Algebraic analogues of resolution of singularities, quasi-hereditary covers and Schur algebras

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte Abhandlung

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Tag der mündlichen Prüfung: 11.10.2021

Institut für Algebra und Zahlentheorie der Universität Stuttgart
2021

Acknowledgements

First and foremost I want to thank my supervisor, Prof. Dr. Steffen Koenig, whose guidance, feedback and encouragement have been fundamental throughout this research project. His willingness to help others and his dedication to mathematics have certainly inspired me. I wish to thank him for his friendship. My gratitude extends to all members of IAZ group and the very friendly environment to study mathematics that they provide.

I wish to thank Prof. Dr. Karin Erdmann and Prof. Dr. Volodymyr Mazorchuk for their time reviewing this Dissertation and for giving me helpful feedback to improve it.

I would like to thank Prof. Dr. Ana Paula Santana and Dr. Ivan Yudin for introducing me to representation theory.

I am indebted to my girlfriend, Margarida, for her patience and emotional support. Many thanks to my friends for your encouragement and enthusiasm. I thank my family, especially my parents, for their understanding and for the opportunities that they gave me making me who I am today.

The financial support of the *Studienstiftung des deutschen Volkes* is gratefully acknowledged.

Eigenständigkeitserklärung

Hiermit bestätige ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe. Die Stellen der Arbeit, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen sind, wurden unter Angabe der Quelle kenntlich gemacht.

Abstract

In algebraic geometry, a resolution of singularities is, roughly speaking, a replacement of a local commutative Noetherian ring of infinite global dimension by a local commutative Noetherian ring of finite global dimension. In representation theory, an analogous problem is asking to resolve algebras of infinite global dimension by algebras of finite global dimension. In addition, such resolutions should have nicer properties to help us study the representation theory of algebras of infinite global dimension. This motivates us to take split quasi-hereditary covers as these algebraic analogues of resolutions of singularities and measure their quality using generalisations of dominant dimension and deformation results based on change of rings techniques.

The Schur algebra, $S_R(n,d)$ with $n \geq d$, together with the Schur functor is a classical example of a split quasi-hereditary cover of the group algebra of the symmetric group, RS_d, for every commutative Noetherian ring R. The block algebras of the classical category $\mathcal{O}$, together with their projective-injective module, are split quasi-hereditary covers of subalgebras of coinvariant algebras.

In this thesis, we study split quasi-hereditary covers, and their quality, of some cellular algebras over commutative Noetherian rings. The quality of a split quasi-hereditary cover can be measured by the fully faithfulness of the Schur functor on standard modules and on m-fold extensions of standard modules. Over fields, the dominant dimension controls the quality of the split quasi-hereditary cover of KS_d formed by the Schur algebra $S_K(n,d)$ and the Schur functor. In particular, this quality improves by increasing the characteristic of the ground field. To understand the integral cases, the classical concept of dominant dimension is not useful since in most cases there are no projective-injective modules.

Using relative homological algebra, we develop and study a new concept of dominant dimension, which we call relative dominant dimension, for Noetherian algebras which are projective over the ground ring making this concept suitable for the integral setup. For simplicity, we call Noetherian algebras which are projective over the ground ring just projective Noetherian algebras. While developing the theory of relative dominant dimension, we generalize the Morita-Tachikawa correspondence for projective Noetherian algebras and we prove that computations of relative dominant dimension over projective Noetherian algebras can be reduced to computations of dominant dimension over finite-dimensional algebras over algebraically closed fields. Using relative dominant dimension, concepts like Morita algebras and gendo-symmetric algebras can be defined for Noetherian algebras.

We compute the relative dominant dimension of Schur algebras $S_R(n,d)$ for every commutative Noetherian ring R. Using such computations together with deformation results that involve the spectrum of the ground ring R we determine the quality of the split quasi-hereditary covers of RS_d, $(S_R(n,d), V^{\otimes d})$ formed by the Schur algebra $S_R(n,d)$ and the Schur functor $\operatorname{Hom}_{S_R(n,d)}(V^{\otimes d}, -)\colon S_R(n,d)\text{-mod} \to RS_d\text{-mod}$ for all regular Noetherian rings. Over local commutative regular rings R, the quality of $(S_R(n,d), V^{\otimes d})$ depends only on the relative dominant dimension and on R containing a field or not. For this cover, the quality improves compared with the finite-dimensional case whenever the local commutative Noetherian ring does not contain a field. This theory is also applied to q-Schur algebras and Iwahori-Hecke algebras of the symmetric group.

In full generality, we prove that the quality of a split quasi-hereditary cover of a finite-dimensional algebra B is bounded above by the number of non-isomorphic simple B-modules.

Other split quasi-hereditary algebras that we study in this thesis are deformations of block algebras of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of a semi-simple Lie algebra. These deformations provide split quasi-hereditary covers of deformations of subalgebras of coinvariant algebras. We compute the relative dominant dimensions of these block algebras and we determine the quality of these covers. In these deformations, the quality dramatically improves compared with the finite-dimensional case.

Using approximation theory to generalize once more the concept of dominant dimension to relative dominant dimension with respect to direct summands of the characteristic tilting module, we find new split quasi-hereditary covers. In particular, the relative dominant dimension of a characteristic tilting module of $S_R(n,d)$ with respect to $V^{\otimes d}$ is a lower bound of the quality of a split quasi-hereditary cover of the cellular algebra $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})^{op}$, independent of the natural numbers n and d. This split quasi-hereditary cover involves the Ringel dual of the Schur algebra $S_R(n,d)$.

Using this technology for deformations of block algebras of the classical BGG category $\mathcal{O}$ of a semi-simple Lie algebra, we obtain a new proof for Ringel self-duality of the blocks of the classical BGG category $\mathcal{O}$ of a complex semi-simple Lie algebra. Here, the uniqueness of split quasi-hereditary covers of deformations of subalgebras of coinvariant algebras with higher quality is the crucial factor to deduce Ringel self-duality.

Zusammenfassung

Eine Singularität aufzulösen bedeutet in der algebraischen Geometrie, sehr vereinfacht gesagt, einen lokalen kommutativen noetherschen Ring unendlicher globaler Dimension durch einen lokalen kommutativen noetherschen Ring endlicher globaler Dimension zu ersetzen. Ein analoges Problem in der Darstellungstheorie fragt nach Auflösungen von Algebren unendlicher globaler Dimension durch Algebren endlicher globaler Dimension. Zusätzlich soll eine solche Auflösung gute Eigenschaften besitzen, die helfen die Darstellungstheorie der gegebenen Algebren unendlicher globaler Dimension zu untersuchen. Dies motiviert die Wahl (split) quasi-erblicher Decken als algebraische Entsprechungen von Auflösungen von Singularitäten, wobei die Qualität einer Decke mit Hilfe von (verallgemeinerter) dominanter Dimension und durch Deformationen in Verbindung mit *change of rings* Methoden bestimmt werden soll.

Ein klassisches Beispiel einer solchen „Auflösung“ ist die Schuralgebra $S_R(n,d)$ mit $n \geq d$, über einem kommutativen noetherschen Ring R, als „Auflösung“ der Gruppenalgebra RS_d der symmetrischen Gruppe S_d, wobei der Tensorraum und der Schurfunktor die Darstellungstheorien der beiden Algebren verbinden. Ein anderes Beispiel sind die Algebren zu den Blöcken der klassischen Bernstein-Gelfand-Gelfand Kategorie $\mathcal{O}$ halbeinfacher komplexer Lie-Algebren, die durch die projektiv-injektiven Moduln split quasi-erbliche Decken von Koinvariantenalgebren sind.

Allgemeiner werden in dieser Dissertation split quasi-erbliche Decken, und deren Qualität, von Klassen zellulärer Algebren über kommutativen noetherschen Ringen untersucht. Die Qualität kann gemessen werden durch die Volltreue des Schurfunktors auf Standardmoduln und auf deren m-fachen Erweiterungen in einem möglichst großen Intervall von Graden. Über Körpern kontrolliert die dominante Dimension diese Qualität und die Qualität verbessert sich mit wachsender Charakteristik des Grundkörpers. Um ganzzahlige Situationen zu verstehen ist die klassische dominante Dimension aber nicht geeignet, da meistens projektiv-injektive Moduln fehlen.

Deshalb wird relative homologische Algebra eingesetzt, um ein neues und allgemeineres Konzept – relative dominante Dimension – zu entwickeln, über projektiv-noetherschen Algebren, das heisst, noetherschen Algebren, die projektiv über dem Grundring sind. Damit werden ganzzahlige Situationen erfasst. Während wir diese Theorie entwickeln erweitern wir die Morita-Tachikawa Korrespondenz entsprechend und zeigen, wie die Berechnung der relativen dominanten Dimension zurückgeführt werden kann auf die Berechnung der dominanten Dimension endlich-dimensionaler Algebren über algebraisch abgeschlossenen Grundkörpern. Über die relative dominante Dimension können auch Konzepte wie Morita-Algebren und gendo-symmetrische Algebren für projektiv-noethersche Algebren definiert werden.

Die relative dominante Dimension der Schuralgebren $S_R(n,d)$ wird über allen kommutativen noetherschen Ringen R berechnet. Die Verbindung solcher Berechnungen mit Deformationsergebnissen bezüglich des Spektrums des Grundrings R erlaubt für alle regulären noetherschen Ringe R die Bestimmung der Qualität der Auflösung der Gruppenalgebra RS_d durch die Schuralgebra $S_R(n,d)$ und den Tensorraum. Über lokalen kom-

mutativen regulären Ringen R hängt diese Qualität nur von der relativen dominanten Dimension ab und davon, ob R einen Körper enthält oder nicht. Wenn R keinen Körper enthält, ist die Qualität der Auflösung besser als im endlich-dimensionalen Fall. Wir wenden die Theorie auch auf q-Schuralgebren und Iwahori-Hecke-Algebren der symmetrischen Gruppen an.

Ganz allgemein wird für endlich-dimensionale Algebren B gezeigt, dass die Qualität jeder split quasi-erblichen Decke von B durch die Anzahl der nichtisomorphen einfachen B-Moduln nach oben beschränkt ist. Andere hier betrachtete split quasi-erbliche Algebren sind Deformationen von Block-Algebren der BGG-Kategorie $\mathcal{O}$ von halbeinfachen komplexen Lie-Algebren. Diese liefern split quasi-erbliche Decken von Deformationen von (Teilalgebren von) Koinvariantenalgebren. Die relative dominante Dimension wird berechnet und die Qualität der Auflösungen bestimmt. Dabei zeigt sich, dass die Qualität im Vergleich zum endlich-dimensionalen Fall dramatisch verbessert wird.

Mit Approximationstheorie kann die dominante Dimension noch ein weiteres Mal verallgemeinert werden zu einer relativen dominanten Dimension bezüglich direkter Summanden des charakteristischen Kippmoduls. Dadurch finden wir neue split quasi-erbliche Decken. Insbesondere ist die relative dominante Dimension des charakteristischen Kippmoduls bezüglich dem Tensorraum eine untere Schranke für die Qualität einer Auflösung des Endomorphismenrings des Tensorraums – unabhängig von n und d. Dabei ist die zur Schuralgebra Ringel-duale Algebra involviert.

Durch eine Anwendung dieser Methoden auf die Deformationen der Block-Algebren der klassischen BGG-Kategorie $\mathcal{O}$ erhalten wir einen neuen Beweis von Soergels Satz, dass diese Blöcke Ringel selbst-dual sind. Hierfür ist die durch den Übergang zur Deformation verbesserte Qualität der Auflösung entscheidend, um Ringelselbstdualität aus der dann vorliegenden Eindeutigkeit einer Auflösung hinreichend hoher Qualität schließen zu können.

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Introduction

Algebraic analogues of resolutions of singularities

An important theme in representation theory of algebras is to transfer properties and information between algebras. Of general interest is Auslander's correspondence [Aus71] connecting finite representation type with homological properties. This correspondence is a particular case of a more general correspondence called Morita-Tachikawa correspondence [Mue68, Theorem 2]. In both of these correspondences, there are projective modules having double centralizer properties and so these correspondences can be formulated using Rouquier's cover theory [Rou08]. Covers were introduced to compare cohomology over split quasi-hereditary algebras with cohomology over endomorphism algebras of projective modules having a double centralizer property via Schur functors. In particular, they provide an abstract framework to formulate Hemmer and Nakano's work [HN04, Corollary 3.9.2] which is the modular representation theory analogue of Schur's results [Sch01] connecting the complex representation theory of the symmetric group with the complex representation theory of a Schur algebra. Some Schur algebras together with their faithful projective-injective module and regular blocks of the BGG category $\mathcal{O}$ of a complex semi-simple Lie algebra with their projective-injective module are classical examples of covers of the group algebra of the symmetric group and the coinvariant algebra, respectively. The prior is a consequence of Schur–Weyl duality [CL74, Gre07, Sch01, Sch27] while the latter is a consequence of [Soe90, Struktursatz 9]. Here, by a projective-injective module we mean a module that is both projective and injective. In both of these examples, dominant dimension controls the quality of these covers [FK11b, Fan08].

Both the group algebra of the symmetric group (over a field) and the coinvariant algebra are symmetric algebras. Therefore, they have infinite global dimension unless they are semi-simple. A local commutative Noetherian ring is regular if and only if it has finite global dimension. Hence, resolving a singularity in commutative algebra translates to the study of an algebra of infinite global dimension through the study of an algebra with finite global dimension. Hence, some Schur algebras and the principal block are algebraic analogues of resolutions of singularities of the symmetric group and the coinvariant algebra, respectively. Several constructions like Dlab-Ringel standardization [DR92], Auslander's construction on the endomorphism algebra of the sum of quotients of the regular module by powers of the Jacobson radical [DR89a, Aus71] and Iyama's construction to prove the finiteness of representation dimension [Iya04, 3.4.1] can be regarded as algebraic analogues of resolution of singularities. Moreover, these can be formulated using Rouquier's cover theory.

Other types of analogues of resolutions of singularities have been attracting attention, for example non-commutative resolutions [DITV15]. Although in most cases, their concept is not a cover, their non-commutative resolutions of commutative self-injective rings coincides with covers of commutative self-injective rings. For orders, the resolution constructed in [Kön91] can also be formulated using covers.

Except for these last two constructions, all the results we have mentioned so far are for finite-dimensional algebras and Artinian algebras. There is much evidence for example in [Rou08, Proposition 4.42] and in [CPS96] that going integrally can improve quality of covers. Going integrally means studying Noetherian versions of the previous covers. The notion of quality will be made precise later on. We aim to study quality of covers, namely split quasi-hereditary covers, by strengthening known connections studying integral versions of such connections and integral analogues of dominant dimension. This approach will help us to obtain further insights in modular representation theory using integral representation theory. A second goal, which is motivated by [KSX01], is to explain Schur–Weyl duality between Schur algebras and symmetric groups without restrictions on parameters using cover theory after going relative and integrally on the concept of dominant dimension. For such aim, going to the integral setup and going relative are crucial techniques. Going relative means using tilting modules instead of projective-injective modules and working with specific classes of exact sequences.

Schur algebras and symmetric groups

In the early years of representation theory, Issai Schur, in his PhD dissertation [Sch01], gave a complete classification of rational representations of the general linear group over the complex numbers using the representation theory of the symmetric groups (over the complex numbers) studied in [Fro00] by his supervisor Frobenius. The crucial step in this classification was the construction of an equivalence of categories from the polynomial representation theory of the complex general linear group of a fixed degree to the representation theory of the complex symmetric group, known today as *Schur functor*. Here, by polynomial representation we mean a representation that sends each element $g \in \mathrm{GL}_n(\mathbb{C})$ to a matrix whose entries are polynomial functions in the entries of g. In 1927, Schur published a paper [Sch27] where he explored the actions of the general linear group $GL_n(\mathbb{C})$ and the symmetric group S_d on the d-th tensor space and reobtained all the results of his PhD dissertation in a very elegant way. Moreover, he proved that these two actions centralize each other. Then Weyl, in the book *Classical Groups* [Wey46], popularized this new approach by his extensive use of double centralizer properties. The double centralizer property involving the subalgebras of the endomorphism algebra of the d-th tensor space generated by the actions of the general linear group $GL_n(\mathbb{C})$ and the symmetric group S_d on the d-th tensor space is called Schur–Weyl duality. These developments were the first step in the study of Schur–Weyl duality and double centralizer properties.

Schur's PhD dissertation came back into focus with Green's monograph [Gre07] where Green extends Schur's ideas to the polynomial representation theory of the general linear group $\mathrm{GL}_n(K)$ for K an infinite field. Over infinite fields, Green established that the polynomial representation theory of the general linear group $\mathrm{GL}_n(K)$ can be reduced to the study of the module categories of Schur algebras.

The Schur algebras can be defined over an arbitrary commutative ring R. Let n,d be natural numbers. The symmetric group S_d acts on the tensor power $V^{\otimes d} := (R^n)^{\otimes d}$ by place permutation:

$$(v_1 \otimes \cdots \otimes v_d)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \quad \sigma \in S_d, \quad v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}.$$

The Schur algebra $S_R(n,d)$ is the endomorphism R-algebra $\mathrm{End}_{RS_d}\left(V^{\otimes d}\right)$ of R-linear endomorphisms of $V^{\otimes d}$ that commute with endomorphisms given by the action of elements $\sigma \in S_d$ in $V^{\otimes d}$. Here, RS_d denotes the group algebra of the symmetric group S_d over the commutative ring R.

Over infinite fields of prime characteristic K, there is no longer, in general, an equivalence between the module category of a Schur algebra and the representation theory of the symmetric group. However, their connection does not disappear and there exists a version of Schur–Weyl duality on $(K^n)^{\otimes d}$ between the Schur algebra and the symmetric group (see [Gre07, (2.6c)] and [dCP76]). Further, under the assumption $n \geq d$, the Schur functor $\mathrm{Hom}_{S_K(n,d)}((K^n)^{\otimes d}, -)$ from the module category of the Schur algebra $S_K(n,d)$ to the representation theory of the symmetric group S_d is well defined and it is just the multiplication by a certain idempotent of $S_K(n,d)$ (see [Gre07, 6.1]). This functor has nice properties like sending the simple modules of the Schur algebra to either zero or to a simple module of the symmetric group and it is an exact functor. It became clear with Green's work that this context belongs to a more general setup. Further, the direction of study was shifted. Nowadays, the usual direction, in this context, is to study first properties of the Schur algebra and then use this knowledge to deduce properties of the symmetric group using a Schur functor. A reason for this comes from the fact that Schur algebras are split quasi-hereditary algebras [Don87, Par89] and also cellular algebras while the group algebras of symmetric groups are only cellular algebras [GL96, 1.2] unless they are semi-simple.

Roughly speaking, split quasi-hereditary algebras over a field form a class of finite-dimensional algebras of finite global dimension for which the regular module has a finite filtration by a collection of modules indexed by a partial order on the set of isomorphism classes of simple modules, called standard modules, whose endomorphism algebras have dimension one over the ground field. Once the partial order is fixed, each standard module $\Delta(\lambda)$ is the largest quotient of its projective cover without simple composition factors indexed by elements greater than λ. Over algebraically closed fields, all quasi-hereditary algebras are split quasi-hereditary algebras and all finite-dimensional algebras over algebraically closed fields with global dimension at most two are split quasi-hereditary. Split quasi-hereditary algebras are quite abundant and they appear frequently in the representation theory of algebraic groups and semi-simple Lie algebras.

Cellular algebras are finite-dimensional algebras characterized by the existence of a basis, called cellular basis, with similar properties as the Kazhdan-Lusztig basis of the group algebra of the symmetric group. Knowing a cellular basis reduces problems like knowing the number of simple modules of the algebra to problems of linear algebra. Further, the cellular basis implies the existence of a filtration of the regular module by a collection of modules, called cell modules. Over cellular algebras, each simple module occurs at the top of some cell module. Many problems like finding the decomposition numbers of cellular algebras can be reduced to problems on split quasi-hereditary algebras via Schur functors. Koenig and Xi proved in [KX99b] that cellular algebras over fields are split quasi-hereditary if and only if they have finite global dimension. So, for cellular algebras which are not split quasi-hereditary, Schur functors connect the study of the module category of an infinite global dimension algebra with the study of the module category of an algebra having finite global dimension, if it exists. In particular, this problem can be seen as an algebraic analogue of resolution of singularities which are studied in algebraic geometry. In fact, the local ring of a variety (over an algebraically closed field) at a singular point, that is, a certain localization of the coordinate ring of a variety, has infinite global dimension whereas the local ring of a smooth variety at any point has finite global dimension (see [Rot09, Example 8.57]).

Main Problems

We will now discuss the abstract framework for our main problems. All rings mentioned in this thesis are rings with identity. As we said, both Schur algebras and group algebras of the symmetric group can be defined over any commutative ring with identity. Not only them but also the concepts of split quasi-hereditary algebras and cellular algebras can be studied over commutative Noetherian rings ([CPS90, GL96]).

Let R be a commutative Noetherian ring and let A be a projective Noetherian R-algebra, that is, an R-algebra that is finitely generated and projective as R-module. A pair (A,P) is called a cover of B if P is a projective (left) A-module, B is the endomorphism algebra $\operatorname{End}_A(P)^{op}$, and the restriction of the functor $\operatorname{Hom}_A(P,-)\colon A\text{-mod} \to B\text{-mod}$ to the full subcategory of finitely generated A-modules which are projective over A is fully faithful. The functor $\operatorname{Hom}_A(P,-)$ is called a Schur functor. The fully faithfulness of the Schur functor on projective A-modules is equivalent to the double centralizer property on the right A-module $\mathrm{Hom}_A(P,A)$ (see [Rou08, Proposition 4.33]). We say that a (left) A-module M has the double centralizer property if the canonical map $A \to \mathrm{End}_B(M)$ is an isomorphism of algebras where B denotes the endomorphism algebra $\mathrm{End}_A(M)^{op}$. By Schur–Weyl duality, $(S_R(n,d), V^{\otimes d})$ is a cover of RS_d for every commutative Noetherian ring assuming that $n \geq d$.

Using the terminology of covers, finding a resolution for a cellular algebra can be formulated in the following way:

- Given a cellular algebra B over a commutative Noetherian ring R, find (if it exists) and study a split quasi-hereditary cover (A,P) of B.

Here, split quasi-hereditary cover means a cover (A,P) so that A is a split quasi-hereditary R-algebra. Split quasi-hereditary algebras over commutative Noetherian rings with finite global dimension have finite global dimension (see [CPS90, 3.6]). Local commutative Noetherian rings with finite global dimension are known as regular rings. Hence, we may ask the following:

(Q1) Are cellular algebras (over commutative Noetherian rings) with finite global dimension split quasi-hereditary?

Due to [DR89a] and [Iya03, Iya04] every finite-dimensional algebra admits a quasi-hereditary cover. In particular, every finite-dimensional algebra over an algebraically closed field admits a split quasi-hereditary cover. Is it possible to say the same in the integral setup? That is,

(Q2) Do all projective Noetherian algebras over a commutative Noetherian ring admit a split quasi-hereditary cover?

The next natural question to pose is how to choose the "best" (if it exists) split quasi-hereditary cover of a given algebra. For this, we use the notion of i-faithfulness of a split quasi-hereditary cover introduced by Rouquier in [Rou08, 4.37].

Let $i \geq 0$. A cover (A,P) is an i-faithful split quasi-hereditary cover of B if A is a split quasi-hereditary algebra and the Schur functor $F = \mathrm{Hom}_A(P,-)$ induces isomorphisms

$$\mathrm{Ext}^j_A(M,N) \to \mathrm{Ext}^j_B(FM,FN), \quad \forall M,N \in \mathscr{F}(\tilde{\Delta}),\ 0 \leq j \leq i.$$

A cover (A,P) is a (-1)-faithful split quasi-hereditary cover of B if A is a split quasi-hereditary algebra and the restriction of the Schur functor to $\mathscr{F}(\tilde{\Delta})$ is faithful. Here, $\mathscr{F}(\tilde{\Delta})$ denotes the full subcategory of (left) A-modules which admit a filtration by standard modules tensored with projective R-modules.

In this sense, the quality of a split quasi-hereditary cover is measured by how exact the right adjoint of the Schur functor is on the syzygies of the image of standard modules under the Schur functor. In Section 3.1, we generalize this measure for resolving subcategories of the module category of A (not being necessarily split quasi-hereditary). In this formulation, the main problem consists of the following:

Main Problem. Given a cellular algebra B over a commutative Noetherian ring R, determine (if it exists) the highest $i \in \mathbb{N} \cup \{-1, 0, +\infty\}$ possible such that (A,P) is an i-faithful split quasi-hereditary cover of B.

Using this terminology, for K an algebraically closed field with characteristic $p > 3$, Hemmer and Nakano [HN04, Corollary 3.9.2], based on the work of Kleshchev and Nakano [KN01], found that $(S_K(n,d), V^{\otimes d})$ is a $p-3$-faithful split quasi-hereditary cover of KS_d for $n \geq d$. In particular, if $p > 3$, this means that there exists an exact equivalence between the full subcategory of $S_K(n,d)$-modules admitting a finite filtration by standard modules and the full subcategory of KS_d-modules admitting a finite filtration by cell modules. The value $p-3$ is actually the optimal value. This value is known in the literature as the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ (with respect to the Schur functor). Given a split quasi-hereditary cover (A,P) we say that the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ (with respect to the Schur functor $\operatorname{Hom}_A(P,A)$) is n if (A,P) is an n-faithful split quasi-hereditary cover of B but it is not an $(n+1)$-faithful split quasi-hereditary cover of B. Later, Fang and Koenig reproved this result for $p>3$ and extended the result to the cases with characteristic two and three using dominant dimension of the algebra and the dominant dimension of the characteristic tilting module of the Schur algebra (see [FK11b]). For this, Mueller's characterization of dominant dimension is crucial in translating the problem from dominant dimension to the Hemmer-Nakano dimension. Unfortunately, for these lower cases $p\in\{2,3\}$ there is no equivalence between the full subcategory of $S_K(n,d)$-modules admitting a finite filtration by standard modules and the full subcategory of KS_d-modules admitting a finite filtration by cell modules.

We say that a module M has dominant dimension at least n if there exists an exact sequence $0\to M\to I_1\to\cdots\to I_n$ with all I_i being projective and injective A-modules. The dominant dimension of the algebra is exactly the dominant dimension of the regular module. Furthermore, split quasi-hereditary algebras with dominant dimension at least two provide an extensive source of split quasi-hereditary covers. With this in mind, an invariant called rigidity dimension was introduced in [CFK+21] to measure, for a given finite-dimensional algebra B, the upper bound of the Hemmer-Nakano dimension of the subcategory of the module category whose modules are projective, running through all the possible covers (formed by a projective-injective module) with finite global dimension of B. In particular, the definition of this invariant is based on the concept of dominant dimension. Further, problems like the finiteness of the rigidity dimension are still unknown in many cases. Here, we are mainly interested in split quasi-hereditary covers, and so, the following question arises:

(Q3) Given an algebra B, is there an upper bound depending only on B, say i, so that for every split quasi-hereditary cover (A,P) of B, the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ is smaller than i?

In the case of the symmetric group, the cell modules are exactly the image of the standard modules of $S_R(d,d)$ under the Schur functor $\operatorname{Hom}_{S_R(d,d)}(V^{\otimes d},-)\colon S_R(d,d)\text{-mod}\to RS_d\text{-mod}$.

(Q4) Do all cellular algebras admit a split quasi-hereditary cover with this extra property?

Rouquier observed that covers with this extra property which are also 1-faithful split quasi-hereditary covers are unique, also in the integral setup (see [Rou08, Corollary 4.46]). Further, (finite-dimensional) 1-faithful split quasi-hereditary covers are exactly the split quasi-hereditary algebras constructed using the Dlab-Ringel standardization [DR92] for a (split) standardizable set.

Going back to Schur algebras and symmetric groups, the results on the Hemmer-Nakano dimension until now discussed are only for the finite-dimensional case. In [CPS96, 4.1.5, 5.2.1], using different terminology, they prove that $(S_{\mathbb{Z}}(n,d),V^{\otimes d})$ is a zero-faithful split quasi-hereditary cover of $\mathbb{Z}S_d$. We may wonder what is the quality of this cover in the remaining cases, considering other commutative Noetherian rings as the ground ring. Proposition 4.42 of [Rou08] states that a zero-faithful split quasi-hereditary cover (A,P) of a projective Noetherian algebra B over a regular commutative Noetherian ring so that A becomes semi-simple under $K\otimes_R -$, for some field K, is a 1-faithful split quasi-hereditary cover of B. Hence, this result motivates us to understand the Hemmer-Nakano dimension in the integral setup, since it appears that integral version of covers might have better properties than their counterparts in the finite-dimensional realm.

(Q5) In particular, do split quasi-hereditary covers over regular rings with higher values of Krull dimension have higher values of faithfulness?

(Q6) What is the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ with respect to the Schur functor $\mathrm{Hom}_{S_R(n,d)}(V^{\otimes d}, -)$ (assuming $n \geq d$) for an arbitrary commutative Noetherian regular ring R?

In [FM19, Theorem 3.13], Fang and Miyachi computed the dominant dimension of quantized Schur algebras. Analogues of Hemmer-Nakano results for q-Schur algebras can also be found in [PS05]. The quantized Schur algebras $S_{R,q}(n,d)$ replace the role of Schur algebras where Iwahori-Hecke algebras of the symmetric replace the role of the group algebra of the symmetric group. So, we can pose the same question:

(Q7) What is the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ with respect to the Schur functor $\mathrm{Hom}_{S_{R,q}(n,d)}(V^{\otimes d}, -)$ (assuming $n \geq d$) for an arbitrary commutative Noetherian regular ring R?

Another object of interest with respect to the main problem is the BGG category $\mathcal{O}$. The regular blocks of the BGG category $\mathcal{O}$ of a semi-simple Lie algebra together with the projective-injective module form a split quasi-hereditary cover of the coinvariant algebra which is a cellular algebra (see [Soe90, Struktursatz 9]). The dominant dimension of the blocks of the category $\mathcal{O}$ was computed in [KSX01, 3.1] and [Fan08, Proposition 4.5]. In 1981, Gabber and Joseph [GJ81] defined and studied integral versions of the BGG category $\mathcal{O}$ over a commutative ring in the context of the Kazhdan-Lusztig conjecture.

(Q8) Does the BGG category $\mathcal{O}$ over a semi-simple Lie algebra over a commutative ring give rise to faithful split quasi-hereditary covers over commutative Noetherian rings?

To discuss the example of Schur algebras and to define the Schur functor we have to impose $n \geq d$. However, Schur–Weyl duality holds between $S_R(n,d)$ and RS_d, independent of the parameters n and d. More precisely, the canonical map $RS_d \to \mathrm{End}_{S_R(n,d)}(V^{\otimes^d})^{op}$ is surjective. This is equivalent to saying that $V^{\otimes^d}$ has the double centralizer property over $S_R(n,d)$. Although, $V^{\otimes d}$ might not be projective and injective, Koenig, Slungård and Xi exploited in [KSX01] the fact that $V^{\otimes d}$ is a partial tilting module in the case $n < d$ to deduce Schur–Weyl duality between $S_K(n,d)$ and KS_d, where K denotes a field. This leads us to the following question:

(Q9) For $n < d$, can Schur–Weyl duality between $S_R(n,d)$ and RS_d be explained using cover theory?

To summarize, the focus of this thesis is to evaluate the quality of known covers and their properties, mainly now in the integral setup, and to study how to construct new covers from known covers. Now, we shall discuss some contributions that this thesis makes to these questions and problems.

Contributions

As we discussed, the main source of examples are algebras that belong to the class of split quasi-hereditary algebras over a commutative Noetherian ring (see Section 1.5) and cellular algebras over a commutative Noetherian ring (see Section 1.6). For split quasi-hereditary algebras, we collect the classical properties in the integral setup clarifying at the same time some confusions present in the literature. For example, we suggest an alternative proof for the fact that opposite algebras of split quasi-hereditary algebras are split quasi-hereditary (see Theorem 1.5.69). We show in Proposition 1.5.80 that two module categories of split quasi-hereditary algebras with an exact equivalence between their subcategories of modules having standard filtrations are equivalent as split highest weight categories. This allows us to give alternative proofs of the results of uniqueness of covers (see Corollary 3.6.6) observed in [Rou08]. We make it precise in Theorem 1.5.58 that deciding whether a Noetherian algebra is split quasi-hereditary can be reduced to deciding whether a finite-dimensional algebra is split quasi-hereditary over algebraically closed fields. In particular, we can regard split quasi-hereditary algebras over a commutative Noetherian ring as deformations of quasi-hereditary algebras over algebraically closed fields. We establish in Theorem 1.5.84 that these split quasi-hereditary algebras are locally semi-perfect.

We apply this fact in Corollary 3.7.3 to obtain a negative answer to **(Q2)**.

Corollary (Corollary 3.7.3)**.** *Let C_3 be the abelian group of order 3. The group algebra $\mathbb{Z}C_3$ over $\mathbb{Z}$ does not have a split quasi-hereditary cover.*

We give detailed proofs of characterizations of the subcategory of modules having filtrations by costandard modules and on the Ringel dual of a split quasi-hereditary algebra over a commutative Noetherian ring complementing and fixing some inaccuracies on the results appearing in [Rou08] (see Theorem 1.5.104). Many results about change of ground ring on homomorphisms between standard modules and costandard modules are improved as well as results on filtrations of these modules (Proposition 1.5.117 and Proposition 1.5.133).

In Theorem 1.6.16 and Theorem 1.6.18, we give a positive answer to **(Q1)**. Furthermore, we prove in Theorem 1.6.19 that cellular algebras with finite global dimension admit a unique split quasi-hereditary structure extending Theorem 2.1.1 of [Cou20] to the integral setup.

In Example 4.6.14, we see that **(Q4)** has a negative answer. In fact:

Example (Example 4.6.14)**.** The bound quiver algebra over an algebraically closed field with characteristic zero

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] \arrow[loop above, "\varepsilon"] & 3 \arrow[l, shift left, "\gamma"] \end{tikzcd}, \quad \beta\alpha = \delta\gamma = \varepsilon\alpha = \beta\varepsilon = \varepsilon\gamma = \delta\varepsilon = 0, \quad \alpha\delta = \varepsilon^2 = \gamma\beta,$$

is a cellular algebra but its cellular structure is not given by a split quasi-hereditary cover. △

For projective Noetherian algebras B over local regular commutative Noetherian rings with Krull dimension at most one, B has a 1-faithful quasi-hereditary cover using a generalization of Dlab-Ringel standardization (Theorem 1.5.83) if it admits a split standardizable set (see Definition 1.5.82) and a certain filtration for the regular module. Hence, the cellular algebras with the cell modules forming a split standardizable set admit split quasi-hereditary covers as asked in **(Q4)**. Also, any split quasi-hereditary cover appearing in the setup of the class $\mathscr{A}$ of [FK11b] gives examples with the extra property in **(Q4)**.

Not all split quasi-hereditary covers are i-faithful split quasi-hereditary covers for some $i \in \mathbb{N} \cup \{-1, 0\}$.

Example (Example 4.6.2 and 4.6.9)**.** Let K be an algebraically closed field. Let A be the following bound quiver K-algebra

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3, \quad \alpha_2\alpha_1 = 0.$$

Let B be the following bound quiver K-algebra

$$2 \xrightarrow{\alpha} 3 \ .$$

Denote by e_i the idempotent of A associated with the vertex i, $i = 1, 2, 3$. Then, $(A, Ae_2 \oplus Ae_3)$ is a split quasi-hereditary cover of B but $(A, Ae_2 \oplus Ae_3)$ is not a (-1)-faithful split quasi-hereditary cover of B. △

Concerning **(Q3)**, the global dimension of the algebra A is always an upper bound independently of A being split quasi-hereditary (see Theorem 3.2.3). In Theorem 3.2.1, we give an upper bound for the level of faithfulness of a split quasi-hereditary cover of B. This value is independent of the cover and it is bounded above by the number of simple B-modules.

The concept of dominant dimension is not suitable to help us in the remaining questions when the ground ring is just Noetherian. For example, the dominant dimension of the integers is zero. Further, the module $(R^n)^{\otimes d}$ is not injective over $S_R(n,d)$ if $n \geq d$ and R is a regular Noetherian commutative ring with positive Krull dimension. This motivates us to use relative homological algebra to modify the definition of dominant dimension. In Chapter 2, we do that by replacing exact sequences with exact sequences of A-modules which split over the ground ring and by replacing injective modules with relative injective modules. In doing so, we dramatically increase the scope of classical theory of dominant dimension to integral representation theory. These new results have interest in their own right, so we will briefly mention some of them. In the relative Morita-Tachikawa correspondence a new condition appears:

Theorem (Theorem 2.4.10 and Corollary 2.5.6)**.** *Let R be a commutative Noetherian ring. There is a bijection:*

$$\left\{ (B,M) \colon \begin{array}{c} B \text{ a projective} \\ \text{Noetherian } R\text{-algebra,} \\ M \text{ a } B\text{-generator } (B,R)\text{-cogenerator,} \\ M \in R\text{-proj}, \\ DM \otimes_B M \in R\text{-proj} \end{array} \right\} \Big/ \sim_1 \longleftrightarrow \left\{ A \colon \begin{array}{c} A \text{ a projective Noetherian} \\ R\text{-algebra with} \\ \operatorname{domdim}(A,R) \geq 2, \end{array} \right\} \Big/ \sim_2$$

In this notation, $A \sim_2 A'$ if and only if A and A' are isomorphic, whereas, $(B,M) \sim_1 (B',M')$ if and only if there is an equivalence of categories $F \colon B\text{-mod} \to B'\text{-mod}$ such that $M' = FM$.

$$(B,M) \mapsto A = \operatorname{End}_B(M)^{op}$$
$$(\operatorname{End}_A(N), N) \leftarrow\!\shortmid A$$

where N is a projective (A,R)-injective-strongly faithful right A-module.

Here, generator (B,R)-cogenerator means a module whose additive closure contains the regular module and the dual of the regular module DB, where D is the standard duality functor $\operatorname{Hom}_R(-,R)$ with respect to R.

The extra condition $DM \otimes_B M \in R$-proj in the relative Morita-Tachikawa correspondence states that only endomorphism algebra of generators relative cogenerators with a base change property are allowed (Proposition 2.5.14). This fact explains the usefulness of relative dominant dimension as a tool to establish characteristic-free proofs of double centralizer properties on projective modules.

Mueller's characterization of dominant dimension becomes, in the integral setup, a characterization of relative dominant dimension in terms of homology over A in terms of homology over the endomorphism algebra of a projective relative injective module.

Theorem (Theorem 2.4.15)**.** *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra with positive relative dominant dimension and V a certain projective right A-module satisfying additional conditions stated in Theorem 2.4.15. Fix $C = \operatorname{End}_A(V)$. For any $M \in A\text{-mod} \cap R\text{-proj}$, the following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq n \geq 2$;

(ii) $\phi_M \colon \operatorname{Hom}_A(V, DM) \otimes_C V \to DM$ *is an isomorphism and* $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM), V) = 0,\ 1 \leq i \leq n-2$.

A particular case of this characterization of relative dominant dimension combined with some considerations on the Krull dimension of the ground ring R is the following:

Theorem (Theorem 3.5.6 for $\mathscr{F}(\tilde{\Delta})$ and Theorem 2.11.1)**.** *Let R be a commutative regular Noetherian ring. Let A be a split quasi-hereditary R-algebra with relative dominant dimension at least two and V a certain projective* *right A-module satisfying additional conditions stated in Theorem 2.4.15. Fix* $n = \operatorname{domdim}_{(A,R)} T$, *where* T *is a characteristic tilting module of A. Then,* $(A, \operatorname{Hom}_A(V,A))$ *is an* $(n-2)$*-faithful split quasi-hereditary cover of* $\operatorname{End}_A(V)$. *Moreover,* $n-2 \leq \operatorname{HNdim}_{\operatorname{Hom}_A(\operatorname{Hom}_A(V,A),-)} \mathscr{F}(\tilde{\Delta}) \leq n + \dim R - 2$.

Computations of relative dominant dimension can be reduced to computation of dominant dimension of modules over algebraically closed fields due to the following:

Theorem (Theorem 2.5.13)**.** *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra with positive relative dominant dimension. Let* $M \in A\text{-mod} \cap R\text{-proj}$. *Then,*

$$\operatorname{domdim}_{(A,R)} M = \inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \colon \mathfrak{m} \textit{ maximal ideal in } R\}.$$

Using relative dominant dimension opens doors to study more concepts over commutative Noetherian rings like Morita and gendo-symmetric algebras (see Theorem 2.9.1 and Theorem 2.10.2).

Using this machinery, we can show that Schur algebras are relative gendo-symmetric and we can compute the relative dominant dimension of Schur algebras:

Theorem (Theorem 4.1.7)**.** *Let R be a commutative Noetherian ring. If* $n \geq d$ *are natural numbers, then*

$$\operatorname{domdim}(S_R(n,d), R) = \inf\{2k \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\} \geq 2.$$

Here, $U(R)$ denotes the set of invertible elements of the commutative ring R.

To answer **(Q6)**, we use the deformation results Theorem 3.3.13 and Corollary 3.3.10 together with a computation of the relative dominant dimension of the Schur algebra. Interestingly, the computation of the Hemmer-Nakano dimension for Schur algebras over local commutative regular Noetherian rings depends on the ground ring containing a field or not.

Theorem (see Subsection 4.1.2)**.** *Let R be a local regular commutative Noetherian ring. Assume that* $n \geq d$. *Then,*

$$\operatorname{HNdim}_{\operatorname{Hom}_{S_R(n,d)}((R^n)^{\otimes d},-)} \mathscr{F}(\tilde{\Delta}) = \begin{cases} \dfrac{\operatorname{domdim}(S_R(n,d),R)}{2} - 2, & \textit{if } R \textit{ contains a field as a subring} \\ \dfrac{\operatorname{domdim}(S_R(n,d),R)}{2} - 1, & \textit{otherwise} \end{cases}.$$

The full answer can be found in Subsection 4.1.2. So, the situation improves for Schur algebras when the ground ring does not contain a field, in comparison to the Schur algebra over the residue field. In particular, we see that $(S_{\mathbb{F}_2}(n,d)), ((\mathbb{F}_2)^n)^{\otimes d})$ is a (-1)-faithful split quasi-hereditary cover of $\mathbb{F}_2 S_d$ while $(S_{\mathbb{Z}_2}(n,d)), ((\mathbb{Z}_2)^n)^{\otimes d})$ is a 0-faithful split quasi-hereditary cover of $\mathbb{Z}_2 S_d$ for $n \geq d$. Here, $\mathbb{Z}_2$ denotes the localization of $\mathbb{Z}$ at $2\mathbb{Z}$. It follows that the connection between the module category over the Schur algebra and the representation theory of the symmetric group does not improve if we consider a polynomial ring as a ground ring. In Theorem 4.1.16, we prove that the integral symmetric group does not admit a better split quasi-hereditary cover mapping the standard modules to the cell modules via a Schur functor than the integral cover formed by the integral Schur algebra. Here, both the level of faithfulness of the cover and the standard modules being mapped to the cell modules play an important role. For example, $\overline{\mathbb{F}_2} S_4$ has two distinct (-1)-faithful split quasi-hereditary covers: the one coming from the Schur algebra $(S_{\overline{\mathbb{F}_2}}(4,4), V^{\otimes 4})$ and $(E, \operatorname{Hom}_E(M,E))$ where $E = \operatorname{End}_{\overline{\mathbb{F}_2}}(M)^{op}$ and $M = \bigoplus_{i=0}^{3} \overline{\mathbb{F}_2} S_4 / \operatorname{rad}^i \overline{\mathbb{F}_2} S_4$. However, the latter is "worse" than the Schur algebra since the Schur functor $\operatorname{Hom}_E(\operatorname{Hom}_E(M,E),-)$ sends all standard modules to the simple modules of $\overline{\mathbb{F}_2} S_4$ (see Example 4.6.8). On the other hand, requiring only the cell modules to be in the image of standard modules of a split quasi-hereditary algebra via a Schur functor is not a strong enough condition to imply uniqueness of covers. For example, the split quasi-hereditary cover constructed using the Auslander algebra of $\overline{\mathbb{F}_3}S_3$ satisfies this condition, however, it is only a (-1)-faithful split quasi-hereditary cover of $\overline{\mathbb{F}_3}S_3$ in contrast with $(S_{\overline{\mathbb{F}}_3}(3,3), V^{\otimes 3})$ (see Example 4.6.7).

Similarly to Schur algebras, we prove that q-Schur algebras are relative gendo-symmetric algebras and the analogous results for q-Schur algebras are developed in Theorems 4.2.8, 4.2.11, 4.2.12 where quantum divisible rings take the place of local rings containing a field. In particular, this solves **(Q7)**.

Let $\mathfrak{g}$ be a finite-dimensional semi-simple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. We can construct a semi-simple Lie algebra over the integers, $\mathfrak{g}_\mathbb{Z}$, so that $\mathbb{C}\otimes_\mathbb{Z}\mathfrak{g}_\mathbb{Z}\simeq\mathfrak{g}$. Then, we can define a Lie algebra, over any commutative Noetherian ring R, $\mathfrak{g}_R := R\otimes_\mathbb{Z}\mathfrak{g}_\mathbb{Z}$. Gabber and Joseph in [GJ81] constructed a category $\mathscr{O}$, based on the work [BGG76], for $\mathfrak{g}_R$ with Cartan subalgebra $\mathfrak{h}_R$. If R is a local commutative ring, this category $\mathscr{O}$ can also be decomposed into blocks similar to the classical case. To each of these blocks, we can associate a module category of a projective Noetherian R-algebra $A_\mathscr{D}$. The details about this algebra given in Subsection 4.4 are self-contained as much as possible. The algebra $A_\mathscr{D}$ is split quasi-hereditary (see Theorem 4.4.43) and $(A_\mathscr{D}, P(\omega))$ is a relative gendo-symmetric R-algebra, where ω is an antidominant weight (see Theorem 4.4.48). We establish an integral version of Soergel's Struktursatz, that is, $(A_\mathscr{D}, P_A(\omega))$ is a split quasi-hereditary cover of a commutative deformation of the coinvariant algebra (see Theorem 4.4.49). Over the complex numbers, the Soergel's combinatorial functor from a given block $\mathscr{C}$ is not fully faithful on standard modules unless the block is semi-simple. However, we can choose blocks of the BGG category of $\mathfrak{g}_R$, for a suitable commutative local ring R, so that the Schur functor $\mathbb{V}_\mathscr{D} = \operatorname{Hom}_{A_\mathscr{D}}(P_A(\omega), -)$ is fully faithful on standard modules and $A_\mathscr{D}(\mathfrak{m})$-mod is equivalent to the block $\mathscr{C}$, where $\mathfrak{m}$ is the unique maximal ideal of R. This fact is a consequence of the following:

Theorem (Theorem 4.4.50)**.** *Fix t a natural number. Let R be the localization of the polynomial ring $\mathbb{C}[X_1,\dots,X_t]$ at the maximal ideal $(X_1,\dots,X_t)$. For each $1\leq s\leq \operatorname{rank}_R\mathfrak{h}_R^*$ and $s\leq t$ there exists a projective Noetherian R-algebra $A_\mathscr{D}$ with $A_\mathscr{D}(X_1,\cdots,X_t)$*-mod *equivalent to a block of the BGG category $\mathscr{O}$ so that*

$$\operatorname{HNdim}_{\mathbb{V}_\mathscr{D}}\mathscr{F}(\Delta_A) = s-1.$$

Concerning **(Q5)**, we saw that this deformation of blocks of the BGG category $\mathscr{O}$, based on the work of Gabber and Joseph, provides split quasi-hereditary covers whose level of faithfulness can improve as much as the Cartan subalgebra allows it compared to the classical category $\mathscr{O}$. Schur algebras and q-Schur algebras, on the other hand, are more static and the quality of the integral covers they provide can be improved in at most one degree, in comparison, with the finite-dimensional case. So, in both cases, we benefit by going integrally. However, there are cases like the integral Auslander algebra of $R[X]/(X^n)$ with no benefit coming, from this perspective, from going integrally. All these three examples fit in our main problem.

The solution to **(Q9)** involves Ringel duality and once again a generalization of dominant dimension. As double centralizer properties on certain projective modules are related to covers, the natural approach is to dualize the concept of cover to obtain an abstract framework to double centralizer properties on arbitrary modules. We call this concept a cocover. More precisely, for a left A-module projective R-module Q, we say that (A,Q) is a cocover if the functor $\operatorname{Hom}_A(Q,-)$ is full and faithful on relative injective modules and $DQ\otimes_A Q$ is a projective R-module. This motivates to measure double centralizer properties using now relative coresolving subcategories and by how far the left adjoint of $\operatorname{Hom}_A(Q,-)$ is from being exact. For this, instead of using dominant dimension we use relative dominant dimension relative to a module (see Definition 2.3.5). This concept generalizes also the faithful dimension defined in [BS98]. In particular, it uses approximation theory. More precisely, we say that a left A-module projective R-module M has relative codominant dimension with respect to Q at least n if there exists an (A,R)-exact sequence $Q_n\to\cdots\to Q_1\to M\to 0$ which remains exact under $\operatorname{Hom}_A(Q,-)$ with all Q_i belonging to the additive closure of Q. The computation of relative codominant dimension of a module T relative to a module Q can also be reduced to computations of relative codominant dimension of a module relative to another module over finite-dimensional algebras over algebraically closed fields (see Theorem 5.3.5 and Lemma 5.3.3).

This invariant behaves like a dominant dimension in the sense that it controls the connection between two module categories. In particular, by taking Q to be a direct summand of a characteristic tilting module of a split quasi-hereditary algebra we obtain the following:

Theorem (Theorem 5.5.1)**.** *Let R be a commutative Noetherian ring. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by R_A the Ringel dual of A, that is $R_A = \operatorname{End}_A(T)^{op}$. Assume that $Q \in \operatorname{add} T$ is a partial tilting module of A and $Q - \operatorname{codomdim}_{(A,R)} T \geq n \geq 2$. Then, $(R_A, \operatorname{Hom}_A(T,Q))$ is an $(n-2)$-faithful split quasi-hereditary cover of $\operatorname{End}_A(Q)^{op}$.*

Furthermore, the relative dominant dimension of the regular module over a split quasi-hereditary algebra with respect to a partial tilting module Q (which coincides with the faithful dimension over finite-dimensional algebras) measures how far the partial tilting module Q is from being a characteristic tilting module (see Section 5.8).

Using deformation results and by truncating covers (see Theorem 3.4.1) we obtain the following construction using Schur algebras:

Theorem (Theorem 6.1.4 and Theorem 6.1.3)**.** *Let R be a commutative Noetherian ring. Denote by $R(S_R(n,d))$ the Ringel dual of the Schur algebra $S_R(n,d)$ (there are no restrictions on the natural numbers n and d). Let T be a characteristic tilting module of $S_R(n,d)$. Then, the following assertions hold.*

(i) $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})^{op}$ *is a cellular algebra.*

(ii) $V^{\otimes d} - \operatorname{domdim}_{(S_R(n,d),R)} T \geq \inf\{k \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\} \geq 1.$

(iii) Then, $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ *is a* $(V^{\otimes d} - \operatorname{domdim}_{(S_R(n,d),R)} T - 2)$*-faithful split quasi-hereditary cover of* $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})^{op}$.

The strategy behind the computation of (ii) is as follows: For higher levels ($i \geq 3$), the Hemmer-Nakano dimension does not drop by truncating the cover $(S_R(d,d), V^{\otimes d})$ (see Theorem 3.4.1). For lower levels, we use the relative Mueller's characterization for the relative dominant dimension relative to a module (Theorem 5.2.2) and Theorem 5.6.1. For (iii), in the case (-1), we need to go through the integers (see Corollary 5.5.6) and then go back to the fields of characteristic two which decreases the level of faithfulness to (-1) again. Now, using that the result holds for all the residue fields of R, it must also hold for R (see Proposition 3.3.6).

The existence of this split quasi-hereditary cover explains why using tilting theory was a successful approach to prove Schur–Weyl duality in [KSX01]. It also gives further insight why making use of the Ringel dual of the Schur algebra helps in the study of the decomposition numbers of the symmetric group [Erd94]. We recall that Ringel duals of Schur algebras are generalized Schur algebras in the sense of Donkin [Don93, 3.11]. Moreover, the Schur algebra $S_R(n,d)$ is Ringel self-dual (see [Don93]) if $n \geq d$ and the split quasi-hereditary cover $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ is equivalent to the cover $(S_R(n,d), V^{\otimes d})$ in case $n \geq d$. In such a case, $\operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d})$ is a projective relative injective and strongly faithful module over $R(S_R(n,d))$. So, the study of the cover $(S_R(n,d), V^{\otimes d})$ together with the approach that we consider in this thesis culminates in the study of the cover $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$.

The value in (iii) is optimal when R is a field. The inequality in (ii) is sharp, in general, since by taking $n = d$ we obtain equality. By fixing $n = 2$, we obtain that the Ringel dual of the Schur algebra provides a split quasi-hereditary cover of a Temperley-Lieb algebra (see Subsection 6.3). The same strategy can be used to obtain an analogue of Theorems 6.1.4 and 6.1.3 for q-Schur algebras.

We remark that a split quasi-hereditary algebra has infinite relative dominant dimension with respect to a partial tilting module Q if and only if Q is a characteristic tilting module (see Subsection 5.8).

As application of our machinery of relative dominant dimension and by making use of quality of covers we reprove a famous result of Soergel. More precisely, using these techniques on the deformation algebra $A_{\mathscr{D}}$, we reprove in Theorem 6.4.1 Ringel self-duality of the BGG category $\mathcal{O}$ without using the semi-regular bimodule (see [Soe97, Corollary 2.3]). The original proof of Soergel exhibits the functor giving Ringel self-duality using properties of a semi-infinite character but it does not make clear which structural properties of the category $\mathcal{O}$ are forcing the blocks of the category $\mathcal{O}$ to be Ringel self-dual. Later, Futorny, König and Mazorchuk in [FKM00, Proposition 4] reproved Ringel self-duality using the Enright completion functor and by studying a full subcategory of an integral block of the category $\mathcal{O}$ whose modules have dominant dimension at least two (see also [KM02]). Our proof illustrates Ringel self-duality of the blocks of the BGG category $\mathcal{O}$ as an instance of uniqueness of covers of deformations of subalgebras of coinvariant algebras.

Methods/Techniques

There are several reasons for us to choose to resolve Noetherian algebras with split quasi-hereditary algebras. For one, the endomorphism algebras of simple modules over cellular algebras over a field k are isomorphic to k. All Noetherian R-algebras discussed here that appear in a cover are constructed using some form of relative dominant dimension, including relative dominant dimension with respect to a (partial) tilting module. Therefore, they are all equipped with a base change property. Hence, these covers should remain covers under change of ground ring to a residue field. In particular, if A has finite global dimension we want that $A(\mathfrak{m})$ to have finite global dimension, as well. This requirement already excludes some choices (for example [Kön91]). At first sight, both the concepts of quasi-hereditary algebras and split quasi-hereditary algebras seem suitable choices to resolve non-cellular algebras. But, the fundamental difference comes from quasi-hereditary algebras being constructed by glueing semi-simple algebras inductively and split quasi-hereditary algebras are constructed by glueing matrix rings over the ground ring inductively. In view of change of ground rings, the second is more appropriate and easier to handle. Moreover, if A is quasi-hereditary algebra then changing the ground ring through localization, truncation to the residue field and finishing with an extension of scalars to its algebraic closure we obtain a split quasi-hereditary algebra $\overline{R(\mathfrak{m})}\otimes_R A$ is a split quasi-hereditary over an algebraically closed field. If the standard modules of $\overline{R(\mathfrak{m})}\otimes_R A$ are defined integrally, that is, there exists a collection of modules Δ so that $\overline{R(\mathfrak{m})}\otimes_R \Delta$ are standard modules of $\overline{R(\mathfrak{m})}\otimes_R A$ for every maximal ideal $\mathfrak{m}$ of R, then A must be a split quasi-hereditary algebra (see Subsection 1.5.5).

All the covers that we study here arise from relative dominant dimension with respect to a projective relative injective module and from relative dominant dimension with respect to a (partial) tilting module. Relative injective modules also called (A,R)-injective are the injective objects, which are projective over the ground ring R, with respect (A,R)-exact sequences. In turn, (A,R)-exact sequences were chosen to extend several concepts from finite-dimensional algebras to Noetherian algebras. These are the exact sequences of A-modules which split as a sequence of R-modules. Given the nature of the generalization of dominant dimension to the relative dominant dimension using (A,R)-exact sequences and projective (A,R)-injective modules the computations can be reduced to finite-dimensional algebras over algebraically closed fields. Further, in our cases, this is reduced to already known cases. On the other hand, relative dominant dimension with respect to a (partial) tilting module is in the majority of cases unknown. However, such a computation can also be reduced to finite-dimensional algebras over algebraically closed fields (see Section 5.3). Further, by using an application of Grothendieck's spectral sequence and the relative analogue of Mueller's characterization of relative dominant dimension with respect to a (partial) tilting module (see Theorem 5.2.5) we obtain the following:

Theorem (Theorem 5.5.1 for finite-dimensional algebras)**.** *Let K be a field. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary K-algebra with a characteristic tilting module T. Denote by R_A the Ringel dual of A. Assume that $Q \in \operatorname{add} T$ is a (partial) tilting module of A. Then, $DQ - \operatorname{domdim}_{(A,R)} DT = Q - \operatorname{codomdim}_{(A,R)} T \geq n \geq 2$ if and only if $(R_A, \operatorname{Hom}_A(T,Q))$ is an $(n-2)$-faithful split quasi-hereditary cover of $\operatorname{End}_A(Q)^{op}$.*

Since the Ringel dual of the Ringel dual of A is Morita equivalent to A all split quasi-hereditary covers of finite-dimensional algebras can be written in the form of the previous theorem. Therefore, the quality of faithful split quasi-hereditary covers of finite-dimensional algebras are controlled by the relative codominant dimension dimension of characteristic tilting modules with respect to a direct summand of characteristic tilting modules.

We then use this Theorem together with the following result about truncation of split quasi-hereditary covers to show that the dominant dimension of the characteristic tilting module of $S_K(n,d)$ with respect to $(K^n)^{\otimes d}$ (with $n < d$) is greater than or equal to the dominant dimension of the characteristic tilting module of $S_K(d,d)$:

Theorem (Theorem 3.4.1)**.** *Let A be a split quasi-hereditary Noetherian R-algebra. Assume that (A,P) is an i-faithful split quasi-hereditary cover of $\operatorname{End}_A(P)^{op}$ for some integer $i \geq 0$. Let J be a split heredity ideal of A. Then, $(A/J, P/JP)$ is an i-faithful split quasi-hereditary cover of $\operatorname{End}_{A/J}(P/JP)^{op}$.*

This result deals with all cases except the characteristic two case which must be treated separately by going integrally. Theorem 3.4.1 gives us that a 0-faithful split quasi-hereditary cover is equipped with double centralizer properties involving each factor of the split heredity chain. However, not all types of truncations have this behaviour. In fact, it is enough to see the influence of the spectrum of the ground ring on the quality of a cover.

Corollary (Corollary 3.3.10 for $\mathscr{F}(\tilde{\Delta})$)**.** *Let R be a commutative Noetherian regular local ring. Let (A,P) be an i-faithful split quasi-hereditary cover of B for some integer $i \geq 0$. Then, $(R/\mathfrak{p} \otimes_R A, R/\mathfrak{p} \otimes_R P)$ is an $(i - \operatorname{ht}(\mathfrak{p}))$-faithful split quasi-hereditary cover of $R/\mathfrak{p} \otimes_R B$ for every prime ideal $\mathfrak{p}$ of R with $\operatorname{ht}(\mathfrak{p}) \leq i+1$.*

Knowing the values of dominant dimensions (with respect to a projective and injective or more generally with respect to a (partial) tilting module), the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ can be determined by combining such values with Corollary 3.3.10 and with the following result:

Theorem (Theorem 3.3.13 for $\mathscr{F}(\tilde{\Delta})$)**.** *Let R be a local commutative regular Noetherian ring with quotient field K. Suppose that (A,P) is a 0-faithful split quasi-hereditary cover of B. Let $i \geq 0$. Assume the following conditions:*

(i) *$(K \otimes_R A, K \otimes_R P)$ is an $i+1$-faithful split quasi-hereditary cover of $K \otimes_R B$;*

(ii) *For each prime ideal $\mathfrak{p}$ of height one, $(R/\mathfrak{p} \otimes_R A, R/\mathfrak{p} \otimes_R P)$ is an i-faithful split quasi-hereditary cover of $R/\mathfrak{p} \otimes_R B$.*

Then, (A,P) is an $i+1$-faithful split quasi-hereditary cover of B.

This procedure can also be applied to compute the Hemmer-Nakano dimension of other resolving subcategories, as well. Having described the general procedure to determine the quality of covers using relative dominant dimensions and deformation techniques, we shall now illustrate the benefits of going integrally. For the construction of the split quasi-hereditary cover of the cellular algebra $\operatorname{End}_{S_K(n,d)}(V^{\otimes d})^{op}$, by our earlier discussions, we see that the characteristic two case is dealt by passing through the integral case. Aside from the characteristic two case, these techniques also help us to understand more about the characteristic three case. In fact, the integral Schur algebra $S_{\mathbb{Z}[\frac{1}{2}]}(n,d)$ away from two, with $n \geq d$, is the unique split quasi-hereditary cover of the integral group algebra of the symmetric group away from two with standard modules being mapped to Specht modules by a Schur functor (see Subsection 4.1.3). Recall that uniqueness cannot be drawn directly over fields of characteristic three or two because the in these cases the subcategories $\mathscr{F}(\Delta)$ and $\mathscr{F}(F\Delta)$ are not equivalent with F denoting the Schur functor. For the BGG category $\mathscr{O}$, there are major benefits of going integrally. By studying some deformations of the blocks of the BGG category $\mathscr{O}$ of a semi-simple Lie algebra whose resolving subcategories have larger values of Hemmer-Nakano dimension, we obtain that each block of the BGG category $\mathscr{O}$ is Ringel self-dual. This application captures quite well how these techniques of this thesis are combined to draw more results and explain situations which were inaccessible before without these techniques. We will just briefly describe the idea of the proof of this result. For a non semi-simple block algebra of the BGG category $\mathscr{O}$ of a complex semi-simple Lie algebra $\mathfrak{g}$, say A, whose simple modules are parametrized by a set of weights $\mathscr{C}$, the dominant dimension of the characteristic tilting module of A is just one. So, the quality of the cover $(A, P(\overline{\omega}))$ is actually negative. Assume that $\mathfrak{g}$ is not $\mathfrak{sl}_2$ whose situation is easier to check directly. So, we can pass over to a deformation $(A_{\mathscr{D}}, P(\omega))$ so that $R(\mathfrak{m}) \otimes_R \mathscr{D} = \mathscr{C}$ over a suitable ring R (with unique maximal ideal $\mathfrak{m}$) making the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ with respect to $(A_{\mathscr{D}}, P(\omega))$ be at least one. This step already requires Theorem 4.4.50 and all the machinery used to prove that result including results involving relative dominant dimension. Then, to obtain a split quasi-hereditary cover involving the Ringel dual of $A_{\mathscr{D}}$ we pass over to finite-dimensional algebras over quotient fields of factors of R by prime ideals $(Q(R/\mathfrak{p}) \otimes_R A_{\mathscr{D}})$. These algebras have larger values of dominant dimension forcing the existence of the desired 0-faithful split quasi-hereditary cover involving the Ringel dual of $A_{\mathscr{D}}$. Now observing that standard modules and costandard modules have the same image by the Schur functor (in this context known as Soergel's combinatorial functor) the results of Rouquier for uniqueness of covers yields that these two previous covers are equivalent. Now, the argument ends by returning to the block algebra we started with through the functor $R(\mathfrak{m}) \otimes_R -$, where $\mathfrak{m}$ is the unique maximal ideal of R.

The same reasoning can be applied to obtain Ringel self-duality of Schur algebras $S_K(d,d)$ over fields of characteristic different from two.

Outline of the thesis

Chapter 1 collects concepts and technical results required to understand the concept of a split quasi-hereditary cover of a cellular algebra over any commutative Noetherian ring. Section 1.1 focus on the ground ring of a Noetherian algebra and change of ground rings techniques. In Section 1.2, we collect results on relative homological algebra following closely the work of [Hat63] and [Hoc56]. In Section 1.3, we gather results on spectral sequences to be used later on. In section 1.4, we discuss the concept of a cover. In Sections 1.5 and 1.6 we collect results and give detailed expositions about split quasi-hereditary algebras and cellular algebras over commutative Noetherian rings. In doing so, we aim to strengthen our knowledge in integral representation theory, by providing more results from representation theory and clarifying, sometimes with different arguments, the existing ones in integral representation theory. Moreover, we establish that cellular algebras with finite global dimension are split quasi-hereditary over commutative Noetherian rings. For the material on split quasi-hereditary algebras, we follow closely the work of [Rou08]. We deviate from his work to establish, for example, that split quasi-hereditary algebras are locally semi-perfect. We clarify that split quasi-hereditary algebras are completely determined by $\mathscr{F}(\tilde{\Delta})$, as in the classical case. Here, the partial tilting modules are not unique (much less indecomposable if the ground ring is not connected) but the Ringel dual can be defined and we give details on its construction in this integral setup. In Section 1.7, we conclude the chapter by collecting some results that justify the usefulness of Schur functors.

As we mentioned, in the latter sections of Chapter 1, there are new results for integral representation theory generalized from the finite-dimensional realm. However, in principle, the reader comfortable with the concepts of split quasi-hereditary algebras, cellular algebras and covers may go directly to Chapter 2 using Chapter 1 as a reference for the subsequent chapters of this thesis.

Chapter 2 introduces both the concepts of relative dominant dimension over (projective) Noetherian algebras and of relative dominant dimension with respect to a module. The first case coincides with relative dominant dimension with respect to a projective relative injective and relative strongly faithful module. The latter concept is introduced here to replace the role of faithful modules in classical dominant dimension theory. For this homological invariant, we establish a relative version of Morita-Tachikawa correspondence and relative Mueller's characterization of relative dominant dimension over a (projective) Noetherian algebras in terms of homology over the endomorphism algebra of a projective relative injective module. The key result for applications is the reduction of the computation of relative dominant dimension over (projective) Noetherian algebras to computations of dominant dimension over finite-dimensional algebras over algebraically closed fields. Several concepts like Morita and gendo-symmetric algebras are brought to the Noetherian realm.

Chapter 3 extends the concept of faithful split quasi-hereditary cover to $\mathscr{A}$-covers by replacing the resolving subcategory $\mathscr{F}(\tilde{\Delta})$ with a resolving subcategory $\mathscr{A}$ of A-mod. To use change of ground rings on these covers, we restrict ourselves to $\mathscr{A}$-covers, where $\mathscr{A}$ is what we call a well behaved resolving subcategory of A-mod. This allows us to study $\mathscr{F}(\tilde{\Delta})$-covers and A-proj-covers, simultaneously. Further, we see how to use change of ground rings on covers to compute the quality of a cover. Such quality is made precise with the study of Hemmer-Nakano dimension of a resolving subcategory (with respect to the Schur functor associated with the cover). Here, we discuss how relative dominant dimension can be used to compute the Hemmer-Nakano dimension over certain covers. Hence, this chapter establishes the framework and technical details to measure quality of covers. We discuss the problem of existence of faithful covers, in particular, giving an example of a group algebra without a faithful cover and the problem of uniqueness of faithful covers clarifying some imprecisions present in the literature.

Chapter 4 contains the study of Schur algebras, q-Schur algebras with parameters $n \geq d$ over any commutative ring and their connection with group algebras of the symmetric group and Iwahori-Hecke algebras of the symmetric group, respectively. We also study deformations of blocks of the BGG category $\mathscr{O}$ of a semisimple Lie algebra following closely the work of [GJ81]. We see that the connection between these deformations of the BGG category $\mathscr{O}$ and deformations of the module category of the coinvariant algebra are stronger than the connection between the BGG category $\mathscr{O}$ and the coinvariant algebra. The last part of chapter 4 contains several examples to illustrate that some assumptions made along the previous chapters cannot be weakened.

Chapter 5 concerns the abstract framework to understanding the study of Schur algebras $S_R(n,d)$ with $n < d$ and the double centralizer property on $V^{\otimes d}$ using cover theory. For this, we study relative codominant dimension with respect to a module. This homological invariant admits an analogue of Mueller's characterization of dominant dimension and it can also be reduced to computations over finite-dimensional algebras over algebraically closed fields. Furthermore, this invariant with respect to partial tilting modules of a quasi-hereditary algebra is deeply connected with quasi-hereditary covers involving the Ringel dual. In this chapter, we obtain lower bounds of relative dominant dimension with respect to a partial tilting module which is the image of a projective and injective over a split quasi-hereditary under a Schur functor.

In Chapter 6, we obtain for the cellular algebra $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})$ (without restrictions on n and d) a faithful split quasi-hereditary cover formed by the Ringel dual of the Schur algebra $S_R(n,d)$ and a projective (not necessarily relative injective) module. This chapter is devoted to understanding this connection. In particular, the quality of this connection does not drop by decreasing the parameter n. We finish using the technology of Chapter 5 to reprove the Ringel self-duality of the BGG category $\mathcal{O}$ of a complex semisimple Lie algebra using only integral representation theory.

List of Symbols

$\operatorname{Add} M$	The full subcategory of the module category over A whose modules are direct summands of direct sums of M
$\operatorname{add} M$	The full subcategory of the module category over A whose modules are direct summands of finite direct sums of M
Set	The category of Sets
$\operatorname{char} R$	The characteristic of a commutative ring R
$\mathbb{1}$	The identicator function
$\operatorname{coht} I$	The coheight of an ideal I
$\operatorname{coker} \phi$	The cokernel of a homomorphism ϕ
$\dim R$	The Krull dimension of a commutative ring R
$\dim_K M$	The vector space dimension of M over a field K
$\operatorname{domdim}_{(A,R)} M$	The relative dominant dimension of M over an R-algebra A
$\operatorname{domdim}(A,R)$	The relative dominant dimension of the regular module of the R-algebra A
$\operatorname{End}_A(M)$	The endomorphism algebra of an A-module M with multiplication given by composition of maps
$\operatorname{Ext}_A^n(M,N)$	The Ext group $\mathrm{R}^n \operatorname{Hom}_A(M,-)(N)$
$\operatorname{Ext}_{(A,R)}^n(M,N)$	The relative Ext group $H^n(\operatorname{Hom}_R(P_M,N))$, where P_M is an (A,R)-projective resolution for M
$\operatorname{GL}_n(K)$	The general linear group of degree n over a field K
$\operatorname{gldim} A$	The global dimension of A
$\operatorname{gldim}_f(A,R)$	The relative global dimension of a Noetherian R-algebra A
$\operatorname{ht} I$	The height of an ideal I
$\operatorname{HNdim}_F \mathscr{A}$	The Hemmer–Nakano dimension of $\mathscr{A}$ with respect to the functor F
$\operatorname{Hom}_A(M,N)$	The set of all A-homomorphisms from M to N
$\operatorname{id} M$	The identity map on M

$\operatorname{im}\phi$	The image of a homomorphism ϕ
$\operatorname{idim}_A N$	The injective dimension of N over A
$\operatorname{idim}_{(A,R)} N$	The relative injective dimension of N over an R-algebra A
$\ker\phi$	The kernel of a homomorphism ϕ
$\mathrm{L}^n F$	The n^{th}-left derived functor of F
$\mathbb{C}$	The field of complex numbers
$\mathbb{F}_p$	The finite field with order p
$\mathbb{N}$	The set of natural numbers $\{1,2,3,\cdots\}$
$\mathbb{Z}$	The ring of integers
$\mathscr{F}(\Delta)$	The full subcategory of A-mod whose objects have finite filtration by objects in the set Δ
$\mathscr{M}(A)$	The set of isomorphism classes of projective R-split A-modules.
$\operatorname{MaxSpec} R$	The set of maximal ideals of a commutative ring R
$\operatorname{Supp}(M)$	The set of prime ideals $\mathfrak{p}$ of R satisfying $M_{\mathfrak{p}} \neq 0$
$\overline{K}$	The algebraic closure of a field K
$\operatorname{pdim}_A M$	The projective dimension of M (over A)
$\phi(\mathfrak{p})$	The homomorphism $R(\mathfrak{p}) \otimes_R \phi$
$\phi_{\mathfrak{p}}$	The localization of a homomorphism ϕ at the prime ideal $\mathfrak{p}$ of R
$\mathrm{R}^n F$	The n^{th}-right derived functor of F
$\operatorname{rad} M$	The Jacobson radical of M
$\operatorname{rank}_R M$	The rank of a free R-module
Mod-A	The category of right A-modules whose objects are all right A-modules and whose morphisms are all right A-homomorphisms
mod-A	The full subcategory of Mod-A whose objects are all finitely generated A-modules
$\operatorname{Spec} R$	The set of prime ideals of a commutative ring R
$\tilde{\Delta}$	The set of modules $\Omega \otimes_R U$, with $\Omega \in \Delta$, $U \in R$-proj
$\operatorname{top} M$	The top of a module M
$\operatorname{Tor}_n^A(M,N)$	The Tor group $\mathrm{L}^n(M \otimes_A -)(N)$
$\operatorname{Tor}_n^{(A,R)}(M,N)$	The relative Tor group $H_n(P_M \otimes_A N)$, where P_M is an (A,R)-projective resolution for M
$\widehat{M}$	The completion of M (with respect to an ideal of a commutative ring R)

A-Mod	The category of left A-modules whose objects are all left A-modules and whose morphisms are all left A-homomorphisms
A-mod	The full subcategory of A-Mod whose objects are all finitely generated A-modules
A-Proj	The full subcategory of A-Mod whose objects are all projective A-modules
A-proj	The full subcategory of A-mod whose objects are all projective A-modules
A^{op}	The opposite algebra of A
D or D_R	The standard duality functor $\mathrm{Hom}_R(-,R)$ with respect to a commutative ring R
$E_a^{i,j} \Rightarrow H^{i+j}$	The spectral sequence E_a converges to H^*
$F \dashv G$	The functor F is left adjoint of G
$H_{R,q}(d)$	The Iwahori-Hecke algebra
$M(\mathfrak{p})$	The module $R(\mathfrak{p}) \otimes_R M$
$M \hookrightarrow N$	injection
$M \twoheadrightarrow N$	surjection
$M_{\mathfrak{p}}$	The localization of the module M at the prime ideal $\mathfrak{p}$ of R
M_A	A right A-module M
$Pic(R)$	The Picard group of a commutative ring R
$Q-\mathrm{codomdim}_{(A,R)} X$	The relative codominant dimension of X with respect to Q
$R(\mathfrak{p})$	The residue field associated with the ideal $\mathfrak{p}$ of R
$R_{\mathfrak{p}}$	The localization of R at the prime ideal $\mathfrak{p}$
RG	The group algebra of a group G over a commutative ring R
S_d	The symmetric group on d letters
$S_R(n,d)$	The Schur algebra $\mathrm{End}_{RS_d}\left((R^n)^{\otimes d}\right)$
$S_{R,q}(n,d)$	The q-Schur algebra
$T-\mathrm{domdim}_{(A,R)} X$	The relative dominant dimension of X with respect to T
$U(R)$	The set of invertible elements of a commutative ring R
$V_R^{\otimes d}$ or $V^{\otimes d}$	The d-fold tensor product $(R^n)^{\otimes d}$ of the free module R^n over a commutative ring R
$Z(A)$	The center of A
${}^{\perp}Q$	The full subcategory of A-mod whose objects X satisfy $\mathrm{Ext}_A^{i>0}(X,Q)=0$
${}_AM$	A left A-module M

Further notation

We write $\otimes_R$ to denote the tensor product over R. We will write just $\otimes$ instead of $\otimes_R$ or $\otimes_A$ when no confusion will arise. We write $M \simeq N$ whenever M and N are isomorphic modules. For two categories $\mathscr{C}_1$ and $\mathscr{C}_2$, we write $\mathscr{C}_1 \simeq \mathscr{C}_2$ whenever the two categories $\mathscr{C}_1$ and $\mathscr{C}_2$ are equivalent. For two rings (resp. two R-algebras) A and B, we write $A \simeq B$ if they are isomorphic as rings (resp. as R-algebras). For two R-algebras A and B, we write $A \overset{\mathsf{Mor}}{\sim} B$ if A and B are Morita equivalent, that is, if the module categories A-Mod and B-Mod are equivalent. We write $F \simeq G$ in case the functors F and G are isomorphic, that is, if there exists a natural isomorphism between F and G. Unless stated otherwise, we read arrows in a path algebra like composition of morphisms, that is, from the right to the left.

Chapter 1

Background

We assume the reader to be familiar with basic concepts in homological algebra such as projective resolutions, Ext and Tor functors, homological dimensions (see for example [Rot09]), and elementary categorical concepts in module theory such as monomorphisms, direct sums, kernels, pullbacks of modules and their duals (see for example [Mac71]) and with representation theory of finite-dimensional algebras (see for example [ASS06] and [ASS06]).

1.1 Basic results on algebras over commutative Noetherian rings

Most of the results contained in this section involve basic notions and facts for Noetherian commutative rings and algebras over commutative Noetherian rings, available in the literature (see for example [Lan02] and [Rot09]). Nevertheless, we will briefly review these subjects either by providing quick proofs or by just pointing out references. This includes the concepts of generators/cogenerators, localization, completion functors, Krull dimension, regular rings, and general techniques for change of ground ring. A reader familiar with these terms and notions can skip this section.

In the following, unless stated otherwise, R is a commutative Noetherian ring. We assume that all rings considered here have an identity. An R-algebra A is known as **Noetherian R-algebra** if R is a Noetherian commutative ring and A is finitely generated as R-module. We assume throughout this thesis that all rings have identity. An R-algebra A is called **projective R-algebra** if A is projective as R-module. An R-algebra A is called **projective Noetherian R-algebra** if it is both a Noetherian R-algebra and a projective R-algebra.

An important fact for Noetherian rings is that we can still decompose a module into indecomposable modules, however this decomposition may not be unique.

Lemma 1.1.1. *Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. Then, any $M \in A$-*mod *is a finite direct sum of indecomposable A-modules.*

Proof. Let $M \in A$-mod. Then, M is a Noetherian module. We proceed by contradiction. Assume that M cannot be written as a finite direct sum of indecomposable A-modules. In particular, M is not indecomposable. So $M \simeq M_0 \bigoplus K$ and K cannot be written as a finite sum of indecomposable modules. If both were a finite sum of indecomposable modules, then M could be written as a finite direct sum of indecomposable modules. Hence applying the same argument for K n times, we obtain

$$M \simeq M_0 \bigoplus \cdots \bigoplus M_n \bigoplus K^{(n)}, \tag{1.1.0.1}$$

where $K^{(n)}$ cannot be written as a direct sum of indecomposable modules. Consider the chain

$$0 \subsetneq M_0 \subsetneq M_0 \bigoplus M_1 \subsetneq M_0 \bigoplus M_1 \bigoplus M_2 \subsetneq \cdots \subsetneq M_0 \bigoplus \cdots \bigoplus M_n \subsetneq \cdots. \tag{1.1.0.2}$$

This chain does not stabilize. This contradicts the fact that M is a Noetherian module. $\square$

The following lemma gives a characterization of projective modules in terms of its dual.

Lemma 1.1.2 (Dual basis Lemma)**.** *Let A be an R-algebra. An A-module P is projective if and only if there exists a generator set $\{a_i\colon i \in I\}$ for P and $\{f_i\colon i \in I\} \subset \operatorname{Hom}_A(P,A)$ such that for any $p \in P$, $f_i(p) = 0$ for almost all $i \in I$ and $p = \sum_{i\in I} f_i(p)a_i$.*

Proof. See [Lam99, §2B 2.9 Dual basis Lemma]. $\square$

One application of this lemma is in the computation of all projective generators over commutative rings (see Proposition 1.4.22).

For Noetherian rings, the category of all injective modules (over a Noetherian ring) has all colimits. The existence of cokernels is immediate from the definition of injective module. The existence of arbitrary coproducts is due to the following result.

Theorem 1.1.3. *Let A be a Noetherian ring. Then, any direct sum of injective modules is injective.*

Proof. See [Rot09, Chapter 3, Proposition 3.31]. $\square$

In fact, this property, characterizes Noetherian rings (see [Rot09, Chapter 3, Theorem 3.39]).

Lemma 1.1.4. *Let A be an algebra over $\mathbb{Z}$. Any A-module M admits an injective hull.*

Proof. See [Rot09, Theorem 3.45]. $\square$

A common philosophy in this work is to deduce properties of algebras going through algebras over commutative Noetherian rings. The justification behind Theorem 1.1.4 follows in line with this idea. In fact, every $\mathbb{Z}$-module can be embedded into some direct sum (possibly infinite) of copies of $\mathbb{Q}/\mathbb{Z}$ which is $\mathbb{Z}$-injective. Denote this injective module by I. Now, every module M over an arbitrary $\mathbb{Z}$-algebra A can be embedded in $\operatorname{Hom}_{\mathbb{Z}}(A,I)$. It turns out that $\operatorname{Hom}_{\mathbb{Z}}(A,I)$ is injective over A. Finally, constructing the maximal essential extension of M in $\operatorname{Hom}_{\mathbb{Z}}(A,I)$ gives the injective hull of M.

Lemma 1.1.5. *Let A be a Noetherian R-algebra Let $M,N \in A$-*mod*. Then,* $\operatorname{Hom}_A(M,N)$ *is finitely generated as R-module.*

Proof. It follows that M and N are finitely generated as R-modules. There is a surjective map $R^n \twoheadrightarrow M \to 0$ for some $n \in \mathbb{N}$. Applying $\operatorname{Hom}_R(-,N)$ yields $0 \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(R^n,N) \simeq N^n$ exact. As R is Noetherian and N^n is finitely generated, it follows that N^n is Noetherian. Since $\operatorname{Hom}_A(M,N) \subset \operatorname{Hom}_R(M,N) \subset N^n$, it follows that $\operatorname{Hom}_A(M,N)$ is finitely generated. $\square$

1.1.1 Generators and Cogenerators

Generators in a category play a very important role as they distinguish morphisms. A generator in a module category of an algebra encodes a lot of information about the module category of an algebra.

Definition 1.1.6. Let $\mathscr{A}$ be an abelian category. An object P is a **generator** of $\mathscr{A}$ if the functor $\operatorname{Hom}_{\mathscr{A}}(P,-)\colon \mathscr{A} \to \mathbf{Set}$ is faithful.

Theorem 1.1.7. *Let A be an R-algebra and $P \in A$-Mod. The following assertions are equivalent.*

(a) P is a generator of A-Mod.

(b) For any module $M \in A$-Mod, there exists an epimorphism $\bigoplus_I P \to M$ for some set I (possibly infinite).

(c) $A \in \operatorname{Add} P$.

Proof. See [Lam99, Theorem 18.8]. □

Using the analogous argument of [Lam99, Theorem 18.8] to A-mod we obtain

Theorem 1.1.8. *Let A be an R-algebra and $P \in A$-mod. The following assertions are equivalent.*

(a) P is a generator of A-mod.

(b) For any $M \in A$-mod, there exists an epimorphism $P^t \to M$ for some $t > 0$.

(c) $A \in \operatorname{add} P$.

Proof. Assume that (c) holds. Let $M \in A$-mod. By definition, there exists a surjective A-homomorphism $A^s \to M$ for some $s > 0$. By assumption, there exists a surjective A-homomorphism $P^t \to A$. Hence, the composition $P^{ts} \to M$ is surjective. So (b) holds.

Assume that (b) holds. Let $M,N \in A$-mod. By assumption, there exists a surjective A-homomorphism $g \in \operatorname{Hom}_A(P^t,M)$. Let k_j and π_j be the canonical homomorphisms arising from the direct sum P^t such that $\mathrm{id}_{P^t} = \sum_j k_j \circ \pi_j$. Let $0 \neq f \in \operatorname{Hom}_A(M,N)$. Then, $\sum_j f \circ g \circ k_j \circ \pi_j \neq 0$. In particular, there exists j such that $f \circ g \circ k_j \neq 0$. Moreover, $\operatorname{Hom}_A(P,N)$ is an abelian group, therefore this shows that P is a generator of A-mod.

Assume now that (a) holds. Let $H \in \operatorname{Hom}_A(\oplus_{g\in \operatorname{Hom}_A(P,A)}P, A)$ such that $H(p_g) = g(p)$, $p \in P$. Let (X,f) be the cokernel of H. In particular, $X \in A$-mod. If $f \neq 0$, then there exists $g \in \operatorname{Hom}_A(P,A)$ such that $fg \neq 0$ by the faithfulness of $\operatorname{Hom}_A(P,-)$. This implies that there exists $p \in P$ such that $fH(p_g) = fg(p) \neq 0$. This contradicts our assumption that f is the cokernel of H. So, $f = 0$, and therefore H is surjective. By considering the preimage of the identity element in A we can choose a finite set I such that the restriction of H to $\oplus_I P$ is surjective. It follows that $A \in \operatorname{add} P$. □

Proposition 1.1.9. *Let A be an R-algebra. If $P \in A$-mod is a generator of A-mod, then it is a generator of A-Mod.*

Proof. If $P \in A$-mod is a generator of A-mod, then we see by Theorem 1.1.8 that $A \in \operatorname{add} P$. By Theorem 1.1.7, P is a generator of A-Mod. □

Being a generator is a categorical property.

Proposition 1.1.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and X be a generator of $\mathscr{A}$. If $F\colon \mathscr{A} \to \mathscr{B}$ is an equivalence, then FX is a generator of $\mathscr{B}$.*

Proof. Applying F gives a bijection $\operatorname{Hom}_{\mathscr{A}}(X,-) \xrightarrow{\simeq} \operatorname{Hom}_{\mathscr{B}}(FX,F-)$. F is essentially surjective, and consequently $\operatorname{Hom}_{\mathscr{B}}(FX,-)$ is faithful. □

Dually, we can define cogenerators.

Definition 1.1.11. Let $\mathscr{A}$ be an abelian category. An object Q is a **cogenerator** of $\mathscr{A}$ if the functor $\mathrm{Hom}_{\mathscr{A}}(-,Q)\colon \mathscr{A} \to \mathbf{Set}$ is faithful.

Theorem 1.1.12. *Let A be an R-algebra and $Q \in A$-mod. Q is a cogenerator of A-Mod if and only if for any $N \in A$-Mod and $0 \neq x \in N$, there exists $g \in \mathrm{Hom}_A(N,Q)$ such that $g(x) \neq 0$ if and only if any $N \in A$-Mod can be imbedded into a direct product (possibly infinite) of copies of Q.*

Proof. See [Lam99, Proposition 19.6] □

Theorem 1.1.13. *Let $\{S_i\colon i \in I\}$ a complete set of simple A-modules. Then, $U = \bigoplus_{i\in I} E(S_i)$ is a cogenerator for R-Mod, where $E(S_i)$ denotes the injective hull of S_i, $i \in I$.*
If A is a Noetherian ring, then U is an injective cogenerator.

Proof. See [Lam99, Theorem 19.10]. Each $E(S_i)$ is an injective module, and therefore $\bigoplus_{i\in I} E(S_i)$ is injective if A is Noetherian. □

Projective generators play an important role in Morita theory (see Section 1.4.4). Injective cogenerators are very important to duality theory. More precisely, given an injective cogenerator Q the functor $\mathrm{Hom}_A(-,Q)$ preserves and reflects exact sequences (see [Lam99, Proposition 4.8] replacing $\mathbb{Q}/\mathbb{Z}$ by any injective cogenerator). In 1.2.4 we will see an application of the existence of injective cogenerators.

1.1.2 Localization, Completion, Residue fields and change of rings

An important tool in commutative algebra is localization of a ring. This technique allows us to reduce problems in module theory of an algebra over a commutative ring to problems in module theories of algebras over commutative local rings.

Definition 1.1.14. Let R be a commutative ring with identity and P a prime ideal of R. Fix $S = R\backslash P$. We define the equivalence relation on $R \times S$, $(a,s) \sim (a',s')\colon \Leftrightarrow \exists u \in S\colon u(as' - a's) = 0$. We denote the equivalence class of (a,s) by $\frac{a}{s}$. Then, the localization of R at P is the set of all equivalence classes

$$R_P = S^{-1}R = \{\frac{a}{s}\colon a \in R, s \in S\}.$$

R_P is a ring with operations $\frac{a}{s} + \frac{a'}{s'} = \frac{as'+a's}{ss'}$, $\frac{a}{s}\frac{a'}{s'} = \frac{aa'}{ss'}$.

The ideals in a localization are characterized in the following way.

Proposition 1.1.15. *Let S be a multiplicative closed subset of a ring R. We have the ring homomorphism $\phi\colon R \to S^{-1}R$, given by $\phi(r) = \frac{r}{1}$.*

1. *For any ideal I of R, $I^e := \{\frac{a}{s}\colon a \in I,\ s \in S\}$ is the ideal of $S^{-1}R$ generated by the image $\phi(I)$.*
2. *For any ideal J of $S^{-1}R$ we have $J^c := \phi^{-1}(J) = \{a \in R\colon \frac{a}{1} \in J\}$ and $(J^c)^e = J$.*
3. *ϕ induces a one to one correspondence $\{\text{prime ideals in } S^{-1}R\} \longleftrightarrow \{\text{prime ideals } I \text{ in } R \text{ with } I \cap S = \emptyset\}$.*

Proof. See [Coh77, Proposition 2, p.396]. □

Corollary 1.1.16. *[Coh77, p.397] Let R be a commutative ring with identity and $\mathfrak{P}$ a prime ideal of R. There is a one to one correspondence*

$$\{\text{prime ideals in } R_{\mathfrak{P}}\} \longleftrightarrow \{\text{prime ideals } I \text{ in } R \text{ with } I \subset \mathfrak{P}\}.$$

Definition 1.1.17. Let S be a multiplicative set of a commutative ring R, and let M be an R-module. We define the equivalence relation on $M \times S$

$$(m,s) \sim (m',s') \Leftrightarrow \exists u \in S\colon\ u(s'm - sm') = 0.$$

We define the localization of M at S, $S^{-1}M = \{\frac{m}{s} = (m,s)\colon m \in M, s \in S\}$ with the usual operations. The localization of M at a prime ideal $\mathfrak{P}$ in R is $S^{-1}M$ with $S = R \backslash \mathfrak{P}$.

Proposition 1.1.18. *Let R be a commutative ring. Let S be a multiplicative set in a commutative ring R and let M be an R-module. Then, $S^{-1}M \simeq S^{-1}R \otimes_R M$ as $S^{-1}R$-modules.*

Proof. Consider the $S^{-1}R$-homomorphism $\phi\colon S^{-1}M \to S^{-1}R \otimes_R M$, given by $\phi(\dfrac{m}{s}) = \frac{1}{s} \otimes m, \dfrac{m}{s} \in S^{-1}M$, and the $S^{-1}R$-homomorphism $\psi\colon S^{-1}R \otimes_R M \to S^{-1}M$, given by $\psi(\dfrac{r}{s} \otimes m) = \dfrac{rm}{s}, \dfrac{r}{s} \otimes m \in S^{-1}R \otimes_R M$. The homomorphisms ψ and ϕ are inverse to each other. $\square$

Proposition 1.1.19. *Let S be a multiplicative set of a commutative ring R. Let A be an R-algebra. We have the localization functor $S^{-1}R \otimes_R -\colon A\text{-Mod} \to S^{-1}R \otimes_R A\text{-Mod}$. For a prime ideal $\mathfrak{P}$ of R, we denote the image of an homomorphism ϕ by this functor $\phi_{\mathfrak{P}}$.*

The following result says that localization is exact (see for example [Coh77, Proposition 3, p.397]).

Proposition 1.1.20. *Let S be a multiplicative set of a commutative ring R. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Then, the localized sequence $0 \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0$ is exact.*

Proof. Since the localization can be written as the $S^{-1}R \otimes_R -$, it is enough to see that localization preserves monomorphisms. Let $\Phi \in \mathrm{Hom}_R(M_1, M_2)$ be a monomorphism. Let $\dfrac{m}{s} \in \ker S^{-1}\Phi$. Then, there exists $u \in S$ such that $u(1\Phi(m) - s \cdot 0) = 0$ which implies $\Phi(um) = 0$. So, $\dfrac{m}{s} = \dfrac{0}{s} = 0$. Every element of $S^{-1}R$ can be written in the form $\dfrac{m}{s}$. Therefore, $S^{-1}\Phi$ is a monomorphism. $\square$

As a corollary, localization at a multiplicative set preserves kernels, quotients, images and naturally direct sums.

Lemma 1.1.21. *Let S be a multiplicative set of a commutative ring R. Every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M.*

Proof. Let K be a submodule of $S^{-1}M$. Consider $x \in K$. Hence, we can write $x = \dfrac{m}{s} = \frac{1}{s}\frac{m}{1}$ for some $m \in M$ and $s \in S$. Thus, K is generated by some set $\{\frac{m_i}{1}\colon i \in I, m_i \in M\}$. Let N be the submodule of M generated by $\{m_i\colon i \in I\}$. Therefore, $S^{-1}N$ has the same generator set as K. So, $S^{-1}N = K$. $\square$

Proposition 1.1.22. *Let R be a commutative Noetherian ring. Let S be a multiplicative set of R. Then, $S^{-1}R$ is a commutative Noetherian ring.*

Proof. Every chain of modules of $S^{-1}R$ induces a chain of submodules of R. Since R is Noetherian, the chain of modules of $S^{-1}R$ must stabilize. $\square$

As a consequence of Lemma 1.1.21, we can deduce that localization is a dense functor (see for example Corollary 4.79 of [Rot09]).

Proposition 1.1.23. *Let S be a multiplicative set of a commutative Noetherian ring R. Let A be a Noetherian R-algebra. If $M \in S^{-1}R \otimes_R A\text{-mod}$, then there exists $N \in A\text{-mod}$ such that $S^{-1}R \otimes_R N \simeq M$ as $S^{-1}A$-modules.*

Proof. We can consider a free $S^{-1}R\otimes_R A$-free presentation $0\to X\to S^{-1}R\otimes_R A^t\to M\to 0$, where X is finitely generated. By Lemma 1.1.21, $X\simeq S^{-1}R\otimes_R X_0$, $X_0\in A$-mod. Let N be the cokernel of $X_0\to A^t$. In particular, $N\in A$-mod. Since $S^{-1}R\otimes_R -$ is exact, $S^{-1}R\otimes_R N\simeq M$. □

For our purposes the following reformulation of [Coh77, Proposition 4, p.398] is more convenient. See also Proposition 4.90 of [Rot09].

Proposition 1.1.24. *Let R be a commutative ring. Let M be an R-module. The following assertions are equivalent.*

(i) $M_{\mathfrak{P}}=0$ *for all* $\mathfrak{P}$ *prime ideals in R;*

(ii) $M_{\mathfrak{m}}=0$ *for all maximal ideals* $\mathfrak{m}$ *in R;*

(iii) $M=0$.

Proof. The implications $(iii)\implies(i)\implies(ii)$ are clear. Assume that $M\neq 0$ and that $M_{\mathfrak{m}}=0$ for all maximal ideals $\mathfrak{m}$ in R. Let $0\neq x\in M$. The ideal

$$\operatorname{Ann}(x)=\{a\in R\colon ax=0\}\tag{1.1.2.1}$$

is a proper ideal of R since $1\notin\operatorname{Ann}(x)$. Hence, $\operatorname{Ann}(x)$ is contained in some maximal ideal $\mathfrak{m}$. By (ii), $M_{\mathfrak{m}}=0$. In particular, $1\in S=R\backslash\mathfrak{m}$. Therefore, $\frac{x}{1}$ is zero in $M_{\mathfrak{m}}$. So, there exists $u\in S$ such that $0=ux$. But this would imply that u belongs to $\operatorname{Ann}(x)\subset\mathfrak{m}$. This contradicts the existence of $u\in S$. Therefore, $M=0$. □

In the same direction, there is the following relation between the annihilator of an element of a module and the prime ideals in the ring. It follows directly by applying the definitions of localization and annihilator.

Lemma 1.1.25. *Let R be a commutative ring. Let M be an R-module and* $x\in M$.

(a) If $\operatorname{Ann}(x)\subset\mathfrak{p}$*, where* $\mathfrak{p}$ *is a prime ideal in R, then the localization element* $x_{\mathfrak{p}}\neq 0$ *in* $M_{\mathfrak{p}}$.

(b) If $\operatorname{Ann}M\subset\mathfrak{p}$ *and* $M\in R$-mod*, where* $\mathfrak{p}$ *is a prime ideal in R, then* $\mathfrak{p}\in\operatorname{Supp}(M)$.

Proof. Assume, by contradiction, that $x_{\mathfrak{p}}=0$. Then, there exists $s\in R\backslash\mathfrak{p}$ such that $sx=0$. This would imply that $s\in\operatorname{Ann}(x)\subset\mathfrak{p}$. Hence, $x_{\mathfrak{p}}\neq 0$.

Let $\mathfrak{p}$ be a prime ideal in R such that $\operatorname{Ann}M\subset\mathfrak{p}$. Recall that $\operatorname{Supp}(M)$ is the set of all prime ideals in R satisfying $M_{\mathfrak{p}}\neq 0$. Assume, by contradiction, that $M_{\mathfrak{p}}=0$. Then, for every $x\in M$, there exists $s\in R\backslash\mathfrak{p}$ such that $sx=0$. Let $\{m_1,\dots,m_t\}$ be an R-generator set of M. Then, there exists, $s_i\in R\backslash\mathfrak{p}$ such that $s_im_i=0$. In particular, $s_1\cdots s_tm_i=0$ for all $i=1,\dots,t$. Hence, $s_1\cdots s_tm=0$ for all $m\in M$. Thus, $s_1\cdots s_t\in\operatorname{Ann}M\subset\mathfrak{p}$. Thus, some s_i should belong to $\mathfrak{p}$. This is a contradiction with the definition of s_i. Therefore, $M_{\mathfrak{p}}\neq 0$. □

An application of this lemma is that we can characterize the exact sequences which split over R using localization.

Lemma 1.1.26. *Let R be a commutative ring. Let*

$$\delta\colon 0\to Y\to M\to X\to 0\tag{1.1.2.2}$$

be an R-exact such that the localization at each maximal ideal in R of δ *is split. Then,* δ *splits over R.*

Proof. $\operatorname{Ext}^1_R(X,Y)$ is an R-module. If δ does not split, then $\delta \neq 0$ in $\operatorname{Ext}^1_R(X,Y)$. In particular, $1_R \neq \operatorname{Ann}(\delta)$. So $\operatorname{Ann}(\delta)$ is contained in some maximal ideal $\mathfrak{m}$ in R. So, $\delta_{\mathfrak{m}} \neq 0$. This contradicts our assumption on δ, and therefore δ splits over R. ☐

Properties that can be reduced to their study over local rings are knows as **local properties**.

Proposition 1.1.27. *(see for example [Bou98, Theorem 1,p.88]) Let R be a commutative ring. Consider a sequence of R-modules*

$$0 \to M_1 \to M_2 \to M_3 \to 0. \tag{1.1.2.3}$$

If the localised sequence of R_m-modules $0 \to (M_1)_m \to (M_2)_m \to (M_3)_m \to 0$ is exact for all maximal ideals m in R, then the exact sequence (1.1.2.3) is exact.

Proof. It suffices to show a sequence $L \xrightarrow{\phi} M \xrightarrow{\psi} M$ is exact if all localizations $L_{\mathfrak{m}} \xrightarrow{\phi_{\mathfrak{m}}} M_{\mathfrak{m}} \xrightarrow{\psi_{\mathfrak{m}}} M_{\mathfrak{m}}$ are exact at every maximal ideal $\mathfrak{m}$ in R.

Note that $(\operatorname{im}\psi \circ \phi)_{\mathfrak{m}} \operatorname{im}(\psi_{\mathfrak{m}} \circ \phi_{\mathfrak{m}}) = 0$ for every maximal ideal $\mathfrak{m}$ in R. Hence, $\psi \circ \phi = 0$. Thus, we can construct the quotient module $\ker\psi/\operatorname{im}\phi$. Further, $(\ker\psi/\operatorname{im}\phi)_{\mathfrak{m}} = (\ker\psi)_{\mathfrak{m}}/(\operatorname{im}\phi)_{\mathfrak{m}} = \ker\psi_{\mathfrak{m}}/\operatorname{im}\phi_{\mathfrak{m}} = 0$, for every maximal ideal $\mathfrak{m}$ in R. Hence, $\ker\psi/\operatorname{im}\phi = 0$. ☐

Corollary 1.1.28. *Let R be a commutative ring. For any R-homomorphism $\phi : M \to N$, the following assertions are equivalent:*

(i) ϕ is injective (surjective);

(ii) $\phi_{\mathfrak{p}}$ is injective (surjective for all prime ideals $\mathfrak{p}$ in R;

(iii) $\phi_{\mathfrak{m}}$ is injective (surjective) for all maximal ideals $\mathfrak{m}$ in R.

Furthermore, $\phi : M \to N$ is an isomorphism if and only if $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is isomorphism for all maximal ideals $\mathfrak{m}$ in R.

All properties that can be expressed in terms of exact sequences are local as well.

Here, it is important that the isomorphism in the localizations arises as the localization of a map defined globally between M and P. In fact, $M_{\mathfrak{m}} \simeq N_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R does not imply $M \simeq N$, in general. Moreover, two modules which are isomorphic at every localization at a prime ideal are said to be in the same genus [CR06, §81]. However, this reasoning is valid if there exists a module which contains both N and M.

Lemma 1.1.29. *Let R be a commutative ring. Let M,N,P be R-modules. Suppose that $N,M \subset P$. Then, $N = M$ if and only if $N_{\mathfrak{m}} = M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $\pi : P \to P/M$ be the canonical surjection. For every maximal ideal $\mathfrak{m}$ in R,

$$\pi(N)_{\mathfrak{m}} = \pi_{\mathfrak{m}}(N_{\mathfrak{m}}) = \pi_{\mathfrak{m}}(M_{\mathfrak{m}}) = 0. \tag{1.1.2.4}$$

Thus, $\pi(N) = 0$. Therefore, $N \subset M$. Symmetrically, we deduce that $M \subset N$. Hence, $M = N$. ☐

We shall now see how the tensor product of two modules and the abelian group of homomorphisms between two modules behave under change of ring (see for example Proposition 4.84 of [Rot09] for the particular case of localization).

Proposition 1.1.30. *Let S be a commutative R-algebra and A be an R-algebra. Let $M \in$ mod-A, $N \in A$-mod. Then, $S \otimes_R (M \otimes_A N) \simeq S \otimes_R M \otimes_{S \otimes_R A} S \otimes_R N$ as S-modules.*

Proof. Consider the map $\psi \colon S \times (M \otimes_A N) \to S \otimes_R M \otimes_{S \otimes_R A} S \otimes_R N$, given by $\psi(s, m \otimes n) = (s \otimes m \otimes 1_S \otimes n)$, $s \in S, m \otimes n \in M \otimes_A N$. ψ is linear in each term. Further, for every $r \in R$,

$$\psi(rs, m \otimes n) = rs \otimes m \otimes 1_S \otimes n = s \otimes rm \otimes 1_S \otimes n = \psi(s, rm \otimes n).$$

So ψ induces uniquely a map $\psi' \in \operatorname{Hom}(S \otimes_R M \otimes_A N, S \otimes_R M \otimes_{S \otimes_R A} S \otimes_R N)$ which maps $s \otimes m \otimes n$ to $s \otimes m \otimes 1_S \otimes n$. Such a map is an S-homomorphism since

$$\begin{aligned}\psi(ls \otimes (m \otimes n)) &= ls \otimes m \otimes 1_S \otimes n = sl \otimes m \otimes 1_S \otimes n = s \otimes m \cdot (l \otimes 1_A) \otimes 1_S \otimes n \\ &= s \otimes m \otimes (l \otimes 1_A) \cdot 1_S \otimes n = s \otimes m \otimes l \otimes n = l\psi(s \otimes m \otimes n), s, l \in S, m \in M, n \in N.\end{aligned}$$

Now, consider the map $\delta \colon S \otimes_R M \times S \otimes_R N \to S \otimes_R M \otimes_A N$, given by $\delta(s \otimes m, s' \otimes n) = ss' \otimes (m \otimes n)$, $m \in M$, $s, s' \in S, n \in N$. It is clear that this map is bilinear. Let $l \otimes a \in S \otimes_R A$. Then,

$$\begin{aligned}\delta(s \otimes m \cdot l \otimes a, s' \otimes n) &= \delta(sl \otimes ma, s' \otimes n) = (sl)s' \otimes (ma \otimes n) = s(ls') \otimes (m \otimes an) = \delta(s \otimes m, ls' \otimes an) \\ &= \delta(s \otimes m, (l \otimes a) \cdot (s' \otimes n)).\end{aligned}$$

So, δ induces uniquely a map $\delta' \in \operatorname{Hom}_S(S \otimes_R M \otimes_{S \otimes_R A} S \otimes_R N, S \otimes_R M \otimes_A N)$. The S-homomorphisms δ' and ψ' are inverse to each other, and thus the result follows. □

Proposition 1.1.31. *Let S be a commutative R-algebra. Let A be an R-algebra. Let $M \in A$-proj and $N \in A$-mod. Then, $S \otimes_R \operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{S \otimes_R A}(S \otimes_R M, S \otimes_R N)$.*

Proof. For each $M \in A$-mod, consider the S-homomorphism $\psi_M \colon S \otimes_R \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S \otimes_R A}(S \otimes_R M, S \otimes_R N)$, given by $\psi_M(s \otimes f)(s' \otimes m) = ss' \otimes f(m)$, $s, s' \in S$, $m \in M$, $f \in \operatorname{Hom}_A(M, N)$. The homomorphism ψ_M is compatible with direct sums. This means that if M admits a decomposition $M = M_1 \oplus M_2$, then there exists a commutative diagram

$$\begin{tikzcd} S \otimes_R \operatorname{Hom}_A(M_1 \oplus M_2, N) \arrow[r, "\psi_{M_1 \oplus M_2}"] \arrow[d, "\simeq"] & \operatorname{Hom}_{S \otimes_R A}(S \otimes_R (M_1 \oplus M_2), S \otimes_R N) \arrow[d, "\simeq"] \\ S \otimes_R \operatorname{Hom}_A(X, N) \oplus S \otimes_R \operatorname{Hom}_A(Y, N) \arrow[r, "\psi_{M_1} \oplus \psi_{M_2}"] & \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R N) \oplus \operatorname{Hom}_{S \otimes_R A}(S \otimes_R Y, S \otimes_R N) \end{tikzcd}.$$

Let $M = A$. Then, there exists a commutative diagram

$$\begin{tikzcd} S \otimes_R \operatorname{Hom}_A(A, N) \arrow[r, "\psi_A"] \arrow[d, "\psi_1", "\simeq"'] & \operatorname{Hom}_{S \otimes_R A}(S \otimes_R A, S \otimes_R N) \arrow[d, "\psi_2", "\simeq"'] \\ S \otimes_R N \arrow[r, equal] & S \otimes_R N \end{tikzcd}.$$

In fact, $\psi_2 \circ \psi_M(s \otimes f) = \psi_2(s \otimes f)(1_S \otimes 1_A) = s1_S \otimes f(1_A) = \psi_1(s \otimes f)$. Therefore, ψ_A is bijective. Since ψ_M is compatible with direct sums it follows that ψ_M is an S-isomorphism whenever $M \in A$-proj. □

Lemma 1.1.32. *Let $f \colon R \to S$ be a surjective R-algebra homomorphism. Let A be an R-algebra. If M and N are A-modules, then $\operatorname{Hom}_{S \otimes_R A}(S \otimes_R M, S \otimes_R N) \simeq \operatorname{Hom}_A(S \otimes_R M, S \otimes_R N) \simeq \operatorname{Hom}_A(M, S \otimes_R N)$.*

Proof. Let $\phi \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R M, S \otimes_R N)$. Then, for any $a \in A, s \otimes_R m \in S \otimes_R M$,

$$\phi(a(s \otimes m)) = \phi(s \otimes am) = \phi((1_S \otimes a)(s \otimes m)) = (1_S \otimes a)\phi(s \otimes m) = a\phi(s \otimes a). \tag{1.1.2.5}$$

Thus, $\phi \in \operatorname{Hom}_A(S\otimes_R M, S\otimes_R N)$. Now consider $\phi \in \operatorname{Hom}_A(S\otimes_R M, S\otimes_R N)$. For any $a\in A$, $m\in M$, $s', s\in S$ we have

$$\phi(s'\otimes as\otimes m) = \phi(s's\otimes am) = \phi(a(s's\otimes m)) = a\phi(f(r')s\otimes m) = a\phi(r'f(1_R)s\otimes m) \tag{1.1.2.6}$$
$$= r'a\phi(1_S s\otimes m) = (1_S\otimes r'a)\phi(s\otimes m) = (f(1_R r')\otimes a)\phi(s\otimes m) = s'\otimes a\phi(s\otimes m), \tag{1.1.2.7}$$

for some $r'\in R$. Hence, $\phi \in \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R N)$. Therefore, the first isomorphism is established. Let $\phi \in \operatorname{Hom}_A(M, S\otimes_R N)$. We extend ϕ to a map $\phi' \in \operatorname{Hom}_A(S\otimes_R M, S\otimes_R N)$ by imposing $\phi'(s\otimes m) = s\phi(m)$. Let $\phi \in \operatorname{Hom}_A(S\otimes_R M, S\otimes_R N)$, we restrict it to $\phi_| \in \operatorname{Hom}_A(M, S\otimes_R N)$ by definining $\phi_|(m) = \phi(1_S\otimes m)$, $m\in M$. Using these two correspondences, we obtain the second isomorphism. ☐

Lemma 1.1.33. *Let $M\in A$-proj. Then, the R-homomorphism $\varsigma_{M,N,U}\colon \operatorname{Hom}_A(M,N)\otimes_R U \to \operatorname{Hom}_A(M, N\otimes_R U)$, given by $g\otimes u\mapsto g(-)\otimes u$ is an R-isomorphism.*

Proof. Consider $M = A$. The following diagram is commutative.

$$\begin{array}{ccc} \operatorname{Hom}_A(A,N)\otimes_R U & \xrightarrow{\varsigma_{A,N,U}} & \operatorname{Hom}_A(M, N\otimes_R U) \\ \downarrow & & \downarrow \\ N\otimes_R U & = & N\otimes_R U \end{array}$$

Both columns are isomorphisms, thus $\varsigma_{A,N,U}$ is an isomorphism. Since this map is compatible with direct sums, it follows that $\varsigma_{M,N,U}$ is an isomorphism for every $M\in A$-proj and any $N\in A$-Mod, $U\in R$-Mod. ☐

By considering the module U to be projective over the ground ring in Lemma 1.1.33, we can drop M being projective over the algebra.

Lemma 1.1.34. *Let $M,N\in A$-mod and $U\in R$-proj. Then, the R-homomorphism $\varsigma_{M,N,U}\colon \operatorname{Hom}_A(M,N)\otimes_R U\to \operatorname{Hom}_A(M,N\otimes_R U)$, given by $g\otimes u\mapsto g(-)\otimes u$ is an R-isomorphism.*

Proof. Since for all modules $U_1, U_2\in R$-mod there are commutative diagrams

$$\begin{array}{ccc} \operatorname{Hom}_A(M,N)\otimes_R U_1 \oplus \operatorname{Hom}_A(M,N)\otimes_R U_2 & \xrightarrow[\varsigma_{M,N,U_1}\oplus\varsigma_{M,N,U_2}]{} & \operatorname{Hom}_A(M,N\otimes_R U_1)\oplus \operatorname{Hom}_A(M,N\otimes_R U_2) \\ \downarrow\simeq & & \downarrow\simeq \\ \operatorname{Hom}_A(M,N)\otimes_R (U_1\oplus U_2) & \xrightarrow{\varsigma_{M,N,U_1\oplus U_2}} & \operatorname{Hom}_A(M,N\otimes_R (U_1\oplus U_2)) \end{array}, \tag{1.1.2.8}$$

it is enough to show that $\varsigma_{M,N,R}$ is an R-isomorphism. But, this isomorphism is obtained by regarding $\varsigma_{M,N,R}$ in the following commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_A(M,N)\otimes_R R & \xrightarrow{\varsigma_{M,N,R}} & \operatorname{Hom}_A(M,N\otimes_R R) \\ \downarrow \mu_{\operatorname{Hom}_A(M,N)} & & \downarrow \operatorname{Hom}_A(M,\mu_N) \\ \operatorname{Hom}_A(M,N) & = & \operatorname{Hom}_A(M,N) \end{array}, \tag{1.1.2.9}$$

where μ_X denotes the multiplication map for any R-module X. In fact, for all $f\in \operatorname{Hom}_A(M,N)$,

$$\operatorname{Hom}_A(M,\mu_N)\circ\varsigma_{M,N,R}(f\otimes 1_R)(m) = \mu_N\circ\varsigma_{M,N,R}(f\otimes 1_R)(m) = \mu_N(f(m)\otimes_R 1_R) = f(m),\ m\in M.$$

Hence, $\varsigma_{M,N,R}$ is an R-isomorphism. ☐

Proposition 1.1.35. *(see for example Lemma 3.3.8 of [Rot09]) Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra and let S be a commutative flat R-algebra. If $M \in A$-mod and $N \in A$-Mod, then* $\operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R N) \simeq S\otimes_R \operatorname{Hom}_A(M,N)$.

Proof. Since $M \in A$-mod we can write a projective presentation

$$P_1 \to P_0 \to M \to 0, \tag{1.1.2.10}$$

where P_1, P_0 are finitely generated projective A-modules. The functor $S\otimes_R -$ is exact and consequently the functors $S\otimes_R \operatorname{Hom}_A(-,N)$ and $\operatorname{Hom}_{S\otimes_R A}(-,S\otimes_R N)\circ S\otimes_R -$ are contravariant left exact. So there exists the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S\otimes_R \operatorname{Hom}_A(M,N) \arrow[r] \arrow[d, "\psi_M"] & S\otimes_R \operatorname{Hom}_A(P_0,N) \arrow[r] \arrow[d, "\psi_{P_0}"] & S\otimes_R \operatorname{Hom}_A(P_1,N) \arrow[d, "\psi_{P_1}"] \\
0 \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R N) \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_0, S\otimes_R N) \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_1, S\otimes_R N)
\end{tikzcd},$$

where the homomorphisms ψ_M are given by Proposition 1.1.31. Using diagram chasing and Proposition 1.1.31 it follows that ψ_M is an isomorphism. ☐

As application of Proposition 1.1.35, we see that Hom commutes with localizations over commutative Noetherian rings.

The functors Ext and Tor also behave well under flat extensions of the ground ring.

Lemma 1.1.36. *(see for example [Rot09, Proposition 3.3.10]) Let R be a commutative Noetherian ring with identity. Let S be a flat commutative R-algebra. Let A be a projective Noetherian R-algebra. Then, the following holds.*

1. *Let $M,N \in A$-mod. Then, $S\otimes_R \operatorname{Ext}^i_A(M,N) \simeq \operatorname{Ext}^i_{S\otimes_R A}(S\otimes_R M, S\otimes_R N)$ for every $i \geq 0$.*
2. *Let $M \in A$-mod and $X \in$ mod-A. Then, $S\otimes_R \operatorname{Tor}^i_A(X,M) \simeq \operatorname{Tor}^i_{S\otimes_R A}(S\otimes_R X, S\otimes_R M)$ for every $i \geq 0$.*

Proof. Let $i \geq 0$. Let

$$P^\bullet\colon \cdots \to P_1 \to P_0 \to M \to 0 \tag{1.1.2.11}$$

be a projective A-resolution of M. Observe that each $S\otimes_R P_i$ is an $S\otimes_R A$-summand of $S\otimes_R A^{t_i} \simeq (S\otimes_R A)^{t_i}$. Hence, applying $S\otimes_R -$ on (1.1.2.11) yields a projective $S\otimes_R A$-resolution of $S\otimes_R M$. Hence,

$$\operatorname{Tor}^i_{S\otimes_R A}(S\otimes_R X, S\otimes_R M) = H_i(S\otimes_R X\otimes_{S\otimes_R A} S\otimes_R P^\bullet) \simeq H_i(S\otimes_R X\otimes_A P^\bullet) \simeq S\otimes_R H_i(X\otimes_A P^\bullet) \tag{1.1.2.12}$$

$$\simeq S\otimes_R \operatorname{Tor}^i_A(X,M). \tag{1.1.2.13}$$

Analogously, we have

$$S\otimes_R \operatorname{Ext}^i_A(M,N) \simeq S\otimes_R H^i(\operatorname{Hom}_A(P^\bullet,N)) \simeq H^i(S\otimes_R \operatorname{Hom}_A(P^\bullet,N)) \tag{1.1.2.14}$$

$$\simeq H^i(\operatorname{Hom}_{S\otimes_R A}(S\otimes_R P^\bullet, S\otimes_R N)) \simeq \operatorname{Ext}^i_{S\otimes_R A}(S\otimes_R M, S\otimes_R N).$$ ☐

In order to deduce results from finite-dimensional algebras over fields to algebras over arbitrary commutative rings, the residue field plays a crucial role.

For local rings, there is only one maximal ideal m. By the **residue field associated with** R (or just the residue field when there is no risk of confusion) we mean the field $R(\mathfrak{m}) = R/\mathfrak{m}$. For arbitrary commutative rings, this notion is defined through localization. For every prime ideal $\mathfrak{p}$ of R, we call $R(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ the **residue field associated with the ideal** $\mathfrak{p}$. For any $M \in R$-mod we will denote by $M(\mathfrak{p})$ the module $R(\mathfrak{p}) \otimes_R M$ for any prime ideal $\mathfrak{p}$ in R. For any Noetherian R-algebra A and any $M \in A$-mod, $M(\mathfrak{p}) \in A(\mathfrak{p})$-mod. For a given A-homomorphism ϕ we will denote by $\phi(\mathfrak{p})$ the image of ϕ under the functor $R(\mathfrak{p}) \otimes_R -$.

Proposition 1.1.37. *Let R be a commutative ring. If $\mathfrak{m}$ is maximal ideal in R, then $R(\mathfrak{m}) \simeq R/\mathfrak{m}$ as R-modules and rings.*

Proof. Let $\mathfrak{m}$ be a maximal ideal in R. Let θ be the composition of canonical R-homomorphisms

$$R \to R_\mathfrak{m} \to R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}. \tag{1.1.2.15}$$

Let $x \in \mathfrak{m}$. Then, $\theta(x) = \frac{x}{1} + \mathfrak{m}_\mathfrak{m} = 0 + \mathfrak{m}_\mathfrak{m}$ because $\frac{x}{1} \in \mathfrak{m}_\mathfrak{m}$. So θ induces an R-homomorphism $\Theta\colon R/m \to R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. Let $r + \mathfrak{m}$ such that $\frac{r}{1} \in \mathfrak{m}_\mathfrak{m}$. Hence, there exists $t \in \mathfrak{m}$, $s \in R\backslash\mathfrak{m}$ such that $\frac{r}{1} = \frac{t}{s}$. So there exists $u \in R\backslash\mathfrak{m}$ satisfying $u(rs-t) = 0$. This implies that $urs \in \mathfrak{m}$. Consequently, $r \in \mathfrak{m}$. So Θ is injective. $R/\mathfrak{m}$ is a field, so for every $s \in R\backslash\mathfrak{m}$, $s + \mathfrak{m}$ has an inverse $t + \mathfrak{m}$ for some $t \in R$. In particular, $st - 1 \in \mathfrak{m}$. Therefore, $\dfrac{t}{1} - \dfrac{1}{s} = \dfrac{st-1}{s} \in \mathfrak{m}_\mathfrak{m}$. So every element $\dfrac{r}{s} + \mathfrak{m}_\mathfrak{m}$ is equal to $\dfrac{rt}{1} + \mathfrak{m}_\mathfrak{m}$ for some $t \in R$. This shows that Θ is also surjective. ☐

The following form of the Nakayama's Lemma will be extensively used throughout this thesis.

Lemma 1.1.38 (Nakayama's Lemma)**.** *Let R be a commutative ring.*

*(a) Denote by J the Jacobson radical of R. Let $M \in R$-*mod*. If $JM = M$, then $M = 0$.*

*(b) Let $M, N \in R$-*mod*. If $\phi\colon M \to N$ is an R-homomorphism such that the quotient $\phi(\mathfrak{m})$ is surjective for every maximal ideal $\mathfrak{m}$ in R, then ϕ is surjective.*

(c) Let A be an R-algebra. If $\phi\colon M \to N$ is a surjective A-homomorphism and $M \simeq N$ as R-modules, then ϕ is an isomorphism.

Proof. For the statement (a) see [Lan02, Lemma 4.1].

(b). ϕ is surjective if and only if $\phi_\mathfrak{m}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. Hence, we can assume without loss of generality that R is a local commutative ring. Let $\mathfrak{m}$ be the unique maximal ideal in R. Observe that $M(\mathfrak{m}) \simeq R/\mathfrak{m} \otimes_R M \simeq M/\mathfrak{m}M$. So,

$$0 = (N/\mathfrak{m}N)/(\operatorname{im}\phi/\mathfrak{m}\operatorname{im}\phi) = (N/\mathfrak{m}N)/(\operatorname{im}\phi/\operatorname{im}\phi \cap \mathfrak{m}N) \tag{1.1.2.16}$$

$$= (N/\mathfrak{m}N)/(\operatorname{im}\phi + \mathfrak{m}N/\mathfrak{m}N) \simeq N/\operatorname{im}\phi + \mathfrak{m}N \implies N = \operatorname{im}\phi + \mathfrak{m}N. \tag{1.1.2.17}$$

Further,

$$N/\operatorname{im}\phi = \operatorname{im}\phi + \mathfrak{m}N/\operatorname{im}\phi \simeq \mathfrak{m}N/\operatorname{im}\phi \cap \mathfrak{m}N \simeq \mathfrak{m}N/\mathfrak{m}\operatorname{im}\phi \simeq \mathfrak{m}(N/\operatorname{im}\phi). \tag{1.1.2.18}$$

By Nakayama's Lemma (a), $N = \operatorname{im}\phi$. So ϕ is surjective. Let $f \in \operatorname{Hom}_R(M,N)$ be an isomorphism. M can be regarded as an $R[x]$-module by imposing $x \cdot m = f \circ \phi(m)$, $m \in M$. Since $f \circ \phi$ is surjective, $R[x]xM = M$. By Nakayama's Lemma, there exists $y \in R[x]$ such that $(1 + xy)M = 0$. Let $u \in \ker\phi$. We have, $0 = (1 + xy)(u) = u + yf \circ \phi(u) = u$. So, ϕ is also injective. ☐

Lemma 1.1.39. *Let R be a commutative ring. Let $N \in R$-*proj*. Let $\psi \in \operatorname{Hom}_R(M,N)$ such that $\psi(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. Then, ψ is an isomorphism.*

Proof. By Nakayama's Lemma 1.1.38, ψ is surjective. Since $N \in R$-proj there exists an R-homomorphism $\delta\colon N \to M$ such that $\psi\circ\delta = \mathrm{id}_N$. In particular, δ is injective. Applying $R(\mathfrak{m})\otimes_R -$, we get $\psi(\mathfrak{m})\circ\delta(\mathfrak{m}) = \mathrm{id}_{N(\mathfrak{m})}$. Thus, $\delta(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, δ is surjective. Thus, δ is an isomorphism, and ψ is an isomorphism as well. $\square$

Lemma 1.1.40. *Let R be a commutative ring. Let $\mathfrak{m}, \mathfrak{Q}$ be maximal distinct ideals in R. Then, the following holds.*

1. *The localizations $\mathfrak{m}_{\mathfrak{Q}} = R_{\mathfrak{Q}}$ coincide and $(R/\mathfrak{m})_{\mathfrak{Q}} = 0$.*
2. $(R_{\mathfrak{m}})_{\mathfrak{Q}} \simeq (R_{\mathfrak{Q}})_{\mathfrak{m}}$.
3. $(R/\mathfrak{m})_{\mathfrak{m}} \simeq R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ *and* $R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}}$ *as* $R_{\mathfrak{m}}$*-modules.*
4. *For any $M \in R$-mod, $M_{\mathfrak{m}} \simeq (M_{\mathfrak{m}})_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules.*

Proof. Since $\mathfrak{m}, \mathfrak{Q}$ are maximal distinct ideals, there exists $x \in \mathfrak{m}$ such that $x \in R\backslash\mathfrak{Q}$. Hence, for every $r,t \in R$, $s \in R\backslash\mathfrak{Q}$

$$(r+\mathfrak{m})\otimes\frac{t}{s} = (r+\mathfrak{m})\otimes\frac{xt}{xs} = (xr+\mathfrak{m})\otimes\frac{t}{xs} = 0. \tag{1.1.2.19}$$

Thus, $(R/\mathfrak{m})_{\mathfrak{Q}} = 0$. The localization at $\mathfrak{Q}$ is exact, hence $\mathfrak{m}_{\mathfrak{Q}} = R_{\mathfrak{Q}}$.

Consider the exact sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$. By localizing at $\mathfrak{m}$ we obtain the exact sequence $0 \to \mathfrak{m}_{\mathfrak{m}} \to R_{\mathfrak{m}} \to (R/\mathfrak{m})_{\mathfrak{m}} \to 0$. By uniqueness of cokernel, $(R/\mathfrak{m})_{\mathfrak{m}} \simeq R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$. Statement 2 is due the following fact $(R_{\mathfrak{m}})_{\mathfrak{Q}} \simeq R_{\mathfrak{m}} \otimes_R R_{\mathfrak{Q}} \simeq (R_{\mathfrak{Q}})_{\mathfrak{m}}$.

Consider the homomorphisms $f\colon R_{\mathfrak{m}} \to R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}}$, given by $f(\frac{t}{s}) = 1_{R_{\mathfrak{m}}} \otimes \frac{t}{s}$, and $g\colon R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}} \to R_{\mathfrak{m}}$, given by $g(\frac{t}{s}\otimes\frac{p}{q}) = \frac{tp}{sq}$. These homomorphisms are inverse to each other because

$$fg(\frac{t}{s}\otimes\frac{p}{q}) = \frac{1}{1}\otimes\frac{tp}{sq} = \frac{ts}{s}\otimes\frac{p}{sq} = \frac{t}{s}\otimes\frac{ps}{sq} = \frac{t}{s}\otimes\frac{p}{q} \in R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}}. \tag{1.1.2.20}$$

It follows also that $(M_{\mathfrak{m}})_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \otimes_R M_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}} \otimes_R M \simeq R_{\mathfrak{m}} \otimes_R M = M_{\mathfrak{m}}$. $\square$

Proposition 1.1.41. *Let R be a commutative Noetherian ring. Let A be an R-algebra.*

(a) *Let $N \in A$-mod. If $N(\mathfrak{m}) = 0$ for any maximal ideal m in R, then $N = 0$.*

(b) *Let $\mathfrak{m}$ be a maximal ideal ideal in R. Then, for any $M \in A(\mathfrak{m})$-mod, $M(\mathfrak{m}) \simeq M$ as $A(\mathfrak{m})$-modules.*

(c) *For any maximal ideal $\mathfrak{Q} \neq \mathfrak{m}$, $M(\mathfrak{m})_{\mathfrak{Q}} = 0$.*

(d) *If $R(\mathfrak{m})$ is flat over R for some maximal ideal $\mathfrak{m}$ in R, then R is a field.*

Proof. Let $N \in A$-mod such that $N(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ in R. $N(\mathfrak{m})$ is isomorphic to $N_{\mathfrak{m}}/\mathfrak{m}N_{\mathfrak{m}}$. By Nakayama's Lemma, $N_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ in R. It follows that $N = 0$ and consequently (a) holds.

Let $M \in A(\mathfrak{m})$-mod. We can regard M as A-module by restriction of scalars. Then, $M(\mathfrak{m}) \simeq R/\mathfrak{m} \otimes_R M \simeq M/\mathfrak{m}M$. But M is an $A/\mathfrak{m}A$-module, and so $\mathfrak{m}M = 0$. So (b) holds.

(c) holds since

$$M(\mathfrak{m})_{\mathfrak{Q}} = M(\mathfrak{m}) \otimes_R R_{\mathfrak{Q}} \simeq M \otimes_R (R/\mathfrak{m})_{\mathfrak{Q}} = 0. \tag{1.1.2.21}$$

Assume that $R(\mathfrak{m})$ is flat over R for some maximal ideal $\mathfrak{m}$. Then, $R(\mathfrak{m})_\mathfrak{m} = R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} = R/\mathfrak{m}$ is flat over $R_\mathfrak{m}$. So we can assume without loss of generality that R is a local ring. Consider the exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0. \tag{1.1.2.22}$$

Applying $R(\mathfrak{m}) \otimes_R -$ yields by assumption the exact sequence

$$0 \to R/\mathfrak{m} \otimes_R \mathfrak{m} \to R/\mathfrak{m} \to R/\mathfrak{m} \otimes_R R/\mathfrak{m} \to 0. \tag{1.1.2.23}$$

Now since $R/m \to R/m \otimes_R R/m$ is the inverse R-homomorphism of the multiplication map it follows that $\mathfrak{m}/\mathfrak{m}^2 = R/\mathfrak{m} \otimes_R \mathfrak{m} = 0$. By Nakayama's Lemma, $\mathfrak{m} = 0$. So $R = R(\mathfrak{m})$ is a field. □

Here, notice that N being finitely generated is fundamental. For example, we can consider $\mathbb{Q}$ as $\mathbb{Z}$-module and $\mathbb{Q}(\mathbb{Z}p) = \mathbb{Q} \otimes_\mathbb{Z} \mathbb{Z}/\mathbb{Z}p = 0$ for every prime element of $\mathbb{Z}$, however $\mathbb{Q} \neq 0$.

Using the following results, we see that projective modules are the objects which are locally free. Moreover, in homological algebra the Nakayama's Lemma may take the following forms.

Proposition 1.1.42. *A finitely generated projective module M over a local commutative ring is free.*

Proof. See, for example, [Wei03, Proposition 4.3.11]. □

Lemma 1.1.43. *Let R be local commutative Noetherian ring with maximal ideal $\mathfrak{m}$. Let $M \in R$-mod. Then,* $\operatorname{pdim}(M) \leq n$ *if and only if* $\operatorname{Tor}_R^{n+1}(M, R/\mathfrak{m}) = 0$.

Proof. See [Rot09, Lemma 8.53]. □

Theorem 1.1.44. *Let R be a commutative Noetherian ring. Let M be a finitely generated R-module. The following assertions are equivalent.*

(i) M is a projective R-module;

(ii) $M_\mathfrak{p}$ is projective $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}$ in R;

(iii) $M_\mathfrak{m}$ is free $M_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ in R;

(iv) $\operatorname{Tor}_1^{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) = 0$ *for every maximal ideal $\mathfrak{m}$ in R;*

(v) $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$ *for every maximal ideal $\mathfrak{m}$ in R.*

Proof. The implications $(i) \implies (ii) \implies (iii)$ are clear. Assume that (iii) holds. $R_\mathfrak{m}$ is a local commutative Noetherian ring, and therefore taking $n = 0$ in Lemma 1.1.43 implies that (iv) is satisfied.

Assume that (iv) holds. Let $\mathfrak{m}, \mathfrak{Q}$ be maximal ideals in R. Then,

$$\operatorname{Tor}_1^R(M, R/\mathfrak{m})_\mathfrak{Q} \simeq \operatorname{Tor}_1^{R_\mathfrak{Q}}(M_\mathfrak{Q}, (R/\mathfrak{m})_\mathfrak{Q}) = \operatorname{Tor}_1^{R_\mathfrak{Q}}(M_\mathfrak{Q}, 0) = 0, \tag{1.1.2.24}$$

unless $\mathfrak{Q}$ is equal to $\mathfrak{m}$. In such a case, $\operatorname{Tor}_1^{R_\mathfrak{Q}}(M_\mathfrak{Q}, (R/\mathfrak{m})_\mathfrak{Q}) = \operatorname{Tor}_1^{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) = 0$ by (iv). Hence, (v) holds.

Using Lemma 1.1.43, the implications $(v) \implies (iv) \implies (iii)$ are clear. Assume that (iii) holds. Let $f \in \operatorname{Hom}_R(X, Y)$ be a surjective map. Then, $f_\mathfrak{m}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By assumption, the map $\operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, f_\mathfrak{m}) = \operatorname{Hom}_R(M, f)_\mathfrak{m}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. Hence, $\operatorname{Hom}_R(M, f)$ is surjective. So $\operatorname{Hom}_R(M, -)$ is exact, and therefore M is projective. □

Similarly, the following is the version of Theorem 1.1.44 for projective R-algebras.

Theorem 1.1.45. *Let R be a commutative Noetherian ring. Let A be a projective R-algebra. The following assertions are equivalent.*

(i) M is a projective A-module;

(ii) The localization $M_\mathfrak{m}$ is a projective $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of R;

(iii) M is a projective R-module and $\operatorname{Ext}^1_A(M,N)=0$ *for all* $N\in R\text{-proj}\cap A\text{-mod}$.

Proof. Assume that (i) holds. Fix $\mathfrak{m}$ a maximal ideal in R. $M_\mathfrak{m}=R_\mathfrak{m}\otimes_R M$ is projective over $R_\mathfrak{m}\otimes_R A=A_\mathfrak{m}$. So, (ii) holds.

Assume that (ii) is satisfied. By assumption, A is projective over R and consequently $A_\mathfrak{m}$ is projective over $R_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ in R. So, for each maximal ideal $\mathfrak{m}$ in R $M_\mathfrak{m}$ is projective over $R_\mathfrak{m}$. In particular, M is projective over R. Let $N\in A\text{-mod}\cap R\text{-proj}$. Then,

$$\operatorname{Ext}^1_A(M,N)_\mathfrak{m}\simeq \operatorname{Ext}^1_{A_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})=0, \forall \mathfrak{m} \text{ maximal ideal} \implies \operatorname{Ext}^1_A(M,N)=0. \tag{1.1.2.25}$$

Hence, (iii) holds.

Assume that (iii) holds. Let $\delta\colon 0\to X\to P\to M\to 0$ be a projective A-presentation of M. Since R is Noetherian, X can be chosen to be finitely generated over A. As M is projective over R, δ splits over R. Therefore, $X\in A\text{-mod}\cap R\text{-proj}$. By assumption, δ splits over A. Thus, (i) follows. ☐

Corollary 1.1.46. *Let R be a local commutative Noetherian ring with maximal ideal $\mathfrak{m}$. Then,* $\operatorname{gldim} R\leq n$ *if and only if* $\operatorname{Tor}^{n+1}_R(R/\mathfrak{m},R/\mathfrak{m})=0$. *In particular,* $\operatorname{gldim} R=\operatorname{pdim}_R(R/\mathfrak{m})$.

Proof. See [Rot09, Theorem 8.55]. ☐

Theorem 1.1.47. *If R is a commutative Noetherian ring, then*

$$\operatorname{gldim}(R)=\sup\{\operatorname{gldim}(R_\mathfrak{m})\colon \mathfrak{m} \textit{ maximal ideal in } R\}.$$

Proof. See [Rot09, Proposition 8.52]. ☐

Another useful tool for commutative rings is completion with respect to ideals. In mathematics, completions of rings are very common. For example, the ring of real numbers is the completion of the ring of rational numbers with respect to the usual norm. For our purposes, it is enough to consider completion of Noetherian rings with respect to maximal ideals. This completion is known as the $\mathfrak{m}$-adic completion. For a detailed exposition of this topic we refer to [GS71].

Let R be a commutative Noetherian ring and $\mathfrak{m}$ be a maximal ideal in R. There is a (possibly infinite) chain

$$\mathfrak{m}\supset\mathfrak{m}^2\supset\cdots\supset\mathfrak{m}^n\supset\cdots. \tag{1.1.2.26}$$

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of R is called a **Cauchy sequence** if for every natural number r, there exists a natural number N such that $x_n-x_s\in\mathfrak{m}^r$ for every $n,s\geq N$. In a complete ring, every Cauchy sequence converges for an element in R. Hence, this motivates the following construction:

For every $n\in N$, $\mathfrak{m}^{n+1}$ is contained in the kernel of the canonical homomorphism $R\to R/\mathfrak{m}^n$. Hence, there exists a map $\pi_n\in\operatorname{Hom}_R(R/\mathfrak{m}^{n+1},R/\mathfrak{m}^n)$ which maps $z+\mathfrak{m}^{n+1}$ to $z+\mathfrak{m}^n$. The **completion of the ring** R (with respect to $\mathfrak{m}$) is the inverse limit $\varprojlim R/\mathfrak{m}^n$ of the sequence of homomorphisms

$$\cdots\xrightarrow{\pi_2}R/\mathfrak{m}^2\xrightarrow{\pi_1}R/\mathfrak{m}. \tag{1.1.2.27}$$

We will denote a completion of the ring R (with respect to $\mathfrak{m}$) by $\widehat{R}$. So,

$$\widehat{R} = \lim_{\leftarrow} R/\mathfrak{m}^n = \{(a_n + \mathfrak{m}^n)_{n\in\mathbb{N}} \in \Pi_n R/\mathfrak{m}^n | \pi_n(a_{n+1} + \mathfrak{m}^{n+1}) = a_n + \mathfrak{m}^n,\ \forall n\}. \tag{1.1.2.28}$$

So the elements of $\widehat{R}$ are sequences $(a_n + \mathfrak{m}^n)_{n\in\mathbb{N}}$ such that $a_{n+1} - a_n \in \mathfrak{m}^n$ for all n. Hence, $\widehat{R}$ can be regarded as an R-module. We can also see that $\widehat{R}$ is a commutative subring of the direct product of commutative rings $R/\mathfrak{m}^n$ over all $n \in \mathbb{N}$.

Analogously, for each $M \in R$-mod, using the chain $\mathfrak{m}N \supset \mathfrak{m}^2M \supset \cdots$ we can define the completion of the module M (with respect to $\mathfrak{m}$). This module is denoted by $\widehat{M}$. Here are some properties of completion.

Proposition 1.1.48. *Let R be a commutative Noetherian ring. Then, the following holds.*

(i) For any $M \in R$-mod, $\widehat{M} = \widehat{R} \otimes_R M$.

(ii) $\widehat{R}$ is a flat commutative Noetherian R-algebra;

(iii) If R is local, then $\widehat{R}$ is faithfully flat local commutative Noetherian R-algebra.

(iv) For any $M \in R$-mod and any maximal ideal $\mathfrak{m}$ in R, $\widehat{M_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}}) = \widehat{M(\mathfrak{m})}$.

Proof. For (i) see [GS71, Theorem 4.6].

By Theorem 5.1 of [GS71], $\widehat{R}$ is a Noetherian R-algebra. By Theorem 4.9 of [GS71], (ii) follows.

If R is a local ring, then Proposition 1.6 of [GS71] implies that R is a Zariski ring. By Theorem 4.9 and Corollary 2.20 of [GS71] (iii) holds.

Let $M \in R$-mod. Then,

$$\begin{aligned}\widehat{M_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}}) &= M_\mathfrak{m} \otimes_{R_\mathfrak{m}} \widehat{R_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}}) \simeq M_\mathfrak{m} \otimes_{R_\mathfrak{m}} \widehat{R_\mathfrak{m}} \otimes_{\widehat{R_\mathfrak{m}}} \widehat{R_\mathfrak{m}}/\widehat{\mathfrak{m}_\mathfrak{m}} \simeq M \otimes_R R_\mathfrak{m} \otimes_{R_\mathfrak{m}} \widehat{R_\mathfrak{m}}/\widehat{\mathfrak{m}_\mathfrak{m}} \qquad (1.1.2.29)\\ &\simeq M \otimes_R \widehat{R_\mathfrak{m}}/\widehat{\mathfrak{m}_\mathfrak{m}} \simeq M \otimes_R \widehat{R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}} \simeq M \otimes_R R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} \otimes_R \widehat{R} \simeq M(\mathfrak{m}) \otimes_R \widehat{R} \simeq \widehat{M(\mathfrak{m})}.\end{aligned}$$ ☐

For us, the main reason to be interested in algebras over local complete rings is the existence of projective covers and over such algebras decompositions into indecomposable modules are essentially unique.

Theorem 1.1.49. *Let R be a local complete commutative Noetherian ring. Let A be a Noetherian R-algebra. Then, A-mod is a Krull-Schmidt category.*

Proof. See [Rei03, pages 88, 89]. ☐

This fact allows us to characterize properties of A-mod through the module categories $A(\mathfrak{m})$-mod where $\mathfrak{m}$ is a maximal ideal in R.

Lemma 1.1.50. *Let R be a commutative Noetherian ring. Let A projective Noetherian R-algebra. Then, M is projective A-module if and only if $\widehat{M_\mathfrak{m}}$ is projective over $\widehat{A_\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ in R. In particular, if $M(\mathfrak{m})$ is projective over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R, then $\widehat{M_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}})$ is projective over $\widehat{A_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Assume that M is a projective A-module. Then, $\widehat{M_\mathfrak{m}} = M_\mathfrak{m} \otimes_R \widehat{R}$ is a projective $A_\mathfrak{m} \otimes_R \widehat{R} = \widehat{A_\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ in R. Assume that $\widehat{M_\mathfrak{m}}$ is projective over $\widehat{A_\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ in R. Let $N \in A_\mathfrak{m}$-mod. Then,

$$\widehat{\operatorname{Ext}^1_{A_\mathfrak{m}}(M_\mathfrak{m}, N)} \simeq \operatorname{Ext}^1_{\widehat{A_\mathfrak{m}}}(\widehat{M_\mathfrak{m}}, \widehat{N}) = 0 \tag{1.1.2.30}$$

As completion over a local ring is faithfully flat, then $\operatorname{Ext}^1_{A_\mathfrak{m}}(M_\mathfrak{m}, N) = 0$. Consequently, $M_\mathfrak{m}$ is projective over $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ in R. Thus, M is projective over A. Let $\mathfrak{m}$ be a maximal ideal in R. Assume that $M(\mathfrak{m})$ is projective over $A(\mathfrak{m})$. Then, $\widehat{M_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}}) \simeq M(\mathfrak{m}) \otimes_R \widehat{R}$ is projective over $A(\mathfrak{m}) \otimes_R \widehat{R} \simeq \widehat{A_\mathfrak{m}}(\widehat{\mathfrak{m}_\mathfrak{m}})$. $\square$

The following result is [CPS90, Lemma 3.3.2].

Theorem 1.1.51. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $M \in A$-mod. Then, M is projective if and only if $M \in R$-proj and $M(\mathfrak{m})$ is projective over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R.*

Proof. Assume that M is projective over A. Then, M is A-summand of A^t for some $t > 0$. In particular, M is an R-summand of A^t. Since A is projective over R, it follows that M is projective over R. On other hand, tensor product commutes with direct sum, hence $A(\mathfrak{m})^t \simeq A^t \otimes_R R(\mathfrak{m}) \simeq M \otimes_R R(\mathfrak{m}) \oplus K$ for every maximal ideal $\mathfrak{m}$ in R. Thus, $M(\mathfrak{m})$ is a projective over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R.

Conversely, assume that $M \in R$-proj and $M(\mathfrak{m})$ is projective over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. In view of Theorem 1.1.45 and Lemma 1.1.50, we can assume without loss of generality that R is a local complete Noetherian ring. So, A is semi-perfect ring. Hence, the endomorphism ring of each projective indecomposable module of A-mod is a local ring. Let P be an indecomposable projective A-module. Then, by Proposition 1.1.31 and Lemma 1.4.32, $\operatorname{End}_{A(\mathfrak{m})}(P(\mathfrak{m})) \simeq \operatorname{End}_A(P)(\mathfrak{m}) \simeq \operatorname{End}_A(P)/\mathfrak{m}\operatorname{End}_A(P)$. Because $\operatorname{End}_A(P)$ has a unique maximal ideal, $\operatorname{End}_{A(\mathfrak{m})}(P(\mathfrak{m}))$ has a unique maximal ideal by the ideal correspondence. In particular, $\operatorname{End}_{A(\mathfrak{m})}(P(\mathfrak{m}))$ is local, and therefore $P(\mathfrak{m})$ is a projective indecomposable $A(\mathfrak{m})$-module.

Let (P, p) be a projective cover of M over A. Applying $R(\mathfrak{m}) \otimes_R -$, it follows that $p(\mathfrak{m})$ is surjective. Since $M(\mathfrak{m})$ is projective over $A(\mathfrak{m})$, $p(\mathfrak{m})$ splits over $A(\mathfrak{m})$. By Krull-Remak-Schmidt theorem, we can write $M(\mathfrak{m})$ into indecomposable projective $A(\mathfrak{m})$-modules. Since the projective indecomposable modules of $A(\mathfrak{m})$-mod are written in the form $P_i(\mathfrak{m})$ for some projective indecomposable module $P_i \in A$-proj, it follows that there exists $k \colon Q \hookrightarrow P$ such that $p(\mathfrak{m}) \circ k(\mathfrak{m})$ is an isomorphism for some summand Q of P. By Nakayama's Lemma, $p \circ k$ is surjective. Since P is the projective cover of M, we must have $Q = P$. Hence, $p(\mathfrak{m})$ is an isomorphism. Since $M \in R$-proj, p is an isomorphism by Lemma 1.1.39. Thus, $M \in A$-proj. $\square$

Theorem 1.1.52. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $M \in A$-mod$\cap R$-proj and $i \geq 0$. Then, $\operatorname{pdim}_A M \leq i$ if and only if $\operatorname{pdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \leq i$ for every maximal ideal $\mathfrak{m}$ in R.*

Proof. Assume that $\operatorname{pdim}_A M \leq i$. Then, there exists a projective A-resolution of length i. As $M \in R$-proj, this resolution is split over R. Therefore, it remains exact under the functor $R(\mathfrak{m}) \otimes_R -$ for every maximal ideal $\mathfrak{m}$ in R. Hence, $\operatorname{pdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \leq i$ for every maximal ideal $\mathfrak{m}$ in R.

Conversely, assume that $\operatorname{pdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \leq i$ for every maximal ideal $\mathfrak{m}$ in R. We will proceed by induction on i to show that $\operatorname{pdim}_A M \leq i$. The case $i = 0$ is Theorem 1.1.51. Assume that the result holds for a certain $t > 0$ and assume that $\operatorname{pdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \leq t + 1$ for every maximal ideal $\mathfrak{m}$ in R. Consider the A-exact sequence

$$0 \to K_t \to P_t \to \cdots \to P_1 \to P_0 \to M \to 0, \tag{1.1.2.31}$$

where each $P_k \in A$-proj, $0 \leq k \leq t$. Again, (1.1.2.31) is split over R and it remains exact under $R(\mathfrak{m}) \otimes_R -$ for every maximal ideal $\mathfrak{m}$ in R. In particular, $K_t \in R$-proj. Because of $\operatorname{pdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \leq t + 1$ and $P_k(\mathfrak{m})$ being projective over $A(\mathfrak{m})$, $K_t(\mathfrak{m}) \in A(\mathfrak{m})$-proj for every maximal ideal $\mathfrak{m}$ in R. By Theorem 1.1.51, $K_t \in A$-proj. Now, the exact sequence (1.1.2.31) gives that $\operatorname{pdim}_A M \leq t + 1$. $\square$

1.1.3 Krull dimension and regular rings

The **Krull dimension** of a commutative ring R is the supremum of the lengths of chains of distinct prime ideals in R. We denote by $\dim R$ the Krull dimension of R. This definition was introduced to provide a notion of dimension of an affine algebraic variety. In fact, the dimension of an affine algebraic variety can be defined as the Krull dimension of its coordinate ring. We refer to [Eis95] for more details.

Lemma 1.1.53. *Let R be a commutative ring. The following assertions are equivalent.*

1. *R is an Artinian ring;*
2. *R is a Noetherian ring with $\dim R = 0$;*
3. *R is a Noetherian ring and any prime ideal in R is maximal.*

Proof. See Theorem 2.14 of [Eis95]. □

It follows that the Krull dimension of a Noetherian ring measures how far the ring is from being Artinian. It is commonly known that the Krull dimension can be computed locally.

Proposition 1.1.54. *If R is a commutative Noetherian ring, then*

$$\dim(R) = \sup\{\dim(R_{\mathfrak{m}})\colon \mathfrak{m} \textit{ maximal ideal in } R\}.$$

Proof. This is consequence of Proposition 1.1.16. In fact, every chain of prime ideals in $R_{\mathfrak{m}}$ is induced by a chain of prime ideals in R which are contained in R. So, $\dim R \geq \dim R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ in R. On the other hand, every chain of prime ideals in R ends with a maximal ideal, say $\mathfrak{m}$. Localizing at $\mathfrak{m}$ gives a chain of prime ideals with the same length in $R_{\mathfrak{m}}$. Thus, the result follows. □

Lemma 1.1.55. *Let R be a local commutative Noetherian ring with maximal ideal $\mathfrak{m}$. Fix $R(\mathfrak{m}) = R/\mathfrak{m}$. Then, the following assertions hold.*

(a) *The number $\dim_{R(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$ is the minimum number of generators for the ideal $\mathfrak{m}$;*

(b) $\dim R(\mathfrak{m}) \leq \dim_{R(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 < \infty$.

Proof. For (a) see Proposition 11.165 of [Rot10]. For statement (b) see Corollary 11.166 of [Rot10]. □

The value $\dim_{R(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$ denoted by $V(R)$ is called the **embedding dimension** of the ring R. The embedding dimension of a local ring R arises as the dimension of a tangent of a point of an algebraic variety (see [Rot09, Example 8.57]).

We say that a local commutative Noetherian ring is **regular** if $\dim R = V(R)$. Geometrically, this means that the dimension of the tangent space at each point is exactly the dimension of the affine variety. In such case, the affine variety is called **smooth**. A commutative Noetherian ring is **regular** if each localization $R_{\mathfrak{m}}$ is a regular ring for every maximal ideal $\mathfrak{m}$ in R.

Proposition 1.1.56. *Let R be a local commutative Noetherian ring with maximal ideal $\mathfrak{m}$. Let $x \in \mathfrak{m}/\mathfrak{m}^2$. Then, $V(R/Rx) = V(R) - 1$.*

Proof. See [Rot09, Proposition 8.56]. □

Lemma 1.1.57. *Let R be a local regular ring. Then, the following hold.*

(a) Let $x \in \mathfrak{m}/\mathfrak{m}^2$. Then, R/Rx is regular Noetherian.

(b) If $\dim R = 0$, then R is a field.

(c) R is an integral domain.

Proof. If $\dim R = 0 = \dim_{R(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$, then $\mathfrak{m}/\mathfrak{m}^2 = 0$. By Nakayama's Lemma $\mathfrak{m} = 0$. Thus, R is a field. For the remaining see, for example, [Wei03, Proposition 4.4.5]. ☐

Note however that not every regular ring is an integral domain. For example, $\mathbb{Z}/6\mathbb{Z}$ is a regular ring but it is not an integral domain.

Lemma 1.1.58. *Let R be a local commutative Noetherian ring. If $x \in \mathfrak{m}/\mathfrak{m}^2$ is a non-zero divisor and R/Rx is regular, then R is regular.*

Proof. It follows from the proof of Theorem 8.62 of [Rot09]. ☐

Theorem 1.1.59. *A local commutative Noetherian ring is regular if and only if $\operatorname{gldim}(R) < \infty$. Moreover, in this case, $\operatorname{gldim}(R) = \dim R = V(R) = \operatorname{pdim}_R(R/\mathfrak{m})$.*

Proof. See [Rot09, Theorem 8.62, Proposition 8.60]. ☐

Theorem 1.1.60. *If R is a regular local ring and $\mathfrak{p}$ is a prime ideal in R, then $R_\mathfrak{p}$ is a regular local ring.*

Proof. See [Rot09, Corollary 8.63]. ☐

In particular, if a commutative Noetherian ring R has finite global dimension, then for every maximal ideal $\mathfrak{m}$ in R the localization $R_\mathfrak{m}$ is a regular local ring.

Proposition 1.1.61. *Let R be a commutative regular ring. Then, $\dim R[X_1, \dots, X_n] = n + \dim R$.*

Proof. See for example Theorem A of [Eis95] for R a field. See, for example, Theorem 8.37 of [Rot09] for the general case. ☐

This gives explicitly that the dimension of an affine space of dimension n has indeed dimension n.

Complete local regular rings are completely described by Cohen's structure theorem [Coh46] (see also [Ive14, Corollary 10.32, Corollary 10.33]). Using this characterization we can state that these rings are faithfully flat over some complete discrete valuation ring or over some field.

Theorem 1.1.62. *Let R be a complete local regular ring. Then, there exists a complete discrete valuation ring or a field k making R a faithfully flat k-algebra.*

Proof. There are two cases. Either R is equicharacteristic, that is, the characteristic of R equals to the characteristici of $R/\mathfrak{m}$, $\mathfrak{m}$ maximal ideal of R, or R is not equicharacteristic.

Assume that R is equicharacteristic. By Corollary 10.32 of [Ive14], there exists a field k such that $R \simeq k[|X_1, \dots, X_n|]$, where n is the Krull dimension of R. It is clear that R is faithfully flat over k.

Assume now that R is not equicharacteristic. By Corollary 10.33 of [Ive14], there exists a discrete valuation ring k with maximal ideal $k\pi$ such that $R \simeq k[|X_1, \dots X_n|]/(a)$, where n is the Krull dimension of R and a is a power series belonging to $(X_1, \dots, X_n, \pi) \backslash (X_1, \dots, X_n, \pi)^2$.

We will start by showing the flatness of R over k. By Theorem 1.1.44, we want to show that $\operatorname{Tor}_1^k(R, k/k\pi) = 0$. In other words, we want to show that $k[|X_1, \dots X_n|]/(a)$ has no π-torsion. Let $g \in k[|X_1, \dots X_n|]$ such that $\pi g \in (a)$.

Since R is an integral domain, (a) is a prime ideal. So, we must have $g \in (a)$ or $\pi \in (a)$. But if the second case holds, then the characteristic of R should be positive which contradicts the fact that R is not equicharacteristic. So, g belongs to (a). This shows that R is flat over k. Now, to show that R is faithfully flat over k, it is enough to show that $\pi k[|X_1, \dots X_n|]/(a) \neq k[|X_1, \dots X_n|]/(a)$ (see [GS71]). Assume that $\pi k[|X_1, \dots X_n|]/(a) = k[|X_1, \dots X_n|]/(a)$. Then, there exists $f \in k[|X_1, \dots X_n|]$ such that $1 - \pi f \in (a)$. Consequently, there exists $l \in k[|X_1, \dots X_n|]$ such that $1 = \pi f + ka \in (X_1, \dots, X_n, \pi)$. This cannot happen since $(X_1, \dots, X_n, \pi)$ is a maximal ideal. Therefore, R is faithfully flat over k. □

1.1.4 Standard duality on Hom and $\otimes$

A commonly known fact which will be extensively used in this and the following chapters is the Tensor-Hom adjunction.

Lemma 1.1.63 (Tensor-Hom adjunction). *Let R, S be two rings. Let N be a left S-module, U an (R,S)-bimodule and W a left R-module. Then, the canonical maps*

$$\sigma\colon \operatorname{Hom}_R(U \otimes_S N, W) \to \operatorname{Hom}_S(N, \operatorname{Hom}_R(U,W)) \text{ and } \rho\colon \operatorname{Hom}_S(N, \operatorname{Hom}_R(U,W)) \to \operatorname{Hom}_R(U \otimes_S N, W)$$

given by

$$\sigma(f)(n)(u) = f(u \otimes n),\ f \in \operatorname{Hom}_R(U \otimes_S N, W),\ n \in N,\ u \in U \tag{1.1.4.1}$$

$$\rho(f)(u \otimes n) = f(n)(u),\ f \in \operatorname{Hom}_S(N, \operatorname{Hom}_R(U,W)),\ u \otimes n \in U \otimes_S N, \tag{1.1.4.2}$$

are inverses of each other.

Proof. See Lemma 1.7.9 of [Zim14]. □

The following results are commonly known for finite-dimensional algebras. The usual arguments carry over to this setting if we restrict to the A-modules which are projective over the ground ring.

Proposition 1.1.64. *Let A be a Noetherian R-algebra. Assume $M, N \in A\text{-mod} \cap R\text{-proj}$ then the map $\psi_{M,N}\colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{A^{op}}(DN, DM)$, given by, $\psi_{M,N}(g)(h) = h \circ g$, $g \in \operatorname{Hom}_A(M,N)$, $h \in DN$, is a $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule isomorphism, where D is the standard duality.*

Proof. Consider the map $e_M\colon M \to DDM$, given by $e_M(m)(g) = g(m)$. This is an $(A, \operatorname{End}_A(M)^{op})$-bimodule homomorphism. Consider $M = R^n$. Then, we have an R-basis for DM $\{e_i^*,\ 1 \le i \le n\}$. We claim that e_M is injective. In fact, let $m = \sum_i \alpha_i e_i \in M$ such that $e_M(m) = 0$. Then,

$$0 = e_M\left(\sum_i \alpha_i e_i\right)(e_j^*) = \alpha_j, \quad \forall 1 \le j \le n \quad \Rightarrow m = 0.$$

Now consider $h \in DDM$. Thus, $h = \sum_i \alpha_i (e_i^*)^*$. Let $m = \sum_i \alpha_i e_i$. Then, $e_M(m)(e_j^*) = \alpha_j = h(e_j^*)$. Thus, e_M is an $(A, \operatorname{End}_A(M)^{op})$-bimodule isomorphism. Now assume M is finitely generated projective over R. There exists $n \in \mathbb{N}$ such that $R^n \simeq M \bigoplus K$. Hence, $DDR^n \simeq DDM \bigoplus DDK$ and we have that the map e_M is compatible with direct sums. So, e_M is $(A, \operatorname{End}_A(M)^{op})$-bimodule isomorphism.

Define the map $\delta\colon \operatorname{Hom}_A(DDM, DDN) \to \operatorname{Hom}_A(M,N)$, given by $\delta(h) = e_N^{-1} \circ h \circ e_M$, $h \in \operatorname{Hom}_A(DDM, DDN)$. This map is bijective since e_M and e_N^{-1} are. Moreover, $\delta(h)$ is given by the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta(h)} & N \\ \downarrow{\scriptstyle e_M} & & \downarrow{\scriptstyle e_N} \\ DDM & \xrightarrow{h} & DDN \end{array}.$$

Now consider $g \in \operatorname{Hom}_A(M,N)$, $m \in M$ and $f \in DN$. We deduce that

$$e_N \circ \delta \circ \psi_{DN,DM} \circ \psi_{M,N}(g)(m)(f) = e_N \circ \delta(\psi_{DN,DM}(\psi_{M,N}(g)))(m)(f) = \psi_{DN,DM}(\psi_{M,N}(g)) \circ e_M(m)(f) = \\ = e_M(m) \circ \psi_{M,N}(g)(f) = e_M(m)(f \circ g) = f \circ g(m). \tag{1.1.4.3}$$

On the other hand, $e_N \circ \mathrm{id}_{\operatorname{Hom}_A(M,N)}(g)(m)(f) = e_N \circ g(m)(f) = e_N(g(m))(f) = f \circ g(m)$.

Therefore, $e_N \circ \mathrm{id}_{\operatorname{Hom}_A(M,N)} = e_N \circ \delta \circ \psi_{DN,DM} \circ \psi_{M,N}$. Hence, $\delta \circ \psi_{DN,DM} \circ \psi_{M,N} = \mathrm{id}_{\operatorname{Hom}_A(M,N)}$. As δ is bijective, $\psi_{DN,DM}$ is surjective. By a symmetric argument, we obtain $\delta' \circ \psi_{DDN,DDM} \circ \psi_{DM,DN} = \mathrm{id}_{\operatorname{Hom}_A(DM,DN)}$. Hence, $\psi_{DM,DN}$ is also an injective map. It follows that $\psi_{M,N}$ is a bijective map. It remains to see that $\psi_{M,N}$ is an $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule homomorphism.

For every $g \in \operatorname{Hom}_A(M,N)$, $h \in DN$, $m \in M$ and $b \in B$ we have

$$\psi_{M,N}(b \cdot g)(h)(m) = h \circ (b \cdot g)(m) = h(g(m \cdot b)) = (b \cdot (h \circ g))(m) = (b \cdot (\psi_{M,N}(g)(h)))(m) \\ = (b \cdot \psi(g))(h)(m).$$

The argument for $\psi_{M,N}$ being a right $\operatorname{End}_A(N)^{op}$-module homomorphism is analogous. □

Proposition 1.1.65. *Let A be a Noetherian R-algebra. Assume $M,N \in A\text{-mod} \cap R\text{-proj}$. Then, the map*

$$\kappa_{M,N} \colon \operatorname{Hom}_A(M,N) \to D(DN \otimes_A M), \text{given by } \kappa(g)(f \otimes m) = f(g(m)),\ g \in \operatorname{Hom}_A(M,N),\ f \in DN,\ m \in M$$

is an $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule isomorphism.

Moreover if $DN \otimes_A M \in R\text{-proj}$ the map

$$\iota_{M,N} \colon DN \otimes_A M \to D\operatorname{Hom}_A(M,N), \text{given by } \iota(f \otimes m)(g) = f(g(m)),\ f \otimes m \in DN \otimes_A M,\ g \in \operatorname{Hom}_A(M,N)$$

is an $(\operatorname{End}_A(N)^{op}, \operatorname{End}_A(M)^{op})$-bimodule isomorphism.

Proof. The map $\operatorname{Hom}_A(M,e_N) \colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(M,DDN)$ is an $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule isomorphism where $e_N \colon N \to DDN$ is the canonical $(A, \operatorname{End}_A(N)^{op})$-bimodule isomorphism. Consider $\rho \colon \operatorname{Hom}_A(M,DDM) \to D(DN \otimes_A M)$ the $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})-$bimodule isomorphism given by tensor-hom adjunction. Then, the composition $\kappa = \rho \circ \operatorname{Hom}_A(M,e_N) \colon \operatorname{Hom}_A(M,N) \to D(DN \otimes_A M)$ is an $(\operatorname{End}_A(M)^{op}, \operatorname{End}_A(N)^{op})$-bimodule isomorphism. Note that

$$k(g)(f \otimes m) = \operatorname{Hom}_A(M,e_N)(g)(m)(f) = e_N \circ g(m)(f) = f(g(m)),\ g \in \operatorname{Hom}_A(M,N),\ f \otimes m \in DN \otimes_A M.$$

Now assume that $DN \otimes_A M \in R$-proj, the map $e_{DN \otimes_A M}$ is an isomorphism. So, the composition $\iota = D\kappa \circ e_{DN \otimes_A M}$ is an $(\operatorname{End}_A(N)^{op}, \operatorname{End}_A(M)^{op})$-bimodule isomorphism and

$$\iota(f \otimes m)(g) = D\kappa \circ e_{DN \otimes_A M}(f \otimes m)(g) = e_{DN \otimes_A M}(f \otimes m) \circ \kappa(g) = \kappa(g)(f \otimes m) \\ = f(g(m)),\ f \otimes m \in DN \otimes_A M,\ g \in \operatorname{Hom}_A(M,N).$$

□

1.2 Relative homological algebra

Every exact sequence of modules over a finite-dimensional algebra over a field K splits as an exact sequence of modules over K. But, this is no longer the case if we replace the field K by a Noetherian commutative ring (which is not a field). This motivates us to focus only on exact sequences of an algebra over a commutative ring R that split as an exact sequence of modules over R and ignore the other exact sequences. This idea is due to [Hoc56]. This concept did not gain as much attention as it should in the context of Noetherian algebras. In this section, we present a detailed exposition of this concept based on the treatment of [Hoc56] and [Hat63]. In this section, unless stated otherwise, R is a commutative ring (not necessarily Noetherian) and A is an R-algebra.

1.2.1 (A,R)-exact sequences

In this subsection, assume only that R is a commutative ring (not necessarily Noetherian) and A an R-algebra.

Definition 1.2.1. Let R be a commutative ring. Let A be an R-algebra. An exact sequence between A-modules

$$\cdots \to M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \to \cdots$$

is called (A,R)**-exact** if, for each i, there exists a map $h_i \in \operatorname{Hom}_R(M_i, M_{i+1})$ such that $h_{i-1}\circ t_i + t_{i+1}\circ h_i = \mathrm{id}_{M_i}$.

That is, we are interested in the exact sequences over A that vanish in the homotopy category of chain complexes $K(R)$.

Proposition 1.2.2. *Let R be a commutative ring. Let A be an R-algebra. An exact sequence between A-modules*

$$\cdots \to M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \to \cdots$$

is (A,R)-exact if and only if for each i, $\ker t_i = \operatorname{im} t_{i+1}$ is a summand of M_i as R-module.

Proof. Let $\cdots \to M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \to \cdots$ be an (A,R)-exact sequence. Then, for each i,

$$t_i = t_i\circ h_{i-1}\circ t_i + t_i\circ t_{i+1}\circ h_i = t_i\circ h_{i-1}\circ t_i. \tag{1.2.1.1}$$

Therefore, $t_i\circ h_{i-1}$ is an idempotent in $\operatorname{End}_R(M_{i-1})$. Hence, $M_{i-1} \simeq t_i\circ h_{i-1}(M_{i-1}) \oplus (\mathrm{id}_{M_{i-1}} - t_i\circ h_{i-1})(M_{i-1})$. Since $t_{i-1}\circ t_i = 0$, $t_i\circ h_{i-1}(M_{i-1}) \subset \ker t_{i-1}$. On the other hand, $\ker t_{i-1} = \operatorname{im} t_i = \operatorname{im} t_i\circ h_{i-1}\circ t_i \subset \operatorname{im} t_i\circ h_{i-1}$. Thus, $\ker t_i$ is an R-summand of M_i.

Conversely, assume that, for each i, $\ker t_i = \operatorname{im} t_{i+1}$ is a summand of M_i as R-module. Thus, the exact sequences $0 \to \ker t_i \xrightarrow{\nu_i} M_i \xrightarrow{\sigma_i} \ker t_{i-1} \to 0$ satisfying, $t_i = \nu_{i-1}\circ\sigma_i$ for all i, split over R. Fix $\pi_i \colon M_i \to \ker t_i$ and $\gamma_i \colon \ker t_{i-1} \to M_i$ the split R-homomorphism of π_i. Define $h_i := \gamma_{i+1}\circ\pi_i$. Then,

$$h_{i-1}\circ t_i + t_{i+1}\circ h_i = \gamma_i\circ\pi_{i-1}\circ t_i + t_{i+1}\circ\gamma_{i+1}\circ\pi_i = \gamma_i pi_{i-1}\nu_i\sigma_i + \nu_i\sigma_{i+1}\gamma_{i+1}\pi_i = \gamma_i\sigma_i + \nu_i\pi_i = \mathrm{id}_{M_i}. \qquad \square$$

In this formulation, we can see that the (A,R)-short exact sequences are exactly the exact sequences of A-modules which are split as a sequence of R-modules. A homomorphism ϕ is called an (A,R)-**monomorphism** if $0 \to M \xrightarrow{\phi} N$ is (A,R)-exact. An homomorphism ϕ is called an (A,R)-**epimorphism** if $M \xrightarrow{\phi} N \to 0$ is (A,R)-exact.

Lemma 1.2.3. *Let A be an algebra over a commutative ring R. By D we denote the standard duality functor $D = \operatorname{Hom}_R(-,R) \colon A\text{-mod} \to A^{op}\text{-mod}$ (with respect to R). Let $\cdots \to M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \to \cdots$ be an (A,R)-*

exact sequence. If the complex

$$\cdots \to DM_{i-1} \xrightarrow{Dt_i} DM_i \xrightarrow{Dt_{i+1}} DM_{i+1} \to \cdots$$

is exact over A^{op}*, then it is* (A^{op},R)*-exact. In particular, every* (A,R)*-exact sequence* $Y_1 \to Y_0 \to X \to 0$ *is sent to an* (A^{op},R)*-exact sequence* $0 \to DX \to DY_0 \to DY_1$.

Proof. Since $\cdots \to M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \to \cdots$ is (A,R)-exact there exists R-homomorphisms $h_i \in \operatorname{Hom}_R(M_i, M_{i+1})$ such that $h_{i-1} \circ t_i + t_{i+1} \circ h_i = \mathrm{id}_{M_i}$. Thus, $Dh_i \in \operatorname{Hom}_R(DM_{i+1}, DM_i)$ and

$$\mathrm{id}_{DM_i} = D(\mathrm{id}_{M_i}) = D(h_{i-1} \circ t_i + t_{i+1} \circ h_i) = D(h_{i-1} \circ t_i) + D(t_{i+1} \circ h_i) = Dt_i \circ Dh_{i-1} + Dh_i \circ Dt_{i+1}. \quad (1.2.1.2)$$

If the complex $\cdots \to DM_{i-1} \xrightarrow{Dt_i} DM_i \xrightarrow{Dt_{i+1}} DM_{i+1} \to \cdots$ is exact over A^{op}, then it is (A^{op},R)-exact by Definition 1.2.1. Since the functor $D = \operatorname{Hom}_R(-,R)$ is contravariant left exact and it preserves the homotopy maps h_i, the second claim follows. ☐

Remark 1.2.4. The functor standard duality $D = \operatorname{Hom}_R(-,R)$ preserves (A,R)-exact sequences of the form

$$0 \to X \to Y \to Z \to 0, \quad Z \in R\text{-proj}. \qquad \triangle$$

The class of (A,R)-exact sequences is Morita invariant. Furthermore, every Schur functor sends (A,R)-exact sequences to (B,R)-exact sequences.

Proposition 1.2.5. *Let* M *be a finitely generated projective left* A*-module. Fix* $B = \operatorname{End}_A(M)^{op}$*. Then, the functor* $F = \operatorname{Hom}_A(M,-)$ *sends* (A,R)*-exact sequences to* (B,R)*-exact sequences.*

Proof. Since $M \in A$-proj, the functor F is exact. Thus, F preserves all A-exact sequences. Let

$$\cdots \to X_{i+1} \xrightarrow{t_{i+1}} X_i \xrightarrow{t_i} X_{i-1} \to \cdots \quad (1.2.1.3)$$

be an (A,R)-exact sequence. In particular,

$$0 \to \ker t_i \xrightarrow{\nu_i} X_i \xrightarrow{\sigma_i} \ker t_{i-1} \to 0 \quad (1.2.1.4)$$

is (A,R)-exact satisfying $t_i = \nu_{i-1} \circ \sigma_i$ for all i. Applying F yields the B-exact sequence

$$0 \to \ker Ft_i \xrightarrow{F\nu_i} FX_i \xrightarrow{F\sigma_i} \ker Ft_{i-1} \to 0, \quad (1.2.1.5)$$

satisfying $Ft_i = F\nu_{i-1} \circ F\sigma_i$. So, it is enough to show that $\ker Ft_i$ is an R-summand of FX_i with split mononomorphism $F\nu_i$. So, it is enough to check that F sends (A,R)-monomorphisms to (B,R)-monomorphisms.

Let $0 \to Y \xrightarrow{\iota} X$ be an (A,R)-monomorphism. In particular, there exists a homomorphism $\pi \in \operatorname{Hom}_R(X,Y)$ satisfying $\pi \circ \iota = \mathrm{id}_Y$. Since $M \in A$-proj, there exists $n \in \mathbb{N}$ and a module K such that $A^n \simeq M \oplus K$. Fix $\pi_M \colon A^n \to M$ and $k_M \colon M \to A^n$ the canonical projection and inclusion, respectively. Let $\pi_i \colon A^n \to A$ and $k_i \colon A \to A^n$ be the canonical projections and inclusions $i = 1, \dots, n$. Denote by ψ_X and ψ_Y^{-1} the usual isomorphisms $\psi_X \colon \operatorname{Hom}_A(A^n, X) \to X^n$ and $\psi_Y^{-1} \colon Y^n \to \operatorname{Hom}_A(A^n, Y)$, respectively.

Consider $\psi := \operatorname{Hom}_A(k_M, Y) \circ \psi_Y^{-1} \circ (\pi, \cdots, \pi) \circ \psi_X \circ \operatorname{Hom}_A(\pi_M, X) \in \operatorname{Hom}_R(FX, FY)$. Let $g \in \operatorname{Hom}_A(M,Y)$ and $m \in M$. Then,

$$\begin{aligned}
\psi \circ \operatorname{Hom}_A(M,\iota)(g)(m) = \psi(\iota \circ g)(m) &= \operatorname{Hom}_A(k_M, Y) \circ \psi_Y^{-1} \circ (\pi, \cdots, \pi) \circ \psi_X \circ \operatorname{Hom}_A(\pi_M, X)(\iota \circ g)(m) \\
&= \psi_Y^{-1}((\pi, \cdots, \pi)(\psi_X(\iota \circ g \circ \pi_M)))(k_M(m))
\end{aligned}$$

$$
\begin{aligned}
&= \psi_Y^{-1}((\pi,\cdots,\pi)(\iota\circ g\circ\pi_M\circ k_1(1_A),\cdots,\iota\circ g\circ\pi_M\circ k_n(1_A))(k_M(m)) \\
&= \psi_Y^{-1}(g\circ\pi_M\circ k_1(1_A),\cdots,g\circ\pi_M\circ k_n(1_A))(k_M(m)) \\
&= \sum_{i=1}^{n}\pi_i(k_M(m))g\circ\pi_M\circ k_i(1_A) = \sum_{i=1}^{n} g\circ\pi_M\circ k_i(\pi_i(k_M(m))) \\
&= g\circ\pi_M\circ k_M(m) = g(m).
\end{aligned}
$$

Therefore, $\psi\circ\mathrm{Hom}_A(M,\iota) = \mathrm{id}_{FY}$. This concludes the proof. ☐

Proposition 1.2.6. *Let V be a finitely generated projective right A-module. Fix $B = \mathrm{End}_A(V)$. Then, the functor $V\otimes_A -\colon A\text{-Mod}\to B\text{-Mod}$ sends (A,R)-exact sequences to (B,R)-exact sequences.*

Proof. By Lemma 1.4.11, the functors $V\otimes_A - \simeq \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),A)\otimes_A - \simeq \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),-)$ are equivalent. Since $\mathrm{Hom}_A(V,A)\in A$-proj, it follows by Proposition 1.2.5, that $V\otimes_A -$ sends (A,R)-exact sequences to (B,R)-exact sequences. ☐

Both Proposition 1.2.5 and Proposition 1.2.6 work also for right modules, using the same arguments.

Corollary 1.2.7. *Let $F\colon A\text{-mod}\to B\text{-mod}$ be an equivalence of categories. Then, F sends (A,R)-exact sequences to (B,R)-exact sequences.*

Proof. By Morita theory (see for example Theorem 1.4.17), there is some projective generator $P\in A$-mod such that $F = \mathrm{Hom}_A(P,-)$ and $B\simeq \mathrm{End}_A(P)^{op}$. By Proposition 1.2.5, the claim follows. ☐

Definition 1.2.8. Let R be a commutative ring. Let A be an R-algebra. An A-module Q is (A,R)**-projective** if every (A,R)-exact sequence $0\to M\to N\to Q\to 0$ splits as a sequence of A-modules.

An A-module Q is (A,R)**-injective** if every (A,R)-exact sequence $0\to Q\to N\to M\to 0$ splits as a sequence of A-modules.

In order to relate the concepts of (A,R)-injective and (A,R)-projective modules to the functor Hom we need the following lemmas.

Lemma 1.2.9. *[Hoc56, Lemma 1, Lemma 2] For every R-module M, consider the left A-module $\mathrm{Hom}_R(A,M)$ and the left A-modules $A\otimes_R M$. Then, the following holds.*

(a) The functor $\mathrm{Hom}_A(-,\mathrm{Hom}_R(A,M))$ is exact on (A,R)-exact sequences.

(b) The functor $\mathrm{Hom}_A(A\otimes_R M,-)$ is exact on (A,R)-exact sequences.

(c) For any $X\in\mathrm{add}(\mathrm{Hom}_R(A,M))$, the functor $\mathrm{Hom}_A(-,X)$ is exact on (A,R)-exact sequences.

(d) For any $X\in\mathrm{add}(A\otimes_R M)$, the functor $\mathrm{Hom}_A(X,-)$ is exact on (A,R)-exact sequences.

(e) Let

$$0\to X\to Y\to Z\to 0 \tag{1.2.1.6}$$

be an A-exact sequence. Assume that (1.2.1.6) remains exact under $\mathrm{Hom}_A(A\otimes_R M,-)$ for every $M\in R\text{-mod}$, then (1.2.1.6) is (A,R)-exact.

Proof. Let

$$0\to U\xrightarrow{p} V\xrightarrow{q} W\to 0 \tag{1.2.1.7}$$

be an (A,R)-exact sequence. In particular, there exists an R-homomorphism $\pi\colon$ $\mathrm{Hom}_R(V,U)$ satisfying $\pi\circ p=\mathrm{id}_U$ and an R-homomorphism $\gamma\in\mathrm{Hom}_R(W,V)$ satisfying $q\circ\gamma=\mathrm{id}_W$. Thus, $\mathrm{Hom}_R(p,M)(f\circ\pi)=f$ for any $f\in\mathrm{Hom}_R(U,M)$. Consequently, $\mathrm{Hom}_R(p,M)$ is surjective. By the commutativity of the following diagram,

$$\begin{tikzcd}
\mathrm{Hom}_A(V,\mathrm{Hom}_R(A,M)) \arrow[r, "\mathrm{Hom}_A(p,\mathrm{Hom}_R(A,M))"] \arrow[d, "\simeq"] & \mathrm{Hom}_A(U,\mathrm{Hom}_R(A,M)) \arrow[d, "\simeq"] \\
\mathrm{Hom}_R(V,M) \arrow[r, "\mathrm{Hom}_R(p,M)"] & \mathrm{Hom}_R(U,M)
\end{tikzcd},$$

$\mathrm{Hom}_A(p,\mathrm{Hom}_R(A,M))$ is surjective. Since $\mathrm{Hom}_A(-,\mathrm{Hom}_R(A,M))$ is left exact, it follows that

$$0\longrightarrow \mathrm{Hom}_A(W,\mathrm{Hom}_R(A,M))\xrightarrow{\mathrm{Hom}_A(q,\mathrm{Hom}_R(A,M))}\mathrm{Hom}_A(V,\mathrm{Hom}_R(A,M))\longrightarrow \mathrm{Hom}_A(U,\mathrm{Hom}_R(A,M))\xrightarrow{\mathrm{Hom}_A(p,\mathrm{Hom}_R(A,M))}0$$

is exact. Hence, (a) follows.

For any $f\in\mathrm{Hom}_R(M,W)$, $\mathrm{Hom}_R(M,q)(\gamma\circ f)=q\circ\gamma\circ f=f$. Hence, $\mathrm{Hom}_R(M,q)$ is surjective. By the diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(M,V) \arrow[r, "\mathrm{Hom}_R(M,q)"] \arrow[d, "\simeq"] & \mathrm{Hom}_R(M,W) \arrow[d, "\simeq"] \\
\mathrm{Hom}_A(A\otimes_R M,V) \arrow[r, "\mathrm{Hom}_A(A\otimes_R M,q)"] & \mathrm{Hom}_A(A\otimes_R M,W)
\end{tikzcd},$$

$\mathrm{Hom}_A(A\otimes_R M,q)$ is surjective. Since $\mathrm{Hom}_A(A\otimes_R M,-)$ is left exact, $\mathrm{Hom}_A(A\otimes_R M,-)$ is exact on (1.2.1.7). So, (b) holds. Let $X\in\mathrm{add}A\otimes_R M$, then $(A\otimes_R M)^t\simeq X\oplus Y$ for some $t>0$. Hence, the functor

$$\mathrm{Hom}_A(X\oplus Y,-)\simeq\mathrm{Hom}_A(A\otimes_R M)^t,-)\simeq\mathrm{Hom}_A(A\otimes_R(M^t),-)\tag{1.2.1.8}$$

is exact on (A,R)-exact sequences. Thus, $\mathrm{Hom}_A(X\oplus Y,q)$ is surjective. Using the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(X\oplus Y,V) \arrow[r, "\mathrm{Hom}_A(X\oplus Y,q)"] \arrow[d, "\simeq"] & \mathrm{Hom}_A(X\oplus Y,W) \arrow[d, "\simeq"] \\
\mathrm{Hom}_A(X,V)\oplus\mathrm{Hom}_A(Y,V) \arrow[r, "\mathrm{Hom}_A(X,q)\oplus\mathrm{Hom}_A(Y,q)"] \arrow[d, two heads] & \mathrm{Hom}_A(X,W)\oplus\mathrm{Hom}_A(Y,W) \arrow[d, two heads] \\
\mathrm{Hom}_A(X,V) \arrow[r, "\mathrm{Hom}_A(X,q)"] & \mathrm{Hom}_A(X,W)
\end{tikzcd},$$

it follows that $\mathrm{Hom}_A(X,q)$ is surjective. Consequently, $\mathrm{Hom}_A(X,-)$ is exact on (A,R)-exact sequences. So, (d) follows. Dually, using contravariant functors (c) follows.

Let

$$0\to X\to Y\xrightarrow{h}Z\to 0\tag{1.2.1.9}$$

be an A-exact sequence. Assume that (1.2.1.9) remains exact under $\mathrm{Hom}_A(A\otimes_R M,-)$ for every $M\in R$-Mod. In particular, it remains exact under $\mathrm{Hom}_A(A\otimes_R Z,-)$. This gives that the following sequence is exact

$$0\to\mathrm{Hom}_A(A\otimes_R Z,X)\to\mathrm{Hom}_A(A\otimes_R Z,Y)\to\mathrm{Hom}_A(A\otimes_R Z,Z)\to 0.\tag{1.2.1.10}$$

This exact sequence is equivalent to

$$0 \to \mathrm{Hom}_R(Z,X) \to \mathrm{Hom}_R(Z,Y) \xrightarrow{\mathrm{Hom}_R(Z,h)} \mathrm{Hom}_R(Z,Z) \to 0. \tag{1.2.1.11}$$

Hence, there exists $g \in \mathrm{Hom}_R(Z,Y)$ such that $\mathrm{id}_Z = h \circ g$. Hence, (1.2.1.9) splits over R. ☐

Proposition 1.2.10. *[Hoc56, 1.] Let A be an R-algebra. The following assertions are equivalent.*

(a) M is (A,R)-injective, that is, every (A,R)-exact sequence

$$0 \to M \to V \to W \to 0$$

is split over A;

(b) The natural homomorphism of A-modules $\varepsilon\colon M \xrightarrow{\simeq} \mathrm{Hom}_A(A,M) \to \mathrm{Hom}_R(A,M)$, $\varepsilon(m)(a) = am$, $\forall a \in A$, $m \in M$, splits over A;

(c) The functor $\mathrm{Hom}_A(-,M)$ is exact on (A,R)-exact sequences;

(d) For every (A,R)-exact sequence $0 \to U \xrightarrow{p} V \xrightarrow{q} W \to 0$ and every A-homomorphism $U \to M$ there exists an A-homomorphism $V \to M$ making the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "p"] \arrow[d, "\forall"'] & V \arrow[r, "q"] \arrow[dl, "\exists"] & W \arrow[r] & 0 \\ & M & & & \end{tikzcd}.$$

Proof. (a) $\Longrightarrow$ (b). Notice that $\varepsilon'\colon \mathrm{Hom}_R(A,M) \to M$, given by $\varepsilon'(f) = f(1_A)$, $f \in \mathrm{Hom}_R(A,M)$, is an R-homomorphism since

$$\varepsilon'(rf) = rf(1_A) = f(1_A r) = r(f(1_A)) = r\varepsilon'(f),\ \forall r \in R, f \in \mathrm{Hom}_R(A,M). \tag{1.2.1.12}$$

Moreover, $\varepsilon' \circ \varepsilon = \mathrm{id}_M$. So, the exact sequence

$$0 \to M \xrightarrow{\varepsilon} \mathrm{Hom}_R(A,M) \to \operatorname{coker} \varepsilon \to 0 \tag{1.2.1.13}$$

is (A,R)-exact. By assumption, it splits over A. In particular, there exists $f \in \mathrm{Hom}_A(\mathrm{Hom}_R(A,M),M)$ satisfying $f \circ \varepsilon = \mathrm{id}_M$. So, (b) follows.

(b) $\Longrightarrow$ (c). By assumption, there exists $f \in \mathrm{Hom}_A(\mathrm{Hom}_R(A,M),M)$ such that $f \circ \varepsilon = \mathrm{id}_M$. Hence, $\varepsilon \circ f$ is an idempotent in $\mathrm{End}_A(\mathrm{Hom}_R(A,M))$. So, M is an A-summand of $\mathrm{Hom}_R(A,M)$. By Lemma 1.2.9, $\mathrm{Hom}_A(-,M)$ is exact on (A,R)-exact sequences.

(c) $\Longrightarrow$ (d). Let

$$0 \to U \xrightarrow{p} V \xrightarrow{q} W \to 0 \tag{1.2.1.14}$$

be an (A,R)-exact sequence. Applying $\mathrm{Hom}_A(-,M)$ to (1.2.1.14) we obtain the exact sequence

$$0 \to \mathrm{Hom}_A(W,M) \to \mathrm{Hom}_A(V,M) \to \mathrm{Hom}_A(U,M) \to 0. \tag{1.2.1.15}$$

So, for every $h \in \mathrm{Hom}_A(U,M)$, there exists $h' \in \mathrm{Hom}_A(V,M)$ such that $h' \circ p = h$.

(d) $\Longrightarrow$ (a) Let

$$0 \to M \xrightarrow{p} V \xrightarrow{q} W \to 0 \tag{1.2.1.16}$$

be an (A,R)-exact sequence. For $\mathrm{id}_M \in \mathrm{Hom}_A(M,M)$ there exists $h \in \mathrm{Hom}_A(V,M)$ such that $h \circ p = \mathrm{id}_M$. $\square$

Corollary 1.2.11. *(A,R)-injective modules are preserved under Morita equivalence.*

Proof. Let $F\colon A\text{-Mod} \to B\text{-Mod}$ be an equivalence of categories. Let G be the quasi-inverse of F. Let M be an (A,R)-injective module. Let

$$0 \to FM \to V \to W \to 0 \tag{1.2.1.17}$$

be a (B,R)-exact sequence. By Corollary 1.2.7, the sequence $0 \to GFM \to GV \to GW \to 0$ is (A,R)-exact. By Proposition 1.2.10, this sequence splits over B since $GFM \simeq M$. Applying F yields the B-split exact sequence

$$0 \to FGFM \to FGV \to FGW \to 0. \tag{1.2.1.18}$$

(1.2.1.17) is equivalent to (1.2.1.18). So, it follows that FM is (B,R)-injective. $\square$

Proposition 1.2.12. *[Hoc56, 1.] Let A be an R-algebra. Let $M \in A$-Mod. The following assertions are equivalent.*

(a) M is (A,R)-projective, that is, every (A,R)-exact sequence

$$0 \to U \to V \to M \to 0$$

is split over A;

(b) The natural epimorphism $\mu\colon A \otimes_R M \to M$, $\mu(a \otimes m) = am$, $\forall a \in A,\ m \in M$, splits over A;

(c) The functor $\mathrm{Hom}_A(M,-)$ is exact on (A,R)-exact sequences;

(d) For every (A,R)-exact sequence $0 \to U \xrightarrow{p} V \xrightarrow{q} W \to 0$ and every A-homomorphism $M \to W$ there exists an A-homomorphism $M \to V$ making the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "p"] & V \arrow[r, "q"] & W \arrow[r] & 0 \\ & & & M \arrow[ul, "\exists"] \arrow[u, "\forall"'] & \end{tikzcd}.$$

Proof. (a) $\implies$ (b) Notice that $\mu'\colon M \to A \otimes_R M$, given by $\mu'(m) = 1_A \otimes_R m$, $m \in M$, is an R-homomorphism since $\mu'(sm) = 1_A \otimes (sm) = s1_A \otimes m = s\mu'(m)$, $\forall m \in M,\ s \in R$. Moreover, $\mu \circ \mu' = \mathrm{id}_M$. So, the exact sequence

$$0 \to \ker\mu \to A \otimes_R M \xrightarrow{\mu} M \to 0 \tag{1.2.1.19}$$

is (A,R)-exact. By assumption, it splits over A. In particular, there exists $f \in \mathrm{Hom}_A(M, A \otimes_R M)$ satisfying $\mu \circ f = \mathrm{id}_M$. So, (b) follows.

(b) $\implies$ (c). By assumption, there exists $f \in \mathrm{Hom}_A(M, A \otimes_R M)$ such that $\mu \circ f = \mathrm{id}_M$. Hence, $f \circ \mu$ is an idempotent in $\mathrm{End}_A(A \otimes_R M)$. So, M is an A-summand of $A \otimes_R M$. By Lemma 1.2.9, $\mathrm{Hom}_A(M,-)$ is exact on (A,R)-exact sequences.

(c) $\implies$ (d). Let $0 \to U \xrightarrow{p} V \xrightarrow{q} W \to 0$ be an (A,R)-exact sequence. Applying $\mathrm{Hom}_A(M,-)$ to (1.2.1) we obtain the exact sequence

$$0 \to \mathrm{Hom}_A(M,U) \to \mathrm{Hom}_A(M,V) \to \mathrm{Hom}_A(M,W) \to 0. \tag{1.2.1.20}$$

So, for every $h \in \mathrm{Hom}_A(M,W)$, there exists $h' \in \mathrm{Hom}_A(M,V)$ such that $q \circ h' = h$.

(d) $\implies$ (a). Let $0 \to U \xrightarrow{p} V \xrightarrow{q} M \to 0$ be an (A,R)-exact sequence. For $\mathrm{id}_M \in \mathrm{Hom}_A(M,M)$ there exists $h \in \mathrm{Hom}_A(M,V)$ such that $q \circ h = \mathrm{id}_M$. $\square$

Remark 1.2.13. In relative homological algebra $A \otimes_R M$ replaces the role of free A-modules. In fact, when R is a field $A \otimes_R M \simeq A^n$ since M is finite-dimensional. $\triangle$

Corollary 1.2.14. *Let M be an A-module. Let $X \in \mathrm{add}(M)$.*

1. *If M is (A,R)-projective, then X is (A,R)-projective.*
2. *If M is (A,R)-injective, then X is (A,R)-injective.*

Proof. Let M be an (A,R)-projective module. Let $X \in \mathrm{add}M$. By Proposition 1.2.12, $M \in \mathrm{add}A \otimes_R M$. Therefore, $X \in \mathrm{add}A \otimes_R M$. By Lemma 1.2.9, $\mathrm{Hom}_A(X,-)$ is exact on (A,R)-exact sequences. Hence, X is (A,R)-projective by Proposition 1.2.12. So, (a) follows. The proof of (b) is analogous. $\square$

The following fact is straightforward and it it is useful to relate relative projective with absolute projective modules.

Lemma 1.2.15. *Let V be a projective R-module. If V is (A,R)-projective, then V is a projective A-module.*

Proof. Since V is (A,R)-projective then V is an A-summand of $A \otimes_R V$. On the other hand, V is an R-summand of R^m for some $m > 0$. Thus, $A \otimes_R V$ is an A-summand of $A \otimes_R R^m \simeq A^m$. It follows that V is projective over A. $\square$

It is also an easy fact that we can relate for every ideal of R, I, the (A,R)-projective modules with $(A/IA, R/I)$-projective modules.

Lemma 1.2.16. *Let R be a commutative ring and let I be an ideal of R. Assume that A is an R-algebra. Then, the following assertions hold.*

(a) If M is an $(A/IA, R/I)$-projective module, then M is an (A,R)-projective module.

(b) If M is an (A,R)-projective module, then M/IM is an $(A/IA, R/I)$-projective module.

Proof. Assume that (a) holds. Thus, $M \in \mathrm{add}_{A/IA} A/IA \otimes_{R/I} M$. Since A/IA-mod is a full subcategory of A-mod, $M \in \mathrm{add}_A A/IA \otimes_{R/I} M$. Now observe that $A/IA \otimes_{R/I} M \simeq A \otimes_R R/I \otimes_{R/I} M \simeq A \otimes_R M$. So, M is (A,R)-projective.

If M is an (A,R)-projective module, then the canonical epimorphism $A \otimes_R M \to M$ splits over A. Applying the functor $R/I \otimes_R -$ yields that the canonical epimorphism $R/I \otimes_R A \otimes_R M \simeq A/IA \otimes_{R/I} M/IM \to M/IM \simeq R/I \otimes_R M$ splits over A/IA. $\square$

By a **projective (left) A-resolution** of $M \in A$-Mod we mean an exact sequence of left A-modules $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where all the modules P_i are projective A-modules. By a (left) (A,R)**-projective resolution** of $M \in A$-Mod we mean an (A,R)-exact sequence of left A-modules $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where all the modules P_i are (A,R)-projective modules. By a (left) (A,R)**-injective resolution** of $M \in A$-Mod we mean an (A,R)-exact sequence of left A-modules $0 \to M \to I_0 \to I_1 \to \cdots$, where all the modules I_i are (A,R)-injective modules.

Corollary 1.2.17.

1. *Every A-module has an (A,R)-projective resolution.*

2. Every A-module has an (A,R)-injective resolution.

Proof. By Proposition 1.2.12 and Lemma 1.2.9, $A\otimes_R M$ is (A,R)-projective for every $M\in R$-Mod. Therefore,

$$\begin{tikzcd}
& 0 \arrow[dr] & & 0 & & & & \\
& & K_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & A\otimes_R K_1 \arrow[ur, "\mu_{K_1}"] \arrow[rr] & & A\otimes_R K_0 \arrow[rr] \arrow[dr, "\mu_{K_0}"] & & A\otimes_R M \arrow[r, "\mu_M"] & M \arrow[r] & 0 \\
& & & & K_0 \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{1.2.1.21}$$

is an (A,R)-projective resolution. Dually,

$$\begin{tikzcd}
& & & & & & 0 & \\
& & & & & C_1 \arrow[ur] \arrow[dr, "\varepsilon_{C_1}"] & & \\
0 \arrow[r] & M \arrow[r, "\varepsilon_M"] & \mathrm{Hom}_R(A,M) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_R(A,C_0) \arrow[rr] \arrow[ur] & & \mathrm{Hom}_R(A,C_1) \arrow[r] & \cdots \\
& & & C_0 \arrow[ur, "\varepsilon_{C_0}"] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 & & &
\end{tikzcd} \tag{1.2.1.22}$$

is an (A,R)-injective resolution. $\square$

The resolutions constructed in Corollary 1.2.17 are called **standard**.

When A is projective as R-module, we may define this notion in terms of the canonical Ext.

Lemma 1.2.18. *[Zim14, Lemma 2.1.2, Proposition 2.1.5] Let R be a commutative ring. Let A be a projective R-algebra. Then, an A-module M is (A,R)-projective if and only if the natural mapping*

$$\mathrm{Ext}^1_A(M,N)\to\mathrm{Ext}^1_R(M,N)$$

is injective for all A-modules N.

Proof. Assume that M is (A,R)-projective. Then, the homomorphism $\varepsilon_M\colon A\otimes_R M\to M$ splits over A. Let $N\in A$-Mod. Notice that $\mathrm{Hom}_R(-,N)\simeq\mathrm{Hom}_R(-,\mathrm{Hom}_A(A,N))\simeq\mathrm{Hom}_A(A\otimes_R -,N)$. Denote this natural transformation by σ. Since for any projective resolution $P^\bullet$ over R of M, $A\otimes_R P^\bullet$ is a projective resolution over $A\otimes_R R\simeq A$ of $A\otimes_R M$ there is a canonical isomorphism $\mathrm{Ext}^{i\geq 0}_R(M,N)\simeq\mathrm{Ext}^{i\geq 0}_A(A\otimes_R M,N)$. Denote this isomorphism by σ^*. Since M is an A-summand of $A\otimes_R M$, it follows that

$$\mathrm{Ext}^1_A(A\otimes_R M,N)\simeq\mathrm{Ext}^1_A(M\oplus\ker\mu,N)\simeq\mathrm{Ext}^1_A(M,N)\oplus\mathrm{Ext}^1_A(\ker\mu,N). \tag{1.2.1.23}$$

Denote the monomorphism $\operatorname{Ext}^1_A(M,N) \to \operatorname{Ext}^1_A(A\otimes_R M,N)$ by μ^*. It is enough to show that the following diagram is commutative since $\sigma^* \circ \mu^*$ is a monomorphism:

$$\begin{tikzcd} \operatorname{Ext}^1_A(M,N) \arrow[r] \arrow[d, "\mu^*"] & \operatorname{Ext}^1_R(M,N) \arrow[d, equal] \\ \operatorname{Ext}^1_A(A\otimes_R M,N) \arrow[r, "\sigma^*"] & \operatorname{Ext}^1_R(M,N) \end{tikzcd} . \tag{1.2.1.24}$$

By assumption, A is projective over R, so every projective A-resolution of M is a projective R-resolution. Let $P^\bullet$ be a projective A-resolution of M. Hence, μ induces a chain map between the A-resolutions

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\alpha_0"] & M \arrow[r] & 0 \\ \cdots \arrow[r] & A\otimes_R P_1 \arrow[r] \arrow[u, "\mu_{P_1}"] & A\otimes_R P_0 \arrow[r, "A\otimes_R \alpha_0"] \arrow[u, "\mu_{P_0}"] & A\otimes_R M \arrow[r] \arrow[u, "\mu_M"] & 0 \end{tikzcd} . \tag{1.2.1.25}$$

It follows that each P_i is an A-summand of $A\otimes_R M$ with canonical epimorphism μ_{P_i}. Thus, applying $\operatorname{Hom}_A(-,N)$ to the diagram (1.2.1.25), we obtain the map

$$\mu^*\colon \operatorname{Ext}^1_A(M,N) = H^1(\operatorname{Hom}_A(P^\bullet,N)) \to H^1(\operatorname{Hom}_A(A\otimes_R P^\bullet,N)) = \operatorname{Ext}^1_A(A\otimes_R M,N). \tag{1.2.1.26}$$

Furthermore, composing σ_{P_i} with the maps $\operatorname{Hom}_A(\mu_{P_i},N)$ induces a cochain complex map between $\operatorname{Hom}_A(P^\bullet,N)$ and $\operatorname{Hom}_R(P^\bullet,N)$. This cochain map induces the map on cohomology $\sigma^*\circ\mu^*$. Now note that for any $g \in \operatorname{Hom}_A(P_i,N)$ and $p \in P_i$,

$$\sigma_{P_i} \circ \operatorname{Hom}_A(\mu_{P_i},N)(g)(p) = \sigma_{P_i}(g\circ\mu_{P_i})(p) = g\circ\mu_{P_i}(1_A\otimes p) = g(p). \tag{1.2.1.27}$$

Thus, $\sigma_{P_i}\circ\operatorname{Hom}_A(\mu_{P_i},N)$ is the restriction map. It follows that $\sigma^*\circ\mu^*$ is the restriction map on Ext^1.

Conversely, assume that $\operatorname{Ext}^1_A(M,N)\to\operatorname{Ext}^1_R(M,N)$ is injective for all A-modules N. Consider the A-exact sequence $0\to\ker\mu\to A\otimes_R M \xrightarrow{\mu} M\to 0$. This sequence splits over R. By hypothesis, as an element in $\operatorname{Ext}^1_A(M,N)$ must be zero. Thus, M is (A,R)-projective by Proposition 1.2.12. □

The assumption on A being projective over R is used to guarantee that the natural function $\operatorname{Ext}^1_A(M,N)\to\operatorname{Ext}^1_R(M,N)$ is an R-linear map since this assumption plays an important role in establishing that (1.2.1.24) is commutative. The author wonders if there might be an example where this fails if we drop the projectivity of A.

Proposition 1.2.19. *Let R be a commutative ring. Let A and B be Morita equivalent R-algebras. Denote by F the equivalence of categories* A-Mod $\to$ B-Mod*. Then, X is (A,R)-projective if and only if FX is (B,R)-projective.*

Proof. Assume that M is an A-progenerator such that $F = N\otimes_A -$ and $N = \operatorname{Hom}_A(M,A)$ the B-progenerator. Assume X is (A,R)-projective. Then, X is an A-summand of $A\otimes_R X$. Thus, FX is a B-summand of

$$N\otimes_A(A\otimes_R X)\simeq N\otimes_A A\otimes_R X \simeq N\otimes_R X.$$

Now, since N is projective over B, $N\otimes_R X$ is a B-summand of $B^t\otimes_R X\simeq (B\otimes X)^t$ which is (B,R)-projective. So, it follows that FM is (B,R)-projective. □

Lemma 1.2.20. *Let $N\in R$*-mod*.*

(a) The functor $-\otimes_R N$ is exact on R-split exact sequences.

(b) *The functor* $\mathrm{Hom}_R(-,N)$ *is exact on R-split exact sequences.*

(c) *The functor* $\mathrm{Hom}_R(N,-)$ *is exact on R-split exact sequences.*

Proof. Let $\cdots \to P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$ be an exact sequence which is R-split. Considering the R-split exact sequences

$$0 \longrightarrow \operatorname{im}\alpha_{i+1} \underset{l_i}{\overset{g_i}{\rightleftarrows}} P_i \underset{h_i}{\overset{\alpha_i}{\rightleftarrows}} \operatorname{im}\alpha_i \longrightarrow 0 \quad \text{for all } i \geq 0. \tag{1.2.1.28}$$

Apply $-\otimes_R N$ to (1.2.1.28). We obtain the exact sequence

$$\operatorname{im}\alpha_{i+1} \otimes_R N \xrightarrow{g_i \otimes_R N} P_i \otimes_R N \xrightarrow{\alpha_i \otimes_R N} \operatorname{im}\alpha_i \otimes_R N \to 0. \tag{1.2.1.29}$$

and note that $l \otimes_R N \circ g \otimes_R N = l \circ g \otimes_R N = \mathrm{id}_{\operatorname{im}\alpha_{i+1}} \otimes_R N = \mathrm{id}_{\operatorname{im}\alpha_{i+1} \otimes_R N}$. Hence, $g_i \otimes_R N$ is, in addition, a monomorphism. By exactness, we have $\operatorname{im}(\alpha_{i+1} \otimes_R \mathrm{id}_N) = \operatorname{im}\alpha_{i+1} \otimes_R N = \ker(\alpha_i \otimes_R \mathrm{id}_N)$ for all i.

Therefore, $\cdots \to P_2 \otimes_R N \to P_1 \otimes_R N \xrightarrow{\alpha_1 \otimes_R N} P_0 \otimes_R N \xrightarrow{\alpha_0 \otimes_R N} M \otimes_R N \to 0$ is exact.

Applying $\mathrm{Hom}_R(-,N)$ to (1.2.1.28) yields the exact sequence

$$0 \to \mathrm{Hom}_R(\operatorname{im}\alpha_i, N) \xrightarrow{\mathrm{Hom}_R(\alpha_i,N)} \mathrm{Hom}_R(P_i,N) \xrightarrow{\mathrm{Hom}_R(g_i,N)} \mathrm{Hom}_R(\operatorname{im}\alpha_{i+1},N). \tag{1.2.1.30}$$

Note that $\mathrm{Hom}_R(g_i,N) \circ \mathrm{hom}_R(l_i,N) = \mathrm{Hom}_R(l_i \circ g_i, N) = \mathrm{Hom}_R(\mathrm{id}_{\operatorname{im}\alpha_{i+1}}, N) = \mathrm{id}_{\mathrm{Hom}_R(\operatorname{im}\alpha_{i+1},N)}$. So, $\mathrm{Hom}_R(g_i,N)$ is also a surjective map. Therefore, $\ker \mathrm{Hom}_R(\alpha_{i+1},N) = \ker \mathrm{Hom}_R(g_i,N) = \operatorname{im}\mathrm{Hom}_R(\alpha_i,N)$. Thus, (b) follows. Symmetrically, (c) follows. $\square$

Note that every (A,R)-exact sequence is R-split. Hence we have,

Corollary 1.2.21. *Let* $N \in R$-mod.

(a) *The functor* $-\otimes_R N$ *is exact on* (A,R)*-exact sequences.*

(b) *The functor* $\mathrm{Hom}_R(-,N)$ *is exact on* (A,R)*-exact sequences.*

(c) *The functor* $\mathrm{Hom}_R(N,-)$ *is exact on* (A,R)*-exact sequences.*

The exact sequences of projective R-modules are R-split. In fact, Let $\cdots \to P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$ be an exact sequence with $P_i, M \in R$-proj for all i. Consider the exact sequence $0 \to \operatorname{im}\alpha_1 \to P_0 \to M \to 0$. Since M is projective over R, then it splits and hence $\operatorname{im}\alpha_1$ is projective over R. Considering the exact sequences

$$0 \to \operatorname{im}\alpha_{i+1} \to P_i \to \operatorname{im}\alpha_i \to 0, \tag{1.2.1.31}$$

it follows, by induction on i, that $\operatorname{im}\alpha_i$ is projective over R for all i.

In particular, the exact sequences of modules belonging to A-mod$\cap R$-proj are (A,R)-exact.

1.2.2 Forgetful functors

We say that we have a relative homological algebra if we choose an abelian category together with a class of exact sequences. A relative abelian category in the sense of Mac Lane [Mac95] consists of the following data: a pair of abelian categories $(\mathscr{A},\mathscr{B})$ together with a covariant additive, exact and faithful functor $F\colon \mathscr{A} \to \mathscr{B}$.

Consider the forgetful functor $F\colon A\text{-Mod} \to R\text{-Mod}$. Since it is a forgetful functor, it is faithful. This functor preserves biproducts, hence it is additive. Consider the functors $G,H\colon R\text{-Mod} \to A\text{-Mod}$, given by $GM = \mathrm{Hom}_R(A,M)$, $HM = A \otimes_R M$, and $Gf = \mathrm{Hom}_R(A,f)$, $Hf = A \otimes_R f$. It follows by Tensor-Hom adjunction that the functor G is a right adjoint of F and H is a left adjoint of F. The existence of left and right adjoint functors imply that F preserves all finite limits and all finite colimits. In particular, kernels and cokernels. Hence F is exact. In view of [Mac95, Chapter 9, 4], a short exact sequence of A-modules

$$0 \to X \to Y \to Z \to 0 \tag{1.2.2.1}$$

is said to be F-allowable if the exact sequence

$$0 \to FX \to FY \to FZ \to 0 \tag{1.2.2.2}$$

splits over R. These are exactly the (A,R)-exact sequences. In Lemma 1.2.9 and Proposition 1.2.12 we saw that the objects for which $\mathrm{Hom}_A(P,-)$ is exact on (A,R)-exact sequences are the modules $A\otimes_R X$, $X \in R$-Mod. Reciprocally, we saw in Lemma 1.2.9 that the class of exact sequences which remains exact under $\mathrm{Hom}_A(A\otimes_R X,-)$ are the (A,R)-exact sequences.

Nowadays, the most common approach to relative homological algebra is to first consider a class of objects $\mathscr{P}$ of an abelian category $\mathscr{A}$. Then, we can compute the class of exact sequences for which the class of objects $\mathscr{P}$ remain exact under $\mathrm{Hom}_{\mathscr{A}}(P,-)$ for every $P \in \mathscr{P}$. The class of (A,R)-exact sequences is closed in the sense of [EM65]. That is, these two approaches are equivalent for (A,R)-exact sequences.

1.2.3 Relative Ext and Relative Tor

We shall next introduce the relative Ext and relative Tor and relative homological dimensions. For the convenience of the reader, we give here some statements and a brief sketch about these topics. Much of these results to be explained, in this subsection, can be obtained from the literature of relative homological algebra taking as the admissible class of sequences the (A,R)-exact sequences, as previously discussed, making the necessary adjustments over the arguments for Artinian algebras to arbitrary algebras.

In order to introduce the relative Ext and relative Tor functor, the following lemmas are essential.

Lemma 1.2.22. *[Hoc56, Proposition 1] Let M be an (A,R)-injective module. Suppose that $U \xrightarrow{\phi} V$ is a homomorphism of A-modules such that* $\mathrm{Hom}_R(\phi,M)$ *is epimorphism. Then, the map* $\mathrm{Hom}_A(\phi,M)$ *is an epimorphism.*

Proof. Since M is (A,R)-injective, M is an A-summand of $\mathrm{Hom}_R(A,M)$. Denote by $\pi\colon \mathrm{Hom}_R(A,M) \to M$ the surjective induced by the direct sum structure. For any $B \in A$-Mod, as π is A-split applying $\mathrm{Hom}_A(B,-)$, we get the surjective map $\mathrm{Hom}_A(B,\pi)\colon \mathrm{Hom}_A(B,\mathrm{Hom}_R(A,M)) \to \mathrm{Hom}_A(B,M)$. By Tensor-Hom adjunction, we have the surjective map $\mathrm{Hom}_R(B,M) \simeq \mathrm{Hom}_A(B,\mathrm{Hom}_R(A,M)) \to \mathrm{Hom}_A(B,M)$ for any $B \in A$-Mod. In particular, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(U,M) \arrow[r, two heads] \arrow[d, "\mathrm{Hom}_R(\phi,M)"] & \mathrm{Hom}_A(U,M) \arrow[d, "\mathrm{Hom}_A(\phi,M)"] \\
\mathrm{Hom}_R(V,M) \arrow[r, two heads] & \mathrm{Hom}_A(V,M)
\end{tikzcd}.$$

It follows that if $\mathrm{Hom}_R(\phi,M)$ is surjective, then $\mathrm{Hom}_A(\phi,M)$ is surjective. ☐

Lemma 1.2.23. *[Hoc56, Proposition 2] Let M be an (A,R)-projective module. Suppose that $V \xrightarrow{\phi} W$ is homomorphism of A-modules such that* $\mathrm{Hom}_R(M,\phi)$ *is an epimorphism. Then,* $\mathrm{Hom}_A(M,\phi)$ *is an epimorphism.*

Proof. Since M is (A,R)-projective, there is an A-split monomorphism $M \xrightarrow{i} A\otimes_R M$. So, for every $B \in A$-Mod, applying $\mathrm{Hom}_A(-,B)$ we get the surjective homomorphism $\mathrm{Hom}_A(A\otimes_R M,B) \xrightarrow{\mathrm{Hom}_A(i,B)} \mathrm{Hom}_A(M,B)$. By

Tensor-Hom adjunction, $\mathrm{Hom}_A(A\otimes_R M, B) \simeq \mathrm{Hom}_R(M,B)$. Therefore, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_R(M,V) \arrow[r, two heads] \arrow[d, "\mathrm{Hom}_R(M,\phi)"] & \mathrm{Hom}_A(M,V) \arrow[d, "\mathrm{Hom}_A(M,\phi)"] \\ \mathrm{Hom}_R(M,W) \arrow[r, two heads] & \mathrm{Hom}_A(M,W) \end{tikzcd}.$$

It follows that if $\mathrm{Hom}_R(M,\phi)$ is surjective, then $\mathrm{Hom}_A(M,\phi)$ is surjective. $\square$

Lemma 1.2.24. *[Hoc56, Proposition 3] Let M be an (A,R)-projective module. Suppose that $\phi\colon U\to V$ is a homomorphism of right A-modules such that the induced map $\phi\otimes_R M\colon U\otimes_R M\to V\otimes_R M$ is a monomorphism. Then, the map $\phi\otimes_A M\colon U\otimes_A M\to V\otimes_A M$ is a monomorphism.*

Proof. Since M is (A,R)-projective, there is an A-split homomorphism $M\xrightarrow{i} A\otimes_R M$. So, for every right A-module B, applying $B\otimes_A -$ yields the injective map $B\otimes_A M\hookrightarrow B\otimes_A A\otimes_R M\simeq B\otimes_R M$. Using the commutative diagram

$$\begin{tikzcd} U\otimes_A M \arrow[r, hook] \arrow[d, "\phi\otimes_A M"] & U\otimes_R M \arrow[d, "\phi\otimes_R M"] \\ V\otimes_A M \arrow[r, hook] & V\otimes_R M \end{tikzcd},$$

it follows that if $\phi\otimes_R M$ is injective, then $\phi\otimes_A M$ is injective. $\square$

The following is based on Section 2 of [Hoc56].

Theorem 1.2.25 (Comparison Theorem for (A,R)-exacts)**.** *[Hoc56, p.250, 251] Given two (A,R)-projective resolutions of M and N, and a map $f\in\mathrm{Hom}_A(M,N)$, we can find a chain map between them. This map is unique up to chain homotopy.*

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r, "h_2"] \arrow[d, "\exists"] & X_1 \arrow[r, "h_1"] \arrow[d, "\exists"] & X_0 \arrow[r, "h_0"] \arrow[d, "\exists"] & M \arrow[r] \arrow[d, "\forall f"] & 0 \\ \cdots \arrow[r] & Y_2 \arrow[r, "g_2"] & Y_1 \arrow[r, "g_1"] & Y_0 \arrow[r, "g_0"] & N \arrow[r] & 0 \end{tikzcd}$$

Proof. First, we show the existence of a chain map from X to Y, that is, the existence of a collection of maps $f_i\colon X_i\to Y_i$ satisfying $g_i\circ f_i = f_{i-1}\circ h_i$. We will proceed by induction on i. Define $K_i=\ker g_i$ and denote by k_i the inclusion map $K_i\to Y_i$. Since Y is an (A,R)-exact sequence then the following exact sequences

$$0\to K_0\to Y_0\xrightarrow{g_0} N\to 0 \tag{1.2.3.1}$$

$$0\to K_i\xrightarrow{k_i} Y_i\xrightarrow{\pi_i} K_{i-1}\to 0,\ i>1 \tag{1.2.3.2}$$

are (A,R)-exact with $k_{i-1}\circ\pi_i=g_i$. Since X_0 is (A,R)-projective, the map $f\circ h_0$ can be lifted to $f_0\in\mathrm{Hom}_A(X_0,Y_0)$ satisfying $f\circ h_0=g_0\circ f_0$. Note that $g_0\circ f_0\circ h_1=f\circ h_0\circ h_1=0$. Hence, there exists a map $x_1\colon X_1\to K_0$ such that $k_0\circ x_1=f_0\circ h_1$. Using the fact that X_1 is (A,R)-projective, we can lift x_1 to a map $f_1\in\mathrm{Hom}_A(X_1,Y_1)$. Hence $\pi\circ f_1=x_1$. Thus,

$$f_0\circ h_1=k_0\circ x_1=k_0\circ\pi_1\circ f_1=g_1\circ f_1. \tag{1.2.3.3}$$

We can repeat this procedure for each $i>1$ and we obtain maps $f_i\in\mathrm{Hom}_A(X_i,Y_i)$ such that $g_i\circ f_i=f_{i-1}\circ h_i$.

Now we shall prove that this chain is unique up to chain homotopy equivalence. Assume that there exists maps $s_i,t_i\in\mathrm{Hom}_A(X_i,Y_i)$ such that $g_i\circ t_i=t_{i-1}\circ h_i$ and $g_i\circ s_i=s_{i-1}\circ h_i$ for every $i\geq 0$. Set $r_i=s_i-t_i$ for every $i\geq 0$.

Thus, $g_i \circ r_i = g_i \circ s_i - g_i \circ t_i = s_{i-1} \circ h_i - t_{i-1} \circ h_i = r_{i-1} \circ h_i$. It is enough to construct maps $l_i \in \mathrm{Hom}_A(X_i, Y_{i+1})$ satisfying $r_i = g_{i+1} \circ l_i + l_{i-1} \circ h_i$, $i \geq 0$ and $l_{-1} = 0$.

Since

$$g_0 \circ r_0 = g_0 \circ s_0 - g_0 \circ t_0 = f \circ h_0 - f \circ h_0 = 0 \tag{1.2.3.4}$$

there exists $x_0 \in \mathrm{Hom}_A(X_0, K_0)$ satisfying $k_0 \circ x_0 = r_0$. As $0 \to K_1 \to Y_1 \to K_0 \to 0$ is (A,R)-exact and X_0 is (A,R)-projective, there exists $l_0 \in \mathrm{Hom}_A(X_0, Y_1)$ such that $g_1 \circ l_0 = r_0$.

Assume that the maps l are defined until level $i-1$. Therefore,

$$\begin{aligned} g_i \circ (r_i - l_{i-1} \circ h_i) &= g_i \circ r_i - g_i \circ l_{i-1} \circ h_i = g_i \circ r_i - (r_{i-1} - l_{i-2} \circ h_{i-1}) \circ h_i & (1.2.3.5)\\ &= g_i \circ r_i - r_{i-1} \circ h_i + l_{i-2} \circ h_{i-1} \circ h_i = g_i \circ r_i - g_i \circ r_i = 0. & (1.2.3.6) \end{aligned}$$

Thus, $k_{i-1} \circ \pi_i \circ (r_i - l_{i-1} h_i) = 0$. As k_{i-1} is injective, $\pi_i(r_i - l_{i-1} h_i) = 0$. Hence, there exists a map $x_i \in \mathrm{Hom}_A(X_i, K_i)$ such that $k_i \circ x_i = r_i - l_{i-1} \circ h_i$. Using the (A,R)-exact sequence $0 \to K_{i+1} \to Y_{i+1} \to K_i \to 0$ and the fact that X_i is (A,R)-projective there exists $l_i \in \mathrm{Hom}_A(X_i, Y_{i+1})$ satisfying $\pi_{i+1} \circ l_i = x_i$. In particular,

$$g_{i+1} \circ l_i = k_i \circ \pi_{i+1} \circ l_i = r_i - l_{i-1} \circ h_i.$$

□

Theorem 1.2.26 (Dual of Comparison Theorem for (A,R)-exacts)**.** *[Hoc56, p.250, 251] Given two (A,R)-injective resolutions of M and N, and a map $f \in \mathrm{Hom}_A(M,N)$, we can find a cochain map between them. This map is unique up to chain homotopy.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots \\ & & \big\downarrow{\scriptstyle \forall f} & & \big\downarrow{\scriptstyle \exists} & & \big\downarrow{\scriptstyle \exists} & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

Proof. It is the dual claim of Theorem 1.2.25. □

Definition 1.2.27. We define $\mathrm{Ext}^n_{(A,R)}(M,N) = H^n(\mathrm{Hom}_R(P_M, N))$, where P_M is an (A,R)-projective resolution for M. We define $\mathrm{Tor}^n_{(A,R)}(M,N) = H_n(P_M \otimes_A N)$, where P_M is an (A,R)-projective resolution for M.

Here we can choose any relative projective resolution.

Proposition 1.2.28. *The functors $\mathrm{Ext}^n_{(A,R)}(M,N)$ and $\mathrm{Tor}^{(A,R)}_n(L,M)$ are independent of the choice of (A,R)-projective resolution for M.*

Proof. Since the comparison theorem holds for (A,R)-projective resolutions the result follows using the same argument as it was used for proving that $\mathrm{Ext}^n_A(M,N)$ and $\mathrm{Tor}^A_n(L,M)$ are independent (see [Wei03, Lemma 2.4.1]) of the choice of the projective A- resolution for M. □

By the same reasoning, we obtain:

Proposition 1.2.29. *For each map $f \in \mathrm{Hom}_A(M,M')$ and each natural number n, there are unique maps $\mathrm{Tor}^{(A,R)}_n(M,N) \to \mathrm{Tor}^{(A,R)}_n(M',N)$ and $\mathrm{Ext}^n_{(A,R)}(M,N) \to \mathrm{Ext}^n_{(A,R)}(M',N)$.*

We can also define Tor and Ext in the second component using relative projective and injective resolutions, respectively. Write these two cases as tor and ext, respectively.

By definition, as in the classical case,

$$\mathrm{Tor}^{(A,R)}_0(-,N) \simeq - \otimes_A N, \quad \mathrm{tor}^{(A,R)}_0(M,-) \simeq M \otimes_A -,$$

$$\mathrm{Ext}^0_{(A,R)}(-,N) \simeq \mathrm{Hom}_A(-,N), \quad \mathrm{Ext}^0_{(A,R)}(M,-) \simeq \mathrm{Hom}_A(M,-).$$

Lemma 1.2.30. *Let M be a left (A,R)-projective, K be a right (A,R)-projective, L be a left (A,R)-injective module. Let $\cdots \to N_1 \to N_0 \to N \to 0$ be an (A,R)-exact sequence of right modules. Let $\cdots \to P_1 \to P_0 \to P \to 0$ be an (A,R)-exact sequence of left modules. Then, the following assertions hold.*

(a) $\cdots \to N_1 \otimes_A M \to N_0 \otimes_A M \to N \otimes_A M \to 0$ *is exact;*

(b) $\cdots \to K \otimes_A P_1 \to K \otimes_A P_0 \to K \otimes_A P \to 0$ *is exact;*

(c) $\cdots \to \mathrm{Hom}_A(M,P_1) \to \mathrm{Hom}_A(M,P_0) \to \mathrm{Hom}_A(M,P) \to 0$ *is exact;*

(d) $0 \to \mathrm{Hom}_A(P,L) \to \mathrm{Hom}_A(P_0,L) \to \mathrm{Hom}_A(P_1,L) \to \cdots$ *is exact.*

Proof. Let $\cdots \to N_1 \xrightarrow{\alpha_1} N_0 \xrightarrow{\alpha_0} N \to 0$ be an (A,R)-exact sequence of right modules. In particular, consider the (A,R)-exact sequences

$$0 \to \operatorname{im}\alpha_{i+1} \xrightarrow{k_i} N_i \xrightarrow{\alpha_i} \operatorname{im}\alpha_i \to 0,\ i \geq 0. \tag{1.2.3.7}$$

By Corollary 1.2.21,

$$0 \to \operatorname{im}\alpha_{i+1} \otimes_R M \xrightarrow{k_i \otimes_R M} N_i \otimes_R M \xrightarrow{\alpha_i \otimes_R M} \operatorname{im}\alpha_i \otimes_R M \to 0,\ i \geq 0. \tag{1.2.3.8}$$

is exact. By Lemma 1.2.24, $k_i \otimes_A M$ is a monomorphism. Therefore,

$$0 \to \operatorname{im}\alpha_{i+1} \otimes_A M \xrightarrow{k_i \otimes_A M} N_i \otimes_R M \xrightarrow{\alpha_i \otimes_A M} \operatorname{im}\alpha_i \otimes_A M \to 0,\ i \geq 0. \tag{1.2.3.9}$$

is exact. In particular, $\operatorname{im}(\alpha_{i+1} \otimes_A M) = \operatorname{im}\alpha_{i+1} \otimes_A M = \ker(\alpha_i \otimes_A M)$. Thus, (a) follows. The argument for (b) is analogous to (a).

Combining Corollary 1.2.21 and Lemma 1.2.23 (c) follows. Combining Corollary 1.2.21 and Lemma 1.2.22 (d) follows. ☐

Corollary 1.2.31. *Let M be an (A,R)-projective module and let N be an (A,R)-injective module. For any A-module X, the following holds.*

(a) $\mathrm{Ext}^n_{(A,R)}(M,X) = 0 = \mathrm{ext}^n_{(A,R)}(M,X)$ *for any $n > 0$.*

(b) $\mathrm{Tor}^{(A,R)}_n(M,X) = 0 = \mathrm{tor}^{(A,R)}_n(M,X)$ *for any $n > 0$.*

(c) $\mathrm{Ext}^n_{(A,R)}(X,N) = 0 = \mathrm{ext}^n_{(A,R)}(X,N)$ *for any $n > 0$.*

Proof. Let M be an (A,R)-projective module. Using the exact sequence $0 \to M \to M \to 0$ we conclude that $\mathrm{Ext}^n_{(A,R)}(M,X) = 0$ for every $n > 0$. Let I denote the standard (A,R)-injective resolution of X. Then, by Lemma 1.2.30, the chain complex $\mathrm{Hom}_A(M,I)$ is exact. Thus, $\mathrm{ext}^n_{(A,R)}(M,X) = 0$. Analogously, (b) and (c) follows. ☐

Since Hom is left exact bifunctor and its right derived functors vanish in (A,R)-projectives and (A,R)-injectives in the first and second component respectively, it follows in the same fashion as in the classical case that Ext $=$ ext and Tor $=$ tor. Hence, we can use resolutions in both entries.

Lemma 1.2.32 (Horseshoe Lemma for (A,R)-exact)**.** *Consider the diagram*

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_1' \arrow[d] & & P_1'' \arrow[d] & \\
 & P_0' \arrow[d, "\varepsilon_0'"] & & P_0'' \arrow[d, "\varepsilon_0''"] & \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d] & M \arrow[r, "p"] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

where the columns are (A,R)-projective resolutions and the row is (A,R)-exact. Then, there exists an (A,R)-projective resolution of M and chain maps so that the column form an exact sequence of complexes.

Proof. Let $K_0' = \ker \varepsilon_0' = \operatorname{im} \varepsilon_1'$ and $K_0'' = \ker \varepsilon_0'' = \operatorname{im} \varepsilon_1''$ and denote by i' and i'' the canonical inclusions $K_0' \hookrightarrow P_0'$ and $K_0'' \hookrightarrow P_0''$, respectively. By induction it suffices to complete the 3x3 diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_0' \arrow[d, "i'"] & & K_0'' \arrow[d, "i''"] & \\
 & P_0' \arrow[d, "\varepsilon_0'"] & & P_0'' \arrow[d, "\varepsilon_0''"] & \\
0 \arrow[r] & M' \arrow[r, "k"] \arrow[d] & M \arrow[r, "p"] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}.$$

Consider $P_0 = P_0' \oplus P_0''$. So, P_0 is (A,R)-projective. Consider $k' \colon P_0' \to P_0$ and $k'' \colon P_0'' \to P_0$ the canonical injections and $p' \colon P_0 \to P_0'$ and $p'' \colon P_0 \to P_0''$ the canonical surjections. Since

$$0 \to M' \xrightarrow{k} M \xrightarrow{p} M'' \to 0 \tag{1.2.3.10}$$

is (A,R)-exact and P_0'' is (A,R)-projective there exists $\sigma \in \operatorname{Hom}_A(P_0'', M)$ such that $p \circ \sigma = \varepsilon_0''$. Define $\varepsilon_0 \in \operatorname{Hom}_A(P_0, M)$ satisfying $\varepsilon_0(x', x'') = k \circ \varepsilon_0'(x') + \sigma(x'')$, for $(x', x'') \in P_0$. In particular,

$$\varepsilon_0 \circ k'(x') = \varepsilon_0(x', 0) = k \circ \varepsilon_0'(x'),\ x' \in P_0' \tag{1.2.3.11}$$

$$p \circ \varepsilon_0(x', x'') = p \circ k \circ \varepsilon_0'(x') + p\sigma(x'') = \varepsilon_0''(x'') = \varepsilon_0'' \circ p''(x', x''), (x', x'') \in P_0. \tag{1.2.3.12}$$

Therefore, the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_0' \arrow[r, "k'"] \arrow[d, two heads, "\varepsilon_0'"] & P_0 \arrow[r, "p''"] \arrow[d, "\varepsilon_0"] & P_0'' \arrow[r] \arrow[d, two heads, "\varepsilon_0''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "k"] & M \arrow[r, "p"] & M'' \arrow[r] & 0
\end{tikzcd} \tag{1.2.3.13}$$

is commutative. By Snake Lemma, ε_0 is surjective and there exists an exact sequence

$$0 \to K_0' \xrightarrow{k_0'} K_0 \xrightarrow{p_0''} K_0'' \to 0, \tag{1.2.3.14}$$

satisfying $k'i' = ik'_0$, $p''i = i''p''_0$ where the pair (K_0, i) is the kernel of ε_0. It remains to show that (1.2.3.14) is (A,R)-exact. Let s be the R-split homomorphism satisfying $\varepsilon'_0 \circ s = \mathrm{id}_{M'}$ and r be the R-split homomorphism satisfying $\varepsilon''_0 \circ r = \mathrm{id}_{M''}$. Then,

$$p \circ (\sigma \circ r) = \varepsilon''_0 \circ r = \mathrm{id}_{M''}\,. \tag{1.2.3.15}$$

Thus, there exists $t \in \mathrm{Hom}_R(M, M')$ such that $\sigma \circ r \circ p + k \circ t = \mathrm{id}_M$.

Let $l = k'' \circ r \circ p + k' \circ s \circ t \in \mathrm{Hom}_R(M, P_0)$. Then,

$$\varepsilon_0 \circ l = \varepsilon_0 k'' rp + \varepsilon_0 k' st = \sigma rp + k\varepsilon'_0 st = \sigma rp + kt = \mathrm{id}_M\,. \tag{1.2.3.16}$$

Therefore, $0 \to K_0 \xrightarrow{i} P_0 \xrightarrow{\varepsilon_0} M \to 0$ is (A,R)-exact. Thus, there exists $l' \in \mathrm{Hom}_R(P_0, K_0)$ such that $i \circ l' + l \circ \varepsilon_0 = \mathrm{id}_{P_0}$. Moreover,

$$i'' \circ p''_0 \circ l' \circ k'' \circ i'' = p'' \circ i \circ l' \circ k'' \circ i'' = p'' \circ k'' \circ i'' - p'' \circ l \circ \varepsilon_0 \circ k'' \circ i'' = i'' - p'' \circ l \circ \sigma \circ i'' \tag{1.2.3.17}$$

$$= i'' \implies p''_0 \circ l' \circ k'' \circ i'' = \mathrm{id}_{K''_0}, \tag{1.2.3.18}$$

where $l' \circ k'' \circ i'' \in \mathrm{Hom}_R(K''_0, K_0)$. This shows that (1.2.3.14) is (A,R)-exact. $\square$

Proposition 1.2.33. *Let $0 \to M' \to M \to M'' \to 0$ be an (A,R)-exact sequence. Then, for any $X \in A\text{-mod}$ and $Y \in \text{mod-}A$, there are long exact sequences*

1. $0 \to \mathrm{Hom}_A(X, M') \to \mathrm{Hom}_A(X, M) \to \mathrm{Hom}_A(X, M'') \to \mathrm{Ext}^1_{(A,R)}(X, M') \to \mathrm{Ext}^1_{(A,R)}(X, M) \to \cdots$
2. $0 \to \mathrm{Hom}_A(M', X) \to \mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(M'', X) \to \mathrm{Ext}^1_{(A,R)}(M', X) \to \mathrm{Ext}^1_{(A,R)}(M, X) \to \cdots$
3. $\cdots \to \mathrm{Tor}_1^{(A,R)}(Y, M) \to \mathrm{Tor}_1^{(A,R)}(Y, M'') \to Y \otimes_A M' \to Y \otimes_A M \to Y \otimes_A M'' \to 0.$

Proof. Choose (A,R)-projective resolutions for M', M''. By the Horseshoe Lemma, we obtain exactness of the sequence of deleted complexes

$$0 \to P_{M'} \to P_M \to P_{M''} \to 0, \tag{1.2.3.19}$$

where P_M is an (A,R)-projective resolution of M. Since in each row the modules are (A,R)-projective and the sequences are (A,R)-exact then they split over A. In particular,

$$0 \to \mathrm{Hom}_A(P_{M''}, X) \to \mathrm{Hom}_A(P_M, X) \to \mathrm{Hom}_A(P_{M'}, X) \to 0 \tag{1.2.3.20}$$

$$0 \to Y \otimes_A P_{M'} \to Y \otimes_A P_M \to Y \otimes_A P_{M''} \to 0 \tag{1.2.3.21}$$

are still exact complexes. The long exact sequences induced by these exact complexes are exactly the ones in 2 and 3. Dualizing the Horseshoe Lemma and applying the same reasoning with (A,R)-injective resolutions 1 follows. $\square$

Corollary 1.2.34. *M is (left) (A,R)-projective if and only if $\mathrm{Ext}^1_{(A,R)}(M, X) = 0$ for all $X \in A\text{-Mod}$. N is (left) (A,R)-injective if and only if $\mathrm{Ext}^1_{(A,R)}(X, N) = 0$ for all $X \in A\text{-Mod}$.*

Proof. Assume that $\mathrm{Ext}^1_{(A,R)}(M, X) = 0$ for all $X \in A\text{-Mod}$. Let $0 \to X \to Y \to Z \to 0$ be an (A,R)-exact sequence. By Proposition 1.2.33, applying $\mathrm{Hom}_A(M, -)$ yields the exact sequence

$$0 \to \mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, Z) \to \mathrm{Ext}^1_{(A,R)}(M, X) = 0. \tag{1.2.3.22}$$

Hence, by Proposition 1.2.12, M is (A,R)-projective. The converse statement is given in Corollary 1.2.31. Similarly, we get the result for (A,R)-injective modules. $\square$

Corollary 1.2.35. *Let A be a projective Noetherian R-algebra. Let $M \in A\text{-mod}\cap R\text{-proj}$. Then, M is (A,R)-injective if and only if $\operatorname{Ext}^1_A(X,M)=0$ for every $X \in A\text{-mod}\cap R\text{-proj}$.*

Proof. Assume that M is (A,R)-injective. Let $X \in A\text{-mod}\cap R\text{-proj}$. Then, $\operatorname{Ext}_{(A,R)}(X,M) = \operatorname{Ext}^1_A(X,M)$ since every (A,R)-projective resolution for X is an projective A-resolution for X. Hence, by assumption, $\operatorname{Ext}^1_A(X,M) = 0$.

Reciprocally, assume $\operatorname{Ext}^1_A(X,M)=0$ for every $X \in A\text{-mod}\cap R\text{-proj}$. Consider the (A,R)-exact sequence

$$0 \to M \xrightarrow{\varepsilon_M} \operatorname{Hom}_R(A,M) \to X \to 0. \tag{1.2.3.23}$$

Since $M \in R$-proj, $\operatorname{Hom}_R(A,M) \in R$-proj. Hence, $X \in R$-proj. By assumption, this exact sequence splits over A. Thus, by Proposition 1.2.10, M is (A,R)-injective. $\square$

Proposition 1.2.36. *Let R be a commutative ring. If $\operatorname{gldim} R = 0$, then $\operatorname{Ext}^n_{(A,R)}(M,N) = \operatorname{Ext}^n_A(M,N)$.*

Proof. By assumption, every $M \in R$-Mod is projective over R. Hence, every (A,R)-projective module $A\otimes_R M$ is projective over A. Hence, the (A,R)-projective resolutions are exactly the projective A-resolutions. $\square$

The meaning of Ext^1 follows from the following theorem.

Theorem 1.2.37. *[Hoc56] We have a one to one correspondence between*

$$\operatorname{Ext}^1_{(A,R)}(M,N) \longleftrightarrow \left\{ \begin{array}{c} \textit{Classes of equivalence of } (A,R)-\textit{exact sequences of the form} \\ 0 \to N \to X \to M \to 0 \end{array} \right\}.$$

Proof. Consider the (A,R)-exact sequence

$$0 \to K_M \xrightarrow{i} A\otimes_R M \xrightarrow{\mu} M \to 0. \tag{1.2.3.24}$$

By Proposition 1.2.33, this exact sequence induces the long exact sequence

$$\operatorname{Hom}_A(A\otimes_R M,N) \xrightarrow{\operatorname{Hom}_A(i,N)} \operatorname{Hom}_A(K_M,N) \xrightarrow{\partial} \operatorname{Ext}^1_{(A,R)}(M,N) \to 0. \tag{1.2.3.25}$$

Hence, $\operatorname{Ext}^1_{(A,R)}(M,N) \simeq \operatorname{im}\partial \simeq \operatorname{Hom}_A(K_M,N)/\ker\partial \simeq \operatorname{Hom}_A(K_M,N)/\operatorname{im}\operatorname{Hom}_A(i,N)$.

Let $\delta\colon 0 \to N \to X \to M \to 0$ be an (A,R)-exact sequence. There is the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_M \arrow[r, "i"] \arrow[d, "\alpha"] & A\otimes_R M \arrow[r, "\mu"] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & X \arrow[r] & M \arrow[r] & 0 \end{tikzcd}, \tag{1.2.3.26}$$

where the existence of f is due to $A\otimes_R M$ being (A,R)-projective and the existence of α is given by the universal property of the kernel of $X \to M$.

We assign δ to $\alpha + \operatorname{im}\operatorname{Hom}_A(i,N)$. This assignment is well defined and we will denote it by ψ.

Conversely, let $\delta \in \operatorname{Ext}^1_{(A,R)}(M,N)$. There exists $f_\delta \in \operatorname{Hom}_A(K_M,N)$ such that $\partial(f_\delta) = \delta$.

Let (X,p,k) be the pushout of the maps (f_δ, i). Explicitly, $X = (N \bigoplus A\otimes_R M)/S$ with

$$S = \{(f_\delta(x), -i(x)) \in N \bigoplus A\otimes_R M \colon x \in K_M\}.$$

Then, there is a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & K_M \arrow[r, "i"] \arrow[d, "f_\delta"] & A\otimes_R M \arrow[r, "\mu"] \arrow[d, "p"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "k"] & X \arrow[r, "\theta"] & M \arrow[r] & 0 \end{tikzcd} . \tag{1.2.3.27}$$

Since the first row is (A,R) is exact, there exists $t\in \operatorname{Hom}_R(M, A\otimes_R M)$ such that $\mu\circ t = \mathrm{id}_M$. Hence,

$$\theta\circ p\circ t = \mu\circ t = \mathrm{id}_M. \tag{1.2.3.28}$$

This shows that the second row in 1.2.3.27 is (A,R)-exact and we can denote ψ^* the assignment δ to this (A,R)-exact sequence.

Now it is clear that the proof that these functions ψ and ψ^* are bijective follows in the same way as in the usual bijection for $\operatorname{Ext}^1_A(M,N)$. ☐

Lemma 1.2.38. *Let A be a projective R-algebra. Let $M,N\in A^{op}$-mod$\cap R$-proj. Then, for any $i\geq 0$, $\operatorname{Ext}^i_{A^{op}}(M,N)\simeq \operatorname{Ext}^i_A(DN,DM)$.*

Proof. Let $M^\bullet$ be a projective left A^{op}-resolution $\cdots\to M_1\xrightarrow{\alpha_1} M_0\xrightarrow{\alpha_0} M\to 0$. Since $M\in R$-proj, all modules $\operatorname{im}\alpha_i$ are projective over R. Hence, $M^\bullet$ is an (A^{op},R)-projective resolution. Moreover, applying D to $M^\bullet$ yields the exact sequence $0\to DM\xrightarrow{D\alpha_0} DM_0\to DM_1\to\cdots$, since $\operatorname{Ext}^i_R(M,R)=0$ for all $i>0$. Each DM_i is (A,R)-injective. Thus, $DM^\bullet$ is an (A,R)-injective resolution of DM. The following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{A^{op}}(M_0,N) \arrow[r, "{\operatorname{Hom}_{A^{op}}(\alpha_1,\, N)}"] \arrow[d, "\psi_{M_0,N}"] & \operatorname{Hom}_{A^{op}}(M_1,N) \arrow[r] \arrow[d, "\psi_{M_1,N}"] & \cdots \\ 0 \arrow[r] & \operatorname{Hom}_A(DN,DM_0) \arrow[r, "{\operatorname{Hom}_A(DN,D\alpha_1)}"] & \operatorname{Hom}_A(DN,DM_1) \arrow[r] & \cdots \end{tikzcd} .$$

In fact, for $g\in\operatorname{Hom}_{A^{op}}(M_0,N), s\in DN,\ m\in M$,

$$\psi_{M,N}\circ\operatorname{Hom}_{A^{op}}(\alpha_1,N)(g)(s)(m) = \psi_{M,N}(g\circ\alpha_1)(s)(m) = s(g\circ\alpha_1(m)) \tag{1.2.3.29}$$

$$\operatorname{Hom}_A(DN,D\alpha_1)\circ\psi_{M_0,N}(g)(s)(m) = D\alpha_1\circ\psi_{M_0,N}(g)(s)(m) = D\alpha_1(s(g))(m) = s\circ g(\alpha_1(m)). \tag{1.2.3.30}$$

Hence,

$$\operatorname{Ext}^i_{A^{op}}(M,N) = H^i(\operatorname{Hom}_{A^{op}}(M^\bullet,N)) = H^i(\operatorname{Hom}_A(DN,DM^\bullet)) = \operatorname{Ext}^i_{(A,R)}(DN,DM). \tag{1.2.3.31}$$

But since every projective A-resolution for $DN\in R$-proj is (A,R)-exact, $\operatorname{Ext}^i_{(A,R)}(DN,DM)=\operatorname{Ext}^i_A(DN,DM)$ for every $i\geq 0$. ☐

1.2.4 Relative dimensions

In relative homological algebra, there are not, in general, minimal resolutions. However, we can use the relative Ext and relative Tor to define the relative versions of projective and injective dimension. Most of the results of this section can be found in section 1 of [Hat63].

1. We say that a module M has **relative projective dimension** $\operatorname{pdim}_{(A,R)} M$ less or equal to n if and only if $\operatorname{Ext}^{n+1}_{(A,R)}(M,N)=0$ for all $N\in A$-Mod.

2. We say that a module N has **relative injective dimension** $\operatorname{idim}_{(A,R)} N$ less or equal to n if and only if $\operatorname{Ext}^{n+1}_{(A,R)}(M,N) = 0$ for all $M \in A$-Mod.
3. We say that a left module M has **relative flat dimension** $\operatorname{flatdim}_{(A,R)}(M)$ less or equal to n if and only if $\operatorname{Tor}^{(A,R)}_{n+1}(L,M) = 0$ for all $L \in$ Mod-A.
4. We say that a right module L has **relative flat dimension** $\operatorname{flatdim}_{(A,R)}(L)$ less or equal to n if and only if $\operatorname{Tor}^{(A,R)}_{n+1}(L,M) = 0$ for all $M \in A$-Mod.
5. We define the **left relative global dimension** as $l.\operatorname{gldim}(A,R) = \sup\{\operatorname{pdim}_{(A,R)}(M) \colon M \in A\text{-Mod}\}$. The right left relative global dimension is defined in the same fashion.
6. We define the **left relative global dimension** of A-mod as

$$l.\operatorname{gldim}_f(A,R) = \sup\{\operatorname{pdim}_{(A,R)}(M) \colon M \in A\text{-mod}\}.$$

The right left relative global dimension is defined in the same fashion.

Lemma 1.2.39. *Let $0 \to N \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} I_1 \xrightarrow{\alpha_2} \cdots$ be an (A,R)-exact sequence. Assume $M \in A$-Mod such that $\operatorname{Ext}^l_{(A,R)}(M,I_i) = 0$ for all $l > 0$ and $i \leq s$. Then, $\operatorname{Ext}^k_{(A,R)}(M,N) \simeq \operatorname{Ext}^1_{(A,R)}(M, \operatorname{im}\alpha_{k-1})$, $2+s \geq k \geq 1$.*

Let $\cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$ be an (A,R)-exact sequence. Assume $N \in A$-Mod such that $\operatorname{Ext}^l_{(A,R)}(P_i,N) = 0$ for all $l > 0$ and $i \leq s$. Then, $\operatorname{Ext}^k_{(A,R)}(M,N) \simeq \operatorname{Ext}^1_{(A,R)}(\operatorname{im}\alpha_{k-1}, N)$, $2+s \geq k \geq 1$.

Proof. For any j there are (A,R)-exact sequences $0 \to \operatorname{im}\alpha_j \to I_j \to \operatorname{im}\alpha_{j+1} \to 0$, where $\operatorname{im}\alpha_0 = N$. Applying the functor $\operatorname{Hom}_A(M,-)$ we get the long exact sequence

$$\cdots \to \operatorname{Ext}^l_{(A,R)}(M,\operatorname{im}\alpha_j) \to \operatorname{Ext}^l_{(A,R)}(M,I_j) \to \operatorname{Ext}^l_{(A,R)}(M,\operatorname{im}\alpha_{j+1}) \to \cdots. \tag{1.2.4.1}$$

Since $\operatorname{Ext}^l_{(A,R)}(M,I_i) = 0,\ i \leq s$ the following are isomorphic

$$\operatorname{Ext}^{l+1}_{(A,R)}(M,\operatorname{im}\alpha_j) \simeq \operatorname{Ext}^l_{(A,R)}(M,\operatorname{im}\alpha_{j+1}), j \leq s,\ l \geq 1. \tag{1.2.4.2}$$

So,

$$\operatorname{Ext}^k_{(A,R)}(M,N) = \operatorname{Ext}^k_{(A,R)}(M,\operatorname{im}\alpha_0) \simeq \operatorname{Ext}^{k-1}_{(A,R)}(M,\operatorname{im}\alpha_1) \simeq \operatorname{Ext}^1_{(A,R)}(M,\operatorname{im}\alpha_{k-1}), 0 \leq k-1 \leq s+1. \tag{1.2.4.3}$$

The other claim follows using the dual argument applying the functor $\operatorname{Hom}_A(-,N)$. $\square$

Proposition 1.2.40. *Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. Then, the following assertions hold.*

1. *$M \in A$-mod is (A,R)-projective if and only if $\operatorname{Ext}^1_{(A,R)}(M,N) = 0$ for all $N \in A$-mod.*
2. *$N \in A$-mod is (A,R)-injective if and only if $\operatorname{Ext}^1_{(A,R)}(M,N) = 0$ for all $M \in A$-mod.*

Proof. Assume that $\operatorname{Ext}^1_{(A,R)}(M,N) = 0$ for all $N \in A$-mod. Consider the exact sequence

$$0 \to K \to A \otimes_R M \to M \to 0. \tag{1.2.4.4}$$

By assumption, both $A \otimes_R M, M \in A$-mod. Since R is Noetherian, $K \in A$-mod. By assumption, this sequence must split over A. The other implication is clear. The argument for (ii) is analogous. $\square$

Proposition 1.2.41. *Let $M \in A$-Mod. The following assertions are equivalent.*

(i) $\mathrm{Ext}^i_{(A,R)}(M,N) = 0$ *for all $i > n$ and all left A-modules N;*

(ii) $\mathrm{pdim}_{(A,R)}(M) \leq n$*, that is,* $\mathrm{Ext}^{n+1}_{(A,R)}(M,N) = 0$ *for all left A-modules N;*

(iii) If $0 \to K_{n-1} \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ *is an (A,R)-exact sequence with all X_i being (A,R)-projective, then K_{n-1} is (A,R)-projective.*

Proof. $(i) \implies (ii)$ is clear. Assume that (ii) holds. Let

$$0 \to K_{n-1} \xrightarrow{\alpha_n} X_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \to X_0 \xrightarrow{\alpha_0} M \to 0 \tag{1.2.4.5}$$

be an (A,R)-exact sequence where all X_i are (A,R)-projective. Since $\mathrm{Ext}^{l>0}_{(A,R)}(X_i,N) = 0$, for all $N \in A$-Mod, it follows

$$0 = \mathrm{Ext}^{n+1}_{(A,R)}(M,N) \simeq \mathrm{Ext}^1_{(A,R)}(\mathrm{im}\,\alpha_n,N) = \mathrm{Ext}^1_{(A,R)}(K_{n-1},N). \tag{1.2.4.6}$$

By Proposition 1.2.40, K_{n-1} is (A,R)-projective. Hence, (iii) follows.

Assume that (iii) holds. Let

$$X^\bullet: \quad \cdots \to X_{n-1} \to \cdots \xrightarrow{\alpha_1} X_0 \xrightarrow{\alpha_0} M \to 0 \tag{1.2.4.7}$$

be the standard (A,R)-projective resolution of M. If, for some $j < n$, X_j is zero, then the cohomology of the associated deleted complex of $\mathrm{Hom}(X^\bullet,N)$ vanishes for degree greater than n. If there is no such j, then we can consider the (A,R)-exact sequence

$$0 \to \ker\alpha_{n-1} \to X_{n-1} \to \cdots \xrightarrow{\alpha_1} X_0 \xrightarrow{\alpha_0} M \to 0. \tag{1.2.4.8}$$

By assumption, $\ker\alpha_{n-1}$ is (A,R)-projective. Now using this (A,R)-projective resolution to compute $\mathrm{Ext}^{i>0}(M,N)$, (i) follows. ☐

Proposition 1.2.42. *Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. For $M \in A$-*mod*, the following are equivalent:*

(i) $\mathrm{Ext}^{n+1}_{(A,R)}(M,N) = 0$ *for all $N \in A$-*mod*;*

(ii) $\mathrm{Ext}^{n+1}_{(A,R)}(M,N) = 0$ *for all $N \in A$-*Mod*.*

Proof. $(ii) \implies (i)$ is clear. Assume that (i) holds. Since M is finitely generated over the Noetherian ring R and $A \otimes_R M$ is finitely generated over A all the modules in the standard (A,R)-projective resolution of M are finitely generated over A. In particular, $K_{n-1} \in A$-mod according to the notation of Proposition 1.2.41. Using the same argument,

$$0 = \mathrm{Ext}^{n+1}_{(A,R)}(M,N) \simeq \mathrm{Ext}^1_{(A,R)}(\mathrm{im}\,\alpha_n,N) = \mathrm{Ext}^1_{(A,R)}(K_{n-1},N), \tag{1.2.4.9}$$

for all $N \in A$-mod. By Proposition 1.2.40, K_{n-1} is (A,R)-projective. Using the (A,R)-projective resolution $0 \to K_{n-1} \to X_{n-1} \to X_0 \to M \to 0$ to compute Ext we conclude (ii). ☐

Proposition 1.2.43. *Let $N \in A$-*Mod*. The following assertions are equivalent.*

1. $\mathrm{Ext}^i_{(A,R)}(M,N) = 0$ *for all $i > n$ and all A-modules M;*
2. $\mathrm{idim}_{(A,R)}(N) \leq n$*;*

3. *If $0 \to N \to Q_0 \to Q_1 \to \cdots \to Q_{n-1} \to N' \to 0$ is (A,R)-exact with all Q_i being (A,R)-injective, then N' is (A,R)-injective.*

Proof. Analogous to Proposition 1.2.41. □

Proposition 1.2.44. *Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. For $N \in A\text{-mod}$, the following are equivalent:*

(i) $\operatorname{Ext}^{n+1}_{(A,R)}(M,N) = 0$ *for all* $M \in A\text{-mod}$;

(ii) $\operatorname{idim}_{(A,R)}(N) \leq n$.

Proof. Analogous to Proposition 1.2.42. □

Corollary 1.2.45. *Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. Let $M \in A\text{-mod} \cap R\text{-proj}$. Then, $\operatorname{idim}_{(A,R)} M \leq n$ if and only if $\operatorname{Ext}^{n+1}_A(X,M) = 0$ for all $X \in A\text{-mod} \cap R\text{-proj}$.*

Proof. Assume $\operatorname{Ext}^{n+1}_A(X,M) = 0$ for all $X \in A\text{-mod} \cap R\text{-proj}$. Consider an (A,R)-injective resolution

$$0 \to M \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} I_1 \to \cdots \to I_{n-1} \xrightarrow{\alpha_n} I_n \to \cdots \tag{1.2.4.10}$$

By Lemma 1.2.39,

$$0 = \operatorname{Ext}^{n+1}_A(X,M) \simeq \operatorname{Ext}^{n+1}_{(A,R)}(X,M) \simeq \operatorname{Ext}^{1}_{(A,R)}(X, \operatorname{im}\alpha_n). \tag{1.2.4.11}$$

Since $M \in R\text{-proj}, \operatorname{im}\alpha_n \in R\text{-proj}$. By Corollary 1.2.35, $\operatorname{im}\alpha_n$ is (A,R)-injective. Hence, $\operatorname{idim}_{(A,R)} M \leq n$.

The other implication is clear since (A,R)-projective resolutions of X are projective A-resolutions. □

Proposition 1.2.46. *For any R-algebra A, we have*

$$\begin{aligned} l.\operatorname{gldim}(A,R) = \sup\{\operatorname{pdim}_{(A,R)}(M) \colon M \in A\text{-Mod}\} &= \sup\{\operatorname{idim}_{(A,R)}(N) \colon N \in A\text{-Mod}\} \\ &= \sup\{n \colon \operatorname{Ext}^n_{(A,R)}(M,N) \neq 0, \ \forall M,N \in A\text{-Mod}\}. \end{aligned}$$

Proof. Let

$$n = l.\operatorname{gldim}(A,R), \ k = \sup\{\operatorname{idim}_{(A,R)}(N) \colon N \in A\text{-Mod}\}, \ s = \sup\{n \colon \operatorname{Ext}^n_{(A,R)}(M,N) \neq 0, \ \forall M,N \in A\text{-Mod}\}.$$

By definition, $n \geq \operatorname{pdim}_{(A,R)} M$ for every $M \in A\text{-Mod}$. By Proposition 1.2.41, for each $M \in A\text{-Mod}$, $\operatorname{Ext}^{i>n}_{(A,R)}(M,N) = 0$ for every $N \in A\text{-Mod}$. Thus, $n \geq s$. By Proposition 1.2.43, it follows $k \geq s$. Let $M \in A\text{-Mod}$. Then, $\operatorname{Ext}^{i>s}_{(A,R)}(M,N) = 0$ for every $N \in A\text{-Mod}$. If $s = +\infty$, then we are done. Assume $s < \infty$.

Let $M \in A\text{-Mod}$. Then, $\operatorname{Ext}^{s+1}_{(A,R)}(M,N) = 0$ for every $N \in A\text{-Mod}$. Thus, $\operatorname{pdim}_{(A,R)} M \leq s$. Since the choice of M is arbitrary $n \leq s$. In the same way, $k \leq s$. Therefore, $n = s = k$. □

Proposition 1.2.47. *[Hat63, Proposition 1.1] For any R-algebra A, the left relative weak global dimension and right relative weak global dimension coincide. Furthermore,*

$$\begin{aligned} l.\operatorname{wgldim}(A,R) &= \sup\{\operatorname{flatdim}_{(A,R)}(M) \colon M \in A\text{-Mod}\} \\ &= \sup\{n \colon \operatorname{Tor}^{(A,R)}_n(L,M) \neq 0, \ M \in A\text{-Mod}, \ L \in \text{Mod-}A\} \\ &= \sup\{\operatorname{flatdim}_{(A,R)}(L) \colon L \in \text{Mod-}A\} = r.\operatorname{wgldim}(A,R). \end{aligned}$$

Proof. The proof is similar to Proposition 1.2.46. □

Proposition 1.2.48. *[Hat63, Proposition 1.1] For any R-algebra we have,*

$$l.\operatorname{wgldim}_f(A,R) = \sup\{\operatorname{flatdim}_{(A,R)}(M)\colon M \in A\text{-mod}\} = \sup\{\operatorname{flatdim}_{(A,R)}(N)\colon N \in \text{mod-}A\}.$$

Proof. Both terms are equal to $\sup\{n\colon \operatorname{Tor}_n^{(A,R)}(L,M) \neq 0,\ M \in A\text{-mod},\ L \in \text{mod-}A\}$. □

Proposition 1.2.49. *[Hat63, Section 1.2] Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. Then,* $\sup\{\operatorname{pdim}_{(A,R)}(M)\colon M \in A\text{-mod}\} = \sup\{\operatorname{idim}_{(A,R)}(N)\colon N \in A\text{-mod}\}$.

Proof. Let $\sup\{\operatorname{pdim}_{(A,R)}(M)\colon M \in A\text{-mod}\} \leq n$. Consider any $N \in A$-mod. Consider the standard (A,R)-injective resolution $0 \to N \xrightarrow{\alpha_0} Q_0 \xrightarrow{\alpha_1} \cdots \to Q_{n-1} \to \operatorname{im}\alpha_n \to 0$. As R is Noetherian, each Q_i is finitely generated, and thus $\operatorname{im}\alpha_n$ is finitely generated. So,

$$0 = \operatorname{Ext}^{n+1}_{(A,R)}(M,N) \simeq \operatorname{Ext}^1_{(A,R)}(M,\operatorname{im}\alpha_n), \tag{1.2.4.12}$$

for every $M \in A$-mod. Therefore, $\operatorname{im}\alpha_n$ is (A,R)-injective. By Proposition 1.2.43, $\operatorname{idim}_{(A,R)} N \leq n$. Thus, $\sup\{\operatorname{idim}_{(A,R)}(N)\colon N \in A\text{-mod}\} \leq n$. By a symmetrical argument, we get the other inequality. □

Proposition 1.2.50. *Let R be a commutative Noetherian ring with identity. Let S be a flat R-algebra and let A be a Noetherian R-algebra. Let $M,N \in A$-mod and $n \geq 0$. Then,*

$$S \otimes_R \operatorname{Ext}^n_{(A,R)}(M,N) \simeq \operatorname{Ext}^n_{(S\otimes_R A,S)}(S\otimes_R M, S\otimes_R N).$$

Proof. Since S is flat R-algebra, the functor $S\otimes_R -\colon A\text{-mod} \to S\otimes_R A\text{-mod}$ is exact. Consider an (A,R)-exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$. By the exactness of $S\otimes_R -$, $S\otimes_R \ker g \simeq \ker(S\otimes_R g)$. Hence, $S\otimes_R \ker g$ is an S-summand of $S\otimes_R N$. Thus, the functor $S\otimes_R -$ sends (A,R)-exact sequences to $(S\otimes_R A,S)$-exact sequences. Moreover,

$$S\otimes_R \operatorname{Hom}_R(A,M) \simeq \operatorname{Hom}_{S\otimes_R R}(S\otimes_R A, S\otimes_R M) \tag{1.2.4.13}$$

is an $(S\otimes_R A,S)$-injective module. So, $S\otimes_R -$ sends (A,R)-injective resolutions to $(S\otimes_R A,S)$-injective resolutions. Let $I^\bullet\colon \cdots \to I_1 \to I_0 \to 0$ be a deleted resolution of the standard (A,R)-injective resolution of N. Then,

$$\begin{aligned} S\otimes_R \operatorname{Ext}^n_{(A,R)}(M,N) &\simeq S\otimes_R H^n(\operatorname{Hom}_A(M,I^\bullet)) \simeq H^n(S\otimes_R \operatorname{Hom}_A(M,I^\bullet)) \\ &\simeq H^n(\operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R I^\bullet)) \simeq \operatorname{Ext}^n_{(S\otimes_R A,S)}(S\otimes_R M, S\otimes_R N). \end{aligned} \tag{1.2.4.14}$$

□

Corollary 1.2.51. *[Hat63, section 1.4] Let $\mathfrak{p}$ be a prime ideal of R and $n \geq 0$. Then,* $\operatorname{Ext}^n_{(A,R)}(M,N)_{\mathfrak{p}} \simeq \operatorname{Ext}^n_{(A_{\mathfrak{p}},R_{\mathfrak{p}})}(M_{\mathfrak{p}},N_{\mathfrak{p}})$.

Proof. It follows from Proposition 1.2.50 by considering $S = R_p$. □

Lemma 1.2.52. *[Hat63, section 1.3] Let A and B be Noetherian R-algebras. Let M be a left A-module, N be an (A,B)-bimodule, Q is a right B-injective. Consider $\sigma_{N,Q,M}\colon$ $\operatorname{Hom}_B(N,Q)\otimes_A M \to \operatorname{Hom}_B(\operatorname{Hom}_A(M,N),Q)$, given by $\sigma(g\otimes m)(f) = g(f(m))$. If $M \in A$-mod, then $\sigma_{N,Q,M}$ is an R-isomorphism.*

Proof. Let N be an (A,B)-bimodule and Q a right B-injective module. First we will show that $\sigma_{N,Q,M}$ is functorial on M.

Let $X,Y \in A$-mod. Denote by $\theta^A_{X,Y,N}$ the canonical isomorphism $\operatorname{Hom}_A(X,N)\bigoplus \operatorname{Hom}_A(Y,N) \to \operatorname{Hom}_A(X\bigoplus Y,N)$ and denote by $\theta^B_{B_1,B_2,Q}$ the canonical isomorphism $\operatorname{Hom}_B(B_1,Q)\bigoplus \operatorname{Hom}_B(B_2,Q) \to \operatorname{Hom}_B(B_1\bigoplus B_2,Q)$ for $B_1,B_2 \in$

B-mod. Put $\theta_2=\theta^B_{\mathrm{Hom}_A(X,N),\mathrm{Hom}_A(Y,N),Q}$ The following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_B(N,Q)\otimes_A(X\bigoplus Y) \arrow[rr, "\sigma_{N,Q,X\bigoplus Y}"] \arrow[dd, "\psi"] & & \mathrm{Hom}_B(\mathrm{Hom}_A(X\bigoplus Y,N),Q) \arrow[d, "\mathrm{Hom}_A(\theta_1,Q)"] \\
& & \mathrm{Hom}_B(\mathrm{Hom}_A(X,N)\bigoplus \mathrm{Hom}_A(Y,N),Q) \\
\mathrm{Hom}_B(N,Q)\otimes_A X\bigoplus \mathrm{Hom}_B(N,Q)\otimes_A Y \arrow[rr, "\sigma_{N,Q,X}\oplus\sigma_{N,Q,Y}"] & & \mathrm{Hom}_B(\mathrm{Hom}_A(X,N),Q)\bigoplus \mathrm{Hom}_B(\mathrm{Hom}_A(Y,N),Q) \arrow[u, "\theta_2"']
\end{tikzcd}, \tag{1.2.4.15}$$

where ψ is the natural isomorphism.

In fact, for $g\otimes(x,y)\in \mathrm{Hom}_B(N,Q)\otimes_A(X\bigoplus Y)$, $(f,h)\in \mathrm{Hom}_A(X,N)\bigoplus \mathrm{Hom}_A(Y,N)$

$$\begin{aligned}
\mathrm{Hom}_A(\theta_1,Q)\circ\sigma_{N,Q,X\oplus Y}(g\otimes(x,y))(f,h) &= \sigma_{N,Q,X\oplus Y}(g\otimes(x,y))\circ\theta_1(f,h)=g(\theta_1(f,h)(x,y)\\
&= g\circ f(x)+g\circ h(y)\\
\theta_2\circ\sigma_{N,Q,X}\oplus\sigma_{N,Q,Y}\circ\psi(g\otimes(x,y))(f,h) &= \theta_2(\sigma_{N,Q,X}\oplus\sigma_{N,Q,Y}(g\otimes x,g\otimes y))(f,h)\\
&= \sigma_{N,Q,X}(g\otimes x)(f)+\sigma_{N,Q,Y}(g\otimes y)(h)=g\circ f(x)+g\circ h(y).
\end{aligned}$$

Define the map $\pi\colon \mathrm{Hom}_A(A,N)\to N$, given by $\pi(f)=f(1_A)$ and denote by μ the multiplication map $\mu\colon \mathrm{Hom}_B(N,Q)\otimes_A A\to \mathrm{Hom}_B(N,Q)$. Hence, $\sigma_{N,Q,A}=\mathrm{Hom}_B(\pi,Q)\circ\mu$ is an isomorphism. Using the diagram (1.2.4.15) we obtain that $\sigma_{N,Q,M}$ is an isomorphism for every $M\in A$-proj.

Let $M\in A$-mod. Then, there is an A-presentation $A^s\to A^t\to M\to 0$. As Q is B-injective, $\mathrm{Hom}_B(-,Q)$ is contravariant exact. On the other hand, $\mathrm{Hom}_A(-,N)$ is contravariant left exact, so the composition functor $\mathrm{Hom}_B(-,Q)\circ\mathrm{Hom}_A(-,N)$ is covariant right exact. The functor $\mathrm{Hom}_B(N,Q)\otimes_A -$ is covariant right exact. Therefore, applying the functors $\mathrm{Hom}_B(-,Q)\circ\mathrm{Hom}_A(-,N)$ and $\mathrm{Hom}_B(N,Q)\otimes_A -$ we obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
\mathrm{Hom}_B(N,Q)\otimes A^s \arrow[r] \arrow[d, "\sigma_{N,Q,A^s}"] & \mathrm{Hom}_B(N,Q)\otimes A^t \arrow[r, two heads] \arrow[d, "\sigma_{N,Q,A^t}"] & \mathrm{Hom}_B(N,Q)\otimes M \arrow[d, "\sigma_{N,Q,M}"] \\
\mathrm{Hom}_B(\mathrm{Hom}_A(A^s,N),Q) \arrow[r] & \mathrm{Hom}_B(\mathrm{Hom}_A(A^t,N),Q) \arrow[r, two heads] & \mathrm{Hom}_B(\mathrm{Hom}_A(M,N),Q)
\end{tikzcd}. \tag{1.2.4.16}$$

By diagram chasing, it follows that $\sigma_{N,Q,M}$ is an isomorphism. ☐

Lemma 1.2.53. *Let R be a commutative Noetherian ring. Let A be a Noetherian R-algebra. Let $M,N\in A$-*mod *and Q an R-injective module. Then,*

$$\mathrm{Tor}_n^{(A,R)}(\mathrm{Hom}_R(N,Q),M)\simeq \mathrm{Hom}_R(\mathrm{Ext}^n_{(A,R)}(M,N),Q).$$

Proof. Let $P^\bullet$ be the standard (A,R)-projective resolution of M. Since R is Noetherian ring and M is finitely generated, every module in $P^\bullet$ belongs to A-mod. Hence,

$$\begin{aligned}
\mathrm{Tor}_n^{(A,R)}(\mathrm{Hom}_R(N,Q),M) &= H_n(\mathrm{Hom}_R(N,Q)\otimes_A P^\bullet)\simeq H_n(\mathrm{Hom}_R(\mathrm{Hom}_A(P^\bullet,N),Q)) \qquad (1.2.4.17)\\
&\simeq \mathrm{Hom}_R(H^n(\mathrm{Hom}_A(P^\bullet,N)),Q)=\mathrm{Hom}_R(\mathrm{Ext}^n_{(A,R)}(M,N),Q).
\end{aligned}$$ ☐

Proposition 1.2.54. *[Hat63, Proposition 1.3] Let R be a commutative Noetherian ring and A a Noetherian R-algebra. Then, $l.\operatorname{gldim}_f(A,R)=\operatorname{wgldim}_f(A,R)=r.\operatorname{gldim}_f(A,R)$.*

Proof. Let $n \in \mathbb{N}$ and $M,N \in A$-mod such that $\operatorname{Ext}^n_{(A,R)}(M,N) \neq 0$. Let Q be an injective cogenerator of R-mod. Then, by Lemma 1.2.53,

$$\operatorname{Tor}_n^{(A,R)}(\operatorname{Hom}_R(N,Q),M) \simeq \operatorname{Hom}_R(\operatorname{Ext}^n_{(A,R)}(M,N),Q) \neq 0. \tag{1.2.4.18}$$

In particular, $\operatorname{wgldim}_f(A,R) \geq l.\operatorname{gldim}_f(A,R)$. If $l.\operatorname{gldim}_f(A,R) = \infty$, then we are done.

Assume that $l.\operatorname{gldim}_f(A,R) \leq n$ for some natural number n. Let $M \in A$-mod. Then, $\operatorname{pdim}_{(A,R)} M \geq n$. So, we can find an (A,R)-projective resolution of length n. So, using such projective resolution to compute Tor we obtain $\operatorname{Tor}_{n+1}^{(A,R)}(L,M) = 0$ for every L right A-module. Therefore, $\operatorname{wgldim}_f(A,R) \leq n$. So, we conclude $l.\operatorname{gldim}_f(A,R) = \operatorname{wgldim}_f(A,R)$. Symmetrically, we obtain $\operatorname{wgldim}_f(A,R) = r.\operatorname{gldim}_f(A,R)$. ☐

For Noetherian R-algebras, we write $\operatorname{gldim}_f(A,R)$ to denote the value $l.\operatorname{gldim}_f(A,R) = r.\operatorname{gldim}_f(A,R)$.

Lemma 1.2.55. *Let A be a projective Noetherian R-algebra. Let I be an (A,R)-injective module and $M \in R\text{-proj} \cap A\text{-mod}$. Then, $\operatorname{Ext}^i_A(M,I) = 0$ for all $i > 0$.*

Proof. Let $i = 1$. We notice that any A-exact sequence $0 \to I \to X \to M \to 0$ splits over R. Since I is (A,R)-injective, it splits over A. Thus, $\operatorname{Ext}^1_A(M,I) = 0$. Consider an projective A-resolution for M, $\cdots \to P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$. In particular, there are exact sequences

$$0 \to \operatorname{im}\alpha_j \to P_{j-1} \to \operatorname{im}\alpha_{j-1} \to 0. \tag{1.2.4.19}$$

Since $M \in R$-proj, (1.2.4.19) is (A,R)-exact. Thus, for every $j \geq 0, \operatorname{im}\alpha_j \in R$-proj. Let $i > 1$. So,

$$\operatorname{Ext}^i_A(M,I) \simeq \operatorname{Ext}^1_A(\operatorname{im}\alpha_{i-1},I) = 0. \qquad ☐$$

Lemma 1.2.56. *Let A be a projective Noetherian R-algebra. Let $M \in A\text{-mod} \cap R\text{-proj}$. Assume that M is an (A,R)-injective. Then, DM is projective over A^{op}. If $P \in A\text{-proj}$, then DP is an (A^{op},R)-injective module.*

Proof. Let P be a projective A-module. Then, DP is an A-summand of $\operatorname{Hom}_R(A^t,R) \simeq \operatorname{Hom}_R(A,R)^t$. Hence, DP is (A^{op},R)-injective.

Let M be an (A,R)-injective module and projective module as R-module. Then, M is an A-summand of $\operatorname{Hom}_R(A,M)$. Note that

$$D\operatorname{Hom}_R(A,M) \simeq \operatorname{Hom}_R(\operatorname{Hom}_R(A,M),R) \simeq \operatorname{Hom}_R(\operatorname{Hom}_R(A,R)\otimes_R M,R) \tag{1.2.4.20}$$
$$\simeq \operatorname{Hom}_R(M,\operatorname{Hom}_R(\operatorname{Hom}_R(A,R),R)) \simeq \operatorname{Hom}_R(M,A) \simeq \operatorname{Hom}_R(M,R)\otimes_R A \tag{1.2.4.21}$$
$$= A \otimes_R DM. \tag{1.2.4.22}$$

Hence, $D\operatorname{Hom}_R(A,M)$ is (A^{op},R)-projective. As $M \in R$-proj, $D\operatorname{Hom}_R(A,M)$ is projective over R. Therefore, $D\operatorname{Hom}_R(A,M) \in A^{op}$-proj. It follows that $DM \in A^{op}$-proj. Thus, $M \simeq DDM$ is (A,R)-injective. ☐

Using this Lemma, we can formulate the dual version of Theorem 1.1.51.

Theorem 1.2.57. *Let A be a projective Noetherian R-algebra. Let $P \in A\text{-mod} \cap R\text{-proj}$. Then, P is (A,R)-injective if and only if $P(\mathfrak{m})$ is $A(\mathfrak{m})$-injective for every maximal ideal $\mathfrak{m}$ in R.*

Proof. Assume that P is (A,R)-injective. Then, DP is (A^{op},R)-injective. Since $P \in R$-proj, $DP \in A^{op}$-proj. Let $\mathfrak{m}$ be a maximal ideal in R. Then, $DP(\mathfrak{m}) = \operatorname{Hom}_R(P,R)(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(P(\mathfrak{m}),R(\mathfrak{m}))$ is projective as right $A(\mathfrak{m})$-module. Thus, $P(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(\operatorname{Hom}_{R(\mathfrak{m})}(P(\mathfrak{m}),R(\mathfrak{m})),R(\mathfrak{m}))$ is $A(\mathfrak{m})$-injective.

Conversely, assume that $P(\mathfrak{m})$ is $A(\mathfrak{m})$-injective for every maximal ideal $\mathfrak{m}$ in R. Then,

$$\mathrm{Hom}_{R(\mathfrak{m})}(P(\mathfrak{m}),R(\mathfrak{m})) \simeq \mathrm{Hom}_R(P,R)(\mathfrak{m})$$

is projective as right module over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Thus, $DP = \mathrm{Hom}_R(P,R)$ is projective over A^{op} since $DP \in R$-proj. Hence, $P \simeq DDP$ is (A,R)-injective. □

In this sense, relative injective modules can be viewed as a natural generalization of injective modules of finite dimensional algebras. For projective Noetherian R-algebras we have one more alternative characterization of (A,R)-injective modules which are projective as R-modules.

Proposition 1.2.58. *Let A be a projective Noetherian R-algebra. Let $I \in A$-mod$\cap R$-proj. I is (A,R)-injective if and only if* $\mathrm{Ext}_A^1(M,I) = 0$ *for all* $M \in A$-mod$\cap R$-proj.

Proof. Assume that $\mathrm{Ext}_A^1(M,I) = 0$ for all $M \in A$-mod$\cap R$-proj. Let

$$\delta\colon \quad 0 \to I \to \mathrm{Hom}_R(A,I) \to X \to 0 \tag{1.2.4.23}$$

be the standard (A,R)-injective copresentation of I. In particular, $\mathrm{Hom}_R(A,I)$ is projective over R. Since δ is R-split exact, X is also a projective R-module. Thus, $\delta \in \mathrm{Ext}_A^1(X,I)$ with $X \in A$-mod$\cap R$-proj. By assumption, δ is split over A. Thus, I is (A,R)-injective. By Lemma 1.2.55, the converse statement is clear. □

In [Rou08], the modules $I \in A$-mod$\cap R$-proj satisfying the property $\mathrm{Ext}_A^1(M,I) = 0$ for all $M \in A$-mod$\cap R$-proj are called relatively R-injective. Therefore, the relatively R-injective modules are exactly the (A,R)-injective modules which are projective over R. Furthermore, this characterization says that the (A,R)-injective modules which are projective over R are exactly the objects X of $\mathscr{A} = A$-mod$\cap R$-proj which make $\mathrm{Hom}_{\mathscr{A}}(-,X)$ exact on $\mathscr{A}$.

1.2.5 Further relative notions

One more evidence that (A,R)-monomorphisms behave like the inclusions between modules over finite dimensional algebras is the following version of Nakayama's Lemma for (A,R)-monomorphisms, dual to Lemma 1.1.38 (c).

Lemma 1.2.59. *Suppose R is a commutative ring. Let A be an R-algebra. If $\phi\colon M \to N$ is (A,R)-monomorphism and $M \simeq N$ as finitely generated R-modules, then ϕ is an isomorphism.*

Proof. Since ϕ is (A,R)-mono, there exists $\varepsilon\colon N \to M$ such that $\varepsilon \circ \phi = \mathrm{id}_M$. Thus, ε is surjective. By Nakayama's Lemma (c), ε is an R-isomorphism. Therefore, $\phi = \varepsilon^{-1} \circ \varepsilon \circ \phi = \varepsilon^{-1}$ is bijective. □

Assume that Q is a cogenerator. Since every finitely generated injective module can be embedded into a finite direct sum of copies of Q (see Theorem 1.1.12), then every injective module belongs to the additive closure of Q. In particular, for Artinian rings, a module is a cogenerator if and only if contains all injective indecomposable modules. However, we are only interested in the relative injective modules which are projective over the ground ring. Thus, for our purposes, the cogenerators can be replaced by modules which additive closure contain all relative injective modules which are projective over the ground ring. This motivates the following definition.

Definition 1.2.60. A module Q is called (A,R)**-cogenerator** if and only if the left A-module $DA_A = \mathrm{Hom}_R(A,R)$ belongs to the additive closure $\mathrm{add}\, Q$.

Definition 1.2.61. Let R be a commutative ring. An R-algebra A is called **semi-simple relative to** R if every finitely generated left A-module is (A,R)-projective.

If the ground ring is semi-simple in the classical sense, relative projectivity coincides with the absolute projectivity. Hence, an algebra semi-simple relative to a semi-simple ring is semi-simple in the classical sense. In particular, if R is a field, relative semi-simple coincides with the classical notion of semi-simplicity. It is also clear that every semi-simple algebra is semi-simple relative to the ground ring. For more details on this concept we refer to [Hat63].

1.3 Spectral sequences

The computation of Ext and Tor groups is not always done directly by the definition using projective and injective resolutions. Instead, spectral sequences provide useful ways to compute homology and cohomology of complexes. For a more detailed approach, we refer to ([Wei03], [Rot09]).

Definition 1.3.1. A **(homology) spectral sequence** (starting with E^a) in an abelian category $\mathscr{A}$ consists of the following data:

- For $r \geq a$, the r-page is a collection of objects of $\mathscr{A}$ $\{E^r_{i,j}\}$, $i,j \in \mathbb{Z}$.
- Maps $d^r_{i,j} \colon E^r_{i,j} \to E^r_{i-r,j+r-1}$ satisfying $d^r_{i,j} \circ d^r_{i+r,j-r+1} = 0$ and $E^{r+1}_{i,j} = \ker d^r_{i,j} / \operatorname{im} d^r_{i+r,j-r+1}$.

If $E^r_{i,j} = 0$ unless $i \geq 0$ and $j \geq 0$, then we say that $\{E^r_{i,j}\}$ is a **first quadrant homology spectral sequence.**

Hence, the $(r+1)$-page consists of the homology of the differential of the r-page. If the value at (i,j)-spot stabilizes from some page on, then we denote this value by $E^\infty_{i,j}$.

Dually, we can define (cohomology) spectral sequences.

Definition 1.3.2. A **(cohomology) spectral sequence** (starting with E_a) in an abelian category $\mathscr{A}$ consists of the following data:

- For $r \geq a$, the r-page is a collection of objects of $\mathscr{A}$ $\{E_r^{i,j}\}$, $i,j \in \mathbb{Z}$.
- Maps $d_r^{i,j} \colon E_r^{i,j} \to E_r^{i+r,j-r+1}$ satisfying $d_r^{i,j} \circ d_r^{i-r,j+r-1} = 0$ and $E^{i,j}_{r+1} = \ker d_r^{i,j} / \operatorname{im} d_r^{i-r,j+r-1}$.

If $E_r^{i,j} = 0$ unless $i \geq 0$ and $j \geq 0$, then we say that $\{E_r^{i,j}\}$ is a **first quadrant cohomology spectral sequence.**

If the value at (i,j)-spot stabilizes from some page on, then we denote this value by $E_\infty^{i,j}$.

We can also see the cohomology spectral sequence as a homology spectral sequence reindexing the (i,j)-spots: $E_r^{i,j} = E^r_{-i,-j}$.

Definition 1.3.3. We say that a (homology) spectral sequence converges to H_*, written as

$$E^a_{i,j} \implies H_{i+j}$$

if we are given a collection of objects H_n of $\mathscr{A}$, each having a finite filtration

$$0 = H^s_n \subset \cdots \subset H^{p-1}_n \subset H^p_n \cdots \subset H^t_n = H_n$$

such that $E^\infty_{p,q} \simeq H^p_{p+q} / H^{p-1}_{p+q}$.

Definition 1.3.4. We say that a (cohomology) spectral sequence converges to H^*, written as

$$E_a^{i,j} \implies H^{i+j}$$

if we are given a collection of objects H^n of $\mathscr{A}$, each having a finite filtration

$$0 = H_t^n \subset \cdots \subset H_{p+1}^n \subset H_p^n \subset \cdots \subset H_s^n = H^n$$

such that $E_\infty^{p,q} \simeq H_p^{p+q}/H_{p+1}^{p+q}$.

The notion of convergence for first quadrant spectral sequence can be stated in a simple way. The majority of spectral sequences here treated are first quadrant homology/cohomology spectral sequences, hence it is worthwhile to restate convergence of first quadrant spectral sequences.

Definition 1.3.5. We say that a first quadrant (homology) spectral sequence converges to H_*, written as

$$E_{i,j}^a \implies H_{i+j}$$

if we are given a collection of objects H_n of $\mathscr{A}$, each having a finite filtration

$$0 = H_n^{-1} \subset H_n^0 \subset H_n^1 \subset \cdots \subset H_n^n = H_n$$

such that $E_{i,n-i}^\infty \simeq H_n^i/H_n^{i-1}$ for $0 \leq i \leq n$.

Definition 1.3.6. We say that a first quadrant (cohomology) spectral sequence converges to H^*, written as

$$E_a^{i,j} \implies H^{i+j}$$

if we are given a collection of objects H^n of $\mathscr{A}$, each having a finite filtration

$$0 = H_{n+1}^n \subset H_n^n \subset H_{n-1}^n \subset \cdots \subset H_1^n \subset H_0^n = H^n$$

such that $E_\infty^{i,n-i} \simeq H_i^n/H_{i+1}^n$ for $0 \leq i \leq n$.

Lemma 1.3.7. *Assume that $E_{i,j}^2 \implies H_{i+j}$ is a first quadrant spectral sequence. Then, there is an exact sequence*

$$H_2 \to E_{2,0}^2 \to E_{0,1}^2 \to H_1 \to E_{1,0}^2 \to 0. \tag{1.3.0.1}$$

Proof. By convergence, we have the filtration

$$0 = H_1^{-1} \subset H_1^0 \subset H_1^1 = H_1. \tag{1.3.0.2}$$

with $E_{1,0}^\infty \simeq H_1^1/H_1^0$ and $E_{0,1}^\infty \simeq H_1^0/H_1^{-1} = H_1^0$. In particular, there is an exact sequence

$$0 \to E_{0,1}^\infty \to H_1 \to E_{1,0}^\infty \to 0. \tag{1.3.0.3}$$

Let $n \geq 2$. Then,

$$\begin{aligned} E_{1,0}^{n+1} &= \ker\left(d_{1,0}^n \colon E_{1,0}^n \to E_{1-n,n-1}^n\right)/\operatorname{im}\left(d_{1+n,1-n}^n \colon E_{1+n,-n+1}^n \to E_{1,0}^n\right) && (1.3.0.4)\\ &= E_{1,0}^n. && (1.3.0.5) \end{aligned}$$

By induction, $E^n_{1,0} = E^2_{1,0}$ for $n \geq 2$. By definition, $E^\infty_{1,0} = E^2_{1,0}$. We will now compute $E^\infty_{0,1}$. For $n \geq 3$,

$$E^{n+1}_{0,1} = \ker d^n_{0,1} / \operatorname{im} d^n_{n,2-n} = \ker d^n_{0,1} = \ker(E^n_{0,1} \to E^n_{-n,n}) = E^n_{0,1}. \tag{1.3.0.6}$$

By induction, it follows that

$$E^\infty_{0,1} = E^3_{0,1} = \ker d^2_{0,1} / \operatorname{im} d^2_{2,0} = E^2_{0,1} / \operatorname{im}(E^2_{2,0} \to E^2_{0,1}) = \operatorname{coker}(E^2_{2,0} \to E^2_{0,1}). \tag{1.3.0.7}$$

Now, $E^\infty_{2,0} = H^2_2 / H^1_2 = H_2 / H^1_2$. For $n \geq 2$,

$$E^{n+1}_{2,0} = \ker d^n_{2,0} / \operatorname{im} d^n_{2+n,1-n} = \ker(E^n_{2,0} \to E^n_{2-n,n-1}). \tag{1.3.0.8}$$

Therefore, $E^\infty_{2,0} = \ker(E^2_{2,0} \to E^2_{0,1})$. We constructed an exact sequence

$$\begin{tikzcd}
H_2 \arrow[rr] \arrow[dr, two heads] & & E^2_{2,0} \arrow[r] & E^2_{0,1} \arrow[rr] \arrow[dr, two heads] & & H_1 \arrow[r] & E^2_{1,0} \arrow[r] & 0 \\
& E^\infty_{2,0} \arrow[ur, hook] & & & E^\infty_{0,1} \arrow[ur, hook] & & &
\end{tikzcd}.$$

□

Lemma 1.3.8. *Assume that $E^{i,j}_2 \implies H^{i+j}$ is a first quadrant spectral sequence. Then, there is an exact sequence*

$$0 \to E^{1,0}_2 \to H^1 \to E^{0,1}_2 \to E^{2,0}_2 \to H^2. \tag{1.3.0.9}$$

Proof. By convergence, we have the filtration

$$0 = H^1_2 \subset H^1_1 \subset H^1_0 = H^1. \tag{1.3.0.10}$$

with $E^{1,0}_\infty \simeq H^1_1 / H^1_2 = H^1_1$ and $E^{0,1}_\infty = H^1 / H^1_1$. In particular, there is an exact sequence

$$0 \to E^{1,0}_\infty \to H^1 \to E^{0,1}_\infty \to 0. \tag{1.3.0.11}$$

Let $n \geq 2$. Then,

$$\begin{aligned} E^{1,0}_{n+1} &= \ker\left(d^{1,0}_n \colon E^{1,0}_n \to E^{1+n,1-n}_n\right) / \operatorname{im}\left(d^{1-n,n-1}_n \colon E^{1-n,n-1}_n \to E^{1,0}_n\right) && (1.3.0.12) \\ &= E^{1,0}_n. && (1.3.0.13) \end{aligned}$$

By induction, $E^{1,0}_n = E^{1,0}_2$ for $n \geq 2$. By definition, $E^{1,0}_\infty = E^{1,0}_2$. We shall compute $E^{0,1}_\infty$. For $n \geq 2$,

$$E^{0,1}_{n+1} = \ker d^{0,1}_n / \operatorname{im} d^{-n,n}_n = \ker d^{0,1}_n = \ker(E^{0,1}_n \to E^{n,2-n}_n). \tag{1.3.0.14}$$

Hence, by induction, $E^{0,1}_n = E^{0,1}_3$. Thus, $E^{0,1}_\infty = \ker(E^{0,1}_2 \to E^{2,0}_2)$. Again, by convergence,

$$E^{2,0}_\infty = H^2_2 / H^2_3 = H^2_2 \subset H^2. \tag{1.3.0.15}$$

For $n \geq 3$,

$$E^{2,0}_{n+1} = \ker d^{2,0}_n / \operatorname{im} d^{2-n,n-1}_n = \ker(E^{2,0}_n \to E^{2+n,1-n}_n) = E^{2,0}_n. \tag{1.3.0.16}$$

Hence,

$$E^{2,0}_\infty = E^{2,0}_3 = E^{2,0}_2 / \operatorname{im}(E^{0,1}_2 \to E^{2,0}_2) = \operatorname{coker}(E^{0,1}_2 \to E^{2,0}_2). \tag{1.3.0.17}$$

Thus, we constructed the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & E_2^{1,0} \arrow[r] & H^1 \arrow[rr] \arrow[dr, two heads] & & E_2^{0,1} \arrow[r] & E_2^{2,0} \arrow[rr] \arrow[dr, two heads] & & H^2 \\
& & & E_\infty^{0,1} \arrow[ur, hook] & & & E_\infty^{2,0} \arrow[ur, hook] &
\end{tikzcd}.$$

□

Lemma 1.3.9. *Assume that $E_2^{i,j} \implies H^{i+j}$ is a first quadrant spectral sequence and $E_2^{i,j} = 0$ for $i > 0$. Then, $E_2^{0,j} \simeq H^j$ for every $j \geq 0$.*

Proof. We claim that $E_s^{i,j} = 0$ for $i > 0$, $s \geq 2$, $j \geq 0$. We shall proceed by induction on s. If $s = 2$, the case follows by assumption. Let $s \geq 2$. Then, for any $i > 0$, $j \geq 0$

$$E_{s+1}^{i,j} = \ker\left(d_s^{i,j} \colon \underbrace{E_s^{i,j}}_{=0,\text{ by induction}} \to E_s^{i+s,j-s+1}\right) / \mathrm{im}\left(d_s^{i-s,j+s-1} \colon E_s^{i-s,j+s-1} \to \underset{\substack{\| \\ 0}}{E_s^{i,j}}\right) = 0. \tag{1.3.0.18}$$

In particular, $E_\infty^{i,j} = 0$ for any $i > 0$, $j \geq 0$.

We now claim that $E_s^{0,j} = E_2^{0,j}$ for any $s \geq 2$, $j \geq 0$. We will proceed by induction. The case $s = 2$ is clear. For $s > 2$, $j \geq 0$,

$$E_s^{0,j} = \ker\left(d_{s-1}^{0,j} \colon E_{s-1}^{0,j} \to E_{s-1}^{s-1,j-s}\right) / \mathrm{im}\left(d_{s-1}^{-s+1,j+s-2}\right) = \ker\left(d_{s-1}^{0,j} \colon E_{s-1}^{0,j} \to E_{s-1}^{s-1,j-s}\right), \tag{1.3.0.19}$$

since $1 - s < 0$, and thus $d_{s-1}^{-s+1,j+s-2} = 0$. By the first claim, $E_{s-1}^{s-1,j-s} = 0$. Hence, $E_s^{0,j} = E_{s-1}^{0,j} = E_2^{0,j}$ for any $s \geq 2$, $j \geq 0$. In particular, $E_\infty^{0,j} = E_2^{0,j}$ for any $j \geq 0$.

Let $j \geq 0$. By convergence, we have a filtration for H^j with $E_\infty^{0,j} \simeq H_0^j / H_1^j = H^j / H_1^j$ for every $j \geq 0$. Furthermore, $0 = E_\infty^{i,j-i} \simeq H_i^j / H_{i+1}^j$, $0 < i \leq j$. Thus, $H_i^j = H_{i+1}^j$, $0 < i \leq j$. Consequently, $H_1^j = H_{j+1}^j = 0$. We conclude, $E_2^{0,j} = E_\infty^{0,j} \simeq H^j$. □

Lemma 1.3.10. *Let $q \geq 1$. Assume that $E_2^{i,j} \implies H^{i+j}$ is a first quadrant spectral sequence and $E_2^{i,j} = 0$ for $1 \leq j \leq q$. Then, $E_2^{i,0} \simeq H^i$, $1 \leq i \leq q$ and there is an exact sequence*

$$0 \to E_2^{q+1,0} \to H^{q+1} \to E_2^{0,q+1} \to E_2^{q+2,0} \to H^{q+2} \tag{1.3.0.20}$$

Proof. We will start by showing by induction that $E_s^{i,j} = 0$ for every $s \geq 2$, $1 \leq j \leq q$ and every $i \geq 0$. Let $1 \leq j \leq q$ and $i \geq 0$. The case $s = 2$ follows by assumption. Assume that $E_l^{i,j} = 0$ for some $s \geq 2$ and $l \leq s$. Then,

$$E_{s+1}^{i,j} = \ker d_s^{i,j} / \mathrm{im}\, d_s^{i-s,j+s-1} = 0, \tag{1.3.0.21}$$

since by induction $\ker d_s^{i,j} \subset E_s^{i,j} = 0$, and thus $\ker d_s^{i,j} = 0$. Therefore, $E_s^{i,j} = 0$ for every $s \geq 2$, $1 \leq j \leq q$ and every $i \geq 0$. In particular,

$$E_\infty^{i,j} = 0,\ 1 \leq j \leq q,\ i \geq 0. \tag{1.3.0.22}$$

Let $s \geq 2$ and $i \geq 0$. Since $1 - s$ is a negative value, $E_s^{i+s,1-s} = 0$, and thus $\ker d_s^{i,0} = E_s^{i,0}$. If $s \leq q + 1$ or $i + 1 \leq s$, then $E_s^{i-s,s-1} = 0$, and therefore, $\mathrm{im}\, d_s^{i-s,s-1} = 0$. For $s \leq q + 1$ or $i + 1 \leq s$, we have

$$E_{s+1}^{i,0} = \ker d_s^{i,0} / \mathrm{im}\, d_s^{i-s,s-1} = E_s^{i,0}. \tag{1.3.0.23}$$

In particular, by an induction argument

$$E_{q+2}^{q+2,0} = E_{q+1}^{q+2,0} = E_2^{q+2,0} \tag{1.3.0.24}$$

$$E_{s+1}^{i,0} = E_s^{i,0} = E_2^{i,0}, \forall s \geq 2,\ 1 \leq i \leq q+1. \tag{1.3.0.25}$$

Thus,

$$E_\infty^{i,0} = E_2^{i,0},\ 1 \leq i \leq q+1. \tag{1.3.0.26}$$

For $s \geq q+3$, we have $E_s^{q+2-s,s-1} = 0$, and thus $\operatorname{im} d_s^{q+2-s,s-1} = 0$. Therefore, we have

$$E_{s+1}^{q+2,0} = \ker d_s^{q+2,0} / \operatorname{im} d_s^{q+2-s,s-1} = E_s^{q+2,0}, s \geq q+3 \text{ and} \tag{1.3.0.27}$$

$$E_\infty^{q+2,0} = E_{q+3}^{q+2,0} = \ker d_{q+2}^{q+2,0} / \operatorname{im} d_{q+2}^{q+2-(q+2),(q+2)-1} = E_{q+2}^{q+2,0} / \operatorname{im}\left(d_{q+2}^{0,q+1} \colon E_{q+2}^{0,q+1} \to E_{q+2}^{q+2,0}\right) \tag{1.3.0.28}$$

$$= E_2^{q+2,0} / \operatorname{im}\left(d_{q+2}^{0,q+1} \colon E_{q+2}^{0,q+1} \to E_{q+2}^{q+2,0}\right). \tag{1.3.0.29}$$

Now we are ready to establish $E_2^{n,0} \simeq H^n$, $1 \leq n \leq q$.

Let $1 \leq n \leq q$ and $1 \leq i \leq n-1$. Then, $1 \leq n-i \leq q$. Hence, by convergence and (1.3.0.22)

$$0 = E_\infty^{i,n-i} \simeq H_i^n / H_{i+1}^n, \text{ and} \tag{1.3.0.30}$$

$$H^n = H_0^n = H_{(n-1)+1}^n = E_\infty^{n,0} \underset{(1.3.0.26)}{=} E_2^{n,0}. \tag{1.3.0.31}$$

Now we shall proceed to construct the desired exact sequence. By the filtration given by convergence, we have for any $n \geq 0$, $E_\infty^{n,0} \simeq H_n^n \subset H^n$.

Since $E_\infty^{0,q+1} = H^{q+1}/H_1^{q+1}$ we have the canonical epimorphism $H^{q+1} \twoheadrightarrow E_\infty^{0,q+1}$ with kernel H_1^{q+1}. For $1 \leq i \leq q$ we have $1 \leq q+1-i \leq q$, and thus by (1.3.0.22) $H_{i+1}^{q+1} = H_i^{q+1}$. Hence,

$$E_2^{q+1,0} \underset{(1.3.0.26)}{=} E_\infty^{q+1,0} = E_\infty^{q+1,(q+1)-(q+1)} \simeq H_{q+1}^{q+1} / H_{q+2}^{q+1} = H_q^{q+1} = H_1^{q+1} \tag{1.3.0.32}$$

So, we have an exact sequence

$$0 \to E_2^{q+1,0} \to H^{q+1} \to E_\infty^{0,q+1} \to 0. \tag{1.3.0.33}$$

We have that $\operatorname{im} d_s^{-s,q+1+s-1} = 0$, hence

$$E_{s+1}^{0,q+1} = \ker\left(d_s^{0,q+1} \colon E_s^{0,q+1} \to E_s^{s,q+2-s}\right) = E_s^{0,q+1},\ s \geq q+3 \tag{1.3.0.34}$$

Therefore,

$$E_\infty^{0,q+1} = E_{q+3}^{0,q+1} = \ker\left(d_{q+2}^{0,q+1} \colon E_{q+2}^{0,q+1} \to E_{q+2}^{q+2,0}\right) \underset{(1.3.0.24)}{\simeq} \ker\left(E_{q+2}^{0,q+1} \to E_2^{q+2,0}\right) \tag{1.3.0.35}$$

Moreover, for $2 \leq s \leq q+1$,

$$E_{s+1}^{0,q+1} = \ker\left(d_s^{0,q+1} \colon E_s^{0,q+1} \to E_s^{s,q+2-s}\right) \simeq \ker\left(E_s^{0,q+1} \to 0\right) = E_s^{0,q+1}. \tag{1.3.0.36}$$

It follows that $E_{q+2}^{0,q+1} = E_2^{0,q+1}$. By (1.3.0.28),

$$\operatorname{coker}\left(E_{q+2}^{0,q+1} \to E_2^{q+2,0}\right) = E_\infty^{q+2,0}. \tag{1.3.0.37}$$

Thus, we have the exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & E_2^{q+1,0} \arrow[r] & H^{q+1} \arrow[rr] \arrow[dr, two heads] & & E_2^{0,q+1} \arrow[r] & E_2^{q+2,0} \arrow[rr] \arrow[dr, two heads] & & H^{q+2} \\
& & & E_\infty^{0,q+1} \arrow[ur, hook] & & & E_\infty^{q+2,0} \arrow[ur, hook] &
\end{tikzcd}
.
$$

□

Lemma 1.3.11. *Let $q \geq 1$. Assume that $E^2_{i,j} \implies H_{i+j}$ is a first quadrant spectral sequence and $E^2_{i,j} = 0$ for $1 \leq j \leq q$. Then, $E^2_{i,0} \simeq H_i$, $1 \leq i \leq q$ and there is an exact sequence*

$$H_{q+2} \to E^2_{q+2,0} \to E^2_{0,q+1} \to H_{q+1} \to E^2_{q+1,0} \to 0. \tag{1.3.0.38}$$

Proof. This proof is essentially the dual argument for Lemma 1.3.10.

We will start by showing by induction that $E^s_{i,j} = 0$ for every $s \geq 2$, $1 \leq j \leq q$ and every $i \geq 0$. Let $1 \leq j \leq q$ and $i \geq 0$. The case $s = 2$ follows by assumption. Assume that $E_l^{i,j} = 0$ for some $s \geq 2$ and $l \leq s$. Then,

$$E^{s+1}_{i,j} = \ker d^s_{i,j} / \operatorname{im} d^s_{i+s,j-s+1} = 0, \tag{1.3.0.39}$$

since by induction $\ker d^s_{i,j} \subset E^s_{i,j} = 0$, and thus $\ker d^s_{i,j} = 0$.

Therefore, $E^s_{i,j} = 0$ for every $s \geq 2$, $1 \leq j \leq q$ and every $i \geq 0$. In particular,

$$E^\infty_{i,j} = 0,\ 1 \leq j \leq q,\ i \geq 0. \tag{1.3.0.40}$$

Since $1-s$ is a negative value, $E^s_{i+s,1-s} = 0$, and thus $\operatorname{im} d^s_{i+s,-s+1} = 0$. If $s \leq q+1$ or $i+1 \leq s$, then $E_s^{i-s,s-1} = 0$. For $s \leq q+1$ or $i+1 \leq s$, we have

$$E^{s+1}_{i,0} = \ker\left(d^s_{i,0} \colon E^s_{i,0} \to E^s_{i-s,s-1}\right) / \operatorname{im} d^s_{i+s,-s+1} = E^s_{i,0}. \tag{1.3.0.41}$$

In particular, by an induction argument

$$E^{q+2}_{q+2,0} = E^{q+1}_{q+2,0} = E^2_{q+2,0} \tag{1.3.0.42}$$

$$E^{s+1}_{i,0} = E^s_{i,0} = E^2_{i,0}, \forall s \geq 2,\ 1 \leq i \leq q+1. \tag{1.3.0.43}$$

Thus,

$$E^\infty_{i,0} = E^2_{i,0},\ 1 \leq i \leq q+1. \tag{1.3.0.44}$$

For $s \geq q+3$, we have $E_s^{q+2-s,s-1} = 0$, and thus $\ker d^s_{q+2,0} = E^s_{q+2,0}$.

Therefore, we have

$$E^{s+1}_{q+2,0} = \ker d^s_{q+2,0} / \operatorname{im} d^s_{q+2+s,-s+1} = E^s_{q+2,0}, s \geq q+3 \text{ and} \tag{1.3.0.45}$$

$$E^\infty_{q+2,0} = E^{q+3}_{q+2,0} = \ker d^{q+2}_{q+2,0} / \operatorname{im} d^{q+2}_{q+2+(q+2),-(q+2)+1} = \ker\left(d^{q+2}_{q+2,0} \colon E^{q+2}_{q+2,0} \to E^{q+2}_{0,q+1}\right) \tag{1.3.0.46}$$

$$\underset{(1.3.0.42)}{=} \ker\left(E^2_{q+2,0} \to E^{q+2}_{0,q+1}\right). \tag{1.3.0.47}$$

Now we are ready to establish $E^2_{n,0} \simeq H_n$, $1 \leq n \leq q$.

Let $1 \leq n \leq q$ and $1 \leq i \leq n-1$. Then, $1 \leq n-i \leq q$. Hence, by convergence and (1.3.0.40)

$$0 = E^\infty_{i,n-i} \simeq H^i_n / H^{i-1}_n, \text{ and} \tag{1.3.0.48}$$

$$H_n^{n-1} = H_n^{n-2} = H_n^0 \simeq E_{0,n}^{\infty} \underset{(1.3.0.40)}{=} 0 \tag{1.3.0.49}$$

$$H_n = H_n^n = H_n^n \; H_n^{n-1} \simeq E_{n,0}^{\infty} \underset{(1.3.0.44)}{=} E_{n,0}^2. \tag{1.3.0.50}$$

Now we shall proceed to construct the desired exact sequence. By the filtration given by convergence, we have for any $n \geq 0$, $E_{n,0}^{\infty} \simeq H_n^n / H_n^{n-1} = H_n / H_n^{n-1}$. Thus, we have a canonical epimorphism $H_n \twoheadrightarrow E_{n,0}^{\infty}$ with kernel H_n^{n-1} for any $n \geq 0$. In particular, we have the exact sequence and the epimorphism

$$0 \to H_{q+1}^q \to H_{q+1} \to E_{q+1,0}^{\infty} \underset{(1.3.0.44)}{=} E_{q+1,0}^2 \to 0, \quad H_{q+2} \twoheadrightarrow E_{q+2,0}^{\infty} \tag{1.3.0.51}$$

For $2 \leq s \leq q+1$, $1 \leq q+2-s \leq q$. Hence, $E_{s,q+2-s}^s = 0$ for $2 \leq s \leq q+1$. Consequently, $\operatorname{im} d_{s,q+2-s}^s = 0$ for $2 \leq s \leq q+1$. Therefore, for $2 \leq s \leq q+1$,

$$E_{0,q+1}^{s+1} = \ker\left(E_{0,q+1}^s \to E_{-s,q+2-s}^s\right) = E_{0,q+1}^s. \tag{1.3.0.52}$$

We conclude that

$$E_{0,q+1}^{q+2} = E_{0,q+1}^{q+1} = E_{0,q+1}^2. \tag{1.3.0.53}$$

In view of (1.3.0.47),

$$E_{q+2,0}^{\infty} = \ker\left(E_{q+2,0}^2 \to E_{0,q+1}^2\right) \tag{1.3.0.54}$$

For $s \geq q+3, \operatorname{im} d_{s,q+2-s}^s = 0$, and thus

$$E_{0,q+1}^{s+1} = \ker d_{0,q+1}^s / \operatorname{im} d_{s,q+2-s}^s = \ker\left(d_{0,q+1}^s \colon E_{0,q+1}^s \to E_{-s,q+s}^s\right) = E_{0,q+1}^s. \tag{1.3.0.55}$$

Therefore, $E_{0,q+1}^{\infty} = E_{0,q+1}^{q+3}$.

By (1.3.0.40) and using the filtration given by convergence for $1 \leq i \leq q$,

$$0 = E_{i,q+1-i}^{\infty} = H_{q+1}^i / H_{q+1}^{i-1}. \tag{1.3.0.56}$$

This gives us

$$\begin{aligned} H_{q+1}^q &= H_{q+1}^{q-1} = H_{q+1}^0 = E_{0,q+1}^{\infty} = E_{0,q+1}^{q+3} = \ker d_{0,q+1}^{q+2} / \operatorname{im} d_{q+2,0}^{q+2} && (1.3.0.57) \\ &= E_{0,q+1}^{q+2} / \operatorname{im}\left(E_{q+2,0}^{q+2} \to E_{0,q+1}^{q+2}\right) \underset{(1.3.0.53, 1.3.0.42)}{=} E_{0,q+1}^2 / \operatorname{im}\left(E_{q+2,0}^2 \to E_{0,q+1}^2\right). && (1.3.0.58) \end{aligned}$$

Combinining (1.3.0.58), (1.3.0.51) and (1.3.0.54) we obtain the exact sequence

$$\begin{tikzcd} H_{q+2} \arrow[rr] \arrow[dr, two heads] & & E_{q+2,0}^2 \arrow[r] & E_{0,q+1}^2 \arrow[rr] \arrow[dr, two heads] & & H_{q+1} \arrow[r] & E_{q+1,0}^2 \arrow[r] & 0 \\ & E_{q+2,0}^{\infty} \arrow[ur, hook] & & & H_{q+1}^q \arrow[ur, hook] & & & \end{tikzcd}.$$

□

Lemma 1.3.12. *Assume that $E_{i,j}^2 \implies H_{i+j}$ is a first quadrant spectral sequence, $E_{i,j}^2 = 0$ for $1 \leq i \leq q$, $j \geq 0$ for some $q \in \mathbb{N}$, and $E_{0,j}^2 = 0$ for $j > 0$. Then, there exists an epimorphism $H_{q+1} \twoheadrightarrow E_{q+1,0}^2$.*

Proof. As we have seen before, by convergence there exists a canonical epimorphism $H_n \twoheadrightarrow E_{n,0}^{\infty}$ for every $n \geq 1$.

We will start by showing by induction that $E_{i,j}^s = 0$ for every $s \geq 2$, $0 \leq i \leq q$ and every $j \geq 0$. Let $0 \leq i \leq q$

and $j \geq 0$. The case $s = 2$ follows by assumption. Assume that $E_l^{i,j} = 0$ for some $s \geq 2$ and $l \leq s$. Then,

$$E_{i,j}^{s+1} = \ker d_{i,j}^s / \operatorname{im} d_{i+s,j-s+1}^s = 0, \tag{1.3.0.59}$$

since by induction $\ker d_{i,j}^s \subset E_{i,j}^s = 0$, and thus $\ker d_{i,j}^s = 0$.

Therefore, $E_{i,j}^s = 0$ for every $s \geq 2$, $0 \leq i \leq q$ and every $j \geq 0$. In particular, $E_{q-s+1,s-1}^s = 0$ since $q-s+1 \leq q-1$ for every $s \geq 2$.

We have, $E_{q+1,0}^{s+1} = \ker d_{q+1,0}^s / \operatorname{im} d_{q+1+s,-s+1}^s = \ker(d_{q+1,0}^s \colon E_{q+1,0}^s \to E_{q-s+1,s-1}^s) = E_{q+1,0}^s$. It follows that $E_{q+1,0}^s = E_{q+1,0}^2$ for every $s \geq 2$. Moreover $E_{q+1,0}^\infty = E_{q+1,0}^2$. □

Lemma 1.3.13. *Assume that $E_2^{i,j} \implies H^{i+j}$ is a first quadrant spectral sequence, $E_2^{i,j} = 0$ for $1 \leq i \leq q$, $j \geq 0$ for some $q \in \mathbb{N}$ and $E_2^{0,j} = 0$ for $j > 0$. Then, there exists a monomorphism $E_2^{q+1,0} \hookrightarrow H^{q+1}$.*

Proof. As we have seen before, by convergence there exists a canonical monomorphism $E_\infty^{n,0} \hookrightarrow H^n$ for every $n \geq 1$.

We will start by showing by induction that $E_s^{i,j} = 0$ for every $s \geq 2$, $0 \leq i \leq q$ and every $j \geq 0$. Let $0 \leq i \leq q$ and $j \geq 0$. The case $s = 2$ follows by assumption. Assume that $E_{i,j}^l = 0$ for some $s \geq 2$ and $l \leq s$. Then,

$$E_{s+1}^{i,j} = \ker d_s^{i,j} / \operatorname{im} d_s^{i-s,j+s-1} = 0, \tag{1.3.0.60}$$

since by induction $\ker d_s^{i,j} \subset E_s^{i,j} = 0$, and thus $\ker d_s^{i,j} = 0$.

Therefore, $E_s^{i,j} = 0$ for every $s \geq 2$, $0 \leq i \leq q$ and every $j \geq 0$. In particular, $E_s^{q-s+1,s-1} = 0$ for every $s \geq 2$ since $q-s+1 \leq q-1$ for every $s \geq 2$. Thus,

$$E_{s+1}^{q+1,0} = \ker d_s^{q+1,0} / \operatorname{im} d_s^{q+1-s,s-1} = \ker d_{q+1,0}^s = E_{q+1,0}^s. \tag{1.3.0.61}$$

It follows that $E_s^{q+1,0} = E_2^{q+1,0}$ for every $s \geq 2$. Moreover $E_\infty^{q+1,0} = E_2^{q+1,0}$. □

Lemma 1.3.14. *(Künneth theorem for cochain complexes)*

Let I be a cochain complex of flat R-modules such that each submodule $\operatorname{im} d^n \subset I^n$ is also R-flat. Then, for every R-module M and every $n \in \mathbb{N}$, there is an exact sequence

$$0 \to H^{n-1}(I) \otimes_R M \to H^{n-1}(I \otimes_R M) \to \operatorname{Tor}_1^R(H^n(I), M) \to 0.$$

Proof. Consider the exact sequence $0 \to \ker d^{n+1} \to I^n \to \operatorname{im} d^{n+1} \to 0$ for every $n \geq 0$. Applying $- \otimes_R M$ yields

$$0 = \operatorname{Tor}_1^R(\operatorname{im} d^{n+1}, M) \to \ker d^{n+1} \otimes_R M \to I^n \otimes_R M \to \operatorname{im} d^{n+1} \otimes_R M \to 0,\ \forall n. \tag{1.3.0.62}$$

Furthermore, by this argument $\operatorname{Tor}_1^R(\ker d^{n+1}, N) = 0$ for every $N \in R$-mod. Hence, $\ker d^{n+1}$ is also R-flat. By the exactness of (1.3.0.62), we have an exact chain of complexes

$$0 \to \ker d \otimes_R M \to I \otimes_R M \to d(I) \otimes_R M \to 0. \tag{1.3.0.63}$$

This yields a long exact sequence

$$\cdots \to H^{n-1}(d(I) \otimes_R M) \xrightarrow{\partial^n} H^n(\ker d \otimes_R M) \to H^n(I \otimes_R M) \to H^n(d(I) \otimes_R M) \xrightarrow{\partial^{n+1}} H^{n+1}(\ker d \otimes_R M) \to \cdots \tag{1.3.0.64}$$

Note that the differentials in $\ker d$ are zero. In fact, they are the restriction of d_i to $\ker d^i$. The differentials in $d(I)$

are also zero, since the n-th differential maps are the restrictions $\operatorname{im} d^n \subset \ker d^{n+1} \to \operatorname{im} d^{n+1}$. In particular,

$$H^{n-1}(d(I)\otimes_R M) = \ker(0\otimes M\colon d^n(I)\otimes M \to d^{n+1}\otimes M) = d^n(I)\otimes_R M \tag{1.3.0.65}$$

$$H^{n-1}(\ker d\otimes_R M) = \ker(0\otimes_R \mathrm{id}_M\colon \ker d^n\otimes_R M \to \ker d^{n+1}\otimes_R M) = \ker d^n\otimes_R M \tag{1.3.0.66}$$

and ∂^n is defined by applying the Snake Lemma on the following diagram

$$\begin{tikzcd}
 & \ker d^n\otimes_R M \arrow[r] \arrow[d, "0"] & I^{n-1}\otimes_R M/(\operatorname{im} d^{n-1}\otimes_R \mathrm{id}) \arrow[r] \arrow[d, "d^n\otimes_R \mathrm{id}"] & d^n(I)\otimes_R M \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \ker d^{n+1}\otimes_R M \arrow[r] & \ker(d^{n+1}\otimes_R \mathrm{id}_M) \arrow[r] & d^{n+1}(I)\otimes_R M &
\end{tikzcd}$$

It follows that $\partial^n = i^n\otimes_R \mathrm{id}$ where i^n is the inclusion map $d^n(I)\to\ker d^{n+1}$.

On the other hand, for each natural number n,

$$0\to d^n(I)\xrightarrow{i^n}\ker d^{n+1}\to H^n(I)\to 0 \tag{1.3.0.67}$$

is exact. Applying $-\otimes_R M$ yields the exact sequence

$$0=\operatorname{Tor}_1^R(\ker d^{n+1},M)\to\operatorname{Tor}_1^R(H^n(I),M)\to d^n(I)\otimes_R M\xrightarrow{i^n\otimes_R\mathrm{id}_M}\ker d^{n+1}\otimes_R M\twoheadrightarrow H^n(I)\otimes_R M \tag{1.3.0.68}$$

Hence,

$$\operatorname{Tor}_1^R(H^n(I),M)=\ker(i^n\otimes_R\mathrm{id}_M)\underset{(1.3.0.64)}{=}\operatorname{im}(H^{n-1}(I\otimes_R M)\to H^{n-1}(d(I)\otimes_R M)). \tag{1.3.0.69}$$

So, we have a canonical surjective map $H^{n-1}(I\otimes_R M)\twoheadrightarrow\operatorname{Tor}_1^R(H^n(I),M)$. Moreover,

$$\begin{aligned}
\ker\left(H^{n-1}(I\otimes_R M)\twoheadrightarrow\operatorname{Tor}_1^R(H^n(I),M)\right) &= \ker\left(H^{n-1}(I\otimes_R M)\to H^{n-1}(d(I)\otimes_R M)\right)\\
&\underset{(1.3.0.64)}{=}\operatorname{im}\left(H^{n-1}(\ker d\otimes_R M)\to H^{n-1}(I\otimes_R M)\right)\\
&\simeq H^{n-1}(\ker d\otimes_R M)/\ker\left(H^{n-1}(\ker d\otimes_R M)\to H^{n-1}(I\otimes_R M)\right)\\
&\underset{(1.3.0.64)}{=} H^{n-1}(\ker d\otimes_R M)/\operatorname{im}\partial^{n-1}\\
&=\ker d^n\otimes_R M/\operatorname{im}(i^{n-1}\otimes_R\mathrm{id})\\
&=\operatorname{coker} i^{n-1}\otimes_R\mathrm{id}\underset{(1.3.0.68)}{=}H^{n-1}(I)\otimes_R M
\end{aligned}$$

□

Lemma 1.3.15. *(Künneth spectral sequence for cochain complexes)*

Let $P^\bullet$ be a flat cochain complex of R-modules $0\to P_0\to P_1\to\cdots$. Let M be an R-module with finite flat R-dimension. Then,

$$E_2^{i,j}=\operatorname{Tor}_{-i}^R(H^j(P),M)\implies H^{i+j}(P\otimes_R M),\quad i,j\in\mathbb{Z}.$$

Proof. Let $Q^\bullet$ be a complex chain $0\to Q_0\to Q_1\to\cdots\to Q_n\to 0$ so that

$$0\to Q_0\to Q_1\to\cdots\to Q_n\to M\to 0 \tag{1.3.0.70}$$

is a flat R-resolution of M. In particular, $H^n(Q^\bullet)=M$ and $H^j(Q^\bullet)=0$ if $j\neq n$. Consider the double complex $P^\bullet\otimes_R Q^\bullet$. Notice that the R-modules $P_p\otimes_R Q_q$ are zero unless $p\geq 0$ and $q\geq 0$. Hence, this is a first quadrant double complex. So, two possible filtrations are possible. One vertically and one horizontally with respect to the double complex $P^\bullet \otimes_R Q^\bullet$.

The spectral sequence associated with the vertical truncation of the total complex $\mathrm{Tot}(P^\bullet \otimes_R Q^\bullet)$ is

$$'E_2^{p,q} = H^p(H^q(P^\bullet \otimes_R Q^\bullet, d''), d') = H^p(H^q(P_p \otimes_R Q_q, d''), d') \tag{1.3.0.71}$$

d' corresponds to the horizontal morphisms $d_P \otimes \mathrm{id}$ and d'' to the vertical morphisms $\mathrm{id} \otimes d_Q$.

P_p is R-flat, so $P_p \otimes_R -$ commutes with cohomology, therefore

$$'E_2^{p,q} = H^p(P_p \otimes_R H^q(Q_q, d''), d') = H^p(P \otimes_R H^q(Q^\bullet), d') \tag{1.3.0.72}$$

$$= \begin{cases} H^p(P^\bullet \otimes_R M) \text{ if } q = n \\ 0, \text{ otherwise} \end{cases} . \tag{1.3.0.73}$$

On the other hand,

$$''E_2^{p,q} = H^p(H^q(P^\bullet \otimes_R Q^\bullet, d'), d'') = H^p(H^q(P_q, d') \otimes_R Q_p, d'') = H^p(H^q(P^\bullet) \otimes_R Q^\bullet) \tag{1.3.0.74}$$

since Q_q is R-flat. Furthermore,

$$''E_2^{p,q} = H^p(H^q(P^\bullet) \otimes_R Q^\bullet) = H_{-p}(H^q(P^\bullet) \otimes_R Q'_\bullet) = \mathrm{Tor}^R_{n-p}(H^q(P^\bullet), M) \tag{1.3.0.75}$$

where $Q'_\bullet$ corresponds to the chain complex obtained from $Q^\bullet$ by reindexing the subscripts ($Q'_i := Q_{-i}$).

By the classical Convergence Theorem (see [Wei03, Classical Convergence Theorem 5.5.1], both of these two spectral sequences converge to the same object.

We claim that

$$'E_s^{p,q} = \begin{cases} H^i(P^\bullet \otimes_R M), \text{ if } q = n \\ 0, \text{otherwise.} \end{cases} \tag{1.3.0.76}$$

for every $s \geq 2$. We shall prove it by induction on s. The case $s = 2$ follows from (1.3.0.73).

If $q \neq n$, then by induction $'E_s^{p,q} = 0$. Thus, $\ker d_s^{p,q} = 0$, which implies $'E_{s+1}^{p,q} = 0$. Assume that $q = n$. For any $s \geq 2$, $'E_{s+1}^{p-s,n+s-1} =' E_s^{p+s,n-s+1} = 0$. Thus, $'E_{s+1}^{p,n} =' E_s^{p,n} = E_2^{p,n}$, by induction.

It follows that

$$'E_\infty^{p,q} = \begin{cases} H^p(P^\bullet \otimes_R M), \text{ if } q = n \\ 0, \text{otherwise.} \end{cases} \tag{1.3.0.77}$$

Thus, for every $p \in \mathbb{Z}$, $H_p^{p+q} = H_{p+1}^{p+q}$ as long as $q \neq n$. Let $l \in \mathbb{N}$. For $s \ll 0$, $H^l = H_s^l = H_{s+1}^l = H_{l-n}^l$. On the other hand, $0 = H_t^l = H_{l-n+1}^l$, $0 \ll t$. Thus,

$$H^l = H_{l-n}^l = H_{l-n}^l / H_{l-n+1}^l =' E_\infty^{l-n,n} = H^{l-n}(P^\bullet \otimes_R M). \tag{1.3.0.78}$$

So,

$$\mathrm{Tor}^R_{n-p}(H^q(P^\bullet), M) \implies H^{p+q-n}(P^\bullet \otimes_R M). \tag{1.3.0.79}$$

The result follows by setting $i = p - n$, and $j = q$. □

We can now state a stronger version of the Künneth theorem for cochain complexes using the Künneth spectral sequence for cochain complexes. This idea goes back to Hashimoto [Has00, Lemma 2.1.2 (Universal coefficient theorem)].

Corollary 1.3.16. *Let $P^\bullet$ be a flat cochain complex of R-modules $0 \to P_0 \to P_1 \to \cdots$. Let M be an R-module with flat R-dimension at most one. Then, for each integer $n \geq 0$, there exists an exact sequence*

$$0 \to H^n(P^\bullet)\otimes_R M \to H^n(P^\bullet \otimes_R M) \to \operatorname{Tor}_1^R(H^{n+1}(P^\bullet), M) \to 0. \tag{1.3.0.80}$$

Proof. By Lemma 1.3.15, there exists a converging spectral sequence

$$E_2^{i,j} = \operatorname{Tor}_{-i}^R(H^j(P^\bullet), M) \implies H^{i+j}(P^\bullet \otimes_R M), \quad i,j \in \mathbb{Z}.$$

Since M has flat dimension at most one, $E_2^{i,j} = 0$ for all $i \leq -2$ and $j \in \mathbb{Z}$. We claim that $E_\infty^{i,j} = E_2^{i,j}$ for all $i,j \in \mathbb{Z}$. Let $l \geq 2$. Recall that $E_{l+1}^{i,j}$ is a quotient of $\ker d_l^{i,j} \subset E_l^{i,j}$. By induction on l, this last is zero if $i \leq -2$ or $i \geq 1$. Hence, $E_\infty^{i,j} = 0$, if $i \leq -2$ or $i \geq 1$. It remains to check the cases $i = 0$ and $i = -1$.

$$E_{l+1}^{0,j} = \ker(E_l^{0,j} \to E_l^{l,j-l+1}) = E_l^{0,j}, \tag{1.3.0.81}$$

since $E_l^{l,j-l+1} = 0$ for $l \geq 2$. By induction,$E_{l+1}^{0,j} = E_2^{0,j}$, and consequently, $E_\infty^{0,j} = E_2^{0,j}$ for any j. As $E_l^{-1+l,j-l+1} = E_l^{-1-l,j+l-1} = 0$ whenever $l \geq 2$ it follows, by induction, the claim for $i = -1$.

Now, using the fact that $E_\infty^{i,j}$ vanishes when $i \leq -2$ or $i \geq 1$ we deduce that $H_i^{i+j} = H_{i+1}^{i+j}$, for $i \leq -2$ or $i \geq 1$. Fix $n \geq 0$. Then, $H_1^n = H_2^n = H_l^n = 0$, $l \gg 0$. Thus, $E_2^{0,n} = E_\infty^{0,n} = H_0^n$. On the other hand,

$$H_{-1}^n = H_{-2}^n = H_l^n = H^n, \ l \ll 0. \tag{1.3.0.82}$$

Hence, $E_2^{-1,n+1} = E_\infty^{-1,n+1} = H_1^n / H_0^n$. We have constructed an exact sequence

$$0 \to E_2^{0,n} \to H^n \to E_2^{-1,n+1} \to 0.$$ □

Lemma 1.3.17. *(Künneth spectral sequence for chain complexes) Let P be a flat chain complex of R-modules $\cdots \to P_1 \to P_0 \to 0$. Let M be an R-module. Then,*

$$E_{i,j}^2 = \operatorname{Tor}_i^R(H_j(P), M) \implies H_{i+j}(P \otimes_R M), \quad i,j \geq 0.$$

Proof. Similar to Lemma 1.3.15. See for example [Wei03, Theorem 5.6.4]. □

1.4 Double centralizer properties and covers

1.4.1 Double centralizer properties

Let A and B be algebras over a commutative ring R. Let M be an (A,B)-bimodule. There are canonical R-algebra homomorphisms

$$\rho\colon A \to \operatorname{End}_B(M), \quad \rho(a)(m) = am, \ a \in A, m \in M$$
$$\psi\colon B \to \operatorname{End}_A(M)^{op}, \quad \psi(b)(m) = mb, \ b \in B, m \in M.$$

Definition 1.4.1. When the maps ρ and ψ are isomorphisms, we say that there is a **double centralizer property** on M between A and B.

Proposition 1.4.2. *If it holds a double centralizer property on M between A and B, then M is a faithful (A,B)-bimodule.*

In the study of Schur algebras, there are two double centralizer properties to keep in mind. Let R be a commutative ring and let n,d be some positive integers. Denote by M the module $(R^n)^{\otimes d}$. Any element $g \in GL_n(R)$ can be viewed as an element in $\mathrm{End}_R(M)$ by the diagonal action:

$$g(v_1 \otimes \cdots v_d) = gv_1 \otimes \cdots \otimes gv_d,$$

$v_1 \otimes \cdots \otimes v_d \in M$. Here, gv is given by the usual action of $GL_n(R)$ on R^n. On the other hand, every element $\sigma \in S_d$ can be viewed as element in $\mathrm{End}_R(M)$ by the place permutation action:

$$(v_1 \otimes \cdots \otimes v_d)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},$$

$v_1 \otimes \cdots \otimes v_d \in M$. Let A be the subalgebra of $\mathrm{End}_R(M)$ generated by the elements $g \in GL_n(R)$. Let B be the subalgebra of $\mathrm{End}_R(M)$ generated by the elements $\sigma \in S_d$. We say that **Schur–Weyl duality** holds if there is a double centralizer property on M between A and B.

Let $S_R(n,d)$ be the Schur algebra with $n \geq d$. Then, there exists a double centralizer property on M between $S_R(n,d)$ and RS_d ([Cru19, Theorem 3.4], [BD09] and fixing $q = 1$ on [DPS98b, 6]).

From now on we will assume $B = \mathrm{End}_A(M)^{op}$, that is we are assuming that ψ is an isomorphism. So, we will say that (A,M) has the double centralizer property if it holds the double centralizer property on M between A and $\mathrm{End}_A(M)^{op}$. When there is no confusion about the algebra A, we will just say that M satisfies the double centralizer property.

Proposition 1.4.3. *[CR06, 26.5] If $M = A$, then the double centralizer property on M holds.*

Lemma 1.4.4. *[CR06, 26.6] Let $V = M^k$ be the direct sum of k copies of a left A-module, $k > 0$. If (A,V) has the double centralizer property, then (A,M) has the double centralizer property.*

Lemma 1.4.5. *[CR06, 59.4] Let A be any ring with identity. Let N be a left A-module and consider the left A-module given by $M = A \oplus N$. Then, (A,M) has the double centralizer property.*

Proposition 1.4.6. *Let $G \in A$-mod. Then, the following assertions hold.*

(i) If $A \in \mathrm{add}\, G$, then (A,G) has the double centralizer property.

(ii) If $DA \in \mathrm{add}\, G$, then it holds a double centralizer property on G between $\mathrm{End}_A(G)$ and A.

Proof. Since $A \in \mathrm{add}\, G$, there exists $t > 0$ such that $G^t \simeq A \oplus K$ for some $K \in A$-mod. By Lemma 1.4.5, (A,G^t) has the double centralizer property. By Lemma 1.4.4, (A,G) has the double centralizer property. Hence, $i)$ follows. If $DA \in \mathrm{add}\, G$, then DG is in the situation i. Thus, (A,DG) has the double centralizer property. Note that $B = \mathrm{End}_A(DG)^{op} = \mathrm{End}_A(G)$ and $\mathrm{End}_B(DG) \simeq \mathrm{End}_B(G)$ as R-algebras. Therefore, it holds the double centralizer property on G between B and A. $\square$

Therefore, any generator or any relative cogenerator satisfies the double centralizer property.

Proposition 1.4.7. *[Tac73, 10.1] Let $F\colon A$-mod $\to B$-mod be an equivalence of categories. Suppose that $M \in A$-mod satisfies the double centralizer property. Then, (B,FM) satisfies the double centralizer property.*

For faithful modules and finite-dimensional K-algebras over a field, we can verify if the double centralizer property holds using other maps than ρ.

Lemma 1.4.8. *[KY14] Let K be a field and A be a finite-dimensional K-algebra. Let M be a faithful A-module. Then, the following assertions are equivalent.*

(i) (A,M) *satisfies the double centralizer property;*

(ii) *For* $B = \operatorname{End}_A(M)^{op}$, $A \simeq \operatorname{End}_B(M)$ *as K-vector spaces;*

(iii) *For* $B = \operatorname{End}_A(M)^{op}$, $A \simeq \operatorname{End}_B(M)$ *as K-algebras;*

Proof. The implications $i) \Rightarrow iii) \Rightarrow ii)$ are clear. We shall prove $ii) \Rightarrow i)$. Consider $\delta\colon \operatorname{End}_B(M) \to A$ the K-vector space isomorphism. Since M is A-faithful, then ρ is a monomorphism. Hence, $\delta \circ \rho : A \to A$ is a K-vector space monomorphism. Since A is finite-dimensional $\delta \circ \rho$ is an isomorphism. In particular, $\rho = \delta^{-1} \circ \delta \circ \rho$ is bijective. So, $i)$ follows. ☐

We will now turn our attention to study Hom functors.

1.4.2 Projectivization

Assume fron now that A is a Noetherian R-algebra over a commutative Noetherian ring R unless stated otherwise. Let $V \in A$-mod and $B = \operatorname{End}_A(V)^{op}$. V is viewed as right B-module with the action $v \cdot b = b(v)$, $v \in V$, $b \in B$. V is an (A,B)-bimodule. For any $X \in A$-Mod, $\operatorname{Hom}_A(V,X)$ is a left B-module with the action $b \cdot h = h \circ b$. So, we have a functor $F = \operatorname{Hom}_A(V,-)\colon A\text{-Mod} \to B\text{-Mod}$. On the other hand, for any $X \in A$-Mod, $\operatorname{Hom}_A(X,V)$ is a right B-module with action $f \cdot b = b \circ f$.

Lemma 1.4.9. *Let* $M_1, M_2, V \in A$-Mod. *Then, the following holds.*

1. *The map* $\gamma\colon \operatorname{Hom}_A(V,M_1) \bigoplus \operatorname{Hom}_A(V,M_2) \to \operatorname{Hom}_A(V, M_1 \bigoplus M_2)$, *given by*
$$\gamma(f,g)(v) = (f(v), g(v)),\ v \in V,$$
is a B-isomorphism.
2. *The map* $\varepsilon\colon \operatorname{Hom}_A(M_1,V) \bigoplus \operatorname{Hom}_A(M_2,V) \to \operatorname{Hom}_A(M_1 \bigoplus M_2, V)$, *given by*
$$\varepsilon(f_1,f_2)(m_1,m_2) = f_1(m_1) + f_2(m_2),\ m_i \in M_i,\ f_i \in \operatorname{Hom}_A(M_i,V),\ i = 1,2,$$
is a B-isomorphism.

Proof. Let $k_i \in \operatorname{Hom}_A(M_i, M_1 \oplus M_2)$ and $\pi_i \in \operatorname{Hom}_A(M_1 \oplus M_2, M_i)$, $i = 1,2$, be the canonical injections and projections given by the direct sum $M_1 \oplus M_2$.

Consider the B-homomorphism $\gamma'\colon \operatorname{Hom}_A(V, M_1 \bigoplus M_2) \to \operatorname{Hom}_A(V,M_1) \bigoplus \operatorname{Hom}_A(V,M_2)$, given by $\gamma'(f) = (\pi_1 \circ f, \pi_2 \circ f)$, $f \in \operatorname{Hom}_A(V, M_1 \bigoplus M_2)$. Then,

$$\gamma \circ \gamma'(f) = \gamma(\pi_1 \circ f, \pi_2 \circ f) = k_1 \circ \pi_1 \circ f + k_2 \circ \pi_2 \circ f = f \in \operatorname{Hom}_A(V, M_1 \bigoplus M_2), \quad (1.4.2.1)$$
$$\gamma' \circ \gamma(h,g) = \gamma'(k_1 \circ h + k_2 \circ g) = (\pi_1 \circ k_1 \circ h, \pi_2 \circ k_2 \circ g) = (h,g) \in \operatorname{Hom}_A(V,M_1) \bigoplus \operatorname{Hom}_A(V,M_2). \quad (1.4.2.2)$$

So, (a) follows.

Consider the B-homomorphism $\varepsilon'\colon \operatorname{Hom}_A(M_1 \bigoplus M_2, V) \to \operatorname{Hom}_A(M_1,V) \bigoplus \operatorname{Hom}_A(M_2,V)$, given by $\varepsilon'(f) = (f \circ k_1, f \circ k_2)$, $f \in \operatorname{Hom}_A(M_1 \bigoplus M_2, V)$. Then,

$$\varepsilon \circ \varepsilon'(f) = \varepsilon(f \circ k_1, f \circ k_2) = f \circ k_1 \circ \pi_1 + f \circ k_2 \circ \pi_2 = f \in \operatorname{Hom}_A(M_1 \bigoplus M_2, V), \quad (1.4.2.3)$$

$$\varepsilon' \circ \varepsilon(f_1, f_2) = \varepsilon'(f_1 \circ \pi_1 + f_2 \circ \pi_2) = ((f_1 \circ \pi_1 + f_2 \circ \pi_2) \circ k_1, (f_1 \circ \pi_1 + f_2 \circ \pi_2) \circ k_2) \tag{1.4.2.4}$$

$$= (f_1, f_2) \in \mathrm{Hom}_A(M_1, V) \bigoplus \mathrm{Hom}_A(M_2, V). \tag{1.4.2.5}$$

Therefore, (b) follows. $\square$

Theorem 1.4.10. *Let A be a Noetherian algebra over a commutative Noetherian ring R. Consider the functor $F = \mathrm{Hom}_A(V, -)$. Then, the following holds.*

1. *For any $X \in A\text{-Mod}$, $Z \in A\text{-Mod}$, the functor F induces the R-homomorphism $F_{Z,X}\colon \mathrm{Hom}_A(Z,X) \to \mathrm{Hom}_B(FZ, FX)$, given by $F_{Z,X}(f)(g) = f \circ g$, $f \in \mathrm{Hom}_A(V,Z)$, $g \in \mathrm{Hom}_A(V,V)$. $F_{Z,X}$ is an R-isomorphism for all $Z \in \mathrm{add}\, V$ and $X \in A\text{-Mod}$.*
2. *If $X \in \mathrm{add}\, V$, then FX is a projective B-module.*
3. *The restriction of F to $\mathrm{add}\, V$ $F_{|_{\mathrm{add}\, V}}\colon \mathrm{add}\, V \to B\text{-proj}$ is an equivalence of categories.*

Proof. In Proposition 2.1 of [ARS95] this Theorem is proved for Artinian algebras. However, their argument does not use any fact only valid for Artinian algebras. Thus, their argument remains true for Noetherian algebras over commutative Noetherian rings. $\square$

Passing from A to $B = \mathrm{End}_A(V)^{op}$ through this functor $\mathrm{Hom}_A(V, -)$ provides a technique for reducing questions about the module V to questions about projective modules.

1.4.3 Schur functor

The Schur functors come from a special class of Hom functors. Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $P \in A\text{-proj}$ and $B = \mathrm{End}_A(P)^{op}$. Consider the functor $F = \mathrm{Hom}_A(P, -)\colon A\text{-Mod} \to B\text{-Mod}$. This functor is called **Schur functor.**

Since P is finitely generated projective, it follows that the Schur functor is an exact functor and preserves arbitrary coproducts.

Lemma 1.4.11. *For any $P \in A\text{-proj}$ the map $\psi_V\colon \mathrm{Hom}_A(P,A) \otimes_A V \to \mathrm{Hom}_A(P,V)$, given by $\psi_V(f \otimes v)(m) = f(m)v$, $f \otimes v \in \mathrm{Hom}_A(P,A) \otimes_A V$, $m \in P$ is a left $\mathrm{End}_A(P)^{op}$-isomorphism. Moreover, the functors $\mathrm{Hom}_A(P,A) \otimes_A -$ and $\mathrm{Hom}_A(P,-)$ are naturally isomorphic. The map ψ_P is an $(\mathrm{End}_A(P)^{op}, \mathrm{End}_A(P)^{op})$-bimodule isomorphism.*

Proof. For the two first statements, we refer to Lemma 4.2.5 of [Zim14]. It remains to check that ψ_P is an $(\mathrm{End}_A(P)^{op}, \mathrm{End}_A(P)^{op})$-bimodule homomorphism. Let $f \otimes v \in \mathrm{Hom}_A(P,A) \otimes_A P$, $m \in P$, $b \in \mathrm{End}_A(P)^{op}$. Then,

$$\psi_P(f \otimes v \cdot b)(m) = \psi_M(f \otimes b(v))(m) = f(m)b(v) = b(f(m)v) = f(m)v \cdot b = \psi_P(f \otimes v)b. \tag{1.4.3.1}$$

So, ψ_P is an $(\mathrm{End}_A(P)^{op}, \mathrm{End}_A(P)^{op})$-bimodule homomorphism. By Lemma 4.2.5 of [Zim14], ψ_P is a left $(\mathrm{End}_A(P))^{op}$-isomorphism, so the claim follows. $\square$

Proposition 1.4.12. *Let $M \in \text{Mod-}A$ and $B = \mathrm{End}_A(M)$. The functor $M \otimes_A -\colon A\text{-Mod} \to B\text{-Mod}$ is left adjoint to $\mathrm{Hom}_B(M, -)\colon B\text{-Mod} \to A\text{-Mod}$.*

Proof. In view of Lemma 1.1.63, it is enough to show that σ is a natural transformation between the bifunctors $\mathrm{Hom}_B(M \otimes_A -, -)$ and $\mathrm{Hom}_A(-, \mathrm{Hom}_B(M, -))$. Indeed, for $f \in \mathrm{Hom}_A(X, X')$, $g \in \mathrm{Hom}_A(Y', Y)$, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}_B(M\otimes_A Y, X) & \xrightarrow{\sigma_{Y,X}} & \mathrm{Hom}_A(Y, \mathrm{Hom}_B(M,X)) \\ \downarrow{\scriptstyle \mathrm{Hom}_B(M\otimes_A g, f)} & & \downarrow{\scriptstyle \mathrm{Hom}_A(g, \mathrm{Hom}_B(M,f))} \\ \mathrm{Hom}_B(M\otimes_A Y', X') & \xrightarrow{\sigma_{Y',X'}} & \mathrm{Hom}_A(Y', \mathrm{Hom}_B(M,X')) \end{array} .$$

In fact, for every $m \in M, y \in Y', h \in \mathrm{Hom}_B(M\otimes_A Y, X)$,

$$\sigma_{Y',X'} \circ \mathrm{Hom}_B(M\otimes_A g, f)(h)(y)(m) = \sigma_{Y',X'}(f\circ h \circ M\otimes_A g)(m\otimes y) = fh(m\otimes g(y)) \tag{1.4.3.2}$$

$$\begin{aligned} \mathrm{Hom}_A(g, \mathrm{Hom}_B(M,f))\circ \sigma_{Y,X}(h)(y)(m) &= \mathrm{Hom}_B(M,f)(\sigma_{Y,X}(h)\circ g)(y)(m) \\ &= f\circ \sigma_{Y,X}(h)\circ g(y)(m) = f(h(m\otimes g(y))). \end{aligned} \tag{1.4.3.3}$$

□

Proposition 1.4.13. *The Schur functor* $\mathrm{Hom}_A(P,-)\colon A\text{-Mod} \to B\text{-Mod}$ *is left adjoint to the functor* $G = \mathrm{Hom}_B(FA,-)\colon B\text{-Mod} \to A\text{-Mod}$.

Proof. By Lemma 1.4.11, $\mathrm{Hom}_A(P,-) \simeq \mathrm{Hom}_A(P,A)\otimes_A -$. By Proposition 1.4.12, the functor $\mathrm{Hom}_A(P,-) \simeq \mathrm{Hom}_A(P,A)\otimes_A -$ is left adjoint to the functor $\mathrm{Hom}_B(FA,-)$. □

We note also the following lemma involving the Schur functor which will be essential to relative dominant dimension. The reader can observe this is a version of the canonical isomorphisms in [Tac73, p.52] without using idempotents.

Lemma 1.4.14. *Let* $V \in A^{op}$*-proj. Let* $C = \mathrm{End}_A(V)$ *and the functors* $F = V\otimes_A -\colon A\text{-mod} \to C\text{-mod}$ $G = \mathrm{Hom}_C(V,-)\colon C\text{-mod} \to A\text{-mod}$. *The composition of functors* $F\circ G\colon C\text{-mod} \to C\text{-mod}$ *is an equivalence of categories. Moreover* $\xi_M\colon V\otimes_A \mathrm{Hom}_C(V,M) \to M$, *given by* $\xi_M(v\otimes\phi) = \phi(v), v\in V, \phi \in \mathrm{Hom}_C(V,M)$ *is a natural isomorphism.*

Proof. Fix $f \in \mathrm{Hom}_C(M,N)$. We have the commutative diagram,

$$\begin{array}{ccc} V\otimes_A \mathrm{Hom}_C(V,M) & \xrightarrow{\xi_M} & M \\ {\scriptstyle V\otimes_A \mathrm{Hom}_C(V,f)}\downarrow & & \downarrow{\scriptstyle f} \\ V\otimes_A \mathrm{Hom}_C(V,N) & \xrightarrow{\xi_N} & N \end{array} .$$

In fact, $\xi_N \circ V\otimes_A \mathrm{Hom}_C(V,f)(v\otimes\phi) = \xi_N(v\otimes f\circ\phi) = f\circ\phi(v)$, whereas $f\circ\xi_M(v\otimes\phi) = f(\phi(v))$ for every $v\otimes\phi \in V\otimes_A \mathrm{Hom}_C(V,M)$.

Consider the diagram

$$\begin{array}{ccc} V\otimes_A \mathrm{Hom}_C(V,M) & \xrightarrow[w\otimes_A \mathrm{id}_{\mathrm{Hom}_C(V,M)}]{} & \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),A)\otimes_A \mathrm{Hom}_C(V,M) \\ \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \psi_{\mathrm{Hom}_C(V,M)}} \\ M & & \mathrm{Hom}_A(\mathrm{Hom}_A(V,A), \mathrm{Hom}_C(V,M)) \\ \downarrow{\scriptstyle \pi^{-1}} & & \downarrow{\scriptstyle \rho} \\ \mathrm{Hom}_C(C,M) & \xrightarrow{\mathrm{Hom}_C(\psi_V, M)} & \mathrm{Hom}_C(V\otimes_A \mathrm{Hom}_A(V,A), M) \end{array} .$$

Here some remarks about these maps are in order. The map $\psi_{\mathrm{Hom}_C(V,M)}$ is an isomorphism by Lemma 1.4.11 since $\mathrm{Hom}_A(V,A) \in A$-proj. The map ρ is the map given by Tensor-hom adjunction 1.1.63, and hence it is an isomorphism. The map ψ_V is given by Lemma 1.4.11 considering right modules, thus $\mathrm{Hom}_C(\psi_V, M)$ is an isomorphism. The map π is the canonical map, so an isomorphism as well. The map w is the evaluation map.

Since V is projective, then w is an isomorphism. We claim that this is a commutative diagram. In fact, for $v\otimes g \in V\otimes_A \mathrm{Hom}_C(V,M), v'\otimes g' \in V\otimes_A \mathrm{Hom}_A(V,A)$, we have

$$\begin{aligned}
\mathrm{Hom}_C(\psi_V,M)\circ\pi^{-1}\circ\eta_M(v\otimes g)(v'\otimes g') &= \pi^{-1}\circ\eta_M(v\otimes g)\circ\psi_V(v'\otimes g')\\
&= \pi^{-1}\circ\eta_M(v\otimes g)(v'g'(-)) = \pi^{-1}(g(v))(v'g'(-))\\
&= v'g'(-)\cdot g(v) = g(v'g'(-)\cdot v) = g(v'g'(v)).\\
\rho\circ\psi_{\mathrm{Hom}_C(V,M)}\circ w\otimes \mathrm{id}_{\mathrm{Hom}_C(V,M)}(v\otimes g)(v'\otimes g') &= \rho\circ\psi_{\mathrm{Hom}_C(V,M)}(w(v)\otimes g)(v'\otimes g')\\
&= \rho(w(v)(-)g)(v'\otimes g') = w(v)()\cdot g(g')(v')\\
&= (w(v)(g')\cdot g)(v') = g'(v)\cdot g(v') = g(v'g'(v)).
\end{aligned}$$

Now by the commutativity of this diagram, it follows that η_M is an isomorphism. ☐

1.4.4 Morita theory

By projectivization, the Schur functor induces an equivalence between $\mathrm{add}P$ and B-proj. Morita completely described when the Schur functor induces an equivalence between the categories A-Mod and B-Mod in terms of progenerators. Furthermore, every equivalence of categories between two module categories arises from a Schur functor of a progenerator. We will present these results since they will be very useful during this exposition, especially in Section 1.5.

We recall that a functor between two categories is said to be an **equivalence of categories** if it is full, faithful and essentially surjective. Properties of modules that can be described using diagrams and in the language of category theory are preserved under equivalence of module categories. For example, it is a short exercise to see that projective modules, monomorphisms or epimorphisms are preserved under equivalence of categories. Also, we already saw in this exposition, some properties which are invariant under equivalence of categories. We call two rings, A and B, **Morita equivalent** if their representation theories are equivalent, in the sense that there is an equivalence of categories $F\colon A\text{-Mod}\to B\text{-Mod}$. The properties preserved under an equivalence of categories between two rings are called **Morita invariant**. Therefore, Morita invariant properties are properties which are completely determined by the representation theory of an algebra.

For example, equivalence of categories preserves finitely generated modules.

Proposition 1.4.15. *Let A be a ring. Let $F\colon A\text{-Mod}\to B\text{-Mod}$ be an equivalence of categories. Then, F preserves finitely generated A-modules. Moreover, F restricts to an equivalence $F_{|A\text{-mod}}\colon A\text{-mod}\to B\text{-mod}$.*

Proof. Let G be the quasi-inverse functor of F. Let $M\in A$-mod. Consider the canonical surjective B-homomorphism $g\colon \bigoplus_{y\in FM} B\to FM$. Applying G yields the surjective A-homomorphism $Gg\colon \bigoplus_{y\in FM} GB\to GFM$. Note that $GFM\simeq M\in A$-mod. Hence, there exists a surjective A-homomorphism $f\in \mathrm{Hom}_A(A^t,GFM)$ for some $t>0$. Denote by $k_j(1_A)$ the element $(0,\cdots,0,1_A,0,\cdots,0)$ where 1_A appears in the j-th component, $1\leq j\leq t$. Since A^t is projective over A, there exists an A-homomorphism $h\in \mathrm{Hom}_A(A^t,\bigoplus_{y\in FM} GB)$ such that $Gg\circ h=f$. For each $1\leq j\leq t$, $hk_j(1_A)\in\bigoplus_{y\in FM} GB$. Furthermore, for each $1\leq j\leq t$, there exists a finite set $I_j\subset FM$ such that $h(k_j(1_A))\in\bigoplus_{y\in I_j} GB$. Thus, h factors through $\bigoplus_{y\in I_1\cup\cdots\cup I_t} GB$. Denote by i the inclusion of $\bigoplus_{y\in I_1\cup\cdots\cup I_t} GB$ into $\bigoplus_{y\in FM} GB$. Then, $Gg\circ i\circ \nu = Gg\circ h = f$, for some map ν. So, $Gg\circ i$ is surjective. Applying F yields a surjective B-homomorphism $\bigoplus_{y\in I_1\cup\cdots\cup I_t} B\to FM$. So, $FM\in B$-mod. This shows that the functor $F_{|A\text{-mod}}\colon A\text{-mod}\to B\text{-mod}$ is well defined. The functor F is full and faithful. In particular, $F_{|A\text{-mod}}$ is full and faithful. Since G is also an equivalence of categories, it preserves finitely generated modules. Consequently, $F_{|A\text{-mod}}$ is also essentially surjective. ☐

Definition 1.4.16. We call a module M an **A-progenerator** if it is a finitely generated projective A-module and a generator of A-Mod.

By Propositions 1.4.15 and 1.1.10 and our previous discussion, a module being progenerator is a Morita invariant property.

Theorem 1.4.17 (Morita ([Mor58])). *Let A be an R-algebra. Then, the following holds.*

1. *Let P be an A-progenerator and $B = \operatorname{End}_A(P)^{op}$. Then, the Schur functor $F = \operatorname{Hom}_A(P,-)\colon A\text{-Mod} \to B\text{-Mod}$ is an equivalence of categories.*
2. *Let B be an R-algebra such that A-Mod is equivalent to B-Mod then there is a Schur functor $A\text{-Mod} \to B\text{-Mod}$ which is an equivalence of categories.*
3. *$A\text{-mod} \simeq B\text{-mod}$ if and only if $A\text{-Mod} \simeq B\text{-Mod}$.*
4. *$A\text{-proj} \simeq B\text{-proj}$ if and only if $A\text{-Mod} \simeq B\text{-Mod}$.*

Proof. For statement 1 see Proposition 4.2.4 of [Zim14]. For statement 2 see Theorem 4.2.8 of [Zim14].

The implication $A\text{-Mod} \simeq B\text{-Mod} \implies A\text{-mod} \simeq B\text{-mod}$ is Proposition 1.4.15. Assume that $A\text{-mod} \simeq B\text{-mod}$. Consider $G\colon B\text{-mod} \to A\text{-mod}$ the equivalence of categories. By Proposition 1.1.10, GB is a generator of A-mod. Using Theorem 1.1.7, we see that GB is a generator of A-Mod. As discussed previously, any equivalence of functors preserves the projective objects. Therefore, $GB \in A\text{-mod}$ is projective. In other words, GB is finitely generated projective A-module. Thus, GB is a progenerator of A-Mod. On the other hand, G is fully faithful. So, we can identify as R-algebras,

$$\operatorname{End}_A(GB)^{op} \simeq \operatorname{End}_B(B)^{op} \simeq (B^{op})^{op} \simeq B. \tag{1.4.4.1}$$

Now statement 1 implies that $A\text{-Mod} \simeq B\text{-Mod}$.

Let $H\colon B\text{-proj} \to A\text{-proj}$ be the equivalence of categories and $F\colon A\text{-proj} \to B\text{-proj}$ its quasi-inverse. By assumption, $HB \in A\text{-proj}$. It remains to show that HB is a generator of A-Mod. Analogously, FA belongs to B-proj. Thus, there exists $K \in B\text{-proj}$ such that $FA \oplus K \simeq B^t$ for some $t > 0$. Applying H yields

$$A \oplus HK \simeq HFA \oplus HK \simeq HB^t \simeq (HB)^t. \tag{1.4.4.2}$$

So, HB is a progenerator of A-Mod. By the same argument $\operatorname{End}_A(GB)^{op} \simeq B$. Hence, $A\text{-Mod} \simeq B\text{-Mod}$. Conversely, in view of Proposition 1.4.15 and any equivalence of categories preserving projective objects, the fully faithful functor $A\text{-Mod} \to B\text{-Mod}$ restricts to the fully faithful functor $A\text{-proj} \to B\text{-proj}$. In the same way, the fully faithful functor $B\text{-Mod} \to A\text{-Mod}$ restricts to the fully faithful functor $B\text{-proj} \to A\text{-proj}$. Thus, statement 4 follows. □

Remark 1.4.18. Recall that every ring can be regarded as an algebra over $\mathbb{Z}$. Let R be a commutative ring. Because of Theorem 1.4.17, given an R-algebra A, every ring B Morita equivalent to A is isomorphic to an endomorphism algebra of an A-module. This endomorphism algebra inherits the R-algebra structure from A. In this way, B becomes an R-algebra. Furthermore, if A is a projective Noetherian R-algebra, then $B = \operatorname{End}_A(P) \in \operatorname{add}_R P$. So, B is a projective Noetherian R-algebra. △

We also note that being Morita equivalent is an equivalence relation.

Proposition 1.4.19. *[Fai73, Chapter 12, pages 447-453] Assume A and B are Noetherian rings. Suppose there is an (A,B)-bimodule M and (B,A)-bimodule N such that there are isomorphisms*

$$M \otimes_B N \xrightarrow{\alpha} A \ \text{ as } (A,A)\text{-bimodules}$$
$$N \otimes_A M \xrightarrow{\beta} B \ \text{ as } (B,B)\text{-bimodules}.$$

Then, M is a progenerator as an A-module, $B \simeq \operatorname{End}_A(M)^{op}$, N is a progenerator as a B-module and $A \simeq \operatorname{End}_B(N)^{op}$, M is a progenerator as a right B-module, $A \simeq \operatorname{End}_B(M)$, N is a progenerator as a right A-module, $B \simeq \operatorname{End}_A(N)$. Moreover,

$$\operatorname{Hom}_A(M,A) \simeq N \simeq \operatorname{Hom}_B(M,B) \ \text{ as } B\text{-modules}$$
$$\operatorname{Hom}_A(N,A) \simeq M \simeq \operatorname{Hom}_B(N,B) \text{ as } A\text{-modules}.$$

Corollary 1.4.20. *[Fai73, Chapter 12, pages 447-453] Assume A-Mod $\simeq$ B-Mod then A^{op}-Mod $\simeq$ B^{op}-Mod.*

Corollary 1.4.21. *[Fai73, Chapter 12, pages 447-453] Assume A and B are Noetherian rings. Suppose there is an A-progenerator M so that $B = \operatorname{End}_A(M)^{op}$. Then, there exists $N \in B$-Mod so that*

$$\operatorname{Hom}_A(M,A) \simeq N \simeq \operatorname{Hom}_B(M,B) \ \text{ as } (B,A)\text{-bimodules}$$
$$\operatorname{Hom}_A(N,A) \simeq M \simeq \operatorname{Hom}_B(N,B) \text{ as } (A,B)\text{-bimodules}.$$

The progenerators over commutative rings are exactly the faithful modules.

Proposition 1.4.22. *Let S be a commutative ring. Suppose P is finitely generated projective S-module. Then, P is S-faithful if and only if P is an S-progenerator.*

Proof. See [Fai73, Proposition 12.2]. ☐

Note that R^n is an R-progenerator for every n, and thus, $M_n(R) \simeq \operatorname{End}_R(R^n)^{op}$ is Morita equivalent to R.

Observe that free modules are not Morita invariant. This fact can be checked, for example, by comparing the free R-modules and the free $M_2(R)$-modules. This is why we deal with finitely generated projective modules over commutative rings (non-fields) instead of free modules.

Corollary 1.4.23. *[KY13, Proposition 1.3] Let A be an R-algebra. Let $M \in A$-mod and $N \in A$-mod such that $\operatorname{add} M = \operatorname{add} N$. Then, $\operatorname{End}_A(M)^{op}$ is Morita equivalent to $\operatorname{End}_A(N)^{op}$.*

Proof. By projectivization, $\operatorname{End}_A(M)^{op}$-proj $\simeq \operatorname{add}(M) = \operatorname{add}(N) \simeq \operatorname{End}_A(N)^{op}$-proj. By Theorem 1.4.17, $\operatorname{End}_A(M)^{op}$ is Morita equivalent to $\operatorname{End}_A(N)^{op}$. ☐

1.4.5 Covers

We saw that when P is both a projective finitely generated A-module and generator then the module categories A-Mod and $\operatorname{End}_A(P)^{op}$-Mod are identical. Hence, the next step is to see what happens to these module categories when we drop one of these conditions on P. In general, when P is just a projective finitely generated A-module the categories A-Mod and $\operatorname{End}_A(P)^{op}$-Mod may be quite different and not being connected at all. Hence, we will proceed by studying the unit and counit of the adjunction pair $(\operatorname{Hom}_A(P,-), \operatorname{Hom}_B(FA,-))$ given by Proposition 1.4.13 to see what properties should P satisfy, in addition, so that we can relate the module categories of A and $\operatorname{End}_A(P)^{op}$.

For our purposes and from now on, we will assume A to be a projective Noetherian R-algebra. Hence, A is a Noetherian ring. Let $P \in A$-proj. Notice also that $B := \mathrm{End}_A(P)^{op}$ is an R-summand of P^t for some $t > 0$. Since P is a finitely generated projective R-module, we deduce that B is finitely generated projective over R. Thus, B is a Noetherian ring as well.

Hence, the categories A-mod and B-mod are abelian and every module belonging either in A-mod or B-mod is finitely presented. Denote by G the right adjoint of the Schur functor $F = \mathrm{Hom}_A(P, -)$.

Proposition 1.4.24. *The Schur functor* $\mathrm{Hom}_A(P, -)\colon A\text{-mod} \to B\text{-mod}$*, denoted by F, and its right adjoint* $\mathrm{Hom}_B(FA, -)\colon B\text{-mod} \to A\text{-mod}$*, denoted by G, are well defined.*

Proof. Let $X \in A$-mod. By Lemma 1.1.5, $\mathrm{Hom}_A(P, X)$ is finitely generated as R-module. Let $\{f_1, \dots f_d\}$ be a finite R-generator set. Then, for any $g \in \mathrm{Hom}_A(P, X)$, there are $r_i \in R$ such that

$$g = \sum_i r_i f_i = \sum_i r_i (1_B f_i) = \sum_i \underbrace{(r_i 1_B)}_{\in B} f_i.$$

Thus, $\{f_1, \dots f_d\}$ is a finite B-generator set for $\mathrm{Hom}_A(P, X)$, that is, $\mathrm{Hom}_A(P, X) \in B$-mod. With the same reasoning, we conclude that $\mathrm{Hom}_B(FA, Y) \in A$-mod for every $Y \in B$-mod. Hence, both functors are well defined. Since A-mod and B-mod are full subcategories of A-Mod and B-Mod, respectively, the restriction functors F and G form also an adjoint pair. $\square$

The unit of the adjunction $F \dashv G$ is the natural transformation $\eta\colon \mathrm{id}_{A\text{-mod}} \to G \circ F$ such that for any module $N \in A$-mod, the A-homomorphism

$$\eta_N\colon N \to \mathrm{Hom}_B(FA, \mathrm{Hom}_A(P, N)) \text{ is given by } \eta(n)(f)(p) = f(p)n,\ n \in N,\ f \in FA, p \in P.$$

The counit of the adjunction $F \dashv G$ is the natural transformation $\varepsilon\colon F \circ G \to \mathrm{id}_{B\text{-mod}}$ such that for any module $M \in B$-mod, the B-homomorphism is given by the following commutative diagram

$$\begin{tikzcd} FA \otimes_A \mathrm{Hom}_B(FA, M) \arrow[r, "\varepsilon'_M"] \arrow[d, "\simeq"] & M \arrow[d, equal] \\ \mathrm{Hom}_A(P, \mathrm{Hom}_B(FA, M)) \arrow[r, "\varepsilon_M"] & M \end{tikzcd}$$

where $\varepsilon'_M\colon FA \otimes_A \mathrm{Hom}_B(FA, M) \to M$ is given by

$$\varepsilon'_M(f \otimes g) = g(f), f \otimes g \in FA \otimes_A \mathrm{Hom}_B(FA, M).$$

From category theory, we know that for each $M \in B$-mod and $N \in A$-mod, the following holds (see [Mac71, Theorem 2, p.81])

$$\mathrm{id}_{FN} = \varepsilon_{FN} \circ F\eta_N \tag{1.4.5.1}$$

$$\mathrm{id}_{GM} = G\varepsilon_M \circ \eta_{GM}. \tag{1.4.5.2}$$

Proposition 1.4.25. *G is fully faithful and ε_M is a B-isomorphism for any $M \in B$-mod.*

Proof. First we will check that ε_M is a B-isomorphism for any $M \in B$-mod. Let $M \in B$-mod and $b \in B$. We have

$$b\varepsilon'_M(g \otimes h) = bh(g) = h(bg) = \varepsilon'_M(bg \otimes h) = \varepsilon'_M(b(g \otimes h)),\ g \otimes h \in FA \otimes_A \mathrm{Hom}_B(FA, M). \tag{1.4.5.3}$$

Now for any $g \in \mathrm{Hom}_A(P, \mathrm{Hom}_B(FA, M))$, we have

$$\varepsilon_M(bg) = \varepsilon_M(b\psi_{\mathrm{Hom}_B(FA,M)}(g')) = \varepsilon_M(\psi_{\mathrm{Hom}_B(FA,M)}(g')) \tag{1.4.5.4}$$

$$= \varepsilon'_M(bg') = b\varepsilon'_M(g') = b\varepsilon_M(\psi_{\mathrm{Hom}_B(FA,M)}(g')) = b\varepsilon_M(g). \tag{1.4.5.5}$$

Here, $\psi_{\mathrm{Hom}_B(FA,M)}$ is the B-isomorphism provided by Lemma 1.4.11. Therefore, ε_M is a B-homomorphism.

We will start by proving that ε'_M is a bijective map. Since $P \in A$-proj, there are canonical projections and injections arising from the respective direct sums (as A-summands):

$$P \underset{\pi_P}{\overset{k_P}{\rightleftarrows}} A^t \underset{k_j}{\overset{\pi_j}{\rightleftarrows}} A \ , 1 \leq j \leq t.$$

We shall need some notation. Denote $\pi_i \circ k_P$ by θ_i, $i = 1, \ldots, t$. For any $f \in \mathrm{Hom}_A(P,A)$, denote $\pi_P \circ k_i \circ f$ by $b_i^f \in B$, $i = 1, \ldots, t$. For any $m \in M$, define $h_m \in \mathrm{Hom}_B(B,M)$ given by $h_m(1_B) = m$. Now define $g_{m,i} \in \mathrm{Hom}_B(FA,M)$ satisfying $g_{m,i}(f) = h_m(b_i^f)$, $f \in FA$. This is well defined since, for any $y \in B$,

$$g_{m,i}(y \cdot f) = g_{m,i}(f \circ y) = h_m(\pi_P \circ k_i \circ f \circ y) = h_m(y \cdot \pi_P \circ k_i \circ f) = y \cdot h_m(b_i^f) = y \cdot g_{m,i}(f), f \in FA. \tag{1.4.5.6}$$

Now define the map $\Theta \colon M \to FA \otimes_A \mathrm{Hom}_B(FA,M)$, given by $\Theta(m) = \sum_l \theta_l \otimes g_{m,l}$, $m \in M$.

Observe that, for any $m \in M$,

$$\sum_i g_{m,i}(\theta_i) = \sum_i h_m(b_i^{\theta_i}) = m \cdot \sum_i b_i^{\theta_i} = m \cdot \sum_i \pi_P \circ k_i \circ \pi_i \circ k_P = m \cdot \mathrm{id}_P = m. \tag{1.4.5.7}$$

Therefore, for any $m \in M$,

$$\varepsilon'_M \Theta(m) = \varepsilon'_M(\sum_l \theta_l \otimes g_{m,l}) = \sum_l g_{m,l}(\theta_l) = m. \tag{1.4.5.8}$$

Hence, ε'_M is surjective. In order to prove that ε'_M is injective we need the following two observations. Let $w \in FA = \mathrm{Hom}_A(P,A)$ and $\chi \in \mathrm{Hom}_B(FA,M)$. Then,

$$w = w \circ \mathrm{id}_P = w \circ \sum_i \pi_P \circ k_i \circ \pi_i \circ k_P = \sum_i w\pi_P k_i \pi_i k_P = \sum_i w\pi_P k_i \theta_i = \sum_i \theta_i \cdot (w\pi_P k_i(1_A)), \tag{1.4.5.9}$$

where the last equality is due to $w\pi_P k_i \in \mathrm{Hom}_A(A,A)$ and $\theta_i(p) \in A$ for any $p \in P$. For any $f \in FA$,

$$g_{\chi(w),i}(f) = h_{\chi(w)}(b_i^f) = b_i^f \cdot \chi(w) = \chi(b_i^f \cdot w) = \chi(w \circ b_i^f) = \chi(w \circ \pi_P \circ k_i \circ f) \tag{1.4.5.10}$$

$$= \chi(f \cdot (w\pi_P k_i(1_A))) = ((w\pi_P k_i(1_A)) \cdot \chi)(f). \tag{1.4.5.11}$$

Thus,

$$\Theta\varepsilon'_M(w \otimes \chi) = \Theta(\chi(w)) = \sum_i \theta_i \otimes g_{\chi(w),i} = \sum_i \theta_i \otimes (w\pi_P k_i(1_A)) \cdot \chi = \sum_i \theta_i \cdot (w\pi_P k_i(1_A)) \otimes \chi = w \otimes \chi.$$

So, ε'_M is also injective, and thus it is an B-isomorphism. In particular, ε_M is an B-isomorphism.

By a general result of Category theory, the counit ε_M is an isomorphism for every $M \in B$-mod if and only if the functor G is full and faithful. For the sake of completeness, we show that G is full and faithful.

Let $f \in \mathrm{Hom}_B(M,M')$ satisfying $Gf = 0$. Then, $f \circ \varepsilon_M = \varepsilon_{M'} \circ FGf = 0$. Thus, $f = 0$. So, G is faithful. Let $h \in \mathrm{Hom}_A(GM,GM')$. We define $f = \varepsilon_{M'} \circ Fh \circ \varepsilon_M^{-1}$. Notice that

$$G\varepsilon_M^{-1} \circ \mathrm{id}_{GM} = G\varepsilon_M^{-1} \circ G\varepsilon_M \circ \eta_{GM} = \eta_{GM}. \tag{1.4.5.12}$$

Then,

$$Gf = G\varepsilon_{M'} \circ GFh \circ G\varepsilon_M^{-1} = G\varepsilon_{M'} \circ GFh \circ \eta_{GM} = G\varepsilon_{M'} \circ \eta_{GM'} \circ h = \mathrm{id}_{GM'} \circ h = h. \tag{1.4.5.13}$$

So, G is full and faithful. □

Lemma 1.4.26. *The unit is compatible with direct sums. In particular, $\eta_{N_1 \bigoplus N_2}$ is mono (surjective) if and only if η_{N_1} and η_{N_2} are mono (surjective) for any $N_1, N_2 \in A$-mod.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} N_1 \bigoplus N_2 & \xrightarrow{\eta_{N_1 \bigoplus N_2}} & \mathrm{Hom}_B(FA, F(N_1 \bigoplus N_2)) \\ \| & & \uparrow{\scriptstyle \mathrm{Hom}_B(FA, \delta_{P,N_1,N_2}) \circ \delta_{FA, FN_1, FN_2}} \\ N_1 \bigoplus N_2 & \xrightarrow[\eta_{N_1} \bigoplus \eta_{N_2}]{} & \mathrm{Hom}_B(FA, FN_1) \bigoplus \mathrm{Hom}_B(FA, FN_2) \end{array}$$

, where δ are the maps given by Lemma 1.4.9.

Since both columns are isomorphisms, the result follows. □

Lemma 1.4.27. *[Rou08, Lemma 4.32] Let $M \in A$-mod. The following assertions are equivalent.*

- *(a) The unit $\eta_M : M \to GFM$ is an isomorphism;*
- *(b) F induces a bijection of abelian groups $\mathrm{Hom}_A(N, M) \to \mathrm{Hom}_B(FN, FM)$, $f \mapsto Ff$ for every $N \in A$-mod.*
- *(c) F induces an isomorphism of A-modules $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_B(FA, FM)$, $f \mapsto Ff$.*
- *(d) M is a direct summand of a module in the image of G.*

Proof. First notice that the map $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_B(FA, FM)$ is an A-homomorphism. Let $f \in \mathrm{Hom}_A(A, M)$, $g \in FA$, $a \in A$ and $p \in P$. Then,

$$F(a \cdot f)(g)(p) = \mathrm{Hom}_A(P, a \cdot f)(g)(p) = a \cdot f \circ g(p) = f(g(p)a) \tag{1.4.5.14}$$

$$= f((ga)(p)) = \mathrm{Hom}_A(P, f)(ga)(p) = (a \cdot \mathrm{Hom}_A(P, f)(g))(p) = aFf(g)(p). \tag{1.4.5.15}$$

$a) \implies b)$. Assume that η_M is an isomorphism. Let $f \in \mathrm{Hom}_A(N, M)$ satisfying $Ff = 0$. Then,

$$\eta_M \circ f = GFf \circ \eta_N = 0 \implies f = 0. \tag{1.4.5.16}$$

Let $g \in \mathrm{Hom}_B(FN, FM)$. Define $f = \eta_M^{-1} \circ Gg \circ \eta_N$. Observe that

$$F\eta_M^{-1} = \mathrm{id}_{FM} F\eta_M^{-1} = \varepsilon_{FM} F\eta_M F\eta_M^{-1} = \varepsilon_{FM}. \tag{1.4.5.17}$$

Hence, $Ff = \varepsilon_{FM} \circ FGg \circ F\eta_A = g \circ \varepsilon_{FA} \circ F\eta_A = g \circ \mathrm{id}_{FA} = g$. So, $b)$ holds.

$b) \implies c)$. By $b)$, $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_B(FA, FM)$ is a bijection. We saw that $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_B(FA, FM)$ is an A-homomorphism, therefore $c)$ is clear.

$c) \implies d)$. $M \simeq \mathrm{Hom}_A(A, M) \simeq \mathrm{Hom}_B(FA, FM) = GFM$ as A-modules. So, M is the image of a module in the image G. In particular $c)$ holds.

$d) \implies a)$. Assume $GD \simeq M \bigoplus K$. Then, $\mathrm{id}_{GD} = G\varepsilon_D \circ \eta_{GD}$. By Proposition 1.4.25, ε_D is an isomorphism. Therefore, $\eta_{GD} = \eta_M \bigoplus \eta_K$ is an isomorphism. By Lemma 1.4.26, η_M is an isomorphism. □

Lemma 1.4.28. *[Rou08, Proposition 4.33] The following assertions are equivalent.*

(i) The canonical map of algebras $A \to \operatorname{End}_B(FA)^{op}$, given by $a \mapsto (f \mapsto f(-)a)$, $a \in A, f \in FA$, is an isomorphism of R-algebras.

(ii) For all $M \in A$-proj, the unit $\eta_M : M \to GFM$ is an isomorphism of A-modules.

(iii) The restriction of F to A-proj is full and faithful.

Proof. $(i) \implies (ii)$. Let $M \in A$-proj. Then, there exists $K \in A$-mod and $t > 0$ such that $A^t \simeq M \bigoplus K$. By i), η_A is an isomorphism. So, M is an A-summand of $A^t \simeq (GFA)^t \simeq G(FA^t)$. By Lemma 1.4.27, η_M is an isomorphism.

$(ii) \implies (iii)$. By Lemma 1.4.27, for every $M \in A$-proj, F induces a bijection of abelian groups $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(FN,FM)$ for every $N \in A$-mod. In particular, the restriction of F to A-proj is full and faithful.

$(iii) \implies (i)$. Clearly, $A \in A$-proj. By assumption, F induces an isomorphism of A-modules $\operatorname{Hom}_A(A,A) \to \operatorname{Hom}_B(FA,FA)$, $f \mapsto Ff$. The composition of the canonical bijection $A \to \operatorname{End}_A(A)^{op}$, and the isomorphism $f \mapsto Ff$ yields the bijection R-homomorphism $A \to \operatorname{End}_B(FA)^{op}$ given by $a \mapsto (f \mapsto f \cdot a)$. We will denote this map by η_A. This maps clearly preserves the identity of A and it preserves the ring multiplication. In fact, for any $a, b \in A$, $f \in FA$,

$$\eta_A(a) \cdot \eta_A(b)(f) = \eta_A(b) \circ \eta_A(a)(f) = (\eta_A(a)(f)) \cdot b = (f \cdot a) \cdot b = f \cdot (ab) = \eta_A(ab)(f). \tag{1.4.5.18}$$

Thus, (i) holds. □

Definition 1.4.29. Let A be a projective Noetherian R-algebra and $P \in A$-proj. We say that (A,P) is a **cover** of B if the restriction of $F = \operatorname{Hom}_A(P,-) \colon A\text{-mod} \to B\text{-mod}$ to A-proj is full and faithful. We also say that $(A\text{-mod}, F)$ is a cover of B-mod.

Notice that since $P \in A$-proj, $B = \operatorname{End}_A(P)^{op} \simeq \operatorname{End}_A(D_A P) = \operatorname{End}_A(\operatorname{Hom}_A(P,A)) = \operatorname{End}_A(FA)$. Hence, by Lemma 1.4.28, (A,P) is a cover of B if and only if it holds a double centralizer property on $\operatorname{Hom}_A(P,A)$ between B and A. So, covers provide a good setup to extend and study double centralizer properties in a more abstract way. On the other hand, by projectivization the functor $\operatorname{Hom}_A(P,-) \colon A\text{-proj} \to B\text{-proj}$ is essentially surjective. So, in a cover situation if the functor $\operatorname{Hom}_A(P,-) \colon A\text{-proj} \to B\text{-proj}$ becomes well defined, then it must be an equivalence of categories. By the Morita theorem, this means that A-mod and B-mod are equivalent categories. Therefore, we can see covers as a good starting point to relate the categories A-mod with B-mod.

It is important to remark that since $P \in A$-proj, $FA = \operatorname{Hom}_A(P,A)$ is a left B-generator. So, if (A,P) is a cover of B, Proposition 1.4.25 is known as the Gabriel-Popescu theorem [Mit81, PG64] for the category B-mod.

Here are some properties of covers under change of ground ring.

Lemma 1.4.30. *The following assertions are equivalent:*

1. *(A,P) is a cover of B;*
2. *$(S \otimes_R A, S \otimes_R P)$ is a cover of $S \otimes_R B$ for every flat commutative R-algebra S;*
3. *$(A_\mathfrak{p}, P_\mathfrak{p})$ is a cover of $B_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R;*
4. *$(A_\mathfrak{m}, P_\mathfrak{m})$ is a cover of $B_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R;*

Proof. $(i) \Rightarrow (ii)$ Assume that (A,P) is a cover of B. Then, η_A is an isomorphism. By Lemma 1.1.36, $S \otimes_R B = S \otimes_R \operatorname{Hom}_A(P,P) \simeq \operatorname{Hom}_{S \otimes_R A}(S \otimes_R P, S \otimes_R P)^{op}$. Consider the following diagram

$$\begin{tikzcd}
S\otimes_R A \arrow[r, "S\otimes_R \eta_A"] \arrow[ddd, equal] & S\otimes_R \mathrm{End}_B(\mathrm{Hom}_A(P,A)) \arrow[d, "\omega_{\mathrm{Hom}_A(P,A)}"] \\
& \mathrm{End}_{S\otimes_R B}(S\otimes_R \mathrm{Hom}_A(P,A)) \\
& \\
S\otimes_R A \arrow[r, "\eta_{S\otimes_R A}"] & \mathrm{End}_{S\otimes_R B}(\mathrm{Hom}_{S\otimes_R A}(S\otimes_R P, S\otimes_R A)) \arrow[uu, "\omega_P^{-1}\circ(-)\circ\omega_P"']
\end{tikzcd}.$$

The maps ω are the canonical maps given by Lemma 1.1.36, hence they are isomorphisms. This is a commutative diagram. In fact, for every $s,s',s''\in S$, $a\in A$, $g\in \mathrm{Hom}_A(P,A)$, $p\in P$, we have

$$\begin{aligned}
\omega_P\circ\omega_P^{-1}\circ(-)\circ\omega_P\circ\eta_{S\otimes_R A}(s\otimes a)(s'\otimes g)(s''\otimes p) &= \eta_{S\otimes_R A}(s\otimes a)\omega_P(s'\otimes g)(s''\otimes p)\\
&= \omega_P(s'\otimes g)(s''\otimes p)s\otimes a = ss's''\otimes g(p)a\\
\omega_P\circ\omega_{\mathrm{Hom}_A(P,A)}\circ S\otimes_R \eta_A(s\otimes a)(s'\otimes g)(s''\otimes p) &= \omega_P\omega_{\mathrm{Hom}_A(P,A)}(s\otimes\eta_A(a))(s'\otimes g)(s''\otimes p)\\
&= \omega_P(ss'\otimes \eta_A(a)(g))(s''\otimes p) = ss's''\otimes g(p)a.
\end{aligned}$$

It follows by Lemma 1.4.28 that $(S\otimes_R A, S\otimes_R P)$ is a cover of $S\otimes_R B$. *ii)* $\Rightarrow$ *iii)* For every prime ideal $\mathfrak{p}$ in R, $R_\mathfrak{p}$ is a flat commutative R-algebra. *iii)* $\Rightarrow$ *iv)* Every maximal ideal is prime. *iv* $\Rightarrow$ *i)* By assumption, the map $A_\mathfrak{m}\to \mathrm{End}_{B_\mathfrak{m}}(\mathrm{Hom}_{A_\mathfrak{m}}(P_\mathfrak{m},A_\mathfrak{m}))\simeq \mathrm{End}_B(\mathrm{Hom}_A(P,A))_\mathfrak{m}$ is an isomorphism for all maximal ideals $\mathfrak{m}$ in R. Therefore, $A\to \mathrm{End}_B(\mathrm{Hom}_A(P,A))$ is an isomorphism. By Lemma 1.4.28, *i)* follows. ☐

Lemma 1.4.31. *Let $M\in A$-mod$\cap R$-proj. If the unit $\eta_{M(\mathfrak{m})}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R, then the unit η_M is (A,R)-monomorphism. If, in addition, $DM\otimes_A P\otimes_B \mathrm{Hom}_A(P,A)\in R$-proj and η_M is (A,R)-monomorphism, then $\eta_{M(\mathfrak{m})}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R.*

Proof. Let $\lambda_M\colon DM\otimes_A P\otimes_B \mathrm{Hom}_A(P,A)\to DM$, given by $\lambda_M(f\otimes p\otimes g) = fg(p)$ for $f\otimes p\otimes g\in DM\otimes_A P\otimes_B \mathrm{Hom}_A(P,A)$. There is a commutative diagram

$$\begin{tikzcd}
DDM \arrow[r, "D\lambda_M"] & D(DM\otimes_A P\otimes_B \mathrm{Hom}_A(P,A)) \\
& D(D\mathrm{Hom}_A(P,M)\otimes_B \mathrm{Hom}_A(P,A)) \arrow[u, "D(\iota\otimes\mathrm{Hom}_A(P,A))"'] \\
M \arrow[r, "\eta_M"] \arrow[uu, "w_M"] & \mathrm{Hom}_B(\mathrm{Hom}_A(P,A),\mathrm{Hom}_A(P,M)) \arrow[u, "\kappa"']
\end{tikzcd} \tag{1.4.5.19}$$

where the isomorphism maps κ and ι are according to Proposition 1.1.65. In fact, for $m\in M$, $f\otimes p\otimes g\in DM\otimes_A P\otimes_B \mathrm{Hom}_A(P,A)$,

$$\begin{aligned}
D(\iota\otimes\mathrm{Hom}_A(P,A))\circ\kappa\circ\eta_M(m)(f\otimes p\otimes g) &= \kappa(\eta_M(m))(\iota\otimes\mathrm{Hom}_A(P,A))(f\otimes p\otimes g)\\
&= \kappa(\eta_M(m))(\iota(f\otimes p)\otimes g) = \iota(f\otimes p)(\eta_M(m)(g))\\
&= f(\eta_M(m)(g)(p)) = f(g(p)m)\\
D\lambda_M\circ w_M(m)(f\otimes p\otimes g) &= w_M(m)(\lambda_M(m)(f\otimes p\otimes g)) = w_M(m)(f\cdot g(p))\\
&= (f\cdot g(p))(m) = f(g(p)m).
\end{aligned}$$

By assumption, $\eta_{M_{(\mathfrak{m})}}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R. According to the commutative diagram (1.4.5.19), $D_{(\mathfrak{m})}\lambda_{M(\mathfrak{m})} = \mathrm{Hom}_{R(\mathfrak{m})}(\lambda_{M(\mathfrak{m})}, R(\mathfrak{m}))$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R.

Hence, $\lambda_{M(\mathfrak{m})}$ is surjective for every maximal ideal in R. In view of the commutative diagram

$$\begin{tikzcd}
DM\otimes_A P\otimes_B \operatorname{Hom}_A(P,A)(\mathfrak{m}) \arrow[rr, "\lambda_M(\mathfrak{m})"] \arrow[d, "\simeq"] & & DM(\mathfrak{m}) \arrow[dd, "\simeq"] \\
DM(\mathfrak{m})\otimes_{A(\mathfrak{m})} P(\mathfrak{m})\otimes_{B(\mathfrak{m})} \operatorname{Hom}_A(P,A)(\mathfrak{m}) \arrow[d, "\simeq"] & & \\
D_{(\mathfrak{m})}M(\mathfrak{m})\otimes_{A(\mathfrak{m})} P(\mathfrak{m})\otimes_{B(\mathfrak{m})} \operatorname{Hom}_{A(\mathfrak{m})}(P(\mathfrak{m}),A(\mathfrak{m})) \arrow[rr, "\lambda_{M(\mathfrak{m})}"] & & D_{(\mathfrak{m})}M(\mathfrak{m})
\end{tikzcd}, \tag{1.4.5.20}$$

$\lambda_M(\mathfrak{m})$ is surjective for every maximal ideal in R. By Nakayama's Lemma, λ_M is surjective. As $DM\in R$-proj, λ_M splits over R, so there is an R-homomorphism t such that $t\circ D\lambda_M=\operatorname{id}_M$. Thus,

$$w_M^{-1}\circ t\circ D(\iota\otimes \operatorname{Hom}_A(P,A))\circ\kappa\circ\eta_M = w_M^{-1}\circ t\circ D\lambda_M\circ w_M = w_M^{-1}\circ w_M=\operatorname{id}_M.$$

Hence, η_M is (A,R)-mono.

Conversely, assume that η_M is (A,R)-mono and $DM\otimes_A P\otimes_B \operatorname{Hom}_A(P,A)\in R$-proj. In view of diagram (1.4.5.19), $D\lambda_M$ is (A,R)-mono. Then, $DD\lambda_M$ is surjective. As $DM\otimes_A P\otimes_B \operatorname{Hom}_A(P,A)\in R$-proj, the map $w_{DM\otimes_A P\otimes_B \operatorname{Hom}_A(P,A)}$ is an isomorphism and consequently, λ_M is surjective. Applying the right exact functor $R(\mathfrak{m})\otimes_R -$, we obtain by diagram (1.4.5.20) that $\lambda_{M(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By the first diagram, it follows that $\eta_{M(\mathfrak{m})}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R. ☐

Lemma 1.4.32. *Let I be an ideal of R. Let $M\in R$-mod. Then, $R/I\otimes_R M\simeq M/IM$.*

Proof. Consider the R-homomorphisms $R/I\otimes_R M\to M/IM$, given by $(r+I)\otimes_R m\mapsto rm+IM$, $r\in R, m\in M$, and $M/IM\to R/I\otimes_R M$, given by $m+IM\mapsto(1_R+I)\otimes m$. These homomorphisms are well defined and are inverse to each other. ☐

Lemma 1.4.33. *Let x be a non-zero divisor of R. The following assertions hold.*

(i) Let $M\in R$-proj. Then, the R-homomorphism $\delta\colon M\to M$, $m\mapsto xm$ is a monomorphism.

(ii) Let B be a projective Noetherian R-algebra and $M\in B$-mod$\cap R$-proj. The map $\operatorname{End}_B(M)\otimes_R R/Rx\to \operatorname{Hom}_B(M,R/Rx\otimes_R M)$, given by $f\otimes r+Rx\mapsto(m\mapsto r+Rx\otimes_R f(m))$, is a monomorphism.

Proof. Let $m\in M$ such that $xm=0$. Since M is projective over R, there exists a natural number n and $K\in R$-mod such that $R^n\simeq M\bigoplus K$. So, there exists $\alpha_i\in R$ satisfying $m=\sum_i\alpha_ie_i$, where $\{e_i : i=1,\ldots,n\}$ is an R-basis for R^n. Therefore, $x\alpha_i=0$ for all $i=1,\ldots,n$. Since x is a non-zero divisor, $\alpha_i=0$ for all i. Hence, $m=0$. Thus, *i)* follows.

Assume that $0=\delta(\sum_i f_i\otimes_R r_i+Rx)=\delta(\sum_i r_if_i\otimes_R 1+Rx)=\delta(f\otimes_R 1+Rx)$,for some $f\in\operatorname{End}_B(M)$. In particular, $1+Rx\otimes_R f(m)=0$ for all $m\in M$. By Lemma 1.4.32, $f(m)\in RxM=xM$ for all $m\in M$. We claim that $f=xg$ for some $g\in\operatorname{End}_B(M)$. By assumption, there is for every $m\in M$ $y_m\in M$ satisfying $f(m)=xy_m$. Note that, any $b\in B$ and $m,m_1,m_2\in M$

$$xy_{bm}=f(bm)=bf(m)=b(xy_m)\implies x(y_{bm}-by_m)=0 \text{ and} \tag{1.4.5.21}$$

$$xy_{m_1+m_2}=f(m_1+m_2)=f(m_1)+f(m_2)=xy_{m_1}+xy_{m_2}. \tag{1.4.5.22}$$

By $i)$, $y_{bm} - by_m = 0$ and $y_{m_1+m_2} = y_{m_1} + y_{m_2}$. Thus, $g\colon M \to M$, given by $g(m) := y_m$ for every $m \in M$ is a well defined element of $\operatorname{End}_B(M)$ satisfying $f = xg$.

Hence, $f \otimes_R (1 + Rx) = xg \otimes 1 + Rx = g \otimes x + Rx = 0$. So, δ is a monomorphism. $\square$

Proposition 1.4.34. *[Rou08, Proposition 4.36] Assume R is a commutative Noetherian regular ring. Let A be a projective Noetherian R-algebra. Let $P \in A\text{-mod} \cap R\text{-proj}$. If $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R, then (A, P) is a cover of B.*

Proof. As $P \in R$-proj and $P(\mathfrak{m})$ is projective over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R, it follows that $P \in A$-proj.

In view of Lemma 1.4.30, we can assume that R is a local commutative Noetherian regular ring. We shall proceed by induction on the Krull dimension of R. Assume that $\dim R = 0$. According to 1.1.57, a local commutative Noetherian regular ring with Krull dimension zero is a field. As $(A, P) = (A(0), P(0))$, there is nothing to prove for $\dim R = 0$.

Assume the result known for rings with Krull dimension less than t. Let R be with $\dim R = t$. Let $x \in \mathfrak{m}/\mathfrak{m}^2$. By the theory of regular rings, a local Noetherian regular commutative ring is an integral domain. Thus, x is a non-zero divisor. Fix $Q = R/Rx$. The ring Q has a unique maximal ideal $\mathfrak{m}/Rx$, by Correspondence theorem for rings. In particular, Q is a local regular Noetherian ring. Moreover, $\dim Q = \dim R/Rx = \dim R - 1 < t$ and

$$Q(\mathfrak{m}/Rx) = Q/(\mathfrak{m}/Rx) \simeq R/Rx/\mathfrak{m}/Rx \simeq R/\mathfrak{m} = R(\mathfrak{m}). \tag{1.4.5.23}$$

By assumption, $(A \otimes_R Q(\mathfrak{m}/Rx), P \otimes_R Q(\mathfrak{m}/Rx))$ is a cover of $B \otimes_R Q(\mathfrak{m}/Rx)$. By induction, $(A \otimes_R Q, P \otimes_R Q)$ is a cover of $B \otimes_R Q$. By Lemma 1.4.28, Lemma 1.1.32 and Proposition 1.1.31, the composition map

$$\begin{tikzcd} A \otimes_R Q \arrow[d] & \operatorname{Hom}_B(\operatorname{Hom}_A(P,A), \operatorname{Hom}_A(P,A) \otimes_R Q) \\ \operatorname{End}_{B\otimes_R Q}(\operatorname{Hom}_{A\otimes_R Q}(P \otimes_R Q, A \otimes_R Q)) \arrow[r] & \operatorname{End}_{B\otimes_R Q}(\operatorname{Hom}_A(P,A) \otimes_R Q) \arrow[u] \end{tikzcd}$$

is an isomorphism. We will denote this map by μ_Q. Explicitly, we have $\mu_Q(a \otimes q) = (f \mapsto f \cdot a \otimes q)$, $a \otimes q \in A \otimes_R Q$.

We have a commutative triangle

$$\begin{tikzcd} A \otimes_R Q \arrow[rr, "\mu_Q"] \arrow[dr, "\eta_A \otimes_R Q"'] & & \operatorname{Hom}_B(\operatorname{Hom}_A(P,A), \operatorname{Hom}_A(P,A) \otimes_R Q) \\ & \operatorname{End}_B(\operatorname{Hom}_A(P,A)) \otimes_R Q \arrow[ur, hook, "\delta"] & \end{tikzcd}$$

with a monomorphism given by Lemma 1.4.33. Since μ_Q is an isomorphism, the monomomorphism δ is also surjective. Thus, $\eta_A \otimes_R Q$ is an isomorphism. Denote the canonical surjective map $Q \to Q/\mathfrak{m}/Rx = R(\mathfrak{m})$ by π. There exists a commutative diagram

$$\begin{tikzcd} A \otimes_R Q \arrow[r, "\eta_A \otimes_R Q"] \arrow[d, "A \otimes_R \pi"] & \operatorname{End}_B(\operatorname{Hom}_A(P,A)) \otimes_R Q \arrow[d, "\operatorname{End}_B(\operatorname{Hom}_A(P,A)) \otimes_R \pi"] \\ A \otimes_R R(\mathfrak{m}) \arrow[r, "\eta_A(\mathfrak{m})"] & \operatorname{End}_B(\operatorname{Hom}_A(P,A)) \otimes_R R(\mathfrak{m}) \end{tikzcd} .$$

In fact, for every $a \in A$,

$$\eta_A(\mathfrak{m}) \circ A \otimes_R \pi(a \otimes_R 1_R + Rx) = \eta_A(\mathfrak{m})(a \otimes_R 1_R + \mathfrak{m}) = \eta_A(a) \otimes_R 1_R + \mathfrak{m}$$

$$\begin{aligned} \operatorname{End}_B(\operatorname{Hom}_A(P,A)) \otimes_R \pi \circ \eta_A \otimes_R Q(a \otimes_R 1_R + Rx) &= \operatorname{End}_B(\operatorname{Hom}_A(P,A)) \otimes_R \pi(\eta_A(a) \otimes_R 1_R + Rx) \\ &= \eta_A(a) \otimes_R 1_R + \mathfrak{m}. \end{aligned}$$

It follows that $\eta_A(\mathfrak{m}) \circ A \otimes_R \pi$ is surjective. In particular $\eta_A(\mathfrak{m})$ is surjective. By Nakayama's Lemma, η_A is surjective. By Lemma 1.4.31, η_A is a monomorphism. By Lemma 1.4.28, the result follows. $\square$

The converse is not necessarily true. In general,

$$\mathrm{End}_{B(\mathfrak{m})}(\mathrm{Hom}_{A(\mathfrak{m})}(P(\mathfrak{m}), A(\mathfrak{m})) \simeq \mathrm{Hom}_{B(\mathfrak{m})}(\mathrm{Hom}_A(P,A)(\mathfrak{m})) \not\simeq \mathrm{Hom}_B(\mathrm{Hom}_A(P,A))(\mathfrak{m}),$$

unless $\mathrm{Hom}_A(P,A) \in B$-proj. But this happens exactly when the Schur functor preserves projectives, hence as it is full and faithful it becomes an equivalence.

It may be tempting to think that $\eta_{M(\mathfrak{m})}$ being an isomorphism is a sufficient condition in view of Nakayama's Lemma to η_M being an isomorphism. However, the argument does not work since $R(\mathfrak{m}) \otimes_R -$ is only right exact when R is a local ring. As a consequence, $\ker \eta_M(\mathfrak{m})$ cannot be viewed as a submodule of $\ker \eta_{M(\mathfrak{m})}$. An example of this failure is actually the canonical projection $\pi\colon R \to R(\mathfrak{m})$. This is not an isomorphism over local rings with Krull dimension higher than zero. However, $\pi(\mathfrak{m})$ is an isomorphism. This justifies the need of Lemma 1.4.31 to conclude the previous Proposition.

As a curiosity, Auslander and Smalø introduced in [AS80] a concept of cover of a subcategory of a module category. If (A,P) is a cover of B (in the sense of Definition 1.4.29), then the full subcategory of B-mod whose objects are the indecomposable summands of $\mathrm{Hom}_A(P,A)$ is a cover of B-mod, in the sense of Auslander and Smalø.

1.4.6 Blocks and covers

Our next goal is to decompose a cover into smaller covers. We recall that a **block** of an algebra is a principal ideal generated by a centrally primitive idempotent. So, each block is an indecomposable ring. First, we will see that the existence of a double centralizer property on an (A,B)-bimodule implies that the number of blocks of A is equal to the number of blocks of B. This proof is based on Corollary 5.38 of [Mat99].

Proposition 1.4.35. *Let A and B be two projective Noetherian R-algebras. Let M be an (A,B)-bimodule. Suppose that there is a double centralizer property on M between A and B. Then, the number of blocks of A is equal to the number of blocks of B.*

Proof. Assume that $B = \prod_{i=1}^k B_i$ is a decomposition of B into block ideals. This gives a decomposition of the identity 1_B into central idempotents such that $B_i = Be_i$ and $e_ie_j = 0$ if $i \neq j$, $i = 1,\dots,k$. Since $Me_i \cap \sum_{j\neq i} Me_j = 0$, $i = 1,\dots,k$, we can write $M \simeq \bigoplus_{i=1}^k Me_i$ as B-modules. By assumption, $B \simeq \mathrm{End}_A(M)^{op}$ as rings. In particular, M is faithful over B. Thus, $Me_i \neq 0$ for all $i = 1,\dots k$. On the other hand, $\mathrm{Hom}_B(Me_i, Me_j) = 0$ if $i \neq j$. Hence,

$$A \simeq \mathrm{End}_B(M) \simeq \mathrm{End}_B(Me_1 \bigoplus \cdots \bigoplus Me_k) \simeq \prod_{i=1}^k \mathrm{End}_{B_i}(Me_i). \tag{1.4.6.1}$$

Each ideal of A, $\mathrm{End}_{B_i}(Me_i)$ is non-zero since $Me_i \neq 0$. So, A can be decomposed as a direct product in at least k ideals. Symmetrically, using the fact that M is faithful over A and $B = \mathrm{End}_A(M)^{op}$ we obtain that the B can be decomposed as a direct product in at least the number of blocks of A. Therefore, the number of blocks of A and B coincide. $\square$

Corollary 1.4.36. *Let (A,P) be a cover of B. Then, the number of blocks of A is equal to the number of blocks of B.*

Proof. By assumption, there is a double centralizer property on $\mathrm{Hom}_A(P,A)$ between B and A. The result now follows by Proposition 1.4.35. $\square$

Proposition 1.4.37. *Let A and B be two projective Noetherian R-algebras such that $B = \operatorname{End}_A(P)^{op}$, for some $P \in A$-proj. Suppose that A admits a decomposition $A = \prod_{i=1}^k A_i$. Then,*

(i) B admits a decomposition $B = \prod_{i=1}^k \operatorname{End}_{A_i}(P_i)$, where $P_i = A_iP$, $i = 1, \dots, k$.

(ii) (A,P) is a cover of B if and only if (A_i,P_i) is a cover of B_i, $i = 1, \dots, k$.

Proof. (i) follows from Proposition 1.4.35. In particular, $1_B = \sum_i b_i$ where each b_i is the idempotent $P \twoheadrightarrow P_i \hookrightarrow P$. We can write

$$\operatorname{End}_B(\operatorname{Hom}_A(P,A)) \simeq \operatorname{End}_B(\operatorname{Hom}_A(\bigoplus_i P_i, A)) \simeq \operatorname{End}_B(\bigoplus_{i,j} \operatorname{Hom}_A(P_i, A_j)) \tag{1.4.6.2}$$

$$\simeq \operatorname{End}_B(\bigoplus_i \operatorname{Hom}_A(P_i,A_i)) \simeq \operatorname{End}_B(\bigoplus_i \operatorname{Hom}_{A_i}(P_i,A_i)) \tag{1.4.6.3}$$

$$\simeq \operatorname{Hom}_B(\bigoplus_i \operatorname{Hom}_{A_i}(P_i,A_i), \bigoplus_j \operatorname{Hom}_{A_j}(P_j,A_j)) \simeq \operatorname{Hom}_B(\bigoplus_i \operatorname{Hom}_{A_i}(P_i,A_i), \operatorname{Hom}_{A_i}(P_i,A_i))$$

$$\simeq \prod_i \operatorname{End}_B(\operatorname{Hom}_{A_i}(P_i,A_i)) \simeq \prod_i \operatorname{End}_{B_i}(\operatorname{Hom}_{A_i}(P_i,A_i)). \tag{1.4.6.4}$$

Assume that (A_i,P_i) is a cover of B_i for every i. Then,

$$\operatorname{End}_B(\operatorname{Hom}_A(P,A)) \simeq \prod_i \operatorname{End}_{B_i}(\operatorname{Hom}_{A_i}(P_i,A_i)) \simeq \prod_i A_i \simeq A. \tag{1.4.6.5}$$

Hence, (A,P) is a cover of B.

Conversely, assume that (A,P) is a cover of B. Then,

$$\prod_i A_i \simeq A \simeq \operatorname{End}_B(\operatorname{Hom}_A(P,A)) \simeq \prod_i \operatorname{End}_{B_i}(\operatorname{Hom}_{A_i}(P_i,A_i)). \tag{1.4.6.6}$$

By assumption, $\operatorname{Hom}_A(P,A)$ is faithful as right A-module. Observe that $\operatorname{Hom}_A(P,A) \simeq \bigoplus_j \operatorname{Hom}_{A_j}(P_j,A_j)$. We note that $\operatorname{Hom}_{A_i}(P_i,A_i)$ is faithful as A_i-module. In fact, since for each $a \in A_j$ and each $a_i \in A_i$, $j \neq i$, $a_ia = 0$, it follows that $\phi \cdot a = 0$ for every $\phi \in \operatorname{Hom}_{A_i}(P_i,A_i)$. In particular, the canonical map $A_i \to \operatorname{End}_{B_i}(\operatorname{Hom}_{A_i}(P_i,A_i))$ is injective. Denote by f_i this map. By (1.4.6.6), $\prod f_i$ is an isomorphism. As f_i is also an R-map, it is enough to check that f_i is an R-epimorphism. Let $\psi \in \operatorname{Hom}_R(\operatorname{End}_{B_i}(\operatorname{Hom}_{A_i}(P_i,A_i)),X)$ such that $\psi f_i = 0$ for some $X \in R$-mod. We can extend ψ to a map $\psi' \in \operatorname{Hom}_R(\prod_i \operatorname{End}_{B_i}(\operatorname{Hom}_{A_i}(P_i,A_i)),X)$ such that $\psi' f_j = 0$, $j \neq i$ and $\psi' f_i = \psi f_i = 0$. So,

$$\psi \circ \prod_j f_j = \psi f_i = 0 \implies \psi' = 0. \tag{1.4.6.7}$$

Hence, $\psi = 0$. Thus, f_i is also surjective, and therefore (A_i,P_i) is a cover of B_i. ☐

1.5 Split quasi-hereditary algebras

Quasi-hereditary algebras, introduced in [CPS88], play an important role in the representation theory of Lie algebras and algebraic groups. Quasi-hereditary algebras have very nice properties, in particular, homological properties like the finiteness of global dimension. An important class of modules for representation theory of algebraic groups is the additive closure of the characteristic tilting module which arise in the context of quasi-hereditary algebras. Using cover theory, properties of quasi-hereditary algebras can provide ways to study many interesting algebras. Here, we are interested in the abstract setting of algebras over commutative Noetherian rings, and therefore we would like to study the concept of quasi-hereditary algebra over a commutative Noetherian ring. During the last 30 years, many approaches have been suggested [DR98, Rou08, CPS90, Du03] to this subject. In this chapter, we will follow Rouquier's approach. We aim to explain some known results in the theory of split quasi-hereditary algebras over fields that can be generalized to arbitrary Noetherian rings including, naturally, the results of Rouquier and Cline, Parshall and Scott.

More precisely, our strategy can be described as follows:

Section 1.5.1 contains the relevant definitions of (resp. split) quasi-hereditary algebras and (resp. split) heredity ideals over commutative Noetherian rings and their differences relative to the field case. We gather here some properties presented in [CPS90] of (resp. split) quasi-hereditary algebras.

In Section 1.5.2, we explore the class of R-split modules for a given algebra A, introduced by Rouquier. This plays a crucial role in the theory of split highest weight categories over commutative Noetherian rings since they generalize the idea behind a standard module with maximal index. We describe how these objects behave under ground ring change and how we can associate to each split R-module a split heredity ideal. The converse statement requires knowledge of Picard groups. In fact, this construction is valid for any commutative Noetherian ring, so we cannot expect a one to one correspondence. However, a one to one correspondence can be deduced using equivalences classes of R-split modules with respect to the Picard group. Hence, elementary properties of the Picard group are also presented. Using such bijection, we prove that the module category of the quotient algebra A/J, with J a split heredity ideal in A, is a Serre subcategory of the module category of A.

In Section 1.5.3, the main goal is to discuss the formal definition of split highest weight category over commutative Noetherian rings and some immediate alternative definitions.

In Section 1.5.4, we start by showing the relation between A-proj and A/J-proj with J a split heredity ideal in A. This relation is crucial to establish that the category A-mod is split highest weight category with n standard modules if and only if there exists a split heredity ideal J in A such that A/J-mod is split highest weight category with $n-1$ standard modules. This result is fundamental for this theory since it allows us to prove several statements and constructions using induction. In contrast with the field case, the category $\mathscr{F}(\Delta)$ does not contain all projective A-modules. It is necessary to consider a larger subcategory denoted by $\mathscr{F}(\tilde{\Delta})$. The latter contains all modules with filtrations by $\Delta\otimes_R U$-modules where $U\in R$-proj.

In Section 1.5.5, we collect criteria to establish that a given module category is a split highest weight category using change of ground rings.

Over finite-dimensional algebras, the standard modules are completely determined, up to isomorphism, once the partial order whose elements index a complete set of projective indecomposable modules is fixed. In Section 1.5.6, we address the analogue problem replacing projective indecomposable modules by projective modules that become indecomposable after tensoring with every residue field.

In Section 1.5.7, we prove that the notion of split highest weight category over a commutative Noetherian ring (in the sense of Rouquier) is equivalent to the notion of split quasi-hereditary algebra (in sense of Cline, Parshall and Scott). Consequently, we show that an algebra is split quasi-hereditary if and only if its opposite algebra is split quasi-hereditary.

In Section 1.5.8, we give an alternative approach to the computation of global dimension of (resp. split) quasi-hereditary algebras over commutative Noetherian regular rings.

In Section 1.5.9, we can reduce the problem of determining whether two split highest weight categories are equivalent to determining if the respective full subcategories whose objects admit a filtration by standard modules are equivalent. We generalize the Dlab-Ringel standardization theorem to Noetherian algebras over regular rings with Krull dimension one. Therefore, our result can be applied to every abelian category with enough projectives and with a certain split standardizable set of objects. Moreover, these categories can be studied using integral split quasi-hereditary algebras.

In Section 1.5.10, we show that every split quasi-hereditary algebra over a local ring is semi-perfect. In particular, the projective modules associated with the standard modules are its projective covers. This situation gives further insight into the reason why the local case of split quasi-hereditary algebras can be approached in several ways. In [DR98], Du and Rui work with standardly full-based algebras. In the local case, they show that standardly full-based algebras over a local commutative Noetherian ring are exactly the split quasi-hereditary algebras over a local ring.

In Section 1.5.13, we construct the dual objects of standard modules called costandard modules. As expected this provides a new characterization of split quasi-hereditary algebras over commutative Noetherian rings in terms of costandard modules. Here, (A,R)-injective modules and relative (A,R)-cogenerators take the place of projective modules and generators, respectively. Rouquier established for the Noetherian case that the standard modules are the Ext-projetive objects of the costandard modules in A-mod$\cap R$-proj and the projective A-modules are the Ext-projective objects of standard modules in $\mathscr{F}(\tilde{\Delta})$. We present detailed proofs of these facts and their dual statements. This characterization allows us to state that $\mathscr{F}(\tilde{\Delta})$ is a resolving subcategory of A-mod$\cap R$-proj.

In Section 1.5.14, we study partial tilting modules. In general, we cannot construct canonical indecomposable (partial) tilting modules but we can still find exact sequences for each $\lambda \in \Lambda$ that relate them with the standards $\Delta(\lambda)$ and the costandards $\Delta(\lambda)$. Furthermore, these exact sequences are (resp. $\mathscr{F}(\tilde{\Delta})$) $\mathscr{F}(\tilde{\nabla})$-approximations of (resp. $\nabla(\lambda)$) $\Delta(\lambda)$. However, these partial tilting modules are indecomposable if R is a connected ring. We will describe the modules in $\mathscr{F}(\tilde{\Delta})$ as the modules with a finite coresolution by partial tilting modules and its dual statement. These statements are crucial to the study of the Ringel dual. In this section, we also find some additional properties to $\mathrm{Hom}_A(M,N)$ with $M \in \mathscr{F}(\tilde{\Delta}), N \in \mathscr{F}(\tilde{\nabla})$. In particular, homomorphisms between partial tilting modules.

In Section 1.5.15, we define the Ringel dual of a split quasi-hereditary algebra and we deduce its uniqueness. Characteristic tilting modules are not unique, however, their endomorphism algebras are Morita equivalent. This is done in several steps. First, we show that the functor $\mathrm{Hom}_A(T,-)$, where T is a characteristic tilting module, induces an exact equivalence between $\mathscr{F}(\nabla)$ and $\mathscr{F}(\mathrm{Hom}_A(T,\nabla))$. Then, we prove that $\mathrm{End}_A(T)$ has a split quasi-hereditary structure and $\mathrm{Hom}_A(T,-)$ sends costandard modules to standard modules. Afterwards, we establish that the partial tilting modules are exactly the additive closure of a characteristic tilting module. We describe how partial tilting modules behave under change of ring. This allows us to generalize the statement that the Ringel dual of an algebra A is Morita equivalent as quasi-hereditary algebra to A for Noetherian rings. Moreover, we can say that two algebras are Ringel dual to each other if there exists an exact equivalence between $\mathscr{F}({}_A\tilde{\Delta})$ and $\mathscr{F}({}_B\tilde{\nabla})$.

In Section 1.5.16, we show that $\mathscr{F}(\tilde{\Delta})$ behave similarly to A-proj in the sense of being a well behaved resolving subcategory, which we will describe later on Definition 3.3.1. This is a known result by Rouquier, however here we present a different approach. We give a criterion for Ringel self-duality for split quasi-hereditary algebras over local commutative Noetherian rings. We also describe when $\mathscr{F}(\tilde{\Delta})$ can be closed under (A,R)-monomorphisms.

1.5.1 Quasi-hereditary algebras and split quasi-hereditary algebras

For the study of quasi-hereditary algebras over fields, we refer to [CPS88], [PS88], [DR89b], [Rin91], [DK94, A], [DR92], [DR89a].

Assume, throughout this section, that R is a commutative Noetherian ring and A is a projective Noetherian

R-algebra.

Definition 1.5.1. Let R be a commutative Noetherian ring. An ideal J in a projective Noetherian R-algebra A is called a **heredity ideal** if

(i) A/J is projective over R;

(ii) J is projective as left ideal over A;

(iii) $J^2 = J$ (idempotent ideal);

(iv) The R-algebra $\mathrm{End}_A({}_AJ)^{op}$ is semi-simple relative to R.

This definition is due to [CPS90]. For our purposes, we are interested in a stronger notion of heredity ideal also used by Rouquier [Rou08].

Definition 1.5.2. Let R be a commutative Noetherian ring and let A be a projective Noetherian R-algebra. Let J be an ideal of A. We call J a **split heredity ideal** of A if

(i) A/J is projective over R;

(ii) J is projective as left ideal over A;

(iii) $J^2 = J$;

(iv) The R-algebra $\mathrm{End}_A({}_AJ)^{op}$ is Morita equivalent to R.

Since semi-simple relative is a Morita invariant property and R is, of course, semi-simple relative to R, it follows that any R-algebra Morita equivalent to R is semi-simple relative to R. In particular, a split heredity ideal is heredity.

Definition 1.5.3. A projective Noetherian R-algebra A is called **quasi-hereditary** if there exists a finite heredity chain of ideals $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$ such that J_i/J_{i+1} is a heredity ideal in A/J_{i+1} for $1 \leq i \leq t$. It is called **split quasi-hereditary** provided that J_i/J_{i+1} is split heredity in A/J_{i+1}.

It follows that a split quasi-hereditary algebra is quasi-hereditary since a split heredity ideal is heredity. Further, for each split quasi-hereditary algebra the regular module A is a faithful module as R-module.

Proposition 1.5.4. *Let R be a field. Then, quasi-hereditary corresponds to the classical concept of quasi-hereditary.*

Proof. Assume that J is an heredity ideal. Since R is a field, $1.5.1(i)$ is trivially checked. $1.5.1(ii)$ and (iii) are conditions on the usual concept of heredity ideal. Since R is a field, there is an idempotent e of A such that $J = AeA$. Since AeA is projective over A, we get $\mathrm{End}_A(AeA)^{op}$ is Morita equivalent to $\mathrm{End}_A(Ae)^{op} \simeq eAe$. Hence, by $1.5.1(iv)$ eAe is semi-simple. This is equivalent to $0 = \mathrm{rad}(eAe) = e\,\mathrm{rad}(A)e$. ☐

Proposition 1.5.5. *Let R be a field splitting for A. Then, split quasi-hereditary corresponds to the classical concept of quasi-hereditary.*

Proof. It is now enough to notice that $R \overset{\text{Mor}}{\sim} \mathrm{End}_A(J)^{op} \simeq \mathrm{End}_A(AeA)^{op} \overset{\text{Mor}}{\sim} \mathrm{End}_A(Ae)^{op} \simeq eAe$. On the other direction, we can assume $J = AeA$ for some primitive idempotent e. Hence, eAe is local algebra, thus $eAe/\mathrm{rad}(eAe) \simeq R$, which follows from the fact that R is a field splitting for A (see for example [ASS06, Lemma 4.6]). Now using the fact that eAe is semi-simple we deduce that $\mathrm{End}_A(J)^{op} \overset{\text{Mor}}{\sim} eAe \simeq R$. ☐

Proposition 1.5.6. *Let A be quasi-hereditary. Then, A/J is quasi-hereditary for J an heredity ideal of A.*

Proof. Assume that A has a heredity chain $0 = J_{t+1} \subset J_t \subset \cdots J_1 = A$. Consider $J = J_t$. The chain of ideals $0 = J_t/J \subset J_{t-1}/J \subset J_1/J = A/J$ in A/J is heredity. In fact, $J_i/J/J_{i+1}/J \simeq J_i/J_{i+1}$ in $A/J/J_{i+1}/J \simeq A/J_{i+1}$. As J_i/J_{i+1} is heredity in A/J_{i+1} and $\mathrm{End}_{A/J/J_{i+1}/J}(J_i/J/J_{i+1}/J) \simeq \mathrm{End}_{A/J_{i+1}}(J_i/J_{i+1})$ the claim follows. □

Proposition 1.5.7. *Let A be split quasi-hereditary. Then, A/J is split quasi-hereditary for J a split heredity ideal of A.*

Proof. The result follows by the same reasoning of Proposition 1.5.6. □

Proposition 1.5.8. *Let A be an R-algebra with J a (resp. split) heredity ideal of A. Assume that A/J is (resp. split) quasi-hereditary. Then, A is (resp. split) quasi-hereditary.*

Proof. By assumption, $0 = I_t \subset I_{t-1} \subset \cdots \subset I_1 = A/J$ is a (resp. split) heredity chain. Now each ideal in A/J can be written as $J_i/J = I_i$, $t \leq i \leq 1$ by the correspondence theorem for quotient rings. Here, $J_i/J_{i+1} \simeq J_i/J/J_{i+1}/J \simeq I_i/I_{i+1}$ as A-modules and $\mathrm{End}_{A/J/J_{i+1}/J}(J_i/J/J_{i+1}/J) \simeq \mathrm{End}_{A/J_{i+1}}(J_i/J_{i+1})$. Therefore, J_i/J_{i+1} is (resp. split) heredity in A/J_{i+1}. So, $0 \subset J \subset J_{t-1} \subset \cdots \subset J_1 = A$ is a (resp. split) heredity chain. □

Cline, Parshall and Scott stated that an idempotent chain of ideals is a heredity chain if it is over every residue field of prime ideals.

Theorem 1.5.9. *[CPS90, Theorem 3.3]Let R be a Noetherian commutative ring. Let A be a projective Noetherian R-algebra. Assume that A admits a chain of idempotent ideals $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$. The algebra A is quasi-hereditary with heredity chain $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$ if and only if for each prime ideal $\mathfrak{p}$ of R, $A(\mathfrak{p})$ is quasi-hereditary $R(\mathfrak{p})$-algebra with heredity chain $0 = J_{t+1}(\mathfrak{p}) \subset J_t(\mathfrak{p}) \subset \cdots \subset J_1(\mathfrak{p}) = A(\mathfrak{p})$.*

As we said, our interest lies in split quasi-hereditary algebras, and therefore we will not use this result in future references. Notice that we cannot deduce for now the same for split quasi-hereditary because of condition 1.5.2(iv). Notice that even if two objects are isomorphic at every localization they are not in general isomorphic. It is needed that they are isomorphic at each localization via a global map.

As a consequence of the previous theorem, we can deduce a known result for classical quasi-hereditary algebras. If an algebra is quasi-hereditary, then its opposite algebra is again quasi-hereditary.

Corollary 1.5.10. *[CPS90, Proposition 3.5] A projective Noetherian R-algebra is quasi-hereditary if and only if A^{op} is quasi-hereditary.*

Proof. Assume A quasi-hereditary. By 1.5.9, $A(\mathfrak{p})$ is quasi-hereditary for every prime ideal $\mathfrak{p}$ of R. Since $R(\mathfrak{p})$ is a field, $A(\mathfrak{p})^{op} \simeq A^{op}(\mathfrak{p})$ is quasi-hereditary for every prime ideal $\mathfrak{p}$ of R [PS88, Theorem 4.3 b)]. Again by 1.5.9, A^{op} is quasi-hereditary. □

Later, we will see that the same result holds for the split case, however, some adjustments are necessary (see Theorem 1.5.69).

The following result established a criterion to verify when a quasi-hereditary algebra is a split quasi-hereditary algebra.

Corollary 1.5.11. *[CPS90, Proposition 3.5] Let R be a regular Noetherian integral domain with quotient field K. Let A be a quasi-hereditary algebra. Then, A is split quasi-hereditary algebra if and only if $K \otimes_R A$ is a split quasi-hereditary algebra.*

The proof of Corollary 1.5.11 uses the theory of maximal orders over Krull Noetherian domains (see [AG60]). Hence, it also follows as an application of Theorem 4.6 of [Hat63]. In particular, this result can be established independently of [CPS90, Theorem (2.1)].

The examples of quasi-hereditary algebras, that we are interested in, are, in fact, split quasi-hereditary. Also, over an algebraically closed field, every quasi-hereditary algebra is split quasi-hereditary. Hence, from now on we will focus only on the study of split quasi-hereditary algebras.

1.5.2 Projective R-split A-modules

In the Artinian case, the notions of quasi-hereditary algebras and highest weight categories are equivalent. Moreover, all heredity ideals can be written in the form of AeA. And without loss of generality, we can deal only with the cases where the idempotent e is primitive. Hence, there is a natural choice for the respective standard module. In such a case, we choose Ae. By [DR89b, Statement 7], if AeA is heredity there is a more precise relation between Ae and AeA. In fact, in such a case, the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism of R-modules with R a field.

In the Noetherian case, this is our starting point for the equivalence of these two notions as well. The first problem we encounter is that projective modules cannot be decomposed into projective modules defined by idempotents. And therefore, the definition of heredity ideals in the form AeA is no longer suitable, and so neither the choice of standard Ae. Observe that in the case that A is a split finite-dimensional algebra over a field K and AeA is heredity then $eAe \simeq K$ for some primitive idempotent $e \in A$. Another important thing to observe is that we want to have A/AeA to be a projective Noetherian R-algebra. This is the same to require that the canonical exact sequence

$$0 \to AeA \to A \to A/AeA \to 0 \tag{1.5.2.1}$$

splits over R. Combining this data with the multiplication map we obtain that the canonical morphism

$$Ae \otimes_R eA \to AeA \to A \tag{1.5.2.2}$$

splits over R with image AeA. This motivates the following definition.

Definition 1.5.12. Let R be a commutative Noetherian ring and let A be a projective Noetherian R-algebra. Let L be a finitely generated projective A-module and faithful over R. L is called projective ***R*-split** A-module if the canonical R-morphism

$$\tau_{L,P} \colon L \otimes_R \operatorname{Hom}_A(L,P) \to P, \quad l \otimes f \mapsto f(l) \tag{1.5.2.3}$$

is a split R-monomorphism for all projective A-modules.

We denote $\mathscr{M}(A)$ the set of isomorphism classes of projective R-split A-modules.

For $P = A$ we can consider the map $\tau_L \colon L \otimes_{\operatorname{End}_A(L)^{op}} \operatorname{Hom}_A(L,P) \to P, \quad l \otimes f \mapsto f(l)$

We will see in the coming sections that the modules (not necessarily indecomposable) in $\mathscr{M}(A)$ are exactly the standard modules with maximal index when A is split quasi-hereditary. Whereas the image of the map $\tau_{L,A}$ is a split heredity ideal for $L \in \mathscr{M}(A)$. For rings R with non-trivial idempotents, we will be able to say that the modules in $\mathscr{M}(A)$ are projective indecomposable.

Lemma 1.5.13. *[Rou08, Lemma 4.3] Let L be a finitely generated projective A-module. Then,* τ_L *is an* (A,A)*-bimodule morphism.* $J = \operatorname{im} \tau_L$ *is an ideal of* A *and* $J^2 = J$.

Proof. Notice that $\operatorname{Hom}_A(L,A)$ is a right A-module with action $(f\cdot a)(l)=f(l)a, f\in \operatorname{Hom}_A(L,A), a\in A, l\in L$. So,

$$\tau_L(a\cdot l\otimes f)=\tau_L((al)\otimes f)=f(al)=af(l)=a\tau_L(l\otimes f)$$
$$\tau_L(l\otimes fa)=(fa)(l)=f(l)a=\tau_L(l\otimes f)a.$$

Thus, the first claim follows. Moreover $a\tau_L(x)=\tau_L(ax)\in \operatorname{im}\tau_L$ and $\tau_L(x)a\in\operatorname{im}\tau_L$, thus $J=\operatorname{im}\tau_L$ is an ideal. Fix $L^*=\operatorname{Hom}_A(L,A)$ and $E=\operatorname{End}_A(L)^{op}$. The map $\psi\colon L^*\otimes_A L\to E,\ f\otimes l'\mapsto (l\mapsto f(l)l')$ is an (E,E)-bimodule morphism. In fact, for any $h\in E,\ f\in L^*, l', l\in L$,

$$\psi(h(f\otimes l'))(l)=\psi(hf\otimes l')(l)=(hf)(l)l'=f(h(l))l'=\psi(f\otimes l')(h(l))=h\psi(f\otimes l')(l)$$
$$\psi(f\otimes l'h)(l)=\psi(f\otimes(l'h))=\psi(f\otimes h(l'))(l)=f(l)h(l')=h(f(l)l')=(f(l)l')h=(\psi(f\otimes l')h)(l).$$

Since L is projective over A, ψ is an isomorphism. Define the map $\delta\colon L\otimes_E E\otimes_E L^*\to A$, $L\otimes_E E\otimes_E L^*\ni l\otimes\phi\otimes f'\mapsto f'(\phi(l))$. The image of δ is exactly J. In fact,

$$\delta(l\otimes \mathrm{id}_L\otimes f')=f'(\mathrm{id}_L(l))=\tau_L(l\otimes f'),\ \forall l\in L, f'\in L^*. \tag{1.5.2.4}$$

Hence, $J\subset\operatorname{im}\delta$. As for any $\phi\in E$, $\delta(l\otimes\phi\otimes f')=\delta(\phi(l)\otimes\mathrm{id}_L\otimes f'), \forall l\in L, f'\in L^*$, $J\supset\operatorname{im}\delta$.

Consider the diagram

$$\begin{tikzcd} L\otimes_E L^*\otimes_A L\otimes_E L^* \arrow[r, "\mathrm{id}_L\otimes\psi\otimes\mathrm{id}_{L^*}"] \arrow[d, "\tau_L\otimes\tau_L"] & L\otimes_E E\otimes_E L^* \arrow[d, "\delta"] \\ A\otimes_A A \arrow[r, "\mu"] & A \end{tikzcd}.$$

This diagram is commutative. In fact, for $l\otimes f\otimes l'\otimes f'\in L\otimes_E L^*\otimes_A L\otimes_E L^*$,

$$\delta\circ\mathrm{id}_L\otimes\psi\otimes\mathrm{id}_{L^*}(l\otimes f\otimes l'\otimes f')=\delta(l\otimes\psi(f\otimes l')\otimes f')=f'(\psi(f\otimes l')(l))=f'(f(l)l')=f(l)f(l')$$
$$\mu\circ\tau_L\otimes\tau_L(l\otimes f\otimes l'\otimes f')=\mu(f(l)\otimes f'(l'))=f(l)f'(l').$$

Since ψ is iso as (E,E)-morphism, μ and $\mathrm{id}_L\otimes\psi\otimes\mathrm{id}_{L^*}$ are isomorphisms. Hence, it follows by the commutativity of the diagram that $J=J^2$. ☐

Lemma 1.5.14. *[Rou08, Lemma 4.3] Let J be an ideal of A. Assume that $J^2=J$. Let M be an A-module. Then* $\operatorname{Hom}_A(J,M)=0$ *if and only if $JM=0$ if and only if $M\in A/J$*-mod.

Proof. Assume that $\operatorname{Hom}_A(J,M)=0$. Consider $m\in M$. We can define the A-homomorphism $f\colon J\to M$, with $f(j)=jm,\ j\in J$. By assumption $f=0$, hence $JM=0$.

Reciprocally, assume $JM=0$. Let $g\in\operatorname{Hom}_A(J,M)$. For any $j\in J$, there exists j_1, j_2 such that $j=j_1j_2$, hence $g(j)=g(j_1j_2)=j_1g(j_2)\in JM=0$. Hence, $g=0$. ☐

Note that the condition $J=J^2$ is fundamental. In fact, assume that if $JM=0$, then $\operatorname{Hom}_A(J,M)=0$. Then, consider $M:=J/J^2$. We have $\operatorname{Hom}_A(J,J/J^2)=0$. In particular, the canonical epimorphism $J\twoheadrightarrow J/J^2$ is zero. Hence, $J=J^2$.

Proposition 1.5.15. *[Rou08, Lemma 4.5] Let L be a finitely generated projective A-module which is a faithful R-module. The following are equivalent:*

(i) $\tau_{L,P}\colon L\otimes_R\operatorname{Hom}_A(L,P)\to P$ *is an (A,R)-monomorphism for all $P\in A$*-proj.

(ii) $\tau_{L,A}\colon L\otimes_R \operatorname{Hom}_A(L,A)\to A$ *is an* (A,R)*-monomorphism.*

(iii) $R\simeq \operatorname{End}_A(L)$ *and given* $P\in A$-proj*, there is a submodule* P_0 *of* P *such that*

- $P/P_0\in R$-proj*,*
- $\operatorname{Hom}_A(L,P/P_0)=0$*,*
- $P_0\simeq L\otimes_R U$ *for some* R*-progenerator* U*.*

Proof. $i) \implies ii)$. Clear since $A\in A$-proj.

$ii) \implies i)$. Notice that for $P=P_1\oplus P_2$,

$$L\otimes_R \operatorname{Hom}_A(L,P)\simeq L\otimes_R(\operatorname{Hom}_A(L,P_1)\oplus\operatorname{Hom}_A(L,P_2))\simeq (L\otimes_R\operatorname{Hom}_A(L,P_1))\oplus L\otimes_R\operatorname{Hom}_A(L,P_2),$$

hence $\tau_{L,P_1\oplus P_2}$ is equivalent to $\tau_{L,P_1}\oplus\tau_{L,P_2}$. So, it follows that $\tau_{L,P}$ is an (A,R)-monomorphism for any $P\in A$-proj.

$i) \implies iii)$. By $i)$, $\tau_{L,L}$ is an (A,R)-monomorphism. Putting $f=\mathrm{id}_L$, we see that $\tau_{L,L}(l\otimes f)=l, l\in L$. Hence, it is an R-isomorphism. Since $L\in A$-proj, it follows that $L\in R$-proj. As R is commutative and L is faithful, L is an R-progenerator [Fai73, Proposition 12.2]. Define $B=\operatorname{End}_R(L)^{op}$. Then, $F=L\otimes_R -\colon R\text{-mod}\to B\text{-mod}$ is an equivalence of categories with adjoint $G=\operatorname{Hom}_B(L,-)\colon B\text{-mod}\to R\text{-mod}$. Notice that $F\operatorname{End}_A(L)=L\otimes_R\operatorname{End}_A(L)\simeq L$. Furthermore, $\operatorname{End}_A(L)\simeq GF\operatorname{End}_A(L)\simeq GL=\operatorname{End}_B(L)\simeq R$, since the double centralizer property holds on generators.

Let P be a finitely generated projective A-module. Define $P_0=\operatorname{im}\tau_{L,P}$. As $\tau_{L,P}$ is an (A,R)-monomorphism we obtain that P_0 is an R-summand of P. Moreover P/P_0 is an R-summand of P, hence it is projective over R. Since $L\in A$-proj, we have the exact sequence

$$0\to\operatorname{Hom}_A(L,P_0)\to\operatorname{Hom}_A(L,P)\to\operatorname{Hom}_A(L,P/P_0)\to 0. \tag{1.5.2.5}$$

However the canonical map $\operatorname{Hom}_A(L,P_0)\to\operatorname{Hom}_A(L,P)$ is surjective: In fact, for each $h\in\operatorname{Hom}_A(L,P)$, define $g\in\operatorname{Hom}_A(L,P_0)$ such that $g(l)=l\otimes h$. Hence, $\tau_{L,P}\circ g=h$. Therefore, by the exactness of (1.5.2.5), $\operatorname{Hom}_A(L,P/P_0)=0$. Since L is faithful over R, it follows that $U=\operatorname{Hom}_A(L,P)$ is faithful over R, and $P_0\simeq L\otimes U$.

$iii) \implies i)$ Let $P\in A$-proj. Consider the exact sequence $0\to P_0\to P\to P/P_0$. Applying $\operatorname{Hom}_A(L,-)$ we obtain the exact sequence $0\to\operatorname{Hom}_A(L,P_0)\to\operatorname{Hom}_A(L,P)\to\operatorname{Hom}_A(L,P/P_0)=0\to 0$. Therefore, the map $\operatorname{Hom}_A(L,P_0)\to\operatorname{Hom}_A(L,P)$ is an isomorphism.

We the following diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(L,P_0) \arrow[r, "w"] & \operatorname{Hom}_A(L,L\otimes_R U) \arrow[r, "z"] & \operatorname{Hom}_A(L,L)\otimes_R U \arrow[d, "\simeq"] \\
 & & R\otimes_R U \arrow[d, "\simeq"] \\
\operatorname{Hom}_A(L,P) \arrow[uu, "\simeq"] \arrow[rr] & & U
\end{tikzcd}.$$

Here, w is an isomorphism since $P_0\simeq L\otimes_R U$ by assumption, and z is an isomorphism since the map $\operatorname{Hom}_A(Q,L)\otimes_R U\to\operatorname{Hom}_A(Q,L\otimes U)$ is an isomorphism for any $Q\in A$-proj.

Therefore,

$$\begin{tikzcd}
L\otimes_R\operatorname{Hom}_A(L,P) \arrow[r, "\tau_{L,P}"] \arrow[d, "\simeq"] & P \\
L\otimes_R U \arrow[r, "\simeq"] & P_0 \arrow[u]
\end{tikzcd}.$$

Now since P_0 is a summand of P, we get that $\tau_{L,P}$ is an (A,R)-monomorphism. $\square$

Remark 1.5.16. Notice that for any $L \in \mathscr{M}(A)$, $R \simeq \operatorname{End}_A(L)$. Hence, $\tau_{L,A} = \tau_L$. Furthermore, $\operatorname{End}_R(L) \simeq \operatorname{Hom}_A(\operatorname{im}\tau_L, A)$. $\triangle$

In fact,

$$\begin{aligned}\operatorname{Hom}_{A^{op}}(\operatorname{im}\tau_L, A) &\simeq \operatorname{Hom}_{A^{op}}(L\otimes_R \operatorname{Hom}_A(L,A),A)\\ &\simeq \operatorname{Hom}_R(L, \operatorname{Hom}_{A^{op}}(\operatorname{Hom}_A(L,A),A)) \simeq \operatorname{End}_R(L).\end{aligned}$$

We can also use the relative projective modules to determine if a given projective module is R-split.

Lemma 1.5.17. *Let $L \in A$-proj which is a faithful R-module. The following are equivalent:*

(i) $\tau_{L,A}\colon L\otimes_R \operatorname{Hom}_A(L,A) \to A$ *is an (A,R)-monomorphism.*

(ii) $\tau_{L,M}\colon L\otimes_R \operatorname{Hom}_A(L,M) \to M$ *is an (A,R)-monomorphism for every (A,R)-projective module M.*

Proof. The implication $(ii) \implies (i)$ is clear since A is (A,R)-projective. Assume that (i) holds. Let M be an (A,R)-projective module. Since $\tau_{L,X_1\oplus X_2}$ is equivalent to $\tau_{L,X_1}\oplus\tau_{L,X_2}$ for every $X_1, X_2 \in A$-mod we can assume that $M = A\otimes_R X$ for some $X \in R$-mod. There is a commutative diagram

$$\begin{tikzcd} L\otimes_R \operatorname{Hom}_A(L,A)\otimes_R X \arrow[r, "\tau_{L,A}\otimes_R X"] \arrow[d, "L\otimes_R \varsigma_{L,A,X}"] & A\otimes_R X \arrow[d, equal] \\ L\otimes_R \operatorname{Hom}_A(L, A\otimes_R X) \arrow[r, "\tau_{L,A\otimes_R X}"] & A\otimes_R X \end{tikzcd} . \tag{1.5.2.6}$$

In fact, following the notation of Proposition 1.1.33, for every $l\in L$, $g\in \operatorname{Hom}_A(L,A)$, $x\in X$,

$$\tau_{L,A\otimes_R X}\circ L\otimes_R \varsigma_{L,A,X}(l\otimes g\otimes x) = \tau_{L,A\otimes_R X}(l\otimes g(-)\otimes x) = g(l)\otimes x = \tau_A\otimes_R X(l\otimes g\otimes x). \tag{1.5.2.7}$$

By assumption, there exists an R-map $t\colon A\to L\otimes_R \operatorname{Hom}_A(L,A)$ satisfying $t\circ\tau_{L,A} = \operatorname{id}_{L\otimes_R \operatorname{Hom}_A(L,A)}$. It follows that $L\otimes_R \varsigma_{L,A,X}\circ t\otimes_R X\circ \tau_{L,A\otimes_R X} = \operatorname{id}_{L\otimes_R \operatorname{Hom}_A(L,A\otimes_R X)}$. $\square$

We can observe that the projective R-split left A-modules determine the projective R-split right A-modules.

Lemma 1.5.18. *If $L\in\mathscr{M}(A)$, then $\operatorname{Hom}_A(L,A)\in\mathscr{M}(A^{op})$.*

Proof. Since $L\in A$-proj, $\operatorname{Hom}_A(L,A)$ is projective as right A-module and $\operatorname{End}_A(\operatorname{Hom}_A(L,A))\simeq\operatorname{End}_A(L)\simeq R$. Further, $\operatorname{Hom}_A(\operatorname{Hom}_A(L,A),A)\simeq L$ as A-modules and the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Hom}_A(\operatorname{Hom}_A(L,A),A)\otimes_R \operatorname{Hom}_A(L,A) \arrow[r, "\tau_{\operatorname{Hom}_A(L,A),A}"] & A \\ L\otimes_R \operatorname{Hom}_A(L,A) \arrow[u, "\simeq"] \arrow[r, "\tau_{L,A}"] & A \arrow[u, equal] \end{tikzcd} . \qquad \square$$

In the following proposition, we determine when a projective R-split A-module is indecomposable.

Proposition 1.5.19. *Assume that R has no non-trivial idempotents. Then, all modules in $\mathscr{M}(A)$ are projective indecomposable A-modules.*

Proof. Let $L\in\mathscr{M}(A)$. By definition, L is projective over A. Assume that L is decomposable, $L\simeq X_1\bigoplus X_2$. Then, we have a non-trivial idempotent $L\twoheadrightarrow X_1\hookrightarrow L$ in $\operatorname{End}_A(L)\simeq R$. So, L must be indecomposable as A-module. $\square$

The following lemma shows that $\mathscr{M}(A)$ behaves well with respect to ground ring change.

Lemma 1.5.20. *[Rou08, Proof of Lemma 4.10] Let L be a finitely generated A-module. Let S be a commutative Noetherian R-algebra. If $L \in \mathscr{M}(A)$, then $S \otimes_R L \in \mathscr{M}(S \otimes_R A)$. Moreover, the following are equivalent:*

(i) $L \in \mathscr{M}(A)$;

(ii) The localization $L_\mathfrak{m} = R_\mathfrak{m} \otimes_R L \in \mathscr{M}(A_\mathfrak{m})$ *for every maximal ideal* $\mathfrak{m}$ *of* R;

(iii) L *is projective over* R *and* $L(\mathfrak{m}) \in \mathscr{M}(A(\mathfrak{m}))$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

Proof. Since $L \in \mathscr{M}(A)$, L is a projective A-module and an R-progenerator. This gives that $S \otimes_R L$ is projective over $S \otimes_R A$ and R is a summand of a finite direct sum of copies of L. Thus, $(S \otimes_R L)^t \simeq S \otimes_R L^t \simeq S \otimes_R R \oplus K \simeq S \oplus S \otimes K$, for some K. Hence, $S \otimes_R L$ is an S-progenerator.

Note that

$$\begin{aligned} S \otimes_R (L \otimes_R \operatorname{Hom}_A(L,A)) \simeq S \otimes_S S \otimes_R (L \otimes_R \operatorname{Hom}_A(L,A)) &\simeq S \otimes_R L \otimes_S S \otimes_R \operatorname{Hom}_A(L,A) \\ &\simeq S \otimes_R L \otimes_S \operatorname{Hom}_{S \otimes_R A}(S \otimes_R L, S \otimes_R A). \end{aligned}$$

Denote this isomorphism by α and its inverse by β. The following diagram is commutative:

$$\begin{tikzcd} S \otimes_R L \otimes_S \operatorname{Hom}_{S \otimes_R A}(S \otimes_R L, S \otimes_R A) \arrow[r, "\tau_{S \otimes_R L, S \otimes_R A}"] \arrow[d, shift left, "\beta"] & S \otimes_R A \\ S \otimes_R (L \otimes_R \operatorname{Hom}_A(L,A)) \arrow[u, shift left, "\alpha"] \arrow[ur, "S \otimes_R \tau_{L,A}"'] & \end{tikzcd} .$$

In fact,

$$\tau_{S \otimes_R L, S \otimes_R A} \circ \alpha(s \otimes l \otimes g) = \tau_{S \otimes_R L, S \otimes_R A}(s \otimes l \otimes 1_S \otimes g) = (1_S \otimes g)(s \otimes l) = s \otimes g(l) = \mathrm{id}_S \otimes_R \tau_{L,A}(s \otimes l \otimes g).$$

Thus, $\tau_{S \otimes_R L, S \otimes_R A}$ is a composition of a split S-mono with an isomorphism, and so it is a split S-mono. By 1.5.15, $S \otimes_R L \in \mathscr{M}(S \otimes_R A)$.

Now assume *i*. *ii* follows putting $S = R_\mathfrak{m}$ for each maximal ideal m of R. *iii* follows putting $S = R(\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ of R. Clearly, in this case, L is projective over R.

Now assume *ii*. $L_\mathfrak{m}$ is faithful for any $\mathfrak{m}$ maximal ideal of R. Hence, $\operatorname{Ann} L_\mathfrak{m} = 0$. Take $r \in \operatorname{Ann} L$, then $s \otimes rs' \otimes l = ss' \otimes rl = ss' \otimes 0 = 0$ for any $s, s' \in R_\mathfrak{m}$, $l \in L$. This means that $s \otimes r \in \operatorname{Ann} L_\mathfrak{m} = 0$. So, any element in $(\operatorname{Ann} L)_\mathfrak{m} = R_\mathfrak{m} \otimes \operatorname{Ann} L$ is zero. Thus, $\operatorname{Ann} L = 0$, that is, L is faithful over R. As $L_\mathfrak{m}$ is projective over $A_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R, it follows that L is projective over A. Now, $\tau_{L_\mathfrak{m}, A_\mathfrak{m}} = (\tau_{L,A})_\mathfrak{m}$ is $R_\mathfrak{m}$-mono, so $\tau_{L,A}$ is R-mono. $R_\mathfrak{m}$ is a flat R-module, hence $(\operatorname{coker} \tau_{L,A})_\mathfrak{m} = \operatorname{coker} \tau_{L_\mathfrak{m}, A_\mathfrak{m}}$ which is projective over $R_\mathfrak{m}$. So, $\operatorname{coker} \tau_{L,A}$ is projective over R. Therefore, $\tau_{L,A}$ is (A,R)-mono and *i* follows.

Finally, assume *iii*. Since L is projective over R and $L(\mathfrak{m})$ is projective over $A(\mathfrak{m})$, it follows that L is projective over A. Consider the canonical map $R \to \operatorname{End}_A(L)$, given by $r \mapsto (l \mapsto rl)$. Denote this map by ϕ. Since $L \in A$-proj, we have $\operatorname{End}_A(L)$ is a projective R-module. By assumption, $L(\mathfrak{m}) \in \mathscr{M}(A(\mathfrak{m}))$ and by Proposition 1.5.15, $\phi(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. According to Lemma 1.1.39, ϕ is R-isomorphism. In particular, L is faithful over R. Let D be the standard duality. According to Proposition 1.1.65, we have isomorphisms $k_1\colon \operatorname{Hom}_R(L, DA \otimes_A L) \to D(D(DA \otimes_A L) \otimes_R L)$ and $k_2\colon \operatorname{Hom}_A(L,A) \to D(DA \otimes_A L)$.

Consider the right A-homomorphism $\vartheta_L\colon DA \to \operatorname{Hom}_R(L, DA \otimes_A L)$, given by, $\vartheta(f)(l) = f \otimes l$, $f \in DA$, $l \in L$.

There is a commutative diagram

$$\begin{tikzcd} DA \arrow[r, "\vartheta_L"] \arrow[d, equal] & \operatorname{Hom}_R(L, DA \otimes_A L) \arrow[d, "D(Dk_2 \otimes_R L)\circ k_1"] \\ DA \arrow[r, "D\tau_{L,A}"] & D(L \otimes_R \operatorname{Hom}_A(L,A)) \end{tikzcd} . \tag{1.5.2.8}$$

In fact, for $f \in DA, \lambda \in L$ and $g \in \operatorname{Hom}_A(L,A)$

$$D(k_2 \otimes_R L) \circ k_1 \circ \vartheta_L(f)(l \otimes g) = k_1(\vartheta_L(l)) \circ k_2 \otimes_R L(l \otimes g) = k_1(\vartheta_L(f))(k_2(g) \otimes l) = \tag{1.5.2.9}$$

$$k_2(g)(\vartheta_L(f)(l)) = k_2(g)(f \otimes l) = f(g(l)) = f \circ \tau_{L,A}(l \otimes g) = D\tau_{L,A}(f)(l \otimes g). \tag{1.5.2.10}$$

By assumption, $\tau_{L(\mathfrak{m}),A(\mathfrak{m})}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R. Denote by $D_{(\mathfrak{m})}$ the standard duality in $R(\mathfrak{m})$. Then, $D_{(\mathfrak{m})}\tau_{L(\mathfrak{m}),A(\mathfrak{m})}$ is surjective for every maximal ideal in R. By the diagram (1.5.2.8), it follows that $\vartheta_{L(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. Using the following commutative diagram

$$\begin{tikzcd} DA(\mathfrak{m}) \arrow[r, "\vartheta_L(\mathfrak{m})"] \arrow[d, "\simeq"] & \operatorname{Hom}_R(L, DA \otimes_A L)(\mathfrak{m}) \arrow[d, "\simeq"] \\ D_{(\mathfrak{m})}A(\mathfrak{m}) \arrow[r, "\vartheta_{L(\mathfrak{m})}"] & \operatorname{Hom}_{R(\mathfrak{m})}(L(\mathfrak{m}), D_{(\mathfrak{m})}A(\mathfrak{m}) \otimes_{A(\mathfrak{m})} L(\mathfrak{m})) \end{tikzcd} \tag{1.5.2.11}$$

we deduce that $\vartheta_L(\mathfrak{m})$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, ϑ_L is surjective. By the commutativity of diagram (1.5.2.8), $D\tau_{L,A}$ is surjective. As $L \in A$-proj, $D(L \otimes_R \operatorname{Hom}_A(L,A)) \in R$-proj. Consequently, $D\tau_{L,A}$ is an (A,R)-epimorphism. Hence, $DD\tau_{L,A}$ is an (A,R)-monomorphism. Taking into account that $L \otimes_R \operatorname{Hom}_A(L,A)$ and $DA \in R$-proj we conclude that $\tau_{L,A}$ is an (A,R)-monomorphism. ☐

The following result completes Lemma 1.5.20 and it reduces the study of projective R-split A-modules to the study of maximal standard modules over finite-dimensional algebras over algebraically closed fields.

Lemma 1.5.21. *Let k be a field and let A be a finite-dimensional k-algebra. Assume that $\overline{k}$ is the algebraic closure of k. Given $L \in A$-mod, if $\overline{k} \otimes_k L \in \mathscr{M}(\overline{k} \otimes_k A)$ then $L \in \mathscr{M}(A)$.*

Proof. It is immediate that L is faithful over k. We will proceed to show that L is projective over A. To see this observe that $\overline{k}$ is faithfully flat over k and

$$\overline{k} \otimes_k \operatorname{Ext}^1_A(L,N) = \operatorname{Ext}^1_{\overline{k}\otimes_k A}(\overline{k} \otimes_k L, \overline{k} \otimes_k N) = 0, \quad \forall N \in A\text{-mod}. \tag{1.5.2.12}$$

It remains to check that the map $\tau_{L,A}$ is injective. By assumption, $\tau_{\overline{k}\otimes_k L, \overline{k}\otimes_k A}$ is injective. Since $\overline{k}$ is faithfully flat over k this implies that $\overline{k} \otimes_k \tau_{L,A}$ is injective and consequently $\tau_{L,A}$ is injective. Therefore, $L \in \mathscr{M}(A)$. ☐

Proposition 1.5.22. *[Rou08, Proposition 4.7] There is a bijection from $\mathscr{M}(A)$ to the set of isomorphism classes of pairs (J,P) where J is a split heredity ideal of A and P is a progenerator for $B := \operatorname{End}_A(J)^{op}$ such that $R \simeq \operatorname{End}_{\operatorname{End}_A(J)^{op}}(P)$. Here the equivalence is given in the following way: $(J,P) \sim (J',P')$ if and only if $J = J'$ and $P \simeq P'$ as B-modules. Explicitly,*

$$\alpha\colon \mathscr{M}(A) \longrightarrow \{\textit{isomorphism classes of pairs } (J,P)\}/_\sim \colon L \mapsto (\operatorname{im}\tau_L, \operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R))$$

$$\beta\colon \{\textit{isomorphism classes of pairs } (J,P)\}/_\sim \longrightarrow \mathscr{M}(A)\colon (J,P) \mapsto J \otimes_B P.$$

Proof. Let $L \in \mathscr{M}(A)$. Let $J = \operatorname{im}\tau_L$ and $B = \operatorname{End}_A(J)^{op}$. By assumption, τ_L is (A,R)-monomorphism, so J is an R-summand of A. Hence, A/J is an R-summand of A. Since L is projective A-module, by Lemma 1.5.13, $J^2 = J$. Notice that $L\otimes_R \operatorname{Hom}_A(L,A) \simeq \operatorname{im}\tau_L = J$ as left A-modules. Since L is faithful over R, it follows that $\operatorname{Hom}_A(L,A)$ is faithful over R. Since L is an R-progenerator, R is a summand of L^s for some $s>0$. Hence, $\operatorname{Hom}_A(L,A)\simeq R\otimes_R \operatorname{Hom}_A(L,A)$ is an R-summand of $L^s\otimes_R \operatorname{Hom}_A(L,A)\simeq J^s$. Hence, $\operatorname{Hom}_A(L,A)$ is projective over R. As R is commutative, $\operatorname{Hom}_A(L,A)$ is a progenerator for R-mod. Now $J\simeq L\otimes_R \operatorname{Hom}_A(L,A)$ is projective over $A\otimes_R R\simeq A$. It remains to show that B is Morita equivalent to R.

By Tensor-Hom adjunction,

$$\operatorname{End}_R(\operatorname{Hom}_A(L,A)) \simeq \operatorname{Hom}_R(\operatorname{Hom}_A(L,A),\operatorname{Hom}_A(L,A)) \simeq \operatorname{Hom}_A(L\otimes_R \operatorname{Hom}_A(L,A),A)\simeq \operatorname{Hom}_A(J,A).$$

Consider the exact sequence

$$0\to J\to A\to A/J\to 0. \tag{1.5.2.13}$$

Applying $\operatorname{Hom}_A(J,-)$ to (1.5.2.13) yields

$$0\to \operatorname{End}_A(J)\to \operatorname{Hom}_A(J,A)\to \operatorname{Hom}_A(J,A/J)\to 0.$$

Now since $J(A/J)=0$ and $J=J^2$, we get $\operatorname{Hom}_A(J,A/J)=0$ by Lemma 1.5.14. It follows that $B^{op}\simeq \operatorname{Hom}_A(J,A)\simeq \operatorname{End}_R(\operatorname{Hom}_A(L,A))$. On the other hand, $\operatorname{Hom}_A(L,A)$ is a progenerator of R-mod, so the functor $\operatorname{Hom}_R(\operatorname{Hom}_A(L,A),-)\colon R\text{-mod}\to B\text{-mod}$ is an equivalence of categories. By Morita theorem, $P:=\operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R)$ is a progenerator for B-mod and $R\simeq \operatorname{End}_B(P)$. Hence, α is well defined.

We claim now that $L\simeq \operatorname{im}\tau_L\otimes_B \operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R)$ as A-modules.

By the Morita theorem for progenerators (see e.g. [Fai73, Proposition 12.10]), $\operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R)\simeq \operatorname{Hom}_B(\operatorname{Hom}_A(L,A),B)$ as (B,R)-bimodules. Note the action of A in $\operatorname{im}\tau_L\otimes_B \operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R)$ is the one induced by A in L. Hence, as left A-modules,

$$\begin{aligned}\operatorname{im}\tau_L\otimes_B \operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R) &\simeq L\otimes_R \operatorname{Hom}_A(L,A)\otimes_B \operatorname{Hom}_B(\operatorname{Hom}_A(L,A),B)\\ &\simeq L\otimes_R \operatorname{Hom}_B(\operatorname{Hom}_A(L,A),\operatorname{Hom}_A(L,A)),\ \text{since } \operatorname{Hom}_A(L,A)\in\text{proj-}B\\ &\simeq L\otimes_R R, \text{since the double centralizer property holds on generators}\\ &\simeq L.\end{aligned}$$

Reciprocally, consider a pair (J,P) such that $R\simeq \operatorname{End}_B(P)^{op}$, where $B=\operatorname{End}_A(J)^{op}$. Let $L=J\otimes_B P$. J is projective over A and P is projective over B, hence L is a projective $A\otimes_B B\simeq A$-module. Notice that for $M\in A$-proj and $M'\in C$-proj there exists a canonical isomorphism

$$\operatorname{Hom}_A(M,N)\otimes_R \operatorname{Hom}_C(M',N')\to \operatorname{Hom}_{A\otimes_R C}(M\otimes_R M',N\otimes_R N').$$

So,

$$\operatorname{End}_A(L)\simeq \operatorname{End}_{A\otimes_B B}(J\otimes_B P)\simeq \operatorname{End}_A(J)\otimes_B \operatorname{End}_B(P)\simeq B\otimes_B \operatorname{End}_B(P)\simeq \operatorname{End}_B(P)\simeq R. \tag{1.5.2.14}$$

Consequently, L is faithful over R.

Let $i\colon J\to A$ be the inclusion A-homomorphism. We can consider $f\in \operatorname{Hom}_B(B,\operatorname{Hom}_A(J,A))$ such that $f(1_B)=i$. Since P is a progenerator of B-mod, B is a summand of some direct sum of copies of P. So, we can extend the map f to $f\in \operatorname{Hom}_B(P^t,\operatorname{Hom}_A(J,A))$ such that there exists $x\in P^t$ with $f(x)=i$. Consider the canonical inclusions and projections $k_j\colon P\to P^t$, $\pi_j\colon P^t\to P$. Define $f_j=f\circ k_j\in \operatorname{Hom}_B(P,\operatorname{Hom}_A(J,A))$. We have $i = f(x) = \sum_j f \circ k_j \circ \pi_j(x) = \sum_j f \circ k_j(x_j) = \sum_j f_j(x_j)$ for some $x_j \in P$.

Consider the adjoint map $\mathrm{Hom}_B(P, \mathrm{Hom}_A(J,A)) \to \mathrm{Hom}_A(J \otimes_B P, A)$, which sends $f \in \mathrm{Hom}_B(P, \mathrm{Hom}_A(J,A))$ to the map $((y \otimes x) \mapsto f(x)(y))$, and let g_j be the image of f_j in $\mathrm{Hom}_A(J \otimes_B P, A)$. Then, for any $y \in J$,

$$\tau_{L=J\otimes_B P}(\sum_j y \otimes x_j \otimes g_j) = \sum_j \tau_L(y \otimes x_j \otimes g_j) = \sum_j g_j(y \otimes x_j) = \sum_j f_j(x_j)(y) = i(y) = y.$$

Therefore, $J \subset \operatorname{im} \tau_L$. Note that $\mathrm{Hom}_A(L, A/J) \simeq \mathrm{Hom}_B(P, \mathrm{Hom}_A(J, A/J)) = 0$, by Tensor-Hom adjunction and by Lemma 1.5.13. The functor $\mathrm{Hom}_A(L,-)$ yields the exact sequence $0 \to \mathrm{Hom}_A(L,J) \to \mathrm{Hom}_A(L,A) \to \mathrm{Hom}_A(L,A/J) = 0$. Thus, we get $\tau_L(L \otimes_R \mathrm{Hom}_A(L,A)) = \tau_L(L \otimes_R \mathrm{Hom}_A(L,J)) \subset J$. We conclude that $\operatorname{im} \tau_L = J$. Since P is a left B-progenerator and $\mathrm{End}_B(P) \simeq R$ then $\mathrm{Hom}_B(P,B) \simeq \mathrm{Hom}_R(P,R)$ as (R,B)-bimodules (see for example Corollary 1.4.21). Now the functor $\mathrm{Hom}_A(J,-)$ yields the exact sequence

$$0 \to B = \mathrm{Hom}_A(J,J) \to \mathrm{Hom}_A(J,A) \to \mathrm{Hom}_A(J,A/J) = 0.$$

Hence, $B \simeq \mathrm{Hom}_A(J,A)$ as left B-modules. Thus, as R-modules,

$$\mathrm{Hom}_A(L,A) \simeq \mathrm{Hom}_B(P, \mathrm{Hom}_A(J,A)) \simeq \mathrm{Hom}_B(P,B) \simeq \mathrm{Hom}_R(P,R). \tag{1.5.2.15}$$

Finally as A-modules,

$$J \simeq J \otimes_B B \simeq J \otimes_B \mathrm{End}_R(P) \simeq J \otimes_B P \otimes_R \mathrm{Hom}_R(P,R) \simeq J \otimes_B P \otimes_R \mathrm{Hom}_A(L,A) \simeq L \otimes_R \mathrm{Hom}_A(L,A).$$

We conclude that the map $\tau_L \colon L \otimes_R \mathrm{Hom}_A(L,A) \to J$ is surjective between two isomorphic finitely generated A-modules. By Nakayama's Lemma, $\tau_L \colon L \otimes_R \mathrm{Hom}_A(L,A) \to J$ is an isomorphism. In particular, $\tau_L \colon L \otimes_R \mathrm{Hom}_A(L,A) \to A$ is injective. Now since A/J is projective over R, the exact sequence $0 \to J \to A \to A/J \to 0$ splits over R. Hence, τ_L is an (A,R)-monomorphism, so $L \in \mathscr{M}(A)$. Thus, β is well defined.

We claim that $\alpha \circ \beta(J,P) = (J,P)$.

In fact, $\alpha \circ \beta(J,P) = \alpha(J \otimes_B P) = (\operatorname{im} \tau_{J\otimes_B P}, \mathrm{Hom}_R(\mathrm{Hom}_A(J \otimes_B P, A), R)) = (J, \mathrm{Hom}_R(\mathrm{Hom}_A(J \otimes_B P, A), R))$. Since $L = J \otimes_B P \in \mathscr{M}(A)$ by the first direction we can regard $\mathrm{Hom}_A(L,A)$ as a right B-module. Recall that the functors $\mathrm{Hom}_B(P,-)$ and $\mathrm{Hom}_R(\mathrm{Hom}_B(P,B),-)$ form an equivalence. Hence, we obtain as left B-modules,

$$\mathrm{Hom}_R(\mathrm{Hom}_A(L,A),R) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(P,R),R) \simeq \mathrm{Hom}_R(\mathrm{Hom}_B(P,B), \mathrm{Hom}_B(P,P)) \tag{1.5.2.16}$$

$$\simeq \mathrm{Hom}_R(\mathrm{Hom}_B(P,B), \mathrm{Hom}_B(P,-))P \simeq P. \tag{1.5.2.17}$$

So, the claim follows.

We have shown also that for any $L \in \mathscr{M}(A)$,

$$\beta \circ \alpha(L) = \beta(\operatorname{im} \tau_L, \mathrm{Hom}_R(\mathrm{Hom}_A(L,A),R)) = \operatorname{im} \tau_L \otimes_{\mathrm{End}_A(\operatorname{im} \tau_L)^{op}} \mathrm{Hom}_R(\mathrm{Hom}_A(L,A),R) \simeq L$$

as A-modules. Hence, $\beta \circ \alpha = \mathrm{id}$. Thus, α and β are bijections. $\square$

Corollary 1.5.23. *[Rou08, Lemma 4.10] For any $L \in \mathscr{M}(A)$, the canonical functor $A/\operatorname{im} \tau_L$-mod $\to$ A-mod induces an equivalence between $A/\operatorname{im} \tau_L$-mod and the full subcategory of A-mod whose objects M satisfy $\mathrm{Hom}_A(L,A) = 0$.*

Proof. For any $L \in \mathscr{M}(A)$, let $J = \operatorname{im} \tau_L$. J is ideal and $J = J^2$. Hence, by Lemma 1.5.13, for $M \in A$-mod, $M \in A/J$-mod if and only if $\mathrm{Hom}_A(J,M) = 0$.

But $L \simeq J \otimes_B P$ for some progenerator P of B by Proposition 1.5.22. By Tensor-Hom adjunction, $\mathrm{Hom}_A(L,M) \simeq$

$\mathrm{Hom}_B(P, \mathrm{Hom}_A(J,M))$.

We claim that $\mathrm{Hom}_A(L,M) \simeq \mathrm{Hom}_B(P, \mathrm{Hom}_A(J,M)) = 0$ if and only if $\mathrm{Hom}_A(J,M) = 0$

Assume that $\mathrm{Hom}_A(J,M) = 0$, then it is clear that $\mathrm{Hom}_A(L,M) \simeq \mathrm{Hom}_B(P, \mathrm{Hom}_A(J,M)) = 0$. Reciprocally, assume that $\mathrm{Hom}_A(L,M) \simeq \mathrm{Hom}_B(P, \mathrm{Hom}_A(J,M)) = 0$. If $\mathrm{Hom}_A(J,M) \neq 0$, then there exists a non-zero B-epimorphism $P^t \to \mathrm{Hom}_A(J,M)$. This would imply that $\mathrm{Hom}_B(P, \mathrm{Hom}_A(J,M)) \neq 0$. The result follows. $\square$

Definition 1.5.24. A full subcategory $\mathscr{A}$ of an abelian category $\mathscr{B}$ is called **Serre subcategory** if for any exact sequence in $\mathscr{B}$

$$0 \to X \to M \to Y \to 0$$

$M \in \mathscr{A}$ if and only if $X, Y \in \mathscr{A}$.

Hence, a Serre subcategory is a subcategory closed under extensions, submodules and quotients.

Corollary 1.5.25. *For any $L \in \mathscr{M}(A)$, let $J = \operatorname{im} \tau_L$. Then, A/J-mod is a Serre subcategory of A-mod.*

Proof. Let $0 \to X \to M \to Y \to 0$ be an exact sequence of A-modules. Applying the functor $\mathrm{Hom}_A(L,-)$ yields

$$0 \to \mathrm{Hom}_A(L,X) \to \mathrm{Hom}_A(L,M) \to \mathrm{Hom}_A(L,Y) \to 0.$$

Thus, $\mathrm{Hom}_A(L,M) = 0$ if and only if $\mathrm{Hom}_A(L,X) = \mathrm{Hom}_A(L,Y) = 0$. By Corollary 1.5.23, the result follows. $\square$

1.5.2.1 Picard Group and invertible modules

To write this bijection in terms of split heredity ideals instead of pairs (J,P) we need the notion of invertible module. The theory of invertible modules can be studied with more detail, for example, in [Fai73].

Definition 1.5.26. Let R be a commutative ring.

A module M is called **invertible** if the functor $M \otimes_R - : R\text{-mod} \to R\text{-mod}$ is an equivalence of categories.

Proposition 1.5.27. *Let R be a commutative Noetherian ring. Let M be a finitely generated R-module. The following assertions are equivalent.*

(a) M is invertible;

(b) There exists an R-module N such that $M \otimes_R N \simeq R$;

(c) $M_\mathfrak{p} \simeq R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R;

(d) $M_\mathfrak{m} \simeq R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R.

Proof. Assume that (a) holds. Since $M \otimes_R -$ is an equivalence of categories its adjoint $\mathrm{Hom}_R(M,-)$ is also an equivalence of categories. Moreover, $R \simeq M \otimes_R \mathrm{Hom}_R(M,-)R = M \otimes_R \mathrm{Hom}_R(M,R)$. Define $N := \mathrm{Hom}_R(M,R)$. So, (b) follows. Since $M \otimes_R -$ is an equivalence of categories it preserves projective modules. In particular, R is projective, so $M \simeq M \otimes_R R$ is projective over R. In the same way, N is projective. So, for every prime ideal $\mathfrak{p}$ of R, $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are free over $R_\mathfrak{p}$, since $R_\mathfrak{p}$ is local. Assume $M_\mathfrak{p} \simeq R_\mathfrak{p}^n$ and $N_\mathfrak{p} \simeq R_\mathfrak{p}^s$. So,

$$R_\mathfrak{p} \simeq (M \otimes_R N)_\mathfrak{p} \simeq M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \simeq R_\mathfrak{p}^n \otimes_{R_\mathfrak{p}} R_\mathfrak{p}^s \simeq R_\mathfrak{p}^{ns} \tag{1.5.2.18}$$

So, we must have $ns = 1$, that is $n = s = 1$. Hence, $M_\mathfrak{p} \simeq R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. Thus, $c)$ follows. $c) \implies d)$ is clear. Assume $d)$. Consider the map $\sigma\colon M\otimes_R \operatorname{Hom}_R(M,R) \to R$, $m\otimes f \mapsto f(m)$. Note that $M_\mathfrak{m}$ is projective over $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$, hence M is projective. We have

$$(M\otimes_R \operatorname{Hom}_R(M,R))_\mathfrak{m} \simeq M_\mathfrak{m}\otimes_{R_\mathfrak{m}} \operatorname{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m},R_\mathfrak{m}) \simeq R_\mathfrak{m}\otimes_{R_\mathfrak{m}} \operatorname{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m},R_\mathfrak{m}) \simeq R_\mathfrak{m}. \qquad (1.5.2.19)$$

Hence, $\sigma_\mathfrak{m}$ is an isomorphism for all maximal ideals $\mathfrak{m}$ of R. So, σ is an isomorphism. So $b)$ follows. Now assume that (b) holds. Suppose that $M\otimes_R N \simeq R$. Then, the functors $F = -\otimes_R M$ and $G = N\otimes_R -$ are quasi-inverse. In fact,

$$\begin{aligned} FGX &= (X\otimes_R N) \simeq X\otimes_R R \simeq X \\ GFX &= N\otimes_R (X\otimes_R M) \simeq X\otimes_R R \simeq X. \end{aligned}$$

So, $a)$ follows. □

Note that for L, L' invertible R-modules, exists N, N' such that $L\otimes_R N \simeq R$ and $L'\otimes_R N' \simeq R$. So,

$$L\otimes_R L'\otimes N\otimes N' \simeq L\otimes_R N\otimes_R L'\otimes_R N' \simeq R\otimes_R R \simeq R.$$

Hence, $L\otimes_R L'$ is invertible. The isomorphism classes of invertible R-modules together with the tensor product form a group. This group is called the **Picard group** of the ring R. We denote it by $Pic(R)$. The unit is the equivalence class of the regular module R and the inverse of M is $\operatorname{Hom}_R(M,R)$.

Example 1.5.28. The Picard group of a field is trivial.

Since R is a field, 0 is the only maximal ideal of R. But $R_0 = (R\setminus 0)R = R$. Let $M \in Pic(R)$. It follows that $M \simeq M_0 \simeq R_0 \simeq R$. △

Example 1.5.29. The Picard group of a local ring is trivial.

Let $M \in Pic(R)$. Then, $M_\mathfrak{m} \simeq R_\mathfrak{m}$ for the maximal ideal $\mathfrak{m}$ of R, hence M is projective. Since R is local, M is free, hence $M \simeq R^n$ for some n. On the other hand, $M_\mathfrak{m} \simeq R_\mathfrak{m}^n$ for the maximal ideal $\mathfrak{m}$. Therefore, we must have $n = 1$, so M is isomorphic to R. △

Now we can see that the Picard group $Pic(R)$ acts on $\mathscr{M}(A)$.

Lemma 1.5.30. *Let $F \in Pic(R)$, $L \in \mathscr{M}(A)$. Then, $L\otimes_R F \in \mathscr{M}(A)$. Moreover, this gives an action of $Pic(R)$ on $\mathscr{M}(A)$.*

Proof. Let $F \in Pic(R)$, $L \in \mathscr{M}(A)$. By Lemma 1.5.20, $L_\mathfrak{m} \in \mathscr{M}(A_\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ of R. Note that for each maximal ideal $\mathfrak{m}$ of R

$$(L\otimes_R F)_\mathfrak{m} \simeq L_\mathfrak{m}\otimes_{R_\mathfrak{m}} F_\mathfrak{m} \simeq L_\mathfrak{m}\otimes_{R_\mathfrak{m}} R_\mathfrak{m} \simeq L_\mathfrak{m} \in \mathscr{M}(A_\mathfrak{m}).$$

Again, by Lemma 1.5.20, $L\otimes_R F \in \mathscr{M}(A)$. Since $R\otimes_R L \simeq L$ and $(F_1\otimes_R F_2)\otimes_R L \simeq F_1\otimes_R (F_2\otimes_R L)$, the second claim follows. □

Note that two elements in $L, L' \in \mathscr{M}(A)$ are in the same orbit if and only if there exists $F \in Pic(R)$ such that $L' \simeq L\otimes_R F$ as A-modules. We denote by $\mathscr{M}(A)/Pic(R)$ the set of orbits of $\mathscr{M}(A)$ under the action of $Pic(R)$.

Proposition 1.5.31. *There is a bijection from $\mathscr{M}(A)/Pic(R)$ to the set of split heredity ideals of A. More precisely,*

$$\delta\colon \mathscr{M}(A)/Pic(R) \to \{\textit{split heredity ideals of } A\},\ L \mapsto \operatorname{im}\tau_L$$
$$\vartheta\colon \{\textit{split heredity ideals of } A\} \to \mathscr{M}(A)/Pic(R),\ J \mapsto J\otimes_B P$$

where $B = \operatorname{End}_A(J)^{op}$ and P an arbitrary B-progenerator that satisfies $\operatorname{End}_B(P)^{op} \simeq R$.

Proof. Consider L and $L\otimes_R F$, $F \in Pic(R)$. For every maximal ideal $\mathfrak{m}$ of R,

$$(L\otimes_R F\otimes_R \operatorname{Hom}_A(L\otimes_R F,A))_\mathfrak{m} \simeq L_\mathfrak{m}\otimes_{R_\mathfrak{m}} F_\mathfrak{m}\otimes_{R_\mathfrak{m}} \operatorname{Hom}_{A_\mathfrak{m}}(L_\mathfrak{m}\otimes_{R_\mathfrak{m}} F_\mathfrak{m}, A_\mathfrak{m})$$
$$\simeq L_\mathfrak{m}\otimes_{R_\mathfrak{m}} R_\mathfrak{m}\otimes_{R_\mathfrak{m}} \operatorname{Hom}_{A_\mathfrak{m}}(L_\mathfrak{m}\otimes_{R_\mathfrak{m}} R_\mathfrak{m}, A_\mathfrak{m}) \simeq L_\mathfrak{m}\otimes_{R_\mathfrak{m}} \operatorname{Hom}_{A_\mathfrak{m}}(L_\mathfrak{m}, A_\mathfrak{m}) \simeq (L\otimes_R \operatorname{Hom}_A(L,A))_\mathfrak{m}$$

Hence, $\tau_{L\otimes_R F\,\mathfrak{m}} = \tau_{L\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, $(\operatorname{im}\tau_{L\otimes_R F})_\mathfrak{m} \simeq (\operatorname{im}\tau_L)_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R. Since $(\operatorname{im}\tau_{L\otimes_R F}),\operatorname{im}\tau_L \subset A$ it follows that $\operatorname{im}\tau_{L\otimes_R F} = \operatorname{im}\tau_L$. So, δ is well defined.

Now we have to see that the image of ϑ is independent of the choice of P. Consider P and Q are B-progenerators such that $\operatorname{End}_B(P)^{op} \simeq R$ and $\operatorname{End}_B(Q)^{op} \simeq R$. Then,

$$P\otimes_B \operatorname{Hom}_B(P,B) \simeq R \text{ as } R\text{-modules and} \tag{1.5.2.20}$$
$$Q\otimes_B \operatorname{Hom}_B(Q,B) \simeq R \text{ as } R\text{-modules.} \tag{1.5.2.21}$$

Fix $P' = \operatorname{Hom}_B(P,B)$ and $Q' = \operatorname{Hom}_B(Q,B)$. By double centralizer property on generators, $Q'\otimes_R Q \simeq B$ and $P'\otimes_R P \simeq B$ as (B,B)-bimodules. It follows as left A-modules,

$$J\otimes_B P \simeq J\otimes_B (B\otimes_B P) \simeq J\otimes_B (Q'\otimes_R Q)\otimes_B P \simeq (J\otimes_B Q)\otimes_R (Q'\otimes_B P) \tag{1.5.2.22}$$

Now $Q'\otimes_B P \in Pic(R)$. In fact,

$$(Q'\otimes_B P)\otimes_R (P'\otimes_B Q) \simeq Q'\otimes_B (P\otimes_R P')\otimes_B Q \simeq Q'\otimes_B B\otimes_B Q \simeq Q'\otimes_B Q \simeq R.$$

Hence, $J\otimes_B P = J\otimes_B Q$ in $\mathscr{M}(A)$. Therefore, ϑ is well defined. Recall the maps α and β from Proposition 1.5.22. Notice that δ is the projection onto the first coordinate of the map α. Denote this projection by π. On the other hand, $\vartheta(J) = \beta(J,P)$ for some B-progenerator P.

Therefore, $\vartheta\circ\delta(L) = \vartheta(\operatorname{im}\tau_L) = \beta(\operatorname{im}\tau_L, \operatorname{Hom}_R(\operatorname{Hom}_A(L,A),R)) = \beta\circ\alpha(L) = L$ for any $L \in \mathscr{M}(A)/Pic(R)$ and $\delta\circ\vartheta(J) = \delta\circ\beta(J,P) = \pi\circ\alpha\circ\beta(J,P) = \pi(J,P) = J$ for any split heredity J. Thus, both ϑ and δ are bijections. □

1.5.3 Split highest weight category over a commutative Noetherian ring

Definition 1.5.32. Let R be a Noetherian commutative ring. Let A be a projective Noetherian R-algebra. Let Λ be a finite preordered set. We say that $(A\text{-mod},\Lambda)$ is a **highest weight category in weak sense** if there exist finitely generated modules $\{\Delta(\lambda)\colon \lambda\in\Lambda\}$ such that

(i) $\Delta(\lambda)$ is a projective R-module;

(ii) If $\operatorname{Hom}_A(\Delta(\lambda'),\Delta(\lambda'')) \neq 0$, then $\lambda' \leq \lambda''$.

(iii) If $N \in A$-mod is such that $\operatorname{Hom}_A(\Delta(\lambda),N) = 0$ for all $\lambda\in\Lambda$, then $N = 0$.

(iv) For each $\lambda \in \Lambda$, there exists a projective A-module $P(\lambda)$ such that there is an exact sequence

$$0 \to C(\lambda) \to P(\lambda) \xrightarrow{\pi_\lambda} \Delta(\lambda) \to 0,$$

where $C(\lambda)$ has a finite filtration by modules of the form $\Delta(\mu) \otimes_R U_\mu$ with $U_\mu \in R$-proj and $\mu > \lambda$.

A highest weight category in weak sense $(A\text{-mod}, \Lambda)$ is **split** if it also satisfies

(v) $\mathrm{End}_A(\Delta(\lambda)) \simeq R$ for all $\lambda \in \Lambda$.

We say also that $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. For simplicity, occasionally we just write Δ to mean the set $\{\Delta(\lambda) \colon \lambda \in \Lambda\}$.

It is important to remark that the set of pairs $\{(P(\lambda), \Delta(\lambda) \colon \lambda \in \Lambda\}$ satisfying conditions 1.5.32$(i)-(iv)$ do not form a standard system in the usual sense. Moreover, there exist highest weight categories in weak sense whose standard modules are not Schurian (see example 1.5.89). In the classical sense, a prerequisite for a pair $(A\text{-mod}, \Lambda)$ to be a highest weight category is the objects Δ being Schurian. Hence, this motivates to not call A-mod together with conditions 1.5.32$(i)-(iv)$ a highest weight category. In addition, even if R is a field, conditions 1.5.32$(i)-(iv)$ do not impose that $P(\lambda)$ are projective indecomposable. On the other hand, this justification could lead us to think that perhaps conditions 1.5.32$(i)-(iv)$ are related to a standardly stratified structure. However, this is again not the case, since we could consider the modules in Example 1.5.89 over the ring $\mathbb{Z}$ and define a new structure with $\Delta'(1) = \Delta(1) \oplus \Delta(1) \oplus \Delta(1)$ and $\Delta'(2) = \Delta(2) \oplus \Delta(2) \oplus \Delta(2)$. This new structure on A implies that the regular module does not have a filtration by standard modules $\Delta(i)$, $i = 1, 2$. (If A had a filtration by standard modules, we could consider a surjective map of A to one of Δ'. As they have the same rank over $\mathbb{Z}$, such a map should be an isomorphism.). However, it satisfies conditions 1.5.32$(i)-(iv)$.

First, we would like to see how in this definition the multiplicities of the standard modules relate to the respective projective modules.

Proposition 1.5.33. *Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a highest weight category in weak sense. Then, for any $\lambda \in \Lambda$, $\mathrm{Hom}_A(P(\lambda), \Delta(\lambda)) \simeq \mathrm{End}_A(\Delta(\lambda))$.*

Proof. Consider $h \in \mathrm{End}_A(\Delta(\lambda))$. By 1.5.32$(iv)$, we have a surjective map $P(\lambda) \xrightarrow{\pi_\lambda} \Delta(\lambda)$, hence we have a map $P(\lambda) \xrightarrow{\pi_\lambda} \Delta(\lambda) \xrightarrow{h} \Delta(\lambda) \in \mathrm{Hom}_A(P(\lambda), \Delta(\lambda))$. Take a map $g \in \mathrm{Hom}_A(P(\lambda), \Delta(\lambda))$.

Consider the following commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & C(\lambda) \arrow[r, "i"] \arrow[dr, "g\circ i"'] & P(\lambda) \arrow[r, "\pi_\lambda"] \arrow[d, "g"] & \Delta(\lambda) \arrow[r] & 0 \\ & & \Delta(\lambda) & & \end{tikzcd} .$$

Assume $g \circ i \neq 0$. Since $C(\lambda)$ has a finite filtration into modules of the form $\Delta \otimes_R X$, $X \in R$-proj, there exists $\Delta(\mu)$ with $\mu > \lambda$ such that $\mathrm{Hom}_A(\Delta(\mu), \Delta(\lambda)) \neq 0$. By 1.5.32$(ii)$ we get that $\mu \leq \lambda$, which contradicts $\mu > \lambda$. Hence, $g \circ i = 0$. So, g induces uniquely a map $g' \in \mathrm{End}_A(\Delta(\lambda))$ such that $g' \circ \pi_\lambda = g$. Notice that $(h \circ \pi_\lambda)'$ satisfies $(h \circ \pi_\lambda)' \circ \pi_\lambda = h \circ \pi_\lambda$. Since π_λ is an epimorphism, we get $(h \circ \pi_\lambda)' = h$. It follows that $\mathrm{Hom}_A(P(\lambda), \Delta(\lambda)) \simeq \mathrm{End}_A(\Delta(\lambda))$ as R-modules. $\square$

Proposition 1.5.34. *Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a highest weight category in weak sense. If $P(\mu)$ is an A-summand of $P(\lambda)$, then $\lambda \leq \mu$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(\lambda) \arrow[r] \arrow[ddr] & P(\lambda) \arrow[r, "\pi_\lambda"] \arrow[d, two heads] & \Delta(\lambda) \arrow[r] & 0 \\
& & P(\mu) \arrow[d, two heads, "\pi_\mu"] & & . \\
& & \Delta(\mu) & &
\end{tikzcd}$$

If the map $C(\lambda) \to \Delta(\mu)$ is non-zero, then exists some module $\Delta(l)$ factor of $C(\lambda)$, hence $l > \lambda$ such that $\operatorname{Hom}_A(\Delta(l), \Delta(\mu)) \neq 0$. By 1.5.32(*ii*), we get $l \leq \mu$, which implies $\lambda < l \leq \mu$.

If the map $C(\lambda) \to \Delta(\mu)$ is zero, then there exists a non-zero A-homomorphism $h\colon \Delta(\lambda) \to \Delta(\mu)$ which makes the diagram commutative. By 1.5.32(*ii*), $\lambda \leq \mu$. ☐

Lemma 1.5.35. *Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ *be a highest weight category in weak sense. If* $\operatorname{Hom}_A(P(\mu), \Delta(\lambda)) \neq 0$*, then* $\mu \leq \lambda$.

Proof. Let $0 \neq \phi \in \operatorname{Hom}_A(P(\mu), \Delta(\lambda))$. Denote by i the inclusion $C(\mu) \hookrightarrow P(\mu)$. If $\phi \circ i = 0$, then ϕ induces a non-zero map in $\operatorname{Hom}_A(\Delta(\mu), \Delta(\lambda))$. By 1.5.32(*ii*), $\mu \leq \lambda$. If $\phi \circ i \neq 0$ then exists by 1.5.32(*iv*) $l > \mu$ such that $\operatorname{Hom}_A(\Delta(l), \Delta(\lambda)) \neq 0$. By 1.5.32(*ii*), $l \leq \lambda$. So, the result follows. ☐

Our goal now is to show that when R is a field and R is a splitting field for A then split highest weight category is the classical notion of highest weight category.

Remark 1.5.36. Let R be a field. Condition 1.5.32(*iii*) ensures that each simple module appears as a top of a standard module. Denote by $\operatorname{rad} A$ the Jacobson radical of A and S a simple A-module. Since S is simple, either $\operatorname{rad} AS = 0$ or $\operatorname{rad} AS = S$. By Nakayama's Lemma, if $\operatorname{rad} AS = S$, then $S = 0$. Thus, $\operatorname{rad} AS = 0$ and hence $\operatorname{top} S = S$. If $\operatorname{Hom}_A(\Delta(\lambda), S) \neq 0$, then $\operatorname{top}\Delta(\lambda) \to \operatorname{top} S = S$ is surjective. In other words, S would appear as a summand of $\operatorname{top}\Delta(\lambda)$. Therefore, if S never occurs as a summand of $\operatorname{top}\Delta(\lambda)$ for some λ, it would follow that $\operatorname{Hom}_A(\Delta(\lambda), S) = 0$ for every λ. So, 1.5.32(*iii*) would have implied $S = 0$. △

Lemma 1.5.37. *Let R be a field. Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ *be a highest weight category in weak sense over R. If* $\operatorname{Hom}_A(P(\mu), \operatorname{rad}\Delta(\lambda)) \neq 0$*, then* $\mu \leq \lambda$. *Furthermore, if additionally* $\dim_R \operatorname{Hom}_A(P(\lambda), \Delta(\lambda)) = 1$*, then* $\mu < \lambda$.

Proof. Let $h \in \operatorname{Hom}_A(P(\mu), \operatorname{rad}\Delta(\lambda))$. Denote by i the inclusion $\operatorname{rad}\Delta(\lambda) \hookrightarrow \Delta(\lambda)$. Applying Lemma 1.5.35 with $i \circ h \neq 0$ it follows that $\mu \leq \lambda$. Now consider additionally that $\dim_R \operatorname{Hom}_A(P(\lambda), \Delta(\lambda)) = 1$. If $\mu = \lambda$, then $i \circ h = \alpha\pi_\lambda$ for some $\alpha \in R$. Thus, $i \circ h$ is surjective. Consequently, i is an isomorphism. By Nakayama's Lemma, we get a contradiction. Hence, the result follows. ☐

Lemma 1.5.38. *Let R be a field. Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ *be a highest weight category in weak sense over R. Let $\lambda \in \Lambda$. If* $\dim_R \operatorname{Hom}_A(P(\lambda), \Delta(\lambda)) = 1$*, then* $\operatorname{top}\Delta(\lambda)$ *is simple. Moreover,* $\Delta(\lambda)$ *is indecomposable.*

Proof. If $\operatorname{top} P(\lambda)$ is simple, then there is nothing to prove. Assume that $\operatorname{top} P(\lambda)$ is not simple. Choose S a simple module summand of $\operatorname{top}\Delta(\lambda)$ which is a summand of $\operatorname{top} P(\lambda)$. Denote by P the projective cover of S. Hence, P is an indecomposable summand of $P(\lambda)$. And so, the canonical map $P \to \operatorname{top} P(\lambda)$ factoring through S is non-zero. We have a commutative diagram

$$\begin{tikzcd}
S = \operatorname{top} P \arrow[r, hook] & \operatorname{top} P(\lambda) \arrow[r] & \operatorname{top}\Delta(\lambda) \\
P \arrow[u] \arrow[rr, "\exists f \neq 0"] & & \Delta(\lambda) \arrow[u, two heads]
\end{tikzcd}$$

Note that the existence of such non-zero map f is due to P being projective and the upper row being a monomorphism. Since $\dim_R \mathrm{Hom}_A(P(\lambda),\Delta(\lambda)) = 1$, there exists $\alpha \in R$ such that $f \circ \pi = \alpha\pi_\lambda$, where π denotes the projection $P(\lambda) \twoheadrightarrow P$ and i denotes the inclusion $P \hookrightarrow P(\lambda)$. If $\alpha = 0$, then $f = f \circ \pi \circ i$ would be zero. Since R is a field, $\alpha\,\mathrm{id}_{\Delta(\lambda)}$ is an isomorphism. Hence, f is surjective. By the commutativity of the diagram, the map $S \hookrightarrow \mathrm{top}\,P(\lambda) \to \mathrm{top}\,\Delta(\lambda)$ is surjective. Since S is simple, it is an isomorphism. Therefore, P is the projective cover of $\Delta(\lambda)$ and $\Delta(\lambda)$ is indecomposable. ☐

We see immediately that even for split quasi-hereditary algebras over fields the conditions in Definition 1.5.32 are not enough to enforce the projectives $P(\lambda)$ to be indecomposable. For example, we can consider a semi-simple algebra with two simple modules say S_1 and S_2 over an algebraically closed field. Fixing $P(1) = S_1 \oplus S_2$ and $P(2) = S_2$ with $\Delta(1) = S_1$ together with the usual order we see that all conditions of Definition 1.5.32 are satisfied. However, $P(1)$ is not indecomposable. As we will see next, for split quasi-hereditary algebras over fields we can replace the projectives $P(\lambda)$ with the projective covers of the standard modules.

Proposition 1.5.39. *(See also [Ari08, Lemma 4.31]). Let R be a splitting field for A. Then, $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category according to Definition 1.5.32 if and only if there is a correspondence between the poset Λ and the isomorphism classes of simple A-modules which we denote by $S(\lambda) = \mathrm{top}\,\Delta(\lambda)$, and for all $\lambda \in \Lambda$, $\Delta(\lambda)$ satisfies*

(I) There is an exact sequence $0 \to K(\lambda) \to \Delta(\lambda) \to S(\lambda) \to 0$ and the composition factors of $K(\lambda)$, $S(\mu)$, satisfy $\mu < \lambda$.

(II) There is an exact sequence $0 \to C(\lambda) \to P_c(\lambda) \to \Delta(\lambda) \to 0$ and $C(\lambda)$ is filtered by modules $\Delta(\mu)$ with $\mu > \lambda$,

where $P_c(\lambda)$ denotes the projective cover of $\Delta(\lambda)$.

Proof. Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a highest weight category in the classical sense, that is, satisfying (I) and (II). Let Λ be the set of isomorphism classes of simple A-modules. (i) is trivially checked since R is a field. By (I) and Lemma 1.5.33, condition 1.5.32(v) holds. Condition 1.5.32(ii) is also satisfied since every non-zero map between standard modules $\Delta(\mu)$ and $\Delta(\lambda)$ can be extended to a non-zero map between the projective cover of $\Delta(\mu)$ and $\Delta(\lambda)$. Consequently, such a case would lead to the multiplicity of $S(\mu)$ in $\Delta(\lambda)$ being positive. By (I), this occurs only if $\mu \leq \lambda$. Define $P_c(\lambda)$ to be the projective cover of $S(\lambda)$. By axiom (II) of highest weight categories it follows that 1.5.32(iv) is satisfied. Assume that $N \in A$-mod such that $\mathrm{Hom}_A(\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda$. If $N \neq 0$, then $\mathrm{soc}\,N \neq 0$. Let $S(\lambda) \subset \mathrm{soc}\,N$. By axiom (I), there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K(\lambda) \arrow[r] & \Delta(\lambda) \arrow[r] \arrow[dr, "\neq 0"] & S(\lambda) \arrow[r] \arrow[d, hook] & 0 \\ & & & N & \end{tikzcd}$$

This contradicts our assumption that $\mathrm{Hom}_A(\Delta(\lambda), N) = 0$. So, $N = 0$ and 1.5.32(iii) holds. So, $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category.

Conversely, assume that $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. Since

$$\dim_R \mathrm{Hom}_A(P(\lambda), \Delta(\lambda)) = \dim_R \mathrm{End}_A(\Delta(\lambda)) = 1$$

all standard modules have a simple top. It can be seen that the category of objects admitting a filtration by standard modules is closed under direct summands for example by using trace filtrations (see [DK94, A.2]).

Hence, $P_c(\lambda)$ satisfies (II). Alternatively, one can also apply Proposition 1.5.48 to see that $P_c(\lambda)$ satisfies (II). By Remark 1.5.36, $|\Lambda|$ is greater than or equal to the number of classes of non-isomorphic simple A-modules. Assume that there exist λ and μ such that $\Delta(\lambda)$ and $\Delta(\mu)$ have the same projective cover. By Proposition 1.5.34 now using P_c instead of P we deduce that $\lambda = \mu$. Hence, $|\Lambda|$ is equal to the number of non-isomorphic simple A-modules. Now $[\Delta(\lambda) \colon S(\lambda)] = \dim_R \operatorname{Hom}_A(P_c(\lambda), \Delta(\lambda)) = 1$. By Lemma 1.5.37, if $[\operatorname{rad}\Delta(\lambda) \colon S(\mu)] \neq 0$, then $\mu < \lambda$. So, axiom (I) holds. $\square$

In addition to the last result, we can observe the following.

Lemma 1.5.40. *Let R be a field and let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split highest weight category over R. Denote by $P(\lambda)$ the projective cover of $\Delta(\lambda)$, $\lambda \in \Lambda$. Then, $\operatorname{End}_A(\operatorname{top} P(\lambda)) \simeq R$.*

Proof. Fix $S(\lambda) = \operatorname{top} P(\lambda)$. Let $g \in \operatorname{End}_A(S(\lambda))$. Denote by π_λ and γ_λ the surjective maps $P(\lambda) \to \Delta(\lambda)$, $\Delta(\lambda) \to S(\lambda)$, respectively. Since $P(\lambda)$ is projective, there is a map $s_g \in \operatorname{Hom}_A(P(\lambda), S(\lambda))$ satisfying $\gamma_\lambda \circ s_g = g \circ \gamma_\lambda \circ \pi_\lambda$. According to Proposition 1.5.33, there is a map $h_g \in \operatorname{End}_A(\Delta(\lambda))$ such that $h_g \circ \pi_\lambda = s_g$. Hence, we have the following diagram

$$\begin{tikzcd} P(\lambda) \arrow[r, "\pi_\lambda"] \arrow[dr, "s_g"] & \Delta(\lambda) \arrow[r, "\gamma_\lambda"] \arrow[d, "h_g"] & S(\lambda) \arrow[d, "g"] \\ & \Delta(\lambda) \arrow[r, "\gamma_\lambda"] & S(\lambda) \end{tikzcd}$$

Note that $\gamma_\lambda \circ h_g \circ \pi_\lambda = \gamma_\lambda \circ s_g = g \circ \gamma_\lambda \circ \pi_\lambda$. Here, π_λ is an epimorphism. Consequently, the diagram is commutative. Because of $\operatorname{End}_A(\Delta(\lambda)) \simeq R$, we have $h_g = r_g \operatorname{id}_{\Delta(\lambda)}$ for some $r_g \in R$. Thus, $r_g \operatorname{id}_{S(\lambda)} \gamma_\lambda = \gamma_\lambda \circ r_g \operatorname{id}_{\Delta(\lambda)} = g \circ \gamma_\lambda$. As γ_λ is an epimorphism, it follows that $g = r_g \operatorname{id}_{S(\lambda)}$. $\square$

We shall go back to the general case. Condition (iii) in the definition of split highest weight category 1.5.32 can be stated in terms of the projective modules $P(\lambda)$. In fact, as we will see next, 1.5.32(iii) occurs if and only if the direct sum of all $P(\lambda)$ constructed in condition 1.5.32(iv) is a progenerator of A-mod.

Proposition 1.5.41. *Let A be a projective Noetherian R-algebra and Λ a poset. Assume that, for each $\lambda \in \Lambda$, there are finitely generated A-modules $\Delta(\lambda)$ and projective A-modules $P(\lambda)$ together with an exact sequence*

$$0 \to C(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0 \tag{1.5.3.1}$$

where $C(\lambda)$ has a finite filtration by modules of the form $\Delta(\mu) \otimes_R U_\mu$, $U_\mu \in R\text{-proj}$, satisfying $\mu > \lambda$. Suppose that $P = \bigoplus_{\lambda\in\Lambda} P(\lambda)$ is a progenerator of A-mod.

If $\operatorname{Hom}_A(\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda$, then $N = 0$.

Proof. First, notice that if $n \neq 0$, then exists an epimorphism $P^t \twoheadrightarrow N$ for some $t > 0$. In particular, $\operatorname{Hom}_A(P,N) \neq 0$ for $N \neq 0$. Assume $\operatorname{Hom}_A(\Delta(\lambda), N) = 0$, $\forall \lambda \in \Lambda$. Consider $0 \neq g \in \operatorname{Hom}_A(P,N)$. Fix $C = \bigoplus_{\lambda\in\Lambda} C(\lambda)$. C has a finite filtration by modules of the form $\Delta(\mu) \otimes_R U_\mu$, $\mu \in \Lambda$. Fix $\Delta = \bigoplus_{\lambda\in\Lambda} \Delta(\lambda)$. We have a commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "k"] \arrow[dr, "g\circ k"'] & P \arrow[r, "\pi"] \arrow[d, "g"] & \Delta \arrow[r] & 0 \\ & & N & & \end{tikzcd}$$

If $g \circ k = 0$, then g induces a non-zero map $\Delta \simeq P/C \to N$, which is a contradiction. Therefore, $g \circ k \neq 0$. Using induction on the filtration of C, by the same reasoning we will obtain eventually a non-zero map $\Delta(\lambda) \to N$, which is a contradiction. Hence, $\operatorname{Hom}_A(P,N) = 0$ and thus, $N = 0$. $\square$

Proposition 1.5.42. *Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split highest weight category over* R. *Let* $P = \bigoplus_{\lambda\in\Lambda} P(\lambda)$. *Then,* P *is a progenerator for* A-mod.

Proof. It is enough to show that there exists an epimorphism $P^t \to A$ for some $t > 0$. First notice that, for any $\lambda \in \Lambda$, $\mathrm{Hom}_A(P(\lambda), A)$ is finitely generated over R because R is Noetherian (see Lemma 1.1.5).

Choose a set of generators $\{f_1^\lambda, \cdots, f_{t_\lambda}^\lambda\}$ for $\mathrm{Hom}_A(P(\lambda), A)$. Then, consider the map

$$h = \bigoplus_{\lambda\in\Lambda}\bigoplus_{j=1}^{t_\lambda} f_j^\lambda : \bigoplus_{\lambda\in\Lambda}\bigoplus_{j=1}^{t_\lambda} f_j^\lambda P(\lambda) \to A.$$

Consider $X = \mathrm{coker}\, h$, with $A \xrightarrow{\pi} X$. Hence, $\pi \circ h = 0$. In particular, $\pi \circ f_j^\lambda = 0$, $\forall \lambda \in \Lambda$, $\forall j = 1, \ldots, t_\lambda$. Now if $X \neq 0$, then by 1.5.32(*iii*) we get that exists some non-zero map in $\mathrm{Hom}_A(\Delta(\lambda), X)$, $\lambda \in \Lambda$. Hence, there is a non-zero map in $\mathrm{Hom}_A(P(\lambda), X), \lambda \in \Lambda$, say g. We have a commutative diagram,

$$\begin{tikzcd}
\bigoplus_{\lambda\in\Lambda}\bigoplus_{j=1}^{t_\lambda} f_j^\lambda P(\lambda) \arrow[r, "h"] & A \arrow[r, "\pi"] & X \arrow[r] & 0 \\
 & & P(\lambda) \arrow[ul, "\exists s"] \arrow[u, "g"'] &
\end{tikzcd}$$

The existence of the map s is due to $P(\lambda)$ being projective. Thus, $\pi \circ s = g \neq 0$. Since $s \in \mathrm{Hom}_A(P(\lambda), A)$ we can write

$$s = \sum_{j=1}^{t_\lambda} \alpha_j f_j^\lambda \quad \text{for some } \alpha_j \in R.$$

But,

$$\pi \circ s = \pi \circ \left(\sum_{j=1}^{t_\lambda} \alpha_j f_j^\lambda\right) = \sum_{j=1}^{t_\lambda} \alpha_j \underbrace{\pi \circ f_j^\lambda}_{=0} = 0.$$

So, we obtained a contradiction. Therefore, $X = 0$, and thus h is an epimorphism. We can extend h canonically to a direct sum of copies of P, hence the result follows. ☐

Combining Propositions 1.5.42 and 1.5.41 we obtain the following.

Corollary 1.5.43. $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *is split highest weight category if and only if the following conditions are satisfied:*

(a) *The modules* $\Delta(\lambda) \in A$-mod *are projective over* R.

(b) *Given* $\lambda, \mu \in \Lambda$, *if* $\mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu)) \neq 0$, *then* $\lambda \leq \mu$.

(c) $\mathrm{End}_A(\Delta(\lambda)) \simeq R$, $\forall \lambda \in \Lambda$.

(d) *Given* $\lambda \in \Lambda$, *there is* $P(\lambda) \in A$-proj *and an exact sequence* $0 \to C(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$ *such that* $C(\lambda)$ *has a finite filtration by modules of the form* $\Delta(\mu) \otimes_R U_\mu$ *with* $U_\mu \in R$-proj *and* $\mu > \lambda$.

(e) $P = \bigoplus_{\lambda\in\Lambda} P(\lambda)$ *is a progenerator for* A-mod.

1.5.4 Filtrations in split highest weight categories

The following lemmas are very useful to construct filtrations, as we will see later.

Lemma 1.5.44. *Let F be a free R-module of finite rank and let $L, Q \in A$-mod with $\operatorname{End}_A(L) \simeq R$. Let $f\colon F \to \operatorname{Ext}^1_A(Q,L)$ be surjective. There is an isomorphism $\operatorname{Hom}_R(F, \operatorname{Ext}^1_A(Q,L)) \to \operatorname{Ext}^1_A(Q, L \otimes_R DF)$. Then, the image of f in $\operatorname{Ext}^1_A(Q, L \otimes_R DF)$*

$$0 \to L \otimes_R DF \to X \to Q \to 0$$

satisfies $\operatorname{Ext}^1_A(X,L) = 0$.

Proof. Note first that there is such isomorphism. Let $Q^\bullet$ be a projective A-resolution for Q. Then,

$$\begin{aligned} \operatorname{Hom}_R(F, \operatorname{Ext}^1_A(Q,L)) &= \operatorname{Hom}_R(F, H^1(\operatorname{Hom}_A(Q^\bullet, L))) \\ &\simeq H^1(\operatorname{Hom}_R(F, \operatorname{Hom}_A(Q^\bullet, L))), \text{ since } \operatorname{Hom}_R(F,-) \text{ is exact} \\ &\simeq H^1(\operatorname{Hom}_R(F,R) \otimes_R \operatorname{Hom}_A(Q^\bullet, L)), \text{ since } F \in R\text{-proj} \\ &\simeq H^1(\operatorname{Hom}_A(Q^\bullet, L \otimes_R DF)) = \operatorname{Ext}^1_A(Q, L \otimes_R DF), \text{since } Q^\bullet \text{is a projective chain.} \end{aligned}$$

Now we need to know how to obtain explicitly the image of f. Consider $F = R^n$, and $\{e_i, 1 \leq i \leq n\}$ a basis. Then, $\{f(e_i)\colon \quad 0 \to L \to X_i \to Q \to 0 \quad | \quad 1 \leq i \leq n\}$ is an R-generator set for $\operatorname{Ext}^1_A(Q,L)$. Note that the previous isomorphism can be viewed as

$$\operatorname{Hom}_R(F, \operatorname{Ext}^1_A(Q,L)) \to \operatorname{Ext}^1_A(Q,L)^n \to \operatorname{Ext}^1_A(Q,L^n) \to \operatorname{Ext}^1_A(Q, \operatorname{Hom}_R(F,L)).$$

Consider a projective presentation for Q, $0 \to N \xrightarrow{k} M \xrightarrow{\pi} Q \to 0$. Apply the functors $\operatorname{Hom}_A(-,L)$ and $\operatorname{Hom}_A(-,L^n)$. We obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_A(N,L)^n \arrow[r, "\partial^n"] \arrow[d, "\simeq"] & \operatorname{Ext}^1_A(Q,L)^n \arrow[r] \arrow[d, "\simeq"] & 0 \\ \operatorname{Hom}_A(N,L^n) \arrow[r, "\partial_n"] & \operatorname{Ext}^1_A(Q,L^n) \arrow[r] & 0 \end{tikzcd}.$$

For every i, since ∂ is surjective there exists a map $s_i \in \operatorname{Hom}_A(N,L)$ such that $\partial(s_i) = f(e_i)$. This map relates with $f(e_i)$ by the following pushout diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "k"] \arrow[d, "s_i"] & M \arrow[r, "\pi"] \arrow[d, "p_i"] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] & X_i \arrow[r] \arrow[d, phantom, "\shortparallel"] & Q \arrow[r] & 0 \\ & & pushout(s_i,k) & & \end{tikzcd}.$$

This description of the map ∂ can be found with more detail in any book of homological algebra (see e.g [HS97, Theorem 2.4]).

Now the image of f in $\operatorname{Ext}^1_A(Q,L^n)$ is just $\partial_n(s_1,\dots,s_n)$. Hence, it is given by the following diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "k"] \arrow[d, "(s_1,\dots,s_n)"] & M \arrow[r, "\pi"] \arrow[d, "p"] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L^n \arrow[r, "g"] & X \arrow[r] \arrow[d, phantom, "\shortparallel"] & Q \arrow[r] & 0 \\ & & pushout((s_1,\dots,s_n),k) & & \end{tikzcd}. \tag{1.5.4.1}$$

Now applying $\mathrm{Hom}_A(-,L)$ to the bottom row of (1.5.4.1) yields

$$\mathrm{Hom}_A(L^n,L) \xrightarrow{\partial'} \mathrm{Ext}^1_A(Q,L) \to \mathrm{Ext}^1_A(X,L) \tag{1.5.4.2}$$

Note that $\mathrm{Hom}_A(L^n,L) \simeq \mathrm{Hom}_A(L,L)^n \simeq R^n = F$. Denote this isomorphism by $h\colon F \to \mathrm{Hom}_A(L^n,L)$. We claim that $f = \partial' \circ h$.

Consider $\pi_j\colon L^n \to L$ the canonical epimorphism. $\partial'(\pi_j)$ is given by

$$\begin{tikzcd}
0 \arrow[r] & L^n \arrow[r, "g"] \arrow[d, "\pi_j"] & X \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & Y_j \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}.$$

$$Y_j = pushout(\pi_j, g)$$

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "k"] \arrow[d, "(s_1,\dots,s_n)"'] \arrow[dd, bend left, "s_j"] & M \arrow[r, "\pi"] \arrow[d, "p"] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L^n \arrow[r, "g"] \arrow[d, "\pi_j"'] & X \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & Y_j \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}.$$

So, the external diagram is a pushout as well. In fact, Y_j is the pushout of (s_j,k). By the universal property of pushouts, it follows that the exact sequences $0 \to L \to Y_i \to Q \to 0$ and $0 \to L \to X_i \to Q \to 0$ are equivalent. Therefore, $\partial'(\pi_j) = f(e_j)$. So, the claim follows, and hence, ∂' is surjective. By the exactness of (1.5.4.2), it follows that $\mathrm{Ext}^1_A(X,L) = 0$. ☐

Lemma 1.5.45. *Let F be a free R-module and let $L,T \in A$-*mod *with* $\mathrm{End}_A(L) \simeq R$*. Let $g\colon F \to \mathrm{Ext}^1_A(L,T)$ be surjective. There is an isomorphism* $\mathrm{Hom}_R(F,\mathrm{Ext}^1_A(L,T)) \to \mathrm{Ext}^1_A(L\otimes_R F,T)$*. Then, the image of f in* $\mathrm{Ext}^1_A(L\otimes_R F,T)$

$$0 \to T \to Y \to L \otimes_R F \to 0$$

satisfies $\mathrm{Ext}^1_A(L,Y) = 0$.

Proof. The proof is the dual version of the previous one. For the sake of completeness, we will write a proof. The isomorphism exists: Let $T^\bullet$ be an injective resolution for T.

$$\begin{aligned}
\mathrm{Hom}_R(F,\mathrm{Ext}^1_A(L,T)) = \mathrm{Hom}_R(F,H^1(\mathrm{Hom}_A(L,T^\bullet))) &\simeq H^1(\mathrm{Hom}_R(F,\mathrm{Hom}_A(L,T^\bullet))) \\
&\simeq H^1(\mathrm{Hom}_A(F\otimes_R L,T^\bullet)), \text{ by Tensor-Hom adjunction} \\
&\simeq \mathrm{Ext}^1_A(F\otimes_R L,T).
\end{aligned}$$

Consider $F = R^n$, and let $\{e_i, 1 \le i \le n\}$ be a basis. Then,

$$\{0 \to T \to Y_i \to L \to 0 | 1 \le i \le n\}.$$

is an R-generator set for $\operatorname{Ext}^1_A(L,T)$. Consider an injective presentation for T

$$0 \to T \xrightarrow{k} M \xrightarrow{\pi} Q \to 0. \tag{1.5.4.3}$$

Applying $\operatorname{Hom}_A(L,-)^n$ and $\operatorname{Hom}_A(L^n,-)$ yields the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(L,Q)^n \arrow[r, "\partial^n"] \arrow[d, "\simeq"] & \operatorname{Ext}^1_A(L,T)^n \arrow[r] \arrow[d, "\simeq"] & 0 \\
\operatorname{Hom}_A(L^n,Q) \arrow[r, "\partial_n"] & \operatorname{Ext}^1_A(L^n,T) \arrow[r] & 0
\end{tikzcd}.$$

For every i, since ∂ is surjective there exists a map $s_i \in \operatorname{Hom}_A(L,Q)$ such that $\partial(s_i) = g(e_i)$,

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "k"] \arrow[d, equal] & M \arrow[r, "\pi"] & Q \arrow[r] & 0 \\
0 \arrow[r] & T \arrow[r] & Y_i \arrow[r] \arrow[u] & L \arrow[r] \arrow[u, "s_i"] & 0
\end{tikzcd}.$$
$$Y_i = pullback(s_i,\pi)$$

Now the image of g in $\operatorname{Ext}^1_A(L^n,T)$ is just $\partial_n(s_1,\ldots,s_n)$. Hence, it is given by the following diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "k"] \arrow[d, equal] & M \arrow[r, "\pi"] & Q \arrow[r] & 0 \\
0 \arrow[r] & T \arrow[r] & Y \arrow[r, "h"] \arrow[u] & L^n \arrow[r] \arrow[u, "(s_1,\ldots,s_n)"] & 0
\end{tikzcd}.$$
$$Y = pullback((s_1,\ldots,s_n),\pi)$$

Now applying $\operatorname{Hom}_A(L,-)$ to the image of g yields

$$\operatorname{Hom}_A(L,L^n) \xrightarrow{\partial'} \operatorname{Ext}^1_A(L,T) \to \operatorname{Ext}^1_A(L,Y). \tag{1.5.4.4}$$

Let w denote the canonical isomorphism $w\colon F \to \operatorname{Hom}_A(L,L^n)$. Now computing $\partial'(k_j)$ for the canonical monomorphisms $k_j\colon L \to L^n$ and comparing with the pullback diagram that gives the image g, it follows again that the induced external diagram is again a pullback. By the universal property of pullbacks, it follows that $g = \partial' \circ w$. So, we conclude by the exactness of (1.5.4.4) that $\operatorname{Ext}^1_A(L,Y) = 0$. $\square$

Our arguments used here in these two lemmas are valid in general abelian R-categories with enough projectives/injectives, respectively. Therefore, the results also hold for general abelian R-categories with enough projectives/injectives respectively. This remark will be useful in the construction of Dlab-Ringel standardization (see Subsection 1.5.9).

Let $L \in \mathscr{M}(A)$. Now we are ready to relate A-proj with A/J-proj for $J = \operatorname{im}\tau_L$.

Lemma 1.5.46. *[Rou08, Lemma 4.9] Let $L \in \mathscr{M}(A)$ and let $J = \operatorname{im}\tau_L$.*

(a) Given $P \in A$-proj, then $\operatorname{im}\tau_{L,P} = JP$ and $P/JP \in A/J$-proj.

(b) Let $Q \in A/J$-proj. Let F be a free R-module and $f\colon F \to \operatorname{Ext}^1_A(Q,L)$ surjective.

Let $0 \to L\otimes_R DF \xrightarrow{g} P \xrightarrow{h} Q \to 0$ be the extension in $\operatorname{Ext}^1_A(Q,L\otimes_R DF)$ corresponding to f via the isomorphism $\operatorname{Hom}_R(F,\operatorname{Ext}^1_A(Q,L)) \to \operatorname{Ext}^1_A(Q,L\otimes_R DF)$. Then, $P \in A$-proj.

Proof. Consider $P = A$. Then, $\operatorname{im}\tau_{L,A} = \operatorname{im}\tau_L = J = JA$ and $P/JP = A/JA = A/J$ which is clearly in A/J-proj.

Put $P = A^s, s > 0$. Then, $\operatorname{im}\tau_{L,P} = (\operatorname{im}\tau_L)^{\oplus s} = J^{\oplus s} = JP$. Now $P/JP = A^{\oplus s}/J^{\oplus s} \simeq (A/J)^{\oplus s} \in A/J$-proj.

Finally, assume P a summand of A^s, say $A^s \simeq P \oplus K$. Then, $JP \oplus JK = JA^s = \operatorname{im}\tau_{L,A^s} = \operatorname{im}\tau_{L,P} \oplus \operatorname{im}\tau_{L,K}$, for some K. Since $\operatorname{im}\tau_{L,P} \subset P$, it follows $\operatorname{im}\tau_{L,P} = JP$. Moreover, $(P \oplus K)/(JP \oplus JK) \simeq P/JP \oplus K/JK$, hence P/JP is a summand of $A^s/JA^s \in A/J$-proj, thus $a)$ follows.

Assume $Q = (A/J)^n$ for some n. Consider the canonical epimorphism $\pi\colon A^n \to Q$. Since A^n is projective over A, π factors through h, that is, there exists $\phi\colon A^n \to P$ such that $\pi = h \circ \phi$. Let $\psi = \phi + g\colon A^n \oplus L \otimes_R DF \to P$. Define $N = \ker\psi$.

Claim 1. ψ is surjective.

Let $p \in P$. Then, $h(p) \in Q$. Since π is surjective, there exists $x \in A^n$ such that $\pi(x) = h(p)$. Note that $h \circ \phi(x) = \pi(x) = h(p)$. Thus, $p - \phi(x) \in \ker h = \operatorname{im} g = L \otimes_R DF$. So, the claim follows.

Claim 2. $N \subset J^{\oplus n} \oplus L \otimes_R DF$.

Notice that $(x, y) \in N$ if and only if $0 = \psi(x, y) = \psi(x) + g(y)$ if and only if $\psi(x) = g(-y)$. In particular, $\pi(x) = h \circ \phi(x) = 0$, hence $x \in J^{\oplus n}$. So, the claim follows.

Now note that for any $x \in J^{\oplus n}$, $h \circ \phi(x) = 0$. So, $\phi(x) \in \ker h = \operatorname{im} g$. Since g is a monomorphism $\operatorname{im} g \simeq L \otimes_R DF \in A$-proj. Therefore, the following sequence is A-split exact

$$0 \to N \xrightarrow{z} J^{\oplus n} \oplus L \otimes_R DF \xrightarrow{w} \operatorname{im} g \to 0$$

where $z(x, y) = (x, y)$ and $w(x, y) = \phi(x) + g(y)$. Since $J^{\oplus n} \oplus L \otimes_R DF$ is projective over A, we obtain that $N \oplus \operatorname{im} g \simeq J^{\oplus n} \oplus L \otimes_R DF$. Furthermore, $J^{\oplus n} \simeq (L \otimes_R \operatorname{Hom}_A(L, A))^n \simeq L \otimes_R V$, $V \in R$-proj. Hence,

$$L \otimes_R (V \oplus DF) \simeq L \otimes_R V \oplus L \otimes_R DF \simeq N \oplus \operatorname{im} g.$$

By Lemma 1.5.44, $\operatorname{Ext}^1_A(P, L) = 0$. Hence, $\operatorname{Ext}^1_A(P, L \otimes_R (V \oplus DF)) = 0$ as $V \oplus DF \in R$-proj. In particular, $\operatorname{Ext}^1_A(P, N) = 0$. Thus, the exact sequence

$$0 \to N \xrightarrow{\psi} A^n \oplus L \otimes_R DF \to P \to 0$$

splits over A. Thus, it follows that P is projective.

Now assume that exists n such that $Q_0 = (A/J)^n \simeq Q \oplus Q_1$. Consider a free R-presentation for $\operatorname{Ext}^1_A(Q_1, L), g\colon R^s \to \operatorname{Ext}^1_A(Q_1, L)$. Therefore, $f \oplus g\colon F \oplus R^s \to \operatorname{Ext}^1_A(Q_0, L)$ is surjective. The image of g in $\operatorname{Ext}^1_A(Q_1, L \otimes_R D(R^s))$ is

$$0 \to L \otimes_R D(R^s) \to P_1 \to Q_1 \to 0. \tag{1.5.4.5}$$

So, the image of $f \oplus g$ in $\operatorname{Ext}^1_A(Q_0, L \otimes_R D(F \oplus R^s))$ is

$$0 \to L \otimes_R D(F \oplus R^s) \to P \oplus P_1 \to Q_0 \to 0. \tag{1.5.4.6}$$

By the previous case, $P \oplus P_1 \in A$-proj. So, we conclude that $P \in A$-proj, so $b)$ follows. ☐

Lemma 1.5.47. *[Rou08, Lemma 4.12] Let A be a projective Noetherian R-algebra. Let $\{\Delta(\lambda)\colon \lambda \in \Lambda\}$ be a finite set of modules in* A-mod *together with a poset structure on Λ. Let α be a maximal element in Λ. Then,* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *is a split highest weight category if and only if $\Delta(\alpha) \in \mathscr{M}(A)$ and* $(A/J\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda\setminus\{\alpha\}}\})$ *is a split highest weight category, where $J = \operatorname{im}\tau_{\Delta(\alpha)}$.*

Proof. Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split highest weight category. Let α be a maximal element in Λ. By (iv) of Definition 1.5.32, $\ker \pi_\alpha = 0$, so $\Delta(\alpha) \simeq P(\alpha) \in A$-proj. By 1.5.32($v$), $\mathrm{End}_A(\Delta(\alpha)) \simeq R$. As $\Delta(\alpha)$ is faithful over $\mathrm{End}_A(\Delta(\alpha))^{op}$, it follows that $\Delta(\alpha)$ is faithful over R. Let $\lambda \in \Lambda \backslash \{\alpha\}$. By iv) of Definition 1.5.32, $C(\lambda)$ has a finite filtration by modules of the form $\Delta(\mu) \otimes_R U_\mu$ with $U_\mu \in R$-proj and $\mu > \lambda$. In particular, $\Delta(\alpha)$ can appear. Note that $\Delta(\alpha)$ is projective over A, so we can rearrange the filtration so that all modules of the form $\Delta(\alpha) \otimes_R U_\alpha$, $U_\alpha \in R$-proj, appear at the bottom of the filtration. In fact, consider the filtration

$$0 \subset X_1 \subset \cdots \subset X_i \subset \cdots \subset X_n = C(\lambda).$$

Assume that $X_i/X_{i-1} \simeq \Delta(\alpha) \otimes_R U_\alpha \in A$-proj for some $U_\alpha \in R$-proj. Thus, $X_i \simeq \Delta(\alpha) \otimes_R U_\alpha \bigoplus X_{i-1}$. So, $X_i/(\Delta(\alpha) \otimes_R U_\alpha) \simeq X_{i-1}$, and hence the filtration until X_{i-1} can be written in the form

$$0 \subset \tilde{X}_1/(\Delta(\alpha) \otimes_R U_\alpha) \subset \cdots \subset \tilde{X}_{i-1}/(\Delta(\alpha) \otimes_R U_\alpha) = X_{i-1}.$$

Notice that $\tilde{X}_j/\tilde{X}_{j-1} \simeq \tilde{X}_j/(\Delta(\lambda) \otimes_R U_\alpha)/\tilde{X}_{j-1}/(\Delta(\lambda) \otimes_R U_\alpha) \simeq X_j/X_{j-1}$. Thus, we obtain a filtration

$$0 \subset \Delta(\alpha) \otimes_R U_\alpha \subset \tilde{X}_1 \subset \cdots \subset \tilde{X}_{i-1} = X_i \subset X_{i+1} \subset \cdots \subset C(\lambda). \tag{1.5.4.7}$$

Hence, we have an exact sequence

$$0 \to \Delta(\alpha) \otimes_R U_\lambda \to C(\lambda) \to X(\lambda) \to 0, \tag{1.5.4.8}$$

where the projective R-module U_λ encodes all the occurrences of $\Delta(\alpha)$ in the filtration of $C(\lambda)$, and consequently in the filtration of $P(\lambda)$. $X(\lambda)$ has a filtration by modules of the form $\Delta(\mu) \otimes_R U_\mu$ with $\mu > \lambda$, $\mu \neq \alpha$. Applying $\mathrm{Hom}_A(\Delta(\alpha), -)$ to the filtration of $P(\lambda)$, we get the exact sequence

$$0 \to \mathrm{Hom}_A(\Delta(\alpha), C(\lambda)) \to \mathrm{Hom}_A(\Delta(\alpha), P(\lambda)) \to \mathrm{Hom}_A(\Delta(\alpha), \Delta(\lambda)) \to 0. \tag{1.5.4.9}$$

By condition (ii) of split highest weight category, we have $\mathrm{Hom}_A(\Delta(\alpha), \Delta(\lambda)) = 0$, since α is maximal. Hence,

$$\mathrm{Hom}_A(\Delta(\alpha), C(\lambda)) \simeq \mathrm{Hom}_A(\Delta(\alpha), P(\lambda)).$$

Applying $\mathrm{Hom}_A(\Delta(\alpha), -)$ to (1.5.4.8), we get the exact sequence

$$0 \to \mathrm{Hom}_A(\Delta(\alpha), \Delta(\alpha) \otimes_R U_\lambda) \to \mathrm{Hom}_A(\Delta(\alpha), C(\lambda)) \to \mathrm{Hom}_A(\Delta(\alpha), X(\lambda)) \to 0. \tag{1.5.4.10}$$

We have $\mathrm{Hom}_A(\Delta(\alpha), X(\lambda)) = 0$. In fact, if $\mathrm{Hom}_A(\Delta(\alpha), X(\lambda)) \neq 0$, then by induction on the size of the filtration of $X(\lambda)$ it would exist $\Delta(\mu)$ with $\mu \neq \alpha$ such that $\mathrm{Hom}_A(\Delta(\alpha), \Delta(\mu)) \neq 0$. Since α is maximal, this cannot happen.

So, $U_\lambda \simeq \mathrm{Hom}_A(\Delta(\alpha), P(\lambda))$.

By Proposition 1.5.42, $P = \bigoplus_{\lambda \in \Lambda} P(\lambda)$ is a progenerator for A-mod. Put $P_0 := \Delta(\alpha) \otimes_R \bigoplus_{\lambda \in \Lambda} U_\lambda = \mathrm{im}\, \tau_{\Delta(\alpha), P} \subset P$. Thus, P/P_0 is an extension of $\Delta(\lambda)$ by $\bigoplus_{\lambda \in \Lambda} X(\lambda)$ with $U_\alpha = R$ and $X(\alpha) = 0$. So, $\mathrm{Hom}_A(\Delta(\alpha), P/P_0) = 0$. Since all standard modules are projective over R, we have that P/P_0 is projective over R. By Proposition 1.5.15, it follows that $\tau_{\Delta(\alpha),P}$ is split R-mono. Since P is a progenerator, it follows by the proof of Proposition 1.5.15, that $\tau_{\Delta(\alpha),A}$ split R-mono, thus $\Delta(\alpha) \in \mathscr{M}(A)$.

Fix $J = \mathrm{im}\, \tau_{\Delta(\alpha)}$. Since $\mathrm{Hom}_A(\Delta(\alpha), \Delta(\lambda)) = 0$ for $\lambda \neq \alpha$ it follows that $\Delta(\lambda) \in A/J$-mod by Corollary 1.5.23. Now we will show that $(A/J\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda \backslash \{\alpha\}}\})$ is a split highest weight category. Condition 1.5.32(i)

is clear. Since A/J-mod is a full subcategory of A-mod, it follows that

$$0 \neq \operatorname{Hom}_{A/J}(\Delta(\lambda'), \Delta(\lambda'')) = \operatorname{Hom}_A(\Delta(\lambda'), \Delta(\lambda'')). \tag{1.5.4.11}$$

By 1.5.32, we get $\lambda' \leq \lambda''$. So, condition 1.5.32(ii) for A/J holds. In the same way,

$$\operatorname{End}_{A/J}(\Delta(\lambda)) = \operatorname{End}_A(\Delta(\lambda)) \simeq R. \tag{1.5.4.12}$$

Let $N \in A/J$-mod satisfying $\operatorname{Hom}_{A/J}(\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda \backslash \{\alpha\}$. Then,

$$\operatorname{Hom}_A(\Delta(\lambda), N) = \operatorname{Hom}_{A/J}(\Delta(\lambda), N) = 0. \tag{1.5.4.13}$$

By Corollary 1.5.23, $\operatorname{Hom}_A(\Delta(\alpha), N) = 0$ since $N \in A/J$-mod. Since $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is split highest weight category $N = 0$, and thus condition 1.5.32(iii) holds.

For any $\lambda \neq \alpha$, define $Q(\lambda) = \Delta(\alpha) \otimes_R U_\lambda$. We have that,

$$\operatorname{im} \tau_{\Delta(\alpha), P(\lambda)} \simeq \Delta(\alpha) \otimes_R \operatorname{Hom}_A(\Delta(\alpha), P(\lambda)) \simeq \Delta(\alpha) \otimes_R U_\lambda = Q(\lambda).$$

By Lemma 1.5.46 (a), $P(\lambda)/Q(\lambda) \in A/J$-proj. Since $Q(\lambda) \subset C(\lambda)$, it follows that the following exact sequence yields condition 1.5.32(iv)

$$0 \to X(\lambda) = C(\lambda)/Q(\lambda) \to P(\lambda)/Q(\lambda) \to \Delta(\lambda) \to 0. \tag{1.5.4.14}$$

Conversely, assume now that $\Delta(\alpha) \in \mathscr{M}(A)$ and $(A/J\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda\backslash\{\alpha\}}\})$ is a split highest weight category.

By Remark 1.5.16, $\operatorname{End}_A(\Delta(\alpha)) \simeq R$. Now by condition 1.5.32(v) of $(A/J\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda\backslash\{\alpha\}}\})$ being a split highest weight category, $R \simeq \operatorname{End}_{A/J}(\Delta(\lambda)) = \operatorname{End}_A(\Delta(\lambda))$ for $\lambda \neq \alpha$. Thus, condition 1.5.32(v)holds for A. By condition 1.5.32(i) of $(A/J\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda\backslash\{\alpha\}}\})$ being split highest weight category, each $\Delta(\lambda)$ is projective over R. By definition of $\Delta(\alpha) \in \mathscr{M}(A)$, $\Delta(\alpha) \in R$-proj. Thus, condition 1.5.32(i) for A holds. Now by Corollary 1.5.23 and the fact that A/J-mod is full subcategory of A-mod, it follows condition 1.5.32(ii) and (iii) for A.

Since $\Delta(\alpha)$ is projective over A, we define $P(\alpha) = \Delta(\alpha)$. Now consider for $\lambda \neq \alpha$ the exact sequences provided by condition 1.5.32(iv) of A/J being a split highest weight category

$$0 \to C'(\lambda) \xrightarrow{i_\lambda^{A/J}} P_{A/J}(\lambda) \xrightarrow{\pi_\lambda^{A/J}} \Delta(\lambda) \to 0. \tag{1.5.4.15}$$

Consider an R-free presentation for $\operatorname{Ext}^1_A(P_{A/J}(\lambda), \Delta(\alpha))$, say $f_\lambda \colon F_\lambda \twoheadrightarrow \operatorname{Ext}^1_A(P_{A/J}(\lambda), \Delta(\alpha))$.

By Lemma 1.5.46 (b), we have an exact sequence

$$0 \to \Delta(\lambda) \otimes_R DF_\lambda \xrightarrow{k_\lambda} P(\lambda) \xrightarrow{h_\lambda} P_{A/J}(\lambda) \to 0, \tag{1.5.4.16}$$

where $P(\lambda) \in A$-proj. So, we have an exact sequence

$$0 \to C(\lambda) \xrightarrow{i_\lambda} P(\lambda) \xrightarrow{\pi_\lambda^{A/J} \circ h_\lambda} \Delta(\lambda) \to 0. \tag{1.5.4.17}$$

We define $\pi_\lambda = \pi_\lambda^{A/J} \circ h_\lambda$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C''(\lambda) \arrow[r, "l_\lambda"] \arrow[d, hook, "\exists w"] & C(\lambda) \arrow[r, "\exists g"] \arrow[d, hook, "i_\lambda"] & C'(\lambda) \arrow[r] \arrow[d, hook, "i_\lambda^{A/J}"] & 0 \\
0 \arrow[r] & \Delta(\lambda)\otimes_R DF_\lambda \arrow[r, "k_\lambda"] & P(\lambda) \arrow[r, "h_\lambda"] \arrow[d, two heads, "\pi_\lambda"] & P_{A/J}(\lambda) \arrow[r] \arrow[d, two heads, "\pi_\lambda^{A/J}"] & 0 \\
& & \Delta(\lambda) \arrow[r, equal] & \Delta(\lambda) &
\end{tikzcd} .$$

Here some observations are in order. The existence of g comes from the fact $\pi_\lambda^{A/J}\circ h_\lambda \circ i_\lambda = \pi_\lambda \circ i_\lambda = 0$. So, $C''(\lambda) = \ker g$. The existence of w comes from the fact $h_\lambda \circ i_\lambda \circ l_\lambda = i_\lambda^{A/J}\circ g \circ l_\lambda = 0$. By Snake Lemma, w is injective. On the other hand, $\pi_\lambda \circ k_\lambda = \pi_\lambda^{A/J}\circ h_\lambda \circ k_\lambda = 0$, so there exists $q_\lambda : \Delta(\lambda)\otimes_R DF_\lambda \to C(\lambda)$ such that $i_\lambda \circ q_\lambda = k_\lambda$. Now note that $i_\lambda^{A/J}\circ g\circ q_\lambda = h_\lambda \circ i_\lambda \circ q_\lambda = h_\lambda\circ k_\lambda = 0$. Since $i_\lambda^{A/J}$ is injective, $g\circ q_\lambda = 0$. Thus, for every $x\in \Delta(\lambda)\otimes_R DF_\lambda$, $k_\lambda(x) = i_\lambda\circ q_\lambda(x) = i_\lambda \circ l_\lambda(y) = k_\lambda(w(y))$ for some $y \in C''(\lambda)$. Thus, w is an isomorphism. So, $C(\lambda)$ has a filtration by standard modules given by the one of $\Delta(\lambda)\otimes_R DF_\lambda$ on the bottom and the filtration of $C'(\lambda)$ on the top. So, it follows that $C(\lambda)$ has a filtration by standard modules where only $\Delta(\mu)\otimes_R X$, with $\mu > \lambda$ and $X\in R$-proj can appear. So, we conclude that $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. ☐

Proposition 1.5.48. *[Rou08, Proposition 4.13] Suppose* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *is split highest weight category over a commutative Noetherian ring R. Let* $P\in A$-proj. *Let* $\Delta \to \{1,\dots,n\}$, $\Delta_i \mapsto i$ *be an increasing bijection. Then, there is a filtration*

$$0 = P_{n+1}\subset P_n \subset \cdots \subset P_1 = P \quad \textit{with} \quad P_i/P_{i+1}\simeq \Delta_i\otimes_R U_i, \quad \textit{for some} \quad U_i \in R\text{-proj}.$$

Proof. We shall proceed by induction on $|\Lambda| = n$. Assume $n = 1$. Consider $\Delta_1 \in \mathscr{M}(A)$. Let $P\in A$-proj. By Proposition 1.5.15, there exists $P_0 = \operatorname{im}\tau_{\Delta_1,P} = \Delta_1\otimes_R U_1 \subset P$, $U_1\in R$-proj and $\operatorname{Hom}_A(\Delta_1, P/P_0) = 0$. Thus, $P/P_0 = 0$. Hence, $0\subset \Delta_1\otimes_R U_1 = P$ is a filtration with the desired properties.

Assume now the result known for $|\Lambda| = n-1$. Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split highest weight category with $|\Lambda| = n$. By Lemma 1.5.47, $\Delta_n \in \mathscr{M}(A)$ and $(A/J\text{-mod}, \{\Delta_{j\, j=1,\dots,n-1}\})$ is a split highest weight category where $J = \operatorname{im}\tau_{\Delta_n}$.

Let $P\in A$-proj. By Proposition 1.5.15, there exists $U_n\in R$-proj such that $\operatorname{im}\tau_{L,P} = \Delta_n \otimes_R U_n$. By Lemma 1.5.46 (a), $JP = \operatorname{im}\tau_{L,P} = \Delta_n\otimes_R U_n$ and $P/JP \in A/J$-proj. By induction, there is a filtration for P/JP:

$$0 = P'_n \subset P'_{n-1}\subset \cdots \subset P'_1 = P/JP, \text{ with } P_i/P_{i+1}\simeq \Delta_i\otimes_R U_i,\ i = 1,\dots,n-1.$$

As the submodules of P/JP are exactly the submodules of P which contain JP, we get a filtration

$$0 = P_{n+1}\subset P_n \subset P_{n-1}\subset \cdots \subset P_1 = P,$$

where $P'_i \simeq P_i/JP$ and $P_n = JP$. Note that $P_i/P_{i+1} \simeq (P'_i/JP)/(P'_{i+1}/JP)$ for $i = 1,\dots n$. Thus, the claim follows. ☐

Notation 1.5.49. Denote by $\mathscr{F}(\Delta)$ the full subcategory of A-mod whose objects have filtration by objects in Δ. Denote $\tilde{\Delta}(\lambda) = \Delta(\lambda)\otimes_R U$, $U\in R$-proj, $\lambda\in\Lambda$. Denote by $\mathscr{F}(\tilde{\Delta})$ the full subcategory of A-mod whose objects have filtrations of the form given in Proposition 1.5.48, that is, filtrations by objects of the form $\Delta(\lambda)\otimes_R U$, $U\in R$-proj, $\lambda\in\Lambda$. Here, we are abusing the notation by writing $\mathscr{F}(\tilde{\Delta})$ instead of $\mathscr{F}(\tilde{\Delta}_{\lambda\in\Lambda})$. Sometimes, we will write $\mathscr{F}_A(\tilde{\Delta})$ instead of just $\mathscr{F}(\tilde{\Delta})$ to recall that $\mathscr{F}(\tilde{\Delta})$ is a full subcategory of A-mod. △

Proposition 1.5.50. *[Rou08, Proposition 4.13] Suppose* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *is a split highest weight categories. Then,*

(a) If $\operatorname{Ext}_A^1(\Delta(\lambda), \Delta(\mu)) \neq 0$*, then* $\lambda < \mu$*.*

(b) If $\operatorname{Ext}_A^i(\Delta(\lambda), \Delta(\mu) \neq 0$ *for some* $i > 0$*, then* $\lambda < \mu$*. In particular,* $\operatorname{Ext}_A^i(\Delta(\lambda), \Delta(\lambda)) = 0,\ i > 0$*.*

Proof. Consider the exact sequence $\delta\colon 0 \to C(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$. Applying $\operatorname{Hom}_A(-, \Delta(\mu))$ we obtain the exact sequence

$$\operatorname{Hom}_A(C(\lambda), \Delta(\mu)) \to \underset{\underset{0}{\not\shortparallel}}{\operatorname{Ext}_A^1(\Delta(\lambda), \Delta(\mu))} \to \underset{\underset{0}{\shortparallel}}{\operatorname{Ext}_A^1(P(\lambda), \Delta(\mu))}.$$

We deduce that $\operatorname{Hom}_A(C(\lambda), \Delta(\mu)) \neq 0$. So, there is a factor of $C(\lambda)$, say $\Delta(\alpha) \otimes_R U_\alpha$ such that $\operatorname{Hom}_A(\Delta(\alpha), \Delta(\mu)) \neq 0$. Thus, $\alpha \leq \mu$. Since $\Delta(\alpha) \otimes_R U_\alpha$ is a factor of $C(\lambda)$ we get that $\alpha > \lambda$. Thus, $\mu \geq \alpha > \lambda$. So, *a*) follows.

Now assume $\operatorname{Ext}_A^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$. Applying $\operatorname{Hom}_A(-, \Delta(\mu))$ to δ we deduce that $0 \neq \operatorname{Ext}_A^i(\Delta(\lambda), \Delta(\mu)) \simeq \operatorname{Ext}_A^{i-1}(C(\lambda), \Delta(\mu))$. Now consider the following filtration of $C(\lambda)$

$$0 \to C_1(\lambda) \to C(\lambda) \to \Delta(\alpha) \otimes_R U_\alpha \to 0. \tag{1.5.4.18}$$

Recall that its factors are of the form $\Delta(\alpha) \otimes_R U_\alpha$ with $\alpha > \lambda$ and $U_\alpha \in R$-proj. Applying again $\operatorname{Hom}_A(-, \Delta(\mu))$ it yields the exact sequence

$$\operatorname{Ext}_A^{i-1}(C_1(\lambda), \Delta(\mu)) \to \operatorname{Ext}_A^{i-1}(C(\lambda), \Delta(\mu)) \to \operatorname{Ext}_A^{i-1}(\Delta(\alpha) \otimes_R U_\alpha, \Delta(\mu)). \tag{1.5.4.19}$$

We can assume that λ is the maximal term that satisfies $\operatorname{Ext}_A^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$. Otherwise, we can consider $\operatorname{Ext}_A^{i-1}(\Delta(\alpha), \Delta(\mu)) \neq 0$ and repeat the process until either $\Delta(\alpha)$ is chosen to be projective or $i - 1 = 1$. Then, we are in situation *a*) and we are done since $\mu > \alpha > \lambda$. Thus, now assume $\operatorname{Ext}_A^{i-1}(\Delta(\alpha), \Delta(\mu)) = 0$. Hence, $\operatorname{Ext}_A^i(C_1(\lambda), \Delta(\mu)) \neq 0$, we can continue the procedure using the factors of $C_1(\lambda)$ until either we get $\operatorname{Ext}_A^l(\Delta(\alpha), \Delta(\mu)) \neq 0$ and $\alpha > \lambda$ which by previous discussion leads to $\mu > \alpha > \lambda$. In case, $\operatorname{Hom}_A(C_1(\lambda), \Delta(\mu)) \neq 0$ we will get $\operatorname{Hom}_A(\Delta(\alpha), \Delta(\mu)) \neq 0$ for some $\Delta(\alpha)$ factor of $C_1(\lambda)$. Thus, $\lambda < \alpha \leq \mu$. In particular, if $\operatorname{Ext}_A^i(\Delta(\lambda), \Delta(\lambda)) \neq 0$ for some $i > 0$, then by *b*) $\lambda < \lambda$ which is an absurd. ☐

Proposition 1.5.51. *Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split highest weight category. If* $M \in \mathscr{F}(\Delta)$*, then* $M \in \mathscr{F}(\tilde{\Delta})$*.*

Proof. Without loss of generality, we can assume that the factors in the filtration of M appear in non-increasing order. Assume

$$0 = M_{t+1} \subset M_t \subset M_{t-1} \subset \cdots \subset M_1 = M \quad \text{where } M_j/M_{j+1} \simeq \Delta_{k_j}. \tag{1.5.4.20}$$

Choose $\lambda \in \Lambda$ maximal. So, there is a maximal index i (possibly $t+1$) such that

$$0 \subset M_t \subset M_{t-1} \subset \cdots \subset M_i \quad \text{with } M_j/M_{j+1} \simeq \Delta(\lambda),\ t \geq j \geq i. \tag{1.5.4.21}$$

Using the fact that $\Delta(\lambda)$ is projective over A, the exact sequences

$$0 \to M_{j+1} \to M_j \to \Delta(\lambda) \to 0,\ i \leq j \leq t,$$

split. So, we deduce that $M_i \simeq \Delta(\lambda)^{t+1-i}$ and this corresponds to the multiplicity of $\Delta(\lambda)$ in the filtration of M.

We shall prove the claim by induction on $n = |\Lambda|$. Assume that $n = 1$. Then, $\Delta(1)$ is maximal, and by the previous discussion, the claim follows. Assume now that the result holds for split quasi-hereditary algebras with $|\Lambda| = n-1$. Let A be a split quasi-hereditary algebra with $|\Lambda| = n$. Let $\lambda \in \Lambda$ maximal. By the previous discussion, $M_i \simeq \Delta(\lambda)^{t+1-i}$. Then, the module M/M_i has a filtration

$$0 \subset M_{i-1}/M_i \subset M_{i-2}/M_i \subset \cdots \subset M/M_i. \tag{1.5.4.22}$$

In particular, M/M_i does not have $\Delta(\lambda)$ in its filtration. It follows that $\mathrm{Hom}_A(\Delta(\lambda), M/M_i) = 0$. By Corollary 1.5.23, $M/M_i \in A/J_\lambda$-mod. $(A/J_\lambda\text{-mod}, \Delta(\mu)_{\mu \neq \lambda})$ is split highest weight category with $|\Lambda \backslash \{\lambda\}| = n-1$. By induction, M/M_i has a filtration

$$0 = F_n \subset \cdots \subset F_1 = M/M_i \quad \text{with } F_j/F_{j+1} \simeq \Delta_j \otimes_R U_j. \tag{1.5.4.23}$$

Here U_j is a free R-module and $\Lambda \backslash \{\lambda\} \to \{1, \dots, n-1\}$ an increasing bijection. Put $\lambda \longleftrightarrow n$. So, the induced map $\Lambda \to \{1, \dots, n\}$ is an increasing bijection. Note that each F_j is written on the form F'_j/M_i. Therefore,

$$0 \subset F'_n = M_i \subset F'_{n-1} \subset \cdots \subset F'_1 = M \tag{1.5.4.24}$$

is a filtration of M such that $F'_n \simeq \Delta_n \otimes_R R^{t+1-i}$ and $F'_j/F'_{j+1} \simeq F_j/M_i/F_{j+1}/M_i \simeq F_j/F_{j+1} \simeq \Delta_j \otimes_R U_j$. This means that $M \in \mathscr{F}(\tilde{\Delta})$. □

Remark 1.5.52. Notice that the modules of $\mathscr{F}(\tilde{\Delta})$ which the U_j are free R-modules are exactly the modules in $\mathscr{F}(\tilde{\Delta})$. △

Remark 1.5.53. Assume that R is a field. Then, clearly $\mathscr{F}(\Delta) = \mathscr{F}(\tilde{\Delta})$. △

Proposition 1.5.54. *Let $F\colon A$-mod $\to B$-mod be an equivalence of categories. Assume $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ is split highest weight category then $(B\text{-mod}, \{F\Delta(\lambda)_{\lambda \in \Lambda}\})$ is a split highest weight category.*

Proof. For every $\lambda \in \Lambda$, $\mathrm{End}_B(F\Delta(\lambda)) \simeq \mathrm{End}_A(\Delta(\lambda)) \simeq R$, since F is full and faithful. Assume that $0 \neq \mathrm{Hom}_B(F\Delta(\lambda'), F\Delta(\lambda'')) \simeq \mathrm{Hom}_A(\Delta(\lambda'), \Delta(\lambda''))$. Thus, $\lambda' \leq \lambda''$. If $N \in B$-mod such that $\mathrm{Hom}_B(F\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda$, then $FM = N$ for some $M \in A$-mod since F is essentially surjective. Therefore,

$$\mathrm{Hom}_A(\Delta(\lambda), M) \simeq \mathrm{Hom}_B(F\Delta(\lambda), FM) = 0, \quad \forall \lambda \in \Lambda. \tag{1.5.4.25}$$

Hence, $M = 0$, and in particular $N = FM = 0$.

Consider the exact sequence $0 \to C(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$. Applying F we get the exact sequence $0 \to FC(\lambda) \to FP(\lambda) \to F\Delta(\lambda) \to 0$. Since F is an equivalence of categories it preserves inclusions and quotients, so a filtration by standard modules is sent to a filtration by modules in $F\tilde{\Delta}$. Since F is an equivalence of categories, there is a progenerator P such that $F = \mathrm{Hom}_A(P, -)$. Therefore, F sends A-mod$\cap R$-proj to B-mod$\cap R$-proj. Therefore, the axioms of split highest weight category are verified. □

1.5.5 Split highest weight categories under change of rings

Proposition 1.5.55. *[Rou08, Proposition 4.14] Let S be a commutative Noetherian R-algebra. Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split highest weight category. Then, $(S \otimes_R A\text{-mod}, \{S \otimes_R \Delta(\lambda)_{\lambda \in \Lambda}\})$ is a split highest weight category. Moreover, $S \otimes_R (A/J)$-mod $\simeq S \otimes_R A/S \otimes_R J$-mod, where J is a split heredity ideal of A.*

Proof. We shall proceed by induction on $t = |\Lambda|$. Assume $t = 1$. Hence, $\Delta(\lambda) \in \mathscr{M}(A)$. By Lemma 1.5.20, $S \otimes_R \Delta(\lambda) \in \mathscr{M}(S \otimes_R A)$. Fix $J = \operatorname{im} \tau_{\Delta(\lambda)}$. By condition 1.5.32(*iii*) of split highest weight category A/J-mod $=$ 0 and $S \otimes_R J = S \otimes_R \operatorname{im} \tau_{\Delta(\lambda)} = \operatorname{im}(S \otimes_R \tau_{\Delta(\lambda)} = J_S$ submodule of $S \otimes_R A$ since $S \otimes_R \tau_{\Delta(\lambda)}$ is an $(S \otimes_R A, S)$-monomorphism.

By Proposition 1.5.15, it follows that A/J is projective over R, thus $\operatorname{Tor}_1^R(S, A/J) = 0$. Thus, the sequence $0 \to J \to A \to A/J \to 0$ remains exact under the functor $S \otimes_R -$. Hence, the sequence

$$0 \to J_S \to S \otimes_R A \to S \otimes_R A/S \otimes_R J \to 0 \tag{1.5.5.1}$$

is exact. Thus, $S \otimes_R A/J \simeq S \otimes_R A/S \otimes_R J$ as S-algebras. Now assume the result is known for $t-1$. Let $\alpha \in \Lambda$ be a maximal element. Then, $\Delta(\alpha) \in \mathscr{M}(A)$ and $(A/J\text{-mod}, \Lambda \backslash \{\lambda\})$ is a split highest weight category with $J = \operatorname{im} \tau_{\Delta(\alpha)}$. Analogous to the case $t = 1$, $S \otimes_R \Delta(\lambda) \in \mathscr{M}(S \otimes_R A)$ and $S \otimes_R A/S \otimes_R J = S \otimes_R A/J$ as S-algebras. By induction, $(S \otimes_R A/J\text{-mod}, \Lambda \backslash \{\alpha\})$ is split highest weight category with standard modules $\{S \otimes_R \Delta(\lambda) \colon \lambda \in \Lambda \backslash \{\alpha\}\}$. By Lemma 1.5.47, $(S \otimes_R A\text{-mod}, \Lambda)$ is a split highest weight category. $\square$

Theorem 1.5.56. *[Rou08, Theorem 4.15]Let A be a projective Noetherian R-algebra and let $\{\Delta(\lambda)_{\lambda \in \Lambda}\}$ be a set of finitely generated A-modules indexed by a poset.* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ *is a split highest weight category if and only if $\Delta(\lambda)$ are projective R-modules, $\lambda \in \Lambda$ and* $(A(\mathfrak{m})\text{-mod}, \{\Delta(\lambda)(\mathfrak{m})_{\lambda \in \Lambda}\})$ *is a split highest weight category for every maximal ideal $\mathfrak{m}$ of R.*

Proof. For every maximal ideal $\mathfrak{m}$ in R, the residue field $R(\mathfrak{m})$ is a Noetherian commutative algebra over R, so by Proposition 1.5.55 $(A(\mathfrak{m})\text{-mod}, \Lambda)$ is split highest weight category with standards $\Delta(\lambda)(\mathfrak{m})$. The modules $\Delta(\lambda)$ are projective over R by definition of A-mod being a split highest weight category.

Conversely, we shall proceed by induction on $t = |\Lambda|$. Let $\mathfrak{m}$ be a maximal ideal of R. Assume $t = 1$. By assumption $\Delta(\lambda)(\mathfrak{m}) \in \mathscr{M}(A(\mathfrak{m}))$. By Lemma 1.5.20, $\Delta(\lambda) \in \mathscr{M}(A)$. Let $M \in A$-mod be such that $\operatorname{Hom}_A(\Delta(\lambda), M) = 0$. Then,

$$\operatorname{Hom}_{A(\mathfrak{m})}(\Delta(\lambda)(\mathfrak{m}), M(\mathfrak{m})) = 0.$$

Since $A(\mathfrak{m})$-mod is a split highest weight category $M(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ in R. Thus, $M = 0$. Therefore, $(A\text{-mod}, \Lambda)$ is a split highest weight category.

Now assume the result known for $t-1$. Let α be a maximal element in Λ. By assumption, $(A(\mathfrak{m})\text{-mod}, \Lambda)$ is a split highest weight category for every maximal ideal $\mathfrak{m}$ in R. By Lemma 1.5.47, $\Delta(\alpha)(\mathfrak{m}) \in \mathscr{M}(A(\mathfrak{m}))$ and $(A(\mathfrak{m})/J(\mathfrak{m})\text{-mod}, \Lambda \backslash \{\alpha\})$ is split highest weight category for every maximal ideal $\mathfrak{m}$ in R. Since $\Delta(\alpha)$ is projective over R, it follows, by Lemma 1.5.20, that $\Delta(\alpha) \in \mathscr{M}(A)$. Here,

$$J(\mathfrak{m}) = \operatorname{im} \tau_{R(\mathfrak{m}) \otimes_R \Delta(\alpha)} = R(\mathfrak{m}) \otimes_R \operatorname{im} \tau_{\Delta(\alpha)} = R(\mathfrak{m}) \otimes_R J.$$

As $\Delta(\alpha) \in \mathscr{M}(A)$, $A/\operatorname{im} \tau_{\Delta(\alpha)} = A/J$ is a projective R-module. So, $\operatorname{Tor}_1^R(R(\mathfrak{m}), A/J) = 0$. We deduce that

$$A/J(\mathfrak{m}) = R(\mathfrak{m}) \otimes_R A/J \simeq R(\mathfrak{m}) \otimes_R A/R(\mathfrak{m}) \otimes_R J = A(\mathfrak{m})/J(\mathfrak{m}). \tag{1.5.5.2}$$

Thus, $(A/J(\mathfrak{m})\text{-mod}, \Lambda \backslash \{\alpha\})$ is split highest weight category for every maximal ideal $\mathfrak{m}$ in R. By induction, $(A/J\text{-mod}, \Lambda \backslash \{\alpha\})$ is a split highest weight category. Finally, by Lemma 1.5.47, the result follows. $\square$

Theorem 1.5.57. *Let A be a projective Noetherian R-algebra and let $\{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a set of finitely generated A-modules indexed by a poset.* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ *is a split highest weight category if and only if* $(A_{\mathfrak{m}}\text{-mod}, \{(\Delta(\lambda)_{\mathfrak{m}})_{\lambda \in \Lambda}\})$ *is a split highest weight category for every maximal ideal $\mathfrak{m}$ of R.*

Proof. The proof is analogous to Theorem 1.5.56. For every maximal ideal $\mathfrak{m}$ in R, $R_{\mathfrak{m}}$ is a Noetherian commutative ring which is an R-algebra. By Proposition 1.5.55, if $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ is a split highest weight category then $(A_{\mathfrak{m}}\text{-mod}, \{(\Delta(\lambda)_{\mathfrak{m}})_{\lambda \in \Lambda}\})$ is a split highest weight category for every maximal ideal $\mathfrak{m}$ in R.

Conversely, we shall proceed by induction on $t = |\Lambda|$. By assumption $\Delta(\lambda)_\mathfrak{m} \in \mathscr{M}(A_\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. By Lemma 1.5.20, $\Delta(\lambda) \in \mathscr{M}(A)$. Let $M \in A$-mod be such that $\operatorname{Hom}_A(\Delta(\lambda), M) = 0$. Then, $\operatorname{Hom}_{A_\mathfrak{m}}(\Delta(\lambda)_\mathfrak{m}, M_\mathfrak{m}) = 0$ which implies that $M_\mathfrak{m} = 0$. Hence, $M = 0$. Therefore, the result holds for $t = 1$.

Assume the result known for $t-1$. Let α be a maximal element in Λ. By Lemma 1.5.47, $\Delta(\alpha)_\mathfrak{m} \in \mathscr{M}(A_\mathfrak{m})$ and $(A_\mathfrak{m}/J_\mathfrak{m}\text{-mod}, \Lambda \backslash \{\alpha\})$ is a split highest weight category for every maximal ideal $\mathfrak{m}$ in R. By Lemma 1.5.20, $\Delta(\alpha) \in \mathscr{M}(A)$. Since $R_\mathfrak{m}$ is flat over R, we deduce that $(A/\operatorname{im}\tau_{\Delta(\alpha)})_\mathfrak{m} = (A/J)_\mathfrak{m} \simeq A_\mathfrak{m}/J_\mathfrak{m}$. By induction, $(A/J\text{-mod}, \Lambda\backslash\{\alpha\})$ is a split highest weight category. By Lemma 1.5.47, the result follows. ☐

Parallelly to Lemma 1.5.21, we can say that an algebra is split quasi-hereditary over some field if this algebra is the restriction of some quasi-hereditary algebra over an algebraically closed field.

Theorem 1.5.58. *Let A be a finite-dimensional k-algebra for some field k and let $\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a set of finitely generated A-modules indexed by a poset. If $\overline{k}$ is the algebraic closure of k, then $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category if and only if $(\overline{k}\otimes_k A\text{-mod}, \{\overline{k}\otimes_k \Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category.*

Proof. The result follows by Proposition 1.5.55, Lemma 1.5.47 and 1.5.21. ☐

Given the formulations of Theorems 1.5.56 to 1.5.58, we can ask whether there is a version involving the quotient field of an integral domain. The following tries to address this question and it aims to generalize Lemma 1.6 of [DPS98a].

Lemma 1.5.59. *Let R be a regular domain with quotient field K. Let A be a projective Noetherian R-algebra. Assume that $\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a set of finitely generated A-modules indexed by a poset and the following conditions hold:*

(i) For $\lambda \in \Lambda$, $\Delta(\lambda) \in R$-proj;

(ii) For each $\lambda \in \Lambda$, there exists a projective A-module $P(\lambda)$ so that there is an exact sequence

$$0 \to C(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0, \tag{1.5.5.3}$$

where $C(\lambda) \in \mathscr{F}(\tilde{\Delta}(\mu)_{\mu>\lambda})$;

(iii) $\bigoplus_{\lambda\in\Lambda} P(\lambda)$ is a projective generator for A-mod.

Then, $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category if and only if $(K\otimes_R A\text{-mod}, K\otimes_R \Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category.

Proof. By Proposition 1.5.55, one of the implications is clear.

Conversely, assume that $(K\otimes_R A\text{-mod}, K\otimes_R \Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. In view of Corollary 1.5.43, it is enough to show that $\operatorname{End}_A(\Delta(\lambda)) \simeq R$ and the condition of non-zero homomorphisms between standard modules. Suppose that $\operatorname{Hom}_A(\Delta(\lambda), \Delta(\mu)) \neq 0$. Then,

$$0 \neq K\otimes_R \operatorname{Hom}_A(\Delta(\lambda), \Delta(\mu)) \simeq \operatorname{Hom}_{K\otimes_R A}(K\otimes_R \Delta(\lambda), K\otimes_R \Delta(\mu)). \tag{1.5.5.4}$$

Hence, $\lambda \leq \mu$. Let $\mathfrak{p}$ be a prime ideal of R with height one. K is the quotient field of $R_\mathfrak{p}$ and $\dim R_\mathfrak{p} = 1$. In particular,

$$K\otimes_{R_\mathfrak{p}} \operatorname{End}_{A_\mathfrak{p}}(\Delta(\lambda)_\mathfrak{p}) \simeq \operatorname{End}_{K\otimes_{R_\mathfrak{p}} A_\mathfrak{p}}(K\otimes_{R_\mathfrak{p}} \Delta(\lambda)_\mathfrak{p}) \simeq \operatorname{End}_{K\otimes_R A}(K\otimes_R \Delta(\lambda)) \simeq K. \tag{1.5.5.5}$$

On the other hand, using the monomorphism $\operatorname{End}_{A_\mathfrak{p}}(\Delta_\mathfrak{p}(\lambda)) \to \operatorname{Hom}_{A_\mathfrak{p}}(P(\lambda)_\mathfrak{p}, \Delta(\lambda)_\mathfrak{p})$ we obtain that $\operatorname{End}_{A_\mathfrak{p}}(\Delta_\mathfrak{p}(\lambda)) \in R_\mathfrak{p}$-proj. Thus, (1.5.5.5) implies that $\operatorname{End}_{A_\mathfrak{p}}(\Delta_\mathfrak{p}(\lambda)) \simeq R_\mathfrak{p}$. This shows that $\operatorname{End}_{A_\mathfrak{p}}(\Delta_\mathfrak{p})$ is a maximal order in K. By Theorem 1.5 of [AG60], $\operatorname{End}_A(\Delta(\lambda))$ is a maximal order in K. By Theorem 4.3 of [AG60], we conclude that $\operatorname{End}_A(\Delta(\lambda)) \simeq R$. ☐

Remark 1.5.60. If, in addition to knowing 1.5.59(i) we know that $\Delta(\lambda)$ is R-faithful, then we can consider another approach without using maximal orders. In fact, $\operatorname{End}_A(\Delta(\lambda))$ is torsion free over R and there exists an exact sequence $0 \to R \to \operatorname{End}_A(\Delta(\lambda)) \to X \to 0$. By Proposition 3.4 of [AB59], if $X \neq 0$, then $X_\mathfrak{p} \neq 0$ for some prime ideal of R with height one. But, as we showed this cannot happen. △

1.5.6 Uniqueness of standard modules with respect to the poset Λ

We are now ready to address some questions concerning the uniqueness of standard modules and the projective modules $P(\lambda)$. Given the existence of $\Delta(\lambda)$ we saw that the projective modules $P(\lambda)$ given by the condition 1.5.32(iv) of split highest weight category are not unique up to isomorphism. However, we saw that for split quasi-hereditary algebras over fields we could replace the projective modules in 1.5.32(iv) with indecomposable projective modules. In the following, we will see a sort of generalization of this phenomenon to general commutative rings.

Proposition 1.5.61. *Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split highest weight category so that the projective modules* $P(\lambda)$ *in 1.5.32(iv) become indecomposable under* $R(\mathfrak{m})\otimes_R -$ *for every maximal ideal* $\mathfrak{m}$ *of* R*. Assume that there exists* $Q(\lambda) \in A$-proj *which becomes indecomposable under* $R(\mathfrak{m})\otimes_R -$ *together with an exact sequence*

$$0 \to S(\lambda) \to Q(\lambda) \xrightarrow{p_\lambda} \Delta(\lambda) \to 0, \quad \textit{such that} \quad S(\lambda) \in \mathscr{F}(\tilde{\Delta}_{\mu>\lambda}).$$

Then, there is an isomorphism $g\colon Q(\lambda) \to P(\lambda)$ *making the following diagram commutative*

$$\begin{tikzcd} Q(\lambda) \arrow[rr, "p_\lambda"] \arrow[dr, "g"'] & & \Delta(\lambda) \\ & P(\lambda) \arrow[ur, "\pi_\lambda"'] & \end{tikzcd}.$$

Proof. Since $P(\lambda)$ and $Q(\lambda)$ are projective A-modules, there are A-homomorphisms f and g making the following diagram commutative:

$$\begin{tikzcd} P(\lambda) \arrow[r, "\pi_\lambda"] \arrow[d, "f"] & \Delta(\lambda) \arrow[r] \arrow[d, equal] & 0 \\ Q(\lambda) \arrow[r, "p_\lambda"] \arrow[d, "g"] & \Delta(\lambda) \arrow[r] \arrow[d, equal] & 0 \\ P(\lambda) \arrow[r, "\pi_\lambda"] & \Delta(\lambda) \arrow[r] & 0 \end{tikzcd} \cdot \tag{1.5.6.1}$$

Applying the right exact functor $R(\mathfrak{m})\otimes_R -$ for every maximal ideal $\mathfrak{m}$ of R we obtain the commutative diagram

$$\begin{tikzcd} P(\lambda)(\mathfrak{m}) \arrow[r, "\pi_\lambda(\mathfrak{m})"] \arrow[d, "f(\mathfrak{m})"] & \Delta(\lambda)(\mathfrak{m}) \arrow[r] \arrow[d, equal] & 0 \\ Q(\lambda)(\mathfrak{m}) \arrow[r, "p_\lambda(\mathfrak{m})"] \arrow[d, "g(\mathfrak{m})"] & \Delta(\lambda)(\mathfrak{m}) \arrow[r] \arrow[d, equal] & 0 \\ P(\lambda)(\mathfrak{m}) \arrow[r, "\pi_\lambda(\mathfrak{m})"] & \Delta(\lambda)(\mathfrak{m}) \arrow[r] & 0 \end{tikzcd} \cdot \tag{1.5.6.2}$$

Note that $g(\mathfrak{m}) \circ f(\mathfrak{m}) = g \otimes \mathrm{id}_{R(\mathfrak{m})} \circ f \otimes \mathrm{id}_{R(\mathfrak{m})} = g \circ f \otimes \mathrm{id}_{R(\mathfrak{m})} = g \circ f(\mathfrak{m})$. For any maximal ideal $\mathfrak{m}$ of R, $(A(\mathfrak{m}), \{\Delta(\lambda)(\mathfrak{m})_{\lambda\in\Lambda}\})$ is a split highest weight category with projectives $P(\lambda)(\mathfrak{m})$ and $Q(\lambda)(\mathfrak{m})$. Further, $(Q(\lambda)(\mathfrak{m}), \pi_\lambda(\mathfrak{m}))$ and $(P(\lambda)(\mathfrak{m}), p_\lambda(\mathfrak{m}))$ are projective covers of $\Delta(\lambda)(\mathfrak{m})$. It follows by diagram (1.5.6.2) that $P(\lambda) \subset \operatorname{im} g \circ f(\mathfrak{m}) + \ker pi_\lambda(\mathfrak{m})$. Since $\ker \pi_\lambda(\mathfrak{m})$ is a superfluous module, it follows that $g \circ f(\mathfrak{m})$ is surjective for every maximal ideal $\mathfrak{m}$ of R. By Nakayama's lemma, $g \circ f$ is surjective. Since $g \circ f \in \mathrm{End}_A(P(\lambda))$, this surjective must be an isomorphism by Nakayama's Lemma for endomorphisms. Since $(Q(\lambda)(\mathfrak{m}), \pi_\lambda(\mathfrak{m}))$ is a projective cover of $\Delta(\lambda)(\mathfrak{m})$, it follows, by symmetry, that $f \circ g$ is an isomorphism. Hence, both f and g are isomorphisms. So, the claim follows. ☐

Before we proceed any further we should pay attention to the following fact.

Observation 1.5.62. Assume that R is a local commutative Noetherian ring with unique maximal ideal $\mathfrak{m}$ and A-mod is a split highest weight category with standard modules $\Delta(\mu)$, $\mu \in \Lambda$. Then, we can pick the projective modules in 1.5.32(iv) so that they become indecomposable under $R(\mathfrak{m}) \otimes_R -$. Such construction can be made by reverse induction. If $\lambda \in \Lambda$ is maximal, then define $P(\lambda) := \Delta(\lambda)$. For the induction step, assume that μ is maximal in $\Lambda\backslash\{\lambda\}$ and λ is maximal in Λ. The Picard group of R is trivial and the multiplicity of $\Delta(\lambda)$ in the projective associated with $\Delta(\mu)$ is controlled by $\mathrm{Ext}_A^1(\Delta(\mu), \Delta(\lambda)) \in R$-mod in view of Lemma 1.5.46. Since all extensions between $\Delta(\mu)$ and $\Delta(\lambda)$ are (A,R)-exact sequences we can pick by Nakayama's Lemma a minimal set of generators for $\mathrm{Ext}_A^1(\Delta(\mu), \Delta(\lambda))$ of size $\dim_{R(\mathfrak{m})} \mathrm{Ext}^1_{A(\mathfrak{m})}(\Delta(\mu)(\mathfrak{m}), \Delta(\lambda)(\mathfrak{m}))$. Using Lemma 1.5.46, this means that we can construct $P(\mu)$ so that the multiplicities of $\Delta(\lambda)$ in $P(\mu)$ and of $\Delta(\lambda)(\mathfrak{m})$ in the projective cover of $\Delta(\mu)(\mathfrak{m})$ over A coincide. Hence, $P(\mu)$ can be constructed so that $P(\mu)(\mathfrak{m})$ is the projective cover of $\Delta(\mu)(\mathfrak{m})$.

In the field case, given an order on Λ, the standard modules when defined are unique (see for example [DK94, A]). This result can be extended to local commutative Noetherian rings in the following way.

Proposition 1.5.63. *Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split highest weight category over a local commutative Noetherian ring. Let* $\Delta \to \{1,\ldots,t\}$, $\Delta_i \mapsto i$ *be an increasing bijection. Choose* $P_i \in A\text{-proj}$ *so that* $P_i(\mathfrak{m})$ *is the projective cover of* $\Delta_i(\mathfrak{m})$ *for all* $i \in \{1,\ldots,t\}$. *Define*

$$U_i = \sum_{j>i} \sum_{f\in \mathrm{Hom}_A(P_j,P_i)} \operatorname{im} f.$$

Then, $\Delta_i \simeq P_i/U_i$.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. By Theorem 1.5.56, $(A(\mathfrak{m}), \{\Delta(\lambda)(\mathfrak{m})_{\lambda\in\Lambda}\})$ is split highest weight category. Since $R(\mathfrak{m})$ is a field, $\Delta_i(\mathfrak{m}) \simeq P_i(\mathfrak{m})/C_i(\mathfrak{m})$. We have,

$$\begin{aligned} C_i(\mathfrak{m}) &\simeq \sum_{j>i} \sum_{f\in \mathrm{Hom}_{A(\mathfrak{m})}(P_j(\mathfrak{m}),P_i(\mathfrak{m}))} \operatorname{im} f \simeq \sum_{j>i} \sum_{f\in \mathrm{Hom}_A(P_j,P_i)(\mathfrak{m})} \operatorname{im} f \\ &\simeq \sum_{j>i} \sum_{f\in \mathrm{Hom}_A(P_j,P_i)} \operatorname{im}(f \otimes_R \mathrm{id}_{R(\mathfrak{m})}) \simeq \left(\sum_{j>i} \sum_{f\in \mathrm{Hom}_A(P_j,P_i)} \operatorname{im} f \right)(\mathfrak{m}) = U_i(\mathfrak{m}) \end{aligned}$$

Claim. $\mathrm{Hom}_A(M, P_i/U_i) = 0$ for $M \in \mathscr{F}(\Delta_{j>i})$. We shall proceed by induction on the size of the filtration of M. Assume $t = 1$. Then, $M \simeq \Delta_j$ for some $j > i$. Let $g \in \mathrm{Hom}_A(\Delta_j, P_i/U_i)$. Since P_j is projective over A we have a commutative diagram

$$\begin{tikzcd} P_j \arrow[r, "\pi_j"] \arrow[rrd, "\exists f"'] & \Delta_j \arrow[r, "g"] & P_i/U_i \\ & & P_i \arrow[u, two heads, "\pi"] \end{tikzcd}.$$

By definition of π and U_i, $0 = \pi \circ f = g \circ \pi_j$. Then, $g = 0$, since π_j is surjective. Now consider the result known for filtrations of size less than t. Assume that M has a filtration with size t. Let $g \in \operatorname{Hom}_A(\Delta_j, P_i/U_i)$. Consider the exact sequence $0 \to M_{t-1} \xrightarrow{i} M \xrightarrow{k} \Delta_j \to 0$, $j > i$. By induction, $\operatorname{Hom}_A(M_{t-1}, P_i/U_i) = 0$. In particular, $g \circ i = 0$. So, g induces a map $g' \in \operatorname{Hom}_A(\Delta_j, P_i/U_i)$ such that $g' \circ k = g$. By $t = 1$, $g' = 0$. Therefore, $g = 0$ and the claim follows.

Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C_i \arrow[r, "k_i"] & P_i \arrow[r, "\pi_i"] \arrow[d, equal] & \Delta_i \arrow[r] & 0 \\ 0 \arrow[r] & U_i \arrow[r, "k"] & P_i \arrow[r, "\pi"] & P_i/U_i \arrow[r] & 0 \end{tikzcd}.$$

Since $C_i \in \mathscr{F}(\Delta_{j>i})$ we get that $\operatorname{Hom}_A(C_i, P_i/U_i) = 0$. In particular, $\pi \circ k_i = 0$. So, the image of k_i is contained in $\ker \pi = \operatorname{im} k$, and thus there exists an A-homomorphism $f \colon C_i \to U_i$ which makes the previous diagram commutative. On the other hand, since $\pi \circ k_i = 0$ there exists a map $\tilde{\pi} \in \operatorname{Hom}_A(\Delta_i, P_i/U_i)$ such that $\tilde{\pi} \circ \pi_i = \pi$. By Snake Lemma, f is injective and $\tilde{\pi}$ is surjective. For every maximal ideal $\mathfrak{m}$ of R, applying the right exact functor $R(\mathfrak{m}) \otimes_R -$ yields the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & C_i(\mathfrak{m}) \arrow[r, "k_i(\mathfrak{m})"] \arrow[d, "f(\mathfrak{m})"] & P_i(\mathfrak{m}) \arrow[r, "\pi_i(\mathfrak{m})"] \arrow[d, equal] & \Delta_i(\mathfrak{m}) \arrow[r] \arrow[d, "\tilde{\pi}(\mathfrak{m})"] & 0 \\ & U_i(\mathfrak{m}) \arrow[r, "k(\mathfrak{m})"] & P_i(\mathfrak{m}) \arrow[r, "\pi(\mathfrak{m})"] & P_i/U_i(\mathfrak{m}) \arrow[r] & 0 \end{tikzcd}.$$

The first row is exact since Δ_i is projective over R. By the commutativity of the diagram, $k(\mathfrak{m}) \circ f(\mathfrak{m}) = k_i(\mathfrak{m})$ is injective, which implies that $f(\mathfrak{m})$ is a monomorphism. Since $C_i(\mathfrak{m}) \simeq U_i(\mathfrak{m})$, we have $\dim_{R(\mathfrak{m})} C_i(\mathfrak{m}) = \dim_{R(\mathfrak{m})} U_i(\mathfrak{m})$, thus $f(\mathfrak{m})$ is an $R(\mathfrak{m})$-isomorphism for every maximal ideal $\mathfrak{m}$ of R. Thus, $f(\mathfrak{m})$ is an $A(\mathfrak{m})$-isomorphism. By Nakayama's Lemma, f is surjective. Hence, f is an isomorphism. By Snake Lemma, $\tilde{\pi}$ is an isomorphism and it follows that $\Delta_i \simeq P_i/U_i$. ☐

Note that this not guarantees uniqueness of standard modules as in the field case, since in Noetherian rings we can have many choices for the projective modules $P(\lambda)$ even when they are indecomposable.

A natural question that arises is whether or not the projective modules $P(\lambda)$ are indecomposable. In the following proposition, we find a positive answer for local rings.

Proposition 1.5.64. *Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split highest weight category. If R has no non-trivial idempotents, then all $\Delta(\lambda)$ are indecomposable. Furthermore, if R is local, then there exists a choice of $P(\lambda)$ satisfying 1.5.32(iv) so that $\operatorname{End}_A(P(\lambda))$ is a local ring.*

Proof. Assume by contradiction that $\Delta(\lambda) = X_1 \oplus X_2$ then $\Delta(\lambda) \twoheadrightarrow X_1 \hookrightarrow \Delta(\lambda)$ is a non-trivial idempotent in $\operatorname{End}_A(\Delta(\lambda))^{op}$. Thus, we have a non-trivial idempotent in R.

Assume that R is local. Let $f \in \operatorname{End}_A(P(\lambda))$. Let $\mathfrak{m}$ be the unique maximal ideal in R. Then, $f(\mathfrak{m}) \in \operatorname{End}_{A(\mathfrak{m})}(P(\lambda)(\mathfrak{m}))$, since $P(\lambda) \in A$-proj. By Observation 1.5.62, we can consider projective modules $P(\lambda)$ so that $P(\lambda)(\mathfrak{m})$ is indecomposable. In view of Proposition 1.5.39, $\operatorname{End}_{A(\mathfrak{m})}(S) \simeq R(\mathfrak{m})$ for all simple $A(\mathfrak{m})$-modules. Thus, the endomorphism ring of a finite-dimensional indecomposable $A(\mathfrak{m})$-module is a local ring.

In particular, $\mathrm{End}_{A(\mathfrak{m})}(P(\lambda)(\mathfrak{m}))$ is a local ring. Hence, if $f(\mathfrak{m})$ is not an isomorphism, then $\mathrm{id}_{P(\mathfrak{m})} - f(\mathfrak{m})$ is an isomorphism. Note that $\mathrm{id}_{P(\mathfrak{m})} - f(\mathfrak{m}) = (\mathrm{id}_P - f)(\mathfrak{m})$. Applying Nakayama Lemma's 1.1.39, it follows that $\mathrm{id}_P - f$ is an isomorphism or f is an isomorphism or both. Thus, $\mathrm{End}_A(P(\lambda))$ is a local ring and $P(\lambda)$ is indecomposable. $\square$

1.5.7 Relation between heredity chains and standard modules

The following result is Theorem 4.16 of [Rou08].

Theorem 1.5.65. *Let A be a projective Noetherian R-algebra. $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category if and only if A is a split quasi-hereditary. Let $\Delta \to \{1,\dots,t\}$, $\Delta_i \mapsto i$ be an increasing bijection. Here the standard modules and the split heredity chain are related in the following way:*

$$\operatorname{im}\tau_{\Delta_i} = J_i/J_{i+1}, \quad J_{t+1} = 0 \subset J_t \subset J_{t-1} \subset \cdots \subset J_1 = A \quad \textit{is a split heredity chain.}$$

Proof. Let A be split quasi-hereditary with split heredity chain: $J_{t+1} = 0 \subset J_t \subset J_{t-1} \subset \cdots \subset J_1 = A$. We shall proceed by induction on the size of the split heredity chain of A to show that A-mod can have a split highest weight category structure.

Assume $t = 1$. Then, $0 \subset A$ is a split heredity chain. So, A is split heredity in A. By Proposition 1.5.31, there is $L \in \mathscr{M}(A)$ such that $\operatorname{im}\tau_L = A$. Put $\Delta(1) = L$ and since $A/\operatorname{im}\tau_L = 0$, it follows by Lemma 1.5.47 that $(A\text{-mod}, \{\Delta(1)\})$ is a split highest weight category.

Assume now that the result holds for $t-1$. Fix $J = J_t$. A/J is split quasi-hereditary with split heredity chain $0 \subset J_{t-1}/J \subset \cdots \subset J_1/J = A/J$. By induction, A/J-mod is a split highest weight category with standards $\Delta(i)$, $1 \leq i \leq t-1$, satisfying $\operatorname{im}\tau_{\Delta(i)} = (J_i/J)/(J_{i+1}/J) \simeq J_i/J_{i+1}$, $1 \leq i \leq t-1$. By Proposition 1.5.31, there is $L \in \mathscr{M}(A)$ such that $\operatorname{im}\tau_L = J$. Put $\Delta(t) = L$. Since each $\Delta(i) \in A/J$-mod, we get that $\mathrm{Hom}_A(\Delta(t), \Delta(i)) = 0$, $1 \leq i \leq t-1$, by Corollary 1.5.23. So, we can consider the usual order $t \geq i$, $1 \leq i \leq t-1$. By Lemma 1.5.47, $(A\text{-mod}, \{\Delta(i)_{i\in\{1,\dots,t\}}\})$ is a split highest weight category.

Now assume that $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. Let $\Lambda \to \{1,\dots,t\}$, $\lambda \mapsto i_\lambda$ be an increasing bijection. We shall proceed by induction on t. If $t = |\Lambda| = 1$, then $L = \Delta(1) \in \mathscr{M}(A)$. By Proposition 1.5.31, there exists J split heredity such that $J = \operatorname{im}\tau_L$. By Corollary 1.5.23, A/J-mod $= 0$. In particular, $A/J = 0$, so $J = A$. Thus, $0 \subset J = A$ is a split heredity chain.

Now assume the result known for $t-1$. Consider a maximal element $\alpha \in \Lambda$ satisfying $i_\alpha = t$. By Lemma 1.5.47, $\Delta(t) \in \mathscr{M}(A)$ and $(A/J\text{-mod}, \{\Delta(i)_{1\leq i\leq t-1}\})$ is a split highest weight category, where $J = \operatorname{im}\tau_{\Delta(t)}$. By induction, there exists a split heredity chain

$$0 \subset I_{t-1} \subset \cdots \subset I_1 = A/J \quad \text{such that } \operatorname{im}\tau_{\Delta(i)} = I_i/I_{i+1}.$$

Fix $J_t = J$. By the correspondence theorem, there are ideals J_i of A such that $I_i = J_i/J$. It follows that $J_i/J_{i+1} \simeq J_i/J/J_{i+1}/J = I_i/I_{i+1} = \operatorname{im}\tau_{\Delta(i)}$ split heredity in $A/J/J_{i+1}/J \simeq A/J_{i+1}$ for $i = 1,\dots,t-1$. Now since J_t is split heredity in A, it follows by the discussed argument above that $0 = J_{t+1} \subset J_t \subset J_{t-1} \subset \cdots \subset J_1 = A$ is a split heredity chain of A satisfying $\operatorname{im}\tau_{\Delta(i)} = J_i/J_{i+1}$. $\square$

Due to Theorem 1.5.65, we can say that $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split quasi-hereditary algebra when $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category without mentioning the split heredity chain.

Note that, by the bijection given in Proposition 1.5.31, the standard modules are not unique in this construction unless the Picard group is trivial.

As we have noted, while constructing the standard modules, in general, we have many choices that can be given by the same heredity chain. Hence, we would like to identify the split highest weight categories that came from the same heredity chain. This motivates the next notion introduced by Rouquier.

Definition 1.5.66. Let (A-mod, $\{\Delta(\lambda)_{\lambda\in\Lambda}\}$) and ($B$-mod, $\{\Omega(\chi)_{\chi\in X}\}$) be two split highest weight categories. A functor $F\colon A\text{-mod}\to B\text{-mod}$ is an **equivalence of split highest weight categories** if

- it is an equivalence of categories;
- there is a bijection $\phi\colon \Lambda\to X$ and invertible R-modules U_λ such that $F(\Delta(\lambda))\simeq\Omega(\phi(\lambda))\otimes_R U_\lambda,\ \lambda\in\Lambda$.

If $Pic(R)$ is trivial, then F is an equivalence if and only if $\{\Omega(\chi)_{\chi\in X}\}=\{F\Delta(\lambda)_{\lambda\in\Lambda}\}$. In particular, if R is a field, the notions of Morita equivalence and split highest weight category equivalence coincide.

Remark 1.5.67. Note that when passing from A to $A(\mathfrak{m})$, there is no confusion if we take equivalent standard modules. That is, $\Delta_i'(\mathfrak{m})=\Delta_i(\mathfrak{m})$. △

In fact, consider $\Delta_i'=\Delta_i\otimes_R F,\ F\in Pic(R)$. Then, there exists G such that $G\otimes_R F\simeq R$. Moreover,

$$F(\mathfrak{m})\otimes_{R(\mathfrak{m})}G(\mathfrak{m})\simeq F\otimes_R R(\mathfrak{m})\otimes_R G\simeq F\otimes_R G\otimes_R R(\mathfrak{m})\simeq R\otimes_R R(\mathfrak{m})\simeq R(\mathfrak{m}).$$

Hence, $F(\mathfrak{m})\in Pic(R(\mathfrak{m}))=\{R(\mathfrak{m})\}$ since $R(\mathfrak{m})$ is a field. Therefore,

$$\Delta_i'(\mathfrak{m})=\Delta_i\otimes_R F(\mathfrak{m})\simeq\Delta_i(\mathfrak{m})\otimes_{R(\mathfrak{m})}F(\mathfrak{m})\simeq\Delta_i(\mathfrak{m}).$$

Now we must observe that Remark 4.18 in [Rou08] is not accurate. Theorem 3.3 in CPS90 does not involve split quasi-hereditary algebras, but instead, it involves (non-split) quasi-hereditary algebras. On the other hand, in general, we cannot construct standard modules Δ just knowing the modules over the residue field. Here the difficulty lies that a priori there is not an R-homomorphism that its image under the functor $-\otimes_R R(\mathfrak{m})$ is the isomorphism. This problem also occurs when dealing with localizations.

So to conclude the split version of Corollary 1.5.10, a direct approach like in its proof might not work in this case. We suggest the following:

Proposition 1.5.68. *Let R be a Noetherian commutative ring and A a projective Noetherian R-algebra. J is a split heredity ideal in A if and only if J is split heredity ideal in A^{op}.*

Proof. For fields R, $J=AeA$ for some idempotent e of A. The result holds for heredity ideals (see [PS88, Theorem 4.3 (b)]). Now assume that $\operatorname{End}_A(AeA)^{op}\overset{\text{Mor}}{\sim}R$. We have that

$$\operatorname{End}_A({}_AAeA)^{op}\overset{\text{Mor}}{\sim}\operatorname{End}_A(Ae)^{op}\simeq eAe$$
$$\operatorname{End}_A(AeA_A)\overset{\text{Mor}}{\sim}\operatorname{End}_A(eA)\simeq eAe.$$

Therefore, $\operatorname{End}_A({}_AAeA)^{op}\overset{\text{Mor}}{\sim}\operatorname{End}_A(AeA_A)$. Hence, $\operatorname{End}_A(AeA_A)\overset{\text{Mor}}{\sim}R$, and the result follows for fields.

Now assume R to be a Noetherian commutative ring and J split heredity ideal in A. Conditions (i) and (ii) of Definition 1.5.2 for J^{op} in A^{op} are clear. Moreover, for any maximal ideal $\mathfrak{m}$ of R, $A/J(\mathfrak{m})\simeq A(\mathfrak{m})/J(\mathfrak{m})$, since A/J is projective over R, and thus $\operatorname{Tor}_1^R(A/J,R(\mathfrak{m}))=0$. We have $J(\mathfrak{m})^2=J\otimes_R R(\mathfrak{m})J\otimes_R R(\mathfrak{m})=J^2\otimes_R R(\mathfrak{m})=J(\mathfrak{m})$, and clearly $J(\mathfrak{m})$ is projective as left $A(\mathfrak{m})$-module.

Since $\operatorname{End}_A({}_AJ)^{op}\overset{\text{Mor}}{\sim}R$, there exists an R-progenerator P such that $\operatorname{End}_A({}_AJ)^{op}\simeq\operatorname{End}_R(P)^{op}$. Therefore,

$$\operatorname{End}_{A(\mathfrak{m})}(J(\mathfrak{m}))^{op}\simeq R(\mathfrak{m})\otimes_R\operatorname{End}_A(J)^{op},\ \text{since } J\in A\text{-proj}$$

$$\simeq R(\mathfrak{m}) \otimes_R \operatorname{End}_R(P)^{op} \simeq \operatorname{End}_{R(\mathfrak{m})}(P(\mathfrak{m}))^{op}, \text{ since } P \in R\text{-proj}.$$

Since the functor $R(\mathfrak{m}) \otimes_R -$ preserves finite direct sums, it preserves the progenerators, hence $P(\mathfrak{m})$ is an $R(\mathfrak{m})$-progenerator and $\operatorname{End}_{A(\mathfrak{m})}(J(\mathfrak{m}))^{op} \overset{\text{Mor}}{\sim} R(\mathfrak{m})$. So, $J(\mathfrak{m})$ is split heredity in $A(\mathfrak{m})$. Since $R(\mathfrak{m})$ is a field, $J(\mathfrak{m})^{op}$ is split heredity in $A(\mathfrak{m})^{op}$. In particular, $J(\mathfrak{m})$ is projective as right $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ of R. By Theorem 1.1.51, J is projective as right A-module.

Consider $L \in \mathscr{M}(A)$ such that $\operatorname{im}\tau_L = J$. By Proposition 1.5.22, $\operatorname{End}_R(\operatorname{Hom}_A(L,A)) \simeq \operatorname{Hom}_A(J,A)$. By Remark 1.5.16, $\operatorname{End}_R(L) \simeq \operatorname{Hom}_{A^{op}}(J^{op},A)$. Since both $\operatorname{Hom}_A(L,A)$ and L are R-progenerators, it follows that $\operatorname{add}_R \operatorname{Hom}_A(L,A) = \operatorname{add}_R L$, thus $\operatorname{End}_R(\operatorname{Hom}_A(L,A))^{op} \overset{\text{Mor}}{\sim} \operatorname{End}_R(L)^{op}$.

Now applying $\operatorname{Hom}_A(J,-)$ and $\operatorname{Hom}_{A^{op}}(J^{op},-)$ to the exact sequence $0 \to J \to A \to A/J \to 0$ yields the following exact sequences

$$0 \to \operatorname{Hom}_A(J,J) \to \operatorname{Hom}_A(J,A) \to \operatorname{Hom}_A(J,A/J) \to 0$$
$$0 \to \operatorname{Hom}_{A^{op}}(J^{op},J^{op}) \to \operatorname{Hom}_{A^{op}}(J^{op},A^{op}) \to \operatorname{Hom}_{A^{op}}(J^{op},A^{op}/J^{op}) \to 0.$$

By Lemma 1.5.14, $\operatorname{Hom}_{A^{op}}(J^{op},A^{op}/J^{op}) = \operatorname{Hom}_A(J,A/J) = 0$. So, we conclude that

$$\operatorname{Hom}_A(J,J) \simeq \operatorname{End}_R(\operatorname{Hom}_A(L,A)) \text{ and } \operatorname{Hom}_{A^{op}}(J^{op},J^{op}) \simeq \operatorname{End}_R(L).$$

Therefore, $R \overset{\text{Mor}}{\sim} \operatorname{End}_A(J)^{op} \simeq \operatorname{End}_R(\operatorname{Hom}_A(L,A))^{op} \overset{\text{Mor}}{\sim} \operatorname{End}_R(L) \simeq \operatorname{End}_{A^{op}}(J^{op})^{op}$. So, J^{op} is split heredity in A^{op}. ☐

Theorem 1.5.69. *A is split quasi-hereditary with split heredity chain $0 \subset J_t \subset \cdots \subset J_1 = A$ if and only if A^{op} is split quasi-hereditary with split heredity chain $0 \subset J_t^{op} \subset \cdots \subset J_1^{op} = A^{op}$.*

Proof. By Proposition 1.5.68, J_i/J_{i+1} is split heredity in A/J_{i+1} if and only if $(J_i/J_{i+1})^{op} = J_i^{op}/J_{i+1}^{op}$ is split heredity in $(A/J_{i+1})^{op} = A^{op}/J_{i+1}^{op}$ for all $1 \leq i \leq t$. ☐

In the following, we want to obtain further insight into what information about split heredity chains can we gain from applying change of rings techniques on split heredity chains.

Lemma 1.5.70. *Let R be a commutative Noetherian ring and let A be a projective Noetherian R-algebra. Assume that A has two split heredity chains*

$$0 \subset J_t \subset J_{t-1} \subset \cdots \subset J_1 = A \tag{1.5.7.1}$$
$$0 \subset I_t \subset I_{t-1} \subset \cdots \subset I_1 = A. \tag{1.5.7.2}$$

If $J_j(\mathfrak{m}) = I_j(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R, then $J_j = I_j$ for all j.

Proof. Let J and I be split heredity ideals of A satisfying $I(\mathfrak{m}) = J(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. Let Δ and L be the modules in $\mathscr{M}(A)$ associated with J and I, respectively. Let $\mathfrak{m}$ be a maximal ideal of R. By Proposition 1.5.31, $\Delta(\mathfrak{m}) \simeq L(\mathfrak{m})$. Therefore, we have surjective A-maps $\pi_\Delta \colon \Delta \to \Delta(\mathfrak{m})$, $\pi_L \colon L \to \Delta(\mathfrak{m})$. In particular, $\pi_\Delta(\mathfrak{m})$ and $\pi_L(\mathfrak{m})$ are $A(\mathfrak{m})$-isomorphisms. Since $L \in A$-proj there exists an A-homomorphism $f \in \operatorname{Hom}_A(L,\Delta)$ satisfying $\pi_\Delta \circ f = \pi_L$. Therefore, $f(\mathfrak{m})$ is an isomorphism. It follows by Lemma 1.1.39, $\Delta_\mathfrak{m} \simeq L_\mathfrak{m}$. The following commutative diagram

$$\begin{array}{ccc} \Delta_\mathfrak{m} \otimes_{R_\mathfrak{m}} \operatorname{Hom}_{A_\mathfrak{m}}(\Delta_\mathfrak{m},A_\mathfrak{m}) & \xrightarrow{\tau_{\Delta_\mathfrak{m}}} & A_\mathfrak{m} \\ {\scriptstyle f_\mathfrak{m} \otimes \operatorname{Hom}_{A_\mathfrak{m}}(f_\mathfrak{m}^{-1},A_\mathfrak{m})}\downarrow & & \| \\ L_\mathfrak{m} \otimes_{R_\mathfrak{m}} \operatorname{Hom}_{A_\mathfrak{m}}(L_\mathfrak{m},A_\mathfrak{m}) & \xrightarrow{\tau_{L_\mathfrak{m}}} & A_\mathfrak{m} \end{array} \tag{1.5.7.3}$$

yields that $I_{\mathfrak{m}} = \operatorname{im}\tau_{L_{\mathfrak{m}}} = \operatorname{im}\tau_{\Delta_{\mathfrak{m}}} = J_{\mathfrak{m}}$. The choice of $\mathfrak{m}$ is arbitrary, thus this equality holds for every maximal ideal $\mathfrak{m}$ of R. By Lemma 1.1.29, $I = J$. As $J_{t-1}/J_t \in R$-proj we can write

$$J_{t-1}/J_t(\mathfrak{m}) \simeq J_{t-1}(\mathfrak{m})/J_t(\mathfrak{m}) \simeq I_{t-1}(\mathfrak{m})/I_t(\mathfrak{m}) \simeq I_{t-1}/I_t(\mathfrak{m}), \tag{1.5.7.4}$$

for every maximal ideal $\mathfrak{m}$ of R. we obtain $J_{t-1}/J_t = I_{t-1}/J_t$. It follows that $J_{t-1} = I_{t-1}$. Continuing this argument, by induction on t, we conclude the result. $\square$

Another interpretation of Observation 1.5.62 is the following statement.

Lemma 1.5.71. *Let R be a commutative Noetherian ring. Let A be a split quasi-hereditary R-algebra and J be a split heredity ideal in A. Then, for each maximal ideal $\mathfrak{m}$ of R, the canonical map*

$$\operatorname{Ext}^1_A(A/J,J)(\mathfrak{m}) \to \operatorname{Ext}^1_{A(\mathfrak{m})}(A(\mathfrak{m})/J(\mathfrak{m}),J(\mathfrak{m})) \tag{1.5.7.5}$$

is an isomorphism.

Proof. Consider the (A,R)-exact sequence $0 \to J \to A \to A/J \to 0$. Applying $\operatorname{Hom}_A(-,J)$ and the tensor product $R(\mathfrak{m}) \otimes_R -$ we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
J(\mathfrak{m}) \arrow[r] \arrow[d, equal] & \operatorname{End}_A(J)(\mathfrak{m}) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Ext}^1_A(A/J,J)(\mathfrak{m}) \arrow[r] \arrow[d] & 0 \\
J(\mathfrak{m}) \arrow[r] & \operatorname{End}_{A(\mathfrak{m})}(J(\mathfrak{m})) \arrow[r] & \operatorname{Ext}^1_{A(\mathfrak{m})}(A(\mathfrak{m})/J(\mathfrak{m}),J(\mathfrak{m})) \arrow[r] & 0
\end{tikzcd} . \tag{1.5.7.6}$$

By diagram chasing, we obtain the result. $\square$

Hence, the extensions between the projective A/J-modules and the projective standard module of A commute with functor $R(\mathfrak{m}) \otimes_R -$.

We would like know to deduce a parallel result to Theorem 1.5.56 now using split heredity chains. But, first, we require the following lemma.

Lemma 1.5.72. *Let K be a field and A a finite-dimensional K-algebra. If AeA is a split heredity ideal of A for some primitive idempotent $e \in A$, then $Ae \in \mathscr{M}(A)$.*

Proof. Since AeA is projective as left ideal of A we obtain that the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism (see Statement 7 [DR89b]). Since $\operatorname{Hom}_A(Ae,A) \simeq eA$ it remains to show that $eAe = K$. Again, as AeA is projective, $AeA \in \operatorname{add}_A Ae$. By projectivization,

$$eA = eAeA = \operatorname{Hom}_A(Ae,AeA) \in \operatorname{End}_A(Ae)\text{-proj} = eAe\text{-proj}. \tag{1.5.7.7}$$

The identification $eA = eAeA$ is obtained by applying the tensor product $eA \otimes_A -$ to the multiplication map $Ae \otimes_{eAe} eA \to AeA$. On the other hand, we can write $eA = eAe \oplus eA(1-e)$ as left eAe-modules. Thus, eA is an eAe-progenerator. By Tensor-Hom adjunction,

$$\operatorname{End}_A(AeA) \simeq \operatorname{Hom}_{eAe}(eA, \operatorname{Hom}_A(Ae, Ae \otimes_{eAe} eA)) \simeq \operatorname{End}_{eAe}(eA). \tag{1.5.7.8}$$

Therefore, eAe is Morita equivalent to $\operatorname{End}_A(AeA)$. By assumption, $\operatorname{End}_A(AeA)$ is Morita equivalent to K. So, eAe is Morita equivalent to K. Since e is primitive, Ae is indecomposable. Thus, eAe is local. So, we must have $eAe = K$. $\square$

Theorem 1.5.73. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that A admits a set of orthogonal idempotents $\{e_1,\dots,e_t\}$ such that for each maximal ideal $\mathfrak{m}$ of R $\{e_1(\mathfrak{m}),\dots,e_t(\mathfrak{m})\}$ becomes a complete set of primitive orthogonal idempotents of $A(\mathfrak{m})$. Then, A is split quasi-hereditary with split heredity chain*

$$0 \subset Ae_tA \subset \cdots \subset A(e_1+\cdots+e_t)A = A \tag{1.5.7.9}$$

if and only if for each maximal ideal $\mathfrak{m}$ of R, $A(\mathfrak{m})$ is split quasi-hereditary with split heredity chain

$$0 \subset A(\mathfrak{m})e_t(\mathfrak{m})A(\mathfrak{m}) \subset \cdots \subset A(\mathfrak{m})(e_1(\mathfrak{m})+\cdots+e_t(\mathfrak{m}))A(\mathfrak{m}) = A(\mathfrak{m}). \tag{1.5.7.10}$$

Proof. Assume that A is split quasi-hereditary. Let $\mathfrak{m}$ be a maximal ideal of R. As $A/Ae_tA \in R$-proj, we can write $Ae_tA(\mathfrak{m}) \simeq A(\mathfrak{m})e_t(\mathfrak{m})A(\mathfrak{m}) \in A(\mathfrak{m})$-proj and $A/Ae_tA(\mathfrak{m}) \simeq A(\mathfrak{m})/A(\mathfrak{m})e_t(\mathfrak{m})A(\mathfrak{m})$. Also, for an $\operatorname{End}_A(Ae_tA)$-progenerator P we can write $R(\mathfrak{m}) \simeq \operatorname{End}_{\operatorname{End}_A(Ae_tA)}(P)(\mathfrak{m}) \simeq \operatorname{End}_{\operatorname{End}_{A(\mathfrak{m})}(A(\mathfrak{m})e_t(\mathfrak{m})A(\mathfrak{m}))}(P(\mathfrak{m}))$. Hence, $A(\mathfrak{m})e_t(\mathfrak{m})A(\mathfrak{m})$ is a split heredity ideal of $A(\mathfrak{m})$. By going through the split heredity chain of A we obtain that $A(\mathfrak{m})$ is split quasi-hereditary a with split heredity chain (1.5.7.10).

Conversely, assume that $A(\mathfrak{m})$ is split quasi-hereditary for every maximal ideal $\mathfrak{m}$ of R with split heredity chain 1.5.7.10. By Lemma 1.5.72, $A(\mathfrak{m})e_t(\mathfrak{m}) \in \mathscr{M}(A(\mathfrak{m}))$ for every maximal ideal $\mathfrak{m}$ of R. Since Ae_t is an A-summand of A, the inclusion $Ae_t \to A$ remains exact under the functor $R(\mathfrak{m})\otimes_R -$. So, $Ae_t(\mathfrak{m}) = A(\mathfrak{m})e_t(\mathfrak{m}) \in \mathscr{M}(A(\mathfrak{m}))$. By Lemma 1.5.20, $Ae_t \in \mathscr{M}(A)$. Hence, $AeA = \operatorname{im}\tau_{Ae}$ is a split heredity ideal of A. In particular, $A/Ae_tA(\mathfrak{m}) \simeq A(\mathfrak{m})/A(\mathfrak{m})e_t(\mathfrak{m})A(\mathfrak{m})$. Continuing the same argument with A/Ae_tA we obtain that A is split quasi-hereditary with split hereditary chain (1.5.7.9). ☐

1.5.8 Global dimension of split quasi-hereditary algebras

We will now show that split quasi-hereditary algebras over a commutative Noetherian ring have finite global dimension. This approach also works with the non-split case.

Lemma 1.5.74. *Let $\cdots \to P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$ be a projective A-resolution. Define $N = \ker\alpha_{k-1}$. Then,* $\operatorname{pdim}_A M \leq k + \operatorname{pdim}_A N$.

Proof. First notice that $\operatorname{Ext}_A^l(M,L) \simeq \operatorname{Ext}_A^{l-k}(\operatorname{im}\alpha_k, L) = \operatorname{Ext}_A^{l-k}(N,L)$ for any $l \geq 0$ and $L \in A$-mod.

If $\operatorname{pdim}_A N < \infty$, then there is nothing to show. Assume $\operatorname{pdim}_A N = s < \infty$. Then,

$$\operatorname{Ext}_A^{s+k+1}(M,L) \simeq \operatorname{Ext}_A^{s+1}(\operatorname{im}\alpha_k, L) = \operatorname{Ext}_A^{s+1}(N,L) = 0,\ \forall L \in A\text{-mod}.$$

Hence, $\operatorname{pdim}_A M \leq s+k = \operatorname{pdim}_A N + k$. ☐

Theorem 1.5.75. *Let A be a quasi-hereditary algebra over a Noetherian commutative ring R with heredity chain $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$. Then,* $\operatorname{gldim} A \leq 2(t-1) + \operatorname{gldim} R$.

Proof. Consider $M \in A$-mod$\cap R$-proj. By Theorem 1.5.9, $A(\mathfrak{m})$ is quasi-hereditary with heredity chain of size t for any maximal ideal $\mathfrak{m}$ of R. By [DR89b, Statement 9], it follows that $\operatorname{gldim} A(\mathfrak{m}) \leq 2(t-1)$. Consider a projective A-resolution for M, $\cdots \to P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$. Let $N = \ker\alpha_{2(t-1)-1}$. Since M is projective over R and P_i is projective over A we obtain $\operatorname{im}\alpha_i \in R$-proj. In particular $N \in R$-proj. Applying $-\otimes_R R(\mathfrak{m})$ we obtain by Lemma 1.2.21 the exact sequence

$$0 \to N(\mathfrak{m}) \to P_{2(t-1)-1}(\mathfrak{m}) \to \cdots \to P_1(\mathfrak{m}) \to P_0(\mathfrak{m}) \to M(\mathfrak{m}) \to 0. \tag{1.5.8.1}$$

Since $\operatorname{pdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \leq 2(t-1)$, $N(\mathfrak{m})$ is projective over $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, N is projective A-module. Hence, $\operatorname{pdim} M \leq 2(t-1)$.

Now consider M an arbitrary module in A-mod. Consider a projective A-resolution for M, $\cdots \to P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$. Define $K = \ker \alpha_{r-1}$, $r = \operatorname{gldim} R$. Then, we must have that K is projective over R as $\operatorname{pdim}_R M \leq r$ and all P_i are projective over R. As we have seen $\operatorname{pdim}_A K \leq 2(t-1)$. By Lemma 1.5.74, it follows that $\operatorname{pdim}_A M \leq 2(t-1) + r$. Therefore, $\operatorname{gldim} A \leq 2(t-1) + \operatorname{gldim} R$. $\square$

Corollary 1.5.76. *Let A be a split quasi-hereditary algebra over a commutative Noetherian ring R with split heredity chain $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$. Then,* $\operatorname{gldim} A \leq 2(t-1) + \operatorname{gldim} R$.

It follows that if R is a regular ring with finite Krull dimension, then a split quasi-hereditary algebra over R has finite global dimension. Of course, this can fail if R has infinite global dimension. In such a case, we just need to consider $A = R$. For rings R with finite global dimension, we can give a precise value of the global dimension of a split quasi-hereditary algebra in terms of the global dimension of the finite-dimensional algebras $A(\mathfrak{m})$.

Theorem 1.5.77. *Let R be a commutative Noetherian ring with finite global dimension. Let A be a split quasi-hereditary R-algebra. Then,*

$$\operatorname{gldim} A = \dim R + \sup\{\operatorname{gldim} A(\mathfrak{m}) \colon \mathfrak{m} \in \operatorname{MaxSpec} R\}.$$

Proof. We can assume that R is a local commutative Noetherian ring with unique maximal ideal $\mathfrak{m}$. By the proof of Theorem 1.5.75, we obtain $\operatorname{gldim} A \leq \dim R + \sup\{\operatorname{gldim} A(\mathfrak{m}) \colon \mathfrak{m} \in \operatorname{MaxSpec} R\}$. Consider the surjective map $A \to A(\mathfrak{m})$. Let $M \in A$-mod. By Theorem 10.75 of [Rot09], we can consider the spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_{A(\mathfrak{m})}(M, \operatorname{Ext}^j_A(A(\mathfrak{m}), A)) \Rightarrow \operatorname{Ext}^{i+j}_A(M, A). \tag{1.5.8.2}$$

As $A \in R$-proj, we can write

$$\operatorname{Ext}^j_A(A(\mathfrak{m}), A) \simeq \operatorname{Ext}^j_{A \otimes_R R}(R(\mathfrak{m}) \otimes_R A, A \otimes_R R) \simeq A \otimes_R \operatorname{Ext}^j_R(R(\mathfrak{m}), R), \ \forall j \geq 0. \tag{1.5.8.3}$$

By Theorem 1.1.59, $\operatorname{Ext}^{\dim R}_R(R(\mathfrak{m}), R) \neq 0$. Since A is faithful as R-module we obtain that $\operatorname{Ext}^{\dim R}_A(A(\mathfrak{m}), A) \neq 0$. Pick $M = DA(\mathfrak{m})$ regarded as A-module. Then, $\operatorname{gldim} A(\mathfrak{m}) = \operatorname{pdim}_{A(\mathfrak{m})} DA(\mathfrak{m})$ and denote by n the value $\operatorname{gldim} A(\mathfrak{m})$. We claim that $E_2^{n,\dim R} \neq 0$. In fact, we can see by induction that $E_k^{n,\dim R} = E_2^{n,\dim R}$ for all $k \geq 2$. Since $\operatorname{Ext}^{\dim R}_R(R(\mathfrak{m}), R) \in R(\mathfrak{m})$-Mod we obtain that $\operatorname{Ext}^{\dim R}_A(A(\mathfrak{m}), A) \in A(\mathfrak{m})$-Proj. Therefore,

$$E_2^{n,\dim R} = \operatorname{Ext}^n_{A(\mathfrak{m})}(DA(\mathfrak{m}), \operatorname{Ext}^{\dim R}_A(A(\mathfrak{m}), A)) \neq 0. \tag{1.5.8.4}$$

Hence, $E_\infty^{n,\dim R} \neq 0$. So, $\operatorname{Ext}^{n+\dim R}_A(DA(\mathfrak{m}), A) \neq 0$. $\square$

Using the next Lemma, we can show that a split quasi-hereditary algebra has finite global dimension if and only if the ground ring has finite global dimension.

Lemma 1.5.78. *Let A be projective Noetherian R-algebra and let J be a split heredity ideal in A. Then,* $\operatorname{Ext}^i_A(M, N) \simeq \operatorname{Ext}^i_{A/J}(M, N)$ *for all $M, N \in A/J$-mod and $i \geq 0$.*

Proof. For $i = 0$, the result is clear since A/J-mod is a full subcategory of A-mod. Consider the exact sequence

$$0 \to J \to A \to A/J \to 0. \tag{1.5.8.5}$$

For any A/J-module N, we deduce $\operatorname{Ext}_A^{i-1}(J,N) \simeq \operatorname{Ext}_A^i(A/J,N)$, $i \geq 2$ by applying the functor $\operatorname{Hom}_A(-,N)$ on (1.5.8.5). J is projective over A, thus $\operatorname{Ext}_A^i(A/J,N) = 0$, $i \geq 2$. Furthermore, by the same argument, the induced map $\operatorname{Hom}_A(J,N) \to \operatorname{Ext}_A^1(A/J,N)$ is surjective. By Lemma 1.5.14, $\operatorname{Hom}_A(J,N) = 0$. This implies that $\operatorname{Ext}_A^1(A/J,N)$ also vanishes. So, we conclude that free A/J-modules are acyclic for the functor $\operatorname{Hom}_A(-,N)$, for every $N \in A/J$-mod. Thus, we can use A/J-free resolutions of $M \in A/J$-mod to compute $\operatorname{Ext}_A^{i\geq 0}(M,N)$. Moreover, let $M^\bullet$ be an A/J-free resolution of M then using the fact that A/J is a full subcategory of A-mod we conclude $\operatorname{Ext}_A^i(M,N) = H^i(\operatorname{Hom}_A(M^\bullet,N)) = H^i(\operatorname{Hom}_{A/J}(M^\bullet,N)) = \operatorname{Ext}_{A/J}^i(M,N)$, $i \geq 0$. ☐

Proposition 1.5.79. *Let A be a split quasi-hereditary algebra over a commutative Noetherian ring R with split heredity chain $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$. If $\operatorname{gldim} A < +\infty$, then $\operatorname{gldim} R < +\infty$. Moreover, R is a regular ring with finite Krull dimension.*

Proof. By definition, $A/J_2 = J_1/J_2$ is a split heredity ideal of A/J_2. Therefore, $\operatorname{End}_{A/J_2}(A/J_2)^{op} \simeq A/J_2$ is Morita equivalent to R. By induction on the split heredity chain together with Lemma 1.5.78 it follows that $\operatorname{gldim} A/J_2 \leq \operatorname{gldim} A$. Thus, the result follows for R. ☐

1.5.9 Dlab and Ringel standardization

The full subcategory of A-mod $\mathscr{F}(\tilde{\Delta})$ completely characterizes the split quasi-hereditary algebra A. In fact, we have the following.

Proposition 1.5.80. *Let $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ and $(B\text{-mod}, \{\Omega(\chi)_{\chi\in X}\})$ be two split highest weight categories. If there is an exact equivalence between $\mathscr{F}(\tilde{\Delta})$ and $\mathscr{F}(\tilde{\Omega})$, then A-mod and B-mod are equivalent as split highest weight categories.*

Proof. Let $H\colon \mathscr{F}(\tilde{\Delta}) \to \mathscr{F}(\tilde{\Omega})$ and $G\colon \mathscr{F}(\tilde{\Omega}) \to \mathscr{F}(\tilde{\Delta})$ be exact equivalences. We claim that there is a bijection $\phi\colon \Lambda \to X$ such that $H\Delta(\lambda) \simeq \Omega(\phi(\lambda)) \otimes_R U_\lambda$ where $U_\lambda \in Pic(R)$ and $HP(\lambda)$ is a projective B-module. We will proceed by induction on $|\Lambda|$. Let $\lambda \in \Lambda$ be a maximal element. By assumption, $H\Delta(\lambda)$ has an Ω-filtration

$$0 = M_{t+1} \subset M_t \subset \cdots \subset M_1 = H\Delta(\lambda), \quad \text{where } M_i/M_{i+1} \simeq \Omega_i \otimes_R F_i, \quad F_i \in R\text{-proj}. \tag{1.5.9.1}$$

Consider the exact sequence

$$0 \to M_t \to H\Delta(\lambda) \to M_1/M_t \to 0. \tag{1.5.9.2}$$

Applying the functor G we obtain the exact sequence (since all elements belong to $\mathscr{F}(\tilde{\Omega})$)

$$0 \to GM_t \to \Delta(\lambda) \xrightarrow{\pi} G(M_1/M_t) \to 0. \tag{1.5.9.3}$$

There are two cases. Either $\pi = 0$ or $\pi \neq 0$. If $\pi = 0$, then $GM_t \simeq \Delta(\lambda)$ and $G(M_1/M_t) = 0$. Thus, $M_1/M_t \simeq HG(M_1/M_t) = 0$ and the filtration (1.5.9.1) collapses to

$$\Omega_t \otimes_R F_t \simeq M_t = \cdots = M_1 = H\Delta(\lambda). \tag{1.5.9.4}$$

Now assume that $\pi \neq 0$.

Consider the Δ-filtration of $G(M_1/M_t)$

$$0 \subset N_\lambda \subset \cdots \subset N_\mu = G(M_1/M_t), \text{where } \mu \text{ is minimal} \tag{1.5.9.5}$$

and $N_\lambda = \Delta(\lambda) \otimes_R U_\lambda$, $U_\lambda \in R$-proj. We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \Delta(\lambda)\otimes_R U_\lambda \arrow[r, "f"] & G(M_1/M_t) \arrow[r, "s"] & G(M_1/M_t)/N_\lambda \arrow[r] & 0 \\ & & \Delta(\lambda) \arrow[u, two heads, "\pi"] & & \end{tikzcd}$$

As λ is maximal $\operatorname{Hom}_A(\Delta(\lambda), G(M_1/M_t)/N_\lambda) = 0$. Moreover if $s \neq 0$, then $s \circ \pi \neq 0$. This means that $G(M_1/M_t)/N_\lambda = 0$. Thus, f is an isomorphism. Let $\mathfrak{m}$ be a maximal ideal in R. Applying $R(\mathfrak{m}) \otimes_R -$ we obtain that $\pi(\mathfrak{m})\colon \Delta(\lambda)(\mathfrak{m}) \to \Delta(\lambda)(\mathfrak{m}) \otimes_{R(\mathfrak{m})} U_\lambda(\mathfrak{m})$ is surjective. By comparing dimensions, we deduce that $U_\lambda(\mathfrak{m}) \simeq R(\mathfrak{m})$ for every maximal ideal in R and $\pi(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R.

As $G(M_1/M_t) \in R$-proj, (1.5.9.3) is (A,R)-exact. Thus, the functor $R(\mathfrak{m}) \otimes_R -$ is exact on (1.5.9.3). Therefore, $GM_t(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ in R. Hence, $GM_t = 0$. Finally applying H, we obtain $M_t = 0$. By going through all modules M_i in the filtration of $H\Delta(\lambda)$ we deduce that $H\Delta(\lambda) = \Omega_x \otimes_R F_\lambda$.

Consider a short exact sequence (given by definition of the standard module Ω_x)

$$0 \to C_x \to Q_x \to \Omega_x \to 0. \tag{1.5.9.6}$$

Applying the exact functor $- \otimes_R F_\lambda$ and then G yields the exact sequence

$$0 \to G(C_x \otimes_R F_\lambda) \to G(Q_x \otimes_R F_\lambda) \to GH\Delta(\lambda) \to 0. \tag{1.5.9.7}$$

Since $GH\Delta(\lambda) \simeq \Delta(\lambda) \in A$-proj, this sequence splits over A. Therefore, by applying H we obtain the B-split exact sequence

$$0 \to HG(C_x \otimes_R F_\lambda) \to HG(Q_x \otimes_R F_\lambda) \to HGH\Delta(\lambda) \to 0 \tag{1.5.9.8}$$

which is equivalent to

$$0 \to C_x \otimes_R F_\lambda \to Q_x \otimes_R F_\lambda \to H\Delta(\lambda) \to 0, \tag{1.5.9.9}$$

and hence it is B-split. In particular, $H\Delta(\lambda)$ is projective over B. Thus, for every maximal ideal $\mathfrak{m}$ in R, $\Omega_x(\mathfrak{m})$ is a $B(\mathfrak{m})$-summand of $\Omega(\mathfrak{m}) \otimes_{R(\mathfrak{m})} F_\lambda(\mathfrak{m}) \simeq \Omega \otimes_R F_\lambda(\mathfrak{m})$ which is projective over $B(\mathfrak{m})$. By Theorem 1.1.51, Ω_x is projective over B since $\Omega_x \in R$-proj. In view of Proposition 1.5.61 and the short exact sequence (1.5.9.6), the module Q_x could have been chosen to be Ω_x and thus $x \in X$ is maximal. Reversing the roles of $\Delta(\lambda)$ and Ω_x and applying the same argument we obtain $G\Omega_x = \Delta(\mu) \otimes_R U_x$, $U_x \in R$-proj. We have

$$H\Delta(\lambda) \simeq \Omega_x \otimes_R F_\lambda \simeq HG\Omega_x \otimes_R F_\lambda \simeq H(\Delta(\mu) \otimes_R U_x) \otimes_R F_\lambda \simeq H(\Delta(\mu) \otimes_R U_x \otimes_R F_\lambda). \tag{1.5.9.10}$$

Therefore, $\Delta(\lambda) \simeq \Delta(\mu) \otimes_R U_x \otimes_R F_\lambda$. It follows that $\mu = \lambda$ and $U_x \otimes_R F_\lambda \simeq R$. Notice that if $n = 1$ there was nothing more to show. Assume that $n > 1$. There is an exact equivalence between $\mathscr{F}(\tilde{\Delta}_{\mu \neq \lambda})$ and $\mathscr{F}(\tilde{\Omega}_{y \neq x})$. Assume by contradiction that Ω_x appears in the filtration of HM for some $M \in \mathscr{F}(\tilde{\Delta}_{\mu \neq \lambda})$. Then, there is an exact sequence with $0 \neq S_x \in R$-proj

$$0 \to \Omega_x \otimes_R S_x \to HM \to HM/\Omega_x \otimes_R S_x \to 0. \tag{1.5.9.11}$$

Applying G yields the exact sequence

$$0 \to \Delta(\lambda) \otimes_R U_x \otimes_R S_x \to M \to G(HM/\Omega_x \otimes_R S_x) \to 0. \tag{1.5.9.12}$$

By assumption, we must have $U_x \otimes_R S_x = 0$, and thus $S_x = 0$ since $\Delta(\lambda)$ is not a factor of M. By induction, there is a bijective map $\phi\colon \Lambda\backslash\{\lambda\} \to X\backslash\{x\}$ satisfying $H\Delta(\mu) \simeq \Omega(\phi(\mu)) \otimes_R U_\mu$ where $U_\mu \in Pic(R)$ and $HP_{A/J}(\mu)$ is a projective B-module. We can extend ϕ to the bijective map $\Lambda \to X$ given by

$$\phi(\mu) = \begin{cases} \phi(\mu) & \text{if } \mu \neq \lambda \\ x & \text{if } \mu = \lambda. \end{cases} \tag{1.5.9.13}$$

It remains to show that $HP(\mu)$ is projective over B. According to Lemma 1.5.46, we have the short exact sequence

$$0 \to \Delta(\lambda) \otimes_R DS_\mu \to P(\mu) \to P_{A/J}(\mu) \to 0. \tag{1.5.9.14}$$

Applying H yields the exact sequence

$$0 \to \Omega_x \otimes_R U_\lambda \otimes_R DS_\mu \to HP(\mu) \to HP_{A/J}(\mu) \to 0. \tag{1.5.9.15}$$

For each maximal ideal $\mathfrak{m}$ in R, we have the exact sequence

$$0 \to (\Omega_x)_\mathfrak{m} \otimes_{R_\mathfrak{m}} DS_{\mu_\mathfrak{m}} \to HP(\mu)_\mathfrak{m} \to HP_{A/J}(\mu)_\mathfrak{m} \to 0. \tag{1.5.9.16}$$

Note that this short exact sequence corresponds to the image of the surjective map

$$S_{\mu_\mathfrak{m}} \twoheadrightarrow \operatorname{Ext}^1_A(P_{A/J}(\mu), \Delta(\lambda))_\mathfrak{m} \simeq \operatorname{Ext}^1_B(HP_{A/J}(\mu), \Omega_x \otimes_R U_\lambda)_\mathfrak{m} \simeq \operatorname{Ext}^1_{B_\mathfrak{m}}(HP_{A/J}(\mu)_\mathfrak{m}, \Omega_{x\mathfrak{m}}). \tag{1.5.9.17}$$

through the isomorphism

$$\begin{aligned} &\operatorname{Hom}_{R_\mathfrak{m}}(S_{\mu_\mathfrak{m}}, \operatorname{Ext}^1_{B_\mathfrak{m}}(HP_{A/J}(\mu)_\mathfrak{m}, \Omega_{x\mathfrak{m}})) \simeq \operatorname{Hom}_R(S_\mu, \operatorname{Ext}^1_B(HP_{A/J}(\mu), \Omega_x \otimes_R U_\lambda))_\mathfrak{m} \\ &\simeq \operatorname{Hom}_R(S_\mu, \operatorname{Ext}^1_A(P_{A/J}(\mu), \Delta(\lambda))_\mathfrak{m} \to \operatorname{Ext}^1_A(P_{A/J}(\mu), \Delta(\lambda) \otimes_R DS_\mu)_\mathfrak{m} \simeq \operatorname{Ext}^1_{B_\mathfrak{m}}(HP_{A/J}(\mu)_\mathfrak{m}, \Omega_{x\mathfrak{m}} \otimes_{R_\mathfrak{m}} DS\mu_\mathfrak{m}). \end{aligned}$$

By Lemma 1.5.46, $HP(\mu)_\mathfrak{m} \in B_\mathfrak{m}$-proj for every maximal ideal $\mathfrak{m}$ in R. Consequently, $HP(\mu) \in B$-proj.

We conclude that $H \bigoplus_{\lambda\in\Lambda} P(\lambda)$ is projective over B. Since $\bigoplus_{\lambda\in\Lambda} P(\lambda)$ is an A-progenerator, there exists $K \in A$-mod and $t > 0$ such that $\left(\bigoplus_{\lambda\in\Lambda} P(\lambda)\right)^t \simeq A \bigoplus K$. Hence, $H\left(\bigoplus_{\lambda\in\Lambda} P(\lambda)\right)^t \simeq HA \bigoplus HK$. Thus, HA is projective over B. In the same way, G preserves projectives. In particular, GB is projective over A. Therefore, $GB \bigoplus K' \simeq A^s$ for some $s > 0$. Applying H yields $B \bigoplus HK' \simeq HA^s$. Therefore, HA is a B-progenerator.

So, the functor $\operatorname{Hom}_B(HA, -)\colon B\text{-mod} \to A\text{-mod}$ is an equivalence of categories. Moreover, for any $x \in X$, $\Omega_x = H\Delta(\phi^{-1}(x)) \otimes_R U_x$. Then,

$$\operatorname{Hom}_B(HA, \Omega_x) \simeq \operatorname{Hom}_B(HA, H\Delta(\phi^{-1}(x)) \otimes_R U_x) \simeq \operatorname{Hom}_B(HA, H\Delta(\phi^{-1}(x))) \otimes_R U_x \tag{1.5.9.18}$$

$$\simeq \operatorname{Hom}_A(A, \Delta(\phi^{-1}(x))) \otimes_R U_x \simeq \Delta(\phi^{-1}(x)) \otimes_R U_x. \tag{1.5.9.19}$$

Thus, the functor $\operatorname{Hom}_B(HA, -)$ is an equivalence of split highest weight categories. □

Remark 1.5.81. The same idea can be used to deduce that if there is an exact equivalence between $\mathscr{F}(\Delta)$ and $\mathscr{F}(\Omega)$, then A and B are equivalent as split highest weight categories. △

Denote by H the exact equivalence $\mathscr{F}(\Delta) \to \mathscr{F}(\Omega)$. By Proposition 1.5.51, there is a filtration (1.5.9.1) with F_i free R-module of finite rank. Using the same argument, we obtain $H\Delta(\lambda) \simeq \Omega_x \otimes_R F_x$ with $F_x(\mathfrak{m}) \simeq R(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R, and thus $F_X \simeq R$. Another difference in the proof of this statement is the choice of S_x. Here S_x is a free R-module of finite rank. In this case, the functor $\operatorname{Hom}_B(HA, -)\colon B\text{-mod} \to A\text{-mod}$ sends Ω_x

to $\Delta(\phi^{-1}(x))$.

Dlab and Ringel showed how to assign a quasi-hereditary algebra to an abelian K-category for some field K (see [DR92]). Here we extend their approach to any abelian R-category, where R can be any commutative Noetherian regular ring with Krull dimension at most one, using the same ideas as in [DR92]. In particular, this applies to all abelian categories that admit a certain collection of objects. This is because we can view every abelian category as an abelian $\mathbb{Z}$-category.

Definition 1.5.82. Let $\mathscr{C}$ be an abelian R-category and $\Theta = \{\theta(i)\colon 1 \leq i \leq n\}$ a finite set of objects of $\mathscr{C}$. The set Θ is said to be **split standardizable** provided the following conditions are satisfied:

(i) $\operatorname{Hom}_{\mathscr{C}}(\theta(i), \theta(j)) = 0$ for $1 \leq j < i \leq n$;

(ii) $\operatorname{Hom}_{\mathscr{C}}(\theta(i), \theta(j)) \in R\text{-proj}$ for $1 \leq i \leq j \leq n$;

(iii) $\operatorname{Ext}^1_{\mathscr{C}}(\theta(i), \theta(j)) = 0$ for $1 \leq j \leq i \leq n$; $\operatorname{Ext}^1_{\mathscr{C}}(\theta(i), \theta(j)) \in R\text{-mod}$ for $1 \leq i, j \leq n$;

(iv) $\operatorname{End}_{\mathscr{C}}(\theta(i)) = R$ for $1 \leq i \leq n$.

Note that for subcategories $\mathscr{C}$ of B-mod for a finite-dimensional K-algebra B over a splitting field K this definition of split standardizable coincides with the usual one of Dlab and Ringel.

We denote $\mathscr{F}(\Theta)$ the full subcategory of $\mathscr{C}$ whose objects have a filtration by objects in Θ.

Theorem 1.5.83. *Let R be a regular ring with Krull dimension at most one. Let Θ be a split standardizable set of objects of an abelian R-category $\mathscr{C}$ with enough projectives. Then, there exists a split quasi-hereditary algebra A, unique up to split highest weight category equivalence, such that the subcategory $\mathscr{F}(\Theta)$ of $\mathscr{C}$ and the category $\mathscr{F}(\Delta)$ are equivalent.*

Proof. The idea used here is essentially the same as in the proof of Dlab-Ringel for the field case ([DR92, Theorem 3]), having, of course, differences regarding the arguments based on the ground ring R. First we construct Ext-projective objects for $\mathscr{F}(\Theta)$, $P_\theta(i)$, together with an exact sequence

$$0 \to K(i) \to P_\theta(i) \to \theta(i) \to 0 \tag{1.5.9.20}$$

and $K(i) \in \mathscr{F}(\theta(i+1), \cdots, \theta(n))$ satisfying $\operatorname{Ext}^1_{\mathscr{C}}(P_\theta(i), \theta(j)) = 0$, $1 \leq j \leq n$ and $\operatorname{Hom}_{\mathscr{C}}(P_\theta(i), \theta(j)) \in R\text{-proj}$. More precisely, we will construct for each $1 \leq i \leq n$, by induction on $m \geq i$, objects $P(i,m)$, $i \leq m \leq n$, such that there is an exact sequence

$$0 \to K(i,m) \to P(i,m) \to \theta(i) \to 0 \tag{1.5.9.21}$$

with $K(i,m) \in \mathscr{F}(\theta(i+1), \cdots, \theta(m))$ and $\operatorname{Ext}^1_{\mathscr{C}}(P(i,m), \theta(j)) = 0, 1 \leq j \leq m$ and $\operatorname{Hom}_{\mathscr{C}}(P(i,m), \theta(j)) \in R\text{-proj}$, $1 \leq j \leq n$.

Assume $m = i$. Let $P(i,i) = \theta(i)$, $K(i,i) = 0$. We have $\operatorname{Ext}^1_{\mathscr{C}}(\theta(i), \theta(j)) = 0$, $j \leq i$ by condition (iii) of split standardizable set. By condition 1.5.82(ii), we have $\operatorname{Hom}_{\mathscr{C}}(P(i,i), \theta(j)) \in R\text{-proj}$ for $1 \leq j \leq n$. Now assume $m > i$ and that $P(i,m-1)$ and $K(i,m-1)$ are already defined. Note that by condition 1.5.82(iv), $\operatorname{End}_{\mathscr{C}}(\theta(i)) \simeq R$.

We need the following observation: $\operatorname{Ext}^1_{\mathscr{C}}(X, \theta(i)) \in R\text{-mod}$ for every $X \in \mathscr{F}(\Theta)$. In fact, we can show it by induction on the size of a filtration of X. If $s = 1$, then it follows by condition 1.5.82(iii). Consider the exact sequence

$$0 \to X' \to X \to \theta(j) \to 0. \tag{1.5.9.22}$$

Applying $\mathrm{Hom}_{\mathscr{C}}(-,\theta(i))$ yields the exact sequence

$$\mathrm{Ext}^1_{\mathscr{C}}(\theta(j),\theta(i)) \to \mathrm{Ext}^1_{\mathscr{C}}(X,\theta(i)) \to \mathrm{Ext}^1_{\mathscr{C}}(X',\theta(i)). \tag{1.5.9.23}$$

This yields an exact sequence

$$0 \to N \to \mathrm{Ext}^1_{\mathscr{C}}(X,\theta(i)) \to M \to 0 \tag{1.5.9.24}$$

where N denotes the image of $\mathrm{Ext}^1_{\mathscr{C}}(\theta(j),\theta(i)) \to \mathrm{Ext}^1_{\mathscr{C}}(X,\theta(i))$ and M denotes the image of $\mathrm{Ext}^1_{\mathscr{C}}(X,\theta(i)) \to \mathrm{Ext}^1_{\mathscr{C}}(X',\theta(i))$. As $\mathrm{Ext}^1_{\mathscr{C}}(\theta(j),\theta(i)) \in R$-mod by exactness there is a surjective map

$$\mathrm{Ext}^1_{\mathscr{C}}(\theta(j),\theta(i)) \twoheadrightarrow N \tag{1.5.9.25}$$

and consequently $N \in R$-mod. By induction, $\mathrm{Ext}^1_{\mathscr{C}}(X',\theta(i)) \in R$-mod. As R is Noetherian, and since there is a mono $M \hookrightarrow \mathrm{Ext}^1_{\mathscr{C}}(X',\theta(i))$, it follows that $M \in R$-mod. Therefore, $\mathrm{Ext}^1_{\mathscr{C}}(X,\theta(i)) \in R$-mod. So, we can consider a free R-module of finite rank $F = R^n$ such that there exists a surjective map $F \twoheadrightarrow \mathrm{Ext}^1_{\mathscr{C}}(P(i,m),\theta(i))$.

By Lemma 1.5.44, there is an exact sequence

$$0 \to \theta(m)^n \xrightarrow{k} P(i,m) \xrightarrow{\pi} P(i,m-1) \to 0 \tag{1.5.9.26}$$

and $\mathrm{Ext}^1_{\mathscr{C}}(P(i,m),\theta(m)) = 0$. Let $1 \leq j < m$. Applying $\mathrm{Hom}_{\mathscr{C}}(-,\theta(j))$ we obtain the exact sequence

$$0 = \mathrm{Hom}_{\mathscr{C}}(\theta(m)^n,\theta(j)) \to \mathrm{Ext}^1_{\mathscr{C}}(P(i,m-1),\theta(j)) \to \mathrm{Ext}^1_{\mathscr{C}}(P(i,m),\theta(j)) \to 0. \tag{1.5.9.27}$$

By induction, $\mathrm{Ext}^1_{\mathscr{C}}(P(i,m-1),\theta(j)) = 0$. Consequently, $\mathrm{Ext}^1_{\mathscr{C}}(P(i,m),\theta(j)) = 0$. Therefore, $\mathrm{Ext}^1_{\mathscr{C}}(P(i,m),\theta(j)) = 0$ for every $1 \leq j \leq m$. By induction, $\mathrm{Hom}_{\mathscr{C}}(P(i,m-1),\theta(j)) \in R$-proj for every j. For each j, applying the functor $\mathrm{Hom}_{\mathscr{C}}(-,\theta(j))$ yields the exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}}(P(i,m-1),\theta(j)) \to \mathrm{Hom}_{\mathscr{C}}(P(i,m),\theta(j)) \to N_j \to 0 \tag{1.5.9.28}$$

where N_j is a submodule of $\mathrm{Hom}_{\mathscr{C}}(\theta(m)^n,\theta(j)) \in R$-proj. Because of $\dim R \leq 1$, $N_j \in R$-proj. It follows that $\mathrm{Hom}_{\mathscr{C}}(P(i,m),\theta(j)) \in R$-proj. Consider the exact sequence

$$0 \to K(i,m-1) \xrightarrow{i_k} P(i,m-1) \xrightarrow{\pi_{m-1}} \theta(i) \to 0, \tag{1.5.9.29}$$

with $K(i,m-1) \in \mathscr{F}(\theta(i+1),\ldots,\theta(m-1))$. Let $(K(i,m),\upsilon)$ be the kernel of $(\pi_{m-1}\circ\pi)$. Now since $\pi_{m-1}\circ\pi\circ\upsilon = 0$, there exists by the uniqueness of kernel of π_{m-1} a unique map $t \in \mathrm{Hom}_{\mathscr{C}}(K(i,m),K(i,m-1))$ such that $\pi\circ\upsilon = i_k\circ t$. Hence, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(i,m) \arrow[r, "\upsilon"] \arrow[d, "t"] & P(i,m) \arrow[r, "\pi"] \arrow[d, two heads, "\pi_{m-1}\circ\pi"] & \theta(i) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K(i,m-1) \arrow[r, "i_k"] & P(i,m-1) \arrow[r, "\pi_{m-1}"] & \theta(i) \arrow[r] & 0
\end{tikzcd}.$$

By Snake Lemma, there exists an exact sequence

$$0 \to \ker\pi \to \ker t \to \operatorname{coker}\mathrm{id} = 0 \to \operatorname{coker} t \to 0. \tag{1.5.9.30}$$

In particular, we have the exact sequence

$$0 \to \theta(m)^n \to K(i,m) \xrightarrow{t} K(i,m-1) \to 0. \tag{1.5.9.31}$$

It follows that $K(i,m) \in \mathscr{F}(\theta(i+1),\dots,\theta(m))$. This finishes the construction of $P(i,m)$. Define $P_\theta(i) = P(i,n)$. Fix $Q = \bigoplus_{i=1}^n P_\theta(i)$.

We claim that, for any $X \in \mathscr{F}(\Theta)$, there exists an exact sequence

$$0 \to X' \to Q_0 \to X \to 0,\ Q_0 \in \operatorname{add} Q,\ X' \in \mathscr{F}(\Theta). \tag{1.5.9.32}$$

We shall proceed by induction on the size of filtration of $X \in \mathscr{F}(\Theta)$. If $s=1$, then $X = \theta(i)$ for some $1 \leq i \leq n$. Then, choose $Q_0 = P_\theta(i)$. Assume $s > 1$ and that the result holds for objects with filtration with a size less than s. Let X be an object which admits a filtration of size s by objects in Θ. Consider the exact sequence

$$0 \to X' \xrightarrow{\iota} X \xrightarrow{\pi} \theta(i) \to 0. \tag{1.5.9.33}$$

By induction, there is an exact sequence

$$0 \to C \to Q_0 \xrightarrow{p} X' \to 0. \tag{1.5.9.34}$$

The reasoning is exactly the same argument as in the Horseshoe's Lemma. By construction of $P_\theta(i)$, $\operatorname{Ext}^1_{\mathscr{C}}(P_\theta(i),X') = 0$ since $X' \in \mathscr{F}(\Theta)$. Applying $\operatorname{Hom}_{\mathscr{C}}(P_\theta(i),-)$ yields the surjective map

$$\operatorname{Hom}_{\mathscr{C}}(P_\theta(i),X) \to \operatorname{Hom}_{\mathscr{C}}(P_\theta(i),\theta(i)) \to \operatorname{Ext}_{\mathscr{C}}(P_\theta(i),X') = 0. \tag{1.5.9.35}$$

Then, there exists $\rho \in \operatorname{Hom}_{\mathscr{C}}(P_\theta(i),X)$ such that $\pi \circ \rho = \pi_i$, where π denotes the map $P_\theta(i) \to \theta(i)$. By the biproduct definition there is a unique map $\varpi \in \operatorname{Hom}_{\mathscr{C}}(Q_0 \oplus P_\theta(i),X)$ making the following diagram commutative

$$\begin{tikzcd} Q_0 \arrow[r, "\iota\circ p"] \arrow[dr, "i_{Q_0}"'] & X & P_\theta(i) \arrow[l, "\rho"'] \arrow[dl, "i_{P_\theta}"] \\ & Q_0 \oplus P_\theta(i) \arrow[u, "\varpi"] & \end{tikzcd}.$$

In particular,

$$\pi \circ \varpi = \pi \circ \varpi \circ i_{P_\theta} \circ \pi_{P_\theta} + \pi \circ \varpi \circ i_{Q_0} \circ \pi_{Q_0} = \pi \circ \rho \circ \pi_{P_\theta} + \pi \circ \iota \circ p \circ \pi_{Q_0} = \pi_i \circ \pi_{P_\theta}. \tag{1.5.9.36}$$

Hence, we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "\iota"] & X \arrow[r, "\pi"] & \theta(i) \arrow[r] & 0 \\ 0 \arrow[r] & Q_0 \arrow[r, "i_{Q_0}"] \arrow[u, "p"] & Q_0 \oplus P_\theta(i) \arrow[r, "\pi_{P_\theta}"] \arrow[u, "\varpi"] & P_\theta(i) \arrow[r] \arrow[u, "\pi_i"] & 0 \end{tikzcd}.$$

By Snake Lemma, there exists an exact sequence

$$0 \to \ker p \to \ker \varpi \to \ker \pi_i \to 0 \to \operatorname{coker} \varpi \to 0. \tag{1.5.9.37}$$

As $K(i) \simeq \ker \pi_i$, and $\ker p = C \in \mathscr{F}(\Theta)$ we obtain $\ker \varpi \in \mathscr{F}(\Theta)$. This completes the proof of our claim.

Let $A = \operatorname{End}_{\mathscr{C}}(Q)$. By construction $\operatorname{Hom}_{\mathscr{C}}(P_\theta(i),\theta(j)) \in R\text{-proj}$. Thus, $\operatorname{Hom}_{\mathscr{C}}(Q,\theta(j)) \in R\text{-proj}$ for every j. Consequently, $\operatorname{Hom}_{\mathscr{C}}(Q,X) \in R\text{-proj}$ for every $X \in \mathscr{F}(\Theta)$. Since $Q \in \mathscr{F}(\Theta)$, A is projective Noetherian R-algebra. Define the functor $G = \operatorname{Hom}_{\mathscr{C}}(Q,-)\colon \mathscr{C} \to A\text{-Mod}$. By this discussion $GX \in R\text{-proj}$ for every $X \in \mathscr{F}(\Theta)$. Again, by the construction of Q,

$$\operatorname{Ext}^1_{\mathscr{C}}(P_\theta(i),\theta(j)) = 0, \forall j \implies \operatorname{Ext}^1_{\mathscr{C}}(Q,\theta(j)) \simeq \oplus \operatorname{Ext}^1_{\mathscr{C}}(P_\theta(i),\theta(j)) = 0. \tag{1.5.9.38}$$

By induction on the size of filtration of $X \in \mathscr{F}(\Theta)$ and applying $\mathrm{Hom}_{\mathscr{C}}(Q,-)$ to the exact sequences arising from such filtration, we deduce that $\mathrm{Ext}^1_{\mathscr{C}}(Q,X)=0$ for all $X \in \mathscr{F}(\Theta)$. Therefore, the functor $G\colon \mathscr{C} \to A\text{-Mod}$ is exact on the exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$ with $X \in \mathscr{F}(\Theta)$.

Define $P_A(i) = GP_\theta(i)$ and $\Delta(i) = G\theta(i)$. In particular, $P_A(i), \Delta(i) \in R\text{-proj}$. Since G is exact on $\mathscr{F}(\Theta)$, it takes objects in $\mathscr{F}(\Theta)$ to modules in $\mathscr{F}(\Delta)$. We shall now prove that the restriction functor $G\colon \mathscr{F}(\Theta) \to \mathscr{F}(\Delta)$ is faithful.

Let $\psi \in \mathrm{Hom}_{\mathscr{F}(\Theta)}(X,Y)$ such that $G\psi = 0$. There exists an exact sequence by (1.5.9.32),

$$\begin{tikzcd} Q_1(X) \arrow[r, "\pi'_X"] \arrow[dr, two heads, "q_x"'] & Q_0(X) \arrow[r, "\pi_X"] & X \arrow[r] & 0 \\ & K_0(X) \arrow[u, hook, "\upsilon_x"] & & \end{tikzcd}, \tag{1.5.9.39}$$

with $K_0(X) \in \mathscr{F}(\Theta), Q_0(X), Q_1(X) \in \mathrm{add}\, Q$. By the same reason we obtain a similar diagram for Y. By projectivization, $G_{|_{\mathrm{add}\,Q}}\colon \mathrm{add}\, Q \to A\text{-proj}$ is an equivalence. Because $\mathrm{Ext}^1_{\mathscr{C}}(Q_0(X), K_0(Y)) = 0$, the homomorphism $\mathrm{Hom}_{\mathscr{C}}(Q_0(X), Q_0(Y)) \to \mathrm{Hom}_{\mathscr{C}}(Q_0(X), Y)$ is surjective. So, there exists $\psi_0 \in \mathrm{Hom}_{\mathscr{C}}(Q_0(X), Q_0(Y))$ such that $\pi_Y \circ \psi_0 = \psi \circ \pi_X$. So, we obtain a commutative diagram

$$\begin{tikzcd} Q_1(X) \arrow[rr] \arrow[dr, two heads, "q_X"] & & Q_0(X) \arrow[r, "\pi_X"] \arrow[ddd, "\psi_0"] & X \arrow[r] \arrow[ddd, "\psi"] & 0 \\ & K_0(X) \arrow[ur, hook, "\upsilon_X"] & & & \\ & K_0(Y) \arrow[dr, hook, "\upsilon_Y"] & & & \\ Q_1(Y) \arrow[rr] \arrow[ur, two heads, "q_Y"] & & Q_0(Y) \arrow[r, "\pi_Y"] & Y \arrow[r] & 0 \end{tikzcd}.$$

Applying G we obtain

$$0 = G\psi \circ G\pi_X = G(\psi \circ \pi_X) = G(\pi_Y \circ \psi_0) = G\pi_Y \circ G\psi_0. \tag{1.5.9.40}$$

Since the sequence

$$0 \to GK_0(Y) \xrightarrow{G\upsilon_Y} GQ_0(Y) \xrightarrow{G\pi_Y} GY \to 0 \tag{1.5.9.41}$$

is exact, then $G\upsilon_Y$ is kernel of $G\pi_Y$. Hence, by the universal property of the kernel, there exists an A-homomorphism $l \in \mathrm{Hom}_A(GQ_0(X), GK_0(Y))$ such that $G\upsilon_Y \circ l = G\psi_0$. By projectivization, $GQ_0(X) \in A\text{-proj}$. Therefore, there exists a map $\varsigma \in \mathrm{Hom}_A(GQ_0(X), GQ_1(Y))$ such that $Gq_Y \circ \varsigma = l$. Since G is full and faithful on $\mathrm{add}\, Q$ there exists a unique map $\varsigma' \in \mathrm{Hom}_{\mathscr{C}}(Q_0(X), Q_1(Y))$ such that $G\varsigma' = \varsigma$. Thus,

$$G(\upsilon_Y \circ q_Y \circ \varsigma') = G(\upsilon_Y \circ q_Y) \circ G\varsigma' = G\upsilon_Y \circ l = G\psi_0. \tag{1.5.9.42}$$

Since G is full and faithful on $\mathrm{add}\, Q$ we get

$$\upsilon_Y \circ q_Y \circ \varsigma' = \psi_0 \implies \psi \circ \pi_X = \pi_Y \circ \psi_0 = \pi_Y \circ \upsilon_Y \circ q_Y \circ \varsigma' = 0 \implies \psi = 0, \tag{1.5.9.43}$$

since π_X is an epimorphism. Thus, $G_{|_{\mathscr{F}(\Theta)}}$ is faithful.

We now shall prove that $G\colon \mathscr{F}(\Theta) \to \mathscr{F}(\Delta)$ is full. Let $X, Y \in \mathscr{F}(\Theta)$ and $f' \in \mathrm{Hom}_A(GX, GY)$. Applying G

to the exact sequence (1.5.9.39) and the one for Y, we obtain projective presentations for GX and GY, respectively. As $GQ_0(X) \in A$-proj, there exists a map $g' \in \operatorname{Hom}_A(Q_0(X), Q_0(Y))$ such that $G\pi_Y \circ g' = f' \circ G\pi_X$. In particular,

$$G\pi_Y \circ g' \circ G\upsilon_X = 0. \tag{1.5.9.44}$$

So, there exists a unique map, by the uniqueness of kernel of $G\pi_Y$, $\tau \in \operatorname{Hom}_A(GK_0(X), GK_0(Y))$ such that $g' \circ G\upsilon_X = G\upsilon_Y \circ \tau$. Since $GQ_1(X) \in A$-proj, there exists a map $h' \in \operatorname{Hom}_A(GQ_1(X), GQ_1(Y))$ such that $Gq_Y \circ h' = \tau \circ Gq_X$. Moreover,

$$g' \circ G(\upsilon_X \circ q_X) = G\upsilon_Y \circ \tau \circ Gq_X = G\upsilon_Y \circ Gq_Y \circ h'. \tag{1.5.9.45}$$

This means that we have constructed the following commutative diagram

$$\begin{tikzcd} GQ_1(X) \arrow[r, "G(\upsilon_X \circ q_X)"] \arrow[d, "h'"] & GQ_0(X) \arrow[r, "G\pi_X"] \arrow[d, "g'"] & GX \arrow[r] \arrow[d, "f'"] & 0 \\ GQ_1(Y) \arrow[r, "G(\upsilon_Y \circ q_Y)"] & GQ_O(Y) \arrow[r, "G\pi_Y"] & GY \arrow[r] & 0 \end{tikzcd}.$$

Since G is full and faithful on $\operatorname{add} Q$ there exists $g \in \operatorname{Hom}_{\mathscr{C}}(Q_0(X), Q_0(Y))$ and $h \in \operatorname{Hom}_{\mathscr{C}}(Q_1(X), Q_1(Y))$ such that $Gg = g'$ and $Gh = h'$. Therefore, the following diagram is commutative:

$$\begin{tikzcd} Q_1(X) \arrow[r, "\upsilon_X \circ q_X"] \arrow[d, "h"] & Q_0(X) \arrow[r, "\pi_X"] \arrow[d, "g"] & X \arrow[r] & 0 \\ Q_1(Y) \arrow[r, "\upsilon_Y \circ q_Y"] & Q_O(Y) \arrow[r, "\pi_Y"] & Y \arrow[r] & 0 \end{tikzcd}.$$

In particular,

$$\pi_Y \circ g \circ \upsilon_X \circ q_X = \pi_Y \circ \upsilon_Y \circ h = 0 \implies \pi_Y \circ g \circ \upsilon_X = 0. \tag{1.5.9.46}$$

As π_X is the cokernel of υ_X, there exists a unique map $f \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$ such that $f \circ \pi_X = \pi_Y \circ g$. Thus,

$$Gf \circ G\pi_X = G\pi_Y \circ Gg = G\pi_Y \circ g' = f' \circ G\pi_X \implies Gf = f'. \tag{1.5.9.47}$$

Hence, $G_{\mathscr{F}(\Theta)}$ is full.

The next step will be to show that $(A\text{-mod}, \Delta)$ is a split highest weight category. As we discussed earlier $\Delta(i) = G\theta(i) \in R$-proj. If $\operatorname{Hom}_A(\Delta(i), \Delta(j)) \neq 0$, then

$$0 \neq \operatorname{Hom}_A(\Delta(i), \Delta(j)) = \operatorname{Hom}_A(G\theta(i), G\theta(j)) \simeq \operatorname{Hom}_{\mathscr{C}}(\theta(i), \theta(j)). \tag{1.5.9.48}$$

So, $i \leq j$ by definition of split standardizable set. Moreover,

$$\operatorname{End}_A(\Delta(i)) \simeq \operatorname{End}_A(G\theta(i)) \simeq \operatorname{End}_{\mathscr{C}}(\theta(i)) \simeq R. \tag{1.5.9.49}$$

Consider the exact sequence

$$0 \to K(i) \to P_\theta(i) \to \theta(i) \to 0 \tag{1.5.9.50}$$

and $K(i) \in \mathscr{F}(\theta(i+1), \ldots, \theta(n))$. Applying G we get the exact sequence

$$0 \to GK(i) \to P_A(i) \to \Delta(i) \to 0 \tag{1.5.9.51}$$

with $GK(i) \in \mathscr{F}(\Delta(i+1),\dots,\Delta(i))$ and $P_A(i) \in A$-proj. Note that $\oplus_{i=1}^n P_A(i) = \oplus_{i=1}^n GP_\theta(i) = GQ$. By projectivization, every projective A-module is given in the form HX, $X \in \operatorname{add} Q$. Therefore, HQ is an A-progenerator. By Corollary 1.5.43, $(A\text{-mod},\Delta)$ is split highest weight category.

It remains to show that $G\colon \mathscr{F}(\Theta) \to \mathscr{F}(\Delta)$ is essentially surjective. Let $0 \neq M \in \mathscr{F}(\Delta)$. Then, M has a Δ-filtration of size s. We shall prove by induction on the size of Δ-filtration of a module that $M \simeq GX$ for some $X \in \mathscr{F}(\Theta)$. Assume $s=1$. Then, $M \simeq \Delta(j) \simeq G\theta(j)$. Assume that the claim holds for modules with filtrations of size $s-1$, $s>1$. Let M have a filtration of size s. Then, there exists an exact sequence

$$0 \to \Delta(j) \xrightarrow{\upsilon} M \xrightarrow{\pi} M' \to 0 \tag{1.5.9.52}$$

and M' has a filtration of size $s-1$. By induction, there exists $X' \in \mathscr{F}(\Theta)$ such that $GX' \simeq M'$. Consider a projective presentation of GX' over A

$$0 \to GK_0(X') \xrightarrow{Gk} GQ_0(X') \xrightarrow{Gp} GX' \to 0,\ Q_0(X') \in \operatorname{add} Q,\ K_0(X') \in \mathscr{F}(\Theta), \tag{1.5.9.53}$$

$p \in \operatorname{Hom}_{\mathscr{C}}(Q_0(X'),X')$ and $k \in \operatorname{Hom}_{\mathscr{C}}(K_0(X'),Q_0(X'))$. Since $GQ_0(X') \in A$-proj there exists $z \in \operatorname{Hom}_A(GQ_0(X'),M)$ such that $\pi \circ z = Gp$. Consider the A-homomorphism $o\colon \Delta(j) \bigoplus GQ_0(X') \to M$, given by $(x,y) \mapsto \upsilon(x)+z(y)$. Note that

$$\pi \circ o(x,y) = \pi \circ \upsilon(x) + \pi \circ z(y) = Gp(y) = Gp \circ \pi_{GQ_0}(x,y),\ (x,y) \in \Delta(j) \oplus GQ_0(X'). \tag{1.5.9.54}$$

Moreover,

$$\pi \circ o \circ i_\Delta = Gp \circ \pi_{GQ_0} \circ i_\Delta = 0 \tag{1.5.9.55}$$

By the uniqueness of kernel of π, there exists a unique map $l \in \operatorname{Hom}_{\mathscr{C}}(\Delta(j),\Delta(j))$ such that $o \circ i_\Delta = \upsilon \circ l$. Hence we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Delta(j) \arrow[r, "\upsilon"] & M \arrow[r, "\pi"] & GX' \arrow[r] & 0 \\
0 \arrow[r] & \Delta(j) \arrow[r, "i_\Delta"] \arrow[u, "l"] & \Delta(j) \bigoplus GQ_0(X') \arrow[r, "\pi_{GQ_0}"] \arrow[u, "o"] & GQ_0(X') \arrow[r] \arrow[u, two heads, "Gp"] & 0
\end{tikzcd}.$$

Since $\upsilon \circ l = o \circ i_\Delta = \upsilon$ and $GX' \in \mathscr{F}(\Delta) \subset R$-proj, it follows that υ is (A,R)-monomorphism. Hence, there is $a \in \operatorname{Hom}_R(M,\Delta(j))$ such that

$$a \circ \upsilon = \mathrm{id}_{\Delta(j)} \implies a \circ \upsilon \circ l = a \circ \upsilon = \mathrm{id}_{\Delta(j)}. \tag{1.5.9.56}$$

Therefore, l is (A,R)-mono. As $\Delta(j) \in R$-proj, it follows by Nakayama's Lemma that l is an isomorphism. By Snake Lemma, there exists an exact sequence

$$0 \to \ker o \to \ker Gp \to 0 \to \operatorname{coker} o \to \operatorname{coker} Gp = 0. \tag{1.5.9.57}$$

Hence, o is surjective and $\ker o \simeq GK_0(X')$. In particular, (as $G_{\mathscr{F}(\Theta)}$ is full and faithful) there exists a monomorphism $\gamma \in \operatorname{Hom}_{\mathscr{C}}(K_0(X'), \theta(j) \bigoplus Q_0(X'))$ such that the following sequence is exact:

$$0 \to GK_0(X') \xrightarrow{G\gamma} \Delta(j) \bigoplus GQ_0(X') \xrightarrow{o} M \to 0. \tag{1.5.9.58}$$

Furthermore, since the isomorphism $GK_0(X') \simeq \ker Gp \simeq \ker o$ arises from the Snake Lemma, we have the following condition $\pi_{GQ_0} \circ G\gamma = Gk \circ \alpha$ where $\alpha \in \operatorname{Hom}_A(GK_0(X'),GK_0(X'))$ is an isomorphism.

Let (X,w) be the cokernel of γ. Applying G, (GX, Gw) is the cokernel of $G\gamma$. By the uniqueness of cokernel of $G\gamma$, it follows that $GX \simeq M$. Since G is full and faithful on $\mathscr{F}(\Theta)$ we obtain $\pi_{Q_0} \circ \gamma = k \circ \alpha'$ with $G\alpha' = \alpha$. By the uniqueness of cokernel of γ there exists a unique map $\beta \in \operatorname{Hom}_{\mathscr{C}}(X,X')$ such that $\beta \circ w = p \circ \pi_{Q_0}$. This means that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_0(X') \arrow[r, "\gamma"] \arrow[d, "\alpha'"] & \theta(j)\bigoplus Q_0(X') \arrow[r, "w"] \arrow[d, "\pi_{Q_0}"] & X \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & K_0(X') \arrow[r, "k"] & Q_0(X') \arrow[r, "p"] & X' \arrow[r] & 0 \end{tikzcd}.$$

By Snake Lemma there exists an exact sequence

$$0 \to \ker \pi_{Q_0} \to \ker \beta \to \operatorname{coker} \alpha' = 0 \to \operatorname{coker} \beta \to 0. \tag{1.5.9.59}$$

Thus, β is an epimorphism and $\ker \beta \simeq \ker \pi_{Q_0} = \theta(j)$. So, we have an exact sequence

$$0 \to \theta(j) \to X \xrightarrow{\beta} X' \to 0. \tag{1.5.9.60}$$

As $X', \theta(j) \in \mathscr{F}(\Theta)$, X belongs in $\mathscr{F}(\Theta)$. This concludes our claim that $G_{\mathscr{F}(\Theta)}$ is essentially surjective. We conclude that $G\colon \mathscr{F}(\Theta) \to (\Delta)$ is an equivalence of categories.

If there exists another split quasi-hereditary algebra A' such that there is an exact equivalence of categories $\mathscr{F}(\Theta) \to \mathscr{F}(\Delta_{A'})$, then there is an exact equivalence $\mathscr{F}(\Delta) \to \mathscr{F}(\Delta_{A'})$. By Remark 1.5.81, A and A' are equivalent as split highest weight categories. ☐

1.5.10 Split quasi-hereditary algebras and the existence of projective covers

Recall that a ring A is called **semi-perfect** if every finitely generated left A-module has a projective cover.

Theorem 1.5.84. *Every split quasi-hereditary algebra over a local commutative Noetherian ring is semi-perfect.*

Proof. According to Proposition 1.5.64, we can choose $P(\lambda)$ in 1.5.32(iv) so that $\operatorname{End}_A(P(\lambda))$ is local. Hence, $\bigoplus_{\lambda \in \Lambda} P(\lambda)$ is a direct sum of modules with local endomorphism rings. Let ${}_AA \simeq Q_0 \bigoplus \cdots \bigoplus Q_t$ be a decomposition into indecomposable A-modules of regular module A. By Corollary 1.5.43, $\bigoplus_{\lambda \in \Lambda} P(\lambda)$ is an A-progenerator. Thus, there is $K \in A$-mod such that

$$\left(\bigoplus_{\lambda \in \Lambda} P(\lambda) \right)^t \simeq A \bigoplus K. \tag{1.5.10.1}$$

By Krull-Schmidt-Remak-Azumaya Theorem [Fac98, Theorem 2.12] any two direct sum decompositions into indecomposable modules of $\left(\bigoplus_{\lambda \in \Lambda} P(\lambda) \right)^t$ are isomorphic. Hence, every Q_i is isomorphic to a projective indecomposable module $P(\lambda_i)$. Hence, ${}_AA$ is a finite direct sum of A-modules with local endomorphism rings. By Theorem 1.5.69, A^{op} is split quasi-hereditary over R, thus by this discussion A_A is a finite direct sum of A-modules with local endomorphism rings. By [Fac98, Proposition 3.14], $A \simeq \operatorname{End}_A(A_A)$ is a semi-perfect ring. ☐

We observe that as a consequence of Theorem 1.5.84, for any $\lambda \in \Lambda$, we can choose $P(\lambda)$ so that $P(\lambda)$ is the projective cover of $\Delta(\lambda)$, when R is a local Noetherian commutative ring.

In fact, assume that R is a local commutative Noetherian ring. By Theorem 1.5.84, there exists a projective cover Q of $\Delta(\lambda)$. Using the surjective homomorphism $\pi_\lambda : P(\lambda) \to \Delta(\lambda)$ given by 1.5.32(iv) with $P(\lambda)$ having a local endomorphism ring, it follows that Q is an A-summand of $P(\lambda)$. As $P(\lambda)$ is indecomposable $P(\lambda) \simeq Q$. By Nakayama's Lemma, we deduce that $(P(\lambda), \pi_\lambda)$ is the projective cover of $\Delta(\lambda)$.

1.5.11 Decomposition of split quasi-hereditary algebras into blocks

This result is widely known, however, we decided to include it here for sake of completeness.

Proposition 1.5.85. *Let R be a local commutative Noetherian ring. Suppose that $A = \prod_{i=1}^n A_i$ is a direct product decomposition of A. The following assertions are equivalent.*

(a) $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *is a split quasi-hereditary algebra.*

(b) *(i) We can decompose Λ as the disjoint union of preordered sets $\Lambda = \dot{\bigcup}_{i=1}^n \Lambda_i$;*

(ii) $(A_i, \{\Delta(\lambda)_{\lambda\in\Lambda_i}\})$ *is a split quasi-hereditary algebra for every $i = 1, \dots, n$.*

Proof. Using induction, if necessary, we can assume without loss of generality that $n = 2$.

Assume that (a) holds. Then, $P(\lambda)$ and $\Delta(\lambda)$ are indecomposable A-modules by Proposition 1.5.64. There are central idempotents e_1, e_2 such that $A_1 = Ae_1A$ and $A_2 = Ae_2A$. So, we can decompose $P(\lambda) = e_1P(\lambda) \bigoplus e_2P(\lambda)$ as A-modules for every $\lambda \in \Lambda$. So, either $e_1P(\lambda) = 0$ or $e_2P(\lambda) = 0$. Moreover, either $P(\lambda) \in A_1$-mod or $P(\lambda) \in A_2$-mod. Since $e_1A \otimes_A -$ and $e_2A \otimes_A -$ are exact functors, there exists a surjective map $e_iP(\lambda) \to e_i\Delta(\lambda)$, for every $\lambda \in \Lambda$, $i = 1, 2$. Hence, $\Delta(\lambda) \in A_i$-mod if $P(\lambda) \in A_i$-mod, $\lambda \in \Lambda$, and $i \in 1, 2$. Define

$$\Lambda_1 = \{\lambda \in \Lambda \colon e_2P(\lambda) = 0\} \tag{1.5.11.1}$$

$$\Lambda_2 = \{\lambda \in \Lambda \colon e_1P(\lambda) = 0\}. \tag{1.5.11.2}$$

In particular, $\Lambda = \Lambda_1 \dot{\bigcup} \Lambda_2$. And, so Λ_i is a preordered set, $i = 1, 2$.

Let $\lambda \in \Lambda_1$. Consider the exact sequence $0 \to X(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$, where $X(\lambda) \in \mathscr{F}(\Delta_{\mu>\lambda})$. Applying $e_2A \otimes_A -$ yields that $e_2X(\lambda) = 0$. Thus, every standard module $\Delta(\mu)$, $\mu > \lambda$ that appears in the filtration of $X(\lambda)$ belongs to A_1-mod. Since A_1-mod is a full subcategory of A-mod $(i), (ii), (iv), (v)$ of Definition 1.5.32 are trivially satisfied. Now assume that there exists $N \in A_1$-mod such that $\operatorname{Hom}_{A_1}(\Delta(\mu), N) = 0$ for all $\mu \in \Lambda_1$. Thus, we can regard N as A-module and $\operatorname{Hom}_A(\Delta(\mu), N) = 0$ for all $\mu \in \Lambda_1$. Using the general fact that $\operatorname{Hom}_A(M, N) = 0$ if $M \in A_2$-mod, we obtain $\operatorname{Hom}_A(\Delta(\lambda), N) = 0$ for every $\lambda \in \Lambda$. So, $N = 0$. This completes (b).

Conversely, assume that (b) holds. Since A_i-mod is a full subcategory of A-mod and $\operatorname{Hom}_A(M, N) = 0$ if $M \in A_i$-mod and $N \in A_j$-mod with $i \neq j$, 1.5.32 $(i), (ii)$ and (v) are satisfied. Let $N \in A$-mod such that $\operatorname{Hom}_A(\Delta(\lambda), N) = 0$ for all $\lambda \in \Lambda$. Then, the decomposition $A = A_1 \times A_2$ induces a decomposition $N \simeq N_1 \oplus N_2$ with $N_i \in A_i$-mod. In particular, $\operatorname{Hom}_A(\Delta(\lambda), N_i) = 0$ for every $\lambda \in \Lambda$ and $i = 1, 2$. So, each N_i is zero, and thus $N = 0$. It is enough to observe that the projective A_i-modules are projective A-modules to obtain 1.5.32(iv). This is the case, since as left A-modules, $A \simeq A_1 \oplus A_2$. This completes the proof of (a). □

This result shows that an algebra is split quasi-hereditary over a local commutative Noetherian ring if and only if each block of the algebra is.

1.5.12 Examples of split quasi-hereditary algebras

A classic example of a split quasi-hereditary algebra is the classical Schur algebra. The quantised Schur algebra is also an example of a split quasi-hereditary algebra (see [CPS90, Theorem 3.7.2]).

Proposition 1.5.86. *Every split relative semi-simple algebra over a commutative Noetherian ring is split quasi-hereditary.*

Proof. By assumption, $A \simeq M_{n_1}(R) \times \cdots \times M_{n_t}(R)$. Define $J_i\colon = M_{n_t}(R) \times \cdots \times M_{n_i}(R),\ i = 1, \dots, t+1$. Then,

$$J_i/J_{i+1} \simeq M_{n_t}(R) \times \cdots M_{n_{i+1}}(R) \times M_{n_i}(R)/M_{n_t}(R) \times \cdots \times M_{n_{i+1}}(R) \simeq M_{n_i}(R) \tag{1.5.12.1}$$

$$A/J_{i+1} \simeq M_{n_1}(R) \times \cdots \times M_{n_t}/M_{n_t}(R) \times \cdots \times M_{i+1}(R) \simeq M_{n_1}(R) \times \cdots \times M_{n_i}(R). \tag{1.5.12.2}$$

Note that A is a projective Noetherian R-algebra. Clearly for each i, $(A/J_{i+1})/(J_i/J_{i+1}) \simeq M_{n_1} \times \cdots \times M_{n_{i-1}}$ is projective over R and for the idempotent $e_i = (0, \cdots, 0, I_{n_i})$ $J_i/J_{i+1} \simeq (A/J_{i+1})e_i \in A/J_{i+1}\text{-proj}$. Note also that

$$\mathrm{End}_{M_{n_1}(R)\times\cdots\times M_{n_i}(R)}(M_{n_i}(R))^{op} \simeq \mathrm{End}_{M_{n_i}(R)}(M_{n_i}(R))^{op} \simeq ((M_{n_i}(R))^{op})^{op} \tag{1.5.12.3}$$

is Morita equivalent to R. ☐

The following example shows that relative hereditary semi-perfect algebras over suitable local commutative Noetherian rings are split quasi-hereditary algebras.

Proposition 1.5.87. *Let A be a projective Noetherian R-algebra over a local commutative Noetherian ring R. Suppose that the following conditions hold.*

(a) A is semi-perfect;

(b) The residue field of R, $R(\mathfrak{m})$, is a splitting field for $A(\mathfrak{m})$;

(c) $\mathrm{gldim}_f(A, R) \leq 1$.

Then, A is a split quasi-hereditary algebra.

Proof. Let $\bigoplus_I P_i^{n_i}$ be a decomposition of A into indecomposable modules for some finite set I. By (a), the modules P_i, $i \in I$, have local endomorphism rings. Let $\mathfrak{m}$ be the unique maximal ideal of R. Let $i \in I$. Observe that

$$\mathrm{End}_{\hat{A}}(\hat{P}_i) \simeq \widehat{\mathrm{End}_A(P_i)} = \lim_n \mathrm{End}_A(P_i)/\mathfrak{m}^n \,\mathrm{End}_A(P_i) = \lim_n \mathrm{End}_A(P_i)/(\mathfrak{m}\,\mathrm{End}_A(P_i))^n. \tag{1.5.12.4}$$

Now, $\lim_n \mathrm{End}_A(P_i)/(\mathfrak{m}\,\mathrm{End}_A(P_i))^n$ is the localization of $\mathrm{End}_A(P_i)$ at the ideal $\mathfrak{m}\,\mathrm{End}_A(P_i)$. Since $\mathrm{End}_A(P_i)$ is local, $\mathfrak{m}\,\mathrm{End}_A(P_i)$ is contained in $\mathrm{rad}\,\mathrm{End}_A(P_i)$. Therefore, $\lim_n \mathrm{End}_A(P_i)/(\mathfrak{m}\,\mathrm{End}_A(P_i))^n$ is a local ring. By (1.5.12.4), $\mathrm{End}_{\hat{A}}(\hat{P}_i)$ is local. Thus, $\hat{P}_i$ is an indecomposable projective $\hat{A}$-module. By Theorem [CR90, (6.7, 6.8)], $P_i(\mathfrak{m}) \simeq \hat{P}_i(\hat{\mathfrak{m}})$ is an indecomposable projective $A(\mathfrak{m})$-module. Recall that all (A,R)-projective modules are summands of $A \otimes_R M$ for some $M \in R$-mod. Hence, $A \otimes_R M(\mathfrak{m}) \simeq A(\mathfrak{m}) \otimes_{R(\mathfrak{m})} M(\mathfrak{m}) \in \mathrm{add}\,A(\mathfrak{m})$. So, (A,R)-projective resolutions are sent to projective $A(\mathfrak{m})$-resolutions under $R(\mathfrak{m}) \otimes_R -$. Using this observation with (c) it follows that $\mathrm{gldim}\,A(\mathfrak{m}) \leq 1$. Thus, $A(\mathfrak{m})$ is an hereditary algebra. Since (b) holds $A(\mathfrak{m})$ is a split quasi-hereditary algebra with poset $(I, <)$. For each $i \in I$, define

$$U_i = \sum_{\substack{i,j \in I \\ j > i}} \sum_{f \in \mathrm{Hom}_A(P(j), P(i))} \mathrm{im}\, f. \tag{1.5.12.5}$$

Let $\Delta(i) = P_i/U_i$, $i \in I$. By Proposition 1.5.63 and construction of I, $\Delta(i) \simeq P_i(\mathfrak{m})/U_i(\mathfrak{m})$ for all $i \in I$. In particular, the monomorphism $U_i \to P_i$ remains injective under $R(\mathfrak{m}) \otimes_R -$. So, $\mathrm{Tor}_1^R(R(\mathfrak{m}), \Delta(i)) = 0$ for every $i \in I$. Thus, $\Delta(i) \in R$-proj for all $i \in I$. By Theorem 1.5.56, $(A, \Delta(i)_{i \in I})$ is a split quasi-hereditary algebra. ☐

Proposition 1.5.88. *Let A be a projective Noetherian R-algebra over a local commutative Noetherian ring R. Suppose that the following conditions hold.*

(a) A is semi-perfect;

(b) The residue field of R, $R(\mathfrak{m})$, is a splitting field for $A(\mathfrak{m})$;

(c) $\operatorname{gldim}_f(A,R) \leq 2$.

Then, A is a split quasi-hereditary algebra.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. Using the same arguments as in the proof of Proposition 1.5.87, it follows that $\operatorname{gldim} A(\mathfrak{m}) \leq 2$ and a decomposition of A into indecomposable modules remains, under $R(\mathfrak{m}) \otimes_R -$, an indecomposable decomposition of $A(\mathfrak{m})$. Since (b) holds and by Theorem [DR89b, Theorem 2], $A(\mathfrak{m})$ is a split quasi-hereditary algebra with poset $(I,<)$. Defining $\Delta(i)$, $i \in I$, in the same way as in Proposition 1.5.87 yields that $(A, \Delta(i)_{i\in I})$ is a split quasi-hereditary algebra. $\square$

Example 1.5.89. Let R be a principal ideal domain. Let $\mathfrak{m} = R\pi$ be a maximal ideal in R. Consider the R-algebra

$$A = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & a' \end{bmatrix} : \quad a,b,c,d,e,a' \in R, \ a - a' \in \mathfrak{m} \right\}$$

with the matrix multiplication and the usual action as R-module. Define $\Delta(1) = \{(x,w) \in R^2 \colon x - w \in \mathfrak{m}\}$ with action

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & a' \end{bmatrix} \cdot (x,w) = (ax, a'w)$$

and $\Delta(2) = R^2$ with action

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & a' \end{bmatrix} \cdot (y,z) = (cy, ey + a'z).$$

Then, $(A\text{-mod}, \{\Delta(1), \Delta(2)\})$ (with $1 < 2$) is a highest weight category in weak sense. Furthermore, $\operatorname{End}_A(\Delta(1))$ is the commutative R-algebra with R-basis $\{\mathrm{id}, \psi\}$ satisfying $\psi^2 = \pi\psi$. $\triangle$

Proof. Let $A_1 = \begin{bmatrix} a_1 & 0 & 0 \\ b_1 & c_1 & 0 \\ d_1 & e_1 & a_1' \end{bmatrix}$, $A_2 = \begin{bmatrix} a_2 & 0 & 0 \\ b_2 & c_2 & 0 \\ d_2 & e_2 & a_2' \end{bmatrix} \in A$. In particular, $a_1 - a_1' = r_1\pi$, $a_2 - a_2' = r_2\pi$. Then,

$$\begin{bmatrix} a_1 & 0 & 0 \\ b_1 & c_1 & 0 \\ d_1 & e_1 & a_1' \end{bmatrix} \cdot \begin{bmatrix} a_2 & 0 & 0 \\ b_2 & c_2 & 0 \\ d_2 & e_2 & a_2' \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 & 0 \\ b_1a_2 + c_1b_2 & c_1c_2 & 0 \\ d_1a_2 + e_1b_2 + a_1'd_2 & e_1c_2 + a_1'e_2 & a_1'a_2' \end{bmatrix} \tag{1.5.12.6}$$

Since

$$a_1a_2 - a_1'a_2' = a_1a_2 - a_1'a_2 + a_1'a_2 - a_1'a_2' = r_1a_2\pi + a_1'r_2\pi \in R\pi, \tag{1.5.12.7}$$

the multiplication in A is well defined.

Let $(y,z) \in \Delta(2)$. Then, $A_1A_2(y,z) = (c_1c_2y, e_1c_2y + a_1'e_2y + a_1'a_2'z) = A_1(A_2(y,z))$. Checking the other computations on the definition of module, it follows that $\Delta(2)$ is an A-module. Similarly $\Delta(1)$ is an A-module.

Consider $P(1) = \{(x,y,z,w) \in R^4 \colon x - w \in R\pi\}$. This is an A-module with action

$$A_1 \cdot (x,y,z,w) = (a_1x, b_1x + c_1y, d_1x + e_1y + a_1'z, a_1'w),\ (x,y,z,w) \in P(1). \tag{1.5.12.8}$$

There are A-isomorphisms $P(1) \to A\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$, given by $P(1) \ni (x,y,z,w) \mapsto \begin{bmatrix}x & 0 & 0\\ y & 0 & 0\\ z & 0 & w\end{bmatrix}$, and $\Delta(2) \to A\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$, given by $\Delta(2) \ni (x,w) \mapsto \begin{bmatrix}0 & 0 & 0\\ 0 & x & 0\\ 0 & w & 0\end{bmatrix}$.

This implies that $P(1), \Delta(2) \in A$-proj and $P(1) \bigoplus \Delta(2) \simeq A$ as left A-modules. There is an A-exact sequence

$$0 \to \Delta(2) \xrightarrow{k} P(1) \xrightarrow{\Pi} \Delta(1) \to 0, \tag{1.5.12.9}$$

where $k\colon \Delta(2) \to P(1)$ is given by $k(x,y) = (0,x,y,0)$, $(x,y) \in \Delta(2)$, and $\Pi\colon P(1) \to \Delta(1)$ is given by $\Pi(x,y,z,w) = (x,w)$. Let $\psi \in \operatorname{Hom}_A(\Delta(2), \Delta(1))$. Then, there exists $x_1, x_2, w_1, w_2 \in R$ with $x_i - w_i \in R\pi$, $i = 1,2$ such that $\psi(1,0) = (x_1, w_1)$ and $\psi(0,1) = (x_2, w_2)$. Fix $e_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $f = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$. Then,

$$(x_1, w_1) = e_1(x_1, w_1) = e_1\psi(1,0) = \psi(e_1(,0)) = \psi(0,0) = (0,0) \tag{1.5.12.10}$$

$$(x_2, w_2) = \psi(0,1) = \psi(f(1,0)) = f\psi(1,0) = f(0,0) = (0,0). \tag{1.5.12.11}$$

We conclude that $\operatorname{Hom}_A(\Delta(2), \Delta(1)) = 0$.

Now, let $N \in A$-mod such that $\operatorname{Hom}_A(\Delta(i), N) = 0$, $i = 1,2$. Applying $\operatorname{Hom}_A(-, N)$ to (1.5.12.9) it follows that $\operatorname{Hom}_A(P(1), N) = 0$. It follows that $N \simeq \operatorname{Hom}_A(A, N) \simeq \operatorname{Hom}_A(P(1) \oplus \Delta(2), N) = 0$. Therefore, A is a highest weight category in weak sense. It remains to compute $\operatorname{End}_A(\Delta(1))$. Let $\psi \in \operatorname{End}_A(\Delta(1))$. The module $\Delta(1)$ has an R-basis $\{(1,1); (0,\pi)\}$. Hence, there are elements $x_i, w_i \in R$, $i = 1,2$ with $x_i - w_i \in R\pi$ such that $\psi(1,1) = (x_1, w_1)$ and $\psi(0,\pi) = (x_2, w_2)$. Fix $h = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \pi\end{bmatrix}$. Then,

$$(x_2, w_2) = \psi(0,\pi) = \psi(h(1,1)) = h\psi(1,1) = h(x_1, w_1) = (0, \pi w_1). \tag{1.5.12.12}$$

Therefore, we can check that every $\psi \in \operatorname{End}_A(\Delta(1))$ is of the form $\psi_{a,r} \in \operatorname{End}_A(\Delta(1))$ with $\psi_{a,r}(x, x + s\pi) = (xa, xa + xr\pi + s\pi a + sr\pi^2)$ for every $(x, x+s\pi) \in \Delta(1)$. Here $\psi_{1,0} = \mathrm{id}_{\Delta(1)}$ and $\psi_{0,1} \cdot \psi_{0,1} = \pi\psi_{0,1}$. As $\psi_{a,r} = a\psi_{1,0} + r\psi_{0,1}$, the claim follows. ☐

1.5.13 Existence and properties of costandard modules

Proposition 1.5.90. *[Rou08, Proposition 4.19] Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then, there is a set $\{\nabla(\lambda)\}_{\lambda\in\Lambda}$ of A-modules, unique up to isomorphism, with the following properties:*

- $(A^{op}$-mod$, \{D\nabla(\lambda)_{\lambda\in\Lambda}\})$ *is a split highest weight category;*
- *Given $\lambda, \beta \in \Lambda$, then* $\operatorname{Ext}_A^i(\Delta(\lambda), \nabla(\beta) = \begin{cases} R & \text{if } i = 0 \text{ and } \lambda = \beta \\ 0 & \text{otherwise}\end{cases}$.

Proof. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra and let $\Lambda \to \{1,\dots,t\}$, $\lambda \mapsto i$ be an increasing bijection. By Theorem 1.5.65, A is split quasi-hereditary with some heredity chain $0 \subset J_t \subset \cdots \subset J_1 = A$. By Theorem 1.5.69, A^{op} is split quasi-hereditary with split heredity chain $0 \subset J_t^{op} \subset \cdots \subset J_1^{op} = A^{op}$. Again by Theorem 1.5.65, $(A^{op}\text{-mod}, \{\Delta^*(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category.

First, we will see how to construct the costandard modules using $\Delta^*(\lambda)$. Assume $\beta \not> \lambda$. Thus, by Definition of split highest weight category 1.5.32 ii), we have $\operatorname{Hom}_A(\Delta^*(\lambda), \Delta^*(\beta)) = 0$. As $\Delta^*(\lambda) \in \mathscr{M}(A^{op}/J^{op}_{i_\lambda+1})$, we obtain that by Corollary 1.5.23 that $\Delta^*(\beta) \in A^{op}/J^{op}_{i_\lambda+1}/J^{op}_{i_\lambda}/J^{op}_{i_\lambda+1}\text{-mod} \simeq A^{op}/J^{op}_{i_\lambda}\text{-mod}$. Thus $D\Delta^*(\beta) \in A/J_{i_\lambda}\text{-mod}$.

Therefore,

$$\begin{aligned}\operatorname{Hom}_A(\Delta(\lambda), D\Delta^*(\beta)) &= \operatorname{Hom}_{A/J_{i_\lambda+1}}(\Delta(\lambda), D\Delta^*(\beta)), \text{ since } \Delta(\lambda), D\Delta^*(\beta) \in A/J_{i_\lambda+1}\text{-mod}\\ &= 0, \text{ since } D\Delta^*(\beta) \in A/J_{i_\lambda}\text{-mod}.\end{aligned}$$

Assume $\lambda \not> \beta$. By symmetry, we have $\operatorname{Hom}_{A^{op}}(\Delta(\lambda), D\Delta^*(\beta)) = 0$. Since $\Delta^*(\beta)$ and $\Delta(\lambda)$ are projective over R, we obtain

$$\operatorname{Hom}_A(\Delta(\lambda), D\Delta^*(\beta)) = \operatorname{Hom}_{A^{op}}(DD\Delta^*(\beta), D\Delta(\lambda)) = \operatorname{Hom}_{A^{op}}(\Delta^*(\beta), D\Delta(\lambda)) = 0, \text{ if } \lambda \neq \beta.$$

Now assume $\lambda = \beta$. Suppose that λ is maximal in Λ. Define $U_\lambda = \operatorname{Hom}_A(\Delta(\lambda), D\Delta^*(\lambda))$. Since $A(\mathfrak{m})$ is quasi-hereditary over a field $R(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R with costandards $D\Delta^*(\lambda)(\mathfrak{m})$ and $\Delta(\lambda)$ is projective over A, we have

$$U_\lambda(\mathfrak{m}) = \operatorname{Hom}_A(\Delta(\lambda)(\mathfrak{m}), D\Delta^*(\lambda)(\mathfrak{m})) \simeq R(\mathfrak{m}). \tag{1.5.13.1}$$

On the other hand, since $\Delta(\lambda)$ is projective over A, U_λ is an R-summand of $\operatorname{Hom}_A(A^n, D\Delta^*(\lambda)) \simeq D\Delta^*(\lambda)^n \in R\text{-proj}$. Hence, $U_\lambda \in R\text{-proj}$. So, for each maximal ideal $\mathfrak{m}$ of R, there exists $n_\mathfrak{m} \geq 0$ such that $(U_\lambda)_\mathfrak{m} \simeq R_\mathfrak{m}^{n_\mathfrak{m}}$. Thus,

$$R(\mathfrak{m}) \simeq U_\lambda(\mathfrak{m}) \simeq (U_\lambda)_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} \simeq (U_\lambda)_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} U_\lambda \simeq R_\mathfrak{m}^{n_\mathfrak{m}}/\mathfrak{m}_\mathfrak{m} R_\mathfrak{m}^{n_\mathfrak{m}} \simeq R_\mathfrak{m}^{n_\mathfrak{m}}/\mathfrak{m}_\mathfrak{m}^{n_\mathfrak{m}} \simeq R(\mathfrak{m})^{n_\mathfrak{m}}.$$

Thus, $n_\mathfrak{m} = 1$ for all maximal ideals $\mathfrak{m}$ of R. By Proposition 1.5.2.1, $U_\lambda \in Pic(R)$.

Now consider $\nabla(\lambda) = DU_\lambda \otimes_R D\Delta^*(\lambda)$. We claim that $\operatorname{Hom}_A(\Delta(\lambda), \nabla(\lambda)) \simeq R$. By Tensor-Hom adjunction, $U_\lambda \simeq D(\Delta^*(\lambda) \otimes_A \Delta(\lambda))$. Thus,

$$\begin{aligned}\operatorname{Hom}_A(\Delta(\lambda), \nabla(\lambda)) &\simeq \operatorname{Hom}_A(\Delta(\lambda), \Delta^*(\lambda) \otimes_A \Delta(\lambda) \otimes_R D\Delta^*(\lambda))\\ &\simeq \operatorname{Hom}_A(\Delta(\lambda), \operatorname{Hom}_R(\Delta^*(\lambda), R) \otimes_R \Delta^*(\lambda) \otimes_A \Delta(\lambda))\\ &\simeq \operatorname{Hom}_A(\Delta(\lambda), \operatorname{Hom}_R(\Delta^*(\lambda), \Delta^*(\lambda)) \otimes_A \Delta(\lambda)), \text{ since } \Delta^*(\lambda) \in R\text{-proj}\\ &\simeq \operatorname{Hom}_A(\Delta(\lambda), \operatorname{Hom}_A(J_{i_\lambda}, A) \otimes_A \Delta(\lambda)), \text{ by Remark 1.5.16}\\ &\simeq \operatorname{Hom}_A(\Delta(\lambda), \operatorname{Hom}_A(J_{i_\lambda}, \Delta(\lambda))) \simeq \operatorname{Hom}_A(J_{i_\lambda} \otimes_A \Delta(\lambda), \Delta(\lambda))\\ &\simeq \operatorname{Hom}_A(\Delta(\lambda) \otimes_R \operatorname{Hom}_A(\Delta(\lambda), A) \otimes_A \Delta(\lambda), \Delta(\lambda))\\ &\simeq \operatorname{Hom}_A(\Delta(\lambda) \otimes_R \operatorname{Hom}_A(\Delta(\lambda), \Delta(\lambda)), \Delta(\lambda)) \simeq \operatorname{End}_A(\Delta(\lambda)) \simeq R.\end{aligned}$$

Suppose that β is a non-maximal element in Λ. Then, $\operatorname{Hom}_A(\Delta(\beta), \nabla(\beta)) \simeq \operatorname{Hom}_{A/J_{i_\beta+1}}(\Delta(\beta), \nabla(\beta)) \simeq R$, since β is maximal in the poset indexing the standards of $A/J_{i_\beta+1}$. The first isomorphism follows from the fact that $A/J_{i_\beta+1}$-mod is a full subcategory of A-mod.

Now, assume that $\lambda \neq \beta$. We obtain, for each maximal ideal $\mathfrak{m}$ of R,

$$\begin{aligned}\operatorname{Hom}_A(\Delta(\lambda),\nabla(\beta))_{\mathfrak{m}} &\simeq \operatorname{Hom}_{A_{\mathfrak{m}}}(\Delta(\lambda)_{\mathfrak{m}},(DU_\beta)_{\mathfrak{m}}\otimes_{R_{\mathfrak{m}}} D\Delta^*(\beta)_{\mathfrak{m}})\\ &\simeq \operatorname{Hom}_{A_{\mathfrak{m}}}(\Delta(\lambda)_{\mathfrak{m}},D\Delta^*(\beta)_{\mathfrak{m}}) \simeq \operatorname{Hom}_A(\Delta(\lambda),D\Delta^*(\beta))_{\mathfrak{m}} = 0.\end{aligned}$$

Thus, $\operatorname{Hom}_A(\Delta(\lambda),\nabla(\beta)) = 0$.

Now we shall prove that $(A^{op}\text{-mod},\{D\nabla(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. We shall proceed by induction on $t=|\Lambda|$. Assume that $t=1$. By Lemma 1.5.47 and $t=1$, $\Delta^*(\lambda)\in\mathscr{M}(A^{op})$ and $A^{op}=J_1^{op}$. For each maximal ideal $\mathfrak{m}$ of R, $D\Delta(\lambda)_{\mathfrak{m}} = \operatorname{Hom}_{R_{\mathfrak{m}}}((DU\lambda)_{\mathfrak{m}},\Delta^*(\lambda)_{\mathfrak{m}}) \simeq \Delta^*(\lambda)_{\mathfrak{m}} \in \mathscr{M}(A^{op}_{\mathfrak{m}})$, by Lemma 1.5.20. Again by Lemma 1.5.20, $D\Delta(\lambda)\in\mathscr{M}(A^{op})$. By Lemma 1.5.47, we conclude the claim for $t=1$.

Now assume that the result holds for $|\Lambda|<t$ for some $t>1$. Assume that $|\Lambda|=t$. Choose $\lambda\in\Lambda$ maximal. As before we have $D\nabla(\lambda)_{\mathfrak{m}}\simeq\Delta^*(\lambda)_{\mathfrak{m}}\in\mathscr{M}(A^{op}_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of R. Thus, $D\nabla(\lambda)\in\mathscr{M}(A^{op})$, by Lemma 1.5.20. By Tensor-Hom adjunction,

$$\begin{aligned}D\nabla(\lambda) &\simeq \operatorname{Hom}_R(DU_\lambda\otimes_R D\Delta^*(\lambda),R)\simeq \operatorname{Hom}_R(D\Delta^*(\lambda),DDU_\lambda)\simeq \operatorname{Hom}_R(D\Delta^*(\lambda),R)\otimes_R DDU_\lambda\\ &\simeq \Delta^*(\lambda)\otimes_R U_\lambda.\end{aligned}$$

Hence, $D\nabla(\lambda)=\Delta^*(\lambda)$ in $\mathscr{M}(A^{op})/Pic(R)$. In particular, $\operatorname{im}\tau_{D\nabla(\lambda)} = \operatorname{im}\tau_{\Delta^*(\lambda)} = J^{op}$.

By hypothesis, $(A^{op}/J^{op}\text{-mod},\{\Delta^*(\mu)_{\mu\in\Lambda\setminus\{\lambda\}}\})$ is a split highest weight category. By induction, $(A^{op}/J^{op}\text{-mod},\{D\nabla(\mu)_{\mu\in\Lambda\setminus\{\lambda\}}\})$ is a split highest weight category.

By Lemma 1.5.47, we conclude that $(A^{op}\text{-mod},\{D\nabla(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category.

Now we shall prove that $\operatorname{Ext}^1_A(\Delta(\lambda),\nabla(\beta)) = 0$, $\forall\lambda,\beta\in\Lambda$. Consider the exact sequence $0\to C(\lambda)\to P(\lambda)\to\Delta(\lambda)\to 0$. Applying $\operatorname{Hom}_A(-,\nabla(\beta))$ we obtain the long exact sequence

$$\operatorname{Hom}_A(P(\lambda),\nabla(\beta))\to\operatorname{Hom}_A(C(\lambda),\nabla(\beta))\to\operatorname{Ext}^1_A(\Delta(\lambda),\nabla(\beta))\to\operatorname{Ext}^1_A(P(\lambda),\nabla(\beta))=0. \tag{1.5.13.2}$$

Assume that $\operatorname{Hom}_A(C(\lambda),\nabla(\beta))\neq 0$. Then, there exists $\alpha<\lambda$ such that $\operatorname{Hom}_A(\Delta(\alpha),\nabla(\beta))\neq 0$. As we have seen, we must have $\alpha=\beta$. Therefore, if $\beta\not<\lambda$, we get $\operatorname{Ext}^1_A(\Delta(\lambda),\nabla(\beta))=0$. We will prove it for all cases by induction on $|\Lambda|$. If $|\Lambda|=1$, it is clear since $1\not<1$. Assume that it holds for $|\Lambda|<t$. Let $|\Lambda|=t$. Choose α maximal in Λ. Then, $\Delta(\alpha)$ is projective over A thus,

$$\operatorname{Ext}^1_A(\Delta(\alpha),\nabla(\beta)) = \operatorname{Ext}^1_A(\Delta(\beta),\nabla(\alpha)=0,\ \forall\beta\in\Lambda. \tag{1.5.13.3}$$

For $J=J_t$ we have that $(A/J\text{-mod},\{\Delta(\lambda)_{\lambda\in\Lambda\setminus\{\alpha\}}\})$ is a split highest weight category. By induction, $\operatorname{Ext}^1_{A/J}(\Delta(\beta),\nabla(\lambda))=0$ for all $\beta,\lambda\in\Lambda\setminus\{\alpha\}$. By Lemma 1.5.78, we have

$$\operatorname{Ext}^1_A(\Delta(\beta),\nabla(\lambda)) = \operatorname{Ext}^1_{A/J}(\Delta(\beta),\nabla(\lambda)) = 0,\ \forall\beta,\lambda\in\Lambda\setminus\{\alpha\}. \tag{1.5.13.4}$$

By (1.5.13.3) and (1.5.13.4), it follows the claim.

Now we shall proceed on induction on $n>0$ to show that $\operatorname{Ext}^n_A(\Delta(\beta),\Delta(\lambda))=0$. The case $n=1$ is already proved. Assume the result known for $n-1$. Consider the exact sequence $0\to C(\beta)\to P(\beta)\to\Delta(\beta)\to 0$. Applying $\operatorname{Hom}_A(-,\nabla(\lambda))$, it yields the exact sequence

$$0=\operatorname{Ext}^{n-1}_A(P(\beta),\nabla(\lambda))\to\operatorname{Ext}^{n-1}_A(C(\beta),\nabla(\lambda))\to\operatorname{Ext}^n_A(\Delta(\beta),\nabla(\lambda))\to\operatorname{Ext}^n_A(P(\beta),\nabla(\lambda))=0.$$

Therefore, $\operatorname{Ext}^n_A(\Delta(\beta),\nabla(\lambda))\simeq\operatorname{Ext}^{n-1}_A(C(\beta),\nabla(\lambda))$. By induction, $\operatorname{Ext}^{n-1}_A(\Delta(\beta),\nabla(\lambda))=0$, $\forall\beta,\lambda\in\Lambda$. By induction on the size of the Δ-filtration of $C(\beta)$, we get that $\operatorname{Ext}^{n-1}_A(C(\beta),\nabla(\lambda))=0$, $\forall\beta,\lambda\in\Lambda$, and the result follows.

It remains to prove the uniqueness part. Assume that $(A^{op}\text{-mod}, \{D\nabla(\lambda)_{\lambda\in\Lambda}\})$ and $(A^{op}\text{-mod}, \{D\nabla'(\lambda)_{\lambda\in\Lambda}\})$ are split highest weight categories and the modules ∇ and ∇' satisfy the given properties. Once again we proceed by induction on $|\Lambda|$ to show that $\nabla'(\lambda)\simeq\nabla(\lambda)$ for any $\lambda\in\Lambda$.

Assume $|\Lambda|=1$ with $\Lambda=\{\beta\}$. Then, $D\nabla(\beta)$ and $D\nabla'(\beta)$ are projective over A^{op}. By Proposition 1.5.48, we can write $D\nabla'(\beta)\simeq D\nabla(\beta)\otimes_R U_\beta$ for some $U_\beta\in R$-proj. By assumption,

$$R\simeq \operatorname{Hom}_A(\Delta(\beta),\nabla'(\beta))\simeq \operatorname{Hom}_{A^{op}}(D\nabla'(\beta),D\Delta(\beta))\simeq \operatorname{Hom}_{A^{op}}(D\nabla(\beta)\otimes_R U_\beta, D\Delta(\beta))$$
$$\operatorname{Hom}_R(U_\beta,\operatorname{Hom}_{A^{op}}(D\nabla(\beta),D\Delta(\beta)))\simeq \operatorname{Hom}_R(U_\beta,\operatorname{Hom}_{A^{op}}(\Delta(\beta),\nabla(\beta)))\simeq \operatorname{Hom}_R(U_\beta,R)=DU_\beta.$$

So, since $\nabla(\beta)\in R$-proj

$$\nabla'(\beta)=\operatorname{Hom}_R(D\nabla(\beta)\otimes_R U_\beta,R)\simeq \operatorname{Hom}_R(D\nabla(\beta),\operatorname{Hom}_R(U_\beta,R))\simeq \operatorname{Hom}_R(D\nabla(\beta),R)\simeq\nabla(\beta).$$

Now assume that for $|\Lambda|=t-1$, the result holds. Consider α maximal in Λ. We want to show that $\nabla'(\alpha)\simeq\nabla(\alpha)$. By Proposition 1.5.48, there is a filtration

$$0=P_{t+1}\subset P_t\subset\cdots\subset P_1=D\nabla'(\alpha)$$

with $P_i/P_{i+1}\simeq D\nabla_i\otimes_R U_i$. Since $D\nabla_i$ is standard in A^{op}, it is projective over R. Thus, $D\nabla_i\otimes_R U_i$ is projective over R. Consider the exact sequence $0\to P_{i+1}\to P_i\to D\nabla_i\otimes_R U_i\to 0$. Applying D we obtain the exact sequence

$$0\to D(D\nabla_i\otimes_R U_i)\to DP_i\to DP_{i+1}\to \operatorname{Ext}^1_R(D\nabla_i\otimes_R U_i,R)=0. \tag{1.5.13.5}$$

Notice that $D\nabla_i\otimes_R U_i$ is an A^{op}-summand of $D\nabla_i\otimes_R R^s\simeq D\nabla_i^s$, and therefore $D(D\nabla_i\otimes_R U_i)$ is an A-summand of ∇_i^s. Let $\beta\in\Lambda$ be an arbitrary index. Hence, $\operatorname{Ext}^1_A(\Delta(\beta),D(D\nabla_i\otimes_R U_i))$ is a summand of $\operatorname{Ext}^1_A(\Delta(\beta),\nabla_i)^s=0$. Moreover, $\operatorname{Hom}_A(\Delta(\beta),D(D\nabla_i\otimes_R U_i))$ is a summand of $\operatorname{Hom}_A(\Delta(\beta),\nabla_i^s)\simeq\begin{cases}R^s, \text{ if } i=i_\beta\\ 0, \text{ otherwise.}\end{cases}$

Applying $\operatorname{Hom}_A(\Delta(\beta),-)$ to (1.5.13.5) yields

$$\operatorname{Hom}_A(\Delta(\beta),DP_j)\simeq \operatorname{Hom}_A(\Delta(\beta),DP_{j+1}),\quad j\neq i_\beta, \tag{1.5.13.6}$$

and therefore $\operatorname{Hom}_A(\Delta(\beta),DP_{i_\beta})\simeq \operatorname{Hom}_A(\Delta(\beta),D(D\nabla_{i_\beta}\otimes_R U_{i_\beta}))$. Hence, putting $\beta=\alpha$,

$$R\simeq \operatorname{Hom}_A(\Delta(\alpha),\nabla'(\alpha))=\operatorname{Hom}_A(\Delta(\alpha),DP_1)\simeq \operatorname{Hom}_A(\Delta(\alpha),DP_{i_\alpha-1+1}) \tag{1.5.13.7}$$
$$\simeq \operatorname{Hom}_A(\Delta(\alpha),D(D\nabla_{i_\alpha}\otimes_R U_{i_\alpha}))\simeq \operatorname{Hom}_{A^{op}}(D\nabla_{i_\alpha}\otimes_R U_{i_\alpha},D\Delta(\alpha)) \tag{1.5.13.8}$$
$$\simeq \operatorname{Hom}_R(U_{i_\alpha},\operatorname{Hom}_{A^{op}}(D\nabla_{i_\alpha},D\Delta(\alpha)))\simeq \operatorname{Hom}_R(U_{i_\alpha},R)=DU_{i_\alpha}. \tag{1.5.13.9}$$

Therefore, $D(D\nabla_{i_\alpha}\otimes_R U_{i_\alpha})\simeq \operatorname{Hom}_R(D\nabla_{i_\alpha},\operatorname{Hom}_R(U_{i_\alpha},R))\simeq\nabla_{i_\alpha}=\nabla(\alpha)$.

Using the same idea, for $\beta\neq\alpha$,

$$0=\operatorname{Hom}_A(\Delta(\beta),\nabla'(\alpha))\simeq \operatorname{Hom}_A(\Delta(\beta),DP_{i_\beta})\simeq \operatorname{Hom}_A(\Delta(\beta),D(D\nabla_{i_\beta}\otimes_R U_{i_\beta}))\simeq DU_{i_\beta}. \tag{1.5.13.10}$$

Therefore, $U_{i_\beta}=0$ for $\beta\neq\alpha$. Thus, $DP_j\simeq DP_{j+1}$ for $j\neq i_\alpha=t$. In particular,

$$\nabla'(\alpha)\simeq DP_1\simeq DP_t\simeq D(D\nabla_t\otimes_R U_t)\simeq\nabla(\alpha). \tag{1.5.13.11}$$

Since $J=\operatorname{im}\tau_{D\nabla(\alpha)}=\operatorname{im}\tau_{D\nabla'(\alpha)}$ we have that $(A^{op}/J,\{D\nabla(\lambda)_{\lambda\in\Lambda\setminus\{\alpha\}}\})$ and $(A^{op}/J,\{D\nabla'(\lambda)_{\lambda\in\Lambda\setminus\{\alpha\}}\})$ are split quasi-hereditary algebras. So, by induction, the uniqueness for all costandards follows. □

Remark 1.5.91. For $\Delta' = \{\Delta(\lambda)\otimes_R F(\lambda)\colon \lambda\in\Lambda,\ F(\lambda)\in Pic(R)\}$ we have $\mathscr{F}(\tilde{\Delta'}) = \mathscr{F}(\tilde{\Delta})$. △

In fact, by definition of the Picard group there exists $G(\lambda)$ such that $F(\lambda)\otimes_R G(\lambda)\simeq R$. Thus, $M_i/M_{i+1}\simeq \Delta_i\otimes_R U_i \simeq \Delta_i'\otimes_R G_i\otimes_R U_i$. Since every element of the Picard group is a projective R-module $G_i\otimes_R U_i\in R$-proj. Hence, $\mathscr{F}(\tilde{\Delta'}) = \mathscr{F}(\tilde{\Delta})$.

Proposition 1.5.92. *With the above notation, if $\Delta_i'\simeq \Delta_i\otimes_R F_i,\ F_i\in Pic(R)$, then $\nabla_i'\simeq \nabla_i\otimes_R F_i$.*

Proof. We will use the same notation as in the proof of Proposition 1.5.90. Denote by $\Delta_i^{*'}$ the standard modules in A^{op} induced by A being split quasi-hereditary with standard modules Δ_i'. In order to define ∇_i', we can observe that Δ_i^* and $\Delta_i^{*'}$ are in the same orbit relative to the action of Picard group in $\mathscr{M}(A)$ since Δ_i and Δ_i' induce the same split heredity chain. So, put $\Delta_i^{*'} = \Delta_i^*\otimes_R G_i$, $G_i\in Pic(R)$ for every $i=1\ldots n$. Therefore,

$$\nabla_i'\simeq D\operatorname{Hom}_A(\Delta_i', D(\Delta_i^*\otimes_R G_i))\otimes_R D(\Delta_i^*\otimes_R G_i). \tag{1.5.13.12}$$

Note that by Tensor-Hom adjunction and since $G_i\in R$-proj,

$$D(\Delta_i^*\otimes_R G_i)\simeq \operatorname{Hom}_R(G_i, D\Delta_i^*)\simeq \operatorname{Hom}_R(G_i,R)\otimes_R D\Delta_i^*. \tag{1.5.13.13}$$

Moreover,

$$\begin{aligned}\operatorname{Hom}_A(\Delta_i', D(\Delta_i^*\otimes_R G_i)) &= \operatorname{Hom}_A(\Delta_i\otimes_R F_i, DG_i\otimes_R D\Delta_i^*)\simeq \operatorname{Hom}_R(F_i,\operatorname{Hom}_A(\Delta_i, DG_i\otimes_R D\Delta_i^*))\\ &\simeq \operatorname{Hom}_R(F_i,\operatorname{Hom}_{A/J_{i+1}}(\Delta_i, DG_i\otimes_R D\Delta_i^*))\\ &\simeq \operatorname{Hom}_R(F_i,\operatorname{Hom}_{A/J_{i+1}}(\Delta_i, D\Delta_i^*)\otimes_R DG_i)\\ &\simeq \operatorname{Hom}_R(F_i,\operatorname{Hom}_A(\Delta_i, D\Delta_i^*)\otimes_R DG_i)\simeq \operatorname{Hom}_R(F_i, DG_i)\otimes_R \operatorname{Hom}_A(\Delta_i, D\Delta_i^*).\end{aligned} \tag{1.5.13.14}$$

So,

$$\begin{aligned}D\operatorname{Hom}_A(\Delta_i', D\Delta_i^*\otimes_R G_i) &\simeq D(\operatorname{Hom}_R(F_i, DG_i)\otimes_R \operatorname{Hom}_A(\Delta_i, D\Delta_i^*)\\ &\simeq D\operatorname{Hom}_R(F_i, DG_i)\otimes_R D\operatorname{Hom}_A(\Delta_i, D\Delta_i^*)\simeq DDF_i\otimes_R G_i\otimes_R D\operatorname{Hom}_A(\Delta_i, D\Delta_i^*).\end{aligned} \tag{1.5.13.15}$$

So, we conclude,

$$\Delta_i'\simeq D\operatorname{Hom}_A(\Delta_i, D\Delta_i^*)\otimes_R F_i\otimes_R G_i\otimes_R DG_i\otimes_R D\Delta_i^*\simeq D\operatorname{Hom}_A(\Delta_i, D\Delta_i^*)\otimes_R D\Delta_i^*\otimes_R F_i\simeq \nabla_i\otimes_R F_i.$$ □

Corollary 1.5.93. *With the above notation, $\mathscr{F}(\tilde{\nabla'}) = \mathscr{F}(\tilde{\nabla})$.*

Corollary 1.5.94. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then, $\{D\Delta(\lambda)\colon \lambda\in\Lambda\}$ are costandard modules in A^{op}.*

Proof. Note that $((A^{op})^{op}\text{-mod}, \{DD\Delta(\lambda)_{\lambda\in\Lambda}\}) = (A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is split highest weight category. In addition, for $\lambda,\beta\in\Lambda$, by Lemma 1.2.38

$$\operatorname{Ext}^i_{A^{op}}(D\nabla(\lambda), D\Delta(\beta))\simeq \operatorname{Ext}_A(\Delta(\beta),\nabla(\lambda)) = \begin{cases} R \text{ if } \lambda=\beta,\ i=0\\ 0 \text{ otherwise.}\end{cases} \tag{1.5.13.16}$$

By the uniqueness of costandard modules in Proposition 1.5.90, the result follows. □

Proposition 1.5.95. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $M \in A$-mod such that M is (A,R)-injective and projective over R. Let $\Lambda \to \{1,\dots,n\}$ be an increasing bijection. Then, there is a filtration*

$$0 \subset I_1 \subset \cdots \subset I_n = M, \quad \textit{with} \quad I_i/I_{i-1} \simeq U_i \otimes_R \nabla_i, \quad \textit{for some } U_i \in R\text{-proj}\,.$$

Proof. By Lemma 1.2.56, DM is a projective A^{op}-module. Recall that $(A^{op}, D\nabla(\lambda))$ is split highest weight category. By Proposition 1.5.48, there is a filtration $0 = P_{n+1} \subset P_n \subset \cdots \subset P_1 = DM$ with $P_i/P_{i+1} \simeq D\nabla_i \otimes_R U_i$, $1 \leq i \leq n$. Applying D yields the exact sequence

$$0 \to D(D\nabla_i \otimes_R U_i) \to DP_i \to DP_{i+1} \to 0. \tag{1.5.13.17}$$

Note that $D(D\nabla_i \otimes_R U_i) \simeq \operatorname{Hom}_R(U_i, \operatorname{Hom}_R(D\nabla_i, R)) \simeq \operatorname{Hom}_R(U_i, \nabla_i) \simeq DU_i \otimes_R \nabla_i$. In particular, $DP_n \simeq DU_n \otimes_R \nabla_n$ and $DP_1 \simeq M$. Now by induction using at each step the filtration of DP_{i+1} and the exact sequence (1.5.13.17) we can construct a ∇-filtration to DP_i

$$0 \subset I_i \subset I_{i+1} \subset \cdots \subset I_n = DP_i, \tag{1.5.13.18}$$

satisfying $I_j/I_{j-1} \simeq DU_j \otimes_R \nabla_j$. Hence, the result follows. ☐

Notation 1.5.96. Denote by $\mathscr{F}(\nabla)$ the subcategory of A-mod whose modules have filtrations

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = M, \quad \text{such that } M_k/M_{k-1} \simeq \nabla(\lambda_k).$$

Let $\Lambda \to \{1,\dots,n\}$ be an increasing bijection. Denote by $\mathscr{F}(\tilde{\nabla})$ the subcategory of A-mod whose modules have filtrations $0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = M$, such that $I_i/I_{i-1} \simeq U_i \otimes_R \nabla_i$ for some $U_i \in R$-proj. △

Proposition 1.5.97. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a commutative Noetherian ring R. Then, the costandard modules satisfy the following properties:*

(i) $\nabla(\lambda) \in A$-mod *are projective over R;*

(ii) If $\operatorname{Hom}_A(\nabla(\alpha), \nabla(\beta)) \neq 0$*, then $\alpha \geq \beta$;*

(iii) If $N \in R$-proj$\cap A$-mod is such that $\operatorname{Hom}_A(N, \nabla(\lambda)) = 0$, $\forall \lambda \in \Lambda$*, then $N = 0$. Furthermore, for any $N \in A$-mod, if* $\operatorname{Hom}_A(N, \nabla(\lambda)) = 0$*, then $DN = 0$.*

(iv) $\operatorname{End}_A(\nabla(\lambda)) \simeq R$*;*

(v) For each $\lambda \in \Lambda$ there exists an (A,R)-injective module and projective over R $I(\lambda)$ together with an exact sequence $0 \to \nabla(\lambda) \to I(\lambda) \to K(\lambda) \to 0$ and $K(\lambda)$ has a finite filtration by modules $\nabla(\mu) \otimes_R U_\mu$ with $\mu > \lambda$ and $U_\mu \in R$-proj.

Proof. *i*) $\nabla(\lambda) = DU_\lambda \otimes_R D\Delta^*(\lambda) \in R$-proj.

ii) Let $\lambda, \beta \in \Lambda$ satisfying $\operatorname{Hom}_A(\nabla(\lambda), \nabla(\beta)) \neq 0$. As both modules are projective over R $\operatorname{Hom}_{A^{op}}(D\nabla(\beta), D\nabla(\lambda)) \neq 0$. Since $D\nabla(\beta)$ and $D\nabla(\lambda)$ are standard modules in A^{op} with poset Λ it follows that $\beta \leq \lambda$.

iii) Assume $N \in R$-proj$\cap A$-mod such that $0 = \operatorname{Hom}_A(N, \nabla(\lambda)) \simeq \operatorname{Hom}_{A^{op}}(D\nabla(\lambda), DN)$ for every $\lambda \in \Lambda$. Then, $DN = 0$. Since $N \in R$-proj we obtain $N = 0$.

iv) $\operatorname{End}_A(\nabla(\lambda)) \simeq \operatorname{End}_{A^{op}}(D\nabla(\lambda)) \simeq R$ for every $\lambda \in \Lambda$.

v) Let $\lambda \in \Lambda$. $D\nabla(\lambda)$ is a standard module, therefore there exists an exact sequence in A^{op}

$$0 \to C_{op}(\lambda) \to P_{op}(\lambda) \to D\nabla(\lambda) \to 0. \tag{1.5.13.19}$$

Hence, $P_{op}(\lambda)$ is a projective right A-module and $C_{op} \in \mathscr{F}(D\tilde{\nabla}(\mu)_{\mu>\lambda})$. Applying D to this exact sequence it yields the exact sequence $0 \to \nabla(\lambda) \to DP_{op}(\lambda) \to DC_{op}(\lambda) \to 0$, since $\nabla(\lambda)$ is projective over R. By Lemma 1.2.56, $DP_{op}(\lambda)$ is an (A,R)-injective left module and it is a projective R-module. By the proof of Proposition 1.5.95, we obtain $DC_{op}(\lambda) \in \mathscr{F}(\tilde{\nabla}_{\mu>\lambda})$. ☐

Remark 1.5.98. If $\lambda \in \Lambda$ is maximal, then $D\operatorname{Hom}_A(P(\lambda),A) \simeq I(\lambda) \simeq \nabla(\lambda)$. In fact, $\operatorname{Hom}_A(P(\lambda),A) \in \mathscr{M}(A^{op})$ by Lemma 1.5.18 and

$$\operatorname{Hom}_A(P(\lambda), D\operatorname{Hom}_A(P(\lambda),A)) \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(P(\lambda),A), DP(\lambda)) \simeq DP(\lambda) \otimes_A P(\lambda) \simeq R. \qquad \triangle$$

Proposition 1.5.99. *Suppose $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split quasi-hereditary algebra. Then, the following holds.*

(a) If $\operatorname{Ext}_A^1(\nabla(\alpha),\nabla(\beta)) \neq 0$, then $\alpha > \beta$.

(b) If $\operatorname{Ext}_A^i(\nabla(\alpha),\nabla(\beta)) \neq 0$ for some $i > 0$, then $\alpha > \beta$. In particular, $\operatorname{Ext}_A^i(\nabla(\alpha),\nabla(\alpha)) = 0,\ i > 0$.

Proof. Consider the exact sequence

$$0 \to \nabla(\beta) \to I_\beta \to K_\beta \to 0, \tag{1.5.13.20}$$

where k_β has a filtration by $\nabla(\mu) \otimes_R U_\mu$, $U_\mu \in R$-proj so that $\mu > \beta$. Apply $\operatorname{Hom}_A(\nabla(\alpha),-)$ to (1.5.13.20). Hence, the following sequence is exact

$$\operatorname{Hom}_A(\nabla(\alpha),K_\beta) \to \operatorname{Ext}_A^1(\nabla(\alpha),\nabla(\beta)) \to \operatorname{Ext}_A^1(\nabla(\alpha),I_\beta). \tag{1.5.13.21}$$

By Lemma 1.2.55, $\operatorname{Ext}_A^1(\nabla(\alpha),I_\beta) = 0$. If $\operatorname{Ext}_A^1(\nabla(\alpha),\nabla(\beta)) \neq 0$, then $\operatorname{Hom}_A(\nabla(\alpha),K_\beta) \neq 0$. As K_β has a ∇-filtration, then $\operatorname{Hom}_A(\nabla(\alpha),\nabla(\mu) \neq 0$ for $\mu > \beta$. This implies that $\alpha \geq \mu > \beta$. So, *a*) follows.

Assume $\operatorname{Ext}_A^i(\nabla(\alpha),\nabla(\beta)) \neq 0$ for some $i > 0$. Choose β being maximal satisfying $\operatorname{Ext}_A^i(\nabla(\alpha),\nabla(\beta)) \neq 0$ for some $i > 0$. By Lemma 1.2.55, applying $\operatorname{Hom}_A(\nabla(\alpha),-)$ to (1.5.13.20), yields

$$0 \neq \operatorname{Ext}_A^i(\nabla(\alpha),\nabla(\beta)) \simeq \operatorname{Ext}_A^{i-1}(\nabla(\alpha),K_\beta). \tag{1.5.13.22}$$

Consider the exact sequence $0 \to K'_\beta \to K_\beta \to \nabla(\mu) \to 0$, $\mu > \beta$. Applying $\operatorname{Hom}_A(\nabla(\alpha),-)$ we get

$$\operatorname{Ext}_A^{i-1}(\nabla(\alpha),K_\beta) \to \operatorname{Ext}_A^{i-1}(\nabla(\alpha),K_\beta) \to \operatorname{Ext}_A^{i-1}(\nabla(\alpha),\nabla(\mu)).$$

Hence, either $\operatorname{Ext}_A^{i-1}(\nabla(\alpha),K'_\beta) \neq 0$ or $\operatorname{Ext}_A^{i-1}(\nabla(\alpha),\nabla(\mu)) \neq 0$. By the maximality of β, $\operatorname{Ext}_A^{i-1}(\nabla(\alpha),K'_\beta) \neq 0$. Applying the same reasoning several times, this leads to $\operatorname{Ext}_A^1(\nabla(\alpha),\nabla(\mu)) \neq 0$ for some $\mu > \beta$. By *a*) we obtain that $\alpha > \mu$. Thus, *b*) follows. ☐

Lemma 1.5.100. *[Rou08, Lemma 4.21]Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $M,N \in$ A-mod. The following holds.*

(a) If $M \in \mathscr{F}(\tilde{\Delta})$, then $\operatorname{Ext}_A^i(M,\nabla(\lambda)) = 0,\ i > 0$.

(b) If $N \in \mathscr{F}(\tilde{\nabla})$, then $\operatorname{Ext}_A^i(\Delta(\lambda),N) = 0,\ i > 0$.

(c) If $M \in \mathscr{F}(\tilde{\Delta})$ and $N \in \mathscr{F}(\tilde{\nabla})$, then $\operatorname{Ext}_A^i(M,N) = 0,\ i > 0$.

Proof. Observe that for $i > 0$ and every $\beta,\lambda \in \Lambda$, $U \in R$-proj $\operatorname{Ext}_A^i(\Delta(\beta) \otimes_R U, \nabla(\lambda))$ is an R-summand of $\operatorname{Ext}_A^i(\Delta(\beta)^t,\nabla(\lambda)) \simeq \operatorname{Ext}_A^i(\Delta(\beta),\nabla(\lambda))^t = 0$ by Proposition 1.5.90. Hence, $\operatorname{Ext}_A^i(\Delta(\beta) \otimes_R U,\nabla(\lambda)) = 0$.

Let $M \in \mathscr{F}(\tilde{\Delta})$. There is a filtration

$$0 = P_{n+1} \subset P_n \subset \cdots \subset P_1 = M, \quad \text{with } P_i/P_{i+1} \simeq \Delta_i \otimes_R U_i. \tag{1.5.13.23}$$

Let $\lambda \in \Lambda$. Applying $\operatorname{Hom}_A(-, \nabla(\lambda))$ to the exact sequence of P_i yields the exact sequence

$$0 = \operatorname{Ext}_A^j(\Delta_i \otimes_R U_i) \to \operatorname{Ext}_A^j(P_i, \nabla(\lambda)) \to \operatorname{Ext}_A^j(P_{i+1}, \nabla(\lambda)) \to \operatorname{Ext}_A^{j+1}(\Delta_i \otimes_R U_i, \nabla(\lambda)) = 0, \ \forall j > 1.$$

We conclude, for $j > 1$,$\operatorname{Ext}_A^j(P_i, \nabla(\lambda)) \simeq \operatorname{Ext}_A^j(P_{i+1}, \nabla(\lambda)) \simeq \operatorname{Ext}_A^j(P_n, \nabla(\lambda)) = \operatorname{Ext}_A^j(\Delta_n, \nabla(\lambda)) = 0$.

The proof of $b)$ is analogous now applying the functor $\operatorname{Hom}_A(\Delta(\lambda), -)$ to the exact sequences given by the filtration of N.

Let $N \in \mathscr{F}(\tilde{\nabla})$. Applying $\operatorname{Hom}_A(-, N)$ to the exact sequences of the filtration $(1.5.13.23)$ we get the long exact sequence $0 = \operatorname{Ext}_A^j(\Delta_i \otimes_R U_i, N) \to \operatorname{Ext}_A^i(P_i, N) \to \operatorname{Ext}_A^j(P_{i+1}, N) \to \operatorname{Ext}_A^j(\Delta_i \otimes_R U_i, N) = 0$.

Therefore, $0 = \operatorname{Ext}_A^j(\Delta \otimes_R U_n, N) = \operatorname{Ext}_A^j(P_n, N) \simeq \operatorname{Ext}_A^j(P_1, N) = \operatorname{Ext}_A^j(M, N)$. ☐

Lemma 1.5.101. *Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary algebra. Let $M \in A$-*mod*. If $M \in \mathscr{F}(\tilde{\Delta})$ or $M \in \mathscr{F}(\tilde{\nabla})$, then $M \in R$-*proj.

Proof. Assume $M \in \mathscr{F}(\tilde{\Delta})$. Then, there are exact sequences $0 \to P_{i+1} \to P_i \to \Delta_i \otimes_R U_i \to 0$. Since $\Delta_i \otimes_R U_i$ is projective over R all these sequences are split over R. Thus, every P_i is projective over R. In particular, $M \in R$-proj. The argument is analogous for $M \in \mathscr{F}(\tilde{\nabla})$. ☐

Proposition 1.5.102. *Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary algebra. If $N \in \mathscr{F}(\nabla)$, the factors of N can always be chosen in increasing order, meaning that the costandard modules with the lowest index appear at the bottom of the filtration.*

Furthermore, if $N \in \mathscr{F}(\nabla)$, then $N \in \mathscr{F}(\tilde{\nabla})$.

Proof. Consider a filtration

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = M. \tag{1.5.13.24}$$

Consider k such that $M_k/M_{k-1} \simeq \nabla_i$, $M_{k+1}/M_k \simeq \nabla_j$ and $i > j$. By Proposition 1.5.99, $\operatorname{Ext}_A^1(\nabla_j, \nabla_i) = 0$. Since $M_{k-1} \subset M_k \subset M_{k+1}$, there is a canonical monomorphism $\nabla_i \simeq M_k/M_{k-1} \hookrightarrow M_{k+1}/M_{k-1}$. As

$$(M_{k+1}/M_{k-1})/(M_k/M_{k-1}) \simeq M_{k+1}/M_k \simeq \nabla_j, \tag{1.5.13.25}$$

we have a short exact sequence $0 \to \nabla_i \to M_{k+1}/M_{k-1} \to \nabla_j \to 0$. Since $\operatorname{Ext}_A^1(\nabla_j, \nabla_i) = 0$, this sequence splits over A. Hence, we have a canonical epimorphism $h\colon M_{k+1} \twoheadrightarrow M_{k+1}/M_{k-1} \simeq \nabla_i \bigoplus \nabla_j \twoheadrightarrow \nabla_i$. Define $\overline{M_k} := \ker(h)$. Thus, $M_{k+1}/\overline{M_k} \simeq \operatorname{im} h \simeq \nabla_i$ and observe that $\overline{M_k}/M_{k-1} \simeq \nabla_j$. In fact, the latter follows applying the Snake Lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{k-1} \arrow[r] \arrow[d, hook] & M_{k+1} \arrow[r] \arrow[d, equal] & \nabla_i \bigoplus \nabla_j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{M_k} \arrow[r] & M_{k+1} \arrow[r] & \nabla_i \arrow[r] & 0
\end{tikzcd}.$$

Therefore, we have a filtration by costandard modules

$$0 \subset M_0 \subset M_1 \subset \cdots M_{k-1} \subset \overline{M_k} \subset M_{k+1} \subset \cdots \subset M. \tag{1.5.13.26}$$

Hence, we order the filtration in such a way that the indexes appear in increasing order. Moreover, we can rearrange every filtration to a filtration $0 = P_0 \subset P_1 \subset \cdots \subset P_n = M$ where $P_i/P_{i-1} \simeq \nabla_i \otimes_R U_i$ and U_i is a free R-module. $\square$

Theorem 1.5.103. *Let A be a projective Noetherian R-algebra and let Λ be a poset. Then, A is split quasi-hereditary if and only if there exist modules $\{\nabla(\lambda)\colon \lambda \in \Lambda\}$ satisfying the following properties:*

(i) The modules $\nabla(\lambda) \in A\text{-mod}$ are projective over R for every $\lambda \in \Lambda$;

(ii) Given $\alpha, \beta \in \Lambda$, if $\operatorname{Hom}_A(\nabla(\alpha), \nabla(\beta)) \neq 0$, then $\alpha \geq \beta$;

(iii) $\operatorname{End}_A(\nabla(\lambda)) \simeq R$, $\lambda \in \Lambda$;

(iv) For each $\lambda \in \Lambda$, there exists an (A,R)-injective module which is projective as R-module $I(\lambda)$ together with an exact sequence $0 \to \nabla(\lambda) \to I(\lambda) \to K(\lambda) \to 0, \quad K(\lambda) \in \mathscr{F}(\tilde{\nabla}_{\mu>\lambda})$.

(v) $DA_A \in \operatorname{add}\left(\bigoplus_{\lambda\in\Lambda} I(\lambda)\right)$.

Proof. Assume that A is split quasi-hereditary. By Theorem 1.5.65, there are standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$ for some poset Λ such that $(A\text{-mod}, \{\Delta(\lambda)\}_{\lambda\in\Lambda})$ is split highest weight category. By Proposition 1.5.97, $i, ii, iii, iv)$ are satisfied. By Proposition 1.5.90 and by Corollary 1.5.43, $\bigoplus_{\lambda\in\Lambda} DI_\lambda = D\bigoplus_{\lambda\in\Lambda} I_\lambda$ is a progenerator in A^{op}-mod. Thus, $A_A \in \operatorname{add} D\bigoplus_{\lambda\in\Lambda} I_\lambda$. This implies $v)$.

Conversely assume there are modules $\{\nabla(\lambda)\colon \lambda \in \Lambda\}$ satisfying the properties above. Then, it is clear at this point that $D\Delta(\lambda)$ satisfy properties $a), b), c)$ of Corollary 1.5.43. Since I_λ is (A,R)-injective and projective as R-module, it follows that DI_λ is projective over A^{op}. Hence, $d)$ is also satisfied. By $v)$ $A^{op} = DDA^{op} \in \operatorname{add} D\bigoplus_{\lambda\in\Lambda} DI_\lambda$. By Corollary 1.5.43, $(A^{op}\text{-mod}, \{D\nabla(\lambda)\}_{\lambda\in\Lambda})$ is split highest weight category. By Theorem 1.5.65, A^{op} is split quasi-hereditary. By Theorem 1.5.69, A is split quasi-hereditary. $\square$

The following result is Lemma 4.21 of [Rou08]. For quasi-hereditary algebras over fields, there are many proofs of this result in the literature. However, for quasi-hereditary algebras over commutative Noetherian rings as far as the author knows this result can only be found in [Rou08] . We present a different approach than the one used by Rouquier. Here, we use a different approach also because it is not clear for the author why M/M_0 is projective over R using Rouquier's approach.

Theorem 1.5.104. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $M \in R\text{-proj} \cap A\text{-mod}$.*

1. *If $\operatorname{Ext}_A^1(M, \nabla(\lambda)) = 0$, $\forall \lambda \in \Lambda$, then $M \in \mathscr{F}(\tilde{\Delta})$.*
2. *If $\operatorname{Ext}_A^1(\Delta(\lambda), M) = 0$, $\forall \lambda \in \Lambda$, then $M \in \mathscr{F}(\tilde{\nabla})$.*

Proof. Assume that $\operatorname{Ext}_A^1(M, \nabla(\lambda)) = 0$ for some $M \in R\text{-proj} \cap A\text{-mod}$. By induction on the size of filtrations of modules in $\mathscr{F}(\tilde{\nabla})$ we deduce that $\operatorname{Ext}_A^1(M,N) = 0$ for every $N \in \mathscr{F}(\tilde{\nabla})$. Let $\lambda \in \Lambda$ be maximal. Thus, $\Delta(\lambda)$ is an R-split A-module. Recall that $\tau_{\Delta(\lambda),A}$ is a left and right (A,R)-monomorphism by Proposition 1.5.15. Analogously, we can consider the left A-homomorphism $\tau_{\Delta(\lambda),M}\colon \Delta(\lambda) \otimes_R \operatorname{Hom}_A(\Delta(\lambda), M) \to M$. If M admits a filtration by standard modules, then it is possible to construct a filtration with $\Delta(\lambda) \otimes_R U_\lambda$ appearing at the bottom, where U_λ is a projective R-module (possibly the zero module). Therefore, we want to show that $\tau_{\Delta(\lambda),M}$ is an (A,R)-monomorphism. If we show in addition that its cokernel belongs to $\mathscr{F}(\Delta(\lambda))$, then we are done.

Claim A. We can relate $\tau_{\Delta(\lambda),A} \otimes_A M$ and $\tau_{\Delta(\lambda),M}$ through the following commutative diagram:

$$\begin{tikzcd} \Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),A)\otimes_A M \arrow[r, "\tau_{\Delta(\lambda),A}\otimes_A M"] \arrow[d, "\simeq"', "\Delta(\lambda)\otimes_R\psi"] & A\otimes_A M \arrow[d, "\simeq"', "\mu_M"] \\ \Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),M) \arrow[r, "\tau_{\Delta(\lambda),M}"] & M \end{tikzcd}, \tag{1.5.13.27}$$

where μ_M is the multiplication map and ψ is given by Lemma 1.4.11.

In fact,

$$\tau_{\Delta(\lambda),M}\Delta(\lambda)\otimes_R \psi(l\otimes f\otimes m) = \tau_{\Delta(\lambda),M}(l\otimes\psi(f\otimes m)) = \psi(f\otimes m)(l) = f(l)m \tag{1.5.13.28}$$

$$\mu_M\circ\tau_{\Delta(\lambda),A}\otimes_A M(l\otimes f\otimes m) = \mu_M(f(l)\otimes m) = f(l)m,\ l\in\Delta(\lambda), f\in \mathrm{Hom}_A(\Delta(\lambda),A), m\in M. \tag{1.5.13.29}$$

Claim B. There are isomorphisms δ and θ making the following diagram commutative

$$\begin{tikzcd} \Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),A)\otimes_A M \arrow[r, "\tau_{\Delta(\lambda),A}\otimes_A M"] \arrow[d, "\simeq"', "\delta"] & A\otimes_A M \arrow[d, "\simeq"', "\theta"] \\ D\mathrm{Hom}_A(M, D(\Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),A)) \arrow[r, "D\mathrm{Hom}_A(M,D\tau_{\Delta(\lambda),A})"] & D\mathrm{Hom}_A(M,DA) \end{tikzcd}. \tag{1.5.13.30}$$

Note that by Tensor-Hom adjunction $D\mathrm{Hom}_A(M,DA) \simeq DDM$. Hence, the map $\theta\in\mathrm{Hom}_A(A\otimes_A M, D\mathrm{Hom}_A(M,DA))$ given by $\theta(a\otimes m)(g) = g(am)(1_A)$ is an isomorphism. Further, as left A-modules,

$$\begin{aligned} D\mathrm{Hom}_A(M, D(\Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),A)) &\simeq DD(\Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),A)\otimes_A M) \\ &\simeq \Delta(\lambda)\otimes_R \mathrm{Hom}_A(\Delta(\lambda),A)\otimes_A M. \end{aligned}$$

Denote by $\delta\in\mathrm{Hom}_A(\Delta(\lambda)\otimes_R\mathrm{Hom}_A(\Delta(\lambda),A)\otimes_A M, D\mathrm{Hom}_A(M,D(\Delta(\lambda)\otimes_R\mathrm{Hom}_A(\Delta(\lambda),A)))$ this isomorphism. Explicitly, for every $l\in\Delta(\lambda), f\in\mathrm{Hom}_A(\Delta(\lambda),A),\ m\in M$,

$$\delta(l\otimes f\otimes m)(g) = g(m)(l\otimes f),\ g\in\mathrm{Hom}_A(M, D(\Delta(\lambda)\otimes_R\mathrm{Hom}_A(\Delta(\lambda),A)).$$

Let $l\otimes f\otimes m\in\Delta(\lambda)\otimes_R\mathrm{Hom}_A(\Delta(\lambda),A)\otimes_A M$, $g\in\mathrm{Hom}_A(M,DA)$. Then,

$$\begin{aligned} D\mathrm{Hom}_A(M,D\tau_{\Delta(\lambda),A})\circ\delta(l\otimes f\otimes m)(g) &= \delta(l\otimes f\otimes m)\mathrm{Hom}_A(M,D\tau_{\Delta(\lambda),A})(g) = \delta(l\otimes f\otimes m)(D\tau_{\Delta(\lambda),A}\circ g) \\ &= D\tau_{\Delta(\lambda),A}g(m)(l\otimes f) = g(m)\circ\tau_{\Delta(\lambda),A}(l\otimes f) = g(m)(f(l)). \end{aligned}$$

On the other hand,

$$\begin{aligned} \theta\tau_{\Delta(\lambda),A}\otimes_A M(l\otimes f\otimes m)(g) &= \theta(\tau_{\Delta(\lambda),A}(l\otimes f)\otimes m)(g) = \theta(f(l)\otimes m)(g) \\ &= g(f(l)m)(1_A) = (f(l)\cdot g(m))(1_A) = g(m)(1_A f(l)) = g(m)(f(l)). \end{aligned}$$

This shows that the diagram (1.5.13.30) is commutative and Claim B follows.

Claim C. The map $D\mathrm{Hom}_A(M,D\tau_{\Delta(\lambda),A})$ is a left (A,R)-monomorphism.

The cokernel of the right (A,R)-monomorphism $\tau_{\Delta(\lambda),A}$ is $A/J\in R$-proj where J is the image of $\tau_{\Delta(\lambda),A}$, and therefore J is a split heredity ideal. Hence, A/J belongs to $\mathscr{F}(\tilde{\Delta}_{op})$. Thus, $D(A/J)$ belongs to $\mathscr{F}(\tilde{\nabla})$. So,

$D = \mathrm{Hom}_R(-, R)$ induces left (A, R)-exact sequence

$$0 \to D(A/J) \to DA \xrightarrow{D\tau_{\Delta(\lambda),A}} D(\Delta(\lambda) \otimes_R \mathrm{Hom}_A(\Delta(\lambda), A)) \to 0. \tag{1.5.13.31}$$

Applying $\mathrm{Hom}_A(M, -)$ yields the exact sequence

$$0 \to \mathrm{Hom}_A(M, D(A/J)) \to \mathrm{Hom}_A(M, DA) \xrightarrow{\mathrm{Hom}_A(M, D\tau_{\Delta(\lambda),A})} \mathrm{Hom}_A(M, D(L \otimes_R \mathrm{Hom}_A(L, A)) \to 0, \tag{1.5.13.32}$$

because $\mathrm{Ext}^1_A(M, D(A/J)) = 0$. Due to $\Delta(\lambda) \in A$-proj, by Tensor-Hom adjunction, we have

$$\begin{aligned}\mathrm{Hom}_A(M, D(\Delta(\lambda) \otimes_R \mathrm{Hom}_A(\Delta(\lambda), A)) &\simeq \mathrm{Hom}_R(\Delta(\lambda) \otimes_R \mathrm{Hom}_A(\Delta(\lambda), A) \otimes_A M, R)\\ &\simeq \mathrm{Hom}_R(\Delta(\lambda) \otimes_R \mathrm{Hom}_A(\Delta(\lambda), M), R) \in R\text{-proj}\,.\end{aligned}$$

This shows that the right A-homomorphism $\mathrm{Hom}_A(M, D\tau_{\Delta(\lambda),A})$ is an (A, R)-epimorphism. Therefore, $D\,\mathrm{Hom}_A(M, D\tau_{\Delta(\lambda),A})$ is a left (A, R)-monomorphism.

Combining Claims A, B and C, we obtain that $\tau_{\Delta(\lambda),M}$ is a left (A, R)-monomorphism.

Let X be the cokernel of $\tau_{\Delta(\lambda),M}$. In particular, $X \in R$-proj and the exact sequence

$$0 \to \Delta(\lambda) \otimes_R \mathrm{Hom}_A(\Delta(\lambda), M) \xrightarrow{\tau_{\Delta(\lambda),M}} M \to X \to 0 \tag{1.5.13.33}$$

is (A, R)-exact. Recall that $U_\lambda := \mathrm{Hom}_A(\Delta(\lambda), M) \in R$-proj. It remains to show that $X \in \mathscr{F}(\tilde{\Delta})$. The exactness of $\mathrm{Hom}_A(\Delta(\lambda), -)$ implies that the map $\mathrm{Hom}_A(\Delta(\lambda), \tau_{\Delta(\lambda),M})$ is injective. We claim that it is also surjective. Let $h \in \mathrm{Hom}_A(\Delta(\lambda), M)$. Then, for any $x \in \Delta(\lambda)$,

$$h(x) = \tau_{\Delta(\lambda),M}(x \otimes h) = \tau_{\Delta(\lambda),M} \circ (- \otimes h)(x), \tag{1.5.13.34}$$

where $- \otimes h \in \mathrm{Hom}_A(\Delta(\lambda), \Delta(\lambda) \otimes_R \mathrm{Hom}_A(\Delta(\lambda), M))$. Consequently, $\mathrm{Hom}_A(\Delta(\lambda), X) = 0$. By Corollary 1.5.23, $X \in A/J$-mod$\cap R$-proj.

We will proceed by induction on $|\Lambda|$ to show that every $Y \in A$-mod$\cap R$-proj satisfying $\mathrm{Ext}^1_A(Y, \nabla(\lambda)) = 0$ for every $\lambda \in \Lambda$ belongs to $\mathscr{F}(\tilde{\Delta})$.

If $|\Lambda| = 1$, then A/J-mod is the zero category, and thus $X = 0$. By (1.5.13.33) $M \in \mathscr{F}(\tilde{\Delta})$. Assume that the result holds for split quasi-hereditary algebras with $|\Lambda| < n$ for some $n > 1$. Assume that $|\Lambda| = n$. By Proposition 1.5.90, $\mathrm{Hom}_A(\Delta(\lambda) \otimes_R U_\lambda, \nabla(\alpha)) \simeq \mathrm{Hom}_R(U_\lambda, \mathrm{Hom}_A(\Delta(\lambda), \nabla(\alpha)) = 0,\ \alpha \neq \lambda$. Let $\alpha \in \Lambda$ distinct of λ. The functor $\mathrm{Hom}_A(-, \nabla(\alpha))$ induces the long exact sequence

$$0 = \mathrm{Hom}_A(\Delta(\lambda) \otimes_R U_\lambda, \nabla(\alpha)) \to \mathrm{Ext}^1_A(X, \nabla(\alpha)) \to \mathrm{Ext}^1_A(M, \nabla(\alpha)) = 0. \tag{1.5.13.35}$$

By induction, $X \in \mathscr{F}(\tilde{\Delta}_{\alpha \neq \lambda})$. By (1.5.13.33) $M \in \mathscr{F}(\tilde{\Delta})$. Now assume that $\mathrm{Ext}^1_A(\Delta(\mu), M) = 0$ for every $\mu \in \Lambda$ and $M \in A$-mod$\cap R$-proj. Since $\Delta(\mu), M \in R$-proj

$$\mathrm{Ext}^1_{A^{op}}(DM, D\Delta(\mu)) = \mathrm{Ext}^1_A(\Delta(\mu), M) = 0,\ \mu \in \Lambda. \tag{1.5.13.36}$$

As $\{D\Delta(\mu)\}$ are costandard modules of A^{op}, we obtain by statement 1. that $DM \in \mathscr{F}_{A^{op}}(D\tilde{\nabla})$. Therefore, $M \in \mathscr{F}(\tilde{\nabla})$. ☐

By Lemma 1.5.101, we see that the condition of $M \in R$-proj cannot be dropped in Theorem 1.5.104. A trivial example to check this situation is the split quasi-hereditary algebra R for some commutative Noetherian ring with positive global dimension and trivial Picard group. Then, $\nabla = \Delta = R$, and therefore $\mathscr{F}(\tilde{\Delta}) = R$-proj while

$\{K \in R\text{-mod}\colon \operatorname{Ext}^1_R(R,K)=0\} = R\text{-mod}$.

So, it follows that $\mathscr{F}(\tilde{\Delta})$ is a resolving subcategory of A-mod$\cap R$-proj, as in the classical case.

Definition 1.5.105. A category χ is said to be **resolving** of a category $\mathscr{A}$ if

- χ contains all projective objects of $\mathscr{A}$;
- χ is closed under direct summands;
- χ is closed under extensions;
- χ is closed under kernels of epimorphisms;

In general, $\mathscr{F}(\tilde{\nabla})$ may not be a coresolving subcategory of A-mod$\cap R$-proj. However, we can introduce a notion of (A,R)-coresolving subcategory such that $\mathscr{F}(\tilde{\nabla})$ is an (A,R)-coresolving subcategory of A-mod$\cap R$-proj.

Definition 1.5.106. A category χ is said to be (A,R)-**coresolving** of a category A-mod$\cap R$-proj if

- χ contains all (A,R)-injective modules of A-mod$\cap R$-proj;
- χ is closed under direct summands;
- χ is closed under extensions;
- χ is closed under cokernels of (A,R)-monomorphisms;

Lemma 1.5.107. *[Rou08, Lemma 4.22] Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $\Delta = \bigoplus\limits_{\lambda\in\Lambda}\Delta(\lambda)$ and let $\nabla = \bigoplus\limits_{\lambda\in\Lambda}\nabla(\lambda)$.*

1. *Let $M \in \mathscr{F}(\tilde{\Delta})$. If $\operatorname{Ext}^1_A(M,\Delta)=0$, then M is projective over A.*
2. *Let $N \in \mathscr{F}(\tilde{\nabla})$. If $\operatorname{Ext}^1_A(\nabla,N)=0$, then N is (A,R)-injective.*

Proof. Let $0\to K\to P\to M\to 0$ be a projective presentation for M. Let $\nabla = \bigoplus\limits_{\lambda\in\Lambda}\nabla(\lambda)$. Applying $\operatorname{Hom}_A(-,\nabla)$ yields $0=\operatorname{Ext}^1_A(P,\nabla)\to\operatorname{Ext}^1_A(K,\nabla)\to\operatorname{Ext}^2_A(M,\nabla)=0$. Hence, $\operatorname{Ext}^1_A(K,\nabla)=0$. By Theorem 1.5.104, $K\in\mathscr{F}(\tilde{\Delta})$. In particular, $\operatorname{Ext}^1_A(M,K)=0$. So, the projective presentation considered splits, therefore M is projective over A.

Consider an (A,R)-injective presentation $0\to N\to I\to X\to 0$. Applying $\operatorname{Hom}_A(\Delta,-)$ yields $\operatorname{Ext}^1_A(\Delta,X)=0$. By Theorem 1.5.104, $X\in\mathscr{F}(\tilde{\nabla})$. Thus, $\operatorname{Ext}^1_A(X,M)=0$. Hence, N is an A-summand of I and consequently, it is (A,R)-injective. ☐

This lemma says that the Ext-projective objects for $\mathscr{F}(\tilde{\Delta})$ belonging to $\mathscr{F}(\tilde{\Delta})$ are exactly the projective A-modules. Recall that in Dlab-Ringel standardization theorem for $\dim R\leq 1$, we constructed projective objects in $\mathscr{F}(\Theta)$ to construct the split quasi-hereditary algebra A. Therefore, the main difference between an algebra having a split standardizable set and an algebra being split quasi-hereditary lies here.

1.5.14 Tilting modules

Characteristic tilting modules of finite-dimensional quasi-hereditary algebras, and their summands known as (partial) tilting modules, are a fundamental tool in order to obtain information about simple modules, and therefore about the structure of A-mod. Here, for the Noetherian case, the (partial) tilting modules behave very similarly to the classical case. Previous uses of partial tilting modules for split quasi-hereditary algebras over commutative Noetherian rings can be found in [Rou08], [Has00, III. 4], [Kra17]. Partial tilting modules for $S_{\mathbb{Z}}(n,d)$, $n \geq d$ were studied in [Don93, section 3].

We shall begin by defining (partial) tilting modules and provide a way to construct them.

Definition 1.5.108. A module $T \in A$-mod is called **(partial) tilting** if $T \in \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$.

Proposition 1.5.109. *[Rou08, Proposition 4.26] $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $M \in \mathscr{F}(\tilde{\Delta})$. There is a partial tilting module T and a monomorphism $i\colon M \to T$ such that* $\operatorname{coker} i \in \mathscr{F}(\tilde{\Delta})$. *Let $\lambda \in \Lambda$. There are exact sequences and a partial tilting module $T(\lambda)$*

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0 \tag{1.5.14.1}$$

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0, \tag{1.5.14.2}$$

where $X(\lambda) \in \mathscr{F}(\Delta_{\mu<\lambda})$, $Y(\lambda) \in \mathscr{F}(\nabla_{\mu<\lambda})$.

Proof. Let $M \in \mathscr{F}(\tilde{\Delta})$. Fix an increasing bijection $\Lambda \to \{1,\dots,n\}$. We construct by induction an object T with a filtration

$$0 = T_{n+1} \subset M = T_n \subset \cdots \subset T_0 = T, \quad T_{i-1}/T_i \simeq \Delta_i \otimes_R U_i,\ U_i \in R\text{-proj}\,. \tag{1.5.14.3}$$

For $n = 1$, there is nothing to show since $\operatorname{Ext}^1_A(\Delta_1,\Delta_1) = 0$. Assume $n > 1$. Assume T_i is defined for some i, $2 \leq i \leq n$. We shall construct T_{i-1}. Let U_i be a free R-module defined by the following map $U_i \xrightarrow{\pi} \operatorname{Ext}^1_A(\Delta_i, T_i)$ being surjective. Consider the extension

$$0 \to T_i \to X \to \Delta_i \otimes_R U_i \to 0 \tag{1.5.14.4}$$

corresponding to π via the isomorphism $\operatorname{Hom}_R(U_i, \operatorname{Ext}^1_A(\Delta_i, T_i)) \to \operatorname{Ext}^1_A(\Delta_i \otimes_R U_i, T_i)$. By Lemma 1.5.45, $\operatorname{Ext}^1_A(\Delta_i, X) = 0$. Define $T_{i-1} = X$. Since $T_i \in \mathscr{F}(\Delta_{j>i})$ we obtain that $T_{i-1} \in \mathscr{F}(\Delta_{j\geq i})$. On the other hand, for $j > i$, applying $\operatorname{Hom}_A(\Delta_j, -)$ to (1.5.14.4) yields

$$\operatorname{Hom}_A(\Delta_j, \Delta_i \otimes_R U_i) \to \operatorname{Ext}^1_A(\Delta_j, T_i) \to \operatorname{Ext}^1_A(\Delta_j, X) \to \operatorname{Ext}^1_A(\Delta_j, \Delta_i \otimes_R U_i) = 0. \tag{1.5.14.5}$$

We can assume by induction that $\operatorname{Ext}^1_A(\Delta_j, T_i) = 0$ for $j > i$. Hence, $\operatorname{Ext}^1_A(\Delta_j, T_{i-1}) = 0$ for $j > i$. Hence, $\operatorname{Ext}^1_A(\Delta_j, T_{i-1}) = 0$ for all $j \geq i$. Hence, by induction, we obtain a module $T \in \mathscr{F}(\Delta)$ with $\operatorname{Ext}^1_A(\Delta_j, T) = 0$ for all j. By Theorem 1.5.104, T is partial tilting.

Now consider $M = \Delta(\lambda) = \Delta_i = T_i$. Notice that we can start the construction of T at i since for $j > i$ we have $\operatorname{Ext}^1_A(\Delta_j, \Delta_i) = 0$. Applying the previous construction we have a filtration

$$0 \subset \Delta_i = T_i \subset T_{i-1} \subset \cdots \subset T_0 = T(i) \tag{1.5.14.6}$$

with $T_{j-1}/T_j \simeq \Delta_j \otimes_R F_j$, F_j an R-free module and $T(i)$ a partial tilting module.

Since $T(i) \in \mathscr{F}(\tilde{\nabla})$ there exists a filtration $0 \subset I_1 \subset \cdots \subset I_n = T(i)$ with $I_j/I_{j-1} \simeq \nabla_j \otimes_R U_j$, $1 \leq j \leq n$. Consider the exact sequences

$$0 \to I_{j-1} \to I_j \to \nabla_j \otimes_R U_j \to 0. \tag{1.5.14.7}$$

Let $1 \leq k \leq n$. Applying the functor $\operatorname{Hom}_A(\Delta_k, -)$ we obtain the exact sequences

$$0 \to \operatorname{Hom}_A(\Delta_k, I_{j-1}) \to \operatorname{Hom}_A(\Delta_k, I_j) \to \operatorname{Hom}_A(\Delta_k, \nabla_j \otimes_R U_j) \to \operatorname{Ext}^1_A(\Delta_k, I_{j-1}) = 0. \tag{1.5.14.8}$$

Hence, for $k \neq j$ we obtain

$$\operatorname{Hom}_A(\Delta_k, I_{j-1}) \simeq \operatorname{Hom}_A(\Delta_k, I_j). \tag{1.5.14.9}$$

For $k = j$, the following is exact

$$0 \to \operatorname{Hom}_A(\Delta_j, I_{j-1}) \to \operatorname{Hom}_A(\Delta_j, I_j) \to U_j \to 0. \tag{1.5.14.10}$$

Combining (1.5.14.9) with (1.5.14.10) we obtain the exact sequence

$$0 \to \operatorname{Hom}_A(\Delta_k, \nabla_1 \otimes_R U_1) \to \operatorname{Hom}_A(\Delta_k, T(i)) \to U_k \to 0. \tag{1.5.14.11}$$

If $i = 1$, it follows by (1.5.14.7)

$$U_1 \simeq \operatorname{Hom}_A(\Delta_1, I_1) \simeq \operatorname{Hom}_A(\Delta_1, I_n) = \operatorname{Hom}_A(\Delta_1, T(1)) = \operatorname{Hom}_A(\Delta_1, \Delta_1) \simeq R. \tag{1.5.14.12}$$

By (1.5.14.11), $U_j = 0$ for $j > 1$. So, the claim follows for $i = 1$. Assume $i > 1$. Note that $\operatorname{Hom}_A(\Delta_i, T(i)) = R$. In fact, using the exact sequence constructed $0 \to \Delta(i) \to T(i) \to X(i) \to 0$, every morphism in $\operatorname{Hom}_A(\Delta_i, T(i))$ factors through Δ_i since $X(i) \in \mathscr{F}(\Delta_{j<i})$. By (1.5.14.11), $U_i = R$. By the same reason, $\operatorname{Hom}_A(\Delta_j, T(i)) = 0$ for $j > i$. Thus, $U_j = 0$, $j > i$ and the result follows. $\square$

We say that $T = \bigoplus_{\lambda \in \Lambda} T(\lambda)$ is a **characteristic tilting module**, where each $T(\lambda)$ is a partial tilting with exact sequences as in Theorem 1.5.109, where we can relax the conditions on $X(\lambda)$ and $Y(\lambda)$ to $X(\lambda) \in \mathscr{F}(\tilde{\Delta}_{\mu<\lambda})$ and $Y(\lambda) \in \mathscr{F}(\tilde{\nabla}_{\mu<\lambda})$. As we will see, a characteristic tilting module is a full tilting module justifying the modules $T(\lambda)$ being called (partial) tilting.

Proposition 1.5.110. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. If R has no non-trivial idempotents, then we can construct the partial tilting modules $T(\lambda)$ to be indecomposable modules.*

Proof. We will use the same notation as in the proof of Proposition 1.5.109. At each step of the filtration of T we can choose $U_i = R^{n_i}$ to be a free R-module with minimal rank n_i such that π is a surjection. Let $\{e_j, j = 1, \dots, n_i\}$ be an R-basis for U_i. Consider the extension

$$0 \to T_i \xrightarrow{i} X \xrightarrow{h} \Delta_i \otimes_R U_i \to 0 \tag{1.5.14.13}$$

corresponding to π. Assume that there exists $\alpha\colon \Delta_i \otimes_R U_i \to X$ such that $h \circ \alpha$ is an idempotent in $\operatorname{End}_A(\Delta_i \otimes_R U_i)$. Denote by π_r the canonical projections $\Delta_i \otimes_R U_i \to \Delta_i \otimes_R R \simeq \Delta_i$ and by i_r the canonical injections $\Delta_i \simeq \Delta_i \otimes_R R \to \Delta_i \otimes_R U_i$, $r = 1, \dots, n_i$. Then, since $h \circ \alpha$ is an idempotent $\sum_r \pi_r h \circ \alpha \circ i_r$ is an idempotent in $\operatorname{End}_A(\Delta_i) \simeq R$. So, either $\sum_r \pi_r h \circ \alpha \circ i_r$ is zero or $\sum_r \pi_r h \circ \alpha \circ i_r = \mathrm{id}_{\Delta_i}$. If $h \circ \alpha$ is a non-zero idempotent, then $\sum_r \pi_r h \circ \alpha \circ i_r = \mathrm{id}_{\Delta_i}$. Applying i_r in both members, it follows that $h \circ \alpha \circ i_r = i_r$. Now observe that for an injective presentation $T_i \xrightarrow{k} I$, X in (1.5.14.13) is the pullback of $((s_1, \dots, s_{n_i}), \operatorname{coker} k)$ where $(s_1, \dots, s_{n_i}) \in \operatorname{Hom}_A(\Delta_i \otimes_R U_i, \operatorname{coker} k)$. It follows by $h \circ \alpha \circ i_r = i_r$ that the exact sequence given by the pullback of $(s_r, \operatorname{coker} k)$ which coincides with the pullback of (h, i_r) splits over A. This exact sequence is the element $\pi(e_r) = 0$. So, we can lower the rank of U_i which contradicts the minimality of U_i. Thus, we conclude that $h \circ \alpha = 0$.

Let $1 \leq i \leq n$. We shall proceed by induction on the filtration to show that each T_j, $j \leq i$, is indecomposable. For $j = i$, $T_i = \Delta_i$ and $\operatorname{End}_A(\Delta_i) \simeq R$ which has no non-trivial idempotents, so Δ_i cannot be decomposable. Assume that for some $k < i$ every T_k is indecomposable. Take $e\colon T_{k-1} \to T_{k-1}$ idempotent. Now, $T_k \in \mathscr{F}(\Delta_{l<k})$, thus $\operatorname{Hom}_A(T_k, \Delta_k \otimes_R U_k) = 0$. Hence, $e|_{T_j}$ has image in T_j. Moreover, $e|_{T_j}$ is an idempotent in $\operatorname{End}_A(T_j)$. By

induction, T_j is indecomposable, so either $e|_{T_j} = 0$ or $e|_{T_j} = \mathrm{id}_{T_j}$. Assume that $e|_{T_j} = 0$. Then, $e \circ i = 0$. So, it induces a map $f \in \mathrm{Hom}_A(\Delta_j \otimes_R U_j, T_{j-1})$ such that $f \circ h = e$. Since $e^2 = e$ and h is surjective

$$f \circ h \circ f \circ h = f \circ h \implies f \circ h \circ f = f \tag{1.5.14.14}$$

$$\implies h \circ f \circ h \circ f = h \circ f. \tag{1.5.14.15}$$

By our initial discussion, $h \circ f \in \mathrm{End}_A(\Delta_j \otimes_R U_j)$ idempotent must be zero. Hence, $f = f \circ h \circ f = 0$. Thus, $e = 0$. If $e_{T_j} = \mathrm{id}_{T_j}$, then the idempotent $(\mathrm{id}_{T_{i-1}} - e)|_{T_j} = 0$. Hence, $e = \mathrm{id}_{T_{i-1}}$. So, there are no non-trivial idempotents in $\mathrm{End}_A(T_{j-1})$, and thus T_{j-1} is indecomposable. ☐

Remark 1.5.111. If R is a field, the partial tilting modules constructed in such a way are unique up to isomorphism. Furthermore, in such a case, each T_i constructed has local endomorphism ring. If R is a commutative Noetherian ring, the uniqueness might fail. For example, the rank of (U_i) at each localization $\mathfrak{m}$ ($\mathfrak{m}$ a maximal ideal of R) might not be constant for some i. △

In practice, the short exact sequences (1.5.109) provide a way for determining the (partial) tilting modules. But, as we will see next, these short exact sequences also give approximations of Δ by ∇ and vice-versa.

Proposition 1.5.112. *Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary algebra. Let $\lambda \in \Lambda$.*

The homomorphism $\Delta(\lambda) \hookrightarrow T(\lambda)$ constructed in Proposition 1.5.109 is an injective left $\mathscr{F}(\tilde{\nabla})$-approximation of $\Delta(\lambda)$. The homomorphism $T(\lambda) \to \nabla(\lambda)$ constructed in Proposition 1.5.109 is a surjective right $\mathscr{F}(\tilde{\Delta})$-approximation of $\nabla(\lambda)$.

Proof. Let $X \in \mathscr{F}(\tilde{\nabla})$. Applying $\mathrm{Hom}_A(-, X)$ to (1.5.14.1) yields the exact sequence

$$0 \to \mathrm{Hom}_A(X(\lambda), X) \to \mathrm{Hom}_A(T(\lambda), X) \to \mathrm{Hom}_A(\Delta(\lambda), X) \to \mathrm{Ext}^1_A(X(\lambda), X). \tag{1.5.14.16}$$

By Lemma 1.5.100, $\mathrm{Ext}^1_A(X(\lambda), X) = 0$ since $X(\lambda) \in \mathscr{F}(\tilde{\Delta})$. Thus, $\mathrm{Hom}_A(T(\lambda), X) \to \mathrm{Hom}_A(\Delta(\lambda), X)$ is surjective. Let $Y \in \mathscr{F}(\tilde{\Delta})$. Applying $\mathrm{Hom}_A(Y, -)$ to (1.5.14.2) yields that the map $\mathrm{Hom}_A(Y, T(\lambda)) \to \mathrm{Hom}_A(Y, \nabla(\lambda))$ is surjective. ☐

There is naturally a version of Corollary 1.5.94 and Proposition 1.5.90 for partial tilting modules.

Lemma 1.5.113. *Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary algebra and let T be a partial tilting module. Then, DT is a partial tilting module in the split highest weight category $(A^{op}, \{D\nabla(\lambda)_{\lambda \in \Lambda}\})$. Moreover, if T is a characteristic tilting module in A, then DT is a characteristic tilting module in A^{op}.*

Proof. By Theorem 1.5.104, $DT \in \mathscr{F}(D\tilde{\Delta}) \cap \mathscr{F}(D\tilde{\nabla})$. Assume that T is a characteristic tilting module. The exact sequences (1.5.14.1) and (1.5.14.2) are (A, R)-exact since $X(\lambda), \nabla(\lambda) \in R$-proj. Applying D, it follows that DT is a characteristic tilting module in A^{op}. ☐

Lemma 1.5.114. *Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary algebra and let $X \in A$-mod. Let T be a characteristic tilting module. Then, $X \in \mathscr{F}(\nabla)$ if and only if X has a finite resolution $0 \to X_r \to \cdots \to X_0 \to X \to 0$, with $X_i \in \operatorname{add} T$.*

Proof. Assume that X has a resolution by partial tilting modules in $\operatorname{add} T$. Since each partial tilting is in $\mathscr{F}(\tilde{\nabla})$ every $X_i \in \mathscr{F}(\tilde{\nabla})$. As $\mathscr{F}(\tilde{\nabla})$ is (A, R)-coresolving in A-mod$\cap R$-proj then, in particular, it is closed under quotients of (A, R)-monomorphisms, so it follows that $X \in \mathscr{F}(\tilde{\nabla})$.

Conversely, we will start by showing by the following lemma:

Lemma 1.5.115. *Let X, Z be modules with a resolution by partial tilting modules in* $\operatorname{add} T$. *Assume there is an exact sequence*

$$0 \to X \xrightarrow{k} Y \xrightarrow{\pi} Z \to 0. \tag{1.5.14.17}$$

Then, Y has a resolution by partial tilting modules in $\operatorname{add} T$.

Proof. Consider the following diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "k"] & Y \arrow[r, "\pi"] & Z \arrow[r] & 0 \\
0 \arrow[r] & T_0' \arrow[u, "p_0'"] \arrow[r, "k_0"] & T_0' \bigoplus T_0'' \arrow[r, "\pi_0"] & T_0'' \arrow[u, "p_0''"] \arrow[r] & 0 \\
 & K_0' \arrow[u] & & K_0'' \arrow[u] & \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

with $T_0', T_0'' \in \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$. Applying $\operatorname{Hom}_A(T_0'', -)$ to the top row yields

$$0 \to \operatorname{Hom}_A(T_0'', X) \to \operatorname{Hom}_A(T_0'', Y) \to \operatorname{Hom}_A(T_0'', Z) \to \operatorname{Ext}_A^1(T_0'', X) = 0. \tag{1.5.14.18}$$

This is an immediate consequence of $T_0'' \in \mathscr{F}(\tilde{\Delta})$ and $X \in \mathscr{F}(\tilde{\nabla})$. Hence, the map p_0'' lifts to $f \in \operatorname{Hom}_A(T_0'', Y)$ such that $p_0'' = \pi \circ f$. Now consider $g \colon T_0' \bigoplus T_0'' \to Y$, given by $g(x, y) = k \circ p_0'(x) + f(y)$, $(x, y) \in T_0' \bigoplus T_0''$. Then, for $(x, y) \in T_0' \bigoplus T_0''$,

$$g \circ k_0(x) = g(x, 0) = k \circ p_0'(x) \tag{1.5.14.19}$$

$$\pi \circ g(x, y) = \pi(k \circ p_0'(x) + f(y)) = \pi \circ f(y) = p_0''(y) = p_0'' \circ \pi_0(x, y). \tag{1.5.14.20}$$

Hence, g makes the previous diagram commutative. By Snake Lemma, g is surjective. Define $K_0 = \ker g$. $k_0|_{K_0'} \colon K_0' \to K_0$ is well defined and it is clearly a monomorphism since

$$g \circ k_0(x) = g(x, 0) = k \circ p_0'(x) = k(0) = 0, \; x \in K_0. \tag{1.5.14.21}$$

Now $\pi_0| \colon K_0 \to K_0''$ is well defined since $p_0'' \circ \pi_0|_{K_0}(x, y) = p_0'' \circ \pi_0(x, y) = \pi \circ g(x, y) = 0$, $(x, y) \in K_0$. Therefore, we have the commutative diagram with exact columns and the two top rows exact,

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "k"] & Y \arrow[r, "\pi"] & Z \arrow[r] & 0 \\
0 \arrow[r] & T_0' \arrow[u, "p_0'"] \arrow[r, "k_0"] & T_0' \bigoplus T_0'' \arrow[u, "g"] \arrow[r, "\pi_0"] & T_0'' \arrow[u, "p_0''"] \arrow[r] & 0 \\
0 \arrow[r] & K_0' \arrow[u] \arrow[r, "k_0|_{K_0'}"] & K_0 \arrow[u] \arrow[r, "\pi_0|_{K_0}"] & K_0'' \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}.$$

Let $y \in K_0''$. Then,

$$\pi \circ g(0, y) = \pi \circ f(y) = p_0''(y) = 0. \tag{1.5.14.22}$$

Thus, $g(0,y) = k(p_0'(t)) = g \circ k_0(t)$ for some $t \in T_0'$. Hence, $(0,y) - k_0(t) \in K_0$ and its image under π_0 is y. Thus, $\pi_0|_{K_0}$ is surjective. Let $(x,y) \in \ker \pi|_{K_0}$. Then, $(x,y) \in K_0 \cap \operatorname{im} k_0$, so there exists $z \in T_0'$ such that $k_0(z) = (x,y)$. Thus,

$$k \circ p_0'(z) = g \circ k_0(z) = 0 \implies p_0'(z) = 0. \tag{1.5.14.23}$$

Thus, $z \in K_0'$. So, the bottom row is also exact.

Now continue with the construction with the bottom row. Note that both K_0', K_0'' have partial tilting resolutions by our choice in them. After a finite number of steps either we must proceed with an exact sequence

$$0 \to K_t' \to K_t \to K_t'' \to 0, \tag{1.5.14.24}$$

with $K_t' \in \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla}), K_t'' \in \mathscr{F}(\tilde{\nabla})$ or $T_{t+1}'' = K_t'' \in \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla}), K_t' \in \mathscr{F}(\tilde{\nabla})$. In the first case, proceed one more step and we end up with $K_{t+1} \simeq K_{t+1}''$. So,

$$0 \to T_r'' \to \cdots \to T_{t+2}'' \to T_{t+1}' \bigoplus T_{t+1}'' \to \cdots \to T_0' \bigoplus T_0'' \to Y \to 0 \tag{1.5.14.25}$$

is a partial tilting resolution for Y. In the second case, $\operatorname{Ext}_A^1(K_t'', K_t') = 0$, so it splits, that is $K_t \simeq K_t'' \bigoplus K_t'$. Hence

$$0 \to T_r' \to \cdots \to T_{t+2}' \to T_{t+1}' \bigoplus T_{t+1}'' \to \cdots \to T_0' \bigoplus T_0'' \to Y \to 0 \tag{1.5.14.26}$$

is a partial tilting resolution for Y. ☐

Now we will show that each costandard module $\nabla(\mu)$ has a partial tilting resolution. If λ is minimal, then $\Delta(\lambda) = T(\lambda) = \nabla(\lambda)$. So, it is clear. Assume by induction that each $\nabla(\mu)$ with $\mu < \lambda$ has a resolution by partial tilting modules. By Lemma 1.5.115, every module in $\mathscr{F}(\nabla_{\mu<\lambda})$ has a finite partial tilting resolution. Hence $Y(\lambda)$, as in Proposition 1.5.109, has a finite partial tilting resolution. Now using the exact sequence

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0 \tag{1.5.14.27}$$

and the partial tilting resolution for $Y(\lambda)$, it follows that $\nabla(\lambda)$ has a finite partial tilting resolution. Applying Lemma 1.5.115, it follows that any module in $\mathscr{F}(\tilde{\nabla})$ has a partial tilting resolution. ☐

We can deduce the dual result for $\mathscr{F}(\tilde{\Delta})$.

Lemma 1.5.116. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra and let $M \in A$-mod. Let T be a characteristic tilting module. Then, $X \in \mathscr{F}(\tilde{\Delta})$ if and only if M has a finite coresolution $0 \to M \to T_0 \to \cdots \to T_r \to 0$, with $T_i \in \operatorname{add} T$.*

Proof. Assume that M admits such finite coresolution. Since $T_i \in \operatorname{add} T$, $T_i \in \mathscr{F}(\tilde{\Delta})$. $\mathscr{F}(\tilde{\Delta})$ is resolving in A-mod$\cap R$-proj, then, in particular, it is closed under kernels of epimorphisms. Hence, by induction on r it follows that $M \in \mathscr{F}(\tilde{\Delta})$.

Conversely, assume that $M \in \mathscr{F}(\tilde{\Delta})$. We will show that each $\Delta(\lambda)$ has a partial tilting coresolution belonging to $\operatorname{add} T$. First, observe the following.

Let $0 \to X \to Y \to Z \to 0$ be an exact sequence where X, Z have a coresolution by partial tilting modules belonging to $\operatorname{add} T$. In particular, $Z \in \mathscr{F}(\tilde{\Delta})$, and thus $Z \in R$-proj. Hence, applying D yields the exact sequence

$$0 \to DZ \to DY \to DX \to 0. \tag{1.5.14.28}$$

By Lemma 1.5.113, every coresolution by partial tilting modules belonging to $\operatorname{add} T$ in A-mod is sent to a resolution by partial tilting modules belonging to $\operatorname{add} DT$ in A^{op}-mod. By Lemma 1.5.115, DY has a resolution by partial tilting modules belonging to $\operatorname{add} DT$ in A^{op}-mod. Since $Y \in R$-proj and $DDT \simeq T$ Y has a coresolution by partial tilting modules belonging to $\operatorname{add} T$ in A.

If λ is minimal, then $\Delta(\lambda)$ is partial tilting. Assume by induction that each $\Delta(\mu)$ with $\mu < \lambda$ has a coresolution by partial tilting modules. Hence, $X(\lambda)$ given by exact sequence (1.5.14.1) admits a coresolution by partial tilting modules belonging to $\operatorname{add} T$. Using the exact sequence (1.5.14.1), it follows that $\Delta(\lambda)$ has a coresolution by partial tilting modules belonging to $\operatorname{add} T$. Now for every $F \in R$-proj, $F \otimes_R -$ is exact, thus $\Delta(\lambda) \otimes_R F$ has a coresolution by partial tilting modules belonging to $\operatorname{add} F \otimes_R T$. But $F \otimes_R T$ is an A-summand of $R^s \otimes_R T \simeq T^s$. Hence, every $\Delta(\lambda) \otimes_R F$ has a coresolution by partial tilting modules belonging to $\operatorname{add} T$. It follows that M has a partial tilting coresolutin belonging to $\operatorname{add} T$. $\square$

Now applying the same idea used in [KSX01] to construct a filtration to $\operatorname{End}_A(T)$, for T a partial tilting we have the following result.

Proposition 1.5.117. *Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary algebra. Let $M \in \mathscr{F}(\tilde{\Delta})$, $L \in \mathscr{F}(\tilde{\nabla})$. Let $\Delta \to \{1, \dots, n\}$, $\Delta_i \mapsto i$ be an increasing bijection. So, there exists $U_i, S_i \in R$-proj such that*

$$0 = M_{n+1} \subset M_n \subset \cdots \subset M_1 = M \text{ with } M_i/M_{i+1} \simeq \Delta_i \otimes_R U_i$$
$$0 = L_{n+1} \subset L_n \subset \cdots \subset L_1 = L \text{ with } L_i/L_{i+1} \simeq \nabla_{n-i+1} \otimes_R S_{n-i+1},\ i = 1, \dots, n.$$

Then, $\operatorname{Hom}_A(M, L)$ *has a filtration*

$$0 = X_{n+1} \subset X_n \subset X_{n-1} \subset \cdots \subset X_1 = X = \operatorname{Hom}_A(M, N),$$
$$X_i = \operatorname{Hom}_A(M/M_{n-i+2}, L_i) = \operatorname{Hom}_{A/J_{n-i+2}}(M/M_{n-i+2}, L_i), \quad X_i/X_{i+1} \simeq \operatorname{Hom}_R(U_{n-i+1}, S_{n-i+1}).$$

Proof. We will proceed by induction on $n = |\Lambda|$. Assume $n = 1$. Then, $M \simeq \Delta_1 \otimes_R U_1$ and $L \simeq \nabla_1 \otimes_R S_1$. Then,

$$\operatorname{Hom}_A(M, N) = \operatorname{Hom}_A(\Delta_1 \otimes_R U_1, \nabla_1 \otimes_R S_1) \simeq \operatorname{Hom}_R(U_1, \operatorname{Hom}_A(\Delta_1, \nabla_1 \otimes_R S_1)) \tag{1.5.14.29}$$
$$\simeq \operatorname{Hom}_R(U_1, \operatorname{Hom}_A(\Delta_1, \nabla_1) \otimes_R S_1) \simeq \operatorname{Hom}_R(U_1, S_1). \tag{1.5.14.30}$$

So, the filtration $0 \subset \operatorname{Hom}_R(U_1, S_1) = X_1$ is the desired one. Assume the result holds for $n-1$. Consider the short exact sequences

$$0 \to \Delta_n \otimes_R U_n \xrightarrow{k_M} M \xrightarrow{\pi_M} M/M_n \to 0 \tag{1.5.14.31}$$
$$0 \to L_2 \xrightarrow{k_L} L \xrightarrow{\pi_L} \nabla_n \otimes_R S_n \to 0. \tag{1.5.14.32}$$

Applying the functor $\operatorname{Hom}_A(M, -)$ to (1.5.14.32) gives

$$0 \to \operatorname{Hom}_A(M, L_2) \xrightarrow{\operatorname{Hom}_A(M, k_L)} \operatorname{Hom}_A(M, L) \xrightarrow{\operatorname{Hom}_A(M, \pi_L)} \operatorname{Hom}_A(M, \nabla_n \otimes_R S_n) \to \operatorname{Ext}^1_A(M, L_2) = 0. \tag{1.5.14.33}$$

Applying $\operatorname{Hom}_A(-, L)$ to (1.5.14.31) gives

$$0 \to \operatorname{Hom}_A(M/M_n, L) \xrightarrow{\operatorname{Hom}_A(\pi_M, L)} \operatorname{Hom}_A(M, L) \xrightarrow{\operatorname{Hom}_A(k_M, L)} \operatorname{Hom}_A(\Delta_n \otimes_R U_n, L) \to \operatorname{Ext}^1_A(M/M_n, L) = 0. \tag{1.5.14.34}$$

Applying the functor $\operatorname{Hom}_A(-,L_2)$ to (1.5.14.31) we get

$$0 \to \operatorname{Hom}_A(M/M_n,L_2) \xrightarrow{\operatorname{Hom}_A(\pi_M,L_2)} \operatorname{Hom}_A(M,L_2) \xrightarrow{\operatorname{Hom}_A(k_M,L_2)} \operatorname{Hom}_A(\Delta_n\otimes_R U_n,L_2) \longrightarrow 0=\operatorname{Ext}^1_A(M/M_n,L_2). \tag{1.5.14.35}$$

Since $L_2\in\mathscr{F}(\nabla_{i<n})$ we obtain $\operatorname{Hom}_A(\Delta_n\otimes_R U_n,L_2)=0$. By (1.5.14.35), $\operatorname{Hom}_A(\pi_M,L_2)$ is an isomorphism. Applying the functor $\operatorname{Hom}_A(-,\nabla_n\otimes_R S_n)$ to (1.5.14.31) yields

$$\begin{aligned}0 &\longrightarrow \operatorname{Hom}_A(M/M_n,\Delta_n\otimes_R S_n) \longrightarrow \operatorname{Hom}_A(M,\Delta_n\otimes_R U_n) \xrightarrow{\operatorname{Hom}_A(k_M,\Delta_n\otimes_R U_n)} \operatorname{Hom}_A(\Delta_n\otimes_R U_n,\nabla_n\otimes_R S_n) \to\\ &\to \operatorname{Ext}^1_A(M/M_n,\Delta_n\otimes_R U_n)=0 \longrightarrow \cdots\end{aligned}.$$

Since $M/M_n\in\mathscr{F}(\tilde{\Delta}_{i<n})$ we obtain $\operatorname{Hom}_A(M/M_n,\nabla_n\otimes_R S_n)=0$. Hence, $\operatorname{Hom}_A(k_M,\Delta_n\otimes_R U_n)$ is an isomorphism. Therefore, we have an exact sequence

$$\operatorname{Hom}_A(M/M_n,L_2) \xhookrightarrow{\operatorname{Hom}_A(M,k_L)\circ\operatorname{Hom}_A(\pi_M,L_2)} \operatorname{Hom}_A(M,L) \xrightarrow{\operatorname{Hom}_A(k_M,\Delta_n\otimes_R U_n)\circ\operatorname{Hom}_A(M,\pi_L)} \operatorname{Hom}_A(\Delta_n\otimes_R U_n,\nabla_n\otimes_R S_n).$$

We have that

$$\begin{aligned}\operatorname{Hom}_A(\Delta_n\otimes_R U_n,\nabla_n\otimes_R S_n) &\simeq \operatorname{Hom}_R(U_n,\operatorname{Hom}_A(\Delta_n,\nabla_n\otimes_R S_n)) \simeq \operatorname{Hom}_R(U_n,\operatorname{Hom}_A(\Delta_n,\nabla_n)\otimes_R S_n)\\ &\simeq \operatorname{Hom}_R(U_n,S_n).\end{aligned} \tag{1.5.14.36}$$

Fix $J_n=\operatorname{im}\tau_{\Delta_n}$. Since $M/M_n\in\mathscr{F}(\Delta_{i<n})$ and $L_2\in\mathscr{F}(\nabla_{i<n})$, we have $\operatorname{Hom}_A(M/M_n,L_2)=\operatorname{Hom}_{A/J_n}(M/M_n,L_2)$. Therefore, $X/\operatorname{Hom}_{A/J_n}(M/M_n,L_2)\simeq\operatorname{Hom}_R(U_n,S_n)$. By induction, $\operatorname{Hom}_{A/J_n}(M/M_n,L_2)$ admits a filtration

$$0\subset X_n\subset X_{n-1}\subset\cdots\subset X_2=\operatorname{Hom}_A(M/M_n,L_2), \tag{1.5.14.37}$$

with $X_i \simeq \operatorname{Hom}_{A/J_n/J_{n-i+2}/J_n}(M/M_{n-i+2},L_i) \simeq \operatorname{Hom}_{A/J_{n-i+2}}(M/M_{n-i+2},L_i)$, $i=2,\dots n$. Thus, $0\subset X_n\subset X_{n-1}\subset\cdots\subset X_2\subset X$ is the desired filtration. ☐

The following result has been observed in the literature several times in particular cases (see for example Lemma 4.2 of [DPS98b]).

Corollary 1.5.118. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $M\in\mathscr{F}(\tilde{\Delta})$ and let $N\in\mathscr{F}(\tilde{\nabla})$. Let Q be a commutative Noetherian R-algebra. Then, $Q\otimes_R\operatorname{Hom}_A(M,N)\simeq\operatorname{Hom}_{Q\otimes_R A}(Q\otimes_R M,Q\otimes_R N)$. In particular,* $\operatorname{Hom}_A(M,N)(\mathfrak{m})\simeq\operatorname{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}),N(\mathfrak{m}))$ *for every maximal ideal* $\mathfrak{m}$ *in R.*

Proof. We shall proceed by induction on $n=|\Lambda|$. Assume $n=1$. Then, $\operatorname{Hom}_A(M,N)\simeq\operatorname{Hom}_R(U_1,S_1)$. So,

$$Q\otimes_R\operatorname{Hom}_A(M,N)\simeq Q\otimes_R\operatorname{Hom}_R(U_1,S_1)\simeq\operatorname{Hom}_{Q\otimes_R R}(Q\otimes_R U_1,Q\otimes_R S_1) \tag{1.5.14.38}$$

$$\simeq\operatorname{Hom}_{Q\otimes_R A}(Q\otimes_R\Delta_1\otimes_{Q\otimes_R R}Q\otimes_R U_1,Q\otimes_R\nabla_1\otimes_{Q\otimes_R R}Q\otimes_R S_1) \tag{1.5.14.39}$$

$$\simeq\operatorname{Hom}_{Q\otimes_R A}(Q\otimes_R M,Q\otimes_R N). \tag{1.5.14.40}$$

Assume that the result holds for $n-1$. Consider A with $|\Lambda|=n$. Consider the exact sequence given by the filtration of $\operatorname{Hom}_A(M,N)$:

$$0\to\operatorname{Hom}_{A/J}(M/M_n,L_2)\to\operatorname{Hom}_A(M,L)\to\operatorname{Hom}_R(U_n,S_n)\to 0. \tag{1.5.14.41}$$

Since $\operatorname{Hom}_R(U_n,S_n)\in R\text{-proj}$, (1.5.14.41) is (A,R)-exact. We will denote by $X(Q)$ the tensor product $Q\otimes_R X$. Applying $Q\otimes_R -$ we get the following commutative diagram with exact rows

$$\begin{tikzcd}
Q\otimes_R \mathrm{Hom}_{A/J}(M/M_n, L_2) \arrow[r, hook] \arrow[d, "\alpha_1"] & Q\otimes_R \mathrm{Hom}_A(M,L) \arrow[r, two heads] \arrow[d, "\alpha"] & Q\otimes_R \mathrm{Hom}_A(\Delta_n\otimes_R U_n, \nabla_n\otimes_R S_n) \arrow[d, "\alpha_2"] \\
\mathrm{Hom}_{A/J(Q)}(M/M_n(Q), L_2(Q)) \arrow[r, hook] & \mathrm{Hom}_{A(Q)}(M(Q), L_2(Q)) \arrow[r, two heads] & \mathrm{Hom}_{A(Q)}(\Delta_n(Q)\otimes_Q U_n(Q), \nabla_n(Q)\otimes_Q S_n(Q))
\end{tikzcd}.$$

Note that the bottom row is exact since we use the same exact sequences given by filtrations of $M(Q)\in\mathscr{F}(\Delta(Q))$ and $L(Q)\in\mathscr{F}(\nabla(Q))$ in view of Proposition 1.5.55. This is admissible because all the modules involved in the filtrations are projective over R. So, the functor $Q\otimes_R -$ preserves the given filtrations. By induction, α_1 is an isomorphism. Since $\Delta_n\otimes_R U_n\in A$-proj, α_2 is an isomorphism. By Snake Lemma, α is an isomorphism.

Let $\mathfrak{m}$ be a maximal ideal in R. Fixing $Q=R(\mathfrak{m})$, the rest of the claim follows. $\square$

Since $\mathrm{Hom}_A(M,N)$ admits a filtration by projective R-modules then it is also projective over R.

Corollary 1.5.119. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $M\in\mathscr{F}(\tilde{\Delta})$ and let $N\in\mathscr{F}(\tilde{\nabla})$. Then,* $\mathrm{Hom}_A(M,N)\in R$-proj.

We should remark that the name tilting module here described in the context of split quasi-hereditary algebras should not be confused with its counterpart tilting module in representation theory. Many representation theorists know tilting modules in the following way:

Definition 1.5.120. Let A be a projective Noetherian R-algebra. A module $T\in A$-mod is **(full generalized) tilting** provided that

(i) T has finite projective dimension over A;

(ii) $\mathrm{Ext}_A^{i>0}(T,T)=0$;

(iii) There is an exact sequence $0\to A\to T_0\to\cdots\to T_r\to 0$ where $T_i\in\operatorname{add}T$ for all $0\leq i\leq r$ for some $r\in\mathbb{N}$.

Although this is not the same concept as partial tilting modules in $\mathscr{F}(\tilde{\Delta})\cap\mathscr{F}(\tilde{\nabla})$, a characteristic tilting module is a generalized tilting module. In fact, condition 1.5.120(i) is clear since A has finite global dimension if R has finite global dimension. Condition 1.5.120(ii) follows from Lemma 1.5.100. Condition 1.5.120(iii) follows from Lemma 1.5.116 since $A\in\mathscr{F}(\tilde{\Delta})$.

1.5.15 Ringel dual and uniqueness of characteristic tilting modules

Lemma 1.5.121. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Assume that $T=\bigoplus_{\lambda\in\Lambda}T(\lambda)$ is a characteristic tilting module. Fix $B=\mathrm{End}_A(T)^{op}$. Then, the functor $G=\mathrm{Hom}_A(T,-)\colon A$-Mod $\to B$-Mod restricts to an exact equivalence between $\mathscr{F}(\tilde{\nabla})$ and $\mathscr{F}(\tilde{\Delta}_B)$ with $\Delta_B(\lambda)=G\nabla(\lambda)$, $\lambda\in\Lambda$. Let $\Delta\to\{1,\dots,n\}$, $\Delta_i\mapsto i$ be an increasing bijection. Here $\mathscr{F}(\tilde{\Delta}_B)$ denotes the subcategory of B-mod whose modules M have a finite filtration*

$$0=P_{n+1}\subset P_n\subset\cdots\subset P_1=M \text{ with } P_i/P_{i+1}\simeq\Delta_B(i)\otimes_R U_i,\ U_i\in R\text{-proj}.$$

Proof. The functor $\mathrm{Hom}_A(T,-)$ is exact on $\mathscr{F}(\tilde{\nabla})$. In fact, this follows from $\mathrm{Ext}_A^1(T,M)=0$ for every $M\in\mathscr{F}(\tilde{\nabla})$ since $T\in\mathscr{F}(\tilde{\Delta})$. Notice that for any $M\in\mathscr{F}(\tilde{\Delta})$, we have $\mathrm{Hom}_A(M,\nabla_j\otimes_R S_j)\simeq\mathrm{Hom}_A(M,\nabla_j)\otimes_R S_j$. In fact, $\mathrm{Hom}_A(M,\nabla_j\otimes_R S_j)$ has a filtration $X^\bullet$ with $X_{n-j+1}/X_{n-j+2}\simeq\mathrm{Hom}_R(U_j,S_j)$ and $X_{n-j+2}=0$. So,

$$\mathrm{Hom}_A(M,\nabla_j\otimes_R S_j)\simeq\mathrm{Hom}_R(U_j,S_j)\simeq\mathrm{Hom}_R(U_j,R)\otimes_R S_j\simeq\mathrm{Hom}_A(M,\nabla_j)\otimes_R S_j. \tag{1.5.15.1}$$

Let $N \in \mathscr{F}(\tilde{\nabla})$. Hence, we have a filtration

$$0 \subset I_1 \subset \cdots \subset I_n = N, \quad I_i/I_{i-1} \simeq I_i \otimes_R U_i, \; i = 1, \dots, n. \tag{1.5.15.2}$$

Applying $\operatorname{Hom}_A(T, -)$ yields the exact sequence

$$0 \to \operatorname{Hom}_A(T, I_{i-1}) \to \operatorname{Hom}_A(T, I_i) \to \operatorname{Hom}_A(T, \nabla_i) \otimes_R U_i \to 0. \tag{1.5.15.3}$$

So, $\operatorname{Hom}_A(T, -)$ sends a module $N \in \mathscr{F}_A(\tilde{\nabla})$ to $\operatorname{Hom}_A(T, N) \in \mathscr{F}_B(\widetilde{\operatorname{Hom}_A(T, \nabla)})$. Fix $\Delta_B(i) = G\nabla_i$. We shall now prove that G is full and faithful on $\mathscr{F}(\tilde{\nabla})$. Let $Y \in A$-mod. Then,

$$\operatorname{Hom}_A(T, Y) \simeq G(Y) = \operatorname{Hom}_B(B, GY) \simeq \operatorname{Hom}_B(\operatorname{Hom}_A(T, T), GY) \simeq \operatorname{Hom}_B(GT, GY). \tag{1.5.15.4}$$

Hence, for any $X \in \operatorname{add} T$, we have $\operatorname{Hom}_A(X, Y) \simeq \operatorname{Hom}_B(GX, GY)$ for all $Y \in A$-mod. Let $X \in \mathscr{F}_A(\tilde{\nabla})$. By Lemma 1.5.115, there is an $\operatorname{add} T$-presentation $T_1 \to T_0 \to X \to 0$. Applying $\operatorname{Hom}_A(-, Y)$ and $\operatorname{Hom}_B(G-, GY)$ we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(X,Y) \arrow[r] \arrow[d] & \operatorname{Hom}_A(T_0,Y) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Hom}_A(T_1,Y) \arrow[d, "\simeq"] \\
0 \arrow[r] & \operatorname{Hom}_B(GX,GY) \arrow[r] & \operatorname{Hom}_B(GT_0,GY) \arrow[r] & \operatorname{Hom}_B(GT_1,GY)
\end{tikzcd}.$$

By diagram chasing, $\operatorname{Hom}_A(X, Y) \simeq \operatorname{Hom}_B(GX, GY)$ for all $X, Y \in \mathscr{F}_A(\tilde{\nabla})$.

Now we claim that $\operatorname{Ext}^1_A(U_i \otimes_R \nabla_i, N) \simeq \operatorname{Ext}^1_B(G(U_i \otimes_R \nabla_i), GN)$ for all $N \in \mathscr{F}_A(\tilde{\nabla})$ and $U_i \in R$-proj.

Consider the exact sequence $0 \to Y_i \to T_i \to \nabla_i \to 0$. Applying $U_i \otimes_R -$ we get the exact sequence

$$0 \to U_i \otimes_R Y_i \to U_i \otimes_R T_i \to U_i \otimes_R \nabla_i \to 0. \tag{1.5.15.5}$$

Let $N \in \mathscr{F}_A(\tilde{\nabla})$. Applying $\operatorname{Hom}_A(-, N)$ we get

$$0 \to \operatorname{Hom}_A(\nabla_i \otimes_R U_i, N) \to \operatorname{Hom}_A(U_i \otimes_R T_i, N) \to \operatorname{Hom}_A(U_i \otimes_R Y_i, N) \to \operatorname{Ext}^1_A(U_i \otimes_R \nabla_i, N) \to 0. \tag{1.5.15.6}$$

Since G is exact on $\mathscr{F}(\tilde{\nabla})$,

$$0 \to \operatorname{Hom}_B(G(U_i \otimes_R \nabla_i), GN) \to \operatorname{Hom}_B(GT_i \otimes_R \nabla_i, GN) \to \operatorname{Hom}_B(G(U_i \otimes_R Y_i), GN) \to \operatorname{Ext}^1_B(G(U_i \otimes_R U_i), GN) \to 0$$

is an exact sequence. Here $\operatorname{Ext}^1_B(G(T_i \otimes_R U_i), GN) = 0$ since $G(T_i \otimes_R U_i)$ is a B-summand of $GT_i^s \in B$-proj. Therefore, there is a commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Hom}_A(T_i \otimes_R U_i, N) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Hom}_A(Y_i \otimes_R U_i, N) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Ext}^1_A(\nabla_i \otimes_R U_i, N) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_B(G(T_i \otimes_R U_i), GN) \arrow[r] & \operatorname{Hom}_B(GY_i \otimes_R U_i, GN) \arrow[r] & \operatorname{Ext}^1_B(G\nabla_i \otimes_R U_i, GN) \arrow[r] & 0
\end{tikzcd}.$$

It follows by diagram chasing that $\operatorname{Ext}^1_B(G(\nabla_i \otimes_R U_i), GN) \simeq \operatorname{Ext}^1_A(\nabla_i \otimes_R U_i, N)$.

Now consider $X \in \mathscr{F}_B(\tilde{\Delta}_B)$. Then, there is a filtration

$$0 \subset X_1 \subset \cdots \subset X_n = X, \quad X_i/X_{i-1} \simeq \Delta_B(i) \otimes_R U_i, \; U_i \in R\text{-proj}. \tag{1.5.15.7}$$

We claim that there exists $N \in \mathscr{F}_A(\tilde{\nabla})$ such that $GN = X$. We will prove it by induction on $n = |\Lambda|$. If $n = 1$, then

$$X = \Delta_B(1) \otimes_R U_1 \simeq G\nabla_1 \otimes_R U_1 \simeq G(\nabla_1 \otimes_R U_1). \tag{1.5.15.8}$$

Assume that the result holds for $n-1$. Consider the exact sequence

$$0 \to X_{n-1} \to X \to \Delta_B(n) \otimes_R U_n \to 0. \tag{1.5.15.9}$$

Here, $\Delta_B(n) \otimes_R U_n \simeq G(\nabla_n \otimes_R U_n)$. By induction, $X_{n-1} \simeq GN_{n-1}$ for some $N_{n-1} \in \mathscr{F}(\tilde{\nabla})$. So, the exact sequence in (1.5.15.9) belongs to $\operatorname{Ext}^1_B(G(\nabla_n \otimes_R U_n), GN_{n-1})$. Hence, there exists an exact sequence

$$0 \to N_{n-1} \to N_n \to \nabla_n \otimes_R U_n \to 0 \tag{1.5.15.10}$$

and its image by G is isomorphic to (1.5.15.9). In particular, $GN_n \simeq X$. □

Theorem 1.5.122. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let T be a characteristic tilting module of $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$. Let $B = \operatorname{End}_A(T)^{op}$. Then, B is split highest weight category with standard modules $\Delta_B(\lambda) = \operatorname{Hom}_A(T, \nabla(\lambda))$, where Λ is ordered in the following way: $\lambda \leq_B \mu$ if and only if $\lambda \geq \mu$.*

Proof. Denote by G the functor $\operatorname{Hom}_A(T, -)$. Since $T \in \mathscr{F}(\tilde{\Delta})$, by Proposition 1.5.117, $\operatorname{Hom}_A(T, \nabla(\lambda))$ has a filtration

$$0 = X_{n+1} \subset X_n \subset X_{n-1} \subset \cdots \subset X_2 \subset X_1 = \operatorname{Hom}_A(T, \nabla(\lambda)), \quad X_i/X_{i+1} \simeq \operatorname{Hom}_R(U_{n-i+1}, S_{n-i+1}) \in R\text{-proj}.$$

Therefore, $\Delta_B(\lambda) \in R$-proj.

Assume that $\operatorname{Hom}_B(\Delta_B(\lambda'), \Delta_B(\lambda'')) \neq 0$. Then, $0 \neq \operatorname{Hom}_B(G\nabla(\lambda'), G\nabla(\lambda'')) \simeq \operatorname{Hom}_A(\nabla(\lambda'), \nabla(\lambda''))$. By Proposition 1.5.97, $\lambda' \geq \lambda''$. Thus, $\lambda' \leq_R \lambda''$.

Assume $N \in B$-mod such that $\operatorname{Hom}_B(\Delta_B(\lambda), N) = 0$ for all $\lambda \in \Lambda$. Consider the exact sequence

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0. \tag{1.5.15.11}$$

Applying $\operatorname{Hom}_B(G-, N)$ (left exact functor on $\mathscr{F}(\tilde{\nabla})$) yields the exact sequence

$$0 \to \operatorname{Hom}_B(\Delta_B(\lambda), N) \to \operatorname{Hom}_B(GT(\lambda), N) \to \operatorname{Hom}_B(GY(\lambda), N). \tag{1.5.15.12}$$

Since $GY(\lambda) \in \mathscr{F}(\tilde{\Delta}_B)$ it holds $\operatorname{Hom}_B(GY(\lambda), N) = 0$. Hence, $\operatorname{Hom}_B(GT(\lambda), N) = 0$ for all $\lambda \in \Lambda$. Therefore,

$$0 = \operatorname{Hom}_B(G(\bigoplus_{\lambda\in\Lambda} T(\lambda)), N) = \operatorname{Hom}_B(\operatorname{Hom}_A(T,T), N) = \operatorname{Hom}_B(B, N) = N. \tag{1.5.15.13}$$

Since $Y(\lambda) \in \mathscr{F}(\tilde{\nabla}_{\mu<\lambda})$, it follows that $GY(\lambda) \in \mathscr{F}(\tilde{\Delta}_{B_{\mu<\lambda}}) = \mathscr{F}(\tilde{\Delta}_{B_{\mu>_B\lambda}})$. As $T(\lambda) \in \operatorname{add} T$, it follows that $GT(\lambda) \in B$-proj. So, the exact sequence

$$0 \to GY(\lambda) \to GT(\lambda) \to \Delta_B(\lambda) \to 0 \tag{1.5.15.14}$$

satisfies *iv*) of Definition 1.5.32. Since G is full and faithful on $\mathscr{F}(\tilde{\nabla})$, the following holds

$$\operatorname{End}_B(\Delta_B(\lambda)) \simeq \operatorname{End}_B(G\nabla(\lambda)) \simeq \operatorname{End}_A(\nabla(\lambda)) \simeq R. \tag{1.5.15.15}$$

Thus, $(B\text{-mod}, \{\Delta_B(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. □

Corollary 1.5.123. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra with T a characteristic tilting module. Then,* $\operatorname{add} T = \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$.

Proof. The inclusion $\operatorname{add} T \subset \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$ is clear. Fix $B = \operatorname{End}_A(T)^{op}$. Consider $X \in \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$. Then, $GX = \operatorname{Hom}_A(T, X) \in \mathscr{F}(\tilde{\Delta}_B)$. Since $\operatorname{Hom}_A(T, -)$ is an exact equivalence from $\mathscr{F}(\tilde{\nabla})$ onto $\mathscr{F}(\tilde{\Delta}_B)$ we obtain

$$\operatorname{Ext}^1_B(GX, \Delta_B(\lambda)) = \operatorname{Ext}^1_B(GX, G\nabla(\lambda)) \simeq \operatorname{Ext}^1_A(X, \nabla(\lambda)) = 0, \quad \forall \lambda \in \Lambda. \tag{1.5.15.16}$$

By Lemma 1.5.107 and Proposition 1.5.122, $GX \in B$-proj. By projectivization, there exists $T' \in \operatorname{add} T$ such that $GX = \operatorname{Hom}_A(T, X) \simeq \operatorname{Hom}_A(T, T') = GT'$. Since G is an equivalence it follows that $X \simeq T' \in \operatorname{add} T$. ☐

Theorem 1.5.124. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Assume there are modules $T(\lambda)$ and $Q(\lambda), \lambda \in \Lambda$ with exact sequences*

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0$$
$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0$$
$$0 \to \Delta(\lambda) \to Q(\lambda) \to X'(\lambda) \to 0$$
$$0 \to Y'(\lambda) \to Q(\lambda) \to \nabla(\lambda) \to 0$$

where $X(\lambda), X'(\lambda) \in \mathscr{F}(\tilde{\Delta}_{\mu<\lambda})$ and $Y(\lambda), Y'(\lambda) \in \mathscr{F}(\tilde{\nabla}_{\mu<\lambda})$. Let $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$, $Q = \bigoplus_{\lambda\in\Lambda} Q(\lambda)$, $B = \operatorname{End}_A(T)^{op}$, $C = \operatorname{End}_A(Q)^{op}$. Then, B and C are Morita equivalent as split quasi-hereditary algebras.

Proof. By Lemma 1.5.121, the functors $\operatorname{Hom}_A(T, -)\colon A\text{-Mod} \to B\text{-Mod}$ and $\operatorname{Hom}_A(Q, -)\colon A\text{-Mod} \to C\text{-Mod}$ restrict to exact equivalences $\mathscr{F}(\tilde{\nabla}_A) \to \mathscr{F}(\tilde{\Delta}_B)$ and $\mathscr{F}(\tilde{\nabla}_A) \to \mathscr{F}(\tilde{\Delta}_C)$, respectively. Moreover, by projectivization $\operatorname{Hom}_A(T, -)$ restricts to an exact equivalence $\operatorname{add} T \to B$-proj and $\operatorname{Hom}_A(Q, -)$ restricts to an exact equivalence $\operatorname{add} Q \to C$-proj. By Corollary 1.5.123,

$$\operatorname{add} T = \mathscr{F}(\tilde{\Delta}_A) \cap \mathscr{F}(\tilde{\nabla}_A) = \operatorname{add} Q. \tag{1.5.15.17}$$

So, B-proj $\simeq C$-proj. Therefore, B and C are Morita equivalent. More precisely, the adjoint is given by $T \otimes_B -\colon B\text{-proj}\colon \to \operatorname{add} T = \operatorname{add} Q$. So, the functor $\operatorname{Hom}_A(Q, -) \circ T \otimes_B -\colon B\text{-proj} \to C\text{-proj}$ is an equivalence of categories. Moreover, $\operatorname{Hom}_A(Q, T) \simeq \operatorname{Hom}_A(Q, T \otimes_B B)$ is a C-progenerator. Therefore, the functor $\operatorname{Hom}_C(\operatorname{Hom}_A(Q, T), -)\colon C\text{-mod} \to B\text{-mod}$ is an equivalence of categories.

Now notice that for any $\lambda \in \Lambda$,

$$\operatorname{Hom}_C(\operatorname{Hom}_A(Q, T), \Delta_C(\lambda)) = \operatorname{Hom}_C(\operatorname{Hom}_A(Q, T), \operatorname{Hom}_A(Q, \nabla(\lambda))) \simeq \operatorname{Hom}_A(T, \nabla(\lambda)) = \Delta_B(\lambda).$$

Therefore, applying $\phi = \operatorname{id}_\Lambda$ in Definition 1.5.66, the result follows. ☐

As a consequence of this theorem, the Ringel dual is well defined over commutative Noetherian rings. We will denote by $R(A)$ the **Ringel dual of** A. We will see afterwards that the Ringel dual of A relates with A in the same way as in the field case.

As in the classical case, the characteristic tilting module characterizes the standard and costandard modules.

Corollary 1.5.125. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then,*

(a) $\mathscr{F}(\tilde{\Delta}) = \{M \in A\text{-mod}\colon \operatorname{Ext}^{i>0}_A(M, T) = 0\}$;

(b) $\mathscr{F}(\tilde{\nabla}) = \{N \in A\text{-mod}\colon \operatorname{Ext}^{i>0}_A(T, N) = 0\}$.

Proof. Let $M \in \mathscr{F}(\tilde{\Delta})$. As $T \in \mathscr{F}(\tilde{\nabla})$ then $\operatorname{Ext}_A^{i>0}(M,T) = 0$ by Lemma 1.5.100.

Conversely, assume that $\operatorname{Ext}_A^{i>0}(M,T) = 0$. Then, $\prod_{\lambda \in \Lambda} \operatorname{Ext}_A^{i>0}(M,T(\lambda)) = 0$ and by consequence for each $\lambda \in \Lambda$, $\operatorname{Ext}_A^{i>0}(M,T(\lambda)) = 0$. We claim that $\operatorname{Ext}_A^{i>0}(M,\nabla(\lambda)) = 0$ for every $\lambda \in \Lambda$. If λ is minimal, then $T(\lambda) = \nabla(\lambda)$, so there is nothing to show. Assume that for all $\mu < \lambda$ $\operatorname{Ext}_A^i(M,\nabla(\mu)) = 0, i > 0$. Then, $\operatorname{Ext}_A^i(M,X) = 0$ for every $X \in \mathscr{F}(\tilde{\nabla}_{\mu<\lambda})$, $i > 0$. Consider the exact sequence

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0. \tag{1.5.15.18}$$

In particular, $\operatorname{Ext}_A^2(M,Y(\lambda)) = 0$ since $Y(\lambda) \in \mathscr{F}(\tilde{\nabla}_{\mu<\lambda})$. Thus, we deduce by applying $\operatorname{Hom}_A(M,-)$ to (1.5.15.18) that $\operatorname{Ext}_A^1(M,\nabla(\lambda)) \simeq \operatorname{Ext}_A^2(M,Y(\lambda)) = 0$. By induction, $\operatorname{Ext}_A^1(M,\nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$. By Proposition 1.5.104, $M \in \mathscr{F}(\tilde{\Delta})$.

By a symmetric argument we obtain statement b). ☐

We will see now that costandard modules and partial tilting modules behave well under ground ring change.

Proposition 1.5.126. *Let S be a commutative Noetherian R-algebra. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then, the following assertions hold.*

(a) $(S\otimes_R A, \{S\otimes_R \Delta(\lambda)_{\lambda\in\Lambda}\})$ has costandard modules $S\otimes_R \nabla(\lambda)\otimes_R U(\lambda)$ for some $U(\lambda) \in Pic(S)$. Moreover, if S is flat over R, then the costandard modules can be written in form $S\otimes_R \nabla(\lambda)$.

(b) Assume that S is flat over R or that S has a trivial Picard group then $S\otimes_R T(\lambda)$ is a partial tilting module (it satisfies (1.5.14.1) and (1.5.14.2)) for $S\otimes_R A$ and $S\otimes_R T$ is a characteristic tilting module.

Proof. By Proposition 1.5.90, $(A^{op}, \{D\nabla(\lambda)_{\lambda\in\Lambda}\})$ is a split highest weight category. By Proposition 1.5.55, $(S\otimes_R A^{op}, \{S\otimes_R D\nabla(\lambda)_{\lambda\in\Lambda}\})$ is split highest weight category. Now note that $(S\otimes_R A)^{op} = S\otimes_R A^{op}$, since S is a commutative ring. Moreover,

$$S\otimes_R D\nabla(\lambda) = S\otimes_R \operatorname{Hom}_R(\nabla(\lambda),R) \simeq \operatorname{Hom}_{S\otimes_R R}(S\otimes_R \nabla(\lambda), S\otimes_R R) = D_S(S\otimes_R \nabla(\lambda)). \tag{1.5.15.19}$$

So, $S\otimes_R \nabla(\lambda)\otimes_S U_\lambda$, $U_\lambda \in Pic(S)$, is a costandard module of $S\otimes_R A$ by Proposition 1.5.90. Now assume that S is a flat R-algebra. Then,

$$\operatorname{Ext}^j_{S\otimes_R A}(S\otimes_R \Delta(\lambda), S\otimes_R \nabla(\beta)) \simeq S\otimes_R \operatorname{Ext}_A^j(\Delta(\lambda),\nabla(\beta)) \simeq \begin{cases} S\otimes_R R & \text{if } \lambda = \beta,\ i=0 \\ 0 & \text{otherwise} \end{cases}. \tag{1.5.15.20}$$

By the uniqueness, $S\otimes_R \nabla(\lambda)$ are costandard modules of $S\otimes_R A$.

Assume that either S is an R-flat or S has trivial Picard group. Then, by (b) the costandard modules of $S\otimes_R A$ are of the form $S\otimes_R \nabla(\lambda)$. Since the exact sequences given by filtrations are all (A,R)-exact, the functor $S\otimes_R -$ is exact on the exact sequences of Proposition 1.5.109. Therefore, $S\otimes_R T$ is a characteristic tilting module for $S\otimes_R A$. ☐

Remark 1.5.127. In view of Remark 1.5.111, we cannot expect that the isomorphism $T(\lambda)(\mathfrak{m}) \simeq T_{(\mathfrak{m})}(\lambda)$ holds in this generality, where $T_{(\mathfrak{m})}(\lambda)$ is a partial tilting indecomposable module of $A(\mathfrak{m})$ for $\mathfrak{m}$ a maximal ideal of R. △

Proposition 1.5.128. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Let $R(A)$ be a Ringel dual of A. Then, $R(R(A))$ is Morita equivalent to A.*

Proof. Define $I = \bigoplus_{\lambda\in\Lambda} I(\lambda)$, where $I(\lambda)$ is the (A,R)-injective module given by Theorem 1.5.103. In particular, each $I(\lambda) \in \mathscr{F}(\tilde{\nabla})$. We will denote by G and B the functor $\operatorname{Hom}_A(T,-)$ and the Ringel dual $R(A)$, respectively. By Theorem 1.5.122, $GI \in \mathscr{F}(\tilde{\Delta}_B)$. By Lemma 1.5.121, $\operatorname{Ext}^1_B(G\nabla(\lambda), GN) \simeq \operatorname{Ext}^1_A(\nabla(\lambda), N)$ for every $N \in \mathscr{F}(\tilde{\nabla}_B)$, for every $\lambda \in \Lambda$. In particular, for $N = I$, and for every $\lambda \in \Lambda$,

$$\operatorname{Ext}^1_B(\Delta_B(\lambda), GI) \simeq \operatorname{Ext}^1_A(\nabla(\lambda), I) \simeq \operatorname{Ext}^1_{(A,R)}(\nabla(\lambda), I) = 0. \tag{1.5.15.21}$$

By Theorem 1.5.104, $GI \in \mathscr{F}(\tilde{\nabla}_B)$. Hence, GI is a partial tilting module. Applying G to the exact sequence

$$0 \to \nabla(\lambda) \to I(\lambda) \to C(\lambda) \to 0 \tag{1.5.15.22}$$

we obtain the exact sequence $0 \to \Delta_B(\lambda) \to GI(\lambda) \to GC(\lambda) \to 0$ with $GC(\lambda) \in \mathscr{F}(\tilde{\nabla}_{B\mu>\lambda}) = \mathscr{F}(\tilde{\nabla}_{B\mu<_B\lambda})$. Therefore, GI is a characteristic tilting module.

$$R(B) = \operatorname{End}_B(GI)^{op} \simeq \operatorname{End}_A(I)^{op} \simeq \operatorname{End}_A(DI) \overset{\text{Mor}}{\sim} \operatorname{End}_A(A) \simeq A. \tag{1.5.15.23}$$

The second identification is due to G being full and faithful on $\mathscr{F}(\tilde{\nabla}_A)$ whereas the fourth identification is due to DI being a right A-progenerator. In particular, $\operatorname{add} DI = \operatorname{add} A_A$. ☐

Note that $R(R(A))$ is isomorphic to $\operatorname{End}_A(DI) \simeq \operatorname{End}_A(P_{op})$ as R-algebras, where P_{op} is the progenerator $\bigoplus_{\lambda\in\Lambda} P_{op}(\lambda)$ making $(A^{op}, D\nabla(\lambda))$ a split quasi-hereditary algebra. So, the equivalence of categories is given by the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(P_{op},A),-)\colon A\text{-mod} \to R(R(A))\text{-mod}$. Denote this functor by H. A natural question that arises is whether this equivalence of categories is also an equivalence as split highest weight categories.

Of course, this is true for split quasi-hereditary algebras over fields. Those can be studied in terms of its simple modules and Λ indexes the set of non-isomorphic classes of simple A-modules (see Proposition 1.5.39). Assume that R is a field. Then, $S_A(\lambda)$ is the top of the projective indecomposable $P_A(\lambda)$ and the socle of the injective indecomposable module $I_A(\lambda)$. By Lemma 1.5.38 and since G is full and faithful

$$S_{R(R(A))}(\lambda) = \operatorname{top}\operatorname{Hom}_{R(A)}(GI, GI(\lambda)) \simeq \operatorname{top}\operatorname{Hom}_A(I, I(\lambda)) \simeq \operatorname{top}\operatorname{Hom}_A(I, D\operatorname{Hom}_A(P_A(\lambda), A)) \tag{1.5.15.24}$$

$$\simeq \operatorname{top}\operatorname{Hom}_A(\operatorname{Hom}_A(P_A(\lambda), A), P_{op}) \simeq \operatorname{top} HP_A(\lambda) = HS_A(\lambda). \tag{1.5.15.25}$$

In particular, H sends $P_A(\lambda)$ to $P_{R(R(A))}(\lambda)$ and $\Delta_A(\lambda)$ to $\Delta_{R(R(A))}(\lambda)$. Therefore, $R(R(A))$ and A are Morita equivalent as split quasi-hereditary algebras over fields. The general case requires a bit more work. The difficulty lies in the fact that we do not know, in general, if $D\operatorname{Hom}_A(P(\lambda), A) \simeq I(\lambda)$ holds nor if the projectives $P(\lambda)$ become indecomposable objects under the functors $R(\mathfrak{m}) \otimes_R -$ for $\mathfrak{m}$ a maximal ideal of R.

Proposition 1.5.129. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a commutative Noetherian ring. Let $R(A)$ be a Ringel dual of A. Then, $R(R(A))$ is Morita equivalent to A as split quasi-hereditary algebras.*

Proof. According to Proposition 1.5.128, it is enough to prove that $\operatorname{Hom}_A(\operatorname{Hom}_A(P_{op}, A), \Delta(\lambda)) \simeq \Delta_{R(R(A))}(\lambda)$, for every $\lambda \in \Lambda$. To do that, we will use induction on $|\Lambda|$. Denote by H the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(P_{op}, A), -)$. By Remark 1.5.98, if $\lambda \in \Lambda$ is maximal, then $D\operatorname{Hom}_A(\Delta(\lambda), A) \simeq I(\lambda)$. Thus,

$$H\Delta(\lambda) \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(\Delta(\lambda), A), P_{op}) \simeq \operatorname{Hom}_A(I, D\operatorname{Hom}_A(\Delta(\lambda), A)) \simeq \operatorname{Hom}_A(I, I(\lambda)) \tag{1.5.15.26}$$

$$\simeq \operatorname{Hom}_{R(A)}(GI, GI(\lambda)) \simeq \Delta_{R(R(A))}(\lambda). \tag{1.5.15.27}$$

Assume that $|\Lambda| > 1$. Let J be the split heredity ideal associated with $\Delta(\lambda)$. Denote by H_J the functor

$$\operatorname{Hom}_A(\operatorname{Hom}_A(\bigoplus_{\mu\in\Lambda\setminus\{\lambda\}} P(\mu)/JP(\mu), A/J), -).$$

By induction, $H_J\Delta(\mu) \simeq \Delta_{R(R(A/J))}(\mu) = \Delta_{R(R(A))}(\mu)$ for every $\mu \neq \lambda$, $\mu \in \Lambda$. Hence, it is enough to check that $H_JX \simeq HX$ for all $X \in A/J$-mod. Since $J = J^2$ $\operatorname{Hom}_A(P(\mu)/JP(\mu), A/J) \simeq \operatorname{Hom}_A(P(\mu), A/J)$ for every $\mu \in \Lambda$ and $\operatorname{Hom}_A(P(\lambda), A/J) = 0$ by Corollary 1.5.23. Therefore,

$$\operatorname{Hom}_A(\bigoplus_{\mu\in\Lambda\setminus\{\lambda\}} P(\mu)/JP(\mu), A/J) \simeq \operatorname{Hom}_A(P_{op}, A/J).$$

Moreover, $\operatorname{Hom}_A(\operatorname{Hom}_A(P_{op}, J), X) = 0$ for all $X \in A/J$-mod. Thus, $HX \simeq H_JX$ for every $X \in A/J$-mod. Hence, H sends $\Delta(\mu)$ to $\Delta_{R(R(A))}(\mu)$ for all $\mu \in \Lambda$. $\square$

Corollary 1.5.130. *Let* $(A\text{-mod}, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *and* $(B\text{-mod}, \{\Omega(\chi)_{\chi\in X}\})$ *be two split highest weight categories. B is a Ringel dual of A if and only if there is an exact equivalence between the categories* $\mathscr{F}(\tilde{\Delta})$ *and* $\mathscr{F}(\tilde{\mho}_B)$*, where* $\mho$ *denotes the set of costandard modules of B.*

Proof. Let $B = R(A)$ be a Ringel dual of A. By Lemma 1.5.121, there is an exact equivalence $\mathscr{F}(\tilde{\nabla}_{R(A)}) \simeq \mathscr{F}(\tilde{\Delta}_{R(R(A))})$. By Proposition 1.5.129, there is an exact equivalence $\mathscr{F}(\tilde{\Delta}_{R(R(A))}) \simeq \mathscr{F}(\tilde{\Delta}_A)$.

Conversely, assume that there is exact equivalence between the categories $\mathscr{F}(\tilde{\Delta}_A)$ and $\mathscr{F}(\tilde{\mho}_B)$. By Lemma 1.5.121, there is an exact equivalence $\mathscr{F}(\tilde{\Delta}_A) \simeq \mathscr{F}(\tilde{\mho}_B) \simeq \mathscr{F}(\tilde{\Omega}_{R(B)})$. In view of Proposition 1.5.80, $R(B)$ and A are equivalent as split highest weight categories. By Proposition 1.5.129, we conclude that B and $R(A)$ are equivalent as split highest weight categories. $\square$

1.5.16 Additional structure on the resolving subcategory $\mathscr{F}(\tilde{\Delta})$ and its dual

Proposition 1.5.131. *[Rou08, Proposition 4.30] Let* $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split quasi-hereditary algebra. Let* $M \in A\text{-mod}$*. Then, the following assertions hold.*

(a) $M \in \mathscr{F}(\tilde{\Delta})$ *if and only if* $M(\mathfrak{m}) \in \mathscr{F}(\Delta(\mathfrak{m}))$ *for all maximal ideals* $\mathfrak{m}$ *of R and M is projective over R.*

(b) $M \in \mathscr{F}(\tilde{\nabla})$ *if and only if* $M(\mathfrak{m}) \in \mathscr{F}(\nabla(\mathfrak{m}))$ *for all maximal ideals* $\mathfrak{m}$ *of R and M is projective over R.*

(c) Let T be a characteristic tilting module. $M \in \operatorname{add} T$ *if and only if* $M(\mathfrak{m}) \in \operatorname{add} T(\mathfrak{m})$ *for all maximal ideals* $\mathfrak{m}$ *of R and M is projective R.*

Proof. Assume that $M \in \mathscr{F}(\tilde{\Delta})$. There is a filtration

$$0 = M_{n+1} \subset M_n \subset \cdots \subset M_1 = M, \quad M_i/M_{i+1} \simeq \Delta_i \otimes_R U_i. \tag{1.5.16.1}$$

All these modules are projective over R, so

$$0 = M_{n+1}(\mathfrak{m}) \subset M_n(\mathfrak{m}) \subset \cdots \subset M_1(\mathfrak{m}) = M(\mathfrak{m}) \tag{1.5.16.2}$$

is a filtration in $\mathscr{F}(\Delta(\mathfrak{m}))$. Hence, $M(\mathfrak{m}) \in \mathscr{F}(\Delta(\mathfrak{m}))$.

Reciprocally, let $M \in A\text{-mod} \cap R\text{-proj}$ such that $M(\mathfrak{m}) \in \mathscr{F}(\Delta(\mathfrak{m}))$ for every maximal $\mathfrak{m}$ in R. We have that $\operatorname{pdim}_A M$ is finite. We shall proceed by induction on $\operatorname{pdim}_A M$.

Assume $\operatorname{pdim}_A M = 0$. Then, M is projective over A, so there is nothing to show. Assume $\operatorname{pdim}_A M > 0$. Consider the projective presentation

$$0 \to L \to P \to M \to 0. \tag{1.5.16.3}$$

Then, $\operatorname{pdim}_A L \leq \operatorname{pdim}_A M - 1$. Let $\mathfrak{m}$ be a maximal ideal in R. Applying $R(\mathfrak{m}) \otimes_R -$, we obtain the exact sequence

$$0 = \operatorname{Tor}_1^R(R(\mathfrak{m}), M) \to L(\mathfrak{m}) \to P(\mathfrak{m}) \to M(\mathfrak{m}) \to 0. \tag{1.5.16.4}$$

By hypothesis, $\operatorname{Ext}^i_{A(\mathfrak{m})}(M(\mathfrak{m}), \nabla(\lambda)(\mathfrak{m})) = 0$ for $i > 0$ and $\lambda \in \Lambda$. Thus, $\operatorname{Ext}^i_{A(\mathfrak{m})}(L(\mathfrak{m}), \nabla(\lambda)(\mathfrak{m})) = 0$ and hence $L(\mathfrak{m}) \in \mathscr{F}(\Delta(\mathfrak{m}))$. By induction, $L \in \mathscr{F}(\tilde{\Delta})$. Let $N \in \mathscr{F}(\tilde{\nabla})$. Applying the functors $R(\mathfrak{m}) \otimes_R \operatorname{Hom}_A(-, N)$ and $\operatorname{Hom}_{A(\mathfrak{m})}(-, N(\mathfrak{m}))$ we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Hom}_A(P,N)(\mathfrak{m}) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Hom}_A(L,N)(\mathfrak{m}) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Ext}^1_A(M,N)(\mathfrak{m}) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_{A(\mathfrak{m})}(P(\mathfrak{m}), N(\mathfrak{m})) \arrow[r] & \operatorname{Hom}_{A(\mathfrak{m})}(L(\mathfrak{m}), N(\mathfrak{m})) \arrow[r] & \operatorname{Ext}^1_{A(\mathfrak{m})}(M(\mathfrak{m}), N(\mathfrak{m})) \arrow[r] & 0
\end{tikzcd}.$$

By Proposition 1.5.118 the two columns on the left are isomorphic maps. By diagram chasing, it follows that $\operatorname{Ext}^1_A(M,N)(\mathfrak{m}) = \operatorname{Ext}_{A(\mathfrak{m})}(M(\mathfrak{m}), N(\mathfrak{m})) = 0$, since $M(\mathfrak{m}) \in \mathscr{F}(\Delta(\mathfrak{m}))$ and $N(\mathfrak{m}) \in \mathscr{F}(\nabla(\mathfrak{m}))$. As $\mathfrak{m}$ is an arbitrary maximal ideal in R we deduce $\operatorname{Ext}^1_A(M,N) = 0$. By Proposition 1.5.104, $M \in \mathscr{F}(\tilde{\Delta})$. Hence, *a*) follows.

Let $N \in \mathscr{F}(\tilde{\nabla})$. Then, $DN \in \mathscr{F}(\tilde{\Delta}_{A^{op}})$ and $N \in R$-proj. By *a*), $DN(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(N(\mathfrak{m}), R(\mathfrak{m})) \in \mathscr{F}(\Delta_{A^{op}}(\mathfrak{m}))$. Thus, $N(\mathfrak{m}) \in \mathscr{F}(\nabla(\mathfrak{m}))$. Conversely, assume that $N \in R$-proj and $N(\mathfrak{m}) \in \mathscr{F}(\nabla(\mathfrak{m}))$ for every maximal ideal $\mathfrak{m}$ in R. Then, $DN(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(N(\mathfrak{m}), R(\mathfrak{m})) \in \mathscr{F}(\Delta_{A^{op}}(\mathfrak{m}))$. By *a*) $DN \in \mathscr{F}(\tilde{\Delta}_{A^{op}})$ hence $N \in \mathscr{F}(\tilde{\nabla})$. As a consequence, *b*) follows.

Applying *a*) and *b*) to Corollary 1.5.123, *c*) follows. ☐

Proposition 1.5.132. *Let* $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split quasi-hereditary algebra. Let* $M \in A$-mod*. Then, the following assertions hold.*

1. *If* $M \in \mathscr{F}(\tilde{\Delta})$ *and* $M(\mathfrak{m}) \simeq \Delta(\lambda)(\mathfrak{m})$ *for some* $\lambda \in \Lambda$ *for every maximal ideal* $\mathfrak{m}$ *of* R*, then* $M \simeq \Delta(\lambda) \otimes F$ *for some* $F \in Pic(R)$.
2. *If* $M \in \mathscr{F}(\tilde{\nabla})$ *and* $M(m) \simeq \nabla(\lambda)(\mathfrak{m})$ *for some* $\lambda \in \Lambda$ *for every maximal ideal* $\mathfrak{m}$ *of* R*, then* $M \simeq \nabla(\lambda) \otimes F$ *for some* $F \in Pic(R)$.

Proof. Since $M \in \mathscr{F}(\tilde{\Delta})$ there is a filtration

$$0 = M_{n+1} \subset M_n \subset \cdots \subset M_1 = M, \quad M_i/M_{i+1} \simeq \Delta_i \otimes_R U_i,\ U_i \in R\text{-proj}. \tag{1.5.16.5}$$

By Proposition 1.5.117,

$$\operatorname{Hom}_A(M, \nabla_i) \simeq \operatorname{Hom}_A(M/M_{i+1}, \nabla_i) \simeq \operatorname{Hom}_R(U_i, R) = DU_i. \tag{1.5.16.6}$$

Let $\lambda \in \Lambda$ be the weight that corresponds to i. Thus, for $\mu \neq \lambda$,

$$\operatorname{Hom}_A(M, \nabla_\mu)(\mathfrak{m}) \simeq \operatorname{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}), \nabla_\mu(\mathfrak{m})) \simeq \operatorname{Hom}_{A(\mathfrak{m})}(\Delta(\lambda)(\mathfrak{m}), \nabla(\mu)(\mathfrak{m})) = 0, \tag{1.5.16.7}$$

for every maximal ideal $\mathfrak{m}$ in R. So, $DU(\mu) \simeq \operatorname{Hom}_A(M, \nabla(\mu)) = 0$. Thus, $U(\mu) = 0$, since $U(\mu) \in R\text{-proj}$. Thus, $M \simeq \Delta(\lambda) \otimes_R U(\lambda)$. We have

$$\operatorname{Hom}_A(M, \nabla(\lambda))(\mathfrak{m}) \simeq \operatorname{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}), \nabla(\lambda)(\mathfrak{m})) \simeq \operatorname{Hom}_{A(\mathfrak{m})}(\Delta(\lambda)(\mathfrak{m}), \nabla(\lambda)(\mathfrak{m})) \simeq R(\mathfrak{m}). \tag{1.5.16.8}$$

On the other hand,

$$\begin{aligned}\operatorname{Hom}_A(M, \nabla(\lambda)) &\simeq \operatorname{Hom}_A(\Delta(\lambda) \otimes_R U(\lambda), \nabla(\lambda)) & (1.5.16.9)\\ &\simeq \operatorname{Hom}_R(U(\lambda), \operatorname{Hom}_A(\Delta(\lambda), \nabla(\lambda))) \simeq DU(\lambda) \in R\text{-proj}. & (1.5.16.10)\end{aligned}$$

Thus, $DU(\lambda)_\mathfrak{m} \simeq R_\mathfrak{m}^{n_\mathfrak{m}}$ for some $n_\mathfrak{m} \geq 0$. We finally deduce that

$$R(\mathfrak{m}) \simeq DU(\lambda)(\mathfrak{m}) \simeq DU(\lambda)_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} \simeq R_\mathfrak{m}^{n_\mathfrak{m}} \otimes_{R_\mathfrak{m}} R(\mathfrak{m}) \simeq R(\mathfrak{m})^{n_\mathfrak{m}} \implies n_\mathfrak{m} = 1, \tag{1.5.16.11}$$

for every maximal ideal $\mathfrak{m}$ in R. Thus, $DU(\lambda) \in Pic(R)$. We conclude that $U(\lambda) \in Pic(R)$. ☐

As a consequence of Proposition 1.5.131, we can provide an alternative proof for Corollary 1.5.119. More precisely, this new approach will give us a stronger result than Corollary 1.5.119.

Proposition 1.5.133. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Suppose that $M \in \mathscr{F}(\tilde{\Delta})$ and $N \in \mathscr{F}(\tilde{\nabla})$. Denote by D the standard duality. Then, the following assertions hold.*

(a) The functor $- \otimes_A M \colon \mathscr{F}(D\tilde{\nabla}) \to R\text{-proj}$ is a well-defined exact functor.

(b) The functor $DN \otimes_A - \colon \mathscr{F}(\tilde{\Delta}) \to R\text{-proj}$ is a well-defined exact functor.

Proof. It is enough to show that $DN \otimes_A M \in R\text{-proj}$ and $\operatorname{Tor}_{i>0}^A(DN, M) = 0$. For each maximal ideal $\mathfrak{m}$ in R, denote by $D_{(\mathfrak{m})}$ the standard duality $\operatorname{Hom}_{R(\mathfrak{m})}(-, R(\mathfrak{m}))$. In particular, $D_{(\mathfrak{m})}$ is an exact functor. Let $M^\bullet$ be a deleted projective (left) A-resolution of M. Since $M \in R\text{-proj}$, $M^\bullet(\mathfrak{m})$ is a deleted projective $A(\mathfrak{m})$-resolution of $M(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Further, each module in the complex $DN \otimes_A M^\bullet$ belong to $\operatorname{add}_R DN$. So, the complex $DN \otimes_A M^\bullet$ is a flat chain complex. Using this flat chain complex and the residue field $R(\mathfrak{m})$ on Lemma 1.3.17 we obtain the Künneth spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^R(\operatorname{Tor}_q^A(DN, M), R(\mathfrak{m})) \implies H_{p+q}(DN \otimes_A M^\bullet \otimes_R R(\mathfrak{m})) = \operatorname{Tor}_{p+q}^{A(\mathfrak{m})}(DN(\mathfrak{m}), M(\mathfrak{m})). \tag{1.5.16.12}$$

Observe that

$$\begin{aligned}\operatorname{Tor}_{i>0}^{A(\mathfrak{m})}(DN(\mathfrak{m}), M(\mathfrak{m})) &= \operatorname{Tor}_{i>0}^{A(\mathfrak{m})}(D_{(\mathfrak{m})}N(\mathfrak{m}), M(\mathfrak{m})) = H_{i>0}(D_{(\mathfrak{m})}N(\mathfrak{m}) \otimes_{A(\mathfrak{m})} M^\bullet(\mathfrak{m})) & (1.5.16.13)\\ &\simeq H_{i>0}(D_{(\mathfrak{m})} \operatorname{Hom}_{A(\mathfrak{m})}(M^\bullet(\mathfrak{m}), N(\mathfrak{m}))) \simeq D_{(\mathfrak{m})} H^{i>0}(\operatorname{Hom}_{A(\mathfrak{m})}(M^\bullet(\mathfrak{m}), N(\mathfrak{m}))) & \\ &\simeq D_{(\mathfrak{m})} \operatorname{Ext}_{A(\mathfrak{m})}^{i>0}(M(\mathfrak{m}), N(\mathfrak{m})) = 0. & (1.5.16.14)\end{aligned}$$

The last equality follows from Proposition 1.5.131 and Lemma 1.5.100.

By Lemma 1.3.7, for each maximal ideal $\mathfrak{m}$ in R, we obtain that

$$0 = E_{1,0}^2 = \operatorname{Tor}_1^R(DN \otimes_A M, R(\mathfrak{m})). \tag{1.5.16.15}$$

Therefore, $DN \otimes_A M \in R\text{-proj}$. Moreover, $E_{i,0}^2 = 0$ for all $i > 0$. Again, by Lemma 1.3.7, it follows that

$$\operatorname{Tor}_1^A(DN, M)(\mathfrak{m}) = E_{0,1}^2 \simeq E_{2,0}^2 = 0. \tag{1.5.16.16}$$

Thus, $\operatorname{Tor}_1^A(DN,M)=0$ and consequently $E^2_{i,1}=0$ for all $i\geq 0$. We can proceed by induction on q to show that $E^2_{i,j}=0$ for all $i\geq 0$, $1\leq j\leq q$. In fact, assume that $E^2_{i,j}=0$ for all $i\geq 0$, $1\leq j\leq q$ for a given q. By Lemma 1.3.11, there exists an exact sequence

$$0=E^2_{q+2,0}\to E^2_{0,q+1}\to H_{q+1}=0. \tag{1.5.16.17}$$

So, $\operatorname{Tor}_{q+1}(DN,M)(\mathfrak{m})=0$. Hence, $\operatorname{Tor}_{q+1}(DN,M)=0$. Therefore, $E^2_{i,q+1}=0$ for all $i\geq 0$. We showed that $E^2_{i,j}=0$ for all $i\geq 0$ and $j\geq 1$. This means that $\operatorname{Tor}_{q>0}(DN,M)=0$. ☐

Using the previous technical results we can give a criterion to deduce Ringel self-duality for split quasi-hereditary algebras over local commutative Noetherian rings.

Lemma 1.5.134. *Let R be a local commutative Noetherian ring. Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra. Then, A is Morita equivalent to its Ringel dual as split quasi-hereditary algebra if and only if $A(\mathfrak{m})$ is Morita equivalent to its Ringel dual as split quasi-hereditary algebra, where $\mathfrak{m}$ is the unique maximal ideal of R.*

Proof. Assume that A is Morita equivalent to its Ringel dual as split quasi-hereditary algebras. That is, there exists a progenerator P of A-mod so that the Ringel dual of A, which we will denote by $R(A)$, is the endomorphism algebra $\operatorname{End}_A(P)^{op}$ and $\operatorname{Hom}_A(P,-)\colon A\text{-mod}\to R(A)\text{-mod}$ satisfies the conditions on Definition 1.5.66, where $R(A)$ takes the place of B. Hence, $P(\mathfrak{m})$ is a progenerator of $A(\mathfrak{m})$ and

$$\operatorname{End}_{A(\mathfrak{m})}(P(\mathfrak{m}))^{op}\simeq \operatorname{End}_A(P)^{op}\otimes_R R(\mathfrak{m})\simeq \operatorname{End}_A(T)^{op}\otimes_R R(\mathfrak{m})\simeq \operatorname{End}_{A(\mathfrak{m})}(T(\mathfrak{m}))^{op}. \tag{1.5.16.18}$$

Moreover, there exists a bijection $\phi\colon \Lambda\to\Lambda^{op}$ such that,

$$\operatorname{Hom}_{A(m)}(P(\mathfrak{m}),\Delta(\lambda)(\mathfrak{m}))\simeq \Delta_{R(A)}(\phi(\lambda))\otimes_R U_\lambda(\mathfrak{m})\simeq \Delta_{R(A)}(\phi(\lambda))(\mathfrak{m}).$$

Here, Λ^{op} is the poset Λ with the reversed order. Hence, $A(\mathfrak{m})$ is Ringel self-dual.

Conversely, assume that $A(\mathfrak{m})$ is Morita equivalent to its Ringel dual as split quasi-hereditary algebras. Since A is semi-perfect we can assume that the projective modules $P(\lambda)$ are the projective covers of $\Delta(\lambda)$. Hence, if $P_{(\mathfrak{m})}$ is the progenerator giving the Morita equivalence between A and its Ringel dual, we can choose $P\in A$-mod so that $P(\mathfrak{m})\simeq P_{(\mathfrak{m})}$. In particular, P is a progenerator of A and for every $\lambda\in\Lambda$,

$$\operatorname{Hom}_A(P,\Delta(\lambda))(\mathfrak{m})\simeq \operatorname{Hom}_{A(\mathfrak{m})}(P_{(\mathfrak{m})},\Delta(\lambda)(\mathfrak{m}))\simeq \Delta_{R(A)}(\phi(\lambda))(\mathfrak{m}). \tag{1.5.16.19}$$

By Proposition 1.5.131, $\operatorname{Hom}_A(P,\Delta(\lambda))\in\mathscr{F}(\tilde{\Delta}_{R(A)})$. By Proposition 1.5.132, $\operatorname{Hom}_A(P,\Delta(\lambda))\simeq\Delta_{R(A)}(\phi(\lambda))$ since the Picard group of R is trivial. Analogously, the adjoint functor of $\operatorname{Hom}_A(P,-)$ also sends $\Delta_{R(A)}(\phi(\lambda))$ to $\Delta(\lambda)$. Therefore, there exists an exact equivalence between $\mathscr{F}(\tilde{\Delta})$ and $\mathscr{F}(\tilde{\Delta}_{R(A)})$. The result now follows from Corollary 1.5.130. ☐

We will now see when $\mathscr{F}(\tilde{\Delta})$ is closed under (A,R)-monomorphisms. For this, we require a notion of relative torsionless. We call a module X **strongly** (A,R)**-torsionless** if there is an (A,R)-monomorphism $X\hookrightarrow P$ with $P\in A$-proj.

Proposition 1.5.135. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then, the following assertions are equivalent.*

(i) The relative injective dimension of any costandard module is at most one, that is, $\operatorname{idim}_{(A,R)}\nabla(\lambda)\leq 1$ *for any $\lambda\in\Lambda$.*

(ii) The relative injective dimension of the characteristic tilting module is at most one, that is, $\operatorname{idim}_{(A,R)} T \leq 1$.

(iii) The subcategory $\mathscr{F}(\tilde{\Delta})$ *is closed under* (A,R)*-monomorphism, that is, if there is an* (A,R)*-monomorphism* $X \hookrightarrow M$ *with* $M \in \mathscr{F}(\tilde{\Delta})$*, then* $X \in \mathscr{F}(\tilde{\Delta})$.

(iv) All strongly (A,R)*-torsionless modules belong to* $\mathscr{F}(\tilde{\Delta})$.

Proof. $i \implies ii$). Let $N \in A\text{-mod}\cap R\text{-proj}$. Consider the exact sequence

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0. \tag{1.5.16.20}$$

This exact sequence is (A,R)-exact since $\nabla(\lambda) \in R$-proj. Every exact sequence of the filtration of $Y(\lambda)$ is also (A,R)-exact. Thus, $\operatorname{idim}_{(A,R)} Y(\lambda) \leq 1$. In particular, $\operatorname{Ext}^2_{(A,R)}(N, Y(\lambda)) = 0$. Applying the functor $\operatorname{Hom}_A(N,-)$ we deduce $\operatorname{Ext}^2_{(A,R)}(N, T(\lambda)) = 0$. Thus, $\operatorname{Ext}^2_{(A,R)}(N,T) = 0$. By Corollary 1.2.45, $\operatorname{idim}_{(A,R)} T \leq 1$.

$ii) \implies iii$). Let $M \in \mathscr{F}(\tilde{\Delta})$. Let $N \hookrightarrow M$ be an (A,R)-monomorphism. Then, the exact sequence

$$0 \to N \to M \to M/N \to 0 \tag{1.5.16.21}$$

is (A,R)-exact. Since $M \in R$-proj, $M/N \in R$-proj. Applying $\operatorname{Hom}_A(-,T)$ yields

$$\operatorname{Ext}^i_{(A,R)}(M,T) \to \operatorname{Ext}^i_{(A,R)}(N,T) \to \operatorname{Ext}^{i+1}_{(A,R)}(M/N,T) \tag{1.5.16.22}$$

for every $i > 0$. As $\operatorname{idim}_{(A,R)} T \leq 1$, then $\operatorname{Ext}^{i+1}_{(A,R)}(M/N,T) = 0$ for all $i > 0$. As $M \in \mathscr{F}(\tilde{\Delta})$, $\operatorname{Ext}^i_{(A,R)}(M,T) = 0$, for all $i > 0$. Thus, $\operatorname{Ext}^i_{(A,R)}(N,T) = 0$ for all $i > 0$. By Corollary 1.5.125, $N \in \mathscr{F}(\tilde{\Delta})$.

$iii) \implies iv$). All projective A-modules belong to $\mathscr{F}(\tilde{\Delta})$. As an (A,R)-strongly torsionless module is an R-summand of a projective A-module, then by iii) every (A,R)-strongly torsionless module belongs to $\mathscr{F}(\tilde{\Delta})$.

$iv) \implies i$). Let $Y \in A\text{-mod}\cap R\text{-proj}$. Consider a projective presentation over A for Y

$$0 \to \ker \pi \to P \xrightarrow{\pi} Y \to 0. \tag{1.5.16.23}$$

As $Y \in R$-proj, this exact sequence is (A,R)-exact. As $\ker\pi$ is strongly (A,R)-torsionless module $\ker\pi \in \mathscr{F}(\tilde{\Delta})$. For each $\lambda \in \Lambda$, applying $\operatorname{Hom}_A(-,\nabla(\lambda))$ to (1.5.16.23) we obtain

$$0 = \operatorname{Ext}^1_{(A,R)}(\ker\pi, \nabla(\lambda)) \to \operatorname{Ext}^2_{(A,R)}(Y, \nabla(\lambda)) \to \operatorname{Ext}^2_{(A,R)}(P, \nabla(\lambda)) = 0 \tag{1.5.16.24}$$

Thus, $\operatorname{Ext}^2_{(A,R)}(Y, \nabla(\lambda)) = 0$. So, i) follows. ☐

Afterwards in Proposition 2.8.2, we will see characterized in terms of relative dominant dimension when the strongly (A,R)-torsionless modules are exactly the modules with a Δ-filtration.

1.6 Cellular algebras

Cellular algebras B are certain algebras characterized by the existence of an involution i with $i^2 = \mathrm{id}_B$ and a certain chain of ideals that provide a filtration of the regular module B. They were introduced by Graham and Lehrer [GL96], to solve such problems as how to obtain the number of non-isomorphic classes of simple modules of Hecke algebras and algebras used in knot theory. In a cellular algebra framework, these problems are reduced to problems in linear algebra. A classical example of a cellular algebra is the group algebra of the symmetric group.

Graham and Lehrer introduced the definition of cellular algebras over commutative rings. However, in applications cellular algebras are considered over a field. Some of the properties we are interested in can be found in [GL96], [KX98], [KX99a], [KX99b], [KX00]. Our aim here is to show that some properties of finite-dimensional cellular algebras remain valid for cellular algebras over commutative Noetherian rings.

Explicitly, the common definition of cellular algebras used for practical purposes is the following:

Definition 1.6.1. Let R be a commutative Noetherian ring. Let A be a free Noetherian R-algebra, that is, A is free as R-module. A is called **cellular** with cell datum (Λ, M, C, ι) if the following holds:

(C1) The finite set Λ is partially ordered. Associated with each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$. The algebra A has an R-basis

$$\{C^{\lambda}_{S,T} \mid S,T \in M(\lambda),\ \lambda \in \Lambda\}. \tag{1.6.0.1}$$

(C2) The map $\iota\colon A \to A$ is an R-linear anti-isomorphism with $i^2 = \mathrm{id}_A$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$, $S,T \in M(\lambda)$, $\lambda \in \Lambda$.

(C3) For each $\lambda \in \Lambda$ and $S,T \in M(\lambda)$ and each $a \in A$ we can write

$$aC^{\lambda}_{S,T} = \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T} + r', \tag{1.6.0.2}$$

where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

Lemma 1.6.2. *Consider the following condition.*

(C3') For each $\lambda \in \Lambda$ and $S,T \in M(\lambda)$ and each $a \in A$ we can write

$$C^{\lambda}_{S,T}a = \sum_{U \in M(\lambda)} r_a(U,T) C^{\lambda}_{S,U} + r', \tag{1.6.0.3}$$

where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,T) \in R$ do not depend on S.

Under conditions (C2) and (C1), condition (C3) is equivalent to (C3').

Proof. Assume that (C3) holds. We can write, for $a = i(x) \in A$, $x \in A$,

$$C^{\lambda}_{S,T}a \overset{(C2)}{=} \iota(C^{\lambda}_{T,S})\iota(x) = \iota(xC^{\lambda}_{T,S}) \overset{(C3)}{=} \iota\left(\sum_{U \in M(\lambda)} r_x(U,T) C^{\lambda}_{U,S} + r'\right) = \sum_{U \in M(\lambda)} r_x(U,T)\iota(C^{\lambda}_{U,S}) + \iota(r') \tag{1.6.0.4}$$

$$= \sum_{U \in M(\lambda)} r_x(U,T) C^{\lambda}_{S,U} + \iota(r'). \tag{1.6.0.5}$$

Since ι only changes the lower indexes, and therefore the upper indexes of the basis elements in the linear combination of $\iota(r')$ are strictly smaller than λ. Putting $r_a(U,T)$ equal to $r_{\iota(a)}(U,T)$, condition (C3') follows. The converse implication is analogous. ☐

The map $\iota\colon A \to A$ is called an **involution** of A.

Corollary 1.6.3. *A is a cellular R-algebra with cell datum (Λ, M, C, ι) if and only if the opposite algebra A^{op} is a cellular R-algebra with cell datum $(\Lambda, M, \iota(C), \iota)$.*

Iwahori-Hecke algebras are a classical example of cellular algebras (see [GL96, Example 1.2], see also Section 4.1). The cell basis is the Kazhdan-Lusztig basis. In fact, the axioms of cellular basis presented in Definition 1.6.1 are based on the Kazhdan-Lusztig basis of Hecke algebras.

There is a more abstract definition of cellular algebras due to Koenig and Xi [KX98] which illustrates better its structural properties.

Definition 1.6.4. Let R be a commutative Noetherian ring and let A be a projective Noetherian R-algebra. Assume that there is an R-linear anti-isomorphism ι on A with $\iota^2 = \mathrm{id}_A$.

- A two-sided ideal J of A is called a **cell ideal** (with respect to ι) if
  (i) $\iota(J) = J$;
  (ii) There exists a left ideal $\theta \in A$-mod, free as R-module, such that $\theta \subset J$;
  (iii) There is an isomorphism of A-bimodules $\alpha\colon J \to \theta \otimes_R \iota(\theta)$ making the following diagram commutative:

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "\iota"] & \theta \otimes_R \iota(\theta) \arrow[d, "x\otimes y \mapsto \iota(y)\otimes\iota(x)"] \\ J \arrow[r, "\alpha"'] & \theta \otimes_R \iota(\theta) \end{tikzcd}.$$

- The algebra A (with involution ι) is called **cellular** if
  (i) There is an R-module decomposition $A = J'_1 \bigoplus \cdots \bigoplus J'_n$ (for some n) with $\iota(J'_j) = J'_j$ for each j;
  (ii) Setting $J_j = \bigoplus_{l=1}^{j} J'_l$ gives a chain of two-sided ideals of A, called **cell chain**: $0 \subset J_1 \subset \cdots \subset J_n = A$ (each of them fixed by ι);
  (iii) For each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by ι on the quotient) of A/J_{j-1}.

In particular, this definition requires that every cell ideal is a free R-module. The modules $\theta(j)$ associated with each cell ideal J'_j are called **cell modules.**

We note that the original Definition in [KX98] requires R to be an integral domain but the arguments easily pass to the general case. For sake of completeness, we will write the proof of equivalence of both notions.

Proposition 1.6.5. *The two definitions of cellular algebras are equivalent.*

Proof. Assume that A is cellular in the sense of Definition 1.6.1. Let λ be a minimal index in Λ. Let $J(\lambda)$ be the R-module with R-basis $\{C^{\lambda}_{S,T}\colon S,T \in M(\lambda)\}$. By condition (C2), $\iota(J) = J$. By conditions (C3) and (C3'), J is an ideal of A. Fix $T \in M(\lambda)$. Let θ be the free R-module with R-basis $\{C^{\lambda}_{S,T}\colon S \in M(\lambda)\}$. By condition (C3), θ is a finitely generated A-module and clearly $\theta \subset J$. Define $\alpha\colon J \to \theta \otimes_R \iota(\theta)$ by mapping $C^{\lambda}_{U,V}$ to $C^{\lambda}_{U,T} \otimes_R \iota(C^{\lambda}_{V,T})$. This map is compatible with the involution. In fact,

$$\alpha\iota(C^{\lambda}_{U,V}) = \alpha(C^{\lambda}_{V,U}) = C^{\lambda}_{V,T} \otimes_R \iota(C^{\lambda}_{U,T}) = \iota\iota(C^{\lambda}_{V,T}) \otimes_R \iota(C^{\lambda}_{U,T}) \tag{1.6.0.6}$$

$$\alpha(C^{\lambda}_{U,V}) = C^{\lambda}_{U,T} \otimes_R \iota(C^{\lambda}_{V,T}). \tag{1.6.0.7}$$

Thus, $J(\lambda)$ is a cell ideal according to Definition 1.6.4. Put $J'_1 = J(\lambda)$. $A/J(\lambda)$ has an R-basis

$$\{C^{\mu}_{S,T} + J(\lambda)\colon \lambda \neq \mu \in \Lambda,\ S,T \in M(\mu)\}$$

and it satisfies condition (C3). So, $A/J(\lambda)$ is again cellular in the sense of Definition 1.6.1. By induction, A is cellular in the sense of Definition 1.6.4.

Conversely, assume that A is cellular according to Definition 1.6.4. Then, there exists a cell chain $0 \subset J_1 \subset \cdots \subset J_n = A$. Let $\{C_S \colon S \in I_1\}$ be an R-basis of θ for some finite set I_1. Define $C_{S,T} \in J_1$ to be the inverse image by α of $C_S \otimes \iota(C_T)$. Denote by ω the twist map on $\theta \otimes_R \iota(\theta)$. By the compatibility of α and ι, we can write

$$\alpha \circ \iota(C_{S,T}) = \omega \circ \alpha(C_{S,T}) = \omega(C_S \otimes \iota(C_T)) = \iota\iota(C_T) \otimes \iota(C_S) = C_T \otimes \iota(C_S). \tag{1.6.0.8}$$

Thus, $\iota(C_{S,T}) = \alpha^{-1}(C_T \otimes \iota(C_S)) = C_{T,S}$. So, condition (C2) holds for the index 1. Put $M(1) = I_1$. Let $a \in A$. Since $aC_S \in \theta$, there are coefficients $r_a(U,S) \in R$ such that $aC_S = \sum_{U \in M(1)} r_a(U,S) C_U$.For $S,T \in M(1)$,

$$\alpha(aC_{S,T}) = a\alpha(C_{S,T}) = aC_S \otimes \iota(C_T) = \sum_{U \in M(1)} r_a(U,S) C_U \otimes \iota(C_T) = \sum_{U \in M(1)} r_a(U,S) \alpha(C_{U,T}) \tag{1.6.0.9}$$

$$= \alpha\left(\sum_{U \in M(1)} r_a(U,S) C_{U,T}\right). \tag{1.6.0.10}$$

Therefore, $aC_{S,T} = \sum_{U \in M(1)} r_a(U,S) C_{U,T}$. By induction, A/J has a cellular basis. Choosing pre-images in A of the elements basis of A/J together with the basis of J gives a cellular basis for A, since A is a direct sum as R-modules of J_t', $t = 1, \dots, n$. □

From the proof of Proposition 1.6.5, we can deduce the following result.

Corollary 1.6.6. *Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). Let $A(<\lambda)$, $\lambda \in \Lambda$, be the free R-module with R-basis*

$$\{C^{\mu}_{S,T} \colon \mu < \lambda,\ S,T \in M(\mu)\}. \tag{1.6.0.11}$$

The (left) cell modules are the A-modules which are free over R with R-basis

$$\theta_l(\lambda) = \{C^{\lambda}_{S,T_0} + A(<\lambda) \colon S \in M(\lambda)\}, \quad \text{for some} \quad T_0 \in M(\lambda), \quad \lambda \in \Lambda. \tag{1.6.0.12}$$

The (right) cell modules are the right A-modules which are free over R with basis

$$\theta_r(\lambda) = \{C^{\lambda}_{S_0,T} + A(<\lambda) \colon T \in M(\lambda)\}, \quad \text{for some} \quad S_0 \in M(\lambda), \quad \lambda \in \Lambda. \tag{1.6.0.13}$$

The statement for right modules follows using condition (C3') instead of (C3).

Proposition 1.6.7. *Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). Let $M \in A$-*mod*. Then, M becomes a right A-module by making $x \cdot_\iota a = \iota(a)x$. Similarly, any $N \in$ *mod-*A becomes a left A-module by making $a \cdot_\iota x = x\iota(a)$. Denote by M^ι the* ***twisted module*** *of M. Moreover,*

(i) $\theta_l(\lambda)^\iota \simeq \theta_r(\lambda)$ as right A-modules, $\lambda \in \Lambda$;

(ii) ${}^\iota\theta_r(\lambda) \simeq \theta_l(\lambda)$ as left A-modules, $\lambda \in \Lambda$.

Proof. Consider the map $\psi \colon \theta_l(\lambda)^\iota \to \theta_r(\lambda)$ that sends $C^{\lambda}_{S,T_0} + A(<\lambda)$ to $C^{\lambda}_{S_0,S} + A(<\lambda)$. Thus, ψ is bijective. We want to show that ψ is an A-isomorphism. To obtain that we can observe that

$$\psi((C^{\lambda}_{S,T_0} + A(<\lambda)) \cdot_\iota \iota(a)) = \psi(\iota^2(a) \cdot (C^{\lambda}_{S,T_0} + A(<\lambda))) = \psi(a \cdot (C^{\lambda}_{S,T_0} + A(<\lambda))) \tag{1.6.0.14}$$

$$= \psi\left(\sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T_0} + A(<\lambda)\right) = \sum_{U \in M(\lambda)} r_a(U,S) \psi(C^{\lambda}_{U,T_0} + A(<\lambda))$$

$$= \sum_{U\in M(\lambda)} r_a(U,S)(C^\lambda_{S_0,U} + A(<\lambda)). \tag{1.6.0.15}$$

On the other hand,

$$\begin{aligned}\psi(C^\lambda_{S,T_0} + A(<\lambda))\cdot\iota(a) &= (C^\lambda_{S_0,S} + A(<\lambda))\iota(a) = \iota(C^\lambda_{S,S_0} + A(<\lambda))\iota(a) = \iota(a\cdot C^\lambda_{S,S_0} + A(<\lambda)) \\ &= \iota(\sum_{U\in M(\lambda)} r_a(U,S)(C^\lambda_{U,S_0} + A(<\lambda))) = \sum_{U\in M(\lambda)} r_a(U,S)(C^\lambda_{S_0,U} + A(<\lambda)).\end{aligned} \tag{1.6.0.16}$$

Therefore, ψ is a right A-isomorphism. Using the map ${}^\iota\theta_r(\lambda) \to \theta_l(\lambda)$, mapping $C^\lambda_{S_0,T} + A(<\lambda)$ to $C^\lambda_{T,T_0} + A(<\lambda)$, (ii) follows. □

We can define a duality functor ${}^\natural(-)\colon A\text{-mod} \to A\text{-mod}$ which sends M to DM^ι and a duality functor $(-)^\natural\colon \text{mod-}A \to \text{mod-}A$ which sends N to $D^\iota N$. The following corollary is an immediate consequence of Proposition 1.6.7.

Corollary 1.6.8. *Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). Let $\lambda \in \Lambda$. Then,*

(i) ${}^\natural\theta_l(\lambda) \simeq D\theta_r(\lambda)$ as left A-modules;

(ii) $\theta_r(\lambda)^\natural \simeq D\theta_l(\lambda)$ as right A-modules.

Proposition 1.6.9. *Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). Then, $A \in \mathscr{F}(\theta_{\lambda\in\Lambda})$.*

Proof. We can consider an increasing bijection between the posets Λ and $\{1,\dots,n\}$. We want to show that there exists a filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = A \tag{1.6.0.17}$$

with $P_i/P_{i-1} \simeq \theta_i \otimes_R U_i$ for some free R-module U_i where the cell module θ_i is associated with the cell ideal J'_i. We shall proceed by induction on n. Assume $n = 1$. Then, A is a cell ideal of A. Thus, there exists $\theta_1 \subset A$ such that $A \simeq \theta_1 \otimes_R \iota(\theta_1)$ and $\iota(\theta_1)$ is R-free. So, $A \in \mathscr{F}(\theta_1)$. Assume now that the result holds for $n-1$. The modules θ_j, $j > 1$, are cell modules of A/J_1. By induction, $A/J \in \mathscr{F}(\theta_{j>1})$. So, there exists a filtration $0 = P'_1 \subset \cdots \subset P'_n = A/J$, $\quad P'_i/P'_{i-1} = \theta_i \otimes_R U_i$, where U_i is a free R-module. Thus, there exists a chain

$$J = P_1 \subset \cdots \subset P_n = A, \quad P_i/P_{i-1} \simeq P_i/J/P_{i-1}/J \simeq P'_i/P'_{i-1}. \tag{1.6.0.18}$$

Since J is a cell ideal, $J \simeq \theta_1 \otimes_R \iota(\theta_1)$. Putting $U_1 = \iota(\theta_1)$, the result follows. □

Cellular algebras have a base change property.

Proposition 1.6.10. *Let S be a commutative Noetherian R-algebra. Let A be a cellular R-algebra with cell datum (Λ, M, C, ι) then $S\otimes_R A$ is cellular with cell datum $(\Lambda, M, 1_S \otimes_R C, \mathrm{id}_S \otimes_R \iota)$.*

Proof. The algebra $S\otimes_R A$ has an S-basis $\{S\otimes_R C^\lambda_{U,T} \mid U,T \in M(\lambda), \lambda\in\Lambda\}$. Hence, condition (C1) holds. Since ι is an anti-isomorphism over R, so it is $S\otimes_R \iota$ over S. Moreover, $(\mathrm{id}_S \otimes_R \iota)^2 = \mathrm{id}_S \otimes_R \iota^2 = \mathrm{id}_S \otimes_R \mathrm{id}_A = \mathrm{id}_{S\otimes_R A}$ and $\mathrm{id}_S \otimes_R \iota(1_S \otimes_R C^\lambda_{U,T}) = 1_S \otimes_R \iota(C^\lambda_{U,T}) = 1_S \otimes_R C^\lambda_{T,U}$ for $U,T \in M(\lambda)$, $\lambda\in\Lambda$. So, condition (C2) holds. It remains to check condition (C3). For $s\otimes_R a \in S\otimes_R A$,

$$(s\otimes_R a)(1_S \otimes C^\lambda_{V,T}) = s\otimes aC^\lambda_{V,T} = s\otimes\left(\sum_{U\in M(\lambda)} r_a(U,V)C^\lambda_{U,T} + r'\right) = \sum_{U\in M(\lambda)} s\otimes r_a(U,V)C^\lambda_{U,T} + s\otimes r' \tag{1.6.0.19}$$

$$= \sum_{U \in M(\lambda)} sr_a(U,V)(1_S \otimes_R C^{\lambda}_{U,T}) + s \otimes r', \tag{1.6.0.20}$$

where $s \otimes r'$ is a linear combination of basis elements $1_S \otimes_R C^{\mu}_{l,t}$ with upper index μ strictly smaller than λ and $\lambda \in \Lambda$, $l,t \in M(\mu)$, $V,T \in M(\lambda)$. ☐

The following result due to [KX98, Proposition 4.3] is fundamental to understand under what conditions an endomorphism algebra of a projective module over a cellular algebra remains cellular. However, we need further assumptions on the ground ring. By a commutative **projective-free** ring R we mean a commutative ring R with every finitely generated projective R-module being free. Properties about these rings can be found in [Lam06].

Proposition 1.6.11. *Let R be a commutative Noetherian projective-free ring. Let A be a cellular R-algebra with involution ι and with cell chain*

$$0 \subset J_1 \subset \cdots \subset J_n = A. \tag{1.6.0.21}$$

Let e be an idempotent of A which is fixed by ι. Then, eAe is a cellular R-algebra with involution $\iota_{|_{eAe}}$ and with cell chain

$$0 \subset eJ_1e \subset \cdots \subset eJ_ne = eAe. \tag{1.6.0.22}$$

Proof. Since ι fixes the idempotent e, the restriction of $\iota\colon A \to A$ to eAe has image in eAe. Thus, $\iota_{|_{eAe}}$ is an involution of eAe. Let J be a cell ideal of A. We claim that eJe is a cell ideal of eAe. Let $j \in J$. By assumption, there exists j' such that $\iota(j') = j$. Hence, $\iota(ej'e) = \iota(e)\iota(ej') = \iota(e)\iota(j')\iota(e) = eje$. This shows that $\iota_{|_{eAe}}(eJe) = eJe$. Let θ be the left ideal associated with J. Then, $e\theta = eA \otimes_A \theta \in \operatorname{add}_R \theta$. Hence, $e\theta \in R$-proj. Since R is projective-free $e\theta$ is R-free and $\iota(e\theta) = \iota(\theta)e$. Applying the functors $eA \otimes_A -$ and $- \otimes_A Ae$ to α we obtain an isomorphism $e\alpha e$ compatible with the desired commutative diagram. So, eJe is a cell ideal. Proceeding by induction, multiplication by e on both sides on a cell chain of A yields a cell chain for eAe. ☐

Of course, $\mathbb{Z}$ is a principal ideal domain, and thus it is a projective-free ring. Due to [Swa78], the Laurent polynomial ring $\mathbb{Z}[X, X^{-1}]$ is projective-free. These observations are important to give proofs of Hecke algebras being cellular using the cellularity of q-Schur algebras using Proposition 1.6.11.

In [KX99a], it is shown that in characteristic two not every projective module can be given by an idempotent fixed by the involution. Hence, cellular algebras are not categorical concepts. The situation becomes even worse for cellular algebras over commutative rings which are not projective-free. Still in [KX99a], they show that cellular algebras over fields of characteristic different from two are preserved under Morita equivalence. This is another evidence that cellular algebras have nicer properties over $\mathbb{Z}[\frac{1}{2}]$ and over Laurent polynomial rings over $\mathbb{Z}[\frac{1}{2}]$.

The following proposition gives a criterion to the problem of finding which split quasi-hereditary algebras are cellular. This is a generalization of Corollary 4.2 of [KX98] to commutative Noetherian rings.

Proposition 1.6.12. *Let R be a commutative Noetherian ring. Let A be a free Noetherian R-algebra. Assume that A admits a set of orthogonal idempotents $\{e_1, \ldots, e_t\}$ such that for each maximal ideal $\mathfrak{m}$ of R $\{e_1(\mathfrak{m}), \ldots, e_t(\mathfrak{m})\}$ becomes a complete set of primitive orthogonal idempotents of $A(\mathfrak{m})$. Suppose that there exists an involution $\iota\colon A \to A$ that fixes the set of orthogonal idempotents $\{e_1, \ldots, e_t\}$. If A is a split quasi-hereditary with split heredity chain*

$$0 \subset Ae_tA \subset \cdots \subset A(e_1 + \cdots + e_t)A = A, \tag{1.6.0.23}$$

then A is a cellular algebra (with respect to ι) and with cell chain (1.6.0.23).

Proof. Put $e = e_t$. Thus, $\iota(AeA) = A\iota(e)A = AeA$. By Theorem 1.5.73, $Ae \in \mathscr{M}(A)$. Moreover, $\operatorname{Hom}_A(Ae, A) = eA = \iota(e)A = \iota(Ae)$. So, the map $\tau_{Ae}\colon Ae \otimes_R \iota(Ae) \to AeA$ is an isomorphism. We can consider the diagram

$$\begin{tikzcd} AeA \arrow[r, "\tau_{Ae}^{-1}"] \arrow[d, "\iota"] & Ae \otimes_R \iota(Ae) \arrow[r, "\tau_{Ae}"] \arrow[d, "\omega"] & AeA \arrow[d, "\iota"] \\ AeA \arrow[r, "\tau_{Ae}^{-1}"] & Ae \otimes_R \iota(Ae) \arrow[r, "\tau_{Ae}"] & AeA \end{tikzcd}, \tag{1.6.0.24}$$

where ω is the usual twist map. We claim that the diagram is commutative. To show that, note that

$$\iota\tau_{Ae}(ae \otimes eb) = \iota(aeb) = \iota(b)e\iota(a) \tag{1.6.0.25}$$

$$\tau_{Ae}\omega(ae \otimes eb) = \tau_{Ae}(\iota(eb) \otimes \iota(ae)) = \tau_{Ae}(\iota(b)e \otimes e\iota(a)) = \iota(b)e\iota(a). \tag{1.6.0.26}$$

It follows that

$$\tau_{Ae}\omega\tau_{Ae}^{-1} = \iota\tau_{Ae}\tau_{Ae}^{-1} = \iota. \tag{1.6.0.27}$$

Thus, all interior squares of the diagram are commutative. In particular, AeA is a cell ideal. Proceeding by induction on the heredity chain, we get that (1.6.0.23) is a cell chain. ☐

We note that if A is split quasi-hereditary with a poset Λ, Λ indexes the cell basis of A but with the reversed order.

Proposition 1.6.12 motivates the following definition of duality for Noetherian algebras.

Definition 1.6.13. Let R be a commutative Noetherian ring. Let A be a free Noetherian R-algebra. Assume that A admits a set of orthogonal idempotents $\mathbf{e} := \{e_1, \dots, e_t\}$ such that for each maximal ideal $\mathfrak{m}$ of R $\{e_1(\mathfrak{m}), \dots, e_t(\mathfrak{m})\}$ becomes a complete set of primitive orthogonal idempotents of $A(\mathfrak{m})$. We say that A has a **duality** $\iota\colon A \to A$ (with respect to $\mathbf{e}$) if ι is an anti-isomorphism with $\iota^2 = \mathrm{id}_A$ fixing the set of orthogonal idempotents $\{e_1, \dots, e_t\}$.

Properties for cellular algebras, when studied over arbitrary commutative rings, are not well understood as compared to finite-dimensional cellular algebras. If one is especially interested in homological properties of cellular algebras, then passing to split quasi-hereditary algebras seems to be the right choice. In fact, if A is cellular, then $A(\mathfrak{m})$ is cellular for every maximal ideal $\mathfrak{m}$ of R and $R(\mathfrak{m})$ is a splitting field for $A(\mathfrak{m})$ (see [GL96, (3.4), (2.6)']). Covers of $\overline{R(\mathfrak{m})} \otimes_R A$ can be chosen to be split quasi-hereditary (see Section 3.7), so two questions immediately arise.

Question A Can all cellular algebras over a commutative Noetherian ring R be realised as endomorphism algebras of projective modules over split quasi-hereditary R-algebras?

Question B Are cellular algebras with finite global dimension over commutative Noetherian rings split quasi-hereditary algebras?

For finite-dimensional cellular algebras, Question B was answered positively in [KX99b]. As we mentioned, Question A is true for finite-dimensional algebras over algebraically closed fields, but, at the moment of writing, the question remains unsolved in the general case. Concerning Question A, we cannot demand, in addition, for the cellular modules to be exactly the image of standard modules through the Schur functor (with the reversed order), that is, $F\Delta(\lambda) = \theta_\lambda$, $\forall \lambda \in \Lambda$. In fact, such a question has a negative answer even for finite-dimensional algebras (see Example 4.6.14). We recall that the group algebra of the symmetric group have a positive answer for this last scenario with the Schur algebra taking the role of the cover. One of the reasons to be interested in such a condition is Corollary 3.6.6. Such a result says that, under these requirement of the standard modules of the cover being sent to cell modules, the cellular algebra can only admit one split quasi-hereditary cover provided the "quality" of the cover is high enough.

Our next goal is to show a positive answer to Question B. The main idea is to show that for a cellular algebra A the simple $A(\mathfrak{m})$-modules arise from a finitely generated B-module which is projective over the ground ring.

To facilitate our life, we will require further notation first. Let A be a cellular algebra over a commutative Noetherian ring R. Denote by $A(\leq \lambda)$ the A-submodule of A with R-basis $\{C^{\mu}_{S,T} \colon \mu \leq \lambda,\ S,T \in M(\mu)\}$ for $\lambda \in \Lambda$. Denote by $A(< \lambda)$ the A-module with R-basis $\{C^{\mu}_{S,T} \colon \mu < \lambda,\ S,T \in M(\mu)\}$. In this notation, $A/A(<\lambda)$ is cellular and $A(\leq \lambda)/A(<\lambda)$ is a cell ideal of $A/A(<\lambda)$.

Using Lemma 1.7 of [GL96], we can define a bilinear form $\phi_\lambda \colon \theta(\lambda) \times \theta(\lambda) \to R$ by $\phi_\lambda(C^{\lambda}_{U,T_0}, C^{\lambda}_{T,T_0}) = \phi_{1_A}(U,T)$ where

$$C^{\lambda}_{U_1,T_1} a C^{\lambda}_{U_2,T_2} - \phi_a(T_1,U_2) C^{\lambda}_{U_1,T_2} \in A(<\lambda), \quad U_1,T_1,U_2,T_2 \in M(\lambda). \tag{1.6.0.28}$$

Let S be a commutative Noetherian R-algebra. $S \otimes_R A$ is cellular S-algebra. So, associated with $S \otimes_R \theta(\lambda)$ there is a bilinear form ϕ^S_λ. We shall relate the bilinear form ϕ^S_λ with ϕ_λ.

By considering the maps that carry the basis of $(S\otimes_R A)(<\lambda)$ (resp. $(S\otimes_R A)(\leq\lambda)$) to $S\otimes_R (A(<\lambda))$ (resp. $S\otimes_R (A(\leq\lambda))$) we obtain $S\otimes_R A$-isomorphjsms

$$(S\otimes_R A)(<\lambda) \simeq S\otimes_R (A(<\lambda)), \quad (S\otimes_R A)(\leq\lambda) \simeq S\otimes_R (A(\leq\lambda)). \tag{1.6.0.29}$$

Now observe that,

$$C^{\lambda}_{U_1,T_1} a C^{\lambda}_{U_2,T_2} - \phi_a(T_1,U_2) C^{\lambda}_{U_1,T_2} \in A(<\lambda), \quad U_1,T_1,U_2,T_2 \in M(\lambda). \tag{1.6.0.30}$$

So, for every $s \in S$,

$$s \otimes (C^{\lambda}_{U_1,T_1} a C^{\lambda}_{U_2,T_2} - \phi_a(T_1,U_2) C^{\lambda}_{U_1,T_2}) \in S\otimes_R A(<\lambda), \quad U_1,T_1,U_2,T_2 \in M(\lambda). \tag{1.6.0.31}$$

Under the isomorphism (1.6.0.29), we obtain that

$$(1_S \otimes C^{\lambda}_{U_1,T_1})(s\otimes a)(1_S \otimes C^{\lambda}_{U_2,T_2}) - \phi_a(T_1,U_2) s (1_S \otimes C^{\lambda}_{U_1,T_2}) \in (S\otimes_R A)(<\lambda), \quad U_1,T_1,U_2,T_2 \in M(\lambda). \tag{1.6.0.32}$$

On the other hand, applying (1.6.0.28) to S and $s\otimes a$ we obtain that

$$(1_S \otimes C^{\lambda}_{U_1,T_1})(s\otimes a)(1_S \otimes C^{\lambda}_{U_2,T_2}) - \phi^S_{s\otimes a}(T_1,U_2) s (1_S \otimes C^{\lambda}_{U_1,T_2}) \in (S\otimes_R A)(<\lambda), \quad U_1,T_1,U_2,T_2 \in M(\lambda). \tag{1.6.0.33}$$

Thus, by comparing basis, $\phi_a(T_1,U_2)s = \phi^S_{s\otimes a}(T_1,U_2)$, $T_1,U_2 \in M(\lambda)$. In particular, $\phi_{1_A}(T_1,U_2)1_S = \phi^S_{1_{S\otimes_R A}}(T_1,U_2)$. We have shown that

Lemma 1.6.14. *For ϕ_λ and ϕ^S_λ the bilinear forms associated with $\theta(\lambda)$ and $S\otimes_R \theta(\lambda)$, respectively, we can write*

$$\phi^S_\lambda(1_S\otimes C^{\lambda}_{U,T_0}, 1_S\otimes C^{\lambda}_{T,T_0}) = \phi_\lambda(C^{\lambda}_{U,T_0}, C^{\lambda}_{T,T_0})1_S, \quad U,T \in M(\lambda). \tag{1.6.0.34}$$

We can now construct modules in A-mod$\cap R$-proj that over the finite-dimensional $A(\mathfrak{m})$ become simple modules as long as $\phi_\lambda^{R(\mathfrak{m})} \neq 0$.

Lemma 1.6.15. *Let R be a local commutative Noetherian ring with maximal ideal $\mathfrak{m}$. Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). For each $\lambda \in \Lambda$, define*

$$\operatorname{rad}(\phi_\lambda) = \{x \in \theta(\lambda) \mid \phi_\lambda(x,y) \in \mathfrak{m}, \quad \forall y \in \theta(\lambda)\}. \tag{1.6.0.35}$$

Then, for each $\lambda \in \Lambda$, there exists an (A,R)-exact sequence

$$0 \to \operatorname{rad}(\phi_\lambda) \to \theta(\lambda) \to X_\lambda \to 0. \tag{1.6.0.36}$$

Proof. Let $\lambda \in \Lambda$. We start by observing that $\operatorname{rad}(\phi_\lambda)$ is an A-module. Since ϕ_λ is a bilinear form, it follows that $\operatorname{rad}(\phi_\lambda)$ is an R-submodule of θ. Let $a \in A$, $x \in \operatorname{rad}(\phi_\lambda)$. By Proposition 2.4 of [GL96],

$$\phi_\lambda(ax, y) = \phi_y(x, \iota(a)y) \in \mathfrak{m},\ \forall y \in \theta(\lambda). \tag{1.6.0.37}$$

Hence, $ax \in \operatorname{rad}(\phi_\lambda)$. We claim now that $\operatorname{rad}(\phi_\lambda^{R(\mathfrak{m})}) = \operatorname{rad}(\phi_\lambda)(\mathfrak{m})$. Suppose, again that $x \in \operatorname{rad}(\phi_\lambda)$. We can write $x = \sum_{V \in M(\lambda)} x_V C^\lambda_{V,T_0}$. By definition,

$$\sum_{V \in M(\lambda)} x_V \phi_\lambda(C^\lambda_{V,T_0}, C^\lambda_{T,T_0}) = \phi_\lambda(x, C^\lambda_{T,T_0}) \in \mathfrak{m},\ \forall T \in M(\lambda). \tag{1.6.0.38}$$

Therefore,

$$0 = \sum_{V \in M(\lambda)} x_V \phi_\lambda(C^\lambda_{V,T_0}, C^\lambda_{T,T_0}) 1_{R(\mathfrak{m})} = \sum_{V \in M(\lambda)} x_V \phi_\lambda^{R(\mathfrak{m})}(1_{R(\mathfrak{m})} \otimes C^\lambda_{V,T_0}, 1_{R(\mathfrak{m})} \otimes C^\lambda_{T,T_0}) \tag{1.6.0.39}$$

$$= \phi_\lambda^{R(\mathfrak{m})}(1_{R(\mathfrak{m})} \otimes x, 1_{R(\mathfrak{m})} \otimes C^\lambda_{T,T_0}), \forall T \in M(\lambda). \tag{1.6.0.40}$$

Hence, $1_{R(\mathfrak{m})} \otimes x \in \operatorname{rad}(\phi_\lambda^{R(\mathfrak{m})})$. So, $\operatorname{rad}(\phi_\lambda)(\mathfrak{m}) \subset \operatorname{rad}(\phi_\lambda^{R(\mathfrak{m})})$. Now consider $y \in \operatorname{rad}(\phi_\lambda^{R(\mathfrak{m})}) \subset \theta(\lambda)(\mathfrak{m})$. So, we can write $y = \sum_{U \in M(\lambda)} y_U 1_{R(\mathfrak{m})} \otimes C^\lambda_{U,T_0}$, with $y_U \in R(\mathfrak{m})$. Further, we can assume that $y_U = r_U 1_{R(\mathfrak{m})}$ for some $r_U \in R$. For every $T \in M(\lambda)$,

$$0 = \phi_\lambda^{R(\mathfrak{m})}(y, 1_{R(\mathfrak{m})} \otimes C^\lambda_{T,T_0}) = \sum_{U \in M(\lambda)} r_U \phi_\lambda^{R(\mathfrak{m})}(1_{R(\mathfrak{m})} \otimes C^\lambda_{U,T_0}, 1_{R(\mathfrak{m})} \otimes C^\lambda_{T,T_0}) \tag{1.6.0.41}$$

$$= \sum_{U \in M(\lambda)} r_U \phi_\lambda(C^\lambda_{U,T_0}, C^\lambda_{T,T_0}) 1_{R(\mathfrak{m})}. \tag{1.6.0.42}$$

Thus,

$$\phi_\lambda(\sum_{U \in M(\lambda)} r_U C^\lambda_{U,T_0}, C^\lambda_{T,T_0}) \in \mathfrak{m}, \forall T \in M(\lambda). \tag{1.6.0.43}$$

It follows that $\sum_{U \in M(\lambda)} r_U C^\lambda_{U,T_0} \in \operatorname{rad}(\phi_\lambda)$. Hence, $y = 1_{R(\mathfrak{m})} \otimes_R \sum_{U \in M(\lambda)} r_U C^\lambda_{U,T_0} \in \operatorname{rad}(\phi_\lambda)(\mathfrak{m})$. This completes our claim.

Let X_λ be cokernel of $\operatorname{rad}(\phi_\lambda) \to \theta(\lambda)$. Applying the functor $R(\mathfrak{m}) \otimes_R -$ yields the long exact sequence

$$0 = \operatorname{Tor}_1^R(\theta(\lambda), R(\mathfrak{m})) \to \operatorname{Tor}_1^R(X_\lambda, R(\mathfrak{m})) \to \operatorname{rad}(\phi_\lambda)(\mathfrak{m}) \to \theta(\lambda)(\mathfrak{m}) \to X_\lambda \to 0. \tag{1.6.0.44}$$

Since $\mathrm{rad}(\phi_\lambda)(\mathfrak{m}) = \mathrm{rad}(\phi_\lambda^{R(\mathfrak{m})}) \subset \theta(\lambda)(\mathfrak{m})$, $\mathrm{Tor}_1^R(X_\lambda, R(\mathfrak{m})) = 0$. So, $X_\lambda \in R$-proj. So, the exact sequence

$$0 \to \mathrm{rad}(\phi_\lambda) \to \theta(\lambda) \to X_\lambda \to 0 \tag{1.6.0.45}$$

is (A,R)-exact. □

Theorem 1.6.16. *Let R be a commutative Noetherian regular ring with finite Krull dimension. Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). Then, $(A, \theta_{\lambda\in\Lambda^{op}})$, with Λ^{op} being the poset Λ with reversed order, is a split quasi-hereditary algebra if and only if A has finite global dimension.*

Proof. By Theorem 1.5.75, if $(A, \{\theta_{\lambda\in\Lambda}\})$ is split quasi-hereditary, then A has finite global dimension. Conversely, assume that A has finite global dimension. Let $\mathfrak{m}$ be a maximal ideal of R. By Proposition 1.1.23, every module in $A_\mathfrak{m}$ can be written in the form $M_\mathfrak{m}$ for some $M \in A$-mod. Thus,

$$\mathrm{Ext}_{A_\mathfrak{m}}^{\mathrm{gldim}A+1}(X_\mathfrak{m}, Y_\mathfrak{m}) = \mathrm{Ext}_A^{\mathrm{gldim}A+1}(X,Y)_\mathfrak{m} = 0. \tag{1.6.0.46}$$

Thus, $\mathrm{gldim}A_\mathfrak{m} \leq \mathrm{gldim}A$. In view of Theorem 1.5.57, we can assume that R is a local regular commutative Noetherian ring. Let L be a simple $A(\mathfrak{m})$-module. By Propositions 3.2 and 3.4 of [GL96], there exists $\lambda \in \Lambda$ such that $\phi_\lambda^{R(\mathfrak{m})} \neq 0$ and $\theta(\lambda)(\mathfrak{m})/\mathrm{rad}(\phi_\lambda^{R(\mathfrak{m})}) \simeq L$. By Lemma 1.6.15, $X_\lambda(\mathfrak{m}) \simeq L$. By assumption, $\mathrm{pdim}_A X_\lambda$ is finite. Since $X_\lambda \in R$-proj, any projective A-resolution of X_λ remains exact under $R(\mathfrak{m}) \otimes_R -$. In particular, $\mathrm{pdim}_{A(\mathfrak{m})} L$ is finite. It follows that $A(\mathfrak{m})$ has finite global dimension. By Theorem 1.1 of [KX99b], $(A(\mathfrak{m}), \theta(\mathfrak{m})_{\lambda\in\Lambda})$ is split quasi-hereditary. By Theorem 1.5.56, $(A, \theta_{\lambda\in\Lambda})$ is a split quasi-hereditary algebra. □

Remark 1.6.17. Every commutative algebra with finite global dimension over an algebraically closed field is a split quasi-hereditary algebra (see Proposition 3.5 of [KX98]). △

We wish to proceed further and give a complete characterization for cellular Noetherian algebras in the similar form as in [KX99b].

Theorem 1.6.18. *Let R be a regular commutative Noetherian ring with finite Krull dimension. Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). The following assertions are equivalent.*

(i) Some cell chain of A is a split heredity chain as well, that is, A is split quasi-hereditary.

(ii) There is a cell chain (with respect to some involution possibly distinct from ι) whose length $|\Lambda|$ equals the number of simple $A(\mathfrak{p})$-modules for every prime ideal $\mathfrak{p}$ of R.

(iii) Any cell chain of A is a split heredity chain of length $|\Lambda|$.

(iv) The algebra A has finite global dimension.

(v) A is locally semi-perfect and the function Cartan: $\mathrm{Spec}\, R \to \mathbb{Z}$, *given by*

$$\mathrm{Cartan}(\mathfrak{p}) = \det[\mathrm{rank}_R \mathrm{Hom}_{A_\mathfrak{p}}(P_i, P_j)],\ \mathfrak{p} \in \mathrm{Spec}\, R,$$

is the constant function 1, where P_i, $i = 1, \dots, r$ for some natural number r, are the projective indecomposable modules of $A_\mathfrak{p}$.

Proof. By Proposition 4.1 of [KX98], if $J^2 \neq 0$, then $J = AeA$ and $Ae = \theta$. Hence, θ_λ are the standard modules of A if A split is quasi-hereditary. In particular, for split quasi-hereditary algebras all split heredity chains have the same size. Together with Theorem 1.6.16, this shows that $(iii) \Leftrightarrow (iv) \Leftrightarrow (i)$. Assume that (iv) holds. Let $\mathfrak{p}$ be a prime ideal of R. Then, $A(\mathfrak{p})$ is a cellular algebra with cell datum (Λ, M, C, ι). In particular, $A(\mathfrak{p})$ has a cell chain (given by the cell datum) of length Λ and $A(\mathfrak{p})$ is split quasi-hereditary by Theorem 1.6.16. Therefore, $|\Lambda|$ is equal to the number of standard modules of $A(\mathfrak{p})$ which is equal to the number of simple $A(\mathfrak{p})$-modules. So, (ii) holds.

Assume that (ii) holds. For every prime ideal, $A(\mathfrak{p})$ has a cell chain whose length equals the number of simple $A(\mathfrak{p})$-modules. Thus, $A(\mathfrak{p})$ is split quasi-hereditary with standard modules $\theta_\lambda(\mathfrak{p})$, $\lambda \in \Lambda$ by Theorem 1.1 of [KX99b]. Therefore, $(A, \theta_{\lambda\in\Lambda^{op}})$ is split quasi-hereditary. So, (i) holds. Assume that (i) holds. By Theorem 1.5.84, $A_\mathfrak{p}$ is semi-perfect for every prime ideal $\mathfrak{p}$ of R. Thus, A is locally semi-perfect. So, we can write $A_\mathfrak{p}$ as a direct sum of unique indecomposable projective module. Moreover, $\mathrm{Hom}_{A_\mathfrak{p}}(P_i, P_j)$ is free over $R_\mathfrak{p}$. Further, $R(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \mathrm{Hom}_{A_\mathfrak{p}}(P_i, P_j) \simeq \mathrm{Hom}_{A(\mathfrak{p})}(P_i(\mathfrak{p}), P_j(\mathfrak{p}))$ and $P_i(\mathfrak{p})$ are the indecomposable projective modules of $A(\mathfrak{p})$. By Theorem 1.1 of [KX99b],

$$1 = \det[\dim_{R(\mathfrak{p})} \mathrm{Hom}_{A(\mathfrak{p})}(P_i(\mathfrak{p}), P_j(\mathfrak{p}))] = \det[\dim_{R(\mathfrak{p})} R(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \mathrm{Hom}_{A_\mathfrak{p}}(P_i, P_j)] = \det[\operatorname{rank} \mathrm{Hom}_{A_\mathfrak{p}}(P_i, P_j)].$$

So, (v) holds. Finally, assume that (v) holds. Let $\mathfrak{p}$ be a prime ideal of R. Applying $R(\mathfrak{p}) \otimes_R -$ we obtain $A(\mathfrak{p})$ is a direct sum of the projective modules $P_i(\mathfrak{p})$ with $i = 1, \dots, r$, and

$$1 = \det[\operatorname{rank}_R \mathrm{Hom}_{A_\mathfrak{p}}(P_i, P_j)] = \det[\dim_{R(\mathfrak{p})} \mathrm{Hom}_{A(\mathfrak{p})}(P_i(\mathfrak{p}), P_j(\mathfrak{p}))]. \tag{1.6.0.47}$$

Moreover, every map between $P_i(\mathfrak{p})$ and $P_j(\mathfrak{p})$ can be lifted to a map between P_i and P_j. Since each $P_j \in R$-proj and by Lemma 1.1.39, $P_i(\mathfrak{p}) \simeq P_j(\mathfrak{p})$ if and only if $P_i \simeq P_j$ if and only if $i = j$. We claim now that each $P_i(\mathfrak{p})$ is indecomposable over $A(\mathfrak{p})$. Since $A_\mathfrak{p}$ is semi-perfect, $\mathrm{End}_{A_\mathfrak{p}}(P_i)$ is a local ring. Furthermore, $\mathfrak{p}_\mathfrak{p} \mathrm{End}_{A_\mathfrak{p}}(P_i)$ is an ideal of $\mathrm{End}_{A_\mathfrak{p}}(P_i)$ and

$$\mathrm{End}_{\widehat{A_\mathfrak{p}}}(\widehat{P_i}) \simeq \widehat{\mathrm{End}_{A_\mathfrak{p}}(P_i)} = \lim_n \mathrm{End}_{A_\mathfrak{p}}(P_i)/\mathfrak{p}_\mathfrak{p}^n \mathrm{End}_{A_\mathfrak{p}}(P_i) = \lim_n \mathrm{End}_{A_\mathfrak{p}}(P_i)/\left(\mathfrak{p}_\mathfrak{p} \mathrm{End}_A(P_i)\right)^n. \tag{1.6.0.48}$$

This last ring is the completion of $\mathrm{End}_{A_\mathfrak{p}}(P_i)$ at the ideal $\mathfrak{p}_\mathfrak{p} \mathrm{End}_{A_\mathfrak{p}}(P_i)$, so it is a local ring. Therefore, $\widehat{P_i}$ is indecomposable. By [CR90, (6.5), (6.7)], $\widehat{P_i}(\widehat{\mathfrak{p}_\mathfrak{p}}) \simeq P_i(\mathfrak{p})$ is indecomposable. By (1.6.0.47), the Cartan matrix of $A(\mathfrak{p})$ has determinant 1. Note that $A(\mathfrak{p})$ is cellular. By Theorem 1.1 of [KX99b], $A(\mathfrak{p})$ is split quasi-hereditary with standard modules $\theta_\lambda(\mathfrak{p})$. Therefore, $r = |\Lambda|$ and since $\mathfrak{p}$ is arbitrary $(A, \theta_{\lambda\in\Lambda^{op}})$ is split quasi-hereditary. □

Cellular algebras over fields which are quasi-hereditary admit, up to equivalence, only one quasi-hereditary structure. This result is due to Coulembier [Cou20, Theorem 2.1.1]. Our focus is now to extend this result to cellular Noetherian algebras. To this end, we need to recall some facts about the ordering of the standard modules in a quasi-hereditary algebra. For finite-dimensional algebras, the order of the split quasi-hereditary algebra is determined by the occurrences of simples $\operatorname{top} P(\mu)$ on $\Delta(\lambda)$ and $\Delta(\lambda)$ on $P(\mu)$ (see for example Proposition 1.5.39). If A has a simple preserving duality $(-)^\natural$, then $\Delta(\mu)^\natural \simeq \nabla(\mu)$, $\mu \in \Lambda$. Further, the number of occurrences of $\Delta(\mu)$ in $P(\lambda)$ is equal to the multiplicity of $\operatorname{top} P(\lambda)$ in $\Delta(\mu)$ (see for example Lemma 2.5 of [DR92]). So, this information can be recovered to some extent by the Grothendieck group of A. The **Grothendieck group** of A, here denoted by $G_0^R(A)$, is the abelian group generated by the symbols $[M]$, $M \in A\text{-mod} \cap R\text{-proj}$ with relations $[M] = [M'] + [M'']$ whenever there exists an (A,R)-exact sequence $0 \to M' \to M \to M'' \to 0$. Therefore, if we have two set of standard modules for a finite-dimensional algebra A with the same image in the Grothendieck group, then we can choose the order so that both sets give the same order in Proposition 1.5.39. By Proposition 1.5.63, these set of standard modules must coincide. Given the existence of a simple preserving duality, Theorem 2.1.1 of [Cou20] implies that every set of standard modules have the same image in the Grothendieck group for finite-dimensional algebras. In particular, if a cellular algebra is split quasi-hereditary, then there is a bijection $\phi\colon \Lambda \to \Lambda$ such that $\Delta(\lambda) \simeq \theta_{\phi_\lambda}$ if A is also split quasi-hereditary with standard modules $\Delta(\lambda)$. Moreover, in view of Theorem 1.5.65, there is a unique split heredity chain of length $|\Lambda|$ for finite-dimensional cellular algebras. Therefore, we can establish the following.

Theorem 1.6.19. *Let R be a commutative regular Noetherian ring. Let A be a cellular R-algebra with cell datum (Λ, M, C, ι). Assume that A has finite global dimension and $(A, \{\Delta(\omega)_{\omega\in\Omega}\})$ is split quasi-hereditary. Then, there exists an equivalence of categories $F\colon A\text{-mod}\to A\text{-mod}$ and a bijective map between posets $\phi\colon \Lambda^{op}\to\Omega$ such that $F\theta_\lambda \simeq \Delta(\phi(\lambda))\otimes_R U_\lambda, \quad \lambda\in\Lambda,\ U_\lambda\in Pic(R)$.*

Proof. Since A is a cellular R-algebra, $A(\mathfrak{m})$ is cellular with cell modules $\theta_\lambda(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R (see Proposition 1.6.10 and Corollary 1.6.6). By Theorem 1.6.16, $(A, \theta_{\lambda\in\Lambda^{op}})$ is split quasi-hereditary. Also $(A(\mathfrak{m}), \{\Delta(\mathfrak{m})(\omega)_{\omega\in\Omega}\})$ is split quasi-hereditary. By the discussion above and Theorem 2.1.1 of [Cou20], these two structures have the same split heredity chain. By Lemma 1.5.70, these two split heredity chains of A must coincide. By Theorem 1.5.65 and Proposition 1.5.31, the result follows. $\square$

1.6.1 Further topics

The following result indicates that endomorphism algebras of partial tilting modules over a split quasi-hereditary algebra with a duality are cellular algebras. The classical case can be found in [AST18] and [BT17, Theorem 1.1].

Theorem 1.6.20. *Let R be a Noetherian commutative ring and A a projective Noetherian R-algebra. Assume that A has a duality ι and that A is split quasi-hereditary with split heredity chain*

$$0\subset Ae_tA\subset\cdots\subset A(e_1+\cdots+e_t)A = A. \tag{1.6.1.1}$$

Let T be a characteristic tilting module of A and let $M=\oplus_{i\in I}T(i)$ (for some subset I of $\{1,\dots,t\}$) be a partial tilting module. Then, $\operatorname{End}_A(M)^{op}$ is a cellular algebra.

Proof. The duality ι induces a functor ${}^\iota(-)\colon A\text{-mod}\to A^{op}\text{-mod}$. In particular, ${}^\iota P(i) = {}^\iota(Ae_i) = e_iA$. Consider the contravariant functor ${}^\natural(-)\colon A\text{-mod}\to A\text{-mod}$ given by $D\circ{}^\iota(-)$. So, ${}^\natural(-)$ is a simple preserving duality and as in Lemma 3.2 of [FK11b] ${}^\natural T(i)\simeq T(i)$. Let $s\colon T\to{}^\natural T$ be an isomorphism of A-modules. Denote by $\alpha\colon \operatorname{End}_A(T)\to\operatorname{End}_A({}^\natural T)$ the isomorphism of R-algebras, given by $\alpha(f) = s\circ f\circ s^{-1}$, $f\in\operatorname{End}_A(T)$ and denote by $\beta\colon \operatorname{End}_A(T)\to\operatorname{End}_A({}^\natural T)$ the anti-isomorphism of R-algebras, given by $\beta(f)(h)(t) = h(f(t))$, $h\in DT$, $t\in T$. Put $\tau=\beta^{-1}\circ\alpha$. By Proposition 2.4 of [FK11b], τ is a duality of the Ringel dual $R_A := \operatorname{End}_A(T)^{op}$. That is, τ fixes all maps $T\twoheadrightarrow T(i)\hookrightarrow T$ for every i, and $\tau^2=\mathrm{id}_{R_A}$. In particular, τ fixes the idempotent f of R_A such that $\operatorname{Hom}_A(T,M)\simeq R_Af$. Observe that R_A is split quasi-hereditary with standard modules $\operatorname{Hom}_A(T,\nabla(i))$ with the reversed order on $\{1,\dots,t\}$. Thus, if we denote by f_i the idempotents $T\twoheadrightarrow T(i)\hookrightarrow T$, R_A has the split heredity chain

$$0\subset R_Af_1R_A\subset\cdots\subset R_A(f_1+\ldots+f_t)R_A = R_A. \tag{1.6.1.2}$$

By Proposition 1.6.12, R_A is a cellular algebra. By Proposition 1.6.11, $\operatorname{End}_A(M)^{op}\simeq\operatorname{End}_{R_A}(\operatorname{Hom}_A(T,M))^{op}$ is a cellular algebra. $\square$

1.7 From A-mod to B-mod

In Corollary 1.4.36, we saw that if (A,P) is a cover of B, then both algebras have the same number of blocks. In this section, we seek to explore and collect more relations between A-mod and B-mod where B is the endomorphism algebra of a finitely generated projective A-module. Again, A will denote a projective Noetherian R-algebra. It is not surprising that for finite-dimensional algebras over a field, we can obtain more properties that are preserved under Schur functors. Hence, in this section, R will be a field unless otherwise stated. For finite-dimensional algebras over a field, the results on covers arising from an idempotent carry over unchanged to the more general situation of covers using projective modules. In fact, we can state the following, which can also be found in [Cru21, Proposition 9].

Proposition 1.7.1. *[Cru21, Proposition 9] Let R be a field. If (A,P) is a cover of B, then there exists an idempotent $e \in A$ such that (A,Ae) is a cover of eAe and eAe is Morita equivalent to B.*

Proof. We can decompose P into a direct sum of projective indecomposable modules $P_1 \oplus \cdots \oplus P_n$. By Krull-Remak-Schmidt Theorem, there is a subset I of $\{1,\ldots,n\}$ so that $Q := \oplus_{i\in I} P_i$ is an A-summand of A and $\operatorname{add} Q = \operatorname{add} P$, where the modules P_i, $i \in I$, are pairwise non-isomorphic. Moreover, there exists an idempotent $e \in A$ such that $Ae \simeq Q$. Hence, the algebras B and eAe are Morita equivalent. By Theorem 1.4.17, the functor $\operatorname{Hom}_B(\operatorname{Hom}_A(P,Ae),-)\colon B\text{-mod} \to eAe\text{-mod}$ is an equivalence of categories. On the other hand, the canonical map $\operatorname{Hom}_A(Ae,A) \to \operatorname{Hom}_B(F(Ae),FA)$ is bijective. Moreover, it is an eAe-isomorphism. Therefore,

$$\begin{aligned} A &\simeq \operatorname{End}_B(\operatorname{Hom}_A(P,A))^{op} \simeq \operatorname{End}_{eAe}(\operatorname{Hom}_B(\operatorname{Hom}_A(P,Ae),\operatorname{Hom}_A(P,A)))^{op} \qquad (1.7.0.1)\\ &= \operatorname{End}_{eAe}(\operatorname{Hom}_B(F(Ae),FA))^{op} \simeq \operatorname{End}_{eAe}(\operatorname{Hom}_A(Ae,A))^{op}. \end{aligned}$$
□

It goes back to the work of Green [Gre07, Theorem 6.2g] and his PhD student T. Martins the classification of simple eAe-modules in terms of Schur functors for a given finite-dimensional algebra A over a field.

Theorem 1.7.2. *Let A be a finite-dimensional R-algebra. Suppose $\{V_\lambda : \lambda \in \Lambda\}$ is a full set of simple modules in A-mod, indexed by a set Λ. Let $\Lambda' = \{\lambda \in \Lambda \colon eV_\lambda \neq 0\}$. Then, $\{eV_\lambda : \lambda \in \Lambda'\}$ is a full set of simple modules in eAe-mod. The simple A/AeA-modules are exactly the simple A-modules, S, with $eS = 0$.*

As we have seen, this determines, in particular, the complete set of simple $\operatorname{End}_A(P)^{op}$-modules whenever (A,P) is a cover of $\operatorname{End}_A(P)^{op}$. The following results are very well known and quite elementary, however, we present a proof for convenience of the reader.

Proposition 1.7.3. *Let (A,Ae) be a cover of eAe for some idempotent $e \in A$. Suppose that $S \in A$-mod is a simple module with projective cover P satisfying $eS \neq 0$. Then, eP is the projective cover of eS. Dually, the Schur functor preserves the injective hull of S.*

Proof. Let f be a primitive idempotent of $B := eAe$ so that Bf is the projective cover of eS. e is the identity of eAe. Thus, $fe = ef = f \in A$. Moreover, $fAf = feAef$ is a local ring. Therefore, f is a primitive idempotent in A. We claim that Af is the projective cover of S. To see this, observe that the following modules are isomorphic as R-modules,

$$\operatorname{Hom}_A(Af,S) \simeq fS = feS \simeq \operatorname{Hom}_{eAe}(eAef,eS) \neq 0. \qquad (1.7.0.2)$$

This implies that there exists a surjective map $Af \to S$. Consequently, $\operatorname{top} Af = S$. This proves that Af is the projective cover of S and $eAf = eAef$ is the projective cover of eS. □

Proposition 1.7.4. *Let $M \in A$-mod and S a simple A-module. If $eS \neq 0$, then $[M\colon S] = [eM\colon eS]$.*

Proof. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$ be a composition series of M. Applying the exact functor $eA \otimes_A -$ yields the filtration $0 = eM_0 \subset eM_1 \subset \cdots \subset eM_s = eM$. In particular, $eM_{i+1}/eM_i \simeq e(M_{i+1}/M_i)$ is either simple or zero, $0 \leq i \leq s-1$. By deleting the redundant modules, we obtain a composition series of eM. Now, the result follows using the composition series of eM and the fact that S is the unique simple module (up to isomorphism) that maps to eS via the Schur functor. ☐

The following two results can be found with more details in the Appendix of [Don98].

Theorem 1.7.5. *Let A be a split quasi-hereditary algebra over a field. Let $\{S(\lambda)\colon \lambda \in \Lambda\}$ be a complete set of simple A-modules. Let e be an idempotent of A and $B = eAe$. Assume that the idempotent e satisfies the following*

$$eS(\lambda) = 0 \iff \lambda \leq \mu \text{ for some } \mu \in \Gamma, \text{ for some fixed subset } \Gamma \subset \Lambda. \tag{1.7.0.3}$$

Fix $\Lambda^* := \{\lambda \in \Lambda\colon eS(\lambda) \neq 0\}$ Then, B is split quasi-hereditary with standard modules $\{e\Delta(\lambda)\colon \lambda \in \Lambda^*\}$ and costandard modules $\{e\nabla\colon \lambda \in \Lambda^*\}$. Moreover, $e\Delta(\lambda) = e\nabla(\lambda) = 0$ for $\lambda \in \Lambda \backslash \Lambda^*$.

Proof. See Proposition A3.11 of [Don98]. The idea of the proof is to use the characterization of quasi-hereditary algebras discussed in Proposition 1.5.39. By applying the Schur functor $\mathrm{Hom}_A(Ae, -)$ on the exact sequences given by Proposition 1.5.39 and by using Propositions 1.7.3, 1.7.4 and Theorem 1.7.2, the result follows. ☐

Idempotents satisfying (1.7.0.3) do exist. For example, the functor $S_R(d,d)$-mod $\rightarrow S_R(n,d)$, defined in [Gre07, 6.5], for $d \geq n$, and R an infinite field, is given by such an idempotent. For future reference, note that this functor is also well defined if we drop the condition that R is an infinite field. Theorem 1.7.5 plays an important role in and it allows us to understand the quasi-hereditary structure of a Schur algebra in cases $n < d$ using bigger Schur algebras. This theorem also gives a sufficient condition for a Schur functor to preserve the quasi-hereditary structure of A.

The subset of Λ whose elements (also called dominant weights) satisfy the statement on the right of (1.7.0.3) is called a saturated set of Λ by Donkin. The set Λ^* is called a cosaturated set of Λ.

Remark 1.7.6. An idempotent in the conditions of Theorem 1.7.5 does not come, in general, from a cover. For example, for $d > n$ $(S_R(d,d), S_R(d,d)\text{-mod} \rightarrow S_R(n,d))$ is not a cover of $S_R(n,d)$. If it was, then it would be true that

$$RS_d \simeq \mathrm{End}_{S_R(d,d)}((R^d)^{\otimes d}) \simeq \mathrm{End}_{S_R(n,d)}((R^n)^{\otimes d}). \tag{1.7.0.4}$$

Consequently, $((R^n)^{\otimes d})$ would become a faithful RS_d-module. This is not true since $n < d$. △

Proposition 1.7.7. *Let A be a split quasi-hereditary algebra over a field. Let e be in the above conditions. Then, the following assertions hold.*

(a) *The Schur functor $\mathrm{Hom}_A(Ae, -)$ preserves (partial) tilting modules.*

(b) *Let $M \in \mathscr{F}(\Delta)$ and $N \in \mathscr{F}(\nabla)$ then the map $\mathrm{Hom}_A(M,N) \rightarrow \mathrm{Hom}_{eAe}(eM, eN)$ is surjective.*

(c) *The partial tilting indecomposable modules of eAe are exactly $\{eT(\lambda)\colon \lambda \in \Lambda^*\}$. Moreover, $eT(\mu) = 0$ for $\mu \in \Lambda \backslash \Lambda^*$.*

Proof. For (b) see Lemma A3.12 of [Don98] or [Erd94, 1.7]. One idea is to observe that the map

$$R \simeq \mathrm{Hom}_A(\Delta(\lambda), \nabla(\lambda)) \rightarrow \mathrm{Hom}_{eAe}(e\Delta(\lambda), e\nabla(\lambda)) \simeq R \tag{1.7.0.5}$$

is non-zero for $\lambda \in \Lambda^*$. In particular, the image under this map of $\Delta(\lambda) \rightarrow T(\lambda) \rightarrow \nabla(\lambda)$ is non-zero. Then, using the filtrations given by Proposition 1.5.117 (b) follows.

For (a) and (c) see Lemma A4.5 of [Don98]. (a) follows directly by observing that $\mathrm{Hom}_A(Ae,-)$ preserves filtrations of standard (resp. costandard) modules due to Theorem 1.7.5. Now, applying (b) to $T(\lambda)$ yields that $\mathrm{End}_{eAe}(eT(\lambda))$ is a local ring if $\lambda \in \Lambda^*$. Hence, it is indecomposable. Using (a) and the uniqueness of indecomposable tilting modules the first part of (c) holds. The second part of (c) follows by the exactness of $\mathrm{Hom}_A(Ae,-)$ and Theorem 1.7.5. $\square$

It is essential that the idempotent e satisfies (1.7.0.3), or in other words, that Λ^* is cosaturated in Λ. If we drop such a condition, Proposition 1.7.7(c) can fail.

Example 1.7.8. Let A be the following bound quiver algebra over an algebraically closed field

$$\begin{tikzcd} & 2 \arrow[dr, "\alpha"] & \\ 1 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 4 \\ & 3 \arrow[ur, "w"'] & \end{tikzcd}, \quad \alpha\beta = \omega\gamma. \tag{1.7.0.6}$$

A is quasi-hereditary with $1 < 2 < 3 < 4$. The projective modules are

$$P(1) = \begin{array}{ccc} & 1 & \\ 2 & & 3 \\ & 4 & \end{array} \quad P(2) = \begin{array}{c} 2 \\ 4 \end{array} \quad P(3) = \begin{array}{c} 3 \\ 4 \end{array} \quad P(4) = 4\,.$$

The injective modules are

$$I(1) = 1 \quad I(2) = \begin{array}{c} 1 \\ 2 \end{array} \quad I(3) = \begin{array}{c} 1 \\ 3 \end{array} \quad I(4) = P(1). \tag{1.7.0.7}$$

Here, $\Delta(i) = S(i)$ and $\nabla(i) = I(i)$ for all $i = 1, \dots, 4$. Hence, the (partial) tilting modules are $T(1) = \Delta(1)$, $T(2) = I(2)$, $T(3) = I(3)$ and $T(4) = P(1)$. Choose $e = e_2 + e_3$. Then, eAe is semi-simple, so every simple module is (partial) tilting indecomposable. However, $eT(4) = 2 \bigoplus 3$ is not indecomposable. $\triangle$

1.7.1 From a cover (A,P) to $\mathrm{End}_A(P)^{op}$

In this subsection, we will give further properties of covers, but now we will assume that R is a Noetherian commutative ring.

The following result, although elementary, does not seem to appear in the literature.

Proposition 1.7.9. *Let R be a commutative Noetherian ring. Suppose that (A,P) is a cover of B. If B is a relative semi-simple R-algebra, then A is a relative semi-simple R-algebra. Conversely, if A is a relative semi-simple R-algebra and DA is the epimorphic image of some module belonging to* $\mathrm{add}\, DA \otimes_A P$ *then B is a relative semi-simple R-algebra.*

Proof. Assume that B is relative semi-simple. Since $P \in A$-proj, $\mathrm{Hom}_A(P,A)$ is a B-generator and projective as R-module. Consequently, $\mathrm{Hom}_A(P,A)$ is a B-progenerator. Therefore, B and A are Morita equivalent. In particular, A is relative semi-simple with respect to R.

Assume that A is relative semi-simple and there exists a surjective homomorphism $\theta \colon DA \otimes_A X \to DA$ for some $X \in \mathrm{add}_A P$. Since $DA \in R$-proj then θ is an (A,R)-epimorphism. So, by assumption, it splits over A, and therefore $DA \in \mathrm{add}\, DA \otimes_A P$. Hence, $\mathrm{Hom}_A(P,A)$ is a projective generator of A-mod. Since (A,P) is a cover of B then A and B are Morita equivalent. $\square$

For finite-dimensional algebras, instead of using techniques on faithful modules, we could have used techniques involving the Jacobson radical. In fact, $\operatorname{rad}(eAe) = e\operatorname{rad}Ae$ (see [Lam01, Theorem (21.10)]).

Covers also give some insights into the classification of indecomposable modules.

Proposition 1.7.10. *Let R be a commutative Noetherian ring. Let (A,P) be a cover of B. If A is an algebra of finite representation type, then B is an algebra of finite representation type.*

Proof. The functor $G\colon B\text{-mod} \to A\text{-mod}$ is fully faithful. Therefore, we can identify B-mod with some full subcategory of A-mod. Since A is of finite type there exists a finite number of indecomposable modules in A-mod. In particular, there exists a finite number of indecomposable objects in any full subcategory of A-mod. It follows that B is of finite representation type. □

The following result states that if there exist projective covers of finitely generated modules for a cover of an algebra B, then the modules belonging to the module category B-mod have also projective covers.

Theorem 1.7.11. *Let R be a commutative Noetherian ring. Let (A,P) be a cover of B. If A is semi-perfect algebra, then B is semi-perfect.*

Proof. By Proposition 3.14 of [Fac98], A is a direct sum of A-modules with local endomorphism rings. By Theorem 2.12 of [Fac98], any two direct sum decompositions of A^t (for any $t > 0$) into indecomposable modules are isomorphic. In particular, P is a direct sum of A-modules with local endomorphism rings. By Proposition 3.14 of [Fac98], $B = \operatorname{End}_A(P)^{op}$ is semi-perfect. □

The converse implication holds if and only if $\operatorname{Hom}_A(P,A)$ is a direct sum of B-modules with local endomorphism rings.

Given an idempotent e of A, the center of eAe and the ring $eZ(A)e$ can be quite different, where $Z(A)$ denotes the center of A.

Example 1.7.12. Consider the $R[t]$-module R satisfying $t1_R = 0$. Let A be the triangular matrix ring $\begin{bmatrix} R[t] & R \\ 0 & R \end{bmatrix}$. Then, the center of A, $Z(A) = R\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is isomorphic to R. But, choosing the idempotent $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ yields $eZ(A)e = R$ while the center of eAe is isomorphic with $R[t]$. △

On the other hand, the center of eAe can be computed using A if there exists a double centralizer property between eAe and A.

Proposition 1.7.13. *Let R be a commutative Noetherian ring. If (A,Ae) is a cover of eAe for some idempotent e in A, then $Z(eAe) = eZ(A)e$.*

Proof. Suppose that (A,Ae) is a cover of eAe for some idempotent e in A. It follows directly from definition that $eZ(A)e \subset Z(eAe)$. Conversely, assume that $x \in Z(eAe)$. Then, the map $eA \to eA$ given by $ea \mapsto xea$ is a left eAe-homomorphism. Denote this map by α_x. Moreover, for any $\phi \in \operatorname{End}_{eAe}(eA)$, $\phi \circ \alpha_x(ea) = \phi(xea) = x\phi(ea) = \alpha_x \circ \phi(ea)$ for all $a \in A$. Hence, $\alpha_x \in Z(\operatorname{End}_{eAe}(eA)^{op})$. Since (A,Ae) is a cover of eAe the canonical map $\psi\colon A \to \operatorname{End}_{eAe}(eA)^{op}$ is an isomorphism of algebras. Therefore, there exists $a \in Z(A)$ such that $\psi(a) = \alpha_x$. Hence, for all $b \in A$, $eba = \psi(a)(eb) = \alpha_x(eb) = xeb = xb$. So, $x = xe = eae \in eZ(A)e$. □

As we saw in Proposition 1.6.12 and 1.6.11, some cellular algebras B appear as the endomorphism algebra of projective modules over split quasi-hereditary algebras with a duality. This motivates us to study the following problem:

> *Problem* 1. For a given cellular algebra B, study a split quasi-hereditary algebra $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ such that (A,P) is a cover of B.

Observe that if B is not a split quasi-hereditary algebra over a regular ring with finite Krull dimension, then according to Theorem 1.6.16 B has infinite global dimension while A has finite global dimension. In algebraic geometry, non-singular varieties are associated with regular rings with finite global dimension (see [Rot09, Theorem 8.62, Proposition 8.60, Example 8.57]). Therefore, asking to realise an algebra B with infinite global dimension as $\mathrm{End}_A(P)^{op}$ with (A,P) being a cover of B and A having finite global dimension is a representation theoretic analogue of resolution of singularities.

The following result illustrates that finding a cover with finite global dimension for finite-dimensional algebras over fields cannot be seen as another technique for resolution of singularities in algebraic geometry in view of Proposition 1.7.1. In fact, we cannot expect a cover of a commutative algebra to be again commutative.

Proposition 1.7.14. *[Cru21, Proposition 10] Let R be a commutative Noetherian ring. Suppose that A is a commutative projective Noetherian R-algebra. If (A,Ae) is a cover of eAe for some idempotent e in A, then A is isomorphic to eAe.*

Proof. Thanks to A being commutative we obtain that eAe is commutative. If (A,Ae) is a cover of eAe, then

$$A \simeq \mathrm{End}_{eAe}(eA) = \mathrm{End}_{eAe}(e^2A) = \mathrm{End}_{eAe}(eAe) \simeq eAe. \qquad \square$$

Studying non-commutative resolutions for commutative rings have been attracting much attention in recent years. See for example [DITV15], for a different perspective and different types of resolutions than the one we use here. Over self-injective commutative Noetherian rings, their concept of resolution coincides with the concept of a cover with finite global dimension. In fact, over self-injective algebras faithful finitely generated modules are exactly the generators. Let R be a commutative self-injective ring. So, if M is a non-commutative resolution of R, in the sense of [DITV15], then $\mathrm{Hom}_N(M,N)$ is a projective (left) N-module where $N := \mathrm{End}_R(M)$. In particular,

$$\mathrm{End}_N(\mathrm{Hom}_N(M,N))^{op} \simeq \mathrm{End}_N(M) \simeq R,$$

and $M \simeq \mathrm{Hom}_N(\mathrm{Hom}_N(M,N),N)$ as (R,N)-bimodules since M is projective over N. So, $(N, \mathrm{Hom}_N(M,N))$ is a cover of R and N has finite global dimension. Conversely, if (N,P) is a cover of a self-injective commutative Noetherian ring R and N has finite global dimension, then $\mathrm{Hom}_N(P,N)$ is faithful over R and $N \simeq \mathrm{End}_R(\mathrm{Hom}_N(P,N))^{op}$ has finite global dimension. Since R is Noetherian, $\mathrm{End}_R(\mathrm{Hom}_N(P,N))$ has finite global dimension, and therefore $\mathrm{Hom}_N(P,N)$ is a non-commutative resolution of R, in the sense of [DITV15].

As we have mentioned, our interest is to resolve cellular algebras (not necessarily being commutative algebras) by covers. Going back to Proposition 1.6.12, we obtain even more information in the passage from (A,P) to a cellular algebra B than we had discussed so far. In particular, the Schur functor sends the standard modules of A to cell modules of B. By imposing this extra condition to Problem 1, we are more strict in choosing (if it exists) the "best" cover of a cellular algebra B so that the connection between A-mod and B-mod is the strongest possible (see Section 3.1). In Section 3.1, we will address how to measure the quality of a cover. Relative dominant dimension, to be studied in the next chapter, will give us both a tool to construct some quasi-hereditary covers and to measure their quality.

Chapter 2

Relative dominant dimension

In this chapter, we will generalize the classical theory of dominant dimension of finite-dimensional algebras to the Noetherian realm. The material to be developed here is intended to be self-contained. In particular, we can recover the classical theory by fixing the ground ring to be a field. Some highlights are the relative Morita-Tachikawa correspondence (Theorem 2.4.10), the relative version of Mueller's characterization of dominant dimension over an algebra in terms of cohomology over a certain endomorphism algebra of a projective-injective module (Theorem 2.4.15). We will also see how this theory is interconnected with the classical theory of dominant dimension of finite-dimensional algebras over algebraically closed fields (Theorem 2.5.13 and Proposition 2.5.10).

Much of the results on dominant dimension of finite-dimensional algebras can be found in [Mue68, Tac73, ARS95, Tac70, Mor58].

2.1 Definition

Unless otherwise stated, in this chapter, all algebras will be projective Noetherian R-algebras for a Noetherian commutative ring R.

Definition 2.1.1. Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $M \in A$-mod. We say that M has **relative dominant dimension at least** $t \in \mathbb{N}$ if there exists an (A,R)-exact sequence of finitely generated left A-modules

$$0 \to M \to I_1 \to \cdots \to I_t \tag{2.1.0.1}$$

where I_i are all projective and (A,R)-injective modules. If M admits no such (A,R)-exact sequence, then we say that M has relative dominant dimension zero. Otherwise, the relative dominant dimension of M is the supremum of the set of all values t such that an (A,R)-exact sequence of the form 2.1.0.1 exists. We denote by $\operatorname{domdim}_{(A,R)} M$ the relative dominant dimension of M.

Analogously, we can define relative dominant dimension for right A-modules.

Proposition 2.1.2. *(A,R)-dominant dimension is invariant under Morita equivalence.*

Proof. Let B be an algebra which is Morita equivalent to A. In view of Remark 1.4.18, B is a projective Noetherian R-algebra. Since (A,R)-exact sequences and (A,R)-injective modules are preserved under Morita equivalence due to Corollaries 1.2.7 and 1.2.11, it follows that (A,R)-dominant dimension is preserved under Morita equivalence. $\square$

Observe that since the zero module is projective and relative injective, if a module admits a finite projective (A,R)-injective coresolution, then it has infinite relative dominant dimension. We can make more precise the case of infinite relative dominant dimension for a module with finite relative injective dimension.

Proposition 2.1.3. *Let $M \in A\text{-mod}\cap R\text{-proj}$ having $\operatorname{idim}_{(A,R)} M < \infty$. The following assertions are equivalent.*

(a) $\operatorname{domdim}_{(A,R)} M = +\infty$*;*

(b) M is a projective and (A,R)-injective module.

Proof. Assume that (b) holds. Consider the (A,R)-exact sequence $0 \to M \to M \to 0$. By Definition 2.1.1, (a) holds.

Assume that (a) holds. In particular, $\operatorname{domdim}_{(A,R)} M \geq t = \operatorname{idim}_{(A,R)} M$, so there exists an (A,R)-exact sequence $0 \to M \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_t} I_t$ with I_i being projective and (A,R)-injective modules (possibly with some of them being zero). By Proposition 1.2.43, $\operatorname{im}\alpha_t$ is (A,R)-injective. So, it is an A-summand of I_t. Thus, it is also projective over A. Further, the exact sequence $0 \to M \to I_0 \to \cdots \to \operatorname{im}\alpha_t \to 0$ splits over A. Hence, $M \in \operatorname{add} I_0$. $\square$

Proposition 2.1.4. *Let $M \in A$-mod. Then, $\operatorname{domdim}_{(A,R)} M > 0$ if and only if M is an (A,R)-submodule of a (left) module that is both projective over A and (A,R)-injective. In particular, $\operatorname{domdim}_{(A,R)} {}_AA \geq 1$ if and only if A is an (A,R)-submodule of a projective (A,R)-injective (left) A-module.*

Proof. Assume that M is not an (A,R)-submodule of a (left) A-module that is both projective and (A,R)-injective. Assume by contradiction that $\operatorname{domdim}_{(A,R)} M > 0$. Then, there exists by definition an (A,R)-monomorphism $M \to I_1$ with $I_1 \in A\text{-proj}\cap(A,R)\text{-inj}$. This contradicts our assumption. Then, $\operatorname{domdim}_{(A,R)} M = 0$. Conversely, assume that $\operatorname{domdim}_{(A,R)} M = 0$. By contradiction assume that M is an (A,R)-submodule of a (left) A-module that is both projective and (A,R)-injective, say N. Then, the monomorphism $M \to N$ is (A,R)-exact and by the definition, we get $\operatorname{domdim}_{(A,R)} M > 0$. $\square$

As a consequence, we see that every module with positive relative dominant dimension is projective over the ground ring. Observe that if $\operatorname{domdim}_{(A,R)} M \geq 1$ for some $M \in A\text{-mod}\cap R\text{-proj}$, then there is an (A,R)-injective left $A\text{-proj}\cap(A,R)\text{-inj}$-approximation of M.

2.2 Strongly faithful modules

The following result is folklore but useful to characterize faithful modules.

Proposition 2.2.1. *Let $M \in A$-Mod. If $0 \to A \to M^t$ for some $t > 0$, then M is faithful. The converse holds for Artinian rings or if M is a finitely generated A-module.*

Proof. Assume that there exists a monomorphism $i\colon A \to M^t$ for some $t > 0$. Let $a \in A$ such that $am = 0$ for all $m \in M$. Then, $ax = 0$ for all $x \in M^t$. Hence, $i(a) = i(a1_A) = ai(1_A) = 0$. So, $a = 0$ and M is faithful. Assume that M is faithful finitely generated. Let $\{m_1, \dots, m_t\}$ be a generator set over A of M. Consider for each $x \in M$ the A-map $l_x\colon A \to M$, given by $l_x(a) = ax, a \in A$. The map $i\colon A \to M^t$, given by $i(a) = (l_{m_1}(a), \dots, l_{m_t}(a))$, is a monomorphism since M is faithful.

Now assume that M is faithful and A is an Artinian ring. Then, for $a \in A$,

$$a = 0 \Leftrightarrow l_x(a) = 0, \forall x \in M \Leftrightarrow a \in \bigcap_{x \in M} \ker l_x. \tag{2.2.0.1}$$

Since A is Artinian the chain $\ker l_{x_1} \supset \ker l_{x_1} \cap \ker l_{x_2} \supset \cdots$ must become stationary. Hence,

$$0 = \bigcap_{x \in M} \ker l_x = \bigcap_{x_1, \dots x_t} \ker l_{x_i}, \tag{2.2.0.2}$$

for some $t > 0$. Then, the A-map $i\colon A \to M^t$, given by $i(a) = (l_{x_1}(a), \dots, l_{x_t}(a))$, is a monomorphism. $\square$

Note that if we drop either the Artinian condition or $M \in A$-mod, then faithfulness cannot be characterized through existence of these exact sequences. In fact, let $R = A = \mathbb{Z}$ and $M = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. M is $\mathbb{Z}$-faithful since for every $a \in \mathbb{Z}$, $1 + (a+1)\mathbb{Z} \in M$ and $a \cdot 1 + (a+1)\mathbb{Z} = a + (a+1)\mathbb{Z} \neq 0$. Now assume that there exists an exact sequence $0 \to \mathbb{Z} \to M^t$ for some $t > 0$. Recall that

$$Q \otimes_{\mathbb{Z}} M \simeq \bigoplus_{n \in \mathbb{N}} Q \otimes_{\mathbb{Z}} Z/n\mathbb{Z} \simeq \bigoplus_{n \in \mathbb{N}} 0 = 0. \tag{2.2.0.3}$$

So, applying $Q \otimes_{\mathbb{Z}} -$ would imply that $0 \to \mathbb{Q} \to 0$ is exact. Hence, such an exact sequence does not exist.

This characterization is essential to deal with cases where the projective faithful modules are not given by an idempotent element. With this characterization, it is also easier to see that the concept of faithful finitely generated modules is Morita invariant.

In relative dominant dimension theory, faithful modules without further properties no longer play a key role in the study of relative dominant dimension of projective Noetherian algebras over commutative Noetherian rings. Here they are replaced by the following concept.

Definition 2.2.2. Let R be a commutative ring. We say that a (left) module M is (A,R)**-strongly faithful** if there is an (A,R)-monomorphism ${}_AA \hookrightarrow M^t$ for some $t > 0$. The definition for right modules is analogous.

If R is a field, then A becomes a finite-dimensional algebra. Thus, (A,R)-strongly faithful coincides with faithful, by Proposition 2.2.1.

For every commutative ring R, any generator of A-mod is (A,R)-strongly faithful. Because of M being a generator of A-mod there exists $t > 0$ such that $M^t \simeq A \otimes K$ as A-modules. In particular, the canonical monomorphism $A \hookrightarrow M^t$ splits over A, and thus is an (A,R)-monomorphism.

Looking back to Proposition 2.2.1, we see that Proposition 2.1.4 says that an algebra has positive relative dominant dimension if and only if it has an (A,R)-strongly faithful, projective (A,R)-injective A-module. By an (A,R)**-injective-strongly faithful module** we mean a module that is simultaneously (A,R)-injective and (A,R)-strongly faithful.

If R is a commutative Noetherian ring, any (A,R)-strongly faithful contains as summand a minimal (A,R)-strongly faithful module in the following sense.

Proposition 2.2.3. *Let R be a commutative Noetherian ring. Let M be a finitely generated projective and (A,R)-injective-strongly faithful A-module. Then, there exists an (A,R)-strongly faithful module $N \in \operatorname{add}_A M$ which does not contain any proper (A,R)-strongly faithful module as A-summand.*

Proof. If M does not contain a proper (A,R)-strongly faithful module as A-summand, then we are done. Otherwise we can write $M \simeq N_0 \bigoplus K_0$ where N_0 is an (A,R)-strongly faithful module and $K_0 \neq 0$. Then, we can apply the same reasoning to N_0. After a finite number of steps, we can construct a proper chain

$$0 \subsetneq K_0 \subsetneq K_1 \bigoplus K_0 \subsetneq \cdots \subsetneq K_n \bigoplus \cdots \bigoplus K_0. \tag{2.2.0.4}$$

Since M is a Noetherian module, this chain must stabilize. Hence, this construction must stop after a finite number of steps, say t. The module N_{t-1} belongs to add$M$ and does not contain any proper (A,R)-strongly faithful module as A-summand. □

Lemma 2.2.4. *Let M be a finitely generated (A,R)-strongly faithful projective and (A,R)-injective A-module. Then, every projective (A,R)-injective A-module belongs to* add$M$*. In particular, all endomorphism rings of modules N being finitely generated (A,R)-strongly faithful, projective over A and (A,R)-injective are Morita equivalent.*

Proof. Let N be projective and (A,R)-injective A-module. Since $N \in A$-proj, then there is n such that $A^n \simeq N \bigoplus L$. Denote by k_N and π_N the canonical injection and projection, respectively. Since M is (A,R)-strongly faithful, there exists $i \in \operatorname{Hom}_A(A, M^t)$ and $\pi \in \operatorname{Hom}_R(M^t, A)$ such that $\pi \circ i = \mathrm{id}_A$. Define $f = (i, \cdots, i) \circ k_N \in \operatorname{Hom}_A(N, M^{tn})$. Then,

$$\pi_N \circ (\pi, \cdots, \pi) \circ f = \pi_N \circ (\pi, \cdots, \pi) \circ (i, \cdots, i) \circ k_N = \pi_N \circ \mathrm{id}_{A^n} \circ k_N = \mathrm{id}_N . \tag{2.2.0.5}$$

Thus, f is an (A,R)-monomorphism. Since N is (A,R)-injective f splits over A. In particular, $N \in \operatorname{add}_A M$.

If N is also (A,R)-strongly faithful, then by reversing the roles of M and N, we obtain $M \in \operatorname{add} N$. Thus, $\operatorname{add} N = \operatorname{add} M$. This concludes the proof. □

2.2.1 Relative self-injective algebras

(A,R)-strongly faithful modules play an important role for relative self-injective algebras in the same fashion that faithful modules play an important role in self-injective Artinian algebras.

Definition 2.2.5. Let R be any commutative ring. An R-algebra B is called **relative (left) self-injective** if ${}_BB$ is (B,R)-injective.

Examples of relative self-injective algebras are quite common. For example, the class of relative self-injective algebras includes the class of group algebras over a commutative ring. The argument follows exactly as in [CR06, (62.1)].

Proposition 2.2.6. *For every finite group G, the group algebra RG is a relative self-injective R-algebra for any commutative ring R.*

Proof. Consider the R-linear map $\pi\colon RG \to R$, given by

$$\pi(g) = \begin{cases} 1_R, & \text{if g=e} \\ 0, & \text{otherwise} \end{cases}, \quad g \in G.$$

Define the RG-map $\phi\colon RG \to DRG$, given by $\phi(g)(h) = \pi(gh)$ for every $h \in RG$. Note that

$$\phi(hg)(x) = \pi((hg)x) = \pi(h(gx)) = \phi(h)(gx) = \phi(h)g(x), \forall g,h,x \in G. \tag{2.2.1.1}$$

Thus, ϕ is an RG-right homomorphism. We claim that ϕ is injective. In fact, let $x = \sum_{g\in G} x_g g \in \ker\phi$. Then, for all $h \in G$,

$$0 = \phi(x)(h) = \pi(xh) = \pi(\sum_{g\in G} x_g gh) = \sum_{g\in G} x_g \mathbb{1}_{\{gh=e\}}(g) = \sum_{g\in G} x_g \mathbb{1}_{\{h^{-1}\}}(g) = x_{h^{-1}}. \tag{2.2.1.2}$$

Thus, $x = 0$.

We shall now prove that ϕ is surjective. Observe that DRG has an R-basis $\{g^* \colon g \in RG\}$, given by $g^*(h) = \mathbb{1}_{\{g\}}(h)$, $h \in G$. Moreover, $g^*(\sum_{g\in G} h_g g) = h_g$. We claim that $\phi(g^{-1}) = g^*$ for every $g \in G$. In fact,

$$\phi(g^{-1})(x) = \pi(g^{-1}\sum_{h\in G} x_h h) = \sum_{h\in G} x_h \mathbb{1}_{\{g^{-1}h=e\}}(h) = \sum_{h\in G} x_h \mathbb{1}_{\{g=h\}}(h) = x_g = g^*(x), \quad \forall x \in RG. \tag{2.2.1.3}$$

Therefore, $RG \simeq D(RG)$ as right RG-modules. Consequently, $RG \simeq DDRG \simeq D(RG)$ as left RG-modules, since $RG \in R$-proj. Hence, RG is (RG, R)-injective. □

Theorem 2.2.7. *Let B be a relative (left and right) self-injective R-algebra. Let M be a (B,R)-strongly faithful module. Then, M is a generator (B,R)-cogenerator and it satisfies a double centralizer property: $A = \operatorname{End}_B(M)^{op}$ and $B = \operatorname{End}_A(M)$.*

Proof. Since M is (B,R)-strongly faithful, there exists a (B,R)-monomorphism $0 \to B \to M^t$. As B is (B,R)-injective, this monomorphism splits over B. Hence, $B \in \operatorname{add} M$. In particular, M is a generator of B-mod. Since double centralizer properties hold on generators, it follows that $B \simeq \operatorname{End}_A(M)$ with $A = \operatorname{End}_B(M)$. Since B is right self-injective algebra then B_B belongs to $\operatorname{add} D_B B$. Consequently, DB_B belongs to $\operatorname{add}_B B \subset \operatorname{add} M$. So, M is a B-generator (B,R)-cogenerator. □

For projective Noetherian R-algebras the notions of relative left and right self-injective R-algebra are equivalent.

Proposition 2.2.8. *Let B be a projective Noetherian R-algebra. B is a relative left self-injective R-algebra if and only if B is a relative right self-injective R-algebra.*

Proof. Assume that B is a relative right self-injective R-algebra. Then, B is (B,R)-injective as a right module. By Theorem 1.2.57, $B(\mathfrak{m})$ is right $B(\mathfrak{m})$-injective for every maximal ideal $\mathfrak{m}$ in R. In particular, every projective right $B(\mathfrak{m})$-module is $B(\mathfrak{m})$-injective. It is well known that this implies that every $B(\mathfrak{m})$-injective is projective over $B(\mathfrak{m})$ ([ARS95, IV. 3]). For the sake of completeness, we will give a proof of this fact: Let S be a simple $B(\mathfrak{m})$-module. Denote by $P(S)$ its projective cover. Then, $\{P(S) \colon S \text{ simple } B(\mathfrak{m})\text{-module}\}$ is a complete set of all non-isomorphic projective indecomposable $B(\mathfrak{m})$-modules. In particular, it is a set of non-isomorphic injective indecomposable $B(\mathfrak{m})$-modules. Since the set of injective hulls of simple modules gives a complete set of non-isomorphic injective indecomposable modules, the number of non-isomorphic injective indecomposable modules is exactly the number of simple modules. Thus, $\{P(S) \colon S \text{ simple } B(\mathfrak{m})\text{-module}\}$ is also a complete set of non-isomorphic injective indecomposable $B(\mathfrak{m})$-modules. So, all right $B(\mathfrak{m})$-injective modules are projective. In particular, $\operatorname{Hom}_{R(\mathfrak{m})}(B(\mathfrak{m}), R(\mathfrak{m}))$ is projective as a right $B(\mathfrak{m})$-module. So, $B(\mathfrak{m})$ is $B(\mathfrak{m})$-injective as a left module for every maximal ideal $\mathfrak{m}$ in R. Again, by Theorem 1.2.57, B is left (B,R)-injective. Thus, B is a relative left self-injective R-algebra. □

Projective Noetherian R-algebras which are relative self-injective over a commutative Noetherian ring were considered several times during the 1960s. The structure of these algebras that have global dimension at most one was determined in [End67].

Note that every relative self-injective R-algebra has infinite relative dominant dimension $\operatorname{domdim}(A,R) = \infty$. Indeed, we can consider the (A,R)-exact sequence $0 \to A \to A \to 0$. In parallel, we conjecture the following relative version of Nakayama conjecture:

Conjecture 2.2.9. *Given a projective Noetherian R-algebra A over any commutative Noetherian ring R satisfying $\operatorname{domdim}_{(A,R)} {}_AA = +\infty$, then A is a relative (left and right) self-injective R-algebra.*

As we will see afterwards, this conjecture is equivalent to the Nakayama conjecture.

2.2.2 Double centralizer properties on strongly faithful modules

For Noetherian algebras over commutative rings, it is easier to check the double centralizer property in the presence of (A,R)-strongly faithful modules. Using Nakayama's Lemma for (A,R)-monomorphisms 1.2.59, we can generalize Lemma 1.4.8 to commutative rings.

Proposition 2.2.10. *Let A be a Noetherian R-algebra. Let M be an (A,R)-strongly faithful and $B = \operatorname{End}_A(M)^{op}$. Then, the following assertions are equivalent.*

(i) (A,M) *satisfies the double centralizer property.*

(ii) $A \simeq \operatorname{End}_B(M)$ *as R-modules.*

(iii) $A \simeq \operatorname{End}_B(M)$ *as R-algebras.*

Proof. $i) \Rightarrow iii) \Rightarrow ii)$ is clear. We shall prove $ii) \Rightarrow i)$. Since M is (A,R)-strongly faithful, there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"'] & M^t \arrow[l, bend right, "\varepsilon"'] \arrow[r, bend right, "\pi_j"'] & M \arrow[l, bend right, "k_j"'] \end{tikzcd}$$

such that $\varepsilon \circ i = \mathrm{id}_A$ and $\sum_j k_j \circ \pi_j = \mathrm{id}_{M^t}$.

Consider $\psi\colon \operatorname{End}_B(M) \to A$, given by $\psi(f) = \sum_j \varepsilon \circ k_j \circ f(\pi_j \circ i(1_A))$, $f \in \operatorname{End}_B(M)$. This is an R-map and

$$\begin{aligned} \psi \circ \rho(a) &= \sum_j \varepsilon \circ k_j \circ \rho(a)(\pi_j \circ i(1_A)) = \sum_j \varepsilon \circ k_j(a\pi_j \circ i(1_A)) \\ &= \sum_j \varepsilon \circ k_j(\pi_j(i(a))) = \varepsilon \circ \sum_j k_k \circ \pi_j i(a) = \varepsilon \circ i(a) = a \end{aligned}$$

Hence, ρ is an (A,R)-monomorphism. By Lemma 1.2.59, since $A \simeq \operatorname{End}_B(M)$ as finitely generated R-modules, it follows that ρ is an isomorphism. By Definition, (A,M) satisfies the double centralizer property. $\square$

Theorem 2.2.7 motivates us to study endomorphism rings of generators-relative cogenerators. For finite dimensional algebras over a field, they can be characterized using dominant dimension. In order to obtain a relative version of this fact for Noetherian algebras, we need first some technical lemmas and it will be useful to introduce another definition of relative dominant dimension and its relation to approximation theory.

2.3 Relative dominant dimension with respect to a module, and approximation theory

Definition 2.3.1. Let $T \in A$-mod. An A-homomorphism $M \to N$ is called a **left** $\operatorname{add} T$**-approximation** of M provided that N belongs to $\operatorname{add} T$ and the induced homomorphism $\operatorname{Hom}_A(N,X) \to \operatorname{Hom}_A(M,X)$ is surjective for every $X \in \operatorname{add} T$.

An A-homomorphism $Y \to M$ is called a **right** $\operatorname{add} T$**-approximation** of M provided that Y belongs to $\operatorname{add} T$ and the induced homomorphism $\operatorname{Hom}_A(X,Y) \to \operatorname{Hom}_A(X,M)$ is surjective for every $X \in \operatorname{add} T$.

Lemma 2.3.2. *Let $N,T \in A\text{-mod}\cap R\text{-proj}$ and $M \in \operatorname{add} T$. Then, an A-homomorphism $f\colon M \to N$ is a right $\operatorname{add} T$-approximation of N if and only if the map $\operatorname{Hom}_A(T,f)\colon \operatorname{Hom}_A(T,M) \to \operatorname{Hom}_A(T,N)$ is surjective.*

Proof. The claim follows from the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(T_1\oplus T_2,M) \arrow[r, "\operatorname{Hom}_A(T_1\oplus T_2,f)"] \arrow[d, "\simeq"', "(-\circ k_1,-\circ k_2)"] & \operatorname{Hom}_A(T_1\oplus T_2,N) \arrow[d, "\simeq"', "(-\circ k_1,-\circ k_2)"] \\
\operatorname{Hom}_A(T_1,M)\oplus\operatorname{Hom}_A(T_2,M) \arrow[r, "\operatorname{Hom}_A(T_1,f)\oplus\operatorname{Hom}_A(T_2,f)"] & \operatorname{Hom}_A(T_1,N)\oplus\operatorname{Hom}_A(T_2,N)
\end{tikzcd},$$

where k_1,k_2 are the canonical injections of the direct sum $T_1\oplus T_2 \in \operatorname{add} T$. □

Lemma 2.3.3. *Let $M,T \in A\text{-mod}\cap R\text{-proj}$ and $N \in \operatorname{add} T$. An A-homomorphism $f\colon M \to N$ is a left $\operatorname{add} T$-approximation of M if and only if $Df\colon DN \to DM$ is a right $\operatorname{add} DT$-approximation of DM.*

Proof. The diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(T,M) \arrow[r, "\operatorname{Hom}_A(T,f)"] \arrow[d, "\simeq"', "\psi_{T,M}"] & \operatorname{Hom}_A(T,N) \arrow[d, "\simeq"', "\psi_{T,N}"] \\
\operatorname{Hom}_A(DM,DT) \arrow[r, "\operatorname{Hom}_A(Df,DT)"] & \operatorname{Hom}_A(DN,DT)
\end{tikzcd},$$

where $\psi_{T,M}$ and $\psi_{T,N}$ are defined according to Proposition 1.1.64, is commutative. In fact,

$$\operatorname{Hom}_A(Df,DT\circ\psi_{T,M}(h)(s) = \psi_{T,M}(h)\circ Df(s) = \psi_{T,M}(h)(s\circ f) = s\circ f\circ h,$$
$$\psi_{T,N}\circ\operatorname{Hom}_A(T,f)(h)(s) = \psi_{T,N}(f\circ h)(s) = s\circ f\circ h,\ h\in\operatorname{Hom}_A(T,M),\ s\in DN.$$

Therefore, $\operatorname{Hom}_A(T,f)$ is surjective if and only if $\operatorname{Hom}_A(Df,DT)$ is surjective. □

Trivial cases are the projective A-modules and relative injective modules. Naturally, every surjective map $P \twoheadrightarrow M$ with $P \in A$-proj is a right $\operatorname{add} A$-approximation of M and every (A,R)-monomorphism $M \to I$, with I being an (A,R)-injective and projective as R-module, is a left $\operatorname{add} DA$-approximation.

Lemma 2.3.4. *Let $M,T \in A\text{-mod}\cap R\text{-proj}$. Let $X_i \in \operatorname{add} T$, $i \geq 0$. The following assertions are equivalent.*

(i) An (A,R)-exact sequence

$$X_t \xrightarrow{\alpha_t} \cdots \to X_1 \xrightarrow{\alpha_1} X_0 \xrightarrow{\alpha_0} M \to 0 \tag{2.3.0.1}$$

remains exact under $\operatorname{Hom}_A(T,-)$ if and only if for every factorization

$$\begin{tikzcd}
X_i \arrow[rr, "\alpha_i"] \arrow[dr, two heads] & & X_{i-1} \\
& \operatorname{im}\alpha_i \arrow[ur, hook] &
\end{tikzcd},$$

the (A,R)-epimorphism $X_i \twoheadrightarrow \operatorname{im}\alpha_i$ and α_0 are right $\operatorname{add} T$-approximations with $i \geq 1$.

(ii) An (A,R)-exact sequence

$$0 \to M \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \to \cdots \to X_t \tag{2.3.0.2}$$

remains exact under $\mathrm{Hom}_A(-,T)$ *if and only if for every factorization*

$$\begin{tikzcd} X_i \arrow[rr, "\alpha_{i+1}"] \arrow[dr, two heads] & & X_{i+1} \\ & \operatorname{im}\alpha_{i+1} \arrow[ur, hook] & \end{tikzcd},$$

the (A,R)-monomorphism $\operatorname{im}\alpha_{i+1} \hookrightarrow X_{i+1}$ *and* α_0 *are left* $\operatorname{add}T$*-approximations with* $i \geq 0$.

Proof. (i). Assume that every factorization $X_i \twoheadrightarrow \operatorname{im}\alpha_i$ is a right $\operatorname{add}T$-approximation. Consider the (A,R)-exact sequences

$$0 \to \operatorname{im}\alpha_{i+1} \to X_i \xrightarrow{\alpha_i} \operatorname{im}\alpha_i \to 0. \tag{2.3.0.3}$$

As α_i is a right $\operatorname{add}T$-approximation, applying $\mathrm{Hom}_A(T,-)$ yields the exact sequence

$$0 \to \mathrm{Hom}_A(T,\operatorname{im}_{\alpha_{i+1}}) \to \mathrm{Hom}_A(T,X_i) \xrightarrow{\mathrm{Hom}_A(T,\alpha_i)} \mathrm{Hom}_A(T,\operatorname{im}\alpha_i) \to 0. \tag{2.3.0.4}$$

Thus, for every i,

$$\ker \mathrm{Hom}_A(T,\alpha_i) = \mathrm{Hom}_A(T,\operatorname{im}_{\alpha_{i+1}}) = \operatorname{im}\mathrm{Hom}_A(T,\alpha_{i+1}),$$

where the last equality follows from $X_{i+1} \xrightarrow{\alpha_{i+1}} \operatorname{im}\alpha_{i+1}$ being a right $\operatorname{add}T$-approximation.

Conversely, assume that (2.3.0.1) remains exact under $\mathrm{Hom}_A(T,-)$. We shall proceed by induction on i to show that $X_i \twoheadrightarrow \operatorname{im}\alpha_i$ is a right $\operatorname{add}T$-approximation. By definition, α_0 is a right $\operatorname{add}T$-approximation. Assume that $X_i \xrightarrow{\alpha_i} \operatorname{im}\alpha_i$ is a right $\operatorname{add}T$-approximation for some $i > 0$. Consider the (A,R)-exact sequence

$$0 \to \operatorname{im}\alpha_{i+1} \to X_i \xrightarrow{\alpha_i} \operatorname{im}\alpha_i \to 0. \tag{2.3.0.5}$$

As $\mathrm{Hom}_A(T,-)$ is left exact and α_i is a right $\operatorname{add}T$-approximation (2.3.0.5) remains exact under $\mathrm{Hom}_A(T,-)$. Hence,

$$\mathrm{Hom}_A(T,\operatorname{im}\alpha_{i+1}) = \ker \mathrm{Hom}_A(T,\alpha_i) = \operatorname{im}\mathrm{Hom}_A(T,\alpha_{i+1}).$$

thus, the image of $\mathrm{Hom}_A(T,X_{i+1} \twoheadrightarrow \operatorname{im}\alpha_{i+1})$ is exactly $\mathrm{Hom}_A(T,\operatorname{im}\alpha_{i+1})$. Hence, (i) follows. The case (ii) is the dual of (i). By Lemma 2.3.3, (ii) follows. ☐

We will now introduce an alternative definition of relative dominant dimension. This will be extremely useful for the arguments in the relative Morita-Tachikawa correspondence. In chapter 5, this definition will be of interest in its own right.

Definition 2.3.5. Let $T, X \in A\text{-mod}\cap R\text{-proj}$. If X does not admit a left $\operatorname{add}T$-approximation which is an (A,R)-monomorphism, then we say that **relative dominant dimension of X with respect to T** is zero. Otherwise, the **relative dominant dimension of X with respect to T**, denoted by $T - \operatorname{domdim}_{(A,R)} X$, is the supremum of all $n \in \mathbb{N}$ such that there exists an (A,R)-exact sequence

$$0 \to X \to T_1 \to \cdots \to T_n \tag{2.3.0.6}$$

which remains exact under $\operatorname{Hom}_A(-,T)$ with all $T_i \in \operatorname{add} T$.

By convention, the empty direct sum is the zero module. So, the existence of a finite relative $\operatorname{add} T$-coresolutions implies that $T-\operatorname{domdim}_{(A,R)} X$ is infinite. In the same way, we can define the relative dominant dimension of a right module with respect to a right module Q.

Definition 2.3.5 generalizes the concept of relative dominant dimension introduced in 2.1.1 as we can see in the following Proposition. Furthermore, this is a generalization of [Tac73, 7.3, 7.7].

Proposition 2.3.6. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra with* $\operatorname{domdim}_{(A,R)} {}_A A \geq 1$ *with projective (A,R)-injective-strongly faithful left A-module P. Then,*

$$P-\operatorname{domdim}_{(A,R)} X = \operatorname{domdim}_{(A,R)} X, \quad X \in A\text{-mod}. \tag{2.3.0.7}$$

Proof. Assume that there exists an (A,R)-exact sequence

$$0 \to X \to X_1 \to X_2 \to \cdots \to X_n \tag{2.3.0.8}$$

with projective (A,R)-injective left A-modules X_i for all $i \geq 1$. Recall that since P is (A,R)-injective, the functor $\operatorname{Hom}_A(-,P)$ is exact on (2.3.0.8). Since all X_i are projective there exists k_i such that $A^{k_i} \simeq X_i \oplus K_i$. Choose $k = \max\{k_1,\dots,k_n\}$. So, each X_i can be embedded in A^k as A-summand. Denote by $f_i\colon X_i \to A^{k_i}$, $g_i\colon A^{k_i} \to A^k$ the canonical injections and denote by $f_i'\colon A^{k_i} \to X_i$, $g_i'\colon A^k \to A^{k_i}$ the canonical projections. Since P is (A,R)-strongly faithful there exists an (A,R)-monomorphism $l\colon A \to P^t$ for some $t>0$. Hence, there exists $\pi \in \operatorname{Hom}_R(V^t, A)$ such that $\pi \circ l = \operatorname{id}_A$. Then, the composition $(\oplus_{j=1}^k l) \circ g_i \circ f_i \in \operatorname{Hom}_A(X_i, P^{tk})$ is an (A,R)-monomorphism. In fact, $f_i' \circ g_i' \circ (\oplus_{j=1}^k \pi) \in \operatorname{Hom}_R(V^{tk}, X_i)$ satisfies

$$f_i' \circ g_i' \circ (\oplus_{j=1}^k \pi) \circ (\oplus_{j=1}^k l) \circ g_i \circ f_i = f_i' \circ g_i' \circ \oplus_{j=1}^k \operatorname{id}_A \circ g_i \circ f_i = f_i' \circ g_i' \circ \operatorname{id}_{A^k} \circ g_i \circ f_i = f_i' \circ \operatorname{id}_{A^{k_i}} \circ f_i = \operatorname{id}_{X_i}.$$

As X_i is (A,R)-injective, then the map $(\oplus_{j=1}^k l) \circ g_i \circ f_i$ splits over A. Therefore, X_i is an A-summand of P^{tk}, hence $X_i \in \operatorname{add} P$.

If $X_i = 0$ for some i, then $\operatorname{domdim}_{(A,R)} X = +\infty = P-\operatorname{domdim}_{(A,R)} X$. This shows that if $\operatorname{domdim}_{(A,R)} X \geq n$ then $P-\operatorname{domdim}_{(A,R)} X \geq n$. Hence, $\operatorname{domdim}_{(A,R)} X \leq P-\operatorname{domdim}_{(A,R)} X$.

Now since each module in $\operatorname{add} P$ is projective (A,R)-injective, it follows that

$$P-\operatorname{domdim}_{(A,R)} X \leq \operatorname{domdim}_{(A,R)} X.$$

This concludes the proof. $\square$

Analogously, we have the right version,

Proposition 2.3.7. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra with* $\operatorname{domdim}_{(A,R)} A_A \geq 1$ *with given projective (A,R)-injective-strongly faithful right A-module V. Then,*

$$V-\operatorname{domdim}_{(A,R)} X = \operatorname{domdim}_{(A,R)} X, \quad X \in \text{mod-}A. \tag{2.3.0.9}$$

Proof. It is analogous to Proposition 2.3.6. $\square$

2.4 Relative Morita-Tachikawa correspondence and relative Mueller's characterization

2.4.1 Modules with relative dominant dimension at least two

Given $X \in A$-mod, $V \in$ mod-A, fix $C = \mathrm{End}_A(V)$ and denote by α_X the map $X \to \mathrm{Hom}_C(V, V \otimes_A X)$ given by $\alpha_X(x)(v) = v \otimes x$, $v \in V$, $x \in X$. This is an $(A, \mathrm{End}_A(X)^{op})$-bimodule homomorphism. In fact,

$$\alpha_X(a \cdot x)(v) = v \otimes ax = va \otimes x = \alpha_X(x)(va) = (a \cdot \alpha_X(x))(v),\ a \in A,\ v \in V,\ x \in X$$
$$\alpha_X(x \cdot b)(v) = \alpha_X(b(x))(v) = v \otimes b(x) = v \otimes (x \cdot b) = (v \otimes x) \cdot b = (\alpha_X(x) \cdot b)(v),\ b \in \mathrm{End}_A(X)^{op},\ v \in V,\ x \in X.$$

Lemma 2.4.1. *Let A be a projective Noetherian R-algebra with* $\mathrm{domdim}_{(A,R)} A_A \geq 1$ *with given (A,R)-strongly faithful projective (A,R)-injective right A-module V. Let F be the Schur functor* $\mathrm{Hom}_A(\mathrm{Hom}(V,A), -) \colon A\text{-mod} \to C\text{-mod}$.

For any $X \in A$-mod, *there exists an isomorphism* $\beta_X \in \mathrm{Hom}_A(\mathrm{Hom}_C(V, V \otimes_A X), \mathrm{Hom}_C(FA, FX))$ *making the following diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & \mathrm{Hom}_C(V, V \otimes_A X) \\ \| & & \downarrow{\scriptstyle \beta_X} \\ X & \xrightarrow{\eta_X} & \mathrm{Hom}_C(FA, FX) \end{array}$$

Proof. Denote by w_V the map $V \to \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),A)$, given by $w(v)(f) = f(v)$. Since V is a projective A-module, this map is an $(\mathrm{End}_A(V), A)$-bimodule isomorphism.

Fix $\psi_X \colon \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),A) \otimes_A X \to \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),X)$ according to Lemma 1.4.11. Then, define $\beta_X = \mathrm{Hom}_C(FA, \psi_X \circ w_V \otimes \mathrm{id}_X) \circ \mathrm{Hom}_C(w_V^{-1}, V \otimes_A X)$. Let $x \in X$. Then,

$$\begin{aligned} \mathrm{Hom}_C(FA, \psi_X \circ w_V \otimes \mathrm{id}_X) \circ \mathrm{Hom}_C(w_V^{-1}, V \otimes_A X)(\alpha_X(x)) &= \mathrm{Hom}_C(FA, \psi_X \circ w_V \otimes \mathrm{id}_X)(\alpha_X(x) \circ w_V^{-1}) \\ &= \psi_X \circ w_V \otimes \mathrm{id}_X \circ \alpha_X(x) \circ w_V^{-1}. \end{aligned} \tag{2.4.1.1}$$

For $v \in V,\ f \in \mathrm{Hom}_A(V,A)$,

$$\psi_X \circ w_V \otimes \mathrm{id}_X \circ \alpha_X(x)(v)(f) = \psi_X \circ w_V \otimes \mathrm{id}_X(v \otimes x)(f) = \psi_X(w_V(v) \otimes x)(f) = w_V(v)(f)x = f(v)x. \tag{2.4.1.2}$$

On the other hand, $\eta_X(x) \circ w_V(v)(f) = w_V(v)(f)x = f(v)x$. Therefore, composing with w_V^{-1} on both sides we conclude

$$\mathrm{Hom}_C(FA, \psi_X \circ w_V \otimes \mathrm{id}_X) \circ \mathrm{Hom}_C(w_V^{-1}, V \otimes_A X)(\alpha_X(x)) = \eta_X(x),\ x \in X.$$ ☐

In particular, since η is well behaved with respect to direct summands α is well behaved with respect to direct summands. The following lemma although technical is crucial for our purposes. This can be seen as the relative version of Proposition 4.8 of [Tac73].

Lemma 2.4.2. *Let P be a projective (A,R)-injective left A-module and let V be a projective right A-module (A,R)-strongly faithful. Fix $C = \mathrm{End}_A(V)$, $B = \mathrm{End}_A(P)^{op}$. Then, the following assertions hold.*

(a) The canonical map $\alpha_P \colon P \to \mathrm{Hom}_C(V, V \otimes_A P)$, given by $\alpha_P(p)(v) = v \otimes p$, $v \in V$, $p \in P$, is an isomorphism of (A,B)-bimodules.

(b) The canonical map $\psi\colon B \to \operatorname{End}_C(V\otimes_A P)^{op}$, given by $\psi(f)(v\otimes p) = v\otimes f(p)$, $f\in B, v\in V, p\in P$, is an isomorphism as left B-modules and as R-algebras.

(c) $V\otimes_A P$ is (C,R)-injective as left C-module.

Proof. We will start by showing that $\alpha := \alpha_P$ is an (A,R)-monomorphism. Since P is projective over A there are maps $k_P \in \operatorname{Hom}_A(P,A^s)$, $\pi_P \in \operatorname{Hom}_A(A^s,P)$ satisfying $\pi_P\circ k_P = \mathrm{id}_P$. Since V is (A,R)-strongly faithful there exists $i\in \operatorname{Hom}_A(A,V^t)$ and $\varepsilon \in \operatorname{Hom}_R(V^t,A)$ such that $\varepsilon\circ i = \mathrm{id}_A$. In addition, consider the A-maps $\nu_j \in \operatorname{Hom}_A(V,V^t)$, $\lambda_j\in\operatorname{Hom}_A(V^t,V)$ satisfying $\lambda_j\circ\nu_j = \mathrm{id}_V$, the multiplication map $\mu\in\operatorname{Hom}_A(V\otimes_A A,V)$ and the canonical maps $\gamma_j\in\operatorname{Hom}_A(V^s,(V^t)^s)$, $\gamma_j(v_1,\dots,v_s) = (\nu_j(v_1),\dots,\nu_j(v_s))$ for $1\leq j\leq t$.

Define τ the R-map $\operatorname{Hom}_C(V,V\otimes_A P)\to P$ given by

$$\tau(h) = \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j\circ\mu^s\circ \mathrm{id}_V\otimes_A k_P\circ h\circ\lambda_j\circ i(1_A),\ h\in\operatorname{Hom}_C(V,V\otimes_A P).$$

Hence,

$$\begin{aligned}
\tau\circ\alpha(p) &= \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j\circ\mu^s\circ\mathrm{id}_V\otimes_A k_P\circ\alpha(p)(\lambda_j\circ i(1_A)) = \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j\circ\mu^s\circ\mathrm{id}_V\otimes_A k_P(\lambda_j\circ i(1_A)\otimes p)\\
&= \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j\circ\mu^s(\lambda_j\circ i(1_A)\otimes k_P(p)) = \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j\circ\mu^s(\lambda_j\circ i(1_A)\otimes(k_P(p)_1,\dots,k_P(p)_s))\\
&= \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j(\lambda_j(i(1_A)k_P(p)_1),\dots,\lambda_j(i(1_A)k_P(p)_s))\\
&= \sum_j \pi_P\circ\varepsilon^s\circ\gamma_j(\lambda_j\circ i(k_P(p)_1),\dots,\lambda_j\circ i(k_P(p)_s))\\
&= \sum_j \pi_P\circ\varepsilon^s\circ(\nu_j\circ\lambda_j\circ i(k_P(p)_1),\dots,\nu_j\circ\lambda_j\circ i(k_P(p)_s)) = \pi_P\circ\varepsilon^s(i(k_P(p)_1),\dots,i(k_P(p)_s))\\
&= \pi_P(k_P(p)_1,\dots,k_P(p)_s) = \pi_P(k_P(p)) = p,\quad p\in P. \qquad (2.4.1.3)
\end{aligned}$$

Thus, $\tau\circ\alpha = \mathrm{id}_P$ and α is an (A,R)-monomorphism.

We claim that α is an essential embedding, that is, $\operatorname{im}\alpha\cap A\beta\neq 0$ if $0\neq\beta\in\operatorname{Hom}_C(V,V\otimes_A P)$.

Denote by $\pi_V\colon A^l\to V$, $k_V\colon V\to A^l$, $\pi_j\in\operatorname{Hom}_A(A^l,A), k_j\in\operatorname{Hom}_A(A,A^l)$ the canonical surjections and injections induced by the direct sum A^l, $1\leq j\leq t$. For each j, define $e_{V,j} = \pi_V\circ k_j(1_A)\in V$ and for each $y\in V$, define $\phi_{y,j}\in\operatorname{End}_A(V) = C$ given by $\phi_{y,j}(x) = y\cdot\pi_j\circ k_V(x)$, $x\in V$. Then,

$$\begin{aligned}
\sum_j \phi_{e_{V,j},j}\cdot v &= \sum_j \phi_{e_{V,j},j}(v) = \sum_j e_{V,j}\cdot\pi_j\circ k_V(v) = \sum_j \pi_V\circ k_j(1_A)\cdot\pi_j\circ k_V(v) = \sum_j \pi_V\circ k_j(1_A\pi_j\circ k_V(v))\\
&= \pi_V\circ k_V(v) = v. \qquad (2.4.1.4)
\end{aligned}$$

Let $0\neq\beta\in\operatorname{Hom}_C(V,V\otimes_A P)$. Hence, there exists $v\in V$ such that $\beta(v)\neq 0$. Moreover, for $y\in V$,

$$\sum_j \pi_j\circ k_V(v)\cdot\beta(y) = \sum_j \beta(y\pi_j\circ k_V(v)) = \sum_j\beta(\phi_{y,j}(v)) = \sum_j\beta(\phi_{y,j}\cdot v) = \sum_j \phi_{y,j}\beta(v). \qquad (2.4.1.5)$$

Assume that $\beta(v) = \sum_i x_i\otimes p_i\in V\otimes_A P$. Then,

$$\begin{aligned}
\sum_j \phi_{y,j}\beta(v) &= \sum_{j,i}\phi_{y,j}x_i\otimes p_i = \sum_{i,j}(\phi_{y,j}\cdot x_i)\otimes p_i = \sum_{i,j}(y\cdot\pi_j\circ k_V(x_i))\otimes p_i = \sum_{i,j} y\otimes\pi_j\circ k_V(x_i)p_i \qquad (2.4.1.6)\\
&= \alpha(\sum_{i,j}\pi_j\circ k_V(x_i)p_i)(y) \implies \alpha(\sum_{i,j}\pi_j\circ k_V(x_i)p_i) = (\sum_j \pi_j\circ k_V(v))\cdot\beta\in\operatorname{im}\alpha\cap A\beta. \qquad (2.4.1.7)
\end{aligned}$$

Since

$$\sum_j \pi_j \circ k_V(v)) \cdot \beta(e_{V,j}) = \sum_j \beta(e_{V,j}\pi_j \circ k_V(v))) = \sum_j \beta(\phi_{e_{V,j},j}v) = \beta(\sum_j \phi_{e_{V,j},j}v) = \beta(v) \neq 0,$$

it follows that α is an essential embedding.

Since P is (A,R)-injective and α is (A,R)-mono, there exists $h \in \mathrm{Hom}_A(\mathrm{Hom}_C(V, V \otimes_A P), P)$ such that $h \circ \alpha = \mathrm{id}_P$. Assume that there exists $0 \neq \beta \in \mathrm{im}(\mathrm{id}_{\mathrm{Hom}_C(V,V\otimes_A P)} - \alpha \circ h)$. As α is an essential embedding, $0 \neq \mathrm{im}\,\alpha \cap A\beta \subset \mathrm{im}\,\alpha \cap \mathrm{im}(\mathrm{id}_{\mathrm{Hom}_C(V,V\otimes_A P)} - \alpha \circ h) = 0$ which lead us to a contradiction. Thus, $\alpha \circ h = \mathrm{id}_{\mathrm{Hom}_C(V,V\otimes_A P)}$. So, α is an isomorphism.

The map ψ given in (b) is an B-homomorphism since

$$\psi(g \circ f)(v \otimes p) = \psi(f \circ g)(v \otimes p) = v \otimes f \circ g(p) = v \otimes f(p \cdot g) = (\mathrm{id}_V \otimes f)(v \otimes p \cdot g) \tag{2.4.1.8}$$
$$= (g \cdot (\mathrm{id}_V \otimes f))(v \otimes p),\ v \otimes p \in V \otimes_A P,\ f, g \in B. \tag{2.4.1.9}$$

The map ψ is a homomorphism of R-algebras since

$$\psi(g \cdot f) = \mathrm{id}_V \otimes_A (f \circ g) = \mathrm{id}_V \otimes_A f \circ \mathrm{id}_V \otimes_A g = \mathrm{id}_V \otimes_A g \cdot \mathrm{id}_V \otimes_A f = \psi(g) \cdot \psi(f),\ f, g \in B \tag{2.4.1.10}$$
$$\psi(\mathrm{id}_P) = \mathrm{id}_{V \otimes_A P}. \tag{2.4.1.11}$$

We claim that ψ is bijective. Towards this goal, our procedure will be as follows. We will construct a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "k_B"] \arrow[d, "\psi"] & & P^s \arrow[d, "\alpha_{P^s}"] \\ \mathrm{Hom}_C(V \otimes_A P, V \otimes_A P) \arrow[rr, "H"] & & \mathrm{Hom}_C(V, V \otimes_A P^s) \end{tikzcd}$$

where H will be a split mono and k_B is the natural injection.

Combining Lemma 1.4.26 with Lemma 2.4.1, we obtain by (a) that α_{P^s} is an isomorphism.

Since $A^s \simeq K \oplus P$ as A-modules we can see that, as right B-modules,

$$P^s \simeq \mathrm{Hom}_A(A^s, P) \simeq \mathrm{Hom}_A(P, P) \oplus \mathrm{Hom}_A(K, P) = B \oplus \mathrm{Hom}_A(K, P). \tag{2.4.1.12}$$

We denote by k_B, k_X the canonical injections of this direct sum (2.4.1.12) and π_B and π_X the canonical surjections, where $X = \mathrm{Hom}_A(K, P)$. So, explicitly, $k_B(b) = b \circ \pi_P(1_A, \dots, 1_A)$. We will by k_K and π_K the canonical injection $K \to A^s$ and the canonical surjection $A^s \to K$, respectively. In order to define H, we first consider the following isomorphism τ given by the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_C(V^s, V \otimes_A P) \arrow[r, "\tau"] \arrow[d, "\simeq"] & \mathrm{Hom}_C(V, V \otimes_A P^s) \\ \mathrm{Hom}_C(V, V \otimes_A P)^s \arrow[r, "\simeq"] & \mathrm{Hom}_C(V, (V \otimes_A P)^s) \arrow[u, "\mathrm{Hom}_C(V,\sigma)"'] \end{tikzcd}$$

where $\sigma(x_1 \otimes p_1, \dots, x_s \otimes p_s) = x_1 \otimes (p_1, 0, \dots, 0) + \dots + x_s \otimes (0, \dots, 0, p_s)$.

Consider $H = \tau \circ \mathrm{Hom}_C(V \otimes_A \pi_P \circ \theta \circ (\mu^{-1})^s, V \otimes_A P)$, where θ is the isomorphism $(V \otimes_A A)^s \to V \otimes_A A^s$. Then,

$$\begin{aligned} H \circ \psi(b)(v) &= \tau(\psi(b) \circ V \otimes_A \pi_P \circ \theta \circ (\mu^{-1})^s)(v) \\ &= \sigma(\psi(b) \circ V \otimes_A \pi_P \circ \theta \circ (\mu^{-1})^s(v, 0, \dots, 0), \dots, \psi(b) \circ V \otimes_A \pi_P \circ \theta \circ (\mu^{-1})^s(0, \dots, 0, v)) \end{aligned}$$

$$= \sigma(\psi(b) \circ V \otimes_A \pi_P \theta(v \otimes 1_A, 0, \dots, 0), \dots, \psi(b) \circ V \otimes_A \pi_P \theta(0, \dots, 0, v \otimes_A 1_A)) \tag{2.4.1.13}$$
$$= \sigma(\psi(b) \circ V \otimes_A \pi_P(v \otimes (1_A, 0, \dots, 0)), \dots, \psi(b) \circ V \otimes_A \pi_P(v \otimes (0, \dots, 0, 1_A))) \tag{2.4.1.14}$$
$$= \sigma(v \otimes b\pi_P(1_A, \dots, 0), \dots, v \otimes b\pi_P(0, \dots, 1_A)) = v \otimes b\pi_P(1_A, \dots, 1_A) \tag{2.4.1.15}$$
$$\alpha_{P^s} \circ k_B(b)(v) = \alpha_{P^s}(b \circ \pi_P(1_A, \dots, 1_A))(v) = v \otimes b\pi_P(1_A, \dots, 1_A),\ v \in V, b \in B. \tag{2.4.1.16}$$

Hence, $H \circ \psi$ is injective. In particular, ψ is injective. Since $V \otimes_A \pi_P \circ \theta \circ (\mu^{-1})^s \in \operatorname{Hom}_C(V^s, V \otimes_A P)$ is the surjection that gives $V \otimes_A P$ as C-summand of V^s the map $\operatorname{Hom}_C(V \otimes_A \pi_P \circ \theta \circ (\mu^{-1})^s, V \otimes_A P)$ is split monomorphism. So, H is a split monomorphism. Thus, there exists a map H' such that $H' \circ H = \mathrm{id}$. In particular, $\psi \circ \pi_B = H' \circ \alpha_{P^s} \circ k_B \circ \pi_B = H' \circ \alpha_{P^s}$ is surjective if $H' \circ \alpha_{P^s} \circ k_X \circ \pi_X = 0$. So, it remains to show that $H' \circ \alpha_{P^s} \circ k_X \circ \pi_X = 0$.

Observe that $H' = \operatorname{Hom}_C(\mu^s \circ \theta^{-1} \circ V \otimes_A k_P, V \otimes_A P) \circ \tau^{-1}$ and in the following $\pi_j^A \in \operatorname{Hom}_A(A^s, A)$, $k_j^A \in \operatorname{Hom}_A(A, A^s)$ will denote the surjections and injections of the direct sum A^s. We remark that the inverse of τ is given by the mapping

$$g \mapsto \left((v_1, \dots, v_s) \mapsto \sum_{i=1}^{s} (\sigma^{-1} \circ g(v_i))_i \right).$$

Thus,

$$\begin{aligned}
H' \circ \alpha_{P^s} \circ k_X \circ \pi_X(p_1, \dots, p_s)(v \otimes p) &= \tau^{-1}(\alpha_{P^s} \circ k_X \circ \pi_X(p_1, \dots, p_s))(\mu^s \circ \theta^{-1} \circ V \otimes_A k_P(v \otimes p)) \\
&= \tau^{-1}(\alpha_{P^s} \circ k_X \circ \pi_X(p_1, \dots, p_s))(v\pi_1^A(k_P(p)), \dots, v\pi_s^A(k_P(p))) \\
&= \sum_{i=1}^{s} \left(\sigma^{-1} \circ \alpha_{P^s} \circ k_X \circ \pi_X(p_1, \dots, p_s)(v\pi_i^A \circ k_P(p))\right)_i \\
&= \sum_{i=1}^{s} \left(\sigma^{-1}(v\pi_i^A \circ k_P(p) \otimes k_X \circ \pi_X(p_1, \dots, p_s))\right)_i \\
&= \sum_{i=1}^{s} v\pi_i^A \circ k_P(p) \otimes (k_X \circ \pi_X(p_1, \dots, p_s))_i \\
&= \sum_{i=1}^{s} v\pi_i^A \circ k_P(p) \otimes \sum_{j=1}^{s} \pi_j^A \circ k_K \circ \pi_K \circ k_i^A(1_A) p_j \\
&= v \otimes \sum_{i,j=1}^{s} \pi_j^A \circ k_K \circ \pi_K \circ k_i^A \circ \pi_i^A(k_P(p)) p_j \\
&= v \otimes \sum_{j=1}^{s} p i_j^A \circ k_K \circ \pi_K \circ k_P(p) p_j = 0, \quad p_i, p \in P,\ v \in V, 1 \leq i \leq s.
\end{aligned}$$

The last equality follows since $\pi_K \circ k_P = 0$. So, (b) follows.

Consider the canonical C-monomorphism $\varepsilon_{V \otimes_A P} \colon V \otimes_A P \to \operatorname{Hom}_R(C, V \otimes_A P)$. The following diagram is commutative

$$\begin{tikzcd}
\operatorname{Hom}_C(V, V \otimes_A P) \arrow[r, "\operatorname{Hom}_C(V,\varepsilon)"] & \operatorname{Hom}_C(V, \operatorname{Hom}_R(C, V \otimes_A P)) \arrow[d, "f"] \\
P \arrow[u, "\alpha_P"] \arrow[r, "\delta"] & \operatorname{Hom}_R(V, V \otimes_A P)
\end{tikzcd}$$

where $\delta \colon P \to \operatorname{Hom}_R(V, V \otimes_A P)$ is the morphism given by $\delta(p)(v) = v \otimes p$, and f is a canonical map given by Tensor-Hom adjunction. We want to show that the map δ is an (A, R)-monomorphism. For that purpose, we need further notation. Define τ' the R-map $\mathrm{Hom}_R(V, V\otimes_A P) \to P$ given by

$$\tau(h) = \sum_j \pi_P \circ \varepsilon^s \circ \gamma_j \circ \mu^s \circ \mathrm{id}_V \otimes_A k_P \circ h \circ \lambda_j \circ i(1_A),\ h \in \mathrm{Hom}_R(V, V\otimes_A P).$$

Using the same computations as in (2.4.1.3), it follows that $\tau'\circ\delta = \mathrm{id}_P$. Since P is (A,R)-injective, it follows that $P \in \mathrm{add}_A \mathrm{Hom}_R(V, V\otimes_A P)$. Therefore, $V\otimes_A P \in \mathrm{add}_C V\otimes_A \mathrm{Hom}_R(V, V\otimes_A P)$. By Lemma 1.4.14 and Lemma 1.1.63,

$$V\otimes_A \mathrm{Hom}_R(V, V\otimes_A P) \simeq V\otimes_A \mathrm{Hom}_C(V, \mathrm{Hom}_R(C, V\otimes_A P)) \simeq \mathrm{Hom}_R(C, V\otimes_A P). \tag{2.4.1.17}$$

Thus, $V\otimes_A P \in \mathrm{add}_C \mathrm{Hom}_R(C, V\otimes_A P)$ and $V\otimes_A P$ is (C,R)-injective. □

Lemma 2.4.3. *Let P be a projective left A-module (A,R)-strongly faithful and let V be a projective right A-module and (A,R)-injective. Fix $C = \mathrm{End}_A(V)$, $B = \mathrm{End}_A(P)^{op}$. Then, the following assertions hold.*

(a) The canonical map $\alpha_V\colon V \to \mathrm{Hom}_B(P, V\otimes_A P)$, given by $\alpha_V(v)(p) = v\otimes p$, $v\in V$, $p\in P$, is an isomorphism of (C,A)-bimodules.

(b) The canonical map $\psi_C\colon C \to \mathrm{End}_B(V\otimes_A P)$, given by $\psi_C(f)(v\otimes p) = f(v)\otimes p$, $f\in B, v\in V, p\in P$, is an isomorphism as left C-modules and as R-algebras.

(c) $V\otimes_A P$ is (B,R)-injective as right B-module.

Proof. It is the dual version of the Lemma 2.4.2. □

2.4.1.1 Relative QF3 algebras

At this point, it is not yet clear that the existence of a projective relative injective strongly faithful left module implies the existence of a projective relative injective strongly faithful right module. In particular, we cannot yet address the problem of left-right symmetry of relative dominant dimension. For this, we will need change of rings techniques. We are interested in the algebras which have positive relative dominant dimension which motivates the following definition.

Definition 2.4.4. Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $P \in A$-mod and $V \in$ mod-A. We call a triple (A,P,V) a **relative QF3 R-algebra, or just RQF3 algebra** provided P is a projective (A,R)-injective-strongly faithful left A-module and V is a projective (A,R)-injective-strongly faithful right A-module.

Given $X \in A$-mod, $V \in$ mod-A, denote by Φ_X the map $\mathrm{Hom}_A(V, DX)\otimes_C V \to DX$ defined by $\Phi_X(g\otimes v) = g(v)$, $v\in V$, $g\in \mathrm{Hom}_A(V, DX)$. This map is an $(\mathrm{End}_A(X)^{op}, A)$-bimodule homomorphism. Let $b \in \mathrm{End}_A(X)^{op}$, $g\otimes v \in \mathrm{Hom}_A(V, DX)\otimes_C V$ and $a\in A$. Then,

$$\Phi_X(b\cdot(g\otimes v)) = \Phi_X(b\cdot g)\otimes v) = (b\cdot g)(v) = bg(v) = b\Phi_X(g\otimes v), \tag{2.4.1.18}$$

$$\Phi_X((g\otimes v)\cdot a) = \Phi_X(g\otimes v\cdot a) = g(v\cdot a) = g(v)a = \Phi_X(g\otimes v)\cdot a. \tag{2.4.1.19}$$

Thus, Φ_X is an $(\mathrm{End}_A(X)^{op}, A)$-bimodule homomorphism.

Dually, we can define the map $\delta_Y\colon P\otimes_B \mathrm{Hom}_A(P, DY) \to DY$, given by $\delta_Y(p\otimes h) = h(p)$, $p\in P$, $h\in \mathrm{Hom}_A(P, DY)$ for any $P\in A$-mod and $Y\in$ mod-A.

In the same manner, δ_Y is an $(A, \mathrm{End}_A(Y))$-bimodule homomorphism.

Lemma 2.4.5. *Let (A,P,V) be a relative QF3 R-algebra. Fix $C=\operatorname{End}_A(V)$, $B=\operatorname{End}_A(P)^{op}$. Then, the following assertions hold.*

(a) $\operatorname{add}_A DV = \operatorname{add}_A P$. *Furthermore, B is Morita equivalent to C.*

(b) $V\otimes_A P$ *satisfies a double centralizer property*

$$\operatorname{End}_B(V\otimes_A P)\simeq C, \quad \operatorname{End}_C(V\otimes_A P)^{op}\simeq B$$

and $V\otimes_A P$ is a left (C,R)-injective-cogenerator and a right (B,R)-injective-cogenerator.

(c) $P\in \text{mod-}B$ *is a B-generator (B,R)-cogenerator and projective over R;*

(d) $V\in C\text{-mod}$ *is a C-generator (C,R)-cogenerator and projective over R.*

(e) The canonical map $\Phi_X\colon \operatorname{Hom}_A(V,DX)\otimes_C V\to DX$, *given by* $\Phi_X(g\otimes v)=g(v)$, $v\in V$, $g\in\operatorname{Hom}_A(V,DX)$, *is an A-isomorphism for any* $X\in\operatorname{add}_A P$.

(f) The canonical map $\delta_Y\colon P\otimes_B \operatorname{Hom}_A(P,DY)\to DY$, *given by* $\delta_Y(p\otimes h)=h(p)$, $p\in P$, $h\in\operatorname{Hom}_A(P,DY)$, *is an A-isomorphism for any* $Y\in\operatorname{add}_A V$.

Proof. By Lemma 1.2.56, DP is a projective (A,R)-injective right A-module and DV is projective (A,R)-injective left A-module. According to Lemma 2.2.4, $DP\in\operatorname{add}V$ and $DV\in\operatorname{add}P$. Hence, $P\in\operatorname{add}DV$ and $C\simeq\operatorname{End}_A(DV)^{op}$ is Morita equivalent to $B=\operatorname{End}_A(P)^{op}$. Thus, (a) follows.

Note that $D(V\otimes_A P)\simeq\operatorname{Hom}_A(P,DV)$. By (a), $P\in\operatorname{add}_A DV$. Hence,

$$_BB=\operatorname{Hom}_A(P,P)\in\operatorname{add}_B\operatorname{Hom}_A(P,DV)=\operatorname{add}_B D(V\otimes_A P). \tag{2.4.1.20}$$

Hence, $DB\in\operatorname{add}_B V\otimes_A P$. So, $V\otimes_A P$ is a right (B,R)-cogenerator. In the same fashion, by (a) $V\in\operatorname{add}_A DP$. Consequently,

$$C_C=\operatorname{Hom}_A(V,V)\in\operatorname{add}_C\operatorname{Hom}_A(V,DP)=\operatorname{add}_C D(V\otimes_A P). \tag{2.4.1.21}$$

Then, $V\otimes_A P$ is a left (C,R)-cogenerator. Now, due to Proposition 1.4.6, there exists a double centralizer property on $V\otimes_A P$ between C and B. By Lemma 2.4.3 (c) and Lemma 2.4.2 (c), (b) follows.

Since $P\in A\text{-proj}$ there exists $s>0$ such that $A^s\simeq P\oplus K$ as left A-modules. Thus, as right A-modules,

$$A^s\simeq\operatorname{Hom}_A(A,A_A)^s\simeq\operatorname{Hom}_A(A^s,A_A)\simeq\operatorname{Hom}_A(P\oplus K,A_A)\simeq\operatorname{Hom}_A(P,A_A)\oplus\operatorname{Hom}_A(K,A_A). \tag{2.4.1.22}$$

Therefore, as right B-modules

$$P^s\simeq A^s\otimes_A P\simeq\operatorname{Hom}_A(P,A_A)\oplus\operatorname{Hom}_A(K,A_A)\otimes_A P\simeq\operatorname{Hom}_A(P,A_A)\otimes_A P\oplus\operatorname{Hom}_A(K,A_A)\otimes_A P \tag{2.4.1.23}$$

$$\simeq\operatorname{Hom}_A(P,P)\oplus\operatorname{Hom}_A(K,A_A)\otimes_A P=B\oplus\operatorname{Hom}_A(K,A_A)\otimes_A P. \tag{2.4.1.24}$$

Hence, P is a right B-generator. In the same fashion, V is a left C-generator.

Since V is a projective right A-module, there exists $t>0$ such that $A^t\simeq V\oplus K'$ as right A-modules. So, as right B-modules,

$$P^t\simeq A^t\otimes_A P\simeq(V\oplus K')\otimes_A P\simeq V\otimes_A P\oplus K'\otimes_A P. \tag{2.4.1.25}$$

Hence, $V\otimes_A P\in\operatorname{add}_B P$. In particular, by (b) P is also a right (B,R)-cogenerator. In the same way, V is a left (C,R)-cogenerator. This completes the proof for (c) and (d).

We claim that Φ_X and δ_X are compatible with direct sums. Let $X = X_1 \oplus X_2 \in A$-mod. Denote by k_i the canonical injections and π_i the canonical projections $i = 1,2$. This follows from the following commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_A(V, D(X_1 \oplus X_2)) \otimes_C V & \xrightarrow{\Phi_{X_1 \oplus X_2}} & D(X_1 \oplus X_2) \\ \Big\downarrow {\scriptstyle (Dk_1 \circ -, Dk_2 \circ -) \otimes_C \mathrm{id}_V} & & \Big\downarrow {\scriptstyle (Dk_1, Dk_2)} \\ \operatorname{Hom}_A(V, DX_1) \otimes_C V \oplus \operatorname{Hom}_A(V, DX_2) \otimes_C V & \xrightarrow{\Phi_{X_1} \oplus \Phi_{X_2}} & DX_1 \oplus DX_2 \end{array} .$$

In fact,

$$\Phi_{X_1} \oplus \Phi_{X_2} \circ (Dk_1 \circ -, Dk_2 \circ -) \otimes_C \mathrm{id}_V (g \otimes v) = \Phi_{X_1} \oplus \Phi_{X_2} (Dk_1 \circ g \otimes v, Dk_2 \circ g \otimes v) \tag{2.4.1.26}$$

$$= (Dk_1 \circ g(v), Dk_2 \circ g(v)) = (g(v) \circ k_1, g(v) \circ k_2) \tag{2.4.1.27}$$

$$(Dk_1, Dk_2) \circ \Phi_{X_1 \oplus X_2} (g \otimes v) = (Dk_1, Dk_2)(g(v)) = (Dk_1(g(v)), Dk_2(g(v))) \tag{2.4.1.28}$$

$$= ((g(v) \circ k_1, g(v) \circ k_2), g \otimes v \in \operatorname{Hom}_A(V, D(X_1 \oplus X_2)) \otimes_C V.$$

Since both columns are isomorphisms it follows our claim. The reasoning for δ_X is analogous.

Now since Φ_{DV} is the isomorphism $\operatorname{Hom}_A(V, DDV) \otimes_C V \simeq \operatorname{Hom}_A(V,V) \otimes_C V \simeq C \otimes_C V \simeq V \simeq DDV$ it follows that Φ_X is an isomorphism for any $X \in \operatorname{add} DV = \operatorname{add} P$. $\square$

We should remark that the statement of Lemma 2.4.5 is a generalization of (5.1) of [Tac73].

Remark 2.4.6. The canonical map $\Phi\colon \operatorname{Hom}_A(V,Y) \otimes_C V \to Y$ is an A-isomorphism for any $Y \in \operatorname{Add}_A(V)$. This follows from the fact that the tensor product commutes with arbitrary coproducts and since V is a finitely generated projective A-module the Hom functor $\operatorname{Hom}_A(V, -)$ commutes with arbitrary coproducts (see [Zim14, Lemma 4.1.9]). Hence, we can apply the same argument as in Lemma 2.4.5. The dual statement also holds for the canonical maps δ. $\triangle$

The importance of these canonical maps Φ_X and α_X stems from the following theorem.

Proposition 2.4.7. *Let (A, P, V) be a relative QF3 R-algebra. Fix $C = \operatorname{End}_A(V)$, $B = \operatorname{End}_A(P)^{op}$.*
Let $X \in A$-mod$\cap R$-proj and let $Y \in$ mod-$A \cap R$-proj, then:

- *(a)* $\operatorname{domdim}_{(A,R)} X \geq 1$ *if and only if the canonical map $\Phi_X\colon \operatorname{Hom}_A(V, DX) \otimes_C V \to DX$ is an epimorphism.*
- *(b) If* $\operatorname{domdim}_{(A,R)} X \geq 1$, *then $\alpha_X\colon X \to \operatorname{Hom}_C(V, V \otimes_A X)$ is an (A,R)-monomorphism. If, in addition,* $\operatorname{Hom}_A(V, DX) \otimes_C V \in R$-proj, *then* $\operatorname{domdim}_{(A,R)} X \geq 1$ *if and only if $\alpha_X\colon X \to \operatorname{Hom}_C(V, V \otimes_A X)$ is an (A,R)-monomorphism.*
- *(c)* $\operatorname{domdim}_{(A,R)} Y \geq 1$ *if and only if the canonical map $\delta_Y\colon P \otimes_B \operatorname{Hom}_A(P, DY) \to DY$ is an epimorphism.*
- *(d) If* $\operatorname{domdim}_{(A,R)} Y \geq 1$, *then $\alpha_Y\colon Y \to \operatorname{Hom}_B(P, Y \otimes_A P)$ is a right (A,R)-monomorphism. If, in addition,* $P \otimes_B \operatorname{Hom}_A(P, DY) \in R$-proj, *then* $\operatorname{domdim}_{(A,R)} Y \geq 1$ *if and only if $\alpha_Y\colon Y \to \operatorname{Hom}_B(P, Y \otimes_A P)$ is a right (A,R)-monomorphism.*
- *(e) The following assertions are equivalent:*
  - *(i)* $\operatorname{domdim}_{(A,R)} X \geq 2$*;*
  - *(ii) The canonical map $\Phi_X\colon$ $\operatorname{Hom}_A(V, DX) \otimes_C V \to DX$ is a right A-isomorphism;*
  - *(iii)* $\operatorname{Hom}_A(V, DX) \otimes_C V \in R$-proj *and the canonical map $\alpha_X\colon X \to \operatorname{Hom}_C(V, V \otimes_A X)$ is a left A-isomorphism.*

(f) The following assertions are equivalent:

(i) $\operatorname{domdim}_{(A,R)} Y \geq 2$*;*

(ii) The canonical map $\delta_Y \colon P \otimes_B \operatorname{Hom}_A(P, DY) \to DY$ *is a left A-isomorphism;*

(iii) $\operatorname{Hom}_A(V, DX) \otimes_C V \in R\text{-proj}$ *and the canonical map* $\alpha_Y \colon Y \to \operatorname{Hom}_B(P, Y \otimes_A P)$ *is a right A-isomorphism.*

Proof. (a). Assume that $\operatorname{domdim}_{(A,R)} X \geq 1$. Then, there exists an (A,R)-monomorphism $f \colon X \to X_0$ with $X_0 \in \operatorname{add} DV = \operatorname{add} P$. In particular, Df is a surjective map. Applying $\operatorname{Hom}_A(V, D-) \otimes_C V$ yields the following diagram with exact rows

$$\begin{tikzcd} \operatorname{Hom}_A(V, DX_0) \otimes_C V \arrow[r, "\operatorname{Hom}_A(V, Df) \otimes_C \operatorname{id}_V"] \arrow[d, "\Phi_{X_0}"] & \operatorname{Hom}_A(V, DX) \otimes_C V \arrow[r] \arrow[d, "\Phi_X"] & 0 \\ DX_0 \arrow[r, "Df"] & DX \arrow[r] & 0 \end{tikzcd}$$

Hence, Φ_X is surjective because $Df \circ \Phi_{X_0}$ is. Conversely, assume that Φ_X is an epimorphism.

Observe that $\operatorname{Hom}_C(V, M)$ is a projective (A,R)-injective left A-module for any (C,R)-injective left module M which is projective over R. In fact, $\operatorname{Hom}_C(V, DC) \simeq \operatorname{Hom}_R(C \otimes_C V, R) \simeq DV$ is a projective (A,R)-injective left A-module. Moreover, every (A,R)-injective projective over R belongs to $\operatorname{add}_C DC$, so $\operatorname{Hom}_C(V, M) \in A\text{-proj} \cap \operatorname{add} DA$.

Consider a projective presentation over C $P_0 \xrightarrow{g} \operatorname{Hom}_A(V, DX) \to 0$. The functor $- \otimes_C V$ is right exact, so $g \otimes_C \operatorname{id}_V$ is surjective. So, $\Phi_X \circ g \otimes_C \operatorname{id}_V \colon P_0 \otimes_C V \to DX$ is surjective, by assumption. As $X \in R\text{-proj}$, $DX \in R\text{-proj}$ and consequently, $\Phi_X \circ g \otimes_C \operatorname{id}_V$ is a right (A,R)-epimorphism. So, applying D yields an (A,R)-monomorphism $X \to D(P_0 \otimes_C V) \simeq \operatorname{Hom}_C(V, DP_0)$. Hence, $\operatorname{domdim}_{(A,R)} X \geq 1$.

(b). We can relate the maps Φ_X and α_X using the following commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_C(V, V \otimes_A X) \arrow[r, "\operatorname{Hom}_C(V, V \otimes_A w_X)", "\simeq"'] & \operatorname{Hom}_C(V, V \otimes_A DDX) \arrow[r, "\operatorname{Hom}_C(V, \iota_{V,DX})", "\simeq"'] & \operatorname{Hom}_C(V, D\operatorname{Hom}_A(V, DX)) \arrow[d, "\kappa_{V, D\operatorname{Hom}_A(V,DX)}", "\simeq"'] \\ & & D(DD\operatorname{Hom}_A(V, DX) \otimes_C V) \arrow[d, "D(w_{\operatorname{Hom}_A(V,\, DX)} \otimes_C \operatorname{id}_V)", "\simeq"'] \\ X \arrow[uu, "\alpha_X"] \arrow[r, "w_X", "\simeq"'] & DDX \arrow[r, "D\Phi_X"] & D(\operatorname{Hom}_A(V, DX) \otimes_C V) \end{tikzcd} \tag{2.4.1.29}$$

Here w_X denotes the natural transformation from the identity to the double dual functor. As $X \in R\text{-proj}$ and $\operatorname{Hom}_A(V, DX) \in R\text{-proj}$ w_X and $w_{\operatorname{Hom}_A(V,DX)}$ are isomorphisms. The isomorphism $\iota_{V,DX}$ and $\kappa_{V, \operatorname{Hom}_A(V,DX)}$ are according to Proposition 1.1.65.

Diagram 2.4.1.29 is commutative because

$$D\Phi_X \circ w_X(x)(f \otimes v) = w_X(x) \circ \Phi_X(f \otimes v) = w_X(x)(f(v)) \tag{2.4.1.30}$$

$$D(w_{\operatorname{Hom}_A(V,DX)} \otimes_C \operatorname{id}_V) \circ \kappa_{V, D\operatorname{Hom}_A(V,DX)} \circ \operatorname{Hom}_C(V, \iota_{V,DX}) \circ \operatorname{Hom}_C(V, V \otimes_A w_X) \circ \alpha_X(x)(f \otimes v) = \tag{2.4.1.31}$$

$$= \kappa_{V, D\operatorname{Hom}_A(V,DX)}(\iota_{V,DX} \circ V \otimes_A w_X \circ \alpha_X(x)) \circ w_{\operatorname{Hom}_A(V,DX)} \otimes_C \operatorname{id}_V(f \otimes v) = \tag{2.4.1.32}$$

$$= w_{\operatorname{Hom}_A(V,DX)}(f)(\iota_{V,DX} \circ V \otimes_A w_X \circ \alpha_X(x)(v)) = w_{\operatorname{Hom}_A(V,DX)}(f)(\iota_{V,DX}(v \otimes w_X(x))) = \tag{2.4.1.33}$$

$$= \iota_{V,DX}(v \otimes w_X(x))(f) = w_X(x)(f(v)), \ x \in X, f \otimes v \in \operatorname{Hom}_A(V, DX) \otimes_C V. \tag{2.4.1.34}$$

Assume that $\operatorname{domdim}_{(A,R)} X \geq 1$. Then, by (a), Φ_X is an (A,R)-epimorphism. Thus, $D\Phi_X$ is an (A,R)-monomorphism. By diagram (2.4.1.29), α_X is an (A,R)-monomorphism. Assume now that α_X is an (A,R)-monomorphism and $\operatorname{Hom}_A(V,DX)\otimes_C V \in R\text{-proj}$. Then, $D\alpha_X$ is an (A,R)-epimorphism. Applying D to (2.4.1.29), we deduce that $DD\Phi_X$ is surjective. Because of $\operatorname{Hom}_A(V,DX)\otimes_C V \in R\text{-proj}$ $w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is an isomorphism. Thus, $w_{DX}\circ\Phi_X = DD\Phi_X \circ w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is surjective. Since $DX \in R\text{-proj}$, Φ_X is surjective. By (a), $\operatorname{domdim}_{(A,R)} X \geq 1$.

The assertions (c) and (d) are analogous to (a) and (b), respectively.

(e). Assume that (i) holds. By definition, there exists an (A,R)-exact sequence $0 \to X \xrightarrow{\varepsilon_0} P_0 \xrightarrow{\varepsilon_1} P_1$ with $P_0, P_1 \in \operatorname{add} P$. Applying D yields the exact sequence

$$DP_1 \xrightarrow{D\varepsilon_1} DP_0 \xrightarrow{D\varepsilon_0} DX \to 0. \tag{2.4.1.35}$$

The functor $\operatorname{Hom}_A(V,-)\otimes_C V$ is right exact, hence applying $\operatorname{Hom}_A(V,-)\otimes_C V$ to (2.4.1.35) yields the following commutative diagram with exact rows

$$\begin{tikzcd}
DP_1 \arrow[r, "D\varepsilon_1"] & DP_0 \arrow[r, "D\varepsilon_0"] & DX \arrow[r] & 0 \\
\operatorname{Hom}_A(V,DP_1)\otimes_C V \arrow[u, "\Phi_{P_1}"] \arrow[r, "\operatorname{Hom}_A(V,D\varepsilon_1)\otimes_C V"] & \operatorname{Hom}_A(V,DP_0)\otimes_C V \arrow[u, "\Phi_{P_0}"] \arrow[r, "\operatorname{Hom}_A(V,D\varepsilon_0)\otimes_C V"] & \operatorname{Hom}_A(V,DX)\otimes_C V \arrow[u, "\Phi_X"] \arrow[r] & 0
\end{tikzcd}.$$

In fact,

$$\Phi_X \circ \operatorname{Hom}_A(V,D\varepsilon_0)\otimes_C V(f\otimes v) = \Phi_X(D\varepsilon_0\circ f\otimes v) = D\varepsilon_0\circ f(v) \tag{2.4.1.36}$$

$$D\varepsilon_0\circ\Phi_{P_0}(f\otimes v) = D\varepsilon_0(f(v)),\ f\otimes v \in \operatorname{Hom}_A(V,DP_0)\otimes_C V. \tag{2.4.1.37}$$

By Lemma 2.4.5, Φ_{P_0}, Φ_{P_1} are isomorphisms. By diagram chasing, we deduce that Φ_X is an isomorphism. So (ii) holds.

Assume that (ii) holds. Φ_X induces the isomorphism as R-modules $\operatorname{Hom}_A(V,DX)\otimes_C V \simeq DX \in R\text{-proj}$. In particular, $D\Phi_X$ is an isomorphism. Using diagram (2.4.1.29), we deduce that α_X is an isomorphism. Thus, (iii) follows. Now consider a projective C-resolution for $\operatorname{Hom}_A(V,DX)$, $P_1 \to P_0 \to \operatorname{Hom}_A(V,DX) \to 0$. Applying $-\otimes_C V$ we obtain the exact sequence

$$P_1\otimes_C V \to P_0\otimes_C V \to \operatorname{Hom}_A(V,DX)\otimes_C V \to 0. \tag{2.4.1.38}$$

Since Φ_X and $X \in R\text{-proj}$ is an isomorphism this yields an (A,R)-exact sequence

$$P_1\otimes_C V \to P_0\otimes_C V \to DX \to 0. \tag{2.4.1.39}$$

Finally, applying D yields an (A,R)-exact sequence

$$0 \to X \to D(P_0\otimes_C V) \to D(P_1\otimes_C V). \tag{2.4.1.40}$$

As we have seen $D(P_i\otimes_C V) \in A\text{-proj}\cap\operatorname{add} DA$, $i = 1,2$, therefore $\operatorname{domdim}_{(A,R)} X \geq 2$. So, (i) holds.

Assume that (iii) holds. By diagram (2.4.1.29), $D\Phi_X$ is an isomorphism. Since $\operatorname{Hom}_A(V,DX)\otimes_C V \in R\text{-proj}$ $w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is an isomorphism. So, $w_{DX}\circ\Phi_X = DD\Phi_X\circ w_{\operatorname{Hom}_A(V,DX)\otimes_C V}$ is an isomorphism. Thus, (ii) follows.

The argument for (f) is analogous to (e). ☐

Here we can see that for a commutative ring, a module having relative dominant dimension at least two is equivalent to a stronger type of the double centralizer property $DV \otimes_C V \simeq DA$, which over fields is exactly the double centralizer property $\mathrm{End}_C(V)^{op} \simeq A$.

This situation raises the question of which situations can the R-module $DV \otimes_C V$ be at least projective over R. The following lemma answers this question for relative QF3 R-algebras with a left or right relative dominant dimension greater than or equal to two.

Lemma 2.4.8. *Let (A, P, V) be a relative QF3 R-algebra. Fix $C = \mathrm{End}_A(V)$ and $B = \mathrm{End}_A(P)^{op}$. If* $\mathrm{domdim}_{(A,R)}\, {}_AA \geq 2$ *or* $\mathrm{domdim}_{(A,R)}\, A_A \geq 2$*, then $DV \otimes_C V \in R$-proj and $P \otimes_B DP \in R$-proj.*

The result is a consequence of the following lemma.

Lemma 2.4.9. *Let X be a left B-progenerator and $C = \mathrm{End}_B(X)^{op}$. Consider the equivalence of categories $F = \mathrm{Hom}_B(X, -)\colon B\text{-mod} \to C\text{-mod}$ and $G = \mathrm{Hom}_B(\mathrm{Hom}_B(X, B), -)\colon \text{mod-}B \to \text{mod-}C$. Then, for any $M \in \text{mod-}B$, $N \in B\text{-mod}$, $\mathrm{add}_R(M \otimes_B N) = \mathrm{add}_R(GM \otimes_C FN)$.*

Proof. By Corollary 1.4.21,

$$\begin{aligned} GM \otimes_C FN &\simeq \mathrm{Hom}_B(\mathrm{Hom}_B(X,B), M) \otimes_C \mathrm{Hom}_B(X,N) \simeq M \otimes_B \mathrm{Hom}_B(\mathrm{Hom}_B(X,M),B) \otimes_C \mathrm{Hom}_B(X,B) \otimes_B N \\ &\simeq M \otimes_B X \otimes_C \mathrm{Hom}_B(X,B) \otimes_B N \simeq M \otimes_B X \otimes_C \mathrm{Hom}_C(X,C) \otimes_B N \simeq M \otimes_B \mathrm{Hom}_C(X,X) \otimes_B N \\ &\simeq M \otimes_B B \otimes_B N \simeq M \otimes_B N. \end{aligned}$$

□

Proof of Lemma 2.4.8. By Lemma 2.4.5(b), $C \simeq \mathrm{End}_B(D(V \otimes_A P))$ with $D(V \otimes_A P)$ a left B-progenerator. Thus, $F = \mathrm{Hom}_B(D(V \otimes_A P), -)$ and $G = \mathrm{Hom}_B(\mathrm{Hom}_B(D(V \otimes_A P), B), -)$. Note that by Lemma 2.4.3(a)

$$FDP = \mathrm{Hom}_B(D(V \otimes_A P), DP) \simeq \mathrm{Hom}_B(P, V \otimes_A P) \simeq V, \tag{2.4.1.41}$$

$$GP \simeq \mathrm{Hom}_B(\mathrm{Hom}_B(D(V \otimes_A P), B), DDP) \simeq \mathrm{Hom}_R(\mathrm{Hom}_B(D(V \otimes_A P), B) \otimes_B DP, R) \tag{2.4.1.42}$$

$$\simeq D\mathrm{Hom}_B(D(V \otimes_A P), DP) \simeq D\mathrm{Hom}_B(P, V \otimes_A P) \simeq DV \tag{2.4.1.43}$$

The last isomorphism follows from Lemma 2.4.3. Consequently,

$$\mathrm{add}_R(P \otimes_B DP) = \mathrm{add}_R(GP \otimes_C FDP) = \mathrm{add}_R(DV \otimes_C V).$$

If $\mathrm{domdim}_{(A,R)}\, {}_AA \geq 2$, then according to Proposition 2.4.7(e),

$$DV \otimes_C V \simeq \mathrm{Hom}_A(V, DA) \otimes_C V \simeq DA \in R\text{-proj}. \tag{2.4.1.44}$$

If $\mathrm{domdim}_{(A,R)}\, A_A \geq 2$, then according to Proposition 2.4.7(f),

$$P \otimes_B DP \simeq P \otimes_B \mathrm{Hom}_A(P, DA) \simeq DA \in R\text{-proj}.$$

□

2.4.2 Relative Morita-Tachikawa correspondence

For finite-dimensional algebras the Morita-Tachikawa states that every finite-dimensional algebra with dominant dimension greater than or equal to two is the endomorphism algebra of a generator-cogenerator. We will present in the following the relative version of this statement now for projective Noetherian algebras.

Theorem 2.4.10 (General case)**.** *Let R be a commutative Noetherian ring. There is a bijection:*

$$\left\{ (B,M)\colon \begin{array}{c} B \textit{ a projective} \\ \textit{Noetherian } R\textit{-algebra,} \\ M \textit{ a } B\textit{-generator } (B,R)\textit{-cogenerator,} \\ M \in R\text{-proj}, \\ DM\otimes_B M \in R\text{-proj} \end{array} \right\} \Big/ \sim_1 \longleftrightarrow \left\{ A\colon \begin{array}{c} A \textit{ a projective Noetherian} \\ R\textit{-algebra with} \\ \operatorname{domdim}_{(A,R)} {}_A A \geq 2, \\ \operatorname{domdim}_{(A,R)} A_A \geq 2 \end{array} \right\} \Big/ \sim_2$$

In this notation, $A \sim_2 A'$ if and only if A and A' are isomorphic, whereas, $(B,M) \sim_1 (B',M')$ if and only if there is an equivalence of categories $F\colon B\text{-mod} \to B'\text{-mod}$ such that $M' = FM$.

$$(B,M) \mapsto A = \operatorname{End}_B(M)^{op}$$
$$(\operatorname{End}_A(N), N) \leftarrow\!\shortmid A$$

where N is a projective (A,R)-injective-strongly faithful right A-module.

Proof. We will start by checking that $\sim_1$ is an equivalence relation. The reflexive property is clear using the identity functor $\mathrm{id}_{A\text{-mod}}$. The symmetry property is also clear using the quasi-inverse functor of F. The transitive property follows using the composition of the equivalence of categories. Let A be a projective Noetherian R-algebra with right and left relative dominant dimension greater than or equal to two. Hence, by definition, there exists $P \in A\text{-mod} \cap R\text{-proj}$ and $V \in \text{mod-}A \cap R\text{-proj}$ such that (A,P,V) is a RQF3 algebra. Let $B = \operatorname{End}_A(V)$. Since V is a projective right A-module B is a projective Noetherian R-algebra. Since R is Noetherian, it follows that B is Noetherian. By Lemma 2.4.5(d), V is a left B-generator (B,R)-cogenerator and projective over R. By Lemma 2.4.8, $DV \otimes_B V \in R\text{-proj}$. Furthermore, by Proposition 2.4.7, there holds the double centralizer property $A \simeq \operatorname{End}_B(V)^{op}$. If there exists another pair (P',V') such that (A,P',V') is RQF3, then we deduce by Lemma 2.2.4 that $\operatorname{add}_A V = \operatorname{add}_A V'$. So, $(\operatorname{End}_A(V'),V') \sim_1 (B,V)$.

Conversely, let (B,M) be a pair such that B is a projective Noetherian R-algebra and M is a B-generator (B,R)-cogenerator satisfying $M, DM\otimes_B M \in R\text{-proj}$. Define $A = \operatorname{End}_B(M)^{op}$. Since $DM \otimes_R M$, it follows that $A = \operatorname{Hom}_B(M,M) \simeq D(DM \otimes_B M) \in R\text{-proj}$. Thus, A is a projective Noetherian R-algebra. As M is a B-generator $M^t \simeq B \oplus K$. In particular, there exists a surjective B-homomorphism $\phi\colon M^t \twoheadrightarrow B$ for some $t > 0$. Let $\pi_j \in \operatorname{Hom}_B(M^t, M)$ and $k_j \in \operatorname{Hom}_B(M, M^t)$, $1 \leq j \leq t$, be the canonical surjections and injections, respectively. In particular, $1_B = \sum_j \phi \circ k_j(m_j)$ for some $m_j \in M$, $1 \leq j \leq t$. For any $x \in M$, define $h_x \in \operatorname{Hom}_B(B,M)$ satisfying $h_x(1_B) = x$. Then, $t_x \circ \phi \circ k_j \in \operatorname{Hom}_B(M,M) = A$, $1 \leq j \leq t$. Then, for any $x \in M$,

$$x = t_x(1_B) = t_x(\sum_j \phi \circ k_j(m_j)) = \sum_j t_x \circ \phi \circ k_j(m_j) = \sum_j m_j \cdot t_x \circ \phi \circ k_j. \tag{2.4.2.1}$$

This shows that M is finitely generated as a right A-module.

As a result of M being a B-generator, we can write

$$A^t \simeq \operatorname{Hom}_B(M,M_A)^t \simeq \operatorname{Hom}_B(M^t,M_A) \simeq \operatorname{Hom}_B(B\oplus K, M_A) \simeq \operatorname{Hom}_B(B,M_A) \oplus \operatorname{Hom}_B(K,M_A) \tag{2.4.2.2}$$
$$\simeq M \oplus \operatorname{Hom}_B(K,M_A). \tag{2.4.2.3}$$

Hence, M is a projective right A-module. On the other hand, as M is a (B,R)-cogenerator, we can write

$$A^s \simeq \operatorname{Hom}_B(M_A,M)^s \simeq \operatorname{Hom}_B(M_A,M^s) \simeq \operatorname{Hom}_B(M_A, DB\oplus K') \simeq \operatorname{Hom}_B(M_A,DB) \oplus \operatorname{Hom}_B(M,K') \tag{2.4.2.4}$$
$$\simeq \operatorname{Hom}_B(B,DM) \oplus \operatorname{Hom}_B(M,K') \simeq DM \oplus \operatorname{Hom}_B(M,K'), \tag{2.4.2.5}$$

for some $s > 0$ and $K' \in B$-mod. Therefore, DM is a projective left A-module, and consequently, M is (A,R)-injective as right module. Hence, M is a projective (A,R)-injective right A-module. Consider a left projective B-presentation for M, $P_1 \to P_0 \to M \to 0$. Due to $DM \otimes_B M \in R$-proj applying $DM \otimes_B -$ yields the (A,R)-exact sequence

$$DM \otimes_B P_1 \to DM \otimes_B P_0 \to DM \otimes_B M \to 0. \tag{2.4.2.6}$$

Now applying D yields the right (A,R)-exact sequence

$$0 \to A \to D(DM \otimes_B P_0) \to D(DM \otimes_A P_1). \tag{2.4.2.7}$$

Observe that $D(DM \otimes_B P_i) \simeq \operatorname{Hom}_B(P_i, M) \in \operatorname{add} M$, $i = 1, 2$. Hence, the (A,R)-monomorphism $A \to D(DM \otimes_B P_0)$ makes M an (A,R)-strongly faithful module and (2.4.2.7) implies $\operatorname{domdim}_{(A,R)} A_A \geq 2$. Consider now a right projective B-presentation for DM, $Q_1 \to Q_0 \to DM \to 0$. Applying $- \otimes_B M$ yields the (A,R)-exact sequence

$$Q_1 \otimes_B M \to Q_0 \otimes_B M \to DM \otimes_B M \to 0. \tag{2.4.2.8}$$

Applying D we obtain the (A,R)-exact sequence

$$0 \to A \to D(Q_0 \otimes_B M) \to D(Q_1 \otimes_B M). \tag{2.4.2.9}$$

Here $D(Q_i \otimes_B M) \simeq \operatorname{Hom}_B(Q_i, DM) \in \operatorname{add} DM$. Therefore, (2.4.2.9) yields that $\operatorname{domdim}_{(A,R)\,A} A \geq 2$ and DM is an (A,R)-strongly faithful module.

As generators satisfy the double centralizer property we have that $B \simeq \operatorname{End}_A(M)$. If $(B,M) \simeq_1 (B',M')$, then by Corollary 1.4.23, $A = \operatorname{End}_B(M)^{op} \simeq \operatorname{End}_{B'}(M')^{op}$. This concludes the proof. $\square$

We should emphasize the importance of R being a commutative Noetherian ring in the proof of the relative Morita-Tachikawa correspondence. Furthermore, we remark that using finitely generated modules in Definition 2.1.1 of relative dominant dimension instead of general modules is no mistake. One of the reasons is that the Hom functors do not preserve in general arbitrary direct sums. Consequently, the techniques employed in relative Morita-Tachikawa correspondence would not hold in such a general setting.

Moreover, it follows from equation 2.4.2.1 the following result which goes back to [Mor58].

Corollary 2.4.11. *Let B be a projective Noetherian R-algebra. Let M be a generator in B-*Mod*. Then, M is finitely generated as an* $\operatorname{End}_B(M)^{op}$*-module.*

Therefore, it is not expected that a version of Morita-Tachikawa correspondence can hold in general for arbitrary commutative non-Noetherian rings. Nonetheless, if such a version happens to exist it should involve at very least compact modules in order to solve the problems of Hom regarding direct sums.

The surprise in this relative version is that we are only interested in the generators relative cogenerators that satisfy $DM \otimes_B M \in R$-proj. Modules are faithful over its endomorphism algebras. The importance of the property $DM \otimes_B M \in R$-proj lies on the fact that this is a sufficient condition for a given B-module M to be strongly faithful over its endomorphism algebra. Later in Proposition 2.5.14, we will see a characterization of this property and what it means for the endomorphism algebra $\operatorname{End}_B(M)$ in terms of base change properties.

2.4.3 Relative Morita-Tachikawa correspondence in case of Krull dimension one

For regular commutative Noetherian rings with Krull dimension less or equal to one, we can drop the condition $DM \otimes_B M \in R$-proj in the relative Morita-Tachikawa correspondence and we can reformulate the relative Morita-Tachikawa correspondence in the following way.

Theorem 2.4.12. *Let R be a commutative regular Noetherian ring with Krull dimension less than or equal to one. There is a bijection between*

$$\left\{ (B,M) : \begin{array}{c} B \textit{ a projective Noetherian} \\ R\textit{-algebra, } M \in R\text{-proj} \\ M \textit{ a } B\textit{-generator } (B,R)\textit{-cogenerator} \end{array} \right\} \Big/ \sim_1 \longleftrightarrow \left\{ A : \begin{array}{c} A \textit{ a projective} \\ \textit{Noetherian } R\textit{-algebra with} \\ \operatorname{domdim}_{(A,R)} {}_A A \geq 1, \\ \operatorname{domdim}_{(A,R)} A_A \geq 1, \\ \textit{all projective} \\ (A,R)\textit{-injective-strongly faithful} \\ \textit{modules satisfy the} \\ \textit{double centralizer property} \end{array} \right\} \Big/ \sim_2$$

In this notation, $A \sim_2 A'$ if and only if A and A' are isomorphic, whereas, $(B,M) \sim_1 (B',M')$ if and only if there is an equivalence of categories $F\colon B\text{-mod} \to B'\text{-mod}$ such that $M' = FM$.

$$(B,M) \mapsto A = \operatorname{End}_B(M)^{op}$$
$$(\operatorname{End}_A(N),N) \hookleftarrow A$$

where N is a projective (A,R)-injective-strongly faithful right A-module.

Proof. Let A be a projective Noetherian R-algebra with $\operatorname{domdim}_{(A,R)} A_A \geq 1$, $\operatorname{domdim}_{(A,R)} {}_A A \geq 1$ and all projective (A,R)-injective-strongly faithful modules satisfy the double centralizer property. Hence, there exists $P \in A$-mod and $V \in$ mod-A such that (A,P,V) is a relative QF3 R-algebra. Define $B = \operatorname{End}_A(V)$. As V is a projective right A-module, B is a projective Noetherian R-algebra. By Lemma 2.4.5, V is a left B-generator (B,R)-cogenerator. By assumption, V satisfies the double centralizer property, thus $A \simeq \operatorname{End}_B(V)^{op}$. By the same argument as in relative Morita-Tachikawa, correspondence, the mapping $\hookleftarrow$ is well defined.

Conversely, let (B,M) with $M \in B\text{-mod} \cap R\text{-proj}$ a B-generator (B,R)-cogenerator. Define $A = \operatorname{End}_B(M)^{op}$. Note that $A = \operatorname{Hom}_B(M,M) \subset \operatorname{Hom}_R(M,M) \in \operatorname{add}_R M$. Since R has Krull dimension less or equal than one, and A is an R-submodule of a projective module then A is projective as R-module. Thus, A is a projective Noetherian R-algebra. As in the proof of Theorem 2.4.10, M is a projective (A,R)-injective finitely generated A-module that satisfies the double centralizer property. Consider a projective presentation for M, $P_1 \to P_0 \to M \to 0$. Applying $DM \otimes_B -$ we get the exact sequence

$$DM \otimes_B P_1 \to DM \otimes_B P_0 \to DM \otimes_B M \to 0. \tag{2.4.3.1}$$

Now, applying D yields the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & D(DM \otimes_B P_0) \arrow[r] \arrow[d, equal] & D(DM \otimes_B P_1) & \\ 0 \arrow[r] & A \arrow[r] & D(DM \otimes_B P_0) \arrow[r] & \text{coker} \arrow[r] \arrow[u] & 0 \end{tikzcd}.$$

By Snake Lemma, the map $\text{coker} \to D(DM \otimes_B P_1) \simeq \operatorname{Hom}_B(P_1,M)$ is a monomorphism and $\operatorname{Hom}_B(P_1,M) \in$

$\operatorname{add} M$. As $\dim R \leq 1$, coker $\in R$-proj. Thus, the monomorphism $A \to D(DM \otimes_B P_0)$ is an (A,R)-monomorphism. It follows that $\operatorname{domdim}_{(A,R)} A_A \geq 1$. Using a projective resolution for DM and applying $D \circ - \otimes_B M$ we deduce that $\operatorname{domdim}_{(A,R)} {}_A A \geq 1$. In particular, (A, DM, M) is a RQF3 algebra and there exists an A-exact sequence

$$0 \to A \to X_0 \to X_1, \tag{2.4.3.2}$$

with $X_0, X_1 \in \operatorname{add} DM$. Now assume that V is another right projective (A,R)-injective-strongly faithful module. Then, (A, DM, V) is a RQF3 algebra. By Lemma 2.4.5(a), $\operatorname{add}_A M = \operatorname{add}_A V$. Then, $(C := \operatorname{End}_A(V), V) \sim_1 (B, M)$. Now applying $\operatorname{Hom}_C(V, V \otimes_A -)$ to (2.4.3.2) yields a commutative diagram where the map $A \to \operatorname{Hom}_C(V,V)$ appears. Combining such diagram with Lemma 2.4.3, we deduce that the canonical map $A \to \operatorname{Hom}_C(V,V)$ is an isomorphism, therefore (A,V) has the double centralizer property. ☐

In general, we know very little about the properties of the natural inclusion

$$\operatorname{End}_C(V) \to \operatorname{End}_R(V) \tag{2.4.3.3}$$

even in the case where V is a left C-generator. In particular, one question that arises is when this map splits over R. A relation between this property and relative dominant dimension can be found in the next proposition.

Proposition 2.4.13. *Let (A, P, V) be a relative QF3 R-algebra. Fix $C = \operatorname{End}_A(V)$.*

(a) If $\operatorname{domdim}(A,R) \geq 2$, then the canonical inclusion

$$i\colon \operatorname{End}_C(V) \hookrightarrow \operatorname{End}_R(V) \tag{2.4.3.4}$$

splits over R.

(b) Assume also that the splitting map $\tau\colon \operatorname{End}_R(V) \to \operatorname{End}_C(V)$ satisfies the following two properties:

$$\tau(h \circ g) = h \circ \tau(g), \quad \tau(g \circ h) = \tau(g) \circ h, \ g \in \operatorname{End}_R(V), h \in \operatorname{End}_C(V). \tag{2.4.3.5}$$

Let $\delta\colon M_{i+1} \to M_i \to M_{i-1}$ be a (C,R)-exact sequence. If $\operatorname{Hom}_C(V, M_{i+1}) \to \operatorname{Hom}_C(V, M_i) \to \operatorname{Hom}_C(V, M_{i-1})$ is exact and $M_i \in R$-proj, then the sequence $\operatorname{Hom}_C(V, \delta)$ is (A,R)-exact.

Proof. By Proposition 2.4.7, $\Phi_A\colon DV \otimes_C V \to DA$ is an isomorphism. In particular, $DV \otimes_C V \in R$-proj. Consider the canonical R-epimorphism $\varepsilon\colon DV \otimes_R V \to DV \otimes_C V$, given by $f \otimes v \mapsto f \otimes v, f \in DV, v \in V$. So, ε splits over R. Using the commutativity of the diagram with bijective columns

$$\begin{tikzcd}
0 \arrow[r] & D(DV \otimes_C V) \arrow[r, "D\varepsilon"] & D(DV \otimes_R V) \\
0 \arrow[r] & \operatorname{Hom}_C(V, DDV) \arrow[r] \arrow[u, "\simeq"] & \operatorname{Hom}_R(V, DDV) \arrow[u, "\simeq"] \\
0 \arrow[r] & \operatorname{Hom}_C(V,V) \arrow[r, "i"] \arrow[u, "\simeq"] & \operatorname{Hom}_R(V,V) \arrow[u, "\simeq"]
\end{tikzcd}$$

we obtain that the natural inclusion i splits over R.

Assume that the splitting map $\tau\colon \operatorname{End}_R(V) \to \operatorname{End}_C(V)$ satisfies the following two properties:

$$\tau(h \circ g) = h \circ \tau(g), \quad \tau(g \circ h) = \tau(g) \circ h, \ g \in \operatorname{End}_R(V), h \in \operatorname{End}_C(V). \tag{2.4.3.6}$$

Let

$$M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \tag{2.4.3.7}$$

be an (C,R)-exact sequence. Hence, there are maps $h_j \in \operatorname{Hom}_R(M_j, M_{j+1})$ satisfying $f_{i+1} \circ h_i + h_{i-1} \circ f_i = \mathrm{id}_{M_i}$, $j = i, i-1$.

Since V is C-generator there exists a surjective $\pi^{(i)} \colon V^{t_i} \to M_i$. As $M_i \in R$-proj, there exists $k^{(i)} \in \operatorname{Hom}_R(M_i, V^{t_i})$ such that $\pi^{(i)} \circ k^{(i)} = \mathrm{id}_{M_i}$. Let $\pi_j^{(i)}$ and $k_j^{(i)}$ be the canonical surjections and inclusions of the direct sum V^{t_i}. Since V is a (C,R)-cogenerator, M_i can be embedded in V^s through a map $l^{(i)}$. Denote by ϕ_z and ν_z the canonical projections and injections of the direct sum V^s. Define the map $H_i \colon \operatorname{Hom}_C(V, M_i) \to \operatorname{Hom}_C(V, M_{i+1})$, given by $H_i(g) = \sum_j \pi^{(i+1)} k_j^{(i+1)} \tau(\pi_j^{(i+1)} k^{(i+1)} h_i g)$ for each $g \in \operatorname{Hom}_C(V, M_i)$. For any $g \in \operatorname{Hom}_C(V, M_i)$,

$$l^{(i)}(\operatorname{Hom}_C(V, f_{i+1} \circ H_i + H_{i-1} \circ \operatorname{Hom}_C(V, f_i)))(g) = l^{(i)}(f_{i+1} \circ H_i(g) + H_{i-1}(f_i \circ g)) \tag{2.4.3.8}$$

$$= \sum_{z,j} \nu_z(\phi_z l^{(i)} f_{i+1} \pi^{(i+1)} k_j^{(i+1)} \tau(\pi_j^{(i+1)} k^{(i+1)} h_i g) + \phi_z l^{(i)} \pi^{(i)} k_j^{(i)} \tau(\pi_j^{(i)} k^{(i)} h_{i-1} f_i g)) \tag{2.4.3.9}$$

$$= \sum_z \nu_z(\tau(\phi_z l^{(i)} f_{i+1} \pi^{(i+1)} \sum_j k_j^{(i+1)} \pi_j^{(i+1)} h_i g) + \tau(\phi_z l^{(i)} \pi^{(i)} \sum_j k_j^{(i)} \pi_j^{(i)} k^{(i)} h_{i-1} f_i g)) \tag{2.4.3.10}$$

$$= \sum_z \nu_z \tau(\phi_z l^{(i)} f_{i+1} h_i g + \phi_z l^{(i)} h_{i-1} f_i g) = \sum_z \nu_z \tau(\phi_z l^{(i)} g) = \sum_z \nu_z \phi_z l^{(i)} g = l^{(i)} g. \tag{2.4.3.11}$$

Therefore, $\operatorname{Hom}_C(V, f_{i+1} \circ H_i + H_{i-1} \circ \operatorname{Hom}_C(V, f_i) = \mathrm{id}_{\operatorname{Hom}_C(V, M_i)}$. Analogously, we can see the same statement holds for the functor $\operatorname{Hom}_C(-, V)$. $\square$

The existence of such a map may not exist in general, otherwise, every module should satisfy the property $\operatorname{Hom}_A(V, DM) \otimes_C V \in R$-proj. However, such a map with the given properties exists for relative separable algebras (see for example [Hat63, 2.2]).

2.4.4 Mueller's characterization of relative dominant dimension

We will now study how to compute the relative dominant dimension of a module in terms of the homology over the endomorphism algebra of a projective relative injective strongly faithful module.

The following technical lemma will be useful for the relative Mueller theorem.

Lemma 2.4.14. *Consider the following commutative diagram with one exact row*

$$\begin{tikzcd} X_0 \arrow[r, "\alpha_0"] & X_1 \arrow[rr, "\alpha_1"] \arrow[dr, "\varepsilon"'] & & X_2 \arrow[r, "\alpha_2"] & X_3 \\ & & Y \arrow[ur, "t"'] & & \end{tikzcd} .$$

The following assertions hold.

(i) If ε is surjective and $\varepsilon \circ \alpha_0 = 0$, then t is mono.

(ii) If t is mono and $\alpha_2 \circ t = 0$, then ε is surjective.

Proof. (i). Let $y \in \ker t$. Since ε is surjective, we can write $y = \varepsilon(x)$ for some $x \in X_1$. Thus, $\alpha_1(x) = t\varepsilon(x) = t(y) = 0$. So, $x \in \operatorname{im} \alpha_0 = \ker \alpha_1$. Hence, $y = \varepsilon(\alpha_0(z)) = 0$ for some $z \in X_0$. Hence, t is injective.

(ii). Let $y \in Y$. Then, $t(y) \in \ker \alpha_2 = \operatorname{im} \alpha_1$. So, we can write $t(y) = \alpha_1(x) = t\varepsilon(x)$ for some $x \in X_1$. As t is injective, $y = \varepsilon(x)$. $\square$

Let $X \in A$-mod. Denote by $\Omega^i(X, P^\bullet)$ the i-th syzygy of X with respect to a projective A-resolution $P^\bullet$. Naturally, $\Omega^0(X, P^\bullet) \simeq X$ for any $P^\bullet$ and $\Omega^i(X, P^\bullet) \in R$-proj whenever $X \in R$-proj.

Theorem 2.4.15. *Let (A, P, V) be a relative QF3 R-algebra. Fix $C = \operatorname{End}_A(V)$. For any projective R-module left A-module M, the following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq n \geq 2$;

(ii) $\phi_M\colon \operatorname{Hom}_A(V, DM) \otimes_C V \to DM$ *is an isomorphism and* $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0,\ 1 \leq i \leq n-2$;

(iii) $\alpha_M\colon M \to \operatorname{Hom}_C(V, V \otimes_A M)$ *is an isomorphism,* $\Omega^i(\operatorname{Hom}_A(V, DM), P^\bullet) \otimes_C V \in R\text{-proj},\ 0 \leq i \leq n-2$ *for every projective C-resolution $P^\bullet$ of* $\operatorname{Hom}_A(V, DM)$ *and* $\operatorname{Ext}_C^i(V, V \otimes_A M) = 0,\ 1 \leq i \leq n-2$.

Proof. $(i) \implies (ii)$. By Proposition 2.4.7, Φ_M is an isomorphism. By definition, there exists an (A, R)-exact sequence

$$0 \to M \xrightarrow{\varepsilon} X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_{n-1}, \tag{2.4.4.1}$$

with projective (A, R)-injective A-modules X_i. The functor $\operatorname{Hom}_A(V, -)$ is exact, and since D preserves (A, R)-exact sequences, applying $\operatorname{Hom}_A(V, D-)$ yields the exact sequence

$$\operatorname{Hom}_A(V, DX_{n-1}) \xrightarrow{\operatorname{Hom}_A(V, Df_{n-1})} \operatorname{Hom}_A(V, DX_{n-2}) \to \cdots \to \operatorname{Hom}_A(V, DX_0) \xrightarrow{\operatorname{Hom}_A(V, D\varepsilon)} \operatorname{Hom}_A(V, DM) \to 0. \tag{2.4.4.2}$$

As $\operatorname{Hom}_A(V, DX_i) \in \operatorname{add} \operatorname{Hom}_A(V, V) = C$-proj, we can extend (2.4.4.2) to a projective C-resolution of $\operatorname{Hom}_A(V, DM)$, $P^\bullet$, where $P_i = \operatorname{Hom}_A(V, DX_i)$, $0 \leq i \leq n-1$. Applying $- \otimes_C V$ we get the following commutative diagram so that the top row is exact.

$$\begin{tikzcd}
DX_{n-1} \arrow[r, "Df_{n-1}"] & DX_{n-2} \arrow[r] & \cdots \arrow[r] & DX_0 \arrow[r, two heads, "D\varepsilon"] & DM \\
\operatorname{Hom}_A(V, DX_{n-1}) \otimes_C V \arrow[u, "\Phi_{X_{n-1}}"] \arrow[r, "\operatorname{Hom}_A(V, Df_{n-1}) \otimes_C V"] & \operatorname{Hom}_A(V, DX_{n-2}) \otimes_C V \arrow[u, "\Phi_{X_{n-2}}"] \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_A(V, DX_0) \otimes_C V \arrow[u, "\Phi_{X_0}"] \arrow[r, two heads, "\operatorname{Hom}_A(V, D\varepsilon) \otimes_C V"] & \operatorname{Hom}_A(V, DM) \otimes_C V \arrow[u, "\Phi_M"]
\end{tikzcd}.$$

According to Lemma 2.4.5, the maps Φ_M and Φ_{X_i}, $i = 1, \ldots, n-1$ are isomorphisms. Thus, the bottom row is exact. Thus,

$$\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = \ker \operatorname{Hom}_A(V, Df_i) \otimes_C V / \operatorname{im} \operatorname{Hom}_A(V, Df_{i+1}) \otimes_C V = 0, \quad 1 \leq i \leq n-2. \tag{2.4.4.3}$$

$(ii) \implies (iii)$. By Proposition 2.4.7, $\operatorname{Hom}_A(V, DM) \otimes_C V \simeq D(V \otimes_A M) \otimes_C V \in R$-proj and α_M is an isomorphism. Let

$$\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} D(V \otimes_A M) \to 0. \tag{2.4.4.4}$$

be an arbitrary projective C-resolution of $D(V \otimes_A M)$. In particular, for every $1 \leq i \leq n-2$, we have the following exact sequence

$$0 \to \Omega^i(\operatorname{Hom}_A(V, DM), P^\bullet) \xrightarrow{k_i} P_{i-1} \xrightarrow{p_{i-1}} P_{i-2} \to \cdots \to P_0 \to D(V \otimes_A M) \to 0, \tag{2.4.4.5}$$

where $P^\bullet$ is the deleted projective resolution of (2.4.4.4). It follows from $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0$, $1 \leq i \leq n-2$ the existence of the following exact sequence and factorization of $p_i \otimes_C V$

$$P_{n-1} \otimes_C V \xrightarrow{p_{n-1} \otimes_C V} P_{n-2} \otimes_C V \to \cdots \to P_0 \otimes_C V \to D(V \otimes_A M) \otimes_C V \to 0, \tag{2.4.4.6}$$

$$\begin{tikzcd}[column sep=small]
P_{i+1}\otimes_C V \arrow[r, "p_{i+1}\otimes_C V"] & P_i\otimes_C V \arrow[rr, "p_i\otimes_C V"] \arrow[dr, two heads, "\varepsilon_i\otimes_C V"'] & & P_{i-1}\otimes_C V \arrow[r, "p_{i-1}\otimes_C V"] & P_{i-2}\otimes_C V \\
& & \Omega^i(\mathrm{Hom}_A(V,DM),P^\bullet)\otimes_C V \arrow[ur, "k_i\otimes_C V"'] & &
\end{tikzcd},$$

where ε_i is the map given in the factorization (epi, mono) $k_i\varepsilon_i = p_i$. For the case $i=1$, we can take $P_{-1} = D(V\otimes_A M)$. Observe that $0 = p_ip_{i+1} = k_i\varepsilon_ip_{i+1}$. Hence, $\varepsilon_ip_{i+1} = 0$ because k_i is a mono. Consequently, $\varepsilon_i\otimes_C V p_{i+1}\otimes_C V = 0$. By Lemma 2.4.14, $k_i\otimes_C V$ is a monomorphism, and thus

$$\Omega^i(\mathrm{Hom}_A(V,DM),P^\bullet)\otimes_C V \simeq \mathrm{im}(p_i\otimes_C V) = \ker(p_{i-1}\otimes_C V)\in R\text{-proj} \tag{2.4.4.7}$$

since $D(V\otimes_A M)\otimes_C V\in R$-proj and every $P_i\in R$-proj. By Tensor-Hom adjunction there exists the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & D(D(V\otimes_A M)\otimes_C V) \arrow[r] \arrow[d, "\simeq"] & D(P_0\otimes_C V) \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r] & D(P_{n-1}\otimes_C V) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}_C(V,V\otimes_A M) \arrow[r] & \mathrm{Hom}_C(V,DP_0) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_C(V,DP_{n-1})
\end{tikzcd}, \tag{2.4.4.8}$$

such that every column is an isomorphism. The upper row is just the exact sequence obtained by applying D to the (A,R)-exact sequence (2.4.4.6), and therefore it is exact. Now, the commutativity of diagram (2.4.4.8) yields that the bottom row of (2.4.4.8) is exact. Taking into account that $0\to V\otimes_A M\to DP_0\to DP_1\to\cdots$ is a (C,R)-injective resolution ($D(V\otimes_A M)\in R$-proj, the exactness of the bottom row of (2.4.4.8) means that $\mathrm{Ext}^i_{(C,R)}(V,V\otimes_A M) = 0$, $1\leq i\leq n-2$. Again, since $V\otimes_A M\in R$-proj and $V\in R$-proj the standard (C,R)-projective resolution of V is a projective C-resolution of V. Therefore,

$$\mathrm{Ext}^i_C(V,V\otimes_A M) = \mathrm{Ext}^i_{(C,R)}(V,V\otimes_A M) = 0,\ 1\leq i\leq n-2.$$

(iii) $\implies$ (i). We shall proceed by induction on k to show that if $\alpha_M\colon M\to \mathrm{Hom}_C(V,V\otimes_A M)$ is an isomorphism, $\Omega^i(\mathrm{Hom}_A(V,DM),P^\bullet)\otimes_C V\in R$-proj, $0\leq i\leq k-2$ for every projective C-resolution $P^\bullet$ of $\mathrm{Hom}_A(V,DM)$ and $\mathrm{Ext}^i_C(V,V\otimes_A M) = 0$, $1\leq i\leq k-2$ then $\mathrm{domdim}_{(A,R)} M\geq k\geq 2$. If $k=2$, then the result holds by Proposition 2.4.7. Assume that the result holds for a given k satisfying $n>k>2$. Assume, in addition, that $\alpha_M\colon M\to \mathrm{Hom}_C(V,V\otimes_A M)$ is an isomorphism, $\Omega^i(\mathrm{Hom}_A(V,DM),P^\bullet)\otimes_C V\in R$-proj, $0\leq i\leq k-1$ for every projective C-resolution $P^\bullet$ of $\mathrm{Hom}_A(V,DM)$ and $\mathrm{Ext}^i_C(V,V\otimes_A M)=0$, $1\leq i\leq k-1$. By induction, $\mathrm{domdim}_{(A,R)} M\geq k$. So, there exists a (A,R)-exact sequence

$$0\to M\xrightarrow{\alpha_0} X_0\xrightarrow{\alpha_1} X_1\to\cdots\to X_{k-1}, \tag{2.4.4.9}$$

with all $X_i\in \mathrm{add}\, DV$. Applying $V\otimes_A -$ yields the (C,R)-exact sequence

$$0\to V\otimes_A M\to V\otimes_A X_0\to V\otimes_A X_1\to\cdots\to V\otimes_A X_{k-1}. \tag{2.4.4.10}$$

Now, observe that $D(V\otimes_A X_i)\simeq \mathrm{Hom}_A(V,DX_i)\in \mathrm{add}\,\mathrm{Hom}_A(V,DDV) = C$-proj. So, we can extend (2.4.4.10) to a (C,R)-injective resolution of $V\otimes_A M$, $I^\bullet$. Furthermore, we have the (epi, mono) factorization

$$\begin{tikzcd}
V\otimes_A X_{k-2} \arrow[rr, "V\otimes_A\alpha_{k-1}"] \arrow[dr, two heads, "\varepsilon"'] & & V\otimes_A X_{k-1} \\
& D\Omega^{k-1}(D(V\otimes_A M),D((V\otimes_A X)^\bullet)) \arrow[ur, hook, "t"'] &
\end{tikzcd},$$

where $(V \otimes_A X)^\bullet$ denotes the deleted (C,R)-injective resolution obtained by $I^\bullet$. Denote by Ω the module $D\Omega^{k-1}(D(V \otimes_A M), D((V \otimes_A X)^\bullet)$. Since $\operatorname{Ext}^i_C(V, V \otimes_A M) = \operatorname{Ext}^i_{(C,R)}(V, V \otimes_A M) = 0$, $i \leq k-1$ applying $\operatorname{Hom}_C(V,-)$ to the (C,R)-injective $I^\bullet$ we obtain the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}_C(V, V \otimes_A M) \arrow[r, hook] & \operatorname{Hom}_C(V, V \otimes_A X_0) \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A X_{k-2}) \arrow[r, "\operatorname{Hom}_C(V, V\otimes_A \alpha_{k-1})"] \arrow[d, "\operatorname{Hom}_C(V,\varepsilon)"'] & \operatorname{Hom}_C(V, V \otimes_A X_{k-1}) \arrow[r, "\operatorname{Hom}_C(V,i_k)"] & \operatorname{Hom}_C(V, I_k) \\
 & & & \operatorname{Hom}_C(V, \Omega) \arrow[ur, hook, "\operatorname{Hom}_C(V,t)"'] & &
\end{tikzcd} \tag{2.4.4.11}$$

where $\operatorname{Hom}_C(V,t)$ is injective and $\ker i_k = \operatorname{im} V \otimes_A \alpha_{k-1}$. Note that $0 = i_k V \otimes -A\alpha_{k-1} = i_k t \varepsilon$. Thus, $i_k t = 0$ since ε is surjective. Now, as $\operatorname{Hom}_C(V, i_k) \circ \operatorname{Hom}_C(V,t) = \operatorname{Hom}_C(V, i_k t) = 0$, it follows by Lemma 2.4.14(ii) that $\operatorname{Hom}_C(V, \varepsilon)$ is surjective. On the other hand,

$$\operatorname{Hom}_C(V, \Omega) \simeq D(\Omega^{k-1}(D(V \otimes_A M), D((V \otimes_A X)^\bullet) \otimes_C V) \in R\text{-proj}. \tag{2.4.4.12}$$

Hence, the exact sequence

$$0 \to \operatorname{Hom}_C(V, V \otimes_A M) \to \operatorname{Hom}_C(V, V \otimes_A X_0) \to \cdots \to \operatorname{Hom}_C(V, V \otimes_A X_{k-2}) \to \operatorname{Hom}_C(V, \Omega) \to 0 \tag{2.4.4.13}$$

is (A,R)-exact. As $M \simeq \operatorname{Hom}_C(V, V \otimes_A M)$ and each $\operatorname{Hom}_C(V, V \otimes_A X_i) \simeq X_i \in \operatorname{add} DV$ it is enough to show that $\operatorname{Hom}_C(V, \Omega)$ has relative dominant dimension greater than or equal to two. In such a case, there exists $Y_0, Y_1 \in \operatorname{add} DV$ and an (A,R)-exact sequence $0 \to \operatorname{Hom}_C(V, \Omega) \to Y_0 \to Y_1$. Combining this (A,R)-exact sequence with (2.4.4.13) we obtain an (A,R)-exact sequence

$$0 \to M \to \operatorname{Hom}_C(V, V \otimes_A X_0) \to \cdots \to \operatorname{Hom}_C(V, V \otimes_A X_{k-2}) \to Y_0 \to Y_1. \tag{2.4.4.14}$$

This would imply that $\operatorname{domdim}_{(A,R)} M \geq k+1$.

We can see that by Lemma 1.4.14 and by assumption on the R-projectivity of the $k-1$ syzygy that

$$\operatorname{Hom}_A(V, D\operatorname{Hom}_C(V, \Omega)) \otimes_C V \simeq D(V \otimes_A \operatorname{Hom}_C(V, \Omega)) \otimes_C V) \simeq D(\Omega) \otimes_C V \tag{2.4.4.15}$$

$$\simeq \Omega^{k-1}(D(V \otimes_A M), D((V \otimes_A X)^\bullet) \otimes_C V \in R\text{-proj}. \tag{2.4.4.16}$$

By Lemma 1.4.14, the map ξ_Ω is an isomorphism. Moreover,

$$\operatorname{Hom}_C(V, \xi_\Omega) \circ \alpha_{\operatorname{Hom}_C(V,\Omega)}(f)(v) = \xi_\Omega(v \otimes f) = f(v), \ f \in \operatorname{Hom}_C(V, \Omega), v \in V. \tag{2.4.4.17}$$

Thus, $\operatorname{Hom}_C(V, \xi_\Omega) \circ \alpha_{\operatorname{Hom}_C(V,\Omega)} = \operatorname{id}_{\operatorname{Hom}_C(V,\Omega)}$. It follows that $\alpha_{\operatorname{Hom}_C(V,\Omega)}$ is an isomorphism. By Proposition 2.4.7, $\operatorname{domdim}_{(A,R)} \operatorname{Hom}_C(V, \Omega) \geq 2$. □

Theorem 2.4.16. *Let (A,P,V) be a relative QF3 R-algebra. Fix $B = \operatorname{End}_A(P)^{op}$. For any right A-module M being projective over R, the following assertions are equivalent.*

- *(a)* $\operatorname{domdim}_{(A,R)} M \geq n \geq 2$;
- *(b)* $\delta_M \colon P \otimes_B \operatorname{Hom}_A(P, DM) \to DM$ *is an isomorphism and* $\operatorname{Tor}_i^B(P, \operatorname{Hom}_A(P, DM)) = 0$, $1 \leq i \leq n-2$*;*
- *(c)* $\alpha_M \colon M \to \operatorname{Hom}_B(P, M \otimes_A P)$ *is an isomorphism,* $P \otimes_B \Omega^i(\operatorname{Hom}_A(P, DM), Q^\bullet) \in R\text{-proj}$, $0 \leq i \leq n-2$ *for every left projective B-resolution $Q^\bullet$ of* $\operatorname{Hom}_A(P, DM)$ *and* $\operatorname{Ext}^i_B(P, M \otimes_A P) = 0$, $1 \leq i \leq n-2$.

Proof. The proof is analogous to Theorem 2.4.15. □

Comparing this version with the Mueller's characterization of dominant dimension over Artinian algebras, we can see that the functors Tor take a more important role than Ext. Furthermore, condition (c) does not seem very practical to use in applications since we have to test every syzygy of a projective resolution of $\mathrm{Hom}_A(V, DM)$. However, if the ground ring is regular, using Ext is still useful provided we know the Krull dimension of the ground ring.

Remark 2.4.17. By Observation 2.4.6 we can deduce as in Theorem 2.4.15 that the existence of an (A,R)-exact sequence

$$Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y \to 0, \tag{2.4.4.18}$$

where $Y_i \in \mathrm{Add}_A V$, $1 \leq i \leq n$, for a given $Y \in \text{Mod-}A$ is equivalent to requiring $\Phi\colon \mathrm{Hom}_A(V,Y) \otimes_C V \to Y$ to be an isomorphism and $\mathrm{Tor}_i^C(\mathrm{Hom}_A(V,Y), V) = 0$, $1 \leq i \leq n-2$. △

Proposition 2.4.18. *Let R be a commutative Noetherian regular ring. Let (A,P,V) be a relative QF3 R-algebra. Fix $C = \mathrm{End}_A(V)$ and $B = \mathrm{End}_A(P)^{op}$. Let $n \geq 2$, $M \in A\text{-mod} \cap R\text{-proj}$, and $N \in \text{mod-}A \cap R\text{-proj}$. The following assertions hold.*

(i) If $\alpha_M\colon M \to \mathrm{Hom}_C(V, V \otimes_A M)$ is an isomorphism and $\mathrm{Ext}_C^i(V, V \otimes_A M) = 0$ for every $1 \leq i \leq n-2$, then $\mathrm{domdim}_{(A,R)} M \geq n - \dim R$.

(ii) If $\alpha_N\colon N \to \mathrm{Hom}_B(P, N \otimes_A P)$ is an isomorphism and $\mathrm{Ext}_B^i(P, N \otimes_A P) = 0$ for every $1 \leq i \leq n-2$, then $\mathrm{domdim}_{(A,R)} N \geq n - \dim R$.

Proof. If $\dim R \geq n$, then there is nothing to prove. Assume that $n > \dim R$. Let $j = n - \dim R$. Let

$$0 \to V \otimes_A M \xrightarrow{\alpha_0} Y_0 \xrightarrow{\alpha_1} Y_1 \to \cdots \tag{2.4.4.19}$$

be a (C,R)-injective resolution of $V \otimes_A M$. The modules Y_i can be chosen to be projective over R as well. Since $\mathrm{Ext}_C^i(V, V \otimes_A M) = 0$, $1 \leq i \leq n-2$, applying $\mathrm{Hom}_C(V,-)$ yields the exact sequence

$$0 \to M \simeq \mathrm{Hom}_C(V, V \otimes_A M) \xrightarrow{\mathrm{Hom}_C(V,\alpha_0)} \mathrm{Hom}_C(V, Y_0) \xrightarrow{\mathrm{Hom}_C(V,\alpha_1)} \cdots \to \mathrm{Hom}_C(V, Y_{n-1}). \tag{2.4.4.20}$$

Note that $\mathrm{Hom}_C(V, Y_i) \in \mathrm{add}\,\mathrm{Hom}_C(V, DC) = \mathrm{add}\, DV = \mathrm{add}\, P$. Let $C_i = \mathrm{im}\,\mathrm{Hom}_C(V, \alpha_i), \forall i$. The exact sequence (2.4.4.20) induces the exact sequence

$$0 \to C_j \to \mathrm{Hom}_C(V, Y_j) \to \cdots \to \mathrm{Hom}_C(V, Y_{n-2}) \to C_{n-1} \to 0. \tag{2.4.4.21}$$

Note that this sequence has length $\dim R + 1$. Furthermore, since $\mathrm{pdim}_R C_{n-1} \leq \dim R$, we must have that C_j is projective over R. This implies that the exact sequence

$$0 \to M \to \mathrm{Hom}_C(V, Y_0) \to \cdots \to \mathrm{Hom}_C(V, Y_{j-1}) \tag{2.4.4.22}$$

is (A,R)-exact. Therefore, it follows that $\mathrm{domdim}_{(A,R)} M \geq j = n - \dim R$. (ii) is analogous to (i). □

When the Krull dimension is at most one, we can formulate the Mueller theorem in the following way.

Theorem 2.4.19. *Let R be a commutative Noetherian regular ring with Krull dimension at most one. Let (A,P,V) be a relative QF3 R-algebra. Fix $C = \mathrm{End}_A(V)$. Let $M \in A\text{-mod} \cap R\text{-proj}$ and $n \geq 2$. The following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq n-1$ *where the* (A,R)*-exact sequence*

$$0 \to M \to X_1 \to \cdots \to X_{n-1}, \tag{2.4.4.23}$$

with (A,R)*-injective projective* A*-modules* X_i *, can be continued to an exact sequence*

$$0 \to M \to X_1 \to \cdots \to X_{n-1} \to Y \tag{2.4.4.24}$$

where Y *is* (A,R)*-injective projective over* A.

(ii) α_M *is an isomorphism and* $\operatorname{Ext}^i_C(V, V \otimes_A M) = 0$*,* $1 \leq i \leq n-2$.

Proof. Assume that (ii) holds. Using Proposition 2.4.18, we see that $\operatorname{domdim}_{(A,R)} M \geq n-1$. Moreover, using the (A,R)-exact constructed there we have

$$0 \to M \to \operatorname{Hom}_C(V, Y_0) \to \cdots \to \operatorname{Hom}_C(V, Y_{n-2}) \to C_{n-1} \to 0. \tag{2.4.4.25}$$

Since C_{n-1} can be embedded into $\operatorname{Hom}_C(V, Y_{n-1})$ (i) follows.

Conversely, assume that (i) holds. Since $n \geq 2$, there exists an exact sequence $0 \to M \to X_1 \to X_2$ where $X_i \in \operatorname{add} DV$. The functor $\operatorname{Hom}_C(V, V \otimes_A -)$ is left exact, so it yields the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\alpha_M"] & X_1 \arrow[r] \arrow[d, "\alpha_{X_1}"] & X_2 \arrow[d, "\alpha_{X_2}"] \\
0 \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A M) \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A X_1) \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A X_2)
\end{tikzcd} \tag{2.4.4.26}$$

By diagram chasing, it follows that α_M is an isomorphism. Applying $V \otimes_A -$ to (2.4.4.24) we obtain the exact sequence

$$0 \to V \otimes_A M \to V \otimes_A X_1 \to \cdots \to V \otimes_A X_{n-1} \to V \otimes_A Y. \tag{2.4.4.27}$$

Note that by deleting $V \otimes_A Y$ we obtain a (C,R)-exact sequence. We can continue such (C,R)-exact to a (C,R)-injective resolution of $V \otimes_A M$. Now consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\alpha_M", "\simeq"'] & \cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d, "\alpha_{X_{n-1}}", "\simeq"'] & Y \arrow[d, "\alpha_Y", "\simeq"'] \\
0 \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A M) \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A X_{n-1}) \arrow[r] & \operatorname{Hom}_C(V, V \otimes_A Y)
\end{tikzcd} \tag{2.4.4.28}$$

It follows that the bottom row is exact. In particular, $\operatorname{Ext}_{(C,R)}(V, V \otimes_A M) = 0$, $1 \leq i \leq n-3$. Notice that by continuing the (C,R)-injective resolution we have the following commutative diagram

$$\begin{tikzcd}
V \otimes_A X_{n-1} \arrow[rr, "\lambda_n"] \arrow[dr, two heads, "\varepsilon"] & & V \otimes_A Y \\
& V \otimes_A \operatorname{coker} \arrow[ur, hook, "t"] \arrow[r, hook, "\nu"] & \tilde{X}_n
\end{tikzcd}. \tag{2.4.4.29}$$

Since $\operatorname{Hom}_C(V, -)$ is left exact,

$$\ker \operatorname{Hom}_C(V, \nu \circ \varepsilon) = \ker \operatorname{Hom}_C(V, \varepsilon) = \ker \operatorname{Hom}_C(V, t \circ \varepsilon) = \operatorname{im} \operatorname{Hom}_C(V, \lambda_{n-1}). \tag{2.4.4.30}$$

This last equality follows from the exactness of (2.4.4.28). This means that

$$0 \to \mathrm{Hom}_C(V, V \otimes_A M) \to \cdots \to \mathrm{Hom}_C(V, V \otimes_A X_{n-1}) \to \mathrm{Hom}_C(V, \tilde{X}_n) \tag{2.4.4.31}$$

is exact. So, (ii) holds. □

This method gives a hint why for Krull dimension one we can say that by continuing an (A,R)-exact sequence of projective relative injectives to a non-(A,R)-exact sequence of projective relative injectives is still enough to recover information about Ext. The method here used requires that at each step to compute the exact sequence we might have to replace the projective (A,R)-injective. This happens in general because we do not have a standard choice here unless the algebra is semiperfect. In such a case, the projective covers can take that role.

Proposition 2.4.20. *Let A be a semi-perfect R-algebra. Let $M \in A\text{-mod}\cap R\text{-proj}$. Let*

$$\cdots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} DM \to 0 \tag{2.4.4.32}$$

be a minimal right projective A-resolution. Then, $\mathrm{domdim}_{(A,R)} M \geq n$ if and only if every P_i, $i = 0, \ldots, n-1$, is right (A,R)-injective.

Proof. One of the implications is clear. Assume that $\mathrm{domdim}_{(A,R)} M \geq n$. Then, there exists an (A,R)-exact sequence

$$0 \to M \xrightarrow{\alpha_0} I_0 \to \cdots \xrightarrow{\alpha_{n-1}} I_{n-1}, \tag{2.4.4.33}$$

with projective (A,R)-injective A-modules I_i. Hence, applying D we obtain an exact sequence

$$DI_{n-1} \xrightarrow{D\alpha_{n-1}} \cdots \to DI_0 \xrightarrow{D\alpha_0} DM \to 0. \tag{2.4.4.34}$$

Since P_0 and DI_0 are projective A-modules there are maps $f_0 \in \mathrm{Hom}_A(P_0, DI_0)$, $g_0 \in \mathrm{Hom}_A(DI_0, P_0)$ satisfying $p_0 \circ g_0 = D\alpha_0$ and $D\alpha_0 \circ f_0 = p_0$. Hence, $p_0 \circ g_0 \circ f_0 = p_0$. Since (P_0, p_0) is the projective cover of DM, it follows that $g_0 \circ f_0 \in \mathrm{End}_A(P_0)$ is an isomorphism. Consequently, g_0 is surjective and thus, P_0 is an A-summand of DI_0. In particular, P_0 is (A,R)-injective. Observe that

$$p_0 \circ g_0 \circ D\alpha_1 = D\alpha_0 \circ D\alpha_1 = 0. \tag{2.4.4.35}$$

Hence, $\mathrm{im}\, g_0 \circ D\alpha_1 \subset \ker p_0$. Let $x \in \ker p_0$. Then, by the surjectivity of g_0, there exists $y \in DI_0$ such that $g_0(y) = x$. Therefore, $D\alpha_0(y) = p_0(x) = 0$. Thus, $y \in \ker D\alpha_0 = \mathrm{im} D\alpha_1$. So, $x \in \mathrm{im}\, g_0 \circ D\alpha_1$. We deduced that the sequence

$$DI_{n-1} \to \cdots \to DI_1 \xrightarrow{g_0 \circ D\alpha_1} P_0 \xrightarrow{p_0} DM \to 0 \tag{2.4.4.36}$$

is exact. Now we can proceed by induction, where in the next step $\ker p_0$ takes the place of DM, to obtain that each P_i is an A-summand of DI_i. □

We shall now see some properties of relative dominant dimension that follow from the relative Mueller theorem. In particular, the relative Mueller characterization applied to A takes the following form. This result is the relative analogue of [Mue68, Lemma 3] and [Tac73, 7.5].

Theorem 2.4.21. *Let (A,P,V) be a relative QF3 R-algebra with $\mathrm{domdim}_{(A,R)}\, {}_AA \geq 2$ and $\mathrm{domdim}_{(A,R)} A_A \geq 2$. For $n \geq 3$, the following are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} {}_AA \geq n$;

(ii) $\operatorname{Tor}_i^C(DV, V) = 0,\ i = 1, \dots, n-2$;

(iii) $\operatorname{Ext}_C^i(V,V) = 0,\ i = 1, \dots, n-2$ *and* $\Omega^j(DV, Q^\bullet) \otimes_C V \in R\text{-proj},\ 0 \leq j \leq n-2$ *for every projective C-resolution* $Q^\bullet$ *of* DV*;*

(iv) $\operatorname{Tor}_i^B(P, DP) = 0\ i = 1, \dots, n-2$;

(v) $\operatorname{Ext}_C^i(P,P) = 0,\ i = 1, \dots, n-2$ *and* $P \otimes_B \Omega^j(DP, Q^\bullet) \in R\text{-proj},\ 0 \leq j \leq n-2$ *for every projective B-resolution* $Q^\bullet$ *of* DP*;*

(vi) $\operatorname{domdim}_{(A,R)} A_A \geq n$.

Proof. The implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ follow from Theorem 2.4.15 and Theorem 2.4.16, respectively. We will, therefore, focus on the implication $(ii) \Leftrightarrow (iv)$.

Consider a left projective B-resolution

$$\cdots \to P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} DP \to 0. \tag{2.4.4.37}$$

Applying the exact functor $\operatorname{Hom}_B(D(V \otimes_A P), -)$ we get the exact sequence

$$\cdots \to \operatorname{Hom}_B(D(V \otimes_A P), P_{n-1}) \to \cdots \to \operatorname{Hom}_B(D(V \otimes_A P), P_0) \to \operatorname{Hom}_B(D(V \otimes_A P), DP) \to 0. \tag{2.4.4.38}$$

Since $D(V \otimes_A P)$ is a B-generator, each $P_i \in \operatorname{add} D(V \otimes_A P)$, therefore $\operatorname{Hom}_B(D(V \otimes_A P), P_i) \in C\text{-proj}$. Also, $\operatorname{Hom}_B(D(V \otimes_A P), DP) \simeq \operatorname{Hom}_B(P, V \otimes_A P) \simeq V$ as left C-modules. Thus, (2.4.4.38) is a projective C-resolution for V.

We recall that in Lemma 2.4.8, we saw that for $F = \operatorname{Hom}_B(D(V \otimes_A P), -)$ and $G = \operatorname{Hom}_B(\operatorname{Hom}_B(D(V \otimes_A P), B), -)$ there was an isomorphism $GM \otimes_C FN \simeq M \otimes_B N$ for every $M \in \text{mod-}B$ and $N \in B\text{-mod}$. Since all the isomorphisms involved are functorial, it follows that there exists a natural isomorphism of bifunctors $\theta\colon G(-) \otimes_C F(-) \to \operatorname{id}(-) \otimes_B \operatorname{id}(-)$. In particular, the following diagram is commutative

$$\begin{tikzcd}
P \otimes_B P_{i-1} \arrow[r, "\mathrm{id}_P \otimes_B f_i"] & P \otimes_B P_{i-2} \arrow[r] & \cdots \arrow[r] & P \otimes_B P_0 \arrow[r, "\mathrm{id}_P \otimes_B f_0"] & P \otimes_B DP \arrow[r] & 0 \\
GP \otimes_B FP_{i-1} \arrow[u, "\theta_{P,P_{i-1}}", "\simeq"'] \arrow[r, "G\mathrm{id}_P \otimes_B F f_i"] & GP \otimes_B FP_{i-2} \arrow[u, "\theta_{P,P_{i-2}}", "\simeq"'] \arrow[r] & \cdots \arrow[r] & GP \otimes_C FP_0 \arrow[u, "\theta_{P,P_0}", "\simeq"'] \arrow[r, "G\mathrm{id}_P \otimes_C F f_0"] & GP \otimes_C FDP \arrow[u, "\theta_{P,DP}", "\simeq"'] \arrow[r] & 0
\end{tikzcd} \tag{2.4.4.39}$$

So, the upper row is exact if and only if the bottom row is exact. Furthermore, the bottom row is exactly the complex obtained by applying $DV \otimes_C -$ to the exact sequence (2.4.4.38). It follows that $\operatorname{Tor}_i^C(DV, V) = 0$ if and only if $\operatorname{Tor}_i^B(P, DP) = 0$. ☐

Corollary 2.4.22. *Let* (A, P, V) *be a relative QF3 R-algebra. Then,* $\operatorname{domdim}_{(A,R)} {}_AA = \operatorname{domdim}_{(A,R)} A_A$.

Proof. Assume that $\operatorname{domdim}_{(A,R)} {}_AA \geq 2$. By Lemma 2.4.5, V is a left C-generator (C,R)-cogenerator. In view of Lemma 2.4.8, $DV \otimes_C V \in R\text{-proj}$. By Theorem 2.4.15, V satisfies the double centralizer property. By relative Morita-Tachikawa correspondence, $\operatorname{End}_C(V) \simeq A$ has left and right relative dominant dimension greater than or equal to two. By Theorem 2.4.21, we have $\operatorname{domdim}_{(A,R)} A_A \geq \operatorname{domdim}_{(A,R)} {}_AA$. Symmetrically, $\operatorname{domdim}_{(A,R)} {}_AA \geq \operatorname{domdim}_{(A,R)} A_A$. ☐

Another consequence of Theorem 2.4.21 is that we can characterize every endomorphism algebra of a generator relative cogenerator such that the generator remains projective over R under tensor product over its dual. In fact, Let B be the endomorphism algebra over A of a generator (A,R)-cogenerator such that $DM\otimes_A M \in R$-proj. By relative Morita-Tachikawa, B has left and right relative dominant dimension greater than or equal to two. Now Theorem 2.4.21 gives that $\operatorname{domdim}(B,R)\geq n+2$ if and only if $\operatorname{Tor}_i^A(DM,M)=0$, $1\leq i\leq n$.

Corollary 2.4.23. *Let (A,P,V) be a relative QF3 R-algebra. Let $M_i\in A\text{-mod}\cap R\text{-proj}$, $i\in I$ for some finite set I. Then,*

$$\operatorname{domdim}_{(A,R)}\bigoplus_{i\in I}M_i=\inf\{\operatorname{domdim}_{(A,R)}M_i\colon i\in I\}. \tag{2.4.4.40}$$

Proof. Since the maps Φ_X are compatible with direct sums, we get that Φ_{M_i} is surjective/bijective for every $i\in I$ if and only if $\Phi_{\bigoplus_{i\in I}}M_i$ is surjective/bijective. Thus, $\operatorname{domdim}_{(A,R)}\bigoplus_{i\in I}M_i\geq 1$ (resp. 2) if and only if $\operatorname{domdim}_{(A,R)}M_i\geq 1$ (resp. 2) for every $i\in I$. Now since for every n

$$\operatorname{Tor}_n^C(\operatorname{Hom}_A(V,D(\bigoplus_{i\in I}M_i)),V)\simeq\operatorname{Tor}_n^C(\operatorname{Hom}_A(V,\bigoplus_{i\in I}DM_i),V)\simeq\bigoplus_{i\in I}\operatorname{Tor}_n^C(\operatorname{Hom}_A(V,DM_i),V), \tag{2.4.4.41}$$

the result follows by Theorem 2.4.15. ◻

Remark 2.4.24. It follows that the value of the relative dominant dimension is independent of the direct sum decomposition of the module. △

The following Lemma is another consequence of relative Mueller characterization. In the field case, this proof is quicker using the relations between dominant dimension and the socle of the regular module and it was first stated in [FK11b, Proposition 3.6].

Lemma 2.4.25. *Let (A,P,V) be a relative QF3 R-algebra. Let $M\in R$-proj and consider the following (A,R)-exact*

$$0\to M_1\to M\to M_2\to 0. \tag{2.4.4.42}$$

Let $n=\operatorname{domdim}_{(A,R)}M$ and $n_i=\operatorname{domdim}_{(A,R)}M_i$. Then, the following holds.

(a) $n\geq\min\{n_1,n_2\}$.

(b) *If $n_1<n$, then $n_2=n_1-1$.*

(c) *(i)* $n_1=n\implies n_2\geq n-1$.

(ii) $n_1=n+1\implies n_2\geq n$.

(iii) $n_1\geq n+2\implies n_2=n$.

(d) $n<n_2\implies n_1=n$.

(e) *(i)* $n=n_2\implies n_1\geq n_2$.

(ii) $n=n_2+1\implies n_1\geq n_2+1$.

(iii) $n\geq n_2+2\implies n_1=n_2+1$.

Proof. Applying D and $\operatorname{Hom}_A(V,D-)\otimes_C V$ we get the commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Tor}_1^C(\operatorname{Hom}_A(V,DM_1),V) \arrow[r] & \operatorname{Hom}_A(V,DM_2)\otimes_C V \arrow[r] \arrow[d, "\Phi_{M_2}"] & \operatorname{Hom}_A(V,DM)\otimes_C V \arrow[r, two heads] \arrow[d, "\Phi_M"] & \operatorname{Hom}_A(V,DM_1)\otimes_C V \arrow[d, "\Phi_{M_1}"] \\
0 \arrow[r] & DM_2 \arrow[r] & DM \arrow[r, two heads] & DM_1
\end{tikzcd}. \tag{2.4.4.43}$$

By Snake Lemma, Φ_M is surjective/bijective if Φ_{M_1} and Φ_{M_2} are surjective/bijective. Thus, $\min\{n_1,n_2\} \geq k$, $k \leq 2$, implies that $n \geq k$. Consider the long exact sequence

$$\operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM_1),V) \to \operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM),V) \to \operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM_2),V) \tag{2.4.4.44}$$

we obtain that if $n_1,n_2 \geq k \geq 2$, $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM_1),V) = \operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM_2),V) = 0$ for $i = 1,\dots,k-2$, then $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM),V) = 0$. Thus, $n \geq \min\{n_1,n_2\}$. By Theorem 2.4.15, (a) follows.

(b). If $n_1 = 0$, then Φ_{M_1} is not surjective. By diagram chasing, if Φ_M is surjective, then Φ_{M_1} is surjective. Thus, $n > 0$ implies that $n_1 > 0$. Assume $n_1 = 1$ and $n > n_1$. Thus, Φ_M is bijective and Φ_{M_1} is surjective. If Φ_{M_2} is surjective, then by Snake Lemma, Φ_{M_1} is also injective. This would imply that $n_1 \geq 2$. So, $n_2 = 0$. Assume now $n_1 \geq 2$. By Snake Lemma, Φ_{M_2} is surjective. So, $n_2 \geq 1$. If $n_2 \geq 2$, then, in particular, Φ_{M_2} is surjective. The exactness of the bottom row of (2.4.4.43) makes $\operatorname{Hom}_A(V,DM_2)\otimes_C V \to \operatorname{Hom}_A(V,DM)\otimes_C V$ injective. Since $\operatorname{Tor}_1^C(\operatorname{Hom}_A(V,DM),V) = 0$, the long exact sequence induces that $\operatorname{Tor}_1^C(\operatorname{Hom}_A(V,DM_1),V) = 0$. This contradicts $n_1 = 2$. Thus, $n_2 = 1$. Now assume that $n_1 \geq 3$. Thus, (2.4.4.43) becomes

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_A(V,DM_2)\otimes_C V \arrow[r] \arrow[d, "\Phi_{M_2}"] & \operatorname{Hom}_A(V,DM)\otimes_C V \arrow[r, two heads] \arrow[d, "\Phi_M"] & \operatorname{Hom}_A(V,DM_1)\otimes_C V \arrow[d, "\Phi_{M_1}"] \\ 0 \arrow[r] & DM_2 \arrow[r] & DM \arrow[r, two heads] & DM_1 \end{tikzcd} . \tag{2.4.4.45}$$

Thus, by Snake Lemma Φ_{M_2} is bijective. Furthermore, using the long exact sequences and as $n > n_1$ we deduce that

$$\operatorname{Tor}_{i+1}^C(\operatorname{Hom}_A(V,DM_1),V) \simeq \operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM_2),V),\ 1 \leq i \leq n_1 - 2. \tag{2.4.4.46}$$

Thus, $n_2 = n_1 - 1$.

Analogously, (c), (d), (e) hold. ☐

2.5 Relative dominant dimension under change of rings

2.5.1 Strongly faithful modules - revisited

Our immediate aim now is to understand how strongly faithful modules behave under change of rings. The proofs of the following two lemmas are technical however they are very useful to characterize strongly faithful modules.

Lemma 2.5.1. *Let A be a projective Noetherian R-algebra . Let $V \in$ mod-$A \cap R$-proj. Consider the A-map $\delta_V\colon \bigoplus_{g\in \operatorname{Hom}_A(DV,DA)} DV \to DA$, given by $\delta_V(f_g) = g(f)$. Then, δ_V is surjective if and only if V is (A,R)-strongly faithful.*

Proof. First, we need to check that δ_V is well defined. Let $g \in \operatorname{Hom}_A(DV,DA)$. Let $\theta_g\colon DV \to DA$ be the map given by $\theta_g(f) = g(f)$, $f \in DV$. This is clearly an A-map since $g \in \operatorname{Hom}_A(DV,DA)$. Taking the direct sum of maps θ_g over $g \in \operatorname{Hom}_A(DV,DA)$ yields the map δ_V. Thus, δ_V is well defined.

Assume that δ_V is surjective. Let $\{f_1,\dots,f_t\}$ be an R-generator set for DA. By assumption, there exists for each $1 \leq i \leq t$ a natural number $s_i > 0$ and elements $w_{i,j} \in DV$, $g_{i,j} \in \operatorname{Hom}_A(DV,DA)$ with $j = 1,\dots,s_i$ such that

$$f_i = \delta_V(\sum_{j=1}^{s_i}(w_{i,j})_{g_{i,j}}). \tag{2.5.1.1}$$

Let $h \in DA$. Then,

$$h = \sum_{i=1}^{t} \alpha_i f_i = \sum_{i=1}^{t} \alpha_i \delta_V (\sum_{j=1}^{s_i} (w_{i,j})_{g_{i,j}}) = \delta_V (\sum_{i=1}^{t} \sum_{j=1}^{s_i} \alpha_i (w_{i,j})_{g_{i,j}}), \quad \alpha_i \in R. \tag{2.5.1.2}$$

Therefore, the restriction of δ_V to the summands indexed by $g_{i,j}$ $1 \leq i \leq t$, $1 \leq j \leq s_i$ is surjective. Denote by o the number of such indexes. Then, we found a surjective A-map $(DV)^o \twoheadrightarrow DA$. As $DA \in R$-proj, this map is an (A,R)-epimorphism. Thus, applying D yields an (A,R)-monomorphism $A \to V^o$. So, V is (A,R)-strongly faithful.

Conversely, assume that V is (A,R)-strongly faithful. Hence, there is an (A,R)-monomorphism $A \to V^t$ for some $t > 0$. Applying D we obtain a surjective map $DV^t \to DA$. Denote this map by ε. Let $k_j \in \operatorname{Hom}_A(DV, DV^t)$ and $\pi_j \in \operatorname{Hom}_A(DV^t, DV)$ be the canonical injections and projections, respectively. Define $g_j = \varepsilon \circ k_j \in \operatorname{Hom}_A(DV, DA)$. For every $h \in DA$, there exists $y \in DV^t$ such that $\varepsilon(y) = h$. Therefore,

$$h = \sum_{j=1}^{t} \varepsilon \circ k_j \circ \pi_j(y) = \delta_V (\sum_{j=1}^{t} \pi_j(y)_{g_j}). \tag{2.5.1.3}$$

So, δ_V is surjective. $\square$

Lemma 2.5.2. *Let A be a projective Noetherian R-algebra. For every commutative R-algebra S, and $X, Y \in$ A-mod there exists a map*

$$\theta_S \colon S \otimes_R \left(\bigoplus_{g \in \operatorname{Hom}_A(X,Y)} X \right) \longrightarrow \bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R Y)} S \otimes_R X,$$

given by $\theta_S(s \otimes x_g) = (s \otimes x)_{1_S \otimes g}$.

Moreover, if $X \in A$-proj, then $\theta_{R(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ in R.

Proof. Consider the map

$$\theta \colon S \times \bigoplus_{g \in \operatorname{Hom}_A(X,Y)} X \to \bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R Y)} S \otimes_R X,$$

given by $\theta(s, x_g) = (s \otimes x)_{1_S \otimes g}$ for $s \in S$, $x \in X$, $g \in \operatorname{Hom}_A(X,Y)$. By definition, this map is linear in each term. Let $r \in R$. Then,

$$\theta(rs, x_g) = (rs \otimes x)_{1_S \otimes g} = (s \otimes rx)_{1_S \otimes g} = \theta(s, (rx)_g). \tag{2.5.1.4}$$

So, θ induces uniquely the S-map θ_S. Assume that $X \in A$-proj. Let $\mathfrak{m}$ be a maximal ideal in R. Then, $\operatorname{Hom}_{A(\mathfrak{m})}(X(\mathfrak{m}), Y(\mathfrak{m})) \simeq \operatorname{Hom}_A(X,Y)(\mathfrak{m})$. Thus, every element in $\operatorname{Hom}_{A(\mathfrak{m})}(X(\mathfrak{m}), Y(\mathfrak{m}))$ can be written in the form $h \otimes (r + m) = (rh) \otimes 1_{R(\mathfrak{m})}$ for $rh \in \operatorname{Hom}_A(X,Y)$. Moreover, every element in $\bigoplus_{h \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R X, S \otimes_R Y)} S \otimes_R X$ is the sum of elements $(1_{R(\mathfrak{m})} \otimes x)_{1_{R(\mathfrak{m})} \otimes h} = \theta_{R(\mathfrak{m})}(1_{R(\mathfrak{m})} \otimes x_h)$, $h \in \operatorname{Hom}_A(X,Y)$ and $S = R(\mathfrak{m})$. This implies that $\theta_{R(\mathfrak{m})}$ is surjective. $\square$

Proposition 2.5.3. *Let A be a projective Noetherian R-algebra. Let $V \in$ mod-$A \cap R$-proj. Then, the following assertions are equivalent.*

(a) V is a projective (A,R)-injective-strongly faithful right A-module.

(b) $S \otimes_R V$ is a projective $(S \otimes_R A, S)$-injective-strongly faithful right $S \otimes_R A$-module for every commutative R-algebra S.

(c) *$V_\mathfrak{m}$ is a projective $(A_\mathfrak{m}, R_\mathfrak{m})$-injective-strongly faithful right $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ in R.*

(d) *$V(\mathfrak{m})$ is projective-injective faithful over right $A(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R.*

Proof. $(i) \implies (ii)$. Let S be a commutative R-algebra. The module V is a right A-summand of A^t for some $t > 0$. Hence, $S \otimes_R V$ is a right $S \otimes_R A$-summand of $S \otimes_R A^t \simeq (S \otimes_R A)^t$. Thus, $S \otimes_R V$ is a right projective $S \otimes_R A$-module. As V is (A,R)-injective, V is an A-summand of $\operatorname{Hom}_R(A,V)$. So, $S \otimes_R V$ is an $S \otimes_R A$-summand of $S \otimes_R \operatorname{Hom}_R(A,V) \simeq \operatorname{Hom}_S(S \otimes_R A, S \otimes_R V)$ since $A \in R$-proj. Hence, $S \otimes_R V$ is projective $(S \otimes_R A, S)$-injective. By Lemma 2.5.1, the map $\delta_V \in \operatorname{Hom}_A(\bigoplus_{g \in \operatorname{Hom}_A(DV,DA)} DV, DA)$ is surjective. Applying the functor $S \otimes_R -$ we have the following commutative diagram

$$\begin{tikzcd}
S \otimes_R \left(\bigoplus\limits_{g \in \operatorname{Hom}_A(DV,DA)} DV \right) \arrow[rr, "S \otimes_R \delta_V"] \arrow[d, "\theta_S"] & & S \otimes_R DA \arrow[dd, "l_S"] \\
\bigoplus\limits_{g \in \operatorname{Hom}_{S \otimes_R A}(S \otimes_R DV, S \otimes_R DA)} S \otimes_R DV \arrow[d, "\kappa_S"] & & \\
\bigoplus\limits_{h \in \operatorname{Hom}_{S \otimes_R A}(\operatorname{Hom}_S(S \otimes_R V, S), \operatorname{Hom}_S(S \otimes_R A, S))} \operatorname{Hom}_S(S \otimes_R V, S) \arrow[rr, "\delta_{S \otimes_R V}"] & & \operatorname{Hom}_S(S \otimes_R A, S)
\end{tikzcd}, \tag{2.5.1.5}$$

where l_S and κ_l are the canonical isomorphisms (as $V, A \in R$-proj). This diagram is commutative since:

$$\delta_{S \otimes_R V} \circ \kappa_S \circ \theta_S (s \otimes x_g) = \delta_{S \otimes_R V} \circ \kappa_S (s \otimes x)_{1_S \otimes g} = \delta_{S \otimes_R V}((s \otimes x)_{1_S \otimes g}) = 1_S \otimes g(s \otimes x) = s \otimes g(x) \tag{2.5.1.6}$$

$$l_S \circ S \otimes_R \delta_V (s \otimes x_g) = l(s \otimes g(x)) = s \otimes g(x), \ s \in S, x \in DV, g \in \operatorname{Hom}_A(DV, DA). \tag{2.5.1.7}$$

The right exactness of $S \otimes_R -$implies that $S \otimes_R \delta_V$ is surjective. Using the commutativity of the diagram $\delta_{S \otimes_R V} \circ \kappa_S \circ \theta_S$ is surjective. Hence, $\delta_{S \otimes_R V}$ is surjective. By Lemma 2.5.1, (ii) follows.

$(ii) \implies (iii)$. For every maximal ideal $\mathfrak{m}$ in R, consider $S = R_\mathfrak{m}$.

$(iii) \implies (iv)$. Let $\mathfrak{m}$ be a maximal ideal in R. Recall that

$$X_\mathfrak{m}(\mathfrak{m}) = X_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} = X \otimes_{R_\mathfrak{m}} R_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} = X \otimes_R R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} = X(\mathfrak{m}). \tag{2.5.1.8}$$

Hence, using the same argument as discussed in $(i) \implies (ii)$ now with $S = R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$ yields that $V(\mathfrak{m})$ is projective $(A(\mathfrak{m}), R(\mathfrak{m}))$-injective-strongly faithful. Since $R(\mathfrak{m})$ is a field, every $(A(\mathfrak{m}), R(\mathfrak{m}))$-injective is $A(\mathfrak{m})$-injective and strongly faithful coincides with faithful. So, (iv) follows.

$(iv) \implies (i)$. Since $V(\mathfrak{m})$ is a projective right $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R and $V \in R$-proj, we deduce that V is a projective right A-module. By Theorem 1.2.57, V is (A,R)-injective. By Lemma 2.5.1, $\delta_{V(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Lemma 2.5.2, $\theta_{R(\mathfrak{m})}$ is surjective. By the commutative diagram (2.5.1.5) with $S = R(\mathfrak{m})$ we get that $l_{R(\mathfrak{m})} \circ R(\mathfrak{m}) \otimes_R \delta_V$ is surjective. Since $l_{R(\mathfrak{m})}$ is bijective, it follows that $R(\mathfrak{m}) \otimes_R \delta_V$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, δ_V is surjective. So, V is also (A,R)-strongly faithful. $\square$

By symmetry, we obtain:

Proposition 2.5.4. *Let A be a projective Noetherian R-algebra. Let $P \in A$-mod$\cap R$-proj. Then, the following assertions are equivalent.*

(a) P is a projective (A,R)-injective-strongly faithful left A-module.

(b) $S\otimes_R P$ is a projective $(S\otimes_R A,S)$-injective-strongly faithful left $S\otimes_R A$-module for every commutative R-algebra S.

(c) $P_\mathfrak{m}$ is a projective $(A_\mathfrak{m},R_\mathfrak{m})$-injective-strongly faithful left $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ in R.

(d) $P(\mathfrak{m})$ is a projective-injective faithful left $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R.

2.5.2 Left-Right symmetry

For finite-dimensional algebras, there exists a left faithful projective-injective if and only if there exists a right faithful projective-injective [Tac63, Theorem 2]. From what we have done so far, the left and right symmetry can be deduced for finite-dimensional algebras once one observes that the dual of a faithful module is again faithful. Although we do not have an argument for (A,R)-strongly faithfulness being preserved under standard duality, we can recover the following statement.

Lemma 2.5.5. *Let A be a projective Noetherian R-algebra. Then,* $\operatorname{domdim}_{(A,R)} A_A \geq 1$ *if and only if* $\operatorname{domdim}_{(A,R)} {}_AA \geq 1$. *In particular, if* $\operatorname{domdim}(A,R)_AA \geq 1$ *or* $\operatorname{domdim}_{(A,R)} A_A \geq 1$*, then there exists P and V such that (A,P,V) is a relative QF3 R-algebra.*

Proof. Assume that $\operatorname{domdim}_{(A,R)} A_A \geq 1$. Then, there exists a right A-module V which is projective (A,R)-injective-strongly faithful. Since $A \in R\text{-proj}$, it follows that $V \in R\text{-proj}$. By Proposition 2.5.3, $V(\mathfrak{m})$ is a projective-injective faithful right $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R. Then, $\operatorname{Hom}_{R(\mathfrak{m})}(V(\mathfrak{m}),R(\mathfrak{m}))$ is a projective-injective left $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R.

Observe that in general if a finitely generated module X over a finite-dimensional algebra B over a field K is faithful, then $\operatorname{Hom}_K(X,K)$ is faithful as left B-module. In fact, let $b \in B$ and assume that $b\cdot f = 0$ for every $f \in \operatorname{Hom}_K(X,K)$. Then, for each $x \in X$,

$$0 = bf(x) = f(xb), \forall f \in \operatorname{Hom}_K(X,K).$$

Since X is finitely generated, we deduce that $xb = 0$. Now using that X is faithful over B yields $b = 0$.

Therefore, $DV(\mathfrak{m}) \simeq \operatorname{Hom}_{R(\mathfrak{m})}(V(\mathfrak{m}),R(\mathfrak{m}))$ is a projective-injective faithful left $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R. By Proposition 2.5.4, DV is a projective (A,R)-injective-strongly faithful left A-module. Thus, $\operatorname{domdim}_{(A,R)} {}_AA \geq 1$. The converse implication is analogous. We also showed that (A,DV,V) is a relative QF3 R-algebra. $\square$

Corollary 2.5.6. *Let A be a projective Noetherian R-algebra. Then,* $\operatorname{domdim}_{(A,R)} A_A = \operatorname{domdim}_{(A,R)} {}_AA$.

Proof. Assume that $\operatorname{domdim}_{(A,R)} A_A \geq n$ for some $n \geq 1$. By Lemma 2.5.5, $\operatorname{domdim}_{(A,R)} {}_AA \geq 1$. By Corollary 2.4.22, $\operatorname{domdim}_{(A,R)} {}_AA \geq n$. Hence, $\operatorname{domdim}_{(A,R)} {}_AA \geq \operatorname{domdim}_{(A,R)} A_A$.

Similarly, $\operatorname{domdim}_{(A,R)} A_A \geq \operatorname{domdim}_{(A,R)} {}_AA$. $\square$

Thus, we will write $\operatorname{domdim}(A,R)$ avoiding the left and right notation.

2.5.3 Computing relative dominant dimension using classical dominant dimension

Proposition 2.5.7. *Let (A,P,V) be a relative QF3 R-algebra. Let $M \in A\text{-mod}\cap R\text{-proj}$. Then, the following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} M \geq 1$.

(ii) $\operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M \geq 1$ *for every commutative R-algebra S which is a Noetherian ring.*

(iii) $\operatorname{domdim}_{(A_\mathfrak{m},R_\mathfrak{m})} M_\mathfrak{m} \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *in R.*

(iv) $\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *in R.*

Proof. Let $C = \operatorname{End}_A(V)$. Denote by D_S the standard duality with respect to S, $\operatorname{Hom}_S(-,S)$. Consider the map $\Phi_M\colon \operatorname{Hom}_A(V,DM)\otimes_C V \to DM$. Applying the functor $S\otimes_R -$ we get the commutative diagram

$$\begin{tikzcd}
S\otimes_R \operatorname{Hom}_A(V,DM)\otimes_C V \arrow[rr, "S\otimes_R\Phi_M"] \arrow[d, "\theta_{S,M}", "\simeq"'] & & S\otimes_R DM \arrow[dd, "l_{S,M}", "\simeq"'] \\
S\otimes_R \operatorname{Hom}_A(V,DM)\otimes_{S\otimes_R C} S\otimes_R V \arrow[d, "\kappa_{S,M}", "\simeq"'] & & \\
\operatorname{Hom}_{S\otimes_R A}(S\otimes_R V, D_S(S\otimes_R M))\otimes_{S\otimes_R C} S\otimes_R V \arrow[rr, "\Phi_{S\otimes_R M}"] & & D_S(S\otimes_R M)
\end{tikzcd}, \tag{2.5.3.1}$$

where the $\theta_{S,M}$, $\kappa_{S,M}$ and $l_{S,M}$ are the natural maps. These are isomorphisms since $V\in A^{op}$-proj and $M\in R$-proj.

$(i) \implies (ii)$. Since Φ_M is an epimorphism, it follows by diagram 2.5.3.1 that $\Phi_{S\otimes_R M}$ is an epimorphism. As $(S\otimes_R A, S\otimes_R P, S\otimes_R V)$ is a relative QF3 S-algebra, (ii) follows by Theorem 2.4.15.

The implication $(ii) \implies (iii)$ follows by using (ii) with $S = R_\mathfrak{m}$. The implication $(iii) \implies (iv)$ follows by using the same argument as in the implication $(i) \implies (ii)$ with $S = R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$ over $R_\mathfrak{m}$.

$(iv) \implies (i)$. By the diagram (2.5.3.1), it follows that $R(\mathfrak{m})\otimes_R \Phi_M$ is surjective for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, Φ_M is surjective. Finally (i) follows by Theorem 2.4.15. ☐

This last Proposition is not surprising since $S\otimes_R -$ is right exact and relative dominant dimension one can be characterized by surjective maps. For the same reason, flat extensions are compatible with relative dominant dimension of a module.

Proposition 2.5.8. *Let* (A,P,V) *be a relative QF3 R-algebra. Let* $M\in A\text{-mod}\cap R\text{-proj}$. *The following assertions are equivalent. Let* $n\in\mathbb{N}$.

(i) $\operatorname{domdim}_{(A,R)} M \geq n \geq 1$.

(ii) $\operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M \geq n \geq 1$ *for every flat commutative R-algebra S which is a Noetherian ring.*

(iii) $\operatorname{domdim}_{(A_\mathfrak{m},R_\mathfrak{m})} M_\mathfrak{m} \geq n \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *in R.*

Proof. By Proposition 2.5.3, $(S\otimes_R A, S\otimes_R P, S\otimes_R V)$ is a relative QF3 S-algebra. Note that

$$S\otimes_R C \simeq S\otimes_R \operatorname{End}_A(V) \simeq \operatorname{End}_{S\otimes_R A}(S\otimes_R V).$$

By Proposition 2.5.7, $\operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M \geq 1$. Assume that $n\geq 2$. Hence, Φ_M is an isomorphism. By the diagram (2.5.3.1), $\Phi_{S\otimes_R M}$ is an isomorphism. So, $\operatorname{domdim}_{(S\otimes_R A,S)} S\otimes_R M \geq 2$. Now assume that $n\geq 3$. Then,

$$\begin{aligned}
0 = S\otimes_R \operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM),V) &= \operatorname{Tor}_i^{S\otimes_R C}(S\otimes_R \operatorname{Hom}_A(V,DM), S\otimes_R V)\\
&= \operatorname{Tor}_i^{S\otimes_R C}(\operatorname{Hom}_{S\otimes_R A}(S\otimes_R V, D_S(S\otimes_R M)), S\otimes_R V),\ 1\leq i\leq n-2.
\end{aligned}$$

Now, (ii) follows by Theorem 2.4.15.

The implication $(ii) \implies (iii)$ follows by applying $S = R_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ in R.

$(iii) \implies (i)$. If $n \geq 1$, then by Proposition 2.5.7, $\operatorname{domdim}_{(A,R)} M \geq 1$. If $n \geq 2$, then $\Phi_{M_\mathfrak{m}}$ is isomorphism for every maximal ideal $\mathfrak{m}$ in R. By the diagram (2.5.3.1), $R_\mathfrak{m} \otimes_R \Phi_M$ is isomorphism for every maximal ideal $\mathfrak{m}$ in R. Hence, Φ_M is an isomorphism. Moreover,

$$\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V)_\mathfrak{m} = \operatorname{Tor}_i^{C_\mathfrak{m}}(\operatorname{Hom}_{A_\mathfrak{m}}(V_\mathfrak{m}, D_\mathfrak{m} M_\mathfrak{m}), V_\mathfrak{m}) = 0, \; 1 \leq i \leq n-2.$$

By Theorem 2.4.15,$\operatorname{domdim}_{(A,R)} M \geq n \geq 1$. □

Proposition 2.5.9. *Let (A, P, V) be a relative QF3 R-algebra. Let $M \in A$-mod$\cap R$-proj. If S is a Noetherian faithfully flat R-algebra, then*

$$\operatorname{domdim}_{(S\otimes_R A, S)} S \otimes_R M = \operatorname{domdim}_{(A,R)} M. \tag{2.5.3.2}$$

Proof. By Proposition 2.5.8, $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq \operatorname{domdim}_{(A,R)} M$. The map $\Phi_{S\otimes_R M}$ is epimorphism (resp. isomorphism) if and only the map $S \otimes_R \Phi_M$ is epimorphism (resp. isomorphism). Recall that since S is faithfully flat an R-module is zero if and only if it is the zero module under the functor $S \otimes_R -$. In particular, the map $\Phi_{S\otimes_R M}$ is epimorphism (resp. isomorphism) if and only if the map Φ_M is epimorphism (resp. isomorphism). By flatness of S,

$$\operatorname{Tor}_i^{S\otimes_R C}(\operatorname{Hom}_{S\otimes_R A}(S \otimes_R V, \operatorname{Hom}_S(S \otimes_R M, S), S \otimes_R V) \simeq S \otimes_R \operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V), \quad \forall i > 0. \tag{2.5.3.3}$$

Therefore, for each natural number i, $\operatorname{Tor}_i^{S\otimes_R C}(\operatorname{Hom}_{S\otimes_R A}(S \otimes_R V, \operatorname{Hom}_S(S \otimes_R M, S), S \otimes_R V) = 0$ if and only if $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, DM), V) = 0$. The result follows Theorem 2.4.15 and Proposition 2.4.7. □

An immediate application of Proposition 2.5.9 is for polynomial rings $R[X_1, \ldots, X_n]$. Further, $R[X_1, \ldots, X_n]$ is free of infinite rank over R, and so it is faithfully flat.

An example of the importance of changing the ground ring to compute dominant dimension is that for finite-dimensional algebras the computation of dominant dimension can be reduced to the computation of dominant dimension over algebraically closed fields. This is a known fact, and it can be found in [Mue68, Lemma 5].

Proposition 2.5.10. *Let A be a finite-dimensional algebra over a field K. Assume that A is QF3-algebra. Then,* $\operatorname{domdim} A = \operatorname{domdim} \overline{K} \otimes_K A$.

Proof. Let $\overline{K}$ be the algebraic closure of K. In particular, $\overline{K}$ can be regarded as K-vector space, hence it is K-free. Furthermore, $\overline{K}$ is faithfully flat over K. By Proposition 2.5.9, the claim follows. □

The idea here used can be generalized to the next Proposition. For the second part of its proof, we will require the following lemma.

Lemma 2.5.11. *Let $f\colon R \to S$ be a surjective R-algebra map. Let A be a projective Noetherian R-algebra. Then, for every $Y \in S \otimes_R A$-mod, $S \otimes_R Y \simeq Y$ as $S \otimes_R A$-modules.*

Proof. Let $Y \in S \otimes_R A$-mod. Y can be regarded as an A-module with action $a \cdot y = (f(1_R) \otimes_R a) \cdot y = (1_S \otimes a) \cdot y$. Consider the multiplication map $\mu\colon S \otimes_R Y \to Y$. We have, for $s' \otimes a \in S \otimes_R A$, $s \otimes y \in S \otimes_R Y$,

$$\mu(s' \otimes a \cdot s \otimes y) = \mu(s's \otimes ay) = s's(ay) = s's(1_S \otimes_a)y = (s's \otimes a)y = (s' \otimes a)(s \otimes 1_A)y = (s' \otimes a)\mu(s \otimes y).$$

Therefore, μ is an $S \otimes_R A$-homomorphism. Consider the map $\nu\colon Y \to S \otimes_R Y$, given by $\nu(y) = 1_S \otimes y$. We have

$$\nu(s \otimes a \cdot y) = 1_S \otimes (s \otimes a) \cdot y = 1_S \otimes ((s \otimes 1_A)(1_S \otimes a)y) = 1_S \otimes ((f(r) \otimes 1_A)ay) = 1_S \otimes ((r1_S \otimes 1_A)ay) \tag{2.5.3.4}$$

$$= r1_S \otimes ay = s \otimes ay = (s \otimes a)(1_S \otimes y) = s \otimes a\nu(y). \tag{2.5.3.5}$$

So, ν is an $S \otimes_R A$-homomorphism. ν and μ are inverse to each other. In fact, since f is surjective

$$\mu \circ \nu(y) = \mu(1_S \otimes y) = 1_S y = y, \ y \in Y$$
$$\nu \circ \mu(s \otimes y) = \nu \circ \mu(f(r_s) \otimes y)) = 1_S \otimes (r_s f(1_R) y) = r_s 1_S \otimes y = s \otimes y, \ s \otimes y \in S \otimes_R Y.$$

It follows that μ is an $S \otimes_R A$-isomorphism. □

Proposition 2.5.12. *Let S be a commutative R-algebra which is a Noetherian ring. Let A be a projective Noetherian algebra over a commutative Noetherian ring R. Let $M \in A\text{-mod} \cap R\text{-proj}$.*

Then, $\operatorname{domdim}_{(A,R)} M \leq \operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M$. *Assume, additionally the following*

- (A,P,V) *is a relative QF3 R-algebra;*
- *there is a surjective map of R-algebras $R \to S$ making S a projective R-module.*

Then, $\operatorname{domdim}_{(A,R)} M = \operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M$.

Proof. Let $\operatorname{domdim}_{(A,R)} M \geq n$. Then, there exists an (A,R)-exact sequence

$$0 \to M \to X_1 \to \cdots \to X_n \tag{2.5.3.6}$$

such that each X_i is a projective (A,R)-injective A-module. Applying D yields the (A,R)-exact sequence

$$DX_n \to DX_{n-1} \to \cdots \to DX_1 \to DM \to 0. \tag{2.5.3.7}$$

The functor $S \otimes_R -$ is exact on (A,R)-exact sequences, so we have the $S \otimes_R A$-exact sequence

$$S \otimes_R DX_n \to S \otimes_R DX_{n-1} \to \cdots \to S \otimes_R DX_1 \to S \otimes_R DM \to 0. \tag{2.5.3.8}$$

Observe that $S \otimes_R DM = S \otimes_R \operatorname{Hom}_R(M,R) \simeq \operatorname{Hom}_{S\otimes_R R}(S \otimes_R M, S \otimes_R R) = D_S(S \otimes_R M)$ and each $S \otimes_R DX_i$ is a projective $(S \otimes_R A, S)$-injective right $S \otimes_R A$-module. As $S \otimes_R M \in S$-proj, (2.5.3.8) is $S \otimes_R A, S)$-exact. Applying D_S yields that $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq n$. This shows that, $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq \operatorname{domdim}_{(A,R)} M$.

Now assume that there is a surjective map of R-algebras $R \to S$. In particular, S can be regarded as an R-module by restriction of scalars. Assume that this map makes S a projective R-module. Let $\operatorname{domdim}_{(S\otimes_R A,S)} S \otimes_R M \geq n$ for some integer $n \geq 0$. Then, there exists an $(S \otimes_R A, S)$-exact sequence

$$0 \to S \otimes_R M \to Y_1 \to \cdots \to Y_n, \tag{2.5.3.9}$$

where Y_i, $1 \leq i \leq n$, is a projective $(S \otimes_R A, S)$-injective $(S \otimes_R A)$-module. Applying D_S we obtain the $(S \otimes_R A, S)$-exact sequence

$$D_S Y_n \to \cdots \to D_S Y_1 \to D_S(S \otimes_R M) \to 0. \tag{2.5.3.10}$$

Observe that $(S \otimes_R A, S \otimes_R P, S \otimes_R V)$ is a relative QF3 S-algebra. Thus, each $D_S Y_i \in \operatorname{add}_{S\otimes_R A} S \otimes_R V$. As S is projective over R, S is an R-summand of $\oplus_I R$ for some set I. Hence, $D_S Y_i$ is an A-summand of $S \otimes_R V^t$ which is an A-summand of $\oplus_I V^t$. Therefore, $D_S Y_i \in \operatorname{Add}_A V$. By Observation 2.4.17, the canonical map $\Phi\colon \operatorname{Hom}_A(V, D_S(S \otimes_R M)) \otimes_C V \to D_S(S \otimes_R M)$ is an isomorphism and $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V, D_S(S \otimes_R M)), V) = 0$,

$1 \leq i \leq n-2$. Now note that

$$D_S(S \otimes_R M) \simeq \operatorname{Hom}_S(S \otimes_R M, S) \simeq S \otimes_R \operatorname{Hom}_R(M,R) = S \otimes_R DM$$

is an A-summand of $\oplus_I DM$. In particular, Φ_M is an isomorphism and $\operatorname{Tor}_i^C(\operatorname{Hom}_A(V,DM),V) = 0$, $1 \leq i \leq n-2$. So, $\operatorname{domdim}_{(A,R)} M \geq n$. This shows that $\operatorname{domdim}_{(A,R)} M \geq \operatorname{domdim}_{(S \otimes_R A, S)} S \otimes_R M$. $\square$

In the following, we will see that we can reduce the computation of relative dominant dimension to computing dominant dimension over fields.

Theorem 2.5.13. *Let R be a commutative Noetherian ring. Let (A,P,V) be a relative QF3 R-algebra. Let $M \in A\text{-mod} \cap R\text{-proj}$. Then,*

$$\operatorname{domdim}_{(A,R)} M = \inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \colon \mathfrak{m} \textit{ maximal ideal in } R\}.$$

Proof. Let $\mathfrak{m}$ be a maximal ideal in R. By Proposition 2.5.12, $\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq \operatorname{domdim}_{(A,R)} M$.

Assume that $\inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \colon \mathfrak{m} \text{ maximal ideal in } R\} \geq n$. We want to show that $\operatorname{domdim}_{(A,R)} M \geq n$. By Proposition 2.5.3, $(A(\mathfrak{m}), P(\mathfrak{m}), V(\mathfrak{m}))$ is a QF3 algebra for every maximal ideal $\mathfrak{m}$ in R. Denote by $D_{(\mathfrak{m})}$ the standard duality with respect to $R(\mathfrak{m})$ and denote $C = \operatorname{End}_A(V)$.

If $n = 0$, there is nothing to show. Assume that $n = 1$. Consider the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})) \otimes_{C(\mathfrak{m})} V(\mathfrak{m}) \arrow[r, "\Phi_{M(\mathfrak{m})}"] & D_{(\mathfrak{m})} M(\mathfrak{m}) \\
R(\mathfrak{m}) \otimes_R \operatorname{Hom}_A(V,DM) \otimes_C V \arrow[u, "\simeq"] \arrow[r, "\Phi_M(\mathfrak{m})"] & DM(\mathfrak{m}) \arrow[u, "\simeq"]
\end{tikzcd} . \tag{2.5.3.11}$$

By assumption, $\Phi_{M(\mathfrak{m})}$ is an epimorphism. Thus, $\Phi_M(\mathfrak{m})$ is an epimorphism for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, Φ_X is an epimorphism. By Proposition 2.4.7, $\operatorname{domdim}_{(A,R)} M \geq 1$.

Assume that $n = 2$. By the commutative diagram (2.5.3.11) $\Phi_M(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. Since Φ_M is an epimorphism and $M \in R$-proj, Φ_M splits over R. That is, there is a map $t \in \operatorname{Hom}_R(DM, \operatorname{Hom}_A(V,DM) \otimes_C V)$ such that $\Phi_M \circ t = \mathrm{id}_{DM}$. In particular, t is a monomorphism. Applying $R(\mathfrak{m}) \otimes_R -$, we get $\mathrm{id}_{DM(\mathfrak{m})} = \Phi_M \circ t(\mathfrak{m}) = \Phi_M(\mathfrak{m}) \circ t(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R. Since $\Phi_M(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R it follows that $t(\mathfrak{m})$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in R. By Nakayama's Lemma, t is surjective. So, t is an R-isomorphism. It follows that Φ_M is bijective. By Proposition 2.4.7, $\operatorname{domdim}_{(A,R)} M \geq 2$.

Assume now that $n \geq 3$. In particular, $\operatorname{domdim}_{(A,R)} M \geq 2$. Hence, $\operatorname{Hom}_A(V,DM) \otimes_C V \simeq DM \in R$-proj. By Theorem 2.4.15, $\operatorname{Tor}_i^{C(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m}), V(\mathfrak{m})) = 0$, $1 \leq i \leq n-2$ for every maximal ideal $\mathfrak{m}$ in R. Let

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to V \to 0 \tag{2.5.3.12}$$

be a projective C-resolution of V. Since $V \in R$-proj, this resolution is (C,R)-exact. Thus,

$$\cdots \to Q_2(\mathfrak{m}) \to Q_1(\mathfrak{m}) \to Q_0(\mathfrak{m}) \to V(\mathfrak{m}) \to 0 \tag{2.5.3.13}$$

is a projective $C(\mathfrak{m})$-resolution of V. Consider the chain complex $P^\bullet = \operatorname{Hom}_A(V,DM) \otimes_C Q^\bullet$, where $Q^\bullet$ denotes the deleted projective resolution (2.5.3.12). Each object $\operatorname{Hom}_A(V,DM) \otimes_C Q_i \in \operatorname{add}_R \operatorname{Hom}_A(V,DM) \subset R$-proj, since $\mathrm{Hom}_A(V,DM) \in R$-proj. By Lemma 1.3.17, we obtain the Künneth Spectral sequence

$$E^2_{i,j} = \mathrm{Tor}_i^R(H_j(\mathrm{Hom}_A(V,DM) \otimes_C Q^\bullet), R(\mathfrak{m})) \implies H_{i+j}(\mathrm{Hom}_A(V,DM) \otimes_C Q^\bullet(\mathfrak{m})). \tag{2.5.3.14}$$

We have that

$$\mathrm{Hom}_A(V,DM) \otimes_C Q^\bullet(\mathfrak{m}) \simeq \mathrm{Hom}_A(V,DM)(\mathfrak{m}) \otimes_{C(\mathfrak{m})} Q(\mathfrak{m})^\bullet \simeq \mathrm{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})) \otimes_{C(\mathfrak{m})} Q(\mathfrak{m})^\bullet, \tag{2.5.3.15}$$

where $Q(\mathfrak{m})^\bullet$ is a projective $C(\mathfrak{m})$-resolution of $V(\mathfrak{m})$. Hence,

$$H_{i+j}(\mathrm{Hom}_A(V,DM) \otimes_C Q^\bullet(\mathfrak{m})) = \mathrm{Tor}_{i+j}^{C(\mathfrak{m})}(\mathrm{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})) \tag{2.5.3.16}$$

and

$$H_j(\mathrm{Hom}_A(V,DM) \otimes_C Q^\bullet) = \mathrm{Tor}_j^C(\mathrm{Hom}_A(V,DM), C). \tag{2.5.3.17}$$

Thus, for every maximal ideal $\mathfrak{m}$ in R,

$$E^2_{i,j} = \mathrm{Tor}_i^R(\mathrm{Tor}_j^C(\mathrm{Hom}_A(V,DM), V), R(\mathfrak{m})) \implies \mathrm{Tor}_{i+j}^{C(\mathfrak{m})}(\mathrm{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})). \tag{2.5.3.18}$$

We shall prove by induction on $1 \leq i \leq n-2$ that $\mathrm{Tor}_j^C(\mathrm{Hom}_A(V,DM), V) = 0$. By Lemma 1.3.7 there is an exact sequence

$$E^2_{2,0} \to E^2_{0,1} \to \mathrm{Tor}_1^{C(\mathfrak{m})}(\mathrm{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})) = 0. \tag{2.5.3.19}$$

As $\mathrm{Hom}_A(V,DM) \otimes_C V \in R$-proj, $E^2_{2,0} = \mathrm{Tor}_2^R(\mathrm{Hom}_A(V,DM) \otimes_C V, R(\mathfrak{m})) = 0$. Thus, for every maximal ideal $\mathfrak{m}$ in R, $0 = E^2_{0,1} = \mathrm{Tor}_1^C(\mathrm{Hom}_A(V,DM), V) \otimes_R R(\mathfrak{m})$. Therefore, $\mathrm{Tor}_1^C(\mathrm{Hom}_A(V,DM), V) = 0$.

Assume now that $\mathrm{Tor}_l^C(\mathrm{Hom}_A(V,DM), V) = 0$ for some $1 \leq l < n-2$. Then,

$$E^2_{i,j} = \mathrm{Tor}_i^R(\mathrm{Tor}_j^C(\mathrm{Hom}_A(V,DM), V), R(\mathfrak{m})) = \mathrm{Tor}_i^R(0, R(\mathfrak{m})) = 0, \ 1 \leq j \leq l, \ i \geq 0. \tag{2.5.3.20}$$

By Lemma 1.3.11, there exists an exact sequence

$$E^2_{l+2,0} \to E^2_{0,l+1} \to \mathrm{Tor}_{l+1}^{C(\mathfrak{m})}(\mathrm{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), DM(\mathfrak{m})), V(\mathfrak{m})) = 0, \tag{2.5.3.21}$$

where $E^2_{l+2,0} = \mathrm{Tor}_{l+2}^R(\mathrm{Hom}_A(V,DM) \otimes_C V, R(\mathfrak{m})) = 0$. Therefore, $E^2_{0,l+1} = \mathrm{Tor}_{l+1}^C(\mathrm{Hom}_A(V,DM), V)(\mathfrak{m}) = 0$, for every maximal ideal $\mathfrak{m}$ in R. Therefore, $\mathrm{Tor}_{l+1}^C(\mathrm{Hom}_A(V,DM), V) = 0$. Hence, we obtain

$$\mathrm{Tor}_i^C(\mathrm{Hom}_A(V,DM), V) = 0, \ 1 \leq i \leq n-2. \tag{2.5.3.22}$$

By Theorem 2.4.15, $\mathrm{domdim}_{(A,R)} M \geq n$. □

Combining this theorem with Proposition 2.5.10, we deduce that the computation of relative dominant dimension of a projective Noetherian R-algebra can be reduced to computing the dominant dimension of finite-dimensional algebras over algebraically closed fields. This shows that the dominant dimension is more static under change of ring than other homological invariants. For example, the global dimension of an algebra can heavily depend on the ground field of the algebra.

2.5.3.1 Base change property

This reduction theorem also explains the meaning behind the generators relative cogenerators which arise in the relative Morita-Tachikawa correspondence. These are the ones that make its endomorphism algebra admit a base change property like the Schur algebra.

Proposition 2.5.14. *Let B be a projective Noetherian algebra over a commutative Noetherian ring R. Let $M \in B$-mod$\cap R$-proj be a B-generator (B,R)-cogenerator. The following assertions are equivalent.*

(i) $DM \otimes_B M \in R$-proj.

(ii) For every commutative R-algebra S, $S \otimes_R \operatorname{End}_B(M)^{op} \simeq \operatorname{End}_{S\otimes_R B}(S\otimes_R M)^{op}$ as S-algebras.

Proof. Assume that $DM \otimes_B M \in R$-proj holds. Let S be a commutative R-algebra. Denote by D_S the standard duality over S. As $S \otimes_R -$ preserves coproducts,

$$D_S(S\otimes_R M)\otimes_{S\otimes_R B} S\otimes_R M = \operatorname{Hom}_S(S\otimes_R M, S)\otimes_{S\otimes_R B} S\otimes_R M \simeq S\otimes_R \operatorname{Hom}_R(M,R)\otimes_B M \in S\text{-proj}\,. \quad (2.5.3.23)$$

Denote by μ the canonical map $S\otimes_R \operatorname{Hom}_B(M,M) \to \operatorname{Hom}_{S\otimes_R B}(S\otimes_R M, S\otimes_R M)$. By Proposition 1.1.30, the canonical map $S\otimes_R DM\otimes_B M \to D_S(S\otimes_R M)\otimes_{S\otimes_R B} S\otimes_R M$ is an isomorphism. Consider the following commutative diagram

$$\begin{tikzcd} D_S \operatorname{Hom}_{S\otimes_R B}(S\otimes_R M, S\otimes_R M) \arrow[r, "D_S\mu"] \arrow[d, "\simeq"] & D_S(S\otimes_R \operatorname{Hom}_B(M,M)) \arrow[d, "\simeq"] \\ D_S(S\otimes_R M)\otimes_{S\otimes_R B} S\otimes_R M & S\otimes_R DM\otimes_B M \arrow[l, "\simeq"'] \end{tikzcd}, \quad (2.5.3.24)$$

where the columns are isomorphisms by Proposition 1.1.65 since

$$DM\otimes_B M \in R\text{-proj}, \quad D_S(S\otimes_R M)\otimes_{S\otimes_R B} S\otimes_R M \in S\text{-proj}\,. \quad (2.5.3.25)$$

Consequently, $D_S\mu$ is an isomorphism. Again, since $D_S(S\otimes_R M)\otimes_{S\otimes_R B} S\otimes_R M \in S$-proj it follows that μ is bijective.

Conversely, assume that (ii) holds. In particular, for every maximal ideal $\mathfrak{m}$ in R, $\operatorname{End}_{B(\mathfrak{m})}(M(\mathfrak{m})) \simeq \operatorname{End}_B(M)(\mathfrak{m})$. Since $R(\mathfrak{m})\otimes_R -$ preserves direct sums, we get that $M(\mathfrak{m})$ is a generator-cogenerator over $B(\mathfrak{m})$. Hence, by Morita-Tachikawa correspondence, $\operatorname{domdim}\operatorname{End}_{B(\mathfrak{m})}(M(\mathfrak{m}))^{op} \geq 2$. Now, for each maximal ideal $\mathfrak{m}$ in R, (ii) yields $\operatorname{domdim}\operatorname{End}_B(M)^{op}(\mathfrak{m}) \geq 2$. By Proposition 2.5.3, M is a projective $(\operatorname{End}_B(M)^{op}, R)$-injective-strongly faithful $\operatorname{End}_B(M)^{op}$-module. By Proposition 2.5.13, $\operatorname{domdim}(\operatorname{End}_B(M)^{op}, R) \geq 2$. By relative Morita-Tachikawa correspondence, $DM\otimes_B M \in R$-proj. ☐

As usual, we can compare this situation with what happens to regular rings with Krull dimension at most one.

Lemma 2.5.15. *Let R be a commutative Noetherian regular ring with Krull dimension at most one. Then, the canonical map $S\otimes_R \operatorname{Hom}_A(M,X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X)$ is a monomorphism for every $M, X \in A$-mod and every commutative R-algebra S.*

Proof. Let $M, X \in A$-mod and let S be a commutative R-algebra. Consider a projective presentation over A

$$P_1 \to P_0 \to M \to 0. \quad (2.5.3.26)$$

The functor $\operatorname{Hom}_{S\otimes_R A}(-, S\otimes_R X)\circ S\otimes_R -\colon A\text{-mod} \to S\otimes_R A\text{-mod}$ is contravariant left exact. So, the induced sequence

$$0 \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_0, S\otimes_R X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_1, S\otimes_R X) \tag{2.5.3.27}$$

is exact. The functor $\operatorname{Hom}_A(-,X)$ is left exact, thus we have the exact sequence

$$0 \to \operatorname{Hom}_A(M,X) \to \operatorname{Hom}_A(P_0,X) \to \operatorname{Hom}_A(P_1,X). \tag{2.5.3.28}$$

Denote by f the map $\operatorname{Hom}_A(M,X) \to \operatorname{Hom}_A(P_0,X)$. By exactness of (2.5.3.28), the cokernel of f is a submodule of $\operatorname{Hom}_A(P_1,X)$. Since $\dim R \leq 1$, the cokernel of f is projective over R. In particular, f is a split R-monomorphism and so it remains a monomorphism under $S\otimes_R -$. Using the commutative diagram

$$\begin{tikzcd} S\otimes_R \operatorname{Hom}_A(M,X) \arrow[r, "S\otimes_R f"] \arrow[d] & S\otimes_R \operatorname{Hom}_A(P_0,X) \arrow[d, "\simeq"] \\ \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X) \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P_0, S\otimes_R X) \end{tikzcd}, \tag{2.5.3.29}$$

we conclude that the canonical map $S\otimes_R \operatorname{Hom}_A(M,X) \to \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M, S\otimes_R X)$ is a monomorphism. $\square$

2.6 Dominant dimension, global dimension and Nakayama's conjecture

In order to compare the relative global dimension with the relative global dimension we need the following result. The argument is essentially Lemma 5.5 of [ARS95, C. VI].

Lemma 2.6.1. *Let A be a projective Noetherian R-algebra. Then,*

$$\operatorname{idim}_{(A,R)} {}_A A = \operatorname{pdim}_A DA = \sup\{m\colon\ \operatorname{Ext}_A^m(DA,A) \neq 0\} \leq \operatorname{idim} A_A. \tag{2.6.0.1}$$

Moreover, if $\operatorname{gldim} A < +\infty$*, then* $\operatorname{gldim} A = \operatorname{idim} A$.

Proof. Since DA is projective over R, it is clear using the standard duality D that $\operatorname{pdim} DA = \operatorname{idim}_{(A,R)} {}_A A$.

Let $M \in A\text{-mod}\cap R\text{-proj}$ with finite projective dimension n. We claim that $\operatorname{Ext}_A^n(M,A) \neq 0$. So, by contradiction assume that $\operatorname{Ext}_A^n(M,A) = 0$. Let

$$0 \to P_n \xrightarrow{h} P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \tag{2.6.0.2}$$

be a projective A-resolution of M. Applying $\operatorname{Hom}_A(-,P_n)$ to (2.6.0.2) we get the surjective map $\operatorname{Hom}_A(P_{n-1},P_n) \to \operatorname{Hom}_A(P_n,P_n)$. So, there exists $f \in \operatorname{Hom}_A(P_{n-1},P_n)$ such that $\operatorname{id}_{P_n} = f\circ h$. So, P_n is an A-summand of P_{n-1}, and therefore we can remove P_n from the projective resolution. This would imply that $\operatorname{pdim}_A M \leq n-1$.

It is clear that $\operatorname{Ext}_A^{\operatorname{pdim}_A DA+i}(DA,A) = 0$ for any $i > 0$. Hence, by the previous argument, it follows that (2.6.0.1) holds.

Assume that $\operatorname{gldim} A$ is finite. Consider the A-module $M = \oplus_{X\in A\text{-mod}} X$. Then, $n = \operatorname{pdim}_A M = \operatorname{gldim} A$. In particular, $\operatorname{Ext}_A^n(M,A) \neq 0$. So, $\operatorname{idim} A \geq \operatorname{gldim} A$. It is clear by definition, that $\operatorname{gldim} A \geq \operatorname{idim} A$. Since left and right global dimension coincide for algebras over Noetherian rings, we get $\operatorname{idim}_A A = \operatorname{idim} A_A$. $\square$

As for finite-dimensional algebras, the relative dominant dimension of Noetherian algebras is bounded by the global dimension.

Proposition 2.6.2. *Let A be a projective Noetherian R-algebra. If* $\operatorname{domdim}(A,R) < \infty$*, then*

$$\operatorname{domdim}(A,R) \leq \operatorname{gldim}_f(A,R), \quad \operatorname{domdim}(A,R) \leq \operatorname{gldim} A.$$

Proof. Assume that $\operatorname{domdim}(A,R) = n < +\infty$. So, there exists an (A,R)-exact sequence

$$0 \to A \to X_0 \to X_1 \to \cdots \to X_{n-1}, \tag{2.6.0.3}$$

with all X_i being (A,R)-injective projective over A. Applying D we obtain the right A-exact sequence

$$DX_{n-1} \to \cdots \to DX_1 \to DX_0 \to DA \to 0. \tag{2.6.0.4}$$

In particular, there exists an exact sequence

$$0 \to K_{n-2} \to DX_{n-2} \to \cdots \to DX_1 \to DX_0 \to DA \to 0. \tag{2.6.0.5}$$

By contradiction, assume that $n > \operatorname{pdim}_A DA$. Since all DX_i are projective over A, it follows that K_{n-2} must be projective over A. Hence, DK_{n-2} is (A,R)-injective and projective over R. Moreover, we have a factorization

$$\begin{tikzcd} X_{n-2} \arrow[rr] \arrow[dr, two heads] & & X_{n-1} \\ & DK_{n-2} \arrow[ur, hook] & \end{tikzcd}, \tag{2.6.0.6}$$

and the monomorphism is an (A,R)-monomorphism since this factorization is given by (2.6.0.3). So, it must split over A, and therefore DK_{n-2} is also projective over A. Applying D to (2.6.0.5), it follows that $\operatorname{domdim}(A,R)$ is infinite. Therefore, we must have

$$\operatorname{gldim}_f(A,R) \geq \operatorname{idim}_{(A,R)\,A} A = \operatorname{pdim}_A DA \geq n = \operatorname{domdim}(A,R)$$
$$\operatorname{gldim} A \geq \operatorname{pdim}_A DA \geq n = \operatorname{domdim}(A,R).$$

□

Theorem 2.6.3. *If the Nakayama conjecture holds for finite-dimensional algebras over a field, then the relative Nakayama Conjecture holds for any projective Noetherian algebra over a commutative Noetherian ring.*

Proof. Assume that $\operatorname{domdim}(A,R) = +\infty$. By Theorem 2.5.13, $\operatorname{domdim} A(\mathfrak{m}) = +\infty$ for every maximal ideal $\mathfrak{m}$ in R. If the Nakayama conjecture holds for finite-dimensional algebras over fields, then $A(\mathfrak{m})$ is $A(\mathfrak{m})$-injective, for every maximal ideal $\mathfrak{m}$ in R. As A is projective when regarded as R-module, it follows that the (left) regular module A is (A,R)-injective by Theorem 1.2.57. In the same way, the right regular module A is (A,R)-injective. Thus, A is a relative self-injective R-algebra. □

2.7 Orders of Finite Lattice Type

When the ground ring R is a Dedekind domain, projective Noetherian R-algebras A are known in the literature as ***R*-orders**. For a more detailed exposure of representation theory of R-orders, we refer to [Rei70]. The modules belonging to $A\text{-mod} \cap R\text{-proj}$ are known as ***A*-lattices**. Let F be the quotient field of R, then $F \otimes_R A$ is a finite-dimensional algebra over F. We can identify A with $1 \otimes_R A$, so A is embedded in the finite-dimensional algebra $F \otimes_R A$. The same idea holds for the A-lattices. Every A-lattice M can be embedded in the vector space $F \otimes_R M$. The (A,R)-monomorphisms also receive special attention in order theory. Given two A-lattices

M,N, M is said to be R-pure A-sublattice of N if there exists an (A,R)-monomorphism $M\to N$. Moreover, the (A,R)-monomorphisms arise as inclusions of $F\otimes_R A$-modules.

Theorem 2.7.1. *[Zas38] Let R be a Dedekind domain and let A be an R-order. Let F be the quotient field of R. Given any A-lattice N, there is a bijection between A-submodules W of $F\otimes_R N$ and R-pure A-sublattices M of N. The correspondence is given by*

$$M = N\cap W, \quad W = F\otimes_R M.$$

Moreover, each $V\in F\otimes_R A\text{-mod}$ is of the form $F\otimes_R N$ for some A-lattice N in V.

We can deduce in this section that the characterization of orders of Finite Lattice-Type by Auslander and Roggenkamp [AR72] is a particular case of relative Morita-Tachikawa correspondence (Theorem 2.4.12). We say that an R-order A has **finite lattice-type** if A has a finite number of indecomposable A-lattices. Otherwise, we say that A is of **infinite lattice-type**.

By [Fad65, Proposition 25.1], if $F\otimes_R A$ is not semi-simple, then A is of infinite lattice type. We remark that semi-simple algebras over algebraic number fields are separable. In [AR72], R is assumed to be a complete discrete valuation ring such that its quotient field is a completion of an algebraic number field. This is due to the following fact:

Theorem 2.7.2. *[Kne66, Jon63] Let R be a Dedekind domain such that its quotient field is an algebraic number field. Let G be a finite group and RG the group algebra of G over R. Then, RG is of finite lattice type if and only if $\hat{RG}_m$ is of finite lattice type for every maximal ideal m in R.*

This reduction technique is useful because for every Noetherian algebra over a commutative Noetherian local complete ring, A, A-mod is a Krull-Schmidt category. In particular, this allowed Jones, Heller and Reiner to completely determine all group algebras of finite type.

Theorem 2.7.3. *Let R be a local complete discrete valuation ring such that its quotient field K is a completion of an algebraic number field. There is a bijection between*

$$\left\{ A\colon \begin{array}{c} \textit{A an R-order in a} \\ \textit{semi-simple K-algebra} \\ \textit{of finite type} \end{array} \right\} \Big/ \sim \longleftrightarrow \left\{ B\colon \begin{array}{c} \textit{B an R-order in a semi-simple K-algebra with} \\ \operatorname{domdim}(B,R)\geq 1, \operatorname{gldim} B\leq 2, \\ \textit{and all minimal } (B,R)\textit{-injective-strongly faithful} \\ \textit{projective satisfy the double centralizer property} \end{array} \right\} \Big/ \mathit{iso}$$

In this notation, $B\sim B'$ if and only if B and B' are Morita equivalent. This is correspondence is given by:

$$A\mapsto B=\operatorname{End}_A(G)^{op}$$
$$(\operatorname{End}_B(N)) \leftarrow\!\shortmid B$$

where N is a projective (B,R)-injective-strongly faithful right B-module and G is an additive generator of $A\text{-mod}\cap R\text{-proj}$.

Proof. Let A be an R-order such that $K\otimes_R A$ is a semi-simple algebra and A is of finite type. Consider $G=\oplus_{i\in I}M_i$, where M_i are all non-isomorphic indecomposable A-lattices for some finite set I. In particular, every module of A-mod belongs to $\operatorname{add} G$. Thus, G is an additive generator of A-mod. So, G is a generator (A,R)-cogenerator. As $A\in R\text{-proj}$, it follows by Theorem 2.4.12 that $B=\operatorname{End}_A(G)^{op}$ has relative dominant dimension $\operatorname{domdim}(B,R)$ greater than or equal to one and all minimal projective (B,R)injective-strongly faithful modules satisfy the double centralizer property between A and B. Since K is flat as R-module B is an R-order in the semi-simple K-algebra

$$K\otimes_R B = K\otimes_R \operatorname{End}_A(G) \simeq \operatorname{End}_{K\otimes_R A}(K\otimes_R G). \tag{2.7.0.1}$$

In fact, $K\otimes_R G$ is a semi-simple module over $K\otimes_R A$ and consequently, its endomorphism algebra is semi-simple by the Wedderburn Theorem. It remains to show that $\operatorname{gldim} B \geq 2$.

Let $X \in B$-mod. Let $P_1 \xrightarrow{h} P_0 \to X \to 0$ be a projective B-presentation of X. By projectivization, the functor $\operatorname{Hom}_A(G,-)\colon A\text{-mod} \to B\text{-mod}$ induces an equivalence between $A\text{-mod}\cap R\text{-proj} = \operatorname{add} G$ and B-proj. Hence, there exist modules $M_0, M_1 \in A\text{-mod}\cap R\text{-proj}$ such that $P_i \simeq \operatorname{Hom}_A(G, M_i)$, $i = 0,1$. Further, there exists a map $f \in \operatorname{Hom}_A(M_1, M_0)$ satisfying $h = \operatorname{Hom}_A(G, f)$. Consider the exact sequence

$$0 \to \ker f \to M_1 \xrightarrow{f} M_0. \tag{2.7.0.2}$$

Applying $\operatorname{Hom}_A(G,-)$ yields the exact sequence

$$0 \to \operatorname{Hom}_A(G, \ker f) \to P_1 \xrightarrow{h} P_0 \to X \to 0. \tag{2.7.0.3}$$

R has Krull dimension one, therefore $\ker f$ is an A-lattice. This shows that $\operatorname{Hom}_A(G,\ker f) \in \operatorname{add}\operatorname{Hom}_A(G,G) = B\text{-proj}$. Hence, $\operatorname{pdim}_B X \geq 2$.

Conversely, assume that B is an R-order in a semi-simple K-algebra $K\otimes_R B$ with $\operatorname{domdim}(B,R)\geq 1$, $\operatorname{gldim} B \leq 2$ and all minimal (B,R)-injective-strongly faithful projective modules M satisfy a double centralizer property between B and $\operatorname{End}_B(M)$. Let M be a B-lattice such that (B, DM, M) is a relative QF3 R-algebra. By Theorem 2.4.12, $A = \operatorname{End}_B(M) \in R$-proj and M is an A-generator (A,R)-cogenerator such that $B \simeq \operatorname{End}_A(M)^{op}$ as R-algebras. So, A is an R-order in the semi-simple K-algebra

$$K\otimes_R A \simeq K\otimes_R \operatorname{End}_B(M) \simeq \operatorname{End}_{K\otimes_R B}(K\otimes_R M). \tag{2.7.0.4}$$

Since A-mod is a Krull-Schmidt category, the number of indecomposable A-lattices summands of M is finite and unique up to isomorphism. Therefore, it is enough to prove that $\operatorname{add}_A M = A\text{-mod}\cap R\text{-proj}$.

Let $X \in A\text{-mod}\cap R\text{-proj}$. Let $0 \to X \to I_0 \to I_1$ be the standard (A,R)-injective resolution of X. Applying the functor $\operatorname{Hom}_A(M,-)$ yields the B-exact sequence

$$0 \to \operatorname{Hom}_A(M,X) \to \operatorname{Hom}_A(M,I_0) \to \operatorname{Hom}_A(M,I_1) \to Y \to 0, \tag{2.7.0.5}$$

for some $Y \in B$-mod. Now, the fact that M is an (A,R)-cogenerator implies that $\operatorname{Hom}_A(M,I_i) \in \operatorname{add}\operatorname{Hom}_A(M,M)$. The projective dimension of Y is at most two, and consequently, $\operatorname{Hom}_A(M,X)$ is projective over B. By projectivization, there exists $M_0 \in \operatorname{add}_A M$ satisfying $\operatorname{Hom}_A(M,X) \simeq \operatorname{Hom}_A(M,M_0)$. Now, thanks to the exactness of $M\otimes_B -$ and the standard (A,R)-injective resolution of X, $M_0 \simeq M\otimes_B \operatorname{Hom}_A(M,X)$ is isomorphic to X. $\square$

2.8 Classification of relative torsionless modules and reflexive modules

Given $M \in A$-mod, we say that M is (A,R)-**torsionless** if there exists a projective module $P \in A$-proj and an (A,R)-monomorphism $M \to P$.

Lemma 2.8.1. *Every strongly (A,R)-torsionless (A,R)-injective module is projective over A.*

Proof. Let M be an (A,R)-torsionless (A,R)-injective module. By definition, there exists an (A,R)-monomorphism $M \to P$ for some projective A-module. Since M is (A,R)-injective, this monomorphism splits over A. Thus,

$M \in \operatorname{add} P$. So, M is projective over A. $\square$

In the next theorem, the **relative injective hull of a module** M, when it exists, is the dual of the projective cover in the opposite algebra of the dual of M. We will denote by $I_R(M)$ the relative injective hull of M. More precisely, given $M \in A$-mod, $I_R(M) = \operatorname{Hom}_R(P(DM), R)$, where $P(DM)$ denotes the projective cover of DM in A^{op}-mod.

Proposition 2.8.2. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a commutative Noetherian ring and relative QF3 R-algebra. The following assertions are equivalent.*

(i) $\operatorname{domdim}_{(A,R)} \Delta(\lambda) \geq 1$ *for every* $\lambda \in \Lambda$.

(ii) $\operatorname{domdim}_{(A,R)} T \geq 1$ *for a characteristic tilting module* T.

(iii) Every module in $\mathscr{F}(\tilde{\Delta})$ is (A,R)-strongly torsionless and $\mathscr{F}(\tilde{\Delta}_{\mathfrak{m}})$ is closed under relative injective hulls for every maximal ideal $\mathfrak{m}$ in R.

Proof. $(i) \implies (iii)$. For every $U_\lambda \in R$-proj, $\Delta(\lambda)\otimes_R U_\lambda \in \operatorname{add}\Delta(\lambda)$. So, $\operatorname{domdim}_{(A,R)} \Delta(\lambda)\otimes_R U_\lambda \geq 1$ for every $\lambda \in \Lambda$ and $U_\lambda \in R$-proj. It follows by Lemma 2.4.25 that $\operatorname{domdim}_{(A,R)} M \geq 1$ for every module $M \in \mathscr{F}(\tilde{\Delta})$. So, there is an (A,R)-monomorphism $M \to P$ for an (A,R)-injective projective module. In particular, every module $M \in \mathscr{F}(\tilde{\Delta})$ is (A,R)-strongly torsionless. Let $\mathfrak{m}$ be a maximal ideal in R. By Theorems 1.5.84 and 1.5.69, $A_{\mathfrak{m}}$ and $A_{\mathfrak{m}}^{op}$ are semi-perfect algebras. Let $M \in \mathscr{F}(\tilde{\Delta}_{\mathfrak{m}})$. Then, $D_{\mathfrak{m}}M \in \mathscr{F}(\tilde{\nabla}_{\mathfrak{m}})$. Let P be the projective cover of $D_{\mathfrak{m}}M$.

By Proposition 2.5.8, $\operatorname{domdim}_{(A_{\mathfrak{m}},R_{\mathfrak{m}})} M \geq 1$. Hence, there exists a projective $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-injective module I and an $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-monomorphism $M \to I$. So, $D_{\mathfrak{m}}I$ is $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-injective and $D_{\mathfrak{m}}I \to D_{\mathfrak{m}}M$ is surjective. Thus, $P \in \operatorname{add} D_{\mathfrak{m}}I$. Consequently P is right $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-injective. Thus, $I_{R_{\mathfrak{m}}}(M) = DP$ is projective over $A_{\mathfrak{m}}$. In particular, $I_{R_{\mathfrak{m}}}(M) \in \mathscr{F}(\tilde{\Delta}_{\mathfrak{m}})$.

$(iii) \implies (ii)$. Let $\mathfrak{m}$ be a maximal ideal in R. Then, $T_{\mathfrak{m}}$ is a characteristic tilting module in $A_{\mathfrak{m}}$. By assumption, $I_{R_{\mathfrak{m}}}(T_{\mathfrak{m}}) \in \mathscr{F}(\tilde{\Delta}_{\mathfrak{m}})$. By localizing at $\mathfrak{m}$, it follows that every module in $\mathscr{F}(\tilde{\Delta}_{\mathfrak{m}})$ is $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-strongly torsionless. Furthermore, the relative injective hull of $T_{\mathfrak{m}}$ $I_{R_{\mathfrak{m}}}(T_{\mathfrak{m}})$ is $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-strongly torsionless. By Lemma 2.8.1, $I_{R_{\mathfrak{m}}}(T_{\mathfrak{m}})$ is a projective $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-injective $A_{\mathfrak{m}}$-module. Using the $(A_{\mathfrak{m}},R_{\mathfrak{m}})$-monomorphism $T_{\mathfrak{m}} \to I_{R_{\mathfrak{m}}}(T_{\mathfrak{m}})$ we deduce that $\operatorname{domdim}_{(A_{\mathfrak{m}},R_{\mathfrak{m}})} T_{\mathfrak{m}} \geq 1$. By Proposition 2.5.7, (ii) follows.

$(ii) \implies (i)$. For every $\lambda \in \Lambda$, there is an (A,R)-monomorphism $\Delta(\lambda) \to T(\lambda)$ since its cokernel belongs to $\mathscr{F}(\tilde{\Delta})$, and therefore it is projective over R. By Corollary 2.4.23, $\operatorname{domdim}_{(A,R)} T(\lambda) \geq 1$. So, there exists an (A,R)-injective projective module P and an (A,R)-monomorphism $T(\lambda) \to P$. Hence, the composition of maps $\Delta(\lambda) \to T(\lambda) \to P$ gives $\operatorname{domdim}_{(A,R)} \Delta(\lambda) \geq 1$. $\square$

In [FKY18] Fang, Kerner and Yamagata showed that the theory of dominant dimension over finite dimensional algebras over a field was related to the exactness of left adjoint of the double dual functor

$$(-)^{**} \colon A\text{-Mod} \to A\text{-Mod},\ M \mapsto \operatorname{Hom}_{A^{op}}(\operatorname{Hom}_A(M,A),A). \tag{2.8.0.1}$$

For relative dominant dimension, the relevant functor to consider is the following functor

$$\mathscr{O} \colon A\text{-Mod} \to \text{Mod-}A, \quad M \mapsto \operatorname{Hom}_A(M,A)\otimes_A DA. \tag{2.8.0.2}$$

Proposition 2.8.3. *Let (A,P,V) be a relative QF3 R-algebra with* $\operatorname{domdim}(A,R) \geq 2$.

Define the natural transformation $\gamma \colon \mathscr{O} \to D$ with morphisms $\gamma_X \colon \operatorname{Hom}_A(X,A)\otimes_A DA \to DX$, given by $\gamma_X(f\otimes g)(x) = g(f(x))$, $f\otimes g \in \operatorname{Hom}_A(X,A)\otimes_A DA$, $x \in X$.

There exists a natural equivalence $\Sigma\colon \operatorname{Hom}_A(V, D-)\otimes_C V \to \mathscr{O}$ *making the following diagram commutative:*

$$\begin{array}{ccc} \operatorname{Hom}_A(V,DX)\otimes_C V & \xrightarrow{\simeq} & \mathscr{O}X \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \gamma_X} \\ DX & = & DX \end{array}, \quad \forall X \in A\text{-Mod}. \tag{2.8.0.3}$$

Proof. Let $X \in A$-mod. By assumption $\Phi_A\colon \operatorname{Hom}_A(V,DA)\otimes_C V \to DA$ is an isomorphism. Consider the C-isomorphism
$\kappa_X\colon \operatorname{Hom}_A(V,DX) \to \operatorname{Hom}_R(V\otimes_A X, R) \to \operatorname{Hom}_A(X,DV)$ given by $\kappa_X(g)(x)(v) = g(v)(x)$, $g \in \operatorname{Hom}_A(V,DX)$, $x \in X$, $v \in V$. By Tensor-Hom adjunction the following composition of C-maps is a C-isomorphism

$$\begin{array}{ccccc} \operatorname{Hom}_A(V,DX) & \xrightarrow{\kappa_X} & \operatorname{Hom}_A(X,DV) & \xrightarrow{\operatorname{Hom}_A(X,w_{DV})} & \operatorname{Hom}_A(X,\operatorname{Hom}_A(\operatorname{Hom}_A(DV,A),A)) \\ & & & & \downarrow{\scriptstyle \rho_{X,\operatorname{Hom}_A(DV,A)}} \\ & & \operatorname{Hom}_A(\operatorname{Hom}_A(DV,A),\operatorname{Hom}_A(X,A)) & \xleftarrow[\sigma_{\operatorname{Hom}_A(DV,A),X}]{} & \operatorname{Hom}_A(\operatorname{Hom}_A(DV,A)\otimes_A X, A) \end{array}. \tag{2.8.0.4}$$

Denote this isomorphism by $\Sigma_X^{(1)}$. By Tensor-Hom adjunction and since $DV \in A^{op}$-proj the following map is an C-isomorphism

$$\operatorname{Hom}_A(X,A)\otimes_A DV \xrightarrow{\operatorname{Hom}_A(X,A)\otimes_A w_{DV}} \operatorname{Hom}_A(X,A)\otimes_A (DV)^{**} \xrightarrow{\psi_{\operatorname{Hom}_A(DV,A)}} \operatorname{Hom}_A(\operatorname{Hom}_A(DV,A),\operatorname{Hom}_A(X,A)).$$

Denote this isomorphism by $\Sigma_X^{(2)}$. We claim that the following diagram is commutative:

$$\begin{array}{ccccc} \operatorname{Hom}_A(X,A)\otimes_A DV\otimes_C V & & \xrightarrow{\operatorname{Hom}_A(X,A)\otimes_A \Phi_A} & & \operatorname{Hom}_A(X,A)\otimes_A DA \\ \downarrow{\scriptstyle \Sigma_X^{(2)}\otimes_C \mathrm{id}_V} & & & & \downarrow{\scriptstyle \gamma_X} \\ \operatorname{Hom}_A(\operatorname{Hom}_A(DV,A),\operatorname{Hom}_A(X,A))\otimes_C V & \xleftarrow{\Sigma_X^{(1)}\otimes_C \mathrm{id}_V} & \operatorname{Hom}_A(V,DX)\otimes_C V & \xrightarrow{\Phi_X} & DX \end{array}. \tag{2.8.0.5}$$

First, note that ${\Sigma_X^{(1)}}^{-1} = \kappa_X^{-1}\circ \operatorname{Hom}_A(X,w_{DV})^{-1}\circ \sigma_{X,\operatorname{Hom}_A(DV,A)}\circ \rho_{\operatorname{Hom}_A(DV,A),X}$. Let $g \in \operatorname{Hom}_A(X,A)$, $f \in DV$, $v \in V$, $x \in X$. Then,

$$\begin{aligned} \Phi_X\circ {\Sigma_X^{(1)}}^{-1}\otimes_C \mathrm{id}_V \circ \Sigma_X^{(2)}\otimes_C \mathrm{id}_V(g\otimes f\otimes v)(x) &= \Phi_X\circ {\Sigma_X^{(1)}}^{-1}\otimes_C \mathrm{id}_V(\psi_{\operatorname{Hom}_A(DV,A)}\circ \operatorname{Hom}_A(X,A)\otimes_A w_{DV})(g\otimes f\otimes v)(x) \\ &= \Phi_X\circ {\Sigma_X^{(1)}}^{-1}\otimes_C \mathrm{id}_V(\psi_{\operatorname{Hom}_A(DV,A)}(g\otimes w_{DV}(f)\otimes v)(x) \quad (2.8.0.6)\\ &= \Phi_X\circ {\Sigma_X^{(1)}}^{-1}\otimes_C \mathrm{id}_V(gw_{DV}(f)(-)\otimes v)(x) \quad (2.8.0.7)\\ &= {\Sigma_X^{(1)}}^{-1}(gw_{DV}(f)(-))(v)(x) \quad (2.8.0.8)\\ &= \kappa_X^{-1}\circ \operatorname{Hom}_A(X,w_{DV})^{-1}\circ \sigma_{X,\operatorname{Hom}_A(DV,A)}\circ \rho_{\operatorname{Hom}_A(DV,A),X}(gw_{DV}(f)(-))(v)(x). \quad (2.8.0.9) \end{aligned}$$

Let $h \in \operatorname{Hom}_A(DV,A)$, then

$$\sigma_{X,\operatorname{Hom}_A(DV,A)}\circ \rho_{\operatorname{Hom}_A(DV,A),X}(gw_{DV}(f)(-))(x)(h) = \rho_{\operatorname{Hom}_A(DV,A),X}(gw_{DV}(f)(-))(h\otimes x) \tag{2.8.0.10}$$

$$= gw_{DV}(f)(-)(h)(x) = (g\cdot w_{DV}(f)(h))(x) = (g\cdot h(f))(x) = g(x)h(f). \tag{2.8.0.11}$$

Note that $\gamma_X \operatorname{Hom}_A(X,A)\otimes_A \Phi_A(g\otimes f\otimes -)(-) \in \operatorname{Hom}_A(V,DX)$. In fact, for $a \in A$,

$$\gamma_X \operatorname{Hom}_A(X,A)\otimes_A \Phi_A(g\otimes f\otimes -)(-)(va)(x) = \gamma_X(g\otimes f(va(-))(x) = f(v\cdot ag(x)) = f(vg(ax)) \tag{2.8.0.12}$$

$$= \gamma_X \operatorname{Hom}_A(X,A)\otimes_A \Phi_A(g\otimes f\otimes -)(-)(v)(ax) \tag{2.8.0.13}$$

$$= (\gamma_X \operatorname{Hom}_A(X,A) \otimes_A \Phi_A(g \otimes f \otimes -)(-) \cdot a)(v)(x). \quad (2.8.0.14)$$

Now, observe that,

$$\begin{aligned}
\operatorname{Hom}_A(X, w_{DV})\kappa_X(\gamma_X \circ \operatorname{Hom}_A(X,A) \otimes_A \Phi_A(g \otimes f \otimes -)(-))(x)(h) &= w_{DV}\,\kappa_X(\gamma_X(g \otimes \Phi_A(f \otimes -)(x)(h) && (2.8.0.15)\\
&= h(\kappa_X(\gamma_X(g \otimes \Phi_A(f \otimes -)(x)) && (2.8.0.16)\\
&= h(\gamma_X(g \otimes \Phi_A(f \otimes -)(x) && (2.8.0.17)\\
&= h(\Phi_A(f \otimes -)(g(x)) && (2.8.0.18)\\
&= h(f(- \cdot g(x)) = h(g(x) \cdot f) = g(x)h(f).
\end{aligned}$$

Therefore, combining (2.8.0.15), (2.8.0.12) and (2.8.0.11) we get

$$\gamma_X \circ \operatorname{Hom}_A(X,A) \otimes_A \Phi_A)(g \otimes f \otimes -)(-) = \Sigma_X^{(1)^{-1}}(g w_{DV}(f)(-)). \quad (2.8.0.19)$$

It follows that the diagram (2.8.0.5) is commutative.

Let Σ_X be the composition $(\operatorname{Hom}_A(X,A) \otimes_A \Phi_A)^{-1} \circ (\Sigma_X^{(2)} \otimes_C \mathrm{id}_V)^{-1} \circ \Sigma_X^{(1)} \otimes_C \mathrm{id}_V$. Since all these maps are functorial then Σ is a natural equivalence between the functors $\operatorname{Hom}_A(V, D-) \otimes_C V$ and $\mathscr{O}$ which satisfies $\gamma_X \circ \Sigma_X = \Phi_X$ for all $X \in A$-mod. ☐

Theorem 2.8.4. *Let (A,P,V) be a relative QF3 R-algebra with* $\operatorname{domdim}(A,R) \geq 2$. *Let $M \in A$-mod$\cap R$-proj. The following assertions are equivalent.*

(i) M is (A,R)-torsionless.

(ii) $\operatorname{domdim}_{(A,R)} M \geq 1$.

(iii) The map $\Phi_M \colon \operatorname{Hom}_A(V, DM) \otimes_C V \to DM$ is surjective.

(iv) The map $\gamma_M \colon \operatorname{Hom}_A(M,A) \otimes_A DA \to DM$ is surjective.

The following assertions are equivalent.

(a) M is A-reflexive and $\operatorname{Hom}_A(M,A) \otimes_A DA \in R$-proj.

(b) $\operatorname{domdim}_{(A,R)} M \geq 2$.

(c) The map $\Phi_M \colon \operatorname{Hom}_A(V, DM) \otimes_C V \to DM$ is bijective.

(d) The map $\gamma_M \colon \operatorname{Hom}_A(M,A) \otimes_A DA \to DM$ is bijective.

Proof. By Proposition 2.8.3, the implications $(iii) \Leftrightarrow (iv)$ and $(c) \Leftrightarrow (d)$ hold. By relative Mueller characterization 2.4.7, $(ii) \Leftrightarrow (iii)$ and $(b) \Leftrightarrow (c)$ follow. Assume that (i) holds. Since $\operatorname{domdim}(A,R) \geq 1$ there exists a projective (A,R)-injective module X such that $A \to X$ is an (A,R)-monomorphism. Using the (A,R)-monomorphism

$$M \to P \to A^t \to X^t \quad (2.8.0.20)$$

(ii) follows. Assume that (ii) holds. Then, there exists an (A,R)-monomorphism of M into a projective (A,R)-injective A-module. In particular, M is (A,R)-torsionless.

It remains to show that (a) is equivalent to (d).

The diagram

$$\begin{tikzcd} DDM \arrow[r, "D\gamma_M"] & D(\mathrm{Hom}_A(M,A)\otimes_A DA) \\ M \arrow[u, "w_M"] \arrow[r, "\tau_M"] & \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A) \arrow[u, "\kappa", "\simeq"'] \end{tikzcd} \tag{2.8.0.21}$$

is commutative. In fact, for $m \in M, f \in \mathrm{Hom}_A(M,A),\ g \in DA$,

$$\kappa\tau_M(m)(f\otimes g) = g(\tau_M(m)(f)) = g(f(m)), \tag{2.8.0.22}$$

$$D\gamma_M \circ w_M(m)(f\otimes g) = \mathrm{Hom}_R(\gamma_M,R)w_M(m)(f\otimes g) = w_M(m)\circ\gamma_M(f\otimes g) = \gamma_M(f\otimes g)(m) \tag{2.8.0.23}$$

$$= g(f(m)). \tag{2.8.0.24}$$

Assume that (a) holds. Then, τ_M is an isomorphism. So, by the diagram (2.8.0.21) $D\gamma_M$ is an isomorphism. Since $\mathrm{Hom}_A(M,A)\otimes_A DA \in R\text{-proj}$, γ_M is an isomorphism. Assume now that (d) holds. As $DM \in R\text{-proj}$, it follows that $\mathrm{Hom}_A(M,A)\otimes_A DA \in R\text{-proj}$. Also, $D\gamma_M$ is an isomorphism. By the diagram (2.8.0.21), τ_M is an isomorphism. So, M is A-reflexive. ☐

2.9 Relative Morita algebras

We shall now introduce a generalization of Morita algebras introduced in [KY13] to algebras over commutative Noetherian rings. This also generalizes [Cru21, Theorem 11] and [FHK21, Proposition 2.9].

Theorem 2.9.1. *Let A be a projective Noetherian algebra over a commutative Noetherian ring R. The following assertions are equivalent.*

(a) *(A,P,DP) is a relative QF3 R-algebra so that* $\mathrm{domdim}\,(A,R)\geq 2$ *and the restriction of the Nakayama functor $DA\otimes_A -\colon\ \mathrm{add}\,P \to \mathrm{add}\,P$ is well defined;*

(b) *(A,P,DP) is a relative QF3 R-algebra so that* $\mathrm{domdim}\,(A,R)\geq 2$ *and* $\mathrm{add}_A\, DA\otimes_A P = \mathrm{add}_A\, P$.

(c) *A is the endomorphism algebra of a generator $M \in B\text{-mod}\cap R\text{-proj}$ satisfying $DM\otimes_B M \in R\text{-proj}$ over a relative self-injective R-algebra, where $B\in R\text{-proj}$.*

(a') *(A,P,DP) is a relative QF3 R-algebra so that* $\mathrm{domdim}\,(A,R)\geq 2$ *and the restriction of the Nakayama functor $-\otimes_A DA\colon\ \mathrm{add}\,DP \to \mathrm{add}\,DP$ is well defined;*

(b') *(A,P,DP) is a relative QF3 R-algebra so that* $\mathrm{domdim}\,(A,R)\geq 2$ *and* $\mathrm{add}_A\, DP\otimes_A DA = \mathrm{add}_A\, DP$.

Proof. We will show $(b) \implies (a) \implies (c) \implies (b)$. The implications $(b') \implies (a') \implies (c) \implies (b')$ are analogous.

The implication $(b) \implies (a)$ is clear since $DA\otimes_A X \in \mathrm{add}\, DA\otimes_A P = \mathrm{add}\, P$ for all $X\in \mathrm{add}_A\, P$.

Assume that (a) holds. By relative Morita-Tachikawa correspondence (see Theorem 2.4.10) $P\otimes_B DP \in R\text{-proj}$, $B = \mathrm{End}_A(P)^{op} = \mathrm{End}_A(DP)$ and $A \simeq \mathrm{End}_B(P)\simeq \mathrm{End}_B(DP)^{op}$. It remains to show that B is relative self-injective. But this follows immediately from observing that

$$B = \mathrm{Hom}_A(P,P)\simeq \mathrm{Hom}_A(P,A)\otimes_A P \simeq D(DA\otimes_A P)\otimes_A P \in \mathrm{add}\, DP\otimes_A P = \mathrm{add}\, DB. \tag{2.9.0.1}$$

Hence, B is (B,R)-injective.

Finally, assume that (c) holds. By the relative Morita-Tachikawa correspondence, domdim$(A,R) \geq 2$ so that (A,DM,M) is a relative QF3 R-algebra and $A = \mathrm{End}_B(M)^{op}$. Moreover,

$$DA \otimes_A DM \simeq DM \otimes_B M \otimes_A DM \simeq DM \otimes_B DB. \tag{2.9.0.2}$$

Since B is a relative self-injective R-algebra DB is a B-progenerator. Hence, $\mathrm{add}_A DM \otimes_B DB = \mathrm{add}_A DM$. This completes the proof. □

The pair (A,P) (or (A,DP) if one prefers to work with right modules) is called a **relative Morita R-algebra** if it satisfies one of the conditions of Theorem 2.9.1.

Using Theorem 2.9.1(c), we see that relative Morita algebras generalize relative self-injective algebras.

2.10 Relative Gendo-symmetric algebras

Definition 2.10.1. Let B be a projective Noetherian algebra over a commutative Noetherian ring R. B is said to be a **relative symmetric R-algebra** if there exists a (B,B)-bimodule isomorphism $DB \simeq B$.

Using the proof of Proposition 2.2.6, we see that group algebras RG are relative symmetric R-algebras for any commutative Noetherian ring R and finite groups G. We refer to [Yam96] for the study of symmetric finite dimensional algebras. We see that over finite-dimensional algebras, the concept of relative symmetric algebra coincides with the concept of symmetric algebra (see [Yam96, Theorem 2.3.1]). A commutative Noetherian ring R is always a relative symmetric R-algebra. So it might happen that a Noetherian algebra is relative symmetric over one commutative Noetherian ring and not being relative symmetric over another unlike finite-dimensional algebras which remain symmetric even if we change the ground field (not necessarily by extension of scalars).

Theorem 2.10.2. *Let A be a projective Noetherian algebra over a commutative Noetherian ring R. The following assertions are equivalent.*

(a) domdim$(A,R) \geq 2$ *and* $V \simeq V \otimes_A DA$ *as* $(\mathrm{End}_A(V),A)$*-bimodules where V is a right projective (A,R)-injective-strongly faithful module.*

(b) domdim$(A,R) \geq 2$ *and* $P \simeq DA \otimes_A P$ *as* $(A, \mathrm{End}_A(P)^{op})$*-bimodules where P is a left projective (A,R)-injective-strongly faithful module.*

(c) A is the endomorphism algebra of a generator $M \in B$-mod$\cap R$-proj satisfying $DM \otimes_B M \in R$-proj over a relative symmetric R-algebra.

Proof. Assume that (a) holds. Let $B = \mathrm{End}_A(V)$. By relative Morita-Tachikawa correspondence 2.4.10, V is a left B-generator satisfying $DV \otimes_B V \in R$-proj and $A = \mathrm{End}_B(V)^{op}$. In particular $DA \simeq DV \otimes_B V$ as (A,A)-bimodules. Furthermore, $DV \simeq D(V \otimes_A DA) \simeq \mathrm{Hom}_A(V,A)$ as (A,B)-bimodules. Thus, as (B,B)-bimodules

$$DB \simeq V \otimes_A DV \simeq V \otimes_A \mathrm{Hom}_A(V,A) \simeq \mathrm{Hom}_A(V,V) \simeq B. \tag{2.10.0.1}$$

Hence, B is a relative symmetric R-algebra. So, (c) follows.

Conversely, assume that (c) holds. Every generator over a relative symmetric algebra is a generator relative cogenerator. By relative Morita-Tachikawa correspondence 2.4.10, $A = \mathrm{End}_B(M)^{op}$ has domdim$(A,R) \geq 2$ and M is a projective (A,R)-injective-strongly faithful right module. In particular, $DA \simeq DM \otimes_B M$ as (A,A)-bimodules. Moreover, as (B,A)-bimodules

$$M \otimes_A DA \simeq M \otimes_A DM \otimes_B M \simeq DB \otimes_B M \simeq B \otimes_B M \simeq M.$$ □

Analogously, one can show the equivalence between (b) and (c)

Let A be a projective Noetherian algebra over a commutative Noetherian ring R. By a **relative gendo-symmetric R-algebra** we mean a pair (A,V) satisfying (a) and (c) of Theorem 2.10.2 or a pair (A,P) satisfying (b) and (c) of Theorem 2.10.2.

Proposition 2.10.3. *Let (A,V) be a relative gendo-symmetric R-algebra. Then,*

(i) $DA\otimes_A DA \simeq DA$ as (A,A)-bimodules.

(ii) $DV \simeq DA\otimes_A DV$ as $(A,\operatorname{End}_A(V))$-bimodules.

Proof. Let $B=\operatorname{End}_A(V)$. We can identify as (A,A)-bimodules

$$DA\otimes_A DA \simeq DV\otimes_B V\otimes_A DV\otimes_B V \simeq DV\otimes_B DB\otimes_B V \simeq DV\otimes_B B\otimes_B V \simeq DV\otimes_B V \simeq DA. \tag{2.10.0.2}$$

So, (i) follows. By assumption, $V\simeq V\otimes_A DA$ as (B,A)-bimodules. Hence, as (A,B)-bimodules

$$DV \simeq D(V\otimes_A DA) \simeq \operatorname{Hom}_A(V,DDA) \simeq \operatorname{Hom}_A(V,A). \tag{2.10.0.3}$$

In particular, there exists an (A,B)-bimodule isomorphism

$$DA\otimes_A DV \simeq DA\otimes_A \operatorname{Hom}_A(V,A) \simeq \operatorname{Hom}_A(V,DA) \simeq \operatorname{Hom}_R(V\otimes_A A,R) \simeq DV. \qquad \square$$

Over fields, these class of algebras were introduced by Fang and Koenig in [FK11a] to give an homological characterization of a class of algebras that generalize Schur algebras and the category $\mathscr{O}$.

Proposition 2.10.3 allows us to construct a comultiplication on A in the same fashion as in [FK16]. The advantage here is of course that the ground ring is any commutative Noetherian ring instead of a field.

A question that arises in this setup is whether the condition (i) in Proposition 2.10.3 is enough to deduce that there exists $V\in\operatorname{proj}(A)$ such that (A,V) is a relative gendo-symmetric R-algebra. The difficulty lies in fact in the construction of V. It is also unclear for the author if an algebra being symmetric can be characterized in terms of closed points.

2.11 Application to class $\mathscr{A}$ of Koenig and Fang

The following is based on Corollary 3.7 of [FK11b].

Theorem 2.11.1. *Let A be a split quasi-hereditary algebra over a commutative Noetherian ring and a relative QF3 R-algebra. Let T be a characteristic tilting module. Then,*

$$\operatorname{domdim}_{(A,R)} T = \min\{\operatorname{domdim}_{(A,R)} \Delta(\lambda)\colon \lambda\in\Lambda\} = \min\{\operatorname{domdim}_{(A,R)} M\colon M\in\mathscr{F}(\tilde{\Delta})\}. \tag{2.11.0.1}$$

Proof. Denote by c the value $\min\{\operatorname{domdim}_{(A,R)}\Delta(\lambda)\colon \lambda\in\Lambda\}$ and λ_0 the index such that $\operatorname{domdim}_{(A,R)}\Delta(\lambda_0)=c$.

Let $M\in\mathscr{F}(\tilde{\Delta})$. By Lemma 2.4.25,

$$\operatorname{domdim}_{(A,R)} M \geq \min\{\operatorname{domdim}_{(A,R)} \Delta(\lambda)\otimes_R U_\lambda\colon \lambda\in\Lambda,\ U_\lambda\in R\text{-proj}\} = c, \tag{2.11.0.2}$$

since $\Delta(\lambda)\otimes_R U_\lambda \in \operatorname{add}\Delta(\lambda)$. Consider the (A,R)-exact sequence

$$0\to\Delta(\lambda_0)\to T(\lambda_0)\to X(\lambda_0)\to 0, \tag{2.11.0.3}$$

given by Proposition 1.5.109. By Lemma 2.4.25,

$$\operatorname{domdim}_{(A,R)} T(\lambda_0) \geq \min\{\operatorname{domdim}_{(A,R)} \Delta(\lambda_0), \operatorname{domdim}_{(A,R)} X(\lambda_0)\}.$$

Since $X(\lambda_0) \in \mathscr{F}(\tilde{\Delta})$ we obtain $\operatorname{domdim}_{(A,R)} X(\lambda_0) \geq c$. Hence, $\min\{\operatorname{domdim}_{(A,R)} \Delta(\lambda_0), \operatorname{domdim}_{(A,R)} X(\lambda_0)\} = c$. Assume that $\operatorname{domdim}_{(A,R)} T(\lambda_0) > c$. Then,

$$\operatorname{domdim}_{(A,R)} X(\lambda_0) = \operatorname{domdim}_{(A,R)} \Delta(\lambda_0) - 1 = c - 1. \tag{2.11.0.4}$$

This contradicts the minimality of c. Thus,

$$\operatorname{domdim}_{(A,R)} T = \min\{\operatorname{domdim}_{(A,R)} T(\lambda) \colon \lambda \in \Lambda\} = c. \qquad \square$$

For any $\lambda \in \Lambda$, we define the length of $\lambda \in \Lambda$ to be the length t of the longest chain $\lambda = x_0 < x_1 < \ldots < x_t$ in Λ and denote it by $d(\Lambda, \lambda)$. Denote by $d(\Lambda)$ to be the maximum value of $d(\Lambda, \lambda)$ over all $\lambda \in \Lambda$.

Note that $\lambda \in \Lambda$ is maximal if and only if $d(\Lambda, \lambda) = 0$ and $d(\Lambda)$ is bounded by $|\Lambda|$. If $z > \lambda$, then $d(\Lambda, \lambda) \geq d(\Lambda, z) + 1$. In the following, we will see how the length of a weight together with the relative dominant dimension of the algebra gives a lower bound to the relative dominant dimension of standard modules.

Proposition 2.11.2. *Let (A, Λ) be a split quasi-hereditary algebra over a commutative Noetherian ring. For any $\lambda \in \Lambda$, $\operatorname{domdim}_{(A,R)} \Delta(\lambda) \geq \operatorname{domdim}(A, R) - d(\Lambda, \lambda)$.*

Proof. We shall prove this result by induction on $d(\Lambda, \lambda)$. If $d(\Lambda, \lambda) = 0$, then λ is maximal in Λ. Thus, $\Delta(\lambda)$ is a projective A-module. By Corollary 2.4.23 and Lemma 2.4.25, $\operatorname{domdim}_{(A,R)} \Delta(\lambda) \geq \operatorname{domdim}(A, R)$.

Suppose now the claim holds for all $\mu \in \Lambda$ with $d(\mu) < t$ for some $t > 1$. Let $\lambda \in \Lambda$ such that $d(\lambda) = t$. Consider the (A, R)-exact sequence

$$0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0, \tag{2.11.0.5}$$

where $K(\lambda) \in \cdot_{\mu > \lambda}$ and $P(\lambda) \in A\text{-proj}$. Comparing lengths, $d(\Lambda, \mu) < d(\Lambda, \lambda) = t$ for $\mu > \lambda$. By induction, $\operatorname{domdim}_{(A,R)} \Delta(\mu) \geq \operatorname{domdim}(A, R) - d(\Lambda, \mu) > \operatorname{domdim}(A, R) - d(\Lambda, \lambda)$. By Lemma 2.4.25, $\operatorname{domdim}_{(A,R)} K(\lambda) > \operatorname{domdim}(A, R) - d(\Lambda, \lambda)$. If $\operatorname{domdim}_{(A,R)} P(\lambda) > \operatorname{domdim}_{(A,R)} K(\lambda)$, then by Lemma 2.4.25, we have

$$\operatorname{domdim}_{(A,R)} \Delta(\lambda) = \operatorname{domdim}_{(A,R)} K(\lambda) - 1 \geq \operatorname{domdim}(A, R) - d(\Lambda, \lambda). \tag{2.11.0.6}$$

If $\operatorname{domdim}_{(A,R)} P(\lambda) \leq \operatorname{domdim}_{(A,R)} K(\lambda)$, then by Lemma 2.4.25, we have

$$\operatorname{domdim}_{(A,R)} \Delta(\lambda) \geq \operatorname{domdim}_{(A,R)} P(\lambda) - 1 \geq \operatorname{domdim}(A, R) - 1 \geq \operatorname{domdim}(A, R) - d(\Lambda, \lambda). \qquad \square$$

Recall that a **duality** ω of an algebra A is an anti-isomorphism $\omega \colon A \to A$ inverse to itself fixing a suitable set of orthogonal idempotents of A. The image of the previous orthogonal idempotents in $A(\mathfrak{m})$ must form a complete set of primitive orthogonal idempotents. We say that $(A, \mathbf{e})$ is a split quasi-hereditary algebra with a duality ω if ω is a duality of A with respect to $\mathbf{e} := \{e_1, \ldots, e_t\}$ and A is split quasi-hereditary with split heredity chain $0 \subset Ae_tA \subset \cdots \subset A(e_1 + \cdots + e_t)A = A$.

Theorem 2.11.3. *Let A be a projective Noetherian R-algebra. Assume that the following holds.*

- *$(A, \boldsymbol{e})$ is split quasi-hereditary algebra with a duality.*
- *(A, Ae) is a relative gendo-symmetric R-algebra for some idempotent e of A.*

Let T be a characteristic tilting module of A. Then,

$$\operatorname{domdim}(A,R) = 2\operatorname{domdim}_{(A,R)} T. \tag{2.11.0.7}$$

Proof. Let $\mathfrak{m}$ be a maximal ideal in R. Let T be a characteristic tilting module of A. By Proposition 1.5.56 and Proposition 1.5.126, $A(\mathfrak{m})$ is a split quasi-hereditary algebra over $R(\mathfrak{m})$ with characteristic tilting module $T(\mathfrak{m})$. Fix $V := eA$. Let θ be the $(\operatorname{End}_A(V), A)$-bimodule isomorphism given by $V \to V \otimes_A DA$. Applying the functor $R(\mathfrak{m}) \otimes_R -$ to θ gives an $(\operatorname{End}_{A(\mathfrak{m})}(V(\mathfrak{m})), A(\mathfrak{m}))$-bimodule isomorphism between $V(\mathfrak{m})$ and $V(\mathfrak{m}) \otimes_{A(\mathfrak{m})} \operatorname{Hom}_{R(\mathfrak{m})}(A(\mathfrak{m}), R(\mathfrak{m}))$. By Theorem 2.5.13, $\operatorname{domdim} A(\mathfrak{m}) \geq \operatorname{domdim}(A,R) \geq 2$. Hence, $(A(\mathfrak{m})$ is a gendo-symmetric algebra. By Theorem [FK11b, Theorem 4.3], $\operatorname{domdim} A(\mathfrak{m}) = 2\operatorname{domdim}_{A(\mathfrak{m})} T(\mathfrak{m})$. Hence, by Theorem 2.5.13

$$\begin{aligned} \operatorname{domdim}(A,R) &= \min\{\operatorname{domdim} A(\mathfrak{m}) \colon \ \mathfrak{m} \text{ is a maximal ideal in } R\} && (2.11.0.8)\\ &= \min\{2\operatorname{domdim} T(\mathfrak{m}) \colon \ \mathfrak{m} \text{ is a maximal ideal in } R\} && (2.11.0.9)\\ &= 2\min\{\operatorname{domdim} T(\mathfrak{m}) \colon \ \mathfrak{m} \text{ is a maximal ideal in } R\} = 2\operatorname{domdim}_{(A,R)} T. && \square \end{aligned}$$

Remark 2.11.4. Although it is not completely clear from the proof of Lemma 3.2 of [FK11b], an algebra A in class $\mathscr{A}$ of Fang and Koenig satisfying Definition 2.1 of [FK11b] is also gendo-symmetric. This becomes clearer by considering also Theorem 3.7 of [MS08] and Theorem 3.2 of [FK11a]. $\triangle$

In Example 4.6.7 we can see that there are quasi-hereditary gendo-symmetric algebras which do not belong to the class $\mathscr{A}$ of Fang and Koenig algebras.

Chapter 3

$\mathscr{A}$-covers and faithful split quasi-hereditary covers

In this chapter, we give the setup to measure the quality of a cover. In particular, we introduce the concept of an $\mathscr{A}$-cover (A,P) for an arbitrary resolving subcategory $\mathscr{A}$ of A-mod. Under this concept, faithful split quasi-hereditary covers are exactly $\mathscr{F}(\tilde{\Delta})$-covers. Some highlights are the upper bounds for the level of faithfulness of a cover (Section 3.2) and results of how $\mathscr{A}$-covers behave under change of ground ring (Section 3.3) and under truncation (Section 3.4) leading to many deformation results. These results are general and valid for $\mathscr{A}$-covers, where $\mathscr{A}$ behaves similarly to $\mathscr{F}(\tilde{\Delta})$ and A-proj. Such resolving subcategories are called here well behaved resolving subcategories. We discuss the problems of existence and uniqueness of covers.

3.1 Definition and properties of $\mathscr{A}$-covers

Unless otherwise stated, in this chapter, all algebras will be projective Noetherian R-algebras for a Noetherian commutative ring R.

By a **split quasi-hereditary cover of** B we mean a cover (A,P) of B such that A is a split quasi-hereditary algebra. By a **quasi-hereditary cover of** B we mean a cover (A,P) of B such that A is a quasi-hereditary algebra.

Definition 3.1.1. Let A be a projective Noetherian R-algebra. Let $\mathscr{A}$ be a resolving subcategory of A-mod. Let $B=\operatorname{End}_A(P)^{op}$ and $i\geq 0$. We say that the pair (A,P) is an $i-\mathscr{A}$ **cover** of B if the Schur functor $F=\operatorname{Hom}_A(P,-)$ induces isomorphisms

$$\operatorname{Ext}_A^j(M,N)\to\operatorname{Ext}_B^j(FM,FN),\quad \forall M,N\in\mathscr{A},\ j\leq i.$$

We say that (A,P) is a ***i*-cover** of B if (A,P) is an $i-A$-proj cover of B.

We say that (A,P) is an $(-1)-\mathscr{A}$ **cover** of B if (A,P) is a cover of B and F induces monomorphisms

$$\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_B(FM,FN),\quad \forall M,N\in\mathscr{A}.$$

Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. As we have noticed earlier $\mathscr{F}(\tilde{\Delta})$ is a resolving subcategory of A-mod$\cap R$-proj. When (A,P) is an $i-\mathscr{F}(\tilde{\Delta})$ cover of B we say that (A,P) is an ***i*-faithful split quasi-hereditary cover** of B for $i\geq -1$. For split quasi-hereditary algebras this definition was introduced by Rouquier in [Rou08]. Over fields, we say that (A,P) is an ***i*-faithful quasi-hereditary cover** of B if (A,P) is an

$i-\mathscr{F}(\Delta)$ cover of B, where A is a quasi-hereditary algebra (not necessarily split).

Notice that if the adjoint functor of $\operatorname{Hom}_A(P,-)$ is exact and (A,P) is a cover of B, then FA is projective over B. Therefore, F sends A-proj to B-proj. Hence, the restriction of F to A-proj induces an equivalence of categories between A-proj and B-proj. Therefore, in such a case, F is an equivalence of categories.

Remark 3.1.2. In our notation, a 0-cover is a cover in the usual sense. △

Taking into account Proposition 1.6.12 and Definition 3.1.1, we can reformulate Problem 1 into:

> *Problem* 2. For a given cellular algebra B with cell datum (Λ, M, C, ι), study a split quasi-hereditary algebra $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ and determine the largest integer i possible such that (A,P) is a faithful quasi-hereditary i-cover of B. Furthermore, if possible, the Schur functor $\operatorname{Hom}_A(P,-)$ should send the standard modules of A into the cell modules of B.

Again, the study of $i-\mathscr{A}$ covers of finite-dimensional algebras over a field with $i\geq 0$ can be reduced to covers coming from idempotents.

Proposition 3.1.3. *Let R be a field and let $i\geq 0$ be an integer. Let $\mathscr{A}$ be a resolving subcategory of A-mod. If (A,P) is an $i-\mathscr{A}$ cover of B, then there exists an idempotent $e\in A$ such that (A,Ae) is an $i-\mathscr{A}$ cover of eAe.*

Proof. By Proposition 1.7.1, there exists an idempotent $e\in A$ such that (A,Ae) is a cover of eAe and eAe is Morita equivalent to B. Denote by H the equivalence of categories B-mod $\to eAe$-mod. For $M,N\in\mathscr{A}$,

$$\operatorname{Ext}_A^j(M,N)\simeq \operatorname{Ext}_B^j(FM,FN)\simeq \operatorname{Ext}_{eAe}^j(HFM,HFN),\ j\leq i. \tag{3.1.0.1}$$

It remains to show that $HFM\simeq \operatorname{Hom}_A(Ae,M)$ for every $M\in\mathscr{A}$. But, this isomorphism holds since (A,P) is a $0-\mathscr{A}$ cover of B. Thus, (A,Ae) is an $i-\mathscr{A}$ cover of eAe. ☐

Lemma 3.1.4. *Let (A,P) be a cover of B. The following holds.*

(a) Let $M\in A$-mod. The map η_M is monomorphism if and only if $\operatorname{Hom}_A(N,M)\to \operatorname{Hom}_B(FN,FM)$ is injective for any $N\in A$-mod if and only if $\operatorname{Hom}_A(A,M)\to\operatorname{Hom}_B(FA,FM)$ is injective.

(b) The map η_M is epimorphism if and only if $\operatorname{Hom}_A(A,M)\to\operatorname{Hom}_B(FA,FM)$ is surjective.

Proof. Assume that $\operatorname{Hom}_A(N,M)\to\operatorname{Hom}_B(FN,FM)$ is injective for any $N\in A$-mod. In particular, $\operatorname{Hom}_A(A,M)\to\operatorname{Hom}_B(FA,FM)$ is injective. Let $m\in M$ such that $\eta_M(m)=0$. Consider $f_m\in\operatorname{Hom}_A(A,M)$, given by $f_m(1_A)=m$. Then, $Ff_m=\eta_M(m)=0$. Thus, $f_m=0$ and $m=0$. So, η_M is a monomorphism. Now assume that η_M is a monomorphism. Let $f\in\operatorname{Hom}_A(N,M)$ satisfying $Ff=0$. Then,

$$\eta_M\circ f = GFf\circ\eta_N=0 \implies f=0. \tag{3.1.0.2}$$

Thus, $a)$ follows.

Assume that η_M is surjective. Let $y\in\operatorname{Hom}_B(FA,FM)=GFM$. There exists $x\in M$ such that $\eta_M(x)=y$. Consider $f_x\in\operatorname{Hom}_A(A,M)$ given by $f_x(1_A)=x$. Then, $Ff_x=\eta_M(x)=y$. Hence, $\operatorname{Hom}_A(A,M)\to\operatorname{Hom}_B(FA,FM)$ is surjective. Reciprocally, assume that $\operatorname{Hom}_A(A,M)\to\operatorname{Hom}_B(FA,FM)$ is surjective. Let $y\in GFM$. There exists $x\in\operatorname{Hom}_A(A,M)$ such that $Fx=y$. We have that $\eta_M(x(1_A))=Fx=y$. ☐

This Lemma gives that (A,P) is a $(-1)-\mathscr{A}$ cover of B if and only if the restriction of $\operatorname{Hom}_A(P,-)$ to $\mathscr{A}$ is faithful if and only if η_M is a monomorphism for every $M\in\mathscr{A}$.

Proposition 3.1.5. *Let (A,P) be a cover of B and A be a split quasi-hereditary algebra over a commutative Noetherian ring. The following assertions are equivalent.*

(i) *(A,P) is a -1-faithful split quasi-hereditary cover of B; that is the restriction of $F = \operatorname{Hom}_A(P,-)$ to $\mathscr{F}(\tilde{\Delta})$ is faithful.*

(ii) *$\eta_{\bigoplus_{\lambda\in\Lambda} \Delta(\lambda)}$ is a monomorphism;*

(iii) *$\eta_{\Delta(\lambda)}$ is a monomorphism for all $\lambda \in \Lambda$;*

(iv) *η_M is a monomorphism for all $M \in \mathscr{F}(\tilde{\Delta})$;*

(v) *η_T is a monomorphism for all (partial) tilting modules T;*

(vi) *Every module of $\mathscr{F}(\tilde{\Delta})$ can be embedded into some module in the image of the functor $G = \operatorname{Hom}_B(FA,-)$.*

Proof. $(i) \implies (ii)$. $A \in \mathscr{F}(\tilde{\Delta})$ and clearly $\bigoplus_{\lambda\in\Lambda} \Delta(\lambda) \in \mathscr{F}(\tilde{\Delta})$.

In view of (i), $\operatorname{Hom}_A(A, \bigoplus_{\lambda\in\Lambda} \Delta(\lambda)) \to \operatorname{Hom}_B(FA, F\bigoplus_{\lambda\in\Lambda} \Delta(\lambda))$ is injective. By Lemma 3.1.4, $\eta_{\bigoplus_{\lambda\in\Lambda} \Delta(\lambda)}$ is a monomorphism.

$(ii) \implies (iii)$. It is clear by Lemma 1.4.26.

$(iii) \implies (iv)$. Every $M \in \mathscr{F}(\tilde{\Delta})$ has a filtration by standard modules. By induction on $|\Lambda|$ and using the Snake Lemma, it follows that η_M is a monomorphism for all $M \in \mathscr{F}(\tilde{\Delta})$.

$(iv) \implies (vi)$. It is clear.

$(vi) \implies (v)$. Every (partial) tilting module belongs to $\mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$. In particular, it belongs to $\mathscr{F}(\tilde{\Delta})$. Thus, given a (partial) tilting T, there exists a monomorphism $\alpha\colon T \to GN$ for some $N \in B$-mod. Since $\mathrm{id}_{GN} = G\varepsilon_N \circ \eta_{GN}$ and ε_N is an isomorphism according to Proposition 1.4.25, it follows that η_{GN} is an isomorphism. Now, $GF\alpha \circ \eta_T = \eta_{GN} \circ \alpha$ is a monomorphism. Thus, η_T is a monomorphism.

$(v) \implies (iv)$. Let $M \in \mathscr{F}(\tilde{\Delta})$. By Proposition 1.5.109, there exists T (partial) tilting module $N \in \mathscr{F}(\tilde{\Delta})$ and an exact sequence $0 \to M \to T \to N \to 0$. Applying $G \circ F = \operatorname{Hom}_B(FA, F-)$ (left exact functor) yields the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\eta_M"] & T \arrow[r] \arrow[d, "\eta_T"] & N \arrow[r] \arrow[d, "\eta_N"] & 0 \\
0 \arrow[r] & GFM \arrow[r] & GFT \arrow[r] & GFN &
\end{tikzcd}.$$

By assumption, η_T is a monomorphism. By Snake Lemma, η_M is a monomorphism.

$(i) \Leftrightarrow (iv)$. By Lemma 3.1.4, η_M is monomorphism for every $M \in \mathscr{F}(\tilde{\Delta})$ if and only if the functor $F_{|_{\mathscr{F}(\tilde{\Delta})}}$ is faithful. ☐

Proposition 3.1.6. *The following assertions are equivalent.*

(a) *(A,P) is a $0-\mathscr{A}$ cover; that is, the restriction of $F = \operatorname{Hom}_A(P,-)$ to $\mathscr{A}$ is full and faithful;*

(b) *η_M is an isomorphism for all $M \in \mathscr{A}$;*

(c) *Every module of $\mathscr{A}$ is in the image of the functor $G = \operatorname{Hom}_B(FA,-)$.*

Proof. $(a) \implies (b)$. Since $\mathscr{A}$ is resolving of A-mod$\cap R$-proj, $A \in \mathscr{A}$. By $a)$ $\operatorname{Hom}_A(A,M) \to \operatorname{Hom}_B(FA,FM)$ is an isomorphism. By Lemma 3.1.4, η_M is an isomorphism.

$(b) \implies (c)$. Let $M \in \mathscr{A}$. By assumption, η_M is an isomorphism. Hence, $M \simeq G(FM)$.

$(c) \implies (b)$. Let $M \in \mathscr{A}$. There exists $N \in B$-mod such that $GN \simeq M$. Since $\mathrm{id}_{GN} = G\varepsilon_N \circ \eta_{GN}$ and ε_N is an isomorphism according to Proposition 1.4.25, it follows that η_{GN} is an isomorphism. Let $\alpha \colon M \to GN$ be an isomorphism. As η_M is the composition of the isomorphisms $GF\alpha^{-1} \circ \eta_{GN} \circ \alpha$, it is an isomorphism.

$(b) \implies (a)$. By Lemma 1.4.27, $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_B(FN,FM)$ for every $N,M \in \mathscr{A}$. $\square$

Proposition 3.1.7. *[Rou08, Proposition 4.40] The following assertions are equivalent.*

(i) *(A,P) is a 0-faithful split quasi-hereditary cover of B; that is, the restriction of $F = \mathrm{Hom}_A(P,-)$ to $\mathscr{F}(\tilde{\Delta})$ is full and faithful;*

(ii) *$\eta_{\bigoplus_{\lambda\in\Lambda} \Delta(\lambda)}$ is an isomorphism;*

(iii) *$\eta_{\Delta(\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$;*

(iv) *η_M is an isomorphism for all $M \in \mathscr{F}(\tilde{\Delta})$;*

(v) *Every module of $\mathscr{F}(\tilde{\Delta})$ is in the image of the functor $G = \mathrm{Hom}_B(FA,-)$;*

(vi) *η_T is an isomorphism for all (partial) tilting modules T;*

(vii) *Let T be a characteristic tilting module. Every module of $\mathrm{add}\, T$ is in the image of the functor $G = \mathrm{Hom}_B(FA,-)$.*

Proof. $(i) \implies (ii)$. Since $A, \bigoplus_{\lambda\in\Lambda} \Delta(\lambda) \in \mathscr{F}(\tilde{\Delta})$, $\mathrm{Hom}_A(A, \bigoplus_{\lambda\in\Lambda} \Delta(\lambda)) \to \mathrm{Hom}_B(FA, F\bigoplus_{\lambda\in\Lambda} \Delta(\lambda))$is an isomorphism, by assumption. By Lemma 3.1.4, $\eta_{\bigoplus_{\lambda\in\Lambda} \Delta(\lambda)}$ is an isomorphism.

$(ii) \implies (iii)$. It is clear by Lemma 1.4.26.

$(iii) \implies (iv)$. Let $M \in \mathscr{F}(\tilde{\Delta})$. There exists a filtration

$$0 = M_{n+1} \subset M_n \subset M_{n-1} \subset \cdots \subset M_1 = M \tag{3.1.0.3}$$

with $M_i/M_{i+1} \simeq \Delta_i \otimes_R U_i$, $U_i \in R$-proj and $n = |\Lambda|$. We will prove by induction on j that η_M is an isomorphism M satisfying $U_i \neq 0$ only if $i \leq j$. Assume that $M \simeq \Delta_1 \otimes_R U_1$. Note that, for $x \otimes u \in \Delta_1 \otimes_R U_1$, $f \in FA$, $p \in P$,

$$\eta_{\Delta_1\otimes_R U_1}(x\otimes u)(f)(p) = f(p)\cdot(x\otimes u) = (f(p)x)\otimes u = \eta_{\Delta_1}(x)(f)(p)\otimes u. \tag{3.1.0.4}$$

So, $\eta_{\Delta_1\otimes_R U_1}(x\otimes u) = \eta_{\Delta_1}(x)\otimes \mathrm{id}_{U_1}(u)$. Thus, η_M is an isomorphism. Assume the result holds for an arbitrary $1 \leq j \leq n-1$. Assume M with a filtration satisfying $U_i \neq 0$ only if $i \leq j+1$. Consider the exact sequence

$$0 \to \Delta_{j+1}\otimes_R U_{j+1} \to M \to M_{j+1}/\Delta_{j+1}\otimes_R U_{j+1} \to 0. \tag{3.1.0.5}$$

Applying GF (left exact functor) we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Delta_{j+1}\otimes_R U_{j+1} \arrow[r] \arrow[d, "\eta_{\Delta_{j+1}\otimes_R U_{j+1}}"] & M \arrow[r] \arrow[d, "\eta_M"] & M_{j+1}/\Delta_{j+1}\otimes_R U_{j+1} \arrow[r] \arrow[d, "\eta_{M_{j+1}/\Delta_{j+1}\otimes_R U_{j+1}}"] & 0 \\
0 \arrow[r] & GF(\Delta_{j+1}\otimes_R U_{j+1}) \arrow[r] & GFM \arrow[r] & GF(M_{j+1}/\Delta_{j+1}\otimes_R U_{j+1}) &
\end{tikzcd}.$$

By induction $\eta_{M_{j+1}/\Delta_{j+1}\otimes_R U_{j+1}}$ is an isomorphism. By assumption, $\eta_{\Delta_{j+1}\otimes_R U_{j+1}}$ is an isomorphism. It follows by Snake Lemma that η_M is an isomorphism. Thus, $iv)$ follows.

$(iv) \implies (v)$. Let $M \in \mathscr{F}(\tilde{\Delta})$. By assumption, η_M is an isomorphism. Hence, $M \simeq G(FM)$. So, (v) follows.

$(v) \implies (vi)$. Let T be a partial tilting module. By (v), there exists $N \in B$-mod such that $GN \simeq T$. Since $\mathrm{id}_{GN} = G\varepsilon_N \circ \eta_{GN}$ and ε_N is an isomorphism according to Proposition 1.4.25, it follows that η_{GN} is an isomorphism. Let $\alpha\colon T \to GN$ be an isomorphism. As η_T is the composition of the isomorphisms $GF\alpha^{-1} \circ \eta_{GN} \circ \alpha$, it is an isomorphism.

$(vi) \implies (vii)$. Let T be a characteristic tilting module. Consequently, every module M belonging to $\operatorname{add} T$ is (partial) tilting. By assumption, η_M is an isomorphism. Thus, $M \simeq GFM$.

$(vii) \implies (vi)$. Let M be a partial tilting module. By Corollary 1.5.123, $\operatorname{add} T = \mathscr{F}(\tilde{\Delta}) \cap \mathscr{F}(\tilde{\nabla})$. In particular, $M \in \operatorname{add} T$. By assumption, there exists $N \in B$-mod such that $M \in GN$. Now applying the same argument used in the implication $(v) \implies (vi)$, (vi) holds.

$(vi) \implies (iv)$. Let $M \in \mathscr{F}(\tilde{\Delta})$. By Proposition 1.5.109, there exists T partial tilting module $N \in \mathscr{F}(\tilde{\Delta})$ and the exact sequence

$$0 \to M \to T \to N \to 0. \tag{3.1.0.6}$$

Applying $G \circ F = \operatorname{Hom}_B(FA, F-)$ (left exact functor) yields the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\eta_M"] & T \arrow[r] \arrow[d, "\eta_T"] & N \arrow[r] \arrow[d, "\eta_N"] & 0 \\ 0 \arrow[r] & GFM \arrow[r] & GFT \arrow[r] & GFN & \end{tikzcd}.$$

By Snake Lemma, η_M is a monomorphism. Since M is arbitrary, η_M is a monomorphism for every $M \in \mathscr{F}(\tilde{\Delta})$. In particular, η_N is a monomorphism. As η_T is an isomorphism, applying again Snake Lemma yields that η_M is an isomorphism.

$(iv) \implies (i)$. By Lemma 1.4.27, $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(FN, FM)$ for every $N, M \in \mathscr{F}(\tilde{\Delta})$. Hence, (i) holds. ☐

Proposition 3.1.8. *Let (A,P) be a cover of B. (A,P) is a 0-faithful split quasi-hereditary cover of B if and only if $\eta_{\Delta(\lambda)}$ is an epimorphism for all $\lambda \in \Lambda$.*

Proof. By Proposition 3.1.7, one implication is clear.

Assume that $\eta_{\Delta(\lambda)}$ is an epimorphism for all $\lambda \in \Lambda$. We claim that η_M is an epimorphism for all $M \in \mathscr{F}(\Delta)$. We will prove it by induction on the size of filtration of M, t. If $t = 1$, then $M \simeq \Delta(\lambda)$ for some λ. So, there is nothing to show. Assume $t > 1$. There is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\eta_{M'}"] & M \arrow[r] \arrow[d, "\eta_M"] & \Delta(\mu) \arrow[r] \arrow[d, "\eta_{\Delta(\mu)}"] & 0 \\ 0 \arrow[r] & GFM' \arrow[r] & GFM \arrow[r] & GF\Delta(\mu) & \end{tikzcd}.$$

By induction $\eta_{M'}$ is an epimorphism. By Snake Lemma, η_M is an epimorphism. Now consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K(\lambda) \arrow[r] \arrow[d, "\eta_{K(\lambda)}"] & P(\lambda) \arrow[r] \arrow[d, "\eta_{P(\lambda)}"] & \Delta(\lambda) \arrow[r] \arrow[d, "\eta_{\Delta(\lambda)}"] & 0 \\ 0 \arrow[r] & GFK(\lambda) \arrow[r] & GFP(\lambda) \arrow[r] & GF\Delta(\lambda) & \end{tikzcd}.$$

Since $K(\lambda) \in \mathscr{F}(\tilde{\Delta})$, $\eta_{K(\lambda)}$ is an epimorphism. By Snake Lemma, there is an exact sequence

$$0 = \ker \eta_{P(\lambda)} \to \ker \eta_{\Delta(\lambda)} \to \operatorname{coker} \eta_{K(\lambda)} = 0. \tag{3.1.0.7}$$

It follows that $\eta_{\Delta(\lambda)}$ is also a monomorphism, and thus $\eta_{\Delta(\lambda)}$ is an isomorphism for every $\lambda \in \Lambda$. By Proposition 3.1.7, the result follows. $\square$

For the resolving subcategory A-proj, 0-covers can be described in the following way.

Proposition 3.1.9. *The following assertions are equivalent.*

(I) (A,P) *is a* 0*-cover of* B*;*

(II) η_M *is an isomorphism for all* $M \in A\text{-proj}$*;*

(III) $F = \operatorname{Hom}_A(P,-)$ *restricts to an equivalence of categories* $A\text{-proj} \to \operatorname{add}_B FA$ *with inverse* $G = \operatorname{Hom}_B(FA,-)$.

Proof. (I)$\Leftrightarrow$ (II) follows from Proposition 3.1.6. Assume that (III) holds. In particular, the functor $F_{A\text{-proj}}$ is full and faithful. By definition, (I) holds.

Assume that (I) holds. Note that $A\text{-proj} \to \operatorname{add} FA$ is well defined since for $M \in A\text{-proj}$, $A^t \simeq M \bigoplus K$, for some K. Hence,

$$(FA)^t \simeq F(A^t) \simeq F(M \bigoplus K) \simeq FM \bigoplus FK \implies FM \in \operatorname{add} FA. \tag{3.1.0.8}$$

By (I), $F_{A\text{-proj}} \colon A\text{-proj} \to \operatorname{add} FA$ is full and faithful. Let $M \in \operatorname{add} FA$. Then, $FA^t \simeq M \bigoplus K$ for some $t > 0$. Since (A,P) is a 0-cover and $A \in A\text{-proj}$, we have $A^t \simeq GFA^t \simeq GM \bigoplus GK$. Hence, $GM \in A\text{-proj}$. Now since the counit $\varepsilon_M \colon FGM \to M$ is an isomorphism, it follows that F is essentially surjective and $G \colon \operatorname{add} FA \to A\text{-proj}$ is well defined. Since $G_{\operatorname{add} FA}$ is right adjoint of $F_{A\text{-proj}}$ and $F_{A\text{-proj}}$ is an equivalence, it follows that $G_{\operatorname{add} FA}$ is its inverse. $\square$

Proposition 3.1.10. *Let* (A,P) *be a* 0-$\mathscr{A}$ *cover of* B*. Then,* (A,P) *is a* $1-\mathscr{A}$ *cover of* B *if and only if* $\mathrm{R}^1 G(FM) = 0$ *for all* $M \in \mathscr{A}$.

Proof. Assume that (A,P) is a $1-\mathscr{A}$ cover of B. Let $M \in \mathscr{A}$. Then,

$$\mathrm{R}^1 G(FM) = \mathrm{R}^1 \operatorname{Hom}_B(FA,-)(FM) = \operatorname{Ext}_B^1(FA,FM) \simeq \operatorname{Ext}_A^1(A,M) = 0. \tag{3.1.0.9}$$

Conversely, assume that $\mathrm{R}^1 G(FM) = 0$ for every $M \in \mathscr{A}$. We will start by showing that the natural correspondence $\operatorname{Ext}_A^1(M,N) \to \operatorname{Ext}_B^1(FM,FN)$, $M,N \in \mathscr{A}$ is injective.

Let

$$0 \to N \to X_1 \xrightarrow{p_1} M \to 0 \tag{3.1.0.10}$$

$$0 \to N \to X_2 \xrightarrow{p_2} M \to 0 \tag{3.1.0.11}$$

be short exact sequences such that they have the same image in $\operatorname{Ext}_B^1(FM,FN)$. Hence, there is the following commutative diagram with exact rows and the columns are isomorphisms

$$\begin{tikzcd} 0 \arrow[r] & FN \arrow[r] \arrow[d, "\vartheta_N"] & FX_1 \arrow[r] \arrow[d, "\vartheta_{X_1}"] & FM \arrow[r] \arrow[d, "\vartheta_M"] & 0 \\ 0 \arrow[r] & FN \arrow[r] & FX_2 \arrow[r] & FM \arrow[r] & 0 \end{tikzcd}.$$

Since $M,N \in \mathscr{A}$ and $\mathscr{A}$ is a resolving subcategory then $X_1, X_2 \in \mathscr{A}$. As (A,P) is a $0-\mathscr{A}$ cover, the functor $\operatorname{Hom}_A(P,-)_{|_{\mathscr{A}}}$ is full and faithful, thus there are A-maps $\theta_{X_1} \colon X_1 \to X_2$, $\theta_N \colon N \to N$, $\theta_M \colon M \to M$ such that

$\vartheta_{X_1} = F\theta_{X_1}$, $\vartheta_M = F\theta_M$, $\vartheta_N = F\theta_N$. Note that since η is a natural transformation, the fact that each ϑ is an isomorphism implies that each θ is an isomorphism. In fact, for X_1 we have

$$G\vartheta_{X_1} \circ \eta_{X_1} = GF\theta_{X_1} \circ \eta_{X_1} = \eta_{X_2} \circ \theta_{X_1}. \tag{3.1.0.12}$$

So, θ_{X_1} is a composition of isomorphisms. Now since the functor $F_{|\mathscr{A}}$ is faithful, it follows that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\theta_N"] & X_1 \arrow[r] \arrow[d, "\theta_{X_1}"] & M \arrow[r] \arrow[d, "\theta_M"] & 0 \\ 0 \arrow[r] & N \arrow[r] & X_2 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}.$$

In fact, we can see

$$F(p_2 \circ \theta_{X_1}) = Fp_2 \circ F\theta_{X_1} = Fp_2 \circ \vartheta_{X_1} = \vartheta_M \circ Fp_1 = F(\theta_M \circ p_1). \tag{3.1.0.13}$$

By the commutativity of the diagram and the columns being isomorphisms, we see that both exact sequences are equivalent, and therefore the map $\mathrm{Ext}^1_A(M,N) \to \mathrm{Ext}^1_B(FM,FN)$, $M,N \in \mathscr{A}$ is injective.

Let $M,N \in \mathscr{A}$. Consider an exact sequence

$$0 \to FN \xrightarrow{k} X \xrightarrow{\pi} FM \to 0 \in \mathrm{Ext}^1_B(FM,FN). \tag{3.1.0.14}$$

Applying the functor G yields the exact sequence

$$0 \to GFN \xrightarrow{Gk} GX \xrightarrow{G\pi} GFM \to \mathrm{R}^1 G(FN) = 0, \tag{3.1.0.15}$$

by assumption. So, we have an exact sequence

$$0 \to N \xrightarrow{Gk \circ \eta_N} GX \xrightarrow{\eta_M^{-1} \circ G\pi} M \to 0 \in \mathrm{Ext}^1_A(M,N). \tag{3.1.0.16}$$

Since $\mathscr{A}$ is resolving, $GX \in \mathscr{A}$. Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & FN \arrow[r, "F(Gk\circ\eta_N)"] \arrow[d, equal] & FGX \arrow[r, "F(\eta_M^{-1}\circ G\pi)"] \arrow[d, "\varepsilon_X"] & FM \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & FN \arrow[r, "k"] & X \arrow[r, "\pi"] & FM \arrow[r] & 0 \end{tikzcd}.$$

This is a commutative diagram where the columns are isomorphisms. In fact,

$$\varepsilon_X \circ FGk \circ F\eta_N = k \circ \varepsilon_{FN} \circ F\eta_N = k \circ \mathrm{id}_{FN}, \tag{3.1.0.17}$$

$$F\eta_M \circ \pi \circ \varepsilon_X = F\eta_M \circ \varepsilon_{FM} \circ FG\pi = FG\pi, \tag{3.1.0.18}$$

since $F\eta_M$ is an isomorphism the equality $\mathrm{id} = \varepsilon_{FM} \circ F\eta_M$ implies $F\eta_M \circ \varepsilon_{FM} = \mathrm{id}$. Thus, the previous exact sequences are equivalent and consequently $\mathrm{Ext}^1_A(M,N) \to \mathrm{Ext}^1_B(FM,FN)$ is also surjective. ☐

Similarly with Proposition 1.7.10, $0-\mathscr{A}$ covers can help us understand the indecomposable objects in B-mod using the indecomposable modules of $\mathscr{A}$.

Proposition 3.1.11. *Let (A,P) be a $0-\mathscr{A}$ cover of B for some resolving subcategory $\mathscr{A}$ of A-*mod*. Then, the Schur functor $F = \mathrm{Hom}_A(P,-)$ preserves the indecomposable objects of $\mathscr{A}$.*

Proof. Let $M \in \mathscr{A}$ be an indecomposable module. Assume that we can write $FM \simeq X_1 \oplus X_2$. Then,

$$M \simeq GFM \simeq GX_1 \oplus GX_2. \tag{3.1.0.19}$$

So, either $GX_1 = 0$ or $GX_2 = 0$. Since FA is a B-generator, there must exist a non-zero epimorphism $FA^t \to X_1$ for some $t > 0$ if X_1 is non-zero. So, if $X_1 \neq 0$, then $GX_1 \neq 0$. Thus, FM is indecomposable. □

We shall now recall the definition of an exact category.

Definition 3.1.12. A category $\mathscr{A}$ is an **exact category** if $\mathscr{A}$ is a full subcategory of some abelian category $\mathscr{C}$ and if $\mathscr{A}$ is closed under extensions. In particular, $\mathscr{A}$ is an exact subcategory of $\mathscr{C}$.

A functor $F\colon \mathscr{A} \to \mathscr{B}$ between exact categories is called exact if F preserves exact sequences. F is said to be an exact equivalence of categories if it is an equivalence of categories and exact.

Proposition 3.1.13. *[Rou08, Proposition 4.41] Let (A,P) be a 0-faithful split quasi-hereditary cover of B. The following assertions are equivalent.*

- *(a) (A,P) is a 1-faithful split quasi-hereditary cover of B;*
- *(b) $F = \mathrm{Hom}_A(P,-)$ restricts to an exact equivalence of categories $\mathscr{F}_A(\tilde{\Delta}) \to \mathscr{F}_B(F\tilde{\Delta})$ with inverse the exact functor $G_{|_{\mathscr{F}_B(F\tilde{\Delta})}} = \mathrm{Hom}_B(FA,-)_{|_{\mathscr{F}_B(F\tilde{\Delta})}}$.*
- *(c) For all $M \in \mathscr{F}_A(\tilde{\Delta})$, we have $\mathrm{R}^1 G(FM) = 0$.*

Proof. By Proposition 3.1.10, $(a) \Leftrightarrow (c)$ holds. Assume that (b) holds. In particular, G is exact on $\mathscr{F}_B(F\tilde{\Delta})$. Let $M \in \mathscr{F}_A(\tilde{\Delta})$. Let

$$0 \to FM \to X \to FA \to 0 \in \mathrm{Ext}^1_B(FA,FM) = \mathrm{R}^1 G(FM). \tag{3.1.0.20}$$

Note that $X/FM \simeq FA \in \mathscr{F}_B(F\tilde{\Delta})$, thus $X \in \mathscr{F}_B(F\tilde{\Delta})$. Consequently, G is exact on this exact sequence. By assumption, (A,P) is a 0-faithful split quasi-hereditary cover of B. In particular, $\eta_A\colon A \to GFA$ is an isomorphism. So, the exact sequence

$$0 \to GFM \to GX \to GFA \to 0 \tag{3.1.0.21}$$

is split over A. We have the commutative diagram making the following exact sequences equivalent

$$\begin{tikzcd} 0 \arrow[r] & FM \arrow[r] & X \arrow[r] & FA \arrow[r] & 0 \\ 0 \arrow[r] & FGFM \arrow[r] \arrow[u, "\varepsilon_{FM}"] & FGX \arrow[r] \arrow[u, "\varepsilon_X"] & FGFA \arrow[r] \arrow[u, "\varepsilon_{FA}"] & 0 \end{tikzcd}.$$

Since the bottom row is split over A the upper row must also be split over A. We conclude that $\mathrm{R}^1 G(FM) = \mathrm{Ext}^1_B(FA,FM) = 0$.

Conversely, assume that (c) holds. It is clear that the image of $\mathscr{F}_A(\tilde{\Delta})$ under F is contained in $\mathscr{F}_B(F\tilde{\Delta})$ since F is exact. By assumption, $F_{|_{\mathscr{F}_A(\tilde{\Delta})}}$ is full and faithful. So, it remains to show that $F_{|_{\mathscr{F}_A(\tilde{\Delta})}}\colon \mathscr{F}_A(\tilde{\Delta}) \to \mathscr{F}_B(F\tilde{\Delta})$ is essentially surjective. Let $U \in \mathscr{F}_B(\tilde{\Delta})$. Then, there is a filtration

$$0 \subset U_t \subset \cdots \subset U_1 = U, \text{ where } U_i/U_{i+1} \simeq F(\Delta_i \otimes_R F_i). \tag{3.1.0.22}$$

We shall prove by induction on t that U can be written as FM for some $M \in \mathscr{F}_A(\tilde{\Delta})$. If $t = 1$, there is nothing to show. Assume $t > 1$. We have an exact sequence

$$0 \to U_t \to U_1 \to U_1/U_t \to 0. \tag{3.1.0.23}$$

Here U_1/U_t has a filtration of size $t-1$. By induction, $U_1/U_t = FM_1$, $M_1 \in \mathscr{F}_A(\tilde{\Delta}_{j\neq t})$. Clearly, $U_t = F(\Delta_t \otimes_R F_t)$. Applying G yields the exact sequence

$$0 \to GU_t \to GU_1 \to GU_1/U_t \to \mathrm{R}^1 G(U_t) = \mathrm{R}^1 G(F(\Delta_t \otimes_R F_t)) = 0 \tag{3.1.0.24}$$

and $GU_t \simeq GF(\Delta_t \otimes_R F_t) \simeq \Delta_t \otimes_R F_t$ and $GU_1/U_t \simeq GFM_1 \simeq M_1 \in \mathscr{F}_A(\tilde{\Delta})$. Therefore, $GU_1 \in \mathscr{F}_A(\tilde{\Delta})$. Now as $\varepsilon_{U_1}\colon FGU_1 \to U_1$ is an isomorphism, (b) follows. □

Here we can see one property that distinguishes faithful covers and $\mathscr{A}$-covers. For 1-faithful covers, the image of $\mathscr{F}_A(\tilde{\Delta})$ under the Schur functor is fully determined by the filtrations of the image of standard modules in B-mod. Furthermore, the image of the resolving subcategory $\mathscr{F}(\tilde{\Delta})$ under F is an exact category.

For the resolving subcategory A-proj we can also describe $1-A$-proj covers in a similar way.

Proposition 3.1.14. *Let (A,P) be a cover of B. The following assertions are equivalent.*

(a) *(A,P) is a 1-cover of B.*

(b) *The category $\operatorname{add}_B FA$ is an exact subcategory of B-mod. Furthermore, $F = \operatorname{Hom}_A(P,-)$ restricts to an exact equivalence of categories A-proj $\to \operatorname{add} FA$ with inverse the exact functor $G_{|\operatorname{add} FA} = \operatorname{Hom}_B(FA,-)_{|\operatorname{add} FA}$.*

(c) *For all $M \in A$-proj, we have $\mathrm{R}^1 G(FM) = 0$.*

Proof. $(a) \Leftrightarrow (c)$ follows from Proposition 3.1.10. The implication $(b) \Leftrightarrow (c)$ is analogous to the argument used in Proposition 3.1.13. Assume that (b) holds.

Let $M \in \operatorname{add} FA$. Let

$$0 \to FM \to X \to FA \to 0 \in \operatorname{Ext}^1_B(FA,FM) = \mathrm{R}^1 G(FM). \tag{3.1.0.25}$$

In particular, $FM, FA \in \operatorname{add} FA$. By assumption, $\operatorname{add} FA$ is closed under extensions. Hence, $X \in \operatorname{add} FA$. Consequently, G is exact on (3.1.0.25). By assumption, (A,P) is a 0-cover of B. In particular, $\eta_A\colon A \to GFA$ is an isomorphism. So, the exact sequence

$$0 \to GFM \to GX \to GFA \to 0 \tag{3.1.0.26}$$

is split over A. We have the commutative diagram making the following exact sequences equivalent

$$\begin{tikzcd} 0 \arrow[r] & FM \arrow[r] & X \arrow[r] & FA \arrow[r] & 0 \\ 0 \arrow[r] & FGFM \arrow[r] \arrow[u, "\varepsilon_{FM}"] & FGX \arrow[r] \arrow[u, "\varepsilon_X"] & FGFA \arrow[r] \arrow[u, "\varepsilon_{FA}"] & 0 \end{tikzcd} .$$

Since the bottom row is split over A the upper row must also be split over A. We conclude that $\mathrm{R}^1 G(FM) = \operatorname{Ext}^1_B(FA,FM) = 0$.

Assume that (c) holds. By Proposition 3.1.9, F restricts to an equivalence of categories A-proj $\to \operatorname{add} FA$ with inverse $G_{|\operatorname{add} FA} = \operatorname{Hom}_B(FA,-)_{|\operatorname{add} FA}$.

Let $0 \to FM \to X \to FN \to 0$ be B-exact with $M,N \in A$-proj. Applying G yields the exact sequence

$$0 \to GFM \to GX \to GFN \to \mathrm{R}^1 G(FM) = 0 \tag{3.1.0.27}$$

Since $GFM \simeq M$ and $GFN \simeq N$, we have $GX \in A$-proj. Hence, $X \simeq FGX \in \operatorname{add} FA$. Thus, $\operatorname{add} FA$ is an exact subcategory of B-mod and G is exact on $\operatorname{add} FA$. This completes the proof. $\square$

We should remark that this exact equivalence does not make the image of the resolving subcategory $\mathscr{A}$ under the Schur functor a resolving subcategory in B-mod$\cap R$-proj. In fact, this only occurs when the Schur functor is an equivalence of categories.

Proposition 3.1.15. *Let $\mathscr{A}$ be a resolving subcategory of* A-mod*. Let (A,P) be a $1-\mathscr{A}$ cover of B. Assume that $\{FM\colon M \in \mathscr{A}\}$ is a resolving subcategory of* B-mod*. Then, $F = \operatorname{Hom}_A(P,-)$ is an exact equivalence.*

Proof. Consider the projective B-presentation

$$\delta\colon 0 \to K \to Q \to FA \to 0. \tag{3.1.0.28}$$

By projectivization, $Q = FX$ for some $X \in \operatorname{add} P$, and consequently $X \in \mathscr{A}$. Because $\{FM\colon M \in \mathscr{A}\}$ is a resolving subcategory, there exists $N \in \mathscr{A}$ such that $K \simeq FN$. Hence,

$$\delta \in \operatorname{Ext}^1_B(FA,K) \simeq \operatorname{Ext}^1_A(A,N) = 0. \tag{3.1.0.29}$$

Therefore, FA is a B-summand of Q. Thus, $FA \in B$-proj. As we have seen before, since (A,P) is a cover of B, this implies that F is an exact equivalence. $\square$

In Example 4.6.15, we can see that $\{FM\colon M \in \mathscr{A}\}$ being a resolving subcategory of B-mod is not a sufficient condition for F to be an equivalence of categories.

In order to determine characterizations for level i with $i \geq 2$ we will use Grothendieck's Spectral sequence applied to the Schur functor F. Indeed, this spectral sequence has been used several times on special cases of Schur functors (see for example [Fan08, Proposition 3.1] and [DEN04, 2.2]).

Lemma 3.1.16. *Let $M \in A$-mod. Suppose $\mathrm{R}^i G(FM) = 0$ for $1 \leq i \leq q$. Then, for any $X \in A$-mod, there are isomorphisms $\operatorname{Ext}^i_A(X,GFM) \simeq \operatorname{Ext}^i_B(FX,FM)$, $0 \leq i \leq q$ and an exact sequence*

$$0 \to \operatorname{Ext}^{q+1}_A(X,GFM) \to \operatorname{Ext}^{q+1}_B(FX,FM) \to \operatorname{Hom}_A(X,\mathrm{R}^{q+1} G(FM)) \to \operatorname{Ext}^{q+2}_A(X,GFM) \to \operatorname{Ext}^{q+2}_B(FX,FM).$$

Proof. Let $X \in A$-mod. For $i = 0$, the result follows from the fact that F is left adjoint to G. Fix, in accordance with Lang's notation, $T := G$, $G := \operatorname{Hom}_A(X,-)$. Both of these functors are left exact covariant. Let I be an injective B-module. Since F is a left adjoint to T we have

$$\operatorname{Hom}_A(X,TI) \simeq \operatorname{Hom}_B(FX,I) = \operatorname{Hom}_B(-,I) \circ F(X) \tag{3.1.0.30}$$

and $\operatorname{Hom}_B(-,I)$ and F are exact functors, thus $\operatorname{Hom}_A(-,TI)$ is an exact functor. Hence, TI is an injective A-module. So, T preserves injective modules. Hence, for any $N \in B$-mod, there is a spectral sequence $\{E_r(N)\}$ such that $E_2^{i,j} \Rightarrow \mathrm{R}^{i+j}(\operatorname{Hom}_A(X,-) \circ T)(N)$. Let $N = FM$. So

$$E_2^{i,j} = ((\mathrm{R}^i)\operatorname{Hom}_A(X,-))(\operatorname{Ext}^j_B(FA,FM)) = \operatorname{Ext}^i_A(X,\operatorname{Ext}^j_B(FA,FM)). \tag{3.1.0.31}$$

On the other hand,

$$\operatorname{Hom}_A(X,-) \circ T(N) = \operatorname{Hom}_A(X,TN) \simeq \operatorname{Hom}_B(FX,N) = \operatorname{Hom}_B(FX,-)(N). \tag{3.1.0.32}$$

So,

$$\mathrm{R}^{i+j}(\mathrm{Hom}_A(X,-)\circ T)(FM) \simeq \mathrm{R}^{i+j}(\mathrm{Hom}_B(FX,-))(FM) = \mathrm{Ext}_B^{i+j}(FX,FM). \tag{3.1.0.33}$$

Therefore, we have

$$E_2^{i,j} = \mathrm{Ext}_A^i(X, \mathrm{Ext}_B^j(FA,FM)) \Rightarrow \mathrm{Ext}_B^{i+j}(FX,FM). \tag{3.1.0.34}$$

By assumption, $\mathrm{Ext}_B^j(FA,FM) = \mathrm{R}^j T(FM) = 0$, $1 \leq j \leq q$. Hence, $E_2^{i,j} = 0$ for $1 \leq i \leq q$. By Lemma 1.3.10, $\mathrm{Ext}_A^i(X,GFM) = E_2^{i,0} = \mathrm{Ext}_B^i(FX,FM)$, $0 \leq i \leq q$ and the result follows. ☐

Proposition 3.1.17. *Let (A,P) be a $0-\mathscr{A}$ cover of B. Let $i \geq 1$. The following assertions are equivalent.*

(a) (A,P) is an $i-\mathscr{A}$ cover of B;

(b) For all $M \in \mathscr{A}$, we have $\mathrm{R}^j G(FM) = 0$, $1 \leq j \leq i$.

Proof. $(a) \implies (b)$. Let $M \in \mathscr{A}$. Let $1 \leq j \leq i$. Then,

$$\mathrm{R}^j G(FM) = \mathrm{R}^j \mathrm{Hom}_B(FA,-)(FM) = \mathrm{Ext}_B^j(FA,FM) \simeq \mathrm{Ext}_A^j(A,M) = 0. \tag{3.1.0.35}$$

$(b) \implies (a)$. Let $M \in \mathscr{A}$. By assumption, $\mathrm{R}^j G(FM) = 0$, $1 \leq j \leq i$. By Lemma 3.1.16, $\mathrm{Ext}_A^j(X,GFM) \simeq \mathrm{Ext}_B^j(FX,FM)$, $0 \leq j \leq i$ for any $X \in A$-mod. Since Let (A,P) is a $0-\mathscr{A}$ cover of B, $\eta_M \colon M \to GFM$ is an A-isomorphism, and thus we have

$$\mathrm{Ext}_A^j(X,M) \simeq \mathrm{Ext}_B^j(FX,FM), \quad 0 \leq j \leq i, \forall X \in A\text{-mod}. \tag{3.1.0.36}$$

The choice of $M \in \mathscr{A}$ is arbitrary, hence (a) follows. ☐

For faithful split quasi-hereditary covers this translates to:

Proposition 3.1.18. *Let (A,P) be a 0-faithful split quasi-hereditary cover of B. Let $i \geq 1$. The following assertions are equivalent.*

(a) (A,P) is an i-faithful split quasi-hereditary cover of B;

(b) For all $M \in \mathscr{F}(\tilde{\Delta})$, we have $\mathrm{R}^j G(FM) = 0$, $1 \leq j \leq i$.

(c) For all $\lambda \in \Lambda$, we have $\mathrm{R}^j G(F\Delta(\lambda)) = 0$, $1 \leq j \leq i$.

Proof. $(a) \Leftrightarrow (b)$ is given by Proposition 3.1.17. The implication $(b) \implies (c)$ is also clear.

Assume that (c) holds. Let $M \in \mathscr{F}(\tilde{\Delta})$. There is a filtration

$$0 = M_{n+1} \subset M_n \subset M_{n-1} \subset \cdots \subset M_1 = M, \quad M_i/M_{i+1} \simeq \Delta_i \otimes_R U_i, \quad 1 \leq i \leq n. \tag{3.1.0.37}$$

We claim that $\mathrm{R}^j G(FM_t) = 0$, for $t = 1,\ldots,n$, $1 \leq j \leq i$. We will prove it by induction on $n-t+1$. Assume that $n-t+1 = 1$. Let $1 \leq j \leq i$. Then, $\mathrm{R}^j G(FM_t) = \mathrm{R}^j G(F(\Delta_t \otimes_R U_t))$ is an R-summand of $\mathrm{R}^j G(F\Delta_t)^s = 0$ for some $s > 0$ since $U_t \in R$-proj. Thus, $\mathrm{R}^j G(FM_t) = 0$. Moreover, $\mathrm{R}^j G(F(\Delta_i) \otimes_R U_i) = 0$ for every $i = 1,\ldots,n$. Assume that the claim holds for $s > t$ for some $n \geq t > 1$. Consider the exact sequence

$$0 \to M_{t+1} \to M_t \to \Delta_t \otimes_R U_t \to 0. \tag{3.1.0.38}$$

Applying the left exact functor $G \circ F$ yields the exact sequence

$$\mathrm{R}^j G(FM_{t+1}) \to \mathrm{R}^j G(FM_t) \to \mathrm{R}^j G(F\Delta_t \otimes_R U_t) = 0. \tag{3.1.0.39}$$

By induction, $\mathrm{R}^j G(FM_{t+1}) = 0$, hence $\mathrm{R}^j G(FM_t) = 0$. Therefore, (b) follows. ☐

Hence, the quality of a 0-faithful split quasi-hereditary cover is given by the value

$$n(\mathscr{F}(\tilde{\Delta})) = \sup\{i \in \mathbb{N}_0 \colon \mathrm{R}^j G(F\Delta(\lambda)) = 0,\ \lambda \in \Lambda,\ 1 \leq j \leq i\}.$$

3.2 Upper bounds for the quality of an $\mathscr{A}$-cover

3.2.1 $\mathscr{F}(\Delta)$

For finite-dimensional algebras over fields, there is an upper bound for the level of faithfulness of a split quasi-hereditary cover.

Theorem 3.2.1. *Let R be a field and let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over R. Let $\{S(\lambda) \colon \lambda \in \Lambda\}$ be a complete set of non-isomorphic simple A-modules.*

We shall denote by Λ^* the set $\{\lambda \in \Lambda \colon FS(\lambda) \neq 0\}$. Recall that $d(\Lambda^*, \lambda)$ denotes the length of λ in the poset Λ^* and $d(\Lambda^*)$ denotes the value $\max\{d(\Lambda^*, \lambda) \colon \lambda \in \Lambda^*\}$ *(see Section 2.11).*

If (A, P) is a split quasi-hereditary $(d(\Lambda^*) + 1)$-faithful cover of B-mod*, then the Schur functor induces by restriction to A*-proj *the functor $F_{|_{A\text{-proj}}} \colon A$*-proj $\to B$-proj*. Moreover, F is an equivalence of categories.*

Proof. Since R is a field we can assume, without loss of generality, that there exists an idempotent $e \in A$ such that $P = Ae$ and $B = eAe$. By Theorem 1.7.2, the simple B-modules can be written in the form FS where S is a simple A-module. Moreover, the simple B-modules are indexed by Λ^*. The set Λ^* is again a poset where its partial order is the one induced by the poset Λ.

Consider M a finitely generated projective A-module. We want to show that FM is a projective B-module. It is enough to show that $\mathrm{Ext}_B^1(FM, S) = 0$ for all simple B-modules S.

We claim that $\mathrm{Ext}_B^j(FM, FS(\lambda)) = 0,\ 1 \leq j \leq d(\Lambda^*, \lambda) + 1,\ \lambda \in \Lambda^*$.

We shall proceed by induction on $n(\lambda) = d(\Lambda^*) - d(\Lambda^*, \lambda)$, $\lambda \in \Lambda^*$. Assume that $n(\lambda) = 0$. Then, λ is minimal in Λ^*. Assume that λ is also minimal in the poset Λ. Then, $\Delta(\lambda) = S(\lambda)$. Hence, $F\Delta(\lambda) = FS(\lambda)$. Now, assume that λ is not minimal in Λ. Consider the short exact sequence

$$0 \to X \to \Delta(\lambda) \to S(\lambda) \to 0, \tag{3.2.1.1}$$

where X has a composition series with composition factors $S(\mu)$ satisfying $\mu < \lambda$. The minimality of λ in Λ^* implies that $FS(\mu) = 0$ for $\mu < \lambda$, $\mu \in \Lambda$. By induction on the length of the composition series of X it follows that $FX = 0$. Applying the functor F to the short exact sequence (3.2.1.1) yields $F\Delta(\lambda) \simeq FS(\lambda)$.

Therefore,

$$\mathrm{Ext}_B^j(FM, FS(\lambda)) = \mathrm{Ext}_B^j(FM, F\Delta(\lambda)) \simeq \mathrm{Ext}_A^j(M, \Delta(\lambda)) = 0, \quad 1 \leq j \leq d(\Lambda^*) + 1 = d(\Lambda^*, \lambda) + 1.$$

The last isomorphism follows from the fact that (A, P) is a $(d(\Lambda^*) + 1)$-faithful cover of B.

Assume that there exists a positive integer k such that the claim holds for all $\lambda \in \Lambda^*$ satisfying $n(\lambda) < k$. Let $\lambda \in \Lambda^*$ such that $n(\lambda) = k$. Consider again the short exact sequence (3.2.1.1). Let $S(\mu)$ be a composition factor of X. Hence, $\mu < \lambda$. If $\mu \notin \Lambda^*$, then $FS(\mu) = 0$. Otherwise, $d(\Lambda^*, \mu) \geq d(\Lambda^*, \lambda) + 1$ and

$$n(\mu) = d(\Lambda^*) - d(\Lambda^*, \mu) \leq d(\Lambda^*) - d(\Lambda^*, \lambda) - 1 = k - 1 < k$$

By induction, $\mathrm{Ext}_B^j(FM, FS(\mu)) = 0$, $1 \leq j \leq d(\Lambda^*, \lambda) + 2$. By induction on the length of the composition series of FX, we obtain $\mathrm{Ext}_B^j(FM, FX) = 0$, $1 \leq j \leq d(\Lambda^*, \lambda) + 2$. Now, applying the functor $\mathrm{Hom}_B(FM, -) \circ F$ to (3.2.1.1) yields the long exact sequence

$$0 = \mathrm{Ext}_B^j(FM, F\Delta(\lambda)) \to \mathrm{Ext}_B^j(FM, FS(\lambda)) \to \mathrm{Ext}_B^{j+1}(FM, FX) = 0, \ 1 \leq j \leq d(\Lambda^*, \lambda) + 1. \tag{3.2.1.2}$$

This completes the proof of our claim. In particular, $\mathrm{Ext}_B^1(FM, FS(\lambda)) = 0$ for all $\lambda \in \Lambda^*$. So, FM is projective over B. By projectivization, since the Schur functor is written in the form $F = \mathrm{Hom}_A(P, -)$, B-proj is equivalent to add(P). Thus, by projectivization, the functor $F_{|A\text{-proj}}$ is essentially surjective. As by definition of cover, the functor $F_{|A\text{-proj}}$ is fully and faithful it follows that the functor $F_{|A\text{-proj}}$ is an equivalence of categories.

So, for any finitely generated projective A-module M, we obtain $FM = \mathrm{Hom}_A(P, M) \cong \mathrm{Hom}_A(P, P')$ for some $P' \in \mathrm{add}(P)$. By applying the adjoint functor G we get that $M \simeq GFM \simeq GFP' \simeq P'$. So, $A \in \mathrm{add}(P)$, which means that P is a progenerator. Hence, by Morita theory, F is an equivalence of categories. ☐

Observe that $d(\Lambda^*) + 1 \leq |\Lambda^*| - 1 + 1 = |\Lambda^*|$ which is exactly the number of non-isomorphic classes of simple B-modules. We have therefore proved that the number of simple B-modules is an upper bound for the level of faithfulness of a split quasi-hereditary cover of B.

3.2.2 A-proj

We can also give upper bounds for A-proj-covers. To do that, we will use another example of resolving subcategories. Let $i \geq 0$ be an integer. Let $\mathscr{P}^i$ be the full subcategory of A-mod whose modules have projective dimension over A less or equal to i. The category $\mathscr{P}^i$ is a resolving subcategory of A-mod$\cap R$-proj. For $i = 0$, $\mathscr{P}^i$ is exactly A-proj. For $i = \mathrm{gldim} A$, $\mathscr{P}^i = A$-mod.

Theorem 3.2.2. *Let $i, j \geq 0$ be integers. If (A, P) is an $i - \mathscr{P}^j$ cover of B, then (A, P) is an $(i-1) - \mathscr{P}^{j+1}$-cover of B.*

Proof. Let X be a module with projective dimension at most $j+1$. We can consider a projective presentation over A for X

$$0 \to Q \to P \to X \to 0, \tag{3.2.2.1}$$

such that $Q \in \mathscr{P}^j$ and $P \in A$-proj. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d, "\eta_Q"] & P \arrow[r] \arrow[d, "\eta_P"] & X \arrow[r] \arrow[d, "\eta_X"] & 0 \\
0 \arrow[r] & GFQ \arrow[r] & GFP \arrow[r] & GFX \arrow[r] & \mathrm{R}^1 G(FQ)
\end{tikzcd} . \tag{3.2.2.2}$$

Due to $i \geq 0$ and $Q, P \in \mathscr{P}^j$, η_Q and η_P are A-isomorphisms. By Snake Lemma, η_X is an monomorphism. So, (A, P) is an $(-1) - \mathscr{P}^{j+1}$-cover of B. If $i \geq 1$, then $\mathrm{R}^1 G(FQ) = 0$. In such a case, the Snake Lemma implies that η_X is an isomorphism. So, the claim holds for $i = 1$. Assume now that $i \geq 2$. Applying GF to (3.2.2.1) yields the long exact sequence

$$0 = \mathrm{R}^l G(FP) \to \mathrm{R}^l G(FX) \to \mathrm{R}^{l+1} G(FQ) = 0, \quad 1 \leq l \leq i-1. \tag{3.2.2.3}$$

Thus, (A,P) is an $(i-1)-\mathscr{P}^{j+1}$-cover of B. $\square$

An immediate consequence of this result is the following bound on A-proj-covers.

Corollary 3.2.3. *Let $i = \operatorname{gldim} A$. Let (A,P) be an $i-A$-proj cover of B. Then, $F = \operatorname{Hom}_A(P,-)\colon A\text{-mod} \to B\text{-mod}$ is an equivalence of categories.*

Proof. Using induction on Theorem 3.2.2, we obtain that (A,P) is a $0-\mathscr{P}^{\operatorname{gldim} A}$-cover of B. Moreover, (A,P) is a $0-A$-mod cover of B. Thus, η_M is an isomorphism for every $M \in A$-mod. This means that the functor $\operatorname{Hom}_A(P,-)\colon A\text{-mod} \to B\text{-mod}$ is full and faithful. Because of (A,P) being a cover of B, the left adjoint of $\operatorname{Hom}_A(P,-)$ is also full and faithful. Therefore, $\operatorname{Hom}_A(P,-)$ is an equivalence of categories. $\square$

3.3 $\mathscr{A}$-covers under change of ground ring

We shall now see how $\mathscr{A}$-covers behave under change of ground ring. Here we need to impose constraints to the resolving subcategories $\mathscr{A}$ we want to work with. As a first step, note that $A\text{-mod} \cap R\text{-proj}$ is a resolving subcategory of A-mod. So, we will restrict our attention to resolving subcategories of $A\text{-mod} \cap R\text{-proj}$ since exact sequences in this category remain exact under extension of scalars. However, this is not sufficient, so we are interested in resolving subcategories which behave well under change of ground ring in the following sense.

Definition 3.3.1. We will call $\mathscr{R}(A)$ a **well behaved resolving subcategory** of $A\text{-mod} \cap R\text{-proj}$ if it is a resolving subcategory of $A\text{-mod} \cap R\text{-proj}$ and the following properties are satisfied:

1. For any commutative Noetherian R-algebra S, there is a resolving subcategory $\mathscr{R}(S \otimes_R A)$ of $S \otimes_R A\text{-mod} \cap S\text{-proj}$ and the functor $H\colon \mathscr{R}(A) \to \mathscr{R}(S \otimes_R A)$, given by $M \mapsto S \otimes_R M$, is well defined with $\langle H\mathscr{R}(A)\rangle = \mathscr{R}(S \otimes_R A)$, where $\langle H\mathscr{R}(A)\rangle$ denotes the smallest subcategory of $S \otimes_R A\text{-mod} \cap S\text{-proj}$ containing $H\mathscr{R}(A)$ closed under direct summands and extensions.
2. $M \in \mathscr{R}(A)$ if and only if $M_\mathfrak{m} \in \mathscr{R}(A_\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ in R.
3. $M \in \mathscr{R}(A)$ if and only if $M(\mathfrak{m}) \in \mathscr{R}(A(\mathfrak{m}))$ for every maximal ideal $\mathfrak{m}$ in R and $M \in R$-proj.

From now on we will consider $\mathscr{R}(A)$ to be a well behaved resolving subcategory of $A\text{-mod} \cap R\text{-proj}$. Here are some examples of well behaved resolving subcategories.

Proposition 3.3.2. *Let A be a projective Noetherian R-algebra. The following assertions hold.*

(I) A-proj *is a well behaved resolving subcategory of* $A\text{-mod} \cap R\text{-proj}$.

(II) *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then, $\mathscr{F}(\tilde{\Delta})$ is a well behaved resolving subcategory of* $A\text{-mod} \cap R\text{-proj}$.

Proof. Clearly, A-proj is a resolving subcategory of $A\text{-mod} \cap R\text{-proj}$. Condition 3.3.1.2 follows from Theorem 1.1.45, whereas condition 3.3.1.3 follows by Theorem 1.1.51. Let $M \in A$-proj. Then, $A^t \simeq M \bigoplus K$ for some $t > 0$ and some module K. Hence, $(S \otimes_R A)^t \simeq S \otimes_R M \bigoplus S \otimes_R K$. So, $S \otimes_R M \in S \otimes_R A$-proj. Thus, the functor H is well defined. Let

$X \in S \otimes_R A$-proj. Hence, $S \otimes_R A^s \simeq (S \otimes_R A)^s \simeq X \bigoplus K$ for some $s > 0$ and $S \otimes_R A^s \in \langle H(A\text{-proj})\rangle$, thus $X \in \langle H(A\text{-proj})\rangle$. So, (I) holds.

By Theorem 1.5.104, $\mathscr{F}(\tilde{\Delta})$ is a resolving subcategory of A-mod$\cap R$-proj. Recall that $0 = \operatorname{Ext}_A^{i>0}(M,T)$ if and only if $\operatorname{Ext}_A^{i>0}(M,T)_{\mathfrak{m}} = 0$ for ever maximal ideal of R if and only if $\operatorname{Ext}_{A_{\mathfrak{m}}}^{i>0}(M_{\mathfrak{m}},T_{\mathfrak{m}}) = 0$ for every maximal ideal of R. By Corollary 1.5.125 and Proposition 1.5.126, Condition 3.3.1.2 follows. Condition 3.3.1.3 follows from Proposition 1.5.131. Since the exact sequences arising from a filtration of $M \in \mathscr{F}(\tilde{\Delta})$ are (A,R)-exact, applying the tensor product $S\otimes_R -$ preserves the filtration and hence $S\otimes_R M \in \mathscr{F}(S\otimes\tilde{\Delta})$. So, the functor H is well defined. Let $X \in \mathscr{F}(S\otimes_R\tilde{\Delta})$. Then, there is a filtration

$$0 = X_{n+1} \subset X_n \subset \cdots \subset X_1 = X,\ X_i/X_{i+1} \simeq S\otimes_R \Delta_i \otimes_S U_i, 1 \leq i \leq n. \tag{3.3.0.1}$$

We will proceed by induction to prove that each X_i belongs to $\langle H\mathscr{R}(A)\rangle$. For $i = n$, $X_n = S\otimes_R \Delta_n \otimes_S U_n$ is an $S\otimes_R A$-summand of $(S\otimes_R \Delta_n)^s \simeq S\otimes_R \Delta_n^s \in \langle H\mathscr{F}(\tilde{\Delta})\rangle$ for some $s > 0$. Since $\langle H\mathscr{F}(\tilde{\Delta})\rangle$ is closed under direct summands, $X_n \in \langle H\mathscr{F}(\tilde{\Delta})\rangle$. Assume that we have proven the result for X_s for $s > i$ for some i. Consider the exact sequence

$$0 \to X_{i+1} \to X_i \to S\otimes_R \Delta_i \otimes_S U_i \to 0. \tag{3.3.0.2}$$

By induction, $X_{i+1} \in \langle H\mathscr{F}(\tilde{\Delta})\rangle$, and since it is closed under extensions, $X_i \in \langle H\mathscr{F}(\tilde{\Delta})\rangle$. Thus, (II) holds. $\square$

As before we will separate the cases -1 and 0 and consider them first.

Proposition 3.3.3. *Let $i \in \{-1,0\}$. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of* A-mod$\cap R$-proj. *The following assertions are equivalent.*

(a) (A,P) is an $i-\mathscr{R}(A)$ cover of B;

(b) $(S\otimes_R A, S\otimes_R P)$ is an $i-\mathscr{R}(S\otimes_R A)$ cover of $S\otimes_R B$ for any commutative flat R-algebra S;

(c) $(A_{\mathfrak{m}}, P_{\mathfrak{m}})$ is an $i-\mathscr{R}(A_{\mathfrak{m}})$ cover of $B_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R;

Proof. Let $M \in A$-mod$\cap R$-proj. Consider the following diagram

$$\begin{tikzcd}
S\otimes_R M \arrow[r, "S\otimes_R \eta_M"] \arrow[dd, equal] & S\otimes_R \operatorname{Hom}_B(\operatorname{Hom}_A(P,A), \operatorname{Hom}_A(P,M)) \arrow[d, "\omega_{\operatorname{Hom}_A(P,A),\operatorname{Hom}_A(P,M)}"] \\
 & \operatorname{Hom}_{S\otimes_R B}(S\otimes_R \operatorname{Hom}_A(P,A), S\otimes_R \operatorname{Hom}_A(P,M)) \\
S\otimes_R A \arrow[r, "\eta_{S\otimes_R M}"] & \operatorname{Hom}_{S\otimes_R B}(\operatorname{Hom}_{S\otimes_R A}(S\otimes_R P, S\otimes_R A), \operatorname{Hom}_{S\otimes_R A}(S\otimes_R P, S\otimes_R M)) \arrow[u, "\omega_{P,M}^{-1}\circ(-)\circ\omega_{P,A}"']
\end{tikzcd} . \tag{3.3.0.3}$$

The maps ω are the canonical maps given by Lemma 1.1.36, hence they are isomorphisms. This is a commutative diagram. In fact, for every $s,s',s'' \in S$, $m \in M$, $g \in \operatorname{Hom}_A(P,A)$, $p \in P$, we have

$$\begin{aligned}
\omega_{P,M}\circ\omega_{P,M}^{-1}\circ(-)\circ\omega_{P,A}\circ\eta_{S\otimes_R M}(s\otimes m)(s'\otimes g)(s''\otimes p) &= \eta_{S\otimes_R M}(s\otimes m)\omega_{P,A}(s'\otimes g)(s''\otimes p) \\
&= \omega_{P,A}(s'\otimes g)(s''\otimes p)s\otimes m = ss's''\otimes g(p)m
\end{aligned}$$

$$\omega_{P,M}\circ\omega_{\operatorname{Hom}_A(P,A),\operatorname{Hom}_A(P,M)}\circ S\otimes_R \eta_M(s\otimes m)(s'\otimes g)(s''\otimes p) =$$

$$\omega_{P,M}\omega_{\operatorname{Hom}_A(P,A),\operatorname{Hom}_A(P,M)}(s\otimes\eta_M(m))(s'\otimes g)(s''\otimes p) = \omega_{P,M}(ss'\otimes\eta_M(m)(g))(s''\otimes p) = ss's''\otimes g(p)m.$$

Assume that (a) holds. By Proposition 1.4.30, $(S\otimes_R A, S\otimes_R P)$ is a cover of $S\otimes_R B$. Let $M \in \mathscr{R}(A)$. By assumption, η_M is mono in case $i = -1$ or it is an isomorphism in case $i = 0$. Applying the exact functor

$S\otimes_R -$, $S\otimes_R \eta_M$ is a monomorphism (if $i=-1$) or $S\otimes_R \eta_M$ is an isomorphism in case $i=0$. In view of the diagram (3.3.0.3), $\eta_{S\otimes_R M}$ is a monomorphism if $i=-1$ and it is an isomorphism if $i=0$. According to Lemma 1.4.26 and Snake Lemma, η_N is a monomorphism/isomorphism $i=-1$ and $i=0$ respectively for any $N\in\langle H\mathscr{R}(A)\rangle=\mathscr{R}(S\otimes_R A)$. By Proposition 3.1.6 and Lemma 3.1.4, (b) follows. $(b) \implies (c)$ is clear.

Assume that (c) holds. By Proposition 1.4.30, (A,P) is a cover of B. Let $M\in\mathscr{R}(A)$. By assumption, $\eta_{M_\mathfrak{m}}$ is a monomorphism in case $i=-1$ or it is an isomorphism in case $i=0$ for every maximal ideal $\mathfrak{m}$ in R. According to the diagram (3.3.0.3), $(\eta_M)_\mathfrak{m}$ is a monomorphism in case $i=-1$ or it is an isomorphism in case $i=0$ for every maximal ideal $\mathfrak{m}$ in R. Therefore, η_M is a monomorphism in case $i=-1$ or it is an isomorphism in case $i=0$. Thus, (a) follows. ☐

Proposition 3.3.4. *Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of* A-mod$\cap R$-proj. *Let (A,P) be a $0-\mathscr{R}(A)$ cover of B. For $i\geq 1$, the following assertions are equivalent.*

(a) (A,P) is an $i-\mathscr{R}(A)$ cover of B;

(b) $(S\otimes_R A, S\otimes_R P)$ is an $i-\mathscr{R}(S\otimes_R A)$ cover of $S\otimes_R B$ for any commutative flat R-algebra S;

(c) $(A_\mathfrak{p},P_\mathfrak{p})$ is an $i-\mathscr{R}(A_\mathfrak{p})$ cover of $B_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R;

(d) $(A_\mathfrak{m},P_\mathfrak{m})$ is an $i-\mathscr{R}(A_\mathfrak{m})$ cover of $B_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R;

Proof. $(a) \implies (b)$. By Proposition 3.3.3, $(S\otimes_R A, S\otimes_R P)$ is a $0-\mathscr{R}(S\otimes_R A)$ cover of $S\otimes_R B$ for any commutative flat R-algebra S. Let $M\in\mathscr{R}(A)$. Let $1\leq j\leq i$. Then,

$$\begin{aligned}\mathrm{R}^j G_S(F_S(S\otimes_R M)) &= \mathrm{Ext}^j_{S\otimes_R B}(\mathrm{Hom}_{S\otimes_R A}(S\otimes_R P, S\otimes_R A), \mathrm{Hom}_{S\otimes_R A}(S\otimes_R P, S\otimes_R M))\\ &= \mathrm{Ext}^j_{S\otimes_R B}(S\otimes_R \mathrm{Hom}_A(P,A), S\otimes_R \mathrm{Hom}_A(P,M))\\ &= S\otimes_R \mathrm{Ext}^j_B(\mathrm{Hom}_A(P,A),\mathrm{Hom}_A(P,M)) = S\otimes_R \mathrm{R}^j G(FM)=0\end{aligned}$$

Using long exact sequences coming from the derived functors $\mathrm{R}^j G_S$ and since it commutes with direct summands we obtain that $\mathrm{R}^j G_S(F_S(N))=0$ for all $N\in\langle H\mathscr{R}(A)\rangle=\mathscr{R}(S\otimes_R A)$. By Proposition 3.1.17, (b) follows. The implications $(b) \implies (c) \implies (d)$ are clear.

Assume that (d) holds. By Proposition 3.3.3, (A,P) is a $0-\mathscr{R}(A)$ cover of B. Let $M\in\mathscr{R}(A)$. Let $1\leq j\leq i$. We have, for every maximal ideal m in R,

$$\begin{aligned}\mathrm{R}^j G(FM)_\mathfrak{m} &= \mathrm{Ext}^j_B(\mathrm{Hom}_A(P,A),\mathrm{Hom}_A(P,M))_\mathfrak{m} \simeq \mathrm{Ext}^j_{B_\mathfrak{m}}(\mathrm{Hom}_{A_\mathfrak{m}}(P_\mathfrak{m},A_\mathfrak{m}),\mathrm{Hom}_{A_\mathfrak{m}}(P_\mathfrak{m},M_\mathfrak{m})) \quad (3.3.0.4)\\ &= \mathrm{R}^j G_\mathfrak{m}(F_\mathfrak{m}M_\mathfrak{m})=0, \quad (3.3.0.5)\end{aligned}$$

since $M_\mathfrak{m}\in\mathscr{R}(A_\mathfrak{m})$. Therefore, $\mathrm{R}^j G(FM)=0$. By Proposition 3.1.17, (A,P) is an $i-\mathscr{R}(A)$ cover of B. ☐

Theorem 3.3.5. *Let R be a regular Artinian ring. Denote by p_B the number of non-isomorphism classes of projective indecomposable of B. Assume $k=\sup\{p_{B_\mathfrak{m}}: \mathfrak{m}$ maximal ideal of $R\}<\infty$. Let (A,P) be an i-faithful split quasi-hereditary cover of B. If $i\geq k$, then A and B are Morita equivalent.*

Proof. Let $\mathfrak{m}$ be a maximal ideal in R. By Theorem 1.1.60, $R_\mathfrak{m}$ is a regular local commutative Noetherian ring and $\dim R_\mathfrak{m}=\operatorname{gldim} R_\mathfrak{m}=0$. By Lemma 1.1.57, $R_\mathfrak{m}$ is a field, and thus $B_\mathfrak{m}$ is a finite-dimensional algebra. By Proposition 3.3.4, $(A_\mathfrak{m},P_\mathfrak{m})$ is $i-\mathscr{F}(\tilde{\Delta}_\mathfrak{m})$ cover of $B_\mathfrak{m}$. By assumption $i\geq k\geq p_{B_\mathfrak{m}}$ which is equal to the number of isomorphism classes of simple $B_\mathfrak{m}$-modules. By Theorem 3.2.1, $\mathrm{Hom}_{A_\mathfrak{m}}(P_\mathfrak{m},-)\colon A_\mathfrak{m}\text{-mod}\to B_\mathfrak{m}\text{-mod}$ is an equivalence of categories. In particular, the functor $\mathrm{Hom}_{A_\mathfrak{m}}(P_\mathfrak{m},-)$ preserves projectives. Let $Q\in A$-proj. Then,

$$\mathrm{Hom}_A(P,Q)_\mathfrak{m}\simeq \mathrm{Hom}_{A_\mathfrak{m}}(P_\mathfrak{m},Q_\mathfrak{m})\in B_\mathfrak{m}\text{-proj} \quad (3.3.0.6)$$

for every maximal ideal $\mathfrak{m}$ in R. By Theorem 1.1.45, $\operatorname{Hom}_A(P,Q) \in B$-proj. Hence, $\operatorname{Hom}_A(P,-)$ preserves projective modules, thus A-proj $\simeq B$-proj. Consequently, A and B are Morita equivalent by Theorem 1.4.17. $\square$

Proposition 3.3.6. *Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of A-mod$\cap R$-proj. Let $P \in A$-mod$\cap R$-proj. Let $(A(\mathfrak{m}), P(\mathfrak{m}))$ be a $(-1)-\mathscr{R}(A(\mathfrak{m}))$ cover of $B(m)$ for every maximal ideal $\mathfrak{m}$ of R. Then, (A,P) is a $(-1)-\mathscr{R}(A)$ cover of B.*

Proof. By Proposition 1.4.34, (A,P) is a cover of B. Let $M \in \mathscr{R}(A)$. By definition, $M(\mathfrak{m}) \in \mathscr{R}(A(\mathfrak{m}))$ for every maximal ideal $\mathfrak{m}$ in R and $M \in R$-proj. By Lemma 3.1.4, $\eta_{M(\mathfrak{m})}$ is a monomorphism. By Lemma 1.4.31, η_M is a monomorphism. Applying again Lemma 3.1.4, it follows that (A,P) is a $(-1)-\mathscr{R}(A)$ cover of B. $\square$

Proposition 3.3.7. *Let R be a regular (commutative Noetherian) ring. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of A-mod$\cap R$-proj. Let $P \in A$-mod$\cap R$-proj. If $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a $0-\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R, then (A,P) is a $0-\mathscr{R}(A)$ cover of B.*

Proof. By Proposition 3.3.6, (A,P) is a $(-1)-\mathscr{R}(A)$ cover of B. Let $\mathfrak{m}$ be a maximal ideal $\mathfrak{m}$ in R. Then, $(A_\mathfrak{m}(\mathfrak{m}_\mathfrak{m}), P_\mathfrak{m}(\mathfrak{m}_\mathfrak{m})) = (A(\mathfrak{m}), P(\mathfrak{m}))$ is a $0-\mathscr{R}(A(\mathfrak{m})) = \mathscr{R}(A_\mathfrak{m}(\mathfrak{m}_\mathfrak{m}))$ cover of $(B(\mathfrak{m})) = B_\mathfrak{m}(\mathfrak{m}_\mathfrak{m})$. So, in view of Proposition 3.3.3, we can assume, without loss of generality, that R is a local regular commutative Noetherian ring. In particular, $\dim R < \infty$. We shall proceed by induction on the Krull dimension of R.

If $\dim R = 0$, then R is a field, so there is nothing to show. Assume the result is known for regular local rings with Krull dimension less than t. Let R be with $\dim R = t$. Let $\mathfrak{m}$ be the unique maximal ideal of R. Let $x \in \mathfrak{m}/\mathfrak{m}^2$ be a non-zero element. By Lemma 1.1.57, $Q := R/Rx$ is a regular commutative Noetherian local ring with $\dim(Q) = t-1$ and $Q(\mathfrak{m}_Q) \simeq R/Rx/\mathfrak{m}/Rx \simeq R/\mathfrak{m} = R(\mathfrak{m})$. Moreover,

$$(Q \otimes_R A)(\mathfrak{m}_Q) = Q(\mathfrak{m}_Q) \otimes_Q Q \otimes_R A \simeq R(\mathfrak{m}) \otimes_R A = A(\mathfrak{m}).$$

Hence, $((Q \otimes_R A)(\mathfrak{m}_Q), (Q \otimes_R P(\mathfrak{m}_Q))$ is a $0-\mathscr{R}((Q \otimes_R A)(\mathfrak{m}_Q))$ cover of $Q \otimes_R B(\mathfrak{m}_Q)$. By induction, $(Q \otimes_R A, Q \otimes_R P)$ is a $0-\mathscr{R}(Q \otimes_R A)$ cover of $Q \otimes_R B$. The remaining of the proof is similar to Proposition 1.4.34. Let $M \in \mathscr{R}(A)$.

By definition, $Q \otimes_R M \in \mathscr{R}(Q \otimes_R A)$. By Lemma 3.1.6, $\eta_{Q \otimes_R M}$ is an isomorphism. By Lemma 1.1.32 and Proposition 1.1.31, the composition map

$$\begin{gathered} Q \otimes_R M \xrightarrow{\eta_{Q\otimes_R M}} \operatorname{Hom}_{Q\otimes_R B}(\operatorname{Hom}_{Q\otimes_R A}(Q \otimes_R P, Q \otimes_R A), \operatorname{Hom}_{Q\otimes_R A}(Q \otimes_R P, Q \otimes_R M)) \\ \longrightarrow \operatorname{Hom}_{Q\otimes_R B}(Q \otimes_R \operatorname{Hom}_A(P,A), Q \otimes_R \operatorname{Hom}_A(P,M)) \longrightarrow \operatorname{Hom}_B(\operatorname{Hom}_A(P,A), Q \otimes_R \operatorname{Hom}_A(P,M)) \end{gathered}$$

is an isomorphism. We will denote this map by $\mu_{Q,M}$.

We have a commutative triangle

$$\begin{tikzcd} Q \otimes_R M \arrow[rr, "\mu_{Q,M}"] \arrow[dr, "Q\otimes_R \eta_M"'] & & \operatorname{Hom}_B(\operatorname{Hom}_A(P,A), Q \otimes_R \operatorname{Hom}_A(P,M)) \\ & Q \otimes_R \operatorname{Hom}_B(\operatorname{Hom}_A(P,A), \operatorname{Hom}_A(P,M)) \arrow[ur, hook, "\delta"'] & \end{tikzcd}$$

with a monomorphism given by Lemma 1.4.33. Since $\mu_{Q,M}$ is an isomorphism, the inclusion map δ is also surjective. Thus, $Q \otimes_R \eta_M$ is an isomorphism. Denote the canonical surjective map $Q \to Q/\mathfrak{m}/Rx = R(\mathfrak{m})$ by π. There is the commutative diagram

$$\begin{array}{ccc} Q\otimes_R M & \xrightarrow{Q\otimes_R \eta_M} & Q\otimes_R \operatorname{Hom}_B(\operatorname{Hom}_A(P,A),\operatorname{Hom}_A(P,M)) \\ \Big\downarrow{\scriptstyle M\otimes_R \pi} & & \Big\downarrow{\scriptstyle \pi\otimes_R \operatorname{Hom}_B(\operatorname{Hom}_A(P,A),\operatorname{Hom}_A(P,M))} \\ R(\mathfrak{m})\otimes_R M & \xrightarrow{\eta_M(\mathfrak{m})} & R(\mathfrak{m})\otimes_R \operatorname{Hom}_B(\operatorname{Hom}_A(P,A),\operatorname{Hom}_A(P,M)) \end{array} .$$

It follows that $\eta_M(\mathfrak{m})\circ M\otimes_R \pi$ is surjective. In particular, $\eta_M(\mathfrak{m})$ is surjective. By Nakayama's Lemma, η_M is surjective. Since (A,P) is a $(-1)-\mathscr{R}(A)$ cover of B, η_M is also a monomorphism, hence η_M is an isomorphism. By Proposition 3.1.6, the result follows. ☐

Proposition 3.3.8. *Let R be a regular (commutative Noetherian) ring. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of* A-mod$\cap R$-proj. *Let* $P\in A$-mod$\cap R$-proj. *Let $i\geq 1$. If $(A(\mathfrak{m}),P(\mathfrak{m}))$ is an $i-\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$ for every maximal ideal m of R, then (A,P) is an $i-\mathscr{R}(A)$ cover of B.*

Proof. $(A(\mathfrak{m}),P(\mathfrak{m}))$ is a $0-\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. By Proposition 3.3.7, (A,P) is a $0-\mathscr{R}(A)$ cover of B. We can assume, without loss of generality, that R is a local regular ring. Let $M\in\mathscr{R}(A)$. Let

$$\operatorname{Hom}_A(P,A)^\bullet\colon\ \cdots\to Q_1\to Q_0\to 0 \tag{3.3.0.7}$$

be a deleted complex chain obtained by deleting $\operatorname{Hom}_A(P,A)$ from a projective B-resolution of $\operatorname{Hom}_A(P,A)$. Consider the cochain complex $P^\bullet=\operatorname{Hom}_B(\operatorname{Hom}_A(P,A)^\bullet,\operatorname{Hom}_A(P,M))$. Note that each module in $\operatorname{Hom}_A(P,A)^\bullet$ is projective over B, so that each module in $P^\bullet$ belongs to $\operatorname{add}_R\operatorname{Hom}_A(P,M)$. In particular, each module in $P^\bullet$ is projective over R.

We claim that $\mathrm{R}^j G(FM)=0$, $1\leq j\leq i$. We shall prove it by induction on $\dim R$.

If $\dim R=0$, there is nothing to show. Assume that $\dim R>0$. Let $x\in\mathfrak{m}/\mathfrak{m}^2$. Then, $\dim(R/Rx)=\dim R-1$. $\mathfrak{m}/Rx$ is the unique maximal ideal of R/Rx and $R/Rx/\mathfrak{m}/Rx\simeq R/\mathfrak{m}$ as R-modules. Hence, for every $X\in A$-mod,

$$R/Rx\otimes_R X(\mathfrak{m}/Rx)=R/Rx/\mathfrak{m}/Rx\otimes_{R/Rx}R/Rx\otimes_R X\simeq R/\mathfrak{m}\otimes_R X=X(\mathfrak{m}). \tag{3.3.0.8}$$

Thus, $(R/Rx\otimes_R A(\mathfrak{m}/Rx),R/Rx\otimes_R P(\mathfrak{m}/Rx))=(A(\mathfrak{m}),P(\mathfrak{m}))$ is an i-$\mathscr{R}(A(\mathfrak{m}/Rx))$ cover of $R/Rx\otimes_R B(\mathfrak{m}/Rx)$. Denote by $F_x\dashv G_x$ the adjoint functors associated with this cover. Therefore, $R^jG_x(F_x(R/Rx\otimes_R M))=0$ for $1\leq j\leq i$.

Observe that $\operatorname{pdim}_R R/Rx\leq 1$. By Corollary 1.3.16, for each $1\leq j\leq i$, there exists an exact sequence

$$0\to R/Rx\otimes_R H^j(P^\bullet)\to H^j(R/Rx\otimes_R P^\bullet). \tag{3.3.0.9}$$

Note that, $H^j(R/Rx\otimes_R P^\bullet)=R^jG_x(F_x(R/Rx\otimes_R M))=0$ and $H^j(P^\bullet)=\operatorname{Ext}^j_B(FA,FM)$ for each $1\leq j\leq i$.

Consider the surjective map $R/Rx\to R/\mathfrak{m}$ induced by the canonical map $R\to R/\mathfrak{m}$. Applying, for each $1\leq j\leq i$, $\operatorname{Ext}^j_B(FA,FM)\otimes_R -$ yields that $\operatorname{Ext}^j_B(FA,FM)(\mathfrak{m})=0$, $1\leq j\leq i$. So, we conclude that $\operatorname{Ext}^j_B(FA,FM)=0$, for $1\leq j\leq i$. ☐

We shall now see that under some conditions truncating a cover, the quality of the cover drops at most by one.

Theorem 3.3.9. *Let R be a commutative Noetherian ring. Let I be an ideal of R such that* $I\in R$-proj *and $i\geq 0$. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of* A-mod$\cap R$-proj. *Let (A,P) be an i-$\mathscr{R}(A)$ cover of B. Then, $(R/I\otimes_R A,R/I\otimes_R P)$ is an $(i-1)$-$\mathscr{R}(R/I\otimes_R A)$ cover of $R/I\otimes_R B$.*

Proof. Denote by Q the commutative ring R/I. Consider the exact sequence

$$0 \to I \to R \to Q \to 0. \tag{3.3.0.10}$$

This exact sequence induces the fully faithful functor $H\colon Q\otimes_R A\text{-mod} \to A\text{-mod}$. Moreover, for every $M \in Q\otimes_R A\text{-mod}$, $Q\otimes_R HM \simeq HM/IHM = HM = M$. Hence, it is enough, to show that if $\eta^Q_{Q\otimes_R M}$ is an isomorphism (resp. a monomorphism) for every $M \in \mathscr{R}(A)$, then $(Q\otimes_R A, Q\otimes_R P)$ is a 0 (resp. -1)-$\mathscr{R}(Q\otimes_R A)$ cover of $Q\otimes_R B$.

Here, η^Q denotes the unit associated with the adjunction

$$F_Q := \operatorname{Hom}_{Q\otimes_R A}(Q\otimes_R P, -) \dashv \operatorname{Hom}_{Q\otimes_R B}(F_Q(Q\otimes_R A), -) := G_Q. \tag{3.3.0.11}$$

First, we will show that for every $M \in \mathscr{R}(A)$ we can relate η_M with $\eta^Q_{Q\otimes_R M}$.

Applying $-\otimes_R M$ and $GF(-\otimes_R M)$ to (3.3.0.10) yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I\otimes_R M \arrow[r] \arrow[d, "\eta_{I\otimes_R M}"] & M \arrow[r] \arrow[d, "\eta_M"] & R/I\otimes_R M \arrow[r] \arrow[d, "\eta_{Q\otimes_R M}"] & 0 \\
0 \arrow[r] & GF(I\otimes_R M) \arrow[r] & GFM \arrow[r] & GF(R/I\otimes_R M) \arrow[r] & \operatorname{Ext}^1_B(FA, F(I\otimes_R M))
\end{tikzcd} \tag{3.3.0.12}$$

with exact rows. Since $I \in R\text{-proj}$, $I\otimes_R M \in \operatorname{add}_A M$. Thus, $I\otimes_R M \in \mathscr{R}(A)$. Hence, $\eta_{I\otimes_R M}$ and η_M are isomorphisms. By Snake Lemma, $\eta_{Q\otimes_R M}$ is a monomorphism. If $\operatorname{Ext}^1_B(FA, F(I\otimes_R M)) = 0$, then $\eta_{Q\otimes_R M}$ is an isomorphism.

On the other hand, there are isomorphisms δ and ψ making the following diagram commutative:

$$\begin{tikzcd}
Q\otimes_R M \arrow[rr, "\eta_{Q\otimes_R M}"] \arrow[d, "\mu"] & & GF(Q\otimes_R M) \\
\operatorname{Hom}_{Q\otimes_R B}(Q\otimes_R FA, Q\otimes_R FM) \arrow[r, "\delta"] & \operatorname{Hom}_B(FA, Q\otimes_R FM) \arrow[ur, "\operatorname{Hom}_B(FA,\psi)"'] &
\end{tikzcd}. \tag{3.3.0.13}$$

By Lemma 1.1.32, δ is an isomorphism. By Proposition 1.1.33, ψ is an isomorphism. We define μ to be the $Q\otimes_R A$-homomorphism that maps $m\otimes q$ to the map $(q_1\otimes f \mapsto qq_1\otimes f(-)m)$. We claim that (3.3.0.13) is commutative. Let $m\in M,\ q\in Q,\ g\in FA,\ p\in P$. Then,

$$\operatorname{Hom}_B(FA,\psi)\delta\mu(q\otimes m)(g)(p) = \psi(\delta\mu(q\otimes m))(g)(p) = \psi(\mu(q\otimes m)(1_Q\otimes g))(p) \tag{3.3.0.14}$$

$$= \psi(q\otimes g(-)m)(p) = q\otimes g(p)m \tag{3.3.0.15}$$

$$= \eta_{Q\otimes_R M}(q\otimes m)(g)(p). \tag{3.3.0.16}$$

Finally, we shall relate μ with $\eta^Q_{Q\otimes_R M}$. There exists a commutative diagram

$$\begin{tikzcd}
Q\otimes_R M \arrow[r, "\mu"] \arrow[d, "\eta^Q_{Q\otimes_R M}"] & \operatorname{Hom}_{Q\otimes_R B}(Q\otimes_R FA, Q\otimes_R FM) \arrow[d, "\operatorname{Hom}_B(Q\otimes_R FA,\varphi_M)"] \\
\operatorname{Hom}_{Q\otimes_R B}(F_Q(Q\otimes_R A), F_Q(Q\otimes_R M)) \arrow[r, "\operatorname{Hom}_{Q\otimes_R B}(\varphi_A, F_Q(Q\otimes_R M))"'] & \operatorname{Hom}_{Q\otimes_R B}(Q\otimes_R FA, F_Q(Q\otimes_R M))
\end{tikzcd}, \tag{3.3.0.17}$$

where φ_X, $X \in A$-mod, is the canonical isomorphism $Q \otimes_R FX \to F_Q Q \otimes_R X$. In fact,

$$\begin{aligned}\mathrm{Hom}_{Q\otimes_R B}(\varphi_A, F_Q(Q\otimes_R M))\eta^Q_{Q\otimes_R M}(q_1\otimes m)(q_2\otimes f)(q_3\otimes p) &= \eta^Q_{Q\otimes_R M}(q_1\otimes m)(\varphi_A(q_2\otimes f))(q_3\otimes p)\\ &= \varphi_A(q_2\otimes f)(q_3\otimes p)(q_1\otimes m)\\ &= q_2q_3\otimes f(p)q_1\otimes m\\ &= q_1q_2q_3\otimes f(p)m\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_B(Q\otimes_R FA, \varphi_M)\mu(q_1\otimes m)(q_2\otimes f)(q_3\otimes p) &= \varphi_M(\mu(q_1\otimes m)(q_2\otimes f))(q_3\otimes p)\\ &= \varphi_M(q_1q_2\otimes f(-)m)(q_3\otimes p)\\ &= q_1q_2q_3\otimes f(-)m(p)\\ &= q_1q_2q_3\otimes f(p)m,\ m\in M, p\in P, f\in FA, q_1,q_2,q_3\in Q.\end{aligned}$$

Thus, using commutative diagrams (3.3.0.13) and (3.3.0.17) we deduce that $\eta^Q_{Q\otimes_R M}$ is a monomorphism. Further, $\eta^Q_{Q\otimes_R M}$ is an isomorphism if $\mathrm{R}^1 G(FM)=0$. So, the result follows for $i\in\{0,1\}$. Assume that $i\geq 1$. Then, $(Q\otimes_R A, Q\otimes_R P)$ is a 0-$\mathscr{R}(Q\otimes_R A)$ cover of $Q\otimes_R B$. The exact sequence (3.3.0.10) yields that $\mathrm{flatdim}_R Q\leq 1$. Let $FA^\bullet$ be a deleted projective B-resolution of FA and $M\in\mathscr{R}(A)$. By Corollary 1.3.16, for each $n\geq 0$, there exists an exact sequence

$$0\to H^n(\mathrm{Hom}_B(FA^\bullet, FM))\otimes_R Q\to H^n(\mathrm{Hom}_B(FA^\bullet, FM)\otimes_R Q)\to \mathrm{Tor}_1^R(H^{n+1}(\mathrm{Hom}_B(FA^\bullet, FM)), Q)\to 0.$$

Notice that $H^n(\mathrm{Hom}_B(FA^\bullet, FM)) = \mathrm{Ext}^n_B(FA, FM)=0,\ 1\leq n\leq i$. Hence,

$$\begin{aligned}0 &= H^n(\mathrm{Hom}_B(FA^\bullet, FM)\otimes_R Q) = H^n(\mathrm{Hom}_{Q\otimes_R B}(Q\otimes_R FA^\bullet, Q\otimes_R FM)) &\quad (3.3.0.18)\\ &= H^n(\mathrm{Hom}_{Q\otimes_R B}(F_Q(Q\otimes_R A)^\bullet, F_Q Q\otimes_R M)) = \mathrm{Ext}^n_{Q\otimes_R B}(F_Q Q\otimes_R A, F_Q Q\otimes_R M),\ 1\leq n\leq i-1. &\quad (3.3.0.19)\end{aligned}$$

It follows that $(Q\otimes_R A, Q\otimes_R P)$ is an $(i-1)$-$\mathscr{R}(Q\otimes_R A)$ cover of $Q\otimes_R B$. ☐

We can describe Theorem 3.3.9 not just for projective ideals of R but also for prime ideals of R in case R is a commutative Noetherian regular local ring.

Corollary 3.3.10. *Let R be a commutative Noetherian regular local ring. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of A-mod$\cap R$-proj. Let (A,P) be an i-$\mathscr{R}(A)$ cover of B for some integer $i\geq 0$. Then, $(R/\mathfrak{p}\otimes_R A, R/\mathfrak{p}\otimes_R P)$ is an $(i-\mathrm{ht}(\mathfrak{p}))$-$\mathscr{R}(R/\mathfrak{p}\otimes_R A)$ cover of $R/\mathfrak{p}\otimes_R B$ for every prime ideal $\mathfrak{p}$ of R with $\mathrm{ht}(\mathfrak{p})\leq i+1$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. Suppose that, for $n=\mathrm{ht}(\mathfrak{p})$,

$$0=\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_n=\mathfrak{p} \quad (3.3.0.20)$$

is the largest chain of distinct prime ideals that are contained in $\mathfrak{p}$. We will proceed by induction on $n=\mathrm{ht}(\mathfrak{p})$.

If $n=0$, there is nothing to show. Assume that $n>0$. By construction, $\mathrm{ht}(\mathfrak{p}_{n-1})=\mathrm{ht}(\mathfrak{p})-1=n-1$, or even, $\mathrm{ht}(\mathfrak{p}/\mathfrak{p}_{n-1})=1$. By induction, $(R/\mathfrak{p}_{n-1}\otimes_R A, R/\mathfrak{p}_{n-1}\otimes_R P)$ is an $(i-\mathrm{ht}(\mathfrak{p}_{n-1}))$-$\mathscr{R}(R/\mathfrak{p}_{n-1}\otimes_R A)$ cover of $R/\mathfrak{p}_{n-1}\otimes_R B$. On the other hand, $R/\mathfrak{p}_{n-1}$ is a local regular ring. Hence, $R/\mathfrak{p}_{n-1}$ is a unique factorization domain. Therefore, every prime ideal of height one is principal. So, $\mathfrak{p}/\mathfrak{p}_{n-1}=R/\mathfrak{p}_{n-1}x\in R/\mathfrak{p}_{n-1}$-proj for some $x\in R/\mathfrak{p}_{n-1}$. Note that, $i-\mathrm{ht}(\mathfrak{p}_{n-1})=i-\mathrm{ht}(\mathfrak{p})+1\geq i-i-1+1=0$. By Theorem 3.3.9,

$$(R/\mathfrak{p}\otimes_R A, R/\mathfrak{p}\otimes_R P) = (R/\mathfrak{p}_{n-1}/\mathfrak{p}/\mathfrak{p}_{n-1}\otimes_{R/\mathfrak{p}_{n-1}} R/\mathfrak{p}_{n-1}\otimes_R A, R/\mathfrak{p}_{n-1}/\mathfrak{p}/\mathfrak{p}_{n-1}\otimes_{R/\mathfrak{p}_{n-1}} R/\mathfrak{p}_{n-1}\otimes_R P)$$

is an $i-\mathrm{ht}(\mathfrak{p})$-$\mathscr{R}(R/\mathfrak{p}\otimes_R A)$ cover of $R/\mathfrak{p}\otimes_R B$. □

Now, we shall see that under some conditions we can obtain a reciprocal statement of Theorem 3.3.9. Furthermore, we want to establish, similar to Rouquier's work, that covers might improve by increasing the Krull dimension of the ground ring.

Theorem 3.3.11. *Let R be a commutative Noetherian regular ring with Krull dimension at least one. Let $i\geq 0$. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of A-mod$\cap R$-proj. Let $P\in A$-mod$\cap R$-proj. Assume that $(K\otimes_R A, K\otimes_R P)$ is an $i+1-\mathscr{R}(K\otimes_R A)$ cover of $K\otimes_R B$ for some Noetherian commutative flat R-algebra K. If $(A(\mathfrak{m}),P(\mathfrak{m}))$ is an $i-\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R, then (A,P) is a $(1+i)-\mathscr{R}(A)$ cover of B.*

Proof. We can assume, without loss of generality, that R is a local commutative Noetherian regular ring. By Proposition 3.3.8, (A,P) is an $i-\mathscr{R}(A)$ cover of B. Let $M\in\mathscr{R}(A)$. It is enough to show that $\mathrm{R}^{i+1}G(FM)=0$. Hence, we want to show that the annihilator of $\mathrm{R}^{i+1}G(FM)$ is R. Assume, by contradiction, that $\mathrm{Ann}_R\,\mathrm{R}^{i+1}G(FM)=0$. In particular, $\mathrm{R}^{i+1}G(FM)$ is a faithful R-module. Thus, there exists an exact sequence

$$0\to R\to\bigoplus_I \mathrm{R}^{i+1}G(FM), \tag{3.3.0.21}$$

for some set (possibly infinite) I. Since K is flat over R we obtain a monomorphism $K\to\bigoplus_I K\otimes_R \mathrm{R}^{i+1}G(FM)$. On the other hand, as $K\otimes_R M\in\mathscr{R}(K\otimes_R A)$,

$$K\otimes_R \mathrm{R}^{i+1}G(FM)\simeq \mathrm{Ext}^{i+1}_{K\otimes_R B}(K\otimes_R FA, K\otimes_R FM)\simeq \mathrm{Ext}^{i+1}_{K\otimes_R B}(F_K(K\otimes_R A), F_K(K\otimes_R M))=0. \tag{3.3.0.22}$$

Here, F_K denotes the functor $\mathrm{Hom}_{K\otimes_R A}(K\otimes_R P,-)$. This would imply that $K=0$. Hence, $\mathrm{R}^{i+1}G(FM)$ cannot be R-faithful. Moreover, there exists a non-zero divisor $x\in R$ such that

$$\mathrm{R}^{i+1}G(FM)[x]:=\{y\in \mathrm{R}^{i+1}G(FM)\colon xy=0\}=\mathrm{R}^{i+1}G(FM). \tag{3.3.0.23}$$

Observe that if $x_1x_2y=0$, then $x_2y\in \mathrm{R}^{i+1}G(FM)[x_1]$ where $y\in \mathrm{R}^{i+1}G(FM)$ and x_1 and x_2 belong to the unique maximal ideal m. Thus, we can assume without loss of generality, that the element x given in (3.3.0.23) belongs to $\mathfrak{m}\backslash\mathfrak{m}^2$. Furthermore, $\mathfrak{m}/Rx$ is the unique maximal ideal of R/Rx so that $(R/Rx\otimes_R A(\mathfrak{m}/Rx), R/Rx\otimes_R P(\mathfrak{m}/Rx))=(A(\mathfrak{m}),P(\mathfrak{m}))$ is an i-$\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$. Therefore, $(R/Rx\otimes_R A, R/Rx\otimes_R P)$ is an i-$\mathscr{R}(R/Rx\otimes_R A)$ cover of $R/Rx\otimes_R B$. Denote by F_x and G_x, with $F_x\dashv G_x$, the adjoint functors associated with this cover. Let

$$\mathrm{Hom}_A(P,A)^\bullet\colon\ \cdots\to Q_1\to Q_0\to 0 \tag{3.3.0.24}$$

be a deleted complex chain obtained by deleting $\mathrm{Hom}_A(P,A)$ from a projective B-resolution of $\mathrm{Hom}_A(P,A)$.

Observe that $\mathrm{pdim}_R R/Rx\leq 1$, so applying $P^\bullet:=\mathrm{Hom}_B(\mathrm{Hom}_A(P,A)^\bullet,\mathrm{Hom}_A(P,M))$ on Corollary 1.3.16 yields exact sequences

$$0\to R/Rx\otimes_R H^n(P^\bullet)\to H^n(R/Rx\otimes_R P^\bullet)\to \mathrm{Tor}_1^R(H^{n+1}(P^\bullet),R/Rx)\to 0,\ \forall n\geq 0. \tag{3.3.0.25}$$

First, assume that $i>0$. Then, $H^i(R/Rx\otimes_R P^\bullet)=\mathrm{R}^iG_x(F_xR/Rx\otimes_R M)=0$. So,

$$\mathrm{R}^{i+1}G(FM)=\mathrm{R}^{i+1}G(FM)[x]=\mathrm{Tor}_1^R(H^{i+1}(P^\bullet),R/Rx)=0. \tag{3.3.0.26}$$

Now, assume that $i=0$. We need to proceed by induction on the Krull dimension of R. If $\dim R=1$, then

$Rx = \mathfrak{m}$. As $R/\mathfrak{m}$ is a field and

$$R/\mathfrak{m} \otimes_R H^0(P^\bullet) = R/\mathfrak{m} \otimes_R GFM \simeq M(\mathfrak{m}) \simeq G_x F_x(M(\mathfrak{m})) = H^0(R/\mathfrak{m} \otimes_R P^\bullet) \tag{3.3.0.27}$$

the exact sequence (3.3.0.25) yields that

$$\mathrm{R}^1 G(FM) = \mathrm{R}^1 G(FM)[x] = \mathrm{Tor}_1^R(H^1(P^\bullet), R/\mathfrak{m}) = 0. \tag{3.3.0.28}$$

Assume that the result holds for all rings with Krull dimension less than t. Let R have Krull dimension t. The Krull dimension of R/Rx is $t-1$. By induction, $\mathrm{R}^1 G_x(F_x R/Rx \otimes_R M) = 0$. The exact sequence (3.3.0.25) implies that $0 = R/Rx \otimes_R H^1(P^\bullet) = R/Rx \otimes_R \mathrm{R}^1 G(FM)$. Applying the functor $\mathrm{R}^1 G(FM) \otimes_R -$ on the surjective map $R/Rx \to R/\mathfrak{m}$ we get that $\mathrm{R}^1 G(FM)(\mathfrak{m}) = 0$. Thus, $\mathrm{R}^1 G(FM) = 0$. This completes the proof. ☐

We remark that Proposition 4.42 of [Rou08] is a particular case of Theorem 3.3.11 by fixing $\mathscr{R}(A) = \mathscr{F}(\tilde{\Delta})$ and $i = 1$. To illustrate, recall that for a flat K R-algebra with $\operatorname{gldim} K \otimes_R A = 0$, every module in $K \otimes_R A$-mod is projective over $K \otimes_R A$. So, $\mathscr{R}(K \otimes_R A) = K \otimes_R A$-mod. By Proposition 3.3.4, $(K \otimes_R A, K \otimes_R P)$ is a 0-$\mathscr{R}(K \otimes_R A)$-cover of $K \otimes_R B$. By Lemma 1.4.27, the functor $\mathrm{Hom}_{K \otimes_R A}(K \otimes_R P, -)\colon K \otimes_R A\text{-mod} \to K \otimes_R B\text{-mod}$ is full and faithful. Consequently, it is an equivalence of categories.

With Theorem 3.3.11, it is natural to ask how much better can a deformed cover be comparing to covers of finite-dimensional algebras.

Example 4.6.3 shows that the assumption on Theorem 3.3.11 is not enough to increase the quality of a deformed cover more than one. This motivates the introduction of flat R-sequences.

3.3.1 Flat R-sequences

Assume, until the end of this section, that R is a local regular commutative Noetherian ring. Let $\mathfrak{m}$ be the unique maximal ideal of R.

Recall that an **R-sequence** of size t is an ordered sequence $\{x_1, \dots, x_t\} \subset \mathfrak{m}$ such that x_1 is a non-zero divisor of R and for $i > 1$ each x_i is not a zero divisor on the module $R/(x_1, \dots, x_{i-1})$. Note that $t \leq \dim R$ and every R-sequence can be extended to a maximal R-sequence of size $\dim R$. This is a well-known fact of the theory of regular rings. In fact, for every non-zero divisor x, $\dim R/Rx = \dim R - 1$.

Given an R-sequence $\mathbf{x} = \{x_1, \dots, x_t\}$ of size $t \geq 1$, we will denote by $R_l^{\mathbf{x}}$ the ring $R/(x_1, \dots, x_l)$, $1 \leq l \leq t$ and $R_0^{\mathbf{x}}$ will denote R. For each R-sequence $\mathbf{x}$ of size t, we can construct an ordered sequence $\{K_0^{\mathbf{x}}, \cdots, K_t^{\mathbf{x}}\}$ so that $K_l^{\mathbf{x}}$ is a flat commutative Noetherian $R_l^{\mathbf{x}}$-algebra, $0 \leq l \leq t$. We will call $\{K_0^{\mathbf{x}}, \cdots, K_t^{\mathbf{x}}\}$ a **flat R-sequence** of $\mathbf{x}$.

Theorem 3.3.12. *Let R be a local regular commutative Noetherian ring with unique maximal ideal $\mathfrak{m}$. Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of A-mod$\cap R$-proj. Let $P \in A$-mod$\cap R$-proj and $i \geq 0$, $0 \leq j \leq \dim R - 1$ be integers. Assume that for every R-sequence $\mathbf{x}$ of size j there exists a flat R-sequence of $\mathbf{x}$ making $(K_l^{\mathbf{x}} \otimes_R A, K_l^{\mathbf{x}} \otimes_R P)$ an $(i+j+1-l)$-$\mathscr{R}(K_l^{\mathbf{x}} \otimes_R A)$ cover of $K_l^{\mathbf{x}} \otimes_R B$ for $0 \leq l \leq j$.*

If $(A(\mathfrak{m}), P(\mathfrak{m}))$ is an i-$\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$, then (A, P) is an $(i+j+1)$-$\mathscr{R}(A)$ cover of B.

Proof. We claim that for every R-sequence $\mathbf{x}$ of size j, $(R_l^{\mathbf{x}} \otimes_R A, R_l^{\mathbf{x}} \otimes_R P)$ is an $(i+j+1-l)$-$\mathscr{R}(R_l^{\mathbf{x}} \otimes_R A)$ cover of $R_l^{\mathbf{x}} \otimes_R B$, $0 \leq l \leq j$.

We shall proceed by induction on $t := j - l$. If $t = 0$, then $l = j$. Let $\mathbf{x}$ be an R-sequence of size j. The unique maximal ideal of $R_j^{\mathbf{x}} = R/(x_1, \dots, x_j)$ is $\mathfrak{m}/(x_1, \dots, x_j)$ and

$$R_j^{\mathbf{x}} \otimes_R M(\mathfrak{m}/(x_1, \dots, x_j)) \simeq R/\mathfrak{m} \otimes_R M, \ \forall M \in R\text{-mod}. \tag{3.3.1.1}$$

Thus, the cover $(R_j^{\mathbf{x}} \otimes_R A, R_j^{\mathbf{x}} \otimes_R P)$ is in the conditions of Theorem 3.3.11. So, $(R_j^{\mathbf{x}} \otimes_R A, R_j^{\mathbf{x}} \otimes_R P)$ is an $(i+1)$-$\mathscr{R}(R_j^{\mathbf{x}}) \otimes_R A)$ cover of $R_j^{\mathbf{x}} \otimes_R B$.

Assume now that the claim holds for a given $t > 0$. We shall prove it for $t+1$. Let $\mathbf{x}$ be an R-sequence of size j. By induction, $(R_{j-t}^{\mathbf{x}} \otimes_R A, R_{j-t}^{\mathbf{x}} \otimes_R P)$ is an $(i+j+1-(j-t)) = (i+1+t)$-$\mathscr{R}(R_{j-t}^{\mathbf{x}} \otimes_R A)$ cover of $R_{j-t}^{\mathbf{x}} \otimes_R B$. Denote by $F_{j-t-1}^{\mathbf{x}} \dashv G_{j-t-1}^{\mathbf{x}}$ the usual adjoint functors of the cover $(R_{j-t-1}^{\mathbf{x}} \otimes_R A, R_{j-t-1}^{\mathbf{x}} \otimes_R P)$.

Since $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a 0-$\mathscr{R}(A(\mathfrak{m}))$ cover of $B(\mathfrak{m})$, it follows that $(R_{j-t-1}^{\mathbf{x}} \otimes_R A, R_{j-t-1}^{\mathbf{x}} \otimes_R P)$ is a 0-$\mathscr{R}(R_{j-t-1}^{\mathbf{x}} \otimes_R A)$ cover of $R_{j-t-1}^{\mathbf{x}} \otimes_R B$.

Note that

$$R_{j-t-1}^{\mathbf{x}} / R_{j-t-1}^{\mathbf{x}}(x_{j-t} + (x_1, \cdots, x_{j-t-1})) \simeq R/(x-1, \cdots, x_{j-t-1}, x_{j-t}) = R_{j-t}^{\mathbf{x}}. \tag{3.3.1.2}$$

In fact, the isomorphisms are the maps induced by the canonical maps

$$R \to R/(x_1, \cdots, x_{j-t-1}) \to R/(x_1, \cdots, x_{j-t-1})/(R/(x_1, \cdots, x_{j-t-1}))(x_{j-t} + (x_1, \cdots, x_{j-t-1})), \tag{3.3.1.3}$$

$$R \to R/(x_1, \cdots, x_{j-t-1}, x_j). \tag{3.3.1.4}$$

Consequently, for every $M \in \mathscr{R}(R_{j-t-1}^{\mathbf{x}} \otimes_R A)$, we have

$$R_{j-t}^{\mathbf{x}} \otimes_{R_{j-t-1}^{\mathbf{x}}} M \in \mathscr{R}(R_{j-t}^{\mathbf{x}} \otimes_{R_{j-t-1}^{\mathbf{x}}} R_{j-t-1}^{\mathbf{x}} \otimes_R A) = \mathscr{R}(R_{j-t}^{\mathbf{x}} \otimes_R A). \tag{3.3.1.5}$$

By Corollary 1.3.16, for each $1 \leq n \leq i+1+t$,

$$\mathrm{R}^n G_{j-t-1}^{\mathbf{x}}(F_{j-t-1}^{\mathbf{x}} M) \otimes_{R_{j-t-1}^{\mathbf{x}}} R_{j-t}^{\mathbf{x}} = 0, \quad \forall M \in \mathscr{R}(R_{j-t-1}^{\mathbf{x}} \otimes_R A). \tag{3.3.1.6}$$

There exists a surjective map $R_{j-t}^{\mathbf{x}} \to R_{j-t-1}^{\mathbf{x}}/(\mathfrak{m}/(x_1, \cdots, x_{j-t-1}))$. For each $M \in \mathscr{R}(R_{j-t-1}^{\mathbf{x}} \otimes_R A)$ and each $1 \leq n \leq i+1+t$, applying the functor $\mathrm{R}^n G_{j-t-1}^{\mathbf{x}}(F_{j-t-1}^{\mathbf{x}}) \otimes_{R_{j-t-1}^{\mathbf{x}}} -$ yields that $\mathrm{R}^n G_{j-t-1}^{\mathbf{x}} F_{j-t-1}^{\mathbf{x}} M = 0$. Now we can use the same argument as in the last part of the proof of Theorem 3.3.11 (replacing K by $K_{j-t-1}^{\mathbf{x}}$) to deduce that

$$\mathrm{R}^{i+2+t} G_{j-t-1}^{\mathbf{x}} F_{j-t-1}^{\mathbf{x}} M[x_j + (x_1, \cdots, x_{j-t-1})] = 0 \tag{3.3.1.7}$$

and $\mathrm{R}^{i+2+t} G_{j-t-1}^{\mathbf{x}} F_{j-t-1}^{\mathbf{x}} M$ is not faithful. Since x_j is arbitrary, this shows that $(R_{j-t-1}^{\mathbf{x}} \otimes_R A, R_{j-t-1}^{\mathbf{x}} \otimes_R P)$ is an $(i+j+1-(j-t-1))$-$\mathscr{R}(R_{j-t-1}^{\mathbf{x}} \otimes_R A)$ cover of $R_{j-t-1}^{\mathbf{x}} \otimes_R B$. Hence, the claim follows. Now the statement follows applying $t = j$. ☐

3.3.2 Quality of a cover and the spectrum of the ground ring

In the same spirit of Theorem 3.3.11, we can obtain a converse statement for Corollary 3.3.10.

Theorem 3.3.13. *Let R be a local commutative regular Noetherian ring with quotient field K. Suppose that (A, P) is a 0-$\mathscr{R}(A)$ cover of B for some resolving subcategory $\mathscr{R}(A)$ of* A-mod$\cap R$-proj. *Let $i \geq 0$. Assume that the following conditions hold:*

(i) $(K \otimes_R A, K \otimes_R P)$ is an $i+1$-$\mathscr{R}(K \otimes_R A)$ cover of $K \otimes_R B$;

(ii) For each prime ideal $\mathfrak{p}$ of height one, $(R/\mathfrak{p} \otimes_R A, R/\mathfrak{p} \otimes_R P)$ is an i-$\mathscr{R}(R/\mathfrak{p} \otimes_R A)$ cover of $R/\mathfrak{p} \otimes_R B$.

Then, (A, P) is an $i+1$-$\mathscr{R}(A)$ cover of B.

Proof. Let $1 \leq j \leq i+1$ and let $M \in \mathscr{R}(A)$. Denote by F_K and G_K the adjoint functors associated with the cover

$(K\otimes_R A, K\otimes_R P)$ and denote by $F_\mathfrak{p}$ and $G_\mathfrak{p}$ the adjoint functors associated with the cover $(R/\mathfrak{p}\otimes_R A, R/\mathfrak{p}\otimes_R P)$, for each prime ideal $\mathfrak{p}$ of R. Assumption (i) implies that

$$K\otimes_R \mathrm{R}^j G(FM)\simeq \mathrm{R}^j G_K(F_K K\otimes_R M)=0. \tag{3.3.2.1}$$

Hence, $\mathrm{R}^j G(FM)$ cannot be R-faithful. Moreover, for each $1\leq j\leq i+1$, there exists a non-zero divisor $x_j\in \mathfrak{m}/\mathfrak{m}^2$ such that

$$\mathrm{R}^j G(FM)[x_j]=\mathrm{R}^j G(FM), \tag{3.3.2.2}$$

where $\mathfrak{m}$ is the unique maximal ideal of R. Since $x_j\in\mathfrak{m}/\mathfrak{m}^2$, R/Rx_j is an integral domain of Krull dimension $\dim R-1$. So, Rx_j is a prime ideal of height one. Like before, applying $P^\bullet:=\mathrm{Hom}_B(\mathrm{Hom}_A(P,A)^\bullet,\mathrm{Hom}_A(P,M))$ on Corollary 1.3.16 we get exact sequences

$$0\to R/Rx_j\otimes_R H^n(P^\bullet)\to H^n(R/Rx_j\otimes_R P^\bullet)\to \mathrm{Tor}_1^R(H^{n+1}(P^\bullet),R/Rx_j)\to 0,\ \forall n\geq 0. \tag{3.3.2.3}$$

Using now assumption (ii) it follows that $H^{j-1}(R/Rx_j\otimes_R P^\bullet)=0$ for $i\geq j>1$. So, $\mathrm{R}^j G(FM)=0$ for $2\leq j\leq i+1$. The case $j=1$ requires a little more work. For each $x\in\mathfrak{m}/\mathfrak{m}^2$, consider the exact sequence

$$0\to R\to R\to R/Rx\to 0, \tag{3.3.2.4}$$

where the first map is multiplication by x. Since $M\in R$-proj, we get an exact sequence

$$0\to M\to M\to M/xM\to 0, \tag{3.3.2.5}$$

where the first map is multiplication by x. Denote by π the projection $M\to M/xM$. Applying $\mathrm{Hom}_B(FA,F-)$ yields a long exact sequence

$$GFM\to \mathrm{Hom}_B(FA,F(M/xM))\to \mathrm{R}^1 G(FM)\to \mathrm{R}^1 G(FM). \tag{3.3.2.6}$$

By Lemma 1.1.32 and Proposition 1.1.33, there exists a commutative diagram

$$\begin{tikzcd} GFM \arrow[r, "\mathrm{Hom}_B(FA,F\pi)"] & \mathrm{Hom}_B(FA,F(M/xM)) \arrow[r, "\simeq"] & G_{Rx}F_{Rx}(M/xM) \\ M \arrow[u, "\eta_M"] \arrow[rr, "\pi"] & & M/xM \arrow[u, "\eta^{Rx}_{M/xM}", "\simeq"'] \end{tikzcd}. \tag{3.3.2.7}$$

Thus, $\mathrm{Hom}_B(FA,F\pi)$ is surjective. By exactness of (3.3.2.6), the map $\mathrm{R}^1 G(FM)\to\mathrm{R}^1 G(FM)$ is injective. Since this map is given by multiplication by x, its kernel is $\mathrm{R}^1 G(FM)[x]=0$. As discussed before, $0=\mathrm{R}^1 G(FM)[x]=\mathrm{R}^1 G(FM)$. Thus, the result follows. ☐

In Example 4.6.4 we can see that the assumptions on Theorem 3.3.13 are optimal.

Observe that the arguments used in the proof of Theorems 3.3.11 and 3.3.8 remain valid if we are interested only in a given module $M\in\mathscr{R}(A)$. Hence, the following corollary follows.

Corollary 3.3.14. *Let $(S\otimes_R A, S\otimes_R P)$ be a cover of $S\otimes_R B$ for any commutative Noetherian R-algebra and $M\in A$-mod$\cap R$-proj. Assume that the following conditions hold.*

1. *The unit $\eta_M\colon M\to GFM$ is an isomorphism;*
2. *$\mathrm{Ext}^j_{B(\mathfrak{m})}(FA(\mathfrak{m}),FM(\mathfrak{m}))=0$ for every maximal ideal $\mathfrak{m}$ of R, where $1\leq j\leq i$ for some $i\geq 0$.*

Then, $\operatorname{Ext}_B^j(FA, FM) = 0$ *for all* $1 \leq j \leq i$. *If, in addition,* $\dim R \leq 1$ *and there exists a Noetherian commutative R-algebra K such that* $\operatorname{Ext}_{K\otimes_R B}^{i+1}(K \otimes_R FA, K \otimes_R FM) = 0$*, then* $\operatorname{Ext}_B^j(FA, FM) = 0$ *for all* $1 \leq j \leq i+1$.

3.4 Truncation of split quasi-hereditary covers

In the previous section, we saw how the quality of a cover relates with the quality of a cover under change of the ground to specific quotient rings. A similar question may be posed to the setup of split quasi-hereditary algebras. More precisely, given a split heredity ideal of A, how covers involving A are related to covers involving the split quasi-hereditary algebra A/J? The following result shows that contrary to Theorem 3.3.9, truncating a split quasi-hereditary cover will induce a new cover with at least the same quality as the original one.

Theorem 3.4.1. *Let A be a split quasi-hereditary Noetherian R-algebra. Assume that* (A, P) *is an* $i-\mathscr{F}(\tilde{\Delta})$ *cover of* $\operatorname{End}_A(P)^{op}$ *for some integer* $i \geq 0$. *Let J be a split heredity ideal of A. Then,* $(A/J, P/JP)$ *is an* $i - \mathscr{F}(\tilde{\Delta}_J)$ *cover of* $\operatorname{End}_{A/J}(P/JP)^{op}$*, where* $\mathscr{F}(\tilde{\Delta}_J) = \mathscr{F}(\tilde{\Delta}) \cap A/J$-mod*.*

Proof. Denote by B the endomorphism algebra $\operatorname{End}_A(P)^{op}$. The map $A \twoheadrightarrow A/J$ induces the fully faithful functor A/J-mod $\to A$-mod. Hence, $\operatorname{End}_{A/J}(P/JP)^{op} \simeq \operatorname{End}_A(P/JP)^{op}$. We wish to express $\operatorname{End}_{A/J}(P/JP)^{op}$ as a quotient of B. To see this, consider the exact sequence of (A,B)-bimodules

$$0 \to JP \to P \to P/JP \to 0. \tag{3.4.0.1}$$

Applying $\operatorname{Hom}_A(P, -)$ yields the exact sequence

$$0 \to \operatorname{Hom}_A(P, JP) \to B \to \operatorname{Hom}_A(P, P/JP) \to 0, \tag{3.4.0.2}$$

while applying $\operatorname{Hom}_A(-, P/JP)$ yields the exact sequence

$$0 \to \operatorname{End}_A(P/JP) \to \operatorname{Hom}_A(P, P/JP) \to \operatorname{Hom}_A(JP, P/JP). \tag{3.4.0.3}$$

Thanks to $J = J^2$ we have $\operatorname{Hom}_A(JP, X) = 0$ for every $X \in A/J$-mod. Combining (3.4.0.3) with (3.4.0.2) we obtain the exact sequence

$$0 \to \operatorname{Hom}_A(P, JP) \to B \to \operatorname{End}_{A/J}(P/JP) \to 0. \tag{3.4.0.4}$$

Since (3.4.0.1) is exact as (A,B)-bimodules the latter is exact as (B,B)-bimodules. Denote by B_J the endomorphism algebra $\operatorname{End}_{A/J}(P/JP)^{op}$. By the previous argument, the functor B_J-mod $\to B$-mod is fully faithful. Denote by G_J the functor $\operatorname{Hom}_{B_J}(\operatorname{Hom}_{A/J}(P/JP, A/J), -) = \operatorname{Hom}_B(\operatorname{Hom}_{A/J}(P/JP, A/J), -)\colon B_J\text{-mod} \to A/J\text{-mod}$ and $F_J = \operatorname{Hom}_{A/J}(P/JP, -) = \operatorname{Hom}_A(P/JP, -)\colon A/J\text{-mod} \to B_J\text{-mod}$.

To assert that the truncated cover is a $0-\mathscr{F}(\tilde{\Delta})$ cover it is enough to compare the restrictions of the functors F and $G \circ F$ to $\mathscr{F}(\tilde{\Delta}) \cap A/J$-mod with the restriction of the functors F_J and $G_J \circ F_J$ to $\mathscr{F}(\tilde{\Delta}_J)$, respectively. For each $X \in A/J$-mod, applying $\operatorname{Hom}_A(-, X)$ to (3.4.0.1) instead of $\operatorname{Hom}_A(-, P/JP)$ yields that $F_J X \simeq FX$. By applying $\operatorname{Hom}_B(-, FX)$ to $0 \to FJ \to FA \to F(A/J) \to 0$ we obtain the exact sequence $0 \to G_J F_J X \to GFX \to \operatorname{Hom}_B(FJ, FX)$. Fixing $X \in \mathscr{F}(\tilde{\Delta}_J)$ we obtain that $\operatorname{Hom}_B(FJ, FX) \simeq \operatorname{Hom}_A(J, X) = 0$ since (A, P) is a $0-\mathscr{F}(\tilde{\Delta})$ cover of B. These isomorphisms are functorial, so if we denote by η^J the unit of the adjunction $F_J \dashv G_J$, then η_X^J is an isomorphism for every $X \in \mathscr{F}(\tilde{\Delta}_J)$. This shows that $(A/J, P/JP)$ is a $0-\mathscr{F}(\tilde{\Delta}_J)$ cover of B_J.

Our aim now is to compute $\mathrm{R}^j G_J(F_J X)$ for $j \leq i$ and every $X \in \mathscr{F}(\tilde{\Delta}_J)$. Hence, fix an arbitrary $X \in \mathscr{F}(\tilde{\Delta}_J)$. Applying $\operatorname{Hom}_B(-, FX)$ to (3.4.0.2) we obtain $\operatorname{Ext}_B^1(B_J, FX) = 0$ and $\operatorname{Ext}_B^l(B_J, FX) \simeq \operatorname{Ext}_B^{l-1}(FJP, FX)$ for every

$l > 1$. Observe that $JP \simeq J \otimes_A P$ as left A-modules since $P \in A$-proj. Moreover, $JP \in \operatorname{add}_A J$, and thus it is projective as left A-module. Thus, $\operatorname{Ext}_B^l(B_J, FX) \simeq \operatorname{Ext}_A^{l-1}(JP, X) = 0$ for every $0 < l-1 \leq i$. Hence, $\operatorname{Ext}_B^l(B_J, FX) = 0$ for every $1 \leq l \leq i+1$. Let $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to F_J A/J \to 0$ be a projective B_J-resolution of $F_J A/J$. Denote by Ω^{j+1} the kernel of $P_j \to P_{j-1}$, with $P_{-1} = \Omega^0 = F_J A/J$. Note that $\operatorname{Ext}_B^l(P_j, FX) = 0$ for $1 \leq l \leq i+1$. Taking into account that B_J-mod $\to B$-mod is a fully faithful functor, applying $\operatorname{Hom}_B(-, FX)$ and $\operatorname{Hom}_{B_J}(-, FX)$ to the B_J projective resolution of FA/J yields

$$\operatorname{Ext}_B^l(\Omega^j, FX) \simeq \operatorname{Ext}_B^{l+1}(\Omega^{j-1}, FX), \quad \operatorname{Ext}_{B_J}^s(\Omega^j, FX) \simeq \operatorname{Ext}_{B_J}^{s+1}(\Omega^{j-1}, FX), \tag{3.4.0.5}$$

for $1 \leq l \leq i$, $s,\ j \geq 1$ and the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_B(P_j, F_J X) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_B(\Omega^{j+1}, F_J X) \arrow[r] \arrow[d, equal] & \operatorname{Ext}_B^1(\Omega^j, F_J X) \arrow[r] & 0 \\ \operatorname{Hom}_{B_J}(P_j, F_J X) \arrow[r] & \operatorname{Hom}_{B_J}(\Omega^{j+1}, F_J X) \arrow[r] & \operatorname{Ext}_{B_J}^1(\Omega^j, F_J X) \arrow[r] & 0 \end{tikzcd}. \tag{3.4.0.6}$$

By the commutative diagram, $\operatorname{Ext}_B^1(\Omega^j, FX)$ is zero if and only if $\operatorname{Ext}_{B_J}^1(\Omega^j, FX)$ is zero. By assumption and the previous discussion, for each $1 \leq l \leq i$,

$$0 = \operatorname{Ext}_B^l(FA/J, FX) = \operatorname{Ext}_B^l(\Omega^0, FX) \simeq \operatorname{Ext}_B^1(\Omega^{l-1}, FX) = \operatorname{Ext}_{B_J}^1(\Omega^{l-1}, F_J X) \simeq \operatorname{Ext}_{B_J}^l(F_J A/J, F_J X). \tag{3.4.0.7}$$

This concludes the proof. ☐

Remark 3.4.2. The module P/JP might not be injective even if P is. △

Remark 3.4.3. It follows from the proof of Theorem 3.4.1 that if (A, P) is a cover of B such that $(A/J, P/JP)$ is a $0 - \mathscr{F}(\tilde{\Delta}_J)$ cover of B_J, then (A, P) is a $(-1) - \mathscr{F}(\tilde{\Delta})$ cover of B. △

This gives another reason to be interested in zero faithful split quasi-hereditary covers. These are exactly the covers for which double centralizer properties occur in every step of the split heredity chain. In particular, this gives another perspective on why zero faithful split quasi-hereditary covers possess so much nicer properties compared to the minus one faithful case.

In Example 4.6.11, we can see that the improvement of the quality of a truncated cover with respect to the original might be dramatic for trivial reasons.

3.5 Relative dominant dimension and covers

Definition 3.5.1. Let (A, P) be a cover of $B = \operatorname{End}_A(P)^{op}$. Let $\mathscr{A}$ be a resolving subcategory of A-mod$\cap R$-proj. The **Hemmer–Nakano dimension** of $\mathscr{A}$ (with respect to P) is the maximal number n such that (A, P) is an $n - \mathscr{A}$ cover of B. We will denote by $\operatorname{HNdim}_F \mathscr{A}$, where F denotes the functor $\operatorname{Hom}_A(P, -)$.

We also say the Hemmer–Nakano dimension of $\mathscr{A}$ (with respect to the functor $F = \operatorname{Hom}_A(P, -)$). When there is no confusion about the functor F, we will just call $\operatorname{HNdim}_F \mathscr{A}$ the Hemmer-Nakano dimension of $\mathscr{A}$. If (A, P) is not a $(-1) - \mathscr{A}$ cover of B, then we say that the Hemmer-Nakano dimension of $\mathscr{A}$ (with respect to P) is $-\infty$.

Proposition 3.5.2. *Let (A, P, V) be a RQF3 algebra over a commutative Noetherian ring R. If* $\operatorname{domdim}(A, R) \geq 2$, *then $(A, \operatorname{Hom}_A(V, A))$ is a cover of $B := \operatorname{End}_A(V)$.*

Proof. Since $\operatorname{domdim}(A,R) \geq 2$ then $\alpha_A \colon A \to \operatorname{Hom}_B(V,V)$ is an isomorphism (see Section 2.4). By Lemma 1.4.28, it follows that $(A, \operatorname{Hom}_A(V,A))$ is a cover of $B = \operatorname{End}_A(\operatorname{Hom}_A(V,A))^{op} \simeq \operatorname{End}_A(V)$. $\square$

Remark 3.5.3. It is essential to consider the projective $\operatorname{Hom}_A(V,A)$ instead of P. Indeed, in Example 4.6.2, we see that there are examples of algebras with dominant dimension two with a projective-injective-faithful module P but the pair (A,P) fails to be a cover of $\operatorname{End}_A(P)^{op}$. $\triangle$

Given Remark 3.5.3, we could ask in what situations (A,P) is a cover of $\operatorname{End}_A(P)^{op}$ for a given (A,P,V) RQF3 algebra. It turns out that this property characterizes Morita algebras.

Theorem 3.5.4. *[Cru21, Theorem 1] Let R be a field. Let (A,P,V) be a QF3 k-algebra. Then, (A,P) is a cover of* $\operatorname{End}_A(P)^{op}$ *if and only if A is a Morita algebra.*

Proof. Assume that A is a Morita algebra. By Theorem 2.9.1, $\operatorname{add} DA \otimes_A P = \operatorname{add} P$. Consequently, $\operatorname{add} \operatorname{Hom}_A(P,A) = \operatorname{add} DP$. Then, $(A, DA \otimes_A P, \operatorname{Hom}_A(P,A))$ is a QF3 algebra. Therefore, in view of Proposition 2.3.7,

$$\operatorname{Hom}_A(P,A) - \operatorname{domdim}_{(A,R)} A_A = DP - \operatorname{domdim}_{(A,R)} A_A = \operatorname{domdim}(A,R) \geq 2. \tag{3.5.0.1}$$

Hence, $\operatorname{End}_B(\operatorname{Hom}_A(P,A))^{op} \simeq A$. This shows that (A,P) is a cover of B.

Conversely, suppose that (A,P) is a cover of $B := \operatorname{End}_A(P)^{op}$. By assumption, there exists a double centralizer property on $\operatorname{Hom}_A(P,A)$. More precisely,

$$\operatorname{End}_A(\operatorname{Hom}_A(P,A)) \simeq B \quad \operatorname{End}_B(\operatorname{Hom}_A(P,A))^{op} \simeq A. \tag{3.5.0.2}$$

In particular, $\operatorname{Hom}_A(P,A)$ is faithful projective as right A-module. Hence, there exists an injective A-homomorphism $A \to \operatorname{Hom}_A(P,A)^s$ for some $s > 0$. Since DP is projective as right A-module, there is a monomorphism $DP \to A^t \to \operatorname{Hom}_A(P,A)^{st}$. DP is injective as right A-module. Hence, $DP \in \operatorname{add}_A \operatorname{Hom}_A(P,A)$.

We claim now that $DA \otimes_A P$ is a projective left A-module. To see this, define P' to be the direct sum of all non-isomorphic indecomposable A-modules that belong to the additive closure of P. So, $\operatorname{add} P = \operatorname{add} P'$ and $P' \in \operatorname{add}_A DA \otimes_A P = \operatorname{add}_A DA \otimes_A P'$. By Krull-Remak-Schmidt theorem, we can write $DA \otimes_A P' \simeq P' \oplus X$ for some A-module X. On the other hand,

$$\operatorname{End}_A(P' \oplus X) \simeq \operatorname{End}_A(DA \otimes_A P')^{op} \simeq \operatorname{End}_A(\operatorname{Hom}_A(P',A)) \simeq \operatorname{End}_A(P')^{op}. \tag{3.5.0.3}$$

So, by comparing R-dimensions X must be the zero module. Hence, $DA \otimes_A P'$ is a projective-injective-faithful module. Consequently, $DA \otimes_A P$ is also a projective-injective-faithful module. Thus, $(A, DA\otimes_A, \operatorname{Hom}_A(P,A))$ is a QF3 algebra. Now, the double centralizer property (3.5.0.2) implies that $\operatorname{domdim}(A,R) \geq 2$. By Lemma 2.2.4, $\operatorname{add}_A P = \operatorname{add}_A DA \otimes_A P$. So, (A,P) is a Morita algebra by Theorem 2.9.1. $\square$

Remark 3.5.5. It remains true that if (A,P) is a relative Morita algebra over a commutative Noetherian ring R, then (A,P) is a cover of B. $\triangle$

Theorem 3.5.6. *Let (A,P,V) be a RQF3 algebra over a commutative Noetherian ring R with* $\operatorname{domdim}(A,R) \geq 2$. *Suppose that $\mathscr{R}(A)$ is a well behaved resolving subcategory of* A-mod$\cap R$-proj. *Let*

$$n = \min\{\operatorname{domdim}_{(A,R)} M \colon M \in \mathscr{R}(A)\}.$$

Then, $(A, \operatorname{Hom}_A(V,A))$ is an $(n-2)-\mathscr{R}(A)$ cover of $\operatorname{End}_A(V)$. *Moreover, the Hemmer–Nakano dimension of $\mathscr{R}(A)$ is less or equal to $n + \dim R - 2$.*

Proof. By Proposition 3.5.2, $(A, \operatorname{Hom}_A(V,A))$ is a cover of $\operatorname{End}_A(V)$. Assume $n = 0$. By contradiction, assume that $(A, \operatorname{Hom}_A(V,A))$ is a $\dim R - 1$-$\mathscr{R}(A)$ cover of $\operatorname{End}_A(V)$. If $\dim R = 0$, then every localization of R at a maximal ideal is a field. In particular, $\eta_{M_\mathfrak{m}}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R for every $M \in \mathscr{R}(A)$. As $R_\mathfrak{m}$ is a field, in view of Lemma 2.4.1, $\operatorname{domdim}_{A_\mathfrak{m}} M_\mathfrak{m} \geq 1$ for every maximal ideal $\mathfrak{m}$ in R. By Proposition 2.5.7, $\operatorname{domdim}_{(A,R)} M \geq 1$ for every $M \in \mathscr{R}(A)$. This is a contradiction with n being zero. If $\dim R \geq 1$, then η_M is an isomorphism for every $M \in \mathscr{R}(A)$ by Proposition 3.1.6. By Proposition 2.4.18 and Lemma 2.4.1, $\operatorname{domdim}_{(A,R)} M \geq 1$ for every $M \in \mathscr{R}(A)$ which contradicts our assumption on n.

Now assume that $n = 1$. For every $M \in \mathscr{R}(A)$, $\operatorname{domdim}_{(A,R)} M \geq 1$. Hence, for every maximal ideal $\mathfrak{m}$ in R $\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq 1$. By Lemma 2.4.1 and Proposition 2.4.7, $\eta_{M(\mathfrak{m})}$ is a monomorphism for every maximal ideal $\mathfrak{m}$ in R. By Lemma 1.4.31, η_M is an (A,R)-mono for every $M \in \mathscr{R}(A)$. By Lemma 3.1.4, we obtain that $(A, \operatorname{Hom}_A(V,A))$ is a $-1 - \mathscr{R}(A)$ cover of $\operatorname{End}_A(V)$. By contradiction, assume that $(A, \operatorname{Hom}_A(V,A))$ is a $\dim R$-$\mathscr{R}(A)$ cover of $\operatorname{End}_A(V)$. Then, in particular, η_M is an isomorphism for every $M \in \mathscr{R}(A)$. By Lemma 2.4.1, α_M is an isomorphism for every $M \in \mathscr{R}(A)$. By Proposition 2.4.18, $\operatorname{domdim}_{(A,R)} M \geq \dim R + 2 - \dim R = 2$ which contradicts our assumption of n.

Finally assume that $n \geq 2$. By Theorem 2.4.15, α_M is an isomorphism for every $M \in \mathscr{R}(A)$ and

$$\begin{aligned} 0 = \operatorname{Ext}_B^i(V, V \otimes_A M) &\simeq \operatorname{Ext}_B^i(\operatorname{Hom}_A(\operatorname{Hom}_A(V,A),A), \operatorname{Hom}_A(\operatorname{Hom}_A(V,A),A) \otimes_A M) \\ &= \operatorname{Ext}_B^i(\operatorname{Hom}_A(\operatorname{Hom}_A(V,A),A), \operatorname{Hom}_A(\operatorname{Hom}_A(V,A),M)) = \operatorname{Ext}_B^i(FA, FM) = \mathrm{R}^i G(FM),\ 1 \leq i \leq n-2. \end{aligned} \tag{3.5.0.4}$$

Hence, $(A, \operatorname{Hom}_A(V,A))$ is an $(n-2) - \mathscr{R}(A)$ cover of B. Using again the Proposition 2.4.18, we see that $(A, \operatorname{Hom}_A(V,A))$ cannot be an $n + 3 - \dim R - \mathscr{R}(A)$ cover of B. ☐

In particular, for A-proj, $(A, \operatorname{Hom}_A(V,A))$ is an i-cover of B for some $\operatorname{domdim}(A,R) - 2 \leq i \leq \operatorname{domdim}(A,R) - 2 + \dim R$. For split quasi-hereditary algebras, $(A, \operatorname{Hom}_A(V,A))$ is an i-$\mathscr{F}(\tilde{\Delta})$ cover of B for some $\operatorname{domdim}_{(A,R)} T - 2 \leq i \leq \operatorname{domdim}_{(A,R)} T - 2 + \dim R$ for T a characteristic tilting module. The idea that computing the Hemmer-Nakano dimension of A-proj (resp. $\mathscr{F}(\tilde{\Delta})$) using the dominant dimension of the regular module (resp. dominant dimension of a characteristic tilting module) goes back to [FK11b].

For algebras admitting additional properties and with Krull dimension larger than one, we can improve the lower bound.

Theorem 3.5.7. *Let R be a commutative Noetherian regular ring with Krull dimension at least one. Let (A,P,V) be a RQF3 algebra over a commutative Noetherian ring R with* $\operatorname{domdim}(A,R) \geq 2$. *Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of* A-mod$\cap R$-proj. *Let* $n = \min\{\operatorname{domdim}_{(A,R)} M \colon M \in \mathscr{R}(A)\} \geq 2$. *Assume that* $\min\{\operatorname{domdim}_{(K \otimes_R A, K)} N \colon N \in \mathscr{R}(K \otimes_R A)\} \geq n+1$ *for some Noetherian commutative flat R-algebra K. Then $(A, \operatorname{Hom}_A(V,A))$ is an $(n-1) - \mathscr{R}(A)$ cover of* $\operatorname{End}_A(V)$. *Moreover, if* $\dim R = 1$ *the Hemmer–Nakano dimension of $\mathscr{R}(A)$ is $n-1$.*

Proof. Assume first that $n \geq 2$. Let B denote $\operatorname{End}_A(V)$. Let $\mathfrak{m}$ be a maximal ideal in R. Fix $F_{(\mathfrak{m})}$ the functor $\operatorname{Hom}_{A(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), A(\mathfrak{m})), -)$ and $G_{(\mathfrak{m})}$ its right adjoint. Since every module in $\mathscr{R}(A(\mathfrak{m}))$ is constructed as a direct summand or via extensions of modules $R(\mathfrak{m}) \otimes_R M$ for $M \in \mathscr{R}(A)$, it is enough to check that $\eta_{M(\mathfrak{m})}$ is an isomorphism and $\mathrm{R}^i G_{(\mathfrak{m})} F_{(\mathfrak{m})}(M(\mathfrak{m})) = 0,\ 1 \leq i \leq n-2$ for every $M \in \mathscr{R}(A)$ to deduce that $(A(\mathfrak{m}), \operatorname{Hom}_{A(\mathfrak{m})}(V(\mathfrak{m}), A(\mathfrak{m})))$ is an $(n-2) - \mathscr{R}(A)$ cover of $B(\mathfrak{m})$. Let $M \in \mathscr{R}(A)$. Then,

$$\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq \operatorname{domdim}_{(A,R)} M \geq 2. \tag{3.5.0.5}$$

By Theorem 2.4.15,

$$0 = \operatorname{Ext}^i_{B(\mathfrak{m})}(V(\mathfrak{m}), V(\mathfrak{m}) \otimes_{A(\mathfrak{m})} M(\mathfrak{m})) = \operatorname{Ext}^i_{B(\mathfrak{m})}(F_{(\mathfrak{m})}A(\mathfrak{m}), F_{(\mathfrak{m})}(M(\mathfrak{m}))), 1 \leq i \leq n-2. \tag{3.5.0.6}$$

In the same way, $(K \otimes_R A, K \otimes_R \operatorname{Hom}_A(V,A))$ is an $(n-1)$-$\mathscr{R}(K \otimes_R A)$ cover of $\operatorname{End}_{K\otimes_R A}(K \otimes_R V)$. By Theorem 3.3.11, $(A, \operatorname{Hom}_A(V,A))$ is an $(n-1) - \mathscr{R}(A)$ cover of $\operatorname{End}_A(V)$. $\square$

Applying this result to A-proj, we see that the double centralizer properties arising from situations of relative dominant dimension greater than or equal to two are stronger in positive Krull dimension. In truth, the Hemmer–Nakano dimension of A-proj is not inferior for algebras with positive Krull dimension than the finite-dimensional algebras over a field.

It may be tempting to think that the Hemmer-Nakano dimension of A-proj, $\operatorname{HNdim}_F A$-proj, is equal to $\operatorname{domdim}(A,R) - 2 + \dim R$. Although, this is not the case, in general. Let $\mathbb{K}$ be an algebraically closed field with characteristic 3. Consider $R = \mathbb{K}[X]$ and assume $n \geq d \geq 3$. Then, the Hemmer-Nakano dimension of $S_{K[X]}(n,d)$-proj with respect to $(K[X]^n)^{\otimes d}$ is 2 (see Example 4.6.1) . Furthermore, in this example, the Hemmer-Nakano dimension of $S_{K[X_1,\ldots,X_r]}(n,d)$-proj is independent of the Krull dimension r. This example also shows that the condition of existence of flat R-algebra K such that the Hemmer-Nakano dimension of $\mathscr{R}(K \otimes_R A)$ in Theorems 3.5.7 and Theorem 3.3.11 cannot be omitted.

If we know that the ground ring is an integral domain, we can use its quotient field to take the role of K. Even better using the quotient field we can improve Theorem 3.5.7 to include the case $n = 1$.

Theorem 3.5.8. *Let R be a commutative Noetherian regular integral domain with Krull dimension at least one and with quotient field K. Let (A,P,V) be a RQF3 algebra over a commutative Noetherian ring R with* $\operatorname{domdim}(A,R) \geq 2$. *Let $\mathscr{R}(A)$ be a well behaved resolving subcategory of* A-mod$\cap R$-proj. *Let*

$$n = \min\{\operatorname{domdim}_{(A,R)} M \colon M \in \mathscr{R}(A)\} \geq 1.$$

Assume that $\min\{\operatorname{domdim}_{(K\otimes_R A,K)} N \colon N \in \mathscr{R}(K \otimes_R A)\} \geq n+1$. *Then, $(A, \operatorname{Hom}_A(V,A))$ is an $(n-1) - \mathscr{R}(A)$ cover of* $\operatorname{End}_A(V)$. *Moreover, if* $\dim R = 1$ *the Hemmer–Nakano dimension of $\mathscr{R}(A)$ is $n-1$.*

Proof. The case $n = 2$ is just a particular case of Theorem 3.5.7. We will now consider the case $n = 1$. Hence, $\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq \operatorname{domdim}_{(A,R)} M \geq 1$ for any $M \in \mathscr{R}(A)$. Taking into account that $\operatorname{domdim}(A,R) \geq 2$ we obtain by Proposition 3.3.6 that (A,P) is a $(-1) - \mathscr{R}(A)$ cover of $\operatorname{End}_A(V)$. Further, by Lemma 2.4.1, Proposition 2.4.7 and 3.5.2, we obtain that the unit map $\eta_M \colon M \to GFM$ is an (A,R)-monomorphism for every $M \in \mathscr{R}(A)$. On the other hand, $K \otimes_R M \in \mathscr{R}(K \otimes_R A)$. Hence, $\eta_{K\otimes_R M}$ is an isomorphism by assumption. Thus, $K \otimes_R \eta_M$ is an isomorphism.

Denote by X the cokernel of η_M. By the flatness of K and $K \otimes_R \eta_M$ being an isomorphism, it follows that $K \otimes_R X = 0$. In particular, X is a torsion R-module. Using the monomorphism

$$GFM \to \operatorname{Hom}_R(V, FM) \in R\text{-proj}, \tag{3.5.0.7}$$

we deduce that GFM is a torsion-free R-module. By a result of Auslander-Buchsbaum (see Proposition 3.4 of [AB59]) if $X \neq 0$, then there exists a prime ideal of height one $\mathfrak{q}$ such that $X_\mathfrak{q} \neq 0$. But $\dim R_\mathfrak{q} = 1$, so the localization $GFM_\mathfrak{q}$ is a projective $R_\mathfrak{q}$-module. Thus, $X_\mathfrak{q}$ is a projective $R_\mathfrak{q}$-module. By applying the tensor product $K \otimes_{R_\mathfrak{q}} -$ it follows that $X_\mathfrak{q} = 0$. So, we must have $X = 0$. Hence, η_M is an isomorphism. This finishes the proof. $\square$

In Section 4.1.1, we can see a complete classification of the Hemmer-Nakano dimension of projective modules and modules admitting a filtration by standard modules for Schur algebras and quantised Schur algebras. We will see that for these algebras the Hemmer-Nakano dimension is either $n-1$ or $n-2$ where n is fixed according to Theorem 3.5.7. This behaviour can be explained due to the presence of a base change property and an integral version for which one can define all Schur algebras and q-Schur algebras.

Proposition 3.5.9. *Let R be a regular local commutative Noetherian ring with Krull dimension greater than or equal to one. Let $(A_\mathbb{Z}, P_\mathbb{Z})$ be a 0-$\mathscr{R}(A_\mathbb{Z})$ cover of $B_\mathbb{Z}$ for some resolving subcategory $\mathscr{R}(A_\mathbb{Z})$ of $A_\mathbb{Z} \cap \mathbb{Z}$-proj. Suppose that $n_\mathfrak{m}$ is the Hemmer-Nakano dimension of $\mathscr{R}(R(\mathfrak{m}) \otimes_\mathbb{Z} A_\mathbb{Z})$, where $\mathfrak{m}$ is the unique maximal ideal of R. Then, the Hemmer-Nakano dimension of $\mathscr{R}(R \otimes_\mathbb{Z} A_\mathbb{Z})$ is at most $n_\mathfrak{m}+1$.*

Proof. Denote by F the functor $\operatorname{Hom}_{A_\mathbb{Z}}(P_\mathbb{Z}, -)\colon A_\mathbb{Z}\text{-mod} \to B_\mathbb{Z}\text{-mod}$. If $\dim R = 1$, the result follows from Corollary 3.3.10. Suppose that $\dim R > 1$. The completion $\hat{R}$ is faithfully flat over R. By Theorem 1.1.62, either $\hat{R}$ is faithfully flat over $\hat{R}(\hat{\mathfrak{m}}) = R(\mathfrak{m})$ (see [GS71, Corollary 2.18]) or $\hat{R}$ is faithfully flat over some complete discrete valuation ring k with residue field $R(\mathfrak{m})$. In the first case, Proposition 3.3.4 says that

$$n_\mathfrak{m} = \operatorname{HNdim}_{R(\mathfrak{m})\otimes_\mathbb{Z} F}(\mathscr{R}(R(\mathfrak{m}) \otimes_\mathbb{Z} A_\mathbb{Z})) \geq \operatorname{HNdim}_{\hat{R}\otimes_\mathbb{Z} F}(\mathscr{R}(\hat{R} \otimes_\mathbb{Z} A_\mathbb{Z})) = \operatorname{HNdim}_{R\otimes_\mathbb{Z} F}(\mathscr{R}(R \otimes_\mathbb{Z} A_\mathbb{Z})) \quad (3.5.0.8)$$

So, it remains to consider the second case. Since $\dim k = 1$ and due to Theorem 3.3.9

$$\operatorname{HNdim}_{k\otimes_\mathbb{Z} F}(\mathscr{R}(k \otimes_\mathbb{Z} A_\mathbb{Z})) \leq \operatorname{HNdim}_{R(\mathfrak{m})\otimes_\mathbb{Z} F}(\mathscr{R}(R(m) \otimes_k k \otimes_\mathbb{Z} A_\mathbb{Z})) + 1 = n_\mathfrak{m} + 1. \quad (3.5.0.9)$$

Since $\hat{R}$ is faithfully flat over k, it follows that

$$\operatorname{HNdim}_{R\otimes_\mathbb{Z} F}(\mathscr{R}(R \otimes_\mathbb{Z} A_\mathbb{Z})) = \operatorname{HNdim}_{\hat{R}\otimes_\mathbb{Z} F}(\mathscr{R}(\hat{R} \otimes_\mathbb{Z} A_\mathbb{Z})) = \operatorname{HNdim}_{k\otimes_\mathbb{Z} F}(\mathscr{R}(k \otimes_\mathbb{Z} A_\mathbb{Z})) \leq n_m + 1. \qquad \square$$

Although this Proposition only considers the case of projective Noetherian $\mathbb{Z}$-algebras, we see the importance of a base change property and an integral version for the Hemmer-Nakano dimension.

In the absence of a base change property, according to Theorems 3.3.9 and 3.3.12, the Hemmer-Nakano dimension can be equal to $\min\{\operatorname{domdim}_{(A,R)} M \colon M \in \mathscr{R}(A)\} + \dim R - 2$ if for every localization of R, $R_\mathfrak{m}$, the cover is compatible with all the possible $R_\mathfrak{m}$-sequences.

We can extend to Noetherian rings the result given in [Fan08] which indicates an upper bound of the faithfulness of a faithful split quasi-hereditary cover in terms of relative dominant dimension of the algebra.

Proposition 3.5.10. *Let A be a projective Noetherian R-algebra. If $\operatorname{domdim}(A,R) \geq d(\Lambda) + 2 + s$ for some $s \geq 1$, then $(A, \operatorname{Hom}_A(V, A))$ is an s-faithful split quasi-hereditary cover of $\operatorname{End}_A(V)$.*

Proof. By Proposition 2.11.2,

$$\operatorname{domdim}_{(A,R)} \Delta(\lambda) \geq \operatorname{domdim}(A,R) - d(\Delta, \lambda) \geq d(\Lambda) - d(\Lambda, \lambda) + 2 + s, \quad (3.5.0.10)$$

for every $\lambda \in \Lambda$. Hence, $\min\{\operatorname{domdim}_{(A,R)} \Delta(\lambda) \colon \lambda \in \Lambda\} \geq 2 + s$. The result follows from Theorem 3.5.6. $\square$

For the split quasi-hereditary algebras satisfying Theorem 2.11.3, the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ is at least $\dfrac{\operatorname{domdim}(A,R)}{2} - 2$.

Proposition 3.5.10 is particularly useful when we have no clear relation between the relative dominant dimension of a characteristic tilting module and the relative dominant dimension of the regular module.

We can use this cover technology to prove some statements involving the Nakayama conjecture.

Proposition 3.5.11. *Let A be a projective Noetherian R-algebra such that*

$$A\text{-proj} = \{Y \in A\text{-mod} \cap R\text{-proj}\colon \operatorname{Ext}_A^{i>0}(Y,A) = 0\}.$$

If (A,P) is an $\infty - A$-proj cover of B, then $F = \operatorname{Hom}_A(P,-)\colon A\text{-mod} \to B\text{-mod}$ is an equivalence of categories. Furthermore, the Nakayama conjecture holds for these class of algebras.

Proof. Consider a projective B-presentation for FA,

$$0 \to K \to Q \to FA \to 0. \tag{3.5.0.11}$$

Applying $\operatorname{Hom}_B(-,FA)$ yields

$$\operatorname{Ext}_B^i(K,FA) \simeq \operatorname{Ext}_B^{i+1}(FA,FA) \simeq \operatorname{Ext}_A^{i+1}(A,A) = 0,\ \forall i > 0. \tag{3.5.0.12}$$

As (A,P) is an $\infty - A$-proj cover of B we have

$$\operatorname{Ext}_A^{i>0}(GK,A) \simeq \operatorname{Ext}_A^{i>0}(GK,GFA) \simeq \operatorname{Ext}_B^{i>0}(FGK,FA) \simeq \operatorname{Ext}_B^{i>0}(K,FA) = 0. \tag{3.5.0.13}$$

By assumption, it follows that $GK \in A$-proj. Hence, $\mathrm{R}^1\, G(K) \simeq \mathrm{R}^1\, G(FGK) = 0$. Thus, G is exact on (3.5.0.11), so we have an exact sequence

$$0 \to GK \to GQ \to GFA \simeq A \to 0. \tag{3.5.0.14}$$

Moreover, this sequence splits over A, and A is a summand of GQ. Thus, FA is a B-summand of $FGQ \simeq Q$. Therefore, $FA \in B$-proj. It follows that F is an equivalence of categories.

By fixing $P = \operatorname{Hom}_A(V,A)$ for algebras with infinite relative dominant dimension with projective (A,R)-injective-strongly faithful right module V we conclude the result. ☐

Observe that every finite global dimension algebra satisfies this property.

In the following, we present evidence using the Nakayama conjecture that for a given projective Noetherian R-algebra B the Hemmer-Nakano dimension of A-proj must be finite, where (A,P) is the cover with finite global dimension of B.

Theorem 3.5.12. *Let R be a Noetherian regular ring with finite Krull dimension. Let (A,P,V) be a relative QF3 R-algebra. Fix $C = \operatorname{End}_A(V)$. If $\operatorname{Ext}_C^i(V,V) = 0$ for all $i > 0$ and $(A, \operatorname{Hom}_A(V,A))$ is a cover of C, then the Nakayama conjecture implies that V is an A-progenerator. In particular, the Nakayama conjecture implies that $C \overset{\text{Mor}}{\sim} A$ is relative self-injective to R and the Schur functor $\operatorname{Hom}_A(\operatorname{Hom}_A(V,A),-)\colon A\text{-mod} \to C\text{-mod}$ is an equivalence of categories.*

Proof. Since $\dim R$ is finite, $\operatorname{Ext}_C^i(V,V) = 0$ for all $i > 0$ and α_A is bijective it follows by Proposition 2.4.18 that $\operatorname{domdim}(A,R) = +\infty$. By Nakayama's conjecture, A is an (A,R)-injective projective A-module. By Lemma 2.2.4, the regular module A belongs to $\operatorname{add} DV$. Hence, $C = \operatorname{End}_A(DV)^{op} = \operatorname{End}_A(V) \simeq \operatorname{End}_A(\operatorname{Hom}_A(V,A))^{op}$ is Morita equivalent to A through the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(V,A),-)$. In particular, $V = \operatorname{Hom}_A(\operatorname{Hom}_A(V,A),A)$ is projective as C-module. ☐

This shows that the Nakayama Conjecture implies that the level of faithfulness of covers arising from RQF3 algebras is finite unless the cover algebra is already a self-injective algebra relative R. In practice, most of the examples that we are interested in are the ones that arise in this way.

We can also reformulate the Nakayama Conjecture in terms of Schur functors in a similar way as the cover property is defined.

Proposition 3.5.13. *Let A be a finite-dimensional algebra over a field. Let V be a projective right A-module. Let $B = \mathrm{End}_A(V) = \mathrm{End}_A(\mathrm{Hom}_A(V,A))^{op}$. If the restriction of the Schur functor $F = \mathrm{Hom}_A(\mathrm{Hom}_A(V,A),-)$ to $\mathrm{add}\, DA \oplus A$ is faithful, then F is an equivalence of categories.*

Proof. By assumption, the map induced by F, $\mathrm{Hom}_A(A,DA) \to \mathrm{Hom}_B(FA,F(DA))$ is injective. By Lemma 3.1.4, $\eta_{DA}\colon DA \to \mathrm{Hom}_B(FA,FDA)$ is a monomorphism. Note that $FA = V$ and $FDA = V \otimes_A DA$. Consider an injective B-presentation of $V \otimes_A DA$, $0 \to V \otimes_A DA \to I_0$. Since $\mathrm{Hom}_B(V,-)$ is left exact, the composition of maps $DA \to \mathrm{Hom}_B(V, V \otimes_A DA) \to \mathrm{Hom}_B(V,I_0)$ is a monomorphism. Observe that $\mathrm{Hom}_B(V,I_0) \in \mathrm{add}\,\mathrm{Hom}_B(V,DB) = \mathrm{add}\, DV$. Hence, $DA \in \mathrm{add}\, DV$ and consequently V is a right A-progenerator. By Morita theory, $\mathrm{Hom}_A(V,A)$ is a left A-progenerator. □

So, we can rewrite the Nakayama Conjecture in the following way:

- If $\mathrm{domdim}\,(A,R) = +\infty$, then (A,DV,V) is a relative QF3 R-algebra such that the restriction of the Schur functor $\mathrm{Hom}_A(\mathrm{Hom}_A(V,A),-)$ to $\mathrm{add}\, DA \oplus A$ is faithful.

3.6 Uniqueness of faithful covers

We will start by introducing a more general concept of equivalence of covers.

Definition 3.6.1. Let A, A', B, B' be projective Noetherian R-algebras and $\mathscr{A}$ and $\mathscr{A}'$ be resolving subcategories of A-mod$\cap R$-proj and A'-mod$\cap R$-proj, respectively.

Assume that (A,P) is a $0-\mathscr{A}$ cover of B and (A',P') is a $0-\mathscr{A}'$ cover of B'. We say that the $\mathscr{A}$-covers (A,P) and (A',P') are **equivalent** if there is an equivalence of categories $H\colon A\text{-mod} \to A'\text{-mod}$, which restricts to an exact equivalence $\mathscr{A} \to \mathscr{A}'$, and an equivalence of categories $L\colon B\text{-mod} \to B'\text{-mod}$ making the following diagram commutative:

$$\begin{tikzcd} A\text{-mod} \arrow[r, "\mathrm{Hom}_A(P,-)"] \arrow[d, "H"'] & B\text{-mod} \arrow[d, "L"] \\ A'\text{-mod} \arrow[r, "\mathrm{Hom}_{A'}(P',-)"'] & B'\text{-mod} \end{tikzcd}.$$

We say that two covers (A,P) and (A',P') are **isomorphic** if they are equivalent with L being the identity functor B-mod $\to B$-mod and $B' \simeq B$ as R-algebras.

The first observation to make is that equivalent covers have the same level of faithfulness.

Proposition 3.6.2. *Let (A,P) be a $0-\mathscr{A}$ cover of B and let (A',P') be a $0-\mathscr{A}'$ cover of B'. Assume that the covers (A,P) and (A',P') are equivalent. If (A,P) is an $i-\mathscr{A}$ cover of B, then (A',P') is an $i-\mathscr{A}'$ cover of B'.*

Proof. Denote the functor $\mathrm{Hom}_{B'}(F'A',-)$ by G'. Let $M \in \mathscr{A}'$. Then, for $1 \leq j \leq i$,

$$\mathrm{R}^j G'(F'M) = \mathrm{Ext}^j_{B'}(F'A',F'M) = \mathrm{Ext}^j_B(F'HQ,F'HX) = \mathrm{Ext}^j_{B'}(LFQ,LFX) = \mathrm{Ext}^j_B(FQ,FX) = \mathrm{Ext}^j_A(Q,X) = 0,$$

for some $X \in \mathscr{A}$ and $Q \in A$-proj. By Proposition 3.1.17, the result follows. □

Rouquier defined equivalence of split quasi-hereditary covers in the following way.

Definition 3.6.3. Two split quasi-hereditary covers (A,P) and (A',P') are **equivalent in the sense of Rouquier** if there is an equivalence of highest weight categories $A\text{-mod} \xrightarrow{\simeq} A'\text{-mod}$ making the following diagram commutative:

$$\begin{tikzcd}
A\text{-mod} \arrow[dr, "Hom_A(P,-)"] \arrow[dd, "\simeq"'] & \\
& B\text{-mod} \cdot \\
A'\text{-mod} \arrow[ur, "Hom_{A'}(P',-)"'] &
\end{tikzcd}$$

We will show next that this definition is a particular case of our definition of isomorphic covers by fixing $\mathscr{A} = \mathscr{F}(\tilde{\Delta})$, $\mathscr{A}' = \mathscr{F}(\tilde{\Delta}')$, and L the identity functor. Moreover, the notion of isomorphic covers for the resolving subcategory $\mathscr{F}(\tilde{\Delta})$ is equivalent with the equivalence of covers of Definition 3.6.3.

Proposition 3.6.4. *Let A,A' be two split quasi-hereditary algebras over a commutative Noetherian ring R. Let (A,P) and (A',B') be split quasi-hereditary covers of B. The covers (A,P) and (A',P') are equivalent in the sense of Definition 3.6.3 if and only if they are isomorphic in the sense of Definition 3.6.1 with respect to the resolvings subcategories $\mathscr{F}(\tilde{\Delta})$ and $\mathscr{F}(\tilde{\Delta}')$.*

Proof. Assume that (A,P) and (A',P') are equivalent in the sense of Definition 3.6.3. Let $H\colon A\text{-mod} \to A'\text{-mod}$ be a highest weight category equivalence such that $\operatorname{Hom}_{A'}(P',-)\circ H = \operatorname{Hom}_A(P,-)$. Since H is an equivalence of highest weight categories, there is a bijection $\phi\colon \Lambda \to \Lambda'$ satisfying $H\Delta(\lambda) = \Delta'(\phi(\lambda))\otimes_R U_\lambda$. As H is exact and $H\Delta(\lambda) \in \mathscr{F}(\Delta')$, the restriction functor $H\colon \mathscr{F}(\tilde{\Delta}) \to \mathscr{F}(\tilde{\Delta}')$ is well defined and it is fully faithful and exact. As $U_\lambda \in Pic(R)$ there is F_λ such that $F_\lambda \otimes_R U_\lambda \simeq R$, thus $\Delta(\lambda') = H\Delta(\phi^{-1}(\lambda'))\otimes_R F_{\lambda'} = H(\Delta(\phi^{-1}(\lambda'))\otimes_R F_{\lambda'})$. Let $M \in \mathscr{F}(\tilde{\Delta}')$. By induction on the filtration of M, we deduce that M is in the image of $H_{|_{\mathscr{F}(\tilde{\Delta})}}$. Therefore, H restricts to an exact equivalence $\mathscr{F}(\tilde{\Delta}) \to \mathscr{F}(\tilde{\Delta}')$ and $B \simeq \operatorname{End}_A(P)^{op} \simeq \operatorname{End}_{A'}(P')^{op}$. Hence, the covers (A,P) and (A',P') are isomorphic in the sense of Definition 3.6.1.

Conversely, assume that the covers (A,P) and (A',P') are isomorphic in the sense of Definition 3.6.1 with respect to the resolving subcategories $\mathscr{F}(\tilde{\Delta})$ and $\mathscr{F}(\tilde{\Delta}')$. Let I be the quasi-inverse of H. Then, HA is a B-progenerator and for any $Y \in B\text{-mod}$,

$$\operatorname{Hom}_B(HA,Y) \simeq \operatorname{Hom}_A(IHA,IY) \simeq \operatorname{Hom}_A(A,IY) \simeq IY. \tag{3.6.0.1}$$

In particular, I commutes with tensor products of projective R-modules, that is, for $Y \in B\text{-mod}$ and $X \in R\text{-proj}$, $I(Y\otimes_R X) = IY \otimes_R X$. In view of the proof of Proposition 1.5.80, there is a bijection $\phi\colon \Lambda \to \Lambda'$ and

$$\operatorname{Hom}_B(HA,\Delta'(\lambda')) = \Delta(\phi^{-1}(\lambda'))\otimes_R U_{\lambda'} = \Delta(\lambda)\otimes_R U_{\lambda'},\ U_{\lambda'} \in Pic(R). \tag{3.6.0.2}$$

Moreover, as A-modules,

$$I(\Delta'(\lambda')\otimes_R F_{\lambda'}) \simeq I\Delta'(\lambda')\otimes_R F_{\lambda'} \simeq \operatorname{Hom}_A(A, I\Delta'(\lambda'))\otimes_R F_{\lambda'} \simeq \operatorname{Hom}_A(IHA, I\Delta'(\lambda'))\otimes_R F_{\lambda'} \tag{3.6.0.3}$$

$$\simeq \operatorname{Hom}_B(HA,\Delta'(\lambda'))\otimes_R F_{\lambda'} \simeq \Delta(\lambda)\otimes_R U_{\lambda'}\otimes_R F_{\lambda'} \simeq \Delta(\lambda). \tag{3.6.0.4}$$

Thus,

$$H\Delta(\lambda) \simeq HI(\Delta'(\lambda')\otimes_R F_{\lambda'}) \simeq \Delta'(\phi(\lambda))\otimes_R F_\lambda, \tag{3.6.0.5}$$

with $F_{\lambda'} = F_\lambda$. Thus, H is an equivalence of highest weight categories, and it follows that (A,P) and (A',P') are equivalent in the sense of Definition 3.6.3. $\square$

The main reason to make the difference between isomorphic covers and equivalent covers is that intuitively the covers constructed in Proposition 3.1.3 should be equivalent although they are not isomorphic. So, from now on, we will use only the concepts in the sense of Definition 3.6.1.

In [Rou08, Proposition 4.45], further assumptions are required to the pair $(Y(M), M)$. For instance, if R is a local ring, then a suitable condition would be requiring $Y(M)$ to be indecomposable. This problem manifests itself in [Ari08, 4.2]. A counter-example to such a pair without further assumptions could be the following:

Example 3.6.5. Let A the path algebra of the quiver

$$2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1$$

modulo the ideal generated by $\alpha\beta$. Pick the partial order $2 > 1$ and $\Delta(2) = P(2)$, $\Delta(1) = 1$. Trivially, $(A, {}_AA)$ is a 1-faithful split quasi-hereditary cover of A. The exact sequence

$$0 \to \Delta(2) \oplus \Delta(2) \to \Delta(2) \oplus P(1) \to \Delta(1) \to 0 \tag{3.6.0.6}$$

also satisfies the conditions required for the pair $(Y(\Delta(1)), p_{\Delta(1)})$. △

However, this does not cause problems to the content of [Rou08, Corollary 4.46] since we can use Proposition 1.5.80 to replace [Rou08, Proposition 4.45].

We will now give an alternative proof of [Rou08, Corollary 4.46].

Corollary 3.6.6. *[Rou08, Corollary 4.46] Let (A, P) and (A', P') be two 1-faithful split quasi-hereditary covers of B. Let $F = \operatorname{Hom}_A(P, -)$ and let $F' = \operatorname{Hom}_{A'}(P', -)$. Assume that there exists an exact equivalence $L\colon B\text{-mod} \to B\text{-mod}$ which restricts to an exact equivalence*

$$\mathscr{F}_B(F\tilde{\Delta}) \to \mathscr{F}_B(F'\tilde{\Delta}'). \tag{3.6.0.7}$$

Then, (A, P) and (A', P') are equivalent as faithful split quasi-hereditary covers of B. If, in addition, the given bijection $\phi\colon \Lambda \to \Lambda'$ associated with the equivalence of categories $H\colon A\text{-mod} \to A'\text{-mod}$ satisfies

$$F\Delta(\lambda) = F'\Delta'(\phi(\lambda)) \otimes_R U_\lambda, \forall \lambda \in \Lambda.$$

where $H\Delta(\lambda) \simeq \Delta'(\phi(\lambda)) \otimes_R U_\lambda$, $U_\lambda \in Pic(R)$, then (A, P) and (A', P') are isomorphic as split quasi-hereditary covers of B.

Proof. By Proposition 3.1.13, there exists an exact equivalence

$$\mathscr{F}_A(\tilde{\Delta}) \xrightarrow{F} \mathscr{F}_B(F\tilde{\Delta}) \xrightarrow{L} \mathscr{F}_B(F'\tilde{\Delta}') \xrightarrow{G'} \mathscr{F}_{A'}(\tilde{\Delta}').$$

By Proposition 1.5.80, A and A' are equivalent as split quasi-hereditary algebras. Furthermore, the equivalence of categories is given by $H = \operatorname{Hom}_A(GL^{-1}F'A', -)\colon A\text{-mod} \to A'\text{-mod}$. Thus, for every $X \in A\text{-proj}$,

$$HX \simeq \operatorname{Hom}_A(GL^{-1}F'A', X) \simeq \operatorname{Hom}_B(FGL^{-1}F'A', FX) \simeq \operatorname{Hom}_B(L^{-1}F'A', FX) \tag{3.6.0.8}$$

$$\simeq \operatorname{Hom}_B(F'A', LFX) \simeq \operatorname{Hom}_{A'}(A', G'LFX) \simeq G'LFX. \tag{3.6.0.9}$$

Therefore, $F'HX \simeq LFX$ for every $X \in A\text{-proj}$. Since all functors involved are exact we conclude that $F'H = LF$.

Assume, in addition,

$$F\Delta(\lambda) = F'\Delta'(\phi(\lambda)) \otimes_R U_\lambda, \forall \lambda \in \Lambda$$

and $H\Delta(\lambda) \simeq \Delta'(\phi(\lambda)) \otimes_R U_\lambda$, $U_\lambda \in Pic(R)$ for some bijection $\phi \colon \Lambda \to \Lambda'$. Thus, for every $\lambda \in \Lambda$, as B-modules,

$$F'H\Delta(\lambda) \simeq F'(\Delta'(\phi(\lambda)) \otimes_R U_\lambda) \simeq F'\Delta'(\phi(\lambda)) \otimes_R U_\lambda \simeq F\Delta(\lambda). \tag{3.6.0.10}$$

Now, using induction on the filtrations by standard modules together with the fact

$$\mathrm{Ext}^1_B(F\Delta(\lambda) \otimes_R F_\lambda, F\Delta(\mu) \otimes_R X_\mu) \simeq \mathrm{Ext}^1_B(F'H\Delta(\lambda) \otimes_R F_\lambda, F'H\Delta(\mu) \otimes_R X_\mu) \simeq \mathrm{Ext}^1_A(\Delta(\lambda) \otimes_R F_\lambda, \Delta(\mu) \otimes_R X_\mu)$$

with F_λ and X_μ being invertible R-modules, we obtain that $FX \simeq F'HX$ for every $X \in \mathscr{F}(\tilde{\Delta})$. In particular, $FA \simeq F'HA$ as B-modules. This means that we can write

$$\mathrm{Hom}_A(H^{-1}P', A) \simeq \mathrm{Hom}_{A'}(P', HA) \simeq \mathrm{Hom}_A(P, A), \tag{3.6.0.11}$$

also as right A-modules. By applying $\mathrm{Hom}_A(-, A)$ we obtain $HP \simeq P'$. Thus, $FX \simeq F'HX$ for every $X \in A$-mod. ☐

For the resolving subcategory A-proj, we only require for A and A' to be Morita equivalent with the projective modules of A and A' being mapped to the same full subcategory of B-mod.

In the same fashion as for quasi-hereditary covers, we can deduce a trivial uniqueness result for the resolving subcategory A-proj.

Corollary 3.6.7. *Let (A, P) and (A', P') be two 1-covers of B. Assume that $\mathrm{add}\, FA = \mathrm{add}\, F'A'$. Then, A and A' are Morita equivalent. If, in addition, $FA = F'A'$, then (A, P) and (A', P') are isomorphic covers of B.*

Proof. By Proposition 3.1.14, there is an exact equivalence

$$A'\text{-proj} \xrightarrow{F'} \mathrm{add}\, F'A' = \mathrm{add}\, FA \xrightarrow{G} A\text{-proj}\,. \tag{3.6.0.12}$$

By Morita theory, $GF'A' = \mathrm{Hom}_B(FA, F'A')$ is an A-progenerator. Hence, the functor $\mathrm{Hom}_A(\mathrm{Hom}_B(FA, F'A'), -) \colon A\text{-mod} \to A'\text{-mod}$ is an exact equivalence of categories. In particular, it restricts to an exact equivalence $A\text{-proj} \to A'\text{-proj}$. Assume that $FA = F'A'$. Let $X \in A$-mod. Then,

$$F' \mathrm{Hom}_A(\mathrm{Hom}_B(FA, F'A'), X) \simeq F' \mathrm{Hom}_A(\mathrm{Hom}_B(FA, FA), X) \simeq F' \mathrm{Hom}_A(\mathrm{Hom}_A(A, A), X) \simeq F'X. \qquad ☐$$

3.7 Existence of faithful covers

Every finite-dimensional algebra over a field has a quasi-hereditary cover (see [DR89a]).

In Example 4.6.8, we will see that the group algebra KS_d might have several (-1)-faithful quasi-hereditary covers. This is not a mere coincidence. In fact, Iyama gave another construction of quasi-hereditary covers in [Iya03, Iya04] to establish the Iyama's finiteness theorem. This construction has better properties than the construction established in [DR89a]. In particular, we have the following result.

Theorem 3.7.1. *Let k be a field. Let B be a finite-dimensional k-algebra. Then, B has a (-1)-faithful (not necessarily split) quasi-hereditary cover (A, P).*

Proof. By Theorem 5(2) of [Rin10], there is a left strongly quasi-hereditary algebra A and an idempotent e of A such that eA is a generator-cogenerator of $eAe = B$ and $A = \mathrm{End}_{eAe}(eA)$. Therefore, (A, Ae) is a cover of $eAe = B$. Now, since A is left strongly quasi-hereditary for a certain poset Λ, there are for each $\lambda \in \Lambda$, exact sequences

$$0 \to X(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0 \tag{3.7.0.1}$$

with both $X(\lambda)$ and $P(\lambda)$ projective A-modules. Let $F = \operatorname{Hom}_A(Ae,-)$ and G its right adjoint. Since, (A,Ae) is a cover of B, η_X is an isomorphism for every $X \in A$-proj. By the commutativity of the diagram

$$\begin{tikzcd} 0 \arrow[r] & X(\lambda) \arrow[r] \arrow[d, "\eta_{X(\lambda)}"] & P(\lambda) \arrow[r] \arrow[d, "\eta_{P(\lambda)}"] & \Delta(\lambda) \arrow[r] \arrow[d, "\eta_{\Delta(\lambda)}"] & 0 \\ 0 \arrow[r] & GFX(\lambda) \arrow[r] & GFP(\lambda) \arrow[r] & GF\Delta(\lambda) & \end{tikzcd}, \tag{3.7.0.2}$$

and the Snake Lemma we deduce that $\eta_{\Delta(\lambda)}$ is a monomorphism for every $\lambda \in \Lambda$. ☐

In Example 4.6.9, we can see that not every split quasi-hereditary cover is a (-1)-faithful quasi-hereditary cover.

We will now focus our attention on covers over commutative Noetherian rings. We can use Dlab-Ringel standardization to construct (if they exist) split quasi-hereditary 1-faithful covers when the ring is regular with Krull dimension at most one.

Theorem 3.7.2. *Let R be a regular commutative Noetherian ring with Krull dimension at most one. Let B be a projective Noetherian R-algebra. Assume that there exists a split standardizable set Θ of B-mod. If $B \in \mathscr{F}(\Theta)$, then there exists a 1-faithful split quasi-hereditary cover (A,P) of B.*

Conversely, assume that (A,P) is a 1-faithful split quasi-hereditary cover of B. Then, there exists a split standardizable set Θ of B-mod with $B \in \mathscr{F}(\Theta)$.

Proof. By Theorem 1.5.83, there exists a split quasi-hereditary cover A and the functor $\operatorname{Hom}_B(Q,-)\colon B\text{-mod} \to A\text{-mod}$ restricts to an exact equivalence between $\mathscr{F}(\Theta)$ and $\mathscr{F}(\Delta)$. Here $Q = \bigoplus_{i=1}^n P_\theta(i)$ as constructed in the proof of Theorem 1.5.83. Since $B \in \mathscr{F}(\Theta)$, there exists by Theorem 1.5.83 (see equation (1.5.9.32 in the proof of Theorem 1.5.83), a surjective map $X \twoheadrightarrow B$ with $X \in \operatorname{add}_B Q$. Hence, $B \in \operatorname{add}_B Q$, and thus Q is a B-generator. In particular, Q satisfies the double centralizer property, so $\operatorname{End}_A(Q) \simeq B$ and $Q_A \simeq \operatorname{Hom}_B(B,Q_A)$ is a right A-summand of $\operatorname{Hom}_B(Q,Q_A) \simeq A_A$. Therefore, Q is a projective right A-module. Thus, $P = \operatorname{Hom}_A(Q,A)$ is a projective left A-module and

$$\operatorname{End}_A(P)^{op} \simeq \operatorname{End}_A(\operatorname{Hom}_A(Q,A))^{op} \simeq \operatorname{End}_A(Q) \simeq B. \tag{3.7.0.3}$$

Fix $F = \operatorname{Hom}_A(P,-)$. Then, by definition of A, $A = \operatorname{End}_B(Q)^{op} = \operatorname{End}_B(FA)^{op}$. So, (A,P) is a cover of B. Let $X \in A$-mod and $Y \in B$-mod. Then,

$$\operatorname{Hom}_B(FX,Y) = \operatorname{Hom}_B(\operatorname{Hom}_A(P,X),Y) \simeq \operatorname{Hom}_B(\operatorname{Hom}_A(P,A)\otimes_A X,Y) \simeq \operatorname{Hom}_B(Q\otimes_A X,Y) \tag{3.7.0.4}$$

$$\simeq \operatorname{Hom}_A(X,\operatorname{Hom}_B(Q,Y)). \tag{3.7.0.5}$$

So, F is left adjoint to $\operatorname{Hom}_B(Q,-)$. In particular,

$$F\Delta(i) = FH\theta(i) \simeq \theta(i), \tag{3.7.0.6}$$

since $\varepsilon_{\theta(i)}$ is an isomorphism. Thus, $\mathscr{F}(F\Delta) = \mathscr{F}(\Theta)$. Moreover, for $X,Y \in \mathscr{F}(\Delta)$, $FX,FY \in \mathscr{F}(\Theta)$, so F induces the isomorphism

$$\operatorname{Hom}_B(FX,FY) \simeq \operatorname{Hom}_A(HFX,HFY) \simeq \operatorname{Hom}_A(X,Y). \tag{3.7.0.7}$$

By Lemma 3.1.4, $\eta_{\Delta(i)}$ is an isomorphism for every $i = 1, \dots, n$. For every $i = 1, \dots, n$

$$R^1H(F\Delta(i)) = R^1H(\theta(i)) = \mathrm{Ext}^1_B(Q, \theta(i)) = 0. \tag{3.7.0.8}$$

By Proposition 3.1.13, the result follows.

Conversely, assume that (A, P) is a 1-faithful split quasi-hereditary cover of B. Define $\theta(i) := F\Delta(\lambda)$. Then, $B = \mathrm{Hom}_A(P, P) = FP$. Since F is exact, it sends the modules belonging to $\mathscr{F}(\Delta)$ to $\mathscr{F}(\Theta)$. Thus, $B \in \mathscr{F}(\Theta)$. We have

$$\mathrm{Hom}_B(\theta(i), \theta(j)) = \mathrm{Hom}_B(F\Delta(i), F\Delta(j)) \simeq \mathrm{Hom}_A(\Delta(i), \Delta(j)), \forall i, j, \tag{3.7.0.9}$$

and

$$\mathrm{Ext}^1_B(\theta(i), \theta(j)) \simeq \mathrm{Ext}^1_A(\Delta(i), \Delta(j)). \tag{3.7.0.10}$$

By Proposition 1.5.50 and definition of split highest weight category, the conditions $(i), (iii)$ and (iv) of split standardizable set are checked.

Now consider the inclusion $\Delta(j) \hookrightarrow T(j)$ given by Proposition 1.5.109. Applying the functor $\mathrm{Hom}_A(\Delta(i), -)$ we obtain the monomorphism $\mathrm{Hom}_A(\Delta(i), \Delta(j)) \to \mathrm{Hom}_A(\Delta(i), T(j))$. By Corollary 1.5.119, $\mathrm{Hom}_A(\Delta(i), T(j)) \in R$-proj. Since $\dim R \leq 1$, $\mathrm{Hom}_B(\theta(i), \theta(j)) \simeq \mathrm{Hom}_A(\Delta(i), \Delta(j)) \in R$-proj. Hence $\Theta = \{\theta(i) \colon 1 \leq i \leq n\}$ is a split standardizable set of B-mod. □

Observe that if B is self-injective relative to R, then Q (according to the notation of the previous Proposition) is also a cogenerator of B-mod. Furthermore, each $B(\mathfrak{m})$ is self injective for every maximal ideal $\mathfrak{m}$ in R, $Q(\mathfrak{m})$ is a generator-cogenerator. Hence, $P(\mathfrak{m})$ is a faithful projective-injective $A(\mathfrak{m})$-module for every maximal ideal $\mathfrak{m}$ in R, and therefore P is a projective (A, R)-injective-strongly faithful module. So, for self-injective algebras relative to R, Dlab–Ringel standardization gives us the construction of a unique 1-faithful quasi-hereditary cover for B which arises from a projective relative injective module, connecting this topic with relative dominant dimension.

We note again that this technique as described is only for rings of Krull dimension one. But if the algebra B admits an integral version, then using ideas similar to Proposition 3.5.9, we can construct a cover of B by changing the ground ring of the cover constructed for the integral version. This would extend this construction for complete local commutative Noetherian rings.

Unfortunately, there is no guarantee that a projective Noetherian algebra over a commutative Noetherian ring possesses a split quasi-hereditary cover.

Corollary 3.7.3. *Let C_3 be the abelian group of order 3. Let $\mathbb{Z}_7$ be the localization of $\mathbb{Z}$ at $7\mathbb{Z}$. The group algebra $\mathbb{Z}_7C_3$ over $\mathbb{Z}_7$ does not have a split quasi-hereditary cover. Moreover, the group algebra $\mathbb{Z}C_3$ does not have a split quasi-hereditary cover.*

Proof. In [Woo74], it was shown that the ring $\mathbb{Z}_7C_3$ is not semi-perfect. The ring $\mathbb{Z}_7$ is a local commutative Noetherian ring. By Theorem 1.5.84, every split quasi-hereditary algebra over $\mathbb{Z}_7$ is semi-perfect. So, $\mathbb{Z}_7C_3$ cannot have a split quasi-hereditary cover because of Theorem 1.7.11. Since any split quasi-hereditary cover remains a split quasi-hereditary cover under localization we obtain that $\mathbb{Z}C_3$ cannot have a split quasi-hereditary cover. □

Observe that any division ring is local, so the split condition is not the reason why the existence of split quasi-hereditary covers of $\mathbb{Z}_7C_3$ fails. We should remark that if we are interested only in covers of finite global dimension, then the construction used by Koenig in [Kön91] can still be applied to these cases when the ground ring is a discrete valuation ring. Again, if the algebra in question admits an integral version, then one can apply the same idea as described for a generalization of Dlab-Ringel standardization for higher Krull dimensions to determine the existence of 1-faithful split quasi-hereditary covers.

Chapter 4

Applications and Examples - Part I

In this chapter, we consider applications of our theory of dominant dimension and covers. In particular, we compute the relative dominant dimension of Schur algebras (and also of q-Schur algebras) $S_R(n,d)$ satisfying $n \geq d$ and of block algebras of a deformation of the BGG category $\mathcal{O}$ in the sense of Gabber and Joseph [GJ81]. We show that both algebras are split quasi-hereditary and together with their projective relative injective modules form split quasi-hereditary covers of certain relative self-injective algebras. We compute Hemmer-Nakano dimensions with respect to these covers, clarifying the interconnections between relative dominant dimension and the Hemmer-Nakano dimension. In addition, we consider additional examples to explain the necessity of assumptions imposed in the above statements of previous chapters.

4.1 Classical Schur algebras

The study of Schur algebras goes back to the PhD thesis of Schur [Sch01]. Using Schur algebras, he connected the polynomial representation theory of the complex general linear group with the representation theory of the symmetric group over the complex numbers. The latter was known at the time due to Frobenius [Fro00]. Nowadays, the connection is used oppositely. A classical reference for the study of Schur algebras (over infinite fields) is [Gre07].

Let R be a commutative ring with identity. Fix natural numbers n,d. The symmetric group on d letters S_d acts by place permutation on the d-fold tensor product $(R^n)^{\otimes d}$, that is,

$$(v_1 \otimes \cdots \otimes v_d)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},\ \sigma \in S_d,\ v_i \in R^n.$$

We will write $V_R^{\otimes d}$ instead of $(R^n)^{\otimes d}$ or simply $V^{\otimes d}$ when the ground ring is well understood. In particular, $V^{\otimes d}$ is a right module over the group algebra RS_d.

Definition 4.1.1. [Gre07] The subalgebra $\operatorname{End}_{RS_d}\left(V^{\otimes d}\right)$ of the endomorphism algebra $\operatorname{End}_R\left(V^{\otimes d}\right)$ is called the **Schur algebra**. We will denote it by $S_R(n,d)$.

We recall some facts about these algebras.

Let $I(n,d)$ be the set of maps $i\colon \{1,\ldots,d\} \to \{1,\ldots,n\}$. We write $i(a) = i_a$. We can associate to $I(n,d)$ a right S_d-action by place permutation. In the same way, S_d acts on $I(n,d) \times I(n,d)$, by setting:

$$(i,j)\sigma = (i\sigma, j\sigma), \quad \forall i,j \in I(n,d), \forall \sigma \in S_d. \tag{4.1.0.1}$$

We will write $(i,j) \sim (f,g)$ if (i,j) and (f,g) belong to the same S_d-orbit. Then, $S_R(n,d)$ has a basis over R $\{\xi_{i,j} \mid (i,j) \in I(n,d) \times I(n,d)\}$ satisfying

$$\xi_{i,j}(e_{s_1} \otimes \cdots \otimes e_{s_d}) = \sum_{\substack{l \in I(n,d) \\ (l,s) \sim (i,j)}} e_{l_1} \otimes \cdots \otimes e_{l_d}, \tag{4.1.0.2}$$

for a given basis $\{e_{s_1} \otimes \cdots \otimes e_{s_d} \colon 1 \leq s_1, \ldots, s_d \leq n\}$ of $V^{\otimes d}$. In particular, $\xi_{i,j} = \xi_{f,g}$ if and only if $(i,j) \sim (f,g)$. In this section, we will sometimes abbreviate $e_{i_1} \otimes \cdots \otimes e_{i_d}$ to e_i, $i \in I(n,d)$.

An immediate consequence of the existence of an R-basis for $S_R(n,d)$ satisfying (4.1.0.2) is the existence of a base change property

$$R \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,d) \simeq S_R(n,d). \tag{4.1.0.3}$$

It also follows that $R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^{\otimes d} \simeq V_R^{\otimes d}$ as $S_R(n,d)$-modules.

For each $i \in I(n,d)$ we can associate a weight $\lambda(i)$. More precisely, a **weight** of an element $i \in I(n,d)$ is the composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of d in at most n parts with $\lambda_j = |\{1 \leq \mu \leq d \colon i_\mu = j\}$. Let $\Lambda(n,d)$ be the set of all weights associated with $I(n,d)$. Then, by (4.1.0.2), for each $\lambda(i) \in \Lambda(n,d)$ there exists an idempotent $\xi_\lambda := \xi_{i,i}$. Let $\Lambda^+(n,d)$ be the subset of $\Lambda(n,d)$ formed by the partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ of d in at most n parts. $\Lambda^+(n,d)$ is partially ordered by the dominance order $\leq$, that is, $\lambda \leq \mu$ if and only if $\lambda_1 + \ldots + \lambda_j \leq \mu_1 + \ldots + \mu_j$, for all j. Let $\Lambda^+(n,d) \to \{1, \ldots, t\}$, $\lambda^k \mapsto k$ be an increasing bijection. Set e^k to be the idempotent $\sum_{l \geq k} \xi_{\lambda^l}$. Put $J_k = S_R(n,d) e^k S_R(n,d)$. Then, with this notation,

Theorem 4.1.2. *For any commutative Noetherian ring R, the Schur algebra $S_R(n,d)$ is a split quasi-hereditary algebra over R with split heredity chain $0 \subset J_t \subset \cdots \subset J_2 \subset J_1 = S_R(n,d)$.*

Proof. The statement for algebraically closed fields follows from Theorem 4.1 of [Par89]. For arbitrary fields see [PW91, Theorem 11.5.2]. The statement for commutative Noetherian rings which are not fields follows from Theorem 3.7.2 of [CPS90]. An alternative proof for this statement without using Theorem 3.7.2 of [CPS90] is to apply Theorem 1.5.73. Using filtrations, this statement for principal ideal domains follows by [Don87, 1.2] together with Theorem 1.5.65. Another proof for the general case of the present statement can be found in [Gre93, 7.2]. ☐

Due to the quasi-hereditary structure on $S_R(n,d)$, if R is a regular ring with finite global dimension, then the Schur algebra $S_R(n,d)$ has finite global dimension. The global dimension of $S_R(n,d)$ was computed in [Tot97]. The standard modules associated with this split heredity chain are called **Weyl modules**. In particular, the Weyl modules are indexed by the partitions of d in at most n parts. Also, the simple $S_K(n,d)$ modules are indexed by the partitions of d in at most n parts for K a field. As of the time of writing, determining simple modules of the Schur algebra remains still an open problem.

In addition to the quasi-hereditary structure, we can associate a cellular structure to the Schur algebra. Consider the R-linear map $\iota \colon S_R(n,d) \to S_R(n,d)$ given by $\iota(\xi_{f,g}) = \xi_{g,f}$, $f,g \in I(n,d)$. We call ι the involution of the Schur algebra. Observe that $\iota(\xi_\lambda) = \xi_\lambda$ for every $\lambda \in \Lambda^+(n,d)$. In particular, ι preserves all idempotents in the split heredity chain of $S_R(n,d)$. Hence, by a version of Corollary 4.2 [KX98] for commutative Noetherian rings (see Proposition 1.6.12), $S_R(n,d)$ is a cellular algebra. Note that, the order of $\Lambda^+(n,d)$ for the definition of cellular algebra is now the reversed order of the dominance order.

We will now focus on the case $n \geq d$. In this case,

$$V^{\otimes d} \simeq S_R(n,d)\xi_{(1,\dots,d),(1,\dots,d)}, \quad DV^{\otimes d} \simeq \xi_{(1,\dots,d),(1,\dots,d)}S_R(n,d). \tag{4.1.0.4}$$

Hence, $V^{\otimes d}$ is a projective $(S_R(n,d),R)$-injective $S_R(n,d)$-module.

Remark 4.1.3. $V^{\otimes d}$ is not an $S_R(n,d)$-injective if R is a regular Noetherian commutative ring with positive Krull dimension. In fact, assume that R is a local commutative Noetherian regular ring with unique (non-zero) maximal ideal $\mathfrak{m}$. Then, $\operatorname{Ext}_R^{\dim R}(R(\mathfrak{m}),R) \neq 0$. Since $V^{\otimes d}$ is a progenerator over R we obtain

$$0 \neq V^{\otimes d} \otimes_R \operatorname{Ext}_R^{\dim R}(R(\mathfrak{m}),R) \simeq \operatorname{Ext}_{S_R(n,d)}^{\dim R}(V^{\otimes d}(\mathfrak{m}),V^{\otimes d}). \qquad \triangle$$

By the Schur functor we mean the functor $F_R = \operatorname{Hom}_{S_R(n,d)}(V^{\otimes d},-)\colon S_R(n,d)\text{-mod} \to RS_d\text{-mod}$ (we will write just F when there is no confusion on the ground ring R). Using Theorem 3.4 of [Cru19] and (4.1.0.4), we can deduce that $(S_R(n,d),V^{\otimes d})$ is a split quasi-hereditary cover of RS_d. Much of the representation theory of symmetric groups can be studied through the representation theory of Schur algebras using the Schur functor. For example, since $\iota(\xi_{(1,\dots,d),(1,\dots,d)}) = \xi_{(1,\dots,d),(1,\dots,d)}$ and

$$RS_d \simeq \operatorname{End}_{S_R(n,d)}(V^{\otimes d}) \simeq \xi_{(1,\dots,d),(1,\dots,d)}S_R(n,d)\xi_{(1,\dots,d),(1,\dots,d)}, \tag{4.1.0.5}$$

F sends the split heredity chain of $S_R(n,d)$ to a cell chain of RS_d. This makes RS_d a cellular algebra. In particular, the Schur functor sends the Weyl modules to the cell modules of RS_d. Usually, the cell modules of RS_d are also called Specht modules. However, a few remarks are in order regarding the cell modules obtained here by this construction and the ones that appear in the literature. By $\theta(\lambda)$ we mean the cell module $F\Delta(\lambda)$, $\lambda \in \Lambda^+(n,d)$. Denote by $S_J(\lambda)$ and $S_M(\lambda)$ the Specht modules defined by James [Jam78] and Mathas [Mat99], respectively. The Weyl modules $\Delta(\lambda)$ coincide with the left Weyl modules of [CPS96]. They work with the functor $\operatorname{Hom}_{RS_d}(-,V^{\otimes d})\colon$ mod-$RS_d \to S_R(n,d)$-mod while in our work the adjoint functor of the Schur functor is $\operatorname{Hom}_{RS_d}(DV^{\otimes d},-)\colon RS_d\text{-mod} \to S_R(n,d)\text{-mod}$. Thus, the left cell modules $F\Delta(\lambda)$ coincide with $DS_J(\lambda)$ while $S_M(\lambda)$ coincide with the twisted modules $F\Delta(\lambda')^\iota$, where λ' is the conjugate partition of λ. For a given natural number p, a partition $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda^+(n,d)$ is called **p-regular** if each λ_i occurs no more than $p-1$ times in λ.

The following result illustrates what are the projective-injective modules over $S_R(n,d)$. This result will play a role to find what is the image of $V^{\otimes d}$ in Schur algebras with indexes $n<d$.

Proposition 4.1.4. *Let K be an algebraically closed field with positive characteristic and $n \geq d$ be natural numbers. Let $\lambda \in \Lambda^+(n,d)$. Then,*

(i) $P(\lambda)$ is projective-injective if and only if λ is a conjugate of a char$K$*-regular partition of d.*

(ii) λ is a char$K$*-regular partition of d if and only if the (partial) tilting module $T(\lambda)$ is projective-injective module.*

Proof. For (i) see [CPS96, (5.2.7), (5.2.8)]. Denote by $R(S_K(n,d))$ the Ringel dual of $S_K(n,d)$ with $\Delta_R(\mu)$ and $P_R(\mu)$ being the standard and projective modules, respectively. By [Don93, (3.8)] there exists an equivalence functor $(-)^\theta\colon R(S_K(n,d))\text{-mod} \to S_K(n,d)\text{-mod}$ satisfying $P_R(\lambda)^\theta = P(\lambda')$ and $\Delta_R(\lambda)^\theta = \Delta(\lambda')$, where λ' denotes the conjugate partition of λ. Assuming that $T(\lambda)$ is projective-injective then $\operatorname{Hom}_{S_K(n,d)}(T,T(\lambda))$ is also a (partial) tilting module. Since $(-)^\theta$ preserves the (partial) tilting modules, $P(\lambda') \simeq P_R(\lambda)^\theta \simeq \operatorname{Hom}_{S_K(n,d)}(T,T(\lambda))^\theta$ is also (partial) tilting. By Lemma 3.2 of [FK11b], $P(\lambda')$ is a projective-injective module. Hence, λ' is a conjugate of a char$K$-regular partition of d. Therefore, λ is a char$K$-regular partition of d. Since there are no more partitions that index projective-injective modules other than the regular ones the converse statement follows. $\square$

Observation 4.1.5. By [Don93, (3.8)] and Lemma 1.5.134, the Schur algebra $S_R(n,d)$ with $n \geq d$ is Ringel self-dual for every commutative Noetherian local ring R.

We now wish to determine the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$, generalizing the results of Hemmer and Nakano [HN04] by completely determining the quality of the correspondence between Weyl filtrations and Specht filtrations. This is achieved through the computation of the relative dominant dimension of $S_R(n,d)$ extending some results of Fang and Koenig [FK11b] contained in the following Theorem.

Theorem 4.1.6. *[FK11b, Theorem 5.1] Let K be a field.*

$$\operatorname{domdim} S_K(n,d) = \begin{cases} 2(\operatorname{char} K - 1) & \text{if } d \geq \operatorname{char} K > 0 \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.1.0.6}$$

In the following, we will show that we can compute the dominant dimension of $S_R(n,d)$ by knowing the invertible elements of R, which we will denote by $U(R)$.

Theorem 4.1.7. *Let R be a commutative Noetherian ring. If $n \geq d$ are natural numbers, then $(S_R(n,d), V^{\otimes d})$ is a relative gendo-symmetric R-algebra and*

$$\operatorname{domdim}(S_R(n,d), R) = \inf\{2k \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\} \geq 2. \tag{4.1.0.7}$$

Proof. $V_K^{\otimes d}$ is a projective-injective faithful $S_K(n,d)$-module for every field K. By Proposition 2.5.4, $(S_R(n,d), V^{\otimes d}, DV^{\otimes d})$ is a relative QF3 R-algebra. Denote by MaxSpec(R) the set of maximal ideals of $\mathfrak{m}$. By Theorem 2.5.13,

$$\operatorname{domdim}(S_R(n,d), R) = \inf\{\operatorname{domdim} S_R(n,d) \otimes_R R(\mathfrak{m}) | \mathfrak{m} \in \operatorname{MaxSpec}(R)\} \tag{4.1.0.8}$$

$$= \inf\{\operatorname{domdim} S_{R(\mathfrak{m})}(n,d) | \mathfrak{m} \in \operatorname{MaxSpec}(R)\} \geq 2. \tag{4.1.0.9}$$

By relative Morita-Tachikawa correspondence, $V^{\otimes d}$ is a generator of RS_d satisfying $V^{\otimes d} \otimes_{RS_d} DV^{\otimes d} \in R$-proj. Therefore, $(S_R(n,d), V^{\otimes d})$ is a relative gendo-symmetric R-algebra because RS_d is a relative symmetric R-algebra.

Let $k \in \mathbb{N}$ such that $(k+1)1_R \notin U(R)$ and $k < d$. Then, $(k+1)1_R \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. In particular, char$R(\mathfrak{m})$ is positive and it is less or equal than $k+1 \leq d$. Hence, domdim$S_{R(\mathfrak{m})}(n,d) \leq 2k$ for some maximal ideal $\mathfrak{m}$ of R. This shows that

$$\operatorname{domdim}(S_R(n,d), R) \leq \inf\{2k \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\}. \tag{4.1.0.10}$$

In particular, if $\operatorname{domdim}(S_R(n,d), R) = +\infty$ there is nothing more to prove. Assume now that $\operatorname{domdim}(S_R(n,d), R) = l \geq 2$. So, there exists $\mathfrak{m} \in \operatorname{MaxSpec}(R)$ such that

$$2(\operatorname{char} R(\mathfrak{m}) - 1) = l, \quad \text{and} \quad \operatorname{char} R(\mathfrak{m}) \leq d. \tag{4.1.0.11}$$

In particular, the image of char$R(\mathfrak{m})1_R$ in $R(\mathfrak{m})$ is zero, and so char$R(\mathfrak{m})1_R \in \mathfrak{m}$. Hence, char$R(\mathfrak{m})1_R \notin U(R)$. Therefore,

$$l \in \{2k \in \mathbb{N} | (k+1)1_R \notin U(R),\ k < d\}. \tag{4.1.0.12}$$

This finishes the proof. □

Once again, we see that the invertible elements of the ground ring determine the quality of a double centralizer property. In [Cru19], a ring having sufficiently many invertible elements under some mild assumptions was a sufficient condition for Schur–Weyl duality to hold. We recall that in that case either the quality of the double centralizer property is the best possible (coming from an equivalence of categories) or is the worst possible (the double centralizer property does not exist at all).

For the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ the relevant dominant dimension to consider is the relative dominant dimension of a characteristic tilting module.

Corollary 4.1.8. *Let R be a commutative Noetherian ring and assume that $n \geq d$. Let T be a characteristic tilting module of $S_R(n,d)$. Then,*

$$\operatorname{domdim}_{(S_R(n,d),R)} T = \inf\{k \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\} \geq 1. \tag{4.1.0.13}$$

Proof. The result follows from applying Theorem 4.1.7 and Theorem 2.11.3. Alternatively, one can reproduce the proof of Theorem 4.1.7 together with Theorem 4.3 of [FK11b]. □

In Theorem 4.1.7, we saw that $V^{\otimes d}$ is an $(S_R(n,d),R)$-strongly faithful module. In general for Noetherian algebras, it is difficult to prove directly that a module is strongly faithful and whenever possible we always prefer to show this property using change of rings techniques. However, it is not difficult to show directly that $V^{\otimes d}$ is strongly faithful. This is the aim of the next example.

Example 4.1.9. Let $\{e_{s_1}\otimes\cdots\otimes e_{s_d} : 1 \leq s_1,\ldots,s_d \leq n\}$ be an R-basis of $V^{\otimes d}$. Choose Λ to be a set of representatives of S_d-orbits on $I(n,d)\times I(n,d)$. Define the R-map $\upsilon \in \operatorname{Hom}_R(S_R(n,d),(V^{\otimes d})^t)$, satisfying

$$\upsilon(\varphi) = \sum_{(i,j)\in\Lambda} \kappa_{i,j}(\varphi(e_{j_1}\otimes\cdots\otimes e_{j_d})), \quad \varphi \in S_R(n,d), \tag{4.1.0.14}$$

with $\kappa_{i,j}$ and $\pi_{i,j}$, $(i,j)\in\Lambda$, being the inclusion and projection mappings of $V^{\otimes}$ into the direct sum $(V^{\otimes d})^t$ as $S_R(n,d)$-modules, respectively, where $t = \binom{n^2+d-1}{d}$. Observe that

$$\upsilon(\eta\varphi) = \sum_{(i,j)\in\Lambda} \kappa_{i,j}(\eta\varphi(e_{j_1}\otimes\cdots\otimes e_{j_d})) = \sum_{(i,j)\in\Lambda} \eta\kappa_{i,j}\varphi(e_{j_1}\otimes\cdots\otimes e_{j_d}) = \eta\upsilon(\varphi),\ \varphi,\eta\in S_R(n,d). \tag{4.1.0.15}$$

Thus, $\upsilon \in \operatorname{Hom}_{S_R(n,d)}(S_R(n,d),V^{\otimes d^t}))$. For each $(i,j)\in\Lambda$, define $f_{i,j}\in\operatorname{Hom}_R(V^{\otimes d},S_R(n,d))$ satisfying

$$f_{i,j}(e_{s_1}\otimes\cdots\otimes e_{s_d}) = \begin{cases} \xi_{i,j} & \text{if } (s_1,\ldots,s_d) = i \\ 0, & \text{otherwise.}\end{cases} \tag{4.1.0.16}$$

Finally, denote by ε the R-map $\sum_{(i,j)\in\Lambda} f_{i,j}\circ\pi_{i,j} \in \operatorname{Hom}_R((V^{\otimes d})^t,S_R(n,d))$. Then, the following holds,

$$\varepsilon\circ\upsilon(\xi_{f,g}) = \varepsilon\left(\sum_{(i,j)\in\Lambda}\kappa_{i,j}\xi_{f,g}(e_{j_1}\otimes\cdots\otimes e_{j_d})\right) = \sum_{(t,u)\in\Lambda}\sum_{(i,j)\in\Lambda} f_{t,u}\pi_{t,u}\kappa_{i,j}\xi_{f,g}(e_{j_1}\otimes\cdots\otimes e_{j_d}) \tag{4.1.0.17}$$

$$= \sum_{(i,j)\in\Lambda} f_{i,j}\xi_{f,g}(e_{j_1}\otimes\cdots\otimes e_{j_d}) = \sum_{(i,j)\in\Lambda} f_{i,j}\left(\sum_{\substack{l\in I(n,d)\\(l,j)\sim(f,g)}} e_{l_1}\otimes\cdots\otimes e_{l_d}\right) \tag{4.1.0.18}$$

$$= \sum_{(i,j)\in\Lambda} \sum_{\substack{l\in I(n,d)\\(l,j)\sim(f,g)}} \mathbb{1}_{\{i\}}(l)\xi_{i,j} = \sum_{(i,j)\in\Lambda} \mathbb{1}_{\{(i,j)\sim(f,g)\}}(i,j)\xi_{i,j} = \xi_{f,g}. \quad (4.1.0.19)$$

Here, $\mathbb{1}_{\{(i,j)\sim(f,g)\}}$ denotes the indicator function. Therefore, υ is an $(S_R(n,d),R)$-monomorphism. $\triangle$

4.1.1 Hemmer-Nakano dimension of $S_R(n,d)$-proj

In sections 4.1.1 and 4.1.2, R is assumed to be a local regular Noetherian commutative ring.

In relative Mueller characterization (see Theorem 2.4.15) we saw that the vanishing of certain Ext groups alone might not give the value of relative dominant dimension, only a lower bound dependent of the Krull dimension of the ground ring. Of course, when the ground ring is a field the dominant dimension can be determined using only Ext groups. Fortunately, for the Schur algebra over a local ring we can completely determine the Hemmer-Nakano dimensions in terms of relative dominant dimensions. We can reduce the problem to local rings R due to Propositions 3.3.4 and 3.3.3. Essentially, the value of Hemmer-Nakano dimension of $S_R(n,d)$-proj in terms of relative dominant dimension divides in two separate cases. Either a local Noetherian regular commutative ring contains a field or not.

4.1.1.1 Case 1 - R contains a field

First, we would like to recall the following elementary result.

Lemma 4.1.10. *A local integral domain R with maximal ideal $\mathfrak{m}$ is equicharacteristic, that is,* $\operatorname{char} R(\mathfrak{m}) = \operatorname{char} R$, *if and only if R contains a field.*

Proof. Assume that R contains a field. If $\operatorname{char} R$ is a prime number, there is nothing to prove. Suppose that $\operatorname{char} R = 0$ and $K \subset R$. In particular, $\operatorname{char} K = 0$. It follows that $\operatorname{char} R/\mathfrak{m} = 0$ by considering the injective map $K \to R \to R/\mathfrak{m}$.

Conversely, assume that R is equicharacteristic. Assume that $\operatorname{char} R/\mathfrak{m} = 0$. Then, the map $\mathbb{Z} \to R \to R/\mathfrak{m}$ is injective. Since R is local, for each $n \in \mathbb{Z}$, $n1_R$ is invertible in R. So, we can embed $\mathbb{Q}$ into R. Assume now that $\operatorname{char} R/\mathfrak{m} = p > 0$ is a prime number. So, the map $\mathbb{Z} \to R \to R/\mathfrak{m}$ factors through $\mathbb{Z}/p\mathbb{Z}$. Moreover, we can embed $\mathbb{F}_p$ into R. $\square$

The idea behind the definition of an equicharacteristic ring comes from the fact that $R/\mathfrak{m}$ is the residue field of every quotient ring of R. In particular, the characteristic of $R/\mathfrak{m}$ divides the characteristic of every quotient ring of R. On the other hand, the characteristic of every quotient ring of R divides the characteristic of R. Since these characteristics are either a prime number or zero it follows that for equicharacteristic rings all these characteristics coincide.

Theorem 4.1.11. *Let R be a local regular commutative Noetherian ring containing a field k as a subring. Assume that $n \geq d$. Then,*

$$\operatorname{HNdim}_F(S_R(n,d)\text{-proj}) = \operatorname{domdim}(S_R(n,d),R) - 2 = \inf\{2(k-1) \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\} \geq 0.$$

Proof. Let K be the quotient field of R. Then, $\operatorname{char} K = \operatorname{char} R = \operatorname{char} R(\mathfrak{m})$, where $\mathfrak{m}$ is the unique maximal ideal of R. In particular, $\operatorname{domdim} S_K(n,d) = \operatorname{domdim}(S_R(n,d),R)$. By Theorem 3.5.6 and the flatness of K over R,

$$\begin{aligned}\operatorname{HNdim}_{F_R}(S_R(n,d\text{-proj})) \geq \operatorname{domdim}(S_R(n,d),R) - 2 = \operatorname{domdim} S_K(n,d) - 2 \quad (4.1.1.1)\\ = \operatorname{HNdim}_{F_K}(S_K(n,d)\text{-proj}) \geq \operatorname{HNdim}_{F_R}(S_R(n,d)\text{-proj}).\end{aligned}$$

By Theorem 4.1.7, the result follows. □

4.1.1.2 Case 2 - R does not contain a field

Theorem 4.1.12. *Let R be a local regular commutative Noetherian ring that does not contain a field as a subring. Assume that $n \geq d$. Then,*

$$\operatorname{HNdim}_F(S_R(n,d)\text{-proj}) = \operatorname{domdim}(S_R(n,d),R) - 1 = \inf\{2k-1 \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k<d\} \geq 1. \tag{4.1.1.2}$$

Proof. Of course, R has Krull dimension greater or equal than one. By assumption, $\operatorname{char} R = 0$ and $\operatorname{char} R(\mathfrak{m})$ is a prime number $p > 0$. In particular, $\operatorname{char} K = 0$ for the quotient field of R. By Theorem 3.5.7, $\operatorname{HNdim}_F(S_R(n,d)\text{-proj}) \geq \operatorname{domdim}(S_R(n,d),R) - 1$. We can assume, without loss of generality, that $p \geq d$. Otherwise, equation (4.1.1.2) is just $+\infty = +\infty = +\infty \geq 1$, and consequently the equality holds. Since R is a local regular ring, R is a unique factorization domain. Therefore, we can write $p1_R = up_1 \cdots p_n$ for some prime elements of R. So, $p1_R$ belongs to a prime ideal $\mathfrak{p}$ of height one. Hence, $\operatorname{char} R/\mathfrak{p}$ is p. Let $Q(R/\mathfrak{p})$ be the quotient field of $R/\mathfrak{p}$. Then, $\operatorname{char} Q(R/\mathfrak{p}) = p$ and $\operatorname{domdim} S_{Q(R/\mathfrak{p})}(n,d) = \operatorname{domdim}(S_R(n,d),R)$. Therefore,

$$\operatorname{HNdim}_{F_{R/\mathfrak{p}}}(S_{R/\mathfrak{p}}(n,d)\text{-proj}) \leq \operatorname{HNdim}_{F_{Q(R/\mathfrak{p})}}(S_{Q(R/\mathfrak{p})}(n,d)\text{-proj}) = \operatorname{domdim}(S_R(n,d),R) - 2. \tag{4.1.1.3}$$

By Corollary 3.3.10, the Hemmer-Nakano dimension of $S_R(n,d)$-proj cannot be higher than $\operatorname{domdim}(S_R(n,d),R) - 1$. The result now follows by Theorem 4.1.7. □

4.1.2 Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$

Because of Corollary 4.1.8, we divide the computation of the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ in two cases, as well.

4.1.2.1 Case 1 - R contains a field

Theorem 4.1.13. *Let R be a local regular commutative Noetherian ring containing a field k as a subring. Assume that $n \geq d$. Let T be a characteristic tilting module of $S_R(n,d)$. Then,*

$$\operatorname{HNdim}_F(\mathscr{F}(\tilde{\Delta})) = \operatorname{domdim}_{(S_R(n,d),R)} T - 2 = \inf\{k-2 \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k<d\} \geq -1. \tag{4.1.2.1}$$

Proof. Let K be the quotient field of R. Since R contains a field, $\operatorname{domdim}_{(S_R(n,d),R)} T = \operatorname{domdim}_{S_K(n,d)} K \otimes_R T$. Note that $K \otimes_R T$ is the characteristic tilting module of $S_K(n,d)$. Again, by Theorem 3.5.6 and the flatness of K,

$$\operatorname{HNdim}_F(\mathscr{F}(\tilde{\Delta})) \geq \operatorname{domdim}_{(S_R(n,d),R)} T - 2 = \operatorname{domdim}_{S_K(n,d)} K \otimes_R T - 2 \tag{4.1.2.2}$$

$$= \operatorname{HNdim}_{F_K}(\mathscr{F}(K \otimes_R \Delta)) \geq \operatorname{HNdim}_F(\mathscr{F}(\tilde{\Delta})). \tag{4.1.2.3}$$

By Corollary 4.1.8, the result follows. □

Remark 4.1.14. We should point out that there is a typo in Corollary 3.9.2 of [HN04]. It should read $p-3$ instead of $p-2$. This typo is a repercussion of a typo in the use of spectral sequences in the published version [KN01, 2.3]. There we should read $0 \leq i \leq t$ instead of $0 \leq i \leq t+1$. The reader can check Lemma 3.1.16 for clarifications. The result [KN01, 2.3] was corrected in Kleshchev's homepage. △

4.1.2.2 Case 2 - R does not contain a field

Theorem 4.1.15. *Let R be a local regular commutative Noetherian ring that does not contain a field as a subring. Assume that $n \geq d$. Let T be a characteristic tilting module of $S_R(n,d)$. Then,*

$$\mathrm{HNdim}_F(\mathscr{F}(\tilde{\Delta})) = \mathrm{domdim}_{(S_R(n,d),R)} T - 1 = \inf\{k-1 \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k<d\} \geq 0. \tag{4.1.2.4}$$

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. By assumption, $\operatorname{char} R = 0$ and $\operatorname{char} R(\mathfrak{m}) = p$ for some prime number. By Theorem 3.5.6, $\mathrm{HNdim}_F(\mathscr{F}(\tilde{\Delta})) \geq \mathrm{domdim}_{(S_R(n,d),R)} T - 2 = p-3$. Hence, if $p<d$, the equation (4.1.2.4) reads $+\infty = +\infty = +\infty$. So, the equality holds. Assume that $p \geq d$ and $p \neq 2$. Let K be the quotient field of R. So, $\operatorname{char} K = 0$ and therefore, $\mathrm{domdim}_{S_K(n,d)} K \otimes_R T = +\infty$. Since $p \neq 2$, $\mathrm{HNdim}_F(\mathscr{F}(\tilde{\Delta})) \geq 0$. Therefore, we are in the conditions of Theorem 3.5.7. Hence,

$$\mathrm{HNdim}_F(\mathscr{F}(\tilde{\Delta})) \geq \mathrm{domdim}_{(S_R(n,d),R)} T - 1. \tag{4.1.2.5}$$

Constructing a prime ideal $\mathfrak{p}$ of height one as in the proof of Theorem 4.1.12, we obtain that $\operatorname{char} R/\mathfrak{p} = p$. Denote by $Q(R/\mathfrak{p})$ the quotient field of $R/\mathfrak{p}$. Therefore,

$$\begin{aligned}\mathrm{HNdim}_{F_{R/\mathfrak{p}}}(\mathscr{F}(R/\mathfrak{p}\otimes_R \tilde{\Delta})) &\leq \mathrm{HNdim}_{F_{Q(R/\mathfrak{p})}}(\mathscr{F}(Q(R/\mathfrak{p}\otimes_R \Delta)) &(4.1.2.6)\\ &= \mathrm{domdim}_{S_{Q(R/\mathfrak{p})}(n,d)} Q(R/\mathfrak{p})\otimes_R T - 2 = p-3 = \mathrm{domdim}_{(S_R(n,d),R)} T - 2.\end{aligned}$$

By Corollary 3.3.10, the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ cannot be higher than $\mathrm{domdim}_{(S_R(n,d),R)} T - 1$. This finishes our claim for $p \neq 2$. It remains to show that the equality holds for $p=2$. In other words, we want to show that $\mathrm{HNdim}_F(\mathscr{F}(\tilde{\Delta})) = 0$ whenever $p=2$. The existence of a prime ideal of height one $\mathfrak{p}$ such that $R/\mathfrak{p}$ has characteristic 2 implies by the same argument as before that the Hemmer-Nakano dimension of $\mathscr{F}(\tilde{\Delta})$ cannot be higher than $\mathrm{domdim}_{(S_R(n,d),R)} T - 1 = 0$.

Applying Theorem 3.5.8 taking into account that $\operatorname{char} K = 0$ and consequently $\mathrm{HNdim}_{F_K}(\mathscr{F}(K\otimes_R \Delta)) = +\infty$ the proof becomes complete. ◻

We note that the situation for $\mathbb{Z}$ is way better than for $\mathbb{F}_2$. In fact,

$$\mathrm{HNdim}_{F_{\mathbb{Z}}}(S_{\mathbb{Z}}(n,d)\text{-proj}) = 1, \quad \mathrm{HNdim}_{F_{\mathbb{F}_2}}(S_{\mathbb{F}_2}(n,d)\text{-proj}) = 0 \tag{4.1.2.7}$$

$$\mathrm{HNdim}_{F_{\mathbb{Z}}}(\mathscr{F}(\tilde{\Delta}_{\mathbb{Z}})) = 0, \quad \mathrm{HNdim}_{F_{\mathbb{F}_2}}(\mathscr{F}(\Delta_{\mathbb{F}_2})) = -1. \tag{4.1.2.8}$$

These results are compatible with the results of [CPS96]. Moreover, these two particular cases were already known for them and they used this knowledge to define the Young modules and Specht modules of the group algebra $\mathbb{F}_2 S_d$ by defining first the Young and Specht modules for the integral group algebra $\mathbb{Z}S_d$ and then applying the functor $\mathbb{F}_2 \otimes_{\mathbb{Z}} -$. This becomes more relevant for Weyl modules over fields of characteristic two since they cannot be reconstructed from Specht modules. That is, the image of a Specht module under the adjoint functor of the Schur functor only contains, in general, a Weyl module.

4.1.3 Uniqueness of covers for RS_d

Considering the localization of $\mathbb{Z}$ away from 2, $\mathbb{Z}[\frac{1}{2}]$, on Theorem 4.1.15 yields that $(S_{\mathbb{Z}[\frac{1}{2}]}(n,d), V_{\mathbb{Z}[\frac{1}{2}]}^{\otimes d})$ is a 1-faithful cover of $\mathbb{Z}[\frac{1}{2}]S_d$. By Corollary 3.6.6, this Schur algebra is the unique cover of RS_d which sends the standard modules (in this case the Weyl modules) to the Specht modules. We remark that this improves the situation for the fields of characteristic 3 since they are algebras over $\mathbb{Z}[\frac{1}{2}]$ and for characteristic 3 the Hemmer-Nakano dimension of $\mathscr{F}(\Delta)$ is only zero.

For the ring of integers, we can already conclude that there is no better cover than the Schur algebra to study the Specht modules over the symmetric group.

Theorem 4.1.16. *Let k be a field of characteristic two and $d \geq 2$. Let $\theta = \{\theta(\lambda) \colon \lambda \in \Lambda^+(d)\}$ be the cell modules of kS_d. Then, (kS_d, θ) does not have a 0-split quasi-hereditary cover. Moreover, there are no 1-faithful split quasi-hereditary covers of $\mathbb{Z}S_d$ satisfying $F\Delta(\lambda) = \theta_{\mathbb{Z}}(\lambda), \lambda \in \Lambda^+(d)$, where F is the Schur functor associated to the cover of $\mathbb{Z}S_d$.*

Proof. Assume, by contradiction, that (A,P) is a 0-faithful quasi-hereditary cover of kS_d satisfying $\operatorname{Hom}_A(P, \Delta(\lambda)) = \theta(\lambda), \lambda \in \Lambda^+(d) := \Lambda^+(d,d)$.

Let ${}^\natural(-) \colon kS_d\text{-mod} \to kS_d\text{-mod}$ be the simple preserving duality of the symmetric group. By Theorem 8.15 of [Jam78],

$$ {}^\natural\theta(1^d) \simeq \theta(d). \tag{4.1.3.1} $$

On the other hand, $\theta(1^d)$ is a simple module, so $\theta(d) \simeq \theta(1^d)$. This implies that

$$ \operatorname{Hom}_A(\Delta(d), \Delta(1^d)) \simeq \operatorname{Hom}_{kS_d}(\theta(d), \theta(1^d)) \simeq \operatorname{Hom}_{kS_d}(\theta(d), \theta(d)) \neq 0. \tag{4.1.3.2} $$

This contradicts A being split quasi-hereditary with the order on the partitions $d > 1^d$. So, kS_d has no such faithful quasi-hereditary cover.

Assume that there exists a 1-faithful split quasi-hereditary cover of $\mathbb{Z}S_d$, say (A,P) such that $\operatorname{Hom}_A(P, \Delta(\lambda)) = \theta(\lambda)$. By Theorem 3.3.9, $(A(2\mathbb{Z}), P(2\mathbb{Z}))$ is a 0-faithful quasi-hereditary cover of $\mathbb{Z}/2\mathbb{Z}S_d = \mathbb{F}_2S_d$ satisfying

$$ \theta_{\mathbb{F}_2}(\lambda) = \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \theta(\lambda) = \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \operatorname{Hom}_A(P, \Delta(\lambda)) \simeq \operatorname{Hom}_{\mathbb{F}_2 \otimes_{\mathbb{Z}} A}(P(2\mathbb{Z}), \Delta(\lambda)(2\mathbb{Z})). \tag{4.1.3.3} $$

By the first part of Theorem, this cannot happen. $\square$

As mentioned, over the integral Schur algebra there is no exact equivalence on the full subcategory of modules admitting a filtration by standard modules. However, we are able to recover an exact equivalence on another resolving subcategory of $S_{\mathbb{Z}}(n,d)$-mod$\cap\mathbb{Z}$-proj other than $S_{\mathbb{Z}}(n,d)$-proj.

Theorem 4.1.17. *Let $\mathscr{A}$ be the following resolving subcategory of $S_{\mathbb{Z}}(n,d)$-mod, $n \geq d$,*

$$ \mathscr{A} := \{X \in S_{\mathbb{Z}}(n,d) \cap \mathbb{Z}\text{-proj} \colon \ \mathrm{R}^1 G(FX) = 0, \ \eta_X \textit{ is bijective } \}. \tag{4.1.3.4} $$

There is an exact equivalence between $\mathscr{A}$ and

$$ \mathscr{B} := \{Y \in \mathbb{Z}S_d\text{-mod}\cap\mathbb{Z}\text{-proj} \colon \ \mathrm{R}^1 G(Y) = 0\}. \tag{4.1.3.5} $$

Proof. The exactness follows by construction. We will start by showing that this correspondence is well-defined. Let $X \in \mathscr{A}$. It is clear that $FX \in \mathscr{B}$. Moreover, $GFX \simeq X$. Let $Y \in \mathscr{B}$. Then, $0 = \mathrm{R}^1 G(Y) = \mathrm{R}^1 G(FGY)$ and $GFGY \simeq GY$. Therefore, $GY \in \mathscr{A}$ with $FGY \simeq Y$. It remains to show that $\mathscr{A}$ is a resolving subcategory of $S_{\mathbb{Z}}(n,d)$-mod$\cap\mathbb{Z}$-proj. Since $\operatorname{domdim}(S_{\mathbb{Z}}(n,d), \mathbb{Z}) = 2$ and the Hemmer-Nakano dimension of $S_{\mathbb{Z}}(n,d)$-proj is one $S_{\mathbb{Z}}(n,d)$-proj $\subset \mathscr{A}$. The unit η is a natural transformation, so it is clear that $\mathscr{A}$ is closed under direct summands. Any exact sequence of $S_{\mathbb{Z}}(n,d)$-modules

$$ 0 \to X \to M \to N \to 0, \tag{4.1.3.6} $$

by applying GF, yields a long exact sequence

$$0 \to GFX \to GFM \to GFN \to \mathrm{R}^1\, G(FX) \to \mathrm{R}^1\, G(FM) \to \mathrm{R}^1\, G(FN). \tag{4.1.3.7}$$

Therefore, if $X, N \in \mathscr{A}$ then $\mathrm{R}^1\, G(FM) = 0$ and by Snake Lemma η_M is iso. Hence, $\mathscr{A}$ is closed under extensions. Assume that $M, N \in \mathscr{A}$ then by Snake Lemma $\mathrm{R}^1\, G(FX) = 0$ and η_X is bijective. This finishes the proof. ☐

4.2 q-Schur algebras

The Hecke algebra of the symmetric group (usually called the Iwahori-Hecke algebra) is obtained by a small perturbation q on the group algebra of the symmetric group. By a small perturbation q we mean replacing the identity of the group algebra in some of its defining relations by a non-trivial root of unity. Although, one usually is more general and defines it for an invertible element q. Usually, the name quantum is referred to q being a small perturbation.

Let R be a commutative ring with identity. Fix natural numbers n, d. Let u be an invertible element of R and put $q = u^{-2}$. The **Iwahori-Hecke algebra** $H_{R,q}(d)$ is the R-algebra with basis $\{T_\sigma : \sigma \in S_d\}$ satisfying the relations

$$T_\sigma T_s = \begin{cases} T_{\sigma s}, & \text{if } l(\sigma s) = l(\sigma) + 1 \\ (u - u^{-1})T_\sigma + T_{\sigma s}, & \text{if } l(\sigma s) = l(\sigma) - 1, \end{cases} \tag{4.2.0.1}$$

where $s \in S := \{(1,2), (2,3), \cdots, (d-1,d)\}$ is a set of transpositions and l is the length function, that is, $l(\sigma)$, $\sigma \in S_d$, is the minimum number of simple transpositions belonging to S needed to write σ.

There are many ways to define Hecke algebras. Here, we are following the definition of Hecke algebras according to Parshall-Wang [PW91] (but we use u instead of q and q instead of h). In [Mat99], they use a different basis for $H_{R,q}(d)$ which is the same as Definition (11.3a) of [PW91]. We would also like to point out that $\mathscr{H}_{R,q}$ in Definition 4.4.1 of [DD91] is exactly $H_{R,q}(d)$ in our notation.

Due to the relations (4.2.0.1), T_s, $s \in S$, generates as algebra $H_{R,q}(d)$.

The Iwahori-Hecke algebra $H_{R,q}(d)$ admits a base change property.

$$H_{R,q}(d) \simeq R \otimes_{\mathbb{Z}[u,u^{-1}]} H_{\mathbb{Z}[u,u^{-1}],u^{-2}}(d) \tag{4.2.0.2}$$

Under this isomorphism of R-algebras $1_R \otimes_{\mathbb{Z}[u,u^{-1}]} T_\sigma$ is mapped to $T_\sigma \in H_{R,q}(d)$.

We can regard $V^{\otimes d}$ as right $H_{R,q}(d)$-module by imposing to an R-basis $\{e_{i_1} \otimes \cdots \otimes e_{i_d} \mid i \in I(n,d)\}$ of $V^{\otimes d}$,

$$e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot T_s = \begin{cases} e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot s & \text{if } i_t < i_{t+1} \\ u e_{i_1} \otimes \cdots \otimes e_{i_d} & \text{if } i_t = i_{t+1}\,, \quad s = (t, t+1) \in S, \\ (u - u^{-1}) e_{i_1} \otimes \cdots \otimes e_{i_d} + e_{i_1} \otimes \cdots \otimes e_{i_d} \cdot s & \text{if } i_t > i_{t+1} \qquad 1 \leq t < d. \end{cases} \tag{4.2.0.3}$$

By considering $q = 1$, we recover the action on $V^{\otimes d}$ of the symmetric group by place permutation.

Definition 4.2.1. The subalgebra $\mathrm{End}_{H_{R,q}(d)}\left(V^{\otimes d}\right)$ of the endomorphism algebra $\mathrm{End}_R\left(V^{\otimes d}\right)$ is called the **q-Schur algebra**. We will denote it by $S_{R,q}(n,d)$.

The q-Schur algebras were introduced by Dipper and James [DJ91, DJ89].

By [Du92, 2.d] (see also [DD91, Lemma 4.4.3]) $S_{R,q}(n,d) = S_{R,u^{-2}}(n,d)$ is isomorphic to the q-Schur algebra of Dipper and James [DJ91].

We should remark, at this point, that $V^{\otimes d}$ can be regarded in many different ways as $H_{R,q}(d)$-module in the literature. However, they are not isomorphic as $H_{R,q}(d)$-modules (unless one changes the action on $V^{\otimes d}$) although they always have isomorphic endomorphism algebras making the q-Schur algebra well defined. Essentially, this is due to a change of basis of $H_{R,q}(d)$ not being compatible with quantum deformation of the general linear group (see Section 4 of [DD91]). A classical reference to q-Schur algebras is [Don98].

Similar to the Schur algebra we will start by recalling some facts about q-Schur algebras. We wish to illustrate an R-basis for $S_{R,q}(n,d)$. In chapter 2, we discussed the importance of the condition $DM \otimes_B M \in R$-proj for some generator $M \in B$-mod. It is now a good opportunity to exhibit an R-basis of $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$. By dualizing such R-basis we will obtain an R-basis for $S_{R,q}(n,d)$. Note, once more, that in general if $\operatorname{End}_B(M)$ has an R-basis nothing can be said about $DM \otimes_B M$, $M \in B$-mod.

Lemma 4.2.2. *Let $\{e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* \mid i \in I(n,d)\}$ be an R-basis of $DV^{\otimes d}$. $DV^{\otimes d}$ is a left $H_{R,q}(d)$-module with action*

$$T_s \cdot e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* = \begin{cases} s \cdot e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* & \text{if } i_t < i_{t+1} \\ u e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* & \text{if } i_t = i_{t+1}\ , \\ (u-u^{-1}) e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* + s \cdot e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* & \text{if } i_t > i_{t+1} \end{cases} \quad \begin{matrix} s = (t,t+1) \in S, \\ 1 \leq t < d. \end{matrix} \tag{4.2.0.4}$$

Proof. Let $e_{k_1} \otimes \cdots \otimes e_{k_d} \in V^{\otimes d}$ be an element basis. Let $s = (t,t+1)$ be a transposition. Then,

$$\begin{aligned}(T_s \cdot e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*)(e_{k_1} \otimes \cdots \otimes e_{k_d}) &= e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*(e_{k_1} \otimes \cdots \otimes e_{k_d} \cdot T_s) \\ &= \begin{cases} e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*(e_{k_1} \otimes \cdots \otimes e_{k_d} \cdot s) & \text{if } k_t < k_{t+1} \\ e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*(u e_{k_1} \otimes \cdots \otimes e_{k_d}) & \text{if } k_t = k_{t+1} \\ e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*((u-u^{-1}) e_{k_1} \otimes \cdots \otimes e_{k_d} + e_{k_1} \otimes \cdots \otimes e_{k_d} \cdot s) & \text{if } k_t > k_{t+1} \end{cases} \\ &= \begin{cases} \mathbb{1}_{\{i\}}(k \cdot s) & \text{if } k_t < k_{t+1} \\ u \mathbb{1}_{\{i\}}(k) & \text{if } k_t = k_{t+1} \\ (u-u^{-1}) \mathbb{1}_{\{i\}}(k) + \mathbb{1}_{\{i\}}(k \cdot s) & \text{if } k_t > k_{t+1} \end{cases}\end{aligned} \tag{4.2.0.5}$$

$$\begin{aligned}&= \mathbb{1}_{\{i\}}(k \cdot s) \mathbb{1}_{\{k_t < k_{t+1}\}}(k) + u \mathbb{1}_{\{i\}}(k) \mathbb{1}_{\{i_t = i_{t+1}\}}(i) + (u-u^{-1}) \mathbb{1}_{\{i\}}(k) \mathbb{1}_{\{k_t > k_{t+1}\}}(k) + \mathbb{1}_{\{i\}}(k \cdot s) \mathbb{1}_{\{k_t > k_{t+1}\}}(k) \\ &= \mathbb{1}_{\{i \cdot s^{-1}\}}(k) \mathbb{1}_{\{i_t > i_{t+1}\}}(i) + u \mathbb{1}_{\{i\}}(k) \mathbb{1}_{\{i_t = i_{t+1}\}}(i) + (u-u^{-1}) \mathbb{1}_{\{i\}}(k) \mathbb{1}_{\{i_t > i_{t+1}\}}(i) + \mathbb{1}_{\{i \cdot s^{-1}\}}(k) \mathbb{1}_{\{i_t < i_{t+1}\}}(i) \\ &= \begin{cases} s \cdot e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*(e_{k_1} \otimes \cdots \otimes e_{k_d}) & \text{if } i_t < i_{t+1} \\ u e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*(e_{k_1} \otimes \cdots \otimes e_{k_d}) & \text{if } i_t = i_{t+1} \\ ((u-u^{-1}) e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* + s \cdot e_{i_1}^* \otimes \cdots \otimes e_{i_d}^*)(e_{k_1} \otimes \cdots \otimes e_{k_d}) & \text{if } i_t > i_{t+1} \end{cases}\end{aligned}$$

Here, $\mathbb{1}$ denotes the indicator function. ☐

We can associate to $I(n,d) \times I(n,d)$ the lexicographical order. Each S_d-orbit of $I(n,d) \times I(n,d)$ has a representative (i,j) satisfying $(i_1, j_1) \leq \cdots \leq (i_d, j_d)$.

Proposition 4.2.3. *$V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$ is a free R-module with basis*

$$\{e_{i_1} \otimes \cdots \otimes e_{i_d} \otimes_{H_{R,q}(d)} e_{j_1}^* \otimes \cdots \otimes e_{j_d}^* : i,j \in I(n,d),\ (i_1, j_1) \leq \cdots \leq (i_d, j_d)\}. \tag{4.2.0.6}$$

Proof. Since $\{e_{i_1}\otimes\cdots\otimes e_{i_d} \mid i\in I(n,d)\}$ is an R-basis of $V^{\otimes d}$ and $\{e^*_{j_1}\otimes\cdots\otimes e^*_{j_d} \mid j\in I(n,d)\}$ is an R-basis of $DV^{\otimes d}$ the set $\{e_{i_1}\otimes\cdots\otimes e_{i_d}\otimes_{H_{R,q}(d)} e^*_{j_1}\otimes\cdots\otimes e^*_{j_d} | i,j\in I(n,d)\}$ generate (over R) $V^{\otimes d}\otimes_{H_{R,q}(d)} DV^{\otimes d}$.

Denote by Λ the set

$$\Lambda := \{(i,j)\in I(n,d)\times I(n,d)\colon (i_1,j_1)\leq\cdots\leq(i_d,j_d)\}. \tag{4.2.0.7}$$

Let $(l,s)\in I(n,d)\times I(n,d)$. Assume that $(l,s)\notin\Lambda$. Then, there exists $1\leq k<d$ such that $(l_k,s_k)\not\leq(l_{k+1},s_{k+1})$. Hence, either $l_k>l_{k+1}$ or $l_k=l_{k+1}$ and $s_k>s_{k+1}$. Assume that $l_k>l_{k+1}$. Take $i=l\cdot(k,k+1)$ and $\omega=(k,k+1)$. Then, $i_k<i_{k+1}$ and

$$e_{l_1}\otimes\cdots\otimes e_{l_d} = (e_{i_1}\otimes\cdots\otimes e_{i_d})\cdot(k,k+1) = e_{i_1}\otimes\cdots\otimes e_{i_d}\cdot T_\omega. \tag{4.2.0.8}$$

Hence,

$$e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} e^*_{s_1}\otimes\cdots\otimes e^*_{s_d} = e_{i_1}\otimes\cdots\otimes e_{i_d}\cdot T_\omega\otimes_{H_{R,q}(d)} e^*_{s_1}\otimes\cdots\otimes e^*_{s_d} \tag{4.2.0.9}$$

$$= e_{i_1}\otimes\cdots\otimes e_{i_d}\otimes_{H_{R,q}(d)} T_\omega e^*_{s_1}\otimes\cdots\otimes e^*_{s_d} \tag{4.2.0.10}$$

Therefore, we can write $e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} e^*_{s_1}\otimes\cdots\otimes e^*_{s_d}$ as a linear combination of elements $e_i\otimes_{H_{R,q}(d)} e^*_f$ where $i_1\leq\ldots i_k\leq i_{k+1}$, $i,f\in I(n,d)$. Now, assume that $l_k=l_{k+1}$ and $s_k>s_{k+1}$ for some k. Put $j=s\cdot\omega$, $\omega=(k,k+1)$. Then, $j_k<j_{k+1}$ and

$$e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} e^*_{s_1}\otimes\cdots\otimes e^*_{s_d} = e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} \omega e^*_{j_1}\otimes\cdots\otimes e^*_{j_d} \tag{4.2.0.11}$$

$$= e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} T_\omega e^*_{j_1}\otimes\cdots\otimes e^*_{j_d} \tag{4.2.0.12}$$

$$= e_{l_1}\otimes\cdots\otimes e_{l_d} T_\omega\otimes_{H_{R,q}(d)} e^*_{j_1}\otimes\cdots\otimes e^*_{j_d} \tag{4.2.0.13}$$

$$= u e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} e^*_{j_1}\otimes\cdots\otimes e^*_{j_d}. \tag{4.2.0.14}$$

So, we can order the elements (for example using Bubble sort) $(l,s)\in I(n,d)\times I(n,d)$ into $(i,j)\in I(n,d)\times I(n,d)$ with $(i,j)\in\Lambda$ and we obtain that each element $e_{l_1}\otimes\cdots\otimes e_{l_d}\otimes_{H_{R,q}(d)} e^*_{s_1}\otimes\cdots\otimes e^*_{s_d}$, $s,l\in I(n,d)$ can be written as a linear combination of elements $e_{i_1}\otimes\cdots\otimes e_{i_d}\otimes_{H_{R,q}(d)} e^*_{j_1}\otimes\cdots\otimes e^*_{j_d}$, $i,j\in\Lambda$. Moreover, the coefficients appearing in this linear combination belong to the image of $\mathbb{Z}[u,u^{-1}]\to R$. Denote these coefficients by $p^{l,s}_{i,j}(u)$. We claim that our desired set is linearly independent. For each $(i,j)\in\Lambda$, we define the map $\psi_{i,j}\colon V^{\otimes d}\times DV^{\otimes d}\to R$ satisfying

$$\psi_{i,j} = \sum_{l,s\in I(n,d)} p^{l,s}_{i,j}(u)(e_l,e^*_s)^*, \tag{4.2.0.15}$$

where $(e_l,e^*_s)^*$ is the dual element of (e_l,e^*_s). So, this map is R-bilinear. By construction, the coefficients $p^{l,s}_{i,j}(u)$ satisfy the following relations: For each $\omega=(k,k+1)$, we have

$$\begin{cases} p^{l,s\omega}_{i,j}(u) = u p^{l\omega,s}_{i,j}(u) & \text{if } l_t=l_{t+1},\ s_t<s_{t+1} \\ p^{l\omega,s}_{i,j}(u) = p^{l,s\omega}_{i,j}(u) & \text{if } l_t<l_{t+1},\ s_t<s_{t+1} \\ p^{l\omega,s}_{i,j}(u) = u p^{l,s}_{i,j}(u) & \text{if } l_t<l_{t+1},\ s_t=s_{t+1} \\ p^{l\omega,s}_{i,j}(u) = (u-u^{-1})p^{l,s}_{i,j}(u) + p^{l,s\omega}_{i,j}(u) & \text{if } l_t<l_{t+1},\ s_t>s_{t+1} \end{cases}. \tag{4.2.0.16}$$

We are now ready to check that $\psi_{i,j}$ satisfies the relation $\psi_{i,j}(e_f T_\omega, e^*_g) = \psi_{i,j}(e_f, T_\omega e^*_g)$ for all $f,g\in I(n,d)$. For

$f, g \in I(n,d)$ and $\omega = (t, t+1)$,

$$\psi(e_f T_\omega, e_g^*) = \begin{cases} \sum_{l,s\in I(n,d)} p_{i,j}^{l,s}(u)\mathbb{1}_{\{f\omega=l, s=g\}}(l,s) & \text{if } f_t < f_{t+1} \\ \sum_{l,s\in I(n,d)} p_{i,j}^{l,s}(u)\mathbb{1}_{\{f=l, g=s\}}(l,s)u & \text{if } f_t = f_{t+1} \\ \sum_{l,s\in I(n,d)} (u-u^{-1})p_{i,j}^{l,s}(u)\mathbb{1}_{\{f=l,g=s\}}(l,s) + p_{i,j}^{l,s}(u)\mathbb{1}_{\{f\omega=l,s=g\}}(l,s) & \text{if } f_t > f_{t+1} \end{cases} \tag{4.2.0.17}$$

$$= \begin{cases} p_{i,j}^{f\omega,g}(u) & \text{if } f_t < f_{t+1} \\ up_{i,j}^{f,g}(u) & \text{if } f_t = f_{t+1} \\ (u-u^{-1})p_{i,j}^{f,g}(u) + p_{i,j}^{f\omega,g}(u) & \text{if } f_t > f_{t+1} \end{cases}. \tag{4.2.0.18}$$

On the other hand,

$$\psi(e_f, T_\omega e_g^*) = \begin{cases} p_{i,j}^{f,g\omega}(u) & \text{if } g_t < g_{t+1} \\ up_{i,j}^{f,g}(u) & \text{if } g_t = g_{t+1} \\ (u-u^{-1})p_{i,j}^{f,g}(u) + p_{i,j}^{f,g\omega}(u) & \text{if } g_t > g_{t+1} \end{cases}. \tag{4.2.0.19}$$

Using the relations (4.2.0.16) we obtain our claim. Hence, $\psi_{i,j}$ induces a unique map $\psi'_{i,j} \colon V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d} \to R$, satisfying

$$\psi'_{i,j}(e_f \otimes_{H_{R,q}(d)} e_g^*) = p_{i,j}^{f,g}(u), \quad f, g \in I(n,d). \tag{4.2.0.20}$$

In particular $\psi'_{i,j}(e_i \otimes_{H_{R,q}(d)} e_j^*) = 1$ and $\psi'_{i,j}(e_f \otimes_{H_{R,q}(d)} e_g^*) = 0$ for all $(f,g) \in \Lambda$ distinct from (i,j). This shows that (4.2.0.6) is an R-basis of $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$. ☐

The dual elements of $e_i \otimes_{H_{R,q}(d)} e_j^*$, $(i,j) \in \Lambda$, denoted by $\xi_{j,i} \in D(V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}) \simeq S_{R,q}(n,d)$, form an R-basis of the q-Schur algebra. Moreover, (by a Tensor-Hom adjunction argument)

$$e_g^*(\xi_{j,i}(e_f)) = \psi'_{i,j}(e_f \otimes_{H_{R,q}(d)} e_g^*) = p_{i,j}^{f,g}(u), \quad f, g \in I(n,d). \tag{4.2.0.21}$$

Thus, we can write

$$\xi_{j,i}(e_f) = \sum_{f\in I(n,d)} p_{i,j}^{f,g}(u)e_g, \quad \forall f \in I(n,d). \tag{4.2.0.22}$$

Using our approach to a basis of the q-Schur algebra it is clear that the q-Schur algebra admits a base change property (see also [DJ89, 2.18(ii)]).

Lemma 4.2.4. *Let R be a commutative ring with an invertible element u. Fix $q = u^{-2}$. For any commutative R-algebra S,*

$$S_{R,q}(n,d) \simeq R \otimes_{\mathbb{Z}[u,u^{-1}]} S_{\mathbb{Z}[u,u^{-1}],u^{-2}}(n,d), \tag{4.2.0.23}$$

$$S_{S,q1_S}(n,d) \simeq S \otimes_R S_{R,q}(n,d). \tag{4.2.0.24}$$

Proof. Since $V^{\otimes d} \otimes_{H_{R,q}(d)} DV^{\otimes d}$ is a free R-module and $H_{R,q}(d)$ admit a base change property the q-Schur algebra $S_{R,q}(n,d)$ has also a base change property:

$$S_{S,q1_S}(n,d) \simeq S \otimes_R S_{R,q}(n,d). \tag{4.2.0.25}$$

The first equation follows by fixing $R = \mathbb{Z}[u, u^{-1}]$. $\square$

According to the notation of proof of Proposition 4.2.3, $p_{i,i}^{f,f} = \mathbb{1}_{\{i\sim f\}}(f)$ for $f \in I(n,d)$. Therefore, for each $(i,i) \in \Lambda$, $\xi_{i,i}$ is an idempotent. Further, we can index these idempotents by the compositions of d in at most n parts, in the same manner as it was to $S_R(n,d)$. Analogously, to the classical case, we can consider an increasing bijection $\Lambda^+(n,d) \to \{1,\ldots,t\}$, $\lambda^k \mapsto k$. Set e^k to be the idempotent $\sum_{l\geq k} \xi_{\lambda^l}$ and define $J_k = S_{R,q}(n,d)e^k S_{R,q}(n,d)$. It follows that $S_{R,q}(n,d)$ is split quasi-hereditary.

Theorem 4.2.5. *For any commutative Noetherian ring R, the q- Schur algebra $S_{R,q}(n,d)$ is a split quasi-hereditary algebra over R with split heredity chain* $0 \subset J_t \subset \cdots \subset J_2 \subset J_1 = S_R(n,d)$.

Proof. The statement for fields follows from [PW91, Theorem 11.5.2]. The statement for Noetherian rings which are not fields follows from Theorem 3.7.2 of [CPS90]. An alternative proof for this statement without using Theorem 3.7.2 of [CPS90] is to apply Theorem 1.5.73. $\square$

In particular, $S_{R,q}(n,d)$ has finite global dimension whenever R has finite global dimension. The standard modules associated with this split heredity chain of the q-Schur algebra are called q-Weyl modules, indexed by the partitions of $d$in at most n parts. To define a cellular structure on the q-Schur algebra we can define the involution ι by assigning to each element basis $\xi_{j,i}$ $(i,j) \in \Lambda$, the image in $S_{R,q}(n,d)$ of $(e_j \otimes_{H_{R,q}(d)} e_i^*)^*$. Observe that $\iota(\xi_\lambda) = \xi_\lambda$ for every $\lambda \in \Lambda^+(n,d)$. In particular, ι preserves all idempotents in the split heredity chain of $S_{R,q}(n,d)$. Hence, by a version of Corollary 4.2 [KX98] for commutative Noetherian rings, $S_{R,q}(n,d)$ is a cellular algebra.

We will now focus on the case $n \geq d$. There are isomorphisms,

$$V^{\otimes d} \simeq S_{R,q}(n,d)\xi_{(1,\ldots,d),(1,\ldots,d)}, \quad DV^{\otimes d} \simeq \xi_{(1,\ldots,d),(1,\ldots,d)}S_{R,q}(n,d). \tag{4.2.0.26}$$

Hence, $V^{\otimes d}$ is a projective $(S_{R,q}(n,d),R)$-injective $S_{R,q}(n,d)$-module. Thus, we can consider the Schur functor $F_{R,q} = \mathrm{Hom}_{S_{R,q}(n,d)}(V^{\otimes d}, -)\colon S_{R,q}(n,d)\text{-mod} \to H_{R,q}(d)\text{-mod}$ (we will write just F_q when there is no confusion on the ground ring R). Note that these facts follow by extending the results of Donkin (see [Don98]) to commutative rings. In particular, the arguments of the results [Don98, Section 2.1 (5), (6),(7)] can easily be extended to commutative rings. Alternatively, we can see these facts as applications of Proposition 1.4.34 and 2.5.3 and Nakayama's Lemma.

Parallel to the classical case, using the representation theory of q-Schur algebras we can obtain information for the representation theory of Hecke algebras. In particular, $(S_{R,q}(n,d), V^{\otimes d})$ is a split quasi-hereditary cover of $H_{R,q}(d)$. Since $\iota(\xi_{(1,\ldots,d),(1,\ldots,d)}) = \xi_{(1,\ldots,d),(1,\ldots,d)}$ and

$$H_{R,q}(d) \simeq \mathrm{End}_{S_{R,q}(n,d)}(V^{\otimes d}) \simeq \xi_{(1,\ldots,d),(1,\ldots,d)}S_{R,q}(n,d)\xi_{(1,\ldots,d),(1,\ldots,d)}, \tag{4.2.0.27}$$

F sends the split heredity chain of $S_{R,q}(n,d)$ to a cell chain of $H_{R,q}(d)$. This makes $H_{R,q}(d)$ a cellular algebra. In particular, the Schur functor sends the q-Weyl modules to the cell modules of $H_{R,q}(d)$. We aim now to determine the connection between q-Weyl modules filtrations and cell filtrations.

At this point, it is not surprising that this is reduced to computing relative dominant dimensions. For the Schur algebra, the dominant dimension is directly related to the characteristics of the residue fields of the ground ring. So, it is natural to consider a quantum version of the characteristic of the ring. This is done by replacing the identity with q on the definition of the characteristic of a ring.

Definition 4.2.6. The **q-characteristic of R**, denoted by q-char, is the smallest positive number s such that $1+q+\cdots+q^{s-1}=0$ if such s exists, and zero otherwise.

We shall refer to $q\text{-}\operatorname{char} R$ as the quantum characteristic of R when there is no misunderstanding about q. Note that $(1-q)(1+q+\cdots+q^{s-1})=1-q^s$ for all $s>0$. So, for integral domains the quantum characteristic is zero if and only if either q is not a root of unity or $q=1$ and $\operatorname{char} R=0$. We refer to [LQ13] for a more detailed discussion of quantum characteristic.

The computation of dominant dimension for quantised Schur algebras over fields is due to Fang and Miyachi.

Theorem 4.2.7. *[FM19, Theorem 3.13] Let K be a field. Assume that $q=u^{-2}$ for some non-zero element $u\in K$ and $n\geq d$.*

$$\operatorname{domdim} S_{K,q}(n,d)=\begin{cases}2(q\text{-}\operatorname{char} K-1) & \text{if } d\geq q\text{-}\operatorname{char} K>0\\ +\infty, & \text{otherwise}.\end{cases} \tag{4.2.0.28}$$

We will now extend this computation for all q-Schur algebras. Further, we can determine the relative dominant dimension of the q-Schur algebra by knowing the invertible elements of R.

Theorem 4.2.8. *Let R be a commutative ring with invertible element $u\in R$. Put $q=u^{-2}$ and assume that $n\geq d$. Then, $(S_{R,q}(n,d),V^{\otimes d})$ is a relative gendo-symmetric R-algebra and*

$$\operatorname{domdim}(S_{R,q}(n,d),R)=\inf\{2s\in\mathbb{N}\mid 1+q+\cdots+q^s\notin U(R),\ s<d\}. \tag{4.2.0.29}$$

Proof. By Proposition 2.5.3, $V^{\otimes d}$ is a projective $(S_{R,q}(n,d))$-injective-strongly faithful module. Hence, $(S_{R,q}(n,d),V^{\otimes d},DV^{\otimes d})$ is a relative QF3 R-algebra. Let $\operatorname{MaxSpec}(R)$ be the set of maximal ideals of R.

By Theorem 2.5.13,

$$\begin{aligned}\operatorname{domdim}(S_{R,q}(n,d),R)&=\inf\{\operatorname{domdim} S_{R,q}(n,d)\otimes_R R(\mathfrak{m})\mid \mathfrak{m}\in\operatorname{MaxSpec}(R)\} && (4.2.0.30)\\ &=\inf\{\operatorname{domdim} S_{R(\mathfrak{m}),q_\mathfrak{m}}(n,d)\mid \mathfrak{m}\in\operatorname{MaxSpec}(R)\}\geq 2, && (4.2.0.31)\end{aligned}$$

where $q_\mathfrak{m}$ is the image of q in $R(\mathfrak{m})$. In particular, $V^{\otimes d}$ is a generator-cogenerator of $H_{R,q}(d)$. Similarly to Proposition 2.2.6, we can define an R-linear map $\pi\colon H_{R,q}(d)\to R$, given by

$$\pi(T_\sigma)=\begin{cases}1_R, & \text{if } \sigma=e\\ 0, & \text{otherwise}\end{cases},\quad \sigma\in S_d.$$

Afterwards, we can define the $H_{R,q}(d)$-isomorphism $\phi\colon H_{R,q}(d)\to DH_{R,q}(d)$, given by $\phi(T_\sigma)(T_\omega)=\pi(T_\sigma T_\omega)$ for every $\sigma,\omega\in S_d$. This yields that the Hecke algebra $H_{R,q}(d)$ is a relative symmetric R-algebra. By Theorem 2.10.2, $(S_{R,q}(n,d),V^{\otimes d})$ is a relative gendo-symmetric R-algebra. First, we will show that

$$\operatorname{domdim}(S_{R,q}(n,d),R)\leq\inf\{2s\in\mathbb{N}\mid 1+q+\cdots+q^s\notin U(R),\ s<d\}. \tag{4.2.0.32}$$

If the right hand side is infinite, then there is nothing to prove. Assume that there exists $s<d$ such that $1+q+\cdots+q^s\notin U(R)$. Then, $1+q+\cdots+q^s$ belongs to some maximal ideal of R, say $\mathfrak{m}$. Therefore, $1+q_\mathfrak{m}+\ldots+q_\mathfrak{m}^s$ is zero in $R(\mathfrak{m})$. Assume that $q_\mathfrak{m}=1$ in $R(\mathfrak{m})$. Then, $0\neq q_\mathfrak{m}\text{-}\operatorname{char} R(\mathfrak{m})=\operatorname{char} R(\mathfrak{m})\leq s+1\leq d-1+1=d$, so $\operatorname{domdim} S_{R(\mathfrak{m}),q_\mathfrak{m}}(n,d)\leq 2s$. Now, assume that $q_\mathfrak{m}\neq 1$. Then,

$$0<q_\mathfrak{m}\text{-}\operatorname{char} R(\mathfrak{m})\leq s+1\leq d-1+1=d. \tag{4.2.0.33}$$

Hence,

$$\operatorname{domdim}(S_{R(\mathfrak{m}),q_{\mathfrak{m}}}(n,d),R) = 2(q_{\mathfrak{m}}\text{-char} - 1) \leq 2s. \tag{4.2.0.34}$$

So, our claim follows. If $\operatorname{domdim}(S_{R,q}(n,d),R)$ is infinite then, of course, that the equality (4.2.0.29) holds. Suppose that $\operatorname{domdim}(S_{R,q}(n,d),R) = l > 0$. So, there exists a maximal ideal $\mathfrak{m}$ of R such that

$$l = \operatorname{domdim} S_{R(\mathfrak{m}),q\mathfrak{m}}(n,d) = 2(q_{\mathfrak{m}}\text{-char} R(\mathfrak{m}) - 1), \tag{4.2.0.35}$$

and $0 < q_{\mathfrak{m}}\text{-char} R(\mathfrak{m}) \leq d$. By definition of quantum characteristic, the image of $1+q+\cdots+q^{q_{\mathfrak{m}}\text{-char} R(\mathfrak{m})-1}$ in $R(\mathfrak{m})$ is zero. So, $1+q+\cdots+q^{q_{\mathfrak{m}}\text{-char} R(\mathfrak{m})-1}$ belongs to $\mathfrak{m}$. Since $q_{\mathfrak{m}}\text{-char} R(\mathfrak{m}) - 1 \leq d-1 < d$ then $l \in \{2s \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}$. This finishes the proof. $\square$

We can now compute $\operatorname{domdim}(S_{\mathbb{Z}[u,u^{-1}],u^{-2}}(n,d), \mathbb{Z}[u,u^{-1}])$. The invertible elements of $\mathbb{Z}[u,u^{-1}]$ are the powers of u and the constants 1 and -1. Hence, $1+q = 1+u^{-2}$ is not invertible. So,

$$\operatorname{domdim}(S_{\mathbb{Z}[u,u^{-1}],u^{-2}}(n,d), \mathbb{Z}[u,u^{-1}]) = 2, \quad d \geq 2. \tag{4.2.0.36}$$

Corollary 4.2.9. *Let R be a commutative ring with invertible element $u \in R$. Put $q = u^{-2}$ and assume that $n \geq d$. Let T be a characteristic tilting module of $S_{R,q}(n,d)$. Then,*

$$\operatorname{domdim}_{(S_{R,q}(n,d),R)} T = \inf\{s \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}. \tag{4.2.0.37}$$

Proof. The result follows from applying Theorem 4.2.8 and Theorem 2.11.3. $\square$

4.2.1 Hemmer-Nakano dimension of $S_{R,q}(n,d)$-proj and $\mathscr{F}(\tilde{\Delta})$

For the Schur algebras, we saw that both the Hemmer-Nakano dimension and the relative dominant dimension are independent of the Krull dimension of the ground ring contrary to other homological invariants like the global dimension. For q-Schur algebras, we expect a similar behaviour. Further, a crucial fact for a better value of the Hemmer-Nakano dimension regarding the relative dominant dimension of $S_R(n,d)$-proj was R not being similar to a field. In particular, R must have Krull dimension bigger or equal to one and it does not contain a field. So, a natural question that arises is

- For what rings R does $S_{R,q}(n,d)$-proj and $\mathscr{F}(\tilde{\Delta})$ have higher Hemmer-Nakano dimension than the respective resolving subcategories over its residue fields?

The following notion based on the work [LQ13, 1.9] gives us the answer to this question.

Definition 4.2.10. Let R be a commutative ring with invertible element q. We call R a **q-divisible ring** (or *quantum divisible ring*) if $1+q+\cdots+q^s \in U(R)$ whenever $1+q+\cdots+q^s \neq 0$ for any $s \in \mathbb{N}$. For a given natural number d, we call R a **d-partial q-divisible ring** (or *d-partial quantum divisible ring*) if $1+q+\cdots+q^s \in U(R)$ whenever $1+q+\cdots+q^s \neq 0$ for any $s<d$.

For example, any field is a quantum divisible ring, and in particular, it is a d-partial quantum divisible ring for any d.

Once again, we can assume that R is a local regular (commutative Noetherian) ring for the computation of Hemmer-Nakano dimension of $S_{R,q}(n,d)$-proj and $\mathscr{F}(\tilde{\Delta})$.

4.2.1.1 Case 1 - R is a d-partial quantum divisible ring

Theorem 4.2.11. *Let R be a local regular d-partial q-divisible ring, where $q = u^{-2}$, $u \in U(R)$. Assume that $n \geq d$. Then,*

$$\begin{aligned}\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) = \mathrm{domdim}(S_{R,q}(n,d),R) - 2 &= \inf\{2(s-1) \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\} \\ &= \inf\{2(s-1) \in \mathbb{N} \mid 1+q+\cdots+q^s = 0,\ s<d\} \geq 0,\end{aligned}$$

Moreover, if T is a characteristic tilting module of $S_{R,q}(n,d)$, then

$$\begin{aligned}\mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta})) = \mathrm{domdim}_{(S_{R,q}(n,d),R)} T - 2 &= \inf\{s-2 \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\} \\ &= \inf\{s-2 \in \mathbb{N} \mid 1+q+\cdots+q^s = 0,\ s<d\} \geq -1.\end{aligned}$$

Proof. By Theorem 3.5.6 and Theorem 4.2.8,

$$\begin{aligned}\mathrm{HNdim}_{F_q}(S_{R,q}(n,d),R\text{-proj}) \geq \mathrm{domdim}\,(S_{R,q}(n,d),R) - 2 &= \inf\{2(s-1) \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}, \\ \mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta})) \geq \mathrm{domdim}_{(S_{R,q}(n,d),R)} T - 2 &= \inf\{s-2 \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}.\end{aligned}$$

If $\inf\{s \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\} = +\infty$, then we are done.

Assume that $\inf\{s \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\}$ is finite. Let K be the quotient field of R. Since R is a d-partial q-divisible ring,

$$\begin{aligned}\inf\{s \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\} &= \inf\{s \in \mathbb{N} \mid 1+q+\cdots+q^s = 0,\ s<d\} && (4.2.1.1)\\ &= q\text{-}\mathrm{char}\,R - 1 = q\text{-}\mathrm{char}\,K - 1 > 0. && (4.2.1.2)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) &\geq (\mathrm{domdim}\,S_{R,q}(n,d),R) - 2 = 2(q\text{-}\mathrm{char}\,K - 1) - 2 && (4.2.1.3)\\ &= \mathrm{domdim}\,S_{K,q}(n,d) - 2 && (4.2.1.4)\\ &= \mathrm{HNdim}_{F_{K,q}}(S_{K,q}(n,d)\text{-proj}) \geq \mathrm{HNdim}_{F_q}(S_{R,q}(n,d),R\text{-proj}). && (4.2.1.5)\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta})) &\geq \mathrm{domdim}_{(S_{R,q}(n,d),R)} T - 2 = q\text{-}\mathrm{char}\,K - 3 && (4.2.1.6)\\ &= \mathrm{domdim}\,S_{K,q}(n,d) K \otimes_R T - 2 = \mathrm{HNdim}_{F_{K,q}}(\mathscr{F}(K \otimes_R \Delta)) \geq \mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta})).\end{aligned}$$

□

4.2.1.2 Case 2 - R is not a d-partial quantum divisible ring

Theorem 4.2.12. *Let R be a local regular ring with invertible element $u \in R$. Put $q = u^{-2}$. Assume that R is not a d-partial q-divisible ring. Assume that $n \geq d$. Then,*

$$\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) = \mathrm{domdim}(S_{R,q}(n,d),R) - 1 = \inf\{2s-1 \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\} \geq 1.$$

Moreover, if T is a characteristic tilting module of $S_{R,q}(n,d)$, then

$$\mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta})) = \mathrm{domdim}_{(S_{R,q}(n,d),R)} T - 1 = \inf\{s-1 \in \mathbb{N} \mid 1+q+\cdots+q^s \notin U(R),\ s<d\} \geq 0.$$

Proof. Since R is not a d-partial q-divisible ring there exists a natural number s smaller than d such that the sum $0 \neq 1+q+\cdots+q^s \notin U(R)$ is a non-zero invertible element of R. Let s be smallest natural number with such a property. Suppose that there exists a natural number l smaller than s satisfying $1+q+\cdots+q^l = 0$. Then,

$$0 \neq q^{l+1}+\cdots+q^s = q^{l+1}(1+q+\cdots q^{s-l-1}) \notin U(R). \tag{4.2.1.7}$$

As $q^{l+1} \in U(R)$ we obtain that

$$0 \neq 1+\cdots+q^{s-l-1} \notin U(R). \tag{4.2.1.8}$$

So, the existence of l contradicts the minimality of s. Therefore,

$$\inf\{t \in \mathbb{N}\colon 1+q+\cdots+q^t = 0, \quad t < d\} > s. \tag{4.2.1.9}$$

Let K be the quotient field of R. By the previous discussion,

$$\mathrm{HNdim}_{F_{K,q}}(S_{K,q}(n,d)\text{-proj}) = \mathrm{domdim}\, S_{K,q}(n,d) - 2 = \inf\{2t \in \mathbb{N}\colon 1+q+\cdots+q^t = 0,\ t<d\} - 2 > 2s-2, \tag{4.2.1.10}$$

$$\mathrm{HNdim}_{F_{K,q}}(\mathscr{F}(K\otimes_R \Delta)) = \mathrm{domdim}_{S_{K,q}(n,d)} K\otimes_R T = \inf\{t \in \mathbb{N}\colon 1+q+\cdots+q^t = 0,\ t<d\} - 2 > s-2. \tag{4.2.1.11}$$

Whereas, by Theorem 3.5.6

$$\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) \geq \mathrm{domdim}(S_{R,q}(n,d),R) - 2 \tag{4.2.1.12}$$

$$= \inf\{2t-2 \in \mathbb{N} \mid 1+q+\cdots+q^t \notin U(R),\ t<d\} = 2s-2 \geq 0 \tag{4.2.1.13}$$

$$\mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta})) \geq \mathrm{domdim}_{(S_{R,q}(n,d),R)} T - 2 \tag{4.2.1.14}$$

$$= \inf\{t-2 \in \mathbb{N}\colon 1+q+\cdots+q^t \notin U(R),\ t<d\} = s-2 \geq -1. \tag{4.2.1.15}$$

Using (4.2.1.13) and (4.2.1.10) on Theorem 3.5.7, we deduce that $\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) \geq 2s-2 = 1 = 2s-1$. On the other hand, R is a unique factorization domain. So, we can write $1+q+\cdots+q^s = xy$ for some prime element $x \in R$. Thus, Rx is a prime ideal of height one. Therefore, the image of $1+q+\cdots+q^s$ in R/Rx is zero. Denote by $Q(R/Rx)$ the quotient field of R/Rx and q_x the image of q in R/Rx. Then,

$$\inf\{2t \in \mathbb{N}\colon 1+q_x+\cdots+q_x^t = 0,\ t<d\} \leq 2s \tag{4.2.1.16}$$

and so,

$$\mathrm{HNdim}_{F_{R/Rx,q_x}}(S_{R/Rx,q_x}(n,d)\text{-proj}) \leq \mathrm{HNdim}_{F_{Q(R/Rx),q_x}}(S_{Q(R/Rx),q_x}(n,d)\text{-proj}) \leq 2s-2. \tag{4.2.1.17}$$

By Corollary 3.3.10, we cannot have $\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) > 2s-1$. Thus, $\mathrm{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) = 2s-1$. If $s>1$, then by applying the same argument as we did for $S_{R,q}(n,d)$-proj for $\mathscr{F}(\tilde{\Delta})$ the result follows. Assume that $s=1$. Then, since

$$\mathrm{HNdim}_{F_{R/Rx,q_x}}(\mathscr{F}(R/Rx\otimes_R\tilde{\Delta})) \leq \mathrm{HNdim}_{F_{Q(R/Rx),q_x}}(\mathscr{F})(Q(R/Rx)\otimes_R \Delta) \leq s-2 = -1 \tag{4.2.1.18}$$

$\mathrm{HNdim}_{F_q}(\mathscr{F}(\tilde{\Delta}))$ cannot be higher than zero. So, it is enough to show that the unit $\eta_T\colon T \to G_qF_qT$ is an isomorphism, where G_q is the right adjoint functor of the Schur functor $F_q\colon S_{R,q}(n,d)\text{-mod} \to H_{R,q}(d)$. Applying Theorem 3.5.8 taking into account the inequality (4.2.1.11) the result follows. $\square$

Observation 4.2.13. If R is a regular integral domain with invertible element $u \in R$ which is not a d-partial u^{-2}-divisible ring, then there exists a maximal ideal $\mathfrak{m}$ so that $0 \neq 1+q+q+\cdots+q^s \in \mathfrak{m}$ for some $s < d$. Then, $R_\mathfrak{m}$ is not a d-partial $q_\mathfrak{m}$-divisible ring and

$$\operatorname{domdim}(S_{R,q}(n,d),R) = \operatorname{domdim}(S_{R_\mathfrak{m},q_\mathfrak{m}}(n,d),R_\mathfrak{m}).$$

Therefore,

$$\begin{aligned}\operatorname{HNdim}_{F_q}(S_{R,q}(n,d)\text{-proj}) \geq \operatorname{domdim}(S_{R,q}(n,d),R) - 1 &= \operatorname{HNdim}_{F_{R_\mathfrak{m},q_\mathfrak{m}}}(S_{R_\mathfrak{m},q_\mathfrak{m}}(n,d)\text{-proj})\\ &\geq \operatorname{HNdim}_{F_{R,q}}(S_{R,q}(n,d)\text{-proj}).\end{aligned}$$

The ring $\mathbb{Z}[u,u^{-1}]$ is not a d-partial q-divisible ring for $d > 2$.

Hence, the previous exposition generalizes many of the results present in [PS05].

4.3 Auslander algebra of $R[X]/(X^n)$ for a commutative Noetherian ring R

Let R be a commutative Noetherian ring. The algebra $B = R[X]/(X^n)$ is a cellular algebra with cellular datum

$$\Lambda = \{0,1,\ldots,n-1\}, \quad M(\lambda) = \{1\}, \quad C^\lambda = C^\lambda_{1,1} = X^\lambda + (X^n),\ \lambda \in \Lambda, \tag{4.3.0.1}$$

where Λ is ordered by the reverse order of the usual ordering.

Let $A = \operatorname{End}_B(\oplus_\Lambda BC^\lambda)$ the Auslander algebra of B. B has the base change property, that is, for any commutative Noetherian R-algebra S

$$S \otimes_R R[X]/(X^n) \simeq S[X]/(X^n), \quad S \otimes_R R[X]/(X^n)(X^\lambda + (X^n)) \simeq S[X]/(X^n)(X^\lambda + (X^n)) \tag{4.3.0.2}$$

Since, for every field K, $\operatorname{Ext}^1_{K[X]/(X^n)}(M,N) \neq 0$ for every arbitrary non-projective modules $M,N \in K[X]/(X^n)$-mod we obtain that

$$\operatorname{domdim} \operatorname{End}_{B(\mathfrak{m})}\left(\bigoplus_{\lambda\in\Lambda} B(\mathfrak{m})C^\lambda\right) = 2, \tag{4.3.0.3}$$

for every maximal ideal $\mathfrak{m}$ of R. This follows by identifying $B(\mathfrak{m})$ (or if necessary $K \otimes_{R(\mathfrak{m})} B(\mathfrak{m})$ with K being the algebraic closure of $R(\mathfrak{m})$) with the bound quiver algebra of the one-loop quiver, and with paths of length greater than or equal to n being zero. Therefore, $\operatorname{domdim}(A,R) = 2$ and A has the base change property. Moreover, consider, for each $j = 0,\ldots,n-1$, the idempotent

$$e_j := \bigoplus_{\lambda\in\Lambda} BC^\lambda \twoheadrightarrow BC^j \hookrightarrow \bigoplus_{\lambda\in\Lambda} BC^\lambda \in A. \tag{4.3.0.4}$$

Then, (A, Ae_0) is a cover of B. We aim to go further and show that this is a split quasi-hereditary cover. To do that, we start by claiming that $Ae_{n-1}A$ is a split heredity ideal of A. It is easier to observe this by viewing A as the matrix algebra

$$\begin{bmatrix} \mathrm{Hom}_B(B,B) & \mathrm{Hom}_B(BC,B) & \cdots & \mathrm{Hom}_B(BC^{n-1},B) \\ \mathrm{Hom}_B(B,BC) & \mathrm{Hom}_B(BC,BC) & \cdots & \mathrm{Hom}_B(BC^{n-1},BC) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Hom}_B(B,BC^{n-1}) & \mathrm{Hom}_B(BC,BC^{n-1}) & \cdots & \mathrm{Hom}_B(BC^{n-1},BC^{n-1}) \end{bmatrix}. \tag{4.3.0.5}$$

For each $t=0,\dots,n-1$, the homomorphism $\phi_t \in \mathrm{Hom}_B(BC^{n-1},BC^t)$ sending C^{n-1} to C^t is linearly independent and generates $\mathrm{Hom}_B(BC^{n-1},BC^t)$. In particular, Ae_{n-1} is the left A-module isomorphic to

$$\begin{bmatrix} 0 & 0 & \cdots & \mathrm{Hom}_B(BC^{n-1},B) \\ 0 & 0 & \cdots & \mathrm{Hom}_B(BC^{n-1},BC) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{Hom}_B(BC^{n-1},BC^{n-1}) \end{bmatrix}. \tag{4.3.0.6}$$

Therefore, Ae_{n-1} is a free R-module with rank n. Analogously, $e_{n-1}A$ is a free R-module with rank n. Now, the elements of $Ae_{n-1}A$ are the morphisms that factor through BC^{n-1}. The entry i,j of a morphism in $Ae_{n-1}A$ is a map in $\mathrm{Hom}_B(BC^j,BC^i)$ which factors through BC^{n-1}. Hence, such map is an R-linear combination of the map sending C^j to C^i. This map is linearly independent. So, this shows that $Ae_{n-1}A$ is a free R-module with rank n^2. Further, the quotient $A/Ae_{n-1}A$ is isomorphic to $\mathrm{End}_{R[X]/(X^{n-2})}(\bigoplus_{\lambda=0}^{n-2} BC^\lambda)$. Therefore, the canonical map $Ae_{n-1}\otimes_R e_{n-1}A \to Ae_{n-1}A \to A$ is an (A,R)-monomorphism. By proceeding on induction we obtain that A is a split quasi-hereditary algebra for the ordering $n-1>n-2>\cdots>0$.

We can also see that the simple standard module of A, $\Delta(0)$ is sent to the simple B-module BC^{n-1} by the Schur functor $\mathrm{Hom}_A(Ae_0,-)$. But, $\mathrm{Hom}_B(\oplus_\Lambda BC^\lambda, BC^{n-1})$ is isomorphic to $\Delta(n-1)=Ae_{n-1}$ as left A-modules. We showed the following:

Proposition 4.3.1. *Let R be a commutative Noetherian ring and let A be as above. Then, (A,Ae_0) is a (-1) faithful quasi-hereditary cover of $R[X]/(X^n)$.*

There are two direct consequences of Proposition 4.3.1. In contrast to the Schur algebras case, here the integral cover versions do not have higher values of the Hemmer-Nakano dimension. Although, the cellular algebras $R[X]/(X^n)$ have split quasi-hereditary covers, from a level of faithfulness-point of view they fit into the extreme situation of having the worst possible resolution into quasi-hereditary covers.

4.4 Deformations of the BGG category $\mathscr{O}$

We will follow closely the material of Gabber and Joseph [GJ81] to study the Bernstein-Gelfand-Gelfand category $\mathscr{O}$ over a commutative ring and use as most as possible the notation and ideas in [Hum08]. We will assume throughout this section that the reader is familiar with Lie algebras and with the material discussed in [Hum08]. We shall start by recalling some facts about root systems in semi-simple complex Lie algebras. The initial motivation to consider a category $\mathscr{O}$ over commutative rings was the study of the Kazhdan-Luzstig conjecture. At the time, this construction did not seem fruitful. However, we will find here that they are very interesting to cover theory.

4.4.1 Root systems

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and associated root system $\Phi \subset \mathfrak{h}^*$, where $\mathfrak{h}^*$ denotes the dual vector space of $\mathfrak{h}$. In particular, $\mathfrak{g}$ admits a direct sum decomposition into weight spaces for $\mathfrak{h}$ of the form $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \colon [h,x] = \alpha(h)x, \ \forall h \in \mathfrak{h}\}$. Let Π be the set of simple roots of Φ, and therefore it is a basis of the root system Φ (see [EW06, Definition 11.9]). It is also a basis of the vector space $\mathfrak{h}^*$. Set $\Phi^+ := \mathbb{Z}_0^+\Pi \cap \Phi$, giving a direct sum decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^\pm := \bigoplus_{\alpha\in\Phi^\pm} \mathfrak{g}_\alpha$. The Lie algebra $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$ is called the **Borel subalgebra** of $\mathfrak{g}$.

Let E be the real span of Φ and $(-,-)$ be the symmetric bilinear form on E induced by the Killing form associated with the adjoint representation of $\mathfrak{g}$. The **Weyl group** associated with the root system Φ which we denote by W is the finite subgroup of $GL(E)$ generated by all reflections s_α, $\alpha \in \Phi$, where $s_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$, $\lambda \in \mathfrak{h}^*$. For each root $\alpha \in \Phi$, we associate the coroot $\alpha^\vee := \frac{2}{(\alpha,\alpha)}\alpha$. Denote by $\Phi^\vee$ the set of all coroots. Hence, the bilinear form induces, in addition, the following map $\langle -,- \rangle \colon \Phi \times \Phi^\vee \to \mathbb{Z}$, given by $\langle \beta, \alpha^\vee \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. This operator is called **Cartan invariant** in [Hum08]. We call $\mathbb{Z}\Phi$ the **root lattice**.

4.4.2 Integral semi-simple Lie algebras

The first step to obtain an integral version of a finite-dimensional algebra, or in this case a finite-dimensional Lie algebra is to find a basis of the algebra which behaves nicely under the ring multiplication or in this case under the Lie bracket operation. By this, we mean that the ring multiplication of two basis elements is an integral linear combination of the elements of the basis under consideration. For Lie algebras, this means that the Lie bracket of two element basis is an integral linear combination of the elements of the basis under consideration.

Let $\{h_\alpha \colon \alpha \in \Pi\} \cup \{x_\alpha \colon \alpha \in \Phi\}$ be a Chevalley basis of the semi-simple Lie algebra $\mathfrak{g}$, where $\{h_\alpha \colon \alpha \in \Pi\}$ is a basis of $\mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$ for each root $\alpha \in \Phi$. In particular, $\alpha(h_\alpha) = 2$ and $h_\alpha = [x_\alpha, x_{-\alpha}]$ for every $\alpha \in \Phi$. Also, $\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha)$, $\alpha, \beta \in \Phi$. Let $\mathfrak{g}_\mathbb{Z}$ be the additive subgroup of $\mathfrak{g}$ with basis $\{h_\alpha \colon \alpha \in \Pi\} \cup \{x_\alpha \colon \alpha \in \Phi\}$. The restriction of the Lie bracket $[-,-]$ to $\mathfrak{g}_\mathbb{Z} \times \mathfrak{g}_\mathbb{Z}$ has image in $\mathfrak{g}_\mathbb{Z}$ making $\mathfrak{g}_\mathbb{Z}$ a Lie algebra.

For each commutative Noetherian ring with identity R, we define the Lie algebra $\mathfrak{g}_R := R \otimes_\mathbb{Z} \mathfrak{g}_\mathbb{Z}$. By construction, $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Z} \mathfrak{g}_\mathbb{Z} \simeq \mathfrak{g}$. Using the Chevalley basis, we define the following integral Lie subalgebras of $\mathfrak{g}_\mathbb{Z}$: $\mathfrak{h}_\mathbb{Z} = \bigoplus_{\alpha\in\Pi} \mathbb{Z}h_\alpha$, $\mathfrak{n}^\pm_\mathbb{Z} = \bigoplus_{\alpha\in\Phi^+} \mathbb{Z}x_{\pm\alpha}$, $\mathfrak{b}_\mathbb{Z} = \mathfrak{n}^+_\mathbb{Z} \oplus \mathfrak{h}_\mathbb{Z}$.

Analogously, we define for each commutative Noetherian ring with identity R, $\mathfrak{h}_R = R \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}$, $\mathfrak{n}^\pm_R = R \otimes_\mathbb{Z} \mathfrak{n}^\pm_\mathbb{Z}$, $\mathfrak{b}_R = R \otimes_\mathbb{Z} \mathfrak{b}_\mathbb{Z}$. Since $\mathfrak{h}_R$ is free over R, its dual $\operatorname{Hom}_R(\mathfrak{h}_R, R)$ which we will denote by $\mathfrak{h}^*_R$ is free over R.

Observe that $\mathfrak{g}_\mathbb{Q}$ is again a semisimple Lie algebra since otherwise every solvable ideal of $\mathfrak{g}_\mathbb{Q}$ could be extended to a solvable ideal of $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{g}$. Therefore, for any field extension $K \supset \mathbb{Q}$, the Lie algebra $\mathfrak{g}_K$ is semisimple.

Let $U(\mathfrak{g}_R)$ be the universal enveloping algebra of $\mathfrak{g}_R$, that is, $U(\mathfrak{g}_R)$ is the quotient $T(\mathfrak{g}_R)/I_R$ of the tensor algebra $T(\mathfrak{g}_R) = R \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$, where I_R is the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x - [x,y]$, $x, y \in \mathfrak{g}_R$. We denote by $S(\mathfrak{g}_R)$ the symmetric algebra of $\mathfrak{g}_R$, that is, $S(\mathfrak{g}_R)$ is the quotient $T(\mathfrak{g}_R)/J_R$ of the tensor algebra and J_R is the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x$, $x, y \in \mathfrak{g}_R$. The symmetric algebra $S(\mathfrak{g}_R)$ is isomorphic to the polynomial algebra

$$R[\{1_R \otimes h_\alpha \colon \alpha \in \Pi\}, \{1_R \otimes x_\alpha \colon \alpha \in \Phi\}].$$

In particular, $R \otimes_\mathbb{Z} S(\mathfrak{g}_\mathbb{Z}) \simeq S(\mathfrak{g}_R)$. The enveloping algebra of $\mathfrak{g}_R$ also has the base change property. Since $\mathfrak{g}_R$ and $T(\mathfrak{g}_R)$ are free over R, with basis elements independent of R, we can identify $R \otimes_\mathbb{Z} T(\mathfrak{g}_\mathbb{Z})$ with $T(\mathfrak{g}_R)$ and $R \otimes_\mathbb{Z} I_\mathbb{Z}$

with I_R. Hence, we have a commutative diagram with exact rows

$$\begin{tikzcd} R\otimes_{\mathbb{Z}} I_{\mathbb{Z}} \arrow[r, hook] \arrow[d, "\simeq"] & R\otimes_{\mathbb{Z}} T(\mathfrak{g}_{\mathbb{Z}}) \arrow[r, two heads] \arrow[d, "\simeq"] & R\otimes_{\mathbb{Z}} U(\mathfrak{g}_{\mathbb{Z}}) \arrow[d] \\ I_R \arrow[r, hook] & T(\mathfrak{g}_R) \arrow[r, two heads] & U(\mathfrak{g}_R) \end{tikzcd} . \tag{4.4.2.1}$$

Therefore, we obtain:

Lemma 4.4.1. *Let R be a commutative Noetherian ring with identity.*
Then, $U(\mathfrak{g}_R) \simeq R\otimes_{\mathbb{Z}} U(\mathfrak{g}_{\mathbb{Z}})$ and $S(\mathfrak{g}_R) \simeq R\otimes_{\mathbb{Z}} S(\mathfrak{g}_{\mathbb{Z}})$.

Since $\mathfrak{g}_R$ is free over R, the PBW theorem (see for example [Hum80, 17.3]) gives the R-isomorphism

$$U(\mathfrak{g}_R) \simeq U(\mathfrak{n}_R^-)\otimes_R U(\mathfrak{h}_R)\otimes_R U(\mathfrak{n}_R^+), \tag{4.4.2.2}$$

and $U(\mathfrak{g}_R)$ has, as an R-module, a monomial basis over the basis elements of $\mathfrak{g}_R$. We call **PBW monomials** such monomials forming the basis of $U(\mathfrak{g}_R)$. Further, it follows that the enveloping algebra of a free Lie algebra is a Noetherian ring (see [MR87, 7.4]).

Since both $U(\mathfrak{n}_R^+)$ and $U(\mathfrak{n}_R^-)$ are free over R, the PBW theorem allows us to view $U(\mathfrak{h}_R)$ as an R-summand of $U(\mathfrak{g}_R)$. Further, denote by π_R the projection $U(\mathfrak{g}_R) \twoheadrightarrow U(\mathfrak{h}_R)$ which sends all PBW monomials with factors either in $\mathfrak{n}_R^+$ or in $\mathfrak{n}_R^-$ to zero.

Let $Z(\mathfrak{g}_R)$ be the center of the universal enveloping algebra $U(\mathfrak{g}_R)$. The restriction of π_R to the center $Z(\mathfrak{g}_R)$ is called the **Harish-Chandra homomorphism**. For details on why this map is an R-algebra homomorphism see for example [GJ81, 1.3.2]. For each $\lambda \in \mathfrak{h}_R^*$, the **central character** associated with λ is the R-algebra homomorphism $\chi_\lambda : Z(\mathfrak{g}_R) \to R$, given by $\chi_\lambda(z) = \lambda(\pi(z))$, $z \in Z(\mathfrak{g}_R)$. For a given semisimple Lie algebra over a splitting field K, the Harish-Chandra theorem (see [Hum08, 1.10]) says that all K-algebra homomorphisms are of the form χ_λ for some $\lambda \in \mathfrak{h}^*$.

4.4.3 BGG category $\mathcal{O}$ over commutative rings

Assume in the remaining of this section, unless stated otherwise, that R is a commutative Noetherian ring and a $\mathbb{Q}$-algebra. In particular, R has characteristic zero and there exists an injective homomorphism of rings $\mathbb{Q} \to R$, $q \mapsto q1_R$. We can extend the map $\langle -,- \rangle$ to $\mathfrak{h}_R^* \times \Phi^\vee \to R$. Let $\{(1\otimes h_\alpha)^* \colon \alpha \in \Pi\}$ denote a basis of $\mathfrak{h}_R^*$. We define $\langle \lambda, \alpha^\vee\rangle_R := \sum_{\beta\in\Pi} t_\beta \langle \beta, \alpha^\vee\rangle$ for $\lambda = \sum_{\beta\in\Pi} t_\beta (1\otimes h_\beta)^* \in \mathfrak{h}_R^*$.

We call the set of integral weights $\Lambda_R := \{\lambda \in \mathfrak{h}_R^* \colon \langle \lambda, \alpha^\vee\rangle_R \in \mathbb{Z},\ \forall \alpha \in \Phi\}$ the **integral weight lattice** associated with Φ with respect to R. For each $M \in U(\mathfrak{g}_R)$-mod and each $\lambda \in \mathfrak{h}_R^*$, we define the weight space $M_\lambda := \{m \in M \colon h\cdot m = \lambda(h)m,\ \forall h \in \mathfrak{h}_R\}$.

For each $\lambda \in \mathfrak{h}_R^*$, we will denote by $[\lambda]$ the set of elements of $\mathfrak{h}_R^*$, μ, that satisfy $\mu - \lambda \in \Lambda_R$. We define an ordering in $[\lambda]$ by imposing $\mu_1 \leq \mu_2$ if and only if $\mu_2 - \mu_1 \in \mathbb{Z}_0^+\Phi^+ \subset \Lambda_R$.

To motivate both the introduction of the notation $[\lambda]$ and the definition of the category $\mathcal{O}$ over commutative rings we need the definition of the category $\mathcal{O}$ for a semi-simple complex Lie algebra $\mathfrak{g} \simeq \mathfrak{g}_{\mathbb{C}}$.

Definition 4.4.2. The BGG category $\mathcal{O}$ (or just the category $\mathcal{O}$) of a semi-simple Lie algebra $\mathfrak{g}$ over a splitting field of characteristic zero is the full subcategory of $U(\mathfrak{g})$-Mod whose modules satisfy the following conditions:

(i) $M \in U(\mathfrak{g})$-mod;

(ii) M is semi-simple over $\mathfrak{h}$, that is, $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$;

(iii) M is locally $\mathfrak{n}^+$-finite, that is, for each $m \in M$ the subspace $U(\mathfrak{n}^+)m$ of M is finite-dimensional.

In a naive look, one could think that the category $\mathscr{O}$ is too large to be considered under the techniques that we studied here for projective Noetherian R-algebras. Especially, since there is an infinite number of Verma modules (which are the standard modules making the category $\mathscr{O}$ a split highest weight category) and even these have infinite vector space dimension. So, instead of generalizing already the Definition 4.4.2 we will first decompose $\mathscr{O}$ into smaller subcategories. In fact, for any $\lambda \in \mathfrak{h}^*$, there is a "block" associated with λ. In the following, we will identify $\Lambda \subset \mathfrak{h}^*$ with $\Lambda_{\mathbb{C}}$ and $[\lambda] \subset \mathfrak{h}^*$.

Lemma 4.4.3. *Let $M \in \mathscr{O}$. For each $\lambda \in \mathfrak{h}^*$, define the vector space $M^{[\lambda]} := \bigoplus_{\mu\in[\lambda]} M_\mu$, where $\mu \in [\lambda]$ if and only if $\mu - \lambda \in \Lambda$. Then, $M^{[\lambda]} \in U(\mathfrak{g})$-mod and $M = \bigoplus_{[\lambda]\in\mathfrak{h}^*/\sim} M^{[\lambda]}$, where $\sim$ denotes the equivalence relation given by $\mu - \lambda \in \Lambda$.*

Proof. Let $\mu, \lambda, \omega \in \mathfrak{h}$ satisfying $\mu \in [\lambda] \cap [\omega]$. Then, $\mu - \lambda, \mu - \omega \in \Lambda$, and so $\omega - \lambda = \mu - \lambda - (\mu - \omega) \in \Lambda$. So, $[\lambda] = [\omega]$. Let $m \in M_\mu$ for $\mu \in [\lambda]$. By PBW theorem, $U(\mathfrak{g})$ is generated by the elements $x_1^{t_1} \cdots x_n^{t_n}$, $t_1, \dots, t_n \geq 0$, where $\{x_1, \dots, x_n\}$ denotes a basis for $\mathfrak{g}$. But the elements $x_1^{t_1} \cdots x_n^{t_n} m$ have weight $l \in \mu + \mathbb{Z}\Phi$. So, $l - \mu \in \mathbb{Z}\Phi \subset \Lambda$. Therefore, $l \in [\lambda]$. This implies that $U(\mathfrak{g})M_\mu \subset \sum_{\lambda\in[\lambda]} M_l$. So, $M^{[\lambda]} \in U(\mathfrak{g})$-mod and since as vector spaces $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda = \bigoplus_{[\lambda]\in\mathfrak{h}^*/\sim} \bigoplus_{\mu\in[\lambda]} M_\mu$, the result follows. $\square$

Definition 4.4.4. [GJ81, 1.4] Let R be a commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\lambda \in \mathfrak{h}_R^*$.

- We define $\mathscr{O}_{[\lambda],(II),R}$ to be the full subcategory of $U(\mathfrak{g}_R)$-Mod whose modules M satisfy $M = \sum_{\mu\in[\lambda]} M_\mu$.
- We define $\mathscr{O}_{[\lambda],(I),R}$ to be the full subcategory of $\mathscr{O}_{[\lambda],(II),R}$ whose modules M are $U(\mathfrak{n}_R^+)$-locally finite, that is, $U(\mathfrak{n}_R^+)m \in R$-mod for every $m \in M$.
- We define $\mathscr{O}_{[\lambda],R}$ to be the full subcategory of $\mathscr{O}_{[\lambda],(I),R}$ whose modules are finitely generated over $U(\mathfrak{g}_R)$.

As we have seen in Lemma 4.4.3, we can reduce the study of the category $\mathscr{O}$ to the categories $\mathscr{O}_{[\lambda],\mathbb{C}}$, where $\lambda \in \mathfrak{h}^*$. Moreover, by a BGG category $\mathscr{O}$ over a commutative ring R we will mean a category $\mathscr{O}_{[\lambda],R}$ for some $\lambda \in \mathfrak{h}_R^*$.

It comes as no surprise that Verma modules can be defined over any ground ring. Let $\mu \in [\lambda]$ and R_μ be the free R-module with rank one together with the $U(\mathfrak{h}_R)$-action $h1_R = \mu(h)1_R$, $h \in \mathfrak{h}_R$. We can extend R_μ to be an $U(\mathfrak{b}_R)$-module by letting $1_R \otimes x_\alpha$, $\alpha \in \Phi^+$, act on R_μ identically as zero. The **Verma module** $\Delta(\mu)$ (associated with μ) is defined to be the $U(\mathfrak{g}_R)$-module $\Delta(\mu) := U(\mathfrak{g}_R) \otimes_{U(\mathfrak{b}_R)} R_\mu$.

Lemma 4.4.5. *Let $\lambda \in \mathfrak{h}_R^*$. If $\mu \in [\lambda]$, then $\Delta(\mu) \in \mathscr{O}_{[\lambda],R}$ and $\Delta(\mu)$ is free as $U(\mathfrak{n}_R^-)$-module.*

Proof. The result follows once we show that the weight modules $\Delta(\mu)_\omega$ are zero unless $\omega \in \mu - \mathbb{Z}_0^+\Pi \subset [\lambda]$. By PBW theorem, we obtain as $U(\mathfrak{n}_R^-)$-modules,

$$\Delta(\mu) = U(\mathfrak{g}_R) \otimes_{U(\mathfrak{b}_R)} R_\mu \simeq U(\mathfrak{n}_R^+) \otimes_R U(\mathfrak{h}_R) \otimes_R U(\mathfrak{n}_R^-) \otimes_{U(\mathfrak{b}_R)} R_\mu \simeq U(\mathfrak{n}_R^-) \otimes_R U(\mathfrak{b}_R) \otimes_{U(\mathfrak{b}_R)} R_\mu \tag{4.4.3.1}$$

$$\simeq U(\mathfrak{n}_R^-) \otimes_R R_\mu \simeq U(\mathfrak{n}_R^-). \tag{4.4.3.2}$$

Moreover, if we denote by $\alpha_1, \dots, \alpha_t$ all the roots in Π, then

$$\{(1_R \otimes x_{-\alpha_1})^{i_1} \cdots (1_R \otimes x_{-\alpha_t})^{i_t} (1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_R) \colon i_1, \dots, i_t \geq 0\}$$

is an R-basis of $\Delta(\mu)$, where the monomials $(1_R \otimes x_{-\alpha_1})^{i_1} \cdots (1_R \otimes x_{-\alpha_t})^{i_t}$ are PBW monomials. Denote by $y := 1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_R \in \Delta(\mu)$. Let $h \in \mathfrak{h}_R$. Then, $hy = (h1_{U(\mathfrak{g}_R)}) \otimes_{U(\mathfrak{b}_R)} 1_R = 1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} h1_R = \mu(h)y$. Also, for $\alpha \in \Phi^+$, we obtain

$$h((1_R \otimes x_{-\alpha})y) = (h(1_R \otimes x_{-\alpha}))y = [h, 1_R \otimes x_{-\alpha}]y + (1_R \otimes x_{-\alpha})hy \tag{4.4.3.3}$$

$$= -\alpha(h)(1_R \otimes x_{-\alpha})y + (1_R \otimes x_{-\alpha})\mu(h)y. \tag{4.4.3.4}$$

Hence, $y \in \Delta(\mu)_\mu$ and $(1_R \otimes x_{-\alpha})y \in \Delta(\mu)_{\mu-\alpha}$, $\alpha \in \Phi^+$. This shows that $\Delta(\mu)_\omega$ is zero unless $\omega \in \mu - \mathbb{Z}_0^+\Pi$ and $\Delta(\mu) = \bigoplus_{\omega \in \mu - \mathbb{Z}_0^+\Pi} \Delta(\mu)_\omega$. Further, $\mathbb{Z}_0^+\Pi \subset \Lambda_R$. Therefore, $\Delta(\mu) \in \mathscr{O}_{[l],(II),R}$.

Again by (4.4.3.4) $(1 \otimes x_\alpha)\Delta(\mu)_\omega \subset \Delta(\mu)_{\omega+\alpha}$ for $\alpha \in \Phi^+$. So, for a large enough i_j depending on $\omega \in \mu - \mathbb{Z}_0^+\Pi$, $(1_R \otimes x_{\alpha_j})^{i_j}\Delta(\mu)_\omega \subset \Delta(\mu)_\nu = 0$ for some $\nu \notin \mu - \mathbb{Z}_0^+\Pi$. Thus, $\Delta(\mu) \in \mathscr{O}_{[\lambda],(I),R}$. Since $\Delta(\mu)$ is finitely generated by y we obtain that $\Delta(\mu) \in \mathscr{O}_{[\lambda],R}$. ☐

We observe that $\Delta(\mu)$ is not finitely generated over R. In the following, we state some known facts about homomorphisms between Verma modules.

Lemma 4.4.6. *Let $\lambda \in \mathfrak{h}_R^*$. Then:*

(i) For every $\mu, \omega \in [\lambda]$, if $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu), \Delta(\omega)) \neq 0$, then $\mu \leq \omega$.

(ii) $\operatorname{End}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu)) \simeq R$ for every $\mu \in [\lambda]$.

(iii) For every $\mu, \omega \in [\lambda]$, any non-zero map in $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu), \Delta(\omega))$ is injective.

Proof. Let $\mu, \omega \in [\lambda]$ such that $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu), \Delta(\omega)) \neq 0$. By Tensor-Hom adjunction,

$$\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu), \Delta(\omega)) \simeq \operatorname{Hom}_{U(\mathfrak{b}_R)}(R_\mu, \Delta(\omega)) \subset \Delta(\omega)_\mu.$$

By assumption, μ is a weight of $\Delta(\omega)$. Hence, $\mu \in \omega - \mathbb{Z}_0^+\Pi$. So, $\mu \leq \omega$.

If $\mu = \omega$, then for any homomorphism $f \in \operatorname{Hom}_{U(\mathfrak{b}_R)}(R_\mu, \Delta(\mu))$, $f(1_R) \in \Delta(\mu)_\mu = R$ and it is annihilated by $\mathfrak{n}_R^+$. Further, for every element $r \in R$, we can define $g \in \operatorname{Hom}_{U(\mathfrak{b}_R)}(R_\mu, \Delta(\mu))$, by imposing $g(1_R) = r(1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_R)$. This shows that $\operatorname{End}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu)) \simeq R$.

For (iii), we can apply the same idea as in the classical case (see [Hum08, 4.2]). In fact, for every $f \in \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\mu), \Delta(\omega))$ we can write $f(1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_{R_\mu}) = u 1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_{R_\omega}$ for some $u \in U(\mathfrak{n}_R^-)$. Using the PBW theorem we can see that $U(\mathfrak{n}_R^-)$ is an integral domain (see [MR87, 7.4]). By identifying f with an endomorphism of $U(\mathfrak{n}_R^-)$ given by $a \mapsto au$, $U(\mathfrak{n}_R^-)$ being an integral domain implies that f is injective. ☐

4.4.4 Properties of (classical) BGG category $\mathscr{O}$

Before we proceed any further, we should recall some properties of the category $\mathscr{O}$ for a given a semisimple Lie algebra $\mathfrak{g}$ over a splitting field K of characteristic zero without giving proofs.

The category $\mathscr{O}$ can be decomposed in finer blocks than the ones described in Lemma 4.4.3 and these can be completely determined by the orbits under the dot action of the Weyl group. In fact, for any $M \in \mathscr{O}$, $M = \bigoplus_\chi M^\chi$, as χ runs over the central characters $Z(\mathfrak{g}) \to K$ and

$$M^\chi := \{m \in M \colon \forall z \in Z(\mathfrak{g})\ \exists n \in \mathbb{N}\ (z - \chi(z))^n m = 0\}. \tag{4.4.4.1}$$

is a module in $\mathscr{O}$. The argument provided in [Hum08, 1.12] requires K to be an algebraically closed field, but we do not need such a condition. We could use instead Gabber and Joseph techniques (see [GJ81, 1.4.2]) together with the Harish-Chandra theorem stating that $\chi_\lambda = \chi_\mu$ if μ and λ are linked by a certain Weyl group and taking into account that the category $\mathscr{O}$ is both Artinian and Noetherian (see [Hum08, 1.11]). To see that this is a finite direct sum is also required to observe that $\Delta(\lambda)^{\chi_\lambda} = \Delta(\lambda)$ and $M \mapsto M^{\chi_\lambda}$ is an exact functor $\mathcal{O} \to \mathcal{O}$ for every $\lambda \in \mathfrak{h}^*$. For each central character χ, denote by $\mathcal{O}_\chi$ the full subcategory of $\mathcal{O}$ whose objects are the modules M satisfying $M = M^\chi$.

The dot action of the Weyl group W is defined as $w \cdot \lambda := w(\lambda + \rho) - \rho$, where ρ is the half-sum of all positive roots. With this, for each $\lambda \in \mathfrak{h}$, one can define another Weyl group $W_{[\lambda]}$ associated with a root system that views λ as an integral weight lattice. Explicitly, $W_{[\lambda]} := \{w \in W : w \cdot \lambda - \lambda \in \mathbb{Z}\Phi\}$.

Theorem 4.4.7. *The following results hold for the category $\mathcal{O}$ of a semisimple Lie algebra over a splitting field of characteristic zero.*

(a) For each $\lambda \in \mathfrak{h}^*$, the Verma module $\Delta(\lambda)$ has a unique simple quotient and every simple module in $\mathcal{O}$ is isomorphic to the simple quotient of some Verma module $\Delta(\lambda)$ which we will denote by $L(\lambda)$.

(b) The simple module $L(\lambda)$ in $\mathcal{O}$ is finite-dimensional if and only if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_0^+$ for every $\alpha \in \Phi^+$. Such weights are known as integral dominant weights. In such a case, $L(\lambda) \simeq U(\mathfrak{g})/J$. Here, J is the left ideal of $U(\mathfrak{g})$ generated by the elements x_α, ($\alpha \in \Phi^+$), $h - \lambda(h)1$, ($h \in \mathfrak{h}$) and $x_\beta^{n_\beta+1}$, ($\beta \in \Pi$), where $n_\beta = \langle \lambda, \beta^\vee \rangle \in \mathbb{Z}_0^+$.

(c) The Verma module $\Delta(\lambda)$ is simple in $\mathcal{O}$ if and only if λ is antidominant, that is, $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \Phi^+$. In particular, λ is minimal and the unique antidominant weight in its $W_{[\lambda]}$-orbit.

(d) The Verma module $\Delta(\lambda)$ is projective in $\mathcal{O}$ if and only if λ is dominant, that is, $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^-$ for all $\alpha \in \Phi^+$. In particular, λ is maximal and the unique dominant weight in its $W_{[\lambda]}$-orbit.

(e) $\mathcal{O}$ has enough projectives, and the projective cover of $\Delta(\lambda)$ (which exists) is injective if and only if λ is antidominant.

(f) The category $\mathcal{O}$ is the direct sum of the subcategories $\mathcal{O}_{\chi_\lambda}$ consisting of modules whose composition factors all have highest weights linked by $W_{[\lambda]}$, as λ runs over all antidominant weights (or alternatively over all dominant weights). In particular, $\chi_\lambda = \chi_\mu$ if μ and λ belong to the same orbit under the Weyl group $W_{[\lambda]}$.

(g) The blocks $\mathcal{O}_{\chi_\lambda}$ with the Verma modules being the standard modules are split highest weight categories with a finite number of standard modules. Here, the ordering is given by $\mu_1 \leq \mu_2$ if and only if $\mu_2 - \mu_1 \in \mathbb{Z}_0^+\Pi$.

Proof. For (a) see [Hum08, p.18]. For (b) see [Hum08, p.21, p.44]. For (c) see [Hum08, p.55,p.77]. For (d) see [Hum08, p.55, p.60]. For (e) see [Hum08, p.60-61, p.149-151] For (f) see [Hum08, p.83]. For (g) see [Hum08, p.64-65, p.68]. ☐

Observe that if a weight λ is both antidominant and dominant, then $\Delta(\lambda)$ is projective and simple. So, the block $\mathcal{O}_{\chi_\lambda}$ is semisimple if and only if λ is both antidominant and dominant.

4.4.5 Properties of BGG category $\mathcal{O}$ over commutative rings

The crucial point of the category $\mathcal{O}$ is that ultimately it can be viewed as a direct sum of module categories over finite-dimensional algebras. This is what we will explore for the BGG category $\mathcal{O}$ over a commutative ring. As we will see later on, we must impose that R is also local so that the classical category $\mathcal{O}$ is obtained as a specialization of a direct sum of module categories of projective Noetherian R-algebras. But for now assume just that R is a commutative Noetherian ring which is a $\mathbb{Q}$-algebra.

Definition 4.4.8. Let S be any commutative ring and $\{J_i\}_{i\in I}$ be a family of two-sided ideals of S such that $J_i+J_j=S$ whenever $i\neq j$. Let $\mathscr{J}$ be the category of S-modules $M=\sum_{i\in I}M_i$ where

$$M_i=\{m\in M\colon \forall x\in J_i\ \exists n\in\mathbb{N}\ x^n m=0\}.$$

Note that $m_1+m_2\in M_i$ whenever $m_1,m_2\in M_i$ since for each $x\in J_i$ we can choose the higher value n_1 and n_2 and then $x^{\max\{n_1,n_2\}}$ kills m_1+m_2. Since S is commutative M_i becomes an S-module.

Lemma 4.4.9. *[GJ81, 1.4.2] $M\in\mathscr{J}$ if and only if each $m\in M$ there exists a finite set $F\subset I$ such that for all $x_i\in J_i$, $i\in F$, there is $n_i\in\mathbb{N}$ satisfying $\prod_{i\in I}x_i^{n_i}m=0$.*

Proof. Let $m\in M=\sum_{i\in I}M_i$. So, there exists a finite set $F\subset I$ such that $m=\sum_{i\in F}m_i$, and $m_i\in M_i$. For all $x_i\in J_i$ there exists $n_i\in\mathbb{N}$ so that $x_i^{n_i}m_i$. Therefore, $\prod\limits_{i\in I}x_i^{n_i}\sum\limits_{i\in F}m_i=0$.

Conversely, let $M\in S$-Mod so that for each $m\in M$ there exists a finite set $F\subset I$ such that for all $x_i\in J_i$, $i\in F$, there is $n_i\in\mathbb{N}$ satisfying $\prod_{i\in I}x_i^{n_i}m=0$. We need to show that $M=\sum_{i\in I}M_i$. Clearly, $\sum_{i\in I}M_i\subset M$. For each finite set $F\subset I$ define $M_F=\{m\in M|\forall x_i\in J_i,\ i\in F,\ \exists n_i\in\mathbb{N}\ \prod_{i\in F}x_i^{n_i}m=0\}$. We claim that M_F is an S-submodule of M. To see that observe that for $m_1,m_2\in M_F$ for any $x_i\in J_i$, $i\in F$ then there exists $n_i^{(1)},n_i^{(2)}\in\mathbb{N}$ so that $\prod_{i\in F}x_i^{n_i^{(1)}}m_1=\prod_{i\in F}x_i^{n_i^{(2)}}m_2=0$. Therefore,

$$\prod_{i\in F}x_i^{n_i^{(1)}+n_i^{(2)}}(m_1+m_2)=0. \tag{4.4.5.1}$$

Since S is commutative, it is clear that M_F is an S-module. Now, we will proceed by induction on $|F|$ to show that $M_F=\sum_{i\in F}M_i$. If $|F|=1$, there is nothing to prove. Assume now that $|F|>1$. Since $J_i+J_j=S$ then $1_S=x_i+x_j$, for some $x_i\in J_i$ and $x_j\in J_j$. Let $m\in M_F$. By assumption, there exists $n_j\in\mathbb{N}$ such that $x_j^{n_j}m\in M_{F\backslash\{j\}}$. By induction, $x_j^{n_j}m=\sum_{i\in F\backslash\{j\}}m_i$, $m_i\in M_i$. So,

$$m-m'=(1_S-x_i)^{n_j}m=\sum_{i\in F\backslash\{j\}}m_i, \tag{4.4.5.2}$$

where m' is a sum of elements of the form $x_i^t m\in M_F$, $t>0$. Proceed now induction with these elements $x_i^t m$. Eventually, we obtain a sum of $x_1^{t_1}\cdots x_i^{t_i}=0$. This shows that $M_F=\sum_{i\in F}M_i$.

Now since each element $m\in M$ belongs to some M_F, where F is a finite subset of I we obtain that $m=\sum_{i\in F}m_i$, where $m_i\in M_i$. Hence, $M\in\mathscr{J}$. ☐

Lemma 4.4.10. *[GJ81, 1.4.3] Let S be any commutative ring and $\{J_i\}_{i\in I}$ be a family of two-sided ideals of S such that $J_i+J_j=S$ whenever $i\neq j$. The following assertions hold.*

(i) If $M\in\mathscr{J}$, then $M=\bigoplus_{i\in I}M_i$.

(ii) The category $\mathscr{J}$ is closed under submodules, quotients and direct sums.

(iii) The functor $\mathscr{J}\to\mathscr{J}$, given by $M\mapsto M_i$, is an exact functor.

Proof. We have already that $M=\sum_{i\in I}M_i$. To prove (a) it remains to show that the elements of M can be written in an unique way as a sum of elements belonging to M_i. For this, it is enough to show that $0=\sum_{i\in F}m_i$ implies that $m_i=0$ for all $i\in F$. If $|F|=1$, the result is clear. Again, $1_S=x_l+x_j$ for some $x_i\in J_i$ and $x_j\in J_j$. So, there exists $n_l\in\mathbb{N}$ satisfying

$$0=x_l^{n_l}\sum_{i\in F}m_i=\sum_{i\in F\backslash\{l\}}x_l^{n_l}m_i=\sum_{i\in F\backslash\{l\}}(1_S-x_j)^{n_l}m_i. \tag{4.4.5.3}$$

By induction, we obtain, for each $i \in F\backslash\{l\}$, $0 = (1_S - x_j)^{n_l} m_i = m_i - m_i'$, where m_i' is a sum of elements of the form $x_j^t m_i$ with $t > 0$. Consider $j \in F$ and the minimal $n_j \in \mathbb{N}$ such that $x_j^{n_j} m_j = 0$. Then, $x_j^{n_j-1} m_j = x_j^{n_j-1} m_j' = 0$, which is a contradiction to the choice of n_j. Hence, $m_j = 0$. Going through all $j \in F$ we obtain that $m_i = 0$ for all $i \in F$.

Let M' be a submodule of $M \in \mathscr{J}$. Since every element of M' satisfies the condition of Lemma 4.4.9 we obtain that $M' \in \mathscr{J}$. Due to Lemma 4.4.9, $\mathscr{J}$ is also closed under direct sums. For each element of the quotient of $M \in \mathscr{J}$ and for every $x_i \in J_i$ we can pick the $n_i \in \mathbb{N}$ which satisfies the condition in Lemma 4.4.9 for its preimage. Hence, $\mathscr{J}$ is also closed under quotients. So, (b) follows. By (ii), for every $M \in \mathscr{J}$, $M_i \in \mathscr{J}$. By (i), each $M \simeq \bigoplus_{i\in I} M_i$ and by the definition of direct sums of modules the functor is exact. □

To get an idea of what Lemma 4.4.10 is doing we can think about central idempotents. For a set of central orthogonal idempotents, $\{e_1, \dots, e_n\}$ of a commutative ring S, define $J_i = S\sum_{j=1, j\neq i}^{n} e_j$. Then, $J_i + J_j = S$ whenever $i \neq j$ and $M_i = e_i M$. Hence, Lemma 4.4.10 is a generalization of the process of decomposing a module in terms of orthogonal idempotents over a commutative ring.

We will now apply Lemma 4.4.10 to the symmetric algebra of the Cartan algebra $S = U(\mathfrak{h}_R) = S(\mathfrak{h}_R)$ and the category $\mathscr{O}_{[\lambda],(II),R}$ taking the role of $\mathscr{J}$. For each $\lambda \in \mathfrak{h}_R^*$, define the R-algebra homomorphism $p_\lambda : S(\mathfrak{h}_R) \to R$, given by $h \mapsto \lambda(h)$, $h \in \mathfrak{h}_R$. This is where R being a $\mathbb{Q}$-algebra is useful.

Lemma 4.4.11. *[GJ81, 1.4.4] Fix $\lambda \in \mathfrak{h}_R^*$. Consider the family of ideals $J_\mu := \ker p_\mu$, $\mu \in [\lambda]$, of the symmetric algebra $S(\mathfrak{h}_R)$. Then, $J_{\mu_1} + J_{\mu_2} = S(\mathfrak{h}_R)$ whenever $\mu_1 \neq \mu_2$.*

Proof. Since $\mu_1, \mu_2 \in [\lambda]$ then $\mu_1 - \mu_2 = \mu_1 - \lambda - (\mu_2 - \lambda) \in \Lambda_R$. Since they are non-zero, there exists $\alpha \in \Phi$ such that

$$\mu_1(h_\alpha) - \mu_2(h_\alpha) = (\mu_1 - \lambda)(h_\alpha) - (\mu_2 - \lambda)(h_\alpha) = \langle \mu_1 - \lambda - (\mu_2 - \lambda), \alpha^\vee \rangle \in \mathbb{Z}\backslash\{0\}. \tag{4.4.5.4}$$

Since R is a commutative $\mathbb{Q}$-algebra the element $(\mu_1(h_\alpha) - \mu_2(h_\alpha))1_R$ is invertible. But, $\mu_i(h_\alpha) - h_\alpha \in J_{\mu_i}$, $i = 1, 2$. Therefore,

$$(\mu_1(h_\alpha) - \mu_2(h_\alpha))1_R = \mu_1(h_\alpha) - h_\alpha - (\mu_2(h_\alpha) - h_\alpha) \in J_{\mu_1} + J_{\mu_2}. \tag{4.4.5.5}$$

It follows that $J_{\mu_1} + J_{\mu_2} = S(\mathfrak{h}_R)$. □

Now combining Lemma 4.4.10 with Lemma 4.4.11 we obtain the following.

Corollary 4.4.12. *For every $M \in \mathscr{O}_{[\lambda],(II),R}$, the following assertions hold:*

1. *$M = \bigoplus_{\mu\in[\lambda]} M_\mu$;*
2. *The assignment $M \mapsto M_\mu$ is an exact functor on $\mathscr{O}_{[\lambda],(II),R}$;*
3. *The category $\mathscr{O}_{[\lambda],(II),R}$ is closed under quotients, submodules and direct sums.*

Of course, we want these properties for the category $\mathscr{O}_{[\lambda],R}$. In fact, these can be transported to our case of interest but it is done in steps.

Lemma 4.4.13. *[GJ81, 1.4.6] Let $M \in \mathscr{O}_{[\lambda],(II),R}$. Then, $M \in \mathscr{O}_{[\lambda],(I),R}$ if and only if for all $m \in M$ there exists a natural number s such that $\mathfrak{n}_R^{+s} m = 0$.*

Proof. Assume that for all $m \in M$ there exists a natural number s such that $\mathfrak{n}_R^{+s} m = 0$. The associative algebra $U(\mathfrak{n}_R^+)$ has an R-basis formed by the monomial elements $(1_R \otimes x_{\alpha_1})^{t_1} \cdots (1_R \otimes x_{\alpha_n})^{t_n}$, where $\alpha_1, \dots, \alpha_n$ are the simple roots. By hypothesis, $(1_R\otimes x_{\alpha_1})^{t_1}\cdots(1_R\otimes x_{\alpha_n})^{t_n}m$ is zero unless $t_1,\dots,t_n\in\mathbb{Z}_0^+$ so that $t_1+\cdots+t_n\leq s$. Hence, $U(\mathfrak{n}_R^+)m\in R$-mod for every $m\in M$.

Conversely, assume that $M\in\mathscr{O}_{[\lambda],(I),R}$. Let $m\in M$. We can choose, without loss of generality, that $m\in M_\mu$ for some $\mu\in[\lambda]$ since $m\in\bigoplus_{\mu\in[\lambda]}M_\mu$. By assumption, $U(\mathfrak{n}_R^+)m\in R$-mod and it is an $U(\mathfrak{h}_R)$-submodule of M. By Corollary 4.4.12, $U(\mathfrak{n}_R^+)m\in\mathscr{O}_{[\lambda],(II),R}$. Further, since $U(\mathfrak{n}_R^+)m\in R$-mod there exists a finite set of $[\lambda]$ such that $U(\mathfrak{n}_R^+)m\subset\sum_{\mu\in F}M_\mu$. Since $(1\otimes x_\alpha)M_\mu\subset M_{\mu+\alpha}$ we can find a natural number s such that $(1\otimes x_{\alpha_1})^{t_1}\cdots(1\otimes x_{\alpha_n})^{t_n}m$ is not contained in $\sum_{\mu\in F}M_\mu$ if $t_1+\cdots+t_n>s$. ☐

Corollary 4.4.14. *[GJ81, 1.4.7] The categories $\mathscr{O}_{[\lambda],(I),R}$ and $\mathscr{O}_{[\lambda],R}$ are closed under submodules, quotients and direct sums.*

Proof. We will just prove the claim for submodules, the others are analogous. Let $M\in\mathscr{O}_{[\lambda],(I),R}$ and $M'\subset M$. By Lemma 4.4.13, we obtain that for all $m\in M'\subset M$ there exists a natural number s such that ${\mathfrak{n}_R^+}^s m=0$. By Corollary 4.4.12, $M'\in\mathscr{O}_{[\lambda],(II),R}$. By Lemma 4.4.13, $M'\in\mathscr{O}_{[\lambda],(I),R}$.

It follows by the PBW theorem and Hilbert basis theorem that $U(\mathfrak{g}_R)$ is a Noetherian ring (see [MR87, 7.4]). So, submodules of finitely generated modules over $U(\mathfrak{g}_R)$ are again finitely generated. Combining this fact with $\mathscr{O}_{[\lambda],(I),R}$ being closed under submodules, we obtain that $\mathscr{O}_{[\lambda],R}$ is closed under submodules. ☐

Lemma 4.4.15. *[GJ81, 1.4.8] Let $M\in\mathscr{O}_{[\lambda],R}$. Then, $M\in U(\mathfrak{n}_R^-)$-mod.*

Proof. By assumption, $M\in U(\mathfrak{g}_R)$-mod. Thus, we can write $M=\sum_{i=1}^t U(\mathfrak{g}_R)m_i$. Moreover, since $M\simeq\bigoplus_{\mu\in[\lambda]}M_\mu$ we can choose the elements m_i to belong to weight spaces M_μ. Thus, $U(\mathfrak{h}_R)$ preserves Rm_i.

By the PBW theorem, $U(\mathfrak{g}_R)\simeq U(\mathfrak{n}_R^-)\otimes_R U(\mathfrak{h}_R)\otimes_R U(\mathfrak{n}_R^+)$. Since $U(\mathfrak{n}_R^+)m_i\in R$-mod we can write $U(\mathfrak{n}_R^+)m_i=\sum_{j=1}^{q_i}Rm_{i,j}$ where each element $m_{i,j}$ belongs to some weight space. Combined all these facts, we obtain that $M=\sum_{i=1}^t\sum_{j=1}^{q_i}U(\mathfrak{n}_R^-)m_{i,j}$. ☐

Taking into account that the Verma modules are free of rank one over $U(\mathfrak{n}_R^-)$, Lemma 4.4.15 can be interpreted as saying that Verma modules are in some sense the building blocks of the category $\mathscr{O}_{[\lambda],R}$ taking the place of projective indecomposable modules. Note once more that for non-local rings the category $\mathscr{O}_{[\lambda],R}$ is very far from being Krull-Schmidt. To make this statement about Verma modules more precise, it is useful to consider an equivalent construction of Verma modules.

For each $\lambda\in\mathfrak{h}_R^*$, the Verma module $\Delta(\lambda)$ is generated by $1_{U(\mathfrak{g}_R)}\otimes_{U(\mathfrak{b}_R)}1_R$ as $U(\mathfrak{g}_R)$-module. Moreover, for every $\alpha\in\Phi^+$, $1_R\otimes x_\alpha$ acts as zero and each $h\in\mathfrak{h}_R$ acts as $\lambda(h)$. Also $\Delta(\lambda)$ is free as $U(\mathfrak{n}_R^-)$-module, therefore the surjective map $U(\mathfrak{g}_R)\to\Delta(\lambda)$ given by $1_{U(\mathfrak{g}_R)}\mapsto 1_{U(\mathfrak{g}_R)}\otimes_{U(\mathfrak{b}_R)}1_R$ has kernel I_λ where I_λ is the ideal generated by $1_R\otimes x_\alpha$, $\alpha\in\Phi^+$ and $h-\lambda(h)1_R$, $h\in\mathfrak{h}_R$. Hence, $\Delta(\lambda)\simeq U(\mathfrak{g}_R)/I_\lambda$.

Lemma 4.4.16. *[GJ81, 1.4.9] If $M\in\mathscr{O}_{[\lambda],R}$, then M has a finite filtration with quotients isomorphic to quotients of $\Delta(\mu)$, $\mu\in[\lambda]$.*

Proof. By the proof of Lemma 4.4.15, we can assume that $M=\sum_{i=1}^s U(\mathfrak{n}_R^-)x_{s-i}$ with $x_i\in M_{\mu_i}$. The labelling is chosen such that $i<j\implies\mu_i\not\geq\mu_j$. In this way, the weights of $U(\mathfrak{n}_R^-)x_i$ are less or equal than μ_i. So, $(U(\mathfrak{n}_R^-)x_i)_{\mu_j}=0$ and also $(U(\mathfrak{n}_R^-)x_i)_\mu=0$ with $\mu\geq\mu_j$. Set

$$F^{s-t}M=\sum_{i=1}^t U(\mathfrak{n}_R^-)x_{s-i},\quad t=1,\dots,s. \tag{4.4.5.6}$$

Hence, $0=F^sM\subset F^{s-1}M\subset\cdots\subset F^1M\subset F^0M=M$ is a filtration of M. We will separate the proof in the following steps:

Claim 1. Each $F^jM \in U(\mathfrak{g}_R)$-Mod for all $j = 0, \dots, s$.

Claim 2. $\forall \alpha \in \Phi^+$, $(1_R \otimes x_\alpha)F^jM \subset F^jM = U(\mathfrak{n}_R^-)x_j + \cdots + U(\mathfrak{n}_R^-)x_{s-1}$.

Claim 3. Each module $F^jM/F^{j+1}M$ is a quotient of a Verma module.

We will start by proving Claim 2. Let $\alpha \in \Phi^+$. Pick an element y of $U(\mathfrak{n}_R^-)x_j$, that is, $y = zx_j$ for some $z \in U(\mathfrak{n}_R^-)$. But, $(1_R \otimes x_\alpha)x_j$ has weight $\alpha + \mu_j$ which cannot be smaller than any μ_i, with $i \leq j$, by the choice of labelling. Thus, $(1_R \otimes x_\alpha)x_j \in F^{j+1}M$. Further,

$$(1_R \otimes x_\alpha)(zx_j) = [1_R \otimes x_\alpha, z]x_j + z((1_R \otimes x_\alpha)x_j)$$

and $z((1_R \otimes x_\alpha)x_j) \in F^{j+1}M$. Now, decomposing $[1_R \otimes x_\alpha, z]$ into a linear combination of PBW monomials we obtain that $[1_R \otimes x_\alpha, z]x_j \in F^jM$. So, Claims 2 and 1 follow. Now, for the last claim, observe the following

$$F^jM/F^{j+1}M = U(\mathfrak{n}_R^-)x_j + \cdots + U(\mathfrak{n}_R^-)x_{s-1}/U(\mathfrak{n}_R^-)x_{j+1} + \cdots + U(\mathfrak{n}_R^-)x_{s-1} \simeq U(\mathfrak{n}_R^-)\overline{x_j}, \tag{4.4.5.7}$$

where $\overline{x_j}$ denotes the image of x_j in the quotient $F^jM/F^{j+1}M$. In particular, $(1_R \otimes x_\alpha)\overline{x_j} = 0$ for all $\alpha \in \Phi^+$. Hence, the surjective $U(\mathfrak{g}_R)$-homomorphism $U(\mathfrak{g}_R) \to F^jM/F^{j+1}M$, given by $1_{U(\mathfrak{g}_R)} \mapsto \overline{x_j}$, factors through $\Delta(\mu_j)$. Hence, there exists a surjective $U(\mathfrak{g}_R)$-homomorphism $\Delta(\mu_j) \to F^jM/F^{j+1}M$. This concludes the proof. $\square$

Corollary 4.4.17. *[GJ81, 1.4.10] For each $M \in \mathscr{O}_{[\lambda],R}$ the weight modules M_μ, $\mu \in [\lambda]$ are finitely generated as R-modules.*

Proof. By Lemma 4.4.16, and the exactness of $M \mapsto M_\mu$ it is enough to show the result for Verma modules $\Delta(\mu)$, $\mu \in [\lambda]$. For each weight δ of $\Delta(\mu)$, there is only a finite number of ways of writing δ as an element in $\lambda - \mathbb{Z}_0^+ \Pi$, since the set of simple roots Π is finite. Therefore, $\Delta(\mu)_\delta = (U(\mathfrak{n}_R^-)(1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_R))_\delta \in R$-mod. $\square$

Another consequence of Lemma 4.4.16 is the fact that endomorphisms of modules belonging to the category $\mathscr{O}$ are finitely generated over the ground ring.

Proposition 4.4.18. *Let $M, N \in \mathscr{O}_{[\lambda],R}$. Then,* $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M,N) \in R$-mod.

Proof. We will proceed by induction on the length of M and N by quotients of Verma modules given in Lemma 4.4.16. Let $Q(\mu)$ be a quotient of $\Delta(\mu)$ and $Q(\omega)$ be a quotient of $\Delta(\omega)$, $\mu, \omega \in [\lambda]$. Applying $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(-, Q(\mu))$ we obtain the monomorphism $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(Q(\omega), Q(\mu)) \to \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega), Q(\mu)) \subset Q(\mu)_\omega$. By Corollary 4.4.17, $Q(\mu)_\omega \in R$-mod. Since R is a Noetherian ring we obtain in this way that $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(Q(\omega), Q(\mu)) \in R$-mod. Assume now that there exists an exact sequence $0 \to M' \to M \to Q(\omega) \to 0$. Again, applying $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(-, Q(\mu))$ we obtain the exact sequence

$$0 \to \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(Q(\omega), Q(\mu)) \to \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M, Q(\mu)) \to X \to 0, \tag{4.4.5.8}$$

where X is a submodule of $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M', Q(\mu)) \in R$-mod by induction.

Thus, $X \in R$-mod and $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M, Q(\mu)) \in R$-mod. Now, using exact sequences $0 \to N' \to N \to Q(\mu) \to 0$ and applying $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M, -))$ we obtain by induction that $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M,N) \in R$-mod. $\square$

Eventually, we would like to reduce the study of $\mathscr{O}_{[\lambda],R}$ into blocks which in turn can be reduced to the study of module categories of projective Noetherian R-algebras. But, for that, we need to first study filtrations by Verma modules and construct projective objects in $\mathscr{O}_{[\lambda],R}$.

4.4.6 Verma flags

For this subsection, we will require in addition that R is a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. We will now discuss modules having filtrations by Verma modules. As usual, we will denote by $\mathscr{F}(\Delta_{[\lambda]})$ the full subcategory of $\mathscr{O}_{[\lambda],R}$ having a filtration by modules $\Delta(\mu)\otimes_R X_\mu$, $X_\mu \in R$-proj, $\mu \in [\lambda]$. In the literature, these filtrations are known as **Verma flags**.

Before we proceed any further and give examples of how such modules having Verma flags appear naturally we need to discuss tensor product of modules in $\mathscr{O}_{[\lambda],R}$.

Let $\mathfrak{a}_R$ be any Lie algebra with finite rank over R. Recall that $L\otimes_R M \in U(\mathfrak{a}_R)$-Mod whenever $L, M \in U(\mathfrak{a}_R)$ with action $a\cdot(l\otimes m) = (al)\otimes m + l\otimes(am)$ and any left $U(\mathfrak{a}_R)$-module L can be regarded as right $U(\mathfrak{a}_R)$-module by taking $l\cdot a := -al$. This is a consequence of the universal enveloping algebra of a Lie algebra being a Hopf algebra. For each left $U(\mathfrak{a}_R)$-module, by L^* we mean the left $U(\mathfrak{a}_R)$, $\mathrm{Hom}_R(L,R)$, which inherits the left action by regarding L as a right $U(\mathfrak{a}_R)$. So, for every $L, M \in U(\mathfrak{a}_R)$ we can also regard $\mathrm{Hom}_R(L,M)$ as an $U(\mathfrak{a}_R)$-module by taking $(a\cdot f)(l) := af(l) - f(al)$, $f\in \mathrm{Hom}_R(L,M)$, $a\in U(\mathfrak{a}_R)$, $l\in L$. In particular, the $U(\mathfrak{a}_R)$-invariants of $\mathrm{Hom}_R(L,M)$ are the elements $f\in\mathrm{Hom}_R(L,M)$ satisfying $a\cdot f = 0$. Therefore, they coincide with $\mathrm{Hom}_{U(\mathfrak{a}_R)}(L,M)$.

So, the tensor Identity as Humphreys calls to the isomorphism in the next lemma also holds for Lie algebras over commutative rings.

Lemma 4.4.19. *For $L\in U(\mathfrak{b}_R)$-Mod, $M\in U(\mathfrak{g}_R)$-Mod, we have the isomorphism*

$$U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)}(L\otimes_R M) \simeq (U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)} L)\otimes_R M. \tag{4.4.6.1}$$

Proof. For any $X\in U(\mathfrak{g}_R)$-Mod, we can write

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g}_R)}(U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)}(L\otimes_R M), X) &\simeq \mathrm{Hom}_{U(\mathfrak{b}_R)}(L\otimes_R M, X) &(4.4.6.2)\\ &\simeq \mathrm{Hom}_{U(\mathfrak{b}_R)}(L, \mathrm{Hom}_R(M,X)) &(4.4.6.3)\\ &\simeq \mathrm{Hom}_{U(\mathfrak{b}_R)}(L, \mathrm{Hom}_{U(\mathfrak{g}_R)}(U(\mathfrak{g}_R), \mathrm{Hom}_R(M,X)) &(4.4.6.4)\\ &\simeq \mathrm{Hom}_{U(\mathfrak{g}_R)}((U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)} L)\otimes_R M, X). &(4.4.6.5)\end{aligned}$$

The first isomorphism is obtained by Tensor-Hom adjunction, the second by Tensor-Hom adjunction and taking on both sides $\mathfrak{b}_R$-invariants and the other ones are again obtained by Tensor-Hom adjunction. So, this provides an isomorphism between these two Hom functors. By taking the image of the identity on $(U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)} L)\otimes_R M$ under the unit of the isomorphism of functors we obtain the desired isomorphism as $U(\mathfrak{g}_R)$-modules. ☐

Remark 4.4.20. The functor $U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)} -\colon U(\mathfrak{b}_R)$-Mod$\cap R$-Proj $\to U(\mathfrak{g}_R)$-Mod is exact. In fact, $U(\mathfrak{g}_R)\simeq U(\mathfrak{n}_R^-)\otimes_R U(\mathfrak{b}_R)\in U(\mathfrak{b}_R)$-Proj when regarded as $U(\mathfrak{b}_R)$-module. △

The following is the generalization of [Hum08, 3.6].

Proposition 4.4.21. *Assume that R is a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $M\in\mathscr{O}_{[\lambda],R}$ which is free over the ground ring R. Then, $\Delta(\omega)\otimes_R M \in \mathscr{F}(\Delta(\mu+\omega)_{\{\mu\in[\lambda]\colon M_\mu\neq 0\}})$.*

Proof. The module M is free of finite rank, and so each M_μ is also free of finite rank. Hence, the basis of M can be picked among the weight vectors of M. The module $N := R_\omega\otimes_R M$ is free with basis elements $v_1,\dots,v_n$ being weight vectors with weights $\nu_1,\dots,\nu_n$, respectively. We can choose them so that $\nu_i\geq\nu_j$ if and only if $i\geq j$. This gives a filtration $0\subset N_n\subset\cdots\subset N_1 = N$, where N_k is the $U(\mathfrak{b}_R)$-submodule generated by $v_k,\dots,v_n$.

Now induction on the rank of M shows that $\Delta(\omega)\otimes_R M \simeq U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)}(R_\omega\otimes_R M)$ has a filtration on $U(\mathfrak{g}_R)$ by modules $\Delta(\nu_i)$, $i=1,\dots,n$.

Observe that the weights of $R_\omega\otimes_R M$ are of the form $\omega+\mu$ where $M_\mu\neq 0$. Further, if μ is a highest weight in M, then $\Delta(\omega)\otimes_R M$ has a submodule $\Delta(\omega+\mu)$ and for the lower weight δ gives that $\Delta(\omega+\delta)$ is a quotient of $\Delta(\omega)\otimes_R M$. ☐

Now, we show that the results in [Hum08, 3.7] also hold in this setup.

Proposition 4.4.22. *Assume that R is a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $M\in\mathscr{F}(\Delta_{[\lambda]})$. The following assertions hold.*

(a) If μ is a maximal weight in M, then $\Delta(\mu)\subset M$ and $M/\Delta(\mu)\in\mathscr{F}(\Delta_{[\lambda]})$.

(b) The category $\mathscr{F}(\Delta_{[\lambda]})$ is closed under direct summands.

(c) M is free as $U(\mathfrak{n}_R^-)$-module.

Proof. By assumption, there is no weight $\omega\geq\mu$ so that M_ω is non-zero. Hence, n_R^+ annihilates every vector in M_μ. Therefore, there exists a non-zero map $f\colon\Delta(\mu)\to M$. Let $0\subset M_1\subset M_2\subset\cdots\subset M$ be a filtration of M. Assume that $\operatorname{im} f\subset M_i$ and $\operatorname{im} f\not\subseteq M_{i-1}$. Then, f induces a non-zero homomorphism $\Delta(\mu)\to M_i/M_{i-1}\simeq\Delta(\mu_i)$, for some weight μ_i. Denote such homomorphism by ψ. This implies that $\mu\leq\mu_i$. On the other hand, $M_i/M_{i-1}\simeq\Delta(\mu_i)$ implies that $(M_i)_{\mu_i}\neq 0$ and consequently $M_{\mu_i}\neq 0$ which contradicts the choice of μ. So, $\mu=\mu_i$. Hence, there exists $r\in R$ such that $\psi=r\,\mathrm{id}_{\Delta(\mu)}$. Further, $\Delta(\mu)$ is free of infinite rank over R, therefore ψ is injective. By Snake Lemma, we obtain that f is injective, as well. Hence, $\Delta(\mu)\subset M_i\subset M$. Applying the Snake Lemma to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_{i-1} \arrow[r] & M_i \arrow[r] & \Delta(\mu) \arrow[r] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & \Delta(\mu) \arrow[r] \arrow[u, hook] & \Delta(\mu) \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \tag{4.4.6.6}$$

we obtain that $M_{i-1}\simeq M_i/\Delta(\mu)$. It follows that $M_{i-1}\subset M/\Delta(\mu)$ and $M/M_i\simeq M/\Delta(\mu)/M_i/\Delta(\mu)\simeq M/\Delta(\mu)/M_{i-1}$. Since both $M/M_i, M_{i-1}$ have filtrations by Verma modules the middle term $M/\Delta(\mu)$ has also a filtration by Verma modules.

In addition, the map $\Delta(\mu)\xrightarrow{\psi} M_i\to M_i/M_{i-1}$ is injective. So, $M_{i-1}\cap\Delta(\mu)=0$. Thus, the canonical map $M_{i-1}\to M/\Delta(\mu)$ is injective and we have an exact sequence

$$0\to M_{i-1}\to M/\Delta(\mu)\to M/M_i\to 0 \tag{4.4.6.7}$$

Assume that $M=M_1\oplus M_2$ has a filtration by Verma modules. If M is a Verma module, then there is nothing to prove since R is local the Verma modules are indecomposable modules. We shall proceed by induction on the size of the filtration of M. Let μ be a maximal weight of M. We have $M_\mu=(M_1)_\mu\oplus(M_2)_\mu$. Assume that $(M_1)_\mu\neq 0$. By (a), $\Delta(\mu)\subset M_1\subset M$ and $M/\Delta(\mu)\simeq M_1/\Delta(\mu)\oplus M_2\in\mathscr{F}(\Delta_{[\lambda]})$. By induction, $M_1/\Delta(\mu)$ has a filtration by Verma modules. So, $M_1\in\mathscr{F}(\Delta_{[\lambda]})$.

By proceeding on induction on the filtration of M and since each Verma module is free as $U(\mathfrak{n}_R^-)$-module we obtain that M is also free as $U(\mathfrak{n}_R^-)$-module. ☐

4.4.7 Duality in BGG categories over commutative rings

The classical category $\mathscr{O}$ admits a simple preserving duality functor. However, since the most interesting modules in the category $\mathscr{O}$ are not finite-dimensional we cannot use the usual standard duality. But, the weight spaces are finite-dimensional, and so this property could be used to define a duality in $\mathscr{O}$ using the standard duality "locally". However, there is another problem in this case. For a general BGG category $\mathscr{O}$ over a commutative local Noetherian ring R which is a $\mathbb{Q}$-algebra we cannot define a duality, even locally, for all modules. We have to focus our attention only on those modules which are free over R. In addition, we have to impose that R is an integral domain.

Define $M^\vee = \bigoplus_{\mu\in[\lambda]} DM_\mu$ for $M \in \mathscr{O}_{[\lambda],R} \cap R\text{-Proj}$. This becomes an $U(\mathfrak{g}_R)$-module by imposing $(g\cdot f)(v) = f(\tau(g)v)$, where $\tau\colon U(\mathfrak{g}_R) \to U(\mathfrak{g}_R)$ is the involution map that fixes $\mathfrak{h}_R$ and sends x_α to $x_{-\alpha}$ for every $\alpha \in \Phi$. Using the fact that R is an integral domain one sees that his action identifies $(DM)_\mu$ with $D(M_\lambda)$ justifying why we changed the action. In fact, any $f \in DM$ with weight μ satisfies

$$f(\mu(h)m) = \mu(h)f(m) = (h\cdot f)(m) = f(\tau(h)m) = f(hm) = f(\omega(h)m),\ \forall m \in M_\omega. \tag{4.4.7.1}$$

Now using that R is an integral domain we would obtain that $f(m) = 0$ for all $m \in M_\omega$ whenever $\omega \neq \mu$. Hence, $M^\vee \in \mathscr{O}_{[\lambda],(II),R}$ whenever $M \in \mathscr{O}_{[l],R}$.

Observe also that for every $\alpha \in \Phi^+$, $f \in (DM)_\mu$, we have

$$h(x_\alpha f) = [h, x_\alpha]f + x_\alpha h f = \alpha(h)x_\alpha f + \mu(h)x_\alpha f = (\alpha+\mu)(h)x_\alpha f,\ \forall h \in \mathfrak{h}_R. \tag{4.4.7.2}$$

Hence, $x_\alpha f \in (DM)_{\alpha+\mu} \simeq D(M_{\alpha+\mu})$. So, $\mathfrak{n}_R^+ f \in R$-mod and consequently, $M^\vee \in \mathscr{O}_{[\lambda],(I),R}$ whenever $M \in \mathscr{O}_{[\lambda],R}$. The problem lies in deciding if $M^\vee \in \mathscr{O}_{[\lambda],R}$, that is if $M^\vee$ is finitely generated as $U(\mathfrak{g}_R)$-module. In the classical case, this is achieved by exploiting the simple modules and the composition series of the modules in $\mathscr{O}$.

Observe that for $M \in \mathscr{O}_{[\lambda],(I),R} \cap R\text{-Proj}$, $(M^\vee)^\vee = \bigoplus_{\mu\in[\lambda]} DDM_\mu \simeq \bigoplus_{\mu\in[\lambda]} M_\mu \simeq M$.

So, in short, we obtained a contravariant exact functor $(-)^\vee\colon \mathscr{O}_{[\lambda],(I),R} \cap R\text{-Proj} \to \mathscr{O}_{[\lambda],(I),R} \cap R\text{-Proj}$ which is self-dual. In particular, it is fully faithful.

4.4.8 Change of rings

It is at this point that our approach will start to diverge with Gabber and Joseph. As the reader may see we are closer to see that $\mathscr{O}_{[\lambda],R}$ is a split highest weight category. But, for that we require further techniques and constructions. In particular, how $\mathscr{O}_{[\lambda],R}$ behaves under change of ground ring.

Concerning Verma modules, we can see that they remain Verma under change of ring. In fact, for every commutative R-algebra S which is a Noetherian ring, and any $\lambda \in \mathfrak{h}_R^*$,

$$S\otimes_R \Delta(\lambda) = S\otimes_R U(\mathfrak{g}_R)\otimes_{U(\mathfrak{b}_R)} R_\lambda \simeq S\otimes_R U(\mathfrak{g}_R)\otimes_{S\otimes_R U(\mathfrak{b}_R)} S\otimes_R R_\lambda \simeq U(\mathfrak{g}_S)\otimes_{U(\mathfrak{b}_S)} S_{1_S\otimes_R\lambda} = \Delta(1_S\otimes_R \lambda).$$

More generally, we can say the following.

Lemma 4.4.23. *Let R be a commutative Noetherian ring which is a $\mathbb{Q}$-algebra and let $\lambda \in \mathfrak{h}_R^*$. For any commutative Noetherian ring S which is an R-algebra, the functor $S\otimes_R -\colon \mathscr{O}_{[\lambda],R} \to \mathscr{O}_{[1_S\otimes_R\lambda],S}$ is well defined and $S\otimes_R \Delta(\mu) \simeq \Delta(1_S\otimes_R \mu)$ for every $\mu \in [\lambda]$. Moreover, $S\otimes_R M_\mu = (S\otimes_R M)_{1_S\otimes\mu}$ for every $\mu \in [\lambda]$.*

Proof. Observe that S is also a $\mathbb{Q}$-algebra, by imposing $q\cdot 1_S = (q1_R)\cdot 1_S$. Let $M \in \mathscr{O}_{[\lambda],R}$. By Lemma 4.4.12, $M = \bigoplus_{\mu\in[\lambda]} M_\mu$. Thus, $S\otimes_R M = \sum_{\mu\in[\lambda]} S\otimes_R M_\mu$ and $S\otimes_R M \in U(\mathfrak{g}_S)$-mod, by identifying $U(\mathfrak{g}_S)$ with $S\otimes_R U(\mathfrak{g}_R)$. By assumption, for all $m \in M$,

$$U(\mathfrak{n}_S^+)(1_S\otimes m) \simeq S\otimes_R U(\mathfrak{n}_R^+)(1_S\otimes m) \simeq S\otimes_R U(\mathfrak{n}_R^+)m \in S\text{-mod}. \tag{4.4.8.1}$$

Since the elements $1_S\otimes m$, $m \in M$, generate $S\otimes_R M$ we obtain that $S\otimes_R M \in \mathscr{O}_{[1_S\otimes_R\lambda],S}$. It remains to show that $S\otimes_R M_\mu = (S\otimes_R M)_\mu$ for every $\mu\in[\lambda]$. Any element of $S\otimes_R M_\mu$ has weight $1_S\otimes_R\mu$. So, $S\otimes_R M_\mu \subset (S\otimes_R M)_{1_S\otimes\mu}$. But, for each $\mu\in[\lambda]$,

$$(S\otimes_R M)_{1\otimes\mu}\subset S\otimes_R M = \sum_{\theta\in[\lambda]} S\otimes_R M_\theta \subset \sum_{\theta\in[\lambda]}(S\otimes_R M)_{1\otimes\theta}. \tag{4.4.8.2}$$

Since $S\otimes_R M\in\mathscr{O}_{[1_S\otimes_R\lambda],S}$, we can write $S\otimes_R M=\bigoplus_{\omega\in[1_S\otimes_R\lambda]}(S\otimes_R M)_\omega$ and consequently the result follows. $\square$

We observe that we cannot apply right way Theorem 1.5.56 since $\mathscr{O}_{[\lambda],R}$ is still too big and contains a finite number of Verma modules. Instead, we will construct projective objects and decompose $\mathscr{O}_{[\lambda],R}$ into smaller subcategories which will allow us to construct projective Noetherian R-algebras with module categories being deformations of the blocks of the category $\mathscr{O}$. To obtain such a statement R being local is crucial. In fact, Gabber and Joseph [GJ81, 1.7] proved that all simple modules are quotients of Verma modules and the number of simple modules for deformations of the category $\mathscr{O}$ that appear as a quotient of a Verma module depends on the number of maximal ideals of the ground ring. So, outside local rings we cannot expect $\mathscr{O}_{[\lambda],R}$ to decompose in the desired way.

As in the classical case, the first step is to see that the center of the universal enveloping algebra behaves well under change of ground ring.

Lemma 4.4.24. *Let R be a commutative Noetherian ring which is a $\mathbb{Q}$-algebra and S a commutative Noetherian ring which is an R-algebra. Then, $S\otimes_R Z(\mathfrak{g}_R)\simeq Z(\mathfrak{g}_S)$.*

Proof. Actually, we just need to observe that

$$R\otimes_{\mathbb{Q}} Z(\mathfrak{g}_{\mathbb{Q}}) = Z(R\otimes_{\mathbb{Q}}\mathfrak{g}_{\mathbb{Q}}). \tag{4.4.8.3}$$

Assume for the moment that (4.4.8.3) holds. Then,

$$S\otimes_R Z(U(\mathfrak{g}_R))\simeq S\otimes_R Z(R\otimes_{\mathbb{Q}} U(\mathfrak{g}_{\mathbb{Q}}))\simeq S\otimes_R R\otimes_{\mathbb{Q}} Z(U(\mathfrak{g}_{\mathbb{Q}}))\simeq S\otimes_{\mathbb{Q}} Z(U(\mathfrak{g}_{\mathbb{Q}}))\simeq Z(S\otimes_{\mathbb{Q}} U(\mathfrak{g}_{\mathbb{Q}}))\simeq Z(U(\mathfrak{g}_S)).$$

Proving (4.4.8.3) is in some sense folklore. The inclusion $R\otimes_{\mathbb{Q}} Z(\mathfrak{g}_{\mathbb{Q}})\subset Z(R\otimes_{\mathbb{Q}}\mathfrak{g}_{\mathbb{Q}})$ is clear. Let $\sum_{i\in F} r_i\otimes a_i\in Z(R\otimes_{\mathbb{Q}} U(\mathfrak{g}_{\mathbb{Q}}))$ for some finite set F. We can assume that $\{r_i\colon i\in F\}$ is a linearly independent set over $\mathbb{Q}$. Otherwise, we can rearrange the sum $\sum_{i\in F} r_i\otimes a_i$. In fact, $r_l=\sum_{i\neq l} b_i r_i$ for some $b_i\in\mathbb{Q}$. This would imply that

$$\sum_{i\in F} r_i\otimes a_i = \sum_{i\in F\backslash\{l\}} r_i\otimes a_i + \sum_{i\in F\backslash\{l\}} b_i r_i\otimes a_l = \sum_{i\in F\backslash\{l\}} r_i\otimes(a_i+b_i a_l), \tag{4.4.8.4}$$

where the elements $a_i+b_i a_l$ can take the place of the previous a_i.

Now, for any $a\in U(\mathfrak{g}_{\mathbb{Q}})$,

$$\sum_{i\in F} r_i\otimes a_i a = \left(\sum_{i\in F} r_i\otimes a_i\right)(1_R\otimes a) = (1_R\otimes a)\left(\sum_{i\in F} r_i\otimes a_i\right) = \sum_{i\in F} r_i\otimes a a_i. \tag{4.4.8.5}$$

Therefore, $\sum_{i\in F} r_i\otimes(aa_i-a_i a)=0$. Since the set $\{r_i\colon i\in F\}$ is assumed to be linearly independent we obtain $aa_i-a_i a=0$ for all $i\in F$. Therefore, $a_i\in Z(\mathfrak{g}_{\mathbb{Q}})$. This shows that $\sum_{i\in F} r_i\otimes a_i\in R\otimes_{\mathbb{Q}} Z(U(\mathfrak{g}_{\mathbb{Q}}))$. $\square$

In light of Lemma 4.4.24, the next natural question is to know what happens to the central characters under change of ring.

Let π_R denote the projection $U(\mathfrak{g}_R)\twoheadrightarrow U(\mathfrak{h}_R)$ and $\lambda\in\mathfrak{h}_R^*$.

By the PBW theorem, for each $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, and each commutative Noetherian ring S which is an R-algebra, we obtain the commutative diagrams

$$\begin{tikzcd} S\otimes_R U(\mathfrak{g}_R) \arrow[r, "S\otimes_R \pi_R"] \arrow[d, "\simeq"] & S\otimes_R U(\mathfrak{h}_R) \arrow[d, "\simeq"] \\ U(\mathfrak{g}_S) \arrow[r, "\pi_S"] & U(\mathfrak{h}_S) \end{tikzcd} \qquad \begin{tikzcd} S\otimes_R U(\mathfrak{h}_R) \arrow[r, "1_S\otimes_R \lambda"] \arrow[d, "\simeq"] & S\otimes_R R \arrow[d, "\simeq"] \\ U(\mathfrak{h}_S) \arrow[r, "1_S\lambda"] & S \end{tikzcd} . \tag{4.4.8.6}$$

Here, $1_S\lambda$ denotes the homomorphism of R-algebras given by $(1_S\lambda)(1_S\otimes_{\mathbb{Z}} h_\alpha) = 1_S\lambda(1_R\otimes_{\mathbb{Z}} h_\alpha) \in S$ for each $\alpha \in \Pi$. In particular, $1_S\lambda \in \mathfrak{h}_S^*$.

By Lemma 4.4.24, and combining all these diagrams we obtain the following commutative diagrams

$$\begin{tikzcd} S\otimes_R Z(\mathfrak{g}_R) \arrow[r] \arrow[d, "\simeq"] & S\otimes_R U(\mathfrak{h}_R) \arrow[d, "\simeq"] \\ Z(\mathfrak{g}_S) \arrow[r] & U(\mathfrak{h}_S) \end{tikzcd} \qquad \begin{tikzcd} S\otimes_R Z(\mathfrak{g}_R) \arrow[r, "S\otimes_R \chi_\lambda"] \arrow[d, "\simeq"] & S\otimes_R R \arrow[d, "\simeq"] \\ Z(\mathfrak{h}_S) \arrow[r, "\chi_{1_S\lambda}"] & S \end{tikzcd} . \tag{4.4.8.7}$$

If I is an ideal of R, there is one more commutative diagram of interest:

$$\begin{tikzcd} R/I\otimes_R U(\mathfrak{h}_R) \arrow[r, "1_{R/I}\otimes_R \lambda"] \arrow[d, "\simeq"] & R/I\otimes_R R \arrow[d, "\simeq"] \\ U(\mathfrak{h}_R)/IU(\mathfrak{h}_R) \arrow[r] & R/I \end{tikzcd} , \tag{4.4.8.8}$$

where the bottom map is given by $1_R\otimes_{\mathbb{Z}} h_\alpha + IU(\mathfrak{h}_R) \mapsto \lambda(1_R\otimes_{\mathbb{Z}} h_\alpha) + I$, $\alpha \in \Pi$. In other words, this is the image of $\lambda \in \mathfrak{h}_R^*$ in $\mathfrak{h}_R^*/I\mathfrak{h}_R^*$.

4.4.9 Decomposition of $\mathscr{O}_{[\lambda],R}$ into blocks

Assume in the remaining of this section that R is a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. To simplify notation, we shall denote by $\overline{\lambda}$ the image of $\lambda \in \mathfrak{h}_R^*$ in $\mathfrak{h}_R^*/\mathfrak{m}\mathfrak{h}_R^*$, where $\mathfrak{m}$ is the maximal ideal of the local ring R, and denote by $\bar{r}$ the image of $r \in R$ in the quotient $R/\mathfrak{m}$. We will also denote by $\bar{z}$ the image of $z \in Z(\mathfrak{g}_R)$ in $Z(\mathfrak{g}_R)/\mathfrak{m}Z(\mathfrak{g}_R)$. Recall that W is the Weyl group associated with the root system Φ. Explicitly, each reflection s_α acts in the following way: $s_\alpha\lambda = \lambda - \langle\lambda, \alpha^\vee\rangle_R\alpha$, where $1_R\alpha$ can be seen as the element in $\mathfrak{h}_R^*$ satisfying $1_R\alpha(1\otimes h_\alpha) = 2$. So, the Weyl group W acts on $\mathfrak{h}_{R(\mathfrak{m})}^* \simeq R(\mathfrak{m})\otimes_R \mathfrak{h}_R^* \simeq \mathfrak{h}_R^*/\mathfrak{m}\mathfrak{h}_R^*$.

Lemma 4.4.25. *For any $w \in W$ and $\lambda \in \mathfrak{h}_R^*$ we have $\overline{w\cdot\lambda} = w\cdot\overline{\lambda}$, under the dot action.*

Proof. Let $\alpha \in \Pi$ and assume that $\lambda = \sum_{\beta\in\Pi} t_\beta(1_R\otimes_{\mathbb{Z}}\beta) \in \mathfrak{h}_R^*$. In the following, we write $\overline{\alpha}$ as $1_{R(\mathfrak{m})}\alpha$. Then,

$$\overline{s_\alpha\cdot\lambda} = \overline{s_\alpha(\lambda+\rho)-\rho} = \overline{\lambda+\rho-\langle\lambda+\rho,\alpha^\vee\rangle\alpha-\rho} = \overline{\lambda} - \overline{\langle\lambda,\alpha^\vee\rangle_R\alpha} - \overline{\alpha} = \overline{\lambda} - \sum_{\beta\in\Pi}\overline{t_\beta}\langle\beta,\alpha^\vee\rangle\alpha - \overline{\alpha}. \tag{4.4.9.1}$$

On the other hand,

$$s_\alpha\cdot\overline{\lambda} = s_\alpha(\overline{\lambda}+\rho)-\rho = \overline{\lambda}+\rho-\langle\overline{\lambda}+\rho,\alpha^\vee\rangle_{R(\mathfrak{m})}\overline{\alpha}-\rho = \overline{\lambda} - \sum_{\beta\in\Pi}\overline{t_\beta}\langle\beta,\alpha^\vee\rangle\overline{\alpha} - \overline{\alpha}. \qquad \square$$

For what follows we are going to need more notation.

Definition 4.4.26. Let K be a field of characteristic zero. For each $\mu \in \mathfrak{h}_K^*$, we can consider the root system making the weight μ integral, that is, $\Phi_\mu := \{\alpha \in \Phi \colon \langle \mu, \alpha^\vee \rangle_K \in \mathbb{Z}\}$ and its associated Weyl group $W_\mu := \{w \in W \colon w \cdot \mu - \mu \in \Lambda_K\}$.

Definition 4.4.27. Let $\lambda \in \mathfrak{h}_R^*$. We call $\mathscr{D} \subset [\lambda]$ a **block** of $[\lambda]$ if $\{\overline{\mu} \colon \mu \in \mathscr{D}\}$ is an orbit under the dot action of the Weyl group W.

Remark 4.4.28.
1. Orbits under the Weyl group are always finite, so a block is always finite.
2. If $\mu_1, \mu_2 \in \mathscr{D}$, then $\mu_1 - \mu_2 \in \Lambda_R$ and since all non-zero integers are invertible in R, we also obtain $\overline{\mu_1} - \overline{\mu_2} \in \Lambda_{R(\mathfrak{m})}$. Further, $\{\overline{\mu} \colon \mu \in \mathscr{D}\}$ is a W-orbit and also an orbit under the subgroup $W_{\overline{\mu_1}}$. △

Lemma 4.4.29. *Let $\lambda \in \mathfrak{h}_R^*$ and let $\mathscr{D} \subset [\lambda]$ be a block. Then, there exists $\mu \in \mathfrak{h}_R^*$, $\nu \in \mathfrak{m}\mathfrak{h}_R^*$ satisfying*

(i) $s_\alpha \mu - \mu \in \mathbb{Z}\alpha$ for all $\alpha \in \Phi_{\overline{\mu}}$;

(ii) $\mathscr{D} = W_{\overline{\mu}} \cdot \mu + \nu$.

Proof. See [GJ81, 1.8.2]. □

Knowing the form of the blocks of $[\lambda]$ it is no surprise as the name indicates that $[\lambda]$ is a disjoint union of its distinct blocks. In fact, assume that $\mu \in [\lambda]$ belongs to two distinct blocks $\mathscr{D}_1 = W_{\overline{\mu_1}} \cdot \mu_1 + \nu_1$ and $\mathscr{D}_2 = W_{\overline{\mu_2}} \cdot \mu_2 + \nu_2$. Then, there exists $w_1, w_2 \in W$ such that $\nu_1 - \nu_2 = w_1\mu_1 - w_2\mu_2$. Hence, $w_1\overline{\mu_1} - w_2\overline{\mu_2} = 0$. Hence, μ_1 and μ_2 are in the same orbit under the Weyl group. By assumption, $\mu_1 - \mu_2 \in \Lambda_R$, therefore $W_{\overline{\mu_1}} = W_{\overline{\mu_2}}$. But then we would obtain $\nu_1 - \nu_2 \in \mathbb{Z}\mu_1$. So, we must have $\nu_1 = \nu_2$. This means that $\mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset$ whenever the blocks are distinct.

Now knowing how to decompose $[\lambda]$, we shall proceed to decompose $\mathscr{O}_{[\lambda],R}$. The idea is similar to how we proved that any module decomposes into its weight modules as $U(\mathfrak{h}_R)$-modules. But, now we will consider the commutative algebra $Z(\mathfrak{g}_R)$. Hence, this will be analogue to finding a suitable set of central orthogonal idempotents. Indeed, their analogue will be the central characters of $Z(\mathfrak{g}_R)$.

Lemma 4.4.30. *[GJ81, 1.8.3] Let $\lambda \in \mathfrak{h}_R^*$. Suppose that $\mu, \omega \in [\lambda]$ belong to distinct blocks.*
Then, $\ker \chi_\omega + \ker \chi_\mu = Z(\mathfrak{g}_R)$.

Proof. Recall the notation used in diagrams (4.4.8.6) and (4.4.8.7). The central characters $\chi_{q_{R(\mathfrak{m})}\mu}$ and $\chi_{1_{R(\mathfrak{m})}\omega}$ are surjective into the field $R(\mathfrak{m})$, so $\ker \chi_{1_{R(\mathfrak{m})}\mu}$ and $\ker \chi_{1_{R(\mathfrak{m})}\omega}$ are maximal ideals of $Z(\mathfrak{g}_{R(\mathfrak{m})})$. These are distinct, otherwise $0 \neq \chi_{1_{R(\mathfrak{m})}\mu} - \chi_{1_{R(\mathfrak{m})}\omega}(z) = \chi_{1_{R(\mathfrak{m})}\mu}(z - \chi_{1_{R(\mathfrak{m})}\omega}(z))$ for some $z \in Z(\mathfrak{g}_{R(\mathfrak{m})})$. Because of $z - \chi_{1_{R(\mathfrak{m})}\omega}(z) \in \ker \chi_{1_{R(\mathfrak{m})}\omega}$, these maximal ideals are distinct.

We can separate the proof in the following two steps.

Claim 1. $Z(\mathfrak{g}_R) = \ker \chi_\omega + \ker \chi_\mu + \mathfrak{m}Z(\mathfrak{g}_R)$.

Claim 2. $\chi_\omega(\ker \chi_\mu) = R$.

By the commutative diagrams (4.4.8.6) and (4.4.8.7) and previous discussion, for each $z \in Z(\mathfrak{g}_R)$ there are $s, t \in Z(\mathfrak{g}_R)$ so that $z - s - t \in \mathfrak{m}Z(\mathfrak{g}_R)$ and $\overline{\chi_\mu(s)} = 0$, $\overline{\chi_\omega(t)} = 0$. Hence, $\chi_\mu(s) + \chi_\omega(t) \in \mathfrak{m}Z(\mathfrak{g}_R)$. So, $z - (s - \chi_\mu(s)) - (t - \chi_\omega(t)) \in \mathfrak{m}Z(\mathfrak{g}_R)$. Thus, Claim 1. follows.

Let $1_R \in Z(\mathfrak{g}_R)$. Then, there exists $t \in \ker \chi_\omega$ and $s \in \ker \chi_\mu$ so that $1_R - (t + s) \in \mathfrak{m}Z(\mathfrak{g}_R)$. Hence, $1_R - \chi_\omega(s) \in \mathfrak{m}$. So, $\chi_\omega(s)$ is invertible in R. So, Claim 2. follows.

Let $z \in Z(\mathfrak{g}_R)$. By Claim 2, we can write $\chi_\omega(z) = \chi_\omega(s)$ for some $s \in \ker \chi_\mu$. Therefore, $z - s \in \ker \chi_\omega$. □

As a consequence, it follows that all central characters which are non-zero $\chi_\mu : Z(\mathfrak{g}_R) \to R$ are surjective for $\mu \in [\lambda]$ since we can always find a weight which belongs to a different block than the one that contains μ.

Now we would like to see what happens to the central characters of weights belonging to the same block.

Lemma 4.4.31. *Let $\lambda \in \mathfrak{h}_R^*$. Suppose that $\mu, \omega \in [\lambda]$ belong to the same block. Then,* $\ker \chi_\omega = \ker \chi_\mu$.

Proof. Let $\mathscr{D} = W_{\overline{\mu}} \cdot \mu + \nu$ be a block of $[\lambda]$. By the commutative diagram (4.4.8.6) and Theorem 4.4.7, the surjective map $\chi_{w\cdot\mu-\mu} = \chi_{w\cdot\mu+\nu} - \chi_{\mu+\nu}$ becomes zero under $R(\mathfrak{m}) \otimes_R -$ for any $w \in W_{\overline{\mu}}$. So, the image of $\chi_{w\cdot\mu-\mu}$ is contained in $\mathfrak{m}$ and the central character is not surjective. But, this can only happen if $\chi_{w\cdot\mu-\mu}$ is the zero map. Now, assume that $x \in \ker \chi_{\mu+\nu}$. Then, for every $w \in W_{\overline{\mu}}$,

$$\chi_{w\cdot\mu+\nu}(x) = w\cdot\mu(\pi_R(x)) + \nu(\pi_R(x)) = w\cdot\mu(\pi_R(x)) - \mu(\pi_R(x)) = \chi_{w\cdot\mu-\mu}(x) = 0. \tag{4.4.9.2}$$

So, x is also an element of $\ker \chi_{w\cdot\mu+\nu}$. ☐

Proposition 4.4.32. *[GJ81, 1.8.4, 1.8.5, 1.8.6] Let R be a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\lambda \in \mathfrak{h}_R^*$. For every $M \in \mathscr{O}_{[\lambda],R}$, $M = \bigoplus_{\mathscr{D}} M^{\mathscr{D}}$, where $\mathscr{D}$ runs over all blocks of $[\lambda]$ and*

$$M^{\mathscr{D}} := \{m \in M : \forall x \in \ker \chi_\mu,\ \mu \in \mathscr{D},\ \exists n \in \mathbb{N}\ x^n m = 0\}.$$

Moreover, the following assertions hold:

(a) $M^{\mathscr{D}}$ is non-zero only for a finite number of blocks $\mathscr{D}$ of $[\lambda]$;

(b) $\Delta(\mu)^{\mathscr{D}} = \Delta(\mu)$ if and only if $\mu \in \mathscr{D}$, otherwise it is zero;

(c) $M \mapsto M^{\mathscr{D}}$ is an exact endofunctor on $\mathscr{O}_{[\lambda],R}$.

Proof. The idea is to apply Lemma 4.4.10 together with Lemma 4.4.30 and 4.4.31. So, first we have to show that we can write $M = \sum_{\mathscr{D}} M^{\mathscr{D}}$ for every $M \in \mathscr{O}_{[\lambda],R}$. To obtain an idea, how to show this we shall consider first the case of M being a Verma module $\Delta(\mu)$, $\mu \in [\lambda]$.

Denote by y the element $1_{U(\mathfrak{g}_R)} \otimes_{U(\mathfrak{b}_R)} 1_R$. Since the actions of $Z(\mathfrak{g}_R)$ and $U(\mathfrak{h}_R)$ commute, for every element $z \in Z(\mathfrak{g}_R)$ the element zy has weight μ. By the PBW theorem, we can write z as a linear combination of PBW monomials. The monomials with factors of elements in $\mathfrak{n}_R^+$ send y to zero. The monomials without factors of elements in $\mathfrak{n}_R^+$ but with factors on $\mathfrak{n}_R^-$ send y to some weight module $\Delta(\mu)_\omega$ with $\omega < \mu$. But since zy must have weight μ, we deduce that $zy = \pi_R(z)y = \mu(\pi_R(z))y = \chi_\mu(z)y$, since $\pi_R(z) \in \mathfrak{h}_R$. So, for every $z \in \ker \chi_\mu$, and $a \in U(\mathfrak{n}_R^-)$, $zay = azy = 0$. This shows by Lemma 4.4.31 that $\Delta(\mu)^{\mathscr{D}} = \Delta(\mu)$. In particular, every element of every quotient of $\Delta(\mu)$ is annihilated by $\ker \chi_\mu$.

We can proceed by induction on the size of a filtration of $M \in \mathscr{O}_{[\lambda],R}$ in quotients of Verma modules to prove that for every $m \in M$ and every $z_i \in \ker \chi_{\mu_i}$, $i = 1, \dots, t$, the product $z_1 \cdots z_t m$ is zero where $\mu_1, \dots, \mu_t$ are weights involved in a filtration of M.

The case of size one is already proved. Consider the exact sequence $0 \to N \to M \to Q(\mu) \to 0$, where $Q(\mu)$ is a quotient of $\Delta(\mu)$. By the above discussion, for every $m \in M$, and every $z \in \ker \chi_\mu$ $zm \in N$. Since N inherits the filtration of M, by induction, $z_1 \cdots z_t zm = 0$. By Lemma 4.4.9, Lemma 4.4.30 and 4.4.31, we obtain $\sum_{\mathscr{D}} M^{\mathscr{D}}$. By Lemma 4.4.10, the first assertion, (b) and (c) follow. By (c) and (b), every quotient of the Verma module $\Delta(\mu)$, say L, satisfies $L^{\mathscr{D}} = L$ and zero in the other blocks. Now by (c) and using the finite filtrations of $M \in \mathscr{O}_{[\lambda],R}$ in quotients of Verma modules, (a) follows. ☐

Definition 4.4.33. Let R be a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\lambda \in \mathfrak{h}_R^*$. For a block $\mathscr{D} \subset [\lambda]$, define $\mathscr{O}_{\mathscr{D}}$ the full subcategory of $\mathscr{O}_{[\lambda],R}$ whose objects satisfy $M = M^{\mathscr{D}}$.

In the classical case, the blocks of the category $\mathscr{O}$ are in a one to one correspondence with the antidominant weights. We can generalize the notion of dominant and antidominant weight to this setup since these notions will help us study the structure of the category $\mathscr{O}_{[\lambda],R}$. We will call $\mu \in \mathfrak{h}_R^*$ a **dominant weight** if $\overline{\mu}$ is a dominant weight. Analogously, we will call μ an **antidominant weight** if $\overline{\mu}$ is an antidominant weight. We call $\mu \in \mathfrak{h}_R^*$ an **integral dominant weight** if $\overline{\mu}$ is an integral dominant weight.

We should remark that the blocks $\mathscr{D}$ of $[\lambda]$ are constructed with the ring R in mind. So, after change of rings these blocks can be refined even further. Moreover, the interested reader can see that typically the Weyl groups associated with elements $\overline{1_S \otimes \mu}$ are subgroups of the Weyl groups associated with elements $\overline{\mu}$. This is the phenomenon that we will exploit on this deformation of the category $\mathscr{O}$, although we will not explore it under the current formulation.

Lemma 4.4.34. *Let $\lambda \in \mathfrak{h}_R^*$. Let $\mathscr{D}$ be a block of $[\lambda]$. Then, $\mathscr{D}$ admits a unique (resp. antidominant) dominant weight μ. In addition μ is (resp. minimal) maximal in $\mathscr{D}$.*

Proof. Assume that $\mathscr{D} = W_{\overline{\theta}}\theta + \nu$, $\nu \in \mathfrak{m}\mathfrak{h}_R^*$ and $s\theta + \nu \in \mathscr{D}$ is a dominant weight, with $s \in W_{\overline{\theta}}$. There exists always one since there is a dominant weight in $\{\overline{\mu} \colon \mu \in \mathscr{D}\}$. Then, for any $w \in W_{\overline{\theta}}$, $\overline{s\theta + \nu - w\theta - \nu} = s\overline{\theta} - w\overline{\theta} \in \mathbb{N}\Pi$. Hence, $s\theta - w\theta \in \mathbb{N}\Pi + \mathfrak{m}\mathfrak{h}_R^*$. Hence, there exists $\nu_1 \in \mathfrak{m}\mathfrak{h}_R^*$ so that $s\theta - w\theta + \nu_1 \in \mathbb{N}\Pi$. But, $s\theta - w\theta \in \Lambda_R$ and consequently ν_1 belongs to Λ_R. However, this only happens if ν_1 is zero since every non-zero integer is invertible in R. We conclude that $s\theta + \nu$ is maximal in $\mathscr{D}$, and therefore it is the unique dominant weight in $\mathscr{D}$. $\square$

It is a natural question to know whether extension of scalars $S \otimes_R -$ preserves dominant (resp. antidominant) weights.

Lemma 4.4.35. *Let R be a local Noetherian integral domain which is a $\mathbb{Q}$-algebra. Assume that S is:*

- *a localization $R_\mathfrak{p}$ of R at some prime ideal $\mathfrak{p}$ of R;*
- *a quotient ring R/I of R for some ideal I.*

If $\lambda \in \mathfrak{h}_R^*$ is a dominant weight, then $1_S \otimes_R \lambda$ is a dominant weight. If $\lambda \in \mathfrak{h}_R^*$ is an antidominant weight, then $1_S \otimes_R \lambda$ is an antidominant weight.

Proof. We will prove the assertion for dominant weights. The other case is analogous. By assumption, $\overline{\lambda}$ is a dominant weight. That is, $\langle \overline{\lambda} + 1_{R(\mathfrak{m})}\rho, \alpha^\vee \rangle_{R(\mathfrak{m})} \notin \mathbb{Z}^-$ for every $\alpha \in \Phi^+$. So, $\langle \lambda + \rho, \alpha^\vee \rangle_R \notin \mathbb{Z}^- + \mathfrak{m}$ for every $\alpha \in \Phi^+$. Assume that $S = R_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R and assume, by contradiction, that $1_{R_\mathfrak{p}} \otimes \lambda$ is not a dominant weight. Hence, there exists $\alpha \in \Phi^+$ such that $\langle 1_S \otimes \lambda + 1_S\rho, 1_S\alpha^\vee \rangle_S \in \mathbb{Z}^- + \mathfrak{p}_\mathfrak{p}$. Further, there exists $t \in \mathbb{Z}^-$ and $s \in R \backslash \mathfrak{p}$ so that $s(\langle \lambda + \rho, \alpha^\vee \rangle_R - t) \in \mathfrak{p}$. Thus, $\langle \lambda + \rho, \alpha^\vee \rangle_R - t \in \mathfrak{p} \subset \mathfrak{m}$ for some $\alpha \in \Phi^+$. The existence of such t contradicts $\overline{\lambda}$ being a dominant weight. So, $1_S \otimes_R \lambda$ is a dominant weight.

Assume now that $S = R/I$ for some ideal I. In particular, $\mathfrak{m}/I$ is the unique maximal ideal of S. Assume, by contradiction, that $1_S \otimes_R \lambda$ is not a dominant weight. Then, there exists $\alpha \in \Phi^+$, $t \in \mathbb{Z}^-$, $x \in \mathfrak{m}$ so that $\langle \lambda + \rho, \alpha^\vee \rangle_R - t - x \in I \subset \mathfrak{m}$. Hence, $\langle \lambda + \rho, \alpha^\vee \rangle_R - t \in \mathfrak{m}$ which contradicts λ being a dominant weight. $\square$

We can now see that there are no homomorphisms between Verma modules that belong to distinct blocks.

Lemma 4.4.36. *Let R be a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\lambda \in \mathfrak{h}_R^*$. Then,* $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(M,N) = 0$ *if $M \in \mathscr{O}_{\mathscr{D}_1} \cap \mathscr{F}(\Delta)$ and $N \in \mathscr{O}_{\mathscr{D}_2} \cap \mathscr{F}(\Delta)$ for distinct blocks $\mathscr{D}_1 \neq \mathscr{D}_2$ of $[\lambda]$.*

Proof. Assume that $\mathscr{D}_1 = W_{\overline{\mu}} \cdot \mu + \nu_1$ and $\mathscr{D}_2 = W_{\overline{\omega}} \cdot \omega + \nu_2$. As usual, we will start with the Verma modules. Suppose that $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(w_1 \cdot \mu + \nu_1), \Delta(w_2 \cdot \omega + \nu_2)) \neq 0$ for $w_1 \in W_{\overline{\mu}}$, $w_2 \in W_{\overline{\omega}}$. Then,

$$0 \neq \operatorname{Hom}_{U(\mathfrak{g}_R)}(U(\mathfrak{g}_R) \otimes_{U(\mathfrak{b}_R)} R_{w_1 \cdot \mu + \nu_1}, \Delta(w_2 \cdot \omega + \nu_2)) \tag{4.4.9.3}$$

$$\simeq \operatorname{Hom}_{U(\mathfrak{b}_R)}(R_{w_1 \cdot \mu + \nu_1}, \Delta(w_2 \cdot \omega + \nu_2)) \subset \Delta(w_2 \cdot \omega + \nu_2)_{w_1 \cdot \mu + \nu_1}. \tag{4.4.9.4}$$

It follows that $w_2 \cdot \omega + \nu_2 - w_1 \cdot \mu - \nu_1 \in \mathbb{Z}_0^+ \Pi$. By Lemma 4.4.23, we obtain that $\Delta(w_2 \cdot \overline{\omega})_{w_1 \cdot \overline{\mu}} \neq 0$. So, also $\Delta(w_2 \cdot \omega)_{w_1 \cdot \mu}(\mathfrak{m}) \simeq \Delta(w_2 \cdot \overline{\omega})_{w_1 \cdot \overline{\mu}} \neq 0$. Since $\Delta(w_2 \cdot \omega)_{w_1 \cdot \mu} \in R$-proj it must be non-zero. Hence, $w_2 \cdot \omega - w_1 \cdot \mu \in \mathbb{Z}_0^+ \Pi$. Therefore, $\nu_2 - \nu_1 \in \mathbb{Z}_0^+ \Pi$. But, this forces $\nu_2 = \nu_1$ since all non-zero integers are invertible in R. By assumption, $\Delta(w_2 \omega + \nu_2)_{w_1 \cdot \mu + \nu_2} \neq 0$ and it contains an element which is killed by $\mathfrak{n}_R^+$. Since this module is free, one of its elements basis of the form $x_{-\alpha_1}^{t_1} \cdots x_{-\alpha_d}^{t_d}(1_{U(\mathfrak{g}_R) \otimes_{U(\mathfrak{b}_R)}} 1_{R_{w_2 \cdot \omega + \nu_2}})$, $\alpha_i \in \Pi$ which does not belong to $\mathfrak{m}\Delta(w_2 \cdot \omega + \nu_2)$ is killed by $\mathfrak{n}_R^+$. Therefore, there exists a non-zero map between $\Delta(w_1 \cdot \overline{\mu})$ and $\Delta(w_2 \cdot \overline{\omega})$. Therefore, both Verma modules $\Delta(w_1 \cdot \overline{\mu})$ and $\Delta(w_2 \cdot \overline{\omega})$ have a common simple module as composition factor, and so they belong to the same block. By Theorem 4.4.7 (f), $W_{\overline{\mu}} = W_{\overline{\omega}}$, and $\overline{\omega} = w \cdot \overline{\mu}$ for some $w \in W_{\overline{\mu}}$. Hence, $\omega - w \cdot \mu = \nu$ for some $\nu \in \mathfrak{m}\mathfrak{h}_R^*$. As we have seen, $w_2 \cdot \omega - w_1 \cdot \mu = w_2 w \cdot \mu + w_2 \cdot \nu - w_1 \cdot \mu \in \mathbb{Z}_0^+ \Pi$. So, also $w_2 \cdot \nu \in \mathbb{Z}_0^+ \Pi$. Therefore, $\nu = 0$. This shows that the blocks $\mathscr{D}_1$ and $\mathscr{D}_2$ coincide.

Now, the claim follows using the (finite) filtrations of M and N by quotients of Verma modules. ☐

4.4.10 Projective objects in $\mathscr{O}_{[\lambda],R}$

At this point, it is difficult to know whether there is information getting out of the blocks of $\mathscr{O}_{[\lambda],R}$, that is, if there are non zero homomorphisms between modules belonging to distinct blocks. For modules with a Verma filtration we saw that such a situation is not possible. But, since we do not know if this is the case for general modules, the classical arguments of construction of projective objects do not carry over to this more general setup. In particular, not knowing if the previous situation might or might not happen makes it difficult to deduce whether the Verma module associated with a dominant weight is a projective object or not. Instead, we will take the advantage of knowing projective objects in $\mathscr{O}_{[\lambda],(II),R}$ to construct projective objects in $\mathscr{O}_{[\lambda],R} \cap R$-Proj, using change of rings techniques.

For each $\mu \in [\lambda]$, define $Q(\mu) := U(\mathfrak{g}_R) \otimes_{U(\mathfrak{h}_R)} R_\mu \in \mathscr{O}_{[\lambda],(II),R}$. These modules are a sort of linearisation of the projective module $U(\mathfrak{g}_R)$. Note that by the PBW theorem, $Q(\mu) \simeq U(\mathfrak{n}_R^-) \otimes_R U(\mathfrak{h}_R) \otimes_R U(\mathfrak{n}_R^+) \otimes_{U(\mathfrak{h}_R)} R_\lambda \simeq U(\mathfrak{n}_R^+) \otimes_R U(\mathfrak{n}_R^-) \in R$-Proj. Also, for every commutative R-algebra which is a Noetherian ring, $S \otimes_R Q(\mu) \simeq Q(1_S \otimes_R \mu)$. The main difference between $Q(\mu)$ and the Verma modules is that $Q(\mu)$ is not annihilated by $\mathfrak{n}_R^+$. But, this feature allows $Q(\mu)$ to detect more information outside $\mathscr{O}$. For instance, for every $M \in U(\mathfrak{g}_R)$-Mod, $\operatorname{Hom}_{U(\mathfrak{g}_R)}(Q(\mu), M) \simeq M_\mu$, and thus the functor $\operatorname{Hom}_{U(\mathfrak{g}_R)}(Q(\mu), -) \colon \mathscr{O}_{[\lambda],(II),R} \to \mathscr{O}_{[\lambda],(II),R}$ is exactly the functor $M \mapsto M_\mu$ which is exact by Corollary 4.4.12. Therefore, $Q(\mu)$ is a projective object in $\mathscr{O}_{[\lambda],(II),R}$. As Gabber and Joseph pointed out, $\mathscr{O}_{[\lambda],(II),R}$ is closed under arbitrary direct sums, hence each weight module M_μ is a quotient of an arbitrary direct sum of copies of $Q(\mu)$, and so each $M \in \mathscr{O}_{[\lambda],(II),R}$ is a quotient of an arbitrary direct sum of copies of modules of the form $Q(\mu)$, where μ runs over all weights in $[\lambda]$. Hence, $\mathscr{O}_{[\lambda],(II),R}$ has enough projectives, and so we can use homological algebra techniques on $\mathscr{O}_{[\lambda],(II),R}$. We obtained so far, the following:

Lemma 4.4.37. *Let R be a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\lambda \in \mathfrak{h}_R^*$. The following assertions hold.*

(a) *The modules $Q(\mu) = U(\mathfrak{g}_R) \otimes_{U(\mathfrak{h}_R)} R_\mu \in \mathscr{O}_{[\lambda],(II),R} \cap R$-Proj are projective objects in $\mathscr{O}_{[\lambda],(II),R}$.*

(b) *The module $\bigoplus_{\mu \in [\lambda]} Q(\mu)$ is a projective generator of $\mathscr{O}_{[\lambda],(II),R}$ and $\mathscr{O}_{[\lambda],(II),R}$ has enough projectives.*

(c) *For every commutative R-algebra S which is a Noetherian ring, $S\otimes_R Q(\mu)\simeq Q(1_S\otimes_R \mu)$, $\mu\in[\lambda]$.*

Now, observe the following: given an exact sequence $0\to Q\to X\to P\to 0\in \operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,Q)$, if $P,Q\in \mathscr{O}_{[\lambda],R}$, then also $X\in U(\mathfrak{g}_R)$-mod and some power of $\mathfrak{n}_R^+$ annihilates X. Hence, $X\in\mathscr{O}_{[\lambda],R}$. Since $\mathscr{O}_{[\lambda],R}$ is a full subcategory of $\mathscr{O}_{[\lambda],(II),R}$, such exact sequence is an exact sequence in $\mathscr{O}_{[\lambda],R}$.

Lemma 4.4.38. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra with unique maximal ideal $\mathfrak{m}$. Let $\lambda\in\mathfrak{h}_R^*$. The following assertions hold.*

(a) *For each $P\in\mathscr{O}_{[\lambda],(I),R}\cap R$-Proj, $\operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,X)=0$ for every $X\in\mathscr{O}_{[\lambda],(I),R}\cap R$-Proj if and only if P is a projective object in $\mathscr{O}_{[\lambda],(I),R}\cap R$-Proj.*

(b) *For each $P\in\mathscr{O}_{[\lambda],R}\cap R$-Proj, $\operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,X)=0$ for every $X\in\mathscr{O}_{[\lambda],R}\cap R$-Proj if and only if P is a projective object in $\mathscr{O}_{[\lambda],R}\cap R$-Proj.*

(c) *If $P\in\mathscr{O}_{[\lambda],R}\cap R$-Proj so that $P(\mathfrak{m})$ is a projective object in $\mathscr{O}_{[\overline{\lambda}],R(\mathfrak{m})}$, then P is a projective object in $\mathscr{O}_{[\lambda],R}\cap R$-Proj.*

Proof. The assertions (a) and (b) follow immediately from the above discussion. To prove (c) we want to apply (b) together with Corollary 1.3.16. In order to do that we have to proceed by induction on the Krull dimension of R. If $\dim R$ is zero, then $R=R(\mathfrak{m})$ and there is nothing to prove. Let $x\in\mathfrak{m}/\mathfrak{m}^2$. Fix $S=R/Rx$, so $\dim S=\dim R-1$ and for any module $X\in U(\mathfrak{g}_R)$-Mod, we can write $X(\mathfrak{m})\simeq(S\otimes_R X)(\mathfrak{m}_S)$, where $\mathfrak{m}_S$ denotes $\mathfrak{m}/Rx$. Hence, by assumption, $S\otimes_R P\in\mathscr{O}_{[1_S\otimes_R\lambda],S}\cap S$-Proj so that $P(\mathfrak{m}_S)\simeq P(\mathfrak{m})$ is a projective object in $\mathscr{O}_{[\overline{\lambda},S(\mathfrak{m}_S)]}$, where $S(\mathfrak{m}_S)=R(\mathfrak{m})$. By induction, $S\otimes_R P$ is a projective object in $\mathscr{O}_{[1_S\otimes_R\lambda],S}\cap S$-Proj.

Consider a projective resolution of P by direct sums of $\bigoplus_{\mu\in[\lambda]}Q(\mu)$ and denote the respective deleted projective resolution by $Q^\bullet$. Since $P\in R$-Proj, each $Q(\mu)\in R$-Proj and the tensor product commutes with arbitrary direct sums we obtain that $S\otimes_R Q^\bullet$ is a deleted projective resolution of $S\otimes_R P$ in $\mathscr{O}_{[1_S\otimes_R\lambda],(II),S}$. Now, for each $X\in\mathscr{O}_{[\lambda],R}\cap R$-Proj,

$$\operatorname{Hom}_{\mathscr{O}_{[\lambda],(II),R}}\Big(\bigoplus_{\mu\in[\lambda]}Q(\mu),X\Big)\simeq\prod_{\mu\in[\lambda]}\operatorname{Hom}_{\mathscr{O}_{[\lambda],(II),R}}(Q(\mu),X)\simeq\prod_{\mu\in[\lambda]}X_\mu. \tag{4.4.10.1}$$

Each X_μ is a flat module, and since every Noetherian ring is coherent, so the arbitrary direct product of flat modules is flat. Hence, the complex $\operatorname{Hom}_{\mathscr{O}_{[\lambda],(II),R}}(Q^\bullet,X)$ satisfies the hypothesis of Corollary 1.3.16. Further, since R is Noetherian and applying Lemma 4.4.23 we obtain

$$S\otimes_R\operatorname{Hom}_{\mathscr{O}_{[\lambda],(II),R}}\Big(\bigoplus_{\mu\in[\lambda]}Q(\mu),X\Big)\simeq S\otimes_R\prod_{\mu\in[\lambda]}X_\mu\simeq\prod_{\mu\in[\lambda]}S\otimes_R X_\mu\simeq\prod_{\mu\in[\lambda]}X_{1_S\otimes_R\mu} \tag{4.4.10.2}$$

$$\simeq\operatorname{Hom}_{\mathscr{O}_{[1_S\otimes_R\lambda],(II),S}}\Big(\bigoplus_{1_S\otimes_R\mu\in[1_S\otimes_R\lambda]}Q(1_S\otimes_R\mu),S\otimes_R X\Big). \tag{4.4.10.3}$$

Therefore, for each integer $i>0$,

$$H^i(\operatorname{Hom}_{\mathscr{O}_{[\lambda],(II),R}}(Q^\bullet,X))=\operatorname{Ext}^i_{\mathscr{O}_{[\lambda],(II),R}}(P,X),$$
$$H^i(S\otimes_R\operatorname{Hom}_{\mathscr{O}_{[\lambda],(II),R}}(Q^\bullet,X))=\operatorname{Ext}^i_{\mathscr{O}_{[1_S\otimes_R\lambda],(II),S}}(S\otimes_R P,S\otimes_R X).$$

By Corollary 1.3.16 and $S\otimes_R P$ being projective in $\mathscr{O}_{[1_S\otimes_R\lambda],(II),S}$ we obtain that $\operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,X)\otimes_R R/Rx=0$. Using the surjective map $R/Rx\to R/\mathfrak{m}$ we obtain that $\operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,X)\otimes_R R/\mathfrak{m}=0$. Observe that P is finitely generated as $U(\mathfrak{g}_R)$-module (for which such generator set can be chosen to be a set of weight vectors). Hence, we can choose only a finite set of weights F so that $\bigoplus_{\mu\in F} Q(\mu) \to P$ is a surjective map. Since R is Noetherian and each weight module of X is finitely generated as R-module, $\operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,X)$ is a quotient of a finitely generated R-module, and so it is finitely generated. By Nakayama's Lemma, $\operatorname{Ext}^1_{\mathscr{O}_{[\lambda],(II),R}}(P,X) = 0$. By (b), P is a projective object in $\mathscr{O}_{[\lambda],R} \cap R$-Proj. $\square$

The construction of projective objects in $\mathscr{O}$ is based on tensoring Verma projective modules with simple modules of finite vector space dimension. These are the simple modules indexed by an integral dominant weight. Their deformations in $\mathscr{O}_{[\lambda],R}$ are similarly obtained. We are expecting them to be free as $U(\mathfrak{n}_R^-)$-modules and consequently free as R-modules. So, the modules taking the place of simple modules should be free over R. Here, we already say that these modules are not the simple modules in $\mathscr{O}_{[\lambda],R}$. The reason for this is that Gabber and Joseph showed that the simple modules in $\mathscr{O}_{[\lambda],R}$, where R is a local commutative Noetherian $\mathbb{Q}$-algebra, are of the form $\Delta(\mu)/N$, where $\mathfrak{m}\Delta(\mu) \subset N$, $\mu \in [\lambda]$ and $\mathfrak{m}$ is the unique maximal ideal of R. Thus, $\mathfrak{m}L(\mu) = 0$, and so $L(\mu)$ would be free over the ground ring if and only if $\mathfrak{m}N = \mathfrak{m}\Delta(\mu)$. The latter condition is something that we just do not know at this point. So, we must consider a different approach.

As we discussed we will try to obtain an integral version of the simple modules indexed by integral dominant weights. Let $\mu \in \mathfrak{h}_R^*$ be an integral dominant weight. We define J_R be the left ideal of $U(\mathfrak{g}_R)$ generated by the set of elements

$$\{x_\alpha \colon \alpha \in \Phi^+\} \cup \{h_\alpha - \mu(h_\alpha)1_R \colon \alpha \in \Pi\} \cup \{x_{-\alpha}^{n_\alpha+1} \colon \alpha \in \Pi\}, \tag{4.4.10.4}$$

where $n_\alpha = \langle \overline{\mu}, \alpha^\vee \rangle_{R(\mathfrak{m})} \in \mathbb{Z}_0^+$.

By the PBW theorem, the monomials generated by this set of elements are linearly independent and also PBW monomials making J_R a free R-module. Moreover, the basis of J_R can be extended to a basis of $U(\mathfrak{g}_R)$, so the canonical inclusion of J_R into $U(\mathfrak{g}_R)$ is an $(U(\mathfrak{g}_R),R)$-monomorphism. Also for any commutative R-algebra S, $S\otimes_R J_R$ is isomorphic to J_S. Let $E(\mu)$ denote the quotient $U(\mathfrak{g}_R)/J_R$. Since $0 \to J_R \to U(\mathfrak{g}_R) \to E(\mu) \to 0$ remains exact under $R(\mathfrak{m})\otimes_R -$ and $U(\mathfrak{g}_R)$ is free over R we obtain $\operatorname{Tor}_1^R(E(\mu), R(\mathfrak{m})) = 0$. Further, $S\otimes_R E(\mu) \simeq E(1_S \otimes_R \mu)$ for every commutative R-algebra S which is a Noetherian ring making $1_S \otimes_R \mu$ an integral dominant weight in $\mathfrak{h}_S^*$. We can also see that $E(\mu)$ is a quotient of $\Delta(\mu)$. Therefore, $E(\mu) \in \mathscr{O}_{[\mu],R}$. In addition, $R(\mathfrak{m})\otimes_R E(\mu) \simeq L(1\otimes_R \mu)$ and $1\otimes_R \mu$ is an integral dominant weight in $\mathfrak{h}_{R(\mathfrak{m})}^*$. Therefore, it is finite-dimensional. By Nakayama's Lemma, $E(\mu)$ is finitely generated over R. By Theorem 1.1.44, $E(\mu)$ is free over R with finite rank. Observe that for each $n \in \mathbb{N}$, by Lemma 4.4.23, the weights of $E(n\rho)$ are weights ranging from $-n\rho$ to $n\rho$. Moreover, the weight modules associated with $-n\rho$ and $n\rho$ are free with rank one.

For each $\mu \in \mathfrak{h}_R^*$, if $\overline{\mu}$ is not a dominant weight, then there is some $\alpha \in \Phi^+$ so that $\langle \overline{\mu}+\rho, \alpha^\vee \rangle_{R(\mathfrak{m})} \in \mathbb{Z}^-$. Since $\langle \rho, \alpha^\vee \rangle_{R(\mathfrak{m})} = 1$, there exists $n \in \mathbb{N}$ so that $\overline{\mu}+n\rho$ is a dominant weight.

Definition 4.4.39. Let $\lambda \in \mathfrak{h}_R^*$ and $\mu \in [\lambda]$. If μ is a dominant weight, define $P(\mu) := \Delta(\mu)$. Otherwise, pick $n \in \mathbb{N}$ minimal so that $\mu + n\rho \in [\lambda]$ is a dominant weight and define $P(\mu) := (\Delta(\mu+n\rho)\otimes_R E(n\rho))^{\mathscr{D}_\mu}$, where $\mathscr{D}_\mu$ is the block of $[\lambda]$ that contains μ.

Theorem 4.4.40. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra with unique maximal ideal $\mathfrak{m}$. Let $\lambda \in \mathfrak{h}_R^*$. The following assertions hold.*

*(a) If $\mu \in [\lambda]$ is a dominant weight, then $\Delta(\mu)$ is projective in $\mathscr{O}_{[\lambda],R} \cap R$-*Proj.

*(b) The modules $P(\mu) \in \mathscr{O}_{\mathscr{D}_\mu}$ are projective objectives in $\mathscr{O}_{[\lambda],R} \cap R$-*Proj*, where $\mathscr{D}_\mu$ is the block of $[\lambda]$ that contains μ.*

(c) *For each $\mu \in [\lambda]$, there exists an exact sequence in $\mathscr{O}_{\mathscr{D}_\mu}$*

$$0 \to C(\mu) \to P(\mu) \to \Delta(\mu) \to 0, \tag{4.4.10.5}$$

where $C(\mu) \in \mathscr{F}(\Delta(\omega)_{\omega>\mu})$ and $\mathscr{D}_\mu$ is the block that contains μ.

(d) *Fix $P = \bigoplus_{\mu \in [\lambda]} P(\mu)$. For each $Q \in \operatorname{add}_{\mathscr{O}_{[\lambda],R}} P$, $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(Q,M) \in R$-proj for every $M \in \mathscr{F}(\Delta)$.*

(e) *Assume that S is:*

- *a localization $R_\mathfrak{p}$ of R at some prime ideal $\mathfrak{p}$ of R;*
- *a quotient ring R/I of R for some ideal I.*

Then, for each $\omega \in [\lambda]$, and $M \in \mathscr{F}(\Delta)$, the canonical map

$$S \otimes_R \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(P(\omega), M) \to \operatorname{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda],S}}(S \otimes_R P(\omega), S \otimes_R M)$$

is an isomorphism.

Proof. If μ is a dominant weight, then $\overline{\mu}$ is dominant. By Theorem 4.4.7(d), $\Delta(\overline{\mu})$ is projective in $\mathscr{O}_{[\overline{\lambda}],R(\mathfrak{m})}$. By Lemma 4.4.38(b), $\Delta(\mu)$ is projective in $\mathscr{O}_{[\lambda],R} \cap R$-Proj.

Since $E(n\rho) \in R$-proj it is clear that $P(\mu) \in R$-Proj. By Theorem 4.4.7(e), $R(\mathfrak{m}) \otimes_R \Delta(\mu + n\rho) \otimes_R E(n\rho) \simeq \Delta(\overline{\mu} + n1_{R(\mathfrak{m})}\rho) \otimes_{R(\mathfrak{m})} L(1_{R(\mathfrak{m})} n\rho)$ is a projective object in $\mathscr{O}_{[\overline{\lambda}],R(\mathfrak{m})}$. By Lemma 4.4.38, $\Delta(\mu + n\rho) \otimes_R E(n\rho)$ is a projective object in $\mathscr{O}_{[\lambda],R} \cap R$-Proj. As $P(\mu)$ is a summand of $\Delta(\mu + n\rho) \otimes_R E(n\rho)$ it is also a projective object in $\mathscr{O}_{[\lambda],R} \cap R$-Proj and also in $\mathscr{O}_{\mathscr{D}_\mu} \cap R$-Proj.

If μ is dominant, the exact sequence on (c) is just the identity map on $\Delta(\mu)$. Assume that μ is not dominant. By Proposition 4.4.21, $\Delta(\mu + n\rho) \otimes_R E(n\rho) \in \mathscr{F}(\Delta(\mu + n\rho + \omega)_{\{\omega \in [\lambda] :\, E(n\rho)_\omega \neq 0\}}$. So, the lowest weight in the filtration of $\Delta(\mu + n\rho) \otimes_R E(n\rho)$ which occurs only once is $\mu + n\rho - n\rho = \mu$ which again by Proposition 4.4.21 appears at the top of the filtration. We obtained in this way an exact sequence

$$0 \to C_1(\mu) \to \Delta(\mu + n\rho) \otimes_R E(n\rho) \to \Delta(\mu) \to 0. \tag{4.4.10.6}$$

The remaining weights are of the form $\mu + n\rho + \gamma$, where $\gamma \in \mathbb{N}\Pi$ not smaller than $-n\rho$. Hence, $\mu + n\rho + \gamma - \mu \in \mathbb{N}\Pi$. So, all weights of $C_1(\mu)$ are greater than μ. Applying $\mathscr{D}_\mu$ to (4.4.10.6) we obtain (c).

By the above discussion for (b), for each $n \in \mathbb{N}$ and each $\omega \in [\lambda]$, the functor $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega) \otimes_R E(n\rho), -)$ is exact on $\mathscr{F}(\Delta)$. Therefore, $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega) \otimes_R E(n\rho), M) \in R$-proj for every $M \in \mathscr{F}(\Delta)$ if and only if $\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega) \otimes_R E(n\rho), \Delta(\mu)) \in R$-proj for every $\mu \in [\lambda]$. By Lemma 4.4.36,

$$\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}((\Delta(\omega) \otimes_R E(n\rho))^{\mathscr{D}}, \Delta(\mu)^{\mathscr{D}}) \simeq \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}((\Delta(\omega) \otimes_R E(n\rho))^{\mathscr{D}}, \Delta(\mu)), \tag{4.4.10.7}$$

which is zero unless $\mu \in \mathscr{D}$. Assuming that $\mu \in \mathscr{D}$, again by Lemma 4.4.36,

$$\operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}((\Delta(\omega) \otimes_R E(n\rho))^{\mathscr{D}}, \Delta(\mu)) \simeq \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}((\Delta(\omega) \otimes_R E(n\rho)), \Delta(\mu)) \simeq \operatorname{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega), E(n\rho)^* \otimes_R \Delta(\mu)).$$

Since ω is dominant $\Delta(\omega)$ is a projective object in $\mathscr{F}(\Delta)$. Hence, if $\Delta(\lambda)$ appears as factor in a Verma filtration of an arbitrary $M \in \mathscr{F}(\Delta)$, then we can assume that all its occurrences appear at the bottom of the filtration. Moreover, all its occurrences can be encoded in a direct sum of copies of $\Delta(\omega)$. Thanks to ω being dominant, by Lemma 4.4.6 and Lemma 4.4.36, homomorphisms from $\Delta(\omega)$ to another Verma module $\Delta(\omega_1)$ are only non-zero if $\omega = \omega_1$.

Therefore, if we fix $M := E(n\rho)^* \otimes_R \Delta(\mu) \in \mathscr{F}(\Delta)$ by Proposition 4.4.21, we obtain

$$\mathrm{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega), M) \simeq \mathrm{Hom}_{\mathscr{O}_{[\lambda],R}}(\Delta(\omega), \Delta(\omega)^j) \simeq R^j, \tag{4.4.10.8}$$

where j denotes the number of occurrences of $\Delta(\omega)$ in M. This shows that $\mathrm{Hom}_{\mathscr{O}_{[\lambda],R}}(P(\omega), \Delta(\mu)) \in R$-proj for all $\mu \in [\lambda]$. By induction on the size of filtrations by Verma modules, we obtain (d). Indeed, if $M \in \mathscr{F}(\Delta)$, then there exists an exact sequence $0 \to M' \to M \to \Delta(\mu) \to 0$ for some $\mu \in [\lambda]$ and $M' \in \mathscr{F}(\Delta)$ having a filtration by Verma modules with lesser length than a filtration by Verma modules of M. By induction, $\mathrm{Hom}_{\mathscr{O}_{[l],R}}(P(\omega), M') \in R$-proj. Since $\mathrm{Hom}_{\mathscr{O}_{[l],R}}(P(\omega), -)$ is exact on $\mathscr{F}(\Delta)$ our claim follows.

Now, we will proceed to prove (e). Let S be a local commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Since any $M \in \mathscr{F}(\Delta)$ is free as R-module (of infinite rank), the filtrations in $\mathscr{F}(\Delta)$ remain exact under $S \otimes_R -$. In particular, assuming that $1_S \otimes_R \omega$ is a dominant weight

$$\begin{aligned} S \otimes_R \mathrm{Hom}_{\mathscr{O}_{[\lambda],R}}((\Delta(\omega) \otimes_R E(n\rho))^{\mathscr{D}}, \Delta(\mu)) &\simeq S \otimes_R R^j \simeq S^j \simeq \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(\Delta(1_S \otimes_R \omega), S \otimes_R E(n\rho)^* \otimes_R \Delta(\mu)) \\ &\simeq \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(\Delta(1_S \otimes_R \omega), E(n1_S\rho)^* \otimes_S \Delta(1_S \otimes_R \mu)) \\ &\simeq \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(\Delta(1_S \otimes_R \omega) \otimes_S E(n1_S\rho), \Delta(1_S \otimes_R \mu)) \\ &\simeq \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(S \otimes_R \Delta(\omega) \otimes_R E(n\rho), \Delta(1 \otimes_R \mu)) \\ &\simeq \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(S \otimes_R P(\omega - n\rho), \Delta(1 \otimes_R \mu)). \end{aligned}$$

Since all these isomorphisms are functorial, we obtain that the canonical map

$$S \otimes_R \mathrm{Hom}_{\mathscr{O}_{[\lambda]},R}(P(\omega - n\rho), \Delta(\mu)) \to \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(S \otimes_R P(\omega - n\rho), S \otimes_R \Delta(\mu))$$

is an isomorphism for every $\mu \in [\lambda]$. Since $P(\omega - n\rho)$ is a projective object in $\mathscr{O}_{[\lambda],R} \cap R$-Proj by using the previous statement there is for every $M \in \mathscr{F}(\Delta)$ a commutative diagram with exact columns

$$\begin{tikzcd} S \otimes_R \mathrm{Hom}_{\mathscr{O}_{[\lambda]},R}(P(\omega - n\rho), M') \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(S \otimes_R P(\omega - n\rho), S \otimes_R M') \arrow[d, hook] \\ S \otimes_R \mathrm{Hom}_{\mathscr{O}_{[\lambda]},R}(P(\omega - n\rho), M) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(S \otimes_R P(\omega - n\rho), S \otimes_R M) \arrow[d] \\ S \otimes_R \mathrm{Hom}_{\mathscr{O}_{[\lambda]},R}(P(\omega - n\rho), \Delta(\mu)) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathscr{O}_{[1_S \otimes_R \lambda]},S}(S \otimes_R P(\omega - n\rho), S \otimes_R \Delta(\mu)) \end{tikzcd}, \tag{4.4.10.9}$$

where $M' \in \mathscr{F}(\Delta)$ which together with $\Delta(\mu)$ gives a Verma filtration to M. Hence, the upper row is obtained by induction. By Snake Lemma, the middle map is also an isomorphism. $\square$

Remark 4.4.41. Using the same argument as in the classical theory of the category $\mathscr{O}$, we could see that the Verma modules associated with dominant weights are projective objects in their blocks. $\triangle$

4.4.11 Noetherian algebra $A_{\mathscr{D}}$ associated with a category $\mathscr{O}_{[\lambda],R}$

Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. Define $P_{\mathscr{D}} := \bigoplus_{\mu \in \mathscr{D}} P(\mu)$.

Definition 4.4.42. Let $\lambda \in \mathfrak{h}_R^*$ and $\mathscr{D}$ a block of $[\lambda]$. We define the R-algebra $A_{\mathscr{D}}$ to be the endomorphism algebra $\mathrm{End}_{\mathscr{O}_{[\lambda],R}}(P_{\mathscr{D}})^{op}$.

By Theorem 4.4.40, $A_{\mathscr{D}}$ is a projective Noetherian R-algebra. By Lemma 4.4.16, we can see that $P_{\mathscr{D}}$ is a generator of $\mathscr{O}_{\mathscr{D}}$. Moreover, since the filtrations involved in Lemma 4.4.16 are finite for each object $X \in \mathscr{O}_{\mathscr{D}}$ there exists an exact sequence $P_{\mathscr{D}}^s \to P_{\mathscr{D}}^t \to X \to 0$. Unfortunately, our methods in Theorem 4.4.40 do not allow us to state already that $P_{\mathscr{D}}$ is a projective generator. However, we can see that the functor $H := \operatorname{Hom}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}}, -)\colon \mathscr{O}_{\mathscr{D}} \to A_{\mathscr{D}}$-mod is fully faithful since $P_{\mathscr{D}}$ is a generator of $\mathscr{O}_{\mathscr{D}}$. It is an equivalence of categories whenever R is a field. Further, the restriction of H to $\mathscr{F}(\Delta)$ is an exact fully faithful functor. This reduces the study of the category $\mathscr{O}_{[\lambda],R}$ to the study of module categories of projective Noetherian R-algebras and its subcategories.

As we have been mentioning throughout this section, the algebras $A_{\mathscr{D}}$ are split quasi-hereditary.

Theorem 4.4.43. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. The algebra $A_{\mathscr{D}}$ is split quasi-hereditary with standard modules $\Delta_A(\mu) := H\Delta(\mu)$, $\mu \in \mathscr{D}$. The set $\mathscr{D}$ is a poset with the partial order $\mu_1 < \mu_2$ if and only if $\mu_2 - \mu_1 \in \mathbb{N}\Pi$.*

Proof. By Theorem 4.4.40 (d), $\Delta_A(\mu) \in R$-proj for all $\mu \in \mathscr{D}$. By Lemma 4.4.6(i) and (ii) and together with H being fully faithful we obtain $\operatorname{End}_{A_{\mathscr{D}}}(\Delta_A(\mu)) \simeq R$ and if $\operatorname{Hom}_{A_{\mathscr{D}}}(\Delta_A(\mu_1), \Delta_A(\mu_2)) \neq 0$, then $\mu_1 \leq \mu_2$. Denote by $P_A(\mu)$ the projective A-modules $\operatorname{Hom}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}}, P(\mu))$. By Theorem 4.4.40 (c),(b), and H being fully faithful we obtain, for each $\mu \in [\lambda]$, an exact sequence

$$0 \to C_A(\mu) \to P_A(\mu) \to \Delta_A(\mu) \to 0, \tag{4.4.11.1}$$

where $C_A(\mu) \in \mathscr{F}(\Delta_A(\omega)_{\omega > \mu})$. Further,

$$\bigoplus_{\mu \in \mathscr{D}} P_A(\mu) = \bigoplus_{\mu \in \mathscr{D}} \operatorname{Hom}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}}, P(\mu)) \simeq \operatorname{Hom}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}}, \bigoplus_{\mu \in \mathscr{D}} P(\mu)) \simeq {}_{A_{\mathscr{D}}}A_{\mathscr{D}}. \tag{4.4.11.2}$$

Hence, this direct sum is a progenerator of $A_{\mathscr{D}}$. By Corollary 1.5.43, the result follows. $\square$

We could wonder given the definition of $P(\mu)$ if there could be other projectives taking its role of mapping surjectively to $\Delta(\mu)$. By Proposition 1.5.61, we see that $P_A(\mu)$ is the right choice and it $P_A(\mu)(\mathfrak{m})$ is actually the projective cover of $\Delta_A(\overline{\mu})$. Hence, the idempotents

$$e_\mu := P_{\mathscr{D}} \twoheadrightarrow P(\mu) \hookrightarrow P_{\mathscr{D}}, \quad \mu \in \mathscr{D},$$

in $\operatorname{End}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}})^{op} = A_{\mathscr{D}}$ form a set of orthogonal idempotents such that their image under $R(\mathfrak{m})$, according to Theorem 4.4.40(e), form a complete set of primitive orthogonal idempotents of $A_{\mathscr{D}}(\mathfrak{m})$. In particular, by Theorem 1.5.73,

$$0 \subset A_{\mathscr{D}} e_\omega A_{\mathscr{D}} \subset \cdots \subset A_{\mathscr{D}}(\sum_{\mu \in \mathscr{D}} e_\mu) A_{\mathscr{D}}$$

is a split heredity chain of $A_{\mathscr{D}}$. Here, ω is the dominant weight of $\mathscr{D}$.

Corollary 4.4.44. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$.*

*(a) The algebra $A_{\mathscr{D}}$ is semi-perfect and $A_{\mathscr{D}}$-*proj *is a Krull-Schmidt category.*

(b) The algebra $A_{\mathscr{D}}$ has finite global dimension.

Proof. By Theorem 1.5.84, (a) follows. R is a regular local ring, so $\operatorname{gldim} R$ is finite. By Corollary 1.5.76, $A_{\mathscr{D}}$ has finite global dimension. $\square$

The category $\mathcal{O}$ has a simple preserving duality, so we expect the algebra $A_{\mathcal{D}}$ to be a cellular algebra as well. We can use the duality functor restricted to the block $(-)^\vee\colon \mathcal{O}_{\mathcal{D}} \cap R\text{-Proj} \to \mathcal{O}_{[\lambda],(I),R} \cap R\text{-Proj}$ to construct the relative injective modules of $A_{\mathcal{D}}$.

Lemma 4.4.45. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathcal{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. For each $\mu \in \mathcal{D}$, the module* $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(P_{\mathcal{D}}, P(\mu)^\vee)$ *is projective over R and $(A_{\mathcal{D}}, R)$-injective.*

Proof. Since $(-)^\vee$ is exact the module $P(\mu)^\vee$ belongs to $\mathcal{F}(\Delta(\omega)^\vee_{\omega\in\mathcal{D}})$ for each $\mu \in \mathcal{D}$. As we saw, by the construction of the duality functor $\Delta(\omega)^\vee \in \mathcal{O}_{[\lambda],(I),R} \cap R$-Proj and each weight module of $\Delta(\omega)^\vee$ is finitely generated as R-module. Using the same arguments as in Lemma 4.4.38, replacing P by $\Delta(\omega_1)$ and X by $\Delta(\omega)^\vee$ and knowing that in the classical case $\Delta(\omega)^\vee$ are the costandard modules making $\mathcal{O}$ a split highest weight category we obtain $\operatorname{Ext}^1_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega_1), \Delta(\omega)^\vee) = 0$ for all $\omega_1, \omega \in \mathcal{D}$. Hence, using induction on finite filtration by Verma modules Δ and on finite filtration by dual Verma modules $\Delta^\vee$ we can reduce the problem of $\operatorname{Hom}_{\mathcal{O}[\lambda],(I),R}(P_{\mathcal{D}}, X)$ being projective over R, with $X \in \mathcal{F}(\Delta(\omega)^\vee_{\omega\in\mathcal{D}})$, to showing that $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega), \Delta(\theta)^\vee) \in R$-proj for all weights $\omega, \theta \in \mathcal{D}$.

Observe that $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega), \Delta(\theta)^\vee) \subset (\Delta(\theta)^\vee)_\omega = D(\Delta(\theta))_\omega$. So, if the homomorphism group is non-zero, then $\omega \leq \theta$. In addition,

$$0 \neq \operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega), \Delta(\theta)^\vee) \simeq \operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\theta), \Delta(\omega)^\vee). \tag{4.4.11.3}$$

So, also $\theta \leq \omega$. Therefore, $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega), \Delta(\theta)^\vee) = 0$ unless $\theta = \omega$. In case, $\theta = \omega$ we obtain

$$\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega), \Delta(\omega)^\vee) \simeq \operatorname{Hom}_{U(\mathfrak{b}_R}(R_\omega, \Delta(\omega)^\vee) \simeq \Delta(\omega)^\vee_\omega \simeq D\Delta(\omega)_\omega \simeq R \in R\text{-proj}. \tag{4.4.11.4}$$

Since $\operatorname{Ext}^1_{\mathcal{O}_{[\lambda],(I),R}}(\Delta(\omega_1), \Delta(\omega)^\vee) = 0$ for all $\omega_1, \omega \in \mathcal{D}$ we can apply the same argument in Proposition 1.5.117 and Corollary 1.5.118 to deduce that the homomorphisms between modules with Verma filtrations and modules with dual Verma filtrations commute with the functor $R(\mathfrak{m}) \otimes_R -$. Since $P(\overline{\mu})^\vee$ is injective and $R(\mathfrak{m}) \otimes_R H$ is an equivalence we obtain, by Theorem 1.2.57, $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(P_{\mathcal{D}}, P(\mu)^\vee)$ is $(A_{\mathcal{D}}, R)$-injective. ☐

Remark 4.4.46. The reader can observe that the modules $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(P_{\mathcal{D}}, \Delta(\mu)^\vee)$, $\mu \in \mathcal{D}$, are the costandard modules of $A_{\mathcal{D}}$. △

It follows that $\operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(P_{\mathcal{D}}, P_{\mathcal{D}}^\vee) \simeq \bigoplus_{\mu\in\mathcal{D}} \operatorname{Hom}_{\mathcal{O}_{[\lambda],(I),R}}(P_{\mathcal{D}}, P(\mu)^\vee) \simeq DA_{\mathcal{D}}$. Using this we can deduce a duality map on $A_{\mathcal{D}}$. In fact, as R-algebras, we have the following commutative diagram

$$
\begin{tikzcd}
A_{\mathscr{D}}(\mathfrak{m}) \arrow[rr, "\simeq"] \arrow[ddd, "\simeq"] & & \operatorname{End}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}})^{op}(\mathfrak{m}) \arrow[rr, "\simeq"] \arrow[ddd, "\simeq"] & & \operatorname{End}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}^{\vee})(\mathfrak{m}) \arrow[ddd] \arrow[ddr] & \\
& A_{\mathscr{D}}^{op}(\mathfrak{m}) \arrow[ddd, "\simeq"] & & & & \\
& \operatorname{End}_{A_{\mathscr{D}}^{op}}(A_{\mathscr{D}})^{op}(\mathfrak{m}) \arrow[u, "\simeq"] \arrow[ddd, "\simeq"] & & \operatorname{End}_{A_{\mathscr{D}}}(DA_{\mathscr{D}})(\mathfrak{m}) \arrow[ll, "\simeq"] \arrow[ddd, "\simeq"] & & \operatorname{End}_{A_{\mathscr{D}}}(\operatorname{Hom}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}, P_{\mathscr{D}}^{\vee}))(\mathfrak{m}) \arrow[ll, "\simeq"] \arrow[ddd, "\simeq"] \\
A(\mathfrak{m})_{\mathscr{D}} \arrow[rr, "\simeq"] & & \operatorname{End}_{\mathscr{O}}(P_{\mathscr{D}}(\mathfrak{m}))^{op} \arrow[rr, "\simeq"] & & \operatorname{End}_{\mathscr{O}}(P_{\mathscr{D}}(\mathfrak{m})^{\vee}) \arrow[ddr, "\simeq"] & \\
& A(\mathfrak{m})_{\mathscr{D}}^{op} & & & & \\
& \operatorname{End}_{A(\mathfrak{m})_{\mathscr{D}}^{op}}(A(\mathfrak{m})_{\mathscr{D}})^{op} \arrow[u, "\simeq"] & & \operatorname{End}_{A(\mathfrak{m})_{\mathscr{D}}}(DA_{\mathscr{D}}(\mathfrak{m})) \arrow[ll, "\simeq"] & & \operatorname{End}_{A(\mathfrak{m})_{\mathscr{D}}}(\operatorname{Hom}_{\mathscr{O}}(P_{\mathscr{D}}(\mathfrak{m}), P_{\mathscr{D}}(\mathfrak{m})^{\vee})) \arrow[ll, "\simeq"]
\end{tikzcd}
$$

The isomorphisms in the diagram are marked with $\simeq$. We required this approach since without it we do not know if the Hom functor on the generator $P_{\mathscr{D}}$ is fully faithful on the additive closure of its dual $P_{\mathscr{D}}^{\vee}$. Since $A_{\mathscr{D}} \in R$-proj by Nakayama's Lemma we obtain that the following composition of maps is an isomorphism of R-algebras

$$
\begin{tikzcd}
A_{\mathscr{D}} \arrow[r] & \operatorname{End}_{\mathscr{O}_{\mathscr{D}}}(P_{\mathscr{D}})^{op} \arrow[r] & \operatorname{End}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}^{\vee}) \arrow[dd] \\
A_{\mathscr{D}}^{op} & & \\
\operatorname{End}_{A_{\mathscr{D}}^{op}}(A_{\mathscr{D}})^{op} \arrow[u] & \operatorname{End}_{A_{\mathscr{D}}}(DA_{\mathscr{D}}) \arrow[l] & \operatorname{End}_{A_{\mathscr{D}}}(\operatorname{Hom}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}, P_{\mathscr{D}}^{\vee})) \arrow[l]
\end{tikzcd}
.
$$

Observe that under this composition of maps, for each $\mu \in \mathscr{D}$,

$$
\begin{aligned}
e_\mu &\mapsto P_{\mathscr{D}} \twoheadrightarrow P(\mu) \hookrightarrow P_{\mathscr{D}} \mapsto P_{\mathscr{D}}^{\vee} \twoheadrightarrow P(\mu)^{\vee} \hookrightarrow P_{\mathscr{D}}^{\vee} \\
&\mapsto \operatorname{Hom}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}, P_{\mathscr{D}}^{\vee}) \twoheadrightarrow \operatorname{Hom}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}, P(\mu)^{\vee}) \hookrightarrow \operatorname{Hom}_{\mathscr{O}_{[\lambda],(I),R}}(P_{\mathscr{D}}, P_{\mathscr{D}}^{\vee}) \\
&\mapsto DA_{\mathscr{D}} \twoheadrightarrow I(\mu) = D(e_\mu A_{\mathscr{D}}) \hookrightarrow DA_{\mathscr{D}} \mapsto A_{\mathscr{D}} \twoheadrightarrow e_\mu A_{\mathscr{D}} \hookrightarrow A_{\mathscr{D}} \mapsto e_\mu.
\end{aligned}
$$

Hence, this gives an involution on $A_{\mathscr{D}}$, denoted by ι, which fixes the set of orthogonal idempotents $\{e_\mu : \mu \in \mathscr{D}\}$. In addition, we can assign a new duality functor on $A_{\mathscr{D}}$ using the duality ι. For each $M \in A_{\mathscr{D}}$-mod, define the right $A_{\mathscr{D}}$-module $M^{\iota} \in$ by imposing $m \cdot_{\iota} a := \iota(a)m,\ m \in M$. The assignment $M \mapsto DM^{\iota}$ is a duality functor on $A_{\mathscr{D}}$-mod$\cap R$-proj, which we will denote by $(-)^{\natural}$.

Theorem 4.4.47. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. The algebra $A_{\mathscr{D}}$ is a cellular algebra with involution ι and cell chain*

$$
0 \subset A_{\mathscr{D}} e_\omega A_{\mathscr{D}} \subset \cdots \subset A_{\mathscr{D}}(\sum_{\mu \in \mathscr{D}} e_\mu) A_{\mathscr{D}} = A_{\mathscr{D}},
$$

where ω is the dominant weight of $\mathscr{D}$.

Proof. The result follows by Proposition 1.6.12. $\square$

4.4.12 The algebra $A_{\mathscr{D}}$ is a relative gendo-symmetric algebra

Our aim now is to compute the relative dominant dimension of the algebra $A_{\mathscr{D}}$ and to prove that it is a relative gendo-symmetric algebra.

Koenig, Slungård and Xi gave a lower bound for the dominant dimension of the blocks of the classical category $\mathcal{O}$ in [KSX01, Theorem 3.2]. Later, Fang proved in [Fan08, Proposition 4.5] that this lower bound was indeed the value of the dominant dimension. Mainly, the dominant dimension sees two cases. Either the algebra associated with a block is semi-simple which obviously gives infinite dominant dimension or the algebra associated with a non semi-simple block has dominant dimension two. The main reason for this situation is that the blocks of the category $\mathcal{O}$ only have one projective-injective module. We will now generalize these results to the Noetherian algebras $A_{\mathscr{D}}$.

Theorem 4.4.48. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra with unique maximal ideal $\mathfrak{m}$. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. For the unique antidominant weight $\mu \in \mathscr{D}$, $(A_{\mathscr{D}}, P_A(\mu), DP_A(\mu))$ is a relative QF3 R-algebra and*

$$\operatorname{domdim}(A_{\mathscr{D}}, R) = \begin{cases} +\infty, & \text{if } |\mathscr{D}| = 1, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. By Lemma 4.4.29, $\mathscr{D}$ is of the form $W_{\overline{\mu}} \cdot \mu + \nu$ for some $\nu \in \mathfrak{h}_R^*$ and $w \cdot \mu - \mu \in \mathbb{Z}\Phi$ for every $w \in W_{\overline{\mu}}$.

By Theorem 4.4.7(e), for an antidominant weight $\omega \in \mathscr{D}$, $P(\omega)$ is the unique projective-injective in $\mathcal{O}_{W_{\overline{\mu}} \cdot \overline{\mu}}$. By Theorem 4.4.43 and 4.4.40, $P_A(\omega)(\mathfrak{m}) \simeq P_{A(\mathfrak{m})}(\overline{\omega})$ is the unique projective-injective of $A_{W_{\overline{\mu}} \cdot \overline{\mu}} \simeq A_{\mathscr{D}}(\mathfrak{m})$. There are now two distinct cases. Assume that $A_{W_{\overline{\mu}} \cdot \overline{\mu}}$ is semi-simple. In particular, $\Delta(\overline{\mu})$ is projective-injective. By Theorem 4.4.7, $\overline{\mu}$ is a dominant and antidominant weight. On the other hand, if there exists a weight in $W_{\overline{\mu}} \cdot \overline{\mu}$ which is both dominant and antidominant, then it is both a maximal and minimal element in $W_{\overline{\mu}} \cdot \overline{\mu}$. Thus, $W_{\overline{\mu}} \cdot \overline{\mu} = \{\overline{\mu}\}$. In such a case, $A_{W_{\overline{\mu}} \cdot \overline{\mu}} \simeq R(\mathfrak{m})$. Further, for any two elements $w_1, w_2 \in W_{\overline{\mu}}$ we have $w_1 \cdot \mu - w_2 \cdot \mu \in \mathfrak{m}\mathfrak{h}_R^*$. By construction of μ, we also have $w_1 \cdot \mu - w_2 \cdot \mu \in \mathbb{Z}\Phi$. So, we deduce that $\operatorname{domdim} A_{W_{\overline{\mu}}} = +\infty$ if and only if the cardinality of $\mathscr{D}$ is one.

Assume now that the cardinality of $\mathscr{D}$ is greater than one. By [Fan08, Proposition 4.5], $\operatorname{domdim} A_{W_{\overline{\mu}} \cdot \overline{\mu}} = 2$ with faithful projective-injective $P_A(\omega)(\mathfrak{m}) \simeq P_{A(\mathfrak{m})}(\overline{\omega})$. By Proposition 2.5.4 and Theorem 2.5.13, the result follows. ☐

It follows from Theorem 4.4.48 and Proposition 2.4.7, the analogue of integral Schur–Weyl duality for the blocks of the category $\mathcal{O}$: There is a double centralizer property

$$C := \operatorname{End}_{A_{\mathscr{D}}}(P_A(\omega))^{op}, \quad A_{\mathscr{D}} \simeq \operatorname{End}_C(P_A(\omega))$$

where ω is the antidominant weight of $\mathscr{D}$. Here, $C(\mathfrak{m})$ is the so-called **coinvariant algebra** $S(\mathfrak{h}_{R(\mathfrak{m})})/I$ whenever $|W_{\overline{\mu}} \cdot \overline{\mu}| = |W_{\overline{\mu}}|$ for the block $\mathscr{D} = W_{\overline{\mu}}\mu + \nu$. Here, I denotes the ideal of the symmetric algebra of $\mathfrak{h}_{R(\mathfrak{m})}^*$ generated by the polynomials which are invariants, under the Weyl group linear action, of positive degree with respect to the grading of the symmetric algebra of $\mathfrak{h}_{R(\mathfrak{m})}^*$. In the other cases, where the stabilizer of $\overline{\mu}$ under the Weyl group $W_{\overline{\mu}}$ is non-trivial, the algebra $C(\mathfrak{m})$ is a subalgebra of invariants of the coinvariant algebra under the elements of the stabilizer of $\overline{\mu}$. In particular, $C(\mathfrak{m})$ is a commutative algebra (see [Soe90, Endomorphismensatz]).

Taking advantage of the previous double centralizer property, we can define the Schur functor

$$\mathbb{V}_{\mathscr{D}} = \operatorname{Hom}_{A_{\mathscr{D}}}(P_A(\omega), -) \colon A_{\mathscr{D}}\text{-mod} \to C\text{-mod}.$$

In the literature, this functor is known as **Soergel's combinatorial functor**. The famous result known as **Skrutursatz** [Soe90, Struktursatz 9] states that $\mathbb{V}_{\overline{\mathscr{D}}}\colon A_{\mathscr{D}}(\mathfrak{m})\text{-mod} \to C(\mathfrak{m})\text{-mod}$ is fully faithful on projectives. To see that, in this more general setup, we start by observing that since $P_A(\omega)$ is projective-injective, it is a (partial) tilting module, so it is self-dual with respect to $(-)^\natural$, that is, $P_A(\omega)^\natural \simeq P_A(\omega)$. So, it follows by [FK11b, Proposition 2.4 and Lemma 3.2] that $C(\mathfrak{m}) \simeq DC(\mathfrak{m})$. We will briefly explain the idea: the arguments are based on bookkeeping the twisted actions and realizing that $P_A(\omega)(\mathfrak{m})$ being self dual implies that the isomorphism of $P_A(\omega)(\mathfrak{m})$ to its dual is also an isomorphism of $C(\mathfrak{m})$ under the twisted action. Then, applying $\operatorname{Hom}_{R(\mathfrak{m})}(-,R(\mathfrak{m}))$ one would obtain an isomorphism between $D(A_{\mathscr{D}}e_\omega)(\mathfrak{m})$ and $e_\omega A_{\mathscr{D}}(\mathfrak{m})$ as left $C(\mathfrak{m})$-modules under the usual action. Now, applying the Schur functor we would obtain the desired isomorphism. Since $C \in C\text{-proj}$ we can complete the diagram

$$\begin{tikzcd} C \arrow[r, two heads] & C(\mathfrak{m}) \arrow[d, "\simeq"] \\ DC \arrow[r, two heads] & DC(\mathfrak{m}) \end{tikzcd} \tag{4.4.12.1}$$

by a C-homomorphism $f\colon C \to DC$. Moreover, $f(\mathfrak{m})$ is an isomorphism. Since $C, DC \in R\text{-proj}$ we obtain that f is an isomorphism as C-modules. This shows that C is a relative self-injective R-algebra. Now, using that C is a commutative R-algebra f yields in addition that C is a relative symmetric R-algebra. To see this observe that the action of the center of the enveloping algebra $Z(\mathfrak{g}_R)$ on $P(\omega)$ (ω the antidominant weight of $\mathscr{D}$) yields a homomorphism of R-algebras $Z(\mathfrak{g}_R) \to \operatorname{End}_{A_{\mathscr{D}}}(HP(\omega))$. Further, we have a commutative diagram

$$\begin{tikzcd} Z(\mathfrak{g}_R)(\mathfrak{m}) \arrow[r] \arrow[d, "\simeq"] & \operatorname{End}_{A_{\mathscr{D}}}(HP(\omega))(\mathfrak{m}) \arrow[d, "\simeq"] \\ Z(\mathfrak{g}_{R(\mathfrak{m})}) \arrow[r, two heads] & \operatorname{End}_{A(\mathfrak{m})_{\mathscr{D}}}(\operatorname{Hom}_{\mathscr{O}}(P_{\mathscr{D}}(\mathfrak{m}), P(\overline{\omega})) \end{tikzcd} . \tag{4.4.12.2}$$

Here, the bottom row is surjective due to Soergel [Soe90, Lemma 5], the left map is an isomorphism by Lemma 4.4.24 while the right map is an isomorphism by Theorem 4.4.40. It follows that the upper map is also a surjective map. Denote by X the cokernel (as R-homomorphisms) of the homomorphism $Z(\mathfrak{g}_R) \to \operatorname{End}_{A_{\mathscr{D}}}(HP(\omega))$. Thus, $X(\mathfrak{m}) = 0$. Since $\operatorname{End}_{A_{\mathscr{D}}}(HP(\omega)) \in R\text{-proj}$, we obtain $X \in R\text{-mod}$ and by Nakayama's Lemma $X = 0$. Hence, $Z(\mathfrak{g}_R) \to C$ is surjective, and therefore C is a commutative R-algebra.

To sum up, we obtained:

Theorem 4.4.49. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra with unique maximal ideal $\mathfrak{m}$. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. Suppose that ω is the antidominant weight of $\mathscr{D}$. The following assertions hold.*

(a) $(A_{\mathscr{D}}, P_A(\omega))$ is a relative gendo-symmetric R-algebra.

(b) $A_{\mathscr{D}}$ is split quasi-hereditary over R with standard modules $\Delta_A(\mu)$, $\mu \in \mathscr{D}$.

(c) $A_{\mathscr{D}}$ is a cellular R-algebra with cell modules $\Delta_A(\mu)$, $\mu \in \mathscr{D}$, with respect to the duality map ι.

*(d) **(Integral Struktursatz)** $(A_{\mathscr{D}}, P_A(\omega))$ is a split quasi-hereditary cover of the commutative R-algebra C.*

(e) C is a cellular R-algebra with cell modules $\mathbb{V}_{\mathscr{D}}\Delta_A(\mu)$, $\mu \in \mathscr{D}$, with respect to the duality map $\iota_{|_{e_\omega A_{\mathscr{D}} e_\omega}}$.

(f) If T is a characteristic tilting module of $A_{\mathscr{D}}$, then $2\operatorname{domdim}_{(A_{\mathscr{D}},R)} T = \operatorname{domdim}(A_{\mathscr{D}}, R)$.

Proof. Statements (b) and (c) are Theorem 4.4.43 and 4.4.47, respectively. By Theorem 2.4.10 and 4.4.48, $P(\omega)$ is a generator as C-module and satisfies $P_A(\omega)\otimes_C DP_A(\omega) \in R$-proj. By Theorem 2.10.2(c) and the discussion above showing that C is a relative symmetric R-algebra (a) follows.

By (a), $\operatorname{Hom}_{A_{\mathscr{D}}}(P_A(\omega), A_{\mathscr{D}}) \simeq DP_A(\omega)$. This fact, together with Theorem 4.4.48 implies the existence of a double centralizer property on $\mathbb{V}A_{\mathscr{D}}$. This shows (d). By Proposition 1.6.11, (e) follows. By Theorem 2.11.3, (f) follows. $\square$

4.4.13 Hemmer-Nakano dimension under $\mathbb{V}_{\mathscr{D}}$

Given that $(A_{\mathscr{D}}, P_A(\omega))$ is a cover of the algebra C, the Schur functor $\mathbb{V}_{\mathscr{D}}$ is fully faithful on projectives. There are two natural questions. One might wonder what happens in higher levels of the Schur functor, that is, how fully faithful is on Ext groups regarding the projectives. Since $A_{\mathscr{D}}$ is split quasi-hereditary, the same question can be posed involving the Verma modules. Lets start by discussing the classical case of complex semi-simple Lie algebras. In that case, the Schur functor $\mathbb{V}$ (on a non semisimple block) restricted to the projective modules cannot induce a bijection on the first Ext groups since otherwise the fact that $A_{W_{\overline{\mu}}\cdot\overline{\mu}}$ is a gendo-symmetric algebra would imply an increase in the dominant dimension to at least three.

Now, regarding the Verma modules, the situation in the classical case is not very promising. Indeed, the vector space dimension of $\mathbb{V}\Delta(w\cdot\overline{\mu})$ is equal to the multiplicity of the simple module $\Delta(\overline{\varpi})$ in the standard module $\Delta(w\cdot\overline{\mu}$ for every $w\in W_{\overline{\mu}}$, where $\overline{\varpi}$ is the unique antidominant weight in the orbit. Since the non-zero homomorphisms between Verma modules are always injective $\Delta(\overline{\varpi})$ only occurs in the socle of $\Delta(w\cdot\overline{\mu})$. Therefore, $\dim_{R(\mathfrak{m})}\mathbb{V}\Delta(w\cdot\overline{\mu}) = 1$. Since the Schur functor $\mathbb{V}$ kills all simple modules which are not in the top of the projective module $P_A(\omega)$ then $\mathbb{V}$ sends all standard modules to the same module with dimension one over C. Therefore, $\mathbb{V}$ is not even fully faithful on Verma modules. It is only faithful on Verma modules.

This is the major difference between the classical case and the Noetherian algebras $A_{\mathscr{D}}$ as we will see now.

Theorem 4.4.50. *Fix t a natural number. Let R be the localization of $\mathbb{C}[X_1,\dots,X_t]$ at the maximal ideal $(X_1,\dots,X_t)$. Denote by $\mathfrak{m}$ the unique maximal ideal of R. Pick $\theta\in\mathfrak{h}^*_{R(\mathfrak{m})}\simeq \mathfrak{h}^*_R/\mathfrak{m}\mathfrak{h}^*_R$ to be an antidominant weight which is not dominant. Define $\mu\in\mathfrak{h}^*_R$ to be a preimage of θ without coefficients in $\mathfrak{m}$ in its unique linear combination of simple roots. Fix s to be a natural number satisfying $1\le s\le \operatorname{rank}_R\mathfrak{h}^*_R$ and $s\le t$. Consider the block $\mathscr{D}=W_{\overline{\mu}}\cdot\mu+\frac{X_1}{1}\alpha_1+\cdots+\frac{X_s}{1}\alpha_s$, where $\alpha_i\in\Pi$ are distinct simple roots, $i=1,\dots,s$ and by $\frac{f}{1}$ we mean the image of $f\in\mathbb{C}[X_1,\dots,X_t]$ in R. Then,*

(i) $\operatorname{HNdim}_{\mathbb{V}_{\mathscr{D}}} A_{\mathscr{D}}\text{-proj} = s$;

(ii) $\operatorname{HNdim}_{\mathbb{V}_{\mathscr{D}}} \mathscr{F}(\Delta_A) = s-1$.

Proof. Denote by $\mathfrak{m}$ the maximal ideal of R. Assume that $\alpha_1,\dots,\alpha_n\in\Pi$ are the simple roots of Φ and $\mu=\sum_{i=1}^n c_i\alpha_i$, where c_i is the image of some complex number in R. Denote by ν the weight $\frac{X_1}{1}\alpha_1+\cdots+\frac{X_s}{1}\alpha_s$. By Theorem 4.4.49, $(A_{\mathscr{D}}, P_A(\mu))$ is a split quasi-hereditary cover of C and it is a relative gendo-symmetric R-algebra.

We will start by showing that s and $s-1$ are upper bounds for the Hemmer-Nakano dimension of $A_{\mathscr{D}}$-proj and $\mathscr{F}(\Delta_A)$, respectively, under the Schur functor $\mathbb{V}_{\mathscr{D}}$. Let T be a characteristic tilting module of $A_{\mathscr{D}}$.

Choose $\mathfrak{p}$ the prime ideal of R generated by the monomials $\frac{X_i}{1}$, with $i=1,\dots,s$. In particular, $\mathfrak{p}$ has height s. Further, $R/\mathfrak{p}\otimes_R\mu$ is an antidominant weight which is not dominant and it has no coefficients belonging to the maximal ideal of $R/\mathfrak{p}$ in its unique linear combination of simple roots and $R/\mathfrak{p}\otimes_R\nu=0$. Therefore, $Q(R/\mathfrak{p})\otimes_R\mathscr{D}$ contains $Q(R/\mathfrak{p})\otimes_R\mu$ which is an antidominant but it is not dominant, where $Q(R/\mathfrak{p})$ denotes the quotient field of $R/\mathfrak{p}$. Therefore, $Q(R/\mathfrak{p})\otimes_R A_{\mathscr{D}}$ contains as direct product the algebra $A_{W_{Q(R/\mathfrak{p})\otimes_R\mu}\cdot Q(R/\mathfrak{p})\otimes_R\mu}$

which is not semi-simple. By Theorem 4.4.48, 3.5.6 and the flatness of $Q(R/\mathfrak{p})$ over $R/\mathfrak{p}$,

$$-1 = \operatorname{HNdim}_{\mathbb{V}_{W_{Q(R/\mathfrak{p})\otimes_R\mu}\cdot Q(R/\mathfrak{p})\otimes_R\mu}} \mathscr{F}(Q(R/\mathfrak{p})\otimes_R \Delta_A) \geq \operatorname{HNdim}_{Q(R/\mathfrak{p})\otimes_R \mathbb{V}_{\mathscr{D}}} \mathscr{F}(Q(R/\mathfrak{p})\otimes_R \Delta_A) \tag{4.4.13.1}$$

$$\geq \operatorname{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}_{\mathscr{D}}} \mathscr{F}(R/\mathfrak{p}\otimes_R \Delta_A). \tag{4.4.13.2}$$

$$0 = \operatorname{HNdim}_{\mathbb{V}_{W_{Q(R/\mathfrak{p})\otimes_R\mu}\cdot Q(R/\mathfrak{p})\otimes_R\mu}} A_{W_{Q(R/\mathfrak{p})\otimes_R\mu}\cdot Q(R/\mathfrak{p})\otimes_R\mu}\text{-proj} \geq \operatorname{HNdim}_{Q(R/\mathfrak{p})\otimes_R \mathbb{V}_{\mathscr{D}}} Q(R/\mathfrak{p})\otimes_R A_{\mathscr{D}}\text{-proj}$$

$$\geq \operatorname{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}_{\mathscr{D}}} R/\mathfrak{p}\otimes_R A_{\mathscr{D}}\text{-proj}. \tag{4.4.13.3}$$

As a consequence of Corollary 3.3.10 and thanks to $\operatorname{ht}(\mathfrak{p}) = s$ we obtain

$$\operatorname{HNdim}_{\mathbb{V}_{\mathscr{D}}} \mathscr{F}(\Delta_A) \leq \operatorname{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}_{\mathscr{D}}} \mathscr{F}(R/\mathfrak{p}\otimes_R \Delta_A) + \operatorname{ht}(\mathfrak{p}) = -1+s \tag{4.4.13.4}$$

$$\operatorname{HNdim}_{\mathbb{V}_{\mathscr{D}}} A_{\mathscr{D}}\text{-proj} \leq \operatorname{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}_{\mathscr{D}}} R/\mathfrak{p}\otimes_R A_{\mathscr{D}}\text{-proj} + \operatorname{ht}(\mathfrak{p}) = s. \tag{4.4.13.5}$$

We claim that this inequality is actually an equality. To show that we will proceed by induction on the coheight of prime ideals $\mathfrak{p}$ of R, that is, on $\dim R - \operatorname{ht}(\mathfrak{p})$ with induction basis step $t-s$ to show that

$$\operatorname{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}} \mathscr{F}(R/\mathfrak{p}\otimes_R \Delta_A) \geq -1+s-\operatorname{ht}(\mathfrak{p})$$

and

$$\operatorname{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}_{\mathscr{D}}} R/\mathfrak{p}\otimes_R A_{\mathscr{D}}\text{-proj} \geq s-\operatorname{ht}(\mathfrak{p}).$$

Let $\mathfrak{p}$ be a prime ideal of R with coheight $t-s$, then it has height s. Since $R/\mathfrak{p}$ has maximal ideal $\mathfrak{m}/\mathfrak{p}$ with residue field $R(\mathfrak{m})$ the claim follows by Theorem 3.5.6, Theorem 4.4.49(f) and Theorem 4.4.48.

Now assume that $\mathfrak{p}$ is a prime ideal of R with coheight greater than $t-s$. Then, $\mathfrak{p}$ has height smaller than s. In particular, $\mathfrak{p}$ cannot contain any prime ideal with height s. Consequently, some monomial $\frac{X_i}{1}$ has non-zero image in $R/\mathfrak{p}$. Moreover, ν has non-zero image in $R/\mathfrak{p}$ and its image belongs to $\mathfrak{m}/\mathfrak{p}$. Therefore, any weight in $R/\mathfrak{p}\otimes_R \mathscr{D}$ when viewed as weight in the quotient field $\mathfrak{h}^*_{Q(R/\mathfrak{p})}$ does not belong to the integral weight lattice. Thus, all weights belonging to $R/\mathfrak{p}\otimes_R \mathscr{D}$ viewed as weights in $\mathfrak{h}^*_{Q(R/\mathfrak{p})}$ are both dominant and antidominant. By the discussion in Theorem 4.4.48, we obtain that

$$\operatorname{domdim} Q(R/\mathfrak{p})\otimes_R A_{\mathscr{D}} = \operatorname{domdim}_{Q(R/\mathfrak{p})\otimes_R A_{\mathscr{D}}} Q(R/\mathfrak{p})\otimes_R T = +\infty. \tag{4.4.13.6}$$

By Theorem 3.5.8, we obtain that the claim holds for prime ideals with coheight $t-s+1$.

Upon these considerations, assume the induction claim known for some prime ideal with coheight $t-s+r$ with $r\geq 1$. Let $\mathfrak{p}$ be a prime ideal of coheight $t-s+r+1$. Then, $\operatorname{ht}(\mathfrak{p}) = t-t+s-1 = s-r-1 < s$ and (4.4.13.6) holds. By Theorem 3.5.8 the assumptions of Theorem 3.3.13 for $R/\mathfrak{p}\otimes_R A_{\mathscr{D}}$ and the resolving subcategories $\mathscr{F}(R/\mathfrak{p}\otimes_R \Delta_A)$ and $R/\mathfrak{p}\otimes_R A_{\mathscr{D}}$ are satisfied. Also, condition (i) of Theorem 3.3.13 is also satisfied. It remains to consider (ii). Let $\mathfrak{q}$ be a prime ideal of $R/\mathfrak{p}$ of height one. Then, we can write $\mathfrak{q} = \mathfrak{q}'/\mathfrak{p}$ for some prime ideal $\mathfrak{q}'$ of R. Furthermore,

$$1 = \operatorname{ht}(\mathfrak{q}'/\mathfrak{p}) = \dim(R/\mathfrak{p}) - \operatorname{coht}(\mathfrak{q}'/\mathfrak{p}) = \operatorname{coht}(\mathfrak{p}) - \operatorname{coht}(\mathfrak{q}'), \tag{4.4.13.7}$$

where the symbol $\operatorname{coht}(\mathfrak{p})$ denotes the coheight of the prime ideal $\mathfrak{p}$. Hence, $\operatorname{coht}(\mathfrak{q}') = \operatorname{coht}(\mathfrak{p}) - 1 = t-s+r$.

By induction,

$$\operatorname{HNdim}_{R/\mathfrak{q}'\otimes_R \mathbb{V}} \mathscr{F}(R/\mathfrak{q}'\otimes_R \Delta_A) \geq -1+s-\operatorname{ht}(\mathfrak{q}') = -1+r \tag{4.4.13.8}$$

$$\operatorname{HNdim}_{R/\mathfrak{q}'\otimes_R \mathbb{V}_{\mathscr{D}}} R/\mathfrak{q}'\otimes_R A_{\mathscr{D}}\text{-proj} \geq s-\operatorname{ht}(\mathfrak{q}') = r. \tag{4.4.13.9}$$

Because of $R/\mathfrak{q}' \simeq R/\mathfrak{p}/\mathfrak{q}'/\mathfrak{p} = R/\mathfrak{p}/\mathfrak{q}$ Theorem 3.3.13 yields

$$\mathrm{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}} \mathscr{F}(R/\mathfrak{p} \otimes_R \Delta_A) \geq r \tag{4.4.13.10}$$

$$\mathrm{HNdim}_{R/\mathfrak{p}\otimes_R \mathbb{V}_{\mathscr{D}}} R/\mathfrak{p} \otimes_R A_{\mathscr{D}}\text{-proj} \geq r+1 \tag{4.4.13.11}$$

This finishes the proof of the claim.

Now considering the prime ideal zero which has height zero, the result follows. □

Hence, for the algebras $A_{\mathscr{D}}$ not only the Schur functor is fully faithful on Verma modules but also the Schur functor behaves quite well on Ext groups of Verma modules. This fact alone justifies studying the category $\mathscr{O}$ under other rings than the complex numbers.

Remark 4.4.51. The non-zero homomorphisms between distinct Verma modules are injective maps but they are not $(A_{\mathscr{D}}, R)$-monomorphisms, in general. Otherwise, we would obtain a (C,R)-monomorphism $\mathbb{V}\Delta(\omega_1) \to \mathbb{V}(\Delta(\omega_2))$ which remains injective under $R(\mathfrak{m}) \otimes_R -$. $\mathbb{V}\Delta(\omega_2)(\mathfrak{m})$ is a simple module, hence the mentioned map must be an isomorphism and by Nakayama's Lemma, so is the map $\mathbb{V}\Delta(\omega_1) \to \mathbb{V}(\Delta(\omega_2))$. By Theorem 4.4.50, we can choose rings R for which such a situation cannot happen. △

4.5 Comparison between Hemmer-Nakano dimension, dominant dimension and Krull dimension

Assume now that R is regular Noetherian commutative local ring. Much focus on Chapter 2 was given to illustrate that the dominant dimension should be measured using Tor groups instead of Ext groups. The reason for this was that the Krull dimension of regular local rings is an obstruction to deduce information on vanishing of Ext groups. By Proposition 2.4.18 for any module $M \in A\text{-mod} \cap R\text{-proj}$ we could obtain the interval

$$n \geq \mathrm{domdim}_{(A,R)} M \geq n - \dim R, \tag{4.5.0.1}$$

where n is the optimal value making $\mathrm{Ext}^i_C(V, V \otimes_A M) = 0$ and $\alpha_M \colon M \to \mathrm{Hom}_C(V, V \otimes_A M)$ an isomorphism for $1 \leq i \leq n$. This leads to the following question: How much information do we lose using Ext groups to compute the relative dominant dimension of a module? To answer this question we can view it from the point of view of covers. For example, fix M to be the regular module A. This question can be translated using Theorem 3.5.6 to how much a cover (A,P) improves with respect to a cover $(A(\mathfrak{m}), P(\mathfrak{m}))$. Further, the interval (4.5.0.1) becomes

$$\mathrm{domdim}(A,R) - 2 \leq \mathrm{HNdim}_F A\text{-proj} \leq \mathrm{domdim}(A,R) + \dim R - 2 \tag{4.5.0.2}$$

Now, knowing what happens with Schur algebras, BBG category $\mathscr{O}$ and Integral Auslander algebras of $R[X]/(X^n)$ we can see that both of these bounds cannot be improved in general since every the relative dominant dimension and the Hemmer-Nakano dimension can take any value in the above intervals. In a nutshell, the behaviour of the Hemmer-Nakano dimension on:

BGG Category $\mathscr{O}$ over R

- Depends on the Krull dimension;
- Depends on the rank of the Cartan subalgebra of the semisimple algebra under study;
- $\mathrm{HNdim}_F A$-proj can take any value in (4.5.0.2) provided the Cartan subalgebra of $\mathfrak{g}$ is big enough.

Schur algebras $S_R(n,d)$ with $n \geq d$

- Depends on whether R contains a field or not;
- $1 \leq \operatorname{domdim}(S_R(n,d),R) - \operatorname{HNdim}_F S_R(n,d)\text{-proj} \leq 2$.

Integral Auslander algebras of $R[X]/(X^n)$

- Does not depend of R;
- $\operatorname{domdim}(A,R) - 2 = \operatorname{HNdim}_F A\text{-proj}$.

4.6 Further examples

Example 4.6.1. *For an algebraically closed field K with characteristic three, the Hemmer-Nakano dimension of $S_{K[X_1,\cdots,X_r]}(n,d)$-proj is two while $\operatorname{domdim} S_{K[X_1,\cdots,X_r]}(n,d)$ is four.*

Let K be an algebraically closed field with characteristic three. Let $r > 0$ be an integer and $n \geq d \geq 3$. Fix $R = K[X_1,\ldots,X_r]$. By Hilbert's Nullstellensatz Theorem, the maximal ideals of R are of the form

$$\mathfrak{m}_{a_1,\ldots,a_r} = (X_1 - a_1,\ldots,X_r - a_r), \quad a_1,\ldots,a_r \in K. \tag{4.6.0.1}$$

Hence, $R/\mathfrak{m}_{a_1,\ldots,a_r} \simeq K$ for every maximal ideal of R. Thus,

$$\operatorname{domdim}(S_R(n,d),R) = \inf\{\operatorname{domdim} S_R(n,d)(\mathfrak{m}_{a_1,\ldots,a_r}) \colon a_1,\ldots,a_r \in K\} \tag{4.6.0.2}$$

$$= \inf\{\operatorname{domdim} S_K(n,d) \colon a_1,\ldots,a_r \in K\} = 2(3-1) = 4. \tag{4.6.0.3}$$

Hence, $(S_R(n,d),(R^n)^{\otimes d})$ is a 2-$S_R(n,d)$-proj cover of RS_d. Assume, by contradiction, that $(S_R(n,d),(R^n)^{\otimes d})$ is a 3-$S_R(n,d)$-proj cover of RS_d. As $K(X_1,\ldots,X_r)$ is flat over R, this would imply that $(S_{K(X_1,\ldots,X_r)}(n,d),(K(X_1,\ldots,X_r)^n)^{\otimes d})$ is a 3-$S_{K(X_1,\ldots,X_r)}(n,d)$-proj cover of $K(X_1,\ldots,X_r)S_d$. But $K(X_1,\ldots,X_r)$ has Krull dimension zero, so, by Proposition 2.4.18, this implies that $\operatorname{domdim} S_{K(X_1,\ldots,X_r)}(n,d)$ is at least 5. On the other hand, $K(X_1,\ldots,X_r)$ is a field with characteristic three. By Theorem 5.1 of [FK11b], $\operatorname{domdim} S_{K(X_1,\ldots,X_r)}(n,d)$ is exactly 4. Therefore, $(S_R(n,d),(R^n)^{\otimes d})$ cannot be a 3-$S_R(n,d)$-proj cover of RS_d. △

Example 4.6.2. (Example 15 of [Cru21]) *For a QF3 algebra (A,P,DP), the pair (A,P) might not be a cover of $\operatorname{End}_A(P)^{op}$.*

Let K be an algebraically closed field. Let A be the following bound quiver K-algebra

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3, \quad \alpha_2\alpha_1 = 0.$$

Denote by $P(i)$ the projective indecomposable module associated with the vertex i and denote by $I(i)$ the indecomposable injective module associated with the vertex i.

The indecomposable projective (left) modules are given by

$$P(1) = I(2) = \begin{matrix} 1 \\ 2 \end{matrix} \qquad P(2) = I(3) = \begin{matrix} 2 \\ 3 \end{matrix} \qquad P(3) = \ 3\ . \tag{4.6.0.4}$$

The exact sequence

$$0 \to A \to P(1) \oplus P(2) \oplus P(2) \to P(1) \to I(1) \to 0 \tag{4.6.0.5}$$

is a minimal injective resolution of A. Denote by P the projective module $P(1)\oplus P(2)$. Hence, (A,P,DP) is a QF3 algebra with $\operatorname{domdim} A\geq 2$. So, $(A,P(2)\oplus P(3))$ is a cover of $\operatorname{End}_A(P)^{op}$. In fact, $P(2)\oplus P(3)\simeq \operatorname{Hom}_A(DA,P)=\operatorname{Hom}_A(DP,A)$ as left A-modules. Here $B=\operatorname{End}_A(P)^{op}$ is the path algebra with quiver

$$1 \xrightarrow{\alpha_1} 2.$$

However, (A,P) is not a cover of B. To see this, observe that

$$\operatorname{Hom}_A(P,A)=\begin{matrix}1\\2\end{matrix}\oplus 2 \tag{4.6.0.6}$$

as B-modules. Thus, $\operatorname{End}_B(\operatorname{Hom}_A(P,A))^{op}$ has only K-dimension 3 whereas the dimension of $A=\operatorname{End}_B(P)$ is 5. △

Example 4.6.3. *If $n\geq d\geq 2$, the Hemmer-Nakano dimension of $S_{\mathbb{Z}[X]}(n,d)$-proj is one while $\operatorname{domdim}(S_{\mathbb{Z}[X]}(n,d),\mathbb{Z}[X])$ is two.*

Let $n\geq d\geq 2$. The ideal generated by 2 and X over $\mathbb{Z}[X]$ is maximal. So,

$$2=\operatorname{domdim} S_{\mathbb{F}_2}(n,d)=\operatorname{domdim} S_{\mathbb{Z}[X]}(n,d)(2,X)\geq \operatorname{domdim}(S_{\mathbb{Z}[X]}(n,d),\mathbb{Z}[X])\geq 2. \tag{4.6.0.7}$$

On the other hand, $\mathbb{Q}(X)$ is a field of characteristic zero flat over $\mathbb{Z}[X]$. By Theorem 3.5.7, $(S_{\mathbb{Z}[X]}(n,d),(\mathbb{Z}[X]^n)^{\otimes d})$ is a 1-cover of $\mathbb{Z}[X]S_d$. Assume, by contradiction, that $(S_{\mathbb{Z}[X]}(n,d),(\mathbb{Z}[X]^n)^{\otimes d})$ is a 2-cover of $\mathbb{Z}[X]S_d$. $\mathbb{Z}[X]2$ is a prime projective ideal of $\mathbb{Z}[X]$ with

$$\mathbb{Z}[X]/\mathbb{Z}[X]2\simeq \mathbb{F}_2[X]. \tag{4.6.0.8}$$

By Theorem 3.3.9, $(S_{\mathbb{F}_2[X]}(n,d),(\mathbb{F}_2[X]^n)^{\otimes d})$ is a 1-cover of $\mathbb{F}_2[X]S_d$. Now, $\mathbb{F}_2(X)$ is a field of characteristic two flat over $\mathbb{F}_2[X]$. Hence, $(S_{\mathbb{F}_2(X)}(n,d),(\mathbb{F}_2(X)^n)^{\otimes d})$ is a 1-cover of $\mathbb{F}_2(X)S_d$. In particular, $\operatorname{domdim} S_{\mathbb{F}_2(X)}(n,d)$ is at least 3. This is a contradiction with Theorem 5.1 of [FK11b]. Therefore, the Hemmer-Nakano dimension of $S_{\mathbb{Z}[X]}(n,d)$-proj is 1. △

Example 4.6.4. Assume the notation of Theorem 3.3.12. *Given a regular local ring R with maximal ideal $\mathfrak{m}$, knowing one system of R-parameters together with the values of $\operatorname{HNdim}_{R/Rx_i\otimes_R F}(\mathscr{R}(R/Rx_i\otimes_R A))$ is not enough to determine $\operatorname{HNdim}_F(\mathscr{R}(A))$. More precisely, it can happen that $\operatorname{HNdim}_{R/Rx_i\otimes_R F}(\mathscr{R}(R/Rx_i\otimes_R A))>\operatorname{HNdim}_{R/Rx\otimes_R F}(\mathscr{R}(R/Rx\otimes_R A))$, $i=1,\dots,n$, where $x\in\mathfrak{m}$ and $\{x_1,\dots,x_n\}$ generate $\mathfrak{m}$.*

Let R_0 be the localization of $\mathbb{Z}_3[X_1,X_2]$ at the maximal ideal $(3,X_1,X_2)$. Let $\overline{f}$ be the image of $f\in\mathbb{Z}_3[X_1,X_2]$ in R_0. Define $R=R_0/(\overline{-3+(X_1+X_2)^2})$. Denote by T_i the image of $\overline{X_i}$ in R, $i=1,2$. Then, T_1,T_2 is a system of R-parameters. By Theorem 4.1.12,

$$\operatorname{HNdim}_F(S_R(3,3)\text{-proj})=3,\quad \operatorname{HNdim}_{R/T_i\otimes_R F}(S_{R/T_i}(3,3)\text{-proj})=3,\ i=1,2. \tag{4.6.0.9}$$

However, if $\operatorname{HNdim}_{R/x\otimes_R F}(S_{R/x}(3,3)\text{-proj})=3$ for any $x\in\mathfrak{m}$, then by the proof of Theorem 3.3.11, $\operatorname{HNdim}_F(S_R(3,3)\text{-proj})$ would be bigger than 3. This is, of course, false. Hence, there exists $x\in\mathfrak{m}$ such that $\operatorname{HNdim}_{R/x\otimes_R F}(S_{R/x}(3,3)\text{-proj})=2$. △

The following example indicates that given a collection of covers of an algebra, evaluating the covers based only on the value of global dimension is not sufficient to select the cover with better properties.

Example 4.6.5. *Given an algebraically closed field with characteristic three K, the group algebra KS_3 admits two distinct 2-covers.* By [Xi92], the algebra KS_3 is Morita equivalent to the path algebra of the quiver

$$2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1$$

modulo the ideal generated by

$$\alpha\beta\alpha, \quad \beta\alpha\beta. \tag{4.6.0.10}$$

In particular, KS_3 is of finite type. The indecomposable projective (left) modules are given by

$$P(1) = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix}, \quad P(2) = \begin{matrix} 2 \\ 1 \\ 2 \end{matrix}. \tag{4.6.0.11}$$

The endomorphism algebras $\operatorname{End}_{KS_3}(KS_3 \bigoplus 1)^{op}$ and $\operatorname{End}_{KS_3}(KS_3 \bigoplus 2)^{op}$ are Morita equivalent to the Schur algebra $S_K(3,3)$. In particular, $S_K(3,3)$ is Morita equivalent to the path algebra of the following quiver

$$3 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 2 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 1$$

modulo the ideal generated by

$$\beta_2\alpha_2,\ \alpha_1\alpha_2,\ \beta_2\beta_1,\ \beta_1\alpha_1 - \alpha_2\beta_2. \tag{4.6.0.12}$$

The indecomposable projective (left) $S_K(3,3)$-modules are given by

$$P(1) = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix}, \quad P(2) = \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix}, \quad P(3) = \begin{matrix} 3 \\ 2 \end{matrix}. \tag{4.6.0.13}$$

The indecomposable injective (left) $S_K(3,3)$-modules are given by

$$P(1) = I(1), \quad I(2) = P(2), \quad I(3) = \begin{matrix} 2 \\ 3 \end{matrix}. \tag{4.6.0.14}$$

The module $(K^3)^{\otimes 3}$ is regarded as $P(1) \oplus P(2)$. Since

$$0 \to S_K(3,3) \to P(1) \oplus P(2) \oplus P(2) \to P(1) \to P(1) \to P(2) \to I(3) \to 0 \tag{4.6.0.15}$$

is a minimal injective resolution of the regular module $S_K(3,3)$ we obtain that $\operatorname{domdim} S_K(3,3) = 4$.

Now, using the following projective resolutions

$$0 \to P(3) \to P(2) \to P(1) \oplus P(3) \to P(2) \to 1 \to 0 \tag{4.6.0.16}$$

$$0 \to P(3) \to P(2) \to P(1) \to 2 \to 0 \tag{4.6.0.17}$$

$$0 \to 1 \to P(3) \to 3 \to 0 \tag{4.6.0.18}$$

we obtain that $\operatorname{gldim} S_K(3,3) = 4$.

On the other hand, the endomorphism algebra $\operatorname{End}_{KS_3}\left(KS_3 \oplus \begin{matrix} 1 \\ 2 \end{matrix}\right)^{op}$ is the path algebra of the following quiver

$$\begin{tikzcd} & 3 \arrow[dl, "\alpha"'] & \\ 1 \arrow[rr, "\beta"'] & & 2 \arrow[ul, "\gamma"'] \end{tikzcd}$$

modulo the ideal generated by

$$\gamma\beta\alpha, \quad \beta\alpha\gamma\beta. \tag{4.6.0.19}$$

We will denote this algebra by C. The indecomposable projective (left) C-modules are given by

$$P(1) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array}, \quad P(2) = \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array}, \quad P(3) = \begin{array}{c} 3 \\ 1 \\ 2 \end{array}, \tag{4.6.0.20}$$

whereas the indecomposable injective C-modules are given by

$$I(1) = P(1), \quad P(2) = I(2), \quad I(3) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}. \tag{4.6.0.21}$$

The regular module C has minimal injective resolution

$$0 \to C \to P(2) \oplus P(1) \oplus P(2) \to P(2) \to P(1) \to I(3) \to 0. \tag{4.6.0.22}$$

Thus, $\operatorname{domdim} C = 4$. Using the exact sequences

$$0 \to P(3) \to P(2) \to 2 \to 0 \tag{4.6.0.23}$$

$$0 \to 2 \to P(2) \to P(1) \to 1 \to 0 \tag{4.6.0.24}$$

$$2 \to P(3) \to P(1) \to P(3) \to 3 \to 0 \tag{4.6.0.25}$$

we conclude that $\operatorname{gldim} C = 4$. In addition, C is not split quasi-hereditary. Assume, by contradiction, that it is split quasi-hereditary. The only projective module that can be standard is $P(3)$. Hence, 3 must be maximal. This implies that $\Delta(2)$ must be 2 and $\Delta(1)$ is a quotient of $\begin{array}{c} 1 \\ 2 \end{array}$. But, $P(1)$ does not have a filtration by these candidates to be standard modules. Thus, C cannot be split quasi-hereditary. $\triangle$

Remark 4.6.6. For our purposes, and according to Proposition 2.3.6 and Lemma 2.2.4, over finite-dimensional algebras we can ignore the multiplicities of $V^{\otimes^d}$ throughout this chapter as we did in the last example. $\triangle$

Example 4.6.7. *Let A be the Auslander algebra of $\overline{\mathbb{F}_3}S_3$, where $\operatorname{Hom}_A(P,A) = \bigoplus_{i>0} \overline{\mathbb{F}_3}S_3/\operatorname{rad}^i \overline{\mathbb{F}_3}S_3$. (A,P) is a (-1)-$\mathscr{F}(\Delta_A)$ cover of $\overline{\mathbb{F}_3}S_3$. On the other hand, $(S_{\overline{\mathbb{F}_3}}(3,3), V^{\otimes 3})$ is a 0-$\mathscr{F}(\Delta)$ cover of $\overline{\mathbb{F}_3}S_3$ and*

$$\mathscr{F}(\operatorname{Hom}_A(P,\Delta_A)) = \mathscr{F}(F\Delta), \tag{4.6.0.26}$$

where $F = \operatorname{Hom}_{S_{\overline{\mathbb{F}_3}}(3,3)}(V^{\otimes 3}, -)$ and $V = \overline{\mathbb{F}_3}^3$. Furthermore, the algebra A is a gendo-symmetric quasi-hereditary algebra without a simple preserving duality. Denote by K the field $\overline{\mathbb{F}_3}$. As we have seen $S_K(3,3)$ is Morita equivalent to the bound quiver algebra defined in (4.6.5). The standard modules of $S_K(3,3)$ are

$$\Delta(1) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ ,\ \Delta(2) = 1\ ,\ \Delta(3) = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\ , \tag{4.6.0.27}$$

with the usual order $3 > 2 > 1$. The costandard modules of $S_K(3,3)$ are

$$\nabla(1) = 1\ ,\ \nabla(2) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\ ,\ \nabla(3) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\ . \tag{4.6.0.28}$$

Applying the Schur functor, we see that the Specht modules of KS_3 are

$$\theta(1) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ ,\ \theta(2) = 1\ ,\ \theta(3) = 2\ . \tag{4.6.0.29}$$

Since $\operatorname{domdim}_{S_K(3,3)} = 4$, $(S_K(3,3), V^{\otimes 3})$ is a 0-$\mathscr{F}(\Delta)$ cover of KS_3. Moreover, under the functor $G = \operatorname{Hom}_{KS_3}(V^{\otimes 3}, -)$ we can see that $G2 = P(3)$ and $G1 = 1$.

Now, the Auslander algebra of KS_3, $A = \operatorname{End}_{KS_3}\left(\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 2\right)^{op}$, is isomorphic to the following bound quiver algebra

$$\begin{tikzcd}
 & & 5 \arrow[dr, "\alpha_2"] \arrow[drr, "\beta_4"] & & \\
3 \arrow[urr, "\beta_3"] & 2 \arrow[ur, "\alpha_1"] & & 6 \arrow[dl, "\alpha_3"] & 1 \arrow[dll, "\beta_1"] \\
 & & 4 \arrow[ul, "\alpha_4"] \arrow[ull, "\beta_2"] & &
\end{tikzcd}\ ,\quad \begin{aligned} \alpha_2\beta_3 = \beta_2\alpha_3 = \beta_1\beta_4\beta_3 = \beta_2\beta_1\beta_4 = 0, \\ \beta_3\beta_2 = \alpha_1\alpha_4,\ \alpha_3\alpha_2 = \beta_1\beta_4 \end{aligned} \tag{4.6.0.30}$$

The projective A-modules are given by

$$P_A(1) = \begin{array}{ccc} & 1 & \\ & | & \\ & 4 & \\ \diagup & & \diagdown \\ 2 & & 3 \\ \diagdown & & \diagup \\ & 5 & \\ & | & \\ & 1 & \end{array},\ P_A(2) = \begin{array}{ccc} & 2 & \\ & | & \\ & 5 & \\ \diagup & & \diagdown \\ 1 & & 6 \\ \diagdown & & \diagup \\ & 4 & \\ & | & \\ & 2 & \end{array},\ P_A(3) = \begin{array}{c} 3 \\ | \\ 5 \\ | \\ 1 \end{array},\ P_A(4) = \begin{array}{ccc} & 4 & \\ \diagup & & \diagdown \\ 2 & & 3 \\ \diagdown & & \diagup \\ & 5 & \\ & | & \\ & 1 & \end{array}, \tag{4.6.0.31}$$

$$P_A(5) = \begin{array}{ccc} & 5 & \\ \diagup & & \diagdown \\ 6 & & 1 \\ \diagdown & & \diagup \\ & 4 & \\ & | & \\ & 2 & \end{array},\quad P_A(6) = \begin{array}{c} 6 \\ | \\ 4 \\ | \\ 2 \end{array}. \tag{4.6.0.32}$$

The injective A-modules are given by

$$I_A(1) = \begin{array}{ccc} & 1 & \\ & | & \\ & 4 & \\ / & & \backslash \\ 2 & & 3 \\ \backslash & & / \\ & 5 & \\ & | & \\ & 1 & \end{array}, \; I_A(2) = \begin{array}{ccc} & 2 & \\ & | & \\ & 5 & \\ / & & \backslash \\ 1 & & 6 \\ \backslash & & / \\ & 4 & \\ & | & \\ & 2 & \end{array}, \; I_A(3) = \begin{array}{c} 1 \\ | \\ 4 \\ | \\ 3 \end{array}, \; I_A(4) = \begin{array}{ccc} & 2 & \\ & | & \\ & 5 & \\ / & & \backslash \\ 1 & & 6 \\ \backslash & & / \\ & 4 & \end{array}, \tag{4.6.0.33}$$

$$I_A(5) = \begin{array}{ccc} & 1 & \\ & | & \\ & 4 & \\ / & & \backslash \\ 2 & & 3 \\ \backslash & & / \\ & 5 & \end{array}, \quad I_A(6) = \begin{array}{c} 2 \\ | \\ 5 \\ | \\ 6 \end{array}. \tag{4.6.0.34}$$

The standard A-modules, with the ordering $3,6 > 4 > 2 > 5 > 1$ making A a split quasi-hereditary algebra, are

$$\Delta_A(1) = \; 1 \;, \; \Delta_A(2) = \begin{array}{c} 2 \\ 5 \\ 1 \end{array}, \; \Delta_A(3) = P_A(3), \; \Delta_A(4) = \begin{array}{c} 4 \\ 2 \end{array}, \; \Delta_A(5) = \begin{array}{c} 5 \\ 1 \end{array}, \; \Delta_A(6) = P_A(6). \tag{4.6.0.35}$$

In fact, there are short exact sequences

$$0 \to \Delta_A(3) \to P_A(4) \to \Delta_A(4) \to 0 \tag{4.6.0.36}$$

$$0 \to \Delta_A(6) \to P_A(1) \to \Delta_A(1) \to 0 \tag{4.6.0.37}$$

$$0 \to \Delta_A(6) \to P_A(5) \to \Delta_A(5) \to 0 \tag{4.6.0.38}$$

and the radical of $P_A(1)$ has a filtration by $\Delta_A(4)$ and $\Delta_A(3)$. The minimal projective-injective A-module $P = P_A(1) \oplus P_A(2)$. Under the Schur functor $F_A = \mathrm{Hom}_A(P, -)$, we obtain

$$F_A\Delta_A(1) = 1, \; F_A\Delta_A(2) = \theta(1), \; F_A\Delta_A(3) = 1, \; F_A\Delta_A(4) = 2, \; F_A\Delta_A(5) = 1, \; F_A\Delta_A(6) = 2. \tag{4.6.0.39}$$

Therefore, $\mathscr{F}(F_A\Delta_A) = \mathscr{F}(F\Delta)$. By the minimal injective resolution of A

$$0 \to A \to P_A(1) \oplus P_A(2) \oplus P_A(2) \oplus P_A(1) \oplus P_A(1) \oplus P_A(2) \tag{4.6.0.40}$$

$$\to P_A(1) \oplus P_A(2) \oplus P_A(1) \oplus P_A(2) \to I_A(3) \oplus I_A(6) \oplus I_A(5) \oplus I_A(4) \to 0, \tag{4.6.0.41}$$

we conclude that $\mathrm{domdim} A = 2$. Let G_A be the right adjoint of F_A. By projectivization, $G_A\theta(1) = P_A(4)$, $G_A(1) = P_A(3)$, $G_A(2) = P_A(6)$. Thus, $\mathrm{domdim}\,\Delta_A = 1$. Therefore, (A, P) is a (-1)-$\mathscr{F}(\Delta_A)$ cover of KS_3.

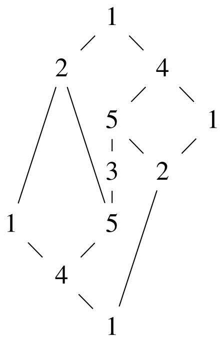
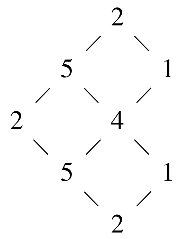
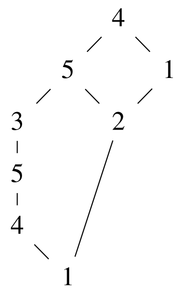
Since KS_3 is a symmetric algebra, A is a gendo-symmetric algebra. If, A had a duality preserving simples then such duality would send $P_A(4)$ to $I_A(4)$. This cannot happen since the simple modules appearing in $P_A(4)$ and $I_A(4)$ are not the same. $\triangle$

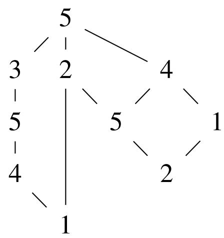
Example 4.6.8. *KS_4 has at least two (-1)-faithful covers (including the Schur algebra), where K is a field of characteristic two. But, contrary to Example 4.6.7, only the standard modules of the Schur algebra are sent to the complete set of cell modules of KS_4.*

By Theorem 3.5 of [Xi93], the Schur algebra $S_K(4,4)$ is Morita equivalent to the bound quiver algebra

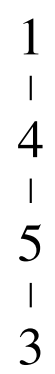
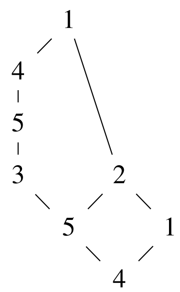
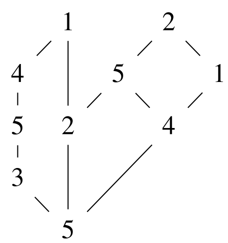
$$\begin{tikzcd} & & 4 \arrow[dl, shift left, "\beta_1"] \arrow[dr, shift left, "\gamma"] & \\ 3 \arrow[r, shift left, "\alpha"] & 5 \arrow[l, shift left, "\alpha_1"] \arrow[ur, shift left, "\beta"] \arrow[dr, shift left, "\varepsilon_1"] & & 1 \arrow[ul, shift left, "\gamma_1"] \arrow[dl, shift left, "\sigma"] \\ & & 2 \arrow[ul, shift left, "\varepsilon"] \arrow[ur, shift left, "\sigma_1"] & \end{tikzcd}, \qquad \begin{gathered} \alpha_1\alpha = \varepsilon_1\alpha = \gamma_1\gamma = \beta\beta_1 = \sigma\sigma_1 = \alpha_1\varepsilon = 0 \\ \sigma\gamma = \varepsilon_1\beta_1, \quad \varepsilon\varepsilon_1 = \beta_1\beta, \quad \beta\varepsilon = \gamma_1\sigma_1, \\ \gamma\beta\alpha\alpha_1 = \sigma_1\varepsilon_1, \quad \alpha\alpha_1\beta_1\gamma_1 = \varepsilon\sigma. \end{gathered} \tag{4.6.0.42}$$

The indecomposable projective $S_K(4,4)$-modules are given by

$P(1) =$  , $P(2) =$  , $P(3) =$  , $P(4) =$  , (4.6.0.43)

$P(5) =$  . (4.6.0.44)

The indecomposable injective $S_K(4,4)$-modules are given by

$I(1) = P(1)$, $I(2) = P(2)$, $I(3) =$  , $I(4) =$  , $I(5) =$  . (4.6.0.45)

Using the minimal injective resolution

$$0 \to S_K(4,4) \to P(1) \oplus P(2) \oplus P(1) \oplus P(1) \oplus P(1) \oplus P(2) \to P(1)^4 \oplus P(2)^2 \to C_2 \to 0, \tag{4.6.0.46}$$

we see that domdim $S_K(4,4) = 2$ since the socle of C_2 contains 4. Similarly, we can see that the global dimension of $S_K(4,4)$ is 6. $S_K(4,4)$ is split quasi-hereditary with $1<2<5<4<3$ and the standard modules

$$\Delta(1)=1,\ \Delta(2)=\begin{array}{c}2\\ \mid\\ 1\end{array},\ \Delta(3)=P(3),\ \Delta(4)=\begin{array}{ccccc} & & 4 & & \\ & / & & \backslash & \\ 5 & & & & 1\\ & \backslash & & / & \\ & & 2 & & \end{array},\ \Delta(5)=\begin{array}{c}5\\ \mid\\ 2\end{array}. \tag{4.6.0.47}$$

From the point of view of the dominance order, we can assign 1 to the partition 1+1+1+1, 2 to 2+1+1, 3 to 4+0, 4 to 3+1 and 5 to 2+2. The respective costandard modules are

$$\nabla(1)=1,\ \nabla(2)=\begin{array}{c}1\\ \mid\\ 2\end{array},\ \nabla(3)=I(3),\ \nabla(4)=\begin{array}{ccccc} & & 2 & & \\ & / & & \backslash & \\ 5 & & & & 1\\ & \backslash & & / & \\ & & 4 & & \end{array},\ \nabla(5)=\begin{array}{c}2\\ \mid\\ 5\end{array}. \tag{4.6.0.48}$$

The indecomposable (partial) tilting $S_K(4,4)$-modules are

$$T(1)=\Delta(1),\ T(2)=\begin{array}{c}1\\ \mid\\ 2\\ \mid\\ 1\end{array},\ T(3)=P(1),\ T(4)=P(2),\ T(5)=\begin{array}{ccccc} & & 2 & & \\ & / & \mid & & \\ 1 & & 5 & & 1\\ & & \mid & / & \\ & & 2 & & \end{array}. \tag{4.6.0.49}$$

The module $P(1)\oplus P(2)$ is the minimal faithful projective-injective module, hence $V^{\otimes d}$ is regarded as $P(1)\oplus P(2)$ as left $S_K(4,4)$-modules. In addition, we can see that $\operatorname{domdim}\Delta(3)=\operatorname{domdim}\Delta(4)=\operatorname{domdim}\Delta(5)=2$ and $\operatorname{domdim}\Delta(1)=\operatorname{domdim}\Delta(2)=1$. Applying the Schur functor $F=\operatorname{Hom}_{S_K(4,4)}(V^{\otimes d},-)$ we obtain that the group algebra KS_4 is Morita equivalent to the bound quiver algebra

$$\gamma\gamma_1 \circlearrowright 1 \underset{\sigma_1}{\overset{\sigma}{\rightleftarrows}} 2 \circlearrowleft \varepsilon_1\varepsilon, \qquad \begin{array}{c}(\gamma\gamma_1)^2=(\varepsilon_1\varepsilon)\sigma=\sigma_1(\varepsilon_1\varepsilon)=\sigma\sigma_1=0,\\ (\varepsilon_1\varepsilon)^2=\sigma(\gamma\gamma_1)\sigma_1,\quad \sigma_1\sigma(\gamma\gamma_1)=(\gamma\gamma_1)\sigma_1\sigma,\end{array} \tag{4.6.0.50}$$

with projective KS_4-modules

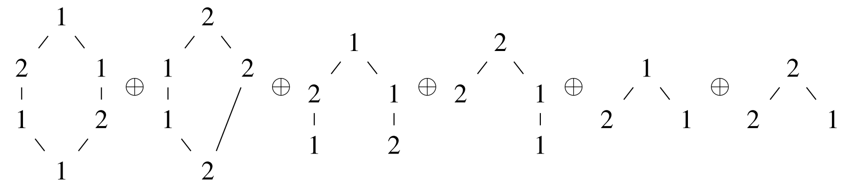
$$P_{S_4}(1)=\begin{array}{ccccc} & & 1 & & \\ & / & & \backslash & \\ 1 & & & & 2\\ \mid & & & & \mid\\ 2 & & & & 1\\ & \backslash & & / & \\ & & 1 & & \end{array},\quad P_{S_4}(2)=\begin{array}{ccccc} & & 2 & & \\ & / & & \backslash & \\ 1 & & & & \\ \mid & & & & 2\\ 1 & & & & \\ & \backslash & & / & \\ & & 2 & & \end{array}. \tag{4.6.0.51}$$

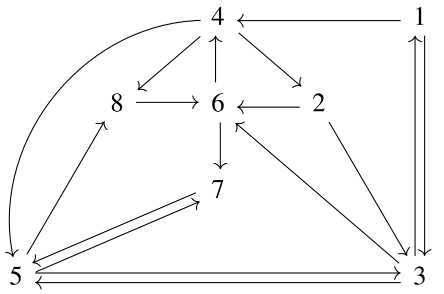
The generator $DV^{\otimes 4}$ is the module

$$P_{S_4}(1)\oplus P_{S_4}(2)\oplus 1\oplus \begin{array}{c}1\\ \mid\\ 2\\ \mid\\ 1\end{array}\oplus \begin{array}{ccc}2 & & 1\\ \mid & \backslash & \mid\\ 1 & & 2\end{array}. \tag{4.6.0.52}$$

Applying the Schur functor, we see that the cell modules of KS_4 are the following:

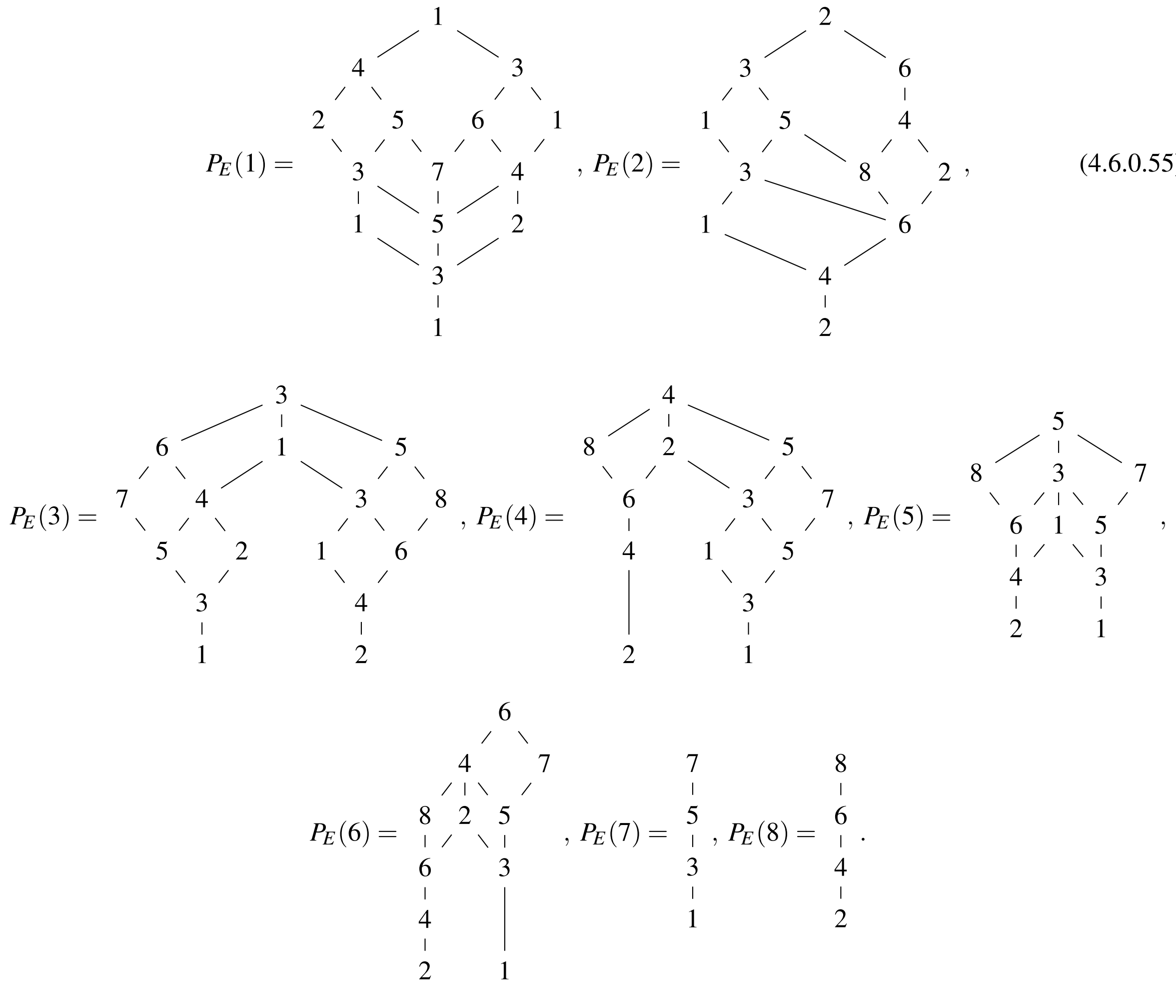
$$\theta(1)=1,\ \theta(2)=\begin{array}{c}2\\ \mid\\ 1\end{array},\theta(3)=1,\ \theta(4)=\begin{array}{c}1\\ \mid\\ 2\end{array},\ \theta(5)=2. \tag{4.6.0.53}$$

Let $M =$  $\oplus 1 \oplus 2$. Consider the algebra $E := \mathrm{End}_{KS_4}(M)^{op}$ with quiver



(4.6.0.54)

and projective modules



(4.6.0.55)

Since M is a generator over KS_4, E has dominant dimension at least two and $P_E(1) \oplus P_E(2)$ is the minimal faithful projective-injective (left) E-module. E is split quasi-hereditary with the usual ordering and the following standard modules

$$\Delta_E(1)=1,\ \Delta_E(2)=2,\ \Delta_E(3)=\begin{matrix}3\\1\end{matrix},\ \Delta_E(4)=\begin{matrix}4\\2\end{matrix},\ \Delta_E(5)=\begin{matrix}5\\3\\1\end{matrix},\ \Delta_E(6)=\begin{matrix}6\\4\\2\end{matrix},\ \Delta_E(7)=P_E(7),\ \Delta_E(8)=P_E(8).$$

Applying the Schur functor $F_E := \operatorname{Hom}_E(P_E(1)\oplus P_E(2),-)$ to the standard modules we obtain as KS_4-modules

$$F_E\Delta_E(1)=F_E\Delta_E(3)=F_E\Delta_E(5)=F_E\Delta_E(7)=1,\quad F_E\Delta_E(2)=F_E\Delta_E(4)=F_E\Delta_E(6)=F_E\Delta_E(8)=2. \tag{4.6.0.56}$$

Hence, by (4.6.0.53) $\mathscr{F}(F\Delta)$ is different from $\mathscr{F}(F_E\Delta_E)$. Let G_E be the right adjoint of the Schur functor F_E. Then, $G_E(1)=P_E(7)$ and $G_E(2)=P_E(8)$. So, the canonical map $\Delta_E\to G_EF_E\Delta_E$ is a monomorphism but not an isomorphism for $\Delta_E=\oplus_{i=1,\dots,8}\Delta_E(i)$. Thus, $(E,P_E(1)\oplus P_E(2))$ is a (-1) faithful quasi-hereditary cover of KS_4. △

Example 4.6.9. *Not every split quasi-hereditary cover is a (-1)-faithful quasi-hereditary cover.* Recall the bound quiver algebra A defined in Example 4.6.2. A is quasi-hereditary for the canonical order $3>2>1$ with the simple modules being the standard modules. Consider the Schur functor $F=\operatorname{Hom}_A(P(2\oplus P(3),-)\colon A\text{-mod}\to C\text{-mod}$ and consider $G=\operatorname{Hom}_C(2\oplus \begin{matrix}2\\3\end{matrix}\oplus 3,-)$, where C is the bound quiver algebra

$$3 \xleftarrow{\alpha} 2\ . \tag{4.6.0.57}$$

We can see that

$$G3=P(3),\ G(2)=P(1),\ G(\begin{matrix}2\\3\end{matrix})=P(2). \tag{4.6.0.58}$$

Therefore, $(A,P(2)\oplus P(3))$ is a cover of C. But $F\Delta(1)=0$. Thus, $\eta_{\Delta(1)}$ is the zero map. So, $(A,P(2)\oplus P(3))$ is not a (-1)-faithful cover of C. △

Denote by $\Lambda^+(d,d,p)$ the set of p-regular partitions of d and $(R,\mathfrak{m})$ a discrete valuation ring. In [CPS96, (4.6.4)] Cline, Parshall and Scott showed that there exists an exact equivalence between the full subcategory $\mathscr{F}(\Delta_{\lambda\in\Lambda^+(d,d,p)})$ of $S_R(d,d)$-mod and $\mathscr{F}(F\Delta_{\lambda\in\Lambda^+(d,d,p)})$, where F is the Schur functor and $p=\operatorname{char}R(\mathfrak{m})$. The following example shows that this exact equivalence cannot be improved for the Schur algebra $S_{\mathbb{Z}_2}(5,5)$ in the sense that this equivalence cannot be extended to a bigger (in terms of inclusion) full subcategory of $\mathscr{F}(\Delta)$. In view of Corollary 3.3.14 it is enough to see that the standard modules of the Schur algebra $S_{\mathbb{F}_2}(5,5)$ associated to partitions which are not 2-regular have dominant dimension one.

Example 4.6.10. *For $n=d=5$, denote by $\Lambda^+(5,5,2)$ the set of 2-regular partitions of 5. All standard modules of $S_{\mathbb{F}_2}(5,5)$, $\Delta(\lambda)$ with $\lambda\in\Lambda^+(5,5,2)$, have dominant dimension one.* By Proposition 3.8 of [Xi93], the basic algebra of the Schur algebra $S_K(5,5)$ is the bound quiver algebra

$$1 \underset{\alpha_1}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta_1}{\overset{\beta}{\rightleftarrows}} 3 \underset{\gamma_1}{\overset{\gamma}{\rightleftarrows}} 4 \underset{\sigma_1}{\overset{\sigma}{\rightleftarrows}} 5 \qquad 6 \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} 7 \tag{4.6.0.59}$$

$$0=\alpha_1\alpha=\beta_1\beta=\gamma_1\gamma=\sigma\sigma_1=\eta\varepsilon=\sigma\gamma\beta\alpha=\alpha_1\beta_1\gamma_1\sigma_1,\quad \gamma_1\sigma_1\sigma\gamma=\beta\beta_1,$$
$$\gamma\beta\alpha\alpha_1=\sigma_1\sigma\gamma\beta,\alpha\alpha_1\beta_1\gamma_1=\beta_1\gamma_1\sigma_1\sigma, \tag{4.6.0.60}$$

where K is an algebraically closed field of characteristic two. Denote by $P(i)$ the projective indecomposable modules. The basic module of $V^{\otimes 5}$ is $P(4)\oplus P(5)\oplus P(6)$. The standard modules of the basic algebra of $S_K(5,5)$ are

$$\Delta(1)=\begin{matrix}1\\2\\3\\4\end{matrix}\ ,\ \Delta(2)=\begin{matrix}2\\3\\4\\5\end{matrix}\ ,\ \Delta(3)=\begin{matrix}3\\4\\5\\4\end{matrix}\ ,\ \Delta(4)=\ 4\ ,\ \Delta(5)=\begin{matrix}5\\4\end{matrix}\ ,\ \Delta(6)=\ 6\ ,\ \Delta(7)=\begin{matrix}7\\6\end{matrix}\ . \tag{4.6.0.61}$$

In particular, $1>2>3>5>4, 7>6$ is the order associated with this quasi-hereditary structure. Therefore, we have the following correspondence between $\{1,\dots,7\}$ and the partitions of 5:

$$1\leftrightarrow 5+0,\ 2\leftrightarrow 3+2,\ 3\leftrightarrow 3+1^2,\ 4\leftrightarrow 1+1+1+1+1,\ 5\leftrightarrow 2+2+1,\ 6\leftrightarrow 2+1+1+1,\ 7\leftrightarrow 4+1 \tag{4.6.0.62}$$

Finally, we can see that

$$\operatorname{domdim}\Delta(1)=\operatorname{domdim}\Delta(2)=\operatorname{domdim}\Delta(7)=2 \tag{4.6.0.63}$$
$$\operatorname{domdim}\Delta(3)=\operatorname{domdim}\Delta(4)=\operatorname{domdim}\Delta(5)=\operatorname{domdim}\Delta(6)=1. \tag{4.6.0.64}$$

$\triangle$

Example 4.6.11. *Truncating a cover might produce a new cover with higher Hemmer-Nakano dimension.* Consider the Schur algebra of finite type A with quiver

$$1 \rightleftarrows 2 \rightleftarrows 3\ , \tag{4.6.0.65}$$

and with projective indecomposables

$$P(1)=\begin{matrix}1\\ \mid\\ 2\\ \mid\\ 1\end{matrix}\ ,\ P(2)=\begin{matrix} & 2 & \\ \swarrow & & \searrow\\ 1 & & 3\\ \searrow & & \swarrow\\ & 2 & \end{matrix}\ ,\ P(3)=\begin{matrix}3\\ \mid\\ 2\end{matrix}\ . \tag{4.6.0.66}$$

A is quasi-hereditary with standard modules $\Delta(3)=P(3)$, $\Delta(2)=\begin{matrix}2\\ \mid\\ 1\end{matrix}$, $\Delta(1)=1$. Fix $P=P(2)\oplus P(3)$. Then, (A,P) is a 0-$\mathscr{F}(\Delta)$ cover of $\operatorname{End}_A(P)^{op}$. Let J be the split heredity ideal associated with $\Delta(3)$. Then, $P/JP\simeq A/J$ as left A/J-modules. Of course, $(A/J,P/JP)$ is an $+\infty$ cover of $\operatorname{End}_{A/J}(P/JP)^{op}$. $\triangle$

Example 4.6.12. *The cover property is not preserved under arbitrary truncations.* By A we will denote the principal block of the basic algebra of $S_K(5,5)$ where $\operatorname{char}K=2$. Let $J=A(e_1+e_2)A$. Then, the algebra A/J is isomorphic to the following bound quiver algebra

$$3 \underset{\gamma_1}{\overset{\gamma}{\rightleftarrows}} 4 \underset{\sigma_1}{\overset{\sigma}{\rightleftarrows}} 5\ , \sigma\sigma_1=\gamma_1\gamma=\gamma_1\sigma_1\sigma\gamma=0. \tag{4.6.0.67}$$

By Example 4.6.10, $(A,P(4)\oplus P(5))$ is a cover of the bound quiver algebra

$$\gamma\gamma_1 \circlearrowright 4 \underset{\sigma_1}{\overset{\sigma}{\rightleftarrows}} 5\ ,\quad \begin{aligned}0=\sigma\sigma_1=\gamma\gamma_1\gamma\gamma_1=\sigma\sigma_1\gamma\gamma_1\sigma_1\sigma\gamma\gamma_1\sigma_1=\sigma\gamma\gamma_1\sigma_1\sigma\gamma\gamma_1\sigma_1\sigma,\\ \sigma_1\sigma\gamma\gamma_1\sigma_1\sigma\gamma\gamma_1=\gamma\gamma_1\sigma_1\sigma\gamma\gamma_1\sigma_1\sigma.\end{aligned}$$

Further, for $P = P(4) \oplus P(5)$, $B_J := \operatorname{End}_{A/J}(P/JP)^{op}$ is a quotient of $\operatorname{End}_A(P(4) \oplus P(5))^{op}$ with projective modules

$$\begin{array}{c} 5 \\ | \\ 4 \\ | \\ 4 \\ | \\ 5 \\ | \\ 4 \end{array} \bigoplus \begin{array}{ccc} & 4 & \\ / & & \backslash \\ 4 & & 5 \\ | & & | \\ 5 & & 4 \\ | & & | \\ 4 & & 4 \\ & & | \\ & & 5 \\ & & | \\ & & 4 \end{array} .$$

We can see that $\operatorname{End}_{B_J}\left(B_J \oplus \begin{array}{c} 4 \\ 5 \\ 4 \end{array}\right)^{op}$ has infinite global dimension. Therefore, this endomorphism algebra cannot be isomorphic to A/J. This shows that $(A/J, P/J)$ is not a cover of B_J. △

Remark 4.6.13. By computing the Iyama generator of the regular module of KS_5, when $\operatorname{char} K$ is 2, we can see that $V^{\otimes 5}$ does not belong to the additive closure of the Iyama generator. Thus, we cannot expect to construct a cover by just selecting summands of the Iyama generator. △

Example 4.6.14. *There are cellular algebras that their cell structure is not given by a quasi-hereditary cover. More precisely, for a given cellular algebra B with cell datum (Λ, M, C, ι) there is not, in general, a split quasi-hereditary cover (A, P) satisfying* $\operatorname{Hom}_A(P, \Delta(\lambda)) = \theta_\lambda, \lambda \in \Lambda$.

Let K be an algebraically closed field with characteristic different from two. By Proposition 3.4 of [AKMW20], the bound quiver algebra, denoted by B,

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] \arrow[loop above, "\varepsilon"] & 3 \arrow[l, shift left, "\gamma"] \end{tikzcd}, \quad \beta\alpha = \delta\gamma = \varepsilon\alpha = \beta\varepsilon = \varepsilon\gamma = \delta\varepsilon = 0, \quad \alpha\delta = \varepsilon^2 = \gamma\beta \tag{4.6.0.68}$$

is a cellular self-injective algebra with poset $\Lambda = \{1,2,3,4,5\}$ together with the reversed order $5 < 4 < 3 < 2 < 1$ and

$$M(i) = \begin{cases} \{1\}, & \text{if } i \in \{1,3,5\}, \\ \{1,2\}, & \text{if } i \in \{2,4\}. \end{cases} \tag{4.6.0.69}$$

Note that in [AKMW20] their definition of cellular algebras uses the reversed order of the original definition of cellular algebras. Denote by $P(i)$ the projective indecomposable module associated with the primitive idempotent e_i and put $S(i) = \operatorname{top} P(i)$. Then,

$$P(1) = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \quad P(2) = \begin{array}{ccccc} & & 2 & & \\ & / & | & \backslash & \\ 2 & & 1 & & 3 \\ & \backslash & | & / & \\ & & 2 & & \end{array}, \quad P(3) = \begin{array}{c} 3 \\ 2 \\ 3 \end{array}. \tag{4.6.0.70}$$

The cell modules are

$$\theta_1 = \begin{matrix} 3 \end{matrix}, \quad \theta_2 = \begin{matrix} 2 \\ 3 \end{matrix}, \quad \theta_3 = \begin{matrix} 2 \end{matrix}, \quad \theta_4 = \begin{matrix} 1 \\ 2 \end{matrix}, \quad \theta_5 = \begin{matrix} 1 \end{matrix}. \tag{4.6.0.71}$$

Assume that there exists a quasi-hereditary cover (A,P) such that $\operatorname{Hom}_A(P,\Delta(i)) = \theta_i$. Note that the order of the quasi-hereditary algebra in A must be $5 > 4 > 3 > 2 > 1$. From now on, this is the order that we consider. Further, there are exact sequences

$$0 \to X_i \to Y_i \to \theta_i \to 0, \quad X_i \in \mathscr{F}(\theta_{j>i}), \tag{4.6.0.72}$$

for all $i = 1,\ldots,5$. Here, $A = \operatorname{End}_B(\oplus_{i=1}^5 Y_i)$, and Y_i are indecomposable modules. Hence, it is clear that Y_5 must be θ_5 and Y_4 must be $P(1)$. In the same way, Y_1 must be $P(3)$. Since $\operatorname{rad} P(2) \notin \mathscr{F}(\theta_3,\theta_4,\theta_5)$ we must have $Y_2 = P(2)$. It remains to consider Y_3. Observe that $X_3 \in \mathscr{F}(\theta_4,\theta_5)$ and θ_5 appears always at the bottom of the filtration. So, if both θ_4 and θ_5 are in the filtration of X_3, then $P(1)$ is a summand of Y_3 which cannot happen. So, either $X_3 \in \mathscr{F}(\theta_5)$ or $X_3 \in \mathscr{F}(\theta_4)$. But since 5 is maximal, $\mathscr{F}(\theta_5) = \operatorname{add}_B \theta_5$. If $Y_3 = \theta_3$, then we can see that the quiver of A has a loop on the vertex 3, and therefore A cannot be split quasi-hereditary. In the remaining cases, we can see that the Cartan matrix of A(with entry i,j equal to $\dim_K \operatorname{Hom}_A(P_A(j),P_A(i)) = \dim_K \operatorname{Hom}_B(Y_j,Y_i)$) has determinant equal to

$$\det \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & \dim_K \operatorname{Hom}_B(Y_3,Y_2) & 1 & 0 \\ 0 & \dim_K \operatorname{Hom}_B(Y_2,Y_3) & \dim_K \operatorname{Hom}_B(Y_3,Y_3) & \dim_K \operatorname{Hom}_B(Y_4,Y_3) & \dim_K \operatorname{Hom}_B(Y_5,Y_3) \\ 0 & 1 & \dim_K \operatorname{Hom}_B(Y_3,Y_4) & 2 & 1 \\ 0 & 0 & \dim_K \operatorname{Hom}_B(Y_3,Y_5) & 1 & 1 \end{bmatrix} =$$

$$= \det \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & [Y_3 : S(2)] & 1 & 0 \\ 0 & [Y_3 : S(2)] & \dim_K \operatorname{Hom}_B(Y_3,Y_3) & [Y_3 : S(1)] & [\operatorname{soc} Y_3 : S(1)] \\ 0 & 1 & [Y_3 : S(1)] & 2 & 1 \\ 0 & 0 & [\operatorname{top} Y_3 : S(1)] & 1 & 1 \end{bmatrix} \tag{4.6.0.73}$$

In any of this cases this value is always bigger than one. So, A cannot be quasi-hereditary. △

Example 4.6.15. *A Schur functor can preserve a resolving subcategory while not being an equivalence.*

Fix $\mathfrak{g}$ to be the complex semisimple Lie algebra $\mathfrak{sl}_2$. Denote by $\mathbb{V}$ the Schur functor between the principal block of the category $\mathscr{O}$, that is the block that contains the simple module with highest weight zero, and the coinvariant algebra C. Then, $\mathbb{V}$ sends all standard modules to the unique simple module of C. Also, it sends all costandard modules to the unique simple module of C. Hence, $\mathscr{F}(F\Delta) = \mathscr{F}(F\nabla) = C\text{-mod}$. But, $\mathbb{V}$ is not even fully faithful on the standard modules. △

Chapter 5

Cocovers and relative codominant dimension with respect to a module

It is now appropriate to look back what we have done so far. So far, our aim was to study covers (A, P) and their quality. In particular, we studied the cover $(S_R(n,d), V^{\otimes d})$ and the Hemmer-Nakano dimension of $\mathscr{F}(\Delta)$ (and of $S_R(n,d)$-proj) among other cases. On the latter part, extending the notion of dominant dimension to Noetherian algebras was crucial. We should emphasize that in all these cases $n \geq d$. The situation for $n < d$ seems to be more mysterious. For $n < d$, the $S_R(n,d)$-module $V^{\otimes d}$ still has the double centralizer property (see for example [Cru19, Theorem 3.4] and [BD09]) but it might not be projective-injective anymore (see for example [KSX01, 3.3] or Example 6.2.7). In [Fan14], Fang shows that the dominant dimension of $S_K(n,d)$, in case $n < d \leq n(\operatorname{char} K - 1)$, is at least two but $V^{\otimes d}$ is not necessarily the projective-injective module over $S_K(n,d)$. Our attempt here will be to understand what happens to $V^{\otimes d}$ rather than computing the dominant dimension of $S_R(n,d)$ for $n < d$ (see [Fan14]). As in the case $n \geq d$, the additional properties of this double centralizer property highly depend on the ground ring. For example, if R is a field of characteristic zero, the double centralizer property between $S_R(n,d)$ and $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})$ comes from a Morita equivalence. So, we would like to attach a measure of quality to this double centralizer property in a similar way to the Hemmer-Nakano dimension and to replace the notion of cover by one suitable for this situation. In [KSX01], they consider a notion of relative dominant dimension of $S_K(n,d)$ (with K being a field) with respect to the tilting module $V^{\otimes d}$. However, there is no version of the Mueller theorem for this dominant dimension. Another approach was taken in [BS98] by introducing the notion of faithful dimension of a module. The relative dominant dimension with respect to a module that we will study in this chapter generalizes this notion although we will not use minimal approximations (see also Definition 2.3.5). We will start by looking for a notion analogue to covers for general double centralizer properties.

Let A be a projective Noetherian R-algebra, where R is a commutative ring. For the moment, assume that $P \in A$-proj. The study of the Schur functor $F = \operatorname{Hom}_A(P, -)$ was made through the derived functor of the right adjoint $G = \operatorname{Hom}_B(FA, -)$, with $B = \operatorname{End}_A(P)^{op}$. Recall that F also has a left adjoint $\mathbb{I} = P \otimes_B -$ by Tensor-Hom adjunction:

$$\begin{tikzcd} A\text{-mod} \arrow[rr, "F"] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & \mathrm{End}_A(P)^{op}\text{-mod} \arrow[ll, bend right=40, "\mathbb{I}"'] \arrow[ll, bend left=40, "G"] \end{tikzcd} \tag{5.0.0.1}$$

The existence of a right adjoint to F is only guaranteed for the case when $P \in A$-proj. In fact, left adjoint functors preserve cokernels. So, we should now focus our attention to the left adjoint of F when P is not necessarily projective. Another fact that we should consider is that for $P \in A$-proj it is clear that $DP \otimes_A P \in R$-proj.

Although we have not mentioned it directly the fact that F is a right adjoint has appeared before in Chapter 2. Indeed, F being a right adjoint is the reason of the appearance of Tor in relative dominant dimension (with respect to a projective relative injective module). Moreover, for covers, the interest lies in which level G fails to be exact on certain subcategories. Turning towards $\mathbb{I}$, it is only natural to be interested in the exactness of the left adjoint of F, $\mathbb{I}$. It is therefore this direction that we will take in this chapter.

5.1 Cocovers

Once again, unless stated otherwise R is a Noetherian commutative ring, A is a projective Noetherian R-algebra. Let $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying $DQ \otimes_A Q \in R$-proj. By F_Q (or just F when no confusion arises) we mean the functor $\mathrm{Hom}_A(Q, -)\colon A\text{-mod} \to B\text{-mod}$, where B is the endomorphism algebra $\mathrm{End}_A(Q)^{op}$. In particular, $B \in R$-proj and B-mod is an abelian category. By $\mathbb{I}_Q$ (or just $\mathbb{I}$ when no confusion arises) we mean the left adjoint of F, $Q \otimes_B -\colon B\text{-mod} \to A\text{-mod}$. We will denote by υ the unit $\mathrm{id}_{B\text{-mod}} \to F\mathbb{I}$ and χ the counit $\mathbb{I}F \to \mathrm{id}_{A\text{-mod}}$. Thus, for any $N \in B$-mod, υ_N is the B-homomorphism $\upsilon_N\colon N \to \mathrm{Hom}_A(Q, Q \otimes_B N)$, given by $\upsilon_N(n)(q) = q \otimes n$, $n \in N, q \in Q$. For any $M \in A$-mod, χ_M is the A-homomorphism $Q \otimes_B \mathrm{Hom}_A(Q, M) \to M$, given by $\chi_M(q \otimes g) = g(q)$, $g \in FM, q \in Q$. By projectivization, the restriction of F to $\operatorname{add} Q$ gives an equivalence between $\operatorname{add} Q$ and B-proj. Moreover, χ_M is an isomorphism for every $M \in \operatorname{add} Q$. If Q has no self-extensions, then this equivalence is exact. Further, there are commutative diagrams

$$\begin{tikzcd} Q \otimes_B \mathrm{Hom}_A(Q, X \oplus Y) \arrow[r, "\chi_{X\oplus Y}"] & X \oplus Y \\ Q \otimes_B \mathrm{Hom}_A(Q, X) \oplus Q \otimes_B \mathrm{Hom}_A(Q, Y) \arrow[u, "\simeq"] \arrow[r, "\chi_X \oplus \chi_Y"] & X \oplus Y \arrow[u, equal] \end{tikzcd}, \tag{5.1.0.1}$$

and

$$\begin{tikzcd} M \oplus N \arrow[r, "\upsilon_{M\oplus N}"] & \mathrm{Hom}_A(Q, Q \otimes_B (M \oplus N)) \\ M \oplus N \arrow[u, equal] \arrow[r, "\upsilon_M \oplus \upsilon_N"] & \mathrm{Hom}_A(Q, Q \otimes_B M) \oplus \mathrm{Hom}_A(Q, Q \otimes_B N) \arrow[u, "\simeq"] \end{tikzcd}, \tag{5.1.0.2}$$

for every $X, Y \in A$-mod, $M, N \in B$-mod.

Remark 5.1.1. Note that, for each $M \in A\text{-mod} \cap R\text{-proj}$, the map χ_M is equivalent with the map δ_{DM} given in Lemma 2.4.5 when $Q = P$ is a projective (A, R)-injective-strongly faithful module. $\triangle$

We shall write χ^r and υ^r for the counit and unit, respectively, of the adjunction $- \otimes_B DQ \dashv \mathrm{Hom}_A(DQ, -)$.

We will now present a version of Lemma 1.4.28 for general double centralizer properties. This result is in some capacity already known in the literature for Artinian algebras (see for example [AS93, Corollary 2.4]).

Lemma 5.1.2. *Let A be a projective Noetherian R-algebra. Let $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying $DQ \otimes_A Q \in R\text{-proj}$ and denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. The following assertions are equivalent.*

(i) The canonical map of algebras $A \to \operatorname{End}_B(Q)^{op}$, given by $a \mapsto (q \mapsto aq)$, is an isomorphism.

(ii) $D\chi_X$ is an isomorphism of right A-modules for all $X \in (A,R)\text{-inj} \cap R\text{-proj}$.

(iii) The restriction of F to $\operatorname{add} DA$ is full and faithful.

Proof. We will start by showing the equivalence $(i) \Leftrightarrow (ii)$. Denote by ψ the canonical map of algebras $A \to \operatorname{End}_B(Q)^{op}$ and denote by ω_X the natural transformation between the identity functor and the double dual DD for $X \in A$-mod. There is a commutative diagram

$$\begin{tikzcd}
DDA \arrow[r, "D\chi_{DA}"] & D(Q \otimes_B \operatorname{Hom}_A(Q,DA)) \arrow[d, "\theta", "\simeq"'] \\
& \operatorname{Hom}_B(Q, D\operatorname{Hom}_A(Q,DA)) \\
& \operatorname{Hom}_B(Q,DDQ) \arrow[u, "\operatorname{Hom}_B(Q,\sigma_Q)"', "\simeq"] \\
A \arrow[uuu, "\omega_A"', "\simeq"] \arrow[r, "\psi"] & \operatorname{End}_B(Q) \arrow[u, "\operatorname{Hom}_A(Q,\omega_Q)"', "\simeq"]
\end{tikzcd} \tag{5.1.0.3}$$

with vertical maps being isomorphisms, where $\sigma_Q \colon DDQ \to D\operatorname{Hom}_A(Q,DA)$ is given by $h \mapsto (f \mapsto h(f(-)(1_A)))$, and θ is the isomorphism given by Tensor-Hom adjunction. In fact,

$$\begin{aligned}
\operatorname{Hom}_B(Q,\sigma_Q) \circ \operatorname{Hom}_A(Q,\omega_Q) \circ \psi(a)(q)(f) &= \sigma_Q \circ \operatorname{Hom}_A(Q,\omega_Q)(\psi(a))(q)(f) = \sigma_Q \circ \omega_Q \circ \psi(a)(q)(f) \\
&= \sigma_Q(\omega_Q(aq))(f) = \omega_Q(aq)(f(-)(1_A)) = f(aq)(1_A) \\
&= af(q)(1_A) = f(q)(1_A a), \\
\theta \circ D\chi_{DA} \circ \omega_A(a)(q)(f) &= \theta(\omega_A(a) \circ \chi_{DA})(q)(f) = \omega_A(a) \circ \chi_{DA}(q \otimes f) = \omega_A(a)(f(q)) \\
&= f(q)(a), \forall a \in A,\ q \in Q,\ f \in \operatorname{Hom}_A(Q,DA).
\end{aligned}$$

By (5.1.0.3), $D\chi_{DA}$ is bijective if and only if ψ is bijective. Taking into account that χ commutes with direct sums, the implication $(i) \Leftrightarrow (ii)$ follows.

Assume that (iii) holds. Therefore, the map $\operatorname{Hom}_A(DA,DA) \to \operatorname{Hom}_B(FDA,FDA)$, given by $f \mapsto Ff = \operatorname{Hom}_A(Q,f)$, is bijective. Denote such map by $F_{DA,DA}$. We can fit ψ into the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\zeta", "\simeq"'] \arrow[d, "\psi"] & \operatorname{Hom}_A(A,A) \arrow[r, "\psi_{A,A}", "\simeq"'] & \operatorname{Hom}_A(DA,DA) \arrow[d, "F_{DA,DA}"] \\
\operatorname{Hom}_B(Q,Q) \arrow[r, "\psi_{Q,Q}", "\simeq"'] & \operatorname{Hom}_B(DQ,DQ) \arrow[r, "\kappa", "\simeq"'] & \operatorname{Hom}_B(FDA,FDA)
\end{tikzcd}. \tag{5.1.0.4}$$

Here, $\psi_{A,A}$ and $\psi_{Q,Q}$ are the isomorphisms provided by Proposition 1.1.64, ζ is given by $\zeta(a)(b) = ab$, $a,b \in A$ and $\kappa = s^{-1} \circ - \circ s$, where s is the isomorphism $\operatorname{Hom}_A(Q,DA) \to DQ$ given by Tensor-Hom adjunction. The diagram (5.1.0.4) is commutative since

$$F_{DA,DA} \circ \psi_{A,A} \circ \zeta(a)(g)(q)(x) = \operatorname{Hom}_A(Q, \psi_{A,A} \circ \zeta(a))(g)(q)(x) = \psi_{A,A} \circ \zeta(a) \circ g(q)(x) \tag{5.1.0.5}$$

$$= \psi_{A,A}(\zeta(a))(g(q))(x) = g(q) \circ \zeta(a)(x) = g(q)(ax), \tag{5.1.0.6}$$

$$\kappa \circ \psi_{Q,Q} \circ \psi(a)(g)(q)(x) = s^{-1} \circ \psi_{Q,Q}(\psi(a))s(g)(q)(x) = \psi_{Q,Q}(\psi(a))(s(g))(xq) \tag{5.1.0.7}$$
$$= s(g) \circ \psi(a)(xq) = s(g)(axq) = g((ax)q)(1_A) \tag{5.1.0.8}$$
$$= axg(q)(1_A) = g(q)(ax), \forall a,x \in A,\ q \in Q,\ g \in FDA. \tag{5.1.0.9}$$

It follows that $\kappa \circ \psi_{Q,Q} \circ \psi$ is bijective. As $\kappa \circ \psi_{Q,Q}$ is bijective, so it is ψ. This shows (i). Assume now that (ii) holds. Let $M,N \in \operatorname{add} DA$. We want to prove that the map $f \mapsto Ff$, $f \in \operatorname{Hom}_A(M,N)$ is bijective. Let $f \in \operatorname{Hom}_A(M,N)$ such that $Ff = 0$. Then, $f \circ \chi_M = 0$, and therefore $Df = 0$ since $D\chi_M$ is an isomorphism. Now thanks to M being projective over R, $DDf = 0$ implying that $f = 0$. So, the desired map is injective. Let $g \in \operatorname{Hom}_B(FM,FN)$. Define $h = D\chi_M^{-1} \circ D(Q \otimes_B g)D\chi_N \in \operatorname{Hom}_A(DN,DM)$ and consider $\psi_{M,N}\colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(DN,DM)$ the isomorphism as given in Proposition 1.1.64. Define $f = \psi_{M,N}^{-1}(h) \in \operatorname{Hom}_A(M,N)$. (iii) follows once we show that $Ff = g$. To do that we shall compute $\omega_N(\operatorname{Hom}_A(Q,\psi_{M,N}^{-1}(h)(t)(q))$ for every $t \in \operatorname{Hom}_A(Q,M)$, $q \in Q$. Let $l \in DN$. Then,

$$\omega_N(\operatorname{Hom}_A(Q,\psi_{M,N}^{-1}(h))(t)(q))(l) = l(\psi_{M,N}^{-1}(h) \circ t(q)) = h(l) \circ t(q) = D\chi_M^{-1} \circ D(Q \otimes_B g)(l \circ \chi_N)(t(q))$$
$$= D\chi_M^{-1}(l \circ \chi_N \circ Q \otimes_B g) \circ \chi_M(q \otimes t) = D\chi_M(D\chi_M^{-1}(l \circ \chi_N \circ Q \otimes_B g))(q \otimes t)$$
$$= l \circ \chi_N \circ Q \otimes_B g(q \otimes t) = l(g(t)(q)) = \omega_N(g(t)(q))(l) \tag{5.1.0.10}$$

Since ω_N is bijective and q and t are arbitrary it follows that $Ff = g$. This concludes the proof. ☐

Remark 5.1.3. The assumption $DQ \otimes_A Q \in R$-proj is only used in Lemma 5.1.2 to deduce that B is a projective Noetherian R-algebra, so the argument provided also gives the result if we drop such a condition. As we will see later on, this condition is true in all cases of interest that we will consider. △

Definition 5.1.4. Let R be a Noetherian commutative ring and let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$. We say that (A,Q) is a **cocover** of $B := \operatorname{End}_A(Q)^{op}$ if the following holds:

(a) $DQ \otimes_A Q \in R$-proj;

(b) The restriction of $F = \operatorname{Hom}_A(Q,-)\colon A\text{-mod} \to B\text{-mod}$ to $(A,R)\text{-inj} \cap R\text{-proj}$ is full and faithful.

Remark 5.1.5. The notion of cocover generalizes the notion of double centralizer property to category theory. △

Combining Lemmas 5.1.2, 1.4.28 and Proposition 3.5.13, we arrive to the following observation:

Observation 5.1.6. Let $P \in A$-proj,

- If $\operatorname{Hom}_A(P,-)$ is fully faithful on $\operatorname{add} A$, then (A,P) is a cover of $\operatorname{End}_A(P)^{op}$;
- If $\operatorname{Hom}_A(P,-)$ is fully faithful on $\operatorname{add} DA$, then there exists a double centralizer property on P between A and $\operatorname{End}_A(P)^{op}$;
- Assume that R is a field. If $\operatorname{Hom}_A(P,-)$ is fully faithful on $\operatorname{add} A \oplus DA$, then P is a left A-progenerator and a right $\operatorname{End}_A(P)^{op}$-progenerator.

By Theorem 3.5.4, for Morita algebras A and projective-injective faithful modules P, (A,P) is simultaneously a cover and a cocover. These are the only algebras with this property if we restrict ourselves to finite-dimensional algebras over a field.

So, in contrast to covers, there is the following symmetry for cocovers.

Proposition 5.1.7. *The pair (A,Q) is a cocover of B if and only if the pair (A^{op},DQ) is a cocover of B^{op}.*

Proof. Since $Q \in R$-proj, $\operatorname{End}_A(Q)^{op} \simeq \operatorname{End}_A(DQ)$ and $\operatorname{End}_B(Q) \simeq \operatorname{End}_B(DQ)^{op}$. Hence, the result follows. $\square$

Remark 5.1.8. If (A,Q) is a cocover of B and Q is a B-generator, then $Q \in A$-proj and $(A, \operatorname{Hom}_A(Q,A))$ is a cover of B. In fact, with these assumptions $Q \simeq \operatorname{Hom}_B(B,Q) \in \operatorname{add}_A \operatorname{Hom}_B(Q,Q) = A$-proj. $\triangle$

Note that for a cocover (A,Q), the module $DQ \otimes_A Q$ is (B,R)-injective.

We wish now to generalize for cocovers, the properties presented in Subsection 1.4.5.

Lemma 5.1.9. *The following assertions are equivalent.*

(i) (A,Q) *is a cocover of* B.

(ii) $(S\otimes_R A, S\otimes_R Q)$ *is a cocover of* $S\otimes_R B$ *for every flat commutative* R*-algebra* S *which is a Noetherian ring.*

(iii) $(A_\mathfrak{p}, Q_\mathfrak{p})$ *is a cocover of* $B_\mathfrak{p}$ *for every prime ideal* $\mathfrak{p}$ *of* R.

(iv) $(A_\mathfrak{m}, Q_\mathfrak{m})$ *is a cocover of* $B_\mathfrak{m}$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

Proof. Let S be a commutative flat R-algebra. Then, there exists the commutative diagram

$$\begin{tikzcd} S\otimes_R A \arrow[r, "S\otimes_R \psi_A"] \arrow[d, equal] & S\otimes_R \operatorname{End}_B(Q) \arrow[d, "c_{Q,Q}"] \\ S\otimes_R A \arrow[r, "\psi_{S\otimes_R A}"] & \operatorname{End}_{S\otimes_R B}(S\otimes_R Q) \end{tikzcd}, \tag{5.1.0.11}$$

where $c_{Q,Q}$ denotes the isomorphism given by Proposition 1.1.35. In fact, for every $s,s' \in S,\ q\in Q,\ a\in A$,

$$c_{Q,Q}\circ S\otimes_R \psi_A(s\otimes a)(s'\otimes q) = ss'\otimes \psi_A(a)(q) = ss'\otimes aq = \psi_{S\otimes_R A}(s\otimes a)(s'\otimes q). \tag{5.1.0.12}$$

If (i) holds, then ψ_A is an isomorphism. By (5.1.0.11), $S\otimes_R \psi_A$ is an isomorphism and, consequently, (ii) holds. The implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear. Assume that (iv) holds. Then, $\psi_{A_\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R. By (5.1.0.11), $\psi_{A\,\mathfrak{m}}$ is an isomorphism, and therefore ψ_A is an isomorphism. So, (i) holds. $\square$

Lemma 5.1.10. *Let* $M \in A$-mod$\cap R$-proj *such that the canonical map* $\operatorname{Hom}_A(Q,M)(\mathfrak{m}) \to \operatorname{Hom}_{A(\mathfrak{m})}(Q(\mathfrak{m}), M(\mathfrak{m}))$ *is an isomorphism for every maximal ideal* $\mathfrak{m}$ *of* R. *Then,* χ_M *is surjective if and only if* $\chi_{M(\mathfrak{m})}$ *is surjective for every maximal ideal* $\mathfrak{m}$ *of* R. *If, in addition,* $Q\otimes_B \operatorname{Hom}_A(Q,M) \in R$-proj, *then* $D\chi_M$ *is an* (A,R)*-monomorphism if and only if* $\chi_{M(\mathfrak{m})}$ *is surjective for every maximal ideal* $\mathfrak{m}$ *of* R.

Proof. As before, it is enough to consider a certain commutative diagram. For each maximal ideal $\mathfrak{m}$ of R there exists a commutative diagram

$$\begin{tikzcd} Q(\mathfrak{m})\otimes_{B(\mathfrak{m})} \operatorname{Hom}_{A(\mathfrak{m})}(Q(\mathfrak{m}), M(\mathfrak{m})) \arrow[r, "\chi_{M(\mathfrak{m})}"] & M(\mathfrak{m}) \arrow[dd, equal] \\ Q(\mathfrak{m})\otimes_{B(\mathfrak{m})} \operatorname{Hom}_A(Q,M)(\mathfrak{m}) \arrow[u, "Q(\mathfrak{m})\otimes_{B(\mathfrak{m})} c_{Q,M}"] & \\ Q\otimes_B \operatorname{Hom}_A(Q,M)(\mathfrak{m}) \arrow[u, "s"] \arrow[r, "\chi_M\otimes_R R(\mathfrak{m})"] & M(\mathfrak{m}) \end{tikzcd}, \tag{5.1.0.13}$$

where $c_{Q,M}$ denotes the usual map and s denotes the isomorphism given by Lemma 1.1.32. In fact,

$$\chi_{M(\mathfrak{m})}\circ Q(\mathfrak{m})\otimes_{B(\mathfrak{m})} c_{Q,M}\circ s(q\otimes g\otimes r+\mathfrak{m}) = \chi_{M(\mathfrak{m})}((q\otimes r+\mathfrak{m})\otimes c_{Q,M}(g\otimes 1_R+\mathfrak{m})) = g(q)\otimes r+\mathfrak{m} \tag{5.1.0.14}$$

$$= \chi_M \otimes_R R(\mathfrak{m})(q \otimes g \otimes r + \mathfrak{m}), \ q \in Q, \ g \in FM, \ r \in R. \quad (5.1.0.15)$$

It follows by the commutative diagram that, for every maximal ideal $\mathfrak{m}$ of R, $\chi_{M(\mathfrak{m})}$ is surjective if and only if $\chi_M(\mathfrak{m})$ is. Thus, the result follows by Nakayama's Lemma and by the right exactness of $R(\mathfrak{m}) \otimes_R -$. Assume, in addition, that $Q \otimes_B \operatorname{Hom}_A(Q, M) \in R\text{-proj}$. If $\chi_{M(\mathfrak{m})}$ is surjective for every maximal ideal $\mathfrak{m}$ of R, then, by the previous discussion, χ_M is surjective onto a projective R-module M. Thus, applying the dual D yields an (A,R)-monomorphism. Conversely, since $Q \otimes_B \operatorname{Hom}_A(Q, M) \in R\text{-proj}$, $DD\chi_M$ is surjective and consequently χ_M is surjective. ☐

Proposition 5.1.11. *Assume that R is a commutative Noetherian regular ring. Let A be a projective Noetherian R-algebra and let $Q \in A\text{-mod} \cap R\text{-proj}$. Assume that $DQ \otimes_A Q \in R\text{-proj}$ and write $B = \operatorname{End}_A(Q)^{op}$. If $(A(\mathfrak{m}), Q(\mathfrak{m}))$ is a cocover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R, then (A, Q) is a cocover of B.*

Proof. The argument is analogous to Proposition 1.4.34. In view of Lemma 5.1.9, we can assume that R is a local commutative Noetherian regular ring. We shall proceed by induction on the Krull dimension of R. If the Krull dimension of R is zero, then it is a field, so there is nothing to prove. Assume that R has positive Krull dimension. Let $x \in \mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the unique maximal ideal of R. Fix $S = R/Rx$ a local commutative regular ring with unique maximal ideal $\mathfrak{m}/Rx$. By assumption, $(S \otimes_R A(\mathfrak{m}/Rx), S \otimes_R Q(\mathfrak{m}/Rx))$ is a cocover of $S \otimes_R B(\mathfrak{m}/Rx)$. By induction, $(S \otimes_R A, S \otimes_R Q)$ is a cocover of $S \otimes_R B$. Thanks to Lemma 1.1.32, the map

$$S \otimes_R A \to \operatorname{End}_{S \otimes_R B}(S \otimes_R Q) \simeq \operatorname{Hom}_B(Q, S \otimes_R Q)$$

is bijective. Denote this map by μ_S. In particular $\mu_Q(s \otimes a)(q) = s \otimes aq$, $a \in A$, $q \in Q$, $s \in S$. Let δ be the monomorphism given in Lemma 1.4.33 and ψ the canonical map between A and $\operatorname{End}_B(Q)$. Then, $\delta \circ S \otimes_R \psi = \mu_S$. Hence, $S \otimes_R \psi$ is bijective. Applying the Nakayama's Lemma together with the commutative diagram

$$\begin{tikzcd} S \otimes_R A \arrow[r, "S \otimes_R \psi"] \arrow[d, two heads] & S \otimes_R \operatorname{End}_B(Q) \arrow[d, two heads] \\ R(\mathfrak{m}) \otimes_R A \arrow[r, "\psi(\mathfrak{m})"] & R(\mathfrak{m}) \otimes_R \operatorname{End}_B(Q) \end{tikzcd} \quad (5.1.0.16)$$

we deduce that ψ is surjective. For the injectivity, we need to observe that by Lemma 5.1.2, $\chi_{D(\mathfrak{m})A(\mathfrak{m})}$ is an isomorphism and, consequently, χ_{DA} is surjective. Thus, $D\chi_{DA}$ is injective. By the proof of Lemma 5.1.2 this implies that ψ is also injective. ☐

Remark 5.1.12. If (A, Q) is a cocover of B, then $(A,R)\text{-inj} \cap R\text{-proj}$ (which is equivalent to $\operatorname{add}_B DQ$) is a cocover of B-proj in the sense of [HU96]. This follows from the fact that for any $X \in B\text{-proj}$, $\mathbb{I}X \in R\text{-proj}$ and it can be embedded into some module in the additive closure of DA. By the left exactness of F and $FDA \simeq DP$, the claim follows. △

5.2 Relative dominant dimension with respect to a module and relative codominant dimension with respect to a module

Recall the definition of relative dominant dimension introduced in Definition 2.3.5. In this section, we will study a version of Mueller theorem for the relative dominant dimension with respect to a module. The arguments for this are based on [AS93, Proposition 2.1].

Theorem 5.2.1. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $DQ \otimes_A Q \in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For $M \in A\text{-mod} \cap R\text{-proj}$, the following assertions hold.*

(i) The counit $\chi_M \colon Q \otimes_B \operatorname{Hom}_A(Q,M) \to M$ is surjective if and only if $DQ - \operatorname{domdim}_{(A,R)} DM \geq 1$.

(ii) The counit $\chi_M \colon Q \otimes_B \operatorname{Hom}_A(Q,M) \to M$ is an isomorphism if and only if $DQ - \operatorname{domdim}_{(A,R)} DM \geq 2$.

Proof. Assume that χ_M is surjective. Since $\operatorname{Hom}_A(Q,M) \in B$-mod there exists $X \in \operatorname{add} Q$ and a surjective map $\operatorname{Hom}_A(Q,X) \twoheadrightarrow \operatorname{Hom}_A(Q,M)$, say g. The functor $Q \otimes_B -$ is right exact, so $Q \otimes_B g$ is surjective as well. Define $f := \chi_M \circ Q \otimes_B g \circ \chi_X^{-1}$. The map f is surjective and we claim that $\operatorname{Hom}_A(Q,f) = g$. To see that, observe first that $q \otimes h = \chi_X^{-1}\chi_X(q \otimes h) = \chi_X^{-1}(h(q))$ for every $q \in Q$ and $h \in \operatorname{Hom}_A(Q,X)$. Now, we can see that for every $h \in \operatorname{Hom}_A(Q,X)$, $q \in Q$,

$$\operatorname{Hom}_A(Q,f)(h)(q) = f \circ h(q) = \chi_M \circ Q \otimes_B g(q \otimes h) = g(h)(q). \tag{5.2.0.1}$$

Applying D yields the (A,R)-monomorphism $DM \to DX$ which remains exact under $\operatorname{Hom}_A(-,DQ)$. Thus, $DQ - \operatorname{domdim}_{(A,R)} DM \geq 1$. Conversely, assume that $DQ - \operatorname{domdim}_{(A,R)} DM \geq 1$. So, there exists $X \in \operatorname{add} Q$ and an (A,R)-monomorphism $f \colon DM \to X$ which is also a left $\operatorname{add} Q$-approximation. Since χ is a natural transformation between $Q \otimes_B \operatorname{Hom}_A(Q,-)$ between $\mathrm{id}_{B\text{-mod}}$ we have $\chi_{DDM} \circ Q \otimes_B \operatorname{Hom}_A(T,Df) = Df \circ \chi_{DX}$ is surjective. In particular, χ_{DDM} is surjective. As $DDM \simeq M$, χ_M is surjective and (i) follows.

Now, assume that $DQ - \operatorname{domdim}_{(A,R)} DM \geq 2$. Then, there exists an (A,R)-exact sequence $0 \to DM \xrightarrow{f_0} X_0 \xrightarrow{f_1} X_1$ which remains exact under $\operatorname{Hom}_A(-,DQ)$. As $Q \otimes_B -$ is right exact, the following diagram is commutative with exact rows

$$\begin{tikzcd} Q \otimes_B \operatorname{Hom}_A(Q,DX_1) \arrow[r, "Q\otimes_B \operatorname{Hom}_A(Q,Df_1)"] \arrow[d, "\chi_{DX_1}", "\simeq"'] & Q \otimes_B \operatorname{Hom}_A(Q,DX_0) \arrow[r, two heads, "Q\otimes_B \operatorname{Hom}_A(Q,Df_0)"] \arrow[d, "\chi_{DX_0}", "\simeq"'] & Q \otimes_B \operatorname{Hom}_A(Q,DDM) \arrow[d, "\chi_{DDM}"] \\ DX_1 \arrow[r, "Df_1"] & DX_0 \arrow[r, two heads, "Df_0"] & DDM \end{tikzcd} . \tag{5.2.0.2}$$

By diagram chasing, χ_{DDM} is an isomorphism. Since $DDM \simeq M$, χ_M is an isomorphism. Conversely, assume that χ_M is an isomorphism. B is a Noetherian R-algebra, so we can consider a projective B-presentation for $\operatorname{Hom}_A(Q,M)$ of the form

$$\operatorname{Hom}_A(Q,Q^m) \xrightarrow{g_1} \operatorname{Hom}_A(Q,Q^n) \xrightarrow{g_0} \operatorname{Hom}_A(Q,M) \to 0, \tag{5.2.0.3}$$

for some integers m,n. Since $\operatorname{Hom}_A(Q,-)_{|\operatorname{add} T}$ is full and faithful there exists $f_1 \in \operatorname{Hom}_A(Q^m,Q^n)$ such that $\operatorname{Hom}_A(Q,f_1) = g_1$. Fix $f_0 = \chi_M \circ Q \otimes_B g_0 \chi_{Q^n}^{-1}$. We have seen previously, that $\operatorname{Hom}_A(Q,f_0) = g_0$. So, the diagram

$$\begin{tikzcd} Q \otimes_B \operatorname{Hom}_A(Q,Q^m) \arrow[r, "Q\otimes_B g_1"] \arrow[d, "\chi_{Q^m}", "\simeq"'] & Q \otimes_B \operatorname{Hom}_A(Q,Q^n) \arrow[r, two heads, "Q\otimes_B g_0"] \arrow[d, "\chi_{Q^n}", "\simeq"'] & Q \otimes_B \operatorname{Hom}_A(Q,M) \arrow[d, "\chi_M"] \\ Q^m \arrow[r, "f_1"] & Q^n \arrow[r, two heads, "f_0"] & M \end{tikzcd} \tag{5.2.0.4}$$

is commutative. Since the vertical maps are isomorphisms and the upper row is exact it follows that the bottom row is exact and by construction it remains exact under $\operatorname{Hom}_A(Q,-)$. As $M \in R$-proj, it is, in addition, (A,R)-exact. By applying the standard duality D we obtain that $DT - \operatorname{domdim}_{(A,R)} DM \geq 2$. ☐

Putting together Theorem5.2.1 and Lemma 5.1.2 we obtain that if the relative dominant dimension

$DQ-\operatorname{domdim}(A,R)$ with respect to DQ is greater than or equal to two for a module Q satisfying $DQ\otimes_A Q\in R$-proj then (A,Q) is a cocover of $\operatorname{End}_A(Q)^{op}$.

Similarly, we can write the dual version of Theorem 5.2.1.

Theorem 5.2.2. *Let R be a commutative Noetherian ring. Let A be projective Noetherian R-algebra. Assume that $Q\in A\text{-mod}\cap R\text{-proj}$ satisfying in addition that $DQ\otimes_A Q\in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For $M\in A\text{-mod}\cap R\text{-proj}$, the following assertions hold.*

(i) The (right) counit $\chi^r_{DM}\colon \operatorname{Hom}_A(DQ,DM)\otimes_B DQ\to DM$ is surjective if and only if $Q-\operatorname{domdim}_{(A,R)} M\geq 1$.

(ii) The counit $\chi^r_{DM}\colon \operatorname{Hom}_A(DQ,DM)\otimes_B DQ\to DM$ is an isomorphism if and only if $Q-\operatorname{domdim}_{(A,R)} M\geq 2$.

It is important to observe that cocovers of relative self-injective algebras arising from higher values of relative dominant dimension with respect to a module is not anything new. Indeed, if $Q-\operatorname{domdim}(A,R)\geq 2$ then $Q\otimes_B DQ\in R$-proj. Now, Q being finitely generated over A means that there exists a surjective A-map $A^s\to Q$, for some $s>0$. Consequently, there exists a surjective map $DQ\otimes_A A^s\to DQ\otimes_A Q\simeq DB$. Since B is relative self-injective it follows that Q is a B-generator (B,R)-cogenerator satisfying $Q\otimes_B DQ\in R$-proj. By the relative Morita-Tachikawa correspondence, Q is a projective-injective A-module and A is a relative Morita algebra.

In order to avoid changing from left to right modules systematically, we can introduce the relative codominant dimension with respect to a module.

Definition 5.2.3. Let A be a projective Noetherian R-algebra. Let $Q,X\in A\text{-mod}\cap R\text{-proj}$. If X does not admit a surjective right $\operatorname{add}Q$-approximation, then we say that **relative codominant dimension of X with respect to Q** is zero. Otherwise, the **relative codominant dimension of X with respect to Q**, denoted by $Q-\operatorname{codomdim}_{(A,R)} X$, is the supremum of all $n\in\mathbb{N}$ such that there exists an (A,R)-exact sequence

$$Q_n\to\cdots\to Q_1\to X\to 0$$

which remains exact under $\operatorname{Hom}_A(Q,-)$ with all $Q_i\in\operatorname{add}Q$. If some $Q_i=0$, then we say that $Q-\operatorname{codomdim}_{(A,R)} X$ is infinite.

In particular, $DQ-\operatorname{domdim}_{(A,R)} DM=Q-\operatorname{codomdim}_{(A,R)} M$ whenever $Q,M\in A\text{-mod}\cap R\text{-proj}$.

With Theorem 5.2.1, we can see that examples of cocovers are very abundant. As we have seen, if Q is projective (A,R)-injective-strongly faithful and $\operatorname{domdim}(A,R)\geq 2$ then (A,Q) is a cocover of $\operatorname{End}_A(Q)^{op}$. By Propositions 1.5.133 and 1.5.109, if Q is a characteristic tilting module of a split quasi-hereditary algebra A then (A,Q) is a cocover of the Ringel dual of A. But, more interesting are the cases where Q is a tilting module (not a characteristic tilting module) of a split quasi-hereditary algebra that have a double centralizer property. Following the work developed in [MS08, 2.2], we will see in following lemma that every split quasi-hereditary algebra has a (partial) tilting module with a double centralizer property. At worst, this (partial) tilting module coincides with the characteristic tilting module.

Lemma 5.2.4. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra and let T be a characteristic tilting module of A. Then, there is an exact sequence $0\to A\to M\to X\to 0$ where M is (partial) tilting and $X\in\mathscr{F}(\tilde{\Delta}_A)$. Moreover, there exists a (partial) tilting module $Q\in A\text{-mod}\cap R\text{-proj}$ such that $DM\in\operatorname{add}Q$ and $Q-\operatorname{domdim}(A,R)\geq 2$. In particular, (A,Q) is a cocover of $\operatorname{End}_A(Q)^{op}$.*

Proof. Denote by R_A the Ringel dual $\operatorname{End}_A(T)^{op}$. Let $P\twoheadrightarrow T$ be a right projective presentation of T over R_A. Then, $P\in\mathscr{F}(\tilde{\Delta}_{R_A^{op}})$. Note that $T\simeq\operatorname{Hom}_A(A,T)\simeq\operatorname{Hom}_{A^{op}}(DT,DA)\in\mathscr{F}(\tilde{\Delta}_{R_A^{op}})$. Since $\mathscr{F}(\tilde{\Delta}_{R_A})$ is resolving, so the kernel of $P \to T$ belongs to $\mathscr{F}(\tilde{\Delta}_{R_A^{op}})$. Since $\operatorname{Hom}_{A^{op}}(DT, -)$ gives an exact equivalence between $\mathscr{F}(\tilde{\nabla}_{A^{op}})$ and $\mathscr{F}(\tilde{\Delta}_{R_A^{op}})$ there exists an exact sequence of right A-modules

$$0 \to K \to M' \to DA \to 0, \tag{5.2.0.5}$$

where M' is a (partial) tilting module and $K \in \mathscr{F}(\tilde{\nabla}_{A^{op}})$. Applying D we obtain the desired exact sequence. By Proposition 1.5.109, since $DK \in \mathscr{F}(\tilde{\Delta}_A)$ there exists an exact sequence $0 \to DK \to M'' \to K'' \to 0$, where M' is a (partial) tilting module and $K' \in \mathscr{F}(\tilde{\Delta}_A)$. Put $Q = DM' \oplus M''$. Hence, Q is (partial) tilting module and the (A,R)-exact sequence $0 \to A \to DM' \to M''$ remains exact under $\operatorname{Hom}_A(Q, -)$. This means that $Q - \operatorname{domdim}(A,R) \geq 2$. By Theorem 5.2.2 and Proposition 5.1.7, (A,Q) is a cocover of $\operatorname{End}_A(Q)^{op}$. ☐

5.2.1 Relative Mueller's characterization of relative dominant dimension with respect to a module

Now, we are ready to formulate the Mueller version for relative codominant dimension with respect to a module.

Theorem 5.2.5. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $DQ \otimes_A Q \in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For $M \in A\text{-mod} \cap R\text{-proj}$, the following assertions hold.*

(i) $Q - \operatorname{codomdim}_{(A,R)} M \geq n \geq 2$ *if and only if* $\chi_M \colon Q \otimes_B \operatorname{Hom}_A(Q,M) \to M$ *is an isomorphism of left A-modules and* $\operatorname{Tor}_i^B(Q, \operatorname{Hom}_A(Q,M)) = 0$, $1 \leq i \leq n-2$.

(ii) $Q - \operatorname{domdim}_{(A,R)} M \geq n \geq 2$ *if and only if* $\chi^r_{DM} \colon \operatorname{Hom}_A(DQ, DM) \otimes_B DQ \to DM$ *is an isomorphism and* $\operatorname{Tor}_i^B(\operatorname{Hom}_A(DQ,DM), DQ) = \operatorname{Tor}_i^B(\operatorname{Hom}_A(M,Q), DQ) = 0$, $1 \leq i \leq n-2$.

Proof. We shall prove (i). The statement (ii) is analogous to (i). Assume that $DQ - \operatorname{domdim}_{(A,R)} DM \geq n \geq 2$. By Theorem 5.2.1, χ_M is an isomorphism. By definition, there exists an (A,R)-exact sequence

$$0 \to DM \to X_0 \to X_1 \to \cdots \to X_{n-1} \tag{5.2.1.1}$$

which remains exact under $\operatorname{Hom}_A(-, DQ)$ with $X_i \in \operatorname{add} DQ$, $i = 0, \ldots, n-1$. In particular, $\operatorname{Hom}_A(X_{n-1}, DQ) \to \cdots \to \operatorname{Hom}_A(X_0, DQ) \to \operatorname{Hom}_A(DM, DQ) \to 0$ is exact and can be continued to a left projective B-resolution of $\operatorname{Hom}_A(Q,M)$. Consider the following commutative diagram

$$\begin{tikzcd} Q \otimes_B \operatorname{Hom}_A(Q, DX_{n-1}) \arrow[r] \arrow[d, "\chi_{DX_{n-1}}", "\simeq"'] & \cdots \arrow[r] & Q \otimes_B \operatorname{Hom}_A(Q, DX_0) \arrow[r, two heads] \arrow[d, "\chi_{DX_0}", "\simeq"'] & Q \otimes_B \operatorname{Hom}_A(Q,M) \arrow[d, "\chi_{DDM}", "\simeq"'] \\ DX_{n-1} \arrow[r] & \cdots \arrow[r] & DX_0 \arrow[r, two heads] & DDM \end{tikzcd} . \tag{5.2.1.2}$$

Observe that the bottom row is exact since the exact sequence (5.2.1.1) is (A,R)-exact. Since all vertical maps are isomorphisms, it follows that the upper row is exact. Thus, $\operatorname{Tor}_i^B(Q, \operatorname{Hom}_A(Q,M)) = 0$, $1 \leq i \leq n-2$.

Conversely, assume that χ_M is an isomorphism and $\operatorname{Tor}_i^B(Q, \operatorname{Hom}_A(Q,M)) = 0$ for $1 \leq i \leq n-2$. Let $\operatorname{Hom}_A(Q, X_{n-1}) \xrightarrow{g_{n-1}} \cdots \to \operatorname{Hom}_A(Q, X_0) \xrightarrow{g_0} \operatorname{Hom}_A(Q,M) \to 0$ be a truncated projective B-resolution of $\operatorname{Hom}_A(Q,M)$ and $X_i \in \operatorname{add}_A Q$. Furthermore, $\operatorname{Hom}_A(Q,-)_{|\operatorname{add} Q}$ is full and faithful, so each map g_i can be written as $\operatorname{Hom}_A(T, f_i)$ including g_0 since χ_M is an isomorphism, where $f_i \in \operatorname{Hom}_A(X_i, X_{i-1})$ and $f_0 \in \operatorname{Hom}_A(X_0, M)$. So, we have a commutative diagram

$$\begin{array}{ccccccc} Q\otimes_B \mathrm{Hom}_A(Q,X_{n-1}) & \xrightarrow{Q\otimes_B \mathrm{Hom}_A(Q,g_{n-1})} \cdots \longrightarrow & Q\otimes_B \mathrm{Hom}_A(Q,X_0) & \xrightarrow{Q\otimes g_0} & Q\otimes_B \mathrm{Hom}_A(Q,M) \\ \simeq\big\downarrow \chi_{X_{n-1}} & & \simeq\big\downarrow \chi_{X_0} & & \simeq\big\downarrow \chi_M \\ X_{n-1} & \xrightarrow{f_{n-1}} \cdots \longrightarrow & X_0 & \xrightarrow{f_0} & M \end{array}. \tag{5.2.1.3}$$

By assumption, $\mathrm{Tor}_i^B(Q,\mathrm{Hom}_A(Q,M))=0$, $1\leq i\leq n-2$. So, the upper row is exact. By the exactness and the vertical maps being isomorphisms the bottom row becomes exact. Since $M\in R$-proj it is also (A,R)-exact and so it remains (A,R)-exact under D. By construction, such the bottom row remains exact under $\mathrm{Hom}_A(Q,-)$, thus $DQ-\mathrm{domdim}_{(A,R)}DM\geq n\geq 2$. ☐

An immediate consequence of Theorems 5.2.5 and 5.2.1 is the following.

Corollary 5.2.6. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q\in A$-mod$\cap R$-proj satisfying in addition that $DQ\otimes_A Q\in R$-proj. Then,*

$$DQ-\mathrm{domdim}\,(A,R)=Q-\mathrm{codomdim}_{(A,R)}DA=Q-\mathrm{domdim}\,(A,R).$$

Proof. By Tensor-Hom adjunction, there are isomorphisms $\mathrm{Hom}_A(Q,DA)\simeq DQ$, given by $f\mapsto (f(-)(1_A))$, and $Q\simeq \mathrm{Hom}_A(DQ,DA)$, given by $q\mapsto (f\mapsto (a\mapsto f(aq)))$. We shall denote the first by ψ and the second isomorphism by ω. So, ψ is a left B-isomorphism while ω is a right B-isomorphism. Moreover, $\chi_{DA}^r\circ\omega\otimes_B\psi=\chi_{DA}$. In fact, for $a\in A$, $q\in Q$, $g\in\mathrm{Hom}_A(Q,DA)$,

$$\chi_{DA}^r\circ\omega\otimes_B\psi(q\otimes g)(a)=\omega(q)(\psi(g))(a)=\psi(g)(aq)=g(aq)(1_A)=(ag(q))(1_A)=g(q)(a)=\chi_{DA}(q\otimes g)(a).$$

By Theorem 5.2.1 and Theorem 5.2.2, $DQ-\mathrm{domdim}_{(A,R)}A_A\geq i$ if and only if $Q-\mathrm{domdim}_{(A,R)}{}_AA\geq i$ for $i=1,2$. Finally, by Theorem 5.2.5, $DQ-\mathrm{domdim}_{(A,R)}A_A=DQ-\mathrm{domdim}_{(A,R)}DDA_A\geq n\geq 2$ if and only if χ_{DA} is an isomorphism and $0=\mathrm{Tor}_i^B(Q,\mathrm{Hom}_A(Q,DA))=\mathrm{Tor}_i^B(Q,DQ)=\mathrm{Tor}_i^B(\mathrm{Hom}_A(DQ,DA),DQ)$, $1\leq i\leq n-2$ if and only if $Q-\mathrm{domdim}_{(A,R)}{}_AA\geq n\geq 2$. ☐

We can obtain a version of Corollary 5.2.6 for (partial) tilting modules. In fact, if the split quasi-hereditary algebra admits a duality functor on A-mod$\cap R$-proj interchanging $\Delta(\lambda)$ with $\nabla(\lambda)$ (or a simple preserving duality if the ground ring is a field), then the relative codominant dimension and the relative dominant dimension of a characteristic tilting module with respect to a partial tilting module are the same.

Proposition 5.2.7. *Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Let $V\in\mathrm{add}_A T$ be a (partial) tilting module and assume that ${}^\natural(-)\colon A$-mod $\to A$-mod is a duality satisfying ${}^\natural\Delta(\lambda)=\nabla(\lambda)$ for all $\lambda\in\Lambda$. Then, $V-\mathrm{domdim}_{(A,R)}T=V-\mathrm{codomdim}_{(A,R)}T$.*

Proof. Assume that $V-\mathrm{domdim}_{(A,R)}T\geq n\geq 1$. By definition, there exists an (A,R)-exact sequence

$$0\to T\xrightarrow{\alpha_0}V_0\xrightarrow{\alpha_1}V_1\to\cdots\to V_{n-1} \tag{5.2.1.4}$$

which remains exact under $\mathrm{Hom}_A(-,V)$, with $V_i\in\mathrm{add}_A V$. Applying the duality $\natural$ we obtain the exact sequence

$${}^\natural V_{n-1}\xrightarrow{{}^\natural\alpha_{n-1}}\cdots\to{}^\natural V_1\xrightarrow{{}^\natural\alpha_1}{}^\natural V_0\xrightarrow{{}^\natural\alpha_0}{}^\natural T\to 0. \tag{5.2.1.5}$$

But all V_i and T are (partial) tilting modules, so ${}^\natural T\simeq T$ and ${}^\natural V_i\simeq V_i$ as A-modules, $i=1,\ldots,n$. In particular, (5.2.1.5) is (A,R)-exact. It remains to show that (5.2.1.5) remains exact under $\mathrm{Hom}_A(V,-)\simeq\mathrm{Hom}_A({}^\natural V,-)$. To show that consider, for each i, the factorization of α_{i+1} through its image $\alpha_{i+1} = \nu_i \circ \pi_i$. Hence, α_0 and ν_i, $i = 0, \dots, n-1$, are left $\operatorname{add} V$-approximations. By the exactness of the contravariant functor $^\natural$, $^\natural(\ker \alpha_{i+1}) \simeq \operatorname{coker}(^\natural \alpha_{i+1})$ and $^\natural \operatorname{im} \alpha_{i+1} = \operatorname{im}(^\natural \alpha_{i+1})$ for all i. Moreover, for every homomorphism $f \in \operatorname{Hom}_A(N, L)$ the maps $\operatorname{Hom}_A(^\natural V, ^\natural f)$ and $\operatorname{Hom}_A(f, V)$ are related by the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_A(^\natural V, ^\natural L) \arrow[r, "\operatorname{Hom}_A(^\natural V, ^\natural f)"] & \operatorname{Hom}_A(^\natural V, ^\natural N) \\ \operatorname{Hom}_A(L, V) \arrow[u, "\simeq"] \arrow[r, "\operatorname{Hom}_A(f,V)"] & \operatorname{Hom}_A(N, V) \arrow[u, "\simeq"] \end{tikzcd} . \tag{5.2.1.6}$$

Hence, for each i, $^\natural \nu_i$ is an surjective right $\operatorname{add} V$-approximation and $^\natural \alpha_{i+1} = {}^\natural \pi_i \circ {}^\natural \nu_i$. The same is true for $^\natural \alpha_0$. By Lemma 2.3.3, (5.2.1.5) remains exact under $\operatorname{Hom}_A(V, -)$. Hence, $V - \operatorname{codomdim}_{(A,R)} T \geq n \geq 1$. Conversely, $V - \operatorname{codomdim}_{(A,R)} T = DV - \operatorname{domdim}_{(A,R)} DT \geq DV - \operatorname{codomdim}_{(A,R)} DT = V - \operatorname{domdim}_{(A,R)} T$. ☐

The existence of Theorem 5.2.5 is the main advantage of this definition compared to [KSX01, Definition 2.1] giving a meaning to what this relative dominant measures. Another point of view that we should refer is the Wakamatsu tilting conjecture (see [Wak88]). In this context, the Wakamatsu tilting conjecture says that if Q has finite projective A-dimension and it admits no self-extensions in any degree, then $Q - \operatorname{domdim}(A, R)$ measures how far Q is from being a tilting module. In particular, for split quasi-hereditary algebras this amounts to saying that for a module Q in the additive closure of a characteristic tilting module, $Q - \operatorname{domdim}(A, R)$ measures how far Q is from being a characteristic tilting module of A. We will come back later to this question once we have better tools to analyse it.

It is worth to point out that this is indeed the case if Q has finite projective dimension over B.

Proposition 5.2.8. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying the following:*

1. $DQ \otimes_A Q \in R\text{-proj}$;
2. *The projective dimensions* $\operatorname{pdim}_A Q$ *and* $\operatorname{pdim}_B Q$ *are finite;*
3. $\operatorname{Ext}_A^{i>0}(Q, Q) = 0$.

If $Q - \operatorname{domdim}(A, R) > \operatorname{pdim}_B Q$, then Q is a tilting A-module (in the sense of Definition 1.5.120).

Proof. Fix $n = \operatorname{pdim}_B Q$. By assumption, $DQ - \operatorname{codomdim}_{(A,R)} DA \geq n+1$, so there exists an exact sequence $X_n \xrightarrow{\alpha_n} X_{n-1} \to \cdots \to X_0 \xrightarrow{\alpha_0} DA \to 0$, with $X_i \in \operatorname{add}_A DQ$ which remains exact under $\operatorname{Hom}_A(DQ, -)$. Denote such a exact sequence by δ. Since $n = \operatorname{pdim}_B Q$ and $\operatorname{Hom}_A(DQ, DA) \simeq Q$ as B-modules the kernel of $\operatorname{Hom}_A(DQ, \alpha_{n-1})$ (which is the image of $\operatorname{Hom}_A(DQ, \alpha_n)$) is projective over B. By projectivization, $\ker \operatorname{Hom}_A(DQ, \alpha_{n-1})$ is isomorphic to $\operatorname{Hom}_A(DQ, Y)$ for some $Y \in \operatorname{add}_A DQ$ and the inclusion of the kernel can be written as $\operatorname{Hom}_A(DQ, f)$ for some $f \in \operatorname{Hom}_A(Y, X_{n-1})$. Applying Lemma 2.4.14 on $Q \otimes_B \operatorname{Hom}_A(DQ, \delta)$ we obtain that $\operatorname{Hom}_A(DQ, Y) \otimes_B DQ \simeq Y$ is the kernel of the map $\operatorname{Hom}_A(DQ, \alpha_{n-1}) \otimes_B DQ$. It follows that Q is a tilting A-module. ☐

5.2.1.1 Behaviour of relative dominant dimension on long exact sequences

Using Theorem 5.2.5 is now easy to obtain how the relative dominant dimension with respect to a module behaves in short exact sequences.

Lemma 5.2.9. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $DQ \otimes_A Q \in R\text{-proj}$. Let $M \in R\text{-proj}$ and consider the following (A,R)-exact*

$$0 \to M_1 \to M \to M_2 \to 0 \tag{5.2.1.7}$$

which remains exact under $\operatorname{Hom}_A(-,Q)$. Let $n = Q - \operatorname{domdim}_{(A,R)} M$ and $n_i = Q - \operatorname{domdim}_{(A,R)} M_i$. Then, the following holds.

(a) $n \geq \min\{n_1, n_2\}$.

(b) *If* $n_1 < n$, *then* $n_2 = n_1 - 1$.

(c) *(i)* $n_1 = n \implies n_2 \geq n - 1$.

(ii) $n_1 = n + 1 \implies n_2 \geq n$.

(iii) $n_1 \geq n + 2 \implies n_2 = n$.

(d) $n < n_2 \implies n_1 = n$.

(e) *(i)* $n = n_2 \implies n_1 \geq n_2$.

(ii) $n = n_2 + 1 \implies n_1 \geq n_2 + 1$.

(iii) $n \geq n_2 + 2 \implies n_1 = n_2 + 1$.

Proof. By assumption, $0 \to \operatorname{Hom}_A(DQ, DM_2) \to \operatorname{Hom}_A(DQ, DM) \to \operatorname{Hom}_A(DQ, DM_1) \to 0$. The remaining of the proof is exactly analogous to Lemma 2.4.25. ☐

In the same manner, it follows that $Q - \operatorname{domdim}_{(A,R)} M \oplus N = \min\{Q - \operatorname{domdim}_{(A,R)} M, Q - \operatorname{domdim}_{(A,R)} N\}$, for any $M \oplus N \in R\text{-proj}$.

Usually, proving that a certain exact sequence remains exact under a certain Hom functor might be difficult. In the following, we show a known result that we can extend an (A,R)-exact sequence if it is only the last homomorphism not being decomposed into an $\operatorname{add} Q$-approximation.

Lemma 5.2.10. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra and Q, $M \in A\text{-mod} \cap R\text{-proj}$. Assume that $Q - \operatorname{domdim}_{(A,R)} M \geq n \geq 1$ where the (A,R)-exact sequence*

$$0 \to M \to X_1 \to \cdots \to X_{n-1}, \tag{5.2.1.8}$$

where $X_i \in \operatorname{add} Q$, which remains exact under $\operatorname{Hom}_A(-,Q)$, can be continued to an (A,R)-exact sequence

$$0 \to M \to X_1 \to \cdots \to X_{n-1} \to Y \tag{5.2.1.9}$$

where $Y \in \operatorname{add} Q$. Then, $Q - \operatorname{domdim}_{(A,R)} M \geq n + 1$.

Proof. Apply D to the exact sequence (5.2.1.8). Denote by α_i the maps $DX_i \to DX_{i-1}$, where we fix $X_{-1} := DM$. Also the map $DY \to DX_{n-1}$ which we will denote by h admits a factorization through $\ker \alpha_{n-1}$, say $\nu \circ \pi$. Since B is a Noetherian R-algebra there exists $Z \in \operatorname{add} DQ$ such that there exists a surjective map $g \colon \operatorname{Hom}_A(DQ, Z) \to \operatorname{Hom}_A(DQ, \ker \alpha_{n-1})$. Further, by projectization, the map $\operatorname{Hom}_A(DQ, \nu) \circ g$ is equal to $\operatorname{Hom}_A(DQ, f)$ for some $f \in \operatorname{Hom}_A(Z, DX_{n-1})$. By construction, the exact sequence $Z \xrightarrow{f} DX_{n-1} \to \cdots \to DX_1 \to DM \to 0$ remains exact under $\operatorname{Hom}_A(DQ, -)$ and if exact it is (A,R)-exact. The remaining of the proof is a routine check that $\ker \alpha_{n-1} = \operatorname{im} f$. First, observe that $\operatorname{Hom}_A(DQ, \alpha_{n-1} \circ f) = \operatorname{Hom}_A(DQ, \alpha_{n-1}) \circ \operatorname{Hom}_A(DQ, \nu) \circ g = 0$.

Thus, $\alpha_{n-1} \circ f\chi^r_Z = \chi^r_{DX_{n-2}} \circ \mathrm{Hom}_A(DQ, \alpha_{n-1} \circ f) = 0$. So, $\alpha_{n-1} \circ f = 0$. By definition of kernel, there exists $s \in \mathrm{Hom}_A(Z, \ker \alpha_{n-1})$ such that $f = \nu \circ s$. Since $\mathrm{Hom}_A(DQ, -)$ is left exact, $g = \mathrm{Hom}_A(DQ, s)$. So, s is a right $\mathrm{add}\, DQ$-approximation of $\ker \alpha_{n-1}$. In particular, there exists $h_1 \in \mathrm{Hom}_A(DY, Z)$ such that $\pi = s \circ h_1$. Consequently, s is surjective, as well. This concludes the proof. ☐

Remark 5.2.11. We can observe that Theorem 2.8 of [KSX01] and consequently also Theorem 2.15 of [KSX01], are particular cases of Lemma 5.2.10 (when $n = 1$) and Theorem 5.2.2. △

Recall that ${}^\perp Q = \{M \in A\text{-mod} \cap R\text{-proj} \mid \mathrm{Ext}^{i>0}_A(M, Q) = 0\}$ is a resolving subcategory of A-mod$\cap R$-proj.

In contrast to Lemma 5.2.10, if we know well the last map in an exact sequence and its cokernel, then we can deduce the value of relative dominant dimension with respect to a module using that exact sequence.

Proposition 5.2.12. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra and $Q \in A\text{-mod} \cap R\text{-proj}$ so that $DQ \otimes_A Q \in R\text{-proj}$ and $\mathrm{Ext}^{i>0}_A(Q, Q) = 0$. Suppose that $M \in {}^\perp Q$. An exact sequence*

$$0 \to M \to Q_1 \to \cdots \to Q_n \tag{5.2.1.10}$$

yields $Q - \mathrm{domdim}_{(A,R)} M \geq n$ if and only if $Q_i \in \mathrm{add}\, Q$ and the cokernel of $Q_{n-1} \to Q_n$ belongs to ${}^\perp Q$.

Proof. Assume that $Q - \mathrm{domdim}_{(A,R)} M \geq n$. By definition, $Q_i \in \mathrm{add}\, Q$ and (5.2.1.10) is (A, R)-exact. Hence, the cokernel of $Q_{n-1} \to Q_n$ belongs to A-mod$\cap R$-proj. Denote by X_i the cokernel of $Q_{i-1} \to Q_i$ and fix $Q_0 = M$. Combining the conditions of $\mathrm{Ext}^{i>0}_A(Q_i, Q) = 0$, $\mathrm{Hom}_A(-, Q)$ being exact on (5.2.1.10) and $M \in {}^\perp Q$, it follows by induction on i that $X_i \in {}^\perp Q$.

Conversely, assume that $Q_i \in \mathrm{add}\, Q$ and the cokernel of $Q_{n-1} \to Q_n$ belongs to ${}^\perp Q$ which we will denote again by X_n. So, $X_n \in R$-proj and (5.2.1.10) is (A, R)-exact. It follows that $\mathrm{Ext}^1_A(X_i, Q) \simeq \mathrm{Ext}^{n-i+1}_A(X_n, Q) = 0$. This means that (5.2.1.10) remains exact under $\mathrm{Hom}_A(-, Q)$. So, the result follows. ☐

We note the following application of Lemma 5.2.9 useful in examples.

Corollary 5.2.13. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $DQ \otimes_A Q \in R\text{-proj}$ and $\mathrm{Ext}^{i>0}_A(Q, Q) = 0$. Let $M \in R\text{-proj}$ and consider the following (A, R)-exact sequence $0 \to M \to Q_1 \to \cdots \to Q_t \to X \to 0$. If $\mathrm{Ext}^i_A(X, Q) = 0$ for $1 \leq i \leq t$, then $Q - \mathrm{domdim}_{(A,R)} M = t + Q - \mathrm{domdim}_{(A,R)} X$.*

Proof. Let C_i be the image of the maps $Q_i \to Q_{i+1}$, $i = 1, \ldots, t-1$. Since $\mathrm{Ext}^{i>0}_A(Q, Q) = 0$, it follows that $\mathrm{Ext}^1_A(C_i, Q) \simeq \mathrm{Ext}^{t-i+1}_A(X, Q) = 0$. So, every exact sequence $0 \to C_i \to Q_{i+1} \to C_{i+1} \to 0$ remains exact under $\mathrm{Hom}_A(-, Q)$ (also if we consider $C_0 = M$ and $C_t = X$). By Lemma 5.2.9 and induction on t, the result follows. ☐

5.3 Change of rings on relative dominant dimension with respect to a module

We will now that, as the usual relative dominant dimension, relative dominant dimension with respect to a module behaves well under change of rings techniques. As usual, the following results also hold for right A-modules and consequently with codominant dimension in place of dominant dimension. For brevity, we will only consider the left versions.

Lemma 5.3.1. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $DQ \otimes_A Q \in R\text{-proj}$. Denote by B the endomorphism algebra $\mathrm{End}_A(Q)^{op}$. Assume that $M \in A\text{-mod} \cap R\text{-proj}$, satisfies the following two conditions:*

1. $\mathrm{Hom}_A(M,Q) \in R\text{-proj}$;
2. *The canonical map* $R(\mathfrak{m}) \otimes_R \mathrm{Hom}_A(M,Q) \to \mathrm{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}), Q(\mathfrak{m}))$ *is an isomorphism for every maximal ideal* $\mathfrak{m}$ *of* R.

Then, the following assertions are equivalent.

(a) $Q - \mathrm{domdim}_{(A,R)} M \geq 1$;

(b) $S \otimes_R Q - \mathrm{domdim}_{(S\otimes_R A, S)} S \otimes_R M \geq 1$ *for every commutative* R*-algebra* S *which is a Noetherian ring;*

(c) $Q_\mathfrak{m} - \mathrm{domdim}_{(A_\mathfrak{m}, R_\mathfrak{m})} M_\mathfrak{m} \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *of* R*;*

(d) $Q(\mathfrak{m}) - \mathrm{domdim}_{A(\mathfrak{m})} M(\mathfrak{m}) \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

Proof. Let S be a commutative R-algebra. Denote by D_S the standard duality with respect to S, $\mathrm{Hom}_S(-,S)$. The result follows from the following commutative diagram:

$$\begin{tikzcd}
S \otimes_R \mathrm{Hom}_A(DQ, DM) \otimes_B DQ \arrow[rr, "S\otimes_R \chi^r_{DM}"] \arrow[d, "\theta_{S,M}", "\simeq"'] & & S \otimes_R DM \arrow[dd, "\simeq"] \\
S \otimes_R \mathrm{Hom}_A(DQ, DM) \otimes_{S\otimes_R B} S \otimes_R DQ \arrow[d, "\varphi_S"] & & \\
\mathrm{Hom}_{S\otimes_R A}(D_S S \otimes_R Q, D_S S \otimes_R M) \otimes_{S\otimes_R B} D_S(S\otimes_R Q) \arrow[rr, "\chi^r_{D_S S \otimes_R M}"] & & D_S S \otimes_R M
\end{tikzcd} \tag{5.3.0.1}$$

where the map $\theta_{S,M}$ is the isomorphism given in Proposition 1.1.30 while φ_S is the tensor product of the canonical map given in Proposition 1.1.31 (which is not claimed at the moment to be an isomorphism) with the one providing the isomorphism $S \otimes_R DQ \simeq D_S S \otimes_R Q$.

The implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate. Assume that (i) holds. Then, χ^r_{DM} is surjective. By the commutative diagram, $\chi^r_{D_S S\otimes_R M}$ is surjective, and so (ii) follows. Assume that (iv) holds. By condition 2, $\varphi_{R(\mathfrak{m})}$ must be an isomorphism for every maximal ideal $\mathfrak{m}$ of R. Thus, by the diagram, $\chi^r_{DM}(\mathfrak{m})$ is surjective for every maximal ideal $\mathfrak{m}$ of R. By Nakayama's Lemma, χ^r_{DM} is surjective and (i) holds. ☐

Lemma 5.3.2. *Let* R *be a commutative Noetherian ring. Let* A *be a projective Noetherian* R*-algebra. Assume that* $Q \in A\text{-mod} \cap R\text{-proj}$ *satisfying in addition that* $DQ \otimes_A Q \in R\text{-proj}$. *Denote by* B *the endomorphism algebra* $\mathrm{End}_A(Q)^{op}$. *For* $M \in A\text{-mod} \cap R\text{-proj}$*, the following assertions are equivalent.*

(a) $Q - \mathrm{domdim}_{(A,R)} M \geq n \geq 1$;

(b) $Q - \mathrm{domdim}_{(S\otimes_R A, S)} S \otimes_R M \geq n \geq 1$ *for every flat commutative* R*-algebra which is a Noetherian ring;*

(c) $Q - \mathrm{domdim}_{(A_\mathfrak{m}, R_\mathfrak{m})} M_\mathfrak{m} \geq n \geq 1$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

Proof. By the flatness of S, the vertical maps of the commutative diagram (5.3.0.1) are isomorphisms. So, by Lemma 5.3.1, the implication $(a) \Rightarrow (b)$ is clear for $n = 1,2$. Again, since S is flat and B is finitely generated projective over R, $S \otimes_R -$ commutes with Tor functors over B. Therefore, (b) follows by Theorem 5.2.5. Analogously, we obtain $(c) \Rightarrow (a)$. ☐

It is no surprise that relative dominant dimension with respect to a module remains stable under extension of scalars to the algebraic closure. For sake of completeness, we give the result.

Lemma 5.3.3. *Let* k *be a field with algebraic closure* $\overline{k}$. *Let* A *be a finite-dimensional* k*-algebra and assume that* $Q \in A\text{-mod}$. *Then,* $\overline{k} \otimes_k Q - \mathrm{domdim}_{\overline{k}\otimes_k A} \overline{k} \otimes_k M = Q - \mathrm{domdim}_A M$.

Proof. Of course, $\overline{k}$ is free over k. Therefore, $\operatorname{Tor}_i^B(\operatorname{Hom}_A(DQ,DM),DQ)=0$ if and only if $\operatorname{Tor}_i^{\overline{k}\otimes_k B}(\operatorname{Hom}_{\overline{k}\otimes_k A}(\overline{k}\otimes_k DQ,\overline{k}\otimes_k DM),\overline{k}\otimes_k DQ)=0$. By the same reason, χ^r_{DM} is surjective (or bijective) if and only if $\chi^r_{\overline{k}\otimes_k DM}$ is surjective (or bijective). $\square$

Lemma 5.3.4. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q\in A\text{-mod}\cap R\text{-proj}$ satisfying in addition that $DQ\otimes_A Q\in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. Let $M\in A\text{-mod}\cap R\text{-proj}$, so that $\operatorname{Hom}_A(M,Q)\in R\text{-proj}$. Assume that S is a commutative R-algebra and a Noetherian ring such that the canonical map $S\otimes_R\operatorname{Hom}_A(M,Q)\to\operatorname{Hom}_{S\otimes_R A}(S\otimes_R M,S\otimes_R Q)$ is an isomorphism. Then, $Q-\operatorname{domdim}_{(A,R)}M\leq S\otimes_R Q-\operatorname{domdim}_{(S\otimes_R A,S)}S\otimes_R M$.*

Proof. Assume that $Q-\operatorname{domdim}_{(A,R)}M\geq n\geq 1$. Then, there exists an (A,R)-exact sequence $0\to M\to X_1\to\cdots\to X_n$ which remains exact under $\operatorname{Hom}_A(-,Q)$, where $X_i\in\operatorname{add}_A Q$. The functor $S\otimes_R-$ preserves R-split exact sequences. Hence, $0\to S\otimes_R M\to S\otimes_R X_1\to\cdots\to S\otimes_R X_n$ is $(S\otimes_R A,S)$-exact and $S\otimes_R X_i\in\operatorname{add}_{S\otimes_R A}S\otimes_R Q$. By assumption,

$$\operatorname{Hom}_A(X_n,Q)\to\operatorname{Hom}_A(X_{n-1},Q)\to\cdots\to\operatorname{Hom}_A(X_1,Q)\to\operatorname{Hom}_A(M,Q)\to 0 \tag{5.3.0.2}$$

is exact. Since $\operatorname{Hom}_A(M,Q)\in R\text{-proj}$ (5.3.0.2) splits over R. Thus, (5.3.0.2) remains exact under $S\otimes_R-$. Using the commutative diagram

$$\begin{tikzcd}
S\otimes_R\operatorname{Hom}_A(X_n,Q) \arrow[r] \arrow[d, "\simeq"] & S\otimes_R\operatorname{Hom}_A(X_{n-1},Q) \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r] & S\otimes_R\operatorname{Hom}_A(M,Q) \arrow[d, "\simeq"] \\
\operatorname{Hom}_{S\otimes_R A}(S\otimes_R X_n,S\otimes_R Q) \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R X_{n-1},S\otimes_R Q) \arrow[r] & \cdots \arrow[r] & \operatorname{Hom}_{S\otimes_R A}(S\otimes_R M,S\otimes_R Q)
\end{tikzcd}$$

it follows that the bottom row is exact. Hence, $S\otimes_R Q-\operatorname{domdim}_{(A,R)}S\otimes_R M\geq n$. $\square$

Finally, we reach the most important result in this section.

Theorem 5.3.5. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q\in A\text{-mod}\cap R\text{-proj}$ satisfying in addition that $DQ\otimes_A Q\in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. Assume that $M\in A\text{-mod}\cap R\text{-proj}$, satisfies the following two conditions*

1. $\operatorname{Hom}_A(M,Q)\in R\text{-proj}$;
2. *The canonical map $R(\mathfrak{m})\otimes_R\operatorname{Hom}_A(M,Q)\to\operatorname{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}),Q(\mathfrak{m}))$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R.*

Then, $Q-\operatorname{domdim}_{(A,R)}M=\inf\{Q(\mathfrak{m})-\operatorname{domdim}_{A(\mathfrak{m})}M(\mathfrak{m})\colon\mathfrak{m}\in\operatorname{MaxSpec}(R)\}$, where $\operatorname{MaxSpec}(R)$ denotes the set of maximal ideals of R.

Proof. By Lemma 5.3.4, $Q(\mathfrak{m})-\operatorname{domdim}_{A(\mathfrak{m})}M(\mathfrak{m})\geq Q-\operatorname{domdim}_{(A,R)}M$ for every maximal ideal $\mathfrak{m}$ of R. Conversely, assume that $Q(\mathfrak{m})-\operatorname{domdim}_{A(\mathfrak{m})}M(\mathfrak{m})\geq n$ for every maximal ideal $\mathfrak{m}$ of R. We want to show that $Q-\operatorname{domdim}_{(A,R)}M\geq n$. If $n=0$, then there is nothing to show. Using the analogue version of the commutative diagram (5.1.0.13) for χ^r we obtain that if $n\geq 1$ ($n\geq 2$), then $\chi^r_{DM}(\mathfrak{m})$ is surjective (is bijective) for every maximal ideal $\mathfrak{m}$ of R. By Nakayama's Lemma χ^r_{DM} is surjective and since $DM\in R\text{-proj}$, χ^r_{DM} is bijective in case $n\geq 2$. So, the inequality holds for $n=1,2$. Assume now that $n\geq 3$. In particular, $Q-\operatorname{domdim}_{(A,R)}M\geq 2$, and therefore $\operatorname{Hom}_A(DQ,DM)\otimes_B DQ\in R\text{-proj}$. By assumption, $\operatorname{Tor}_i^{B(\mathfrak{m})}(\operatorname{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}),Q(\mathfrak{m})),D_{(\mathfrak{m})}Q(\mathfrak{m}))=0$, where $D_{(\mathfrak{m})}=\operatorname{Hom}_{R(\mathfrak{m})}(-,R(\mathfrak{m}))$ and $1\geq i\geq n-2$ for every maximal ideal $\mathfrak{m}$ of R. Let $Q^\bullet$ be a deleted B-projective resolution of DQ. So the chain complex $P^\bullet=\operatorname{Hom}_A(M,Q)\otimes_B Q^\bullet$ is a projective complex over R since

$\operatorname{Hom}_A(M,Q) \in R$-proj. By Lemma 1.3.17, we have the Künneth spectral sequence for chain complexes

$$E^2_{i,j} = \operatorname{Tor}^R_j(H_i(\operatorname{Hom}_A(M,Q) \otimes_B Q^\bullet), R(\mathfrak{m})) \Rightarrow H_{i+j}(\operatorname{Hom}_A(M,Q) \otimes_B Q^\bullet(\mathfrak{m})). \tag{5.3.0.3}$$

Since $\operatorname{Hom}_A(M,Q) \otimes_B DQ \in R$-proj, $\operatorname{Hom}_A(M,Q) \otimes_B Q^\bullet(\mathfrak{m})$ becomes a deleted projective $B(\mathfrak{m})$-resolution of $DQ(\mathfrak{m})$. We shall proceed by induction on $1 \leq i \leq n-2$ to show that $\operatorname{Tor}^B_i(\operatorname{Hom}_A(M,Q), DQ) = 0$. By Lemma 1.3.7, $\operatorname{Tor}^B_1(\operatorname{Hom}_A(M,Q), DQ) \otimes_R R(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ of R. Hence, $\operatorname{Tor}^B_1(\operatorname{Hom}_A(M,Q), DQ) = 0$. Assume now that $\operatorname{Tor}^B_i(\operatorname{Hom}_A(M,Q), DQ) = 0$ for all $1 \leq i \leq l$ with $1 \leq l \leq n-2$ for some l. Then, $E^2_{i,j} = 0$, for $1 \leq i \leq l$, $j \geq 0$ and $E^2_{0,j} = 0$, $j > 0$. By Lemma 1.3.12, it follows that $\operatorname{Tor}^B_{l+1}(\operatorname{Hom}_A(M,Q), DQ)(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, $\operatorname{Tor}^B_i(\operatorname{Hom}_A(M,Q), DQ) = 0$ for $1 \leq i \leq n-2$. By Theorem 5.2.5, the result follows. ☐

Remark 5.3.6. The condition $DQ \otimes_A M \in R$-proj implies both of the conditions required in Theorem 5.3.5. △

Combining Theorem 5.3.5 with Lemma 5.3.3, we obtain that, in most applications, the computations of relative dominant dimension with respect to a module over a commutative ring can be reduced to computations of relative dominant dimension with respect to a module in the setup of algebraically closed fields.

It may seem unnatural the condition $DQ \otimes_A M \in R$-proj but we should refer once again that projective modules, or more generally tilting modules of split quasi-hereditary algebras do satisfy such a condition. The following result explains why we should expect that there are many modules with such a condition (see also [CPS96, 1.5.2(e), (f)]).

Lemma 5.3.7. *Let R be a commutative Noetherian ring and let A be a projective Noetherian R-algebra. Let $Q \in A$-mod$\cap R$-proj. If $\operatorname{Ext}^1_{A(\mathfrak{m})}(Q(\mathfrak{m}), Q(\mathfrak{m})) = 0$ for every maximal ideal $\mathfrak{m}$ of R, then $DQ \otimes_A Q \in R$-proj.*

Proof. For each maximal ideal $\mathfrak{m}$ of R,

$$\operatorname{Tor}^{A(\mathfrak{m})}_1(DQ(\mathfrak{m}), Q(\mathfrak{m})) = \operatorname{Hom}_{R(\mathfrak{m})}(\operatorname{Ext}^1_{A(\mathfrak{m})}(Q(\mathfrak{m}), Q(\mathfrak{m})), R(\mathfrak{m})) = 0. \tag{5.3.0.4}$$

Let $Q^\bullet$ be a deleted projective A-resolution of Q. Since $Q \in R$-proj, $Q^\bullet(\mathfrak{m})$ is a deleted projective $A(\mathfrak{m})$-resolution of $Q(\mathfrak{m})$. Thus, applying Lemma 1.3.17 with $P = DQ \otimes_A Q^\bullet$ we obtain

$$\begin{aligned} E^2_{i,j} = \operatorname{Tor}^R_i(\operatorname{Tor}^A_j(DQ,Q), R(\mathfrak{m})) &= \operatorname{Tor}^R_i(H_j(DQ \otimes_A Q^\bullet), R(\mathfrak{m})) \\ &\Rightarrow H_{i+j}(DQ \otimes_A Q^\bullet \otimes_R R(\mathfrak{m})) = \operatorname{Tor}^{A(\mathfrak{m})}_{i+j}(DQ(\mathfrak{m}), Q(\mathfrak{m})). \end{aligned} \tag{5.3.0.5}$$

By Lemma 1.3.7, $E^2_{1,0} = \operatorname{Tor}^R_1(DQ \otimes_A Q, R(\mathfrak{m})) = 0$ for every maximal ideal $\mathfrak{m}$ of R. Hence, $DQ \otimes_A Q \in R$-proj. ☐

5.4 Quality of cocovers on coresolving subcategories

Again, assume throughout this section that R is a commutative Noetherian ring, A is a projective Noetherian R-algebra, Q belongs to A-mod$\cap R$-proj with $DQ \otimes_A Q \in R$-proj and B is the endomorphism algebra $\operatorname{End}_A(Q)^{op}$.

We wish now to compare how high values of relative dominant dimension of A-modules with respect to a module Q influence the quality of the cocover (A,Q). In particular, our aim is to show that the higher the relative dominant dimension the more properties we can attach to the cocover. Ultimately, this will allow us to use cocovers to construct new covers.

Lemma 5.4.1. *Let $\mathscr{A}$ be a resolving subcategory of* A-mod$\cap R$-proj. *Then, the full subcategory* $D\mathscr{A} := \{DX : X \in \mathscr{A}\}$ *of* A-mod$\cap R$-proj *is a relative coresolving subcategory of* A-mod$\cap R$-proj.

Proof. Every (A,R)-injective module which is projective over the ground ring can be written as DP for some projective A-module P. Hence, $P \in \mathscr{A}$. $D\mathscr{A}$ is closed under direct summands since if $X \oplus Y \simeq DM \in D\mathscr{A}$, then $M \in R$-proj and $DX \oplus DY \simeq DDM \simeq M \in \mathscr{A}$. Thus, both $DX, DY \simeq \mathscr{A}$ and consequently $X \simeq DDX \in D\mathscr{A}$. It is closed under extensions since every extension is an (A,R)-exact sequence, and therefore it remains exact under D and the middle term is also projective over the ground ring. So, $\mathscr{A}$ being closed under extensions immediately implies that $D\mathscr{A}$ is closed under extensions. It remains to prove that $D\mathscr{A}$ is closed under cokernels of monomorphisms. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in A-mod$\cap R$-proj with $Y \simeq DY'$ and $Z \simeq DZ'$ where $Y', Z' \in \mathscr{A}$. So, the exact sequence under consideration is (A,R)-exact. Applying D and the fact that $\mathscr{A}$ is closed under kernels of epimorphisms it follows that $DZ \in \mathscr{A}$. Hence, $Z \simeq DDZ \in D\mathscr{A}$. $\square$

The definition of cocover motivates us to study faithfullness in relative coresolving subcategories instead of resolving subcategories. Here, relative coresolving subcategories because we just want the relative injective modules instead of the "absolute" injective modules. Hence, it is natural to make the following definition.

Definition 5.4.2. Let A be a projective Noetherian R-algebra. Let $\mathscr{C}$ be a relative coresolving subcategory of A-mod$\cap R$-proj. Let $B = \operatorname{End}_A(Q)^{op}$ and $i \geq 0$. We say that the pair (A,Q) is an $i-\mathscr{C}$ **cocover** of B if the functor $F = \operatorname{Hom}_A(Q,-)$ induces isomorphisms

$$\operatorname{Ext}_A^j(M,N) \to \operatorname{Ext}_B^j(FM,FN), \quad \forall M,N \in \mathscr{C}, j \leq i.$$

We say that (A,Q) is an $(-1)-\mathscr{C}$ cover of B if (A,Q) is a cocover of B and F induces monomorphisms

$$\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(FM,FN), \quad \forall M,N \in \mathscr{C}.$$

In the following, by an i-cocover we mean an $i-(A,R)$-inj$\cap R$-proj cocover.

Lemma 5.4.3. *Let $\mathscr{X}$ be the full subcategory of* A-mod$\cap R$-proj *whose modules X satisfy* $DQ-\operatorname{domdim}_{(A,R)} DX \geq 2$. *Then,* $\operatorname{Hom}_A(Q,-)$ *is fully faithful on $\mathscr{X}$.*

Proof. By Theorem 5.2.1, χ_X is an isomorphism for every $X \in \mathscr{X}$. Fix $F = \operatorname{Hom}_A(Q,-)$ and $\mathbb{I}$ its left adjoint. Then, if $Ff = 0$ for some $f \in \operatorname{Hom}_{\mathscr{X}}(M,N)$, we obtain $f \circ \chi_M = \chi_N \circ \mathbb{I}Ff = 0$. Hence, in such a case, $f = 0$. So, F is faithful. To show fullness, let $g \in \operatorname{Hom}_B(FM,FN)$ with $M,N \in \mathscr{X}$. Fixing $h = \chi_N \circ \mathbb{I}g \circ \chi_M^{-1}$ we get $Fh = g$. $\square$

Lemma 5.4.4. *Let $M \in A$-mod. Suppose that* $\operatorname{Tor}_i^B(Q,FM) = \mathrm{L}_i\mathbb{I}(FM) = 0$ *for $1 \leq i \leq q$. For any $X \in Q^{\perp} := \{Y \in A\text{-mod} \mid \operatorname{Ext}_A^{i>0}(Q,Y) = 0\}$, there are isomorphisms* $\operatorname{Ext}_A^i(\mathbb{I}FM,X) \simeq \operatorname{Ext}_B^i(FM,FX)$, $0 \leq i \leq q$, *and an exact sequence*

$$\begin{aligned}0 \to \operatorname{Ext}_A^{q+1}(\mathbb{I}FM,X) \to \operatorname{Ext}_B^{q+1}(FM,FX) \to \operatorname{Hom}_A(\operatorname{Tor}_{q+1}^B(Q,FM),X) \\ \to \operatorname{Ext}_A^{q+2}(\mathbb{I}FM,X) \to \operatorname{Ext}_B^{q+2}(FM,FX). \quad (5.4.0.1)\end{aligned}$$

Proof. Let $X \in A$-mod such that $\operatorname{Ext}_A^{i>0}(Q,X) = 0$. Fix $i = 0$. Then, by Tensor-Hom adjunction,

$$\operatorname{Hom}_A(\mathbb{I}FM,X) \simeq \operatorname{Hom}_B(FM,FX). \quad (5.4.0.2)$$

To obtain the result for higher values we will use Theorem 10.49 of [Rot09]. So, fix $f = \mathrm{Hom}_A(-,X)$ and $g = Q \otimes_B -$. f is a contravariant left exact and g is covariant. We note that gP is f-acyclic for any $P \in B$-proj. In fact, $\mathrm{R}^{j>0} f(gP) = \mathrm{Ext}_A^{j>0}(gP, X) = 0$, since $gP = Q \otimes_B P \in \mathrm{add}_A Q$. So, for each $a \in B$-mod, there is a spectral sequence

$$E_2^{i,j} = (\mathrm{R}^i f)(L_j g)(a) \Rightarrow \mathrm{R}^{i+j}(f \circ g)(a). \tag{5.4.0.3}$$

By Tensor-Hom adjunction $f \circ g(N) = \mathrm{Hom}_A(Q \otimes_B N, X) \simeq \mathrm{Hom}_B(N, \mathrm{Hom}_A(Q,X)) = \mathrm{Hom}_B(-, FX)(N)$, for every $N \in B$-mod. Hence, we can rewrite the previous spectral sequence into

$$E_2^{i,j} = \mathrm{Ext}_A^i(\mathrm{Tor}_j^B(Q,a), X) \Rightarrow \mathrm{Ext}_B^{i+j}(a, FX). \tag{5.4.0.4}$$

For each $M \in A$-mod, fix $a = FM$. By assumption, $\mathrm{Tor}_i^B(Q, FM) = 0$ for $1 \leq i \leq q$. Hence, $E_2^{i,j} = 0$, $1 \leq i \leq q$. By Lemma 1.3.10, the result follows. ☐

Theorem 5.4.5. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A$-mod$\cap R$-proj satisfying in addition that $DQ \otimes_A Q \in R$-proj. If $Q - \mathrm{codomdim}_{(A,R)} DA \geq n \geq 2$, then (A, Q) is an $(n-2)$-cocover of $\mathrm{End}_A(Q)^{op}$.*

Proof. Denote by B the endomorphism algebra $\mathrm{End}_A(Q)^{op}$. By Lemma 1.2.58, any (A,R)-injective module belongs to $Q^{\perp}$. By Proposition 5.1.2 and Theorem 5.2.1, (A,Q) is a cocover of $\mathrm{End}_A(Q)^{op}$. Further, it is also clear that $\mathrm{Hom}_A(M,X) \simeq \mathrm{Hom}_B(FM, FX) \in R$-proj for every (A,R)-injective projective R-modules M, X. The result now follows from Lemma 5.4.4. ☐

Note that this value is optimal if $\mathrm{Tor}_{n-1}^B(Q, DQ)$ is not just non-zero but also a projective R-module.

5.5 Relations between Ringel duality and cover theory

In Section 3.1, we were always comparing the quality of covers, through the computation of Hemmer-Nakano dimensions on certain resolving subcategories like A-proj and $\mathscr{F}(\tilde{\Delta})$ (in case A is split quasi-hereditary). So, the focus for cocovers should rely on (A,R)-inj$\cap R$-proj and $\mathscr{F}(\tilde{\nabla})$. In particular, we may wonder what information does the "level of faithfulness" on $\mathscr{F}(\tilde{\nabla})$ of a functor $\mathrm{Hom}_A(Q,-)$ provide for a given tilting module Q of a split quasi-hereditary algebra. It turns out that this pursuit will lead us back to cover theory. Moreover, this approach will show us a connection between Ringel duality and cover theory.

Recall that for a given set (possibly infinite) of modules Θ in A-mod$\cap R$-proj, $\mathscr{F}(\Theta)$ denotes the full subcategory of A-mod$\cap R$-proj whose modules admit a filtration by the modules in Θ.

Theorem 5.5.1. *Let R be a commutative Noetherian ring. Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by R_A the Ringel dual of A, that is $R_A = \mathrm{End}_A(T)^{op}$. Assume that $Q \in \mathrm{add}\, T$ is a (partial) tilting module of A. Then, the following assertions hold.*

(a) $Q - \mathrm{codomdim}_{(A,R)} \mathscr{F}(\tilde{\nabla}) = Q - \mathrm{codomdim}_{(A,R)} T = Q - \mathrm{codomdim}_{(A,R)} \bigoplus_{\lambda \in \Lambda} \nabla(\lambda)$.

(b) If $Q - \mathrm{codomdim}_{(A,R)} T \geq n \geq 2$, *then* (A,Q) *is an* $(n-2) - \mathscr{F}(\tilde{\nabla})$ *cocover of* $\mathrm{End}_A(Q)^{op}$.

(c) If $Q - \mathrm{codomdim}_{(A,R)} T \geq 3$, *then the functor* $F = \mathrm{Hom}_A(Q,-)$ *induces an exact equivalence between* $\mathscr{F}(\tilde{\nabla})$ *and* $\mathscr{F}(F\tilde{\nabla})$.

(d) *If* $Q - \operatorname{codomdim}_{(A,R)} T \geq n \geq 2$*, then* $(R_A, \operatorname{Hom}_A(T,Q))$ *is an* $(n-2)$-$\mathscr{F}(\tilde{\Delta}_{R_A})$ *split quasi-hereditary cover of* $\operatorname{End}_A(Q)^{op}$.

Proof. The proof of (a) is analogous to Theorem 2.11.1. Thanks to $\operatorname{Hom}_A(Q,-)$ and D being exact on short exact sequences of modules belonging to $\mathscr{F}(\tilde{\nabla})$ we obtain that we can apply Lemma 5.2.9 to the filtrations by costandard modules. Further, for every $X \in R$-proj so that $R^t \simeq X \oplus Y$

$$Q - \operatorname{codomdim}_{(A,R)} \nabla(\lambda) = Q - \operatorname{codomdim}_{(A,R)} \nabla(\lambda)^t \tag{5.5.0.1}$$

$$= \min\{Q - \operatorname{codomdim}_{(A,R)} \nabla(\lambda) \otimes_R X, Q - \operatorname{codomdim}_{(A,R)} \nabla(\lambda) \otimes_R Y\}. \tag{5.5.0.2}$$

Therefore, $Q - \operatorname{codomdim}_{(A,R)} \bigoplus_{\lambda \in \Lambda} \nabla(\lambda) = Q - \operatorname{codomdim}_{(A,R)} \mathscr{F}(\tilde{\nabla})$. Now using the exact sequences (1.5.14.2) together with Lemma 5.2.9 and the reasoning of Theorem 2.11.1, assertion (a) follows.

By Proposition 1.5.133, $DQ \otimes_A Q \in R$-proj. As before, By Lemma 1.2.58, any (A,R)-injective module belongs to $Q^{\perp}$. By Proposition 5.1.2 and Theorem 5.2.1, (A,Q) is a cocover of $\operatorname{End}_A(Q)^{op}$. By Theorem 5.2.5, $\operatorname{Tor}_i^B(Q,FM) = 0$, $1 \leq i \leq n-2$ for every $M \in \mathscr{F}(\tilde{\nabla})$. By Lemma 5.4.4, $\operatorname{Ext}_B^i(FM,FX) \simeq \operatorname{Ext}_A^i(\mathbb{I}FM,X)$ for $0 \leq i \leq n-2$, where $M,X \in \mathscr{F}(\tilde{\nabla})$. Since χ_M is an isomorphism for every $M \in \mathscr{F}(\tilde{\nabla})$, (b) follows.

By the exactness of F on $\mathscr{F}(\tilde{\nabla})$ and according to Lemma 1.1.33, $F(\nabla(\lambda) \otimes_R X) \simeq F\nabla(\lambda) \otimes_R X$ for every $\lambda \in \Lambda$ and $X \in R$-proj, the restriction of the functor F on $\mathscr{F}(\tilde{\nabla})$ has image in $\mathscr{F}(F\tilde{\nabla})$. By (b), it is enough to prove that for each module M in $\mathscr{F}(F\tilde{\nabla})$, there exists $N \in \mathscr{F}(\tilde{\nabla})$ so that $FN \simeq M$. By (b), the functor $\mathbb{I} = Q \otimes_B -$ is exact on short exact sequences of modules belonging to $\mathscr{F}(F\tilde{\nabla})$. Thanks to $Q \otimes_B F\nabla(\lambda) \otimes_R X \simeq \nabla(\lambda) \otimes_R X$ for every $X \in R$-proj and $\lambda \in \Lambda$, we obtain that $\mathbb{I}$ sends $\mathscr{F}(F\tilde{\nabla})$ to $\mathscr{F}(\tilde{\nabla})$. So, (c) follows.

Assume now that $Q - \operatorname{codomdim}_{(A,R)} T \geq n \geq 2$. Fix $B = \operatorname{End}_A(Q)^{op}$. By Lemma 1.5.121, for each $M \in \mathscr{F}(\tilde{\nabla})$,

$$\operatorname{Hom}_{R_A}(\operatorname{Hom}_A(T,Q), \operatorname{Hom}_A(T,M)) \simeq \operatorname{Hom}_A(Q,M) \tag{5.5.0.3}$$

as (B,R_A)-bimodules. In particular, $\operatorname{Hom}_{R_A}(\operatorname{Hom}_A(T,Q),R_A) \simeq \operatorname{Hom}_A(Q,T)$ as (B,R_A)-bimodules. By (c), F is fully faithful on $\mathscr{F}(\tilde{\nabla})$. Hence,

$$\operatorname{End}_B(\operatorname{Hom}_A(Q,T))^{op} \simeq \operatorname{End}_A(T)^{op} \tag{5.5.0.4}$$

and $\operatorname{End}_{R_A}(\operatorname{Hom}_A(T,Q))^{op} \simeq \operatorname{End}_A(Q)^{op}$. So, $(R_A, \operatorname{Hom}_A(T,Q))$ is a split quasi-hereditary cover of B. Now, by (b) and Lemma 1.5.121, for each $M \in \mathscr{F}(\tilde{\nabla})$,

$$\operatorname{Ext}_B^i(\operatorname{Hom}_A(Q,T), \operatorname{Hom}_{R_A}(\operatorname{Hom}_A(T,Q), \operatorname{Hom}_A(T,M))) \simeq \operatorname{Ext}_B^i(\operatorname{Hom}_A(Q,T), \operatorname{Hom}_A(Q,M)) \tag{5.5.0.5}$$

$$= \operatorname{Ext}_B^i(FT,FM) = 0, \quad 1 \leq i \leq n-2. \tag{5.5.0.6}$$

By Theorem 1.5.122 and Proposition 3.1.18, we conclude the proof. □

We note that since every projective module is the image of a (partial) tilting under the Ringel dual functor, every quasi-hereditary cover can be recovered/discovered using this approach. More precisely, every split quasi-hereditary algebra A is Morita equivalent to the Ringel dual of its Ringel dual R_{R_A} and every projective over R_{R_A} can be written as $\operatorname{Hom}_{R_A}(T_{R_A}, Q)$ for some $Q \in \operatorname{add} T_{R_A}$, where T_{R_A} is a characteristic tilting module of R_A. Hence, every split quasi-hereditary cover can be written in the form $(R_A, \operatorname{Hom}_A(T,Q))$ for some split quasi-hereditary algebra A, T a characteristic tilting module and $Q \in \operatorname{add} T$. This dramatically increases the scope of the theory of quasi-hereditary covers since before the main tools to construct these covers were the classical dominant dimension (covers related with a projective-injective module) and Dlab-Ringel standardization for 1-faithful quasi-hereditary covers. Further, if the split quasi-hereditary algebra A also has a duality the Ringel dual of A is a cover of the cellular algebra $\mathrm{End}_A(Q)^{op}$ whenever Q is a (partial) tilting module (that is, $Q \in \mathrm{add}\, T$ for a characteristic tilting module T) having a double centralizer property. Therefore, this description is our main theoretical example for our main problem of studying split quasi-hereditary covers of cellular algebras.

In Example 6.2.7, we can see an example where a quasi-hereditary cover can be constructed using relative dominant dimension with respect to a (partial) tilting (non projective-injective) and it cannot be constructed using Dlab-Ringel standardization. Moreover, in such an example $\mathrm{Hom}_A(T,Q)$ is not injective.

Remark 5.5.2. $T-\mathrm{codomdim}_{(A,R)}\mathscr{F}(\tilde{\nabla}) = T-\mathrm{codomdim}_{(A,R)} T = +\infty$ for a characteristic tilting module T. Of course, the Ringel dual is an infinite cover of itself. △

Remark 5.5.3. The cover constructed in Theorem 5.5.1 makes the following diagram commutative

$$\begin{tikzcd} \mathscr{F}(\tilde{\nabla}_A) \arrow[rr, "\mathrm{Hom}_A(T,-)"] \arrow[dr, "\mathrm{Hom}_A(Q,-)"'] & & \mathscr{F}(\tilde{\Delta}_{R_A}) \arrow[dl, "\mathrm{Hom}_{R_A}(\mathrm{Hom}_A(T,Q),-)"] \\ & \mathscr{F}(F\tilde{\nabla}_A) & \end{tikzcd} . \tag{5.5.0.7}$$

△

Corollary 5.5.4. *Let k be a field. Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary k-algebra with a characteristic tilting module T. Denote by R_A the Ringel dual of A, that is $R_A = \mathrm{End}_A(T)^{op}$. Assume that $Q \in \mathrm{add}\, T$ is a (partial) tilting module of A and $n \geq 2$ is a natural number. Then, $Q-\mathrm{codomdim}_{(A,R)} T \geq n \geq 2$ if and only if $(R_A, \mathrm{Hom}_A(T,Q))$ is an $(n-2)$-$\mathscr{F}(\tilde{\Delta}_{R_A})$ split quasi-hereditary cover of $\mathrm{End}_A(Q)^{op}$.*

Proof. It follows by Theorem 5.5.1, equations 5.5.0.3 and 5.5.0.6 together with Theorem 5.2.5. ☐

Therefore, the previous results say that the quality of faithful split quasi-hereditary covers of finite-dimensional algebras are controlled by the relative codominant dimension of characteristic tilting modules with respect to (partial) tilting modules.

Also, Theorem 5.5.1 says that for bound quiver algebras with dominant and codominant dimension larger than one with respect to a projective-injective module we can see which order should we choose (in case there is more than one) so that the algebra is split quasi-hereditary from a cover point of view.

As application of Theorem 5.5.1, we will establish in Theorem 6.1.4 one of the main findings of this PhD thesis. That is, we will construct a split quasi-hereditary cover (over any commutative ring) of the cellular algebra $\mathrm{End}_{S_R(n,d}(V^{\otimes d})$ without restrictions on n and d.

5.5.1 An analogue of Lemma 5.1.2 for Ringel duality

Lemma 5.5.5. *Let R be a commutative Noetherian ring. Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by R_A the Ringel dual of A, that is $R_A = \mathrm{End}_A(T)^{op}$. Assume that $Q \in \mathrm{add}\, T$ is a (partial) tilting module of A and fix $B = \mathrm{End}_A(Q)^{op}$. Then, the following assertions hold.*

(a) If $D\chi_T \colon DT \to \mathrm{Hom}_B(\mathrm{Hom}_A(Q,T),DQ)$ is an isomorphism, then $(R_A, \mathrm{Hom}_A(T,Q))$ is a split quasi-hereditary cover of B.

(b) If $D\chi^r_{DT} \colon DDT \to \mathrm{Hom}_B(\mathrm{Hom}_A(DQ,DT),DDQ)$ is an isomorphism, then $\mathrm{Hom}_A(T,Q)$ satisfies a double centralizer property between R_A and B.

Proof. By projectivization, $\mathrm{Hom}_A(T,Q) \in R_A$-proj and $\mathrm{End}_{R_A}(\mathrm{Hom}_A(T,Q))^{op} \simeq \mathrm{End}_A(Q)^{op} = B$. By (a) and Proposition 1.5.133, we have as (R_A, R_A)-bimodules

$$R_A = \mathrm{Hom}_A(T,T) \simeq \mathrm{Hom}_{A^{op}}(DT,DT) \simeq \mathrm{Hom}_{A^{op}}(DT, \mathrm{Hom}_B(\mathrm{Hom}_A(Q,T),DQ)) \tag{5.5.1.1}$$
$$\simeq \mathrm{Hom}_{A^{op}}(DT, \mathrm{Hom}_{B^{op}}(Q, D\mathrm{Hom}_A(Q,T))) \simeq \mathrm{Hom}_{B^{op}}(DT \otimes_A Q, D\mathrm{Hom}_A(Q,T)) \tag{5.5.1.2}$$
$$\simeq \mathrm{Hom}_B(\mathrm{Hom}_A(Q,T), \mathrm{Hom}_A(Q,T)). \tag{5.5.1.3}$$

Since $FR_A = \mathrm{Hom}_{R_A}(\mathrm{Hom}_A(T,Q), \mathrm{Hom}_A(T,T)) \simeq \mathrm{Hom}_A(Q,T)$ assertion (a) follows.

Now using the isomorphism χ^r_{DT} and Proposition 1.5.133 we obtain

$$R_A = \mathrm{Hom}_A(T,T) \simeq \mathrm{Hom}_A(T, \mathrm{Hom}_{B^{op}}(\mathrm{Hom}_A(T,Q),Q)) \simeq \mathrm{Hom}_A(T, \mathrm{Hom}_B(DQ, D\mathrm{Hom}_A(T,Q))) \tag{5.5.1.4}$$
$$\simeq \mathrm{Hom}_B(DQ \otimes_A T, D\mathrm{Hom}_A(T,Q)) \simeq \mathrm{Hom}_B(\mathrm{Hom}_A(T,Q), \mathrm{Hom}_A(T,Q)). \qquad \square$$

We did not yet address the case of $Q-\mathrm{codomdim}_{(A,R)} T = 1$. For this case, we can recover the Ringel dual being a cover using deformation theory.

Corollary 5.5.6. *Let R be a commutative regular Noetherian domain with quotient field K. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by R_A the Ringel dual of A, that is $R_A = \mathrm{End}_A(T)^{op}$. Assume that $Q \in \mathrm{add}\, T$ is a (partial) tilting module of A so that $Q-\mathrm{codomdim}_{(A,R)} T \geq 1$ and $K\otimes_R Q-\mathrm{codomdim}_{(K\otimes_R A)} K\otimes_R T \geq 2$. Then, $(R_A, \mathrm{Hom}_A(T,Q))$ is a split quasi-hereditary cover of $\mathrm{End}_A(Q)^{op}$. Moreover, $(R_A, \mathrm{Hom}_A(T,Q))$ is a $0-\mathscr{F}(\tilde{\Delta})$ split quasi-hereditary cover of $\mathrm{End}_A(Q)^{op}$.*

Proof. If $Q-\mathrm{codomdim}_{(A,R)} T \geq 2$, then this is nothing more than Theorem 5.5.1. Assume that $Q-\mathrm{domdim}_{(A,R)} T = 1$. By Theorem 5.2.1, χ_T is surjective. In view of Lemma 5.5.5, it is enough to prove that $D\chi_T$ is an isomorphism. Since $T \in R$-proj, χ_T is an (A,R)-epimorphism, and therefore $D\chi_T$ is an (A,R)-monomorphism. By assumption, $K\otimes_R Q-\mathrm{codomdim}_{(K\otimes_R A)} K\otimes_R T \geq 2$. Hence, thanks to the flatness of K, $K\otimes_R D\chi_T$ is an isomorphism.

Denote by X the cokernel of $D\chi_T$. As we saw, $K\otimes_R X = 0$. In particular, X is a torsion R-module. We cannot deduce right away that $\mathrm{Hom}_B(\mathrm{Hom}_A(Q,T),DQ) \in R$-proj but we can embed $\mathrm{Hom}_B(\mathrm{Hom}_A(Q,T),DQ)$ into $\mathrm{Hom}_R(\mathrm{Hom}_A(Q,T),DQ)$ which is projective over R due to both $\mathrm{Hom}_A(Q,T)$ and DQ being projective over R. So, $\mathrm{Hom}_B(\mathrm{Hom}_A(Q,T),DQ)$ is a torsion free R-module. As we did in Theorem 4.1.2.4, applying Proposition 3.4 of [AB59] to $D\chi_T$ we obtain that X must be zero, and consequently $D\chi_T$ is an isomorphism. Denote by F_R the Schur functor and G_R its adjoint of this cover. Observe that $\mathrm{Hom}_A(T,DA)$ is a characteristic tilting module of $R(A)$. Since $D\chi_T$ is a monomorphism and

$$\mathrm{Hom}_B(\mathrm{Hom}_A(Q,T),DQ) \simeq \mathrm{Hom}_B(\mathrm{Hom}_A(Q,T), \mathrm{Hom}_A(Q,DA)) \tag{5.5.1.5}$$
$$\simeq \mathrm{Hom}_B(\mathrm{Hom}_{R(A)}(\mathrm{Hom}_A(T,Q), \mathrm{Hom}_A(T,T)), \mathrm{Hom}_{R(A)}(\mathrm{Hom}_A(T,Q), \mathrm{Hom}_A(T,DA)))$$
$$\simeq G_R F_R \mathrm{Hom}_A(T,DA), \tag{5.5.1.6}$$

the claim follows by Proposition 3.1.5. $\square$

5.5.2 Ringel self-duality and uniqueness of covers

We will now see how can we relate Ringel self-duality with uniqueness of covers.

Corollary 5.5.7. *Let (A,P,V) be a relative Morita R-algebra. Assume that $\mathrm{domdim}_{(A,R)} T, \mathrm{codomdim}_{(A,R)} T \geq 3$, for a characteristic tilting module T. Then, there exists an exact equivalence $\mathscr{F}(F\tilde{\Delta}_A) \to \mathscr{F}(F\tilde{\nabla}_A)$ if and only if*

A is Morita equivalent as split quasi-hereditary algebra to its own Ringel dual.

Proof. By Theorems 5.5.1, 3.5.6 and 3.5.4, (A,P) is a 1-faithful split quasi-hereditary cover of $\operatorname{End}_A(P)^{op}$ and $(R_A, \operatorname{Hom}_A(T,P))$ is a 1-faithful split quasi-hereditary cover of $\operatorname{End}_A(P)^{op}$. As illustrated in Remark 5.5.3, F restricts to exact equivalences $\mathscr{F}(\tilde{\nabla}_A) \to \mathscr{F}(F\tilde{\nabla}_A)$ and $\mathscr{F}(\tilde{\Delta}_A) \to \mathscr{F}(F\tilde{\Delta}_A)$. Therefore, there exists an exact equivalence between $\mathscr{F}(F\tilde{\Delta}_A)$ and $\mathscr{F}(F\tilde{\nabla}_A)$ if and only if there exists an exact equivalence between $\mathscr{F}(\tilde{\Delta}_A)$ and $\mathscr{F}(\tilde{\nabla}_A)$. By Corollary 1.5.130, the latter is equivalent to A being Ringel self-dual. ☐

This is an indication that the phenomenon of Ringel self-duality behaves better the larger the dominant dimension of the characteristic tilting module. As before, for deformations we can weaken the conditions on the dominant and codominant dimension of the characteristic tilting module.

Corollary 5.5.8. *Let R be an integral regular domain with quotient field K. Let (A,P,V) be a relative Morita R-algebra. Fix $B = \operatorname{End}_A(P)^{op}$. Assume the following conditions hold.*

(i) $(K\otimes_R A, K\otimes_R P)$ *is a 1-faithful split quasi-hereditary cover of B;*

(ii) $(K\otimes_R A, K\otimes_R P)$ *is a $1-\mathscr{F}(\tilde{\nabla}_{K\otimes_R A})$ cocover of B;*

(iii) $\operatorname{domdim}_{(A,R)} T$, $\operatorname{codomdim}_{(A,R)} T \geq 2$ *for a characteristic tilting module T;*

(iv) There exists an exact equivalence $\mathscr{F}(F\tilde{\Delta}_A) \to \mathscr{F}(F\tilde{\nabla}_A)$.

Then, A is Morita equivalent as split quasi-hereditary algebra to its own Ringel dual.

Proof. Observe that $\operatorname{domdim}_{A(\mathfrak{m})} T(\mathfrak{m}) \geq 2$ and $\operatorname{domdim}_{A^{op}(\mathfrak{m})} DT(\mathfrak{m}) = \operatorname{codomdim}_{A(\mathfrak{m})} T(\mathfrak{m}) \geq 2$ for every maximal ideal $\mathfrak{m}$ of R. By Theorem 5.5.1 and Corollary 1.5.118,

$$(R_{K\otimes_R A}, \operatorname{Hom}_{K\otimes_R A}(K\otimes_R T, K\otimes_R P)) = (K\otimes_R R_A, K\otimes_R \operatorname{Hom}_A(T,P))$$

is a 1-faithful split quasi-hereditary cover of $K\otimes_R B$, and $(R_{A(\mathfrak{m})}, \operatorname{Hom}_A(T,P)(\mathfrak{m}))$ is a 0-faithful split quasi-hereditary cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. By Theorem 3.5.6, $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a 0-faithful split quasi-hereditary cover of $B(\mathfrak{m})$ for every maximal ideal $\mathfrak{m}$ of R. By Theorem 3.3.11, $(R_A, \operatorname{Hom}_A(T,P))$ and (A,P) are 1-faithful split quasi-hereditary covers of B. By Remark 5.5.3 and Proposition 3.1.13, there exists an exact equivalence,

$$\mathscr{F}(\tilde{\Delta}_A) \to \mathscr{F}(F\tilde{\Delta}_A) \xrightarrow{(iv)} \mathscr{F}(F\tilde{\nabla}_A) \to \mathscr{F}(\tilde{\nabla}_A). \tag{5.5.2.1}$$

By Corollary 1.5.130, A being Ringel self-dual. ☐

As an application of Corollary 5.5.8 we obtain a new proof for the fact that the Schur algebras $S_{\mathbb{Z}[\frac{1}{2}]}(n,d)$ are Ringel self-dual for $n \geq d$.

So far, the author has not been able to find an example of split quasi-hereditary algebra, not being Ringel self-dual, with a characteristic tilting module having relative dominant and codominant dimension bigger than 2. In view of Corollary 5.5.7, such an example would provide a case where there are at least two covers of B having a large level of faithfulness if one drops the condition about the filtrations of Corollary 3.6.6.

5.6 Relative dominant dimension with respect to a module in the image of a Schur functor preserving the highest weight structure

Many split quasi-hereditary algebras can be written as endomorphism algebras of certain projective modules Ae over a bigger quasi-hereditary algebra A. This is the case for Schur algebras $S_K(n,d)$ when $n<d$ (recall Theorem 1.7.5 and Proposition 1.7.7). Further, if the bigger algebra A has large relative dominant dimension with respect to a projective-(A,R)-injective module P, then one can ask if this can be used to compute the relative dominant dimension of eAe with respect to the partial tilting module eP.

As we saw in Theorem 5.3.5, we can restrict ourselves to the finite-dimensional algebras for the computations of relative dominant dimension of costandard modules with respect to a partial tilting module.

Theorem 5.6.1. *Let k be a field and $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over k. Assume that there exists an idempotent e of A such that both e and A satisfy the conditions of Theorem 1.7.5. Suppose that P is a projective-injective faithful module. Let $M\in\mathscr{F}(\nabla)$. If* $\operatorname{codomdim}_A M\geq i$*, then* $eP-\operatorname{codomdim}_{eAe} eM\geq i$ *for $i\in\{1,2\}$.*

Proof. Denote by $B=\operatorname{End}_A(P)^{op}$ and by $C=\operatorname{End}_{eAe}(eP)^{op}$. Since P is a (partial) tilting module the map given by multiplication by e, $B\to C$ is surjective according to Proposition 1.7.7. Thus, C is a quotient of B. In particular, C-mod is a full subcategory of B-mod. Again by Proposition 1.7.7, the map $\operatorname{Hom}_A(P,M)\to\operatorname{Hom}_{eAe}(eP,eM)$ is a surjective left B-homomorphism. Denote such a map by φ_M. We can consider the following commutative diagram

$$\begin{tikzcd}
e\cdot(P\otimes_B\operatorname{Hom}_A(P,M)) \arrow[r, equal] \arrow[d, "e\delta_{DM}"] & (eP)\otimes_B\operatorname{Hom}_A(P,M) \arrow[r, "(e\cdot P)\otimes_B\varphi_M"] & (eP)\otimes_B\operatorname{Hom}_{eAe}(eP,eM) \arrow[r, equal] & (eP)\otimes_C\operatorname{Hom}_{eAe}(eP,eM) \arrow[d, "\chi_{eM}"] \\
eM \arrow[rrr, equal] & & & eM
\end{tikzcd}$$

with the composition of the upper rows being surjective (see also Remark 5.1.1). In fact, thanks to the C-mod being a full subcategory of B-mod we have the isomorphisms

$$D((eP)\otimes_C\operatorname{Hom}_{eAe}(eP,eM))\simeq\operatorname{Hom}_C(\operatorname{Hom}_{eAe}(eP,eM),D(eP))=\operatorname{Hom}_B(\operatorname{Hom}_{eAe}(eP,eM),D(eP)) \quad (5.6.0.1)$$

$$\simeq D((eP)\otimes_B\operatorname{Hom}_{eAe}(eP,eM)). \quad (5.6.0.2)$$

Since $\operatorname{domdim}_{A^{op}} DM=\operatorname{codomdim}_A M\geq 1$(resp. 2) if and only if δ_{DM} is surjective (resp. isomorphism) we obtain that $e\delta_{DM}$ is surjective if $i=1$ and bijective if $i=2$. So, if $i=1$ it follows that χ_{eM} is surjective, by the commutative diagram. Assume that $i=2$. Then, $(e\cdot P)\otimes_B\varphi_M$ must be injective, and so it is an isomorphism. This implies that χ_{eM} is also an isomorphism. ☐

For larger values of relative dominant dimension the most natural approach to consider is to see when the exact sequence giving the value of dominant dimension under the Schur functor $eA\otimes_A-$ gives information about the relative dominant dimension of eAe with respect to eP. As we know, we can focus only in what happens over finite-dimensional algebras over a field.

Proposition 5.6.2. *Let k be a field. Let $(A,\{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary k-algebra. Suppose that A has dominant dimension at least n with exact sequence*

$$0\to A\to P_0\to\cdots\to P_{n-1}, \quad (5.6.0.3)$$

which we will denote by δ, where $P_i \in \operatorname{add} P$, $i = 0, \dots, n-1$ for a projective-injective module P. Assume, in addition, the existence of an idempotent e in the conditions of Theorem 1.7.5. Then, the exact sequence $e\delta$ remains exact under $\operatorname{Hom}_{eAe}(-, eP)$ if and only if $P \in \operatorname{add} D(eA)$. In particular, if $e\delta$ remains exact under $\operatorname{Hom}_{eAe}(-, eP)$, then eP is a projective-injective eAe-module.

Proof. Assume that $P \in \operatorname{add}_A D(eA)$. Then, $eP \in \operatorname{add}_{eAe} D(eAe)$, that is, eP is injective over eAe. It is clear that the functor $\operatorname{Hom}_{eAe}(-, eP)$ is exact.

Conversely, suppose that $e\delta$ remains exact under $\operatorname{Hom}_{eAe}(-, eP)$. Let X_0 be the cokernel of $A \to P_0$. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(P_1,P) \arrow[rr] \arrow[dr, two heads] \arrow[ddd, two heads] & & \operatorname{Hom}_A(P_0,P) \arrow[r, two heads] \arrow[ddd, two heads] & \operatorname{Hom}_A(A,P) \arrow[ddd, two heads] \\
& \operatorname{Hom}_A(X_0,P) \arrow[ur, hook] \arrow[d] & & \\
& \operatorname{Hom}_{eAe}(eX_0,eP) \arrow[dr, hook] & & \\
\operatorname{Hom}_{eAe}(eP_1,eP) \arrow[rr] \arrow[ur] & & \operatorname{Hom}_{eAe}(eP_0,eP) \arrow[r, two heads] & \operatorname{Hom}_{eAe}(eA,eP)
\end{tikzcd} \tag{5.6.0.4}$$

The vertical maps are surjective maps due to Proposition 1.7.7. By assumption, the bottom row of (5.6.0.4) is exact. Hence, the lower triangle is a epi-mono factorization. Therefore, $\operatorname{Hom}_A(X_0, P) \to \operatorname{Hom}_{eAe}(eX_0, eP)$ is surjective. By Snake Lemma, we obtain that the map $\operatorname{Hom}_A(A, P) \to \operatorname{Hom}_{eAe}(eA, eP)$ is in addition to being surjective an injective map. Since eA has a filtration by standard modules over eAe, $\operatorname{Ext}^{i>0}_{eAe}(eA, eP)$. By Lemma 2.10 of [GK15] for every $M \in A$-mod,

$$\operatorname{Hom}_A(M,P) \simeq \operatorname{Hom}_A(M, \operatorname{Hom}_{eAe}(eA, eP)) \simeq \operatorname{Hom}_{eAe}(eM, eP). \tag{5.6.0.5}$$

By Theorem 3.10 of [Psa14], this means that there exists an exact sequence $0 \to P \to \operatorname{Hom}_{eAe}(eA, D(eAe)) \simeq D(eA)$. Since P is injective, this exact sequence splits and we obtain that $P \in \operatorname{add}_A D(eA)$. ☐

For Schur algebras, this is only true in case $V^{\otimes d}$ is projective-injective module since it is a partial tilting module. We can however give a lower bound to the relative dominant dimension with respect to $V^{\otimes d}$ based on its injective dimension.

Corollary 5.6.3. *Let k be a field and A a finite-dimensional k-algebra. Let $Q \in A$-mod with $\operatorname{Ext}^{i>0}_A(Q,Q) = 0$. Suppose that $M \in {}^{\perp}Q$ and assume that there exists an A-exact sequence*

$$0 \to M \to Q_1 \to \cdots \to Q_n, \tag{5.6.0.6}$$

with $Q_i \in \operatorname{add} Q$. Then, $Q - \operatorname{domdim}_A M \geq n - \operatorname{idim}_A Q$.

Proof. Assume that $n > \operatorname{idim}_A Q$, otherwise there is nothing to prove. Denote by X_i the cokernel of $Q_{i-1} \to Q_i$ where by convention we consider $Q_0 := M$. By dimension shifting,

$$\operatorname{Ext}^{i>0}_A(X_{n-\operatorname{idim}_A Q}, Q) \simeq \operatorname{Ext}^{i+1>0}_A(X_{n-\operatorname{idim}_A Q+1}, Q) \simeq \operatorname{Ext}^{i+\operatorname{idim}_A Q>0}_A(X_n, Q) = 0. \tag{5.6.0.7}$$

So, the exact sequence $0 \to M \to Q_1 \to \cdots \to Q_{n-\operatorname{idim}_A Q}$ satisfies the assumptions of Proposition 5.2.12. ☐

We should mention that, for Schur algebras, even the global dimension remains an open problem for the case $n < d$.

Other approach is to consider the homology over $\operatorname{End}_{eAe}(eP)$ by regarding the algebra as a quotient of $\operatorname{End}_A(P)$. This surjective map is not, in general, an homological epimorphism (see Remark 6.2.6). As it turns out, we do not need such assumption on the map $\operatorname{End}_A(P)^{op} \to \operatorname{End}_{eAe}(eP)^{op}$ to give lower bounds of codominant dimension with respect to eP using the codominant dimension with respect to P. We can use, instead, the techniques of truncation of covers. This techniques are only fruitful for values of Hemmer-Nakano dimension greater than or equal to zero but this poses no problem in our situation since the lower cases can be treated using Theorem 5.6.1.

Theorem 5.6.4. *Let k be a field and $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary k-algebra with a duality ω. Assume that there exists an idempotent e of A such that both e and A satisfy the conditions of Theorem 1.7.5. If A is a gendo-symmetric algebra with faithful projective-injective Af and $\omega(f) = f$, then*

$$eAf - \operatorname{domdim}_{eAe} eT \geq \operatorname{domdim}_A T,$$

where T is the characteristic tilting module of A.

Proof. We can assume without loss of generality that A is a basic algebra and Af is also basic. If $\operatorname{domdim}_A T = 1$, then the result follows from Theorem 5.6.1. Assume that $\operatorname{domdim}_A T \geq 2$.

By Proposition 1.7.7, eT is the characteristic tilting module of eAe. Hence, the endomorphism algebra $\operatorname{End}_{eAe}(eT)^{op}$ is the Ringel dual of eAe which we will denote by R_{eAe}. Also, by Proposition 1.7.7 there exists an exact sequence $0 \to X \to R_A \to R_{eAe} \to 0$ where X is an ideal of the Ringel dual of A. More precisely, X is the set of all endomorphisms $g \in \operatorname{End}_A(T)$ satisfying $eg = 0$. Fix $P = Af$. We claim that $X \operatorname{Hom}_A(T,P)$ is the kernel of the surjective map $\operatorname{Hom}_A(T,P) \to \operatorname{Hom}_{eAe}(eT, eP)$. Denote this surjection by ψ. Let $g \in X$ and $l \in \operatorname{Hom}_A(T,P)$ then $e(lg) = (el)(eg) = 0$. So, it is clear that $X \operatorname{Hom}_A(T,P) \subset \ker \psi$. Now, let $l \in \operatorname{Hom}_A(T,P)$ such that $el = 0$, that is, $l \in \ker \psi$. By assumption, we can write $i \circ \pi = \mathrm{id}_P$, where $\pi \in \operatorname{Hom}_A(T,P)$. So, $e(i \circ l) = ei \circ el = 0$. This means that $i \circ l \in X$. Now $l = \pi \circ i \circ l = (i \circ l) \cdot \pi \in X \operatorname{Hom}_A(T,P)$. Now, a k-basis of $\operatorname{End}_A(T)$ can be constructed using its filtration by modules $\operatorname{Hom}_A(\Delta(\nu), \nabla(\nu))$, $\nu \in \Lambda$ and the liftings of $\Delta(\lambda) \hookrightarrow T(\lambda) \twoheadrightarrow \nabla(\lambda)$ along these filtrations (see Proposition 1.5.117). In particular, these maps factor through $T(\lambda)$, $\lambda \in \Lambda$. By assumption, $eS(\lambda) = 0$ if and only if $\lambda < \mu$ for a fixed $\mu \in \Lambda$, and so $eT(\lambda) = 0$ if and only if $\lambda < \mu$. Analogously, $\operatorname{End}_{eAe}(eT)$ has a k-basis of the maps factoring through $eT(\lambda) \neq 0$. So, X has a basis whose maps $T \to T$ factor through $T(\lambda)$, $\lambda < \mu$. Let g_λ denote the idempotent $T \twoheadrightarrow T(\lambda) \hookrightarrow T$ and $g_e = \sum_{\lambda<\mu} g_\lambda$. Then, we showed that $X = R_A g_e R_A$. In particular, X has a filtration by split heredity ideals of quotients of R_A.

As $\operatorname{codomdim}_A T \geq 2$, Theorem 5.5.1 implies that $(R_A, \operatorname{Hom}_A(T,P))$ is a $(\operatorname{codomdim}_A T - 2) - \mathscr{F}(\Delta_{R_A})$ cover of $\operatorname{End}_A(P)^{op}$. By induction on the filtration of X by split heredity ideals and using Theorem 3.4.1, we obtain that

$$(\operatorname{End}_{eAe}(eT), \operatorname{Hom}_{eAe}(eT, eP)) \simeq (R_A/X, \operatorname{Hom}_A(T,P)/X \operatorname{Hom}_A(T,P)) \tag{5.6.0.8}$$

is a $\operatorname{codomdim}_A T - 2 - \mathscr{F}(\Delta_{R_{eAe}})$ cover of $\operatorname{End}_{R_A/X}(\operatorname{Hom}_A(T,P)/X \operatorname{Hom}_A(T,P))^{op}$ which is isomorphic to $\operatorname{End}_{R_{eAe}}(\operatorname{Hom}_{eAe}(eT, eP))^{op} \simeq \operatorname{End}_{eAe}(eP)^{op}$. By Lemma 5.4.4 and 5.2.5, it follows that

$$eP - \operatorname{codomdim}_{eAe} eT = eAf - \operatorname{domdim}_{eAe} eT \geq \operatorname{domdim}_A T = \operatorname{codomdim}_A T.$$

□

5.7 The reduced grade with respect to a module

For this section, we return to the general case of R being a Noetherian commutative ring and A a projective Noetherian R-algebra. In [GK15], Koenig and Gao compared the Auslander-Bridge grade with dominant dimension. We will now see that the same method also works for relative dominant dimension over any ring with respect to a module once we replace the Ext in the notion of grade by Tor, giving rise to the name cograde. Also, this technique has the advantage of avoiding to deal with approximations. There is, however, another modification that needs to be considered. We are not interested in the case of grade being zero, and so we will instead talk about the dual notion of reduced grade (see for example [Hos90]). Roughly speaking, the reduced grade will coincide with the notion of grade if the grade is non-zero otherwise the reduced grade is bigger than the grade.

Definition 5.7.1. Let R be a Noetherian commutative ring and A a projective Noetherian R-algebra. Let $X \in \text{mod-}A \cap R\text{-proj}$ and $M \in A\text{-mod} \cap R\text{-proj}$. The reduced cograde of X with respect to M, written as $\operatorname{rcograde}_M X$, is defined as the value

$$\operatorname{rcograde}_M X = \inf\{i > 0 | \operatorname{Tor}_i^A(X, M) \neq 0\}.$$

Analougously, we can define the reduced cograde of a right module with respect to a left module.

The following is based on Theorem 2.3 of [GK15].

Theorem 5.7.2. *Let R be a Noetherian commutative ring and A a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod}\cap R\text{-proj}$ satisfying in addition that $DQ \otimes_A Q \in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For any $Y \in A\text{-mod}\cap R\text{-proj}$ with an exact sequence*

$$Q_1 \xrightarrow{f} Q_0 \to Y \to 0, \tag{5.7.0.1}$$

define $X = \operatorname{coker} \operatorname{Hom}_A(f, Q) \in \text{mod-}B$. Then, $Q-\operatorname{domdim}_{(A,R)} Y \geq n \geq 1$ if and only if $\operatorname{rcograde}_{DQ} X \geq n+1$.

Proof. Applying the functor $\operatorname{Hom}_A(-, Q)$ yields the exact sequence

$$0 \to \operatorname{Hom}_A(Y, Q) \to \operatorname{Hom}_A(Q_0, Q) \xrightarrow{\operatorname{Hom}_A(f,Q)} \operatorname{Hom}_A(Q, Q) \twoheadrightarrow X. \tag{5.7.0.2}$$

Denote by C the kernel of $\operatorname{Hom}_A(Q, Q) \to X$ which is the same as the image of $\operatorname{Hom}_A(f, Q)$. Since $\operatorname{Hom}_A(Q, Q) \in \text{proj-}B$ by applying $-\otimes_B DQ$ we obtain the exact sequence

$$0 \to \operatorname{Tor}_1^B(X, DQ) \to C \otimes_B DQ \xrightarrow{\iota} \operatorname{Hom}_A(Q_1, Q) \otimes_B DQ \to X \otimes_B DQ \to 0 \tag{5.7.0.3}$$

and $\operatorname{Tor}_{i+1}^B(X, DQ) = \operatorname{Tor}_i^B(C, DQ)$, $i \geq 1$. Since $Y \in R$-proj we can consider the (A,R)-exact sequence $0 \to DY \to DQ_0 \to D\operatorname{im} f \to 0$. Applying $\operatorname{Hom}_A(DQ, -)$ to such exact sequence we obtain the commutative diagram

$$\begin{tikzcd}
\operatorname{Tor}_1^B(C, DQ) \arrow[r, hook] & \operatorname{Hom}_A(DQ, DY) \otimes_B DQ \arrow[r] \arrow[d, "\chi_{DY}^r"] & \operatorname{Hom}_A(DQ, DQ_0) \otimes_B DQ \arrow[r, two heads, "\pi_2"] \arrow[d, "\chi_{DQ_0}^r", "\simeq"'] & C \otimes_B DQ \\
0 \arrow[r] & DY \arrow[r] & DQ_0 \arrow[r, two heads] & D\operatorname{im} f
\end{tikzcd} \tag{5.7.0.4}$$

and $\operatorname{Tor}_{i+1}^B(C, DQ) \simeq \operatorname{Tor}_i^B(\operatorname{Hom}_A(DQ, DY), DQ)$ for all $i \geq 1$. Thus, $\operatorname{Tor}_i^B(\operatorname{Hom}_A(DQ, DY), DQ) \simeq \operatorname{Tor}_{i+2}^B(X, DQ)$, for all $i \geq 1$. By the commutativity of the diagram (5.7.0.4) we can complete the diagram with a map $g\colon C \otimes_B DQ \to D\operatorname{im} f$. By Snake Lemma, there exists an exact sequence $\ker \chi_{DY}^r \to 0 \to \ker g \to \operatorname{coker} \chi_{DY}^r \to 0 \to \operatorname{coker} g \to 0$. Also, by the diagram (5.7.0.4) we obtain that $\ker \chi_{DY}^r \simeq \operatorname{Tor}_1^B(C, DQ)$.Therefore, $\ker \chi_{DY}^r \simeq \operatorname{Tor}_2^B(X, DQ)$. So, it remains to show that $\operatorname{coker} \chi_{DY}^r = 0$ if and only if $\operatorname{Tor}_1^B(X, DQ) = 0$. For that, consider the diagram

$$\begin{tikzcd} DQ_0 \arrow[r, two heads, "\pi_1"] \arrow[dr, phantom, "\diamond"] & \operatorname{Dim} f \arrow[r, hook, "k"] \arrow[dr, phantom, "\star"] & DQ_1 \\ \operatorname{Hom}_A(DQ, DQ_0)\otimes_B DQ \arrow[u, "\chi^r_{DQ_0}"] \arrow[r, two heads, "\pi_2"] & C\otimes_B DQ \arrow[u, "g"] \arrow[r, "\iota"] & \operatorname{Hom}_A(DQ, DQ_1)\otimes_B DQ \arrow[u, "\chi^r_{DQ_1}", "\simeq"'] \end{tikzcd} . \tag{5.7.0.5}$$

By construction of g, $\diamond$ is a commutative diagram. Since C is isomorphic to the image of $\operatorname{Hom}_A(DQ, Df)$, $\operatorname{Hom}_A(DQ, Df)\otimes_B DQ$ factors through $C\otimes_B DQ$. More precisely, $\iota\circ\pi_2 = \operatorname{Hom}_A(DQ, Df)\otimes_B DQ$. Observe that $k\circ\pi_1 = Df$. Hence, the external diagram is commutative. Therefore,

$$k\circ g\circ\pi_2 = k\circ\pi_1\circ\chi^r_{DQ_0} = Df\chi^r_{DQ_0} = \chi^r_{DQ_1}\circ\operatorname{Hom}_A(DQ, Df)\otimes_B DQ = \chi^r_{DQ_1}\circ\iota\circ\pi_2. \tag{5.7.0.6}$$

By the surjectivity of π_2, the diagram $\star$ is commutative. Now, assume that, $\operatorname{coker}\chi^r_{DY} = \ker g = 0$, then the diagram $\star$ implies that ι is injective. By (5.7.0.3), $\operatorname{Tor}_1^B(X, DQ) = 0$. Conversely, suppose that $\operatorname{Tor}_1^B(X, DQ) = 0$. Then, ι is injective and $k\circ g = \chi^r_{DQ_1}\circ\iota$ is injective. Thus, g is injective and χ^r_{DY} is surjective. ☐

5.8 Wakamatsu tilting conjecture for quasi-hereditary algebras

In this section, we apply Theorem 5.5.1 to deduce that a Wakamatsu tilting module which is also a (partial) tilting module over a quasi-hereditary algebra must be a characteristic tilting module.

Theorem 5.8.1. *Let R be a Noetherian commutative ring and $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra. Assume that T is a characteristic tilting module and $Q\in\operatorname{add}_A T$ is a partial tilting module.*

If $Q - \operatorname{domdim}(A, R) = +\infty$, then Q is a characteristic tilting module of A.

Proof. Consider first that R is a field. By assumption, $DQ - \operatorname{codomdim}_{(A^{op}, R)} DA = +\infty$. Since $\operatorname{Hom}_{A^{op}}(DQ, -)$ is exact on $\mathscr{F}(\tilde{\nabla}_{A^{op}})$ we obtain by Lemma 5.2.9 that $DQ - \operatorname{codomdim}_{(A^{op}, R)}\mathscr{F}(\tilde{\nabla}_{A^{op}}) = +\infty$. By Theorem 5.5.1, $(\operatorname{End}_{A^{op}}(DT)^{op}, \operatorname{Hom}_{A^{op}}(DT, DQ))$ is an $+\infty$ faithful split quasi-hereditary cover of $\operatorname{End}_{A^{op}}(DQ)^{op}$. By Corollary 3.2.3, $\operatorname{End}_{A^{op}}(DT)^{op}$ is Morita equivalent to $\operatorname{End}_{A^{op}}(DQ)^{op}$. In particular, by projectization, DT and DQ have the same number of indecomposable modules. Therefore, $\operatorname{add}_{A^{op}} DQ = \operatorname{add}_{A^{op}} DT$, and so Q is a characteristic tilting module. Assume now that R is an arbitrary Noetherian commutative ring. If $Q - \operatorname{domdim}_{(A,R)} = +\infty$, then we have $Q(\mathfrak{m}) - \operatorname{domdim}_{A(\mathfrak{m})} = +\infty$ for every maximal ideal $\mathfrak{m}$ of R. Hence, $Q(\mathfrak{m})$ is a characteristic tilting module for every maximal ideal $\mathfrak{m}$ of R. By Proposition 1.5.131, we conclude that Q is a characteristic tilting module. ☐

Historical Remarks

Many generalizations of dominant dimension have been proposed over the years. Here, we are proposing one for the setup of projective Noetherian algebras over a Noetherian commutative ring. The essence of the dominant dimension is being an invariant that controls the connection between the representation theory of two algebras. Some desired properties are its left-right symmetry, the existence of a version of a Mueller characterization of dominant dimension (including the relation of dominant dimension of an algebra and double centralizer properties, see [Mue68]) and the ground ring should not be an obstacle to dominant dimension of an algebra. This last means that a regular module over any commutative Noetherian algebra (which is an algebra over itself) should have infinite dominant dimension.

Since projective-injective modules rarely exist for projective Noetherian algebras over Noetherian commutative rings, there were some approaches to extending dominant dimension by replacing the projective modules for flat (see [Hos89]) or even torsionless (see [Kat68]) modules. Neither of these approaches seemed successful in the long run. Other approach was the introduction of the so-called U-dominant dimension (see [Mor70]). Basically, here one replaces projective-injectives by modules in the additive closure of U. This approach seems more fruitful compared to the previous ones, especially if one is interested in double centralizer properties. Such ideas were exploited in [KSX01]. Note that for the cases where U is projective-injective, one recovers the original definition of dominant dimension. An extension of both U-dominant dimension and dominant dimension based on flat modules was introduced in [Hua06]. Unfortunately, most of these concepts lack a similar characterization theorem for dominant dimension of the form developed in [Mue68] and in some of these notions even a left-right symmetry is not guaranteed. Another variation of U-dominant dimension was proposed in [Hua05].

Our attempt here is to find a generalization of dominant dimension that captures the properties that dominant dimension should have, and, in particular, to also introduce a notion of dominant dimension with respect to a module that coincides with the original when the module is projective-injective. In doing so, we also generalize the notion of faithful dimension introduced in [BS98]. Moreover, the faithful dimension of a module Q coincides with the dominant dimension of the regular module (of a finite-dimensional algebra over a field) with respect to the module Q. Here, one could ask why not calling it relative faithful dimension to this new generalization. One of the reasons is that faithful modules do not play a role in the relative setup (of algebras over a Noetherian ring). They are replaced by relative strongly faithful modules. Another reason is the notion introduced here of relative dominant dimension with respect to a module Q really controls in some sense the connection of the algebras A and the endomorphism algebra of Q over A. In particular, our focus lies more in evaluating how much the functor $\mathrm{Hom}_A(Q,-)$ (or $\mathrm{Hom}_A(\mathrm{Hom}_A(Q,-),-)$ depending of the context that we are working in) is fully faithful than evaluating how much Q is faithful. Recently, the U-dominant dimension has attracted some interest where U is chosen to be an injective module (see [LZ21]). This case is a particular case of the relative dominant dimension with respect to U here proposed giving more evidence that the generalization of dominant dimension here studied is the right one to consider.

In the literature, the words of cover and cocover appear very often (even in representation theory) but one must be careful not to confuse concepts. However, we would like to remark that there are some resemblances of the notions cover/cocover that appear for example in [HU96, AS80] and the covers (in the sense of Rouquier) and cocovers that we introduce here. If (A,P) is a cover of B, then $\mathrm{add}_B \mathrm{Hom}_A(P,A)$ (which is equivalent to A-proj under the cover assumption) is a cover of B-mod in the terminology of [AS80]. On the other hand, if (A,Q) is a cocover of B, then $\mathrm{add}_B DQ$ (which is equivalent to (A,R)-inj$\cap R$-proj) is a cocover of B-proj in the sense of [HU96].

Chapter 6

Applications and Examples - Part II

The main examples in this thesis are the class of Schur algebras and the BGG category $\mathcal{O}$ of semi-simple Lie algebras, in both cases over commutative rings. Applying the results of the previous chapters culminates in new proofs of Ringel self-duality of Schur algebras $S_K(n,d)$ with parameters $n \geq d$ where K is a field of characteristic distinct from two and of Ringel self-duality of blocks of the BGG category $\mathcal{O}$ of complex semi-simple Lie algebras. Both cases illustrate the advantage of going integrally and relative to split quasi-hereditary covers with better quality. For the class of Schur algebras $S_R(n,d)$ with parameters $n < d$ our approach culminates in the discovery of new split quasi-hereditary covers and lower bounds for their quality.

6.1 Generalized Schur algebras in the sense of Donkin

We considered so far only the Schur algebras $S_R(n,d)$ with parameters $n \geq d$ mainly because, once we drop $n \geq d$, the pair $(S_R(n,d), V^{\otimes d})$ is no longer, in general, a split quasi-hereditary cover of $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})^{op}$. At first sight, one could think that this is just a small technicality since the definition of Schur algebras does not make a distinction between these two cases. But, if we look from the symmetric group side of the definition of Schur algebra we see already a big change. If $n \geq d$, the RS_d-module $V^{\otimes d}$ is faithful whereas if $n < d$ it is not. Further, the higher the difference between d and n with $d > n$ the farther is $V^{\otimes d}$ from being faithful as RS_d-module. In fact, the annihilator of $V^{\otimes d}$ over RS_d grows by decreasing the parameter n. Taking into account KS_d being a self-injective algebra over a field K, it becomes clear that the first case involves classical dominant dimension while the second case seems more complicated. We aim to unify these two cases by using the relative dominant dimension theory. For us, to make such a distinction between these cases will be similar to the the study of classical Schur–Weyl duality between the general linear group and S_d. First, one deals with algebraically closed fields/infinite fields. Second, we try to transfer the results from algebras over infinite fields to algebras over finite fields and finally we transfer the information from algebras over fields to algebras over arbitrary commutative rings. Typically, this involves using the known particular case to understand the more general case. In this situation, this strategy reads as follows: apply the known results for the case $n \geq d$ into the case $n < d$. Such technique was made possible using Schur functors by Green [Gre07] to transfer properties from $S_R(d,d)$ to $S_R(n,d)$ with $n < d$. In [KSX01] this relation between these two Schur algebras was exploited to deduce Schur–Weyl duality between $S_R(n,d)$ and RS_d in both cases $n \geq d$ and $n < d$ for arbitrary fields without using invariant theory. Here, we will use this relation to regard the cover $(S_R(n,d), V^{\otimes d})$ with $n \geq d$ studied in Section 4.1 as a particular case of a more general cover that contains both situations $n \geq d$ and $n < d$. This gives an additional motivation to compute the relative dominant dimension studied in Chapter 5. This phenomenon also explains why [KSX01] were successful since their situation is a special case of relative dominant dimension that we use here.

For the benefit of the reader, we shall start by recalling the connection between $S_R(d,d)$ and $S_R(n,d)$ if $n<d$. For simplicity, we will focus on the case of Schur algebras but all these results in this section for Schur algebras have analogue versions for q-Schur algebras.

Theorem 6.1.1. *[Gre07, 6.5]Let n,d be natural numbers so that $d>n$. Define*

$$\Lambda(d,d)^n := \{\beta \in \Lambda(d,d)\colon \beta_{n+1} = \cdots = \beta_d = 0\}.$$

Let R be a commutative Noetherian ring. Define the idempotent $f = \sum\limits_{\beta\in\Lambda(d,d)^n} \xi_\beta$. Then,

$$fS_R(d,d)f = S_R(n,d).$$

Moreover, $f(R^d)^{\otimes d} \simeq (R^n)^{\otimes d}$ as $S_R(n,d)$-modules.

Proof. Recall that $S_R(d,d)$ has an R-basis $\{\xi_{i,j}\colon i,j\in I(d,d)\}$ and $S_R(n,d)$ has an R-basis $\{\xi_{i,j}\colon i,j\in I(n,d)\}$. Consider the injective map $\Upsilon\colon \Lambda(n,d)\to\Lambda(d,d)$, given by $\alpha\mapsto(\alpha_1,\cdots,\alpha_n,0,\cdots,0)$. The image of Υ is exactly $\Lambda(d,d)^n$. Note that if $i\in I(d,d)$ has weight $\beta\in\Lambda(d,d)^n$, then $i\in I(n,d)$. By Equation (4.1.0.2), the following holds for each $i,j\in I(d,d)$

$$f\xi_{i,j} = \sum_{\beta\in\Lambda(d,d)^n} \xi_\beta\xi_{i,j} = \sum_{\beta\in\Lambda(d,d)^n}\sum_{[k]\in I(d,d)/\sim} \mathbb{1}_{\{k\in I(d,d)\colon\, \omega(k)=\beta\}}(k)\xi_{k,k}\xi_{i,j} \tag{6.1.0.1}$$

$$= \sum_{\beta\in\Lambda(d,d)^n}\sum_{[k]\in I(d,d)/\sim} \mathbb{1}_{\{k\in I(d,d)\colon\, \omega(k)=\beta\}}(k)\mathbb{1}_{\{i\sim k\}}(k)\xi_{i,j} = \begin{cases}\xi_{i,j}, \text{ if } i\in I(n,d)\\ 0, \text{ otherwise.}\end{cases} \tag{6.1.0.2}$$

Here, $\omega(k)$ means the weight of k. Analogously,

$$\xi_{i,j}f = \begin{cases}\xi_{i,j}, \text{ if } j\in I(n,d)\\ 0, \text{ otherwise.}\end{cases} \tag{6.1.0.3}$$

This shows that $fS_R(d,d)f = S_R(n,d)$. By (4.1.0.4), $(R^d)^{\otimes d}\simeq S_R(d,d)\xi_\lambda$ with $\lambda=(1,\cdots,1)\in\Lambda(d,d)$. So,

$$f(R^d)^{\otimes d} \simeq fS_R(d,d)\xi_\lambda. \tag{6.1.0.4}$$

Hence, the left $S_R(n,d)$-module $f(R^d)^{\otimes d}$ is generated by $\{f\xi_{i,j}\xi_\lambda : i,j\in I(d,d)\}$ and for $i,j\in I(d,d)$

$$f\xi_{i,j}\xi_\lambda = f\xi_{i,j}\mathbb{1}_{\{\omega(j)=\lambda\}}(j) = \begin{cases}\xi_{i,j}, \text{ if } i\in I(n,d) \text{ and } \omega(j)=\lambda,\\ 0, \text{ otherwise.}\end{cases} \tag{6.1.0.5}$$

Moreover, $\xi_{i,j}(e_1\otimes\cdots\otimes e_d) = e_{i_1}\otimes\cdots\otimes e_{i_d}$ for $i\in I(n,d)$ and $\omega(j)=\lambda$. Therefore, $fS_R(d,d)\xi_\lambda\to(R^n)^{\otimes d}$, given by $f\xi_{i,j}\xi_\lambda\mapsto f\xi_{i,j}\xi_\lambda(e_1\otimes\cdots e_d)$ is an $S_R(n,d)$-isomorphism. □

The following result can also be found in 4.2 and 3.9 of [Erd94].

Corollary 6.1.2. *Let n,d be natural numbers so that $d>n$. Then, the following assertions hold.*

(a) For any field K, the idempotent $f=\sum\limits_{\beta\in\Lambda(d,d)^n}\xi_\beta\in S_K(d,d)$ satisfies the hypothesis of Theorem 1.7.5.

(b) $V^{\otimes d}$ *is a partial tilting module of* $S_R(n,d)$.

Further, if $R = K$ *is a field then,*

(i) The (partial) tilting indecomposable modules of $S_K(n,d)$ *are the image of the (partial) tilting indecomposable modules of* $S_K(d,d)$ *under the Schur functor* $\mathrm{Hom}_{S_R(d,d)}(S_R(d,d)f,-)$;

(ii) The partial tilting module $T(\lambda)$ *is a summand of* $V^{\otimes d} \in S_R(n,d)$-mod *if and only if* λ *is a* $\mathrm{char}\,K$*-regular partition of d in at most n parts.*

Proof. By Theorem 4.1.2 and applying $fS_R(d,d)$ to the split heredity chain of $S_R(d,d)$ we obtain $f\Delta(\mu) = 0$ for every partition μ of d in m parts with $m > n$. Now fix an arbitrary field K, since $\Delta(\mu) \twoheadrightarrow S(\mu)$ the idempotent $f = \sum_{\beta \in \Lambda(d,d)^n} \xi_\beta$ satisfies the condition fixing Γ to be the subset of all partitions of d in at least $n+1$ parts. Since $V^{\otimes d}$ is the image of a projective-injective module by the Schur functor $\mathrm{Hom}_{S_R(d,d)}(S_R(d,d)f,-)$, (b) follows by Proposition 1.7.7. The remaining follows by Proposition 4.1.4, Theorem 6.1.1 and Proposition 1.7.7. ☐

Theorem 6.1.3. *Let R be a commutative Noetherian ring and n,d be natural numbers. Let T be a characteristic tilting module of* $S_R(n,d)$. *Then,*

$$V^{\otimes d} - \mathrm{domdim}_{(S_R(n,d),R)}\, T \geq \inf\{k \in \mathbb{N} \mid (k+1)\cdot 1_R \notin U(R),\ k < d\} \geq 1.$$

Proof. The result follows from Theorems 5.6.4, 6.1.1, 5.3.5 and Corollary 4.1.8. ☐

This inequality is sharp in general since this becomes an equality in case $n \geq d$. Although the lower bound might be just one, this is already quite a strong statement giving that the rational Schur algebra is semi-simple. This brings us to one of the main theorems of this chapter.

Theorem 6.1.4. *Let R be a commutative Noetherian ring. Denote by* $R(S_R(n,d))$ *the Ringel dual of the Schur algebra* $S_R(n,d)$ *(there are no restrictions on the natural numbers n and d). Then, the following assertions hold.*

(i) $\mathrm{End}_{S_R(n,d)}(V^{\otimes d})^{op}$ *is a cellular algebra.*

(ii) Let T be a characteristic tilting module of $S_R(n,d)$. *Then,* $(R(S_R(n,d)), \mathrm{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ *is a* $(V^{\otimes d} - \mathrm{domdim}_{(S_R(n,d),R)}\, T - 2) - \mathscr{F}(\tilde{\Delta}_{R(S_R(n,d))})$ *split quasi-hereditary cover of* $\mathrm{End}_{S_R(n,d)}(V^{\otimes d})^{op}$.

Proof. The assertion (i) follows from Theorems 1.6.20 and 4.1.2 together with the existence of the standard duality on Schur algebras.

The assertion (ii) follows from Theorems 6.1.3, 5.5.1 and Proposition 5.2.7, if $V^{\otimes d} - \mathrm{domdim}_{(S_R(n,d),R)}\, T \geq 2$. For $R = \mathbb{Z}$, it follows that $(R(S_R(n,d)), \mathrm{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ is a $0 - \mathscr{F}(\tilde{\Delta}_{R(S_R(n,d))})$ split quasi-hereditary cover of $\mathrm{End}_{S_R(n,d)}(V^{\otimes d})^{op}$ because of $S_{\mathbb{Q}}(n,d)$ being semi-simple and Corollary 5.5.6. By Theorem 3.3.9, the assertion (ii) holds for $R = \mathbb{F}_2$. Since all fields of characteristic two are faithfully flat over $\mathbb{F}_2$ we obtain (ii) with R being a field of characteristic two. By Proposition 3.3.6, the result follows. ☐

With this formulation, we give meaning to the Schur–Weyl duality between $S_R(n,d)$ and RS_d without restrictions on the parameters n and d. Moreover, this generalizes the results of Hemmer and Nakano in [HN04] and [FK11b, Theorem 3.9] on the Schur algebra ($S_R(n,d)$ with parameters $n \geq d$). In fact, if $n \geq d$, then by [Don93, Proposition 3.7] the Ringel dual of $S_R(n,d)$ is the opposite algebra of $S_R(n,d)$ and we can identify the projective-injective module $\mathrm{Hom}_{S_R(n,d)}(T, V^{\otimes d})$ with $DV^{\otimes d}$. In this case, Schur–Weyl duality gives $\mathrm{End}_{S_R(n,d)}(V^{\otimes d}) \simeq RS_d^{op}$. So, for $n \geq d$, Theorem 6.1.4 is translated to $(S_R(n,d)^{op}, DV^{\otimes d})$ being a split quasi-hereditary cover of RS_d^{op}. More precisely, the cover $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ is equivalent to $(S_R(n,d), V^{\otimes d})$. Let $H\colon S_R(n,d)\text{-mod} \to R(S_R(n,d))$ be the equivalence of categories given by Ringel self-duality in [Don93, Proposition 3.7]. Denote by F' the Schur functor associated with the cover $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$. Then, $F\Delta(\lambda) = \xi_{(1,\ldots,d),(1,\ldots,d)}\Delta(\lambda)$ and by [CPS96, Lemma 1.6.12],

$$F'H\Delta(\lambda) \simeq F' \operatorname{Hom}_{S_R(n,d)}(T, \nabla(\lambda')) \simeq F\nabla(\lambda') = \xi_{(1,\ldots,d),(1,\ldots,d)} D\Delta(\lambda')^{\iota} \tag{6.1.0.6}$$

$$\simeq D(\xi_{(1,\ldots,d),(1,\ldots,d)}\Delta(\lambda'))^{\xi_{(1,\ldots,d),(1,\ldots,d)}\iota\xi_{(1,\ldots,d),(1,\ldots,d)}} \simeq \operatorname{sgn} \otimes_K \xi_{(1,\ldots,d),(1,\ldots,d)}\Delta(\lambda). \tag{6.1.0.7}$$

Here sgn is the free module R with the S_d-action $\sigma \cdot 1_R = \operatorname{sgn}(\sigma)1_R$ and M^{ι} is the right module M with right action $m \cdot a = \iota(a)m$, $m \in M$ and $a \in S_R(n,d)$. The same notation is used for modules over RS_d. Thus, there exists a commutative diagram

$$\begin{tikzcd} S_R(n,d)\text{-mod} \arrow[r, "F"] \arrow[d, "H"] & RS_d\text{-mod} \arrow[d, "\operatorname{sgn}\otimes_R -"] \\ R(S_R(n,d))\text{-mod} \arrow[r, "F'"] & RS_d\text{-mod} \end{tikzcd} .$$

Therefore, for $n \geq d$, this statement is nothing new and since $S_R(n,d)$ is relative gendo-symmetric this cover is the cover studied in (4.1.0.4 and 4.1.0.5). The novelty lies in the case $n < d$.

In case $n < d$, the Ringel dual of $S_R(n,d)$ is no longer, in general, a Schur algebra; it is instead a generalized Schur algebra in the sense of Donkin. The construction of the Ringel dual of $S_R(n,d)$ is as follows: let $U_{\mathbb{Z}}$ be the Konstant $\mathbb{Z}$-form of the enveloping algebra of the semi-simple complex Lie algebra $\mathfrak{sl}_d(\mathbb{C})$. That is, $U_{\mathbb{Z}}$ is the subring of the enveloping algebra of $\mathfrak{sl}_d(\mathbb{C})$ generated by the elements

$$\frac{e_{i,j}^m}{m!}, \quad 1 \leq i \neq j \leq d,\ m \geq 0,$$

where $e_{i,j}$, $1 \leq i \neq j \leq d$ denote the generators of the enveloping algebra of $\mathfrak{sl}_d(\mathbb{C})$. Then, the Ringel dual of $S_{\mathbb{Z}}(n,d)$ is the free Noetherian $\mathbb{Z}$-algebra $U_{\mathbb{Z}}/I_{\mathbb{Z}}$, where $I_{\mathbb{Z}}$ is the largest ideal of $U_{\mathbb{Z}}$ so that the simple modules of $\mathbb{Q} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}/I_{\mathbb{Z}}$ are isomorphic to the Weyl modules indexed by the weights belonging to $\Lambda^+(n,d)$ (see [Don86, 3.1] and [Don93, Proposition 3.11]). For an arbitrary commutative ring R, $R \otimes_{\mathbb{Z}} U_{\mathbb{Z}}/I_{\mathbb{Z}}$ is the Ringel dual of $S_R(n,d)$ known as **generalized Schur algebra** associated with $\mathfrak{sl}_d$ and the set $\Lambda^+(n,d)$. Since the Ringel dual of the Schur algebra is a quotient of $R \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$, Theorem 6.1.4 suggests why Schur–Weyl duality between $S_R(n,d)$ and RS_d can be deduced by studying the action of the Konstant $\mathbb{Z}$-form on $V^{\otimes d}$.

The existence of the quasi-hereditary cover described in Theorem 6.1.4 makes that the multiplicities of simple modules of the cellular algebra $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})$ which is a quotient of RS_d can be studied through the multiplicities of simple modules in the Ringel dual of the Schur algebra. In particular, this explains the background for the techniques used in [Erd94] to determine decomposition numbers in the symmetric group. For example, [Erd94, 4.5] can be deduced using the Schur functor constructed in Theorem 6.1.4, the Ringel duality functor and BGG reciprocity.

If R is a field, the value of the cover in Theorem 6.1.4 is optimal. But, as we saw even for the case $n \geq d$ the situation can be improved in some cases. We will not pursue this direction now, instead, we will try to understand a bit more what values $V^{\otimes d} - \operatorname{domdim}_{(S_R(n,d),R)} T$ can take. For this, it is enough to consider R a field.

6.2 Some particular cases of relative dominant dimension over Schur algebras

The Schur algebras of finite type already offer us some glimpses of what happens to $V^{\otimes d} - \operatorname{domdim} S_K(n,d)$ and some of its behaviours compared to the classical dominant dimension.

Example 6.2.1. *For any $m \geq 2$, and any algebraically closed field K, let $\mathscr{A}_m$ be the following bound quiver K-algebra*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \quad \cdots \quad m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m \ , \qquad \begin{array}{c} \alpha_i\alpha_{i-1} = \beta_{i-1}\beta_i = \beta_1\alpha_1 = 0, \\ \beta_i\alpha_i = \alpha_{i-1}\beta_{i-1}, \ 2 \leq i \leq m-1. \end{array} \tag{6.2.0.1}$$

Assume that P denotes the minimal faithful projective-injective module of $\mathscr{A}_m$ and e_i denotes the primitive idempotent associated with the vertex i, $i = 1, \ldots, m$. Then, for each $i < m$, $\varepsilon_i P - \operatorname{domdim} \varepsilon_i \mathscr{A}_m \varepsilon_i = +\infty$, where $\varepsilon_i = e_1 + \cdots + e_i$, and *$\varepsilon_i \mathscr{A}_m \varepsilon_i = \mathscr{A}_i$.*

The indecomposable projective $\mathscr{A}_m$-modules are

$$P(1) = \begin{array}{c} 1 \\ 2 \end{array}, \ P(2) = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array}, \ \cdots, \ P(m-1) = \begin{array}{ccc} & m-1 & \\ m-2 & & m \\ & m-1 & \end{array}, \ P(m) = \begin{array}{c} m \\ m-1 \\ m \end{array}. \tag{6.2.0.2}$$

The indecomposable injective $\mathscr{A}_m$-modules are

$$I(1) = \begin{array}{c} 2 \\ 1 \end{array}, \ I(2) = P(2), \ \cdots, \ I(m-1) = P(m-1), \ I(m) = P(m). \tag{6.2.0.3}$$

We can see that $\operatorname{domdim} \mathscr{A}_m = 2(m-1)$. Together with the partial order $1 > 2 > \cdots > m$, $\mathscr{A}_m$ is split quasi-hereditary with standard modules

$$\Delta(1) = P(1), \ \Delta(2) = \begin{array}{c} 2 \\ 3 \end{array}, \ \cdots, \ \Delta(m-1) = \begin{array}{c} m-1 \\ m \end{array}, \ \Delta(m) = \ m \ . \tag{6.2.0.4}$$

Hence, the partial tilting modules are

$$T(1) = P(2), \ T(2) = P(3), \ \cdots, \ T(m-1) = P(m), \ T(m) = \ m \ . \tag{6.2.0.5}$$

Fix $i < m$. By Theorems 1.7.5 and 1.7.7, $\varepsilon_i \mathscr{A}_m \varepsilon_i$ is split quasi-hereditary with characteristic tilting module $\varepsilon_i T(1) \oplus \varepsilon_i T(2) \cdots \oplus \varepsilon_i T(i) = \varepsilon_i P$ and the result follows. △

Corollary 6.2.2. *Let K be an algebraically closed field of characteristic $p > 0$. Then,*

$$V^{\otimes p} - \operatorname{domdim} S_K(n,p) = +\infty$$

whenever $n < p$.

Proof. By [Xi92], the non-simple block of $S_K(p,p)$ is of the form of Example 6.2.1. □

Observation 6.2.3. By [Erd93], all blocks of Schur algebras of finite type are of the form of Example 6.2.1. By Proposition 1.7.7, we are killing the simple tilting modules m in each block whenever we lower the value of n.

Hence, by Example 6.2.1, for a fixed natural number d, $(V^{\otimes d} - \operatorname{domdim} S_K(n,d))_{n\in\mathbb{N}}$ is a decreasing sequence on n with lower bound $2(\operatorname{char} K - 1)$ if $S_K(d,d)$ is an algebra of finite type.

In particular, if $S_K(d,d)$ is of finite type and $n < d$ the blocks of $S_K(n,d)$ are of the form $\varepsilon_i \mathscr{A}_m \varepsilon_i = \mathscr{A}_i$, where $\mathscr{A}_m$ is a block of $S_K(d,d)$. Moreover, in such a case,

$$\operatorname{domdim} S_K(n,d) \leq \operatorname{domdim} S_K(d,d) = 2(\operatorname{char} K - 1) \leq V^{\otimes d} - \operatorname{domdim} S_K(n,d).$$

In general, the partial tilting module $V^{\otimes d}$ contains all indecomposable projective-injective $S_K(n,d)$-modules since $V^{\otimes d} - \operatorname{domdim} S_K(n,d) \geq 1$. So, we expect the following to happen:

Conjecture 6.2.4. *For all $n,d \in \mathbb{N}$, and for any commutative ring R*

$$V^{\otimes d} - \operatorname{domdim}(S_R(n,d),R) \geq \operatorname{domdim}(S_R(n,d),R).$$

Using the Schur algebras of finite type, we can also see that, in general, the lower bound in Theorem 5.6.4 is sharp.

For the following, we can ignore the multiplicities of $V^{\otimes^d}$ as we observed in Remark 4.6.6 taking now into account that according to Definition 2.3.5, the value of relative dominant dimension with respect to $V^{\otimes^d}$ does not change if we ignore multiplicities.

Example 6.2.5. *Let K be an algebraically closed field of characteristic three. Then,*

$$V^{\otimes 5} - \operatorname{domdim} S_K(4,5) = 4 = \operatorname{domdim} S_K(5,5). \tag{6.2.0.6}$$

Further,

$$V^{\otimes 5} - \operatorname{domdim}_{S_K(4,5)} Q = 2 = \operatorname{domdim}_{S_K(5,5)} T, \tag{6.2.0.7}$$

where Q and T are the characteristic tilting modules of $S_K(4,5)$ and $S_K(5,5)$, respectively. In view of [Don94], the Schur algebra $S_K(5,5)$ has two non-simple blocks, each containing 3 simple modules. As $S_K(5,5)$ is of finite type, it must be Morita equivalent to $\mathscr{A}_3 \times \mathscr{A}_3 \times \mathscr{A}_1$ in the notation of Example 6.2.1, where $\mathscr{A}_1$ denotes a simple block corresponding to the partition $(3,1,1)$. Therefore, $S_K(4,5)$ is Morita equivalent to $\mathscr{A}_2 \times \mathscr{A}_3 \times \mathscr{A}_1$ and $(K^2)^{\otimes 5}$ corresponds to the module $Q_1 \times P_2 \times A_1$, where Q_1 is the characteristic tilting module of $\mathscr{A}_2$, P_2 is the minimal faithful projective-injective module of $\mathscr{A}_3$ and A_1 is the regular module of the simple block $\mathscr{A}_1$. Hence, $(K^2)^{\otimes 5} - \operatorname{domdim} S_K(4,5) = \operatorname{domdim} \mathscr{A}_3 = 4$ and $(K^2)^{\otimes 5} - \operatorname{domdim}_{S_K(4,5)} Q = \operatorname{domdim}_{\mathscr{A}_3} Q_2 = 2$, where Q_2 is the characteristic tilting module of $\mathscr{A}_3$. △

To prove this bound we wondered if using for example homological epimorphisms could yield an alternative argument. But, as we see in the next remark this is not the case.

Remark 6.2.6. Even for a field K, the surjective map $\psi\colon KS_d \twoheadrightarrow \operatorname{End}_{S_K(n,d)}(V^{\otimes d})$ may not be a homological epimorphism if $n < d$. Indeed, by Proposition 2.2(a) of [dlPX06], ψ is a homological epimorphism if and only if $\ker \psi$ is an idempotent ideal and $\operatorname{Tor}_{i>0}^{KS_d}(\ker \psi, KS_d/\ker \psi) = 0$. Fix $n = 2$, $d = 3$ and K a field of characteristic three. Then, $\ker \psi$ is the ideal generated by $a := e + (132) + (123) - (12) - (13) - (23)$. As $a^2 = 0$, $\ker \psi$ is not an idempotent ideal. △

We shall now consider the cases of $d = 4$, $\operatorname{char} K = 2$, $n \in \{2,3\}$.

Example 6.2.7. *Let K be an algebraically closed field of characteristic two. Let T_3 and T_2 be the characteristic* *tilting module of* $S_K(3,4)$ *and of* $S_K(2,4)$*, respectively. Then*

$$V^{\otimes 4} - \operatorname{domdim} S_K(3,4) = 2V^{\otimes 4} - \operatorname{domdim}_{S_K(3,4)} T_3 = 2 \tag{6.2.0.8}$$

$$V^{\otimes 4} - \operatorname{domdim} S_K(2,4) = 2V^{\otimes 4} - \operatorname{domdim}_{S_K(2,4)} T_2 = 4 \tag{6.2.0.9}$$

By Theorem 1.7.5 and [Gre07, 6.5] on Example 4.6.8, $S_K(3,4)$ is Morita equivalent to the following bound quiver algebra

$$\begin{tikzcd} & & 4 \arrow[dl, shift left, "\beta_1"] \\ 3 \arrow[r, shift left, "\alpha"] & 5 \arrow[l, shift left, "\alpha_1"] \arrow[ur, shift left, "\beta"] \arrow[dr, shift left, "\varepsilon_1"] & \\ & & 2 \arrow[ul, shift left, "\varepsilon"] \end{tikzcd}, \quad \begin{matrix} \alpha_1\alpha = \varepsilon_1\alpha = \beta\beta_1 = \alpha_1\varepsilon = 0 \\ \varepsilon\varepsilon_1\beta_1 = 0, \quad \varepsilon\varepsilon_1 = \beta_1\beta. \end{matrix} \tag{6.2.0.10}$$

Moreover, the (partial) tilting modules of $S_K(3,4)$ are

$$T_3(2) = 2,\ T_3(3) = \begin{tikzcd}[column sep=small] 2 \arrow[ddd, dash] & & 4 \arrow[d, dash] \\ & & 5 \arrow[dl, dash] \arrow[dd, dash] \\ & 3 \arrow[dl, dash] & \\ 5 \arrow[d, dash] & & 2 \\ 4 & & \end{tikzcd},\ T_3(4) = \begin{tikzcd}[column sep=small] & 2 \arrow[d, dash] & \\ & 5 \arrow[dl, dash] \arrow[dr, dash] & \\ 2 \arrow[dr, dash] & & 4 \arrow[dl, dash] \\ & 5 \arrow[d, dash] & \\ & 2 & \end{tikzcd},\ T_3(5) = \begin{matrix} 2 \\ | \\ 5 \\ | \\ 2 \end{matrix}, \tag{6.2.0.11}$$

and $V^{\otimes 4}$ is the module $T_3(3) \oplus T_3(4)$. Therefore, $V^{\otimes 4}$ is not projective-injective. We can see that the cokernel of a inclusion of 2 to $V^{\otimes 4}$ has socle either 5 or it is a quotient of $T_3(3)$. In both cases, this cokernel cannot be embedded into a module in the additive closure of $V^{\otimes 4}$. Since the cokernel of $P(3) \hookrightarrow T_3(3)$ has summand 2, the result follows from Theorem 6.1.3.

We shall now compute $S_K(2,4)$. By Theorem 1.7.5 and [Gre07, 6.5] on Example 4.6.8, $S_K(2,4)$ is Morita equivalent to the following bound quiver algebra

$$\begin{tikzcd} 3 \arrow[r, shift left, "\alpha"] & 5 \arrow[l, shift left, "\alpha_1"] \arrow[r, shift left, "\beta"] & 4 \arrow[l, shift left, "\beta_1"] \end{tikzcd}, \quad \begin{matrix} \alpha_1\alpha = \beta\beta_1 = 0 \\ \beta_1\beta\alpha = \alpha_1\beta_1\beta = 0. \end{matrix} \tag{6.2.0.12}$$

The indecomposable projective $S_K(2,4)$-modules are

$$P_2(3) = \begin{matrix} 3 \\ | \\ 5 \\ | \\ 4 \end{matrix},\ P_2(4) = \begin{matrix} 4 \\ | \\ 5 \\ | \\ 3 \\ | \\ 5 \\ | \\ 4 \end{matrix},\ P_2(5) = \begin{tikzcd}[column sep=small] & 5 \arrow[dl, dash] \arrow[dr, dash] & \\ 3 \arrow[d, dash] & & 4 \arrow[d, dash] \\ 5 \arrow[d, dash] & & 5 \\ 4 & & \end{tikzcd}. \tag{6.2.0.13}$$

The (partial) tilting $S_K(2,4)$-modules are

$$T_2(3) = P_2(4),\ T_2(4) = \begin{matrix} 5 \\ | \\ 4 \\ | \\ 5 \end{matrix},\ T_2(5) = 5. \tag{6.2.0.14}$$

$V^{\otimes 4}$ is the module $T_2(3)\oplus T_2(4)$.

To compute $V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} T_2$, we only need to compute $V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} 5$. We want to use Theorem 5.2.2 and 5.2.5 to compute this value. We denote by B the endomorphism algebra $\operatorname{End}_{S_K(2,4)}(V^{\otimes 4})^{op}$. The algebra B is Morita equivalent to the following bound quiver algebra

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \circlearrowleft t, \quad \alpha\beta=\beta t=t\alpha=t^2=0, \tag{6.2.0.15}$$

where the simple associated with the vertex 1 is the top of $\operatorname{Hom}_{S_K(2,4)}(V^{\otimes 4},T_2(3))$ and the simple associated with the vertex 2 is the top of $\operatorname{Hom}_{S_K(2,4)}(V^{\otimes 4},T_2(4))$. Moreover, $\operatorname{Hom}_{S_K(2,4)}(V^{\otimes 4},T_2(3))$ is projective-injective. We can see that $DV^{\otimes 4}\simeq \operatorname{Hom}_{S_K(2,4)}(V^{\otimes 4},DS_K(2,4))$ is isomorphic to $\operatorname{Hom}_{S_K(2,4)}(V^{\otimes 4},T_2(3))\oplus 1\oplus \begin{smallmatrix} 1 & & 2 & \\ & 2 & & 1 \end{smallmatrix}$.

Observe that $D\operatorname{Hom}_{S_K(2,4)}(T_2(5),V^{\otimes 4})=D\operatorname{Hom}_{S_K(2,4)}(T_2(5),T_2(4))$ is a simple module, so it must coincide with the top of $\operatorname{Hom}_{S_K(2,4)}(V^{\otimes 4},T_2(4))$. Therefore,

$$\operatorname{Hom}_B(DV^{\otimes 4},D\operatorname{Hom}_{S_K(2,4)}(T_2(5),V^{\otimes 4}))\simeq \operatorname{Hom}_B\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\oplus 1\oplus \begin{smallmatrix} 1 & & 2 & \\ & 2 & & 1 \end{smallmatrix}, 2\right)$$

is a simple module. Since $D\chi^r_{DT_2(5)}$ is injective into a simple module it must be also surjective. By Theorem 5.2.2, $V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} 5\geq 2$. It cannot be higher than two since the simple 2 over B has extensions with the simple module 1. Hence, $V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} T_2=2$.

Using the exact sequence (which remains exact under $\operatorname{Hom}_{S_K(2,4)}(-,V^{\otimes 4})$)

$$0\to \begin{smallmatrix}3\\5\\4\end{smallmatrix}\to P_2(4)\to T_2(4)\to 5\to 0, \tag{6.2.0.16}$$

together with Corollary 5.2.13 and Lemma 5.2.9, we obtain that

$$V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} P_2(5)=V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} \begin{smallmatrix}3\\5\\4\end{smallmatrix} =2+V^{\otimes 4}-\operatorname{domdim}_{S_K(2,4)} 5=4. \qquad \triangle$$

Example 6.2.8. Assume the same notation as in Example 6.2.7. *The Ringel dual of $S_K(2,4)$ has dominant dimension zero and $(R(S_K(2,4)),\operatorname{Hom}_{S_K(2,4)}(T_2,V^{\otimes 4}))$ is a $0-\mathscr{F}(\Delta_{R(S_K(2,4))})$ quasi-hereditary cover of B which is not a $1-\mathscr{F}(\Delta_{R(S_K(2,4))})$ cover of B.*

The latter follows by Example 6.2.7, Theorem 5.5.1 and Theorem 6.1.4.

To check the claim about dominant dimension. Observe that the Ringel dual of $S_K(2,4)$ is Morita equivalent to the following bound quiver algebra

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\theta}{\overset{\gamma}{\rightleftarrows}} 3, \quad \gamma\alpha=\beta\theta=\alpha\beta=\gamma\theta=0. \tag{6.2.0.17}$$

Only the projective $\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}$ is injective. So, the projective $\begin{matrix} 3 \\ 2 \end{matrix}$ has dominant dimension zero. Moreover, $\operatorname{Hom}_{S_K(2,4)}(T_2, V^{\otimes 4})$ is isomorphic to $R_A e_1 \oplus R_A e_2$, where e_i denotes the idempotent associated with the vertex i. △

Example 6.2.9. *Let V be the free $\mathbb{Z}$-module $\mathbb{Z}^2$. Then, $V^{\otimes 4}$ is not isomorphic to $DS_{\mathbb{Z}}(2,4) \otimes_{S_{\mathbb{Z}}(2,4)} V^{\otimes 4}$ as $(S_{\mathbb{Z}}(2,4)), \mathbb{Z}S_4)$-bimodules.* Assume, by contradiction, that such isomorphism holds. Then, as bimodules,

$$DS_{\mathbb{Z}}(2,4) \otimes_{S_{\mathbb{Z}}(2,4)} DS_{\mathbb{Z}}(2,4) \simeq DS_{\mathbb{Z}}(2,4) \otimes_{S_{\mathbb{Z}}(2,4)} V^{\otimes 4} \otimes_{\mathbb{Z}S_4} DV^{\otimes 4} \simeq V^{\otimes 4} \otimes_{\mathbb{Z}S_4} DV^{\otimes 4} \simeq DS_{\mathbb{Z}}(2,4). \quad (6.2.0.18)$$

This bimodule isomorphism remains exact under $\mathbb{Z}(2\mathbb{Z}) \otimes_{\mathbb{Z}} -$. The existence of such isomorphism is a contradiction with the fact that $\operatorname{domdim} S_{\mathbb{F}_2}(2,4) = 0$. △

We will now consider the case $d = 6$ to see what happens and still the characteristic two case to see what happens.

Example 6.2.10. *Let K be an algebraically closed field of characteristic two. Let T be the characteristic tilting module of $S_K(2,6)$. Then,*

$$V^{\otimes 6} - \operatorname{domdim} S_K(2,6) = 2V^{\otimes 6} - \operatorname{domdim}_{S_K(2,6)} T = 6 \quad (6.2.0.19)$$

By [DEMN99, p. 153], $S_K(2,6)$ is Morita equivalent to the following bound quiver algebra

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4\ , \qquad \begin{matrix} \beta_1\alpha_1 = \beta_2\alpha_2 = \alpha_3\beta_3 = \alpha_3\alpha_2\alpha_1 = \beta_1\beta_2\beta_3 = 0, \\ \alpha_2\alpha_1\beta_1 = \beta_3\alpha_3\alpha_2,\ \beta_2\beta_3\alpha_3 = \alpha_1\beta_1\beta_2. \end{matrix} \quad (6.2.0.20)$$

The indecomposable projective $S_K(2,6)$-modules are

$$P(1) = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\ ,\ P(2) = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ 2 & & 4 \\ & 3 & \end{array}\ ,\ P(3) = \begin{array}{cccc} & & 3 & \\ & 2 & & 4 \\ 3 & 1 & & 3 \\ 4 & & 2 & \\ & 3 & & \end{array}\ ,\ P(4) = \begin{matrix} 4 \\ 3 \\ 2 \\ 3 \\ 4 \end{matrix} \quad (6.2.0.21)$$

and $P(3)$ together with $P(4)$ are the projective-injective modules. We can see that the dominant dimension of $S_K(2,6)$ is exactly two. The standard modules of $S_K(2,6)$ with respect to the order $1 > 2 > 4 > 3$ are

$$\Delta(1) = P(1),\ \Delta(2) = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\ ,\ \Delta(3) = 3\ ,\ \Delta(4) = \begin{matrix} 4 \\ 3 \end{matrix}\ . \quad (6.2.0.22)$$

The (partial) tilting modules of $S_K(2,6)$ are $T(1) = P(3),\ T(2) = P(4),\ T(3) = 3,\ T(4) = P(3)/P(2)$. The module $(K^6)^{\otimes 6}$ corresponds to the module $T(6) \oplus T(5+1) \oplus T(4+2) \oplus T(3+2+1)$ in $S_K(6,6)$ since these are the 2-regular partitions of 6. But, in $S_K(2,6)$ only the partitions in at most 2 parts appear. Hence, $6 > 5+1 > 4+2 > 3+3$. So, for $V = K^2$, $V^{\otimes 6}$ is the module $T(1) \oplus T(2) \oplus T(4)$. The endomorphism algebra

$\operatorname{End}_{S_K(2,6)}(V^{\otimes 6})^{op}$ which we will denote by B has quiver

$$\circlearrowleft 3 \underset{\longleftarrow}{\overset{\longrightarrow}{}} 1 \underset{\longleftarrow}{\overset{\longrightarrow}{}} 2\ , \tag{6.2.0.23}$$

with projective modules

$$P_B(1) = \begin{array}{ccccc} & & 1 & & \\ & 3 & & 2 & \\ & 1 & & 1 & \\ & 2 & 3 & & \\ & & 1 & & \end{array}, \quad P_B(2) = \begin{array}{c} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{array}, \quad P_B(3) = \begin{array}{ccc} & 3 & \\ 3 & & 1 \\ & & 2 \\ & 1 & \end{array}.$$

Now, $DV^{\otimes 6}$ is the left B-module

$$\operatorname{Hom}_{S_K(2,6)}(S_K(2,6), DV^{\otimes 6}) \simeq \operatorname{Hom}_{S_K(2,6)}(V^{\otimes 6}, DS_K(2,6)) \simeq P_B(1) \oplus P_B(2) \oplus 1 \oplus \begin{array}{c} 1 \\ 2 \\ 1 \end{array}. \tag{6.2.0.24}$$

So, $DV^{\otimes 6}$ has four non-isomorphic indecomposable summands as left B-module. This value is consistent with the theory because since $DV^{\otimes 6}$ has a double centralizer property as B-module, the number of non-isomorphic summands as left B-module is equal to the number of non-isomorphic projective indecomposable $S_K(2,6)$-modules which is four. Note that the exact sequence $0 \to P(2) \to P(3) \to T(4) \to 0$ gives that $V^{\otimes 6} - \operatorname{domdim} P(2) = +\infty$. By Corollary 5.2.13 the exact sequence $0 \to P(1) \to P(3) \to P(4) \oplus T(4) \to T(4) \to 3 \to 0$ gives $V^{\otimes 6} - \operatorname{domdim} S_K(2,6) = 3 + V^{\otimes 6} - \operatorname{domdim}_{S_K(2,6)} T(3)$. To compute $V^{\otimes 6} - \operatorname{domdim}_{S_K(2,6)} T(3)$ we can see that $\operatorname{Hom}_{S_K(2,6)}(T(3), V^{\otimes 6}) = \begin{array}{c} 3 \\ 1 \end{array}$ and $\operatorname{Hom}_B(DV^{\otimes 6}, D\operatorname{Hom}_{S_K(2,6)}(T(3), V^{\otimes 6})) = \operatorname{Hom}_B(P_B(1) \oplus P_B(2) \oplus 1 \oplus \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \begin{array}{c} 1 \\ 3 \end{array})$ is a simple module, therefore $V^{\otimes 6} - \operatorname{domdim}_{S_K(2,6)} T(3) \geq 2$. So, it is enough to check when the module

$$\operatorname{Ext}_B^i(DV^{\otimes 6}, D\operatorname{Hom}_{S_K(2,6)}(T(3), V^{\otimes 6})) = \operatorname{Ext}_B^i(DV^{\otimes 6}, \begin{array}{c} 1 \\ 3 \end{array}) = \operatorname{Ext}_B^i(1 \oplus \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \begin{array}{c} 1 \\ 3 \end{array})$$

is zero. Since $\Omega^1(\begin{array}{c} 1 \\ 2 \\ 1 \end{array}) = P_B(3)$ and the map $P_B(3) \hookrightarrow P_B(1) \twoheadrightarrow \begin{array}{c} 1 \\ 3 \end{array}$ is a basis of $\operatorname{Hom}_B(P_B(3), \begin{array}{c} 1 \\ 3 \end{array})$ we obtain that $\operatorname{Ext}_B^1(\begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \begin{array}{c} 1 \\ 3 \end{array}) = 0$. The simple 1 in B has no self-extensions of degree one, therefore $\operatorname{Ext}_B^1(1, \begin{array}{c} 1 \\ 3 \end{array})$ is also zero. However, $\Omega^2(1) = \begin{array}{c} 3 \\ 1 \\ 2 \end{array}$ and the map $\Omega^2(1) \hookrightarrow P_B(3) \oplus P_B(2) \to \begin{array}{c} 1 \\ 3 \end{array}$ is zero. Thus, $\operatorname{Ext}_B^2(1, \begin{array}{c} 1 \\ 3 \end{array}) \neq 0$. Hence,

$V^{\otimes 6} - \operatorname{domdim}_{S_K(2,6)} T(3) = 3.$ △

The previous Examples 6.2.7 and 6.2.10 together with the case $n \geq d$ motivates us to conjecture the following:

Conjecture 6.2.11. *Let R be a commutative Noetherian ring with identity. Let n and d be natural numbers and T be a characteristic tilting module of the Schur algebra $S_R(n,d)$. Then,*

$$V^{\otimes d} - \operatorname{domdim} S_R(n,d) = 2V^{\otimes d} - \operatorname{domdim}_{(S_R(n,d),R)} T.$$

6.3 Temperley-Lieb algebras

In this section, we will focus on the Temperley-Lieb algebras, introduced in [TL71], and their relations with Schur algebras. The crucial and first step was done by Jones when he established the Temperley Lieb algebras as quotients of Iwahori-Hecke algebras [Jon83] and [Jon87].

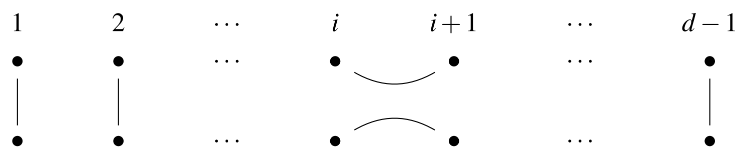
Definition 6.3.1. Let R be a commutative ring. The **Temperley-Lieb algebra**, denoted by $TL_{R,d}(-2)$ is the R-algebra generated by elements $U_1, \ldots, U_{d-1}$with defining relations $i = 1, \ldots, d-1$,

$$U_i U_{i\pm 1} U_i = U_i \tag{6.3.0.1}$$

$$U_i U_j = U_j U_i, \quad |i-j| > 1 \tag{6.3.0.2}$$

$$U_i^2 = -2U_i \tag{6.3.0.3}$$

The elements U_i can be represented by the diagram



(6.3.0.4)

and the multiplication can be viewed as the concatenation of diagrams (replacing the internal loops by the element (-2)).

Although the arguments to be provided in the following results, up to some modifications, can be used to general Temperley-Lieb algebras and consequently with its relations to the quantized Schur algebras, we will focus, for simplicity, in the cases $TL_{R,d}(-2)$. As we have mentioned, it is commonly known that this algebra is a quotient of the group algebra RS_d (see [Wes95, 7] or [Jon83, Jon87]. Moreover, we will approach the study of this algebra and its properties from this point of view.

Lemma 6.3.2. *There exists a surjective R-algebra homomorphism $\Phi \colon RS_d \to TL_{R,d}(-2)$ that maps $T_i = (i\, i+1)$ to $U_i + 1$, $i = 1, \ldots, d-1$.*

Proof. Recall that RS_d is the R-algebra generated by $T_1, \ldots, T_{d-1}$ with defining relations $i = 1, \ldots, d-1$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{6.3.0.5}$$

$$T_i T_j = T_j T_i, \quad |i-j| > 1 \tag{6.3.0.6}$$

$$T_i^2 = 1. \tag{6.3.0.7}$$

Hence, Φ is well defined if and only if Φ preserves the defining relations of RS_d. To this end, observe that

$$\Phi(T_i)\Phi(T_{i+1})\Phi(T_i) = (U_i+1)(U_{i+1}+1)(U_i+1) = U_i + U_{i+1}U_i + U_iU_{i+1} + U_{i+1} + 1 \tag{6.3.0.8}$$
$$= (U_{i+1}+1)(U_i+1)(U_{i+1}+1) = \Phi(T_{i+1})\Phi(T_i)\Phi(T_{i+1}). \tag{6.3.0.9}$$

Let i, j be two elements satisfying $|i-j|>1$. Then,

$$\Phi(T_i)\Phi(T_j) = (U_i+1)(U_j+1) = U_iU_j + U_i + U_j + 1 = U_jU_i + U_i + U_j + 1 = (U_j+1)(U_i+1) = \Phi(T_j)\Phi(T_i).$$

Also,

$$\Phi(T_i)\Phi(T_i) = (U_i+1)(U_i+1) = U_i^2 + 2U_i + 1 = 1. \tag{6.3.0.10}$$

Therefore, Φ is well defined. It is clear that Φ is surjective. In fact, any element $\sum_i \alpha_i U_i \in TL_{R,d}(-2)$ can be written as

$$\sum_i \alpha_i U_i = \sum_i \alpha_i \Phi(T_i - T_i^2) = \Phi(\sum_i \alpha_i (T_i - T_i^2)). \qquad \square$$

In the following, we wish to compute the kernel of the map Φ. This goes back to [Jon87, p.364].

Theorem 6.3.3. *For each $i = 1, \dots, d-2$, define $x_i := T_iT_{i+1}T_i - T_iT_{i+1} - T_{i+1}T_i + T_i + T_{i+1} - 1$. Let I be the ideal of RS_d generated by the elements x_i, $i = 1, \dots, d-2$. Then, there exists an exact sequence*

$$0 \to I \to RS_d \xrightarrow{\Phi} TL_{R,d}(-2) \to 0 \tag{6.3.0.11}$$

Proof. Using (6.3.0.8), we can see that $\Phi(x_i) = 0$ for each $i = 1, \dots, d-2$. Hence, $I \subset \ker\Phi$ and we can write the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker\Phi \arrow[r] & RS_d \arrow[r, "\Phi"] & TL_{R,d}(-2) \arrow[r] & 0 \\ & \cup\!\mathsf{I} & & & \\ 0 \arrow[r] & I \arrow[r] & RS_d \arrow[r] \arrow[uu, equal] & RS_d/I \arrow[r] \arrow[uu, two heads, "\pi"] & 0 \end{tikzcd}, \tag{6.3.0.12}$$

where π maps $T_i + I$ to $\Phi(T_i) = U_i + 1$. Now, consider the map $\pi' \colon TL_{R,d}(-2) \to RS_d/I$ by setting $\pi'(U_i) = T_i - 1 + I$. We must check that π' is well-defined. Again, for this it is enough to check π' preserves the defining relations of $TL_{R,d}(-2)$. In fact,

$$\pi'(U_i)\pi'(U_i) + 2\pi'(U_i) = (T_i-1)(T_i-1) + 2T_i - 2 + I = 0 + I,$$
$$\pi'(U_i)\pi'(U_j) = (T_i - 1 + I)(T_j - 1 + I) = T_iT_j - T_i - T_j + 1 + I = T_jT_i - T_i - T_j + I = \pi'(U_j)\pi'(U_i),\ |i-j|>1,$$
$$\pi'(U_i)\pi'(U_{i+1})\pi'(U_i) - \pi'(U_i) = x_i + I = 0,$$
$$\pi'(U_i)\pi'(U_{i-1})\pi'(U_i) - \pi'(U_i) = x_{i-1} + I = 0.$$

Finally, we can observe that

$$\pi'(\pi(T_i + I)) = \pi'(U_i + 1) = T_i - 1 + 1 + I = T_i + I, \tag{6.3.0.13}$$
$$\pi\pi'(U_i) = \pi(T_i - 1 + I) = -1 + U_i + 1 = U_i,\ \forall i. \tag{6.3.0.14}$$

Therefore, $I = \ker\Phi$. $\square$

Note that if $d \leq 2$, then Φ is an isomorphism. It is due to Martin [Mar92] and Jimbo [Jim86] that Temperley Lieb algebras can be interpreted as the centralizer algebras of quantum groups $\mathfrak{sl}_2$ in the endomorphism algebra of a tensor power. In the following, we adapt this statement to our situation with the Schur algebras $S_R(2,d)$ replacing $\mathfrak{sl}_2$. It can also be found in [DPS98b, Theorem 6.2] with a different approach.

Theorem 6.3.4. *Let R be a commutative ring and denote by V the free R-module R^2. $(R^2)^{\otimes d}$ is a $TL_{R,d}(-2)$-module where U_i acts in $(R^2)^{\otimes d}$ as $\mathrm{id}^{\otimes^{(i-1)}} \otimes \tau \otimes \mathrm{id}^{\otimes^{(d-i-1)}}$. Here, $\tau\colon (R^2)^{\otimes 2} \to (R^2)^{\otimes 2}$ is defined by $\tau(v_1 \otimes v_2) = v_2 \otimes v_1 - v_1 \otimes v_2$, $v_1, v_2 \in (R^2)^{\otimes 2}$. Moreover, there is an isomorphism of R-algebras $TL_{R,d}(-2) \to \operatorname{End}_{S_R(2,d)}(V^{\otimes d})$.*

Proof. For $d = 1,2$ the map is well-defined since $TL_{R,d}(-2) \simeq RS_d$ and $RS_d \to \operatorname{End}_{S_R(2,d)}(V^{\otimes d})$ is an isomorphism. Let $d > 2$ and $1 \leq i \leq d-2$.

Consider the surjective homomorphism of R-algebras $\psi\colon RS_d \to \operatorname{End}_{S_R(2,d)}(V^{\otimes d})$. Let x_i be the generator of I with index i. Observe that

$$\begin{aligned}\psi(x_i)(e_{k_1} \otimes \cdots \otimes e_{k_d}) &= \psi(T_iT_{i+1}T_i - T_iT_{i+1} - T_{i+1}T_i + T_i + T_{i+1} - 1)(e_{k_1} \otimes \cdots \otimes e_{k_d}) \\ &= e_{k_1} \otimes \cdots \otimes (e_{k_{i+2}} \otimes e_{k_{i+1}} \otimes e_{k_i} - e_{k_{i+2}} \otimes e_{k_i} \otimes e_{k_{i+1}} - e_{k_{i+1}} \otimes e_{k_{i+2}} \otimes e_{k_i} + e_{k_{i+1}} \otimes e_{k_i} \otimes e_{k_{i+2}} \\ &\qquad - e_{k_i} \otimes e_{k_{i+1}} \otimes e_{k_{i+2}} + e_{k_i} \otimes e_{k_{i+2}} \otimes e_{k_{i+1}}) \otimes \cdots \otimes e_{k_d}. \end{aligned} \tag{6.3.0.15}$$

Since $k_{i+1}, k_i, k_{i+2} \in \{1,2\}$ there must be two indexes that are the same. Assume, without loss of generality, that $k_i = k_{i+1}$. Then, either $k_{i+2} = k_i$ or $k_{i+2} \neq k_i$. In both of these cases, it is immediate by (6.3.0.15) that $\psi(x_i)(e_{k_1} \otimes \cdots \otimes e_{k_d}) = 0$. Hence, the surjective homomorphism of R-algebras ψ factors through $TL_{R,d}(-2)$, that is, there exists a surjective homomorphism $\overline{\psi}\colon TL_{R,d}(-2) \to \operatorname{End}_{S_R(2,d)}(V^{\otimes d})$ satisfying $\overline{\psi} \circ \Phi = \psi$. In particular, $\overline{\psi}$ maps U_i to $\psi(T_i - 1) = (i\ i+1) - \mathrm{id} = \mathrm{id}^{\otimes^{(i-1)}} \otimes \tau \otimes \mathrm{id}^{\otimes^{(d-i-1)}} \in \operatorname{End}_{S_R(2,d)}(V^{\otimes d})$. It remains to show that $\overline{\psi}$ is injective. Let $\sum_i \alpha_i U_i \in TL_{R,d}(-2)$ such that

$$0 = \overline{\psi}(\sum_i \alpha_i U_i) = \sum_i \alpha_i \,\mathrm{id}^{\otimes^{(i-1)}} \otimes \tau \otimes \mathrm{id}^{\otimes^{(d-i-1)}}. \tag{6.3.0.16}$$

Consider $y_k := e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes e_1 \otimes \cdots \otimes e_1$, where e_2 appears only in the position $k+1$ for a certain k. Then,

$$0 = \overline{\psi}(\sum_i \alpha_i U_i)(y_k) = \alpha_k e_1 \otimes \cdots \otimes \tau(e_1 \otimes e_2) \otimes e_1 \otimes \cdots \otimes e_1 + \alpha_{k+1} e_1 \otimes \cdots \otimes e_1 \otimes \tau(e_2 \otimes e_1) \otimes \cdots \otimes e_1 \tag{6.3.0.17}$$

The last equality follows from the fact that $\tau(e_1 \otimes e_1) = 0$. Since these are element basis, it follows that $\alpha_k = 0 = \alpha_{k+1}$. As k is arbitrary we obtain that $\ker \overline{\psi} = 0$. ☐

Theorem 6.3.4 turns $V^{\otimes d}$ a central piece to understand the structure of Temperley-Lieb algebras $TL_{R,d}(-2)$. Since it is a partial tilting module over the Schur algebra it follows immediately that the Temperley-Lieb algebra is a cellular algebra. Moreover, the relative dominant dimension with respect to $V^{\otimes d}$ gives meaning to the connection between Schur algebras and Temperley-Lieb algebras by illustrating that this scenario fits in our main problem of quasi-hereditary covers of cellular algebras.

Corollary 6.3.5. *Let R be a commutative Noetherian ring. Let d be a natural number. Denote by $R(S_R(2,d))$ the Ringel dual of the Schur algebra $S_R(2,d)$. Then, the following assertions hold.*

(i) $TL_{R,d}(-2)$ is a cellular algebra.

(ii) Let T be a characteristic tilting module of $S_R(2,d)$. Then, $(R(S_R(2,d)), \operatorname{Hom}_{S_R(2,d)}(T, V^{\otimes d}))$ is a $(V^{\otimes d} - \operatorname{domdim}_{(S_R(2,d),R)} T - 2) - \mathscr{F}(\tilde{\Delta}_{R(S_R(2,d))})$ split quasi-hereditary cover of $TL_{R,d}(-2)$.

Proof. The result follows by Theorem 6.3.4 and Theorem 6.1.4. $\square$

This, of course, motivates us to compute $V^{\otimes d} - \operatorname{domdim} S_R(2,d)$ to obtain information about the structure of Temperley-Lieb algebras and it establishes this relative dominant dimension as a new point of view to the representation theory of Temperley-Lieb algebras. Further, we now see that the Schur–Weyl duality between Schur algebras and Temperley-Lieb algebras is a manifestation of cover theory similar to the Schur–Weyl duality between Schur algebras and the symmetric groups (in case $n \geq d$). Assuming that R is a field, the main difference, however, is that the cover for symmetric groups is composed of a Schur algebra and a projective-injective module and the cover for Temperley-Lieb algebras is composed of the Ringel dual of a Schur algebra and a projective module (not necessarily injective). On special cases which were completely determined in [EH02], the Schur algebra is Ringel self-dual. Hence, on those cases, the Schur algebra $S_R(2,d)$ together with the projective module $\operatorname{Hom}_{S_R(2,d)}(T, V^{\otimes d})^\iota$ are a split quasi-hereditary cover of the Temperley-Lieb algebra $TL_{d,R}(-2)$. By understanding the value $V^{\otimes d} - \operatorname{domdim} S_R(2,d)$ we understand what are the Hemmer-Nakano versions of [HN04] replacing the symmetric groups by Temperley-Lieb algebras.

Corollary 6.3.6. *Let $n=2$ and d an odd number. Let R be a commutative Noetherian ring. The (partial) tilting modules of $S_R(2,d)$ are summands of $V^{\otimes d}$. Moreover, $V^{\otimes d}$ is a characteristic tilting module of $S_R(2,d)$. In particular, $V^{\otimes d} - \operatorname{domdim}(S_R(2,d), R) = +\infty$.*

Proof. Assume first that $R = k$ is an algebraically closed field. By Proposition 1.7.7, the indecomposable (partial) tilting modules of $S_k(2,d)$ are exactly $\{eT(\lambda)\colon \lambda \in \Lambda^+(2,d)\}$, where e is the idempotent making $eS_k(d,d)e \simeq S_k(2,d)$ and $T(\lambda)$ are (partial) tilting modules of $S_k(d,d)$. Since d is odd it cannot be written as $d = \lambda_1 + \lambda_2$, $\lambda_1 = \lambda_2$. Hence, all partitions in exactly 2-parts of d are 2-regular partitions of d. Hence, $T(\lambda)$ is projective-injective for $\lambda \in \Lambda^+(2,d)$. Therefore, $T(\lambda)$ is a summand of $(k^n)^{\otimes d}$. It follows that $eT(\lambda)$ is a summand of $V^{\otimes d}$ and $V^{\otimes d}$ is the characteristic tilting module of $S_k(2,d)$. Let R be a commutative Noetherian ring. It is clear that $R(\mathfrak{m})$ has a trivial Picard group and a flat algebraic closure. Furthermore, $(R^2)^{\otimes d} \in R\text{-proj}$. By Propositions 1.5.126 and 1.5.131, the result follows. The last part is clear since $V^{\otimes d}$ is a characteristic tilting module of $S_R(2,d)$. $\square$

Corollary 6.3.7. *Let R be a Noetherian commutative ring. Let d be an odd number. The Temperley-Lieb algebra $TL_{d,R}(-2)$ is split quasi-hereditary and it is the Ringel dual of $S_R(2,d)$.*

Proof. By Corollary 6.3.6, $V^{\otimes d}$ is a characteristic tilting module of $S_R(2,d)$. In view of Theorem 6.3.4, $TL_{d,R}(-2)$ is the Ringel dual of $S_R(2,d)$. $\square$

For d an even number, the Temperley-Lieb algebra is no longer split quasi-hereditary, in general. Based on the Examples 6.2.7 and 6.2.10 we are lead to believe that the characteristic two case is completely classified in the following way.

Conjecture 6.3.8. *Let K be a field of characteristic two and d a natural number. Then,*

$$(K^2)^{\otimes d} - \operatorname{domdim}_{S_K(2,d)} T = \begin{cases} +\infty & \text{if } d \text{ is an odd number,} \\ d & \text{if } \frac{d}{2} \text{ is an even number,} \end{cases}$$

where T is a characteristic tilting module of $S_K(2,d)$. In particular, $(R(S_K(2,d)), \operatorname{Hom}_{S_K(2,d)}(T, V^{\otimes d}))$ is a $\frac{d}{2} - 2$-faithful split quasi-hereditary cover of $TL_{K,d}(-2)$, where $R(S_K(2,d))$ denotes the Ringel dual of the Schur algebra $S_K(2,d)$.

6.4 Relative dominant dimension as a tool for Ringel self-duality

6.4.1 Ringel self-duality of BGG category $\mathcal{O}$

It is well known that the blocks of classical BGG category $\mathcal{O}$ are Ringel self-dual. This goes back to the work of Soergel [Soe98, Corollary 2.3]. This fact was then reproved independently in [FKM00, Proposition 4] using the Enright completion functor. Using the methodology introduced here we can establish a new proof of this fact without using the so-called semi-regular bimodule and without using Enright's completions.

Theorem 6.4.1. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathscr{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. The split quasi-hereditary R-algebra $A_{\mathscr{D}}$ (defined in Definition 4.4.42) is Ringel self-dual.*

Proof. Let $\mu \in \mathscr{D}$. Both $\Delta(\mu)$ and $\nabla(\mu)$ belong to $A_{\mathscr{D}}/J$-mod where J is an ideal admitting a filtration by split heredity ideals and such that $\Delta(\mu)$ is a projective $A_{\mathscr{D}}/J$-module. Further, since $\nabla(\mu)(\mathfrak{m})$ is the dual of $\Delta(\mu)(\mathfrak{m})$ its socle coincides with the top of $\Delta(\mu)(\mathfrak{m})$. Denote by f the non-zero $A_{\mathscr{D}}/J(\mathfrak{m})$-homomorphism $\Delta(\mu)(\mathfrak{m}) \twoheadrightarrow \operatorname{top}\Delta(\mu)(\mathfrak{m}) \hookrightarrow \nabla(\mu)(\mathfrak{m})$. As $\Delta(\mu)$ is a projective object in $A_{\mathscr{D}}/J$-mod there exists an $A_{\mathscr{D}}$-homomorphism $\overline{f}$ making the following diagram commutative:

$$\begin{tikzcd} \Delta(\mu) \arrow[r, two heads] \arrow[d, "\overline{f}"] & \Delta(\mu)(\mathfrak{m}) \arrow[d, "f"] \\ \nabla(\mu) \arrow[r, two heads] & \nabla(\mu)(\mathfrak{m}) \end{tikzcd} . \tag{6.4.1.1}$$

Consider the Schur functor $F = \operatorname{Hom}_{A_{\mathscr{D}}}(P_A(\omega), -)$, where ω is the antidominant weight. Applying F, we obtain the commutative diagram

$$\begin{tikzcd} F\Delta(\mu) \arrow[r, two heads] \arrow[d, "F\overline{f}"] & F\Delta(\mu)(\mathfrak{m}) \arrow[d, "Ff"] \\ F\nabla(\mu) \arrow[r, two heads] & F\nabla(\mu)(\mathfrak{m}) \end{tikzcd} . \tag{6.4.1.2}$$

Recall that by Lemma 1.1.32 for any $X \in A_{\mathscr{D}}$-mod,

$$F(X(\mathfrak{m})) = \operatorname{Hom}_{A_{\mathscr{D}}}(P_A(\omega), X(\mathfrak{m})) \simeq \operatorname{Hom}_{A_{\mathscr{D}(\mathfrak{m})}}(P_{A_{\mathscr{D}(\mathfrak{m})}}(\overline{\omega}), X(\mathfrak{m})) \tag{6.4.1.3}$$

and Ff is isomorphic to the map $\operatorname{Hom}_{A_{\mathscr{D}(\mathfrak{m})}}(P_{A(\mathfrak{m})}(\overline{\omega}), f)$ which is non-zero since $\operatorname{top} P_{A(\mathfrak{m})}(\overline{\omega})$ is the image of f. Moreover, $F\Delta(\mu)(\mathfrak{m}) \simeq R(\mathfrak{m})$ and $F\nabla(\mu)(\mathfrak{m}) \simeq R(\mathfrak{m})$. Hence, Ff is an isomorphism. Applying $R(\mathfrak{m}) \otimes_R -$ to the diagram (6.4.1.2) we obtain that $F\overline{f}(\mathfrak{m})$ is an isomorphism. Since both $F\Delta(\mu), F\nabla(\mu) \in R$-proj, Nakayama's Lemma yields that $F\overline{f}$ is an isomorphism. This shows that

$$F\Delta(\mu) \simeq F\nabla(\mu), \quad \forall \mu \in \mathscr{D}. \tag{6.4.1.4}$$

The results in Theorem 4.4.50 hold if we replace the complex numbers by any field of characteristic zero. Fix, for a moment $R = K[X_1, X_2]_{(X_1, X_2)}$ and $\mathscr{D}$ to be the block $W_{\overline{\mu}}\mu + \frac{X_1}{1}\alpha_1 + \frac{X_2}{1}\alpha_2$, where α_1, α_2 are distinct simple roots (so we are excluding the case $\mathfrak{g} = \mathfrak{sl}_2$), where $\mu \in \mathfrak{h}_R^*$ is a preimage of an antidominant weight in $\mathfrak{h}_{R(\mathfrak{m})}^*$ which is not dominant without coefficients in $\mathfrak{m}$ in its unique linear combination of simple roots. Hence, we are excluding the simple blocks which are trivially Ringel self-dual. By Theorem 4.4.50, $(A_{\mathscr{D}}, P_A(\omega))$ is a $1 - \mathscr{F}(\tilde{\Delta})$ cover of C.

Let T be a characteristic tilting module of $A_{\mathscr{D}}$. We claim that $(R(A_{\mathscr{D}}), \operatorname{Hom}_{A_{\mathscr{D}}}(T, P(\omega))$ is a $1 - \mathscr{F}(\tilde{\Delta}_R)$ cover of C, where $R(A_{\mathscr{D}})$ denotes the Ringel dual of $A_{\mathscr{D}}$. In the proof of Theorem 4.4.50 (replacing $\mathbb{C}$ by K), we observe that, for any prime ideal $\mathfrak{p}$ of R with height at most one, $Q(R/\mathfrak{p}) \otimes_R A_{\mathscr{D}}$ is semi-simple. Here, $Q(R/\mathfrak{p})$ denotes the quotient field of $R/\mathfrak{p}$. Therefore, the Ringel dual of $Q(R/\mathfrak{p}) \otimes_R A_{\mathscr{D}}$, which is $Q(R/\mathfrak{p}) \otimes_R R(A_{\mathscr{D}})$ according to Propositions 1.5.126 and 1.5.133, is semi-simple for any prime ideal $\mathfrak{p}$ of R with height at most one. Therefore,

$$Q(R/\mathfrak{p}) \otimes_R P(\omega) - \operatorname{codomdim}_{Q(R/\mathfrak{p}) \otimes_R A_{\mathscr{D}}} Q(R/\mathfrak{p}) \otimes_R T = +\infty \tag{6.4.1.5}$$

and $(Q(R/\mathfrak{p}) \otimes_R R(A_{\mathscr{D}}), Q(R/\mathfrak{p}) \otimes_R \operatorname{Hom}_{A_{\mathscr{D}}}(T, P(\omega))$ is a $+\infty-$faithful split quasi-hereditary cover of $Q(R/\mathfrak{p}) \otimes_R C$ for every prime ideal $\mathfrak{p}$ of R with height at most one. By Theorems 4.4.48 and 4.4.49 and Proposition 5.2.7, $P(\omega) - \operatorname{codomdim}_{(A_{\mathscr{D}}, R)} T = 1$. By Corollary 5.5.6, $(R(A_{\mathscr{D}}), \operatorname{Hom}_{A_{\mathscr{D}}}(T, P(\omega)))$ is a $0 - \mathscr{F}(\tilde{\Delta}_R)$ cover of C. So, we can apply Theorem 3.3.13 to obtain that $(R(A_{\mathscr{D}}), \operatorname{Hom}_{A_{\mathscr{D}}}(T, P(\omega)))$ is a $1 - \mathscr{F}(\tilde{\Delta}_R)$ cover of C. Now, rewriting (6.4.1.4) we obtain

$$F\Delta(\theta) \simeq F\nabla(\theta) = \operatorname{Hom}_{A_{\mathscr{D}}}(P(\omega), \nabla(\theta)) \simeq \operatorname{Hom}_{R(A_{\mathscr{D}})}(\operatorname{Hom}_{A_{\mathscr{D}}}(T, P(\omega)), \operatorname{Hom}_{A_{\mathscr{D}}}(T, \nabla(\theta))). \tag{6.4.1.6}$$

By Corollary 3.6.6, $A_{\mathscr{D}}$ is Ringel self-dual. That is, there exists an equivalence of categories $H \colon A_{\mathscr{D}}\text{-mod} \to R(A_{\mathscr{D}})\text{-mod}$ preserving the highest weight structure. Applying $R(\mathfrak{m}) \otimes_R -$ to H we obtain that $A_{\mathscr{D}}(\mathfrak{m})$ is Ringel self-dual. That is, the blocks of category $\mathscr{O}$ over a field of characteristic zero are Ringel self-dual. We excluded the case $\mathfrak{g} = \mathfrak{sl}_2$. But the non-simple blocks of the category $\mathscr{O}$ associated with $\mathfrak{sl}_2$ are Morita equivalent to $\mathscr{A}_2$ according to Example 6.2.1 which is Ringel self-dual.

Return to the general case of R being an arbitrary local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra and $\mathscr{D}$ an arbitrary block. Since $R(\mathfrak{m})$ is a field of characteristic zero, $A_{\mathscr{D}}(\mathfrak{m})$ is Ringel self-dual. By Lemma 1.5.134, $A_{\mathscr{D}}$ is Ringel self-dual. ☐

6.4.2 Ringel self-duality of Schur algebras

The approach presented in Theorem 6.4.1 can also be applied to Schur algebras. However, we have to exclude the case of characteristic two for similar reasons why we excluded the case $\mathfrak{sl}_2$ for the general cases in the category $\mathscr{O}$.

Theorem 6.4.2. *Assume that $n \geq d$ are natural numbers. The Schur algebra $S_{\mathbb{Z}[\frac{1}{2}]}(n,d)$ is Ringel self-dual.*

Proof. The quotient field of $\mathbb{Z}[\frac{1}{2}]$ is $\mathbb{Q}$. So, for $S_{\mathbb{Z}[\frac{1}{2}]}(n,d)$ conditions (i) and (ii) of Corollary 5.5.8 hold. Condition (iii) follows by Proposition 5.2.7 and Corollary 4.1.8. Now we will focus on the image of the dual of Weyl modules under the Schur functor. Fix $R = \mathbb{Z}[\frac{1}{2}]$. We can see that, for any $\lambda \in \Lambda^+(n,d)$

$$F\nabla(\lambda) \simeq \xi_{(1,\dots,d),(1,\dots,d)}\nabla(\lambda) \simeq \xi_{(1,\dots,d),(1,\dots,d)} D\Delta(\lambda)^\iota \simeq D(\xi_{(1,\dots,d),(1,\dots,d)}\Delta(\lambda))^\iota \tag{6.4.2.1}$$

$$\simeq D\theta(\lambda)^\iota \simeq \operatorname{sgn} \otimes_R \theta(\lambda'). \tag{6.4.2.2}$$

The last isomorphism is [CPS96, Lemma 1.6.12] and λ' is the conjugate partition of λ. Here sgn is the free R-module with rank one with the action the sign of the permutation $\sigma \cdot 1_R = \operatorname{sgn}(\sigma)$, $\sigma \in S_d$. Moreover, RS_d acts on $\operatorname{sgn} \otimes_R M$ through the diagonal action. Hence, $\operatorname{sgn} \otimes_R \otimes_R M \simeq M$ for any $M \in RS_d$-mod. Hence, $\operatorname{sgn} \otimes_R - \colon RS_d\text{-mod} \to RS_d\text{-mod}$ is an isomorphism of categories. Therefore,

$$F\nabla(\lambda) \simeq \operatorname{sgn} \otimes_R \theta(\lambda') \simeq \operatorname{sgn} \otimes_R F\Delta(\lambda'),\ \forall \lambda \in \Lambda^+(n,d), \tag{6.4.2.3}$$

and $\mathscr{F}(F\nabla) \simeq \mathscr{F}(F\Delta)$. By Corollary 5.5.8, the result follows. ☐

6.4.3 Uniqueness of covers

As we can use the point of view of cocovers to prove Ringel self-duality, this could be exploited to construct an example of a cover (if it exists) with a level of faithfulness as big as we want (always smaller than the global dimension) which is not unique, emphasizing the importance of the condition (3.6.0.7) on Corollary 3.6.6. The author believes that the following example is a first step towards such a construction.

Example 6.4.3. *For any $m \geq 2$, the bound quiver algebra, which we will denote by A_m,*

$$\begin{tikzcd} & & & 1 \arrow[dlll, "\alpha_1"'] & \\ m \arrow[r, "\alpha_m"'] & m-1 \arrow[r] & \cdots \arrow[r] & 3 \arrow[r, "\alpha_3"'] & 2 \arrow[ul, "\alpha_2"'] \end{tikzcd} \quad , \quad \alpha_i\alpha_{i+1} = 0,\ i = 1, \dots m-1, \tag{6.4.3.1}$$

is a quasi-hereditary algebra not being Ringel self-dual with a characteristic tilting module having positive dominant and codominant dimension. The indecomposable projective A_m-modules are

$$P_m(1) = \begin{array}{c} 1 \\ m \\ m-1 \end{array},\ P_m(2) = \begin{array}{c} 2 \\ 1 \end{array},\ P_m(3) = \begin{array}{c} 3 \\ 2 \end{array},\ \cdots,\ P_m(m) = \begin{array}{c} m \\ m-1 \end{array}. \tag{6.4.3.2}$$

The indecomposable injective A_m-modules are

$$I_m(1) = P_m(2),\ I_m(2) = P_m(3),\ \cdots,\ I_m(m-2) = P_m(m-1),\ I_m(m-1) = P_m(1),\ I_m(m) = \begin{array}{c} 1 \\ m \end{array}. \tag{6.4.3.3}$$

Therefore, $P_m := P_m(1) \oplus \cdots \oplus P_m(m-1)$ is a projective-injective module and the exact sequence

$$0 \to P_m(m) \to P_m(1) \to P_m(2) \to \cdots P_m(m-1) \to P_m(1) \to I_m(m) \to 0 \tag{6.4.3.4}$$

is both a minimal injective resolution of $P_m(m)$ and a minimal projective resolution of $I_m(m)$. In particular, $\operatorname{domdim} A_m = m$.

With the order $m > 1 > 2 > \cdots > m-1$ and standard modules

$$\Delta(1) = \operatorname{top} P_m(1), \cdots,\ \Delta(m-1) = \operatorname{top} P_m(m-1),\ \Delta(m) = P_m(m), \tag{6.4.3.5}$$

A_m is a split quasi-hereditary algebra. For this order, the costandard modules are

$$\nabla(1) = I_m(1), \cdots,\ \nabla(m-2) = I_m(m-2),\ \nabla(m-1) = \operatorname{top} P_m(m-1),\ \nabla(m) = I_m(m). \tag{6.4.3.6}$$

So, $P_m \oplus \Delta(m-1)$ is the characteristic tilting module. Further, the exact sequences

$$0 \to P_m(m) \to P_m(1) \to P_m(2) \to \cdots \to P_m(m-1) \to m-1 \to 0 \tag{6.4.3.7}$$

$$0 \to m-1 \to P_m(1) \to I_m(m) \to 0 \tag{6.4.3.8}$$

give that $\operatorname{domdim} T = m-1$ and $\operatorname{codomdim} T = 1$. So, (A_m, P_m) is a (-1)-faithful quasi hereditary cover of $\operatorname{End}_{A_m}(P_m)^{op}$ and $(R(A_m), \operatorname{Hom}_{A_m}(T, P_m))$ is an $m-3$-faithful quasi hereditary cover of $\operatorname{End}_{A_m}(P_m)^{op}$. We can observe that the endomorphism algebra $\operatorname{End}_{A_m}(P_m)^{op}$ has radical square zero and it is self-injective while the Ringel dual functor does not send costandard modules to simple modules (except the maximal one). So, A_m is not Ringel self-dual. $\triangle$

Bibliography

[AB59] M. Auslander and D. A. Buchsbaum. On ramification theory in Noetherian rings. *Am. J. Math.*, 81:749–765, 1959.

[AG60] M. Auslander and O. Goldman. Maximal orders. *Trans. Am. Math. Soc.*, 97:1–24, 1960.

[AKMW20] S. Ariki, R. Kase, K. Miyamoto, and K. Wada. Self-injective cellular algebras whose representation type are tame of polynomial growth. *Algebr. Represent. Theory*, 23(3):833–871, 2020.

[AR72] M. Auslander and K. W. Roggenkamp. A characterization of orders of finite lattice type. *Inventiones Mathematicae*, 17:79–84, 1972.

[Ari08] S. Ariki. Finite dimensional Hecke algebras. In *Trends in representation theory of algebras and related topics. Proceedings of the 12th international conference on representations of algebras and workshop (ICRA XII), Toruń, Poland, August 15–24, 2007*, page 1–48. Zürich: European Mathematical Society (EMS), 2008.

[ARS95] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras.*, volume 36. Cambridge: Cambridge University Press, 1995.

[AS80] M. Auslander and S. O. Smalø. Preprojective modules over Artin algebras. *J. Algebra*, 66:61–122, 1980.

[AS93] M. Auslander and Ø. Solberg. Relative homology and representation theory. III: Cotilting modules and Wedderburn correspondence. *Commun. Algebra*, 21(9):3081–3097, 1993.

[ASS06] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1: Techniques of representation theory*, volume 65. Cambridge: Cambridge University Press, 2006.

[AST18] H. H. Andersen, C. Stroppel, and D. Tubbenhauer. Cellular structures using U_q-tilting modules. *Pacific J. Math.*, 292(1):21–59, 2018.

[Aus71] M. Auslander. Representation dimension of Artin algebras. With the assistance of Bernice Auslander. Queen Mary College Mathematics Notes. London: Queen Mary College. 179 p. , 1971.

[BD09] D. Benson and S. Doty. Schur-Weyl duality over finite fields. *Arch. Math.*, 93(5):425–435, 2009.

[BGG76] I. N. Bernshteĭn, I. M. Gel'fand, and S. I. Gel'fand. Category of $\mathfrak{g}$ modules. *Funct. Anal. Appl.*, 10:87–92, 1976.

[Bou98] N. Bourbaki. *Elements of mathematics. Commutative algebra. Chapters 1–7. Transl. from the French. Softcover edition of the 2nd printing 1989*. Berlin: Springer, softcover edition of the 2nd printing 1989 edition, 1998.

[BS98] A. B. Buan and Ø. Solberg. Relative cotilting theory and almost complete cotilting modules. In *Algebras and modules II. Eighth international conference on representations of algebras, Geiranger, Norway, August 4–10, 1996*, page 77–92. Providence, RI: American Mathematical Society, 1998.

[BT17] G. Bellamy and U. Thiel. Cores of graded algebras with triangular decomposition, 2017. https://arxiv.org/abs/1711.00780.

[CFK+21] H. Chen, M. Fang, O. Kerner, S. Koenig, and K. Yamagata. Rigidity dimension of algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 170(2):417–443, 2021.

[CL74] R. W. Carter and G. Lusztig. On the modular representations of the general linear and symmetric groups. Proc. 2nd internat. Conf. Theory of Groups, Canberra 1973, Lect. Notes Math. 372, 218-220 (1974)., 1974.

[Coh46] I. S. Cohen. On the Structure and Ideal Theory of Complete Local Rings. *Transactions of the American Mathematical Society*, 59(1):54–106, 1946.

[Coh77] P. M. Cohn. *Algebra. Volume 2*. John Wiley & Sons, Ltd., 1st edition, 1977.

[Cou20] K. Coulembier. The classification of blocks in BGG category $\mathscr{O}$. *Math. Z.*, 295(1-2):821–837, 2020.

[CPS88] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[CPS90] E. Cline, B. Parshall, and L. Scott. Integral and graded quasi-hereditary algebras, I. *Journal of Algebra*, 131(1):126–160, 1990.

[CPS96] E. Cline, B. Parshall, and L. Scott. *Stratifying endomorphism algebras.*, volume 591. Providence, RI: American Mathematical Society (AMS), 1996.

[CR90] C. W. Curtis and I. Reiner. *Methods of representation theory with applications to finite groups and orders. Volume 1. Paperback edition.* New York etc.: John Wiley &— Sons, paperback edition edition, 1990.

[CR06] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1962 original.

[Cru19] T. Cruz. Schur-Weyl duality over commutative rings. *Commun. Algebra*, 47(4):1619–1628, 2019.

[Cru21] T. Cruz. A characterisation of Morita algebras in terms of covers. *Algebras and Representation Theory*, 2021. https://doi.org/10.1007/s10468-021-10064-8.

[dCP76] C. de Concini and C. Procesi. A characteristic free approach to invariant theory. *Adv. Math.*, 21:330–354, 1976.

[DD91] R. Dipper and S. Donkin. Quantum GL_n. *Proc. Lond. Math. Soc. (3)*, 63(1):165–211, 1991.

[DEMN99] S. R. Doty, K. Erdmann, S. Martin, and D. K. Nakano. Representation type of Schur algebras. *Math. Z.*, 232(1):137–182, 1999.

[DEN04] S. R. Doty, K. Erdmann, and D. K. Nakano. Extensions of Modules over Schur Algebras, Symmetric Groups and Hecke Algebras. *Algebras and Representation Theory*, 7(1):67–99, Mar 2004.

[DITV15] H. Dao, O. Iyama, R. Takahashi, and C. Vial. Non-commutative resolutions and Grothendieck groups. *J. Noncommut. Geom.*, 9(1):21–34, 2015.

[DJ89] R. Dipper and G. James. The q-Schur algebra. *Proc. Lond. Math. Soc. (3)*, 59(1):23–50, 1989.

[DJ91] R. Dipper and G. James. q-tensor space and q-Weyl modules. *Trans. Am. Math. Soc.*, 327(1):251–282, 1991.

[DK94] Y. A. Drozd and V. V. Kirichenko. *Finite-dimensional algebras*. Springer-Verlag, Berlin, 1994. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab.

[dlPX06] J. A. de la Peña and C. Xi. Hochschild cohomology of algebras with homological ideals. *Tsukuba J. Math.*, 30(1):61–79, 2006.

[Don86] S. Donkin. On Schur algebras and related algebras. I. *J. Algebra*, 104(2):310–328, 1986.

[Don87] S. Donkin. On Schur algebras and related algebras. II. *J. Algebra*, 111:354–364, 1987.

[Don93] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[Don94] S. Donkin. On Schur algebras and related algebras. IV: The blocks of the Schur algebras. *J. Algebra*, 168(2):400–429, 1994.

[Don98] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[DPS98a] J. Du, B. Parshall, and L. Scott. Cells and q-Schur algebras. *Transform. Groups*, 3(1):33–49, 1998.

[DPS98b] J. Du, B. Parshall, and L. Scott. Quantum Weyl reciprocity and tilting modules. *Commun. Math. Phys.*, 195(2):321–352, 1998.

[DR89a] V. Dlab and C. M. Ringel. Every semiprimary ring is the endomorphism ring of a projective module over a quasi-hereditary ring. *Proc. Am. Math. Soc.*, 107(1):1–5, 1989.

[DR89b] V. Dlab and C. M. Ringel. Quasi-hereditary algebras. *Illinois Journal of Mathematics*, 33(2):280–291, 1989.

[DR92] V. Dlab and C. M. Ringel. The module theoretical approach to quasi-hereditary algebras. In *Representations of algebras and related topics. Proceedings of the Tsukuba international conference, held in Kyoto, Japan, 1990*, page 200–224. Cambridge: Cambridge University Press, 1992.

[DR98] J. Du and H. Rui. Based algebras and standard bases for quasi-hereditary algebras. *Trans. Am. Math. Soc.*, 350(8):3207–3235, 1998.

[Du92] J. Du. The modular representation theory of q-Schur algebras. *Trans. Am. Math. Soc.*, 329(1):253–271, 1992.

[Du03] J. Du. Finite-dimensional algebras and standard systems. *Algebr. Represent. Theory*, 6(5):461–474, 2003.

[EH02] K. Erdmann and A. Henke. On Ringel duality for Schur algebras. *Math. Proc. Camb. Philos. Soc.*, 132(1):97–116, 2002.

[Eis95] D. Eisenbud. *Commutative algebra. With a view toward algebraic geometry.*, volume 150. Berlin: Springer-Verlag, 1995.

[EM65] S. Eilenberg and J. C. Moore. *Foundations of relative homological algebra.*, volume 55. Providence, RI: American Mathematical Society (AMS), 1965.

[End67] S. Endo. Completely faithful modules and quasi-Frobenius algebras. *J. Math. Soc. Japan*, 19:437–456, 1967.

[Erd93] K. Erdmann. Schur algebras of finite type. *Quart. J. Math. Oxford Ser. (2)*, 44(173):17–41, 1993.

[Erd94] K. Erdmann. Symmetric groups and quasi-hereditary algebras. In *Finite dimensional algebras and related topics. Proceedings of the NATO Advanced Research Workshop on Representations of algebras and related topics. Ottawa, Canada, August 10-18, 1992*, page 123–161. Dordrecht: Kluwer Academic Publishers, 1994.

[EW06] K. Erdmann and M. J. Wildon. *Introduction to Lie algebras*. Berlin: Springer, 2006.

[Fac98] A. Facchini. *Module theory*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1998. Endomorphism rings and direct sum decompositions in some classes of modules.

[Fad65] D. K. Faddeev. Introduction to multiplicative theory of modules of integral representations. *Proc. Steklov Inst. Math.*, 80:164–210, 1965.

[Fai73] C. Faith. *Algebra: rings, modules and categories. I.* Springer-Verlag, New York-Heidelberg, 1973. Die Grundlehren der mathematischen Wissenschaften, Band 190.

[Fan08] M. Fang. Schur functors on QF-3 standardly stratified algebras. *Acta Math. Sin., Engl. Ser.*, 24(2):311–318, 2008.

[Fan14] M. Fang. Permanents, Doty coalgebras and dominant dimension of Schur algebras. *Adv. Math.*, 264:155–182, 2014.

[FHK21] M. Fang, W. Hu, and S. Koenig. On derived equivalences and homological dimensions. *J. Reine Angew. Math.*, 770:59–85, 2021.

[FK11a] M. Fang and S. Koenig. Endomorphism algebras of generators over symmetric algebras. *J. Algebra*, 332(1):428–433, 2011.

[FK11b] M. Fang and S. Koenig. Schur functors and dominant dimension. *Trans. Am. Math. Soc.*, 363(3):1555–1576, 2011.

[FK16] M. Fang and S. Koenig. Gendo-symmetric algebras, canonical comultiplication, bar cocomplex and dominant dimension. *Trans. Am. Math. Soc.*, 368(7):5037–5055, 2016.

[FKM00] V. Futorny, S. König, and V. Mazorchuk. $\mathscr{S}$-subcategories in $\mathscr{O}$. *Manuscripta Math.*, 102(4):487–503, 2000.

[FKY18] M. Fang, O. Kerner, and K. Yamagata. Canonical bimodules and dominant dimension. *Trans. Am. Math. Soc.*, 370(2):847–872, 2018.

[FM19] M. Fang and H. Miyachi. Hochschild cohomology and dominant dimension. *Trans. Am. Math. Soc.*, 371(8):5267–5292, 2019.

[Fro00] G. Frobenius. *Über die Charaktere der symmetrischen Gruppe*. Königliche Gesellschaft der Wissenschaften, Göttingen, 1900.

[GJ81] O. Gabber and A. Joseph. Towards the Kazhdan-Lusztig conjecture. *Ann. Sci. Éc. Norm. Supér. (4)*, 14:261–302, 1981.

[GK15] N. Gao and S. Koenig. Grade, dominant dimension and Gorenstein algebras. *J. Algebra*, 427:118–141, 2015.

[GL96] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[Gre93] J. A. Green. Combinatorics and the Schur algebra. *J. Pure Appl. Algebra*, 88(1-3):89–106, 1993.

[Gre07] J. A. Green. *Polynomial representations of GL_n. With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, J. A. Green and M. Schocker. 2nd corrected and augmented edition.*, volume 830. Berlin: Springer, 2nd corrected and augmented edition edition, 2007.

[GS71] S. Greco and P. Salmon. Topics in $\mathfrak{m}$-adic topologies. Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 58. Berlin-Heidelberg-New York: Springer Verlag (1971)., 1971.

[Has00] M. Hashimoto. *Auslander-Buchweitz approximations of equivariant modules.*, volume 282. Cambridge: Cambridge University Press, 2000.

[Hat63] A. Hattori. Semisimple algebras over a commutative ring. *Journal of the Mathematical Society of Japan*, 15(4):404–419, 1963.

[HN04] D. J. Hemmer and D. K. Nakano. Specht filtrations for Hecke algebras of type A. *J. Lond. Math. Soc., II. Ser.*, 69(3):623–638, 2004.

[Hoc56] G. Hochschild. Relative homological algebra. *Transactions of the American Mathematical Society*, 82(1):246–269, 1956.

[Hos89] M. Hoshino. On dominant dimension of Noetherian rings. *Osaka J. Math.*, 26(2):275–280, 1989.

[Hos90] M. Hoshino. Syzygies and Gorenstein rings. *Arch. Math. (Basel)*, 55(4):355–360, 1990.

[HS97] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[HU96] D. Happel and L. Unger. Modules of finite projective dimension and cocovers. *Math. Ann.*, 306(3):445–457, 1996.

[Hua05] Z. Huang. On U-dominant dimension. *J. Algebra*, 285(2):669–681, 2005.

[Hua06] Z. Huang. Wakamatsu tilting modules, U-dominant dimension, and k-Gorenstein modules. In *Abelian groups, rings, modules, and homological algebra. Selected papers of a conference on the occasion of Edgar Earle Enochs' 72nd birthday, Auburn, AL, USA, September 9–11, 2004*, page 183–202. Boca Raton, FL: Chapman & Hall/CRC, 2006.

[Hum80] J. E. Humphreys. *Introduction to Lie algebras and representation theory. 3rd printing, rev*, volume 9. Springer, New York, NY, 1980.

[Hum08] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94. Providence, RI: American Mathematical Society (AMS), 2008.

[Ive14] B. Iversen. *Lecture notes on local rings. Edited by Holger Andreas Nielsen.* Hackensack, NJ: World Scientific, 2014.

[Iya03] O. Iyama. Finiteness of representation dimension. *Proc. Am. Math. Soc.*, 131(4):1011–1014, 2003.

[Iya04] O. Iyama. Representation dimension and Solomon zeta function. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry. Proceedings from the 10th international conference on algebras and related topics, ICRA X, Toronto, Canada, July 15–August 10, 2002*, page 45–64. Providence, RI: American Mathematical Society (AMS), 2004.

[Jam78] G. D. James. *The representation theory of the symmetric groups.*, volume 682. Springer, Cham, 1978.

[Jim86] M. Jimbo. A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys.*, 11:247–252, 1986.

[Jon63] A. Jones. Groups with a finite number of indecomposable integral representations. *Mich. Math. J.*, 10:257–261, 1963.

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72:1–25, 1983.

[Jon87] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. Math. (2)*, 126:335–388, 1987.

[Kat68] T. Kato. Rings of dominant dimension ≥ 1. *Proc. Japan Acad.*, 44:579–584, 1968.

[KM02] S. König and V. Mazorchuk. Enright's completions and injectively copresented modules. *Trans. Amer. Math. Soc.*, 354(7):2725–2743, 2002.

[KN01] A. S. Kleshchev and D. K. Nakano. On comparing the cohomology of general linear and symmetric groups. *Pac. J. Math.*, 201(2):339–355, 2001.

[Kne66] M. Kneser. Einige Bemerkungen über ganzzahlige Darstellungen endlicher Gruppen. *Arch. Math.*, 17:377–379, 1966.

[Kön91] S. König. Every order is the endomorphism ring of a projective module over a quasi-hereditary order. *Commun. Algebra*, 19(8):2395–2401, 1991.

[Kra17] H. Krause. Highest weight categories and strict polynomial functors. With an appendix by Cosima Aquilino. In *Representation theory – Current trends and perspectives. In part based on talks given at the last joint meeting of the priority program in Bad Honnef, Germany, in March 2015*, page 331–373. Zürich: European Mathematical Society (EMS), 2017.

[KSX01] S. König, I. H. Slungård, and C. Xi. Double centralizer properties, dominant dimension, and tilting modules. *Journal of Algebra*, 240(1):393–412, 2001.

[KX98] S. König and C. Xi. On the structure of cellular algebras. In *Algebras and modules II. Eighth international conference on representations of algebras, Geiranger, Norway, August 4–10, 1996*, page 365–386. Providence, RI: American Mathematical Society, 1998.

[KX99a] S. König and C. Xi. Cellular algebras: Inflations and Morita equivalences. *J. Lond. Math. Soc., II. Ser.*, 60(3):700–722, 1999.

[KX99b] S. König and C. Xi. When is a cellular algebra quasi-hereditary? *Math. Ann.*, 315(2):281–293, 1999.

[KX00] S. König and C. Xi. A self-injective cellular algebra is weakly symmetric. *J. Algebra*, 228(1):51–59, 2000.

[KY13] O. Kerner and K. Yamagata. Morita algebras. *J. Algebra*, 382:185–202, 2013.

[KY14] O. Kerner and K. Yamagata. Morita theory, revisited. In *Expository lectures on representation theory*, volume 607 of *Contemp. Math.*, page 85–96. Amer. Math. Soc., Providence, RI, 2014.

[Lam99] T. Y. Lam. *Lectures on modules and rings.*, volume 189. New York, NY: Springer, 1999.

[Lam01] T. Y. Lam. *A first course in noncommutative rings. 2nd ed.*, volume 131. New York, NY: Springer, 2nd ed. edition, 2001.

[Lam06] T. Y. Lam. *Serre's problem on projective modules*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.

[Lan02] S. Lang. *Algebra. 3rd revised ed.*, volume 211. New York, NY: Springer, 3rd revised ed. edition, 2002.

[LQ13] B. Le Stum and A. Quirós. On quantum state of numbers, 2013.

[LZ21] S. Li and S. Zhang. Algebras with finite relative dominant dimension and almost n-precluster tilting modules. *J. Pure Appl. Algebra*, 225(2):18, 2021. Id/No 106498.

[Mac71] S. Mac Lane. Categories for the Working Mathematician, 1971.

[Mac95] S. Mac Lane. *Homology. Reprint of the 3rd corr. print. 1975.* Berlin: Springer-Verlag, reprint of the 3rd corr. print. 1975 edition, 1995.

[Mar92] P. P. Martin. On Schur-Weyl duality, A_n Hecke algebras and quantum $sl(N)$ on $\otimes^{n+1}\mathbb{C}^N$. *Int. J. Mod. Phys. A*, 7:645–673, 1992.

[Mat99] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group.*, volume 15. Providence, RI: American Mathematical Society, 1999.

[Mit81] B. Mitchell. A quick proof of the Gabriel-Popescu theorem. *J. Pure Appl. Algebra*, 20:313–315, 1981.

[Mor58] K. Morita. Duality for modules and its applications to the theory of rings with minimum condition. *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A*, 6:83–142, 1958.

[Mor70] K. Morita. Localizations in categories of modules. I. *Math. Z.*, 114:121–144, 1970.

[MR87] J. C. McConnell and J. C. Robson. Noncommutative Noetherian rings. With the cooperation of L. W. Small. Pure and Applied Mathematics. A Wiley-Interscience Publication. Chichester etc.: John Wiley & Sons. XV, 596 p.; £65.00 (1987)., 1987.

[MS08] V. Mazorchuk and C. Stroppel. Projective-injective modules, Serre functors and symmetric algebras. *J. Reine Angew. Math.*, 616:131–165, 2008.

[Mue68] B. J. Mueller. The classification of algebras by dominant dimension. *Can. J. Math.*, 20:398–409, 1968.

[Par89] B. J. Parshall. Finite dimensional algebras and algebraic groups. Classical groups and related topics, Proc. Conf., Beijing/China 1987, Contemp. Math. 82, 97-114 (1989)., 1989.

[PG64] N. Popescu and P. Gabriel. Caractérisation des categories abeliennes avec générateurs et limites inductives exactes. *C. R. Acad. Sci., Paris*, 258:4188–4190, 1964.

[PS88] B. Parshall and L. Scott. Derived categories, quasi-hereditary algebras, and algebraic groups. *Carlton Univ. Lecture Notes in Math.*, 3(3):1–104, 1988.

[PS05] B. J. Parshall and L. L. Scott. Quantum Weyl reciprocity for cohomology. *Proc. Lond. Math. Soc. (3)*, 90(3):655–688, 2005.

[Psa14] C. Psaroudakis. Homological theory of recollements of abelian categories. *J. Algebra*, 398:63–110, 2014.

[PW91] B. Parshall and J. Wang. *Quantum linear groups.*, volume 439. Providence, RI: American Mathematical Society (AMS), 1991.

[Rei70] I. Reiner. A survey of integral representation theory. *Bull. Am. Math. Soc.*, 76:159–227, 1970.

[Rei03] I. Reiner. *Maximal orders. Reprint of the 1975 original.* Oxford: Oxford University Press, reprint of the 1975 original edition, 2003.

[Rin91] C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991.

[Rin10] C. M. Ringel. Iyama's finiteness theorem via strongly quasi-hereditary algebras. *J. Pure Appl. Algebra*, 214(9):1687–1692, 2010.

[Rot09] J. J. Rotman. *An introduction to homological algebra. 2nd ed.* Berlin: Springer, 2nd ed. edition, 2009.

[Rot10] J. J. Rotman. *Advanced modern algebra. 2nd ed.*, volume 114. Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2010.

[Rou08] R. Rouquier. q-Schur algebras and complex reflection groups. *Moscow Mathematical Journal*, 8(1):119–158, 184, 2008.

[Sch01] I. Schur. Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen. Diss. Berlin. 76 S., 1901.

[Sch27] I. Schur. Über die rationalen Darstellungen der allgemeinen linearen Gruppe. *Sitzungsber. Preuß. Akad. Wiss., Phys.-Math. Kl.*, 1927:58–75, 1927.

[Soe90] W. Soergel. Kategorie $\mathscr{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Am. Math. Soc.*, 3(2):421–445, 1990.

[Soe97] W. Soergel. Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren. *Represent. Theory*, 1:115–132, 1997.

[Soe98] W. Soergel. Character formulas for tilting modules over Kac-Moody algebras. *Represent. Theory*, 2:432–448, 1998.

[Swa78] R. G. Swan. Projective modules over Laurent polynomial rings. *Trans. Am. Math. Soc.*, 237:111–120, 1978.

[Tac63] H. Tachikawa. A characterization of QF-3 algebras. *Proc. Am. Math. Soc.*, 13:701–703, 1963.

[Tac70] H. Tachikawa. Double centralizers and dominant dimensions. *Math. Z.*, 116:79–88, 1970.

[Tac73] H. Tachikawa. *Quasi-Frobenius rings and generalizations. QF-3 and QF-1 rings. Notes by Claus Michael Ringel*, volume 351. Springer, Cham, 1973.

[TL71] H. V. N. Temperley and E. H. Lieb. Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem. *Proc. R. Soc. Lond., Ser. A*, 322:251–280, 1971.

[Tot97] B. Totaro. Projective resolutions of representations of $\mathrm{GL}(n)$. *J. Reine Angew. Math.*, 482:1–13, 1997.

[Wak88] T. Wakamatsu. On modules with trivial self-extensions. *J. Algebra*, 114(1):106–114, 1988.

[Wei03] C. A. Weibel. *An introduction to homological algebra*. Number 38 in Cambridge studies in advanced mathematics. Cambridge Univ. Press, Cambridge, reprint. 1997, transf. to digital print. edition, 2003.

[Wes95] B. W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, 219(4):539–565, 1995.

[Wey46] H. Weyl. *The classical groups, their invariants and representations. 2nd ed.* Princeton, NJ: Princeton University Press, 2nd ed. edition, 1946.

[Woo74] S. M. Woods. Some results on semi-perfect group rings. *Can. J. Math.*, 26:121–129, 1974.

[Xi92] C. Xi. The structure of Schur algebras $S\_k(n,p)$ for $n \geq p$. *Can. J. Math.*, 44(3):665–672, 1992.

[Xi93] C. Xi. On representation types of q-Schur algebras. *J. Pure Appl. Algebra*, 84(1):73–84, 1993.

[Yam96] K. Yamagata. Frobenius algebras. In *Handbook of algebra. Volume 1*, page 841–887. Amsterdam: North-Holland, 1996.

[Zas38] H. Zassenhaus. Neuer Beweis der Endlichkeit der Klassenzahl bei unimodularer Äquivalenz endlicher ganzzahliger Substitutionsgruppen. *Abh. Math. Semin. Univ. Hamb.*, 12:276–288, 1938.

[Zim14] A. Zimmermann. *Representation theory*, volume 19 of *Algebra and Applications*. Springer, Cham, 2014. A homological algebra point of view.