK11, JK13] and stochastic uncertainty [251]. In addition, experimental results, extensions to safe learning under stochastic uncertainty, numerical comparisons, and an application to safe approximate MPC can be found in [JK30], [273], [39] and [JK8, JK30].

Overall, incremental system properties are a natural tool to study uncertainty propagation and develop (tube-based) robust MPC formulations, which naturally generalize standard linear MPC approaches based on polytopic and ellipsoidal sets [153].

Detectability and output-feedback

An i-IOSS Lyapunov function V_o verifying the stage cost detectability in Chapter 4 can be computed using results for differential detectability [245]. Furthermore, since i-IOSS is a special case of incremental dissipativity, the method in [270] based on differential dissipativity could be used to verify such system properties. In addition, contraction metrics can be used to construct nonlinear observers based on *universal detectability* [179, 246, 288], which can be used for output-feedback MPC (cf. [JK18]). Alternatively, a nonlinear observer can be designed using moving horizon estimation (MHE), which is an optimization-based method for state estimation (dual to MPC), compare [12, 205, 233]. The theoretical stability results for MHE are typically based on an i-IOSS assumption (cf. Ass. 4.24), which can be verified offline using, e.g., [245, 270].

System identification and incremental stability/stabilizability priors

Often the system model f is unknown and needs to be estimated from data. From a control perspective, it is beneficial if the learned/identified model has certain system theoretic properties, which may be known a priori, such as (incremental) stability. Recent neural network identification methods with guaranteed i-ISS and a corresponding MPC formulation exploiting this property in the design of the terminal ingredients can be found in [22, 263]. Identification of incrementally stabilizable systems using CCMs and incrementally stable systems using contraction metrics can be found in [240, 259].

Bibliography

[1] H. Abbas, J. Hanema, R. Tóth, J. Mohammadpour, and N. Meskin. "An improved robust model predictive control for linear parameter-varying input-output models." In: *Int. J. Robust and Nonlinear Control* 28.3 (2018), pp. 859–880.

[2] A. Aboudonia, A. Eichler, and J. Lygeros. "Distributed Model Predictive Control with Reconfigurable Terminal Ingredients for Reference Tracking." In: *arXiv preprint arXiv:2007.00427* (2020).

[3] A. P. Aguiar, J. P. Hespanha, and P. V. Kokotovic. "Path-following for nonminimum phase systems removes performance limitations." In: *IEEE Trans. Automat. Control* 50.2 (2005), pp. 234–239.

[4] A. P. Aguiar, J. P. Hespanha, and P. V. Kokotović. "Performance limitations in reference tracking and path following for nonlinear systems." In: *Automatica* 44.3 (2008), pp. 598–610.

[5] C. O. Aguilar and A. J. Krener. "Model predictive regulation." In: *Proc. 19th IFAC World Congress*. 2014, pp. 3682–3689.

[6] A. A. Ahmadi and P. A. Parrilo. "Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems." In: *Proc. 47th IEEE Conf. Decision and Control (CDC)*. 2008, pp. 614–621.

[7] M. Alamir and G. Bornard. "Stability of a truncated infinite constrained receding horizon scheme: the general discrete nonlinear case." In: *Automatica* 31.9 (1995), pp. 1353–1356.

[8] M. Alamir and G. Pannochia. "A new formulation of Economic Model Predictive Control without terminal constraint." In: *Automatica* 125 (2021), p. 109420.

[9] T. Alamo, A. Ferramosca, A. H. González, D. Limon, and D. Odloak. "A gradient-based strategy for the one-layer RTO+ MPC controller." In: *J. Proc. Contr.* 24.4 (2014), pp. 435–447.

[10] A. Alessandretti, A. P. Aguiar, and C. N. Jones. "On convergence and performance certification of a continuous-time economic model predictive control scheme with time-varying performance index." In: *Automatica* 68 (2016), pp. 305–313.

[11] A. Alessandretti, A. P. Aguiar, and C. N. Jones. "On the design of discrete-time Economic Model Predictive Controllers." In: *Proc. 55th IEEE Conf. Decision and Control (CDC)*. 2016, pp. 3196–3201.

[12] A. Alessandri, M. Baglietto, and G. Battistelli. "Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes." In: *Automatica* 44.7 (2008), pp. 1753–1765.

[13] D. A. Allan and J. B. Rawlings. *An input/output-to-state stability converse theorem for closed positive invariant sets*. Tech. rep. 2018-01. Texas – Wisconsin – California Control Consortium (TWCCC), 2018.

[14] D. A. Allan, J. B. Rawlings, and A. R. Teel. *Nonlinear Detectability and Incremental Input/output-to-state Stability*. Tech. rep. 2020-01. Texas – Wisconsin – California Control Consortium (TWCCC), 2020.

[15] F. Amato, M. Mattei, and A. Pironti. "Gain scheduled control for discrete-time systems depending on bounded rate parameters." In: *Int. J. Robust and Nonlinear Control* 15.11 (2005), pp. 473–494.

[16] R. Amrit, J. B. Rawlings, and D. Angeli. "Economic optimization using model predictive control with a terminal cost." In: *Annual Reviews in Control* 35 (2011), pp. 178–186.

[17] J. A. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl. "CasADi: a software framework for nonlinear optimization and optimal control." In: *Mathematical Programming Computation* 11.1 (2019), pp. 1–36.

[18] D. Angeli. "A Lyapunov approach to incremental stability properties." In: *IEEE Trans. Automat. Control* 47 (2002), pp. 410–421.

[19] D. Angeli, R. Amrit, and J. B. Rawlings. "On average performance and stability of economic model predictive control." In: *IEEE Trans. Automat. Control* 57 (2012), pp. 1615–1626.

[20] D. Angeli, A. Casavola, and F. Tedesco. "Theoretical advances on economic model predictive control with time-varying costs." In: *Annual Reviews in Control* 41 (2016), pp. 218–224.

[21] P. Apkarian and H. D. Tuan. "Parameterized LMIs in control theory." In: *SIAM J. Control Optim.* 38 (2000), pp. 1241–1264.

[22] L. B. Armenio, E. Terzi, M. Farina, and R. Scattolini. "Model Predictive Control Design for Dynamical Systems Learned by Echo State Networks." In: *IEEE Control Systems Letters* 3.4 (2019), pp. 1044–1049.

[23] E. Aydiner, M. A. Müller, and F. Allgöwer. "Periodic reference tracking for nonlinear systems via model predictive control." In: *Proc. European Control Conf. (ECC)*. 2016, pp. 2602–2607.

[24] J. Bailey, F. Horn, and R. Lin. "Cyclic operation of reaction systems: Effects of heat and mass transfer resistance." In: *AIChE Journal* 17 (1971), pp. 818–825.

[25] G. Basile and G. Marro. *Controlled and conditioned invariants in linear system theory*. Prentice Hall Englewood Cliffs, NJ, 1992.

[26] T. Bäthge, M. Kögel, S. Di Cairano, and R. Findeisen. "Contract-based Predictive Control for Modularity in Hierarchical Systems." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2018, pp. 499–504.

[27] T. Bäthge, S. Lucia, and R. Findeisen. "Exploiting models of different granularity in robust predictive control." In: *Proc. 55th IEEE Conf. Decision and Control (CDC)*. 2016, pp. 2763–2768.

[28] I. Batkovic, M. Ali, P. Falcone, and M. Zanon. "Safe Trajectory Tracking in Uncertain Environments." In: *arXiv preprint arXiv:2001.11602* (2020).

[29] F. Bayer, M. Bürger, and F. Allgöwer. "Discrete-time incremental ISS: A framework for robust NMPC." In: *Proc. European Control Conf. (ECC)*. 2013, pp. 2068–2073.

[30] F. A. Bayer, M. Lorenzen, M. A. Müller, and F. Allgöwer. "Robust economic Model Predictive Control using stochastic information." In: *Automatica* 74 (2016), pp. 151–161.

[31] F. A. Bayer, M. A. Müller, and F. Allgöwer. "Tube-based robust economic model predictive control." In: *J. Proc. Contr.* 24 (2014), pp. 1237–1246.

[32] A. Bemporad, A. Casavola, and E. Mosca. "Nonlinear control of constrained linear systems via predictive reference management." In: *IEEE Trans. Automat. Control* 42.3 (1997), pp. 340–349.

[33] A. Bemporad and M. Morari. "Robust model predictive control: A survey." In: *Robustness in identification and control*. Springer, 1999, pp. 207–226.

[34] S. Bernhard and J. Adamy. "Optimal Output Regulation for Square, Overactuated and Underactuated Linear Systems." In: *IEEE Trans. Autom. Control* 65.10 (2020), pp. 4416–4423.

[35] D. P. Bertsekas, D. P. Bertsekas, D. P. Bertsekas, and D. P. Bertsekas. *Dynamic programming and optimal control*. Vol. 1. 2. Athena scientific Belmont, MA, 1995.

[36] G. Betti, M. Farina, and R. Scattolini. "A robust MPC algorithm for offset-free tracking of constant reference signals." In: *IEEE Trans. Automat. Control* 58.9 (2013), pp. 2394–2400.

[37] A. Boccia, L. Grüne, and K. Worthmann. "Stability and feasibility of state constrained MPC without stabilizing terminal constraints." In: *Systems & Control Letters* 72 (2014), pp. 14–21.

[38] N. M. Boffi, S. Tu, and J.-J. E. Slotine. "Regret Bounds for Adaptive Nonlinear Control." In: *Proc. 3rd Conf. Learning for Dynamics & Control (L4DC)*. 2021, pp. 471–483.

[39] A. D. Bonzanini, T. L. Santos, and A. Mesbah. "Tube-based Stochastic Nonlinear Model Predictive Control: A Comparative Study on Constraint Tightening." In: *Proc. 12th IFAC Symp. on Dynamics and Control of Process Systems*. 2019, pp. 598–603.

[40] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*. SIAM, 1994.

[41] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[42] P. Braun, L. Grüne, and C. M. Kellett. "Feedback design using nonsmooth control Lyapunov functions: A numerical case study for the nonholonomic integrator." In: *Proc. 56th IEEE Conf. Decision and Control (CDC)*. 2017, pp. 4890–4895.

[43] T. J. Broomhead, C. Manzie, R. C. Shekhar, and P. Hield. "Robust periodic economic MPC for linear systems." In: *Automatica* 60 (2015), pp. 30–37.

[44] T. Brüdigam, J. Teutsch, D. Wollherr, and M. Leibold. "Combined Robust and Stochastic Model Predictive Control for Models of Different Granularity." In: *Proc. 21st IFAC World Congress*. 2020, pp. 7123–7129.

[45] M. Bujarbaruah, X. Zhang, M. Tanaskovic, and F. Borrelli. "Adaptive Stochastic MPC under Time Varying Uncertainty." In: *IEEE Trans. Automat. Control* 66.6 (2020), pp. 2840–2845.

[46] E. Bullinger and F. Allgöwer. "Adaptive λ-tracking for nonlinear higher relative degree systems." In: *Automatica* 41.7 (2005), pp. 1191–1200.

[47] C. I. Byrnes and A. Isidori. "Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation." In: *IEEE Trans. Automat. Control* 48.10 (2003), pp. 1712–1723.

[48] C. I. Byrnes, F. D. Priscoli, and A. Isidori. *Output regulation of uncertain nonlinear systems*. Springer Science & Business Media, 2012.

[49] R. Cagienard, P. Grieder, E. C. Kerrigan, and M. Morari. "Move blocking strategies in receding horizon control." In: *J. Proc. Contr.* 17.6 (2007), pp. 563–570.

[50] C. Cai and A. R. Teel. "Input–output-to-state stability for discrete-time systems." In: *Automatica* 44.2 (2008), pp. 326–336.

[51] H. M. Calderón, P. S. Cisneros, and H. Werner. "qLPV predictive control-A benchmark study on state space vs input-output approach." In: *Proc. 3rd IFAC Workshop on Linear Parameter Varying Systems (LPVS)*. 2019, pp. 146–151.

[52] O. L. Carrapiço and D. Odloak. "A stable model predictive control for integrating processes." In: *Comp. & Chem. Eng.* 29.5 (2005), pp. 1089–1099.

[53] B. Castillo, S. Di Gennaro, S. Monaco, and D. Normand-Cyrot. "Nonlinear regulation for a class of discrete-time systems." In: *Systems & Control Letters* 20 (1993), pp. 57–65.

[54] H. S. Che, E. Levi, M. Jones, W.-P. Hew, and N. A. Rahim. "Current control methods for an asymmetrical six-phase induction motor drive." In: *IEEE Trans. Power Electronics* 29 (2014), pp. 407–417.

[55] H. Chen and F. Allgöwer. "A Quasi-Infinite Horizon Nonlinear Model Predictive Control Scheme with Guaranteed Stability." In: *Automatica* 34 (1998), pp. 1205–1217.

[56] L. Chisci, P. Falugi, and G. Zappa. "Gain-scheduling MPC of nonlinear systems." In: *Int. J. Robust and Nonlinear Control* 13.3-4 (2003), pp. 295–308.

[57] L. Chisci, P. Falugi, and G. Zappa. “Predictive tracking control of constrained nonlinear systems.” In: *IEE Proceedings-Control Theory and Applications* 152.3 (2005), pp. 309–316.

[58] L. Chisci and G. Zappa. “Dual mode predictive tracking of piecewise constant references for constrained linear systems.” In: *Int. J. Control* 76.1 (2003), pp. 61–72.

[59] P. S. Cisneros and H. Werner. “A Velocity Algorithm for Nonlinear Model Predictive Control.” In: *IEEE Trans. on Control Systems Technology* (2020).

[60] C. Conte, C. N. Jones, M. Morari, and M. N. Zeilinger. “Distributed synthesis and stability of cooperative distributed model predictive control for linear systems.” In: *Automatica* 69 (2016), pp. 117–125.

[61] C. Conte, M. N. Zeilinger, M. Morari, and C. N. Jones. “Cooperative distributed tracking MPC for constrained linear systems: Theory and synthesis.” In: *Proc. 52nd IEEE Conf. on Decision and Control (CDC)*. 2013, pp. 3812–3817.

[62] A. Cotorruelo, M. Hosseinzadeh, D. R. Ramirez, D. Limon, and E. Garone. “Reference Dependent Invariant Sets: Sum of Squares Based Computation and Applications in Constrained Control.” In: *Automatica* 129 (2021), p. 109614.

[63] A. Cotorruelo, D. Limon, E. Garone, and D. R. Ramirez. “Tracking MPC with non-convex steady state admissible sets.” In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2018, pp. 153–156.

[64] A. Cotorruelo, D. R. Ramirez, D. Limon, and E. Garone. “Nonlinear MPC for Tracking for a Class of Non-Convex Admissible Output Sets.” In: *IEEE Transactions on Automatic Control* (2020).

[65] J. Coulson, J. Lygeros, and F. Dörfler. “Data-enabled predictive control: In the shallows of the DeePC.” In: *Proc. 18th European Control Conf. (ECC)*. 2019, pp. 307–312.

[66] J. Coulson, J. Lygeros, and F. Dörfler. “Distributionally Robust Chance Constrained Data-enabled Predictive Control.” In: *IEEE Trans. Automat. Control* (2021).

[67] L. D’Alto and M. Corless. “Incremental quadratic stability.” In: *Numerical Algebra, Control & Optimization* 3.1 (2013), pp. 175–201.

[68] A. D'Jorge, A. Anderson, A. H. González, and A. Ferramosca. "A robust economic MPC for changing economic criterion." In: *Int. J. Robust and Nonlinear Control* 28.15 (2018), pp. 4404–4423.

[69] J. Daafouz and J. Bernussou. "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties." In: *Systems & Control Letters* 43.5 (2001), pp. 355–359.

[70] T. Damm, L. Grüne, M. Stieler, and K. Worthmann. "An exponential turnpike theorem for dissipative discrete time optimal control problems." In: *SIAM J. Control Optim.* 52.3 (2014), pp. 1935–1957.

[71] M. L. Darby and M. Nikolaou. "MPC: Current practice and challenges." In: *Control Engineering Practice* 20.4 (2012), pp. 328–342.

[72] E. Davison. "The robust control of a servomechanism problem for linear time-invariant multivariable systems." In: *IEEE Trans. Automat. Control* 21.1 (1976), pp. 25–34.

[73] E. Davison and S. Wang. "Properties and calculation of transmission zeros of linear multivariable systems." In: *Automatica* 10.6 (1974), pp. 643–658.

[74] G. De Nicolao, L. Magni, and R. Scattolini. "Stabilizing receding-horizon control of nonlinear time-varying systems." In: *IEEE Trans. Automat. Control* 43.7 (1998), pp. 1030–1036.

[75] M. C. De Oliveira, J. Bernussou, and J. C. Geromel. "A new discrete-time robust stability condition." In: *Systems & Control Letters* 37.4 (1999), pp. 261–265.

[76] J. De Schutter, M. Zanon, and M. Diehl. "TuneMPC - A Tool for Economic Tuning of Tracking (N)MPC Problems." In: *IEEE Control Systems Letters* 4.4 (2020), pp. 910–915.

[77] S. Di Cairano and F. Borrelli. "Reference tracking with guaranteed error bound for constrained linear systems." In: *IEEE Trans. Automat. Control* 61.8 (2016), pp. 2245–2250.

[78] M. Diehl, R. Amrit, and J. B. Rawlings. "A Lyapunov function for economic optimizing model predictive control." In: *IEEE Trans. Autom. Control* 56 (2011), pp. 703–707.

[79] M. Diehl, R. Findeisen, F. Allgöwer, H. G. Bock, and J. P. Schlöder. "Nominal stability of real-time iteration scheme for nonlinear model predictive control." In: *IEE Proceedings-Control Theory and Applications* 152.3 (2005), pp. 296–308.

[80] M. Diehl, L. Magni, and G. De Nicolao. "Efficient NMPC of unstable periodic systems using approximate infinite horizon closed loop costing." In: *Annual Reviews in Control* 28.1 (2004), pp. 37–45.

[81] Z. Dong and D. Angeli. "Analysis of economic model predictive control with terminal penalty functions on generalized optimal regimes of operation." In: *Int. J. Robust and Nonlinear Control* 28.16 (2018), pp. 4790–4815.

[82] Z. Dong and D. Angeli. "Homothetic tube-based robust offset-free economic Model Predictive Control." In: *Automatica* 119 (2020), p. 109105.

[83] A. Dutta, E. Hartley, J. Maciejowski, and R. De Keyser. "Certification of a class of industrial predictive controllers without terminal conditions." In: *Proc. 53nd IEEE Conf. Decision and Control (CDC)*. 2014, pp. 6695–6700.

[84] C. Ebenbauer and F. Allgöwer. "A dissipation inequality for the minimum phase property." In: *IEEE Trans. Automat. Control* 53.3 (2008), pp. 821–826.

[85] C. Ebenbauer and F. Allgöwer. "Analysis and design of polynomial control systems using dissipation inequalities and sum of squares." In: *Comp. & Chem. Eng.* 30.10-12 (2006), pp. 1590–1602.

[86] W. Esterhuizen, K. Worthmann, and S. Streif. "Recursive feasibility of continuous-time model predictive control without stabilising constraints." In: *IEEE Control Systems Letters* 5.1 (2020), pp. 265–270.

[87] L. Fagiano and A. R. Teel. "Generalized terminal state constraint for model predictive control." In: *Automatica* 49 (2013), pp. 2622–2631.

[88] P. Falugi. "Model predictive control for tracking randomly varying references." In: *Int. J. Control* 88 (2015), pp. 745–753.

[89] P. Falugi and D. Q. Mayne. "Tracking a periodic reference using nonlinear model predictive control." In: *Proc. 52nd IEEE Conf. Decision and Control (CDC)*. 2013, pp. 5096–5100.

[90] M. Farina and R. Scattolini. "Distributed predictive control: A non-cooperative algorithm with neighbor-to-neighbor communication for linear systems." In: *Automatica* 48.6 (2012), pp. 1088–1096.

[91] T. Faulwasser. "Optimization-based solutions to constrained trajectory-tracking and path-following problems." PhD thesis. Otto-von-Guericke-Universität Magdeburg, 2012.

[92] T. Faulwasser and D. Bonvin. "Exact turnpike properties and economic NMPC." In: *European J. Control* 35 (2017), pp. 34–41.

[93] T. Faulwasser and R. Findeisen. "A model predictive control approach to trajectory tracking problems via time-varying level sets of Lyapunov functions." In: *Proc. 50th IEEE Conf. Decision and Control (CDC), European Control Conf. (ECC)*. 2011, pp. 3381–3386.

[94] T. Faulwasser and R. Findeisen. "Nonlinear model predictive control for constrained output path following." In: *IEEE Trans. Automat. Control* 61.4 (2015), pp. 1026–1039.

[95] T. Faulwasser and R. Findeisen. "Predictive path following without terminal constraints." In: *Proc. of 20th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*. 2012.

[96] T. Faulwasser, L. Grüne, and M. A. Müller. "Economic Nonlinear Model Predictive Control." In: *Foundations and Trends® in Systems and Control* 5 (2018), pp. 1–98.

[97] T. Faulwasser, V. Hagenmeyer, and R. Findeisen. "Constrained reachability and trajectory generation for flat systems." In: *Automatica* 50.4 (2014), pp. 1151–1159.

[98] T. Faulwasser, M. Korda, C. N. Jones, and D. Bonvin. "On turnpike and dissipativity properties of continuous-time optimal control problems." In: *Automatica* 81 (2017), pp. 297–304.

[99] T. Faulwasser and G. Pannocchia. "Toward a Unifying Framework Blending Real-Time Optimization and Economic Model Predictive Control." In: *Industrial & Engineering Chemistry Research* 58.30 (2019), pp. 13583–13598.

[100] A. Ferramosca, D. Limon, A. González, D. Odloak, and E. Camacho. "MPC for tracking zone regions." In: *J. Proc. Contr.* 20 (2010), pp. 506–516.

[101] A. Ferramosca, A. H. González, and D. Limon. "Offset-free multi-model economic model predictive control for changing economic criterion." In: *J. Proc. Contr.* 54 (2017), pp. 1–13.

[102] A. Ferramosca, D. Limon, and E. F. Camacho. "Economic MPC for a changing economic criterion for linear systems." In: *IEEE Trans. Autom. Control* 59 (2014), pp. 2657–2667.

[103] A. Ferramosca, D. Limón, I. Alvarado, T. Alamo, and E. F. Camacho. "MPC for tracking with optimal closed-loop performance." In: *Automatica* 45 (2009), pp. 1975–1978.

[104] A. Ferramosca, D. Limón, I. Alvarado, and E. F. Camacho. "Cooperative distributed MPC for tracking." In: *Automatica* 49.4 (2013), pp. 906–914.

[105] R. Findeisen, H. Chen, and F. Allgöwer. "Nonlinear predictive control for setpoint families." In: *Proc. American Control Conf. (ACC)*. 2000, pp. 260–264.

[106] R. Findeisen, L. Imsland, F. Allgöwer, and B. A. Foss. "State and output feedback nonlinear model predictive control: An overview." In: *European J. Control* 9.2-3 (2003), pp. 190–206.

[107] F. A. Fontes. "A general framework to design stabilizing nonlinear model predictive controllers." In: *Systems & Control Letters* 42.2 (2001), pp. 127–143.

[108] F. Forni and R. Sepulchre. "A differential Lyapunov framework for contraction analysis." In: *IEEE Trans. Automat. Control* 59.3 (2013), pp. 614–628.

[109] B. A. Francis and W. M. Wonham. "The internal model principle of control theory." In: *Automatica* 12.5 (1976), pp. 457–465.

[110] V. Fromion and G. Scorletti. "A theoretical framework for gain scheduling." In: *Int. J. Robust and Nonlinear Control* 13.10 (2003), pp. 951–982.

[111] V. Fromion and G. Scorletti. "Connecting nonlinear incremental Lyapunov stability with the linearizations Lyapunov stability." In: *Proc. 44th IEEE Conf. Decision and Control (CDC)*. 2005, pp. 4736–4741.

[112] P. Gahinet, P. Apkarian, and M. Chilali. "Affine parameter-dependent Lyapunov functions and real parametric uncertainty." In: *IEEE Trans. Automat. Control* 41.3 (1996), pp. 436–442.

[113] E. Garone, S. Di Cairano, and I. Kolmanovsky. "Reference and command governors for systems with constraints: A survey on theory and applications." In: *Automatica* 75 (2017), pp. 306–328.

[114] E. G. Gilbert and K. T. Tan. "Linear systems with state and control constraints: The theory and application of maximal output admissible sets." In: *IEEE Trans. Automat. Control* 36 (1991), pp. 1008–1020.

[115] R. Gondhalekar and J.-i. Imura. "Least-restrictive move-blocking model predictive control." In: *Automatica* 46.7 (2010), pp. 1234–1240.

[116] P. S. González Cisneros and H. Werner. "Nonlinear model predictive control for models in quasi-linear parameter varying form." In: *Int. J. Robust and Nonlinear Control* 30.10 (2020), pp. 3945–3959.

[117] D. W. Griffith, L. T. Biegler, and S. C. Patwardhan. "Robustly stable adaptive horizon nonlinear model predictive control." In: *J. Proc. Contr.* 70 (2018), pp. 109–122.

[118] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel. "Examples when nonlinear model predictive control is nonrobust." In: *Automatica* 40.10 (2004), pp. 1729–1738.

[119] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel. "Model predictive control: For want of a local control Lyapunov function, all is not lost." In: *IEEE Trans. Automat. Control* 50.5 (2005), pp. 546–558.

[120] L. Grüne. "Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems." In: *IEEE Trans. on Control Systems Technology* 48.2 (2009), pp. 1206–1228.

[121] L. Grüne. "Approximation properties of receding horizon optimal control." In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 118.1 (2016), pp. 3–37.

[122] L. Grüne. "Economic receding horizon control without terminal constraints." In: *Automatica* 49 (2013), pp. 725–734.

[123] L. Grüne. "NMPC without terminal constraints." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2012, pp. 1–13.

[124] L. Grüne and R. Guglielmi. "Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems." In: *SIAM J. Control Optim.* 56.2 (2018), pp. 1282–1302.

[125] L. Grüne and V. G. Palma. "Robustness of performance and stability for multistep and updated multistep MPC schemes." In: *Discrete & Continuous Dynamical Systems - Series A* 35.9 (2015), p. 4385.

[126] L. Grüne and J. Pannek. *Nonlinear Model Predictive Control*. Springer, 2017.

[127] L. Grüne, J. Pannek, M. Seehafer, and K. Worthmann. "Analysis of unconstrained nonlinear MPC schemes with time varying control horizon." In: *SIAM J. Control Optim.* 48.8 (2010), pp. 4938–4962.

[128] L. Grüne and S. Pirkelmann. "Closed-loop performance analysis for economic model predictive control of time-varying systems." In: *Proc. 56th IEEE Conf. Decision and Control (CDC)*. 2017, pp. 5563–5569.

[129] L. Grüne and S. Pirkelmann. "Economic model predictive control for time-varying system: Performance and stability results." In: *Optimal Control Applications and Methods* 41.1 (2020), pp. 42–64.

[130] L. Grüne, S. Pirkelmann, and M. Stieler. "Strict dissipativity implies turnpike behavior for time-varying discrete time optimal control problems." In: *Control Systems and Mathematical Methods in Economics*. Springer, 2018, pp. 195–218.

[131] L. Grüne and A. Rantzer. "On the infinite horizon performance of receding horizon controllers." In: *IEEE Trans. Automat. Control* 53.9 (2008), pp. 2100–2111.

[132] L. Grüne and M. Stieler. "Asymptotic stability and transient optimality of economic MPC without terminal conditions." In: *J. Proc. Contr.* 24 (2014), pp. 1187–1196.

[133] J. Gutekunst, H. G. Bock, and A. Potschka. "Economic NMPC for averaged infinite horizon problems with periodic approximations." In: *Automatica* 117 (2020), p. 109001.

[134] H. Haimin, X. Feng, R. Quirynen, M. E. Villanueva, and B. Houska. "Real-Time Tube MPC Applied to a 10-State Quadrotor Model." In: *Proc. American Control Conf. (ACC)*. 2018, pp. 3135–3140.

[135] M. Hirche, P. N. Köhler, M. A. Müller, and F. Allgöwer. "Distributed Model Predictive Control for Consensus of Constrained Heterogeneous Linear Systems." In: *Proc. 59th IEEE Conf. Decision and Control (CDC)*. 2020, pp. 1248–1253.

[136] M. Höger and L. Grüne. "On the Relation Between Detectability and Strict Dissipativity for Nonlinear Discrete Time Systems." In: *IEEE Control Systems Letters* 3.2 (2019), pp. 458–462.

[137] M. Hou and R. J. Patton. "Input observability and input reconstruction." In: *Automatica* 34.6 (1998), pp. 789–794.

[138] B. Houska and M. A. Müller. "Cost-to-travel functions: A new perspective on optimal and model predictive control." In: *Systems & Control Letters* 106 (2017), pp. 79–86.

[139] R. Huang, E. Harinath, and L. T. Biegler. "Lyapunov stability of economically oriented NMPC for cyclic processes." In: *J. Proc. Contr.* 21 (2011), pp. 501–509.

[140] A. Ilchmann and E. P. Ryan. "Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise." In: *Automatica* 30.2 (1994), pp. 337–346.

[141] A. Isidori. *Lectures in feedback design for multivariable systems*. Vol. 3. Springer, 2017.

[142] A. Isidori. *Nonlinear Control Systems*. Springer, 2013.

[143] A. Isidori. "The zero dynamics of a nonlinear system: From the origin to the latest progresses of a long successful story." In: *European J. Control* 19.5 (2013), pp. 369–378.

[144] A. Isidori and C. I. Byrnes. "Output regulation of nonlinear systems." In: *IEEE Trans. Automat. Control* 35.2 (1990), pp. 131–140.

[145] A. Jadbabaie and J. Hauser. "On the stability of receding horizon control with a general terminal cost." In: *IEEE Trans. Automat. Control* 50.5 (2005), pp. 674–678.

[146] C. M. Kellett. "A compendium of comparison function results." In: *Mathematics of Control, Signals, and Systems* 26 (2014), pp. 339–374.

[147] S. Knüfer and M. A. Müller. "Time-Discounted Incremental Input/Output-to-State Stability." In: *Proc. 59th IEEE Conf. Decision and Control (CDC)*. 2020, pp. 5394–5400.

[148] P. J. Koelewijn, R. Tóth, and H. Nijmeijer. "Linear parameter-varying control of nonlinear systems based on incremental stability." In: *Proc. 3rd IFAC Workshop on Linear Parameter Varying Systems (LPVS)*. 2019, pp. 38–43.

[149] M. Kögel and R. Findeisen. "On MPC based trajectory tracking." In: *Proc. European Control Conf. (ECC)*. 2014, pp. 121–127.

[150] P. N. Köhler, M. A. Müller, and F. Allgöwer. "A distributed economic MPC framework for cooperative control under conflicting objectives." In: *Automatica* 96 (2018), pp. 368–379.

[151] J. Kong, M. Pfeiffer, G. Schildbach, and F. Borrelli. "Kinematic and dynamic vehicle models for autonomous driving control design." In: *Proc. IEEE Intelligent Vehicles Symp. (IV)*. 2015, pp. 1094–1099.

[152] M. Korda and C. N. Jones. "Stability and performance verification of optimization-based controllers." In: *Automatica* 78 (2017), pp. 34–45.

[153] B. Kouvaritakis and M. Cannon. *Model predictive control*. Springer, 2016.

[154] R. Kumar, M. J. Wenzel, M. J. Ellis, M. N. ElBsat, K. H. Drees, and V. M. Zavala. "A Stochastic Dual Dynamic Programming Framework for Multiscale MPC." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2018, pp. 493–498.

[155] H. C. La, A. Potschka, and H. G. Bock. "Partial stability for nonlinear model predictive control." In: *Automatica* 78 (2017), pp. 14–19.

[156] K. Leung and I. R. Manchester. "Nonlinear stabilization via control contraction metrics: A pseudospectral approach for computing geodesics." In: *Proc. American Control Conf. (ACC)*. 2017, pp. 1284–1289.

[157] Y. Li, X. Chen, and N. Li. "Online optimal control with linear dynamics and predictions: Algorithms and regret analysis." In: *Advances in Neural Information Processing Systems*. 2019, pp. 14887–14899.

[158] D. Liberzon, A. S. Morse, and E. D. Sontag. "Output-input stability and minimum-phase nonlinear systems." In: *IEEE Trans. Automat. Control* 47.3 (2002), pp. 422–436.

[159] D. Limon, I. Alvarado, T. Alamo, and E. Camacho. "Robust tube-based MPC for tracking of constrained linear systems with additive disturbances." In: *J. Proc. Contr.* 20 (2010), pp. 248–260.

[160] D. Limon et al. "Input-to-state stability: A unifying framework for robust model predictive control." In: *Nonlinear Model Predictive Control: Towards New Challenging Applications*. Vol. 384. Springer, 2009, pp. 1–26.

[161] D. Limon and T. Alamo. "Tracking model predictive control." In: *Encyclopedia of Systems and Control* (2015), pp. 1475–1484.

[162] D. Limon, T. Alamo, F. Salas, and E. F. Camacho. "On the stability of constrained MPC without terminal constraint." In: *IEEE Trans. Automat. Control* 51 (2006), pp. 832–836.

[163] D. Limon, I. Alvarado, T. Alamo, and E. F. Camacho. "MPC for tracking piece-wise constant references for constrained linear systems." In: *Automatica* 44 (2008), pp. 2382–2387.

[164] D. Limon, A. Ferramosca, I. Alvarado, and T. Alamo. "Nonlinear MPC for Tracking Piece-Wise Constant Reference Signals." In: *IEEE Trans. Autom. Control* 63 (2018), pp. 3735–3750.

[165] D. Limon, M. Pereira, D. M. De La Peña, T. Alamo, and J. M. Grosso. "Single-layer economic model predictive control for periodic operation." In: *J. Proc. Contr.* 24 (2014), pp. 1207–1224.

[166] D. Limon, M. Pereira, D. M. de la Peña, T. Alamo, C. N. Jones, and M. N. Zeilinger. "MPC for tracking periodic references." In: *IEEE Trans. Autom. Control* 61 (2016), pp. 1123–1128.

[167] S. Liu and J. Liu. "Economic model predictive control with extended horizon." In: *Automatica* 73 (2016), pp. 180–192.

[168] J. Löfberg. "Oops! I cannot do it again: Testing for recursive feasibility in MPC." In: *Automatica* 48.3 (2012), pp. 550–555.

[169] W. Lohmiller and J.-J. E. Slotine. "On contraction analysis for non-linear systems." In: *Automatica* 34 (1998), pp. 683–696.

[170] B. T. Lopez, J.-J. E. Slotine, and J. P. How. "Dynamic Tube MPC for nonlinear systems." In: *Proc. American Control Conf. (ACC)*. 2019, pp. 1655–1662.

[171] J. Z. Lu. "Closing the gap between planning and control: A multiscale MPC cascade approach." In: *Annual Reviews in Control* 40 (2015), pp. 3–13.

[172] S. Lucia, M. Kögel, and R. Findeisen. "Contract-based predictive control of distributed systems with plug and play capabilities." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2015, pp. 205–211.

[173] U. Maeder and M. Morari. "Offset-free reference tracking with model predictive control." In: *Automatica* 46.9 (2010), pp. 1469–1476.

[174] L. Magni, G. De Nicolao, and R. Scattolini. "Output feedback and tracking of nonlinear systems with model predictive control." In: *Automatica* 37.10 (2001), pp. 1601–1607.

[175] L. Magni, G. De Nicolao, L. Magnani, and R. Scattolini. "A stabilizing model-based predictive control algorithm for nonlinear systems." In: *Automatica* 37.9 (2001), pp. 1351–1362.

[176] L. Magni and R. Scattolini. "On the solution of the tracking problem for non-linear systems with MPC." In: *Int. J. of Systems Science* 36 (2005), pp. 477–484.

[177] I. R. Manchester and T. L. Chaffey. "An Amendment to" Control Contraction Metrics: Convex and Intrinsic Criteria for Nonlinear Feedback Design"." In: *arXiv preprint arXiv:1711.08128* (2017).

[178] I. R. Manchester and J.-J. E. Slotine. "Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design." In: *IEEE Trans. Automat. Control* 62 (2017), pp. 3046–3053.

[179] I. R. Manchester and J.-J. E. Slotine. "Output-feedback control of nonlinear systems using control contraction metrics and convex optimization." In: *Proc. 4th Australian Control Conf. (AUCC)*. 2014, pp. 215–220.

[180] J. Manzano, J. Calliess, D. Munoz de la Pena, and L. Daniel. "Online learning robust MPC: an exploration-exploitation approach." In: *Proc. 21st IFAC World Congress*. 2020, pp. 5292–5297.

[181] J. Manzano, D. Muñoz de la Peña, J. Calliess, and D. Limon. "Online learning constrained model predictive controller based on double prediction." In: *Int. J. Robust and Nonlinear Control* (2020).

[182] J. M. Manzano, D. Limon, D. M. de la Peña, and J. Calliess. "Output feedback MPC based on smoothed projected kinky inference." In: *IET Control Theory & Applications* 13.6 (2019), pp. 795–805.

[183] J. M. Manzano, D. Limon, D. M. de la Peña, and J.-P. Calliess. "Robust learning-based MPC for nonlinear constrained systems." In: *Automatica* 117 (2020), p. 108948.

[184] W.-J. Mao. "Robust stabilization of uncertain time-varying discrete systems and comments on "an improved approach for constrained robust model predictive control"." In: *Automatica* 39 (2003), pp. 1109–1112.

[185] L. Marconi, A. Isidori, and A. Serrani. "Non-resonance conditions for uniform observability in the problem of nonlinear output regulation." In: *Systems & Control Letters* 53.3-4 (2004), pp. 281–298.

[186] T. Martin, P. N. Köhler, and F. Allgöwer. "Dissipativity and economic model predictive control for optimal set operation." In: *Proc. American Control Conf. (ACC)*. 2019, pp. 1020–1026.

[187] L. Matino. "Numerical verification of incremental stability for nonlinear systems." MA thesis. Università del Salento, 2020.

[188] D. Mayne. "An apologia for stabilising terminal conditions in model predictive control." In: *Int. J. Control* 86.11 (2013), pp. 2090–2095.

[189] D. Mayne and P. Falugi. "Generalized stabilizing conditions for model predictive control." In: *J. of Optimization Theory and Applications* 169 (2016), pp. 719–734.

[190] D. Q. Mayne. “Model predictive control: Recent developments and future promise.” In: *Automatica* 50.12 (2014), pp. 2967–2986.

[191] D. Q. Mayne, E. C. Kerrigan, E. Van Wyk, and P. Falugi. “Tube-based robust nonlinear model predictive control.” In: *Int. J. Robust and Nonlinear Control* 21 (2011), pp. 1341–1353.

[192] D. Q. Mayne and H. Michalska. “Receding horizon control of nonlinear systems.” In: *IEEE Trans. Automat. Control* 35.7 (1990), pp. 814–824.

[193] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert. “Constrained model predictive control: Stability and optimality.” In: *Automatica* 36 (2000), pp. 789–814.

[194] M. W. Mehrez, K. Worthmann, G. K. Mann, R. G. Gosine, and T. Faulwasser. “Predictive path following of mobile robots without terminal stabilizing constraints.” In: *Proc. 20th IFAC World Congress*. 2017, pp. 9852–9857.

[195] P. Mhaskar, N. H. El-Farra, and P. D. Christofides. “Stabilization of nonlinear systems with state and control constraints using Lyapunov-based predictive control.” In: *Systems & Control Letters* 55.8 (2006), pp. 650–659.

[196] H. Michalska and D. Q. Mayne. “Robust receding horizon control of constrained nonlinear systems.” In: *IEEE Trans. Automat. Control* 38.11 (1993), pp. 1623–1633.

[197] S. Monaco and D. Normand-Cyrot. “Minimum-phase nonlinear discrete-time systems and feedback stabilization.” In: *Proc. 26th IEEE Conf. Decision and Control (CDC)*. 1987, pp. 979–986.

[198] S. Monaco and D. Normand-Cyrot. “Zero dynamics of sampled nonlinear systems.” In: *Systems & Control Letters* 11.3 (1988), pp. 229–234.

[199] P. R. B. Monasterios and P. A. Trodden. “Model predictive control of linear systems with preview information: Feasibility, stability, and inherent robustness.” In: *IEEE Trans. Automat. Control* 64.9 (2018), pp. 3831–3838.

[200] V. F. Montagner, R. C. Oliveira, V. J. Leite, and P. L. Peres. “Gain scheduled state feedback control of discrete-time systems with time-varying uncertainties: an LMI approach.” In: *Proc. 44th IEEE Conf. Decision and Control (CDC)*. 2005, pp. 4305–4310.

[201] M. Morari and U. Maeder. “Nonlinear offset-free model predictive control.” In: *Automatica* 48.9 (2012), pp. 2059–2067.

[202] M. M. Morato, J. E. Normey-Rico, and O. Sename. “Model predictive control design for linear parameter varying systems: A survey.” In: *Annual Reviews in Control* (2020).

[203] M. A. Müller. “Dissipativity in economic model predictive control: beyond steady-state optimality.” In: *Recent Advances in Model Predictive Control*. Springer, 2021, pp. 27–43.

[204] M. A. Müller. “Distributed and economic model predictive control: beyond setpoint stabilization.” PhD thesis. Universität Stuttgart, 2014.

[205] M. A. Müller. “Nonlinear moving horizon estimation in the presence of bounded disturbances.” In: *Automatica* 79 (2017), pp. 306–314.

[206] M. A. Müller, D. Angeli, and F. Allgöwer. “Economic model predictive control with self-tuning terminal cost.” In: *European J. Control* 19 (2013), pp. 408–416.

[207] M. A. Müller, D. Angeli, and F. Allgöwer. “On necessity and robustness of dissipativity in economic model predictive control.” In: *IEEE Trans. Automat. Control* 60.6 (2015), pp. 1671–1676.

[208] M. A. Müller, D. Angeli, and F. Allgöwer. “On the performance of economic model predictive control with self-tuning terminal cost.” In: *J. Proc. Contr.* 24 (2014), pp. 1179–1186.

[209] M. A. Müller, D. Angeli, F. Allgöwer, R. Amrit, and J. B. Rawlings. “Convergence in economic model predictive control with average constraints.” In: *Automatica* 50 (2014), pp. 3100–3111.

[210] M. A. Müller and L. Grüne. “Economic model predictive control without terminal constraints for optimal periodic behavior.” In: *Automatica* 70 (2016), pp. 128–139.

[211] M. A. Müller, L. Grüne, and F. Allgöwer. “On the role of dissipativity in economic model predictive control.” In: *Proc. 5th IFAC Conf. Nonlinear Model Predictive Control*. 2015, pp. 110–116.

[212] M. A. Müller and K. Worthmann. “Quadratic costs do not always work in MPC.” In: *Automatica* 82 (2017), pp. 269–277.

[213] K. R. Muske and T. A. Badgwell. “Disturbance modeling for offset-free linear model predictive control.” In: *J. Proc. Contr.* 12.5 (2002), pp. 617–632.

[214] M. Nonhoff and M. A. Müller. "An online convex optimization algorithm for controlling linear systems with state and input constraints." In: *Proc. American Control Conf. (ACC)*. 2021, pp. 2523–2528.

[215] J. Nubert. "Learning-based Approximate Model Predictive Control with Guarantees: Joining Neural Networks with Recent Robust MPC." MA thesis. ETH Zürich, 2020.

[216] S. L. de Oliveira Kothare and M. Morari. "Contractive model predictive control for constrained nonlinear systems." In: *IEEE Trans. Automat. Control* 45.6 (2000), pp. 1053–1071.

[217] G. Pannocchia. "An economic MPC formulation with offset-free asymptotic performance." In: *Proc. 10th IFAC Symp. on Advanced Control of Chemical Processes (ADCHEM)*. 2018, pp. 393–398.

[218] G. Pannocchia and J. B. Rawlings. "Disturbance models for offset-free model-predictive control." In: *AIChE journal* 49.2 (2003), pp. 426–437.

[219] D. Papadimitriou, U. Rosolia, and F. Borrelli. "Control of Unknown Nonlinear Systems with Linear Time-Varying MPC." In: *Proc. 59th Conf. on Decision and Control (CDC)*. 2020, pp. 2258–2263.

[220] P. A. Parrilo. "Semidefinite programming relaxations for semialgebraic problems." In: *Mathematical programming* 96 (2003), pp. 293–320.

[221] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer. "Convergent dynamics, a tribute to Boris Pavlovich Demidovich." In: *Systems & Control Letters* 52.3-4 (2004), pp. 257–261.

[222] A. V. Pavlov, N. van de Wouw, and H. Nijmeijer. *Uniform output regulation of nonlinear systems: A convergent dynamics approach*. Springer Science & Business Media, 2006.

[223] M. Pereira, D. M. de la Peña, D. Limon, I. Alvarado, and T. Alamo. "Robust model predictive controller for tracking changing periodic signals." In: *IEEE Trans. Autom. Control* 62 (2017), pp. 5343–5350.

[224] M. Pereira, D. Limon, D. M. de la Peña, L. Valverde, and T. Alamo. "Periodic economic control of a nonisolated microgrid." In: *IEEE Trans. Ind. Electron.* 62.8 (2015), pp. 5247–5255.

[225] S. Pirkelmann, D. Angeli, and L. Grüne. "Approximate computation of storage functions for discrete-time systems using sum-of-squares techniques." In: *Proc. 11th IFAC Symp. Nonlinear Control Systems (NOLCOS)*. 2019, pp. 508–513.

[226] I. G. Polushin and H. J. Marquez. "On the existence of a continuous storage function for dissipative systems." In: *Proc. 40th IEEE Conf. Decision and Control (CDC)*. 2001, pp. 431–435.

[227] J. A. Primbs. "The analysis of optimization based controllers." In: *Automatica* 37.6 (2001), pp. 933–938.

[228] F. D. Priscoli, A. Isidori, and L. Marconi. "A dissipativity-based approach to output regulation of non-minimum-phase systems." In: *Systems & Control Letters* 58.8 (2009), pp. 584–591.

[229] S. J. Qin and T. A. Badgwell. "A survey of industrial model predictive control technology." In: *Control Engineering practice* 11.7 (2003), pp. 733–764.

[230] T. Raff, C. Ebenbauer, and P. Allgöwer. "Nonlinear model predictive control: A passivity-based approach." In: *Assessment and future directions of nonlinear model predictive control*. Vol. 358. Springer, 2007, pp. 151–162.

[231] T. Raff, S. Huber, Z. K. Nagy, and F. Allgöwer. "Nonlinear model predictive control of a four tank system: An experimental stability study." In: *Proc. IEEE Int. Conf. Control Applications*. 2006, pp. 237–242.

[232] C. Rajhans, D. W. Griffith, S. C. Patwardhan, L. T. Biegler, and H. K. Pillai. "Terminal region characterization and stability analysis of discrete time quasi-infinite horizon nonlinear model predictive control." In: *J. Proc. Contr.* 83 (2019), pp. 30–52.

[233] C. V. Rao, J. B. Rawlings, and D. Q. Mayne. "Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations." In: *IEEE Trans. Automat. Control* 48.2 (2003), pp. 246–258.

[234] J. B. Rawlings, D. Bonné, J. B. Jorgensen, A. N. Venkat, and S. B. Jorgensen. "Unreachable setpoints in model predictive control." In: *IEEE Trans. Autom. Control* 53 (2008), pp. 2209–2215.

[235] J. B. Rawlings, N. R. Patel, M. J. Risbeck, C. T. Maravelias, M. J. Wenzel, and R. D. Turney. "Economic MPC and real-time decision making with application to large-scale HVAC energy systems." In: *Comp. & Chem. Eng.* 114 (2018), pp. 89–98.

[236] J. B. Rawlings, D. Q. Mayne, and M. Diehl. *Model Predictive Control: Theory, Computation, and Design*. third printing. Nob Hill Publishing, 2017.

[237] M. Reble and F. Allgöwer. “Unconstrained model predictive control and suboptimality estimates for nonlinear continuous-time systems.” In: *Automatica* 48 (2012), pp. 1812–1817.

[238] M. Reble, D. E. Quevedo, and F. Allgöwer. “A unifying framework for stability in MPC using a generalized integral terminal cost.” In: *Proc. American Control Conf. (ACC)*. 2012, pp. 1211–1216.

[239] M. Reble, D. E. Quevedo, and F. Allgöwer. “Improved stability conditions for unconstrained nonlinear model predictive control by using additional weighting terms.” In: *Proc. 51st IEEE Conf. Decision and Control (CDC)*. 2012, pp. 2625–2630.

[240] M. Revay, R. Wang, and I. R. Manchester. “Convex Sets of Robust Recurrent Neural Networks.” In: *arXiv preprint arXiv:2004.05290* (2020).

[241] M. J. Risbeck, C. T. Maravelias, and J. B. Rawlings. “Unification of Closed-loop Scheduling and Control: State-space Formulations, Terminal Constraints, and Nominal Theoretical Properties.” In: *Comp. & Chem. Eng.* 129 (2019), p. 106496.

[242] M. J. Risbeck and J. B. Rawlings. “Economic model predictive control for time-varying cost and peak demand charge optimization.” In: *IEEE Trans. Automat. Control* 65.7 (2020), pp. 2957–2968.

[243] W. J. Rugh and J. S. Shamma. “Research on gain scheduling.” In: *Automatica* 36 (2000), pp. 1401–1425.

[244] I. J. Sánchez, A. D’Jorge, A. C. Limache, A. H. González, and A. Ferramosca. “Model Predictive Periodic Output Path Following.” In: *Proc. Argentine Conf. on Automatic Control (AADECA)*. 2020, pp. 1–6.

[245] R. G. Sanfelice and L. Praly. “Convergence of Nonlinear Observers on $\mathbb{R}^n$ With a Riemannian Metric (Part I).” In: *IEEE Trans. Automat. Control* 57.7 (2012). revised version on arXiv:1412.6730, pp. 1709–1722.

[246] R. G. Sanfelice and L. Praly. “Convergence of Nonlinear Observers on $\mathbb{R}^n$ With a Riemannian Metric (Part III).” In: *arXiv preprint arXiv:2102.08340* (2021).

[247] T. L. Santos and V. M. Cunha. “Robust MPC for linear systems with bounded disturbances based on admissible equilibria sets.” In: *Int. J. Robust and Nonlinear Control* 31.7 (2021), pp. 2690–2711.

[248] C. W. Scherer. "Gain-scheduling control with dynamic multipliers by convex optimization." In: *SIAM J. Control Optim.* 53 (2015), pp. 1224–1249.

[249] C. W. Scherer. "LPV control and full block multipliers." In: *Automatica* 37 (2001), pp. 361–375.

[250] C. W. Scherer and I. E. Kose. "Gain-scheduled control synthesis using dynamic *D*-scales." In: *IEEE Trans. Automat. Control* 57.9 (2012), pp. 2219–2234.

[251] H. Schlüter and F. Allgöwer. "A Constraint-Tightening Approach to Nonlinear Stochastic Model Predictive Control for Systems under General Disturbances." In: *Proc. 21st IFAC World Congress*. 2020, pp. 7130–7135.

[252] M. Schulze Darup and M. Cannon. "A missing link between nonlinear MPC schemes with guaranteed stability." In: *Proc. 54th IEEE Conf. Decision and Control (CDC)*. 2015, pp. 4977–4983.

[253] P. O. Scokaert, D. Q. Mayne, and J. B. Rawlings. "Suboptimal model predictive control (feasibility implies stability)." In: *IEEE Trans. Autom. Control* 44 (1999), pp. 648–654.

[254] G. Scorletti, V. Fromion, and S. De Hillerin. "Toward nonlinear tracking and rejection using LPV control." In: *Proc. 1st IFAC Workshop on Linear Parameter Varying Systems (LPVS)*. Elsevier, 2015, pp. 13–18.

[255] G. Serale, M. Fiorentini, A. Capozzoli, D. Bernardini, and A. Bemporad. "Model Predictive Control (MPC) for Enhancing Building and HVAC System Energy Efficiency: Problem Formulation, Applications and Opportunities." In: *Energies* 11.3 (2018), p. 631.

[256] D. Simon, J. Löfberg, and T. Glad. "Reference tracking MPC using dynamic terminal set transformation." In: *IEEE Trans. Autom. Control* 59.10 (2014), pp. 2790–2795.

[257] S. Singh, B. Landry, A. Majumdar, J.-J. Slotine, and M. Pavone. "Robust Feedback Motion Planning via Contraction Theory." In: *Int. J. Robotics Research* (2019). submitted.

[258] S. Singh, A. Majumdar, J.-J. Slotine, and M. Pavone. "Robust online motion planning via contraction theory and convex optimization." In: *Proc. Int. Conf. on Robotics and Automation (ICRA)*. 2017, pp. 5883–5890.

[259] S. Singh, S. M. Richards, V. Sindhwani, J.-J. E. Slotine, and M. Pavone. "Learning stabilizable nonlinear dynamics with contraction-based regularization." In: *Int. J. Robotics Research* (2020).

[260] E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, 2013.

[261] J. F. Sturm. "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones." In: *Optimization methods and software* 11 (1999), pp. 625–653.

[262] D. Sun, S. Jha, and C. Fan. "Learning certified control using contraction metric." In: *arXiv preprint arXiv:2011.12569* (2020).

[263] E. Terzi, F. Bonassi, M. Farina, and R. Scattolini. "Learning model predictive control with long short-term memory networks." In: *Int. J. Robust and Nonlinear Control* (2021).

[264] D. N. Tran, B. S. Rüffer, and C. M. Kellett. "Convergence properties for discrete-time nonlinear systems." In: *IEEE Trans. Automat. Control* 64.8 (2019). extended version on arxiv:1612.05327v3, pp. 3415–3422.

[265] D. N. Tran, B. S. Rüffer, and C. M. Kellett. "Incremental stability properties for discrete-time systems." In: *Proc. 55th IEEE Conf. Decision and Control (CDC)*. 2016, pp. 477–482.

[266] H. Tsukamoto and S.-J. Chung. "Neural Contraction Metrics for Robust Estimation and Control: A Convex Optimization Approach." In: *IEEE Control Systems Letters* 5.1 (2021), pp. 211–216.

[267] S. E. Tuna, M. J. Messina, and A. R. Teel. "Shorter horizons for model predictive control." In: *Proc. American Control Conf. (ACC)*. 2006, pp. 863–868.

[268] J. Veenman and C. W. Scherer. "A synthesis framework for robust gain-scheduling controllers." In: *Automatica* 50.11 (2014), pp. 2799–2812.

[269] J. Vergara-Diertrich, A. Ferramosca, V. Mirasierra, J. E. Normey-Rico, and D. Limon. "A Modifier-Adaptation Approach to the One-Layer Economic MPC." In: *Proc. 21st IFAC World Congress*. 2020, pp. 6957–6962.

[270] C. Verhoek, P. J. W. Koelewijn, and R. Toth. "Convex Incremental Dissipativity Analysis of Nonlinear Systems." In: *arXiv preprint arXiv::2006.1420* (2020).

[271] M. E. Villanueva, E. De Lazzari, M. A. Müller, and B. Houska. "A set-theoretic generalization of dissipativity with applications in Tube MPC." In: *Automatica* 122 (2020), p. 109179.

[272] K. P. Wabersich, F. A. Bayer, M. A. Müller, and F. Allgöwer. "Economic model predictive control for robust periodic operation with guaranteed closed-loop performance." In: *Proc. European Control Conf. (ECC)*. 2018, pp. 507–513.

[273] K. P. Wabersich and M. N. Zeilinger. "A predictive safety filter for learning-based control of constrained nonlinear dynamical systems." In: *Automatica* 129 (2021), p. 109597.

[274] Z. Wan and M. V. Kothare. "Efficient scheduled stabilizing model predictive control for constrained nonlinear systems." In: *Int. J. Robust and Nonlinear Control* 13 (2003), pp. 331–346.

[275] Z. Wan and M. V. Kothare. "Efficient scheduled stabilizing output feedback model predictive control for constrained nonlinear systems." In: *IEEE Trans. Automat. Control* 49 (2004), pp. 1172–1177.

[276] F. Wang and V. Balakrishnan. "Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems." In: *IEEE Trans. Automat. Control* 47.5 (2002), pp. 720–734.

[277] R. Wang and I. R. Manchester. "Continuous-time dynamic realization for nonlinear stabilization via control contraction metrics." In: *Proc. American Control Conf. (ACC)*. 2020, pp. 1619–1624.

[278] R. Wang and I. R. Manchester. "Robust Contraction Analysis of Nonlinear Systems via Differential IQC." In: *Proc. 58th IEEE Conf. Decision and Control (CDC)*. 2019, pp. 6766–6771.

[279] R. Wang, I. R. Manchester, and J. Bao. "Distributed economic MPC with separable control contraction metrics." In: *IEEE Control Systems Letters* 1.1 (2017), pp. 104–109.

[280] R. Wang, R. Tóth, and I. R. Manchester. "A Comparison of LPV Gain Scheduling and Control Contraction Metrics for Nonlinear Control." In: *Proc. 3rd IFAC Workshop on Linear Parameter Varying Systems (LPVS)*. 2019, pp. 44–49.

[281] R. Wang, R. Tóth, and I. R. Manchester. "Virtual Control Contraction Metrics: Convex Nonlinear Feedback Design via Behavioral Embedding." In: *arXiv preprint arXiv:2003.08513* (2020).

[282] Y. Wang, J. R. Salvador, D. M. de la Peña, V. Puig, and G. Cembrano. "Economic model predictive control based on a periodicity constraint." In: *J. Proc. Contr.* 68 (2018), pp. 226–239.

[283] K. Worthmann, M. W. Mehrez, G. K. Mann, R. G. Gosine, and J. Pannek. "Interaction of open and closed loop control in MPC." In: *Automatica* 82 (2017), pp. 243–250.

[284] K. Worthmann, M. W. Mehrez, M. Zanon, G. K. Mann, R. G. Gosine, and M. Diehl. "Model predictive control of nonholonomic mobile robots without stabilizing constraints and costs." In: *IEEE Trans. on Control Systems Technology* 24.4 (2015), pp. 1394–1406.

[285] F. Wu and K. Dong. "Gain-scheduling control of LFT systems using parameter-dependent Lyapunov functions." In: *Automatica* 42.1 (2006), pp. 39–50.

[286] H. Yang, H. Zhao, Y. Xia, and J. Zhang. "Nonlinear MPC with time-varying terminal cost for tracking unreachable periodic references." In: *Automatica* 123 (2021), p. 109337.

[287] B. Yi, R. Wang, and I. R. Manchester. "On necessary conditions of tracking control for nonlinear systems via contraction analysis." In: *Proc. 59th IEEE Conf. Decision and Control (CDC)*. 2020, pp. 2000–2005.

[288] B. Yi, R. Wang, and I. R. Manchester. "Reduced-Order Nonlinear Observers via Contraction Analysis and Convex Optimization." In: *arXiv preprint arXiv:2012.06158* (2020).

[289] S. Yu, C. Böhm, H. Chen, and F. Allgöwer. "Robust model predictive control with disturbance invariant sets." In: *Proc. American Control Conf. (ACC)*. 2010, pp. 6262–6267.

[290] M. Zamani, N. van de Wouw, and R. Majumdar. "Backstepping controller synthesis and characterizations of incremental stability." In: *Systems & Control Letters* 62.10 (2013), pp. 949–962.

[291] M. Zanon and T. Faulwasser. "Economic MPC without terminal constraints: Gradient-correcting end penalties enforce asymptotic stability." In: *J. Proc. Contr.* 63 (2018), pp. 1–14.

[292] M. Zanon, S. Gros, and M. Diehl. "A Lyapunov function for periodic economic optimizing model predictive control." In: *Proc. 52nd IEEE Conf. Decision and Control (CDC)*. 2013, pp. 5107–5112.

[293] M. Zanon, S. Gros, and M. Diehl. "A tracking MPC formulation that is locally equivalent to economic MPC." In: *J. Proc. Contr.* 45 (2016), pp. 30–42.

[294] M. Zanon, S. Gros, and M. Diehl. “A periodic tracking MPC that is locally equivalent to periodic economic MPC.” In: *Proc. 20th IFAC World Congress*. 2017, pp. 10711–10716.

[295] M. Zanon, L. Grüne, and M. Diehl. “Periodic optimal control, dissipativity and MPC.” In: *IEEE Trans. Autom. Control* 62 (2017), pp. 2943–2949.

[296] M. N. Zeilinger, M. Morari, and C. N. Jones. “Soft constrained model predictive control with robust stability guarantees.” In: *IEEE Trans. Automat. Control* 59.5 (2014), pp. 1190–1202.

Publications of the Author

[JK1] J. Berberich, J. Köhler, F. Allgöwer, and M. A. Müller. "Dissipativity properties in constrained optimal control: a computational approach." In: *Automatica* 114 (2020), p. 108840.

[JK2] J. Berberich, J. Köhler, F. Allgöwer, and M. A. Müller. "Indefinite linear quadratic optimal control: Strict dissipativity and turnpike properties." In: *IEEE Control Systems Letters* 2.3 (2018), pp. 399–404.

[JK3] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer. "Data-Driven Model Predictive Control with Stability and Robustness Guarantees." In: *IEEE Trans. Automat. Control* 66.4 (2021), pp. 1702–1717.

[JK4] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer. "Data-Driven Tracking MPC for Changing Setpoints." In: *Proc. 21st IFAC World Congress*. 2020, pp. 6923–6930.

[JK5] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer. "On the design of terminal ingredients for data-driven MPC." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2021, pp. 259–265.

[JK6] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer. "Robust Constraint Satisfaction in Data-Driven MPC." In: *Proc. 59th Conf. on Decision and Control (CDC)*. 2020, pp. 1260–1267.

[JK7] J. Bongard, J. Berberich, J. Köhler, and F. Allgöwer. "Robust stability analysis of a simple data-driven model predictive control approach." In: *arXiv preprint arXiv:2103.00851* (2021).

[JK8] M. Hertneck, J. Köhler, S. Trimpe, and F. Allgöwer. "Learning an approximate model predictive controller with guarantees." In: *IEEE Control Systems Letters* 2.3 (2018), pp. 543–548.

[JK9] A. Koch, J. Berberich, J. Köhler, and F. Allgöwer. "Determining optimal input-output properties: A data-driven approach." In: *arXiv preprint arXiv:2002.03882* (2020).

[JK10] J. Köhler and F. Allgöwer. "Stability and performance in MPC using a finite-tail cost." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2021, pp. 166–171.

[JK11] J. Köhler, E. Andina, R. Soloperto, M. A. Müller, and F. Allgöwer. "Linear robust adaptive model predictive control: Computational complexity and conservatism." In: *Proc. 58th IEEE Conf. Decision and Control (CDC)*. 2019, pp. 1383–1388.

[JK12] J. Köhler, C. Enyioha, and F. Allgöwer. "Dynamic Resource Allocation to Control Epidemic Outbreaks A Model Predictive Control Approach." In: *Proc. American Control Conf. (ACC)*. 2018, pp. 1546–1551.

[JK13] J. Köhler, P. Kötting, R. Soloperto, F. Allgöwer, and M. A. Müller. "A robust adaptive model predictive control framework for nonlinear uncertain systems." In: *Int. J. Robust and Nonlinear Control* (2020). extended version: arXiv:1911.02899.

[JK14] J. Köhler, M. Manderla, and M. Florian. "Embedded Model Predictive Direct Switching Control for High Performance Electrical Drives - A Quantitative Comparison." In: *Proc. 20th IFAC World Congress*. 2017, pp. 11871–11876.

[JK15] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear model predictive control framework using reference generic terminal ingredients." In: *IEEE Trans. Automat. Control* 65.8 (2020). extended version: arXiv:1909.12765, pp. 3576–3583.

[JK16] J. Köhler, M. A. Müller, and F. Allgöwer. "A nonlinear tracking model predictive control scheme for unreachable dynamic target signals." In: *Automatica* 118 (2020). extended version: arXiv:1911.03304, p. 109030.

[JK17] J. Köhler, M. A. Müller, and F. Allgöwer. "A novel constraint tightening approach for nonlinear robust model predictive control." In: *Proc. American Control Conf. (ACC)*. 2018, pp. 728–734.

[JK18] J. Köhler, M. A. Müller, and F. Allgöwer. "A simple framework for nonlinear robust output-feedback MPC." In: *Proc. European Control Conf. (ECC)*. 2019, pp. 793–798.

[JK19] J. Köhler, M. A. Müller, and F. Allgöwer. "Constrained nonlinear output regulation using Model Predictive Control." In: *IEEE Trans. Automat. Control* (2021). extended version: arXiv:2005.12413.

[JK20] J. Köhler, M. A. Müller, and F. Allgöwer. "Distributed model predictive control—Recursive feasibility under inexact dual optimization." In: *Automatica* 102 (2019), pp. 1–9.

[JK21] J. Köhler, M. A. Müller, and F. Allgöwer. "Implicit solutions to constrained nonlinear output regulation using MPC." In: *Proc. 59th Conf. on Decision and Control (CDC)*. 2020, pp. 4604–4609.

[JK22] J. Köhler, M. A. Müller, and F. Allgöwer. "MPC for nonlinear periodic tracking using reference generic offline computations." In: *Proc. IFAC Conf. Nonlinear Model Predictive Control*. 2018, pp. 656–661.

[JK23] J. Köhler, M. A. Müller, and F. Allgöwer. "Nonlinear Reference Tracking with Model Predictive Control: An Intuitive Approach." In: *Proc. European Control Conf. (ECC)*. 2018, pp. 1355–1360.

[JK24] J. Köhler, M. A. Müller, and F. Allgöwer. "Nonlinear reference tracking: An economic model predictive control perspective." In: *IEEE Trans. Automat. Control* 64.1 (2019), pp. 254–269.

[JK25] J. Köhler, M. A. Müller, and F. Allgöwer. "On periodic dissipativity notions in economic model predictive control." In: *IEEE Control Systems Letters* 2.3 (2018), pp. 501–506.

[JK26] J. Köhler, M. A. Müller, and F. Allgöwer. "Periodic optimal control of nonlinear constrained systems using economic model predictive control." In: *J. Proc. Contr.* 92 (2020). extended version: arXiv:2005.05245, pp. 185–201.

[JK27] J. Köhler, M. A. Müller, N. Li, and F. Allgöwer. "Real time economic dispatch for power networks: A distributed economic model predictive control approach." In: *Proc. 56th Conf. on Decision and Control (CDC)*. 2017, pp. 6340–6345.

[JK28] J. Köhler, L. Schwenkel, A. Koch, J. Berberich, P. Pauli, and F. Allgöwer. "Robust and optimal predictive control of the COVID-19 outbreak." In: *Annual Reviews in Control* 51 (2021), pp. 525–539.

[JK29] J. Köhler, R. Soloperto, M. A. Müller, and F. Allgöwer. "A computationally efficient robust model predictive control framework for uncertain nonlinear systems." In: *IEEE Trans. Automat. Control* 66.2 (2021). extended version: arXiv:1910.12081, pp. 794–801.

[JK30] J. Nubert, J. Köhler, V. Berenz, F. Allgöwer, and S. Trimpe. "Safe and Fast Tracking on a Robot Manipulator: Robust MPC and Neural Network Control." In: *IEEE Robotics and Automation Letters* 5.2 (2020), pp. 3050–3057.

[JK31] P. Pauli, J. Köhler, J. Berberich, A. Koch, and F. Allgöwer. "Offset-free setpoint tracking using neural network controllers." In: *Proc. 3rd Conf. Learning for Dynamics & Control (L4DC)*. 2021, pp. 992–1003.

[JK32] A. Romer, J. Berberich, J. Köhler, and F. Allgöwer. "One-Shot Verification of Dissipativity Properties From Input–Output Data." In: *IEEE Control Systems Letters* 3.3 (2019), pp. 709–714.

[JK33] M. Rosenfelder, J. Köhler, and F. Allgöwer. "Stability and performance in transient average constrained economic MPC without terminal constraints." In: *Proc. 21st IFAC World Congress*. 2020, pp. 6943–6950.

[JK34] L. Schwenkel, J. Köhler, M. A. Müller, and F. Allgöwer. "Dynamic uncertainties in MPC: Guaranteed stability for constrained linear systems." In: *Proc. 59th Conf. on Decision and Control (CDC)*. 2020, pp. 1235–1241.

[JK35] L. Schwenkel, J. Köhler, M. A. Müller, and F. Allgöwer. "Robust Economic Model Predictive Control without Terminal Conditions." In: *Proc. 21st IFAC World Congress*. 2020, pp. 7097–7104.

[JK36] R. Soloperto, J. Köhler, M. A. Müller, and F. Allgöwer. "Collision avoidance for uncertain nonlinear systems with moving obstacles using robust Model Predictive Control." In: *Proc. European Control Conf. (ECC)*. 2019, pp. 811–817.

[JK37] R. Soloperto, J. Köhler, M. A. Müller, and F. Allgöwer. "Dual Adaptive MPC for output tracking of linear systems." In: *Proc. 58th IEEE Conf. Decision and Control (CDC)*. 2019, pp. 1377–1382.

[JK38] R. Soloperto, J. Köhler, and F. Allgöwer. "Augmenting MPC schemes with active learning: Intuitive tuning and guaranteed performance." In: *IEEE Control Systems Letters* 4.3 (2020), pp. 713–718.

[JK39] J. Venkatasubramanian, J. Köhler, J. Berberich, and F. Allgöwer. "Robust Dual Control based on Gain Scheduling." In: *Proc. 59th Conf. on Decision and Control (CDC)*. 2020, pp. 2270–2277.

[JK40] J. Walter et al. "Comparative Study of a Biomechanical Model-based and Black-box Approach for Subject-Specific Movement Prediction." In: *Proc. 42nd Int. Conf. of the IEEE Engineering in Medicine and Biology Society (EMBC)*. 2020, pp. 4775–4778.