Absence of the Efimov Effect in Dimensions One and Two

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Abstract

In this thesis we consider the Schrödinger operator

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} + \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j) \tag{0.0.1}$$

corresponding to a system of one- or two-dimensional quantum particles which interact via non-vanishing short-range potentials V_{ij}. We consider the operator in the so-called center of mass frame and denote the associated operator by H.

Our goal is to investigate virtual levels of the operator H, by which we mean that $H = -\Delta + V \ge 0$ and the essential spectrum is stable under small perturbations, but for any $\varepsilon > 0$ the perturbed operator $-(1-\varepsilon)\Delta + V$ has a negative eigenvalue. We prove that in this case there exists a weak solution ψ of the equation

$$H\psi = 0$$

for which we show that it is an eigenfunction of H if the system consists of $N \ge 3$ one-dimensional or $N \ge 4$ two-dimensional particles. We also provide estimates for the rate of decay of the function ψ.

Later, we study multi-particle Schrödinger operators for particles in dimension one or two with respect to the Efimov effect. This effect is a phenomenon which appears in systems of three three-dimensional particles, namely the three-body Hamiltonian has an infinite number of bound states provided the Hamiltonians of the two-body subsystems have a virtual level. We deal with the question whether an Efimov type effect can occur in systems of $N \ge 3$ one- or two-dimensional particles. We prove that this is not the case if $N = 3$ or if $N \ge 4$ and the particles are one-dimensional or if $N \ge 5$ and the particles are two-dimensional.

Zusammenfassung

In dieser Arbeit betrachten wir den Schrödinger-Operator

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} + \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j), \tag{0.0.2}$$

der ein System von ein- oder zwei-dimensionalen Quantenteilchen beschreibt, deren Wechselwirkung durch Potentiale V_{ij} beschrieben wird. Wir betrachten den Operator im Schwerpunktsystem und bezeichnen ihn im Folgenden mit H.

Unser Ziel ist es, virtual levels des Operators H zu untersuchen. Darunter verstehen wir, dass $H = -\Delta + V$ nicht-negativ ist, dass das wesentliche Spektrum von H stabil unter kleinen Störungen ist, aber dass für jedes $\varepsilon > 0$ der gestörte Operator $-(1-\varepsilon)\Delta + V$ einen negativen Eigenwert besitzt. Wir beweisen, dass in diesem Fall eine schwache Lösung ψ der Gleichung

$$H\psi = 0$$

existiert und dass sie eine Eigenfunktion von H ist, wenn das System aus $N \ge 3$ eindimensionalen oder $N \ge 4$ zweidimensionalen Teilchen besteht. Außerdem geben wir eine Abschätzung für die Abfallrate dieser Lösung im Unendlichen an.

Anschließend untersuchen wir Mehrteilchen-Schrödingeroperatoren für ein- oder zwei-dimensionale Teilchen im Hinblick auf den Efimov-Effekt. Dieser beschreibt ein Phänomen, das in Systemen von drei drei-dimensionalen Teilchen auftritt. Der Operator, der das Dreiteilchensystem beschreibt, hat unendlich viele Eigenwerte, falls die Operatoren, die zu den Zweiteilchen-Teilsystemen gehören, ein virtual level besitzen. Wir zeigen, dass ein solcher Effekt in Systemen von drei ein- oder zweidimensionalen Teilchen, $N \ge 4$ eindimensionalen Teilchen oder $N \ge 5$ zweidimensionalen Teilchen nicht auftritt.

Contents

1. Introduction

1.1. Spectral theory of Schrödinger operators

In this thesis we investigate a multi-particle Schrödinger operator

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{x_i} + \sum_{1\le i<j\le N} V_{ij}(x_i - x_j) \tag{1.1.1}$$

which acts on $L^2(\mathbb{R}^{dN})$ and corresponds to a system of $N \ge 3$ one- or two-dimensional quantum particles. Here, $d \in \{1,2\}$ is the spatial dimension of the particles, $m_i > 0$ is the mass of the ith particle, x_i describes its position in $\mathbb{R}^d$, Δ_{x_i} is the Laplace operator on $\mathbb{R}^d$ describing the dynamics of the ith particle and the potentials $V_{ij} : \mathbb{R}^{dN} \to \mathbb{R}$ are real-valued functions which describe the pair interaction between the particles. Our goal is to study the discrete spectrum of the associated Schrödinger operator with removed center of mass of the system. Before we explain the context of our studies and discuss the main results of this thesis let us introduce some basics.

Spectral theory of Schrödinger operators has become a large field in mathematical physics and investigates qualitative and quantitative properties of the spectrum of operators

$$H = -\Delta + V \tag{1.1.2}$$

acting on $L^2(\mathbb{R}^d)$. Under broad conditions the operator H is self-adjoint on $L^2(\mathbb{R}^d)$, see for example the famous work of T. Kato [38].

The spectrum of H consists of two parts, namely the discrete spectrum, which consists of isolated eigenvalues of finite multiplicity, and the essential spectrum. Typical questions concerning the discrete spectrum of H are whether it is finite or infinite, properties of the eigenvalue counting function or of the eigenfunctions. Concerning the essential spectrum one is interested for example in the spectral type (densely pure point, singular continuous or absolutely continuous spectrum) and in the behavior of solutions corresponding to the time-dependent Schrödinger equation, which is related to scattering theory.

For the Schrödinger operator $H = -\Delta + V$ corresponding to a single particle with a potential V decaying at infinity, it is well known that the essential spectrum is given by $[0,\infty)$. In this case the answer to the question whether the operator has a finite or an infinite number of negative eigenvalues depends on the sign and the rate of decay of V. If $|V(x)| \le C|x|^{-2-\delta}$ for some constants $C, \delta > 0$ and large values of $|x|$, then H has a finite number of negative eigenvalues. On the other hand, if there exist constants $C, \delta > 0$, such that $V(x) \le -C|x|^{-2+\delta}$ for large values of $|x|$, then H has infinitely many negative eigenvalues. We refer to [40, 58] for a more detailed discussion.

For the case of multi-particle systems the situation is quite different. Even if the pair potentials are compactly supported, the potential V, which in this case is the sum of the pair potentials, does not necessarily decay as $|x| \to \infty$. This makes the spectral analysis for multi-particle Schrödinger operators much more complicated. In general the essential spectrum does not coincide with the semi-axis $[0,\infty)$ and its determination is a challenging problem. The famous HVZ theorem, named after W. Hunziker, C. van Winter and G. Zhislin, determines the bottom of the essential spectrum of multi-particle Schrödinger operators. The question whether the discrete spectrum is finite or infinite is in general difficult to answer and non-intuitive effects, such as the Efimov effect, can appear.

Concerning the quantum mechanical point of view we only want to give a few terms which seem to be important for this work. The Schrödinger operator H describes the dynamics of a single particle or of a system of several particles. A function $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$ is called a state of the system and according to Max Born $|\psi(x)|^2$ can be interpreted as the probability to find the particle(s) at the position $x \in \mathbb{R}^d$. The total energy of the system is given by the quadratic form

$$\int_{\mathbb{R}^d} |\nabla\psi|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d} V|\psi|^2 \, \mathrm{d}x, \tag{1.1.3}$$

where the first integral describes the kinetic energy and the second integral the potential energy. Negative eigenvalues of H are interpreted as the possible energy levels and the corresponding eigenfunctions are bound states of the system.

1.2. The Efimov effect

In this work we consider the operator H_N given in (1.1.1) with regard to the Efimov effect. In this section we describe this effect and give a brief overview over its history. One of the goals of this thesis is to investigate the occurrence of the Efimov effect for systems of one- or two-dimensional particles.

1.2.1. Efimov's prediction

In 1970, V. Efimov predicted a counter-intuitive phenomenon which can be stated as follows: A system of three quantum particles in dimension three, interacting through attractive short-range potentials, has an infinite number of bound states if the Hamiltonians of the two-body subsystems do not have negative spectrum and at least two of them are resonant [16]. By a zero energy resonance we mean that there exists a (weak) solution ψ of the Schrödinger equation $H\psi = 0$ which is not square-integrable. A physical interpretation of resonances is that particles are "close to bind", i.e., they spend some time together (they resonate), before they separate again.

Heuristically, the Efimov effect can be explained as follows: Although any two particles interact via a short-range potential (i.e., fast decaying potential), the third particle leads to an effective long-range potential which decays as $\frac{C}{|x|^2}$ for some $C < 0$ and thus leads to the infinitude of the discrete spectrum.

One of the special features of the Efimov effect is its universality. This does not mean that the effect occurs in any system, but any system satisfying the conditions for the occurance has the same universal features [47]. For example, for the eigenvalues it was predicted that they tend exponentially to the accumulation point zero, namely

$$E_n \sim \exp\left(-\frac{2\pi n}{s_0}\right) \quad \text{as} \quad n \to \infty, \tag{1.2.1}$$

where $s_0 > 0$ is a universal constant [16]. These universal features depend only on a few general properties, e.g., the masses of the particles. In particular, *the effect does not depend on the form of two-body forces - it is only their resonant character that we require.*[1]

[1] V. Efimov, [16]

1.2.2. Mathematical proofs of the Efimov effect

The first mathematical proof of the effect was given by D.R. Yafaev [78]. He assumed that the Hamiltonians h corresponding to the two-body subsystems have a virtual at zero, i.e., $h \geq 0$ and any small negative perturbation of the potential leads to a negative eigenvalue. In this case zero is a resonance of the operator h. Using symmetrized Faddeev equations and investigating integral operators he showed that the three-body Hamiltonian H has infinitely many eigenvalues if for at least two of the subsystem the corresponding Hamiltonian has a virtual level. He also proved that the effect does not occur if at most one of the Hamiltonians corresponding to the subsystems has a virtual level [80].

Later, A.V. Sobolev [69] completed the mathematical proof by deriving the asymptotics

$$\lim_{z \to 0-} \frac{N(z)}{|\ln(|z|)|} = \mathscr{A}_0 > 0, \tag{1.2.2}$$

for the counting function $N(z)$ of the eigenvalues below z, where the constant $\mathscr{A}_0$ does not depend on the potentials. This asymptotics is in accordance with the prediction (1.2.1).

In the years after Yafaev's proof of the Efimov effect this topic attracted many mathematicians and different techniques were developed and many results obtained. We can not discuss all of them, but we want to mention two which seem to be of special interest.

First, in the papers of Yu.N. Ovchinnikov and I.M. Sigal [55] and of H. Tamura [71] it is demonstrated how the long-range interaction in the three-body system appears, which leads to the infiniteness of the discrete spectrum.

Second, it was shown by S.A. Vugalter and G.M. Zhislin that the existence of the Efimov effect depends on the nature of the virtual levels in the two-body subsystems. Precisely, it was proved in [74] that if the virtual levels in the two-body subsystems correspond to eigenvalues, then the Hamiltonian of the three-particle system has only a finite number of eigenvalues. In particular, their technique shows that the effect does not occur in systems of three three-dimensional fermions. This, together with the result of Yafaev [80] underlines the important role of virtual levels, concerning both their existence and their behaviour. The technique developed in [74] plays an important role in the work at hand.

1.2.3. Physical experiments and new interest on the effect

For a long time the Efimov effect was regarded by many as a theoretical peculiarity. Because it is very difficult to create and to control resonant short-range interactions, it took more than 30 years before in 2006 it was verified in an ultra-cold gas of caesium atoms by a group of phycisists in Innsbruck [41]. This experiment was a milestone and opened the way to many further experiments in different systems of ultra-cold atoms in many laboratories all over the world [19, 29, 9].
In addition, it lead to a resurgence of interest to the Efimov effect and today one no longer speaks only of the Efimov effect, but even of Efimov physics. In the last years many scientists have worked in different directions of this subject and generalizations of the Efimov effect to different systems were investigated. Some of them are presented in the following. For a more detailed discussion and to underline the richness of this subject we refer to the review of P. Naidon and S. Endo [47] which contains 400 references.

1.3. Generalizations of the Efimov effect

It is a natural and interesting question to ask whether the Efimov effect can occur if we vary

- the spatial dimension of the particles,
- the number of particles

or if we restrict the operator to subspaces of certain symmetries, e.g., if we consider fermionic systems. Investigations of such questions involve several difficulties. The low-energy behavior of Schrödinger operators, and therefore the behavior of virtual levels, depends strongly on the dimension. As we have described before, the nature of virtual levels has a big impact on the existence of the Efimov effect. For systems consisting of three or more particles the potential V, which is the sum of all pair interactions, does not decay in all directions of the configuration space. This causes great difficulties on a technical level and requires new methods.

1.3.1. Absence of the Efimov effect in higher dimensions and for multi-particle systems

It is well known, see for example [5, 22], that for dimension $d \geq 5$ virtual levels of two-body Schrödinger operators correspond to eigenvalues, which implies the absence of the Efimov effect [5]. For dimension $d = 4$ virtual levels correspond to resonances. It was shown in [5] that also in this case the Efimov effect is absent. The case of one- or two-dimensional particles will be explained below.

A few years after Efimov's discovery of the effect, R.D. Amado and F.C. Greenwood predicted that it does not exist for systems consisting of $N \geq 4$ three-dimensional particles [3]. As already mentioned, such a problem is difficult to solve because the potential does not decay and Faddeev equations, which were used in the proofs for the three-particle case, become very complicated in the multi-particle case. The first mathematical proof was provided by D. Gridnev by the use of generalized Faddeev equations [25]. Recently, a proof based on variational methods was given in [7] under less restrictive conditions on the pair potentials. The work [7] also covers the case of particles in dimension $d \geq 4$. A major step in the proof of the absence of the effect for such systems is to show that virtual levels of the Hamiltonians of the subsystems consisting of $N - 1$ particles are eigenvalues.

For fermionic systems of three three-dimensional particles the Efimov effect does not occur, see for example [74]. However, for two-dimensional fermions a so-called super Efimov effect appears, which is explained later.

1.3.2. On the Efimov effect for systems of one- or two-dimensional particles

In recent years quite a lot of attention has been paid to the question whether an Efimov type effect might occur for systems of one- or two-dimensional particles. Before presenting some of the known results and new predictions, let us point out a fundamental difference between systems of particles in dimension one or two and in dimension $d \geq 3$ which makes investigation of systems of one- or two-dimensional particles interesting and challenging.

A crucial difference between dimension one and two and higher dimensions

It is well known that for dimension $d \geq 3$ Hardy's inequality holds, namely

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx, \qquad u \in C_0^\infty(\mathbb{R}^d). \tag{1.3.1}$$

This means that for sufficiently fast decaying potentials V, e.g., $V \in C_0^\infty(\mathbb{R}^d)$, the potential energy is controlled by the kinetic energy. In other words, sufficiently weak interactions can not bind a particle. For dimensions one and two this is different. Hardy's inequality does not hold for these dimensions and one can show that the Laplace operator is critical, i.e., for every potential $V \in C_0^\infty(\mathbb{R}^d)$ with $V \neq 0$ and $\int_{\mathbb{R}^d} V(x) \, dx < 0$ the operator $-\Delta + \lambda V$ has a negative eigenvalue for any $\lambda > 0$, see for example [64].

Absence of the Efimov effect in dimension one and two and occurance of the super Efimov effect

It was shown by S. Vugalter and G. Zhislin by the use of variational methods that for systems of three one- or two-dimensional particles the Efimov effect does not exist [73]. This result was proved under very restrictive assumptions on the pair potentials. Later, in [76] these restrictions were relaxed, but unfortunately Lemma 1 in [76] contains a mistake. This mistake will be corrected in this work.

In 2013 the physicists Y. Nishida, S. Moroz and D.T. Son predicted the existence of a so-called super Efimov effect for systems of three two-dimensional spinless fermions [51]. More precisely, they claimed that if the two-body subsystems are resonant, then the Hamiltonian of the three-body system has infinitely many bound states. The denomination *super Efimov effect* is motivated by the observation that the eigenvalues tend to zero much faster than in the three-dimensional case. Namely, it was predicted in [51] that the eigenvalues E_n asymptotically behave as

$$E_n \sim \exp\left(-2\exp\left(\frac{3\pi n}{4} + \theta\right)\right) \quad \text{as} \quad n \to \infty \tag{1.3.2}$$

for a constant θ depending on the potentials. Thus, for the eigenvalues one has a double exponential scaling, while for the original Efimov effect it is exponential.

The first mathematical proof for the existence of this effect was given by D.K. Gridnev in 2014 [26], together with the asymptotics

$$\lim_{z \to 0} \frac{N(z)}{|\ln|\ln(z^2)||} = \frac{8}{3\pi} \tag{1.3.3}$$

for the eigenvalue counting function. This asymptotics is in accordance with the prediction (1.3.2) given in [51]. The proof of Gridnev is based on methods similar to [69] and [78], namely on an application of the Birman-Schwinger principle, the use of symmetrized Faddeev equations and a reduction of the problem to counting eigenvalues of an integral operator.

Besides the difference in the asymptotics of the eigenvalues there is another difference between dimensions two and three. Namely, the Efimov effect exists for systems of three three-dimensional particles without symmetry restrictions and is absent for a system of three fermions. This is exactly the other way round in dimension two.

The question whether there is an Efimov type effect for systems of more than three one- or two-dimensional particles was open and is an important part of this thesis.

Recent predictions for systems of one- or two-dimensional particles

To underline the richness of the Efimov physics we conclude this overview chapter by presenting some recent predictions which can be found in the physics literature.

It is predicted that Efimov type effects occur for several systems of N one- or two-dimensional particles under the assumption that interaction in subsystems with less than $N-1$ particles is absent, there is an effective $N-1$ particle short-range interaction and the Hamiltonians corresponding to subsystems consisting of $N-1$ particles have a resonance at zero. For example, it is expected that such an effect exists for systems of four two-dimensional [52] or five one-dimensional particles [50, 53].

Further investigations deal with the question whether a so-called confinement-induced Efimov effect can be observed. This means that the particles, or at least some of them, are confined in certain subspaces of the three-dimensional space, possibly with different dimensions. There are several systems for which a confinement-induced Efimov effect is expected, see for example [53] or the review [47] for a list of possible systems.

To our knowledge, from a mathematical point of view these questions are still open.

1.4. Discussion of the main results of this thesis

This thesis is based on the following two articles.

1. S. Barth, A. Bitter and S. Vugalter, *On the Efimov effect in systems of one- or two-dimensional particles*, submitted to Journal of Mathematical Physics (2020)

2. S. Barth, A. Bitter and S. Vugalter, *Decay properties of zero-energy resonances of multi-particle Schrödinger operators and why the Efimov effect does not exist for systems of $N \geq 4$ particles*, submitted to Reviews in Mathematical Physics (2020)

The main part of the thesis is based on the first of these articles and only a small part of the second article is a constituent of this thesis. Our main goal is to investigate the existence and non-existence of the Efimov effect for systems consisting of $N \geq 3$ one- or two-dimensional particles with short-range pair interactions.

1.4.1. Results on virtual levels of Schrödinger operators

As mentioned before, the investigation of the existence of the Efimov effect requires a detailed study of the virtual levels of the Schrödinger operators corresponding to the subsystems. We say that a Schrödinger operator $H = -\Delta + V$ has a virtual level at zero if $H \geq 0$ and for any sufficiently small $\varepsilon > 0$

$$\sigma_{\text{ess}}(H + \varepsilon\Delta) = [0, \infty) \qquad \text{and} \quad \inf \sigma(H + \varepsilon\Delta) < 0. \tag{1.4.1}$$

As a warm-up we start by the study of virtual levels of one-particle Schrödinger operators in dimension one or two with a short-range potential and prove that there exists a weak solution φ_0 of the equation $H\varphi_0 = 0$ which is not in $L^2(\mathbb{R}^d)$, i.e., zero is a resonance of H. We also prove that for $d \in \{1, 2\}$ and operators $H \geq 0$ the condition $\inf \sigma(H + \varepsilon\Delta) < 0$ for any $\varepsilon > 0$ is equivalent to

$$\inf \sigma\left(H - \varepsilon(1 + |x|)^{-2}\right) < 0 \tag{1.4.2}$$

for any $\varepsilon > 0$. This equivalence is obvious for dimensions $d \geq 3$ but not for $d = 1$ or $d = 2$ and is important in some of the proofs of the multi-particle results.

We extend our studies to the case of multi-particle Schrödinger operators corresponding to a system of $N \geq 3$ one- or two-dimensional particles with masses $m_i > 0$. These operators are considered in the center of mass frame, i.e., acting on $L^2(X_0)$, where

$$X_0 = \left\{ x = (x_1, \dots x_N) \in \mathbb{R}^{dN} : \sum_{i=1}^{N} m_i x_i = 0 \right\}. \tag{1.4.3}$$

Our main result concerning virtual levels is a sufficient condition in terms of a Hardy type constant, such that a virtual level of a system of N d-dimensional particles corresponds to a simple eigenvalue. Namely, let

$$\mathcal{M} = \left\{ \psi \in C_0^1(X_0 \setminus \{0\}) : \psi(x) = 0 \text{ for } x_i = x_j, 1 \leq i, j \leq N, \ i \neq j \right\} \tag{1.4.4}$$

and

$$\tilde{C}_H(X_0) = \inf_{0 \neq \psi \in \mathcal{M}} \frac{\|\nabla \psi\|}{\| |x|^{-1} \psi \|}. \tag{1.4.5}$$

We prove that virtual levels correspond to eigenvalues if $\tilde{C}_H(X_0) > 1$ and we give estimates for the decay rates of the corresponding eigenfunctions in dependence of the constant $\tilde{C}_H(X_0)$.

As an application of this result we show that the condition $\tilde{C}_H(X_0) > 1$ is fulfilled for systems of $N \geq 3$ one-dimensional or $N \geq 4$ two-dimensional particles and for systems of $N \geq 3$ one- or two-dimensional fermions. Hence, for such systems virtual levels are eigenvalues.

It is remarkable that concerning the decay rate of the eigenfunctions corresponding to a virtual level the case of one-dimensional particles is different from that of higher dimensions. For particles with spatial dimension $d \geq 3$ the decay rate depends on the Hardy constant $C_H = \frac{d(N-1)-2}{2}$, which coincides with the constant $\tilde{C}_H(X_0)$ if the dimension of the particles is $d \geq 2$. Since for one-dimensional particles the sets $\{x \in X_0 : x_i = x_j \text{ for some } i \neq j\}$ have co-dimension one in X_0, we have $\tilde{C}_H(X_0) > C_H$ in this case. This leads to a higher decay rate. We discuss the case of one-dimensional particles separately.

The case of four two-dimensional particles is a special case which is not covered by the above given criterion. Here we do not know whether a virtual level corresponds to an eigenvalue. However, we show that there exists a weak solution φ_0 of the equation $H\varphi_0 = 0$ which is possibly not square integrable.

1.4.2. Results on the absence of the Efimov effect

The results obtained for virtual levels of Schrödinger operators are applied to prove the absence of the Efimov effect for systems of $N \geq 4$ one-dimensional or $N \geq 5$ two-dimensional particles. We also prove that for systems of $N \geq 4$ one- or two-dimensional fermions the Efimov effect does not occur. Finally, we prove the non-existence of the effect for systems of three one- or two-dimensional particles interacting via short-range potentials. The only case of multi-particle systems with short-range pair interactions for which the question of existence or non-existence of the Efimov effect remains open is the case of four two-dimensional particles. With these results we can add some examples to the list of systems for which the Efimov effect does not exist, see Table 1.1.

Number and dimension of particles	No symmetry restrictions	Fermions
$d = 3,\ N = 3$	✓ [78]	✗ [74]
$d \geq 3,\ N \geq 4$	✗ [25, 7]	✗ [7]
$d \geq 4,\ N \geq 3$	✗ [5, 7]	✗ [5, 7]
$d = 1,\ N = 3$	✗ [73, 8]	✗ [8]
$d = 1,\ N \geq 4$	✗ [8]	✗ [7]
$d = 2,\ N = 3$	✗ [73, 8]	✓ [26]
$d = 2,\ N \geq 5$	✗ [8]	✗ [7]
$d = 2,\ N = 4$	**open**	✗[7]

Table 1.1.: Existence and non-existence of the Efimov effect for systems of N d-dimensional particles. The marked cases are considered in this work.

1.4.3. Techniques used for the main results

The main result concerning virtual levels, namely the criterion that virtual levels correspond to eigenvalues, is based on the following two steps:

(i) A generalization of the one-particle result to the case of non-decaying, but form bounded potentials, i.e., potentials V satisfying for any $\varepsilon > 0$

$$\langle |V|\psi, \psi\rangle \leq \varepsilon \|\nabla \psi\|^2 + C(\varepsilon)\|\psi\|^2, \qquad \psi \in H^1(\mathbb{R}^d). \tag{1.4.6}$$

Under this condition we show that virtual levels correspond to eigenvalues if there exist constants $\gamma_0 > 0,\ \alpha_0 > 1$ such that for any $\psi \in H^1(\mathbb{R}^d)$ supported away from the origin we have

$$(1-\gamma_0)\|\nabla\psi\|^2 - \alpha_0\||x|^{-1}\psi\|^2 \geq 0. \tag{1.4.7}$$

(ii) Use of geometric methods to show that condition (1.4.7) is fulfilled. These methods include an appropriate partition of unity of the configuration space by which we separate regions where particles are in groups which are moved apart.

This strategy is similar to the one used in [7], where virtual levels of systems of particles in spatial dimension $d \geq 3$ have been studied. We extend this technique to dimension $d \in \{1,2\}$. On this way, we have to overcome several differences and difficulties, most of which arise from the lack of Hardy's inequality in the one- and two-dimensional space. Some of them are fundamental, others are on a technical level. Let us give two examples here. First, when studying virtual levels the so-called homogenous Sobolev space $\dot{H}^1(\mathbb{R}^d)$ plays an important role. For $d \geq 3$ this space can be defined as the space of all functions $\psi \in L^1_{\text{loc}}(\mathbb{R}^d)$ with $\left|\{x \in \mathbb{R}^d : |\psi(x)| > \mu\}\right| < \infty$ for all $\mu > 0$ and $\nabla\psi \in L^2(\mathbb{R}^d)$, equipped with the norm $\|\psi\|_{\dot{H}^1} = \|\nabla\psi\|$. However, the straightforward generalization to dimension $d = 1$ or $d = 2$ does not lead to a function space and one has to find a different appropriate space. A second difficulty which we want to mention here occurs when we construct the partition of unity to separate the particles. This method requires an appropriate estimate of the resulting localization error. For particles in space dimension $d \geq 3$ one can use an estimate given in [74]. Due to the lack of Hardy's inequality in lower dimensions this estimate does not work in our case and we need to find an improved one.

The key idea for the proof of the absence of the Efimov effect is due to [74] where the finiteness of the negative spectrum was proved for systems of three three-dimensional particles if virtual levels in the two-body subsystems correspond to eigenvalues. This idea was generalized in [7] to multi-particle systems of d-dimensional particles with $d \geq 3$. We extend this strategy to the one- and two-dimensional case. Again, the absence of Hardy's inequality causes several difficulties which have to be overcome.

1.4.4. Outline of the thesis

This thesis is organized as follows.

In the second chapter we consider Hardy type inequalities. Since such inequalities play an important role in this thesis and there are fundamental differences between dimension one or two and higher dimensions, we discuss this in detail. We give counter-examples for the one- and two-dimensional case and present several inequalities which are important for us. Furthermore, we introduce homogeneous Sobolev spaces $\dot{H}^1(\mathbb{R}^d)$ for $d \geq 3$ and analogous spaces $\tilde{H}^1(\mathbb{R}^d)$ for dimensions one and two.

In the third chapter we introduce Schrödinger operators by perturbation methods. We also introduce the center of mass frame of multi-particle Schrödinger operators and provide several tools which will be used later.

In the fourth chapter we study virtual levels of Schrödinger operators. We start by considering one-particle Schrödinger operators with short-range potentials and extend the investigations to the multi-particle case. We prove that for systems of $N \geq 3$ one-dimensional or $N \geq 4$ two-dimensional particles virtual levels correspond to eigenvalues.

In the fifth chapter we prove that the Efimov effect does not occur in systems of $N = 3$ one- or two-dimensional, $N \geq 4$ one-dimensional or $N \geq 5$ two-dimensional particles.

2. Hardy type inequalities and homogeneous Sobolev spaces

In this chapter we collect several integral inequalities of the Hardy type, which are of great importance in the further course of this work. We start by presenting Hardy's inequality for the semi-axis and its extensions to dimension $d \geq 3$. We point out the differences between dimensions $d \geq 3$ and lower dimensions. We also present some inequalities which are similar to Hardy's inequality being helpful in later sections. At the end of this chapter we introduce homogeneous Sobolev spaces $\dot{H}^1(\mathbb{R}^d)$ for dimension $d \geq 3$ and similar spaces $\tilde{H}^1(\mathbb{R}^d)$ for dimensions $d = 1$ and $d = 2$.

We point out that all of the following results are known and can be found in the literature. At the same time, the list is by far incomplete. There are many generalizations of the Hardy inequality in different directions, e.g., for L^p norms, domains, pseudo-differential operators or Schrödinger operators with magnetic field. We restrict the demonstration to the cases which are relevant for this work and refer to [10, 15, 54] for further generalizations and discussions. We also recommend to the interested reader the beautiful article [42] which gives an overview over the history of the development of Hardy's inequality.

2.1. Hardy type inequalities

2.1.1. Hardy's inequality for the semi axis

We start by presenting some inequalities for functions on the semi axis $\mathbb{R}_+ = (0, \infty)$, one of which is Hardy's inequality for the semi axis. Later, these inequalities will be used to derive Hardy's inequality for higher dimensions.

Theorem 2.1.1 (Hardy's inequality for the semi axis)**.**

(i) *For any function* $u \in H_0^1(\mathbb{R}_+)$ *we have*

$$\int_0^\infty t^{-2}|u(t)|^2\,\mathrm{d}t \le 4\int_0^\infty |u'(t)|^2\,\mathrm{d}t. \tag{2.1.1}$$

(ii) *Let* $\alpha > -1$. *Then for any function* $u \in H^1(\mathbb{R}_+)$ *with* $u' \in L^2(\mathbb{R}_+, t^{\alpha+2}\,\mathrm{d}t)$

$$\int_0^\infty t^\alpha |u(t)|^2\,\mathrm{d}t \le \frac{4}{(\alpha+1)^2}\int_0^\infty t^{\alpha+2}\left|u'(t)\right|^2\mathrm{d}t. \tag{2.1.2}$$

Remark 2.1.2. **(i)** Inequality (2.1.1) is often reffered to as Hardy's original inequality on the half line (actually Hardy proved an integral version of this inequality, see [31]).

(ii) The constants 4 and $\frac{4}{(\alpha+1)^2}$ in inequalities (2.1.1) and (2.1.2) are sharp. There are no minimizers of the inequalities, i.e., the inequalities are strict unless $u = 0$.

(iii) The condition $\lim_{t\to 0} u(t) = 0$ is necessary for (2.1.1) to hold, see e.g., [12]. Indeed, for $n \in \mathbb{N}$ let the function $u_n : \mathbb{R}_+ \to \mathbb{R}$ be given by

$$u_n(t) = \left(1 - \frac{t}{n}\right)_+. \tag{2.1.3}$$

Then $u_n \in H^1(\mathbb{R}_+)$ with $|u_n'(t)| = \frac{1}{n}$ for $t < n$ and $u_n'(t) = 0$ for $t > n$. Therefore, we have

$$\int_0^\infty (u_n'(t))^2\,\mathrm{d}t = \frac{1}{n} \to 0 \quad \text{as } n \to \infty. \tag{2.1.4}$$

On the other hand, $u_n(t) \to 1$ as $n \to \infty$, uniformly on every compact set. Hence, for large n inequality (2.1.1) is violated. This shows that the condition $\lim_{t\to 0} u(t) = 0$ is necessary for (2.1.1) to hold.

Let us give the simple

Proof of Theorem 2.1.1. Assume that $u \in C_0^1(\mathbb{R}_+)$. Then we can write

$$\int_0^\infty \frac{|u|^2}{t^2}\,\mathrm{d}t = -\left.\frac{u^2}{t^2}\right|_{t=0}^{t=\infty} + 2\mathrm{Re}\int_0^\infty \frac{\overline{u}u'}{t}\,\mathrm{d}t = 2\mathrm{Re}\int_0^\infty \frac{\overline{u}u'}{t}\,\mathrm{d}t. \tag{2.1.5}$$

Hence, by the Cauchy-Bunjakowski-Schwarz inequality we get

$$\int_0^\infty \frac{|u|^2}{t^2}\,\mathrm{d}t \le 2\left(\int_0^\infty \frac{|u|^2}{t^2}\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^\infty |u'|^2\,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.1.6}$$

This proves statement **(i)** for $C_0^1(\mathbb{R}_+)$-functions. Now assume that $u \in H_0^1(\mathbb{R}_+)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C_0^1(\mathbb{R}_+)$ which converges to u in $H^1(\mathbb{R}_+)$. Then, due to

$$\int_0^\infty \frac{|u_n - u_m|^2}{t^2}\,\mathrm{d}t \le 4\int_0^\infty |\nabla(u_n - u_m)|^2\,\mathrm{d}t, \quad n, m \in \mathbb{N}, \tag{2.1.7}$$

the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $L^2\left(\mathbb{R}_+, t^{-2}\mathrm{d}t\right)$ and thus converges to a function $v \in L^2\left(\mathbb{R}_+, t^{-2}\mathrm{d}t\right)$. On the other hand, we have $u_n \to u$ in $H^1(\mathbb{R}_+)$ and therefore $u(t) = v(t)$ for almost every $t \in \mathbb{R}_+$. This completes the proof of assertion **(i)**. The proof of statement **(ii)** is a simple modification.

□

2.1.2. Hardy's inequality in higher dimensions

In this section we present how Hardy's inequality for dimension $d \ge 3$ can be derived from the scalar inequalities given in Theorem 2.1.1. We also give an example which shows why Hardy's inequality does not hold for dimension $d = 2$ and give the proof of a Hardy type inequality under additional assumptions in this dimension. We start with a short parenthesis on spherical harmonics which is useful to extend Hardy type inequalities to higher dimensions and which we will come back to from time to time later in this thesis.

Spherical harmonics

We recall some basics about spherical harmonics, which can be found for example the book [66] of B. Simon. For $d \ge 2$ we denote by $-\Delta_{\mathbb{S}^{d-1}}$ the Laplace-Beltrami operator on $L^2(\mathbb{S}^{d-1})$. If we use polar coordinates $r = |x|$ and $\omega = \frac{x}{|x|}$ we will also write $-\Delta_\omega := -\Delta_{\mathbb{S}^{d-1}}$. The Laplace operator $-\Delta$ on $L^2(\mathbb{R}^d)$ can be represented in polar coordinates as

$$-\Delta u = -\left(\partial_r^2 u + \frac{d-1}{r}\partial_r u + \frac{1}{r^2}\Delta_\omega u\right) \tag{2.1.8}$$

and we have

$$|\nabla u|^2 = |\partial_r u|^2 + \frac{1}{r^2}\,|\nabla_\omega u|^2. \tag{2.1.9}$$

The operator $-\Delta_{\mathbb{S}^{d-1}}$ is non-negative, self-adjoint and has compact resolvent. Its eigenvalues are given by $\mu_l = l(l+d-2)$, $l \in \mathbb{N}_0$ with multiplicity $\nu_0 = 1$ and

$$\nu_l = \begin{cases} 2 & \text{if } d = 2, \\ \frac{(2l+d-2)(l+d-3)}{(d-2)!\,l!} & \text{if } d \geq 3 \end{cases} \tag{2.1.10}$$

for $l \geq 1$. Eigenfunctions Y_l of $-\Delta_{\mathbb{S}^{d-1}}$ corresponding to the eigenvalue μ_l are called spherical harmonics of degree l. We find an orthonormal basis $\{Y_{l,m} : l \geq 0,\, m = 1,\dots,\nu_l\}$ of $L^2(\mathbb{S}^{d-1})$ consisting of spherical harmonics $Y_{l,m}$ of degree l. Let $u \in L^2(\mathbb{R}^d)$ and let $r = |x|$ and $\omega = \frac{x}{|x|}$ be polar coordinates. Then $u(r\cdot) \in L^2(\mathbb{S}^{d-1})$ for almost every $r > 0$ and we have

$$u(x) = \sum_{l,m} u_{l,m}(r)\,Y_{l,m}(\omega), \quad \text{where} \quad u_{l,m}(r) := \int_{\mathbb{S}^{d-1}} u(r\omega)\,\overline{Y_{l,m}(\omega)}\,\mathrm{d}\omega. \tag{2.1.11}$$

Since the functions $Y_{l,m}$ are orthonormal in $L^2(\mathbb{S}^{d-1})$, we get

$$\int_{\mathbb{S}^{d-1}} |u(r\omega)|^2\,\mathrm{d}\omega = \sum_{l,m} |u_{l,m}(r)|^2. \tag{2.1.12}$$

Moreover, if $u \in H^1(\mathbb{R}^d)$, then $u(r\cdot) \in H^1(\mathbb{S}^{d-1})$ for almost every $r > 0$ and by (2.1.9) and the definition of $Y_{l,m}$ we have

$$\int_{\mathbb{S}^{d-1}} |\nabla u(r\omega)|^2\,\mathrm{d}\omega = \sum_{l,m}\left(\left|u_{l,m}'(r)\right|^2 + \frac{l(l+d-2)}{r^2}\left|u_{l,m}(r)\right|^2\right). \tag{2.1.13}$$

A different way to define spherical harmonics is via harmonic homogeneous polynomials. A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic if $\Delta u = 0$ and homogeneous if there exists $\lambda > 0$, such that $u(\rho x) = \rho^\lambda u(x)$ for any $\rho > 0$ and $x \in \mathbb{R}^d$. Let p be a harmonic homogeneous polynomial of degree $l > 0$. Then its restriction to $\mathbb{S}^{d-1}$ is a spherical harmonic of degree l.

Hardy's inequality for dimension $d \geq 3$

Now we use Theorem 2.1.1 and the facts about spherical harmonics to derive the following result, which is often referred to as the classical Hardy inequality.

Theorem 2.1.3 (Hardy's inequality for $d \geq 3$)**.** *Let $d \in \mathbb{N}$, $d \geq 3$ and $u \in H^1(\mathbb{R}^d)$. Then we have*

$$\int_{\mathbb{R}^d} |x|^{-2}|u|^2 \,\mathrm{d}x \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x. \tag{2.1.14}$$

We sketch the proof.

Proof. For the proof we use spherical coordinates $r = |x|$ and $\omega = \frac{x}{|x|}$. Then, for almost every $\omega \in \mathbb{S}^{d-1}$ the function $r \mapsto u(r\omega)$ is weakly differentiable and the derivative is in $L^2(\mathbb{R}_+, r^{d-1}\,\mathrm{d}r)$. By (2.1.9) we get

$$\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x \geq \int_{\mathbb{S}^{d-1}} \int_0^\infty r^{d-1} |\partial_r u(r\omega)|^2 \,\mathrm{d}r \,\mathrm{d}\omega. \tag{2.1.15}$$

Applying Theorem 2.1.1 for fixed $\omega \in \mathbb{S}^{d-1}$ and with $\alpha = d-3$ we get

$$\int_0^\infty r^{d-1} |\partial_r u(r\omega)|^2 \,\mathrm{d}r \geq \frac{(d-2)^2}{4} \int_0^\infty r^{d-3} |u(r\omega)|^2 \,\mathrm{d}\omega. \tag{2.1.16}$$

Integration over $\mathbb{S}^{d-1}$ completes the proof. □

Hardy's inequality for dimension two

Note that for dimension $d = 2$ the constant on the right hand side of (2.1.14) is infinite and therefore the inequality is meaningless in this case. There exists no finite constant $C > 0$, such that

$$\int_{\mathbb{R}^2} |x|^{-2}|u|^2 \,\mathrm{d}x \leq C \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \tag{2.1.17}$$

holds for all functions $u \in H^1(\mathbb{R}^2)$. This can be seen in the following example which can be found in [12].

Example 2.1.4. For $n \in \mathbb{N}$, $n \geq 2$ let the function $u_n : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$u_n(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ (\ln(n))^{-1}(\ln(n) - \ln(|x|))_+ & \text{else.} \end{cases} \tag{2.1.18}$$

Then we have $u_n \in H^1(\mathbb{R}^2)$ and

$$|\nabla u_n| = \begin{cases} 0 & \text{if } |x| < 1, \\ (\ln(n))^{-1}|x|^{-1} & \text{if } 1 < |x| < n, \\ 0 & \text{if } |x| > n. \end{cases} \tag{2.1.19}$$

Therefore, $\int_{\mathbb{R}^2} |\nabla u_n|^2 \,\mathrm{d}x \to 0$ for $n \to \infty$. At the same time, we have $u_n(x) \to 1$ for $n \to \infty$, uniformly on every compact set. This example shows that there exists no $C > 0$, such that inequality (2.1.17) holds for all functions $u \in H^1(\mathbb{R}^2)$.

By adding an additional, logarithmic so-called Hardy weight one gets the following Hardy type inequality for dimension two. It can be found in a more general form in [70].

Theorem 2.1.5 (Hardy type inequality for $d = 2$)**.** *Let $u \in H^1(\mathbb{R}^2)$ be represented in polar coordinates $r = |x|$ and $\omega = \frac{x}{|x|}$ as*

$$u(x) = \sum_{l=-\infty}^{\infty} u_l(r) \frac{\mathrm{e}^{\mathrm{i}l\omega}}{\sqrt{2\pi}}. \tag{2.1.20}$$

If $u_0(1) = \int_{\mathbb{S}^1} u(\omega)\,\mathrm{d}\omega = 0$, then the following inequality holds.

$$\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|u_0|^2}{|x|^2 \ln^2(|x|)} \,\mathrm{d}x + \int_{\mathbb{R}^2} \frac{|u - u_0|^2}{|x|^2} \,\mathrm{d}x. \tag{2.1.21}$$

Remark 2.1.6. **(i)** Note that u_0 is a radial function. We have seen that for such functions the logarithmic term in Hardy's inequality is necessary. Theorem 2.1.5 shows that for functions which are orthogonal to radial functions the logarithmic term can be omitted. These are exactly those functions which satisfy $\int_{\mathbb{S}^1} u(R\omega)\,\mathrm{d}\omega = 0$ for all $R > 0$.

(ii) For the radial part u_0 in the expansion of u the condition $u_0(1) = 0$ is necessary. This can be seen by the following example which can be found in [70]. Let $\varphi \in C^1(\mathbb{R})$ be a function satisfying $0 \le \varphi(t) \le 1$ and

$$\varphi(t) = 1 \text{ for } -1 \le t \le 1 \quad \text{and} \quad \varphi(t) = 0 \text{ for } |t| > 2. \tag{2.1.22}$$

Furthermore, for $n \in \mathbb{N}$ let

$$u_n(x) = \varphi\left(\frac{\ln(|x|)}{n}\right). \tag{2.1.23}$$

Then we have

$$\int_{\mathbb{R}^2} \frac{|u_n|^2}{|x|^2\left(1+\ln^2(|x|)\right)}\,\mathrm{d}x = n\int_{-\infty}^{\infty} \frac{\varphi^2(t)}{1+n^2t^2}\,\mathrm{d}t. \tag{2.1.24}$$

On the other hand,

$$\int_{\mathbb{R}^2} |\nabla u_n|^2\,\mathrm{d}x = \frac{1}{n}\int_{-\infty}^{\infty} \left|\varphi'(t)\right|^2\,\mathrm{d}t. \tag{2.1.25}$$

By sending n to infinity we see that (2.1.21) does not hold for all of the functions u_n. Therefore, we find that the condition $u(1) = 0$ is necessary for radial functions.

(iii) Let u be given as in Theorem 2.1.5. Then, as an immediate consequence of Theorem 2.1.5 we have

$$\int |\nabla u|^2\,\mathrm{d}x \ge \frac{1}{4}\int \frac{|u|^2}{|x|^2(1+\ln^2(|x|))}\,\mathrm{d}x. \tag{2.1.26}$$

The proof of Theorem 2.1.5 is based on the following integral inequality for functions $u : \mathbb{R}_+ \to \mathbb{C}$.

Proposition 2.1.7. *For any function $u \in H^1(\mathbb{R}_+)$ with $u' \in L^2(\mathbb{R}_+, t\,\mathrm{d}t)$ and $u(1) = 0$*

$$\int_0^{\infty} t^{-1}(\ln(t))^{-2}|u(t)|^2\,\mathrm{d}t \le 4\int_0^{\infty} t|u'(t)|^2\,\mathrm{d}t. \tag{2.1.27}$$

Proof. Let u be as in the proposition and $v(t) = u(e^t),\ t \in \mathbb{R}_+$. Then $v \in H^1_0(\mathbb{R}_+)$ with $v'(t) = u'(e^t)e^t$. Therefore,

$$\int_0^\infty |v'(t)|^2 \,dt = \int_0^\infty |u'(e^t)|^2 e^{2t} \,dt = \int_1^\infty |u'(s)|^2 \,s ds. \tag{2.1.28}$$

Hence, by the one-dimensional Hardy inequality (2.1.1) we get

$$\begin{aligned}\int_1^\infty |u'(s)|^2 \,s ds &= \int_0^\infty |v'(t)|^2 \,dt \\ &\geq \frac{1}{4}\int_0^\infty |v(t)|^2 t^{-2} \,dt = \frac{1}{4}\int_1^\infty |u(s)|^2 s^{-1} (\ln s)^{-2} \,ds.\end{aligned} \tag{2.1.29}$$

If we define $v(t) = u(e^{-t}),\ t \in \mathbb{R}_+$, we get analogously

$$\int_0^1 |u'(s)|^2 \,s ds \geq \frac{1}{4}\int_0^1 |u(s)|^2 s^{-1} (\ln s)^{-2} \,ds. \tag{2.1.30}$$

This completes the proof. □

Now we turn to the

Proof of Theorem 2.1.5. Due to $u_0(1) = 0$, Proposition 2.1.7 can be applied to the function u_0 and yields

$$\int_0^\infty |u_0'(r)|^2 \,r dr \geq \frac{1}{4}\int_0^\infty r^{-1} (\ln(r))^{-2} |u_0(r)|^2 \,dr. \tag{2.1.31}$$

Therefore, by the use of (2.1.12) and (2.1.13) we get

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u|^2 \,dx &= \int_0^\infty \sum_{l=-\infty}^{\infty} \left(|u_l'(r)|^2 + \frac{l^2}{r^2}|u_l(r)|^2 \right) r dr \\ &\geq \frac{1}{4}\int_0^\infty r^{-2} (\ln(r))^{-2} |u_0(r)|^2 \,r dr + \int_0^\infty \sum_{l\neq 0} \frac{1}{r^2}|u_l(r)|^2 \,r dr \\ &= \frac{1}{4}\int_{\mathbb{R}^2} \frac{|u_0|^2}{|x|^2 \ln^2(|x|)} \,dx + \int_{\mathbb{R}^2} \frac{|u-u_0|^2}{|x|^2} \,dx.\end{aligned} \tag{2.1.32}$$

This completes the proof of Theorem 2.1.5. □

2.1.3. Further Hardy type inequalities

In the following we collect some further inequalities which will be used later in this thesis.

Hardy type inequality for functions being orthogonal to radial functions

The following inequality can be found for example in [17].

Lemma 2.1.8. *Let $d \geq 2$. Then for all functions $u \in H^1(\mathbb{R}^d)$ satisfying the condition $\int_{\mathbb{S}^{d-1}} u(r\omega)\,\mathrm{d}\omega = 0$ for all $r \geq 0$ we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leq \frac{4}{d^2}\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x. \tag{2.1.33}$$

Proof. Let $u \in H^1(\mathbb{R}^d)$. By substituting $u = |x|^{\frac{2-d}{2}} v$ and using polar coordinates we get

$$\begin{aligned}\int_{\mathbb{R}^d}\left(|\nabla u|^2 - \frac{(d-2)^2}{4}\frac{|u|^2}{|x|^2}\right)\mathrm{d}x &= \int_{\mathbb{R}^d} |\nabla v|^2 |x|^{2-d}\,\mathrm{d}x \\ &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(\left|\frac{\partial v}{\partial r}\right|^2 + \frac{|\nabla_\omega v|^2}{r^2}\right)\mathrm{d}\omega\, r\,\mathrm{d}r \\ &\geq \int_0^\infty \int_{\mathbb{S}^{d-1}} r^{-1} |\nabla_\omega v|^2\,\mathrm{d}\omega\,\mathrm{d}r.\end{aligned} \tag{2.1.34}$$

By assumption, for almost every $r > 0$ the function $v(r\cdot)$ is orthogonal to the eigenfunction corresponding to the first eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. Since the second eigenvalue is given by $d-1$, we have

$$\int_{\mathbb{S}^{d-1}} |\nabla_\omega v(r\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}r \geq (d-1)\int_{\mathbb{S}^{d-1}} |v(r\omega)|^2\,\mathrm{d}\omega \tag{2.1.35}$$

and therefore

$$\begin{aligned}\int_0^\infty \int_{\mathbb{S}^{d-1}} r^{-1} |\nabla_\omega v(r\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}r &\geq (d-1)\int_{\mathbb{R}^d} |x|^{-d} |v|^2\,\mathrm{d}x \\ &= (d-1)\int_{\mathbb{R}^d} |x|^{-2}|u|^2\,\mathrm{d}x.\end{aligned} \tag{2.1.36}$$

This, together with (2.1.34) and $\frac{(d-2)^2}{4} + d - 1 = \frac{d^2}{4}$ completes the proof. □

Hardy's inequality for sectors

The following Hardy type inequality can be found in [48, Proposition 4.1].

Lemma 2.1.9. *Let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a cone with vertex at the origin, i.e.,*

$$\Omega = \{(r, \omega) : r \geq 0,\ \omega \in \Sigma\}, \tag{2.1.37}$$

where $\Sigma \subset \mathbb{S}^{d-1}$ is a Lipschitz domain. Then for any function $u \in H_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \geq \left(\left(\frac{d-2}{2}\right)^2 + \lambda_0(\Sigma)\right) \int_\Omega |x|^{-2}|u|^2 \,\mathrm{d}x, \tag{2.1.38}$$

where $\lambda_0(\Sigma)$ is the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on Σ. Moreover, the constant in (2.1.38) is sharp.

Proof. We use spherical coordinates $r = |x|$ and $\omega = \frac{x}{|x|}$ and expand the function u as

$$u(x) = \sum_{k=1}^{\infty} u_k(r)\varphi_k(\omega), \tag{2.1.39}$$

where $\{\varphi_k : k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Sigma)$ consisting of eigenfunctions of the Laplace-Beltrami operator on Σ. Then, similar to (2.1.13) we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x &= \int_0^\infty \sum_{k=1}^\infty \left(|u_k'(r)|^2 + \frac{\lambda_k(\Sigma)}{r^2}|u_k(r)|^2\right) r^{d-1}\mathrm{d}r \\ &\geq \int_0^\infty \sum_{k=1}^\infty \left(|u_k'(r)|^2 + \frac{\lambda_0(\Sigma)}{r^2}|u_k(r)|^2\right) r^{d-1}\mathrm{d}r,\end{aligned} \tag{2.1.40}$$

where $\lambda_k(\Sigma)$ is the k-th eigenvalue of the Dirichlet Laplacian on Σ. For $d \geq 3$ we apply Theorem 2.1.1 **(ii)** with $\alpha = d-3$ to the function u_k. This yields

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \geq \int_0^\infty \sum_{k=1}^\infty \left(\frac{(d-2)^2}{4r^2}|u_k(r)|^2 + \frac{\lambda_0(\Sigma)}{r^2}|u_k(r)|^2\right) r^{d-1}\mathrm{d}r. \tag{2.1.41}$$

Using

$$\int_0^\infty \sum_{k=1}^\infty r^{-2}|u_k(r)|^2\, r^{d-1}\mathrm{d}r = \int_\Omega |x|^{-2}|u(x)|^2 \,\mathrm{d}x \tag{2.1.42}$$

completes the proof. □

Hardy's inequality for exterior domains

Now we give a Hardy type inequality for integrals over sets $\{x : |x| \geq v\}$ for some $v > 0$.

Lemma 2.1.10. *The following assertions hold.*

(i) *Let* $d = 2$ *and* $u \in H^1(\mathbb{R}^2)$ *with* $\operatorname{supp}(u) \subset \{x : |x| < 1\}$ *or* $\operatorname{supp}(u) \subset \{x : |x| > 1\}$. *Then, for any constant* $v > 0$ *we have*

$$\int_{\{|x| \geq v\}} \frac{|u|^2}{|x|^2(1+\ln^2(|x|))} \, \mathrm{d}x \leq 4 \int_{\{|x| \geq v\}} |\nabla u|^2 \, \mathrm{d}x. \tag{2.1.43}$$

(ii) *Let* $d \geq 3$. *Then for any function* $u \in H^1(\mathbb{R}^d)$ *and* $v \geq 0$ *we have*

$$\int_{\{|x| \geq v\}} \frac{|u|^2}{|x|^2} \, \mathrm{d}x \leq \frac{4}{(d-2)^2} \int_{\{|x| \geq v\}} |\nabla u|^2 \, \mathrm{d}x. \tag{2.1.44}$$

Proof of Lemma 2.1.10. Since the function $v(x) = |x|^{-2}\left(1 + (\ln(|x|))^2\right)^{-1}$ is radial, it suffices to show that the inequality holds for radial functions. Let $u \in H^1(\mathbb{R}^2)$ be radial with $\operatorname{supp}(u) \subset \{x : |x| < 1\}$. Then the function $\tilde{u}$, given by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } |x| \geq v, \\ u(v) & \text{if } |x| < v, \end{cases} \tag{2.1.45}$$

is also an element of $H^1(\mathbb{R}^2)$. Applying the two-dimensional Hardy inequality (2.1.26) to the function $\tilde{u}$, using that $\tilde{u}$ is constant for $|x| \leq v$ and that $\tilde{u}$ and u coincide for $|x| \geq v$ completes the proof of statement **(i)**. The proof of statement **(ii)** is similar. □

Poincaré-Friedrichs's inequality

The following inequality can be found for example in [1, Theorem 6.30].

Theorem 2.1.11 (Poincaré-Friedrichs's inequality)**.** *Let* $\Omega \subseteq \mathbb{R}^d$ *be a bounded domain with diameter* $D > 0$. *Then for any function* $u \in H_0^1(\Omega)$ *we have*

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \geq \frac{2}{D^2} \int_\Omega |u|^2 \, \mathrm{d}x. \tag{2.1.46}$$

Proof. Without loss of generality we assume that Ω lies between the lines $x_d = 0$ and $x_d = D$. Then, we have

$$u(x) = \int_0^{x_d} \frac{\mathrm{d}}{\mathrm{d}t} u(x', t)\,\mathrm{d}t, \tag{2.1.47}$$

where $x = (x', x_d)$ with $x' = (x_1, \ldots, x_d)$. Therefore, by the use of Hölder's inequality we get

$$\begin{aligned} \int_\Omega |u|^2\,\mathrm{d}x &= \int_{\mathbb{R}^d} \mathrm{d}x' \int_0^D |u(x', x_d)|^2\,\mathrm{d}x_d \\ &\le \int_{\mathbb{R}^d} \mathrm{d}x' \int_0^D x_d\,\mathrm{d}x_d \int_0^D |\nabla u(x', t)|^2\,\mathrm{d}t \\ &\le \frac{D^2}{2} \int_{\mathbb{R}^d} \mathrm{d}x' \int_0^D |\nabla u(x', t)|^2\,\mathrm{d}t = \frac{D^2}{2} \int_\Omega |\nabla u|^2\,\mathrm{d}x. \end{aligned} \tag{2.1.48}$$

This completes the proof. □

2.2. Homogeneous Sobolev spaces

Now we introduce function spaces which come into play when we deal with virtual levels of Schrödinger operators. We refer to [20] for the proofs of the statements and a more detailed discussion.

For dimensions $d \ge 3$ the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d)$ is defined as the space of all functions $u : \mathbb{R}^d \to \mathbb{C}$ satisfying the conditions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\nabla u \in L^2(\mathbb{R}^d)$ and $\mathrm{meas}(\{\tau : |u(x)| > \tau\}) < \infty$ for all $\tau > 0$, equipped with the norm

$$\|u\|_{\dot{H}^1} = \left(\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.2.1}$$

One can show that for functions in $\dot{H}^1(\mathbb{R}^d)$, $d \ge 3$, Hardy's inequality (2.1.14) holds. This can be used to prove that $\dot{H}^1(\mathbb{R}^d)$ is a Hilbert space and that $C_0^\infty(\mathbb{R}^d)$ is dense in $\dot{H}^1(\mathbb{R}^d)$. Note that functions $u \in \dot{H}^1(\mathbb{R}^d)$ do not have to be elements of $L^2(\mathbb{R}^d)$.

If we try to extend the definition of $\dot{H}^1(\mathbb{R}^d)$ to dimensions $d = 1$ and $d = 2$, we face problems which are caused by the lack of Hardy's inequality in these dimensions. To point this out, we note that for $d \ge 3$ it follows immediately from Hardy's inequality that a sequence $(u_n)_{n\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^d)$ functions with $\nabla u_n \to 0$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$

converges to zero in $L^2(\mathbb{R}^d, |x|^{-2}\,\mathrm{d}x)$ and in particular in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. However, for dimensions one and two we have seen examples of sequences $(u_n)_{n\in\mathbb{N}}$ with

$$\nabla u_n \to 0 \text{ in } L^2(\mathbb{R}^d) \quad \text{and} \quad u_n \to 1 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^d), \tag{2.2.2}$$

see Remark 2.1.2 and Example 2.1.4. This shows that the form $\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x$ degenerates on constant functions and that the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to this form does not lead to a function space. To get rid of this degeneration we add a local L^2 norm to $\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x$. Precisely, we define

$$\|u\|_{\tilde{H}^1} = \left(\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x + \int_{\{|x|\le 1\}} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \tag{2.2.3}$$

and the space $\tilde{H}^1(\mathbb{R}^d)$ as

$$\tilde{H}^1(\mathbb{R}^d) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\ \nabla u \in L^2(\mathbb{R}^d)\right\} \tag{2.2.4}$$

equipped with the norm (2.2.3). Then $\tilde{H}^1(\mathbb{R}^d)$ is a function space. Moreover, the one- and two-dimensional Hardy inequalities (2.1.1) and (2.1.21) can be extended to functions in $\tilde{H}^1(\mathbb{R}^d)$. This can be used to prove the following inequalities.

Proposition 2.2.1. *Let $d \in \{1,2\}$. There exists a constant $C > 0$ such that for all functions $u \in \tilde{H}^1(\mathbb{R}^d)$*

$$\int_0^\infty \frac{|u|^2}{1+x^2}\,\mathrm{d}x \le C\|u\|^2_{\tilde{H}^1} \qquad \text{if } d = 1, \tag{2.2.5}$$

$$\int_{\mathbb{R}^2} \frac{|u|^2}{1+|x|^2\ln^2(|x|)}\,\mathrm{d}x \le C\|u\|^2_{\tilde{H}^1} \qquad \text{if } d = 2. \tag{2.2.6}$$

This can be used to show the following important result.

Proposition 2.2.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\tilde{H}^1(\mathbb{R}^d)$ which converges weakly in $\tilde{H}^1(\mathbb{R}^d)$ to a function $u \in \tilde{H}^1(\mathbb{R}^d)$. Then for any bounded measurable set $A \subset \mathbb{R}^d$ we have $\chi_A u_n \to \chi_A u$ in $L^2(\mathbb{R}^d)$.*

Another important consequence of Proposition 2.2.1 is the following

Proposition 2.2.3. *$\tilde{H}^1(\mathbb{R}^d)$ is a Hilbert space and $C_0^\infty(\mathbb{R}^d)$ is dense in $\tilde{H}^1(\mathbb{R}^d)$.*

3. Multi-particle Schrödinger operators

3.1. Schrödinger operators

In this section we introduce the Schrödinger operator

$$H = -\Delta + V \tag{3.1.1}$$

as a self-adjoint operator on $L^2(\mathbb{R}^d)$, where $d \in \mathbb{N}$, by the use of perturbation methods. Here, Δ is the Laplace operator and V is the operator of multiplication with a real-valued potential, also denoted by V. Perturbation theory for Schrödinger operators has been intensively studied and two common ways to define the Schrödinger operator with such methods have emerged: The definition as a sum of operators or as the sum of closed quadratic forms. Later we will mainly use the quadratic form approach, but we present them both.

3.1.1. Schrödinger operators defined as the sum of operators

This way goes back to the famous work of T. Kato [38] using the Kato-Rellich theorem. We sketch this method here.

Definition 3.1.1 (Relative operator boundedness, see [67], p.528)**.** Let A, B be densely defined operators on a Hilbert space $\mathscr{H}$. We say that B is A-bounded if

(i) $D(A) \subseteq D(B)$,

(ii) there exist constants $a, b > 0$, such that for any $u \in D(A)$ we have

$$\|Bu\| \leq a\|Au\| + b\|u\|. \tag{3.1.2}$$

For an operator B which is A-bounded we call the infimum over all a for which there exists a $b > 0$ such that condition (3.1.2) holds, the A-bound for B. If this is zero, we say B is infinitesimally A-bounded.

Theorem 3.1.2 (Kato-Rellich-Theorem, cf. Theorem 7.1.14 in [67])**.** *Let A be a self-adjoint operator and suppose that B is a symmetric operator which is relatively A-bounded with A-bound $a < 1$. Then the operator $A + B$ defined on $D(A)$ is self-adjoint. If A is bounded below, so is $A + B$.*

For the concrete case of Schrödinger operators, in which we are interested here, we give the following

Example 3.1.3 ($-\Delta$-bounded potentials)**.** Let $A = -\Delta$ on $L^2(\mathbb{R}^d)$. We collect some conditions for a potential V to be $-\Delta$-bounded with relative bound less than one. In these cases the Kato-Rellich Theorem applies and $H = -\Delta + V$ is self-adjoint on $L^2(\mathbb{R}^d)$.

(i) Assume that $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with

$$\begin{aligned} p &\geq 2 && \text{if } d = 1,2,3, \\ p &> 2 && \text{if } d = 4, \\ p &\geq \frac{d}{2} && \text{if } d \geq 5. \end{aligned} \tag{3.1.3}$$

Then V is $-\Delta$-bounded with relative bound zero, see for example [67, Theorem 7.1.18.]

(ii) In terms of $L^p_{\text{loc}}(\mathbb{R}^d)$ requirements the conditions (3.1.3) are necessary. Other conditions have been formulated in terms of convolutions, e.g., the so-called Stummel conditions, see [67] or [68] for details.

3.1.2. Schrödinger operators introduced via quadratic forms

From the physical definition of observables it is sufficient to define the sum $C = -\Delta + V$ in an expectation value sense, i.e., $\langle u, Cu\rangle = \langle u, -\Delta u\rangle + \langle u, Vu\rangle$. The following approach was developed by B. Simon [63], defining the Schrödinger operator via quadratic forms by the use of the KLMN theorem. It turns out that the sum of the quadratic form is associated with a self-adjoint operator for a larger class of potentials than given by (3.1.2). For variational techniques used in this work the approach via quadratic forms is the more natural one.

Let us brievly repeat some basic facts of quadratic forms and the connection with self-adjoint operators. It is well known that there is a one-to-one correspondence between closed semi-bounded forms and self-adjoint semi-bounded operators. Following [11], we repeat some fundamental facts.

Definition 3.1.4 (Quadratic forms associated with self-adjoint operators)**.** Let $\mathscr{H}$ be a Hilbert space. We say that a closed form $Q : d[Q] \times d[Q] \to \mathbb{R}$ is associated with an operator A if

(i) $D(A) \subseteq d[Q]$,

(ii) $(Au, v) = Q[u, v] \quad u \in D(A),\ v \in d[Q]$.

Theorem 3.1.5. *Let $\mathscr{H}$ be a Hilbert space.*

(i) *Assume that $A = A^*$ is a self-adjoint bounded below operator. Then there exists a unique closed, bounded below quadratic form Q which is associated with A.*

(ii) *Let Q be a closed and bounded below quadratic form. Then there exists a unique self-adjoint bounded below operator $A = A^*$, such that Q is associated with A. Its domain is given by*

$$D(A) = \{x \in d[Q] : \exists h \in \mathscr{H}\, Q[u, v] = (h, v) \textit{ for all } v \in d[Q]\}. \tag{3.1.4}$$

Now we introduce the concept of relative form-boundedness.

Definition 3.1.6 (Relative form boundedness, cf. [67], p. 578)**.** Let $Q : d[Q] \times d[Q] \to \mathbb{R}$ be a quadratic form and P a sesquilinear form on $d[Q]$. We say that P is form bounded with respect to Q if there exist constants $a, b > 0$ with

$$|P[u,u]| \leq aQ[u] + b\|u\|^2, \qquad u \in d[Q]. \tag{3.1.5}$$

Given a form P which is form bounded with respect to Q, we call the infimum over all $a > 0$ for which there is a $b > 0$ such that (3.1.5) holds, the relative form bound.

The following theorem is known as the KLMN theorem (named after Kato [39], Lions [44], Lax-Milgram [43] and Nelson [49]).

Theorem 3.1.7 (KLMN theorem, see [63], Theorem 2)**.** *Let $A \geq 0$ be a self-adjoint operator associated with a quadratic form Q and assume that P is a symmetric bilinear form which is form bounded with respect to Q with relative form bound $a < 1$. Then the quadratic form $u \mapsto \langle u, Au\rangle + P[u]$, $u \in d[Q]$, is associated with a self-adjoint bounded below operator C.*

This theorem can be used to introduce Schrödinger operators via quadratic forms. Let the quadratic form Q, given by

$$d[Q] = H^1(\mathbb{R}^d), \qquad Q[u] = \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x,\ u \in d[Q], \tag{3.1.6}$$

be associated with the Laplacian on $L^2(\mathbb{R}^d)$. Assume that V is a real-valued potential, such that the form

$$P[\varphi,\psi] = \int_{\mathbb{R}^d} V\varphi\overline{\psi}\,\mathrm{d}x \tag{3.1.7}$$

is form bounded with respect to Q with bound less than one. In this case we sometimes simply say that V is form bounded with respect to $-\Delta$. Then, by Theorem 3.1.7 there exists a self-adjoint operator H associated with the form $u \mapsto Q[u] + P[u,u]$. Its domain is given by

$$D(H) = \{u \in H^1(\mathbb{R}^d) : -\Delta u + Vu \in L^2(\mathbb{R}^d)\} \tag{3.1.8}$$

and for $u \in D(H)$ we have

$$Hu = -\Delta u + Vu \tag{3.1.9}$$

in the distributional sense [63]. In the following we denote by $H = -\Delta + V$ the operator given by (3.1.9), where we note that this is not necessarily the sum of operators and therefore an abuse of notation. If not stated otherwise, we will consider the Schrödinger operator in the sense of quadratic forms.

Remark 3.1.8. **(i)** The operator $-\Delta + V$ defined as quadratic form is an extension of the operator defined in the sense of an operator sum. Its domain may be larger than that of the operator sum.

(ii) Obviously, $-\Delta u + Vu \in L^2(\mathbb{R}^d)$ is fulfilled if both summands are in $L^2(\mathbb{R}^d)$, but also if there are cancellations.

(iii) For dimensions $d \in \{1,2,3\}$ the KLMN theorem, compared with the Kato-Rellich theorem, allows stronger local singularities on the potential V, cf. [32]. While the Kato-Rellich theorem requires essentially $V \in L^2_{\text{loc}}(\mathbb{R}^d)$, a potential V is form bounded with respect to $-\Delta$ with bound less than one if $V \in L^p_{\text{loc}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for all p with

$$p \geq 1 \text{ if } d = 1, \quad p > 1 \text{ if } d = 2, \quad \text{and} \quad p \geq \frac{3}{2} \text{ if } d = 3, \tag{3.1.10}$$

see for example [13].

We complete this abstract part with a remark which can be found in a similar form in [35, Theorem 2.2] and which will be useful for introducing multi-particle Schrödinger operators later.

Remark 3.1.9. **(i)** The set of potentials V which are relatively $-\Delta$-bounded on $\mathbb{R}^d$ in the sense of Definition 3.1.1 (respectively Definition 3.1.6) is a real vector space.

(ii) Let $\mathbb{R}^d = X_1 \bigoplus X_2$ be an orthogonal decomposition of $\mathbb{R}^d$ with corresponding coordinates $x = x_1 + x_2$. Assume that $V : \mathbb{R}^d \to \mathbb{R}$ depends only on x_1. Then V is relatively $-\Delta$-bounded on $\mathbb{R}^d$ in the sense of Definition 3.1.1 (respectively Definition 3.1.6) if and only if it is relatively $-\Delta$-bounded on X_1 in the sense of Definition 3.1.1 (respectively Definition 3.1.6).

3.1.3. Eigenvalues and resonances

Later we will deal with the question whether there is an eigenvalue at the edge of the spectrum. The following abstract theorem gives a criterion for this.

Theorem 3.1.10 (Minimizers are eigenfunctions, see Theorem 10.2.2 in [11])**.** *Assume that A is a self-adjoint, bounded below operator, associated with the quadratic form Q. If there exists a minimizer of Q, i.e., a function $u_0 \in d[Q]$ with $\|u_0\| = 1$ and*

$$Q[u_0] = \inf_{u \in d[Q], \|u\|=1} Q[u], \tag{3.1.11}$$

then u_0 is an eigenfunction of A corresponding to the ground state eigenvalue.

Now let $H = -\Delta + V$ be associated with the quadratic form Q with form domain $H^1(\mathbb{R}^d)$,

$$Q[u] = \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d} V|u|^2 \, \mathrm{d}x. \tag{3.1.12}$$

If the infimum of $Q[u]$ over all $u \in H^1(\mathbb{R}^d)$ is attained, then by Theorem 3.1.10 it is the ground state eigenvalue of the operator H. If the infimum is not attained by a function in $H^1(\mathbb{R}^d)$, but the potential V is chosen in such a way that the form Q can be defined for functions in $\dot{H}^1(\mathbb{R}^d)$, respectively $\tilde{H}^1(\mathbb{R}^d)$, and there exists a minimizer of $Q[u]$ in this space, then we say that H has a ground state resonance.

Another Hardy type inequality

Now we give an estimate for potentials which are relatively form bounded and decay at infinity. This estimate is a simple application of the Hardy type inequality given in Proposition 2.2.1 and will play an important role later in this thesis.

Lemma 3.1.11. *Assume that V is relatively form bounded and satisfies*

$$|V(x)| \leq C(1+|x|)^{-2-\nu}, \quad |x| \geq A \tag{3.1.13}$$

for some $\nu, A, C > 0$. Then there exists a constant $\tilde{C} > 0$, such that for all $u \in \tilde{H}^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |V(x)||u(x)|^2 \, \mathrm{d}x \leq \tilde{C} \|u\|^2_{\tilde{H}^1}. \tag{3.1.14}$$

3.2. Multi-particle Schrödinger operators

Now we consider a system of $N \geq 2$ one- or two-dimensional quantum particles with masses $m_i > 0$ and position vectors $x_i \in \mathbb{R}^d$, $i = 1, \dots, N$ with $d \in \{1, 2\}$. Such a system is described by the Hamiltonian

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} + \sum_{1 \leq i < j \leq N} V_{ij}(x), \tag{3.2.1}$$

acting on $L^2(\mathbb{R}^{dN})$. Here, the operator

$$H_0 = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} \tag{3.2.2}$$

describes the kinetic part and the potentials V_{ij} describe the pair interactions between the particles indicated by i and j. We assume that the potentials V_{ij} are not identically zero and that they can be represented by functions $v_{ij} : \mathbb{R}^d \to \mathbb{R}$ as

$$V_{ij}(x) = v_{ij}(x_{ij}) \quad \text{with} \quad x_{ij} = x_i - x_j. \tag{3.2.3}$$

We will always assume that the functions v_{ij} are relatively form bounded with relative bound zero, i.e., for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that for every $\psi \in H^1(\mathbb{R}^d)$ we have

$$\langle |v_{ij}| \psi, \psi \rangle \leq \varepsilon \|\nabla \psi\|^2 + C(\varepsilon) \|\psi\|^2. \tag{3.2.4}$$

Moreover, we assume that there exist constants $A, C, \nu > 0$, such that

$$|v_{ij}(x)| \leq C\,(1 + |x|)^{-2-\nu}, \qquad x \in \mathbb{R}^d, \quad |x| \geq A. \tag{3.2.5}$$

3.2.1. Separation of the center of mass of the system

The center of mass of the system is a constant of motion, i.e., it moves with constant velocity. It is common practice to fix the center of mass at the origin and to consider instead of the operator H_N an operator which describes the dynamics of the system relative to the center of mass. In the following we separate the center of mass of the system, following [21, 62]. We introduce on $\mathbb{R}^{dN}$ the scalar product $\langle\cdot,\cdot\rangle_m$, given by

$$\langle x, y\rangle_m = 2\sum_{i=1}^{N} m_i\langle x_i, y_i\rangle, \qquad |x|_m^2 = \langle x, x\rangle_m, \qquad x, y \in \mathbb{R}^{dN}. \tag{3.2.6}$$

Here, we denote by $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^d$. The index m in the definition of the scalar product $\langle\cdot,\cdot\rangle_m$ is chosen in order to emphasize that it is weighted with the masses of the particles. Let X be the space $\mathbb{R}^{dN}$ equipped with the scalar product $\langle\cdot,\cdot\rangle_m$ and let

$$X_0 = \left\{ x = (x_1, \dots, x_N) \in X \,:\, \sum_{i=1}^{N} m_i x_i = 0 \right\} \tag{3.2.7}$$

be the space of relative positions of the particles and let $X_c = X \ominus X_0$ be the space of the center of mass position of the system. We denote by P_0 and P_c the orthogonal projections from X on X_0 and X_c, respectively. Furthermore, we introduce Δ, Δ_0 and Δ_c as the Laplace-Beltrami operators on $L^2(X)$, $L^2(X_0)$ and $L^2(X_c)$, respectively.

Remark 3.2.1. The choice of the scalar product $\langle\cdot,\cdot\rangle_m$ is natural and appropriate in the sense that the Hamiltonian H_0, defined in (3.2.2), is (minus) the Laplace-Beltrami operator on $\mathbb{R}^{dN}$ with respect to $\langle\cdot,\cdot\rangle_m$. Put differently, if we choose an orthogonal basis $\{e_1, \dots, e_{dN}\}$ of $X = (\mathbb{R}^{dN}, \langle\cdot,\cdot\rangle_m)$ and corresponding coordinates $y_1, \dots y_{dN}$, then we have

$$H_0 = -\sum \frac{\partial^2}{\partial y_i^2} = -\Delta. \tag{3.2.8}$$

Corresponding to the decomposition $L^2(X) = L^2(X_0) \otimes L^2(X_c)$ we find

$$-\Delta = -\Delta_0 \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\Delta_c). \tag{3.2.9}$$

Moreover, since for every $x \in X$ we have

$$(P_0 x)_i - (P_0 x)_j = x_i - x_j, \tag{3.2.10}$$

the potential $V(x) = \sum_{1\le i<j\le N} v_{ij}(x_{ij})$ satisfies

$$V(x) = V(P_0 x). \tag{3.2.11}$$

Therefore, H_N is unitarily equivalent to the operator

$$H \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\Delta_c), \tag{3.2.12}$$

where the operator H, acting on $L^2(X_0)$ is given by

$$H = -\Delta_0 + V. \tag{3.2.13}$$

In view of (3.2.12) the center of mass of the system moves like a free particle and the operator H corresponds to the relative motion of the system. The operator H is the main subject of the following studies.

Remark 3.2.2. **(i)** Under the above assumptions on the potentials the operator H is self-adjoint on $L^2(X_0)$, which follows from the KLMN theorem and Remark 3.1.9.

(ii) Separating the center of mass reduces the problem of an N-particle system to a problem of a system consisting of $N-1$ particles. In particular, the two-body Schrödinger operator in the center of mass frame can be considered as a one-particle operator.

3.2.2. Clusters and Cluster Hamiltonians

Now we introduce the concept of so-called cluster Hamiltonians, c.f. for example [59]. A cluster C of the system is defined as a non-empty subset of $\{1,\dots,N\}$ and we denote by $|C|$ the number of particles contained in C. To decouple a cluster C from the whole system we define

$$X_0[C] = \{x \in X_0 : x_i = 0 \text{ if } i \not\in C\}. \tag{3.2.14}$$

Note that for $x \in X_0[C]$

$$\sum_{i\in C} m_i x_i = 0, \tag{3.2.15}$$

i.e., $X_0[C]$ is the space of the relative positions of the particles in the cluster C. Let $\Delta_0[C]$ be the Laplace-Beltrami operator on $L^2(X_0[C])$ and

$$V[C] = \sum_{i,j\in C,\ i<j} V_{ij} \tag{3.2.16}$$

the potential of the pair interactions between the particles in the cluster C. Then for $1 < |C| < N$ the cluster Hamiltonian with reduced center of mass, acting on $L^2(X_0[C])$, is given by

$$H[C] = -\Delta_0[C] + V[C] \tag{3.2.17}$$

and describes the internal dynamics of the cluster C. For $C = \{1,\dots,N\}$ we have $X_0[C] = X_0$, so we set $H[C] = H$. For $|C| = 1$ we have $X_0[C] = \{0\}$ and we set $H[C] = 0$. Clusters C with $1 < |C| < N$ are called non-trivial clusters.

3.2.3. Partitions of the system

For $p \geq 2$ we say that $Z = \{C_1,\dots,C_p\}$ is a cluster decomposition or partition of the system of order $|Z| = p$ if the $C_i \in Z$ are non-empty, disjoint clusters whose union is the whole system. Let

$$X_0(Z) = \bigoplus_{C_k\in Z} X_0[C_k], \qquad X_c(Z) = X \ominus X_0(Z). \tag{3.2.18}$$

This gives rise to the decomposition

$$L^2(X_0(Z)) = \bigotimes_{C_k\in Z} L^2(X_0[C_k]). \tag{3.2.19}$$

By abuse of notation we denote the operators

$$\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}\otimes(-\Delta_0[C_k])\otimes\mathrm{Id}\otimes\cdots\otimes\mathrm{Id} \tag{3.2.20}$$

and

$$\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}\otimes H[C_k]\otimes\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}, \tag{3.2.21}$$

acting on $L^2(X_0(Z))$, by $-\Delta_0[C_k]$ and $H[C_k]$, respectively. Let $\Delta_0(Z)$ be the Laplace-Beltrami operator on $L^2(X_0(Z))$, then

$$-\Delta_0(Z) = \sum_{C_k \in Z} -\Delta_0[C_k]. \tag{3.2.22}$$

The cluster decomposition Hamiltonian of the partition Z is defined by

$$H(Z) = \sum_{C_k \in Z} H[C_k] \tag{3.2.23}$$

and describes the joint internal dynamics of the clusters in Z. We denote the potential of the inter-cluster interaction by

$$I(Z) = V - \sum_{C_k \in Z} V[C_k]. \tag{3.2.24}$$

Then, corresponding to the decomposition $L^2(X_0) = L^2(X_0(Z)) \otimes L^2(X_c(Z))$, the Hamiltonian of the whole system can be written as

$$H = H(Z) \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\Delta_c(Z)) + I(Z), \tag{3.2.25}$$

where $\Delta_c(Z)$ is the Laplace-Beltrami operator on $L^2(X_c(Z))$.

3.2.4. Systems of fermions and bosons

Now we consider a system of N identical one- or two-dimensional particles, i.e., $m_i = m_j$ for all $1 \le i, j \le N$ and the potentials satisfy

$$v_{ij}(x) = v_{ij}(-x), \qquad v_{ij}(x) = v_{kl}(x), \quad x \in \mathbb{R}^d \tag{3.2.26}$$

for $i \neq j$, $k \neq l$.

In quantum physics identical particles are considered to be indistinguishable, which means that if we interchange two particles, the new wave function should describe the same physical state. Two unit vectors in the quantum Hilbert space describe the same physical state if and only if they differ by a constant of absolute value

1. Hence, for identical particles we have

$$u(x_1,\dots,x_i,\dots,x_j,\dots,x_N) = \lambda u(x_1,\dots,x_j,\dots,x_i,\dots,x_N) \tag{3.2.27}$$

for some λ with $|\lambda| = 1$. Two values of λ are of special interest, namely $\lambda = \pm 1$. We say that a system is fermionic if $\lambda = -1$ and bosonic if $\lambda = +1$. In other words, for fermions the operator H_N is restricted to the space of functions which are anti-symmetric with respect to exchange of particles, i.e., H_N acts on $L^2_{\text{as}}(\mathbb{R}^{dN})$ which is the space of all functions $\psi \in L^2(\mathbb{R}^{dN})$ satisfying

$$\psi(x_1,\dots,x_N) = \operatorname{sgn}(\pi)\psi(x_{\pi(1)},\dots,x_{\pi(N)}) \tag{3.2.28}$$

for all permutations π of the set $\{1,\dots,N\}$. Here, $\operatorname{sgn}(\pi)$ is the signature of π. For a system of bosons the operator H_N is restricted to the space of functions which are symmetric with respect to permutation of particles, i.e., it acts on $L^2_{\text{sym}}(\mathbb{R}^{dN})$, i.e., the space of all functions in $L^2(\mathbb{R}^{dN})$ which are symmetric with respect to permutations of particles.

It is worth mentioning that any two fermions can not have the same quantum state, which is known as the Pauli exclusion principle. Bosons and fermions differ in their internal angular momentum (spin). Fermions have a half-integer spin, while bosons obey an integer spin. For our considerations the spin does not play a role and we shall not discuss it further. We refer to [30] for more details.

Similarly as in the case of Hamiltonians without symmetry restrictions we can introduce the Hamiltonians H_{as} and H_{sym} which are the Hamiltonians of a fermionic, repectively bosonic system in the center of mass frame, acting on $L^2_{\text{as}}(X_0)$ and $L^2_{\text{sym}}(X_0)$, respectively. Analogously we can define the cluster Hamiltonians $H_{\text{as}}[C]$ and $H_{\text{sym}}[C]$ for clusters of fermions and bosons, respectively.

3.2.5. Relative coordinates q and ξ and separation of clusters

For a cluster C we denote by $P_0[C]$ the projection from X_0 on $X_0[C]$ and for $x \in X_0$ we denote

$$q[C] = P_0[C]x. \tag{3.2.29}$$

We notice that $q[C] = (q_1, \ldots, q_N)$ is a vector in $\mathbb{R}^{dN}$ and the component $q_i \in \mathbb{R}^d$ is zero if $i \notin C$ and

$$q_i = x_i - x_C \quad \text{if} \quad i \in C, \tag{3.2.30}$$

where

$$x_C = \frac{1}{M[C]} \sum_{j \in C} m_j x_j \tag{3.2.31}$$

is the center of mass position of the cluster C. Here, $M[C] = \sum_{i \in C} m_i$ is the total mass of the system. In other words, if $i \in C$, then q_i describes the position of the particle i relative to the center of mass of C, cf. Figure 3.1.

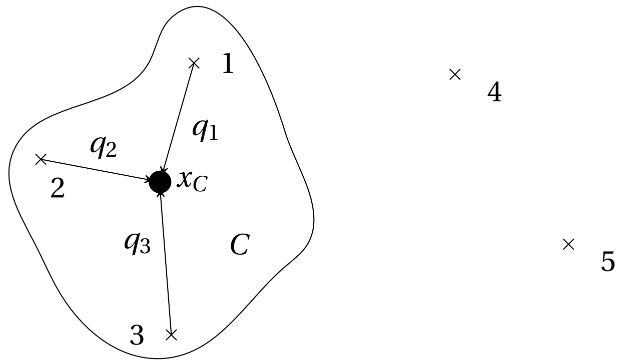

Figure 3.1.: If the particle i is in the cluster C, then q_i describes its position relative to the center of mass of the cluster C. If $i \notin C$, then $q_i = 0$.

Now let $Z = \{C_1, \ldots, C_p\}$ be a partition of the system. We introduce the projections $P_0(Z)$ and $P_c(Z)$ from X_0 to $X_0(Z)$ and $X_c(Z)$, respectively. For $x \in X_0$ let

$$q(Z) = P_0(Z)x, \qquad \xi(Z) = P_c(Z)x. \tag{3.2.32}$$

Note that the ith components of $q(Z)$ and $\xi(Z)$ are vectors $q_i \in \mathbb{R}^d$ and $\xi_i \in \mathbb{R}^d$ given by

$$q_i = x_i - x_{C_l}, \qquad \xi_i = x_{C_l} \tag{3.2.33}$$

where C_l is the cluster which contains the particle i. It is obvious that

$$q(Z) = \sum_{C_k \in Z} q[C_k]. \tag{3.2.34}$$

Now we introduce several subsets of X_0 which will be important later.

Definition 3.2.3. For $\kappa > \kappa' > 0$, $R > 0$ and partitions Z with $1 < |Z| < N$ we define the regions

$$\begin{aligned} B(R) &= \{x \in X_0 \,:\, |x|_m \le R\}, \\ K(Z,\kappa) &= \left\{x \in X_0 \,:\, |q(Z)|_m \le \kappa |\xi(Z)|_m\right\} \\ K_R(Z,\kappa) &= \left\{x \in X_0 \,:\, |q(Z)|_m \le \kappa |\xi(Z)|_m,\ |x|_m \ge R\right\} \\ K_R(Z,\kappa',\kappa) &= K_R(Z,\kappa) \setminus K_R(Z,\kappa') \end{aligned} \tag{3.2.35}$$

Using these sets we can separate regions in X_0 where the particles are divided into several groups which are moved apart. Since these objects play an important role later, let us brievly discuss the coordinates $q(Z)$ and $\xi(Z)$ and the regions $K(Z,\kappa)$. For further discussions see for example [4, 75]. By (3.2.33) and the definition of the scalar product $\langle\cdot,\cdot\rangle_m$ we find

$$|q(Z)|_m^2 = 2\sum_{l=1}^{p}\sum_{i\in C_l} m_i |x_i - x_{C_l}|^2, \tag{3.2.36}$$

i.e., $|q(Z)|_m$ is a weighted average of the distance of the particles from the center of mass of the cluster to which they belong. It follows immediately from (3.2.36) that if the particle i belongs to the cluster C_l, then

$$|x_i - x_{C_l}| \le (2m_0)^{-\frac{1}{2}} |q(Z)|_m, \tag{3.2.37}$$

where $m_0 = \min\{m_l : l = 1,\dots,N\}$. Now assume that $|Z| = 2$, i.e., $Z = \{C_1, C_2\}$. Then

$$\begin{aligned} |\xi(Z)|_m^2 &= 2\sum_{i\in C_1} m_i |x_{C_1}|^2 + 2\sum_{j\in C_2} m_j |x_{C_2}|^2 \\ &= 2M[C_1]|x_{C_1}|^2 + 2M[C_2]|x_{C_2}|^2 \end{aligned} \tag{3.2.38}$$

Since the center of mass of the whole system is fixed at the origin, it follows

$$|\xi(Z)|_m^2 = \frac{2M[C_1]M[C_2]}{M[C_1]+M[C_2]} |x_{C_1} - x_{C_2}|^2, \tag{3.2.39}$$

i.e., $|\xi(Z)|_m^2$ is a weighted distance between the centers of mass of the two clusters.

We keep the assumption $|Z| = 2$ and describe the region $K(Z,\kappa)$. Let $x \in K(Z,\kappa)$ and $i, j \in C_1,\ k \in C_2$. Then, by (3.2.37) and the definition of $K(Z,\kappa)$ we have

$$|x_i - x_{C_1}| \le (2m_0)^{-\frac{1}{2}} |q(Z)|_m \le (2m_0)^{-\frac{1}{2}} \kappa |\xi(Z)|_m \tag{3.2.40}$$

and analogously

$$|x_k - x_{C_2}| \le (2m_0)^{-\frac{1}{2}} \kappa |\xi(Z)|_m \tag{3.2.41}$$

Hence, for $\kappa > 0$ small enough we get by the use of (3.2.39)

$$|x_i - x_k| = |x_i - x_{C_1} + x_{C_1} - x_{C_2} + x_{C_2} - x_j| \ge c(m,\kappa)|\xi(Z)|_m, \tag{3.2.42}$$

where the constant $c(m,\kappa)$ is given by

$$c(m,\kappa) = \left(\frac{M_1 + M_2}{2M_1 M_2}\right)^{\frac{1}{2}} - 2\kappa\,(2m_0)^{-\frac{1}{2}}. \tag{3.2.43}$$

and depends on the masses of the particles and κ. On the other hand, we have

$$|x_i - x_j| \le 2\kappa\,(2m_0)^{-\frac{1}{2}}\,|\xi(Z)|_m. \tag{3.2.44}$$

This shows that for small $\kappa > 0$ and any $x \in K(Z,\kappa)$ the distance between any two particles in the same cluster is small compared to the distance between particles from different clusters. This means that $x \in K(Z,\kappa)$ corresponds to the situation that the two clusters in the partition Z are separated.

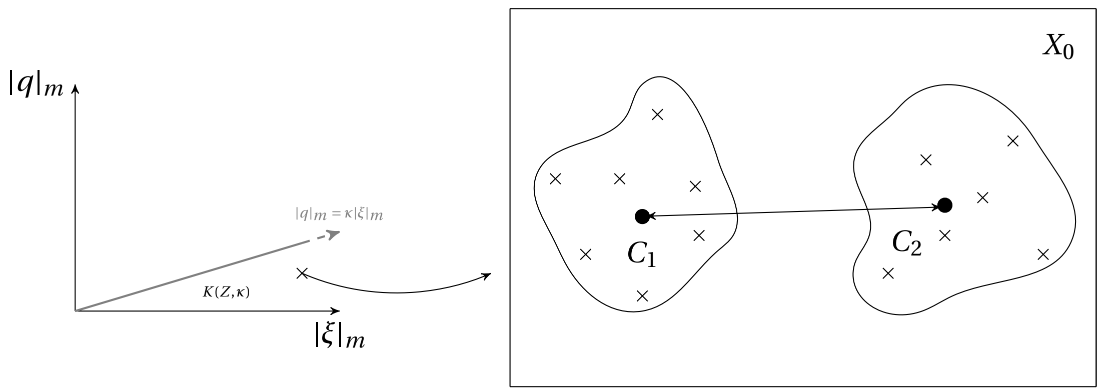

Figure 3.2.: $x \in K(Z,\kappa)$, $|Z| = 2$, describes two separated clusters.

From the above observations we get the following

Lemma 3.2.4. *There exists a constant* $\kappa_0 > 0$*, such that for all* $0 < \kappa < \kappa_0$ *and any pair of partitions* $Z \neq Z'$ *with* $|Z| = |Z'| = 2$ *we have*

$$K(Z,\kappa) \cap K(Z',\kappa) = \{0\}. \tag{3.2.45}$$

This result can be generalized to partitions consisting of more than two clusters, namely

Theorem 3.2.5. *For any system of* $N \geq 3$ *particles there exist constants* $\kappa(2), \dots, \kappa(N-1)$ *and* $\kappa'(2), \dots, \kappa'(N-1)$ *with* $\kappa(l) > \kappa'(l)$ *for* $1 < l < N$*, such that for any* $1 < l < N$ *and any pair of partitions* $Z \neq Z'$ *with* $|Z| = |Z'| = l$ *we have*

$$K((Z,\kappa(l)) \cap K\left((Z',\kappa(l)\right) \subseteq \bigcup_{\tilde{Z}:|\tilde{Z}|<l} K\left(\tilde{Z},\kappa'(|\tilde{Z}|)\right). \tag{3.2.46}$$

This theorem was proved in an appendix of M. Antonets, G. Zhislin, and I. Shereshevskijto [4] to the book of K. Jörgens and J. Weidmann [37]. The appendix exists in Russian only, an English version can be found in [7].

3.2.6. Further important tools

To study the spectrum of multi-particle Schrödinger operators we will make use of geometric methods. In this section we collect some tools which will be useful later.

The IMS localization formula

An important tool for these methods is the use of a partition of unity and the IMS localization formula.

Definition 3.2.6 (Partition of Unity)**.** A partition of unity of a space X is a finite family of functions $J_\alpha : X \to [0,1]$ with bounded distributional derivatives and $\sum_\alpha J_\alpha^2 = 1$.

Remark 3.2.7. **(i)** Note that sometimes for a partition of unity $\{J_\alpha\}$ one demands $\sum_\alpha J_\alpha = 1$ (without the squares). We require that the sum of squares of the functions J_α equals one because this is more convenient for our purpose.

(ii) In the literature often it is required that a partition of unity consists of smooth functions J_α, see for example [14]. Since we consider the operators in the sense of quadratic forms, the requirements of the above given definition are sufficient, cf. for example [65]. Later, our concretely chosen partitions of unity consist of non-smooth functions.

Theorem 3.2.8 (IMS Localization Formula, cf. [14], Theorem 3.2, [65])**.** *Let* $\{J_\alpha\}$ *be a partition of unity and let* $H = H_0 + V$ *for a potential* V *satisfying* (3.2.4)*. Then*

$$H = \sum_\alpha J_\alpha H J_\alpha - \sum_\alpha |\nabla J_\alpha|^2. \tag{3.2.47}$$

Remark 3.2.9. **(i)** The localization formula was first derived by R. Ismagilov [36], rediscovered by J. Morgan [45] and used in J. Morgan and B. Simon [46]. Later, its importance was discovered by I. M. Sigal [61]. This explains why it is known as the IMS formula.

(ii) For apparent reasons the term $\sum_\alpha |\nabla J_\alpha|^2$ is called localization error.

(iii) Later we will choose an appropriate partition of unity which separates regions in the configuration space where particles are in several groups that are moved apart. Then the IMS formula allows us to estimate the Schrödinger operator of the whole system by studying the cluster Hamiltonians and estimating the localization error. A crucial part of the work will be to find an appropriate estimate for the localization error. Such an estimate can be found in Section 4.3.3.

The HVZ theorem

Recall that for a one-body Schrödinger operator $h = -\Delta + V$ with a potential V decaying at infinity the essential spectrum coincides with the semi-axis $[0, \infty)$. In the case of multi-particle systems the localization of the essential spectrum is a challenging problem. The famous HVZ theorem, named after W. Hunziker [34], C. van Winter [72] and G.M. Zhislin [81] gives an answer to this question. We formulate the theorem with the conditions for the pair potentials V_{ij} for which we will apply it later.

Theorem 3.2.10 (HVZ Theorem, cf. [58], Theorem XIII.17, and [18])**.** *Let* H *be the Hamiltonian of a system of* $N \geq 3$ *particles in the center of mass frame, where the po-*

tentials V_{ij} *satisfy* (3.2.4) *and* (3.2.5). *For a partition* Z *let* $\Sigma(Z) = \inf\sigma(H(Z))$ *and* $\Sigma := \min_{Z:|Z|\geq 2} \Sigma(Z)$. *Then the essential spectrum of* H *is given by*

$$\sigma_{\text{ess}}(H) = [\Sigma, \infty). \tag{3.2.48}$$

Remark 3.2.11. **(i)** Let $Z = \{C_1, \ldots, C_p\}$ be a partition of the system with $p > 1$. Then

$$\sigma(H(Z)) = \sum_{k=1}^{p} \sigma\left(H[C_k]\right). \tag{3.2.49}$$

(ii) Assume that $\inf\sigma_{\text{ess}}(H) < 0$, then by (3.2.49) and the HVZ theorem there exists a cluster, such that the corresponding cluster Hamiltonian has negative spectrum. On the other hand, if there exists a cluster C with $|C| < N-1$, such that $\inf\sigma(H[C]) < 0$, then $\inf\sigma_{\text{ess}}(H) < 0$. Indeed, let Z be the partition which consists of the cluster C and such that the remaining clusters in Z contain only one particle each. Then $\sigma(H(Z)) = \sigma(H[C])$ and therefore by (3.2.49) and the HVZ theorem $\inf\sigma_{\text{ess}}(H) < 0$. This observation will be used later in the thesis.

3.2.7. Notation and Convention

We conclude this chapter by briefly explaining some notations which will be used in the further course of this thesis.

Given a partition $Z = \{C_1, \ldots, C_p\}$ with coordinates $q = q(Z)$ and $\xi = \xi(Z)$ we will write $-\Delta_q$ instead of $-\Delta_0(Z)$ and $-\Delta_\xi$ instead of $-\Delta_c(Z)$. Corresponding to a decomposition $X_0 = X_1 \bigoplus X_2$ and $L^2(X_0) = L^2(X_1) \otimes L^2(X_2)$ we can consider operators A_i on $L^2(X_i)$ as operators on $L^2(X_0)$, namely as

$$A_1 \otimes \text{Id} \qquad \text{and} \qquad \text{Id} \otimes A_2, \tag{3.2.50}$$

respectively. In this sense we will sometimes consider the cluster Hamiltonians $H[C]$ and the cluster decomposition Hamiltonian $H(Z)$ as operators on $L^2(X_0)$ without changing the notation. From time to time we will also consider the operators as operators acting on $L^2(\mathbb{R}^k)$, where $k \in \mathbb{N}$ is the dimension of the corresponding subspace of X_0.

4. Virtual levels of Schrödinger operators

4.1. Introduction

The behavior of Schrödinger operators at the bottom of the essential spectrum has attracted many mathematicians. In particular, the solvability of the Schrödinger equation

$$(-\Delta + V)\psi = 0 \tag{4.1.1}$$

and properties of such solutions are of interest for Schrödinger operators $H = -\Delta + V$ which are critical in the sense that $\sigma_{\text{ess}}(H) = [0, \infty)$ and the negative spectrum of the operator is empty, but any negative perturbation creates a negative eigenvalue. Such problems have been studied for example in [24, 56, 57, 79].

Concerning the behaviour at infinity of solutions of the equation (4.1.1) for a critical operator $H = -\Delta + V$ several scenarios are possible. While it is well known that eigenfunctions corresponding to eigenvalues below the essential spectrum decay exponentially at infinity [2], the situation at the threshold of the essential spectrum is quite different. It is not even ensured that in this case the solution of the Schrödinger equation $H\psi = 0$ is an eigenfunction, it can also be a resonant state, as for example in the case of a one-particle Schrödinger operator in dimension three [79]. In some cases the decay of solutions of (4.1.1) is sub-exponential [7, 33] or only polynomial, see [7].

In the context of the Efimov effect, which we are interested in, it is important to understand the behavior of the Schrödinger operator near the bottom of its essential spectrum. In particular, the question whether so-called virtual levels correspond to eigenvalues or to threshold resonances is important.

By a virtual level of a Schrödinger operator $H = -\Delta + V$ we mean that

- $H \geq 0$,
- $\inf \sigma(H + \varepsilon\Delta) < 0$ for any $\varepsilon > 0$,
- $\inf \sigma_{\text{ess}}(H + \varepsilon\Delta) = 0$ for any sufficiently small $\varepsilon > 0$,

i.e., any small negative perturbation of the operator leads to a negative eigenvalue while the essential spectrum is stable under small perturbations. Recently, virtual levels of Hamiltonians corresponding to systems of $N \geq 3$ particles with space dimension $d \geq 3$ have been studied in [7] and [24]. There, it has been proved that in the presence of a virtual level the solution of (4.1.1) is an eigenfunction corresponding to the eigenvalue zero. This observation was the key to show the absence of the Efimov effect for systems of $N \geq 4$ particles in dimension $d \geq 3$.

The main goal of this chapter is to study virtual levels of Schrödinger operators corresponding to systems of one- or two-dimensional particles and in particular to answer the question whether they correspond to eigenvalues or to resonances. In doing so we have to bear in mind the following difference concerning criticality of Schrödinger operators between dimensions $d \geq 3$ and $d = 1$ or $d = 2$. In the first case, Hardy's inequality implies that for any bounded and sufficiently fast decaying potential V we have

$$h(\lambda) = -\Delta + \lambda V \geq 0 \tag{4.1.2}$$

for all sufficiently small $\lambda > 0$. However, for dimension $d = 1$ or $d = 2$ the Laplace operator is critical, namely the following result holds.

Theorem 4.1.1 (The Laplacian is critical for dimensions one and two, see, e.g., [64])**.** *Let $V \neq 0$. For $d = 1$ assume that $\int (1 + x^2)|V(x)|\,\mathrm{d}x < \infty$ and for $d = 2$ assume that for $\delta > 0$ small enough we have*

$$\int |V(x)|^{1+\delta}\,\mathrm{d}x < \infty \quad \textit{and} \quad \int (1 + |x|^2)^{1+\delta}|V(x)|\,\mathrm{d}x < \infty. \tag{4.1.3}$$

Then the operator $h(\lambda) = -\Delta + \lambda V$ has a negative eigenvalue for any $\lambda > 0$ if and only if $\int V(x)\,\mathrm{d}x \leq 0$.

Remark 4.1.2. **(i)** Besides the critical behavior of the Laplacian, in [64] also the behavior of the eigenvalues in case of $\int V(x)\,\mathrm{d}x < 0$ has been studied. The observation that the Laplacian is critical will be important in our following studies.

(ii) As mentioned above, for dimension $d \geq 3$ the Laplace operator is not critical due to Hardy's inequality. If we subtract a Hardy term i.e., consider the operator

$$H = -\Delta - \frac{(d-2)^2}{4}|x|^{-2}, \tag{4.1.4}$$

we get a critical operator. In [77] a necessary and sufficient condition is given for potentials V, such that $H(\lambda) = -\Delta - \frac{(d-2)^2}{4}|x|^{-2} + \lambda V$ has a negative eigenvalue for any $\lambda > 0$.

Let us give a short outline of this chapter. We start in Section 4.2 with the case of one-particle Schrödinger operators, which is rather for the sake of completeness. Some of the results of this section are ingredients in the proofs of the multi-particle results, but might be of independent interest. In Section 4.3 we extend the studies to multi-particle systems. Our main result is Theorem 4.3.3, where we present a criterion that virtual levels correspond to a zero eigenvalue. We also show that this criterion is fulfilled for systems of $N \geq 3$ one-dimensional or $N \geq 4$ two-dimensional particles.

4.2. Virtual levels of one-particle Schrödinger operators

4.2.1. Introduction

In this section, which is based on [8], we consider the Schrödinger operator

$$h = -\Delta + V \tag{4.2.1}$$

acting on $L^2(\mathbb{R}^d)$ with $d \in \{1,2\}$. We assume that the potential V is not identically zero and that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that for every $\psi \in H^1(\mathbb{R}^d)$

we have

$$\langle |V|\psi,\psi\rangle \le \varepsilon\|\nabla\psi\|^2 + C(\varepsilon)\|\psi\|^2. \tag{4.2.2}$$

Moreover, we assume that the potential is short-range, i.e.,

$$|V(x)| \le C(1+|x|)^{-2-\nu}, \qquad x\in\mathbb{R}^d, \quad |x|\ge A \tag{4.2.3}$$

for some constants $A, C, \nu > 0$. We use the small letter h for the Schrödinger operator to emphasize that we deal with a one-particle operator. For $\varepsilon \in (0,1)$ we write

$$h_\varepsilon = h + \varepsilon\Delta = -(1-\varepsilon)\Delta + V. \tag{4.2.4}$$

Definition 4.2.1. Assume that the potential V satisfies (4.2.2) and (4.2.3). We say that the operator h, defined in (4.2.1), has a virtual level at zero if $h \ge 0$ and for any $\varepsilon \in (0,1)$

$$\inf\sigma(h_\varepsilon) < 0 \tag{4.2.5}$$

For $d \ge 3$ it is an immediate consequence of Hardy's inequality that for short-range potentials the operator h has a virtual level at zero if and only if $h \ge 0$ and for any $\varepsilon > 0$ the operator $\tilde{h} = -\Delta + V - \varepsilon(1+|x|^2)^{-1}$ has a discrete eigenvalue below zero. For dimension one or two, where Hardy's inequality does not hold, this equivalence is not evident. On the other hand, since the Laplace operator is critical in these dimensions, see Theorem 4.1.1, the condition $h \ge 0$ implies $\int_{-\infty}^{\infty} V(x)\,\mathrm{d}x > 0$. This observation is a crucial ingredient in the proof of the following theorem.

Theorem 4.2.2 (Necessary and sufficient condition for a virtual level)**.** *Let $d = 1$ or $d = 2$. We assume that $V \ne 0$ satisfies* (4.2.2) *and* (4.2.3) *and that $h \ge 0$. Furthermore, let $\mathcal{U}$ be a bounded, strictly negative potential satisfying for $|x| \ge A$ the condition*

$$|\mathcal{U}(x)| \le C|x|^{-2} \text{ if } d=1 \quad \text{and} \quad |\mathcal{U}(x)| \le C|x|^{-2}\ln^{-2}(|x|) \text{ if } d=2 \tag{4.2.6}$$

for some $C > 0$, $A > 1$. Then h has a virtual level at zero if and only if for any $\varepsilon > 0$ and for any function $\psi \in H^1(\mathbb{R}^d)$ we have

$$\inf\sigma(h+\varepsilon\mathcal{U}) < 0. \tag{4.2.7}$$

Before we turn to the proof of Theorem 4.2.2, let us collect some consequences that will be useful later in this thesis for the multi-particle case.

Corollary 4.2.3. *Assume that $V \neq 0$ satisfies* (4.2.2) *and* (4.2.3), $h = -\Delta + V \geq 0$ *and that h does not have a virtual level at zero. Then for small $\varepsilon_0 > 0$ the operator $h_{\varepsilon_0} \geq 0$ also does not have a virtual level. Fixing such an $\varepsilon_0 > 0$ and applying Theorem 4.2.2 to the operator h_{ε_0} we find for $\mathscr{U}$ given by* (4.2.6) *an $\varepsilon_1 > 0$, such that*

$$(1-\varepsilon_0)\|\nabla\psi\|^2 + \langle V\psi,\psi\rangle + \varepsilon_1\langle \mathscr{U}\psi,\psi\rangle \geq 0 \tag{4.2.8}$$

for any function $\psi \in H^1(\mathbb{R}^d)$.

Corollary 4.2.4. *Assume that $h \geq 0$ does not have a virtual level and that the potential $V \neq 0$ satisfies* (4.2.2) *and* (4.2.3). *Then by choosing $\mathscr{U}$ according to Theorem 4.2.2 with $\mathscr{U}(x) = -1$ for $|x| \leq 1$ we obtain from* (4.2.8) *that for any $\psi \in \tilde{H}^1(\mathbb{R}^d)$*

$$\|\psi\|^2_{\tilde{H}^1} \leq \frac{1+\varepsilon_1-\varepsilon_0}{\varepsilon_1}\|\nabla\psi\|^2 + \frac{1}{\varepsilon_1}\langle V\psi,\psi\rangle. \tag{4.2.9}$$

Proof of Theorem 4.2.2. Here we only prove that the absence of a virtual level of h implies that (4.2.7) does not hold. The other direction follows from Theorem 4.2.6, which we will prove later, and the variational principle.

Let $d = 1$. Without loss of generality we can assume that $\mathscr{U}(x) = -(1+|x|)^{-2}$. For $\psi \in H^1(\mathbb{R})$ we write

$$\psi_0(x) = \psi(x) - \psi(0). \tag{4.2.10}$$

Then we have

$$\int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \leq 2|\psi(0)|^2 \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x + 2\int_{-\infty}^{\infty} \frac{|\psi_0(x)|^2}{1+x^2}\,\mathrm{d}x. \tag{4.2.11}$$

Computing the first integral and applying the Hardy inequality for the semi axis for the second integral, which is possible because $\psi_0(0) = 0$, we get

$$\int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2} \leq 2\pi|\psi(0)|^2 + 8\|\psi'\|^2, \tag{4.2.12}$$

where we used that $\psi' = \psi_0'$. Let us estimate $|\psi(0)|$, which is done in the following

Lemma 4.2.5. *Assume that the conditions of Theorem 4.2.2 are fulfilled. Then there exist constants* $C_0,\ C_1 > 0$ *which are independent of* $\psi \in \tilde{H}^1(\mathbb{R}^d)$*, such that*

$$|\psi(0)|^2 \le C_0^{-1}\langle V\psi,\psi\rangle + C_1\|\psi'\|^2. \tag{4.2.13}$$

Proof of Lemma 4.2.5. From the identity

$$\begin{aligned}\langle V\psi,\psi\rangle = \int_{-\infty}^{\infty} V(x)|\psi(0)|^2\,\mathrm{d}x + \int_{-\infty}^{\infty} V(x)|\psi_0(x)|^2\,\mathrm{d}x \\ + 2\mathrm{Re}\int_{-\infty}^{\infty} V(x)\psi(0)\psi_0(x)\,\mathrm{d}x\end{aligned} \tag{4.2.14}$$

we get

$$\begin{aligned}\langle V\psi,\psi\rangle \ge \int_{-\infty}^{\infty} V(x)|\psi(0)|^2\,\mathrm{d}x + \int_{-\infty}^{\infty} V(x)|\psi_0(x)|^2\,\mathrm{d}x \\ - 2\int_{-\infty}^{\infty} |V(x)||\psi(0)\psi_0(x)|\,\mathrm{d}x.\end{aligned} \tag{4.2.15}$$

Note that for any $\delta > 0$

$$2|\psi(0)\psi_0(x)| \le \delta|\psi(0)|^2 + \delta^{-1}|\psi_0(x)|^2, \tag{4.2.16}$$

which together with (4.2.15) implies

$$\begin{aligned}\langle V\psi,\psi\rangle &\ge |\psi(0)|^2\int_{-\infty}^{\infty} (V(x) - \delta|V(x)|)\,\mathrm{d}x \\ &\quad + \int_{-\infty}^{\infty} |\psi_0(x)|^2\left(V(x) - \delta^{-1}|V(x)|\right)\mathrm{d}x \\ &\ge |\psi(0)|^2\int_{-\infty}^{\infty} (V(x) - \delta|V(x)|)\,\mathrm{d}x \\ &\quad - (1+\delta^{-1})\int_{-\infty}^{\infty} |V(x)||\psi_0(x)|^2\,\mathrm{d}x.\end{aligned} \tag{4.2.17}$$

Using Lemma 3.1.11 to estimate the last term on the r.h.s of (4.2.17) we get

$$\langle V\psi,\psi\rangle \geq |\psi(0)|^2 \int_{-\infty}^{\infty} (V(x)-\delta|V(x)|)\,\mathrm{d}x - C(1+\delta^{-1})\|\psi_0\|^2_{\tilde{H}^1} \tag{4.2.18}$$

for some $C>0$. Due to $\psi_0(0)=0$ we have $\|\psi_0\|^2_{\tilde{H}^1} \leq C'\|\psi_0'\|^2$ for some $C'>0$. This, together with (4.2.18) yields

$$\langle V\psi,\psi\rangle \geq |\psi(0)|^2 \int_{-\infty}^{\infty} (V(x)-\delta|V(x)|)\,\mathrm{d}x - C(\delta)\int_{-\infty}^{\infty} |\psi_0'(x)|^2\,\mathrm{d}x. \tag{4.2.19}$$

Since $\int_{-\infty}^{\infty} V(x)\,\mathrm{d}x > 0$, we can choose the constant $\delta>0$ sufficiently small, such that

$$\int_{-\infty}^{\infty} (V(x)-\delta|V(x)|)\,\mathrm{d}x \geq \frac{1}{2}\int_{-\infty}^{\infty} V(x)\,\mathrm{d}x =: C_0 > 0. \tag{4.2.20}$$

This, together with (4.2.19) and $\psi'(x)=\psi_0'(x)$ implies

$$|\psi(0)|^2 \leq C_0^{-1}\langle V\psi,\psi\rangle + C_1(\delta)\|\psi'\|^2 \tag{4.2.21}$$

for some constant $C_1(\delta)>0$ which depends on V and δ only. This completes the proof of Lemma 4.2.5. □

Combining (4.2.12) with (4.2.13) yields

$$\int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \leq \left(2\pi C_0^{-1}\langle V\psi,\psi\rangle + C_2(\delta)\|\psi'\|^2\right), \tag{4.2.22}$$

where $C_2(\delta)=C_1(\delta)+8$. Now let $0\neq\psi\in H^1(\mathbb{R}^d)$ be fixed. We distinguish between two cases:

(i) If $2\pi C_0^{-1}\langle V\psi,\psi\rangle < C_2(\delta)\|\psi'\|^2$, then (4.2.22) yields

$$\varepsilon_1\int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\,\mathrm{d}x \leq 2\varepsilon_1 C_2(\delta)\|\psi'\|^2. \tag{4.2.23}$$

Now since h does not have a virtual level, for $\varepsilon > 0$ small enough we find

$$\langle h\psi, \psi \rangle - \varepsilon \|\psi'\|^2 \geq 0, \tag{4.2.24}$$

where ε can be chosen independently of ψ. Hence, in view of (4.2.23) we can choose $\varepsilon_1 > 0$ sufficiently small to conclude that

$$\langle h\psi, \psi \rangle - \varepsilon_1 \int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\, \mathrm{d}x \geq 0. \tag{4.2.25}$$

(ii) If $2\pi C_0^{-1} \langle V\psi, \psi \rangle \geq C_2(\delta) \|\psi'\|^2$, we have in particular $\langle V\psi, \psi \rangle > 0$ and by (4.2.22)

$$\varepsilon_1 \int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\, \mathrm{d}x \leq 4\varepsilon_1 \pi C_0^{-1} \langle V\psi, \psi \rangle. \tag{4.2.26}$$

By choosing $0 < \varepsilon_1 < (4\pi)^{-1} C_0$ we obtain

$$\begin{aligned} &\langle h\psi, \psi \rangle - \varepsilon_1 \int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\, \mathrm{d}x \\ &\qquad = \|\psi'\|^2 + \langle V\psi, \psi \rangle - \varepsilon_1 \int_{-\infty}^{\infty} \frac{|\psi(x)|^2}{1+x^2}\, \mathrm{d}x \geq \|\psi'\|^2 \geq 0. \end{aligned} \tag{4.2.27}$$

This implies (4.2.25) in both cases and therefore, because ε_1 can be chosen independently of ψ, the statement of Theorem 4.2.2 for the case $d = 1$.

Now we assume that $d = 2$. For $\psi \in H^1(\mathbb{R}^2)$ we write $\psi_0(x) = \psi(x) - a_0$, where

$$a_0 = \frac{1}{2\pi} \int_{\mathbb{S}^1} \psi(\omega)\, \mathrm{d}\omega. \tag{4.2.28}$$

Then $\int_{\mathbb{S}^1} \psi_0(\omega)\, \mathrm{d}\omega = 0$ and thus we can apply the two-dimensional Hardy inequality (2.1.26) to the function ψ_0. Proceeding as in the proof of the one-dimensional case yields the statement for $d = 2$ and therefore completes the proof of Theorem 4.2.2. $\square$

4.2.2. Solutions of the Schrödinger equation in the presence of a virtual level

For the case $d = 3$ it was shown by D. Yafaev that the one-particle Schrödinger operator h with $h \ge 0$ and a short-range potential V has a virtual level at zero if and only if the equation $h\psi = 0$ has a solution in $\dot{H}^1\left(\mathbb{R}^3\right)$. This solution does not belong to $L^2\left(\mathbb{R}^3\right)$ and decays as $|x|^{-1}$ for $|x| \to \infty$ and therefore virtual levels of one-particle Schrödinger operators in dimension three correspond to resonances, see [79]. For dimension $d = 4$ virtual levels also correspond to resonances [5], while for $d \ge 5$ they correspond to eigenvalues, see for example [22]. We investigate the analogue problem for the cases $d = 1$ and $d = 2$. We prove the following

Theorem 4.2.6. *Assume that $d \in \{1,2\}$ and that $V \neq 0$ satisfies the conditions* (4.2.2) *and* (4.2.3). *If h has a virtual level at zero, then the following assertions hold:*

(i) *There exists a solution $\varphi_0 \in \tilde{H}^1(\mathbb{R}^d)$, $\varphi_0 \neq 0$, of the equation $-\Delta\varphi_0 + V\varphi_0 = 0$, i.e., for all $\psi \in \tilde{H}^1(\mathbb{R}^d)$*

$$\langle \nabla\varphi_0, \nabla\psi\rangle + \langle V\varphi_0, \psi\rangle = 0. \tag{4.2.29}$$

(ii) *If $d = 1$, then for the functions φ_0 satisfying* (4.2.29) *we have*

$$(1+|\cdot|)^{-\frac{1}{2}-\varepsilon}\varphi_0 \in L^2(\mathbb{R}) \qquad \text{for any } \varepsilon > 0. \tag{4.2.30}$$

(iii) *Let $d = 2$. Then for the functions φ_0 satisfying* (4.2.29) *we have*

$$(1+|\cdot|)^{-1}\,(1+|\ln(|\cdot|)|)^{-\frac{1}{2}-\varepsilon}\,\varphi_0 \in L^2(\mathbb{R}^2) \qquad \text{for any } \varepsilon > 0. \tag{4.2.31}$$

(iv) *If in addition the potential V is relatively $-\Delta$-bounded in the sense of operators i.e., there exists a constant $C > 0$, such that*

$$\|V\psi\|^2 \le C\left(\|\Delta\psi\|^2 + \|\psi\|^2\right) \tag{4.2.32}$$

holds for all functions $\psi \in H^2(\mathbb{R}^d)$, then there exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^d)$ satisfying $\langle\nabla\psi, \nabla\varphi_0\rangle = 0$

$$\langle h\psi, \psi\rangle \ge \delta_0\|\nabla\psi\|^2. \tag{4.2.33}$$

Remark 4.2.7. **(i)** Note that the left-hand side of (4.2.29) is well-defined due to conditions (4.2.2) and (4.2.3) and inequalities (2.2.5) and (2.2.6).

(ii) In Theorem 4.2.6 we give a lower bound on the decay rate of solutions of the Schrödinger equation corresponding to virtual levels. It is easy to see that if the potentials are compactly supported and $V(x) = V(|x|)$ for $d = 2$, then the estimates (4.2.30) and (4.2.31) are almost sharp. It is also easy to see that in this case the solution φ_0 is constant outside of the support of V and therefore it can not be an eigenfunction - zero is a resonance of h.

(iii) Theorem 4.2.6 provides the owing direction in the proof of Theorem 4.2.2, namely that the existence of a virtual level implies

$$\inf \sigma\left(h + \varepsilon \mathscr{U}\right) < 0 \tag{4.2.34}$$

for any function $\mathscr{U}$ which satisfies (4.2.6) and any $\varepsilon > 0$.

Proof of Theorem 4.2.6

Since for any $\varepsilon > 0$ we have $\inf \sigma_{\text{disc}}(h_\varepsilon) < 0$, we find a sequence of eigenfunctions $\psi_n \in H^1(\mathbb{R}^d)$ corresponding to a sequence of eigenvalues $E_n < 0$ of the operator $h_{n^{-1}}$, i.e.,

$$-\left(1 - n^{-1}\right)\Delta\psi_n + V\psi_n = E_n\psi_n. \tag{4.2.35}$$

We normalize the functions ψ_n by the condition $\|\psi_n\|_{\tilde{H}^1} = 1$. Then there exists a subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges weakly in $\tilde{H}^1(\mathbb{R}^d)$ to a function $\varphi_0 \in \tilde{H}^1(\mathbb{R}^d)$. We show that this function fulfills the properties stated in the theorem.

We break down the proof into several steps. At first, we show that φ_0 satisfies

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0. \tag{4.2.36}$$

This is done in Lemma 4.2.8. Then, in Lemma 4.2.9, we show that the sequence $(\psi_n)_{n\in\mathbb{N}}$ satisfies an estimate for a weighted $L^2(\mathbb{R}^d)$ norm which is uniform in $n \in \mathbb{N}$. From this we conclude the estimates (4.2.30) and (4.2.31) for φ_0, see Corollary 4.2.11. Finally, in Lemma 4.2.12 we prove uniqueness of the solution.

Lemma 4.2.8. *Assume that h has a virtual level at zero and that V satisfies* (4.2.2) *and* (4.2.3). *Then the function φ_0 defined above is not zero and for any $\psi \in \tilde{H}^1(\mathbb{R}^d)$*

$$\langle \nabla\varphi_0, \nabla\psi\rangle + \langle V\varphi_0, \psi\rangle = 0. \tag{4.2.37}$$

Proof of Lemma 4.2.8. At first, we show that $\langle V\psi_n, \psi_n\rangle$ converges to $\langle V\varphi_0, \varphi_0\rangle$ for $n \to \infty$, which is the difficult part of the proof. To do this we consider the behavior of $V\psi_n$ for small and large values of $|x|$ separately. For small $|x|$ we use that $\psi_n \to \varphi_0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ and for large $|x|$ we use the fast decay of the potential V at infinity. Let $R > 0$ be fixed and

$$\langle V\psi, \psi\rangle_{B(R)} = \int_{B(R)} V|\psi|^2 \,\mathrm{d}x, \qquad \psi \in \tilde{H}^1(\mathbb{R}^d), \tag{4.2.38}$$

where $B(R) = \{x \in \mathbb{R}^d : |x| \le R\}$. We write

$$\langle V\psi_n, \psi_n\rangle_{B(R)} - \langle V\varphi_0, \varphi_0\rangle_{B(R)} = \langle V(\psi_n - \varphi_0), \psi_n\rangle_{B(R)} + \langle V\varphi_0, \psi_n - \varphi_0\rangle_{B(R)} \tag{4.2.39}$$

and show that both summands on the r.h.s. tend to zero as $n \to \infty$. Let $\chi : \mathbb{R}^d \to [0,1]$ be a differentiable function, such that

$$\nabla\chi \text{ is bounded}, \quad \chi(x) = 1 \text{ if } x \in B(R), \quad \chi(x) = 0 \text{ if } x \notin B(R+1). \tag{4.2.40}$$

Then we get by monotony and by the Cauchy-Bunjakowski-Schwarz inequality

$$\begin{aligned} \langle |V||\psi_n - \varphi_0|, |\psi_n|\rangle_{B(R)} &\le \langle |V|^{\frac12}|\psi_n - \varphi_0|\chi, |V|^{\frac12}|\psi_n|\chi\rangle \\ &\le \left(\langle |V||\psi_n - \varphi_0|\chi, |\psi_n - \varphi_0|\chi\rangle\right)^{\frac12} \left(\langle |V|\psi_n\chi, \psi_n\chi\rangle\right)^{\frac12}. \end{aligned} \tag{4.2.41}$$

We estimate the two factors on the r.h.s. of (4.2.41) separately. By assumption (4.2.2) we get

$$\langle |V||\psi_n - \varphi_0|\chi, |\psi_n - \varphi_0|\chi\rangle \le \varepsilon \|\nabla\left(|\psi_n - \varphi_0|\chi\right)\|^2 + C(\varepsilon)\|(\psi_n - \varphi_0)\chi\|^2, \tag{4.2.42}$$

where $\varepsilon > 0$ can be chosen arbitrarily small, independently of $n \in \mathbb{N}$. Because of $\|\nabla\psi_n\| \le 1$, $\|\nabla\varphi_0\| \le 1$, the boundedness of χ and $\nabla\chi$ and because χ is compactly supported, the first term on the r.h.s. of (4.2.42) gets arbitrarily small, uniformly in $n \in \mathbb{N}$, if $\varepsilon > 0$ is small enough. The second term tends to zero as $n \to \infty$ because χ

has compact support and $\psi_n \to \varphi_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Hence, we obtain

$$\langle |V||\psi_n - \varphi_0|\chi, |\psi_n - \varphi_0|\chi\rangle \to 0 \qquad (n \to \infty). \tag{4.2.43}$$

Similarly, we can show that $\langle |V|\psi_n\chi, \psi_n\chi\rangle$ is uniformly bounded for $n \in \mathbb{N}$ and therefore in view of (4.2.41) we conclude that $\langle |V||\psi_n - \varphi_0|, |\psi_n|\rangle_{B(R)}$ tends to zero for $n \to \infty$. Analogously we get $\langle V\varphi_0, \psi_n - \varphi_0\rangle_{B(R)} \to 0$ as $n \to \infty$. We conclude that for any fixed $R > 0$

$$\int_{\{|x|\le R\}} V(x)|\psi_n(x)|^2 \,\mathrm{d}x \longrightarrow \int_{\{|x|\le R\}} V(x)|\varphi_0(x)|^2 \,\mathrm{d}x \quad \text{as} \quad n \to \infty. \tag{4.2.44}$$

Now we consider the behavior of $\int_{\{|x|\ge R\}} V(x)|\psi_n(x)|^2 \,\mathrm{d}x$ for sufficiently large $R > 0$. By taking $R > A$, condition (4.2.3) together with the Hardy type inequality (2.2.5) for $d = 1$ and (2.2.6) for $d = 2$, respectively, implies

$$\begin{aligned} \int_{\{|x|>R\}} |V(x)||\psi_n(x)|^2 \,\mathrm{d}x &\le C \int_{\{|x|>R\}} \frac{|\psi_n(x)|^2}{(1+|x|)^{2+\nu}} \,\mathrm{d}x \\ &\le CR^{-\frac{\nu}{2}} \|\psi_n\|^2_{\tilde{H}^1} = CR^{-\frac{\nu}{2}}, \end{aligned} \tag{4.2.45}$$

where the constant C does not depend on $n \in \mathbb{N}$. Due to the semi-continuity of the norm we have $\|\varphi_0\|_{\tilde{H}^1} \le 1$, and therefore we get similarly to (4.2.45)

$$\int_{\{|x|>R\}} |V(x)||\varphi_0(x)|^2 \,\mathrm{d}x \le CR^{-\frac{\nu}{2}}. \tag{4.2.46}$$

We see by (4.2.45) and (4.2.46) that by taking $R > 0$ large enough both integral tails $\int_{\{|x|>R\}} |V(x)||\psi_n(x)|^2 \,\mathrm{d}x$ and $\int_{\{|x|>R\}} |V(x)||\varphi_0(x)|^2 \,\mathrm{d}x$ can be done arbitrarily small, uniformly in $n \in \mathbb{N}$. This, together with (4.2.44) implies

$$\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle \qquad \text{as } n \to \infty. \tag{4.2.47}$$

Now we prove that $\varphi_0 \ne 0$ and that φ_0 satisfies the equation

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0. \tag{4.2.48}$$

By definition of the functions ψ_n and due to the convergence of ψ_n to φ_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ we have

$$\begin{aligned}\langle V\psi_n,\psi_n\rangle &\le -\left(1-n^{-1}\right)\|\nabla\psi_n\|^2\\ &= -\left(1-n^{-1}\right)\left(1-\int_{\{|x|\le 1\}}|\psi_n|^2\,\mathrm{d}x\right)\\ &\longrightarrow -1+\int_{\{|x|\le 1\}}|\varphi_0|^2\,\mathrm{d}x\end{aligned}\tag{4.2.49}$$

as $n\to\infty$. Due to (4.2.47) this yields

$$\langle V\varphi_0,\varphi_0\rangle \le -1+\int_{\{|x|\le 1\}}|\varphi_0|^2\,\mathrm{d}x = -1-\|\nabla\varphi_0\|^2+\|\varphi_0\|^2_{\tilde{H}^1}\tag{4.2.50}$$

and therefore

$$\|\nabla\varphi_0\|^2+\langle V\varphi_0,\varphi_0\rangle \le -1+\|\varphi_0\|^2_{\tilde{H}^1}.\tag{4.2.51}$$

This inequality together with $h\ge 0$ implies $\|\varphi_0\|_{\tilde{H}^1}\ge 1$. On the other hand, we have

$$\|\varphi_0\|_{\tilde{H}^1}\le \liminf_{n\to\infty}\|\psi_n\|_{\tilde{H}^1}=1.\tag{4.2.52}$$

Therefore, we conclude $\|\varphi_0\|_{\tilde{H}^1}=1$ and

$$\|\nabla\varphi_0\|^2+\langle V\varphi_0,\varphi_0\rangle=0.\tag{4.2.53}$$

Standard arguments show that φ_0 satisfies (4.2.29). □

By Lemma 4.2.8 we have proved statement **(i)** of Theorem 4.2.6. Let us continue and prove the estimates (4.2.30) and (4.2.31) for the weighted $L^2(\mathbb{R}^d)$ norm of φ_0. First, we prove an estimate for the functions ψ_n, namely the following lemma.

Lemma 4.2.9. *Assume that V satisfies* (4.2.2) *and* (4.2.3) *and that h has a virtual level at zero. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of eigenfunctions corresponding to negative eigenvalues $E_n<0$ of the operator $h_{n^{-1}}$, normalized as $\|\psi_n\|_{\tilde{H}^1}=1$. Then the following assertions hold:*

(i) *If $d=1$, then for any $0\le\alpha<\frac{1}{2}$ there exists a $C>0$, such that for all $n\in\mathbb{N}$*

$$\|\nabla\left(|\cdot|^\alpha\psi_n\right)\|\le C\quad\textit{and}\quad\|(1+|\cdot|)^{\alpha-1}\psi_n\|\le C.\tag{4.2.54}$$

(ii) *If $d = 2$, then for any $0 \le \alpha < \frac{1}{2}$ there exists a $C > 0$, such that for all $n \in \mathbb{N}$*

$$\|\nabla\left(|\ln(|\cdot|)|^{\alpha}\psi_n\right)\| \le C \tag{4.2.55}$$

and

$$\|(1+|\cdot|)^{-1}(1+|\ln(|\cdot|)|)^{\alpha-1}\psi_n\| \le C. \tag{4.2.56}$$

Remark 4.2.10. The idea of the proof stems from [2], where exponential decay has been proved for eigenfunctions corresponding to eigenvalues below the essential spectrum. The estimates for the decay rate depend on the distance of the eigenvalue to the essential spectrum. We learned the idea from [27, 28]. In our case each of the eigenfunctions ψ_n corresponds to an eigenvalue below the essential spectrum, which implies that each of the ψ_n decays exponentially. However, the sequence of eigenvalues E_n tends to zero as $n \to \infty$, which is the threshold of the essential spectrum of h. This is the reason why we do not have exponential decay with a uniform exponential decay constant for all ψ_n. In [7] a similar statement of the lemma was proved for the case $d \ge 3$. The proof for $d \in \{1,2\}$ is based on the same ideas. However, due to the lack of Hardy's inequality in these dimensions, the proof differs in some places and we have to work more carefully. Therefore, we give the complete proof.

Proof of Lemma 4.2.9. Let G be a differentiable, bounded, real-valued function with bounded derivative. Then, the eigenvalue equation

$$-(1-n^{-1})\Delta\psi_n + V\psi_n = E_n\psi_n \tag{4.2.57}$$

yields

$$\left(1-n^{-1}\right)\langle\nabla\psi_n, \nabla\left(G^2\psi_n\right)\rangle + \langle V\psi_n, G^2\psi_n\rangle = E_n\|G\psi_n\|^2 < 0. \tag{4.2.58}$$

Note that

$$\langle\nabla\psi_n, \nabla\left(G^2\psi_n\right)\rangle + \langle\nabla\left(G^2\psi_n\right), \nabla\psi_n\rangle - 2\|\nabla\left(G\psi_n\right)\|^2 = -2\langle\psi_n, |\nabla G|^2\psi_n\rangle, \tag{4.2.59}$$

which is the basis for the IMS formula. Since

$$\langle\nabla\left(G^2\psi_n\right),\nabla\psi_n\rangle+\langle\nabla\psi_n,\nabla\left(G^2\psi_n\right)\rangle=2\mathrm{Re}\langle\nabla\psi_n,\nabla\left(G^2\psi_n\right)\rangle, \tag{4.2.60}$$

we get

$$\mathrm{Re}\langle\nabla\psi_n,\nabla\left(G^2\psi_n\right)\rangle-\langle\nabla(G\psi_n),\nabla(G\psi_n)\rangle=-\langle\psi_n,|\nabla G|^2\psi_n\rangle. \tag{4.2.61}$$

On the other hand, by (4.2.58) we see that $\langle\nabla\psi_n,\nabla\left(G^2\psi_n\right)\rangle$ is real and therefore

$$\langle\nabla\psi_n,\nabla(G^2\psi_n)\rangle=\|\nabla(G\psi_n)\|^2-\|\psi_n\nabla G\|^2. \tag{4.2.62}$$

Substituting this identity into (4.2.58) implies

$$\left(1-n^{-1}\right)\left(\|\nabla(G\psi_n)\|^2-\|\psi_n\nabla G\|^2\right)+\langle VG\psi_n,G\psi_n\rangle<0. \tag{4.2.63}$$

Proof of statement (i). Let $d=1$ and $0\le\alpha<\frac{1}{2}$ be fixed. For $\varepsilon>0$ and $R>1$ we define

$$G(x)=G_{\varepsilon,R}(x)=\frac{|x|^\alpha}{1+\varepsilon|x|^\alpha}\chi_R(x), \tag{4.2.64}$$

where χ_R is a smooth function with

$$\chi_R(x)=\begin{cases}0, & |x|\le R,\\ 1, & |x|\ge 2R.\end{cases} \tag{4.2.65}$$

Then by definition, the function $G_{\varepsilon,R}$ vanishes in the region $\{x\in\mathbb{R}^d:|x|<R\}$. For $|x|>2R$ we can estimate the gradient of $G_{\varepsilon,R}$ as

$$\left|\nabla G_{\varepsilon,R}(x)\right|=\frac{\alpha|x|^{\alpha-1}}{(1+\varepsilon|x|^\alpha)^2}=\frac{\alpha|x|^{-1}}{1+\varepsilon|x|^\alpha}|G_{\varepsilon,R}|\le\alpha|x|^{-1}|G_{\varepsilon,R}| \tag{4.2.66}$$

and for $R<|x|<2R$ we have

$$\left|\nabla G_{\varepsilon,R}(x)\right|\le\left|\frac{\alpha|x|^{\alpha-1}}{(1+\varepsilon|x|^\alpha)^2}\chi_R(x)\right|+\left|G_{\varepsilon,R}(x)\nabla\chi(x)\right|\le C(R), \tag{4.2.67}$$

for some $C(R)$ depending on R only, and not on ε. By (4.2.67) we get

$$\int_{\{R\le|x|\le 2R\}} |\nabla G_{\varepsilon,R}|^2 |\psi_n|^2 \,\mathrm{d}x \le C_0 \int_{\{R\le|x|\le 2R\}} |\psi_n|^2 \,\mathrm{d}x, \tag{4.2.68}$$

for some $C_0 > 0$ which depends on R only. Now we use inequality (2.2.5) to estimate the r.h.s. of (4.2.68). We get

$$\begin{aligned}\int_{\{R\le|x|\le 2R\}} |\psi_n|^2 \,\mathrm{d}x &\le (1+4R^2) \int_{\{R\le|x|\le 2R\}} \frac{|\psi_n|^2}{1+x^2} \,\mathrm{d}x \\ &\le C_H(1+4R^2)\|\psi_n\|^2_{\tilde{H}^1},\end{aligned} \tag{4.2.69}$$

where C_H is a Hardy type constant in (2.2.5). This, together with (4.2.68) and the normalization $\|\psi_n\|_{\tilde{H}^1} = 1$ implies

$$\int_{\{R\le|x|\le 2R\}} |\nabla G_{\varepsilon,R}|^2 |\psi_n|^2 \le C_1 \tag{4.2.70}$$

for some $C_1 > 0$ which is independent of $n \in \mathbb{N}$ and $\varepsilon > 0$. Substituting (4.2.66) and (4.2.70) into (4.2.63) we obtain

$$\begin{aligned}&\left(1-n^{-1}\right)\|\nabla(G_{\varepsilon,R}\psi_n)\|^2 + \langle V G_{\varepsilon,R}\psi_n, G_{\varepsilon,R}\psi_n\rangle \\ &\qquad - \alpha^2 \int_{\{|x|>2R\}} \frac{|G_{\varepsilon,R}\psi_n|^2}{|x|^2} \,\mathrm{d}x \le C_2,\end{aligned} \tag{4.2.71}$$

where $C_2 > 0$ does not depend on $n \in \mathbb{N}$ or $\varepsilon > 0$. The function $G_{\varepsilon,R}\psi_n$ is supported outside the ball with radius R. Therefore, choosing $R > A$ we can use (4.2.3) and apply Hardy's inequality for the semi axis, which yields

$$(1-\gamma_0)\|\nabla(G_{\varepsilon,R}\psi_n)\|^2 + \langle V G_{\varepsilon,R}\psi_n, G_{\varepsilon,R}\psi_n\rangle - \alpha^2 \langle |x|^{-2} G_{\varepsilon,R}\psi_n, G_{\varepsilon,R}\psi_n\rangle \ge 0 \tag{4.2.72}$$

for all α with $\alpha^2 < \frac{1}{4}$ and all $\gamma_0 < (1-4\alpha^2)$. For $n > 2\gamma_0^{-1}$ the estimates (4.2.71) and (4.2.72) imply

$$\frac{\gamma_0}{2}\|\nabla(G_{\varepsilon,R}\psi_n)\|^2 \le C_2. \tag{4.2.73}$$

Taking the limit $\varepsilon \to 0$ yields

$$\|\nabla\left(|\cdot|^{\alpha}\psi_n\right)\| \le C \tag{4.2.74}$$

for some $C > 0$. Applying Hardy's inequality for the semi axis to the function $G_{\varepsilon,R}\psi_n$, using (4.2.74) and taking the limit $\varepsilon \to 0$ implies

$$\|(1+|\cdot|)^{\alpha-1}\psi_n\| \le C. \tag{4.2.75}$$

This completes the proof of Lemma 4.2.9 for $d = 1$.

Proof of statement (ii). Let $d = 2$ and $0 < \alpha < \frac{1}{2}$ be fixed. For any $\varepsilon > 0$ and $R > 1$ we define the function

$$\tilde{G}_{\varepsilon,R}(x) = \frac{|\ln(|x|)|^\alpha}{1+\varepsilon|\ln(|x|)|^\alpha}\chi_R(x), \tag{4.2.76}$$

where $\chi_R : \mathbb{R}^d \to \mathbb{R}$ is a smooth function with

$$\chi_R(x) = \begin{cases} 0, & |x| \le R, \\ 1, & |x| \ge 2R. \end{cases} \tag{4.2.77}$$

Similarly to (4.2.66), for $|\nabla \tilde{G}_{\varepsilon,R}|$ we have

$$|\nabla \tilde{G}_{\varepsilon,R}| = \frac{\alpha \ln^{\alpha-1}(|x|)}{|x|(1+\varepsilon \ln^\alpha(|x|))^2} \le \alpha \frac{\tilde{G}_{\varepsilon,R}}{|x|\ln(|x|)}, \qquad |x| > 2R, \tag{4.2.78}$$

while for $R < |x| < 2R$ the gradient $\nabla \tilde{G}_{\varepsilon,R}$ is bounded by a constant which depends on R only. Now the proof goes along the same line as the proof of statement **(i)**. For the sake of completeness we present it in detail. By the use of inequality (2.2.6) we have

$$\begin{aligned} \int_{\{R\le|x|\le 2R\}} |\psi_n|^2\,\mathrm{d}x &\le (1+4R^2\ln^2(4R)) \int_{\{R\le|x|\le 2R\}} \frac{|\psi_n|^2}{1+|x|^2\ln^2(|x|)}\,\mathrm{d}x \\ &\le C_H(1+4R^2\ln^2(4R))\|\psi_n\|^2_{\tilde{H}^1}, \end{aligned} \tag{4.2.79}$$

where C_H is a Hardy type constant in (2.2.6). Due to the boundedness of $|\nabla \tilde{G}_{\varepsilon,R}|$ in the region $R < |x| < 2R$ we find

$$\int_{\{R\le|x|\le 2R\}} |\nabla \tilde{G}_{\varepsilon,R}|^2 |\psi_n|^2 \le C_1 \tag{4.2.80}$$

for some $C_1 > 0$ depending on R only. Substituting (4.2.78) and (4.2.80) into (4.2.63) we obtain

$$\begin{aligned}\left(1-n^{-1}\right)\|\nabla(\psi_n\tilde{G}_{\varepsilon,R})\|^2 &+ \langle V\tilde{G}_{\varepsilon,R}\psi_n, \tilde{G}_{\varepsilon,R}\psi_n\rangle \\ &- \alpha^2\int_{\{|x|>2R\}}\frac{|\tilde{G}_{\varepsilon,R}\psi_n|^2}{|x|^2\ln^2(|x|)}\,\mathrm{d}x \le C_2,\end{aligned} \tag{4.2.81}$$

where $C_2 > 0$ does not depend on $n \in \mathbb{N}$ or $\varepsilon > 0$. Since the function $\tilde{G}_{\varepsilon,R}\psi_n$ is supported outside the ball with radius R, by choosing $R > A$, using (4.2.3) and the two-dimensional Hardy inequality we get

$$\begin{aligned}(1-\gamma_0)\|\nabla(\tilde{G}_{\varepsilon,R}\psi_n)\|^2 &+ \langle V\tilde{G}_{\varepsilon,R}\psi_n, \tilde{G}_{\varepsilon,R}\psi_n\rangle \\ &- \alpha^2\int_{\{|x|>2R\}}\frac{|\tilde{G}_{\varepsilon,R}\psi_n|^2}{|x|^2\ln^2(|x|)}\,\mathrm{d}x \ge 0\end{aligned} \tag{4.2.82}$$

for all $\alpha^2 < \frac{1}{4}$ and $\gamma_0 < (1-4\alpha^2)$. For $n > 2\gamma_0^{-1}$ the estimates (4.2.81) and (4.2.82) imply

$$\frac{\gamma_0}{2}\|\nabla(\tilde{G}_{\varepsilon,R}\psi_n)\|^2 \le C_2. \tag{4.2.83}$$

Taking the limit $\varepsilon \to 0$ yields

$$\|\nabla\left(|\ln(|\cdot|)|^\alpha\,\psi_n\right)\| \le C \tag{4.2.84}$$

Applying the two-dimensional Hardy inequality and taking the limit $\varepsilon \to 0$ completes the proof as in the one-dimensional case. □

Now we use Lemma 4.2.9 to derive the claimed estimates (4.2.30) and (4.2.31) of the weighted $L^2(\mathbb{R}^d)$ norm of the function φ_0.

Corollary 4.2.11. *The weak limit φ_0 of the sequence $(\psi_n)_{n\in\mathbb{N}}$ has the following properties.*

(i) *If $d = 1$, then*

$$(1+|\cdot|)^{\alpha-1}\varphi_0 \in L^2(\mathbb{R}) \qquad \text{for any } \alpha < \frac{1}{2}. \tag{4.2.85}$$

(ii) *If $d = 2$, then*

$$(1+|\cdot|)^{-1}\,(1+\ln(|\cdot|))^{\alpha-1}\,\varphi_0 \in L^2(\mathbb{R}^2) \qquad \text{for any } \alpha < \frac{1}{2}. \tag{4.2.86}$$

Proof of Corollary 4.2.11. Let $d = 1$ and let $\alpha < \frac{1}{2}$ be fixed. We show that ψ_n converges to φ_0 in $L^2\left(\mathbb{R}^d, (1+|x|)^{2(\alpha-1)}\,\mathrm{d}x\right)$. For $R > 0$ and $\alpha_0 \in \left(\alpha, \frac{1}{2}\right)$ we have

$$\begin{aligned}\int_{\{|x|\geq R\}} &(1+|x|)^{2(\alpha-1)}\,|\psi_n|^2\,\mathrm{d}x\\ &\leq (1+R)^{2(\alpha-\alpha_0)}\int_{\{|x|\geq R\}}(1+|x|)^{2(\alpha_0-1)}\,|\psi_n|^2\,\mathrm{d}x.\end{aligned} \tag{4.2.87}$$

Since by Lemma 4.2.9 we have $\|(1+|\cdot|)^{\alpha_0-1}\psi_n\| \leq C$ uniformly for $n \in \mathbb{N}$ and due to $\alpha - \alpha_0 < 0$, the r.h.s. tends to zero as $R \to \infty$. Since ψ_n converges to φ_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, this yields the proof of part **(i)** of the Corollary. The case $d = 2$ follows analogously. □

By Corollary 4.2.11 we have proved assertions **(ii)** and **(iii)** of Theorem 4.2.6 and move on to the proof of assertion **(iv)**. We prove it in the following Lemma.

Lemma 4.2.12. *Assume that h has a virtual level at zero and that the potential V satisfies* (4.2.2) *and* (4.2.3). *Then, there exists a $\delta_0 > 0$, such that for any $\psi \in \tilde{H}^1(\mathbb{R}^d)$ with $\langle \nabla\psi, \nabla\varphi_0\rangle = 0$*

$$\langle h\psi, \psi\rangle \geq \delta_0 \|\nabla\psi\|^2. \tag{4.2.88}$$

Proof of Lemma 4.2.12. The proof is a straightforward modification of the proof of Lemma 2.10 in [7]. Assume for a contradiction that (4.2.88) does not hold i.e., there exists a sequence of functions $\tilde{\psi}_n \in H^1(\mathbb{R}^d)$ with $\|\tilde{\psi}_n\|_{\tilde{H}^1} = 1$, $\langle \nabla\tilde{\psi}_n, \nabla\varphi_0\rangle = 0$ and $\left(1-n^{-1}\right)\|\nabla\tilde{\psi}_n\|^2 + \langle V\tilde{\psi}_n, \tilde{\psi}_n\rangle < 0$. Repeating the arguments of Lemmas 4.2.8-4.2.11 we see that there exists a function $\varphi_1 \in \tilde{H}^1(\mathbb{R}^d) \cap L^2\left(\mathbb{R}^d, (1+|x|)^{-2}\,\mathrm{d}x\right)$ with $\|\nabla\varphi_1\|^2 + \langle V\varphi_1, \varphi_1\rangle = 0$, such that $\tilde{\psi}_k \to \varphi_1$ in $L^2\left(\mathbb{R}^d, (1+|x|)^{-2}\,\mathrm{d}x\right)$. Since φ_0 and φ_1 both satisfy (4.2.29), any linear combination of φ_0 and φ_1 is also a minimizer of the quadratic form of h. Due to the orthogonality $\langle \nabla\tilde{\psi}_k, \nabla\varphi_0\rangle = 0$ and the convergence of $\tilde{\psi}_k$ to φ_1 the functions φ_0 and φ_1 are linearly independent. Therefore, the subspace of linear combinations of φ_0 and φ_1 contains two non-trivial functions which are orthogonal in the sense of the weighted L^2 scalar product with weight $(1+|x|)^{-2}$. This implies that there exists a minimizer of the quadratic form of h which has non-trivial positive and non-trivial negative part and each of them is also a minimizer of the quadratic form of h. This contradicts the unique continuation theorem [60, Theorem 2.1] and the Lemma is proved. □

This completes the proof of Theorem 4.2.6.

4.3. Virtual levels of multi-particle Schrödinger operators

4.3.1. Introduction

Now we turn to the multi-particle case. We consider the Hamiltonian H corresponding to a system of $N \geq 2$ one- or two-dimensional particles in the center of mass frame as it was introduced in Section 3.2.1. We assume that the potentials V_{ij} describing the pair interactions between the particles are given by

$$V_{ij}(x) = v_{ij}(x_{ij}), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^{dN}, \quad x_{ij} = x_i - x_j \tag{4.3.1}$$

with v_{ij} satisfying $v_{ij} \neq 0$ and (4.2.2) and (4.2.3). In the following we shall not distinguish between V_{ij} and v_{ij} and sometimes we write $V_{ij}(x_{ij})$ instead of $V_{ij}(x)$. Recall that after reduction of the center of mass the case $N = 2$ coincides with the one-particle case considered in Section 4.2. Hence, we will only consider the case $N \geq 3$. Let us extend the definition of virtual levels to multi-particle Schrödinger operators.

Definition 4.3.1. Assume that the potentials V_{ij} satisfy (4.2.2) and (4.2.3). For a cluster $C \subseteq \{1, \dots, N\}$ we say that the cluster Hamiltonian $H[C]$ has a virtual level at zero if $H[C] \geq 0$ and

(i) there exists a constant $\varepsilon_0 > 0$, such that

$$\inf \sigma_{\text{ess}} \left(H[C] + \varepsilon_0 \Delta_0[C]\right) = 0, \tag{4.3.2}$$

(ii) for any $\varepsilon \in (0,1)$

$$\inf \sigma \left(H[C] + \varepsilon \Delta_0[C]\right) < 0. \tag{4.3.3}$$

Remark 4.3.2. Assume $H[C] \geq 0$ for a cluster C. Then condition (4.3.2) is fulfilled if and only if for no subcluster $\tilde{C} \subset C$ with $1 < |\tilde{C}| < |C|$ the cluster Hamiltonian $H[\tilde{C}]$ has a virtual level at zero. Indeed, if there exists such a cluster $\tilde{C}$, for which the corresponding Hamiltonian has a virtual level, then we have $\inf \sigma \left(H[\tilde{C}] + \varepsilon \Delta_0[\tilde{C}]\right) < 0$ for any $\varepsilon \in (0,1)$ and according to the HVZ theorem condition (4.3.2) can not be fulfilled for $H[C]$.

On the other hand, assume that (4.3.2) does not hold for some cluster C and any $\varepsilon \in (0,1)$. We show that there exists a cluster C_0, such that $H[C_0]$ has a virtual level at zero. By the HVZ theorem there exists a subcluster $\tilde{C}$ of C with $\tilde{C} \neq C$, such that

$$\inf \sigma \left(H[\tilde{C}] + \varepsilon \Delta_0[\tilde{C}] \right) < 0 \tag{4.3.4}$$

holds for any $\varepsilon > \in (0,1)$. Among these clusters we choose one with the smallest number of particles and denote it by C_0. If $|C_0| = 2$, the Hamiltonian $H[C_0]$ can be considered as a one-particle operator with short-range potential, and therefore we have $\inf \sigma_{\text{ess}} (H[C_0] + \varepsilon \Delta_0[C_0]) = 0$ for any $\varepsilon \in (0,1)$. Hence, $H[C_0]$ has a virtual level at zero. If $|C_0| \geq 3$, then, because C_0 is the smallest cluster for which (4.3.4) holds for any $\varepsilon \in (0,1)$, for any subcluster $C' \subsetneq C_0$ with $|C'| > 1$ inequality (4.3.4) does not hold for all $\varepsilon \in (0,1)$ i.e., we have

$$\inf \sigma \left(H[C'] + \varepsilon \Delta_0[C'] \right) = 0 \tag{4.3.5}$$

for some $\varepsilon > 0$. Since C has only a finite number of particles, we can choose this $\varepsilon > 0$, such that (4.3.5) holds for any subcluster of C_0. Thus, by the HVZ theorem we have for some $\varepsilon_0 > 0$

$$\inf \sigma_{\text{ess}} (H[C_0] + \varepsilon_0 \Delta_0[C_0]) = 0 \tag{4.3.6}$$

Since in addition $\inf \sigma (H[C_0] + \varepsilon \Delta_0[C_0]) < 0$ for any $\varepsilon > 0$, $H[C_0]$ has a virtual level.

It was shown in [24] that for systems of $N \geq 3$ particles in dimension three virtual levels correspond to eigenvalues. In [7] this result was extended to space dimension $d \geq 3$ and for the corresponding eigenfunction φ_0 the following estimate was given,

$$\nabla_0 \left(|\cdot|_m^\alpha \varphi_0 \right) \in L^2(X_0) \quad \text{and} \quad (1 + |\cdot|_m)^{\alpha - 1} \varphi_0 \in L^2(X_0) \tag{4.3.7}$$

for any $0 \leq \alpha < \frac{d(N-1)-2}{2}$.

Our goal is to generalize these results to systems of one- and two-dimensional particles. In the next paragraph we give a criterion in terms of a Hardy type constant ensuring that virtual levels correspond to eigenvalues. This criterion, which is the main result of this chapter, applies for systems of $N \geq 3$ one-dimensional or $N \geq 4$ two-dimensional particles. Later, in Sections 4.3.4-4.3.7 we discuss some special cases and prove some further results. If not stated otherwise, the results are from [8].

4.3.2. The main result: A sufficient condition that virtual levels correspond to eigenvalues

Let

$$\mathcal{M} = \left\{\psi \in C_0^1(X_0 \setminus \{0\}) : \psi(x) = 0 \text{ for } x_i = x_j, 1 \le i, j \le N,\ i \ne j\right\} \tag{4.3.8}$$

and

$$\tilde{C}_H(X_0) = \inf_{0 \ne \psi \in \mathcal{M}} \frac{\|\nabla_0 \psi\|}{\||x|_m^{-1}\psi\|}. \tag{4.3.9}$$

The main theorem of this chapter is the following

Theorem 4.3.3. *Let H be the Hamiltonian of a system of $N \ge 3$ d-dimensional particles with $d \ge 1$, where the potentials $V_{ij} \ne 0$ satisfy* (4.2.2) *and* (4.2.3)*. Assume that H has a virtual level at zero and for the constant $\tilde{C}_H(X_0)$ defined in* (4.3.9) *we have $\tilde{C}_H(X_0) > 1$. Then*

(i) *zero is a simple eigenvalue of H and for the corresponding eigenfunction φ_0 we have*

$$\nabla_0\left(|\cdot|_m^{\alpha}\varphi_0\right) \in L^2(X_0) \quad \textit{and} \quad (1 + |\cdot|_m)^{\alpha-1}\varphi_0 \in L^2(X_0) \tag{4.3.10}$$

for any $0 \le \alpha < \tilde{C}_H(X_0)$.

(ii) *There exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(X_0)$ satisfying $\langle \nabla_0\varphi_0, \nabla_0\psi\rangle = 0$ we have*

$$(1 - \delta_0)\|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle \ge 0. \tag{4.3.11}$$

This theorem has the following immediate consequences for systems of one- or two-dimensional particles.

Corollary 4.3.4 (Virtual levels for systems of $N \ge 4$ two-dimensional particles)**.** *If $d = 2$ and $N \ge 4$, then $\tilde{C}_H(X_0)$ coincides with the Hardy constant for the* $2(N-1)$ *dimensional space X_0 i.e., $\tilde{C}_H(X_0) = N - 2 > 1$. Therefore, Theorem 4.3.3 can be applied. In particular, it shows that in this case the solution φ_0 of the Schrödinger equation corresponding to the virtual level is a non-degenerate eigenfunction satisfying*

$$(1 + |\cdot|_m)^{\alpha-1}\varphi_0 \in L^2(X_0) \qquad \textit{for any } \alpha < N - 2. \tag{4.3.12}$$

Corollary 4.3.5 (Virtual levels for systems of $N \geq 4$ one-dimensional particles)**.** *If $d = 1$ and $N \geq 4$, each of the hyperplanes $\{x_i = x_j\}$ divides the space X_0 into two half-spaces. Taking one of these hyperplanes and using that the Hardy constant for the half-space is given by $\frac{N-1}{2}$, see for example [48, Proposition 4.1], we get $\tilde{C}_H(X_0) \geq \frac{N-1}{2} > 1$. Hence, Theorem 4.3.3 can be applied. This implies that zero is a simple eigenvalue of H and the corresponding eigenfunction φ_0 satisfies*

$$(1+|\cdot|_m)^{\alpha-1}\varphi_0 \in L^2(X_0) \qquad \text{for any } \alpha < \frac{N-1}{2}. \tag{4.3.13}$$

Remark 4.3.6. **(i)** The case of three particles is not covered by Corollary 4.3.4 and Corollary 4.3.5. For systems of three one-dimensional particles virtual levels correspond to eigenvalues as we will see in Section 4.3.4. For the case of three two-dimensional particles the condition given in Theorem 4.3.3 is not fulfilled. We will discuss this case in Section 4.3.5.

(ii) Note that in Theorem 4.3.3 and Corollaries 4.3.4 and 4.3.5 we give estimates for the decay rates of the solution φ_0. It was shown in [6] that for systems of particles in dimension $d \geq 3$ the solution φ_0 decays with the same rate as the fundamental solution of the Laplace operator in $L^2(\mathbb{R}^k)$ with $k = d(N-1)$. There, the estimate for the decay rate given in (4.3.7) was combined with the representation of the solution φ_0 as convolution with the Green function. This method can not directly be extended to the one- or two-dimensional case due to the different behavior of the Green function, and it seems to be more difficult to obtain the exact decay rate. In fact, there are not only technical differences, but also a different decay behavior occurs for one-dimensional particles as we will see later (see Section 4.3.4).

4.3.3. Proof of Theorem 4.3.3

The proof consists of several steps. Recall that for multi-particle systems the potential V does not decay in all directions even if the pair potentials V_{ij} are compactly supported. In the first step we generalize Theorem 4.2.6, where we studied virtual levels of Schrödinger operators with short-range potentials, to potentials which do not decay in all directions. Later, we will show that the Hamiltonian H of the system, considered as an operator on $L^2(\mathbb{R}^{d(N-1)})$, satisfies the conditions of this theorem. To do this we use geometric methods which include a partition of unity of the configuration space. Since the estimates for the localization error which can be found in the literature are not appropriate in dimensions one and two, we will need improved estimates.

Virtual levels of one-particle Schrödinger operators with non-decaying potentials

Theorem 4.3.7. *Consider the operator $h = -\Delta + V$ acting on $L^2(\mathbb{R}^k)$, $k \in \mathbb{N}$, where the potential V satisfies* (4.2.2). *Suppose that h has a virtual level at zero and that there exist constants $\alpha_0 > 1$, $b > 0$ and $\gamma_0 \in (0,1)$, such that for any function $\psi \in H^1(\mathbb{R}^k)$ with* $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^d : |x| \geq b\}$ *we have*

$$\langle h\psi, \psi\rangle - \gamma_0 \|\nabla\psi\|^2 - \langle \alpha_0^2 |x|^{-2}\psi, \psi\rangle \geq 0. \tag{4.3.14}$$

Then zero is a simple eigenvalue of h. The eigenfunction φ_0 satisfies

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^k) \tag{4.3.15}$$

and for any $\alpha < \alpha_0$

$$\begin{aligned} &(1+|\cdot|)^{\alpha-1}\varphi_0 \in L^2(\mathbb{R}^k) && \textit{if } k \neq 2, \\ \textit{and} \quad &(1+|\cdot|)^{\alpha-1}(1+\ln(|\cdot|))^{-1}\varphi_0 \in L^2(\mathbb{R}^k) && \textit{if } k = 2. \end{aligned} \tag{4.3.16}$$

Moreover, there exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^k)$ with $\langle \nabla\psi, \nabla\varphi_0\rangle = 0$

$$\langle h\psi, \psi\rangle \geq \delta_0 \|\nabla\psi\|^2. \tag{4.3.17}$$

Remark 4.3.8. **(i)** For dimensions $k \geq 3$ the statement of Theorem 4.3.7 has been proved in [7, Theorem 2.1]. Here, we generalize it to dimensions $k = 1$ and $k = 2$. Since the proof is very similar to the one of Theorem 2.1 in [7], supplemented by arguments of the proof of Theorem 4.2.6, we skip it here. We give the proof of Theorem 4.3.7 for dimensions $k = 1$ and $k = 2$ in Appendix A.1.

(ii) Later, we will apply this theorem with dimension $k = d(N-1)$ for the multi-particle Schrödinger operator corresponding to a system of N d-dimensional particles.

(iii) For dimension $k = 1$ or $k = 2$ Theorem 4.3.7 considers the case which is in some sense complementary to the one studied in Theorem 4.2.6. In Theorem 4.2.6 we assumed that the potential V decays fast at infinity. In Theorem 4.3.7 we do not have decay of the potential at infinity. Instead of this we use inequality (4.3.14) for functions ψ which are supported far away from the origin. This condition can not be fulfilled for $k = 1$ and $k = 2$ if V decays fast at infinity, because the term $-\langle |x|^{-2}\psi, \psi\rangle$ can not be controlled by the kinetic energy. Moreover, under the conditions of Theorem 4.3.7 virtual levels correspond to eigenvalues of h. In contrast to that, under the conditions of Theorem 4.2.6 they correspond to resonances.

Estimate of the localization error

As already mentioned, to prove Theorem 4.3.3 we apply Theorem 4.3.7 to the multi-particle operator H. To show that the conditions of this theorem are fulfilled we use geometric methods which are similar as in [7]. A central element of these methods is a separation of clusters of the particles in the system. This is done by a partition of unity of the configuration space which separates regions $K(Z,\kappa)$ corresponding to different partitions Z. An important step is to find an appropriate estimate of the resulting localization error.

If the dimension of the particles is $d \geq 3$, one can use an estimate given in [74, Lemma 5.1]. This estimate shows that when we separate a cone $K(Z,\kappa)$, then the localization error can be estimated as $\varepsilon|q(Z)|_m^{-2}$ with an arbitrarily small $\varepsilon > 0$. For particles in dimension $d \geq 3$ one can then use Hardy's inequality to control this term by a small part of the kinetic energy. However, in our case, where the particles are one- or two-dimensional, the estimate given in [74] cannot be used, because Hardy's inequality for dimension two needs a different Hardy weight. Therefore, we need a significant improvement of the estimate of the localization error. This is given in the following theorem.

Theorem 4.3.9. *Given $\varepsilon > 0$ and $0 < \kappa < 1$, for each partition Z with $|Z| \geq 2$ one can find a constant $0 < \kappa' < \kappa$ and piece-wise differentiable functions $u_Z, v_Z : X_0 \to \mathbb{R}$, such that*

$$u_Z^2 + v_Z^2 = 1, \qquad u_Z(x) = \begin{cases} 1 & \text{if } x \in K\left(Z,\kappa'\right), \\ 0 & \text{if } x \notin K(Z,\kappa) \end{cases} \tag{4.3.18}$$

and

$$|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 < \varepsilon \left[|v_Z|^2 |x|_m^{-2} + |u_Z|^2 |q|_m^{-2} \ln^{-2}\left(|q|_m |\xi|_m^{-1}\right)\right] \tag{4.3.19}$$

for $x \in K\left(Z,\kappa',\kappa\right)$. Here, $q = q(Z)$ and $\xi = \xi(Z)$.

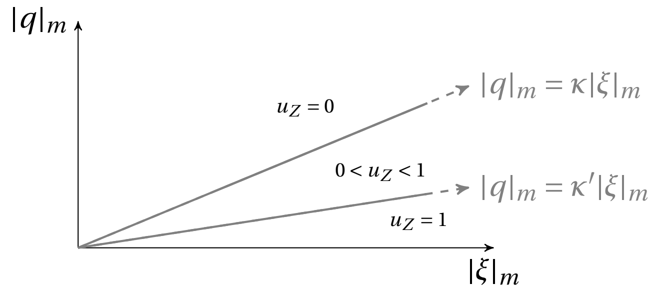

Figure 4.1.: The function u_Z

To prove Theorem 4.3.9 we will use an auxiliary result for scalar functions, namely the following

Lemma 4.3.10. *For any $\varepsilon > 0$ and any $0 < \beta < 1$ one can find a constant $0 < \alpha < \beta^2$ and a non-increasing function $u \in H^1(\alpha,\beta) \cap C([\alpha,\beta])$ which is piece-wise continuously differentiable, such that $u(\alpha) = 1,\ u(\beta) = 0$ and*

$$(u'(t))^2 \leq \varepsilon t^{-2} \ln^{-2}(t), \qquad \alpha \leq t \leq \beta. \tag{4.3.20}$$

Proof of Lemma 4.3.10. Let $\varepsilon > 0$ and $\beta \in (0,1)$ be fixed. For any $0 < \gamma < 1$ and $\alpha \in (0,\beta^2)$ let $u_{\alpha,\gamma} : [\alpha,\beta] \to \mathbb{R}$ be given by

$$u_{\alpha,\gamma}(t) := \begin{cases} \left|\ln(\alpha\beta^{-1})\right|^{-\gamma} \left|\ln(t\beta^{-1})\right|^{\gamma} & \text{if } \alpha \leq t \leq \beta^2, \\ \left|\ln(\alpha\beta^{-1})\right|^{-\gamma} \left|\ln\beta\right|^{\gamma-1} \left|\ln(t\beta^{-1})\right| & \text{if } \beta^2 \leq t \leq \beta. \end{cases} \tag{4.3.21}$$

It is evident that u is continuous and piecewise continuously differentiable and satisfies $u_{\alpha,\gamma}(\alpha) = 1$ and $u_{\alpha,\gamma}(\beta) = 0$. We will show that for appropriately chosen $\alpha, \gamma > 0$ the function $u'_{\alpha,\gamma}$ satisfies the estimate claimed in the lemma. At first, we show that the claimed estimate holds on the interval (α, β^2), uniformly for $0 < \alpha < \beta^2$ if $\gamma > 0$ is small enough. Afterwards, we fix such $\gamma > 0$ and choose $\alpha > 0$ sufficiently small, such that we get the claimed estimate on the interval (β^2, β) as well. For $\alpha < t < \beta^2$ we have

$$\left(u'_{\alpha,\gamma}(t)\right)^2 = \gamma^2 \left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln(t\beta^{-1})\right|^{2(\gamma-1)} t^{-2}. \tag{4.3.22}$$

Note that due to $\alpha\beta^{-1} < 1$ and $t\beta^{-1} < 1$ for $\alpha \le t \le \beta^2$ we have $\left|\ln(\alpha\beta^{-1})\right| \ge \left|\ln(t\beta^{-1})\right|$. This yields

$$\left(u'_{\alpha,\gamma}(t)\right)^2 \le \gamma^2 \left|\ln(t\beta^{-1})\right|^{-2} t^{-2}, \qquad \alpha < t < \beta^2. \tag{4.3.23}$$

Since for $t, \beta \ge 0$

$$t \le \beta^2 \quad \Longleftrightarrow \quad \sqrt{t} \le \beta \quad \Longleftrightarrow \quad t\beta^{-1} \le \sqrt{t} \tag{4.3.24}$$

and because of $\beta < 1$, we have $\left|\ln(t\beta^{-1})\right| \ge \left|\ln\sqrt{t}\right| = \frac{1}{2}|\ln t|$. This implies

$$\left(u'_{\alpha,\gamma}(t)\right)^2 \le 4\gamma^2 |\ln t|^{-2} t^{-2}, \qquad \alpha < t < \beta^2. \tag{4.3.25}$$

Choosing γ such that $0 < \gamma < \frac{\sqrt{\varepsilon}}{2}$ we get

$$\left(u'_{\alpha,\gamma}(t)\right)^2 \le \varepsilon |\ln t|^{-2} t^{-2}, \qquad \alpha < t < \beta^2. \tag{4.3.26}$$

Now we fix $0 < \gamma < \frac{\sqrt{\varepsilon}}{2}$ and estimate $(u'_{\alpha,\gamma}(t))^2$ for $\beta^2 < t < \beta$. In this case we have

$$\left(u'_{\alpha,\gamma}(t)\right)^2 = \left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln\beta\right|^{2(\gamma-1)} t^{-2}. \tag{4.3.27}$$

Since $\beta < 1$, we have $|\ln\beta^2| \ge |\ln t|$ for $\beta^2 \le t \le \beta$ and therefore

$$\begin{aligned} \left(u'_{\alpha,\gamma}(t)\right)^2 &\le \left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln\beta\right|^{2(\gamma-1)} |\ln\beta^2|^2 |\ln t|^{-2} t^{-2} \\ &= 4 \left|\ln(\alpha\beta^{-1})\right|^{-2\gamma} \left|\ln\beta\right|^{2\gamma} |\ln t|^{-2} t^{-2} \le \varepsilon |\ln t|^{-2} t^{-2} \end{aligned} \tag{4.3.28}$$

if α is chosen small enough. This completes the proof of Lemma 4.3.10. □

Now we turn to the

Proof of Theorem 4.3.9. Let Z be a partition of the system with $|Z| \geq 2$ and let $\varepsilon > 0$ and $0 < \kappa < 1$ be fixed. We construct functions u_Z, v_Z which satisfy the conditions of Theorem 4.3.9. This is done in several steps. For the sake of convenience we write q and ξ instead of $q(Z)$ and $\xi(Z)$, respectively.

Step 1: Definition of the functions u_Z and v_Z in the vicinity of $|q|_m = \kappa|\xi|_m$.
Let $v_1 \in H^1(\mathbb{R}_+)$ be a function satisfying

- v_1 is non-decreasing on $\mathbb{R}_+$,
- $v_1(t) = 1$ for $t \geq \kappa$ and $0 \leq v_1(t) < 1$ for $t < \kappa$,
- $v_1'(t)(1 - v_1^2(t))^{-\frac{1}{2}} \to 0$ as $t \to \kappa_-$.

For $x \in X_0,\ x = q + \xi$, let

$$v_Z(x) = v_1\left(\frac{|q|_m}{|\xi|_m}\right), \qquad u_Z(x) = \sqrt{1 - v_Z^2(x)}. \tag{4.3.29}$$

Then for $x \in K(Z,\kappa)$ we have

$$\begin{aligned} |\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 &= |\nabla_0 v_Z|^2 \left(1 - v_Z^2\right)^{-1} \\ &= (v_1'(t))^2 \left(1 - v_1^2(t)\right)^{-1} \left(1 + |q|_m^2 |\xi|_m^{-2}\right) |\xi|_m^{-2}, \end{aligned} \tag{4.3.30}$$

where $t = |q|_m |\xi|_m^{-1}$. For $x \in K(Z,\kappa)$ we have $|\xi|_m^{-2} \leq \left(1 + \kappa^2\right) |x|_m^{-2}$ and $\frac{|q|_m}{|\xi|_m} \leq \kappa$. This implies

$$|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 \leq (v_1'(t))^2 \left(1 - v_1^2(t)\right)^{-1} (1 + \kappa^2)^2 |x|_m^{-2}. \tag{4.3.31}$$

Since $v_1'(t)(1 - v_1^2(t))^{-\frac{1}{2}} \to 0$ as $t \nearrow \kappa$, we can find $0 < \kappa'' < \kappa$ so close to κ that

$$(v_1'(t))^2 \left(1 - v_1^2(t)\right)^{-1} (1 + \kappa^2)^2 \leq \varepsilon v_1^2(t), \qquad \kappa'' \leq t < \kappa. \tag{4.3.32}$$

This implies

$$|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2 \leq \varepsilon v_Z^2 |x|_m^{-2}, \qquad x \in K(Z,\kappa) \setminus K(Z,\kappa''). \tag{4.3.33}$$

Step 2: Definition of v_Z for $x \in K(Z,\kappa'')$.
Now we define v_Z for $x \in K(Z,\kappa'')$. By Lemma 4.3.10, for given $\tilde{\varepsilon} > 0$ we find a constant $0 < \kappa' < \kappa''$ and a non-decreasing function v_2, such that

- $v_2(\kappa'') = v_1(\kappa'')$,
- $v_2(\kappa') = 0$
- $\left(v_2'(t)\right)^2 \le \tilde{\varepsilon}|t|^{-2}\ln^{-2} t \quad \text{for} \quad \kappa' < t < \kappa''.$

We choose v_2 in such a way that v_2 is strictly increasing on (κ',κ''). For $x \in K(Z,\kappa'')$ with $x = q + \xi$, let

$$v_Z(x) = v_2\left(\frac{|q|_m}{|\xi|_m}\right), \qquad u_Z(x) = \sqrt{1 - v_Z^2(x)}. \tag{4.3.34}$$

Then, similar to (4.3.30) we have

$$\left(|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2\right) u_Z^{-2} = (v_2'(t))^2 \left(1 - v_2^2(t)\right)^{-1} u_Z^{-2} \cdot \left(1 + t^2\right)|\xi|_m^{-2}, \tag{4.3.35}$$

where $t = |q|_m|\xi|_m^{-1}$. Since v_2 is increasing and $u_Z = \sqrt{1 - v_Z^2}$, we have

$$(v_2'(t))^2(1 - v_2^2(t))^{-1} u_Z^{-2} \le \left(v_2'(t)\right)^2 \left(1 - v_2^2(k'')\right)^{-2}, \qquad t \le \kappa''. \tag{4.3.36}$$

Substituting this estimate into (4.3.35) and using $t \le \kappa''$ we get

$$\left(|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2\right) u_Z^{-2} \le (v_2'(t))^2 \left(1 - v_2(k'')^2\right)^{-2} \left(1 + (\kappa'')^2\right)|\xi|_m^{-2}. \tag{4.3.37}$$

Recall that $v_2(\kappa'')$ is possibly close to one, but strictly less then one. Because of $\left(v_2'(t)\right)^2 \le \tilde{\varepsilon}|t|^{-2}\ln^{-2} t$ we get

$$\left(|\nabla_0 u_Z|^2 + |\nabla_0 v_Z|^2\right) u_Z^{-2} \le \tilde{\varepsilon}\, t^{-2}\ln^{-2} t \left(1 - v_2(k'')^2\right)^{-2} \left(1 + (\kappa'')^2\right)|\xi|_m^{-2}. \tag{4.3.38}$$

Choosing $\tilde{\varepsilon} > 0$ so small that $\tilde{\varepsilon}\left(1 - v_2(k'')^2\right)^{-2}\left(1 + (\kappa'')^2\right) < \varepsilon$ and using $t = |q|_m|\xi|_m^{-1}$ completes the proof of Theorem 4.3.9. □

Proof of Theorem 4.3.3

Now we turn to the proof of Theorem 4.3.3. It is an application of Theorem 4.3.7 and we use geometric methods to prove that all conditions of the latter theorem are fulfilled. Since the pair potentials V_{ij} are relatively form bounded, so is $V = \sum_{1\leq i<j\leq N} V_{ij}(x_{ij})$. Hence, we only need to show that condition (4.3.14) is fulfilled for any $0 \leq \alpha < \tilde{C}_H(X_0)$. This is done in the following

Lemma 4.3.11. *Let $d \in \{1,2\}$ and $N \geq 3$. Assume that the potentials V_{ij} satisfy (4.2.2) and (4.2.3). Furthermore, suppose that H has a virtual level at zero. Then for any $0 \leq \alpha < \tilde{C}_H(X_0)$ there exist constants γ_0, $R > 0$, such that for any function $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{x \in X_0 : |x|_m \geq R\}$ we have*

$$L[\varphi] := (1-\gamma_0)\|\nabla_0\varphi\|^2 + \langle V\varphi,\varphi\rangle - \alpha^2\||x|_m^{-1}\varphi\|^2 \geq 0. \tag{4.3.39}$$

To explain the strategy we give the proof for the case $N = 3$ first.

Proof of Lemma 4.3.11 for $N = 3$. In the proof we use the idea of the proof of Theorem 4.4 in [7], where an analogue statement was proved for $d \geq 3$. We take $\kappa > 0$ so small that cones $K(Z,\kappa)$ and $K(Z',\kappa)$ do not overlap for different partitions $Z \neq Z'$ of the system into two clusters. This is possible because of Lemma 3.2.4. For given $\varepsilon > 0$ we choose functions u_Z according to Theorem 4.3.9. Recall that in (4.3.39) we only consider functions φ which are supported outside the ball $B(R)$ with radius $R > 0$. Since the cones $K(Z,\kappa)$ do not overlap, the functions u_Z with $|Z| = 2$ and $\mathcal{V} = \sqrt{1-\sum_{Z:|Z|=2} u_Z^2}$, restricted to $X_0 \setminus B(R)$, are a partition of unity of $X_0 \setminus B(R)$. Therefore, we get by Theorem 4.3.9

$$L[\varphi] \geq \sum_{Z:|Z|=2} L_2[\varphi u_Z] + L_2'[\mathcal{V}\varphi], \tag{4.3.40}$$

where the functionals $L_2, L_2' : H^1(X_0) \to \mathbb{R}$ are given by

$$\begin{aligned} L_2[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - \alpha^2\||x|_m^{-1}\psi\|^2 \\ &\qquad - \varepsilon\||q(Z)|_m^{-1}\ln^{-1}\left(|q(Z)|_m|\xi(Z)|_m^{-1}\right)\psi\|^2_{K_R(Z,\kappa',\kappa)}, \\ L_2'[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\psi\|^2. \end{aligned} \tag{4.3.41}$$

We recall that by Theorem 4.3.9 the constants $\varepsilon > 0$ and $\kappa > 0$ in (4.3.41) can be chosen arbitrarily small if $\kappa' > 0$ is sufficiently small. Note also that the terms $\varepsilon \| |x|_m^{-1} \psi \|^2$ and $\varepsilon \| |q(Z)|_m^{-1} \ln^{-1} \left(|q(Z)|_m |\xi(Z)|_m^{-1} \right) \psi \|^2_{K_R(Z,\kappa',\kappa)}$ come from the estimate of the localization error given in Theorem 4.3.9. We estimate the functional L by estimating $L_2[\varphi u_Z]$ and $L_2'[\mathcal{V}\varphi]$ separately.

Estimate of the functional $L_2[\varphi u_Z]$.

Let $Z = \{C_1, C_2\}$ be an arbitrary partition of the system into two clusters. Our goal is to prove that $L_2[\varphi u_{Z_2}] \geq 0$ holds for any function $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{|x|_m \geq R\}$, whenever $R > 0$ is large enough and the constants $\varepsilon, \kappa > 0$ in the definition of L_2 are sufficiently small. Since we consider a fixed breaking Z, we can omit the index Z in the following computations. We write q and ξ instead of $q(Z)$ and $\xi(Z)$, respectively and denote $\psi = \varphi u_Z$. We rewrite

$$
\begin{aligned}
L_2[\psi] =& \langle H(Z)\psi, \psi \rangle - \gamma_0 \|\nabla_q \psi\|^2 + \left(1 - \gamma_0\right) \left\| \nabla_\xi \psi \right\|^2 + \langle I(Z)\psi, \psi \rangle \\
& - \alpha^2 \left\| |x|_m^{-1} \psi \right\|^2 - \varepsilon \left\| |q|_m^{-1} \ln^{-1} \left(|q|_m |\xi|_m^{-1} \right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)}.
\end{aligned}
\tag{4.3.42}
$$

First, we estimate the inter-cluster potential $I(Z)$. We assume that $|C_1| = 2$ and denote the particles in the cluster C_1 by i, j and the particle in C_2 by k. Then by (3.2.42) for $x \in K(Z, \kappa)$

$$
|x_i - x_k| \geq c|\xi|_m \quad \text{and} \quad |x_j - x_k| \geq c|\xi|_m \tag{4.3.43}
$$

for some $c > 0$ depending on the masses of the particles and the constant κ only. Since for $x \in K_R(Z, \kappa)$ we have $|\xi|_m \geq \left(1 + \kappa^2\right)^{-\frac{1}{2}} |x|_m$, we can choose $R > 0$ so large that $c|\xi|_m \geq A$ holds for all $x \in K_R(Z, \kappa)$, where A is the constant in the condition (4.2.3). Therefore, we can estimate

$$
|I(Z)(x)| \leq |V_{ik}(x)| + |V_{jk}(x)| \leq C|\xi|_m^{-2-\nu} \leq \varepsilon |\xi|_m^{-2}, \tag{4.3.44}
$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $R > 0$ is large enough. Furthermore, on the support of ψ we have $|q|_m \leq \kappa |\xi|_m$ and therefore the Poincaré-Friedrichs inequality (Theorem 2.1.11) yields

$$
\gamma_0 \|\nabla_q \psi\|^2 \geq \frac{\gamma_0}{2\kappa^2} \| |\xi|_m^{-1} \psi \|^2. \tag{4.3.45}
$$

By choosing $\kappa > 0$ small enough this together with (4.3.44) implies

$$\gamma_0 \|\nabla_q \psi\|^2 + \langle I(Z)\psi, \psi\rangle - \alpha^2 \left\| |x|_m^{-1} \psi \right\|^2 \geq 0 \tag{4.3.46}$$

and therefore

$$L_2[\psi] \geq \langle H(Z)\psi, \psi\rangle - 2\gamma_0 \|\nabla_q \psi\|^2 - \varepsilon \left\| |q|_m^{-1} \ln^{-1}\left(|q|_m |\xi|_m^{-1}\right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)}. \tag{4.3.47}$$

To estimate the r.h.s. of (4.3.47) we distinguish between the case of one-dimensional and the case of two-dimensional particles.
(i) If the particles are one-dimensional, we have $\dim(X_0(Z)) = 1$ and due to $|C_2| = 1$

$$\langle H(Z)\psi, \psi\rangle = \langle H[C_1]\psi, \psi\rangle \quad \text{and} \quad \|\nabla_{q(Z)}\psi\| = \|\nabla_{q[C_1]}\psi\|, \tag{4.3.48}$$

where here we consider the operators $H(Z)$ and $H[C_1]$ as operators on $L^2(X_0)$. We will use the letters q for $q(Z)$ and $q[C_1]$ and ξ for $\xi(Z)$ and $\xi[C_1]$ simultaneously. We estimate the last term on the r.h.s. of (4.3.47) by

$$\varepsilon \left\| |q|_m^{-1} \ln^{-1}\left(|q|_m |\xi|_m^{-1}\right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)} \leq \varepsilon \|(1+|q|_m)^{-1}\psi\|^2_{K_R(Z,\kappa',\kappa)}, \tag{4.3.49}$$

which is true for $\kappa > 0$ small enough and $R > 0$ sufficiently large. This yields

$$L_2[\psi] \geq \langle H[C_1]\psi, \psi\rangle - 2\gamma_0 \|\nabla_q \psi\|^2 - \varepsilon \|(1+|q|_m)^{-1}\psi\|^2_{K_R(Z,\kappa',\kappa)}. \tag{4.3.50}$$

Since C_1 consists of two particles and $H[C_1]$ is the operator with the center of mass of C_1 removed, we can consider $H[C_1]$ as one-particle Schrödinger operator on $L^2(\mathbb{R})$. Moreover, by Remark 4.3.2 the operator $H[C_1]$ does not have a virtual level. Therefore, we can use Corollary 4.2.3 to conclude that $L_2[\psi] \geq 0$ for $\varepsilon > 0$ and $\gamma_0 > 0$ small enough and $R > 0$ sufficiently large.
(ii) If the particles are two-dimensional, we have $\dim(X_0(Z)) = 2$. In this case we use that since $H[C_1]$ does not have a virtual level, we have

$$\langle H[C_1]\psi, \psi\rangle \geq 3\gamma_0 \|\nabla_{q[C_1]}\psi\|^2 \tag{4.3.51}$$

for $\gamma_0 > 0$ small enough. This, together with (4.3.47) and (4.3.48) implies

$$L_2[\psi] \geq \gamma_0 \|\nabla_q \psi\|^2 - \varepsilon \left\| |q|_m^{-1} \ln^{-1}\left(|q|_m |\xi|_m^{-1}\right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)}. \tag{4.3.52}$$

To estimate the r.h.s. of (4.3.52) we introduce the new variable $y = \frac{q}{|\xi|_m}$. Then we get

$$\begin{aligned} &\gamma_0 \|\nabla_q \psi\|^2 - \varepsilon \left\| |q|_m^{-1} \ln^{-1}\left(|q|_m |\xi|_m^{-1}\right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)} \\ &\geq \int \int_{\{\kappa'|\xi|_m \leq |q|_m \leq \kappa |\xi|_m\}} \left(\gamma_0 |\nabla_q \psi|^2 - \varepsilon |q|_m^{-2} \left|\ln^{-2}\left(|q|_m |\xi|_m^{-1}\right)\right| |\psi|^2\right) \mathrm{d}q\, \mathrm{d}\xi \\ &= \int \frac{1}{|\xi|_m^2} \int_{\{\kappa' \leq |y|_m \leq \kappa\}} \left(\gamma_0 |\nabla_y \tilde{\psi}(y,\xi)|^2 - \varepsilon |y|_m^{-2} \left|\ln^{-2}\left(|y|_m\right)\right| |\tilde{\psi}(y,\xi)|^2\right) \mathrm{d}y\, \mathrm{d}\xi, \end{aligned} \tag{4.3.53}$$

where $\tilde{\psi}(y,\xi) = \psi(y|\xi|_m, \xi)$. Note that $\tilde{\psi}(y,\xi) = 0$ for $|y|_m \geq \kappa$. By choosing $\kappa < 1$ we have $(\ln|y|_m)^{-2} \leq C(1 + (\ln|y|_m)^2)^{-1}$ for some $C > 0$ and $|y|_m \leq \kappa$. Therefore, applying the two-dimensional Hardy type inequality, given in Lemma 2.1.10, to the function $\tilde{\psi}(y,\xi)$ for fixed ξ shows that the r.h.s. of (4.3.53) is non-negative for sufficiently small $\varepsilon > 0$. This proves $L_2[\psi] \geq 0$.

Estimate of $\mathbf{L}_2'[\mathcal{V}\psi]$.

Note that $\mathcal{V}\varphi$ is supported in the region where all particles are separated. More precisely, supp $(\mathcal{V}\varphi) \subseteq X_0 \setminus K(Z,\kappa')$ for all partitions Z with $|Z| = 2$. Let i, j, k be pairwise distinct and $Z^{(k)} = \{\{i,j\},\{k\}\}$. Then for $x \in X_0 \setminus K(Z^{(k)},\kappa')$ we have

$$|x_i - x_j|^2 = \frac{m_1 m_2}{m_1 + m_2} |q(Z^{(k)})|_m^2 \geq \frac{m_1 m_2}{m_1 + m_2} \left(1 + (\kappa')^{-2}\right)^{-1} |x|_m^2. \tag{4.3.54}$$

Hence, for $x \in$ supp $(\mathcal{V}\varphi)$ we have $|x_i - x_j| \geq A$ if $R > 0$ is large enough and thus

$$|V_{ij}(x)| \leq c|x|_m^{-2-\nu} \leq \varepsilon |x|_m^{-2}. \tag{4.3.55}$$

Inserting this estimate in the definition of $L'[\mathcal{V}\varphi]$ we get

$$L_2'[\mathcal{V}\varphi] \geq (1-\gamma_0)\|\nabla_0(\mathcal{V}\varphi)\|^2 - (\alpha^2 + 4\varepsilon)\| |x|_m^{-1} \mathcal{V}\varphi\|^2. \tag{4.3.56}$$

Since $\mathcal{V}\varphi$ can be approximated (in the norm of $H^1(X_0)$) by functions in $\mathcal{M}$, we get

$$\|\nabla_0(\mathcal{V}\varphi)\|^2 \ge \left(\tilde{C}_H(X_0)\right)^2 \||x|_m^{-1}\mathcal{V}\varphi\|^2. \tag{4.3.57}$$

Due to $0 \le \alpha < \tilde{C}_H(X_0)$ we obtain $L_2'[\mathcal{V}\varphi] \ge 0$ for $\gamma_0 > 0$ and $\varepsilon > 0$ small enough. This completes the proof of Lemma 4.3.11 for the case $N = 3$. □

Remark 4.3.12. Note that for particles with space dimension $d \ge 3$ one can directly use the absence of virtual levels for the cluster Hamiltonians and Hardy's inequality to estimate the r.h.s of (4.3.47). For the case of two-dimensional particles we used the two-dimensional Hardy type inequality, which required an improved estimate of the localization error and the substitution (4.3.53). For the one-dimensional case Hardy's inequality can not be used because the function $\varphi(\cdot,\xi)$ does not vanish for $q = 0$. In this case we used $\int_{\mathbb{R}} V(x)\,\mathrm{d}x > 0$ to compensate for the localization error.

Now we extend the proof of Lemma 4.3.11 to systems of more than three particles.

Proof of Lemma 4.3.11 for $N \ge 4$. The proof is a generalization of the one for the three-particle case. We estimate the functional L in cones $K(Z,\kappa)$ corresponding to partitions Z with increasing order $|Z|$. Let $\kappa_2 > 0$ be so small that cones $K(Z,\kappa_2)$ and $K(Z',\kappa_2)$ do not overlap for different partitions $Z \ne Z'$ with $|Z| = |Z'| = 2$. Then analogously to the three-particle case we get

$$L[\varphi] \ge \sum_{Z:|Z|=2} L_2[\varphi u_Z] + L_2'[\mathcal{V}^{(2)}\varphi], \tag{4.3.58}$$

where $\mathcal{V}^{(2)} = \sqrt{1 - \sum_{Z:|Z|=2} u_Z^2}$ for $|x|_m \ge R$ and the functionals $L_2, L_2' : H^1(X_0) \to \mathbb{R}$ are defined analogously to (4.3.41). By estimating the inter-cluster potential $I(Z)$ as in the case $N = 3$ we arrive at

$$L_2[\psi] \ge \langle H(Z)\psi,\psi\rangle - 2\gamma_0\|\nabla_q\psi\|^2 - \varepsilon\left\||q|_m^{-1}\ln^{-1}\left(|q|_m|\xi|_m^{-1}\right)\psi\right\|^2_{K_R(Z,\kappa',\kappa)}. \tag{4.3.59}$$

Since the Hamiltonians $H[C]$ with $|C| > 1$ do not have virtual levels, we have

$$\langle H(Z)\psi,\psi\rangle \ge 3\gamma_0\|\nabla_q\psi\|^2 \tag{4.3.60}$$

for sufficiently small $\gamma_0 > 0$ and therefore, by (4.3.59),

$$L_2[\psi] \geq \gamma_0 \|\nabla_q \psi\|^2 - \varepsilon \left\| |q|_m^{-1} \ln^{-1}\left(|q|_m |\xi|_m^{-1}\right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)} \tag{4.3.61}$$

if $\gamma_0 > 0$ is small enough. If $d = 1$ and $N = 4$, we have $\dim(X_0(Z)) = 2$ and we can proceed as in the case of three two-dimensional particles. In all other cases we have $\dim(X_0(Z)) \geq 3$ and we estimate

$$\varepsilon \left\| |q|_m^{-1} \ln^{-1}\left(|q|_m |\xi|_m^{-1}\right) \psi \right\|^2_{K_R(Z,\kappa',\kappa)} \leq \varepsilon \|(1+|q|_m)^{-1}\psi\|^2_{K_R(Z,\kappa',\kappa)} \tag{4.3.62}$$

and use the Hardy type inequality (2.1.44) to derive $L_2[\psi] \geq 0$ for small $\varepsilon > 0$.

To estimate $L_2'[\mathcal{V}^{(2)}\varphi]$ we make a partition of unity of the set

$$\mathscr{S} = X_0 \setminus \Big(B(R) \bigcup_{Z:|Z|=2} K(Z,\kappa_2')\Big), \tag{4.3.63}$$

which contains the support of $\mathcal{V}^{(2)}\varphi$. Let $\kappa_3 \in (0,1)$ be so small that $K(Z,\kappa_3)$ and $K(\tilde{Z},\kappa_3)$ for partitions $Z \neq \tilde{Z}$ with $|Z| = |\tilde{Z}| = 3$ do not overlap on the set $\mathscr{S}$ given in (4.3.63). Such a constant κ_3 exists due to Theorem 3.2.5. By applying Theorem 4.3.9 we get

$$L_2'[\mathcal{V}^{(2)}\varphi] \geq \sum_{Z:|Z|=3} L_3[\mathcal{V}^{(2)}\varphi u_Z] + L_3'[\mathcal{V}^{(3)}\varphi], \tag{4.3.64}$$

where $\mathcal{V}^{(3)} = \mathcal{V}^{(2)}\sqrt{1-\sum_{Z:|Z|=3} u_Z^2}$ on $\mathscr{S}$ and the functionals $L_3, L_3' : H^1(X_0) \to \mathbb{R}$ are given by

$$\begin{aligned} L_3[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\psi\|^2 \\ &\qquad - \varepsilon\||q(Z)|_m^{-1}\ln^{-1}\left(|q(Z)|_m|\xi(Z)|_m^{-1}\right)\psi\|^2_{K_R(Z,\kappa_3',\kappa_3)}, \qquad (4.3.65) \\ L_3'[\psi] &= (1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi,\psi\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\psi\|^2 \end{aligned}$$

for some $\varepsilon > 0$ which can be chosen arbitrarily small. By the same arguments as for partitions Z with $|Z| = 2$ we can prove $L_3[\mathcal{V}^{(2)}\varphi u_Z] \geq 0$ for all partitions Z into three clusters. If $N \geq 5$, we continue this process for all partitions Z with $|Z| \leq N-1$ and finally arrive at the point where it remains to estimate the functional

$$L'[\tilde{\psi}] := (1-\gamma_0)\|\nabla_0\tilde{\psi}\|^2 + \langle V\tilde{\psi},\tilde{\psi}\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\tilde{\psi}\|^2 \geq 0 \tag{4.3.66}$$

for functions $\tilde{\psi} := \mathcal{V}^{(N-1)}\varphi$ supported in the region where all particles are separated from each other, namely there exists a constant $c > 0$, such that we have $|x_{ij}| \geq c|x|_m$ for $x \in \operatorname{supp}(\mathcal{V}^{(N-1)}\varphi)$. Therefore, we can estimate

$$|V(x)| \leq C(1+|x|_m)^{-2-\nu} \leq \varepsilon(1+|x|_m)^{-2} \tag{4.3.67}$$

if $R > 0$ is large enough. This implies

$$L'[\mathcal{V}^{(N-1)}\varphi] \geq (1-\gamma_0)\|\nabla_0\big(\mathcal{V}^{(N-1)}\varphi\big)\|^2 - (\alpha^2+2\varepsilon)\||x|_m^{-1}\mathcal{V}^{(N-1)}\varphi\|^2. \tag{4.3.68}$$

Similarly to (4.3.57) we have

$$\left\|\nabla_0\left(\mathcal{V}^{(N-1)}\varphi\right)\right\|^2 \geq \left(\tilde{C}_H(X_0)\right)^2\left\||x|_m^{-1}\mathcal{V}^{(N-1)}\varphi\right\|^2. \tag{4.3.69}$$

Since $0 \leq \alpha < \tilde{C}_H(X_0)$, we can choose $\gamma_0 > 0$ and $\varepsilon > 0$ sufficiently small to obtain $L'[\mathcal{V}^{(N-1)}\varphi] \geq 0$. This completes the proof of Lemma 4.3.11 and therefore the proof of Theorem 4.3.3. □

4.3.4. The special case of one-dimensional particles

Recall that in Theorem 4.3.3 we proved a condition, such that virtual levels correspond to eigenvalues. Concerning the decay behavior of the corresponding eigenfunction φ_0 the case of one-dimensional particles is special. Recall that if the dimension of the particles is $d \geq 3$, then φ_0 decays with the same rate as the fundamental solution of the Laplacian in $\mathbb{R}^k$ with $k = d(N-1)$, see [6]. By the estimate (4.3.13) given in Corollary 4.3.5 we see that for one-dimensional particles φ_0 decays faster than the fundamental solution in $\mathbb{R}^{N-1}$. This can be explained by the fact that for one-dimensional particles the sets $\{x_i = x_j\}$ are hyperplanes in X_0, which implies that the constant $\tilde{C}_H(X_0)$ is larger than the Hardy constant $C_H(X_0)$ for the whole space.

For the estimate given in Corollary 4.3.5 we used that functions in $\mathcal{M}$ are zero on the hyperplanes $\{x_i = x_j\}$ and applied the Hardy inequality for the half space. In fact, we only used that these functions vanish on *one* of the hyperplanes $\{x_i = x_j\}$. By taking into account that they vanish on all of them we can improve the estimate given in Corollary 4.3.5. We demonstrate this exemplarily for two cases.

Theorem 4.3.13. ***(i)*** *For a system of three one-dimensional particles with masses* $m_1, m_2, m_3 > 0$ *and pairwise distinct* $i, j, k \in \{1,2,3\}$ *let*

$$\theta_i = \arccos\left(\frac{\sqrt{m_j m_k}}{\sqrt{m_i + m_j}\sqrt{m_i + m_k}}\right). \tag{4.3.70}$$

Then

$$\tilde{C}_H(X_0) = \frac{\pi}{\theta_0}, \quad \textit{where} \quad \theta_0 = \max\{\theta_i,\ i = 1,2,3\}. \tag{4.3.71}$$

In particular, for systems of three one-dimensional particles virtual levels of the operator H correspond to simple eigenvalues.

(ii) *For a system of four identical one-dimensional particles we have* $\tilde{C}_H(X_0) = \frac{13}{2}$.

Remark 4.3.14. **(i)** It is easy to see that $\frac{\pi}{3} \le \theta_0 \le \frac{\pi}{2}$. Therefore, in case of three one-dimensional particles the constant $\tilde{C}_H(X_0)$ takes its maximal value $\tilde{C}_H(X_0) = 3$ for $\theta_0 = \frac{\pi}{3}$, which corresponds to the case $m_1 = m_2 = m_3$. On the other hand, if one of the masses m_i tends to infinity while the other masses are bounded, then $\theta_0 \to \frac{\pi}{2}$ and therefore $\tilde{C}_H(X_0) \to 2$.

(ii) Note that by Theorem 4.3.13 **(ii)** we get a significantly improved estimate for the decay rate compared to the one given in Corollary 4.3.5, where $\tilde{C}_H(X_0)$ was estimated by $\frac{3}{2}$.

Proof of Theorem 4.3.13 (i)

The proof consist of two steps: First, we prove some geometric properties of the space X_0 which is planar in case of three one-dimensional particles. Later, we will use the Hardy type inequality for sectors, given in Theorem 2.1.9, to derive the value of $\tilde{C}_H(X_0)$.

Lemma 4.3.15. *Let* $d = 1$ *and* $N = 3$. *Then the lines* $x_1 = x_2,\ x_1 = x_3$ *and* $x_2 = x_3$ *divide the space* X_0 *into six sectors* $S_1, S_2, \dots, S_6$ *with angles* $\theta_1 = \theta_4, \theta_2 = \theta_5$ *and* $\theta_3 = \theta_6$ *where* $\theta_i,\ i = 1,2,3$ *is given by*

$$\theta_i = \arccos\left(\frac{\sqrt{m_j m_k}}{\sqrt{m_i + m_j}\sqrt{m_i + m_k}}\right). \tag{4.3.72}$$

Proof of Lemma 4.3.15. At first, we describe the half lines $x_1 = x_2 \geq 0$, $x_1 = x_3 \leq 0$ and $x_2 = x_3 \geq 0$. It is clear that they are spanned by the vectors

$$\begin{aligned} u_{12} &= \left(1, 1, -\frac{m_1+m_2}{m_3}\right)^\top, \quad u_{13} = \left(-1, \frac{m_1+m_3}{m_2}, -1\right)^\top, \\ \text{and} \quad u_{23} &= \left(-\frac{m_2+m_3}{m_1}, 1, 1\right)^\top, \quad \text{respectively.} \end{aligned} \tag{4.3.73}$$

For a better understanding of the following steps we illustrate the half lines in a coordinate system. To this end we choose an orthogonal basis $\{v_1, v_2\}$ of X_0 with

$$v_1 = \left(1, 1, -\tfrac{(m_1+m_2)}{m_3}\right)^\top, \qquad v_2 = \left(-m_2, m_1, 0\right)^\top \tag{4.3.74}$$

and denote by (α, β) the coordinates corresponding to this basis. In this coordinate system the vectors u_{ij} have the following coordinates $(\alpha_{ij}, \beta_{ij})$:

$$\begin{aligned} (\alpha_{12}, \beta_{12}) = (1, 0), \quad (\alpha_{13}, \beta_{13}) &= \left(\frac{m_3}{m_1+m_2}, \frac{m_1+m_2+m_3}{m_2(m_1+m_2)}\right), \\ (\alpha_{23}, \beta_{23}) &= \left(-\frac{m_3}{m_1+m_2}, \frac{m_1+m_2+m_3}{m_1(m_1+m_2)}\right) \end{aligned} \tag{4.3.75}$$

Hence, we can illustrate the half lines $x_1 = x_2 \geq 0$, $x_1 = x_3 \leq 0$ and $x_2 = x_3 \geq 0$ in the following picture:

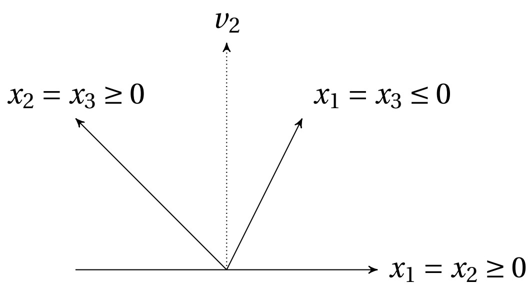

Figure 4.2.: The half lines $x_1 = x_2 \geq 0$, $x_1 = x_3 \leq 0$ and $x_2 = x_3 \geq 0$

Let S_1 be the sector between the half lines $x_1 = x_2 \geq 0$ and $x_1 = x_3 \leq 0$, S_2 the sector between the half lines $x_1 = x_2 \leq 0$ and $x_2 = x_3 \geq 0$ and S_3 the sector between the half lines $x_2 = x_3 \geq 0$ and $x_1 = x_3 \leq 0$, where we always choose the one sector with angle $0 < \theta_i < \pi$, cf. Figure 4.3.

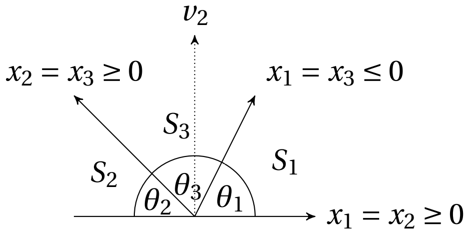

Figure 4.3.: The sectors $S_1,\ S_2,\ S_3$

For $i \in \{4,5,6\}$ we define the sector $S_i = -S_{i-3}$ i.e., S_i is the sector which we get by reflecting the sector S_{i-3} at the origin. It is obvious that for the angle θ_i of the sector $S_i,\ i \in \{4,5,6\}$, we have $\theta_i = \theta_{i-3}$.

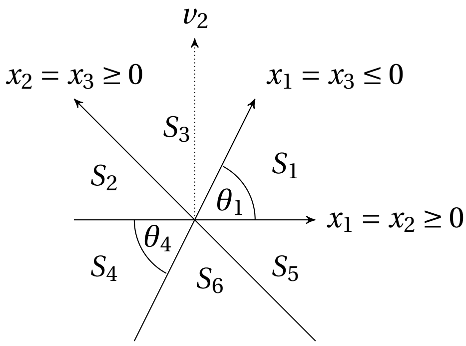

Figure 4.4.: Definition of the sectors S_i

Now we compute the angles θ_i of the sectors $S_i,\ i = 1,2,3$. They are given by the formula

$$\cos(\theta_i) = \frac{\langle u_{ij}, u_{ik}\rangle_m}{|u_{ij}|_m |u_{jk}|_m}, \quad i \neq j,k,\ j \neq k. \tag{4.3.76}$$

For $i = 1$ we get

$$\langle u_{12}, u_{13}\rangle_m = -m_1 + m_2 \cdot \frac{m_1 + m_3}{m_2} + m_3 \cdot \frac{m_1 + m_2}{m_3} = m_1 + m_2 + m_3. \tag{4.3.77}$$

Moreover, we have

$$|u_{12}|_m = \sqrt{m_1 + m_2 + m_3 \cdot \frac{(m_1 + m_2)^2}{m_3^2}} = \frac{\sqrt{m_1 + m_2}\sqrt{m_1 + m_2 + m_3}}{\sqrt{m_3}} \tag{4.3.78}$$

and analogously

$$|u_{13}|_m = \frac{\sqrt{m_1+m_3}\sqrt{m_1+m_2+m_3}}{\sqrt{m_2}}. \tag{4.3.79}$$

This yields

$$\cos(\theta_i) = \frac{\sqrt{m_2 m_3}}{\sqrt{m_1+m_2}\sqrt{m_1+m_3}}. \tag{4.3.80}$$

Therefore, the angle θ_1 satisfies (4.3.72). The other angles can be computed analogously. □

Lemma 4.3.16. *Let S_i be one of the sectors given by Lemma 4.3.15 with angle θ_i and assume that $\psi \in H_0^1(S_i)$. Then we have*

$$\|\nabla_0 \psi\| \geq \frac{\pi}{\theta_i} \| |x|_m^{-1} \psi \|. \tag{4.3.81}$$

and the constant $\frac{\pi}{\theta_i}$ is sharp.

Proof of Lemma 4.3.16. According to Theorem 2.1.9, functions $v \in H^1(\mathbb{R}^2)$ supported in a sector $S \subset \mathbb{R}^2$ satisfy

$$\|\nabla v\| \geq (\Lambda(G))^{\frac{1}{2}} \| |x|^{-1} v \|, \tag{4.3.82}$$

were $\Lambda(G)$ is the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in $G = S \cap \mathbb{S}^1$. The Laplace-Beltrami operator on G can be identified with the Laplacian on the interval $(0,\theta)$ where θ is the angle of S. The Dirichlet eigenvalues of the Laplacian on an interval of length $l > 0$ are given by $\lambda_k = \left(\frac{k\pi}{l}\right)^2$, $k \in \mathbb{N}$. Therefore, we have $\Lambda(G) = \left(\frac{\pi}{\theta}\right)^2$, which implies that for any function $v \in H^1(\mathbb{R}^2)$ supported in S we have

$$\|\nabla v\| \geq \frac{\pi}{\theta} \| |x|^{-1} v \|. \tag{4.3.83}$$

The sharpness of the constant $\frac{\pi}{\theta_i}$ follows from the sharpness of the Hardy type constant in Theorem 2.1.9. This completes the proof of Lemma 4.3.15. □

The proof of Theorem 4.3.13 **(i)** is a direct combination of Lemma 4.3.15 and Lemma 4.3.16.

Proof of Theorem 4.3.13 (ii)

The idea is to find a homogeneous harmonic polynomial p which vanishes on the sets $x_i = x_j$. Recall that we have $\dim(X_0) = 3$ for systems of four one-dimensional particles. We choose the following orthonormal basis $\{v_1, v_2, v_3\}$ of X_0, given by

$$v_1 = \frac{1}{\sqrt{2}}(1,-1,0,0)^\top, \quad v_2 = \frac{1}{\sqrt{6}}(1,1,-2,0)^\top, \quad v_3 = \frac{1}{\sqrt{12}}(1,1,1,-3)^\top.$$

with coordinates $\lambda_1, \lambda_2, \lambda_3$. Then the hyperplanes $\{x_i = x_j\}$ can be represented as follows:

$$\begin{aligned}
x_1 = x_2 &\iff \lambda_1 = 0,\\
x_1 = x_3 &\iff \sqrt{3}\lambda_1 + 3\lambda_2 = 0,\\
x_1 = x_4 &\iff \sqrt{6}\lambda_1 + \sqrt{2}\lambda_2 + 4\lambda_3 = 0,\\
x_2 = x_3 &\iff -\sqrt{3}\lambda_1 + 3\lambda_2 = 0,\\
x_2 = x_4 &\iff -\sqrt{6}\lambda_1 + \sqrt{2}\lambda_2 + 4\lambda_3 = 0,\\
x_3 = x_4 &\iff -2\sqrt{2}\lambda_2 + 4\lambda_3 = 0.
\end{aligned} \tag{4.3.84}$$

Let the polynomial $p : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\begin{aligned}
p(\lambda_1,\lambda_2,\lambda_3) = \lambda_1 \left(\sqrt{3}\lambda_1 + 3\lambda_2\right)\left(\sqrt{6}\lambda_1 + \sqrt{2}\lambda_2 + 4\lambda_3\right)\left(-\sqrt{3}\lambda_1 + 3\lambda_2\right)\\
\cdot \left(-\sqrt{6}\lambda_1 + \sqrt{2}\lambda_2 + 4\lambda_3\right)\left(-2\sqrt{2}\lambda_2 + 4\lambda_3\right).
\end{aligned} \tag{4.3.85}$$

Then p is a homogeneous harmonic polynomial of degree six which by (4.3.84) vanishes if and only if $x_i = x_j$ for some $i \neq j$. Hence, its restriction to $\mathbb{S}^2$ is a spherical harmonic of degree six and therefore, by the discussion of spherical harmonics in Chapter 2, an eigenfunction of the Laplace-Beltrami operator on $\mathbb{S}^2$ corresponding to the eigenvalue $6 \cdot 7 = 42$. Moreover, p does not change sign inside the sectors S_i. Hence, its restriction to $S_i \cap \mathbb{S}^2$ is an eigenfunction to the first eigenvalue of the Laplace-Beltrami on $S_i \cap \mathbb{S}^2$, equipped with Dirichlet boundary conditions. By Theorem 2.1.9 we get

$$\tilde{C}_H(X_0) = \left(\frac{1}{4} + \frac{168}{4}\right)^{\frac{1}{2}} = \frac{13}{2}, \tag{4.3.86}$$

which completes the proof of Theorem 4.3.13 **(ii)**.

4.3.5. The case of three two-dimensional particles

Note that for systems of three two-dimensional particles we have $\tilde{C}_H(X_0) = 1$ and therefore Theorem 4.3.3 does not apply in this case. We can not say whether virtual levels correspond to eigenvalues or to resonances, but we prove the following

Theorem 4.3.17. *Let H be the Hamiltonian of a system of three two-dimensional particles. Assume that the potentials $V_{ij} \neq 0$ satisfy* (4.2.2) *and* (4.2.3) *and that H has a virtual level at zero. Then there exists a function $\varphi_0 \in \tilde{H}^1(X_0)$, $\varphi_0 \neq 0$, satisfying*

$$\|\nabla_0 \varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0 \tag{4.3.87}$$

and

$$(1 + |\cdot|_m)^{-\alpha}\, \varphi_0 \in L^2(X_0) \qquad \textit{for any } \alpha > 0. \tag{4.3.88}$$

Proof of Theorem 4.3.17

To prove Theorem 4.3.17 we take a sequence $(\psi_n)_{n\in\mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $E_n < 0$ of the operator $H + n^{-1}\Delta_0$, i.e.,

$$-\left(1 - n^{-1}\right)\Delta_0\psi_n + V\psi_n = E_n\psi_n. \tag{4.3.89}$$

We normalize the functions ψ_n by $\|\nabla_0\psi_n\| = 1$. Note that due to $\dim(X_0) = 4$ the space $\dot{H}^1(X_0)$ is a Hilbert space. Therefore, there exists a subsequence of $(\psi_n)_{n\in\mathbb{N}}$, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges weakly in $\dot{H}^1(X_0)$ to a function $\varphi_0 \in \dot{H}^1(X_0)$. Due to the Rellich-Kondrachov thereon we have convergence of ψ_n to φ_0 in $L^2_{\text{loc}}(X_0)$. We split the proof into several steps.

Step 1: $\varphi_0 \neq 0$ and it satisfies the decay property (4.3.88).
Due to Lemma 4.3.11 there exist constants $\gamma_0 > 0$ and $R > 0$, such that for every function $\psi \in H^1(X_0)$, supported in the region $\{|x|_m \geq R\}$, we have

$$(1-\gamma_0)\|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle \geq 0. \tag{4.3.90}$$

Applying Lemma 2.3 in [7] we see that the weak limit $\varphi_0 \in \dot{H}^1(X_0)$ of the sequence $(\psi_n)_{n\in\mathbb{N}}$ of eigenfunctions normalized by $\|\nabla_0\psi_n\| = 1$ is not zero.

In the next step we show that φ_0 satisfies the estimate (4.3.88) on the decay rate. To do this we first give the following estimate for a weighted L^2 norm of the functions ψ_n.

Lemma 4.3.18. *Let H be the Hamiltonian of a system of three two-dimensional particles. Assume that the potentials V_{ij} satisfy* (4.2.2) *and* (4.2.3) *and that H has a virtual level at zero. Then, for any $0 \le \alpha < 1$ there exists a constant $C > 0$, such that for all $n \in \mathbb{N}$ we have*

$$\|\nabla_0 \left(|\cdot|_m^\alpha \psi_n \right)\| \le C \qquad \textit{and} \qquad \|(1 + |\cdot|_m)^{\alpha-1} \psi_n\| \le C. \tag{4.3.91}$$

Proof. The proof is an easy modification of the proof of Lemma A.1.1 in the Appendix, together with the observation that $\tilde{C}_H(X_0) = 1$. □

By Lemma 4.3.18 we get convergence of $(\psi_n)_{n\in\mathbb{N}}$ to φ_0 in $L^2(X_0, (1+|x|_m)^{-\alpha} \mathrm{d}x)$ for any $\alpha > 0$. This shows that the function φ_0 satisfies (4.3.88).

Step 2: $\langle V\varphi_0, \varphi_0 \rangle$ is well-defined.
Note that in contrast to Theorem 4.2.6 and Theorem 4.3.7 we neither have fast decay of V nor do we know whether φ_0 is in $L^2(X_0)$, which makes this part difficult. We prove that $\langle V_{ij}\varphi_0, \varphi_0 \rangle$ is well-defined for each pair of particles $\beta = (i, j)$. Since each V_{ij} satisfies (4.2.2) and (4.2.3), by Lemma 3.1.11 we have

$$\int |V_{ij}(q_\beta)| \cdot |\varphi_0(q_\beta, \xi_\beta)|^2 \, \mathrm{d}q_\beta \le C \|\varphi_0(\cdot, \xi_\beta)\|^2_{\tilde{H}^1(X_0[C])}, \quad \text{a.e. } \xi_\beta, \tag{4.3.92}$$

where here and in the following for a pair of particles $\beta = (i, j)$ we denote by q_β, ξ_β the variables $q[C], \xi[C]$ with $C = \{i, j\}$. In view of (4.3.92), to prove well-definedness of $\langle V_{ij}\varphi_0, \varphi_0 \rangle$ it suffices to show

$$\int \int_{\{|q_\beta|_m \le 1\}} |\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \infty. \tag{4.3.93}$$

In other words, it is enough to prove that the restriction of the function φ_0 to cylindrical regions $\{(q_\beta, \xi_\beta) : |q_\beta|_m \le 1\}$, $\beta \in \{(1,2), (1,3), (2,3)\}$, is square-integrable.

Let $\chi_1 : \mathbb{R}_+ \to [0,1]$ be a function with $\chi_1 \in C^1(\mathbb{R}_+)$ and $(1-\chi_1^2)^{\frac{1}{2}} \in C^1(\mathbb{R}_+)$, satisfying

$$\chi_1(t) = 0,\, 0 \le t \le 1, \qquad \chi_1(t) = 1,\, t \ge 2. \tag{4.3.94}$$

For $b > 0$ and $x \in X_0$ let $\chi(x) = \chi_1\left(\frac{|x|_m}{b}\right)$. The first step to prove that $\langle V_{ij}\varphi_0, \varphi_0\rangle$ is well-defined is the following

Lemma 4.3.19. *Let ψ_n and χ be defined as above. Then, for any $\varepsilon > 0$ we can find $b > 0$ and $n_0 \in \mathbb{N}$, such that for all $n > n_0$ we have*

$$\textit{(i)}\quad \|\nabla_0\left(\chi\psi_n\right)\| < \varepsilon, \qquad \textit{(ii)}\quad \langle V_{ij}\chi\psi_n, \chi\psi_n\rangle < \varepsilon, \quad i,j \in \{1,2,3\}. \tag{4.3.95}$$

Proof of Lemma 4.3.19. For $\psi \in H^1(X_0)$ let

$$L[\psi] = \|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle. \tag{4.3.96}$$

Then, by the IMS localization formula we get

$$\begin{aligned} L[\psi_n] = L[(1-\chi^2)^{\frac{1}{2}}\psi_n] + L[\chi\psi_n] \\ - \int_{X_0} \left(|\nabla_0\chi|^2 + |\nabla_0(1-\chi^2)^{\frac{1}{2}}|^2\right)|\psi_n|^2\,\mathrm{d}x. \end{aligned} \tag{4.3.97}$$

Since χ is supported in the region $\{|x|_m \ge b\}$, by Lemma 4.3.11 with $\alpha = 0$ we get

$$L[\chi\psi_n] \ge \gamma_0\|\nabla_0\left(\chi\psi_n\right)\|^2 \tag{4.3.98}$$

for some $\gamma_0 > 0$ if $b > 0$ is large enough and therefore by (4.3.97)

$$\begin{aligned} \gamma_0\|\nabla_0\left(\chi\psi_n\right)\|^2 \le L[\psi_n] - L[(1-\chi^2)^{\frac{1}{2}}\psi_n] \\ + \int_{X_0} \left(|\nabla_0\chi|^2 + |\nabla_0(1-\chi^2)^{\frac{1}{2}}|^2\right)|\psi_n|^2\,\mathrm{d}x. \end{aligned} \tag{4.3.99}$$

We estimate the terms on the r.h.s of (4.3.99) separately. By definition of the functions ψ_n the first term can be estimated as

$$L[\psi_n] \le \frac{1}{n}\|\nabla_0\psi_n\|^2 = \frac{1}{n}. \tag{4.3.100}$$

Due to $H \geq 0$ the second term on the r.h.s. of (4.3.99) is non-positive. Now we estimate the last term on the r.h.s. of (4.3.99). Note that $\nabla_0 \chi$ and $\nabla_0 \left(1-\chi^2\right)^{\frac{1}{2}}$ are supported in the region $\{b \leq |x| \leq 2b\}$ and satisfy

$$|\nabla_0 \chi|^2 + |\nabla_0 \left(1-\chi^2\right)^{\frac{1}{2}}|^2 \leq \frac{C}{b^2} \tag{4.3.101}$$

for some $C > 0$ which does not depend on b. This, together with the estimate (4.3.91) on the decay rate of ψ_n yields, uniformly in $n \in \mathbb{N}$,

$$\int_{X_0} \left(|\nabla_0 \chi|^2 + |\nabla_0 (1-\chi^2)^{\frac{1}{2}}|^2\right) |\psi_n|^2 \, \mathrm{d}x \leq 4C \int_{\{|x|_m \geq b\}} \frac{|\psi_n|^2}{|x|_m^2} \, \mathrm{d}x \leq \varepsilon_1(b) \tag{4.3.102}$$

for some $\varepsilon_1(b)$ with $\varepsilon_1(b) \to 0$ as $b \to \infty$. Combining this with (4.3.99) and (4.3.100) we obtain

$$\gamma_0 \|\nabla_0 \left(\chi \psi_n\right)\|^2 \leq \frac{1}{n} + \varepsilon_1(b). \tag{4.3.103}$$

Therefore, for fixed $\varepsilon > 0$ we can choose $n_0 \in \mathbb{N}$ and $b > 0$ large enough, such that $\|\nabla_0 \left(\chi \psi_n\right)\|^2 \leq \varepsilon$ holds uniformly for $n \geq n_0$. This completes the proof of statement **(i)** of the Lemma.

Now we turn to the proof of assertion **(ii)**. We fix a pair of particles (i_0, j_0) and note that

$$\langle V_{i_0 j_0} \chi \psi_n, \chi \psi_n \rangle = L[\chi \psi_n] - \|\nabla_0 \left(\chi \psi_n\right)\|^2 - \sum_{(i,j) \neq (i_0, j_0)} \langle V_{ij} \chi \psi_n, \chi \psi_n \rangle, \tag{4.3.104}$$

i.e., $\langle V_{i_0 j_0} \chi \psi_n, \chi \psi_n \rangle$ can be estimated by estimating the r.h.s. of (4.3.104). For the first term we get by (4.3.97) and (4.3.102)

$$L[\chi \psi_n] \leq L[\psi_n] + C \int_{\{|x|_m \geq b\}} \frac{|\psi_n|^2}{|x|_m^2} \, \mathrm{d}x. \tag{4.3.105}$$

Now, by using $L[\psi_n] \leq \frac{1}{n}$ and the estimate (4.3.91) for the functions ψ_n we obtain

$$L[\chi \psi_n] \leq \frac{1}{n} + \varepsilon_2(b), \tag{4.3.106}$$

where $\varepsilon_2(b) \to 0$ as $b \to \infty$. Substituting this in (4.3.104) we get

$$\langle V_{i_0 j_0} \chi \Psi_n, \chi \Psi_n \rangle \leq \frac{1}{n} + \varepsilon_2(b) - \sum_{(i,j) \neq (i_0, j_0)} \langle V_{ij} \chi \Psi_n, \chi \Psi_n \rangle. \tag{4.3.107}$$

Let us estimate the last term on the r.h.s. of (4.3.107). Since the Hamiltonians of the clusters consisting of two particles do not have negative spectrum, we have

$$\langle V_{ij} \chi \Psi_n, \chi \Psi_n \rangle \geq -\| \nabla_0 \left(\chi \Psi_n \right) \|^2 \tag{4.3.108}$$

and therefore by statement **(i)** of the Lemma

$$\langle V_{ij} \chi \Psi_n, \chi \Psi_n \rangle \geq -\| \nabla_0 \left(\chi \Psi_n \right) \|^2 \geq -\varepsilon, \tag{4.3.109}$$

where the constant $\varepsilon > 0$ can be chosen arbitrarily small if $b > 0$ and $n \in \mathbb{N}$ are sufficiently large. Inserting this in (4.3.107) we get

$$\langle V_{i_0 j_0} \chi \Psi_n, \chi \Psi_n \rangle \leq \frac{1}{n} + \varepsilon_2(b) + 2\varepsilon, \tag{4.3.110}$$

which completes the proof of Lemma 4.3.19. □

Now we use Lemma 4.3.19 to prove the well-definedness of $\langle V_{ij} \varphi_0, \varphi_0 \rangle$. Recall that for this purpose we want to show that

$$\int \int_{\{|q_\beta|_m \leq 1\}} |\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \infty. \tag{4.3.111}$$

Since $V_{ij} \neq 0$ and the cluster Hamiltonians for non-trivial clusters do not have virtual levels, by Corollary 4.2.4 we get

$$\int \int_{\{|q_\beta|_m \leq 1\}} |\chi \Psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \leq C_1 \| \nabla_{q_\beta} (\chi \Psi_n) \|^2 + C_2 \langle V_{ij} \chi \Psi_n, \chi \Psi_n \rangle \tag{4.3.112}$$

for some $C_1, C_2 > 0$ and any pair of particles $\beta = (i, j)$. Now by Lemma 4.3.19 we see that the r.h.s. of (4.3.112) can be done arbitrarily small if the constant $b > 0$ in the definition of the function χ and $n \in \mathbb{N}$ are sufficiently large. Hence, for any $\varepsilon > 0$ we can estimate

$$\int\int_{\{|q_\beta|_m\le 1\}} |\chi\psi_n|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta \le \varepsilon. \tag{4.3.113}$$

Recall that for $|\xi_\beta|_m > 2b$ we have $\chi(x) = 1$. Therefore, (4.3.113) yields

$$\int_{\{|\xi_\beta|_m\ge 2b\}}\int_{\{|q_\beta|_m\le 1\}} |\psi_n(x)|^2 \,\mathrm{d}x = \int_{\{|\xi_\beta|_m\ge 2b\}}\int_{\{|q_\beta|_m\le 1\}} |\chi\psi_n(x)|^2 \,\mathrm{d}x \le \varepsilon \tag{4.3.114}$$

for $b>0$ and $n\in\mathbb{N}$ large enough. Furthermore, we have $\psi_n \to \varphi_0$ in $L^2_{\mathrm{loc}}(X_0)$. Hence,

$$\int_{\{|\xi_\beta|_m\le 2b\}}\int_{\{|q_\beta|_m\le 1\}} |\psi_n|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta \to \int_{\{|\xi_\beta|_m\le 2b\}}\int_{\{|q_\beta|_m\le 1\}} |\varphi_0|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta. \tag{4.3.115}$$

This, together with (4.3.114) shows that the integral

$$\int\int_{\{|q_\beta|_m\le 1\}} |\varphi_0|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta \tag{4.3.116}$$

is bounded and thus $\langle V_{ij}\varphi_0, \varphi_0\rangle$ is well-defined.

Step 3: $\langle V_{ij}\psi_n, \psi_n\rangle$ converges to $\langle V_{ij}\varphi_0, \varphi_0\rangle$.
At first, we consider the integral

$$\int_{\{|\xi_\beta|_m\ge 2b\}}\int |V_{ij}||\psi_n|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta \tag{4.3.117}$$

and prove that it can be done arbitrarily small if $b>0$ and $n\in\mathbb{N}$ are large enough. By Lemma 3.1.11 we have

$$\begin{aligned}&\int_{\{|\xi_\beta|_m\ge 2b\}}\int |V_{ij}||\psi_n|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta \\ &\qquad \le C\int_{\{|\xi_\beta|_m\ge 2b\}}\left(\int |\nabla_{q_\beta}\psi_n|^2 \,\mathrm{d}q_\beta + \int_{\{|q_\beta|_m\le 1\}} |\psi_n|^2 \,\mathrm{d}q_\beta\right)\mathrm{d}\xi_\beta.\end{aligned} \tag{4.3.118}$$

Note that by Lemma 4.3.19 we get for arbitrary $\varepsilon>0$

$$\int_{\{|\xi_\beta|_m\ge 2b\}}\int |\nabla_{q_\beta}\psi_n|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta = \int_{\{|\xi_\beta|_m\ge 2b\}}\int |\nabla_{q_\beta}(\chi\psi_n)|^2 \,\mathrm{d}q_\beta \,\mathrm{d}\xi_\beta \le \varepsilon \tag{4.3.119}$$

if $b > 0$ and $n \in \mathbb{N}$ are large enough. Substituting this inequality and (4.3.114) in (4.3.118) yields

$$\int_{\{|\xi_\beta|_m \geq 2b\}} \int |V_{ij}||\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \leq 2\varepsilon. \tag{4.3.120}$$

Due to

$$\int |V_{ij}||\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \infty, \tag{4.3.121}$$

which is true as we have seen in the previous steps of the proof, we also obtain

$$\int_{\{|\xi_\beta|_m \geq 2b\}} \int |V_{ij}||\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta \leq \varepsilon \tag{4.3.122}$$

for $b > 0$ large enough. Now we consider the region $\{|\xi_\beta|_m \leq 2b\}$. Due to the decay property (4.2.3) of the potentials V_{ij} and the estimates (4.3.91) and (4.3.88) for ψ_n and φ_0 we get

$$\int_{\{|\xi_\beta|_m \leq 2b\}} \int_{\{|q_\beta|_m \geq b_1\}} |V_{ij}||\psi_n|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \varepsilon \tag{4.3.123}$$

and

$$\int_{\{|\xi_\beta|_m \leq 2b\}} \int_{\{|q_\beta|_m \geq b_1\}} |V_{ij}||\varphi_0|^2 \, \mathrm{d}q_\beta \, \mathrm{d}\xi_\beta < \varepsilon, \tag{4.3.124}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $b_1 > 0$ is large enough and estimate (4.3.123) holds uniformly for $n \in \mathbb{N}$.

Estimates (4.3.120) and (4.3.122) - (4.3.124) show us that to prove the convergence $\langle V_{ij}\psi_n, \psi_n\rangle \to \langle V_{ij}\varphi_0, \varphi_0\rangle$ it suffices to show that $\langle V_{ij}\psi_n, \psi_n\rangle_\Omega \to \langle V_{ij}\varphi_0, \varphi_0\rangle_\Omega$ for the compact set

$$\Omega := \{x \in X_0 : x = q_\beta + \xi_\beta, \| q_\beta|_m \leq b_1, \ |\xi_\beta|_m \leq 2b\}. \tag{4.3.125}$$

We write

$$\langle V_{ij}\psi_n, \psi_n\rangle_\Omega - \langle V_{ij}\varphi_0, \varphi_0\rangle_\Omega = \langle V_{ij}(\psi_n - \varphi_0), \psi_n\rangle_\Omega + \langle V_{ij}\varphi_0, (\psi_n - \varphi_0)\rangle_\Omega. \tag{4.3.126}$$

Since ψ_n converges to φ_0 in $L^2_{\mathrm{loc}}(X_0)$, $\|\nabla_{q_\beta}\psi_n\| \leq 1$, $\|\nabla_{q_\beta}\varphi_0\| \leq 1$ and the potential V_{ij} satisfies (4.2.2), both summands on the r.h.s. of (4.3.126) tend to zero as $n \to \infty$. This, together with the estimates (4.3.120) - (4.3.124) yields $\langle V_{ij}\psi_n, \psi_n\rangle \to \langle V_{ij}\varphi_0, \varphi_0\rangle$ for every pair (i, j) of particles and therefore $\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle$ as $n \to \infty$.

Step 4: φ_0 satisfies $\|\nabla_0\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0$.
Since by definition of the functions ψ_n

$$\langle V\psi_n, \psi_n\rangle \le -\left(1 - n^{-1}\right) \tag{4.3.127}$$

and $\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle$, we get $\langle V\varphi_0, \varphi_0\rangle \le -1$. Moreover, we have

$$\|\varphi_0\|_{\dot{H}^1} \le \liminf_{n\to\infty} \|\psi_n\|_{\dot{H}^1} = 1. \tag{4.3.128}$$

On the other hand, $H \ge 0$ and therefore

$$\|\nabla_0\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0, \tag{4.3.129}$$

which completes the proof of Theorem 4.3.17.

4.3.6. Virtual levels of fermionic systems

Now we consider a system of $N \ge 3$ one- or two-dimensional identical fermions with corresponding Hamiltonian H_{as}. Our goal is to prove that virtual levels correspond to eigenvalues for systems of $N \ge 3$ one-or two-dimensional particles. This section is based on [7].

Definition 4.3.20. Assume that the potentials V_{ij} satisfy (3.2.26), (4.2.2) and (4.2.3). Let $C \subseteq \{1, \dots, N\}$ be a cluster of the system. We say that the cluster Hamiltonian $H_{\text{as}}[C]$ has a virtual level at zero if $H_{\text{as}}[C] \ge 0$ and

(i) there exists a constant $\varepsilon_0 > 0$, such that

$$\inf \sigma_{\text{ess}}\left(-(1-\varepsilon_0)\Delta_0[C] + V[C]\right) = 0, \tag{4.3.130}$$

(ii) for any $\varepsilon > 0$ we have

$$\inf \sigma\left(H[C] + \varepsilon\Delta_0[C]\right) < 0. \tag{4.3.131}$$

Here, the operators in (4.3.130) and (4.3.131) are considered as operators restricted to the subspace of functions which are anti-symmetric with respect to permutations of particles in the cluster C.

A crucial role in the proof of Theorem 4.3.3 is played by Hardy's inequality. Since by definition the wave-function for a system of fermions is anti-symmetric with respect to permutation of particles and therefore orthogonal to constant functions, we can use the Hardy type inequality (2.1.33) for fermionic systems and prove the following

Theorem 4.3.21. *Let H_{as} be the Hamiltonian corresponding to a system of $N \geq 3$ one- or two-dimensional identical fermions. Assume that the potentials V_{ij} satisfy the conditions* (3.2.26), (4.2.2) *and* (4.2.3) *and that H_{as} has a virtual level at zero. Then zero is an eigenvalue of H_{as} and the corresponding eigenspace $\mathscr{W}_0$ is finite-dimensional. For any $\varphi_0 \in \mathscr{W}_0$*

$$\nabla_0 \left(|\cdot|_m^\alpha \varphi_0 \right) \in L^2(X_0) \quad \textit{and} \quad (1 + |\cdot|_m)^{\alpha-1} \varphi_0 \in L^2(X_0) \tag{4.3.132}$$

for any $0 \leq \alpha < \alpha_0$, where

$$\alpha_0 = \begin{cases} 3 & \textit{if } d = 1,\ N = 3, \\ \frac{d(N-1)}{2} & \textit{else.} \end{cases} \tag{4.3.133}$$

There exists a constant $\delta_0 > 0$, such that for any $\psi \in H^1(X_0)$ satisfying $\langle \nabla_0 \varphi_0, \nabla_0 \psi \rangle = 0$ for all $\varphi_0 \in \mathscr{W}_0$

$$(1 - \delta_0) \|\nabla_0 \psi\|^2 + \langle V\psi, \psi \rangle \geq 0. \tag{4.3.134}$$

Remark 4.3.22. **(i)** Note that in contrast to Theorem 4.3.3, where the operator H was considered without symmetry restrictions, in the fermionic case zero is not a simple eigenvalue. However, its multiplicity is still finite.

(ii) Comparing the estimates of the decay rate with the ones without symmetry restrictions we see that for fermions we get better ones. This is not surprising because the Hardy type constant for anti-symmetric functions is larger than for functions without symmetry restrictions.

Proof of Theorem 4.3.21

The proof is based on the following abstract Theorem 4.3.23 which is a straightforward modification of Theorem 4.2.6 to Schrödinger operators with symmetry restrictions.

Theorem 4.3.23. *Let $k \in \mathbb{N}$ and let $h_{\mathrm{as}} = -\Delta + V$ be the Schrödinger operator acting on the subset of $L^2(\mathbb{R}^k)$ which consists of functions which are orthogonal (w.r.t. the $L^2(\mathbb{R}^k)$ scalar product) to all functions which depend on $|x|$ only. Suppose that V satisfies* (4.2.2). *Furthermore, assume that*

$$h_{\mathrm{as}} \geq 0 \qquad \textit{and} \qquad \inf \sigma\,(h_{\mathrm{as}} + \varepsilon\Delta) < 0 \tag{4.3.135}$$

for any $\varepsilon > 0$. If there exist constants $\alpha_0 > 1$, $b > 0$ and $\gamma_0 > 0$, such that for any ψ in the form domain of h_{as} with $\mathrm{supp}(\psi) \subset \{x \in \mathbb{R}^k : |x| \geq b\}$ we have

$$\langle h_{\mathrm{as}}\psi, \psi\rangle - \gamma_0 \|\nabla\psi\|^2 - \alpha_0^2 \langle |x|^{-2}\psi, \psi\rangle \geq 0, \tag{4.3.136}$$

then zero is an eigenvalue of h_{as} with finite multiplicity. Let $\mathcal{W}_0$ be the corresponding eigenspace. Then for any $\varphi_0 \in \mathcal{W}_0$ we have

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^k) \quad \textit{and} \quad (1 + |\cdot|)^{\alpha-1}\,\varphi_0 \in L^2(\mathbb{R}^k) \tag{4.3.137}$$

for any $\alpha < \alpha_0$. Moreover, there exists a constant $\delta_0 > 0$, such that for any function ψ in the form domain of h_{as} with $\langle \nabla\psi, \nabla\varphi_0\rangle = 0$ for all $\varphi_0 \in \mathcal{W}_0$ we have

$$\langle h_{\mathrm{as}}\psi, \psi\rangle \geq \delta_0 \|\nabla\psi\|^2. \tag{4.3.138}$$

We shall not prove this theorem, but turn to the

Proof of Theorem 4.3.21. We show that the conditions of Theorem 4.3.23 are fulfilled for any $\alpha < \alpha_0$ with α_0 given by (4.3.133). To do this we use similar geometric methods as in the proof of Theorem 4.3.3 to show that for any $0 \leq \alpha < \alpha_0$ we find constants $\gamma_0, R > 0$, such that for any function $\varphi \in H^1_{\mathrm{as}}(X_0)$ with $\mathrm{supp}\,(\varphi) \subset \{x \in X_0 : |x|_m \geq R\}$ we have

$$L[\varphi] := (1 - \gamma_0)\|\nabla_0\varphi\|^2 + \langle V\varphi, \varphi\rangle - \alpha^2 \||x|_m^{-1}\varphi\|^2 \geq 0. \tag{4.3.139}$$

Then the statement of Theorem 4.3.21 follows from Theorem 4.3.23. Let $\kappa > 0$ be so small that cones $K(Z,\kappa)$ and $K(Z',\kappa)$ do not overlap for different partitions $Z \neq Z'$ of the system into two clusters and for given $\varepsilon > 0$ we choose functions u_Z according to

Theorem 4.3.9. Then we have

$$L[\varphi] \geq \sum_{Z:|Z|=2} L_2[\varphi u_Z] + L_2'[\mathcal{V}\varphi], \tag{4.3.140}$$

where the functionals $L_2, L_2' : H^1_{\text{as}}(X_0) \to \mathbb{R}$ are defined as in (4.3.41). Note that for $\varphi \in H^1_{\text{as}}(X_0)$ the functions φu_Z and $\mathcal{V}\varphi$ are also elements of $H^1_{\text{as}}(X_0)$. To show $L_2[\varphi u_Z] \geq 0$ we note that since φu_Z is anti-symmetric with respect to permutations of particles, it is orthogonal to functions depending on $|q(Z)|_m$ only, and so is φu_Z. Therefore, we can apply the Hardy type inequality (2.1.33) to the function $\varphi u_Z(\cdot,\xi)$ for fixed ξ in the $d(N-2)$-dimensional space $L^2(X_0(Z))$, which yields

$$\|\nabla_q\left(\varphi u_Z\right)\|^2 \geq \frac{(d(N-2))^2}{4}\left\||q(Z)|_m^{-1}\varphi u_Z\right\|^2. \tag{4.3.141}$$

The rest of the proof of $L_2[\varphi u_Z] \geq 0$ is a straightforward modification of the proof of Lemma 4.3.11.

To prove that $L_2'[\mathcal{V}\varphi] \geq 0$ we distinguish between several cases. If $d = 1$ and $N = 3$, we have $\dim(X_0) = 2$ and since the particles are identical, the lines $x_i = x_j$ cut the space X_0 into six congruent sectors, each of angle $\frac{\pi}{3}$. Therefore, as in the proof of Theorem 4.3.13 we have

$$\|\nabla_0\left(\mathcal{V}\varphi\right)\|^2 \geq 9\||x|_m^{-1}\left(\mathcal{V}\varphi\right)\|^2. \tag{4.3.142}$$

Repeating the same arguments as in the proof of Lemma 4.3.11 yields $L_2'[\mathcal{V}\varphi]$ for the case $d = 1$ and $N = 3$. For $d = 2$ and $N = 3$ we use the observation that the function $\mathcal{V}\varphi$ is orthogonal (with respect to the $L^2(X_0)$ scalar product) to all functions depending on $|x|_m$ only. Therefore, we can apply the Hardy type inequality (2.1.33) for dimension four, which yields

$$\|\nabla_0\left(\mathcal{V}\varphi\right)\|^2 \geq 4\||x|_m^{-1}\mathcal{V}\varphi\|^2. \tag{4.3.143}$$

Now $L_2'[\mathcal{V}\varphi] \geq 0$ follows in the same way as in the proof of Lemma 4.3.11. If the system consists of more than three particles, we continue to estimate the functional $L_2'[\mathcal{V}\varphi]$ in cones corresponding to partitions Z with $|Z| = 3,\dots,N-1$. Finally, we arrive at the point where it remains to estimate the functional

$$L'[\tilde{\psi}] := (1-\gamma_0)\|\nabla_0\tilde{\psi}\|^2 + \langle V\tilde{\psi},\tilde{\psi}\rangle - (\alpha^2+\varepsilon)\||x|_m^{-1}\tilde{\psi}\|^2 \geq 0 \tag{4.3.144}$$

for functions $\tilde{\psi} := \mathcal{V}^{(N-1)}\varphi \in H^1_{\mathrm{as}}(X_0)$ supported in the region where all particles are separated from each other. Since $\tilde{\psi}$ is orthogonal to all functions which depend on $|x|_m$ only, we can use the Hardy type inequality (2.1.33) to estimate

$$\left\|\nabla_0\left(\mathcal{V}^{(N-1)}\varphi\right)\right\|^2 \geq \frac{(d(N-1))^2}{4}\||x|_m^{-1}\mathcal{V}\varphi\|^2. \tag{4.3.145}$$

Now we complete the proof of Theorem 4.3.21 in the same way as in the proof of Lemma 4.3.11.

□

4.3.7. A sufficient and necessary condition for virtual levels of multi-particle Schrödinger operators

In this paragraph we extend the equivalent definition of virtual levels for one-particle Schrödinger operators with short-range potentials, given in Theorem 4.2.2, to the multi-particle case.

Theorem 4.3.24. *Let H be the Hamiltonian corresponding to a system of $N \geq 3$ one- or two-dimensional particles, where the potentials $V_{ij} \neq 0$ satisfy* (4.2.2) *and* (4.2.3) *and assume that $H \geq 0$. Then H has a virtual level at zero if and only if the following two assertions hold.*

(i) *There exists an $\varepsilon_0 > 0$, such that for any cluster C with $1 < |C| < N$ we have*

$$H[C] - \varepsilon_0\left(1+|q[C]|_m^2(\ln(|q[C]|_m))^2\right)^{-1} \geq 0. \tag{4.3.146}$$

(ii) *For any $\varepsilon > 0$ we have*

$$\inf\sigma\left(H - \varepsilon\left(1+|x|_m^2\ln^2(|x|_m)\right)^{-1}\right) < 0. \tag{4.3.147}$$

Proof of Theorem 4.3.24. The proof consists of four steps.
Step 1: If H has a virtual level at zero, then condition **(i)** of Theorem 4.3.24 holds, i.e., there exists an $\varepsilon_0 > 0$, such that for any cluster C with $1 < |C| < N$ we have

$$H[C] - \varepsilon_0 \left(1 + |q[C]|_m^2 (\ln(|q[C]|_m))^2\right)^{-1} \geq 0. \tag{4.3.148}$$

We prove this by induction over the number of particles $N \geq 2$ in the whole system. The base case $N = 2$ was considered in Theorem 4.2.2. Assuming that the statement is true for systems consisting of N particles, we show that it also holds for systems of $N+1$ particles. Assume for a contradiction that the Hamiltonian of a system of $N+1$ particles has a virtual level at zero and that there exists a cluster C with $1 < |C| < N+1$, such that for all $\varepsilon_0 > 0$ inequality (4.3.148) does not hold. Among such clusters we choose one with the smallest number of particles and denote it by C_0. If C_0 consists of only two particles, then by Theorem 4.2.2 the condition

$$\inf \sigma \left(H[C_0] - \varepsilon \left(1 + |q[C_0]|_m^2 (\ln(|q[C_0]|_m)^2\right)^{-1} \right) < 0 \tag{4.3.149}$$

for any $\varepsilon \in (0,1)$ implies that C_0 has a virtual level at zero. This is a contradiction to Remark 4.3.2 Hamiltonians, because the Hamiltonian of the whole system has a virtual level at zero. Therefore, C_0 must consist of at least three particles. Thus, by definition C_0 satisfies (4.3.149) for any $\varepsilon > 0$ and for each cluster $\tilde{C} \subsetneq C_0$ with $|\tilde{C}| > 1$

$$H[\tilde{C}] - \varepsilon_0 \left(1 + |q[\tilde{C}]|_m^2 (\ln(|q[\tilde{C}]|_m))^2\right)^{-1} \geq 0 \tag{4.3.150}$$

for an $\varepsilon_0 > 0$. Hence, by induction assumption $H[C_0]$ has a virtual level at zero. Again, this is a contradiction to the assumption that H has a virtual level at zero. We conclude that if H has a virtual level at zero, then condition **(i)** of Theorem 4.3.24 holds.
Step 2: If H has a virtual level at zero, then condition **(ii)** of Theorem 4.3.24 holds, i.e., for any $\varepsilon > 0$

$$\inf \sigma \left(H - \varepsilon \left(1 + |x|_m^2 \ln^2(|x|_m)\right)^{-1} \right) < 0. \tag{4.3.151}$$

Recall that in case of $d = 1$, $N \geq 3$ or $d = 2$, $N \geq 4$ zero is an eigenvalue of H. Taking the corresponding eigenfunction as a trial function shows that (4.3.151) is fulfilled for any $\varepsilon > 0$. For $d = 2$, $N = 3$ we do not know whether zero is an eigenvalue. However, by Theorem 4.3.17 we know that there is a function $\varphi_0 \in \tilde{H}^1(X_0)$ with $\|\nabla_0\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0$ and therefore we get (4.3.151) in this case as well. Hence, if H has a virtual level at zero, then condition **(ii)** of Theorem 4.3.24 is also true.

Step 3: If conditions **(i)** and **(ii)** of Theorem 4.3.24 are fulfilled, then condition **(i)** of Definition 4.3.1 holds, i.e., there exists a constant $\varepsilon_0 > 0$, such that

$$\inf \sigma_{\text{ess}}(H + \varepsilon_0 \Delta_0) = 0. \tag{4.3.152}$$

Recall that according to Remark 4.3.2 it is sufficient to prove the absence of virtual levels for $H[C]$ for all clusters C with $1 < |C| < N+1$. Assume for a contradiction that there exists a cluster C_1 with $1 < |C_1| < N+1$, such that $H[C_1]$ has a virtual level at zero. Then, as we proved in Step 2, we have

$$\inf \sigma\left(H[C_1] - \varepsilon\left(1 + |q[C_1]|_m^2 (\ln(|q[C_1]|_m))^2\right)^{-1}\right) < 0. \tag{4.3.153}$$

This is a contradiction to condition **(ii)** of Theorem 4.3.24. Hence, condition **(i)** of Definition 4.3.1 is fulfilled.

Step 4: If conditions **(i)** and **(ii)** of Theorem 4.3.24 are fulfilled, then condition **(ii)** of Definition 4.3.1 holds, i.e.,

$$\inf \sigma(H + \varepsilon \Delta_0) < 0 \tag{4.3.154}$$

for any $\varepsilon > 0$. Assume that (4.3.147) is fulfilled for any $\varepsilon > 0$. If $\dim(X_0) \geq 3$, we can use Hardy's inequality to conclude that (4.3.154) holds. If $\dim(X_0) < 3$, i.e., the system consists of three one-dimensional particles, we take a sequence of eigenfunctions ψ_n corresponding to negative eigenvalues of the operator $H - n^{-1}\left(1 + |x|_m^2 (\ln(|x|_m))^2\right)^{-1}$, normalized by $\|\psi_n\|_{\tilde{H}^1} = 1$. Applying the same arguments as in the proof of Theorem 4.3.13 we see that ψ_n converges in $L^2(X_0)$ to a function ψ_0 which is an eigenfunction of the operator H corresponding to the eigenvalue zero. For this function we have

$$(1-\varepsilon)\|\nabla_0\psi_0\|^2 + \langle V\psi_0, \psi_0\rangle = -\varepsilon\|\nabla_0\psi_0\|^2 < 0 \tag{4.3.155}$$

for any $\varepsilon > 0$. This proves that condition **(ii)** of Definition 4.3.1 is fulfilled and completes the proof of Theorem 4.3.24.

□

5. Absence of the Efimov effect for systems of one- or two-dimensional particles

5.1. Introduction and results

In this chapter we prove the absence of the Efimov effect for systems of $N = 3$ one- or two-dimensional particles and for systems of $N \geq 4$ one-dimensional or $N \geq 5$ two-dimensional particles. We also prove that it does not occur in systems of $N \geq 4$ one- or two-dimensional fermions.

For systems consisting of more than three particles the absense of the effect will be proved by adapting the technique of [74] and [7]. The main ingredient in this proof is the fact that virtual levels of the cluster Hamiltonians correspond to eigenvalues as it was shown in the previous chapter.

For systems of three one- or two-dimensional particles the absence of the effect was proved in [73] under very restrictive assumptions on the potentials, namely they had to be short-range and negative at infinity or compactly supported. Later, in [76] the restrictions on the potentials were relaxed. There, the proof was given for the case of two-dimensional fermions only. Unfortunately, it contains a mistake. In fact, for three two-dimensional particles the super Efimov effect exists, see [26]. Below we use the strategy of [76], correct the mistake and give the complete proof for the case of one-dimensional as well as for two-dimensional particles.

The main part of this chapter is based on [8]. Theorem 5.1.3, which shows the absence of the Efimov effect for fermionic systems, is based on [7]. Our main result of this chapter is the following

Theorem 5.1.1 (Absence of the Efimov effect for multi-particle systems)**.** *Let $d = 1$ and $N \geq 4$ or $d = 2$ and $N \geq 5$. Suppose that every pair potential $V_{ij} \neq 0$ satisfies* (4.2.3) *and is operator bounded with respect to $-\Delta$ with relative bound zero, i.e., for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that*

$$\|V_{ij}\psi\|^2 \leq \varepsilon\|\Delta\psi\|^2 + C(\varepsilon)\|\psi\|^2, \qquad \psi \in H^2(\mathbb{R}^d). \tag{5.1.1}$$

Moreover, assume that $H[C] \geq 0$ for all clusters C with $|C| = N-1$ and that there exists $\varepsilon \in (0,1)$, such that

$$\sigma_{\text{ess}}(-(1-\varepsilon)\Delta_0[C] + V[C]) = [0,\infty). \tag{5.1.2}$$

Then the discrete spectrum of H is finite.

Remark 5.1.2. **(i)** Note that in Theorem 5.1.1 the cluster Hamiltonian $H[C]$ with $|C| = N-1$ may have a virtual level at zero. For clusters C' with $1 < |C'| < N-1$ however, the Hamiltonian $H[C']$ is not allowed to have a virtual level, which is a consequence of (5.1.2) and the HVZ theorem.

(ii) The question whether the Efimov effect can appear in systems of four two-dimensional particles remains open. This is related to the question whether virtual levels in systems of three two-dimensional particles correspond to eigenvalues or to resonances. If they correspond to eigenvalues, it is a signal for the absence of the effect. However, even if they correspond to resonances it might be that the effect is absent, as it is for example the case for systems of three four-dimensional particles [5].

We also prove the absence of the Efimov effect for systems of $N \geq 4$ one- or two-dimensional fermions, namely the following

Theorem 5.1.3 (Absence of the effect for fermionic systems)**.** *Let H_{as} be the Hamiltonian corresponding to a system of $N \geq 4$ one- or two-dimensional particles, where the potentials satisfy* (3.2.26)*,* (4.2.3) *and* (5.1.1)*. Furthermore, assume that $H_{\text{as}}[C] \geq 0$ for all clusters with $|C| = N-1$ and there exists an $\varepsilon \in (0,1)$, such that*

$$\sigma_{\text{ess}}(H_{\text{as}}[C] + \varepsilon\Delta_0[C]) = [0,\infty). \tag{5.1.3}$$

Then the discrete spectrum of H_{as} is finite.

For systems of three one- or two-dimensional particles we prove the following results.

Theorem 5.1.4 (Absence of the effect for three one-dimensional particles)**.** *Let H be the Hamiltonian corresponding to a system of $N = 3$ one-dimensional particles. Suppose that $H[C] \geq 0$ for any two-particle cluster C and that each pair potential V_{ij} satisfies* (4.2.2) *and* (4.2.3)*. Then the discrete spectrum of H is finite. The same is true for the operator H_{as} corresponding to a system of three identical one-dimensional fermions.*

Theorem 5.1.5 (Absence of the effect for three two-dimensional particles)**.** *Let H be the Hamiltonian corresponding to a system of $N = 3$ two-dimensional particles. Assume that $H[C] \geq 0$ for any two-particle cluster C and that the pair potentials V_{ij} satisfy* (4.2.2) *and* (4.2.3) *and that they are radially symmetric, i.e., $V_{ij}(x_{ij}) = V_{ij}(|x_{ij}|)$. Then the discrete spectrum of H is finite.*

Remark 5.1.6. Recall that for three two-dimensional fermions the so-called super Efimov effect appears. This, together with Theorem 5.1.5 shows that systems of two-dimensional particles behave very different than systems of three-dimensional particles. While for system of three three-dimensional particles without symmetry restrictions the Efimov effect occurs and is absent for systems of fermions, for two-dimensional particles it is just the other way round.

5.2. Proofs of the results

5.2.1. A criterion for the finiteness of the discrete spectrum

To prove the finiteness of the discrete spectrum of the operator H we show that for some constants $\varepsilon > 0$ and $\beta > 0$

$$\langle H\psi, \psi\rangle - \varepsilon \| |x|_m^{-\beta} \psi \|^2 \geq 0 \tag{5.2.1}$$

holds for all functions $\psi \in H^1(X_0)$ which are supported far away from the origin. This criterion is due to G. Zhislin who used it in a slightly different form in [82]. A precise formulation of this criterion is given in the following

Lemma 5.2.1 (Criterion for the finiteness of the discrete spectrum)**.** *Let $h = -\Delta + V$ in $L^2(\mathbb{R}^k)$, $k \in \mathbb{N}$, where V satisfies* (4.2.2)*. Assume there exist constants β, ε, $b > 0$, such that*

$$\langle h\psi, \psi\rangle - \varepsilon \| |x|^{-\beta} \psi\|^2 \geq 0 \tag{5.2.2}$$

for any $\psi \in H^1(\mathbb{R}^k)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^k,\ |x| \geq b\}$*. Then the following assertions hold.*

(i) $\inf \sigma_{\text{ess}}(h) \geq 0$.

(ii) *The operator h has at most a finite number of negative eigenvalues.*

(iii) *Zero is not an infinitely degenerate eigenvalue of h.*

Remark 5.2.2. **(i)** Again, we use the small letter h for the Hamiltonian in this abstract case. The capital letter H is reserved for the multi-particle operator.

(ii) In [82] condition (5.2.2) was formulated and used with $\beta = 1$. In our proofs of the absence of the Efimov effect the parameter $\beta = 1$ is not sufficient because at some point we use Hardy's inequality to compensate for the term $-\varepsilon \| |x|_m^{-\beta} \psi\|^2$. For the case of two-dimensional particles this is not possible if $\beta = 1$ because the two-dimensional Hardy inequality requires an additional logarithmic factor, see Section 2.1.2. We will show that the operator H, considered as an operator on $L^2(\mathbb{R}^k)$ with $k = d(N-1)$ satisfies condition (5.2.2) for some $\beta > 1$.

To prove Lemma 5.2.1 we use the following

Lemma 5.2.3. *Assume that V satisfies* (4.2.2) *and let $h = -\Delta + V$, acting on $L^2(\mathbb{R}^k)$. Let $\beta > 0$, $\varepsilon > 0$ and $\tilde{b} > b > 0$. Then there exist a function $\chi_1 \in C^1(\mathbb{R}^k)$, $0 \leq \chi_1 \leq 1$, with*

$$\chi_1(x) = \begin{cases} 1, & |x| \leq b, \\ 0, & |x| \geq \tilde{b} \end{cases} \tag{5.2.3}$$

and a constant $C > 0$, such that for all $\psi \in H^1(\mathbb{R}^k)$ we have

$$\begin{aligned} \langle h\psi, \psi\rangle \geq \langle h\psi\chi_1, \psi\chi_1\rangle - C\|\psi\chi_1\|^2 \\ + \langle h\psi\chi_2, \psi\chi_2\rangle - \varepsilon \| |x|^{-\beta} \psi\chi_2\|^2_{\{b \leq |x| \leq \tilde{b}\}}, \end{aligned} \tag{5.2.4}$$

where $\chi_2 = \sqrt{1 - \chi_1^2}$.

Proof of Lemma 5.2.3. Let $\beta, \varepsilon > 0$ and $b, \tilde{b} > 0$ with $\tilde{b} > b$ be fixed. Furthermore, let $u : \mathbb{R}_+ \to [0,1]$ be a C^1-function, such that $u(t) = 1$ for $t \le b$ and $u(t) = 0$ for $t \ge \tilde{b}$. We assume that u is strictly monotonically decreasing on $(b, \tilde{b})$. Let $v = \sqrt{1-u^2}$. We choose u in such a way that $v'(t)(1-v^2(t))^{-\frac{1}{2}} \to 0$ as $t \to \tilde{b}_-$. For $x \in \mathbb{R}^k$ let

$$\chi_1(x) = u(|x|), \qquad \chi_2(x) = v(|x|). \tag{5.2.5}$$

Then we have

$$|\nabla\chi_1|^2 + |\nabla\chi_2|^2 = \frac{|\nabla\chi_2|^2}{1-\chi_2^2} = \frac{(v'(|x|))^2}{1-v^2(|x|)}. \tag{5.2.6}$$

Since $v'(|x|)(1-v^2(|x|))^{-\frac{1}{2}} \to 0$ as $|x| \to \tilde{b}_-$ and $v(|x|)$ is close to one for $|x|$ close to $\tilde{b}$, we can choose $b < b' < \tilde{b}$ so close to $\tilde{b}$ that

$$\frac{(v'(|x|))^2}{1-v^2(|x|)} \le \varepsilon v^2(|x|)|x|^{-2\beta}, \qquad b' \le |x| \le \tilde{b}. \tag{5.2.7}$$

This, together with (5.2.6) implies

$$|\nabla\chi_1|^2 + |\nabla\chi_2|^2 \le \varepsilon\chi_2^2(x)|x|^{-2\beta}, \qquad b' \le |x| \le \tilde{b}. \tag{5.2.8}$$

Now we estimate $|\nabla\chi_1|^2 + |\nabla\chi_2|^2$ for $b \le |x| \le b'$. Recall that for $b < t < b'$ we have $u(t) > u(b') > 0$ and $0 < v(t) < v(b') < 1$. Hence, we get

$$|\nabla\chi_1|^2 + |\nabla\chi_2|^2 = \frac{(v'(|x|))^2}{1-v^2(|x|)} \le Cu^2(|x|)|x|^{-2\beta}, \qquad b \le |x| \le b' \tag{5.2.9}$$

for some $C > 0$. Due to the IMS formula we have

$$\langle h\psi, \psi\rangle = \langle h\psi\chi_1, \psi\chi_1\rangle + \langle h\psi\chi_2, \psi\chi_2\rangle - \int \left(|\nabla\chi_1|^2 + |\nabla\chi_2|^2\right)|\psi|^2\,\mathrm{d}x. \tag{5.2.10}$$

This, together with (5.2.8) and (5.2.9) completes the proof of Lemma 5.2.3. □

Now we turn to the

Proof of Lemma 5.2.1. We construct a finite-dimensional subspace $M \subset L^2(\mathbb{R}^k)$, such that $\langle h\psi, \psi\rangle > 0$ holds for any $\psi \in H^1(\mathbb{R}^k)$ with $\psi \ne 0$ which is orthogonal to M. Let $\varepsilon, \beta, b > 0$, such that (5.2.2) is fulfilled and let χ_1 and χ_2 be functions according to

Lemma 5.2.3. Then by condition (5.2.2) for any function $\psi \in H^1(\mathbb{R}^k)$

$$\begin{aligned}\langle h\psi,\psi\rangle &\geq \langle h\psi\chi_1,\psi\chi_1\rangle - C\|\psi\chi_1\|^2 + \langle h\psi\chi_2,\psi\chi_2\rangle - \varepsilon\||x|^{-\beta}\psi\chi_2\|^2_{\{b\leq|x|\leq\tilde{b}\}}\\ &\geq \langle h\psi\chi_1,\psi\chi_1\rangle - C\|\psi\chi_1\|^2,\end{aligned}$$

because $\mathrm{supp}\,(\chi_2) \subset \{x \in \mathbb{R}^k : |x| \geq b\}$. Thus, to prove statements **(i)**-**(iii)** it suffices to show that

$$\langle h\psi\chi_1,\psi\chi_1\rangle - C\|\psi\chi_1\|^2 \geq 0 \tag{5.2.11}$$

holds for any function $\psi \in H^1(\mathbb{R}^k)$ with $\psi \perp M$ (with respect to the $L^2(\mathbb{R}^k)$ scalar product) for some finite-dimensional space $M \subset H^1(\mathbb{R}^k)$. By condition (4.2.2) we get

$$\langle h\psi\chi_1,\psi\chi_1\rangle - C\|\psi\chi_1\|^2 \geq (1-\varepsilon)\|\nabla\left(\psi\chi_1\right)\|^2 - C'\|\psi\chi_1\|^2 \tag{5.2.12}$$

for some $C' > 0$. For $l \in \mathbb{N}$ let

$$M_l := \left\{\varphi_1\chi_1,\ldots,\varphi_l\chi_1\right\}, \tag{5.2.13}$$

where $\{\varphi_1,\ldots,\varphi_l\}$ is an orthonormal set of eigenfunctions corresponding to the l lowest eigenvalues of the Laplacian, acting on $L^2\left(\{|x| \leq \tilde{b}\}\right)$ with Dirichlet boundary conditions. For $\psi \perp M_l$ we have $\psi\chi_1 \perp \varphi_1,\ldots\varphi_l$, which for sufficiently large l implies

$$\|\nabla(\psi\chi_1)\|^2 \geq (1-\varepsilon)^{-1} C'\|\psi\chi_1\|^2. \tag{5.2.14}$$

Therefore, we conclude $L[\psi\chi_1] > 0$. This proves statements **(i)**-**(iii)** of Lemma 5.2.1. □

5.2.2. Proof of Theorem 5.1.1

The idea of the proof stems from [74], where it was proved that the Hamiltonian of a system consisting of three three-dimensional particles has a finite number of negative eigenvalues provided virtual levels for the two-body Hamiltonians correspond to eigenvalues. In [7] this strategy was extended to prove the absence of the Efimov effect for systems consisting of $N \geq 4$ particles in dimension $d \geq 3$. We generalize this to the case of one-or two-dimensional particles.

For $\varepsilon > 0$ we define the functional $L: H^1(X_0) \to \mathbb{R}$ as

$$L[\varphi] := \langle H\varphi, \varphi\rangle - \varepsilon \| |x|_m^{-2} \varphi\|^2 \tag{5.2.15}$$

and prove that $L[\varphi] \geq 0$ for any function $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{|x|_m \geq R\}$ if $R > 0$ is large enough and $\varepsilon > 0$ is small enough. Then the finiteness of the discrete spectrum of H follows from Lemma 5.2.1. We notice that the choice of $\beta = 2$ as exponent in $\| |x|_m^{-\beta} \varphi\|^2$ is somewhat random and the proof works for any $\beta > 1$.
In what follows we always assume $\kappa > 0$ to be so small that $K_R(Z, \kappa) \cap K_R(Z', \kappa) = \emptyset$ for all partitions $Z \neq Z'$ with $|Z| = |Z'| = 2$ and any $R > 0$. By applying Theorem 4.3.9 we get

$$L[\varphi] \geq \sum_{Z:|Z|=2} L_2[\varphi u_Z] + L_2'[\varphi \mathcal{V}], \tag{5.2.16}$$

where we recall that $\operatorname{supp}\big(\varphi\big) \subset X_0 \setminus B(R)$, $\mathcal{V} = \sqrt{1 - \sum_{Z:|Z|=2} u_Z^2}$ on $X_0 \setminus B(R)$ and the functionals L_2 and L_2' are defined by

$$\begin{aligned} L_2[\psi] &:= \langle H\psi, \psi\rangle - \varepsilon \| |x|_m^{-2} \psi\|^2 \\ &\qquad - \varepsilon_1 \left\| |q(Z)|_m^{-1} |\ln(|q(Z)|_m |\xi(Z)|_m^{-1})|^{-1} \psi \right\|^2_{K_R(Z,\kappa',\kappa)}, \\ L_2'[\psi] &:= \langle H\psi, \psi\rangle - (\varepsilon + \varepsilon_1) \| |x|_m^{-2} \psi\|^2. \end{aligned} \tag{5.2.17}$$

We recall that the constants $\kappa > 0$ and $\varepsilon_1 > 0$ can be chosen arbitrarily small and $\kappa' \in (0, \kappa)$ depends on κ and ε_1 only. At first, we prove $L_2[\varphi u_Z] \geq 0$. We distinguish between the following two types of partitions $Z = \{C_1, C_2\}$:

(i) Neither $H[C_1]$ nor $H[C_2]$ has a virtual level at zero,

(ii) $H[C_1]$ or $H[C_2]$ has a virtual level at zero.

In the first case there exists a constant $\mu_0 > 0$, such that

$$\langle H(Z)\psi, \psi\rangle \geq \mu_0 \|\nabla_{q(Z)} \psi\|^2 \tag{5.2.18}$$

holds for any function $\psi \in H^1(X_0)$. Repeating the arguments which were used in the proof of Lemma 4.3.11 we get $L_2[\varphi u_Z] \geq 0$.

We turn to case **(ii)** and assume that $H[C_1]$ has a virtual level. Then by the assumption of the theorem we have $|C_1| = N-1$ and $|C_2| = 1$. Recall that $H[C_1]$ is the Hamiltonian of a cluster which is decoupled from the system and can therefore be considered as the Hamiltonian of a system of $N-1$ particles with removed center of mass. Therefore, according to the results of Chapter 4 zero is a simple eigenvalue of $H[C_1]$. Let $\varphi_0 \in H^1(X_0[C_1])$ be the corresponding eigenfunction normalized by $\|\varphi_0\|_{L^2(X_0[C_1])} = 1$. Note that because of $|C_2| = 1$ we have

$$\langle H(Z)\psi, \psi\rangle = \langle H[C_1]\psi, \psi\rangle \quad \text{and} \quad \|\nabla_{q(Z)}\psi\|^2 = \|\nabla_{q[C_1]}\psi\|^2, \tag{5.2.19}$$

where we consider the operators $H(Z)$ and $H[C_1]$ as operators on $L^2(X_0)$. We will use the letters q for $q(Z)$ and $q[C_1]$, and ξ for $\xi(Z)$ and $\xi[C_1]$ simultaneously. Similar to [74] we define

$$f(\xi) := \|\nabla_q \varphi_0\|^{-2} \langle \nabla_q \left(\varphi u_Z(\cdot,\xi)\right), \nabla_q \varphi_0\rangle_{L^2(X_0(Z))} \tag{5.2.20}$$

and

$$g(q,\xi) := \varphi u_Z(q,\xi) - \varphi_0(q) f(\xi). \tag{5.2.21}$$

Then we have

$$\varphi u_Z = \varphi_0 f + g \quad \text{and} \quad \langle \nabla_q g(\cdot,\xi), \nabla_q \varphi_0\rangle_{L^2(X_0(Z))} = 0 \tag{5.2.22}$$

for almost every ξ. We write

$$\begin{aligned} L_2[\varphi u_Z] = {} & \langle H[C_1] g, g\rangle + \langle H[C_1]\varphi_0 f, \varphi_0 f\rangle + 2\,\mathrm{Re}\langle g, H[C_1]\varphi_0 f\rangle \\ & + \|\nabla_\xi \left(\varphi u_Z\right)\|^2 + \langle I(Z)\varphi u_Z, \varphi u_Z\rangle - \varepsilon \||x|_m^{-2} \varphi u_Z\|^2 \\ & - \varepsilon_1 \||q|_m^{-1} |\ln(|q|_m |\xi|_m^{-1})|^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}, \end{aligned} \tag{5.2.23}$$

where we recall that $I(Z)$ is the inter-cluster potential describing the interaction between particles from different clusters in the partition Z. Due to $H[C_1]\varphi_0 = 0$ the second term and the third term on the r.h.s. of (5.2.23) are zero. We estimate the term $\langle I(Z)\varphi u_Z, \varphi u_Z\rangle$. Recall that u_Z is supported in the cone $K(Z,\kappa)$ and φ is supported in the region $\{x \in X_0 : |x|_m \geq R\}$. Therefore,

$$|\xi|_m \geq (1+\kappa^2)^{-\frac{1}{2}} |x|_m \geq \frac{R}{2}, \qquad x \in \mathrm{supp}\,(\varphi u_Z) \tag{5.2.24}$$

if $\kappa > 0$ is small enough. This, together with the assumption that the pair potentials V_{ij} satisfy (4.2.3), implies that for fixed $\varepsilon_2 > 0$ we have

$$|I(Z)(x)| \le C|\xi|_m^{-2-\nu} \le \frac{\varepsilon_2}{4}|\ln(|\xi|_m)|^{-2}|\xi|_m^{-2} \tag{5.2.25}$$

for $x \in K_R(Z,\kappa)$ if $R > 0$ is large enough. Since by (5.2.24) $\varphi u_Z(q,\xi) = 0$ for $|\xi|_m \le \frac{R}{2}$, we can apply the one- or two-dimensional Hardy type inequality in the ξ-variable (without loss of generality we can assume that $R > 2$, so the conditions for the one- or two-dimensional Hardy inequality are fulfilled) to obtain

$$|\langle I(Z)\varphi u_Z, \varphi u_Z\rangle| \le \frac{\varepsilon_2}{4}\||\ln(|\xi|_m)|^{-1}|\xi|_m^{-1}\varphi u_Z\|^2 \le \varepsilon_2\|\nabla_\xi(\varphi u_Z)\|^2. \tag{5.2.26}$$

This, together with (5.2.23) implies

$$\begin{aligned} L_2[\varphi u_Z] \ge &\langle H[C_1]g,g\rangle + (1-\varepsilon_2)\|\nabla_\xi\left(\varphi u_Z\right)\|^2 - \varepsilon\||x|_m^{-2}\varphi u_Z\|^2 \\ &- \varepsilon_1\||\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}. \end{aligned} \tag{5.2.27}$$

Since

$$\left\||x|_m^{-2}\varphi u_Z\right\|^2 \le \left\||\xi|_m^{-1}(\ln^{-1}|\xi|_m)\varphi u_Z\right\|^2 \tag{5.2.28}$$

for $|x|_m > 1$ and we have $|\xi|_m \ge \frac{R}{2}$ on the support of φu_Z, we get by the one- or two-dimensional Hardy inequality

$$4\varepsilon\|\nabla_\xi\left(\varphi u_Z\right)\|^2 - \varepsilon\||x|_m^{-2}\varphi u_Z\|^2 \ge 0. \tag{5.2.29}$$

Note that here is the place where we need the assumption $\beta > 1$ in Lemma 5.2.1. Substituting this inequality into (5.2.27) yields

$$\begin{aligned} L_2[\varphi u_Z] \ge &\langle H[C_1]g,g\rangle + (1-\varepsilon_3)\|\nabla_\xi\left(\varphi u_Z\right)\|^2 \\ &- \varepsilon_1\||\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}, \end{aligned} \tag{5.2.30}$$

where $\varepsilon_3 = \varepsilon_2 + 4\varepsilon$. Now we estimate the term

$$\langle H[C_1]g,g\rangle - \varepsilon_1\||\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}. \tag{5.2.31}$$

This is done in the following

Lemma 5.2.4. *Let* $1 < \alpha < \tilde{C}_H(X_0)$*, where the constant* $\tilde{C}_H(X_0)$ *is given by* (4.3.9)*, and let* C_1 *be a cluster with* $|C_1| = N-1$*. Furthermore, let the functions* f *and* g *be defined by* (5.2.20) *and* (5.2.21)*. Then for* $\varepsilon_1 > 0$ *small enough and* $R > 0$ *sufficiently large we have*

$$\begin{aligned}\langle H[C_1]g, g\rangle - \varepsilon_1 \| |\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \\ \geq -\int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha}|f(\xi)|^2\,\mathrm{d}\xi.\end{aligned} \tag{5.2.32}$$

Proof of Lemma 5.2.4. Due to Theorem 4.3.3 the orthogonality in (5.2.22) implies

$$\langle H[C_1]g, g\rangle \geq \delta_0 \|\nabla_q g\|^2 \tag{5.2.33}$$

for some $\delta_0 > 0$. Therefore, we have

$$\begin{aligned}\langle H[C_1]g, g\rangle - \varepsilon_1 \| |\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \\ \geq \delta_0\|\nabla_q g\|^2 - \varepsilon_1 \| |\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)}.\end{aligned} \tag{5.2.34}$$

Since $\varphi u_Z = \varphi_0 f + g$, we also have

$$|\nabla_q(\varphi u_Z)|^2 = |\nabla_q(\varphi_0 f + g)|^2 \leq 2|\nabla_q \varphi_0 f|^2 + 2|\nabla_q g|^2, \tag{5.2.35}$$

which yields

$$\|\nabla_q g\|^2_{K_R(Z,\kappa',\kappa)} \geq \frac{1}{2}\|\nabla_q(\varphi u_Z)\|^2_{K_R(Z,\kappa',\kappa)} - \|\nabla_q \varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}. \tag{5.2.36}$$

If $d = 1$ and $N = 4$, we have $\dim(X_0(Z)) = 2$. In this case we use that $\varphi u_Z = 0$ for $|q|_m = \kappa|\xi|_m$ and apply the two-dimensional Hardy type inequality, similarly as in the proof of Lemma 4.3.11, to get

$$\frac{\delta_0}{2}\|\nabla_q(\varphi u_Z)\|^2_{K_R(Z,\kappa',\kappa)} - \varepsilon_1 \| |\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \geq 0 \tag{5.2.37}$$

if $\varepsilon_1 > 0$ is small enough. In all other cases we have $\dim(X_0(Z)) \geq 3$ and we can apply Hardy's inequality to get (5.2.37). Combining (5.2.37) with the inequalities (5.2.36) and (5.2.34) yields

$$\begin{aligned} \langle H[C_1]g,g\rangle - \varepsilon_1 \| |\ln(|q|_m|\xi|_m^{-1})|^{-1}|q|_m^{-1}\varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \\ \geq -\delta_0\|\nabla_q\varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}. \end{aligned} \tag{5.2.38}$$

Now we estimate the term $\|\nabla_q\varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}$. Recall that by Theorem 4.3.3 we have

$$\left|\nabla_q\left(|q|_m^\alpha\varphi_0\right)\right| \in L^2(X_0(Z)) \quad \text{and} \quad (1+|q|_m)^{\alpha-1}\varphi_0 \in L^2(X_0(Z)) \tag{5.2.39}$$

for any $0 \leq \alpha < \tilde{C}_H(X_0)$. This implies

$$|q|_m^\alpha\left|\nabla_q\varphi_0\right| \in L^2(X_0(Z)). \tag{5.2.40}$$

Due to $f(\xi) = 0$ for $|\xi|_m \leq \frac{R}{2}$ we have

$$\begin{aligned} \int_{K_R(Z,\kappa',\kappa)} |\nabla_q\varphi_0 f|^2\,\mathrm{d}x &= \int_{\{|\xi|_m\geq\frac{R}{2}\}} |f(\xi)|^2 \int_{\{\kappa'|\xi|_m\leq|q|_m\leq\kappa|\xi|_m\}} |\nabla_q\varphi_0|^2\,\mathrm{d}q\,\mathrm{d}\xi \\ &= \int_{\{|\xi|_m\geq\frac{R}{2}\}} |f(\xi)|^2 \int_{\{\kappa'|\xi|_m\leq|q|_m\leq\kappa|\xi|_m\}} |q|_m^{-2\alpha}|q|_m^{2\alpha}|\nabla_q\varphi_0|^2\,\mathrm{d}q\,\mathrm{d}\xi \\ &\leq (\kappa')^{-2\alpha}\int_{\{|\xi|_m\geq\frac{R}{2}\}} |\xi|_m^{-2\alpha}|f(\xi)|^2 \int_{\{\kappa'|\xi|_m\leq|q|_m\leq\kappa|\xi|_m\}} |q|_m^{2\alpha}|\nabla_q\varphi_0|^2\,\mathrm{d}q\,\mathrm{d}\xi, \end{aligned} \tag{5.2.41}$$

where in the last inequality we used $|q|_m \geq \kappa'|\xi|_m$ for $x \in K_R(Z,\kappa',\kappa)$. Since by (5.2.40) $|q|_m^\alpha|\nabla_q\varphi_0| \in L^2(X_0(Z))$, we can choose $R > 0$ so large that

$$\int_{\{\kappa'|\xi|_m\leq|q|_m\leq\kappa|\xi|_m\}} |q|_m^{2\alpha}|\nabla_q\varphi_0|^2\,\mathrm{d}q \leq (\kappa')^{2\alpha}\delta_0^{-1} \tag{5.2.42}$$

for $|\xi|_m \geq \frac{R}{2}$. This yields

$$-\delta_0\|\nabla_q\varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)} \geq -\int_{\{|\xi|_m\geq\frac{R}{2}\}} |\xi|_m^{-2\alpha}|f(\xi)|^2\,\mathrm{d}\xi, \tag{5.2.43}$$

which completes the proof of Lemma 5.2.4. □

Remark 5.2.5. Note that for particles in dimension $d \geq 3$ one can apply Hardy's inequality to the function g to get

$$\begin{aligned} \delta_0 \|\nabla_q g\|^2 - \varepsilon_1 \| |\ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi u_Z\|^2_{K_R(Z,\kappa',\kappa)} \\ \geq -2\varepsilon_1 \| |\ln(|q|_m |\xi|_m^{-1})|^{-1} |q|_m^{-1} \varphi_0 f\|^2_{K_R(Z,\kappa',\kappa)}. \end{aligned} \tag{5.2.44}$$

Then, the term on the r.h.s. of (5.2.44) can be estimated by the use of the decay rate of φ_0 given in Theorem 4.3.3. This was the strategy in [7]. However, if $\dim(X_0(Z)) = 2$, which is the case if $d = 1$ and $N = 4$, Hardy's inequality can not be applied to the function g because the condition $g(q,\cdot) = 0$ for $|q|_m = 1$ is not fulfilled. Therefore, we estimated the term $\langle H[C_1]g, g\rangle$ differently. We created the zero condition artificially by adding the function $\varphi_0 f$, applied the two-dimensional Hardy inequality to the function φu_Z and estimated the resulting error $\|\nabla_q \varphi_0 f\|$.

We continue to estimate the functional $L_2[\varphi u_Z]$. Combining (5.2.30) and (5.2.32) yields

$$L_2[\varphi u_Z] \geq (1-\varepsilon_3)\|\nabla_\xi(\varphi u_Z)\|^2 - \varepsilon_1 \int_{\{|\xi|_m \geq \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \,\mathrm{d}\xi. \tag{5.2.45}$$

In the next step we estimate the term $\|\nabla_\xi(\varphi u_Z)\|^2$. This is done in the following lemma which is based on [74, Lemma 5.3].

Lemma 5.2.6. *Let $\delta > 0$. There exists a constant $\omega > 0$ which depends on $\|\varphi_0\|$, $\|\nabla_q \varphi_0\|$ and $\|\Delta_q \varphi_0\|$ only, such that*

$$\|\nabla_\xi \left(\varphi u_Z\right)\|^2 \geq \omega \left(\| |\xi|_m^{-1-\delta} \varphi_0 f\|^2 + \| |\xi|_m^{-1-\delta} g\|^2 \right). \tag{5.2.46}$$

Proof of Lemma 5.2.6. Since $\varphi u_Z(q,\xi) = 0$ for $|\xi|_m \leq \frac{R}{2}$, we can apply the one- or two-dimensional Hardy inequality in the space $X_c(Z)$ to $\varphi u_Z(q,\cdot)$ for fixed q. This implies

$$\begin{aligned} \|\nabla_\xi \left(\varphi u_Z\right)\|^2 &\geq \frac{1}{4} \| |\xi|_m^{-1-\delta} \varphi u_Z\|^2 = \frac{1}{4} \| |\xi|_m^{-1-\delta} \varphi_0 f + |\xi|_m^{-1-\delta} g\|^2 \\ &\geq \frac{1}{4} \left(\| |\xi|_m^{-1-\delta} \varphi_0 f\|^2 + \| |\xi|_m^{-1-\delta} g\|^2 \right) \\ &\qquad - \frac{1}{2} |\langle |\xi|_m^{-1-\delta} \varphi_0 f, |\xi|_m^{-1-\delta} g\rangle| \end{aligned} \tag{5.2.47}$$

for any $\delta > 0$. Since $\langle \nabla_q \varphi_0, \nabla_q g(\cdot,\xi)\rangle_{L^2(X_0(Z))} = 0$ for almost every ξ, we have

$$\langle \nabla_q |\xi|_m^{-1-\delta} \varphi_0 f, \nabla_q |\xi|_m^{-1-\delta} g\rangle = 0. \tag{5.2.48}$$

Moreover, the condition on the potentials implies that the domain of the operator $H[C_1]$ is given by $H^2(X_0[C_1])$. Using this together with the orthogonality (5.2.48), we can use Lemma 5.3 in [74] and find a constant $\omega > 0$ depending on $\|\varphi_0\|$, $\|\nabla_q \varphi_0\|$ and $\|\Delta_q \varphi_0\|$ only, such that

$$\left|\langle |\xi|_m^{-1-\delta} \varphi_0 f, |\xi|_m^{-1-\delta} g\rangle\right| \le \frac{1}{2}(1-4\omega)\left(\| |\xi|_m^{-1-\delta}\varphi_0 f\|^2 + \| |\xi|_m^{-1-\delta} g\|^2\right). \tag{5.2.49}$$

Substituting this inequality in (5.2.47) yields

$$\|\nabla_\xi \left(\varphi u_Z\right)\|^2 \ge \omega \left(\| |\xi|_m^{-1-\delta}\varphi_0 f\|^2 + \| |\xi|_m^{-1-\delta} g\|^2\right), \tag{5.2.50}$$

which completes the proof of Lemma 5.2.6. □

By combining (5.2.45) with (5.2.46) and using $\|\varphi_0\| = 1$ we get

$$\begin{aligned} L_2[\varphi u_Z] \ge (1-\varepsilon_3)\omega \int_{\{|\xi|_m \ge \frac{R}{2}\}} |\xi|_m^{-2-2\delta} |f(\xi)|^2 \, \mathrm{d}\xi \\ - \varepsilon_1 \int_{\{|\xi|_m \ge \frac{R}{2}\}} |\xi|_m^{-2\alpha} |f(\xi)|^2 \, \mathrm{d}\xi. \end{aligned} \tag{5.2.51}$$

Choosing $\delta \le \alpha - 1$ and $\varepsilon_1, \varepsilon_3 > 0$ small enough yields $L_2[\varphi u_Z] \ge 0$.

To complete the proof of Theorem 5.1.1 it remains to show $L_2'[\varphi \mathcal{V}] \ge 0$ for every $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{x \in X_0 : |x|_m \ge R\}$, where L_2' is the functional defined in (5.2.17). Note that for all partitions $Z = \{C_1, \ldots, C_p\}$ with $p = 3, 4, \ldots, N-1$ the Hamiltonians $H[C_i]$ do not have a virtual level if $|C_i| > 1$. Hence, we can estimate the functional $L_2'[\mathcal{V}\varphi] \ge 0$ in cones corresponding to partitions Z with $|Z| \ge 3$ in the same way as in the proof of Lemma 4.3.11. In the region which remains after separation of the cones corresponding to all partitions Z with $|Z| \le N-1$ we have $|V_{ij}(x_{ij})| \le |x|_m^{-2-\nu}$ for all $i \ne j$. Using Hardy's inequality in the space X_0 yields $L[\psi] \ge 0$ and applying Lemma 5.2.1 completes the proof of Theorem 5.1.1.

5.2.3. Proof of Theorem 5.1.3

The proof of Theorem 5.1.3 goes along the same line as that of Theorem 5.1.1. The only difference is that for clusters C with $|C| = N-1$ virtual levels correspond to eigenvalues which are not simple, but of finite multiplicity. Therefore, we find a decomposition which is similar to (5.2.20) – (5.2.22) with a function g which is orthogonal to the corresponding eigenspace $\mathscr{W}_0$ of the operator $H^{\mathrm{as}}[C]$ corresponding to the eigenvalue zero. Namely, let φ_i, $i = 1,\dots,\dim \mathscr{W}_0$ be eigenfunctions satisfying $\langle \nabla_q \varphi_i, \nabla_q \varphi_j \rangle = 0$ for $i \neq j$. Furthermore, for $k = 1,\dots,\dim(\mathscr{W}_0)$ let

$$f_k(\xi) = \|\nabla_q \varphi_k\|^{-2} \langle \nabla_q \left(\varphi u_Z(\cdot,\xi)\right), \nabla_q \varphi_k \rangle_{L^2(X_0(Z))} \tag{5.2.52}$$

and

$$g(q,\xi) = \varphi u_Z(q,\xi) - \sum_k f_k(\xi)\varphi_k(q). \tag{5.2.53}$$

Then we have $\langle \nabla_q g, \nabla_q \varphi_k \rangle = 0$ for all k. Using the Hardy type inequality (2.1.33) for functions which are orthogonal to all functions depending on $|x|_m$ only and repeating the arguments of the proof of Theorem 5.1.1 proves Theorem 5.1.3.

5.2.4. Proof of Theorem 5.1.4

Recall that in the proof of Theorem 5.1.1 we used that virtual levels of cluster Hamiltonians $H[C]$ with $|C| = N-1$ correspond to eigenvalues. This is not the case if the system consists of three particles and we need a different strategy. We start by proving several lemmas for one-dimensional Schrödinger operators. The first one gives an estimate of the corresponding quadratic form, restricted to an interval $(-b,b)$.

Lemma 5.2.7. *Consider the Schrödinger operator $h = -\Delta + V$ on $L^2(\mathbb{R})$, such that $h \geq 0$ and the potential V satisfies* (4.2.2) *and* (4.2.3). *Then there exists a constant $C > 0$, such that for any $b_0 > A$ and any function $\psi \in H^1(\mathbb{R})$*

$$\begin{aligned} J[\psi, b_0] :=& \int_{-b_0}^{b_0} \left(|\psi'(t)|^2 + V(t)|\psi(t)|^2\right) \mathrm{d}t \\ &\geq -C b_0^{-1-\nu} \left(|\psi(b_0)|^2 + |\psi(-b_0)|^2\right). \end{aligned} \tag{5.2.54}$$

Here, ν and A are constants given by (4.2.3).

Proof of Lemma 5.2.7. Let $\psi \in H^1(\mathbb{R})$ and $b_0 > A$. For $n \geq 2$ we define the function ψ_n as

$$\psi_n(t) = \begin{cases} \psi(t) & \text{if } -b_0 \leq t \leq b_0, \\ 0 & \text{if } t < -nb_0 \text{ or } t > nb_0, \\ \psi(-b_0)\frac{nb_0+t}{b_0(n-1)} & \text{if } -nb_0 < t < -b_0, \\ \psi(b_0)\frac{nb_0-t}{b_0(n-1)} & \text{if } b_0 < t < nb_0. \end{cases} \tag{5.2.55}$$

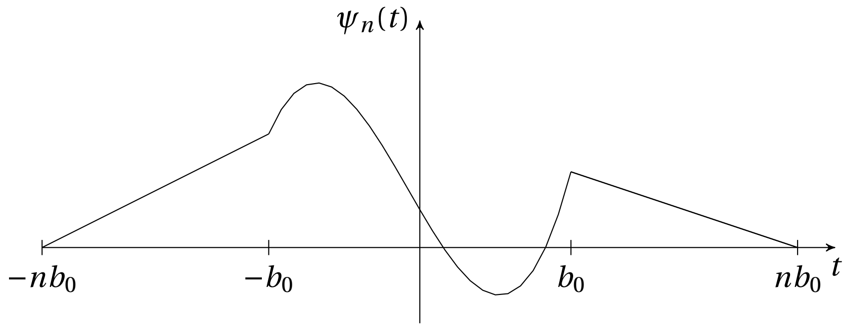

Figure 5.1.: Function ψ_n

Since ψ and ψ_n coincide for $-b_0 \leq t \leq b_0$, we have

$$\begin{aligned} \langle h\psi_n, \psi_n\rangle \leq & \int_{-b_0}^{b_0} \left(|\psi'(t)|^2 + V(t)|\psi(t)|^2\right) \mathrm{d}t \\ & + \int_{-nb_0}^{-b_0} \left(|\psi_n'(t)|^2 + |V(t)||\psi_n(t)|^2\right) \mathrm{d}t \\ & + \int_{b_0}^{nb_0} \left(|\psi_n'(t)|^2 + |V(t)||\psi_n(t)|^2\right) \mathrm{d}t. \end{aligned} \tag{5.2.56}$$

Let us estimate the two last integrals of the r.h.s of (5.2.56). Since

$$\psi_n'(t) = \frac{\psi(-b_0)}{b_0(n-1)} \quad \text{for} \quad t \in (-nb_0, -b_0), \tag{5.2.57}$$

we get

$$\int_{-nb_0}^{-b_0} |\psi_n'(t)|^2 \, \mathrm{d}t = \frac{|\psi_n(-b_0)|^2}{b_0(n-1)} < \varepsilon, \tag{5.2.58}$$

where $\varepsilon > 0$ is a constant which can be chosen arbitrarily small if $n \in \mathbb{N}$ is large enough. Due to $0 \le \frac{nb_0+t}{b_0(n-1)} \le 1$ for $t \in (-nb_0, -b_0)$ we get

$$\int_{-nb_0}^{-b_0} |V(t)||\psi_n(t)|^2 \mathrm{d}t \le |\psi(-b_0)|^2 \int_{-nb_0}^{-b_0} |V(t)|\mathrm{d}t \tag{5.2.59}$$

Analogously, we find

$$\int_{b_0}^{nb_0} |\psi_n'(t)|^2 \,\mathrm{d}t = \frac{|\psi_n(b_0)|^2}{b_0(n-1)} < \varepsilon \tag{5.2.60}$$

and

$$\int_{b_0}^{nb_0} |V(t)||\psi_n(t)|^2 \mathrm{d}t \le |\psi(b_0)|^2 \int_{b_0}^{nb_0} |V(t)|\mathrm{d}t. \tag{5.2.61}$$

Substituting these estimates in (5.2.56) yields

$$\begin{aligned} \langle h\psi_n, \psi_n \rangle \le J[\psi, b_0] + |\psi(-b_0)|^2 \int_{-nb_0}^{-b_0} |V(t)|\mathrm{d}t \\ + |\psi(b_0)|^2 \int_{b_0}^{nb_0} |V(t)|\,\mathrm{d}t + 2\varepsilon. \end{aligned} \tag{5.2.62}$$

For any $0 < \delta < \nu$ we have $-1 - \nu + \delta < -1$. Since $b_0 \ge A$, we get by (4.2.3)

$$\int_{b_0}^{nb_0} |V(t)|\,\mathrm{d}t \le c b_0^{-1-\delta} \int_A^\infty t^{-1-\nu+\delta}\,\mathrm{d}t \le c_1 b_0^{-1-\delta} \tag{5.2.63}$$

for some constants $c, c_1 > 0$. Analogously we have

$$\int_{-nb_0}^{-b_0} |V(t)|\,\mathrm{d}t \le c_1 b_0^{-1-\delta}. \tag{5.2.64}$$

Due to $h \ge 0$ we conclude from (5.2.62), (5.2.63) and (5.2.64) that

$$J[\psi, b_0] \ge -c b_0^{-1-\delta} \left(|\psi(b_0)|^2 + |\psi(-b_0)|^2 \right) - 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this completes the proof. □

The next lemma is an easy application of the Hardy inequality for the semi-axis.

Lemma 5.2.8. *Let* $C_0 > 0$. *Then for any sufficiently large* $b > 0$ *and for any* $\psi \in H^1(\mathbb{R})$ *we have*

$$\int_b^\infty \left(|\psi'(t)|^2 - C_0 t^{-2-\nu}|\psi(t)|^2\right) \mathrm{d}t \geq -2C_0 b^{-1-\nu}|\psi(b)|^2. \tag{5.2.65}$$

Proof of Lemma 5.2.8. Let $b > 0$, $\psi \in H^1(\mathbb{R})$ and $\tilde{\psi}(t) = \psi(t) - \psi(b)$. Then $\tilde{\psi}'(t) = \psi'(t)$ and we have

$$\begin{aligned} &\int_b^\infty \left(|\psi'(t)|^2 - C_0 t^{-2-\nu}|\psi(t)|^2\right) \mathrm{d}t \\ &\quad \geq \int_b^\infty \left(|\tilde{\psi}'(t)|^2 - 2C_0 t^{-2-\nu}|\tilde{\psi}(t)|^2\right) \mathrm{d}t - 2C_0 \int_b^\infty t^{-2-\nu}|\psi(b)|^2 \,\mathrm{d}t. \end{aligned} \tag{5.2.66}$$

Since $\tilde{\psi}(b) = 0$, we can use the one-dimensional Hardy inequality, which for sufficiently large $b > 0$ yields

$$\int_b^\infty \left(|\tilde{\psi}'(t)|^2 - 2C_0 t^{-2-\nu}|\tilde{\psi}(t)|^2\right) \mathrm{d}t \geq 0. \tag{5.2.67}$$

This, together with (5.2.66) implies

$$\int_b^\infty \left(|\psi'(t)|^2 - C_0 t^{-2-\nu}|\psi(t)|^2\right) \mathrm{d}t \geq -2C_0|\psi(b)|^2 \int_b^\infty t^{-2-\nu} \,\mathrm{d}t. \tag{5.2.68}$$

Computing the integral on the r.h.s. of (5.2.68) completes the proof. □

The following lemma is the one-dimensional analogue of Lemma 2 in [76].

Lemma 5.2.9. *Let* $b_2 > b_1$. *Then for any* $\psi \in H^1(\mathbb{R})$ *and* $i = 1, 2$ *we have*

$$|\psi(b_i)|^2 \leq 2(b_2 - b_1)^{-1} \int_{b_1}^{b_2} |\psi(t)|^2 \,\mathrm{d}t + 2(b_2 - b_1) \int_{b_1}^{b_2} |\psi'(t)|^2 \,\mathrm{d}t. \tag{5.2.69}$$

Proof of Lemma 5.2.9. For $t \in (b_1, b_2)$ we write

$$\psi(t) = \int_{b_1}^t \psi'(s) \,\mathrm{d}s + \psi(b_1). \tag{5.2.70}$$

Therefore, we have

$$|\psi(b_1)|^2 \le 2|\psi(t)|^2 + 2\left(\int_{b_1}^{b_2} |\psi'(s)|\,\mathrm{d}s\right)^2, \quad t \in (b_1, b_2). \tag{5.2.71}$$

Applying the Cauchy-Bunjakovsky-Schwarz inequality to the integral on the r.h.s. of (5.2.71) yields

$$|\psi(b_1)|^2 \le 2|\psi(t)|^2 + 2(b_2 - b_1)\int_{b_1}^{b_2} |\psi'(s)|^2\,\mathrm{d}s, \quad t \in (b_1, b_2). \tag{5.2.72}$$

Integrating both sides of (5.2.72) over (b_1, b_2) and dividing by $(b_2 - b_1)$ implies

$$|\psi(b_1)|^2 \le 2(b_2 - b_1)^{-1}\int_{b_1}^{b_2} |\psi(t)|^2\,\mathrm{d}t + 2(b_2 - b_1)\int_{b_1}^{b_2} |\psi'(t)|^2\,\mathrm{d}t, \tag{5.2.73}$$

which yields (5.2.69) for $i = 1$. To prove the statement for b_2 we can use the identity

$$\psi(t) = -\int_t^{b_2} \psi'(s)\,\mathrm{d}s + \psi(b_2) \tag{5.2.74}$$

and proceed as in the proof for b_1. This completes the proof of Lemma 5.2.9. □

Now we turn to the

Proof of Theorem 5.1.4. As in the proof of Theorem 5.1.1 we show that

$$L[\varphi] := \int \left(|\nabla_0 \varphi|^2 + V|\varphi|^2 - \varepsilon |x|_m^{-4}|\varphi|^2\right)\mathrm{d}x \ge 0 \tag{5.2.75}$$

holds for all functions $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{|x|_m \ge R\}$ if $\varepsilon > 0$ is small enough and $R > 0$ is sufficiently large. Then the statement of the theorem follows from the criterion given in Lemma 5.2.1. In the following we assume that $\kappa > 0$ is so small that the cones $K(Z,\kappa)$ and $K(Z',\kappa)$ corresponding to different partitions do not overlap. Let $Z = \{C_1, C_2\}$ be a partition of the system into two clusters, where we assume that $|C_1| = 2$. At first, we estimate the part of the quadratic form L corresponding to the cone $K(Z,\kappa)$, i.e., in the region where two particles are close to another and the third particle is separated from them. We denote the particles in the cluster C_1 by i and j

and the third particle by k.

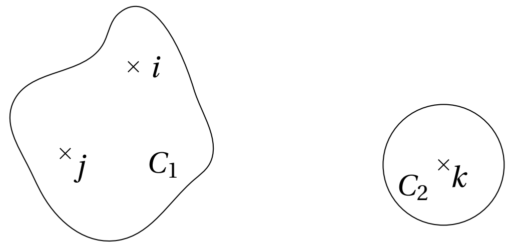

Figure 5.2.: Partition $Z = \{C_1, C_2\}$

Because we will need subtle geometric arguments in the following, we introduce a basis of X_0 and work with the corresponding coordinates. Recall that $\dim(X_0(Z)) = 1$ and $\dim(X_c(Z)) = 1$. Choosing a vector $u_1 \in X_0(Z)$ and a vector $u_2 \in X_c(Z)$, both normalized with respect to the norm $|u_i|_m = 1$, we get an orthonormal basis of X_0. Denote by $\tilde{q}$ and $\tilde{\xi}$ the coordinates corresponding to the basis $\{u_1, u_2\}$. Then we have $|q|_m = |\tilde{q}|$, $|\xi|_m = |\tilde{\xi}|$ and we can represent $K_R(Z,\kappa)$ as

$$K_R(Z,\kappa) = \left\{(\tilde{q},\tilde{\xi}) \in \mathbb{R}^2 : |\tilde{q}| \le \kappa|\tilde{\xi}|,\ |\tilde{q}|^2 + |\tilde{\xi}|^2 \ge R^2\right\} \tag{5.2.76}$$

and $\varphi = \varphi(\tilde{q},\tilde{\xi})$ as a function of $\tilde{q}$ and $\tilde{\xi}$. We estimate the integral

$$\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \tag{5.2.77}$$

in several steps. First, we estimate it by an integral over the edge of $K_R(Z,\kappa)$.

Step 1: Estimate of (5.2.77) by an integral over $\partial K_R(Z,\kappa)$
We write

$$\begin{aligned}
&\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \\
&\qquad = \int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x \\
&\qquad\quad + \int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{\xi}}\varphi|^2 + (V_{ik} + V_{jk})|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x
\end{aligned} \tag{5.2.78}$$

and estimate the two integrals on the r.h.s of (5.2.78) separately. Let $x^* = (\tilde{q}, \tilde{\xi})$ be a point of intersection of the ball $B(R)$ with the set $\{|\tilde{q}| = \kappa|\tilde{\xi}|\}$. Then we have $|\tilde{\xi}| = \left(1+\kappa^2\right)^{-\frac{1}{2}} R =: \gamma$ and

$$x = (\tilde{q}, \tilde{\xi}) \in K_R(Z, \kappa) \quad \Longrightarrow \quad |\tilde{\xi}| \geq \gamma,\ |\tilde{q}| \leq \kappa|\tilde{\xi}|. \tag{5.2.79}$$

At the same time we have for x^* that $|\tilde{q}| = \left(1+\kappa^{-2}\right)^{-\frac{1}{2}} R =: \eta$ and

$$x = (\tilde{q}, \tilde{\xi}) \in K_R(Z, \kappa) \quad \Longleftrightarrow \quad |\tilde{\xi}| \geq \begin{cases} \kappa^{-1}|\tilde{q}| & \text{if } |\tilde{q}| \geq \eta, \\ \sqrt{R^2 - |\tilde{q}|^2} & \text{if } |\tilde{q}| \leq \eta, \end{cases} \tag{5.2.80}$$

cf. Figure 5.3.

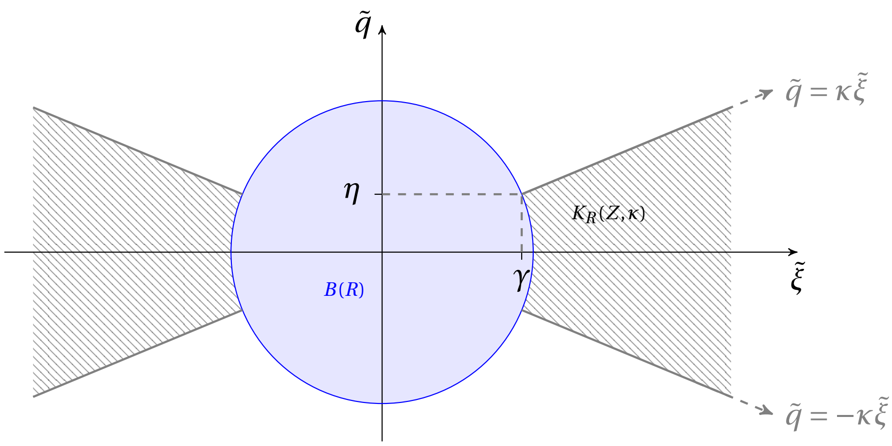

Figure 5.3.: The set $K_R(Z, \kappa)$

Since $\varphi(x) = 0$ for $|x|_m \leq R$, by (5.2.79) we have

$$\int_{K_R(Z,\kappa)} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x = \int_{\{|\tilde{\xi}| \geq \gamma\}} \int_{\{|\tilde{q}| \leq \kappa|\tilde{\xi}|\}} \left(|\partial_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{\xi}. \tag{5.2.81}$$

By applying Lemma 5.2.7 to the function $\varphi(\cdot,\tilde{\xi})$ for fixed $\tilde{\xi}$ we get

$$\begin{aligned}&\int_{\{|\tilde{\xi}|\geq\gamma\}}\int_{\{|\tilde{q}|\leq\kappa|\tilde{\xi}|\}}\left(|\partial_{\tilde{q}}\varphi|^2+V_{ij}|\varphi|^2\right)\mathrm{d}\tilde{q}\,\mathrm{d}\tilde{\xi}\\&\quad\geq -C\int_{\{|\tilde{\xi}|\geq\gamma\}}|\tilde{\xi}|^{-1-\nu}\left(|\varphi(\kappa\tilde{\xi},\tilde{\xi})|^2+|\varphi(-\kappa\tilde{\xi},\tilde{\xi})|^2\right)\mathrm{d}\tilde{\xi}\end{aligned}\tag{5.2.82}$$

for some $C>0$.

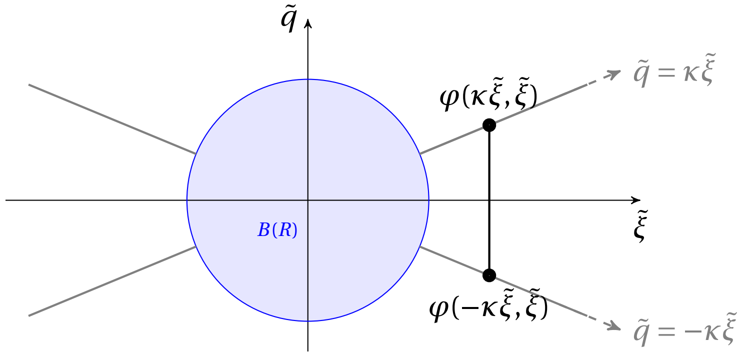

Figure 5.4.: Estimate of the integral over the set $\{\tilde{q}:-\kappa\tilde{\xi}\leq\tilde{q}\leq\kappa\tilde{\xi}\}$ for fixed $\tilde{\xi}$ by the value of φ at the points $(-\kappa\tilde{\xi},\tilde{\xi})$ and $(\kappa\tilde{\xi},\tilde{\xi})$.

Now we estimate the second integral on the right hand side of (5.2.78). Since the particle k is separated from the particles i and j and the potentials satisfy (4.2.3), we can estimate

$$|V_{ik}(x_{ik})|+|V_{jk}(x_{jk})|\leq c|\tilde{\xi}|^{-2-\nu}\tag{5.2.83}$$

for some $c>0$ if $R>0$ is large enough. This, together with $|x|_m^{-1}\leq|\tilde{\xi}|^{-1}$ implies

$$\begin{aligned}&\int_{K_R(Z,\kappa)}\left(|\partial_{\tilde{\xi}}\varphi|^2+(V_{ik}+V_{jk})|\varphi|^2-|x|_m^{-4}|\varphi|^2\right)\mathrm{d}x\\&\qquad\geq\int_{K_R(Z,\kappa)}\left(|\partial_{\tilde{\xi}}\varphi|^2-C|\tilde{\xi}|^{-2-\nu}|\varphi|^2\right)\mathrm{d}x\end{aligned}\tag{5.2.84}$$

for some $C>0$, where without loss of generality we assumed that $\nu<2$. To estimate the integral on the r.h.s. of (5.2.84) we first integrate over the variable $\tilde{\xi}$ for fixed $\tilde{q}$. In view of (5.2.80) we have

$$\begin{aligned}\int_{K_R(Z,\kappa)} &\left(|\partial_{\tilde\xi}\varphi|^2 - c|\tilde\xi|^{-2-\nu}|\varphi|^2\right) \mathrm{d}x \\ &= \int_{\{|\tilde q|<\eta\}} \int_{\left\{|\tilde\xi|\geq\sqrt{R^2-|\tilde q|^2}\right\}} \left(|\partial_{\tilde\xi}\varphi|^2 - c|\tilde\xi|^{-2-\nu}|\varphi|^2\right) \mathrm{d}\tilde\xi\,\mathrm{d}\tilde q \\ &\quad + \int_{\{|\tilde q|\geq\eta\}} \int_{\{|\tilde\xi|\geq\kappa^{-1}|\tilde q|\}} \left(|\partial_{\tilde\xi}\varphi|^2 - c|\tilde\xi|^{-2-\nu}|\varphi|^2\right) \mathrm{d}\tilde\xi\,\mathrm{d}\tilde q.\end{aligned} \tag{5.2.85}$$

Recall that $\varphi(x)=0$ for $|x|\leq R$ and thus $\varphi(\tilde q,\tilde\xi)=0$ if $|\tilde q|\leq\eta$ and $|\tilde\xi|\leq\sqrt{R^2-|\tilde q|^2}$. Hence, we can apply the one-dimensional Hardy inequality for fixed $\tilde q$, which implies

$$\int_{\{|\tilde q|<\eta\}} \int_{\left\{|\tilde\xi|\geq\sqrt{R^2-|\tilde q|^2}\right\}} \left(|\partial_{\tilde\xi}\varphi|^2 - c|\tilde\xi|^{-2-\nu}|\varphi|^2\right) \mathrm{d}\tilde\xi\,\mathrm{d}\tilde q \geq 0 \tag{5.2.86}$$

if $R>0$ is large enough. To estimate the second integral on the r.h.s of (5.2.85) we apply Lemma 5.2.8 with $b=\kappa^{-1}|\tilde q|$ and with the analogue statement for $b=-\kappa^{-1}|\tilde q|$, which yields

$$\begin{aligned}\int_{\{|\tilde q|\geq\eta\}} \int_{\{|\tilde\xi|\geq\kappa^{-1}|\tilde q|\}} &\left(|\partial_{\tilde\xi}\varphi|^2 - c|\tilde\xi|^{-2-\nu}|\varphi(x)|^2\right) \mathrm{d}\tilde\xi\,\mathrm{d}\tilde q \\ &\geq -C\int_{\{|\tilde q|\geq\eta\}} |\tilde q|^{-1-\nu}\left(|\varphi(\tilde q,\kappa^{-1}|\tilde q|)|^2 + |\varphi(\tilde q,-\kappa^{-1}|\tilde q|)|^2\right) \mathrm{d}\tilde q\end{aligned} \tag{5.2.87}$$

for some $C>0$.

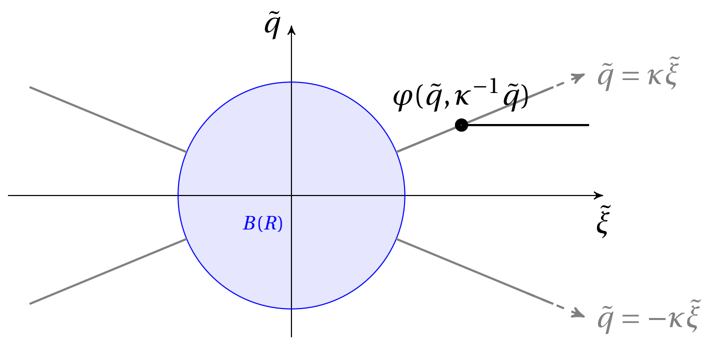

Figure 5.5.: Estimate of the integral over the set $\{\tilde\xi:\tilde\xi\geq\kappa^{-1}\tilde q\}$ for fixed $\tilde q$ by the value of φ at the point $(\tilde q,\kappa^{-1}\tilde q)$

By combining (5.2.82) and (5.2.87) with (5.2.78) we get

$$\begin{aligned}\int_{K_R(Z,\kappa)} & \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \\ & \geq -C\int_{\{|\tilde{\xi}|\geq\gamma\}} |\tilde{\xi}|^{-1-\nu}\left(|\varphi(\kappa|\tilde{\xi}|,\tilde{\xi})|^2 + |\varphi(-\kappa|\tilde{\xi}|,\tilde{\xi})|^2\right) \mathrm{d}\tilde{\xi} \\ & \quad - C\int_{\{|\tilde{q}|\geq\eta\}} |\tilde{q}|^{-1-\nu}\left(|\varphi(\tilde{q},\kappa^{-1}|\tilde{q}|)|^2 + |\varphi(\tilde{q},-\kappa^{-1}|\tilde{q}|)|^2\right) \mathrm{d}\tilde{q}.\end{aligned} \tag{5.2.88}$$

Note that the integrals on the r.h.s of (5.2.88) are in fact integrals of the function φ over the edges of the cone $K(Z,\kappa)$. In the following we estimate these integrals.

Step 2: Estimate of the boundary integrals over $\partial K_R(Z,\kappa)$
We estimate the integrals on the r.h.s. of (5.2.88) over the edges of $K_R(Z,\kappa)$ by an integral over the set $K_R(Z,\kappa,\kappa')$ with some $\kappa' > \kappa$. At first, we consider the integral

$$\int_{\{|\tilde{\xi}|\geq\gamma\}} |\tilde{\xi}|^{-1-\nu}\left(|\varphi(\kappa|\tilde{\xi}|,\tilde{\xi})|^2 + |\varphi(-\kappa|\tilde{\xi}|,\tilde{\xi})|^2\right) \mathrm{d}\tilde{\xi}, \tag{5.2.89}$$

which is comprised of the integrals over the four edges of $K_R(Z,\kappa)$ given by

$$\left\{\tilde{q} = \pm\kappa\tilde{\xi},\ \tilde{\xi}\geq\gamma\right\}, \qquad \left\{\tilde{q} = \pm\kappa\tilde{\xi},\ \tilde{\xi}\leq-\gamma\right\}. \tag{5.2.90}$$

We introduce polar coordinates (ρ,ω) in the two-dimensional space X_0 and denote $\omega_0 = \arctan(\kappa) \in \left(0,\frac{\pi}{2}\right)$. Then the integral over the straight line $\left\{\tilde{q} = \kappa\tilde{\xi},\ \tilde{\xi}\geq\gamma\right\}$ can be represented as

$$\int_{\{\tilde{\xi}\geq\gamma\}} \tilde{\xi}^{-1-\nu}|\varphi(\kappa\tilde{\xi},\tilde{\xi})|^2\,\mathrm{d}\tilde{\xi} = \cos(\omega_0)\int_R^\infty \rho^{-1-\nu}|\varphi(\rho,\omega_0)|^2\,\mathrm{d}\rho. \tag{5.2.91}$$

We choose $\kappa' > \kappa$ such that $K(Z,\kappa')$ and $K(Z',\kappa')$ do not overlap for any pair of two-cluster partitions $Z \neq Z'$ and denote $\omega_1 = \arctan(\kappa') \in \left(0,\frac{\pi}{2}\right)$. By applying Lemma 5.2.9 to the function $\varphi(\rho,\cdot)$ for fixed ρ and with $b_1 = \omega_0$ and $b_2 = \omega_1$ we find

$$|\varphi(\rho,\omega_0)|^2 \leq C(\omega_0,\omega_1)\int_{\omega_0}^{\omega_1}\left(|\varphi(\rho,\omega)|^2 + |\partial_\omega\varphi(\rho,\omega)|^2\right)\mathrm{d}\omega \tag{5.2.92}$$

for some $C(\omega_0,\omega_1)>0$. Substituting inequality (5.2.92) into (5.2.91) we get

$$\begin{aligned}\int_R^\infty &\rho^{-1-\nu}|\varphi(\rho,\omega_0)|^2\,\mathrm{d}\rho\\ &\le C(\omega_0,\omega_1)\int_R^\infty\int_{\omega_0}^{\omega_1}\rho^{-1-\nu}\left(|\varphi(\rho,\omega)|^2+|\partial_\omega\varphi(\rho,\omega)|^2\right)\mathrm{d}\omega\,\mathrm{d}\rho\\ &= C(\omega_0,\omega_1)\int_{\omega_0}^{\omega_1}\int_R^\infty\rho^{-1-\nu}\left(|\varphi(\rho,\omega)|^2+|\partial_\omega\varphi(\rho,\omega)|^2\right)\mathrm{d}\rho\,\mathrm{d}\omega.\end{aligned} \tag{5.2.93}$$

Applying inequality (2.1.27) for fixed $\omega\in(\omega_0,\omega_1)$ yields

$$C(\omega_0,\omega_1)\int_R^\infty\rho^{-1-\nu}|\varphi(\rho,\omega)|^2\,\mathrm{d}\rho\le\varepsilon\int_R^\infty|\partial_\rho\varphi(\rho,\omega)|^2\rho\,\mathrm{d}\rho \tag{5.2.94}$$

where $\varepsilon>0$ can be chosen arbitrarily small if $R>0$ is large enough. Substituting this inequality into (5.2.93) and using

$$\left|\frac{\partial\varphi}{\partial\rho}\right|^2+\frac{1}{\rho^2}\left|\frac{\partial\varphi}{\partial\omega}\right|^2\le|\nabla_0\varphi|^2 \tag{5.2.95}$$

we obtain

$$\int_R^\infty\rho^{-1-\nu}|\varphi(\rho,\omega_0)|^2\,\mathrm{d}\rho\le\varepsilon\int_{K_R(Z,\kappa,\kappa')}|\nabla_0\varphi|^2\,\mathrm{d}x \tag{5.2.96}$$

for sufficiently large $R>0$. Inserting this inequality in (5.2.91) yields

$$\int_{\{\tilde\xi\ge\gamma\}}\tilde\xi^{-1-\nu}|\varphi(\kappa\tilde\xi,\tilde\xi)|^2\,\mathrm{d}\tilde\xi\le\varepsilon\int_{K_R(Z,\kappa,\kappa')}|\nabla_0\varphi|^2\,\mathrm{d}x. \tag{5.2.97}$$

The other integrals on the r.h.s. of (5.2.88) can be estimated in the same way. Therefore, we obtain

$$\int_{K_R(Z,\kappa)}\left(|\nabla_0\varphi|^2+V|\varphi|^2-\varepsilon|x|_m^{-4}|\varphi|^2\right)\mathrm{d}x\ge-8\varepsilon C\int_{K_R(Z,\kappa,\kappa')}|\nabla_0\varphi|^2\,\mathrm{d}x. \tag{5.2.98}$$

Since in (5.2.88) C is a fixed constant and $\varepsilon>0$ can be chosen arbitrarily, we will assume $C=1$ in the following. Summing inequality (5.2.98) over all partitions Z with $|Z|=2$, inserting the resulting inequality into the definition of L and using that the cones $K(Z,\kappa')$ and $K(Z',\kappa')$ do not overlap, we get

$$L[\varphi] \geq \int_{K_R^c(\kappa)} \left((1-8\varepsilon)|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x, \tag{5.2.99}$$

where $K_R^c(\kappa) = X_0 \setminus \left(B(R) \cup \bigcup_{Z:|Z|=2} K(Z,\kappa)\right)$. Note that $K_R^c(\kappa)$ is a region in X_0, where all particles are far away from each other and we can estimate $|V(x)| \leq C|x|_m^{-2-\nu}$. Moreover, we can assume $\nu < 2$ and thus $|x|_m^{-4} \leq |x|_m^{-2-\nu}$. Hence, we get

$$L[\varphi] \geq \int_{K_R^c(\kappa)} \left((1-8\varepsilon)|\nabla_0\varphi|^2 - (C+\varepsilon)|x|_m^{-2-\nu}|\varphi|^2\right) \mathrm{d}x. \tag{5.2.100}$$

Using polar coordinates (ρ,ω) and $|\nabla_0\varphi| \geq \left|\frac{\partial\varphi}{\partial\rho}\right|$ we find

$$L[\varphi] \geq \int_{\omega\in I}\int_R^\infty \left((1-8\varepsilon)\left|\partial_\rho\varphi\right|^2 - (C+\varepsilon)\rho^{-2-\nu}|\varphi|^2\right)\rho\,\mathrm{d}\rho\mathrm{d}\omega, \tag{5.2.101}$$

where $I \subset [0,2\pi]$ is the set of angles corresponding to the region $K_R^c(\kappa)$. Now since $\varphi(\rho,\omega) = 0$ for $\rho \leq R$, we can apply inequality (2.1.27) to the function $u(\rho) = \varphi(\rho,\omega)$ for fixed $\omega \in I$. Choosing R sufficiently large yields $L[\varphi] \geq 0$ and thus the finiteness of the discrete spectrum of H. All arguments used above can be applied to the operator H_{as}, which completes the proof of Theorem 5.1.4. □

5.2.5. Proof of Theorem 5.1.5

In the proof of this theorem we follow the same strategy as in the proof of Theorem 5.1.4. First, we give some auxiliary Lemmas which are the two-dimensional analogues to Lemma 5.2.7 and Lemma 5.2.8.

Lemma 5.2.10. *Let $d = 2$ and consider the operator $h = -\Delta + V$ acting on $L^2(\mathbb{R}^2)$, where we assume that $h \geq 0$ and $V(x) = V(|x|)$ satisfies* (4.2.2) *and* (4.2.3). *Then there exists a constant $c > 0$, such that for any $b_0 > A$ and for any function $\psi \in H^1(\mathbb{R}^2)$*

$$\begin{aligned} J[\psi,b_0] &:= \int_{\{|x|\leq b_0\}} \left(|\nabla\psi(x)|^2 + V(x)|\psi(x)|^2\right) \mathrm{d}x \\ &\geq -cb_0^{-\nu}\int_0^{2\pi} |\psi(b_0,\omega)|^2\,\mathrm{d}\omega. \end{aligned} \tag{5.2.102}$$

Remark 5.2.11. Lemma 5.2.10 does not hold if we restrict the operator h to anti-symmetric functions. This is the reason why our proof of Theorem 5.1.5 does not work for a fermionic system, where the super Efimov effect is known to exist.

Proof of Lemma 5.2.10. Let $\psi \in H^1(\mathbb{R}^2)$ and $b_0 > A$. We introduce polar coordinates $x = (\rho, \omega)$ and write $\psi(x) = \sum_{n=-\infty}^{\infty} \psi_n(x)$ with $\psi_n(x) = R_n(\rho) e^{in\omega}$. For $k \in \mathbb{N}$, $k \geq 2$, let

$$R_n^k(\rho) := \begin{cases} R_n(\rho) & \text{if } \rho \leq b_0, \\ R_n(b_0) \ln\left(k b_0 \rho^{-1}\right) (\ln k)^{-1} & \text{if } b_0 < \rho \leq k b_0, \\ 0 & \text{if } \rho > k b_0. \end{cases} \tag{5.2.103}$$

We set $\psi_n^k : \mathbb{R}^2 \to \mathbb{C}$, $\psi_n^k(x) = R_n^k(|x|) \mathrm{e}^{in\omega}$. Then we have $J[\psi_n^k, b_0] = J[\psi_n, b_0]$ and therefore

$$J[\psi, b_0] = \sum_{n=-\infty}^{\infty} J[\psi_n^k, b_0] \quad \text{for any } k \geq 2. \tag{5.2.104}$$

Now we estimate $J[\psi_n^k, b_0]$ for fixed $k, n \in \mathbb{N}$ with $k \geq 2$. Due to

$$|\nabla \psi|^2 = |\frac{\partial \psi}{\partial \rho}|^2 + \frac{1}{\rho^2} |\frac{\partial \psi}{\partial \omega}|^2 \geq |\frac{\partial \psi}{\partial \rho}|^2 \tag{5.2.105}$$

and $V(x) = V(|x|)$ we can estimate

$$J[\psi_n^k, b_0] \geq 2\pi \int_0^{b_0} \left(|\partial_\rho R_n^k(\rho)|^2 + V(\rho) |R_n^k(\rho)|^2 \right) \rho \mathrm{d}\rho. \tag{5.2.106}$$

Using $\langle h \tilde{\psi}_n^k, \tilde{\psi}_n^k \rangle \geq 0$ for the radial function $\tilde{\psi}_n^k(x) = R_n^k(|x|)$ and (5.2.106) yields

$$J[\psi_n^k, b_0] \geq -2\pi \int_{b_0}^{\infty} \left(|\partial_\rho R_n^k(\rho)|^2 + V(\rho) |R_n^k(\rho)|^2 \right) \rho \mathrm{d}\rho. \tag{5.2.107}$$

Easy computation shows that

$$\partial_\rho R_n^k(\rho) = \begin{cases} -R_n(b_0) (\ln(k))^{-1} \rho^{-1} & \text{if } b_0 < \rho < k b_0, \\ 0 & \text{if } \rho > k b_0. \end{cases} \tag{5.2.108}$$

This implies

$$\int_{b_0}^{\infty} |\partial_\rho R_n^k(\rho)|^2 \rho \mathrm{d}\rho \leq |R_n(b_0)|^2 (\ln(k))^{-2} \int_{b_0}^{kb_0} \rho^{-1} \mathrm{d}\rho = |R_n(b_0)|^2 (\ln(k))^{-1}. \quad (5.2.109)$$

Since $|V(\rho)| \leq C\left(1+\rho\right)^{-2-\nu}$ for $\rho > b_0$, we get

$$\begin{aligned}\int_{b_0}^{\infty} |V(\rho)||R_n^k(\rho)|^2 \rho \mathrm{d}\rho &\leq C|R_n(b_0)|^2 (\ln(k))^{-2} \int_{b_0}^{kb_0} (1+\rho)^{-2-\nu} \left(\ln(kb_0\rho^{-1})\right)^2 \rho \mathrm{d}\rho \\ &\leq C|R_n(b_0)|^2 \int_{b_0}^{kb_0} (1+\rho)^{-2-\nu} \rho \mathrm{d}\rho, \quad (5.2.110)\end{aligned}$$

where for the last inequality we used that $(\ln(k))^{-2} \left(\ln(kb_0\rho^{-1})\right)^2 \leq 1$ for $\rho \in (b_0, kb_0)$. By inserting

$$\int_{b_0}^{kb_0} (1+\rho)^{-2-\nu} \rho \mathrm{d}\rho \leq \int_{b_0}^{\infty} \rho^{-1-\nu} \mathrm{d}\rho = b_0^{-\nu} \quad (5.2.111)$$

in inequality (5.2.110) we find

$$\int_{b_0}^{\infty} |V(\rho)||R_n^k(\rho)|^2 \rho \mathrm{d}\rho \leq C|R_n(b_0)|^2 b_0^{-\nu}. \quad (5.2.112)$$

Combining (5.2.107) with (5.2.109) and (5.2.112) we obtain

$$J[\psi_n^k, b_0] \geq -2\pi C|R_n(b_0)|^2 b_0^{-\nu} - 2\pi |R_n(b_0)| (\ln(k))^{-1}. \quad (5.2.113)$$

Recall that the left hand side of (5.2.113) coincides with $J[\psi_n, b_0]$ and in particular does not depend on k. Therefore, sending k to infinity and using

$$2\pi \sum_{n=-\infty}^{\infty} |R_n(b_0)|^2 = \int_0^{2\pi} |\psi(b_0,\omega)|^2 \mathrm{d}\omega \quad (5.2.114)$$

completes the proof of Lemma 5.2.10. □

The following lemma is analogous to Lemma 5.2.8.

Lemma 5.2.12. *Let $C_0 > 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R}^2)$*

$$\int_{\{|x|\geq b\}} \left(|\nabla\psi(x)|^2 - C_0|x|^{-2-\nu}|\psi(x)|^2\right) \mathrm{d}x \geq -2C_0 b^{-\nu} \int_0^{2\pi} |\psi(b,\omega)|^2 \,\mathrm{d}\omega. \tag{5.2.115}$$

Proof of Lemma 5.2.12. Let $\psi \in H^1(\mathbb{R}^2)$. We write $\psi = \psi_0 + \psi_1$ with $\psi_0 = \mathscr{P}^{m=0}\psi$ and $\psi_1 = \mathscr{P}^{|m|\geq 1}\psi$, where $\mathscr{P}^m$ is the projection onto the space of functions with angular momentum m. Then for ψ_1 we have

$$|\nabla\psi_1|^2 = |\partial_\rho\psi_1|^2 + \frac{1}{\rho^2}|\partial_\omega\psi_1|^2 \geq \frac{1}{\rho^2}|\psi_1|^2 \tag{5.2.116}$$

and therefore

$$\int_{\{|x|\geq b\}} \left(|\nabla\psi_1(x)|^2 - C_0|x|^{-2-\nu}|\psi_1(x)|^2\right) \mathrm{d}x \geq 0 \tag{5.2.117}$$

if $b > 0$ is sufficiently large. Hence, it suffices to prove (5.2.115) for the radial function ψ_0. For the sake of convenience we shall not distinguish between $\psi_0(x)$ and $\psi_0(|x|)$. For $|x| \geq b$ let $\tilde{\psi}(x) = \psi_0(x) - \psi_0(b)$. Then $\tilde{\psi}$ is also radial and $\tilde{\psi}(b) = 0$. We extend $\tilde{\psi}$ with zero to the region $\{|x| < b\}$. Similarly to the one-dimensional case we obtain

$$\begin{aligned} &\int_{\{|x|\geq b\}} \left(|\nabla\psi_0(x)|^2 - C_0|x|^{-2-\nu}|\psi_0(x)|^2\right) \mathrm{d}x \\ &\quad \geq \int_{\{|x|\geq b\}} \left(|\nabla\tilde{\psi}(x)|^2 - 2C_0|x|^{-2-\nu}|\tilde{\psi}(x)|^2\right) \mathrm{d}x \\ &\qquad\qquad - \int_{\{|x|\geq b\}} 2C_0|x|^{-2-\nu}|\psi_0(b)|^2 \,\mathrm{d}x. \end{aligned} \tag{5.2.118}$$

Since $\tilde{\psi}(x) = 0$ for $|x| \leq b$, we can apply the two-dimensional Hardy inequality to the function $\tilde{\psi}$, which implies that the first integral on the r.h.s of (5.2.118) is non-negative. Hence, we arrive at

$$\int_{\{|x|\geq b\}} \left(|\nabla\psi_0(x)|^2 - C_0|x|^{-2-\nu}|\psi_0(x)|^2\right) \mathrm{d}x \geq -2C_0 \int_{\{|x|\geq b\}} |x|^{-2-\nu}|\psi_0(b)|^2 \,\mathrm{d}x. \tag{5.2.119}$$

Computing the integral on the r.h.s. of (5.2.119) completes the proof of Lemma 5.2.12. □

Proof of Theorem 5.1.5. Let

$$L[\varphi] := \int \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x. \tag{5.2.120}$$

We show that $L[\varphi] \geq 0$ for all functions $\varphi \in H^1(X_0)$ with $\operatorname{supp}(\varphi) \subset \{|x|_m \geq R\}$ if $\varepsilon > 0$ is small enough and $R > 0$ is sufficiently large. First, we estimate the part of $L[\varphi]$ corresponding to the cone $K(Z,\kappa)$ for an arbitrary partition Z into two clusters. Assume that $Z = \{C_1, C_2\}$ with $C_1 = \{i, j\}$ and $C_2 = \{k\}$. Note that the spaces $X_0(Z)$ and $X_c(Z)$ are both two-dimensional. We choose orthonormal bases of $X_0(Z)$ and $X_c(Z)$ and denote by $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{\xi}_1$, $\tilde{\xi}_2$ the corresponding coordinates. We write $\tilde{q} = (\tilde{q}_1, \tilde{q}_2)$, $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$ and $\varphi = \varphi(\tilde{q}, \tilde{\xi})$. Similarly to (5.2.78) we write

$$\begin{aligned}\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x &= \int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x \\ &+ \int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde{\xi}}\varphi|^2 + (V_{ik}+V_{jk})|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x.\end{aligned} \tag{5.2.121}$$

To estimate the integrals on the r.h.s of (5.2.121) we introduce the polar coordinates $\tilde{q} = (\rho_1, \beta_1)$ and $\tilde{\xi} = (\rho_2, \beta_2)$ in the planar spaces $X_0(Z)$ and $X_c(Z)$. For the first integral on the r.h.s. of (5.2.121) we use Lemma 5.2.10 for fixed $\tilde{\xi}$ with $b_0 = \kappa|\tilde{\xi}|$. Then similarly to (5.2.82) we get

$$\begin{aligned}\int_{K_R(Z,\kappa)} \left(|\nabla_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x &= \int_{\{|\tilde{\xi}|\geq\gamma\}} \int_{\{|\tilde{q}|\leq\kappa|\tilde{\xi}|\}} \left(|\nabla_{\tilde{q}}\varphi|^2 + V_{ij}|\varphi|^2\right) \mathrm{d}x \\ &\geq -C \int_{\{|\tilde{\xi}|\geq\gamma\}} |\tilde{\xi}|^{-\nu} \int_0^{2\pi} |\varphi(\kappa|\tilde{\xi}|, \beta_1, \tilde{\xi})|^2 \,\mathrm{d}\beta_1 \,\mathrm{d}\tilde{\xi}\end{aligned} \tag{5.2.122}$$

for some $C > 0$, where $\gamma = \left(1+\kappa^2\right)^{-\frac{1}{2}} R$ is analogous to the proof of Theorem 5.1.4. For the second integral on the r.h.s. of (5.2.121) we use Lemma 5.2.12 for fixed $\tilde{q}$ and with $b = \kappa^{-1}|\tilde{q}|$, which similarly to (5.2.87) yields

$$\begin{aligned}\int_{K_R(Z,\kappa)} &\left(|\nabla_{\tilde{\xi}}\varphi|^2 + (V_{ik}+V_{jk})|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \\ &\geq -C \int_{\{|\tilde{q}|\geq\eta\}} |\tilde{q}|^{-\nu} \int_0^{2\pi} |\varphi(\tilde{q}, \kappa^{-1}|\tilde{q}|, \beta_2)|^2 \,\mathrm{d}\beta_2 \,\mathrm{d}\tilde{q}\end{aligned} \tag{5.2.123}$$

for some $C>0$, where $\eta=\left(1+\kappa^{-2}\right)^{-1}R$ is analogous to the proof of Theorem 5.1.4. Combining (5.2.122) and (5.2.123) with (5.2.121) implies

$$\begin{aligned}\int_{K_R(Z,\kappa)} &\left(|\nabla_0\varphi|^2+V|\varphi|^2-\varepsilon|x|_m^{-4}|\varphi|^2\right)\mathrm{d}x\\ &\geq -C\int_{\{|\tilde{\xi}|\geq\gamma\}}|\tilde{\xi}|^{-\nu}\int_0^{2\pi}|\varphi(\kappa|\tilde{\xi}|,\beta_1,\tilde{\xi})|^2\,\mathrm{d}\beta_1\,\mathrm{d}\tilde{\xi}\\ &\qquad -C\int_{\{|\tilde{q}|\geq\eta\}}|\tilde{q}|^{-\nu}\int_0^{2\pi}|\varphi(\tilde{q},\kappa^{-1}|\tilde{q}|,\beta_2)|^2\,\mathrm{d}\beta_2\,\mathrm{d}\tilde{q}.\end{aligned}\tag{5.2.124}$$

In the set $\{(|\tilde{q}|,|\tilde{\xi}|)\in\mathbb{R}_+\times\mathbb{R}_+\}$ we introduce the polar coordinates (ρ,ω), where $\rho^2=|\tilde{q}|^2+|\tilde{\xi}|^2=|x|_m^2$ and $\omega=\arctan\left(\frac{|\tilde{q}|}{|\tilde{\xi}|}\right)\in[0,\frac{\pi}{2})$. Then for $\rho_1=|\tilde{q}|$ and $\rho_2=|\tilde{\xi}|$ we have $\rho_1=\rho\sin(\omega)$ and $\rho_2=\rho\cos(\omega)$. Furthermore, we represent the function $\varphi(x)$ as a function $\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)$. Note that the integrals on the r.h.s of (5.2.124) are integrals of the function $|\varphi(x)|^2$ over the set where $|\tilde{q}|=\kappa|\tilde{\xi}|$, i.e., where $\omega=\omega_0=\arctan(\kappa)$. Hence, for the first integral on the r.h.s of (5.2.124) we get

$$\begin{aligned}&\int_{\{|\tilde{\xi}|\geq\gamma\}}|\tilde{\xi}|^{-\nu}\int_0^{2\pi}|\varphi(\kappa|\tilde{\xi}|,\beta_1,\tilde{\xi})|^2\,\mathrm{d}\beta_1\,\mathrm{d}\tilde{\xi}\\ &=\int_\gamma^\infty\int_0^{2\pi}\int_0^{2\pi}\rho_2^{-\nu}|\varphi(\kappa\rho_2,\beta_1,\rho_2,\beta_2)|^2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\rho_2\mathrm{d}\rho_2\\ &=c\int_R^\infty\int_0^{2\pi}\int_0^{2\pi}\rho^{-\nu}|\tilde{\varphi}(\rho,\omega_0,\beta_1,\beta_2)|^2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\rho\mathrm{d}\rho,\end{aligned}\tag{5.2.125}$$

where $c>0$ is a constant which comes from the transformation of variables if we represent the function $\rho_2\mapsto\varphi(\kappa\rho_2,\beta_1,\rho_2,\beta_2)$ as function $\rho\mapsto\tilde{\varphi}(\rho,\omega_0,\beta_1,\beta_2)$, where $\omega_0=\arctan(\kappa)$. In the first equality in (5.2.125) we used that $\dim(X_c(Z))=2$, which implies that the Jacobian of the transformation to polar coordinates in $X_c(Z)$ gives a factor ρ_2. In the last equality of (5.2.125) we used that the function $\tilde{\varphi}$ is zero for $\rho<R$. Similarly we get

$$\begin{aligned}&\int_{\{|\tilde{q}|\geq\eta\}}|\tilde{q}|^{-\nu}\int_0^{2\pi}|\varphi(\tilde{q},\kappa^{-1}|\tilde{q}|,\beta_2)|^2\,\mathrm{d}\beta_2\,\mathrm{d}\tilde{q}\\ &=c'\int_R^\infty\int_0^{2\pi}\int_0^{2\pi}\rho^{-\nu}|\tilde{\varphi}(\rho,\omega_0,\beta_1,\beta_2)|^2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\rho\mathrm{d}\rho\end{aligned}\tag{5.2.126}$$

for some $c' > 0$. By combining (5.2.125) and (5.2.126) with (5.2.124) we obtain

$$\begin{aligned}&\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right) \mathrm{d}x \\ &\qquad \geq -C\int_R^\infty \int_0^{2\pi}\int_0^{2\pi} \rho^{1-\nu}|\tilde{\varphi}(\rho,\omega_0,\beta_1,\beta_2)|^2 \,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\rho\end{aligned} \tag{5.2.127}$$

for some $C > 0$. Now as in the proof of Theorem 5.1.4 we estimate the integral on the r.h.s. of inequality (5.2.127), which is an integral over the edge of $K_R(Z,\kappa)$ given by $\{|\tilde{q}| = \kappa|\tilde{\xi}|,\ |x|_m \geq R\}$, by an integral over the set $K(Z,\kappa,\kappa')$ for some κ' which is slightly larger than κ. For this purpose let $\kappa' > \kappa$ be so small that the cones $K_R(Z,\kappa')$ and $K_R(Z',\kappa')$ do not overlap for partitions $Z \neq Z'$ with $|Z| = |Z'| = 2$ and denote $\omega_1 = \arctan(\kappa')$. We apply Lemma 5.2.9 to the function $\varphi(\rho,\cdot,\beta_1,\beta_2)$ for fixed ρ,β_1,β_2 and with $b_1 = \omega_0,\ b_2 = \omega_1$. Then we get

$$\begin{aligned}&\int_R^\infty \int_0^{2\pi}\int_0^{2\pi} \rho^{1-\nu}|\tilde{\varphi}(\rho,\omega_0,\beta_1,\beta_2)|^2 \,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\rho \\ &\quad \leq C(\omega_0,\omega_1)\int_R^\infty \int_0^{2\pi}\int_0^{2\pi}\int_{\omega_0}^{\omega_1} \rho^{1-\nu}|\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 \,\mathrm{d}\omega\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\rho \\ &\quad + C(\omega_0,\omega_1)\int_R^\infty \int_0^{2\pi}\int_0^{2\pi}\int_{\omega_0}^{\omega_1} \rho^{1-\nu}|\partial_\omega\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 \,\mathrm{d}\omega\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\rho,\end{aligned} \tag{5.2.128}$$

where $C(\omega_0,\omega_1)$ depends on ω_0 and ω_1 only. Applying the Hardy type inequality (2.1.2) to the function $u(\rho) = \tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)$ with $\alpha = 1-\nu$ (where without loss of generality we assume $\nu < 2$, so $\alpha > -1$), we obtain

$$\int_R^\infty \rho^{1-\nu}|\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 \,\mathrm{d}\rho \leq C\int_R^\infty \rho^{3-\nu}|\partial_\rho\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 \,\mathrm{d}\rho \tag{5.2.129}$$

for some $C > 0$. Therefore, we get

$$\begin{aligned}&\int_R^\infty \rho^{1-\nu}\left(|\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 + |\partial_\omega\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2\right) \mathrm{d}\rho \\ &\qquad \leq R^{-\nu}\int_R^\infty \rho^3\left(|\partial_\rho\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 + \frac{|\partial_\omega\tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2}{\rho^2}\right) \mathrm{d}\rho.\end{aligned} \tag{5.2.130}$$

Recall that (ρ,ω) are the polar coordinates corresponding to $(|\tilde{q}|,|\tilde{\xi}|)$, which implies

$$|\partial_\rho \tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 + \frac{|\partial_\omega \tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2}{\rho^2} = |\partial_{|\tilde{q}|}\varphi|^2 + |\partial_{|\tilde{\xi}|}\varphi|^2 \le |\nabla_0\varphi|^2. \tag{5.2.131}$$

This yields

$$R^{-\nu}\int_R^\infty \int_0^{2\pi}\int_0^{2\pi}\int_{\omega_0}^{\omega_1} \rho^3 \left(|\partial_\rho \tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2 + \frac{|\partial_\omega \tilde{\varphi}(\rho,\omega,\beta_1,\beta_2)|^2}{\rho^2}\right) \mathrm{d}\omega\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\rho$$
$$\le \varepsilon \int_{K_R(Z,\kappa,\kappa')} |\nabla_0\varphi|^2\,\mathrm{d}x, \tag{5.2.132}$$

where $\varepsilon > 0$ can be chosen arbitrarily small if $R > 0$ is sufficiently large. Here we used that the Jacobian of the transformation from the coordinates $x = (\tilde{q}_1,\tilde{q}_2,\tilde{\xi}_1,\tilde{\xi}_2)$ to the variables $(\rho,\omega,\beta_1,\beta_2)$ is given by $\rho^3\sin(\omega)\cos(\omega)$ and that we can estimate

$$0 < \sin(\omega_0)\cos(\omega_0) \le \sin(\omega)\cos(\omega) \tag{5.2.133}$$

for any $\omega \in (\omega_0,\omega_1)$ if $0 < \kappa < \kappa' < 1$. Combining (5.2.132) with (5.2.130), (5.2.128) and (5.2.127) we get

$$\int_{K_R(Z,\kappa)} \left(|\nabla_0\varphi|^2 + V|\varphi|^2 - \varepsilon|x|_m^{-4}|\varphi|^2\right)\mathrm{d}x \ge -\varepsilon\int_{K_R(Z,\kappa,\kappa')} |\nabla_0\varphi|^2\,\mathrm{d}x. \tag{5.2.134}$$

This inequality is analogous to inequality (5.2.96) in the proof of Theorem 5.1.4. Now we can complete the proof of Theorem 5.1.5 by repeating the same steps as in the proof of Theorem 5.1.4 if we replace the scalar form of the two-dimensional Hardy inequality by the scalar form of the four-dimensional one, i.e., (2.1.2) with $\alpha = 1$. □

A. Appendix

A.1. Proof of Theorem 4.3.7

The proof of Theorem 4.3.7 is a modification of the proof of Theorem 2.1 in [7]. As in the proof of Theorem 4.2.6 we take a sequence of eigenfunctions $\psi_n \in H^1(\mathbb{R}^k)$ which correspond to eigenvalues $E_n < 0$ of the operator $h_{n^{-1}}$, normalized by $\|\psi_n\|_{\tilde{H}^1} = 1$. Then we find a function φ_0 which is the weak limit of a subsequence of $(\psi_n)_{n\in\mathbb{N}}$ and which will turn out to be the eigenfunction stated in the theorem.

Let us briefly explain how the strategy of the proof of Theorem 4.3.7 differs from that of Theorem 4.2.6. In the latter we used the fast decay of the potential to show that $\langle V\psi_n, \psi_n\rangle$ converges $\langle V\varphi_0, \varphi_0\rangle$ and concluded that φ_0 is a minimizer of the quadratic form of h. Here we will use the assumption (4.3.14) to derive a uniform estimate for a weighted $L^2(\mathbb{R}^k)$ norm of the functions ψ_n, which guarantees $L^2(\mathbb{R}^k)$ convergence of the sequence $(\psi_n)_{n\in\mathbb{N}}$, from which we conclude convergence of $\langle V\psi_n, \psi_n\rangle$ to $\langle V\varphi_0, \varphi_0\rangle$. We start by providing the following uniform estimate of the decay rate of ψ_n.

Lemma A.1.1. *Assume that h has a virtual level at zero, where V satisfies* (4.2.2) *and suppose that* (4.3.14) *holds for some $\alpha_0 > 1$. Then there exists a constant $C > 0$, such that for any eigenfunction $\psi_n \in H^1(\mathbb{R}^k)$ corresponding to a negative eigenvalue of the operator $h_{n^{-1}}$, normalized by $\|\psi_n\|_{\tilde{H}^1} = 1$, we have*

$$\|\nabla(|\cdot|^{\alpha_0}\psi_n)\| \leq C \tag{A.1.1}$$

and

$$\begin{aligned} &\|(1+|\cdot|)^{\alpha_0-1}\psi_n\| \leq C && \text{if } k=1 \\ \text{and} \quad &\|(1+|\cdot|)^{\alpha_0-1}(1+|\ln(|\cdot|)|)^{-1}\psi_n\| \leq C && \text{if } k=2. \end{aligned} \tag{A.1.2}$$

Proof of Lemma A.1.1. The proof is a straightforward modification of the proof of Lemma 4.2.9. Let $k = 1$ and

$$G_{\varepsilon,R}(x) = \frac{|x|^{\alpha_0}}{1+\varepsilon|x|^{\alpha_0}}\chi_R(x), \tag{A.1.3}$$

with a smooth cutoff function χ_R satisfying

$$\chi_R(x) = \begin{cases} 0, & |x| \leq R, \\ 1, & |x| \geq 2R. \end{cases} \tag{A.1.4}$$

By repeating the same computations as in the proof of Lemma 4.2.9 we arrive at

$$\begin{aligned}\left(1-n^{-1}\right)\|\nabla(\psi_n G_{\varepsilon,R})\|^2 + \langle V G_{\varepsilon,R}\psi_n, G_{\varepsilon,R}\psi_n\rangle \\ - \alpha_0^2 \int_{\{|x|>2R\}} \frac{|G_{\varepsilon,R}\psi_n|^2}{|x|^2}\,\mathrm{d}x \leq C_2,\end{aligned} \tag{A.1.5}$$

where the constant $C_2 > 0$ does not depend on $n \in \mathbb{N}$ or $\varepsilon > 0$. The function $G_{\varepsilon,R}\psi_n$ is supported outside the ball with radius R. Therefore by assumption (4.3.14) we have

$$(1-\gamma_0)\|\nabla(G_{\varepsilon,R}\psi_n)\|^2 + \langle V G_{\varepsilon,R}\psi_n, G_{\varepsilon,R}\psi_n\rangle - \alpha_0^2\langle |x|^{-2} G_{\varepsilon,R}\psi_n, G_{\varepsilon,R}\psi_n\rangle \geq 0 \tag{A.1.6}$$

for some $\alpha_0 > 1$ and $\gamma_0 > 0$. For $n > 2\gamma_0^{-1}$ the estimates (A.1.5) and (A.1.6) imply

$$\frac{\gamma_0}{2}\|\nabla(G_{\varepsilon,R}\psi_n)\|^2 \leq C_2. \tag{A.1.7}$$

Taking the limit $\varepsilon \to 0$ yields $\|\nabla\left(|\cdot|^{\alpha_0}\psi_n\right)\| \leq C$ for some $C > 0$. Applying Hardy's inequality for the half-line to the function $G_{\varepsilon,R}\psi_n$ and taking the limit $\varepsilon \to 0$ implies

$$\|(1+|\cdot|)^{\alpha_0-1}\psi_n\| \leq C. \tag{A.1.8}$$

This completes the proof of Lemma 4.2.9 for $k = 1$. The proof for $k = 2$ goes along the same line.

□

Since $(\psi_n)_{n\in\mathbb{N}}$ converges weakly in $\tilde{H}^1(\mathbb{R}^k)$ to φ_0, we have strong convergence

$\psi_n \to \varphi_0$ in $L^2_{\text{loc}}(\mathbb{R}^k)$. This, together with the uniform estimate (A.1.2) for some $\alpha_0 > 1$ implies strong convergence in $L^2(\mathbb{R}^k)$ and that φ_0 satisfies

$$(1+|\cdot|)^{\alpha-1}\varphi_0 \in L^2(\mathbb{R}^k) \tag{A.1.9}$$

for any $\alpha < \alpha_0$. It remains to prove that φ_0 is an eigenfunction. Note that $\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle$ as $n \to \infty$, which follows from assumption (4.2.2), the $L^2(\mathbb{R}^k)$ convergence of ψ_n to φ_0 and due to the boundedness of $\|\nabla\psi_n\|$ and $\|\nabla\varphi_0\|$. Now as in the proof of Lemma 4.2.8 we get

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0 \tag{A.1.10}$$

and $\|\varphi_0\|_{\tilde{H}^1} = 1$. In conclusion, we have $\psi_n \rightharpoonup \varphi_0$ in $\tilde{H}^1(\mathbb{R}^k)$, $\psi_n \to \varphi_0$ in $L^2(\mathbb{R}^k)$ and $\|\psi_n\|_{\tilde{H}^1} \to \|\varphi_0\|_{\tilde{H}^1}$. This implies $\psi_n \to \varphi_0$ in $H^1(\mathbb{R}^k)$. Hence, by (A.1.10) and Theorem 3.1.10 φ_0 is an eigenfunction corresponding to the eigenvalue zero. Repeating the arguments of the proof of Lemma A.1.1 for the function φ_0 shows $\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^k)$.

To complete the proof of Theorem 4.3.7 it remains to prove (4.3.17). We prove the assertion in the following lemmas which are analogouos to Lemmas 2.6-2.8 in [7]. We adapt them to the case of $k = 1, 2$.

Lemma A.1.2. *Assume that V satisfies* (4.2.2)*, that h has a virtual level at zero and that* (4.3.14) *holds for some $\alpha_0 > 1$. Let $a(n) \to 0$ be a sequence of real numbers with $a(n) \neq 0$ for all $n \in \mathbb{N}$ and $(\psi_n)_{n\in\mathbb{N}}$ a sequence of real-valued eigenfunctions of the operator $h_{a(n)}$ corresponding to some negative eigenvalues, normalized as $\|\psi_n\|_{\tilde{H}^1} = 1$. Then $(\psi_n)_{n\in\mathbb{N}}$ converges in $H^1(X_0)$ to the function φ_0 defined above. In other words: All such sequences of eigenfunctions converge to the same limit.*

Proof of Lemma A.1.2. Assume for a contradiction that $(\psi_n)_{n\in\mathbb{N}}$ does not converge to φ_0. Then, there exists a constant $\mu > 0$ and a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$, such that $\|\psi_{n_k} - \varphi_0\|_{H^1} \geq \mu$. Due to $\|\psi_{n_k}\|_{\tilde{H}^1} = 1$ there exists a subsequence of $(\psi_{n_k})_{k\in\mathbb{N}}$, also denoted by $(\psi_{n_k})_{k\in\mathbb{N}}$, which converges weakly to a function $\varphi_1 \in \tilde{H}^1(\mathbb{R}^k)$. Repeating the arguments above, we see that φ_1 is an eigenfunction of h corresponding to the eigenvalue zero with $\|\varphi_1\|_{\tilde{H}^1} = 1$ and we have convergence of $(\psi_{n_k})_{k\in\mathbb{N}}$ to φ_1 in $H^1(\mathbb{R}^k)$. Since an eigenvalue of a Schrödinger operator coinciding with the bottom of the spectrum cannot be degenerate [23], we have $\varphi_1 = \varphi_0$. This is a contradiction to the choice of $(\psi_{n_k})_{k\in\mathbb{N}}$. □

Lemma A.1.3. *For any sufficiently small $\varepsilon > 0$ the operator h_ε has only one negative eigenvalue and this is non-degenerate.*

Proof of Lemma A.1.3. Suppose for a contradiction that there exists a sequence of real numbers $a(n) \in (0,1)$ with $a(n) \to 0$ as $n \to \infty$, such that for any $n \in \mathbb{N}$ the operator $h_{a(n)} = -(1-a(n))\Delta + V$ has at least two eigenvalues. Recall that the lowest eigenvalue of $h_{a(n)}$ is non-degenerate. We consider two sequences of eigenfunctions $\psi_n^{(1)}$ and $\psi_n^{(2)}$ of $h_{a(n)}$, normalized by $\|\psi_n^{(1)}\|_{\tilde{H}^1} = \|\psi_n^{(2)}\|_{\tilde{H}^1} = 1$, where $\psi_n^{(1)}$ corresponds to the lowest eigenvalue and $\psi_n^{(2)}$ corresponds to the second eigenvalue. Then $\psi_n^{(1)}$ and $\psi_n^{(2)}$ are orthogonal in $L^2(\mathbb{R}^k)$. On the other hand, by Lemma A.1.2 $\psi_n^{(1)}$ and $\psi_n^{(2)}$ both converge in $L^2(\mathbb{R}^k)$ to φ_0, which is a contradiction. □

Lemma A.1.4. *There exists a constant $\delta_0 > 0$, such that for every function $\psi \in H^1(\mathbb{R}^k)$ with $\langle \nabla\psi, \nabla\varphi_0\rangle = 0$*

$$(1-\delta_0)\|\nabla\psi\|^2 + \langle V\psi, \psi\rangle \geq 0. \tag{A.1.11}$$

Proof. We prove the Lemma by contradiction. Assume that there is no such constant $\delta_0 > 0$. Then there exists a sequence of functions $g_n \in H^1(\mathbb{R}^k)$ with

$$\langle \nabla g_n, \nabla\varphi_0\rangle = 0 \qquad \text{and} \qquad \langle h_{n^{-1}} g_n, g_n\rangle < 0. \tag{A.1.12}$$

Note that for $c_1, c_2 \in \mathbb{C}$ we have

$$\begin{aligned} &\langle h_{n^{-1}}(c_1 g_n + c_2\varphi_0), (c_1 g_n + c_2\varphi_0)\rangle \\ &\qquad = c_1^2\langle h_{n^{-1}} g_n, g_n\rangle + c_2^2\langle h_{n^{-1}}\varphi_0, \varphi_0\rangle + 2\,\mathrm{Re}\, c_1\overline{c_2}\langle h_{n^{-1}} g_n, \varphi_0\rangle. \end{aligned} \tag{A.1.13}$$

Furthermore, it is easy to see that

$$\mathrm{Re}\langle h_{n^{-1}} g_n, \varphi_0\rangle = \mathrm{Re}\langle g_n, h\varphi_0\rangle - n^{-1}\,\mathrm{Re}\langle \nabla g_n, \nabla\varphi_0\rangle = 0 \tag{A.1.14}$$

and

$$\langle h_{n^{-1}}\varphi_0, \varphi_0\rangle = \langle h\varphi_0, \varphi_0\rangle - n^{-1}\|\nabla\varphi_0\|^2 = -n^{-1} \tag{A.1.15}$$

for every $n \in \mathbb{N}$. Hence, we conclude that for any linear combination $c_1 g_n + c_2\varphi_0$ with c_1, $c_2 \neq 0$ we have

$$\langle h_{n^{-1}}(c_1 g_n + c_2\varphi_0), (c_1 g_n + c_2\varphi_0)\rangle < 0. \tag{A.1.16}$$

Since by (A.1.12) the functions φ_0 and g_n are linearly independent, for any $n \in \mathbb{N}$ we can find a linear combination f_n of φ_0 and g_n, such that f_n is orthogonal to the ground state of $h_{n^{-1}}$. According to Lemma A.1.3 for sufficiently large $n \in \mathbb{N}$ the operator $h_{n^{-1}}$ has only one negative eigenvalue, which yields $\langle h_{n^{-1}} f_n, f_n \rangle \geq 0$. This is a contradiction to (A.1.16). $\square$

Lemma A.1.3 completes the proof of Theorem 4.3.7.

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