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Bachelorarbeit

A Tool for the Estimation of Lattice Parameters

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Abstract

This work introduces a new tool that can be used to find secure parameters for schemes based on the Learning with Errors (LWE) and the Short Integer Solution (SIS). Since the proposal of worst-case to average-case reductions from certain hard lattice problems to SIS and LWE respectively, both SIS and LWE have led to a plethora of cryptograpic schemes. Lattice based cryptography is highly in demand particularly in the context of post-quantum cryptography, as the era of quantum computing will render several widely applied schemes, such as RSA, insecure. To use LWE and SIS in practice, we first must establish their concrete hardness. Given a set of paramters, we can estimate the hardness of a problem instance by estimating the runtime of the currently best known algorithms that solve the respective problem instance. This has been done for LWE in the LWE Estimator by Albrecht et al. (JMC 2015) and subsequent works. However, at this point, there is no unified tool that provides estimates for both LWE and SIS as well as their ring and module variants.

We aim to close this gap with a new Python library. Our tool includes previous estimates for LWE from the LWE Estimator and adds new attack estimates for SIS. In this thesis, we give an overview of the LWE and SIS problems and describe various algorithms that that can be used to solve them. In addition, we present current popular cost models that many estimates rely on from the literature. At the core of our tool is a generic paramter search function that is both simple to use and allows for extensive customization. Our tool supports important problem variants in several p-norms, norm bound estimates and distribution classes.

Contents

List of Figures

List of Tables

List of Algorithms

1 Introduction

The rise of quantum computing promises great progress in areas that are considered to be very difficult for classical computers. At the same time, it poses a threat to cybersecurity, as some widely used cryptographic schemes that are assumed to be secure for classical computers no longer are secure in the era of quantum computing. The idea of quantum computing began in the early 1980's with a quantum mechanical model of a Turing machine by Paul Benioff [24]. It soon became clear that quantum computing has a great advantage over classical computing in certain applications. In 1994, Peter Shor proposed algorithms to solve the problems of integer factorization and discrete logarithms in polynomial time [89]. As a result, cryptographic schemes that rely on the hardness of either of those assumptions, such as RSA, can be efficiently broken by quantum computers. Another popular algorithm that we will refer to later on, Grover's search algorithm (1996), leads to a quadratic speedup of search on unstructured data [43]. Recent years have shown increasing interest and investments into quantum computing both by startups and large corporations like Google and IBM. For example, only a few months ago, Europe's most powerful 27-Qubit IBM Quantum System One was inaugurated in Ehningen, Germany [71]. The current standard of technology, however, is still at a relatively immature stage and therefore, quantum computers are mainly used for academic purposes. The realization of fully capable quantum computers is not expected to happen for at least another five to ten years. Nevertheless, particularly in the area of cryptography, we need new secure encryption schemes that can replace previous schemes and are based on different and stronger hardness assumptions. One solution that has gained great popularity over the last two decades is lattice-based cryptography.

The concept of lattices has been applied in cryptography since the end of the last century. Lattice reduction algorithms, such as the LLL algorithm, resulted in a number of new practical attacks on popular cryptosystems [73]. Then, in 1996, the landmark work of Ajtai [2] laid a foundation for a new family of cryptosystems. He showed that any instance of a certain lattice problem, namely, the approximate Shortest Vector Problem (SVP_γ), can be reduced to a randomized instance of the Short Integer Solution (SIS) problem. This implies that an average-case SIS instance is at least as hard as a worst-case instance of SVP_γ, resulting in very strong security guarantees. Later, Regev came up with a similar worst-case to average-case reduction for the Learning with Errors (LWE) problem [82]. SIS and LWE can be applied to create advanced cryptographic primitives, even some that seemed infeasible before. The SIS problem can be used to build one-way functions [2, 68], collision resistant hash functions [41], identity-based encryption schemes [60] and digital signature schemes [39, 61]. LWE is even more powerful than SIS and leads to a number of public-key encryption schemes [64, 77, 82], identity-based encryption schemes [1, 39], somewhat and fully homomorphic encryption schemes [30, 38, 40] and more. In addition to good security guarantees and an abundance of applications, the required key sizes are relatively small (quasilinear) and we obtain good practical runtimes.

To use LWE and SIS in practice, we first need to find a set of parameters to parameterize the problem instance used in a scheme such that the chosen parameters satisfy a certain security requirement. We can measure the security of an instance by means of a bit security level `sec`. A security level of `sec` = 128 means that an attacker has to perform at least 2^{128} operations to solve the respective problem instance. In general, it is difficult to find hard lower bounds on the number of operations required to break a scheme. Even if they exist, they may not be realistic in practice and lead to larger key sizes and therefore less efficient schemes than really needed. We therefore need a way to estimate the concrete hardness of LWE and SIS. This can be achieved by examining the runtime complexity of the currently best attack algorithms. Albrecht et al. [13] and follow-up works such as [4, 26] performed an in-depth study of the practical hardness of LWE and encapsulated their findings in the LWE Estimator[1], which we will subsequently call *estimator*. However, the *estimator* does not include estimates for SIS, and specifying distributions is not very user-friendly as of now. In addition, it only returns costs for a fixed parameter set. A user must specify a search function by himself.

In this work, we aim to fill this gap with a new Python library, which we call Lattice Parameter Estimation[2] and will subsequently refer to simply as *tool*. Our tool can be used to generically search for secure parameter sets given a set of problem instances. We provide classes for LWE and SIS, their ring and module variants, and statistically secure variants where applicable. We also included more convenient distribution classes and classes for various ℓ_p-norms and the canonical embedding, since several attacks use bounds in some ℓ_p norm and cryptographic schemes oftentimes require some guarantees on the bounds of operands. We use the *estimator* for cost estimates for LWE instances and add two new estimation algorithms for SIS instances.

1.1 Organization

In Chapter 2, we introduce lattices, lattice problems and their variants and a number of mathematical tools that will be needed later on. In Chapter 3, we describe the main approaches and algorithms that we use in our tool to solve LWE and SIS. We remark on the complexity of the algorithms, whenever feasible in the scope of our work, and present several lattice reduction cost models from the literature. Finally, in Chapter 4, we explain our tool in more detail and give an overview of the various functionalities and configuration options. The discussion in previous chapters, in particular, the presentation of reduction cost models in Section 3.1.3, aims to help users of our tool to be aware of the impact of configuration choices and to understand estimate results of our parameter search.

[1] `https://bitbucket.org/malb/lwe-estimator/`

[2] `https://github.com/krebsni/a-tool-for-the-estimation-of-lattice-parameters`

2 Preliminaries

2.1 Notation

In the following, we denote vectors by bold lower-case letters $\mathbf{v}$ and matrices by bold upper-case letters $\mathbf{M}$. We interchangeably use matrix notation and sets of column vectors $[\mathbf{v}_1 \cdots \mathbf{v}_n] = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Unless specified otherwise, by $\| \cdot \|$ or simply *norm* we refer to the Euclidean norm. By $[n]$ we denote the set $\{1, \ldots, n\}$ for $n \in \mathbb{N} \setminus \{0\}$. We denote the logarithm base 2 by log and the natural logarithm with base e by ln. In order to avoid confusion, throughout this work, we let σ denote the standard deviation, and set $s = \sigma\sqrt{2\pi}$ and $\alpha = \frac{s}{q} = \frac{\sqrt{2\pi}\sigma}{q}$ for other commonly used Gaussian parameters.

2.2 Norms

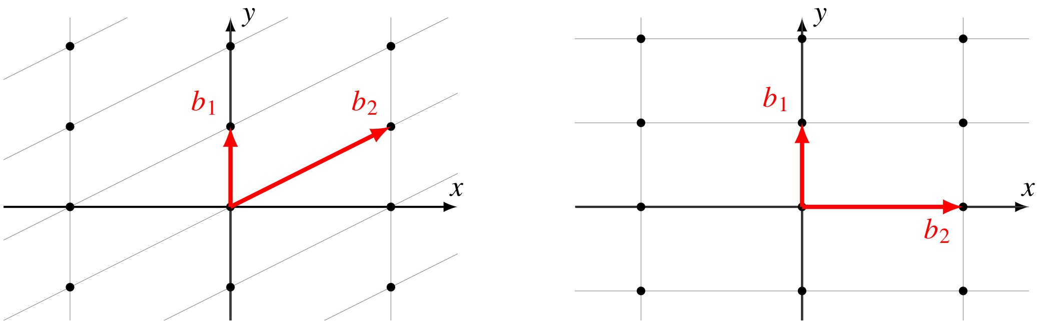
We can define the standard ℓ_p-norms on the vector space $\mathbb{R}^n$. In this work, we will also consider rings and modules (see Section 2.5.3) and require a way to bound the length of ring and module elements. Let $\mathcal{R}_q = \mathbb{Z}_q[X]/\langle X^n + 1 \rangle$ be a quotient ring as defined in [21] and $f \in \mathcal{R}_q$ with $f = \sum_i f_i X^i$. Following [21], we define the norms

$$\begin{aligned}
\ell_1 &: \|f\|_1 = \sum_i |f_i|, \\
\ell_2 &: \|f\|_2 = \sqrt{\sum_i |f_i|^2}, \\
\ell_p &: \|f\|_p = \left(\sum_i |f_i|^p \right)^{\frac{1}{p}} \quad \text{and} \\
\ell_\infty &: \|f\|_\infty = \max_i |f_i|.
\end{aligned} \tag{2.1}$$

For standard vector spaces, the definitions are essentially equivalent to the above, except that f is a vector in $\mathbb{R}^n$ or, in our case, $\mathbb{Z}^n$ with coefficients f_i. For a module element $\mathbf{f} \in \mathcal{R}_q^d$, we can simply view $\mathbf{f}$ as a $n \cdot d$-dimensional vector.

2.3 Lattices

We now present the mathematical definition of lattices and related tools that we will use later on. Most of the background theory is based on material from [65] with some adaptions.



(a) Lattice with basis $\mathbf{b}_1 = (0, 1)^\top,\ \mathbf{b}_2 = (2, 1)^\top$ **(b)** Lattice with basis $\mathbf{b}_1 = (0, 1)^\top,\ \mathbf{b}_2 = (2, 0)^\top$

Figure 2.1: Lattice Examples

A *lattice* Λ is a discrete additive subgroup of the vector space $\mathbb{R}^m$. Every lattice can be defined by a basis $\mathbf{B}$ of n linearly independent basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^m$ with $m \geq n$. A lattice is then the set of all integer linear combinations of the basis vectors in $\mathbf{B}$. Figure 2.1a shows a lattice with basis vectors $\mathbf{b}_1 = (0, 1)^\top$ and $\mathbf{b}_2 = (2, 1)^\top$.

Definition 2.3.1 (Lattice)
Given a basis $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n] \in \mathbb{R}^{m \times n}$*, we call* Λ *a lattice generated by the column vectors of* $\mathbf{B}$ *if*

$$\Lambda(\mathbf{B}) = \left\{ \mathbf{x} \in \mathbb{R}^m \;\middle|\; \exists c_1, \ldots, c_n \in \mathbb{Z} : \mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{b}_i \right\}. \tag{2.2}$$

We call n the *rank* of the lattice. A lattice has full rank if the $n = m$. The basis of a lattice $\mathbf{B}$ is not unique. The two example lattices in Figure 2.1 are identical, but have different bases. If $\mathbf{B}$ is a basis of a lattice Λ, then for any unimodular matrix $\mathbf{U} \in \mathbb{Z}^{n \times n}$ with determinant ±1, the basis $\mathbf{B} \cdot \mathbf{U}$ is also a basis of Λ. In cryptographic applications, we usually begin with a lattice basis $\mathbf{B}$ with long basis vectors that are not very orthogonal to each other. For example, the problem of finding a short vector (SVP or SVP$_\gamma$, see Definition 2.3.2) in $\Lambda(\mathbf{B})$ turns out to be very difficult. If we can somehow find a "nicer" basis for $\Lambda(\mathbf{B})$ with shorter and more orthogonal basis vectors, finding a solution becomes a lot easier. We can improve the quality of a basis or "reduce" a basis in such a way by means of lattice basis reduction. The basis in Figure 2.1b is ideal, i.e., it cannot be further reduced, whereas the basis in Figure 2.1a can be reduced. In this particular case, the reduction is quite trivial. We can simply compute $\mathbf{b}_2' = \mathbf{b}_2 - \mathbf{b}_1$ and take $\{\mathbf{b}_1, \mathbf{b}_2'\}$ as our new basis. But more to that later.

Similar to a quotient ring $\mathbb{Z}/q\mathbb{Z}$ of integers modulo some positive integer q with cosets $c + q\mathbb{Z}$, we can define the quotient group $\mathbb{R}^m/\Lambda$ with *cosets*

$$\mathbf{c} + \Lambda = \{\mathbf{c} + \mathbf{v} \mid \mathbf{v} \in \Lambda\} \tag{2.3}$$

where $\mathbf{c} \in \mathbb{R}^m$ [75].

An important concept is the *fundamental domain* of a lattice. The fundamental domain is a subset of $\mathbb{R}^m$ that contains exactly one representative of every coset of $\mathbb{R}^m/\Lambda$. The most commonly considered fundamental domain of a lattice with basis $\mathbf{B}$ is the (shifted) *fundamental parallelepiped*

$$\mathcal{P}_{\frac{1}{2}}(\mathbf{B}) = \mathbf{B} \cdot \left[-\frac{1}{2}, \frac{1}{2}\right)^n = \left\{ \mathbf{x} \in \mathbb{R}^m \;\middle|\; \mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{b}_i,\ c_i \in \left[-\frac{1}{2}, \frac{1}{2}\right) \right\} \tag{2.4}$$

Another region that is often used is the *Voronoi region* $\mathcal{V}$ [44] and is defined as

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{y} \in \Lambda : \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|\} \,. \tag{2.5}$$

The n-dimensional volume of the fundamental parallelepiped of a lattice $\Lambda(\mathbf{B})$ is equivalent to the *determinant* of the lattice $\det(\Lambda(\mathbf{B})) = \sqrt{\det(\mathbf{B}^\intercal\mathbf{B})}$. For a full-ranked lattice, the determinant becomes $\det(\Lambda(\mathbf{B})) = |\det(\mathbf{B})|$. The determinant is independent from the used basis, which can be easily verified by looking at $\det(\Lambda(\mathbf{UB}))$.

The *minimum distance* $\lambda_1(\Lambda)$ of a lattice is the length of its shortest nonzero vector

$$\lambda_1(\Lambda) = \min_{v \in \Lambda \setminus \{0\}} \|\mathbf{v}\|. \tag{2.6}$$

Furthermore, we can define *ith successive minima* $\lambda_i(\Lambda)$ by considering an m-dimensional ball $\mathcal{B}(\mathbf{0}, r)$ with increasing radius $r \in \mathbb{R}$ and center $\mathbf{0} \in \mathbb{R}^m$ at the origin of Λ. Then, $\lambda_i(\Lambda)$ is the smallest radius r such that the ball $\mathcal{B}(\mathbf{0}, r)$ contains exactly i linearly independent lattice vectors. Note that an optimally reduced basis with basis vectors $\mathbf{b}_1 \leq \mathbf{b}_2 \cdots \leq \mathbf{b}_n$ satisfies $\lambda_i(\Lambda) = \|\mathbf{b}_i\|$ for all $i \in [n]$.

In general, it is hard to determine the exact values of $\lambda_i(\Lambda(\mathbf{B}))$ for a given basis $\mathbf{B}$. *Minkowski's theorem* states that $\lambda_1(\Lambda) \leq \sqrt{n} \cdot (\det(\Lambda))^{\frac{1}{n}}$ given that Λ has rank n. The *Gaussian heuristic* is commonly used to estimate the minimum distance λ_1 of a lattice Λ given the determinant $\det \Lambda$:

$$\lambda_1(\Lambda) \approx \frac{\Gamma(1 + n/2)^{1/n}}{\sqrt{\pi}} \det(\Lambda)^{1/n} \tag{2.7}$$

By applying Stirling's formula to estimate the value of the Γ-function as described in [42], we can simply the estimate to

$$\lambda_1(\Lambda) \approx \sqrt{\frac{n}{2\pi e}} \det(\Lambda)^{1/n} \tag{2.8}$$

The dual $\Lambda^\perp$ of a lattice $\Lambda(\mathbf{B})$ is defined as the set of vectors $\mathbf{y}$ in the span of $\mathbf{B}$, such that the inner product $\langle \mathbf{y}, \mathbf{v} \rangle$ is an integer for all $\mathbf{v} \in \Lambda(\mathbf{B})$. The basis of the dual of a lattice with basis $\mathbf{B}$ is given by $\mathbf{B}' = \mathbf{B}(\mathbf{B}^\intercal\mathbf{B})^{-1}$.

In cryptography, we are mainly interested in modular integer (or q-ary) lattices. A q-ary lattice is a lattice Λ_q such that $q\mathbb{Z}^m \subseteq \Lambda \subseteq \mathbb{Z}^m$ given $q \in \mathbb{N} \setminus \{0\}$. This means that a vector $\mathbf{x} \in \mathbb{Z}^m$ is in Λ if and only if $\mathbf{x} \bmod q$ also is in Λ.

We now look at two important ways of specifying a q-ary lattice given a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ [26].

$$\Lambda_q(\mathbf{A}^\intercal) = \{\mathbf{v} \in \mathbb{Z}^m \mid \exists \mathbf{y} \in \mathbb{Z}^n : \mathbf{v} = \mathbf{A}^\intercal\mathbf{y} \mod q\} \tag{2.9}$$

$\Lambda_q(\mathbf{A}^\intercal)$ is commonly referred to as the *(primal) LWE lattice*, since finding a short vector in $\Lambda_q(\mathbf{A}^\intercal)$ corresponds to solving LWE. The second, referred to as the *(dual) SIS lattice*, is given by

$$\Lambda_q^\perp(\mathbf{A}) = \{\mathbf{v} \in \mathbb{Z}^m \mid \mathbf{A}\mathbf{v} = \mathbf{0} \mod q\} \,. \tag{2.10}$$

Finding a short vector in $\Lambda_q^\perp(\mathbf{A})$ corresponds to solving the Short Integer Solution problem.

In many scenarios, it is convenient to have an explicit formula that describes the relationship between the determinant of the above two lattices and the dimensions of $\mathbf{A}$. For q prime and m sufficiently larger than n, we have that the rank of $\mathbf{A}$ is n, since the rows of $\mathbf{A}$ are linearly independent with high probability. As a result, the lattice $\Lambda_q(\mathbf{A}^\top)$ has q^n points in $\mathbb{Z}_q^m$. Consider the fundamental domain $D = \mathcal{P}_{1/2}(\Lambda_q(\mathbf{A}^\top))$ and the fact that $(\Lambda_q(\mathbf{A}^\top) \bmod q) + (D \bmod q) = \mathbb{R}^m/q\mathbb{R}^m$ is a partition [48]. The volume of $\mathbb{R}^m/q\mathbb{R}^m$ is given by $q^m = |(\Lambda_q(\mathbf{A}^\top) \bmod q)||D \bmod q|$ and thus

$$\det(\Lambda_q(\mathbf{A}^\top)) = |D \bmod q| = \frac{q^m}{|(\Lambda_q(\mathbf{A}^\top) \bmod q)|} = \frac{q^m}{q^n} = q^{m-n}. \tag{2.11}$$

Furthermore, we know that the dual lattice $\Lambda_q^\perp(\mathbf{A})$ has q^{m-n} points in $\mathbb{Z}_q^m$ as the dimension of the kernel of $\mathbf{A}$ is $m-n$. Analogously, we obtain

$$\det(\Lambda_q(\mathbf{A}^\top)^\perp) = |D \bmod q| = \frac{q^m}{|(\Lambda_q(\mathbf{A}^\top) \bmod q)|} = \frac{q^m}{q^{m-n}} = q^n. \tag{2.12}$$

Another useful tool it the *Gram-Schmidt orthogonalization* . Given a basis $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_n] \in \mathbb{Z}_q^{m\times n}$, we write $\pi_{\mathrm{span}(\mathbf{B})}(\mathbf{t})$ for the projection of a vector $\mathbf{t}$ onto the span of the vectors in $\mathbf{B}$. Define $\tilde{\mathbf{b}}_i$ as follows: $\tilde{\mathbf{b}}_1 = \mathbf{b}_1$. For $i \in \{2, \ldots, n\}$, let $\tilde{\mathbf{b}}_i$ be the component of $\mathbf{b}_i$ that is orthogonal to the span of $\{\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}\}$. In other words,

$$\tilde{\mathbf{b}}_i = \mathbf{b}_i - \pi_{\mathrm{span}(\mathbf{b}_1,\ldots,\mathbf{b}_{i-1})}(\mathbf{b}_i). \tag{2.13}$$

Then, $\tilde{\mathbf{B}} = \left[\tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n\right]$ is called the Gram-Schmidt orthogonalization of the basis $\mathbf{B}$. We define the Gram-Schmidt coefficients as follows:

$$\mu_{i,j} = \frac{\langle \tilde{\mathbf{b}}_j, \mathbf{b}_i \rangle}{\langle \tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j \rangle} \tag{2.14}$$

We define $\mathrm{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$, where $\Lambda(\mathbf{B}) \subset \mathbb{R}^m$, as the *distance* of a vector $\mathbf{t} \in \mathbb{R}^m$ to the closest lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$, i.e., $\mathrm{dist}(\mathbf{t}, \Lambda(\mathbf{B})) = \min_{\mathbf{v}\in\Lambda(\mathbf{B})} \|\mathbf{t} - \mathbf{v}\|$.

2.3.1 Lattice Problems

In the previous section, we already mentioned the Shortest Vector Problem or short SVP. In this section, we want to briefly list several lattice problems that are used to show the hardness of LWE and SIS.

Definition 2.3.2 (SVP$_\gamma$)
Given a basis $\mathbf{B}$ of a lattice Λ, the (approximate) Shortest Vector Problem (SVP$_\gamma$) is the problem of finding a short lattice vector $v \in \Lambda$ such that $0 < \|v\| \leq \gamma\lambda_1(\Lambda)$.

The corresponding decision version is the GapSVP$_\gamma$ problem, in which we are asked to decide whether $\lambda_1(\Lambda) \leq 1$ or $\lambda_1(\Lambda) \geq \gamma$ given a basis $\mathbf{B}$ of Λ. If neither is the case, any answer is accepted. In an alternative version of GapSVP$_\gamma$, we have to decide between $\lambda_1(\Lambda) \leq d$ and $\lambda_1(\Lambda) \geq \gamma d$ for some positive real number d [62].

Definition 2.3.3 (SIVP$_\gamma$)
Given a basis $\mathbf{B}$ of a lattice Λ of rank n, the (approximate) Shortest Independent Vector Problem (SIVP) is the problem of finding n linearly independent lattice vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \Lambda$ such that $\|\mathbf{v}_i\| \leq \gamma \cdot \lambda_n(\Lambda)$ for all $i \in \{1, \ldots, n\}$.

Both GapSVP$_\gamma$ and SIVP$_\gamma$ are NP-hard for any constant approximation factor γ [27, 52].

2.4 Discrete Gaussian Distribution

Oftentimes in lattice cryptography, we work with samples or vectors drawn from a discrete uniform or Gaussian distribution. We can define a discrete Gaussian distribution as follows:

Definition 2.4.1 (Discrete Gaussian Distribution [44])
The discrete Gaussian distribution $D_{\Lambda,s,\mathbf{c}}$ over an m-dimensional lattice Λ with width parameter $s > 0$ and center $\mathbf{c}$ is the probability distribution we obtain by assigning each vector $\mathbf{x} \in \Lambda$ a probability proportional to $e^{-\pi\|\mathbf{x}-\mathbf{c}\|^2/s^2}$. If $\mathbf{c} = \mathbf{0}$ we simply write $D_{\Lambda,s}$.

A discrete Gaussian sampler over a lattice can be efficiently realized (see [39]). Recall that σ denotes the standard deviation and we use width parameter $s = \sqrt{2\pi}\sigma$. Furthermore, $\alpha = \frac{s}{q} = \frac{\sqrt{2\pi}\sigma}{q}$.

2.5 LWE and SIS

In Chapter 1, we informally introduced the Learning with Errors (LWE) problem and the Short Integer Solution (SIS) problem that constitute the main focus of this work. Both problems have given rise to a plethora of cryptosystems with strong underyling hardness assumptions, particularly in the context of the imminent advent of quantum computing. Hence, their importance in modern cryptography cannot be understated. We will now continue to describe LWE and SIS in more detail.

2.5.1 Learning with Errors (LWE)

The *Learning with Errors* (LWE) problem asks us to recover some secret vector $\mathbf{s} \in \mathbb{Z}_q^n$ from a sequence of perturbed random linear equations on $\mathbf{s}$. The perturbed linear equations, also called samples, are of the form $\langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \bmod q$, where $\mathbf{a}_i$ are randomly chosen over $\mathbb{Z}_q^n$ and e_i are error terms. We obtain a total of m samples and can thus also express the equation system that is to be solved as $\mathbf{z} = \mathbf{A}^\intercal \mathbf{s} + \mathbf{e} \bmod q$, where $\mathbf{A} \in \mathbb{Z}^{n \times m}$ and $\mathbf{e} \in \mathbb{Z}^m$. Note that we only know $\mathbf{z}$ and $\mathbf{A}$. A formal definition follows.

Definition 2.5.1 (LWE Distribution [84])
Given an integer $n \geq 1$, a modulus $q \geq 2$, an error distribution χ on $\mathbb{Z}_q$, and a fixed secret vector $\mathbf{s}$, let $\mathcal{A}_{\mathbf{s},\chi}$ be the probability distribution over $\mathbb{Z}_q^n \times \mathbb{Z}_q$ by choosing a vector $\mathbf{a}_i \in \mathbb{Z}_q^n$ uniformly at random and $e_i \in \mathbb{Z}_q$ according to χ. $\mathcal{A}_{\mathbf{s},\chi}$ outputs pairs of

$$(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \ mod\ q) \in \mathbb{Z}_q^n \times \mathbb{Z}_q. \tag{2.15}$$

Additions are performed in $\mathbb{Z}_q$. In the case of $q = 2$, the LWE problem corresponds to the *Learning Parity with Noise* (LPN) problem. We distinguish between two versions of LWE.

Definition 2.5.2 (Search-LWE$_{n,q,m,\chi}$)
The Search-LWE$_{n,q,m,\chi}$ asks for the recovery of the secret vector $\mathbf{s}$*, given m independent samples* $(\mathbf{a}_i, z_i) \leftarrow \mathcal{A}_{\mathbf{s},\chi}$

Definition 2.5.3 (Decision-LWE$_{n,q,m,\chi}$)
Given m samples, the Decision-LWE$_{n,q,m,\chi}$ asks to distinguish whether the samples were drawn from $\mathcal{A}_{\mathbf{s},\chi}$ or from a uniform distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$.

Intuitively, Search-LWE is at least as hard as Decision-LWE as a solution to Search-LWE trivially solves Decision-LWE. The other direction, however, also holds true [83] for a prime modulus $q = \text{poly}(n)$. The equivalence of the search and decision versions is convenient when we consider common attacks against LWE, some of which solve Decision-LWE and others solve Search-LWE (see Chapter 3).

LWE as a Decoding Problem

We request m samples $(\mathbf{a}_1, z_1), \ldots, (\mathbf{a}_m, z_m)$ where $z_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \in \mathbb{Z}_q$ and $\mathbf{s} \in \mathbb{Z}_q^n$. Let $A = [\mathbf{a}_1 \cdots \mathbf{a}_m] \in \mathbb{Z}_q^{n \times m}$, $\mathbf{z} = [z_1, \ldots, z_m]^\intercal$ and $e = [e_1, \ldots, e_n]^\intercal \in \mathbb{Z}_q^n$. Hence, we can formulate LWE as a decoding problem as in [44]:

$$\mathbf{z} = \mathbf{A}^\intercal \mathbf{s} + \mathbf{e} \tag{2.16}$$

with generator matrix $\mathbf{A}$ for a linear code over $\mathbb{Z}_q$ and $\mathbf{z}$ as the received word (see Appendix A.1 for more details about linear codes). Finding the secret vector $\mathbf{z}$ is equivalent to finding the codeword $\mathbf{y} = \mathbf{A}^\intercal \mathbf{s}$ with minimum distance $\|\mathbf{y} - \mathbf{z}\|$.

We can transform an LWE$_{n,q,m,\chi}$ instance with a secret vector $\mathbf{s}$ chosen according to a uniform distribution into an LWE$_{n,q,m-n,\chi}$ instance with a secret vector $\hat{\mathbf{s}}$ chosen according to the error distribution χ at a loss of n samples as follows [44]: Let $\mathbf{A}_0 = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the first n columns of $\mathbf{A}$. We introduce new variables $\hat{\mathbf{s}} = \mathbf{A}_0^\intercal \mathbf{s} - [z_1, \ldots, z_n]^\intercal = [e_0, \ldots, e_n]^\intercal$ and $\hat{\mathbf{A}} = \mathbf{A}_0^{-1} \mathbf{A} = [\mathbf{I}\ \hat{\mathbf{a}}_{n+1} \cdots \hat{\mathbf{a}}_m]$ and compute $\hat{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{A}}^\intercal\ [z_1, \ldots, z_n]^\intercal = [\mathbf{0}, \hat{z}_{n+1} \cdots \hat{z}_m]^\intercal$. Our new LWE instance has samples $(\hat{\mathbf{a}}_{n+1}, \hat{z}_{n+1}), \ldots, (\hat{\mathbf{a}}_m, \hat{z}_m)$.

LWE as a BDD Problem

Solving LWE also corresponds to solving the *Bounded Distance Decoding problem* (BDD) in the lattice $\Lambda(\mathbf{A}^\intercal) = \{\mathbf{x} \in \mathbb{Z}_q^m \mid \exists \mathbf{s} \in \mathbb{Z}_q^n : \mathbf{x} = \mathbf{A}^\intercal \mathbf{s} \mod q\}$, where the m columns of $\mathbf{A}$ correspond to the vectors $\mathbf{a}_i \in \mathbb{Z}_q^n$ of m independent LWE samples $(\mathbf{a}_i, z_i) \leftarrow \mathcal{A}_{\mathbf{s},\chi}$ and the components z_i correspond to a perturbed lattice point in $\Lambda(\mathbf{A}^\intercal)$.

Hardness

The most important hardness results for LWE come from [82] and [76]. Regev [82] showed that there exists a polynomial-time quantum reduction from worst-case GapSVP_γ and SIVP_γ in the ℓ_2-norm to the (average-case) search version of LWE. That means that LWE is quantumly at least as hard as GapSVP_γ and SIVP_γ. A similar classical probabilistic polynomial-time reduction from worst-case GapSVP_γ was later proposed by Peikert [76] for a sufficiently large modulus $q \geq 2^n$ in any ℓ_p-norm for $p \geq 2$. The Shortest Vector Problem and its variants are well studied problems that are NP-hard for certain approximation problems (see Section 2.3.1). To this point, it is assumed that no efficient quantum algorithms exist that can solve NP-hard problems. It is thus conjectured that cryptosystems based on the worst-case to average-case reductions from Regev and Peikert are secure, even in the context of quantum computing. The currently best algorithms solve LWE in $2^{O(n)}$ time.

2.5.2 Short Integer Solution (SIS)

The dual problem to LWE is the *Short Integer Solution problem* (SIS). In the SIS problem, we again have a set of uniformly random vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}_q^n$ and want to find an integer linear combination of the vectors such that $c_1\mathbf{a}_1 + \cdots + c_m\mathbf{a}_m = \mathbf{0} \mod q$ with small coefficients c_i. It is not difficult to find a linear combination that sums to the zero vector. The hardness of the problem comes from the restriction to small coefficients. A formal definition follows.

Definition 2.5.4 (SIS Problem, adapted from [54])
The problem $SIS_{n,q,m,\beta}$ is defined as follows: Given a uniformly random matrix $\mathbf{A}^{n\times m}$, find a vector $\mathbf{s} \in \mathbb{Z}^m$ such that $\mathbf{A} \cdot \mathbf{s} = \mathbf{0} \mod q$ and $0 < \|\mathbf{s}\| \leq \beta$.

Finding such a vector corresponds to solving SVP_γ in the scaled q-ary dual lattice $\Lambda_q^\perp(\mathbf{A}) = \{\mathbf{v} \in \mathbb{Z}^m \mid \mathbf{A}\mathbf{v} = \mathbf{0} \mod q\}$. Note that the problem becomes trivial for sufficiently large bounds β. If $\beta \geq q$ in any ℓ_p-norm, we can simply choose $\mathbf{s} = [q, 0, \dots, 0]^\top$. In some scenarios, the vector $\mathbf{s}$ is restricted to $\mathbb{Z}_q^m$. However, depending on the used norm for the length of $\mathbf{s}$, we can still efficiently find $\mathbf{s}$ with $\|\mathbf{s}\|_p$ by using some standard linear equation solver $\|\mathbf{s}\|_p$, if $\{\mathbf{s} \in \mathbb{Z}_q^m \mid \|\mathbf{s}'\|_p \leq \beta\} = \mathbb{Z}_q^m$. For example, in the ℓ_∞-norm we have that $\beta = q$ is large enough. In Section 4.2, we will define norm bound inequalities to be able to express relationships between different ℓ_p-norms.

As a simple example, SIS can be used to create collision-resistant hash functions. The matrix $\mathbf{A}$ defines a linear transformation that maps elements in $\mathbb{Z}^m$ to $\mathbb{Z}^n$, where $n < m$. The function $h_\mathbf{A}(\mathbf{v}) = \mathbf{A}\mathbf{v} \mod q$ is thus a hash function. We now restrict the domain of $h_\mathbf{A}$ by a bound β such that $\|\mathbf{v}\|_\infty \leq \beta$. Consider a collision $\mathbf{v}' \neq \mathbf{v}$ such that $h_\mathbf{A}(\mathbf{v}') = h_\mathbf{A}(\mathbf{v}) \mod q$ or equivalently, $h_\mathbf{A}(\mathbf{v}' - \mathbf{v}) = \mathbf{0}$. Obviously, it holds that $\|\mathbf{v}' - \mathbf{v}\| \leq 2\beta$. Thus, finding a collision also solves SIS and, in particular, is as hard as solving SIS.

We obtain similar hardness results as for LWE. Micciancio and Regev [68] show a reduction from SIVP_γ in the worst-case to average-case SIS problem.

2.5.3 Ring and Module Variants

In practice, the base variants of LWE and SIS are not used due to large key sizes required for secure cryptographic schemes. Consider the matrix $\mathbf{A} \in \mathbb{Z}_q^{m\times n}$. We usually require $m \in \Omega(n)$, which gives us at least a quadratic space complexity. We therefore want to decrease our key sizes. A way to do this is by using rings and modules as the underlying structure of LWE and SIS. Explaining the details regarding the ring and module variants of both problems would go beyond the scope of this work. Hence, we will only provide a rough intuition following [84] and define the various problem variants of LWE and SIS. All of them are included in our tool.

Consider the following choice for $\mathbf{A}$. We set $n = 2^k$ for some $k > 0$ and choose the column vectors $\mathbf{a}$ of $\mathbf{A}$ in groups of size n. In each group, we sample $\mathbf{a}_1 = [a_1, \ldots, a_n]^\intercal$ from the uniform distribution over $\mathbb{Z}_q^n$. The remaining columns are rotations $\mathbf{a}_i = [a_i, \ldots, a_n, -a_1, \ldots, -a_{i-1}]^\intercal$ of the first vector in the group. Hence, a block of $n \times n$ only needs $O(n)$ memory, reducing the size of $\mathbf{A}$ by a factor of n. In addition, it is possible to achieve speedups in the operations by means of the Fast Fourier Transformation.

In the ring variant, instead of vectors of the group $\mathbb{Z}_q^n$, the columns of $\mathbf{A}$ are chosen as elements of the ring $\mathbb{Z}_q\,[x]\,/\langle x^n + 1\rangle$, which we call $\mathcal{R}_q$. We ensure that the $x^n + 1$ is irreducible over the rationals by letting n be a power of two. The disadvantage of powers of two is that our key size is rather coarse-grained, which may result in a larger key size for a secure instance than needed up to a factor of two. Modules present themselves as a solution to this problem. Let $K = \mathbb{Q}(\theta)$ be a number field, where θ is an algebraic number with degree n, as defined in [54]. A $\mathcal{R}$-module $\mathcal{M} \subseteq K^d$ with dimension or rank d is a generalization of rings and vector spaces and is closed under addition and multiplication by elements of $\mathcal{R}$. We only consider the module $\mathcal{R}^d$ with ring dimension n and module rank d. For more details on the mathematical background, please see [54].

Definition 2.5.5 (RSIS [54])
The Ring-SIS problem RSIS$_{n,q,m,\beta}$ is defined as follows: Given $a_1, \ldots, a_m \in \mathcal{R}_q$ chosen independently from the uniform distribution, find $s_1, \ldots, s_m \in \mathcal{R}$ such that $\sum_{i=1}^m \mathbf{a}_i \cdot s_i = 0 \mod q$ and $0 < \|\mathbf{s}\| \leq \beta$, where $\mathbf{s} = [s_1, \ldots, s_m]^\intercal \in \mathcal{R}^m$.

We can interpret a ring element $r \in \mathcal{R}$ as an n dimensional vector with coefficients r_i such that $r = \sum_{i=0}^{n-1} r_i x^i$. If we compare RSIS to standard SIS, each a_i in RSIS corresponds to a $n \times n$ block in the standard SIS matrix. The n columns are obtained by rotation, similar to what we described above. We thus, for example, have $\mathbf{A} = [\mathrm{Rot}(a_1) \mid \cdots \mid \mathrm{Rot}(a_m)]$ where each block Rot is given by

$$\mathrm{Rot}(a) = \begin{pmatrix} a_0 & -a_{n-1} & \cdots & -a_1 \\ a_1 & a_0 & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}$$

An RSIS$_{n,q,m,\beta}$ instance can thus be seen as an SIS$_{n,q,m\cdot n,\beta}$ instance.

Definition 2.5.6 (MSIS [54])
The Module-SIS problem MSIS$_{n,d,q,m,\beta}$ is defined as follows: Given $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathcal{R}_q^d$ chosen independently from the uniform distribution, find $s_1, \ldots, s_n \in \mathcal{R}$ such that $\sum_{i=1}^m a_i \cdot s_i = 0 \mod q$ and $0 < \|\mathbf{s}\| \leq \beta$, where $\mathbf{s} = [s_1, \ldots, s_m]^\intercal \in \mathcal{R}^m$.

As for RSIS, we can view the vectors $\mathbf{a}_i$ in MSIS as blocks of $\mathbf{A}$ in SIS. Each vector $\mathbf{a}_i$ has d coefficients and corresponds to a $n \cdot d \times n$ block in $\mathbf{A}$, as follows:

$$n \cdot d \left\{ \overbrace{\begin{bmatrix} \mathrm{Rot}(\boldsymbol{a}_{1,1}) & \mathrm{Rot}(\boldsymbol{a}_{1,2}) & \cdots & \mathrm{Rot}(\boldsymbol{a}_{1,m}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Rot}(\boldsymbol{a}_{d,1}) & \mathrm{Rot}(\boldsymbol{a}_{d,2}) & \cdots & \mathrm{Rot}(\boldsymbol{a}_{d,m}) \end{bmatrix}}^{n \cdot m} \right.$$

An $\mathrm{MSIS}_{n,d,q,m,\beta}$ instance thus becomes an $\mathrm{SIS}_{n \cdot d,q,m \cdot n,\beta}$ instance.

For LWE, we will only list the definition of the ring and module variant. We are aware that we have not introduced the underlying math and refer the reader to [54]. Let us just note here that $\mathcal{R}^{\perp}$ refers to the dual of $\mathcal{R}$.

Definition 2.5.7 (Ring-LWE (RLWE) Distribution [54])
Let χ be the error distribution on $\mathbb{T}_{\mathcal{R}^{\perp}} = K_{\mathbb{R}}/\mathcal{R}^{\perp}$ and $s \in \mathcal{R}^{\perp}$ be the secret. Then, we define $\mathcal{A}^{(\mathcal{R})}_{q,s,\chi}$ as the RLWE distribution on $\mathcal{R}_q \times \mathbb{T}_{\mathcal{R}^{\perp}}$ obtained by choosing $a \in \mathbb{R}_q$ uniformly at random and an error term $e \in \mathbb{T}_{\mathcal{R}^{\perp}}$ according to χ, and returning samples $(a, (a \cdot s)/q + e)$.

The search version of $\mathrm{RLWE}_{n,q,m,\chi}$ asks us to find s given m samples from $\mathcal{A}^{(\mathcal{R})}_{q,s,\chi}$ with modulus $q \geq 2$ and n the degree of the polynomial of $\mathcal{R}$. The decision version asks us to distinguish between m samples from $\mathcal{A}^{(\mathcal{R})}_{q,s,\chi}$ and m independent samples from the uniform distribution over $\mathcal{R}_q \times \mathbb{T}_{\mathcal{R}^{\perp}}$.

Definition 2.5.8 (Module-LWE (MLWE) Distribution [54])
Let χ be the error distribution on $\mathbb{T}_{\mathcal{R}^{\perp}}$ and $\mathbf{s} \in (\mathcal{R}^{\perp})^d$ be the secret vector. Then, we define $\mathcal{A}^{(\mathcal{M})}_{q,\mathbf{s},\chi}$ as the MLWE distribution on $(\mathcal{R}_q)^d \times \mathbb{T}_{\mathcal{R}^{\perp}}$ obtained by choosing $\mathbf{a} \in (\mathbb{R}_q)^d$ uniformly at random and an error term $e \in \mathbb{T}_{\mathcal{R}^{\perp}}$ according to χ, and returning samples $(\mathbf{a}, \frac{1}{q}\langle \mathbf{a}, \mathbf{s} \rangle + e)$.

Analogously, the search version of $\mathrm{MLWE}_{n,d,q,m,\chi}$ asks us to find $\mathbf{s}$ given m samples from $\mathcal{A}^{(\mathcal{M})}_{q,\mathbf{s},\chi}$ with modulus $q \geq 2$, n the degree of the polynomial of $\mathcal{R}$ and modulus rank d. The decision version asks us to distinguish between m samples from $\mathcal{A}^{(\mathcal{M})}_{q,\mathbf{s},\chi}$ and m independent samples from the uniform distribution over $\mathcal{R}_q^d \times \mathbb{T}_{\mathcal{R}^{\perp}}$.

We can construct a matrix $\mathbf{A}$ of the a_i of RLWE and $\mathbf{a}_i$ of MLWE, as for standard LWE, to formulate the problems as a decoding problem, as in Section 2.5.1. We interpret $\mathrm{RLWE}_{n,q,m,\chi}$ and $\mathrm{MLWE}_{n,d,q,m,\chi}$ as instances of $\mathrm{LWE}_{n,q,m \cdot d,\chi'}$ and $\mathrm{RLWE}_{n \cdot d,q,m \cdot n,\chi'}$ respectively. For a summary of our results, see Section 4.3.

3 Algorithms and Estimates

The main functionality of our tool relies on runtime cost estimation procedures for several popular algorithms that can be used to solve SIS or LWE. The estimates for LWE are provided by the *estimator* [13]. In addition, we included cost estimates of two different approaches for SIS. Many of the algorithms we use rely on subroutines, especially lattice reduction algorithms, and there exist more realistic and more conservative estimates. Depending on the choice of attack algorithms and reduction cost models, the returned cost may differ significantly. Also, not all attack estimates perform well in practice. It is thus of interest to us to gain a basic understanding of how the respective algorithms work.

3.1 Lattice Basis Reduction

In Section 2.5, we showed how LWE and SIS can be viewed as lattice problems. However, the corresponding lattice basis that we obtain are rather “ugly” lattice bases with long basis vectors. Solving lattice problems in such a basis is infeasible. Our goal is thus to find a better basis with shorter and more orthogonal basis vectors. A family of algorithms that achieve just this is called lattice reduction algorithms.

First, we need to define a measure to evaluate the quality of a given basis. The standard measure in the literature is the (root) Hermite factor.

Definition 3.1.1 (Root Hermite Factor [57])
Given a basis $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ *with* $\|\mathbf{b}_1\| \leq \cdots, \leq \|\mathbf{b}_n\|$, *then an n-dimensional lattice* $\Lambda(\mathbf{B})$ *has root Hermite factor* δ *if*

$$\|\mathbf{b}_1\| \approx \delta^n \det(\Lambda)^{1/n}. \tag{3.1}$$

We will later use a result that follows from the Geometric Series Assumption (GSA) as in [42]. The GSA estimates the length of the Gram-Schmidt vectors $\tilde{\mathbf{b}}_i$ as follows [87]:

$$\|\tilde{\mathbf{b}}_i\| \approx \alpha^{i-1} \|\mathbf{b}_1\|, \tag{3.2}$$

for $0 < \alpha < 1$. If we combine Equation (3.1) with Equation (3.2), we obtain $\|\tilde{\mathbf{b}}_i\| \approx \alpha^{i-1}\delta^n \det(\Lambda)^{1/n}$. Furthermore, we know that $\prod_{i-1}^{n} \|\tilde{\mathbf{b}}_i\| = \det(\Lambda)$ and get

$$\begin{aligned}
\prod_{i-1}^{n} \|\tilde{\mathbf{b}}_i\| \approx \prod_{i-1}^{n} \alpha^{i-1}\delta^n \det(\Lambda)^{1/n} &\iff \det(\Lambda) \approx \delta^{2n} \det(\Lambda) \prod_{i-1}^{n} \alpha^{i-1} \\
&\iff \delta^{-2n} \approx \alpha^{\frac{n(n-1)}{2}} \\
&\iff \delta^{-2} \approx \alpha^{(n-1)/n}
\end{aligned}$$

Hence, $\alpha \approx \delta^{-2}$ and

$$\|\tilde{\mathbf{b}}_i\| \approx \delta^{-2(i-1)+n} \det(\Lambda)^{1/n} \tag{3.3}$$

We can see that for smaller i, the length of the Gram-Schmidt vectors $\tilde{\mathbf{b}}_i$ decreases, whereas for larger i, the length increases, resulting in a long and skinny fundamental parallelepiped $\mathcal{P}_{1/2}(\tilde{\mathbf{B}})$.

In the following, we will focus on two related methods for lattice reduction. First, we define some reduction criteria following [6]. A basis $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ is *size-reduced* if its Gram-Schmidt coefficients (see Section 2.3) satisfy $|\mu_{i,j}| \leq \frac{1}{2}$ for all $0 \leq j < i < n$. If the first basis vector $\mathbf{b}_1$ of $\mathbf{B}$ is the shortest lattice vector, i.e., $\|\mathbf{b}_1\| = \lambda_1(\Lambda(\mathbf{B}))$, we call $\mathbf{B}$ *SVP-reduced.* If a basis $\mathbf{B}$ is size-reduced and in addition, each block $\{\mathbf{b}_i, \ldots, \mathbf{b}_n\}$ for $i = 1, \ldots, n$ of basis vectors is SVP-reduced, then $\mathbf{B}$ is *HKZ-reduced.* We will see in the next section that size reduction is closely related to the LLL algorithm, and that a special case of the BKZ reduction outputs an HKZ-reduced basis.

3.1.1 The LLL Algorithm

The LLL algorithm was proposed by Lenstra, Lenstra and Lovász [55] and can be considered as a generalization of the two dimensional Lagrange reduction. The Lagrange reduction reduces a basis of two basis vectors such that the output basis satisfies $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$ and $|\langle \mathbf{b}_1, \mathbf{b}_2 \rangle| / \|\mathbf{b}_1\|^2 = |\mu_{2,1}| \leq \frac{1}{2}$. Intuitively, a multiple of the shorter vector $\mathbf{b}_1$ is subtracted from the longer vector $\mathbf{b}_2$, such that the resulting vector $\mathbf{b}_2'$ is as orthogonal to $\mathbf{b}_0$ as possible, i.e., $\mathbf{b}_1' = \mathbf{b}_1 - \lfloor \mu_{1,0} \rceil \mathbf{b}_0$. We set $\mathbf{b}_2 = \mathbf{b}_2'$ and repeat until nothing changes.

A θ-LLL reduced basis ensures two criteria [55]:

1. Size-reduced: $|\mu_{i,j}| \leq \frac{1}{2}$ for $1 \leq i \leq n$ and $j < i$
2. Lovász condition: $\theta\|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i}\tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$ for $1 \leq i < n$

Recall the definition of the Gram-Schmidt coefficients $\mu_{i,j} = \langle \tilde{\mathbf{b}}_j, \mathbf{b}_i \rangle / \langle \tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j \rangle$. The LLL algorithm shown in Algorithm 1 follows the notation in [80]. We start by computing the Gram-Schmidt orthogonalization of the input basis (Line 2) and continue with a reduction step in which we update every basis vector $\mathbf{b}_i$ by pairwise comparing and subtracting lower indexed basis vector, just as in the Lagrange reduction (Line 5) to ensure Criteria 1. Finally, vectors violating the Lovász condition are swapped (Line 7) and the process is repeated until nothing changes. The LLL algorithm can be used to find short vectors of at most $2^{n/2}\lambda_1(\Lambda)$ in polynomial time. Several floating-point variants have been suggested that can significantly speed up the runtime of LLL. For example, L^2 runs in $O(n^2 \log^2 B)$, where B is a bound on the norm of the input basis vectors [72].

The proven upper bound of the output quality of the LLL algorithm is $\delta^n = (4/3)^{\frac{n-1}{4}}$ or, equivalently, $\delta \approx 1.075$ [55]. In practice, we get much better results of $\delta \approx 1.021$ on average [31]. Given an input basis in which the length of all basis vectors is bounded by B, the runtime of LLL is proven to be in $O\left(n^4 m \log B(n + \log B)\right)$ [72] and heuristically known to be $O(n^3 \log^2 B)$ [13].

Algorithm 1: The LLL Algorithm [55]

```
function $\theta$-LLL($\mathbf{B} \in \mathbb{Z}^{m \times n}$)
    Compute $\tilde{\mathbf{B}}$
    for $i = 2, \dots, n$ do
        for $j = i - 1, \dots, 1$ do
            $\mathbf{b}_i = \mathbf{b}_i - \lfloor \mu_{i,j} \rceil \mathbf{b}_j$
    if $\exists i$ such that $\theta \|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i}\tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$ then
        Swap $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$
        Return $\theta$-LLL($\mathbf{B}$)
    else
        Return $\mathbf{B}$
```

3.1.2 The BKZ Algorithm

The Block Korkin-Zolotarev (BKZ) algorithm was proposed by Schnorr in 1987 and adapted by Schnorr and Euchner in [88] and represents a family of lattice reduction algorithm. Essentially, BKZ iteratively divides the input basis into blocks of a lower dimension k and calls an SVP oracle on each block. The output of the oracle is then used to obtain a basis of improved quality.

Algorithm 2 presents the main concept of BKZ and follows the description in [32] with some adjustments. Initially, we run an LLL reduction on the input basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and update the basis. In each jth iteration, we consider a block of k basis vectors $\mathbf{b}_j, \dots, \mathbf{b}_{j+k-1}$. The vectors of the current block are projected onto the orthogonal complement of span $(\{\mathbf{b}_i \mid i \in [j-1]\})$, that is, the span of vectors from previous iterations (Line 6ff; we skip this step, if the span is empty). The orthogonal complement $A^\perp$ of a subspace A is defined as the set of all vectors that are orthogonal to every vector in A. We then run an SVP oracle on the projected block to obtain a shortest vector $\mathbf{b}'_{\text{new}}$ in the projected lattice (Line 12) and reconstruct a lattice vector $\mathbf{b}_{\text{new}}$ of which $\mathbf{b}'_{\text{new}}$ is a projection (Line 13). Note that in practice, the SVP oracle should include this step. If $\mathbf{b}_{\text{new}}$ is a new vector, we insert it in our list of basis vectors before $\mathbf{b}_j$. Otherwise, since nothing changed, we increment a counter z. Finally, we run LLL on all basis vectors up to index $j + i$ (including the possibly newly added vector). If no new lattice vectors can be found in n iterations, the reduction terminates. After n iterations, j is reset to start over at the first block. The output of the algorithm is a BKZ$_k$-reduced basis. For $k = 2$, we obtain an LLL-reduced basis in polynomial time, and for $k = n$, an optimally HKZ-reduced basis in at least exponential time.

Several improvements have been suggested. The total number of rounds until termination is unknown and can be quite large. Hanrot et al. [45] show an *early termination* of BKZ still yields a very good output basis quality and propose $\frac{n^2}{k^2} \log n$ rounds as a bound.

Local preprocessing increases the quality of the current block basis by recursively calling BKZ with smaller block size. A variant known as *progressive BKZ* applies the recursion globally [16].

Algorithm 2: The BKZ Algorithm [88]

```
function BKZ($\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, k \in [n] \setminus \{1\}$)
    z = 0; j = 0
    $\mathbf{B}$ = LLL($\mathbf{B}$)
    while $z < n - 1$ do
        $j = (j \mod (n-1)) + 1; l = \min(j + k - 1, n); h = \min(l + 1, n)$
        $A = \text{span}(\{\mathbf{b}_i | i \in [j-1]\})$
        for $i \in \{j, ..., l\}$ do
            if $A \neq \emptyset$ then
                $\mathbf{b}'_i = \pi_{A^\perp}(\mathbf{b}_\mathbf{i})$
            else
                $\mathbf{b}'_i = \mathbf{b}_i$
        $\mathbf{b}'_{\text{new}}$ = SVP-Oracle($\mathbf{b}'_j, \dots, \mathbf{b}'_l$)
        Reconstruct $\mathbf{b}_{\text{new}} = \sum_{i=j}^{l} \alpha_i \mathbf{b}_i$ with $\alpha_i \in \mathbb{Z}$ such that $\mathbf{b}'_{\text{new}} = \pi_{(\text{span}(\mathbf{b}_j,\dots,\mathbf{b}_l))^\perp}(\mathbf{b}_{\text{new}})$
        if $\mathbf{b}'_{new} \neq \tilde{\mathbf{b}}_j$ then
            $z = 0; \{\mathbf{b}_j, \dots, \mathbf{b}_h\} = \text{LLL}(\{\mathbf{b}_j, \dots, \mathbf{b}_{j-1}, \mathbf{b}_{\text{new}}, \mathbf{b}_j, \dots, \mathbf{b}_h\})$
        else
            $z = z + 1; \{\mathbf{b}_j, \dots, \mathbf{b}_h\} = \text{LLL}(\{\mathbf{b}_j, \dots, \mathbf{b}_h\})$
```

If enumeration is used as an SVP oracle, the size of the search space can be reduced by means of *pruning* techniques. Nodes closer to the edges of the enumeration tree are less likely to result in short lattice vectors. For more details on enumeration, we refer to Section 3.1.3. Gama et al. [37] show that applying a variant of this known as *extreme pruning* can reduce the running time by a much larger factor than the success probability. Repeating the search yields the desired speedup.

In addition, [32] optimizes the *enumeration radius* by using experimental results. BKZ 2.0 incorporates a number of these techniques [32].

It is difficult to find hard runtime bounds for BKZ. The upper bound on the number of rounds is superexponential in the dimension n for a fixed block size [36, 45] before BKZ terminates, given that no early termination strategy is used. In addition, calls to the SVP oracle in all dimensions $k' \leq k$ must be considered. Albrecht et al. [13] ignore these intricacies and estimate the cost of BKZ in clock cycles as $\rho \cdot n \cdot t_k$, where ρ is the number of rounds needed and t_k is the cost (in block cycles) of calling the SVP oracle on a block of dimension k. The value ρ is set to 8 in the *estimator* and is derived from experiments in [31] that indicate that the most significant progress happens in the first 7 – 9 rounds.

The output quality of BKZ is closely related to the used block size k. The *estimator* uses a limiting value from [31] to estimate the root Hermite factor δ for a given k:

$$\lim_{n \to \infty} \delta \approx \left(\frac{k(\pi k)^{\frac{1}{k}}}{2\pi e} \right)^{\frac{1}{2(k-1)}} \tag{3.4}$$

For smaller block sizes $k \leq 40$, the *estimator* uses fixed experimental values for δ.

3.1.3 Cost Models for Lattice Reduction

In this section, we will look at various high-level ideas to realize an SVP solver that can be used as a subroutine in BKZ and present up-to-date cost models from the literature. SVP is known to be NP-complete even for large constant approximation factors [3, 52]. An exponential approximation factor can be achieved in polynomial time, but is mostly insufficient for practical purposes [55]. We will mainly focus on two classes of (nearly) exact SVP solvers, namely, enumeration algorithms and sieving algorithms. Enumeration algorithms can solve SVP in a lattice of dimension k in $2^{O(k \log k)}$ time and polynomial space. Sieving algorithms only need $2^{O(k)}$ time, however, at the cost of exponential memory complexity. Only recently, progress in sieving strategies has given rise to BKZ implementations relying on sieving (e.g., the General Sieve Kernel (G6K) implementation [9, 35]) that outperform enumeration implementations already in relatively small dimensions $\gtrsim 70$ in the classical setting [6]. On the other hand, quantum speedups for enumeration are greater than for sieving. Aono et al. [15] show a quadratic cost reduction for enumeration, while the speedup is much lower for sieving, even with idealized assumptions [53]. The authors of [14] argue that due to a lower bound $2^{0.2075k}$ on the required size of the building lists, future quantum sieving algorithms are not expected to achieve an asymptotic runtime below $2^{0.2075k}$.

A selection of the most relevant cost models for cryptographic purposes can be seen in Table 3.2. All of these cost models are supported in our tool (see Section 4.5).

Enumeration

Enumeration aims to find the shortest vector by enumerating all lattice vectors within some bounded region. In general, we start with reducing the lattice basis to improve the basis quality. We then define a bound and iteratively project the lattice to the span of its Gram-Schmidt vectors beginning from $\tilde{\mathbf{b}}_n$ until we arrive at the lowest level of a one-dimensional subspace. We continue by enumerating all vectors of norm less than r in the projected lattice and "lift" each of these vectors to the level above and repeat this process until we arrive at the level from which we started. The search space can be thought of as a large tree of (projected) vectors on which we apply depth-first search. Note that the root of the tree here is at the lowest level and the leaves are the lattice vectors in our target lattice. The low memory cost of enumeration is due to its similarities to depth-first search.

A very early but very efficient variant was suggested by Kannan [50] with a proven worst-case runtime of $2^{O(k \log k)}$. BKZ$_k$ using Kannan's enumeration algorithm as SVP oracle yields a short lattice vector of norm approximately $\left(k^{1/(2k)}\right)^n \cdot \det(\Lambda)^{1/n}$, or equivalently achieves a root Hermite factor of $k^{1/(2k)}$ [5, 46].

In [5], we find a more concrete experimental cost model of $\text{poly}(n) \cdot 2^{(k \log k)/(2e) - 0.995k + 16.25}$ for BKZ 2.0 (see Section 3.1.2), where $\text{poly}(n)$ is the number of calls to the enumeration subroutine. BKZ 2.0 achieves a root Hermite factor of $\left(k/(2\pi e) \cdot (\pi k)^{1/k}\right)^{1/(2(k-1))}$ [31].

The FastEnum algorithm in Albrecht et al. [5] incorporates an idea called "extended preprocessing" and simulations achieve a root Hermite factor of $k^{(1+o(1))/(2k)}$ in $\text{poly}(n) \cdot 2^{0.125k \log k - 0.050k + 56}$ time. The corresponding quantum algorithm reduces the runtime from $2^{(k \log k)/8 + o(k)}$ to $2^{(k \log k)/16 + o(k)}$. In extended preprocessing, instead of preprocessing the current projected basis block of size k, the BKZ-reduction is applied to a block of higher dimension $\lceil (1+c) \cdot k \rceil$ for some constant c. Enumeration is faster on the first basis vectors, as their Gram-Schmidt norms closely follow the Geometric Series Assumption [70].

A tradeoff of runtime and success probability for "relaxing" the approximation and extreme pruning turns out to exponentially speed up the search [56] and was combined with extended preprocessing by Albrecht et al. [5] to further reduce the experimental runtime of BKZ to $\mathrm{poly}(n) \cdot 2^{(k \log k)/8 - 0.654k + 25.84}$ for a root Hermite factor of $k^{1/(2k)}$.

Sieving

The second group of SVP solvers are sieving algorithms [9, 20, 22, 23, 47, 69, 74]. In sieving, initially, we create a long list of randomly selected lattice points. The points in the list are then combined or "reduced" in some way to find points of smaller length. One way to achieve this is by finding a minimal sublist of "center" points in the initial list, such that spheres centered at these points cover all list points. Subtracting the center points yields short lattice points. ListSieve [69] uses a smaller initial list to divide the space into two half-spaces, one closer to the center and one closer to the respective point. The list is then used to reduce the length of newly sampled points as much as possible by subtracting each list vector, such that the result is located in the half-space closer to the center. Once two points with a distance less than the target distance are found, they are subtracted, and the result is returned.

Algorithm	Runtime complexity	Memory complexity
List Sieve [69]	$2^{0.3199n+o(n)}$	$2^{0.1325+o(n)}$
NV-sieve [9, 74]	$2^{0.415n+o(n)}$	$2^{0.2075n+o(n)}$
NV-sieve (quantum) [9, 74]	$2^{0.311n+o(n)}$	$2^{0.2075n+o(n)}$
Gauss sieve [47, 69]	$2^{0.415n+o(n)}$	$2^{0.2075n+o(n)}$
BGJ-sieve [23]	$2^{0.311n+o(n)}$	$2^{0.2075n+o(n)}$
3-sieve [20, 47]	$2^{0.3962n+o(n)}$	$2^{0.1887n+o(n)}$
BDGL-sieve [22]	$2^{0.292n+o(n)}$	$2^{0.2075n+o(n)}$
BDGL-sieve (quantum) [22]	$2^{0.265n+o(n)}$	$2^{0.2075n+o(n)}$

Table 3.1: Overview of Popular Sieving Algorithms

Table 3.1 presents a list of currently best sieving algorithms. Note that some runtime and space complexities are only conjectured and not proven yet.

In the Nguyen-Vidick sieve [74], we iteratively reduce a pair of list points whose combined length is smaller than the longest list vector. The longest vector is then replaced by the result. The list length is fixed. In the Gauss sieve [69], we start with an empty list and a stack. In each step, a new point is either sampled or taken from the stack. We then attempt to reduce the new point with all points in the list. If a reduction is successful, the longer vector of the pair is replaced. If the longer vector was the list point, the replacement is inserted in the stack. If no reduction is possible, the stack points are moved back to the list. If the stack is empty, all list points are reduced pairwise. In practice, the Gauss sieve outperforms Nguyen-Vidick sieve. The Becker-Gama-Joux sieve [23] exploits coding theory to find vectors that are likely to be nearest neighbors. Similar vectors are

Name	Reference	Cost model
Sieving		
Q-Sieve (paranoid lower bound)	[14]	$2^{0.2075k}$
Q-Sieve	[12, 14, 53]	$2^{0.265k}$
Q-Sieve + O(1)	[90]	$2^{0.265k+16}$
Q-Sieve (min space)	[86]	$2^{0.2975k}$
Sieve	[12, 14, 22]	$2^{0.292k}$
Sieve + O(1)	[90]	$2^{0.292k+16}$
Sieve (min space)	[86]	$2^{0.368k}$
Enumeration		
Lotus	[8, 78]	$2^{0.125k \log k - 0.755k + 2.254}$
Enum + O(1)	[8, 31, 86]	$2^{0.187k \log k - 1.019k + 16.1}$
Q-Enum + O(1)	[8, 31, 86]	$2^{0.0936k \log k - 0.51k + 8.05}$
BCLV-Enum (quadratic fit) + O(1)	[25]	$2^{0.000784k^2 + 0.366k + 0.875}$
BKZ2.0-Enum	[5, 31, 32]	$2^{0.184k \log k - 0.995k + 16.25}$
ABF-Enum	[5]	$2^{0.125k \log k}$
ABF-Enum + O(1)	[5]	$2^{0.125k \log k - 0.547k + 10.4}$
Q-ABF-Enum	[5]	$2^{0.0625k \log k}$
ABLR-Enum + O(1)	[6]	$2^{0.125k \log k - 0.654k + 25.84}$

Table 3.2: SVP Cost Models Overview (Based on [8, Table 4])

stored in the same bucket to speed up the search for reduction candidates. The 3-sieve [20, 47] reduces the required list size by using triples instead of pairs of points for combination. Finally, the Becker-Ducas-Gama-Laarhoven sieve [22] applies locality sensitive hashing to create buckets of points in near neighborhood similar to the Becker-Gama-Joux sieve.

3.2 Algorithms for Solving LWE

In this section, we present a number of popular attacks on LWE. The *estimator* includes cost estimate algorithms for each of these attacks. In our treatise, we will ignore special cases of LWE that can be exploited to obtain more efficient attack variants and only present the main algorithms. For more details, we refer the reader to [13] and [26].

3.2.1 Lattice Problems

Some of the following attacks rely on a reduction of LWE to different lattice problems. We first define three important lattice problems in the context of LWE.

Definition 3.2.1 (HSVP$_\delta$)
Given a basis $\mathbf{B}$ *of a lattice* $\Lambda(\mathbf{B}) \in \mathbb{R}^m$*, the (approximate) Hermite Shortest Vector Problem (HSVP$_\delta$) is the problem of finding a nonzero lattice vector* $\mathbf{v} \in \Lambda$ *such that* $\|\mathbf{v}\| \leq \delta \cdot det(\Lambda)^{\frac{1}{n}}$.

Definition 3.2.2 (uSVP$_\gamma$)
Given a lattice Λ such that $\lambda_2(\Lambda) > \gamma\lambda_1(\Lambda)$, the (approximate) Unique Shortest Vector Problem (uSVP$_\gamma$) is the problem of finding the shortest nonzero vector in $\mathbf{v} \in \Lambda$ with $\|\mathbf{v}\| = \lambda_1(\Lambda)$.

Definition 3.2.3 (BDD$_\gamma$)
Given a lattice $\Lambda \subset \mathbb{R}^m$ and a target vector $\mathbf{t} \in \mathbb{R}^m$ such that dist$(\mathbf{t}, \Lambda) < \gamma\lambda_1(\Lambda)$, the (approximate) Bounded Distance Decoding (BDD$_\gamma$) is the problem of finding the closest lattice vector $\mathbf{v} \in \Lambda$, i.e., $\mathbf{v} = \arg\min_{\mathbf{v}' \in \Lambda} \|\mathbf{v}' - \mathbf{t}\|$.

3.2.2 Approaches

Figure 3.1 gives an overview of the three main approaches to solving LWE. The most natural approach is to recover the secret in LWE directly, e.g., by exhaustively searching the search space. In general, however, the search space is quite large, resulting in high attack costs. The *estimator* incorporates two direct attacks, the Meet-in-the-Middle attack [19] and an attack due to Arora and Ge [17].

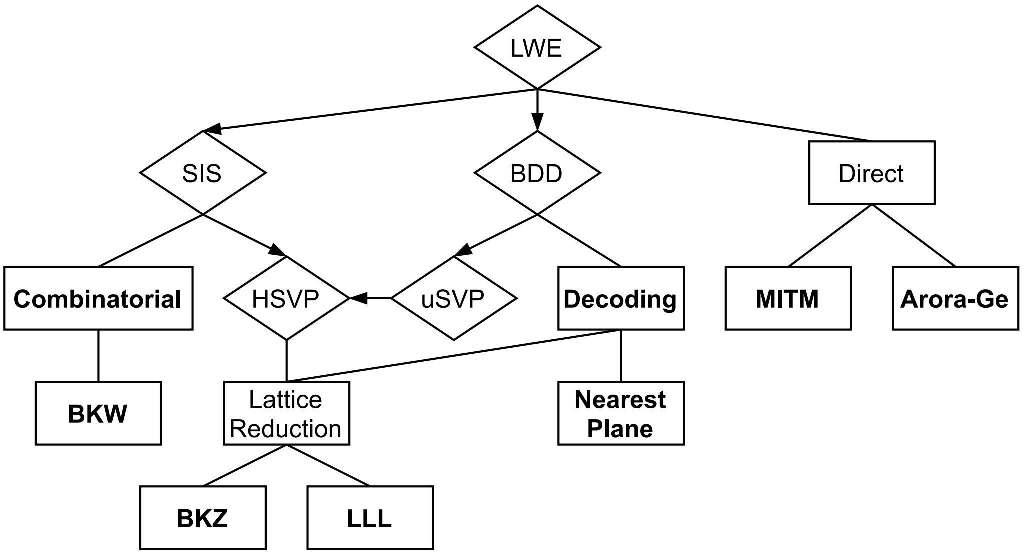


Figure 3.1: An overview of algorithms and related problems for soling LWE in this work. The diamond shaped boxes represent problems and arrows reductions between problems. Rectangular boxes represent approaches, classes of algorithms or algorithms. A connection between a problem and a rectangular box indicates that the latter can be used as a solver for the respective problem. Bold names indicate algorithms that can be used in our tool.

Primal Attack

The second approach aims to solve LWE in the primal LWE lattice by a reduction to the BDD problem. Consider an LWE instance with samples $\mathbf{A}, \mathbf{z}$ and the corresponding lattice $\Lambda_q(\mathbf{A}^\top)$. We know that for the secret vector $\mathbf{s}$ and error vector $\mathbf{e}$, it holds that $\mathbf{z} = \mathbf{A}^\top\mathbf{s} + \mathbf{e} \mod q = \mathbf{A}^\top\mathbf{s} + \mathbf{e} + q\mathbf{x}$

for some vector $\mathbf{x} \in \mathbb{Z}^m$. Obviously, $\mathbf{A}^\top\mathbf{s} + q\mathbf{x}$ is a lattice vector in the lattice $\Lambda_q(\mathbf{A}^\top)$. We know that $\text{dist}(\mathbf{z}, \Lambda_q(\mathbf{A}^\top) = \|\mathbf{e}\|$ and in general, it holds that $\|\mathbf{e}\| < \gamma\lambda_1(\Lambda_q(\mathbf{A}^\top))$. We thus have a reduction from LWE to BDD. By finding closest lattice point to $\mathbf{z}$ or, in other words, solving BDD in the lattice $\Lambda_q(\mathbf{A}^\top)$ we obtain $\mathbf{A}^\top\mathbf{s} + q\mathbf{x}$ and can recover the secret $\mathbf{s}$.

Dual Attack

Finally, we can take advantage of the close relationship of LWE and SIS and solve LWE by finding a short vector in the dual SIS lattice [57]. More precisely, we want to find a short nonzero vector $\mathbf{v} \in \mathbb{Z}_q^m$ in the scaled q-ary lattice $\Lambda_q(\mathbf{A}^\top)^\perp = \{\mathbf{y} \in \mathbb{Z}^m \mid \mathbf{A}\mathbf{y} = \mathbf{0} \mod q\}$. Obviously, we have that $\langle \mathbf{v}, \mathbf{z} \rangle = \langle \mathbf{v}, \mathbf{A}^\top\mathbf{s} \rangle + \langle \mathbf{v}, \mathbf{e} \rangle = \langle \mathbf{v}\mathbf{A}^\top, \mathbf{s} \rangle + \langle \mathbf{v}, \mathbf{e} \rangle \mod q$ and $\langle \mathbf{v}, \mathbf{z} \rangle$ roughly corresponds to a sample drawn from a Gaussian with parameter $s' = \|\mathbf{v}\| \cdot s$, given that $\mathbf{e}$ is distributed according to a Gaussian with parameter s. We then test whether $\langle \mathbf{v}, \mathbf{z} \rangle \mod q$ approximates the Gaussian. If $\mathbf{z}$ is instead drawn from a uniform distribution, the test accepts exactly with probability 0.5 [57]. If s' is not much larger than q, the advantage of distinguishing uniform from LWE samples is very close to $\exp(-\pi(\|\mathbf{v}\|s/q)^2)$. An optimal attack cost can be achieved by balancing the advantage with the computational effort required to solve SVP_γ on the SIS lattice, where $\gamma = \|\mathbf{v}\|$.

3.2.3 Decoding Attack [57]

The decoding attack falls into the regime of primal attacks that solve LWE by solving BDD in the primal LWE lattice (as described in Section 3.2.2). In the decoding attack, we first apply a reduction algorithm to the input basis $\mathbf{A}^\top$ to improve the basis quality. We then enumerate a number of candidate lattice points close to our target $\mathbf{z}$ by running a variant of Babai's Nearest Planes algorithm and simply choose the closest point.

We begin with the original Nearest Planes algorithm due to Babai [18], as shown in Algorithm 3. Our algorithm follows the notation in [81]. The goal of the algorithm is to find a lattice vector relatively close to the target vector. The procedure used is similar to the procedure in the inner loop of LLL. We first project $\mathbf{t}$ to the span of all basis vectors $\text{span}(\mathbf{B})$, where B denotes an LLL-reduced basis, to eliminate irrelevant dimensions and obtain a projected vector which we call $\mathbf{b}$. Next, we iterate over $i = n, \ldots, 1$. In each step, we want to find the closest hyperplane $c_i\tilde{\mathbf{b}}_i + \text{span}(\mathbf{b}_1, \ldots, \mathbf{b}_{i-1})$ to the projected vector $\mathbf{b}$. We can compute c_i by a simple projection of b onto the Gram-Schmidt vector $\tilde{\mathbf{b}}_i$ and dividing by the square length of $\tilde{\mathbf{b}}_i$. We subtract $c_i\mathbf{b}_i$ from b and replace $\mathbf{b}$ with the result. Notice that in the last step, the hyperplane is simply a point. We return $\mathbf{t} - \mathbf{b}$. If we indeed found the closest lattice vector for target $\mathbf{z}$, the vector $\mathbf{b}$ will be our error vector $\mathbf{e}$ and $\mathbf{t} - \mathbf{b}$ will be the desired lattice vector $\mathbf{A}^\top\mathbf{s} + q\mathbf{x}$, from which we can recover our secret.

In general, however, the output is a lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$ in the fundamental parallelepiped of the Gram-Schmidt basis $\mathcal{P}_{1/2}(\tilde{\mathbf{B}})$ and we only get the guarantee of $\|\mathbf{v} - \mathbf{t}\| \le 2^{n/2}\text{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$. To apply the algorithm on LWE, either $\mathbf{e}$ must be in $\mathcal{P}_{1/2}(\tilde{\mathbf{B}})$ or we must increase our search radius. The former case is unlikely as the last Gram-Schmidt vectors in a reduced basis because of the long and skinny structure of $\mathcal{P}_{1/2}(\tilde{\mathbf{B}})$ (see Section 3.1. The latter case is exactly what the generalized algorithm by [57] does.

Algorithm 3: Babai's Nearest Planes Algorithm [18]

```
function NearestPlanes(LLL-reduced basis $\mathbf{B} \in \mathbb{R}^{m \times n}$, target $\mathbf{t} \in \mathbb{R}^m$)
    $\mathbf{b} = \pi_{\mathrm{span}(\mathbf{B})}(\mathbf{t})$
    for $i = n, \ldots, 1$ do
        $c_i = \mathrm{round}(\langle \mathbf{b}, \tilde{\mathbf{b}}_i \rangle / \langle \tilde{\mathbf{b}}_i, \tilde{\mathbf{b}}_i \rangle)$
        $\mathbf{b} = \mathbf{b} - c_i \mathbf{b}_i$
    output $\mathbf{t} - \mathbf{b}$
```

Instead of choosing only the nearest plane in each iteration step, the Lindner and Peikert's variant that can be seen in Algorithm 4 selects a variable amount d_k of distinct planes in each step. As a consequence, the fundamental parallelepiped of the Gram-Schmidt basis is stretched in the direction of $\tilde{\mathbf{b}}_k$. The values of $\mathbf{d}$ should be chosen such that the covered area is approximately the same in each direction (i.e., by maximizing $\min_i(d_i \cdot \|\tilde{\mathbf{b}}_i\|)$). In particular, this implies that the d_k are larger for larger k, as the Gram-Schmidt vectors have a smaller length. Compared to Algorithm 3, the runtime increases by a factor $\prod_{i \in \{1,\ldots,d_k\}} d_i$; however, the recursion step can be fully parallelized.

Algorithm 4: Generalized Nearest Planes Algorithm [57]

```
function GeneralizedNearestPlanes($\mathbf{B} \in \mathbb{R}^{m \times k}, \mathbf{t} \in \mathbb{R}^m, \mathbf{d} \in (\mathbb{Z}^+)^k$)
    if $k = 0$ then
        Return $\mathbf{0}$
    else
        Compute projection $\mathbf{v}$ of $\mathbf{t}$ onto span($\mathbf{B}$)
        Compute the $d_k$ distinct integers $c_1, \ldots, c_{d_k}$ closest to $\langle \mathbf{v}, \tilde{\mathbf{b}}_k \rangle / \langle \tilde{\mathbf{b}}_k, \tilde{\mathbf{b}}_k \rangle)$
        Return $\bigcup_{i \in \{1,\ldots,d_k\}} (c_i \cdot \mathbf{b}_k +$
          GeneralizedNearestPlanes$(\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}, (d_1, \ldots, d_{k-1}), \mathbf{v} - c_i \cdot \mathbf{b}_k))$
```

It is evident that a lower quality of the reduced input basis can be compensated by increasing the values of $\mathbf{d}$. The input parameters of the lattice reduction and the generalized Nearest Planes algorithm should hence be adjusted such that the overall runtime is minimized. The success probability of the Generalized Nearest Planes algorithm is given by

$$\Pr\left[\mathbf{e} \in \mathcal{P}_{\frac{1}{2}}(\tilde{\mathbf{B}} \cdot \mathrm{diag}(\mathbf{d}))\right] = \prod_{i=1}^{m-1} \mathrm{erf}\left(\frac{d_i \|\tilde{\mathbf{b}}_i\| \sqrt{\pi}}{2\alpha q}\right).$$

The runtime of the decoding attack cannot be explicitly stated, as the reduction phase and the decoding phase have to be balanced for a given problem instance. This is done by iteratively varying the required basis quality δ for the reduction phase and stepwise increasing the components of $\mathbf{d}$ until the success probability is reached, such that the total runtime is minimized. The final runtime is given by $t_{\mathrm{DEC}} = \rho \cdot (t_{\mathrm{BKZ}} + t_{\mathrm{GNP}})$ where t_{BKZ} is the runtime of BKZ in the reduction phase and t_{GNP} is the runtime of the Generalized Nearest Planes algorithm in the decoding phase. The number of rounds ρ must be greater than $\log(1 - \epsilon')/\log(1 - \epsilon)$ to achieve a target success probability ϵ' if one decoding round has success probability ϵ (more details at the end of Section 3.2.4.

3.2.4 Primal uSVP [14, 19]

In the Primal uSVP attack, we again view the $\mathrm{LWE}_{n,q,m,\chi}$ instance $(\mathbf{A}, \mathbf{z})$ as a BDD instance in the q-ary lattice $\Lambda_q(\mathbf{A}^\intercal) = \{\mathbf{y} \mid \exists \mathbf{x} \in \mathbb{Z}_q^n : \mathbf{y} = \mathbf{A}^\intercal \mathbf{x} \mod q\}$ generated by rows of LWE instance, as in Section 2.5.1. The target vector is $\mathbf{z}$.

Recall the γ-uSVP problem. Given a lattice Λ where $\lambda_2(\Lambda) > \gamma\lambda_1(\Lambda)$, we are asked to find shortest nonzero vector in Λ. In the primal attack, instead of directly solving BDD, we reduce BDD to uSVP, i.e., we reduce a BDD instance to a γ-uSVP instance. By solving γ-uSVP, we obtain a solution to BDD. To do this, we apply Kannan's embedding technique [51]. Kannan's embedding creates a lattice with uSVP structure. We know that $\mathbf{A}^\intercal \mathbf{s} \mod q$ is the closest vector to the target $\mathbf{z} = \mathbf{A}^\intercal \mathbf{s} + \mathbf{e}^\intercal \mod q$ in $\Lambda(\mathbf{A}^\intercal)$. We now add a linearly independent basis vector $(\mathbf{z}, \mu)$ and append a zero coefficient to each basis vector of the original lattice (i.e., the rows of $\mathbf{A}$). Thereby, we ensure that the new lattice contains the vector $[-\mathbf{e}, -\mu]^\intercal$ as $[\mathbf{A} \mid \mathbf{0}]^\intercal \mathbf{s} - 1 \cdot [\mathbf{z}^\intercal, \mu] = [-\mathbf{e}, -\mu]^\intercal$.

More formally, let $\mathbf{B} \in \mathbb{Z}^{m \times m}$ be the basis of an m dimensional lattice derived from the LWE instance and $\mu = \mathrm{dist}(\mathbf{z}, \Lambda(\mathbf{B})) = \|\mathbf{z} - \mathbf{s}\|$ be the embedding factor, where $\mathbf{s}$ is the secret vector of the LWE instance. For more details on how to compute $\mathbf{B}$, we refer to [11].

We now embed $\Lambda(\mathbf{B})$ into $\Lambda(\mathbf{B}')$ with γ-uSVP structure as follows:

$$\mathbf{B}' = \begin{pmatrix} \mathbf{B} & \mathbf{z} \\ \mathbf{0}^\intercal & \mu \end{pmatrix} \tag{3.5}$$

If $\gamma \geq 1$ and $\mu < \frac{\lambda_1(\Lambda(\mathbf{B})}{2\gamma}$ (or equivalently, $(\Lambda(\mathbf{A}^\intercal), \mathbf{z})$ a $\mathrm{BDD}_{1/(2\gamma)}$-instance), then $\Lambda(\mathbf{B}')$ contains a γ-unique shortest vector $\mathbf{z}' = [(\mathbf{A}^\intercal \mathbf{s} - \mathbf{z})^\intercal, -\mu]^\intercal = [-\mathbf{e}^\intercal, -\mu]^\intercal$. The statement can be proven by showing by contradiction that all vectors $\mathbf{v} \in \Lambda(\mathbf{B}')$ that are independent of $\mathbf{z}'$ satisfy $\|\mathbf{v}\| \geq \lambda_1 \Lambda(\mathbf{B})/\sqrt{2} > \sqrt{2}\gamma\mu = \gamma\|\mathbf{z}'\|$ (see [62, Section 4] for more details). Note that the reduction can be done in polynomial time [62, Theorem 4.1]. The length of $\mathbf{z}'$ is given by $\|\mathbf{z}'\| = \sqrt{\|\mathbf{e}\|^2 + |\mu|^2} = \sqrt{m\alpha^2 q^2/(2\pi) + |\mu|^2}$ [26]. Given $\mathbf{z}'$, we can easily recover the error vector $\mathbf{e}$ and thereby the secret vector $\mathbf{s} = \mathbf{z} - \mathbf{e} \mod q$.

A solution to γ-uSVP can be found by reducing it to δ-HSVP, where $\gamma = \delta^2$ [13]. Various algorithms, in particular, lattice reduction algorithms, exist to solve δ-HSVP. If we are able to solve a linear number of δ-HSVP instances that correspond to a δ^2-approximate SVP instance, we can construct a solution of the latter (see [58, Section 1.2.21] for more details). Consider any lattice with uSVP structure. In exactly one direction, that is, in the direction of its unique shortest vector, the lattice has vectors that are significantly smaller than in other directions. A lattice reduction algorithm that yields a sufficiently good output basis quality, therefore, must return some small vector in the desired direction. Let $\mathbf{v}$ be a solution to SVP_{δ^2}, i.e., $\|\mathbf{v}\| \leq \delta^2 \lambda_1(\Lambda(\mathbf{B}'))$. All other vectors $\mathbf{w} \in \Lambda(\mathbf{B}')$ that are not multiples of a shortest vector have length $\|\mathbf{w}\| \geq \lambda_2(\Lambda(\mathbf{B}')) > \delta^2 \lambda_1(\Lambda(\mathbf{B}'))$. Thus, we obtain a solution to γ-uSVP and, as shown above, we can reconstruct the secret vector to solve LWE.

The attack is successful with high probability if $\lambda_2(\Lambda(\mathbf{B}'))/\lambda_1(\Lambda(\mathbf{B}')) \geq \tau\delta^m$ [11]. We assume that the determinant of $\Lambda_q(\mathbf{A}^\intercal)$ is given by q^{m-n} (see Equation (2.11)). The length of the error is $\|\mathbf{e}\| \approx \sqrt{m \cdot \left(\alpha q/(\sqrt{2\pi})\right)^2} \approx \sqrt{m/(2\pi)}\alpha q$, and in practice, $\mu = 1$ is used [13]. Next, we apply the simplified Gaussian heuristic as in [42] on $\lambda_2(\Lambda(\mathbf{B}')) = \lambda_2(\Lambda(\mathbf{A}^\intercal))$ (see Equation (2.8)) to obtain

$$\begin{aligned}\tau\delta^m &\leq \frac{\sqrt{\frac{m}{2\pi e}}\det(\Lambda(\mathbf{A}^\intercal))}{\lambda_1(\Lambda(\mathbf{B}'))} \approx \frac{\sqrt{m}q^{1-\frac{n}{m}}}{\sqrt{2\pi e}\lambda_1(\Lambda(\mathbf{B}'))}\\ &\leq \frac{\sqrt{m}q^{1-\frac{n}{m}}}{\sqrt{2\pi e}\|\mathbf{e}\|} \approx \frac{q^{1-\frac{n}{m}}}{\sqrt{2\pi e}\frac{\alpha q}{\sqrt{2\pi}}} \approx \frac{q^{1-\frac{n}{m}}}{\sqrt{e}\alpha q}\end{aligned}$$

The success probability of the attack is thus non-negligible if

$$\delta \leq \left(\frac{q^{1-\frac{n}{m}}}{\tau\sqrt{e}\alpha q}\right)^{\frac{1}{m}} \tag{3.6}$$

Experiments show that $\tau \leq 0.4$ achieves a success rate of about $\epsilon = 0.1$ [11]. In order to increase the success probability to some fixed target $\epsilon' > \epsilon$, we can simply repeat the algorithm ρ times and obtain a success probability of $\epsilon' \leq 1 - (1-\epsilon)^\rho$ [26]. Consequently, we need at least $\rho \geq \log(1-\epsilon')/\log(1-\epsilon)$ rounds for a successful attack.

3.2.5 The BKW Algorithm [28]

The Blum, Kalai and Wasserman (BKW) algorithm was originally designed to solve the Learning Parity with Noise problem (LPN) [28], which, as we pointed out in Section 2.5.1, is a subproblem of LWE. Albrecht et al. [7] adapted the algorithm to LWE. The runtime and memory complexity of BKW is in $2^{O(n)}$ for an LWE instance with secret dimension n prime modulus $q \in \text{poly}(n)$. The number of samples m must be sufficiently large (in $\Omega(n \log n)$).

BKW falls into the regime of dual attacks, that is, it solves LWE by finding a short vector $\mathbf{s}$ in the scaled dual lattice $\Lambda(\mathbf{A}^\intercal)^\perp$.

Albrecht et al. [7] divide the algorithm into three stages, namely, sample reduction, hypothesis testing and back substitution.

Sample Reduction. Algorithm 5 shows the sample reduction part of the BKW algorithm. The notation is inspired by the textual description in [44] with minor adjustments.

For the algorithm, we use the matrix notation of LWE as in Equation (2.16), i.e., $\mathbf{z} = \mathbf{A}^\intercal\mathbf{s} + \mathbf{e}$. BKW consists of a series of BKW steps that iteratively reduce the dimension of input matrix $\mathbf{A}$ by finding collisions of its column vectors in the currently examined block of b entries. We start from the last b entries of $\mathbf{A}^{(1)} = \mathbf{A}$. In every step i, we maintain a collision table $\mathbf{T}^{(i)}$ and loop over the columns $\mathbf{a}_k^{(i)}$ of $\mathbf{A}^{(i)}$ and distinguish between the following cases: (1) If $\mathbf{a}_k^{(i)}$ only has zero entries in the examined block, pass $\mathbf{a}_k^{(i)}$ and $z_k^{(i)}$ to the next step, (2) if no match of $\mathbf{a}_k^{(i)}$ or the negation of $\mathbf{a}_k^{(i)}$ can be found in the collision table, add $\mathbf{a}_k^{(i)}$ to the collision table, and (3) if a match $\mathbf{a}_l^{(i)}$ is found, compute $\mathbf{a}_l^{(i)} + \mathbf{a}_k^{(i)}$, or in the case of a negation, match $\mathbf{a}_l^{(i)} - \mathbf{a}_k^{(i)}$ (all operations are modulo q) such that the last b nonzero entries cancel out. By exploiting the symmetry of $\mathbb{Z}_q$ in this way, in

Algorithm 5: BKW (Sample Reduction)

```
function BKW($\mathbf{A} \in \mathbb{Z}^{n\times m}, \mathbf{z} \in \mathbb{Z}^{m}, b \in \mathbb{Z}, d \in \mathbb{Z}$)
    $i = 1$
    $\mathbf{A}^{(i)} = \mathbf{A}$
    $\mathbf{z}^{(i)} = \mathbf{z}$
    while the last $n - d$ coefficients of the columns of $\mathbf{A}^{(i)}$ are nonzero do
        // BKW step
        $j = 1$
        $\mathbf{T}^{(i)} = []$                                              // Collision table
        for $k = 1, \dots, m^{(i)}$ do
            // $m^{(i)}$ is number of columns in $\mathbf{A}^{(i)}$
            if last $(i \cdot b)$ coefficients of $\mathbf{a}_k^{(i)}$ are zero then
                $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)}$
                $z_j^{(i+1)} = z_k$
                $j = j + 1$
            else if no match for $\mathbf{a}_k^{(i)}$ in $\mathbf{T}$ then
                $\mathbf{T} = \mathbf{T} + \left[\mathbf{a}_k^{(i)}\right]$                         // append to collision set
            else if match $\mathbf{a}_l^{(i)}$ for $\mathbf{a}_k^{(i)}$ is found then
                if $\mathbf{a}_l^{(i)}$ matches $\mathbf{a}_k^{(i)}$ in the last $(i \cdot b)$ components then
                    $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)} - \mathbf{a}_l^{(i)}$;        // last $i \cdot b$ coefficients of $\mathbf{a}_j^{(i+1)}$ are now zero
                    $z_j^{(i+1)} = z_k^{(i)} - z_l^{(i)} = y_j^{(i)} + e_j^{(i)}$, where $y_j^{(i)} = \left\langle \mathbf{s}, \mathbf{a}_j^{(i)} \right\rangle$ and $e_j^{(i)} = e_k^{(i)} - e_l^{(i)}$
                    $j = j + 1$
                else if the negation of $\mathbf{a}_l^{(i)}$ in $\mathbb{Z}_q^n$ matches $\mathbf{a}_k^{(i)}$ in the last $(i \cdot b)$ components then
                    $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)} + \mathbf{a}_l^{(i)}$
                    $z_j^{(i+1)} = z_k^{(i)} + z_l^{(i)} = y_j^{(i)} + e_j^{(i)}$, where $y_j^{(i)} = \left\langle \mathbf{s}, \mathbf{a}_j^{(i)} \right\rangle$ and $e_j^{(i)} = e_k^{(i)} + e_l^{(i)}$
                    $j = j + 1$
        $i = i + 1$
        // Calculate input for next BKW step
        $\mathbf{A}^{(i)} = (\mathbf{a}_1^{(i)} \cdots \mathbf{a}_{j-1}^{(i)})$
        $\mathbf{z}^{(i)} = (z_1^{(i)}, \dots, z_{j-1}^{(i)})$
    Return $(\mathbf{A}^{(i)}, \mathbf{z}^{(i)})$
```

every step we obtain at most $(q^b - 1)/2$ columns with distinct coefficients in the current b entries. We also make note of "observed symbols" $z_j^{(i)}$ that represent the combination of two samples given their respective matching columns (see Lines 20 and 24 for more details).

In each BKW step, the number of columns (and samples) decreases by at least $(q^b - 1)/2$ (size of the collision set) and the variance of the error distribution σ^2 increases by a factor of two. The algorithm terminates after $t = \lceil b/(n-d) \rceil$ steps returns a a set of observed symbols $\mathbf{z}^{(t)}$ and a corresponding reduced matrix $\mathbf{A}^{(t)}$ in which only the first d rows have nonzero entries. The parameter d should be set to 1, as in the original BKW algorithm, or 2 for the best performance [7].

The remaining part $\mathbf{s}'$ of the secret vector $\mathbf{s}$ is then guessed by means of hypothesis testing. After t steps the error term $\left(\mathbf{z}_j^{(t)} - \left\langle \mathbf{s}', \mathbf{a}_j^{(t)} \right\rangle\right)$ with $j \in [m']$ of the m' remaining observed symbols follows a Gaussian distribution χ with noise $\sigma'^2 = 2^t \cdot \sigma^2$ (for more details, see [7, Lemma 1]). We can test the noise of the error term for all $\mathbf{s}'' \in \mathbb{Z}_q^d$ against the hypothesized noise σ'^2 by means of the log-likelihood ratio (for details we again refer to [7]) and are thus able to determine $\mathbf{s}'$ given sufficiently many samples m'.

Finally, we can apply back substitution to recover all elements of $\mathbf{s}$. We again apply a similar procedure as in Algorithm 5 to reduce a number of columns from the collision tables computed in the Sample Reduction step and obtain m' columns with $d + d'$ nonzero entries and their corresponding "observed symbols". Next, we substitute the part of $\mathbf{s}$ that was recovered in the previous steps and recover the next part of $\mathbf{s}$ by hypothesis testing and repeat the process until we have found $\mathbf{s}$.

Theorem 1 (BKZ Complexity [7, Corollary 2])
Let $(\mathbf{a}_i, z_i)$ be samples following $\mathcal{A}_{\mathbf{s},\chi}$, set $a = \lfloor \log_2(1/(2\alpha)^2) \rceil$, $b = n/a$ and q a prime. Let d be a small constant $0 < d < \log_2(n)$. Assume α is such that $q^b = q^{n/a} = q^{n/\lfloor \log_2(1/(2\alpha)^2) \rceil}$ is superpolynomial in n. Then, given these parameters the cost of the BKW algorithm to solve Search-LWE is

$$\left(\frac{q^b - 1}{2}\right) \cdot \left(\frac{a(a-1)}{2} \cdot (n+1)\right) + \left\lceil \frac{q^b}{2} \right\rceil \cdot \left(\left\lceil \frac{n}{d} \right\rceil + 1\right) \cdot d \cdot a + poly(n) \approx (a^2 n) \cdot \frac{q^b}{2} \tag{3.7}$$

operations in $\mathbb{Z}_q$. Furthermore,

$$a \cdot \left\lceil \frac{q^b}{2} \right\rceil + poly(n) \text{ calls to } \mathcal{A}_{\mathbf{s},\chi} \text{ and storage of } \left(a \cdot \left\lceil \frac{q^b}{2} \right\rceil \cdot n\right) \text{ elements in } \mathbb{Z}_q \text{ are needed.} \tag{3.8}$$

The first summand of Equation (3.7) roughly corresponds to the cost of creating the collision tables and the second summand is the cost of back substitution. For a more detailed cost analysis, see [7, Theorem 2].

Coded-BKW [44]

The *estimator* uses a more efficient variant of BKW, called Coded-BKW [44], that makes use of coding theory. We will only present the high-level idea of Coded-BKW.

The main advantage of the algorithm comes from an improved "coded" BKW step. The improved step removes more column entries at the cost of an additional noise term. To avoid the noise increasing too much, the final algorithm combines original BKW steps with the "coded" BKW step. Let I be an index set and $\mathbf{x}_I$ the part of $\mathbf{x}$ with entries indexed by I.

In a "coded" step i, we set I as the set of b positions to be removed and fix some q-ary linear $[N_i, b]$ code C_i with q^b codewords. We then search for the closest codeword $\mathbf{c}_I \in C$ for every input vector $\mathbf{a}_I$ such that $\mathbf{a}_I = \mathbf{c}_I + \mathbf{e}_I$, where the error part $\mathbf{e}_I \in \mathbb{Z}_q^{N_i}$ is minimized by a decoding procedure. Finally, we subtract two vectors and their corresponding samples and pass the result to the next BKW step. Consider the inner product $\langle \mathbf{s}_I, \mathbf{a}_I \rangle = \langle \mathbf{s}_I, \mathbf{c}_I \rangle + \langle \mathbf{s}_I, \mathbf{e}_I \rangle$. In the subtraction, only the error part $\langle \mathbf{s}_I, \mathbf{e}_I \rangle$ remains.

The runtime estimate for Coded-BKW is slightly more complicated than for the standard variant of BKW. We therefore refer an interested reader to [44].

3.2.6 Dual Attack [67]

The dual attack [67] falls into the regime of solving LWE in the dual SIS lattice (see Section 3.2.2). We will present the attack and show how it can be applied to LWE in Section 3.3.1.

3.2.7 Other Algorithms

In the previous subsections, we gave an overview the most relevant algorithms in practice. The list is by no means exhaustive. We indicated that many improvements and variants have been suggested within the last two decades. In particular, for special cases LWE in which the components of the secret vector are sampled from a small set $\mathcal{S} \subset \mathbb{Z}$, it is possible to speed up the algorithms (for example, by using a technique called modulus switching for BKW [10]). Small secret variants also give rise to a different approach due to the drastically decreasing search space. In the Meet-In-The-Middle attack, we create a sorted list of $\mathbf{A}^\top \mathbf{s}'$ for all $\mathbf{s}' \in \left\{\mathbf{v} \in \mathcal{S}^n \mid v_i = 0 \text{ for } \frac{n}{2} < i \leq n\right\}$. We then iterate over all $\mathbf{s}'' \in \left\{\mathbf{v} \in \mathcal{S}^n \mid v_i = 0 \text{ for } 0 \leq i \leq \frac{n}{2}\right\}$ and check if $\mathbf{z} - \mathbf{A}^\top \mathbf{s}''$ matches any value in the list. The basic version of this algorithm has a runtime and memory complexity of about $|\mathcal{S}|^{n/2}$.

The *estimator* also includes estimates for an algorithm due to Arora and Ge [17] that has subexponential runtime for a sufficiently narrow Gaussian distribution. In practical cryptographic scenarios however, the algorithm has a much higher cost than other algorithms and is thus irrelevant for our purposes.

3.3 Algorithms for Solving SIS

Recall that the $\mathrm{SIS}_{n,q,m,\beta}$ problem asks to find a short vector $\mathbf{s} \in \mathbb{Z}_q^m$ of norm $\|\mathbf{s}\| \leq \beta$ such that such that $\mathbf{A} \cdot \mathbf{s} = 0 \bmod q$ for some uniformly distributed matrix $\mathbf{A}^{n \times m}$. Solving SIS is equivalent to finding a short vector in the dual lattice $\Lambda(\mathbf{A}^\top)^\perp = \{\mathbf{y} \in \mathbb{Z}^m \mid \mathbf{A}\mathbf{y} = \mathbf{0} \mod q\}$.

3.3.1 Lattice Reduction

MR Variant [67]

Our first approach to solving SIS, sometimes referred to as the "dual attack", follows quite naturally. Given $\mathbf{A}$, we can efficiently compute the basis $\mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}$ of the dual lattice $\Lambda(\mathbf{A}^\top)^\perp$ in polynomial time using Gauss-Jordan elimination or other more modern algorithms.

We can then apply a lattice reduction algorithm and obtain a basis with root Hermite factor δ. The first basis $\mathbf{b}_1$ vector of the reduced basis has length $\|\mathbf{b}_1\| = \delta^m \det(\Lambda(\mathbf{A}^\top)^\perp)^{1/m}$. We can see that δ depends on the subdimension m, which we want to be ideal, in order to minimize the cost of the lattice reduction by relaxing δ.

We further assume that $\det(\Lambda(\mathbf{A}^\top)^\perp) = \mathrm{Vol}(\Lambda(\mathbf{A}^\top)^\perp) = q^n$ (see Equation (2.12)). Our first equation then becomes

$$\|\mathbf{b}_1\| = \delta^m q^{\frac{n}{m}}, \tag{3.9}$$

which is minimal for $m = \sqrt{n \log q / \log \delta}$.

Theorem 2 (Optimal subdimension *m* [67])
Given a q-ary scaled dual lattice $\Lambda(\mathbf{A}^\top)^\perp$ defined by a matrix $\mathbf{A} \in \mathbb{Z}^{n\times m}$ with m sufficiently larger than n and a prime q. Then a lattice reduction algorithm yields an optimal output if performed in subdimension

$$m' = \sqrt{\frac{n \log q}{\log \delta}}. \tag{3.10}$$

Higher dimensions increase the complexity of the reduction algorithms and lower dimensions may cause a lack of sufficiently short lattice vectors [67]. In contexts in which Equation (3.9) does not hold, we may still choose m as in Equation (3.10) heuristically. Removing columns from $\mathbf{A}$ does not have a great impact on our results, since we can simply set the corresponding components of the secret vector $\mathbf{s}$ to zero. We reformulate Equation (3.9) a bit:

$$\begin{aligned}\|\mathbf{b}_1\| = \delta^m q^{\frac{n}{m}} &\iff \log\beta = m\log\delta + \frac{n\log q}{m}\\ &\iff \log\delta = \frac{\log\beta}{m} - \frac{n\log q}{m^2}\end{aligned} \tag{3.11}$$

We continue by plugging Equation (3.10) into Equation (3.11):

$$\begin{aligned}\log\delta = \frac{\log\beta}{\sqrt{\frac{n\log q}{\log\delta}}} - \frac{n\log q}{\left(\sqrt{\frac{n\log q}{\log\delta}}\right)^2} &\iff \log\delta = \frac{\log\beta}{\sqrt{\frac{n\log q}{\log\delta}}} - \log\delta\\ &\iff 2\log\delta = \frac{\log\beta}{\sqrt{\frac{n\log q}{\log\delta}}}\\ &\iff \log\delta = \frac{\log^2\beta}{4n\log q}\end{aligned} \tag{3.12}$$

Theorem 3 (log δ in optimal subdimension)
Given a q-ary scaled dual lattice $\Lambda(\mathbf{A}^\top)^\perp$ defined by a matrix $\mathbf{A} \in \mathbb{Z}^{n\times m}$ with m sufficiently larger than n and a prime q. Then a lattice reduction algorithm performed in its optimal subdimension achieves a log root Hermite factor of

$$\log\delta = \frac{\log^2\beta}{4n\log q}. \tag{3.13}$$

To estimate the cost of the lattice reduction for SIS, we call a function from the *estimator* to find the required block size k such that BKZ achieves root Hermite factor δ and apply a cost model with the optimal subdimension m' and block size k.

Note that for LWE we have α, q as input parameters instead of a bound. The advantage of distinguishing $\langle \mathbf{v}, \mathbf{e}\rangle$ from uniformly random $\mod q$ is given by $\epsilon = e^{-\pi(\|\mathbf{v}\|\alpha)^2}$ [57]. We can thus convert α to a required bound $\beta = \frac{1}{\alpha}\sqrt{\ln(\frac{1}{\epsilon})/\pi}$, such that the success probability of solving an LWE instance is given by ϵ [13, Corollary 2]. The *estimator* uses a rinse and repeat strategy to find the best tradeoff between runtime and success probability.

RS Variant [85]

A similar approach is described in [85]. The optimal subdimension and required root Hermite factor are given by a slightly different expression. Apart from that, the attack works as described in Section 3.3.1.

Theorem 4 (Optimal subdimension m [85, Conjecture 2]))
For every $n \geq 128$, constant $c \geq 2, q \geq n^c, m = \Omega(n\log_2(q))$ and $\beta < q$, the best known approach to solve SIS with parameters (n, m, q, β) involves solving δ-HSVP in dimension $m' = \min(x : q^{2n/x} \leq \beta)$ with $\delta = \sqrt{d}\beta/q^{n/m'}$.

We reformulate the expression for m'

$$\begin{aligned} q^{2n/m'} &\leq \beta \\ \frac{2n}{m'\log(q)} &\leq \log(\beta) \\ m' &\geq \frac{2n\log(q)}{\log(\beta)} \end{aligned} \tag{3.14}$$

and obtain $m' = \left\lceil \frac{2n\log(q)}{\log(\beta)} \right\rceil$.

If $m' > m$, we take $m' = m$. The root Hermite factor δ must be larger than 1 for the reduction to be tractable. From $\delta = \sqrt{d}\beta/q^{n/m} \geq 1$, it follows that we need that $m \geq n\log_2(q)/\log_2(\beta)$ for the original dimension.

3.3.2 Combinatorial Attack [67]

Micciancio and Regev also describe a combinatorial method for solving SIS [67] that is similar to BKW.

Given a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ and the dual lattice $\Lambda(\mathbf{A}^\top)^\perp$, we want to find a lattice vector $\mathbf{v} \in \Lambda(\mathbf{A}^\top)^\perp$ with coefficients bounded by β. Expressed differently, we want to find $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = 0 \mod q$ and $\|\mathbf{v}_i\| \leq b$ for all $i \in [m]$.

We begin by dividing the columns of $\mathbf{A}$ into 2^k for some k. Each set contains $m/2^k$ column vectors. We now compute all linear combinations $c_1\mathbf{b}_1 + \cdots + c_{m/2^k}\mathbf{b}_{m/2^k}$, where $\mathbf{b}_i$ denote the indexed column vectors in each set, such that $|c_i| \leq b$ and obtain 2^k new sets $\mathbf{A}_j^{(k)}$ of $L = (2b+1)^{m/2^k}$ vectors, $j \in \left[2^k\right]$. Remember that each c_i represents a coeffient of the lattice vector. By means of these new sets, we satisfy the shorteness criteria of the output vector.

Next, we continue iteratively for $i = k$ to $i = 0$ as follows. In each step, we merge pairs of two sets $\mathbf{A}_j^{(i)}, \mathbf{A}_{j+1}^{(i)}$. If a vector $\mathbf{x}$ in the first set of the pair can be combined with each vector $\mathbf{y}$ in the second set such that the first $\log_q L$ components in $\mathbf{x} \pm \mathbf{y}$ are zero, we put the result in combined set in $\mathbf{A}_j^{(i-1)}$. The size of the combined sets is at most L, as we consider a part of the vectors with $\log_q L$ components that can take at most $q^{\log_q L} = L$ different values. We start from $j = 0$ and increment j by 2 after each merge for $j < 2^i$. After the merge, we have 2^{i-1} sets.

We choose k such that

$$n \approx (k+1)\log_q L = (k+1)\log_q (2\beta+1)^{m/2^k} \iff \frac{2^k}{k+1} \approx \frac{m\log(2\beta+1)}{n\log(q)} \tag{3.15}$$

After k steps, the first $k \log_q L$ entries of the columns in the result set are cancelled out. We expect that of the remaining $\approx \log_q L$ entries in result set, we should find at least one zero vector, as at there again are most L different vectors. The zero vector represents the linear combination with entries bounded by β and we can easily reconstruct the short lattice vector $\mathbf{v}$ with $\|\mathbf{v}\|_\infty \leq \beta$.

To find an optimal k, we iterate over k starting from $k = 1$ to minimize the following expression:

$$\Delta = \operatorname{abs}\left(\frac{2^k}{k+1} - \frac{m\log(2\beta+1)}{n\log(q)}\right). \tag{3.16}$$

When Δ does not decrease we for 10 iteration steps, we stop and return k for the lowest value of Δ.

In our tool, we include two estimates of the cost of the algorithm. The overall runtime is dominated by the size of the sets L. Variants of the algorithm may speed up various steps and hence in our first, more conservative, estimate algorithm `algorithms.combinatorial_conservative()` we neglect the cost of single operations in the algorithm and just set the cost to the list size L. We obtain a more realistic estimate by considering the number of operations needed to create the initial sets. Each of the 2^k lists contains L vectors. The cost for any operation on a list element is at least $\log_2(q) \cdot n$. Hence, the total cost is $2^k \cdot L \cdot \log_2(q) \cdot n$. This second estimate is included in our tool in `algorithms.combinatorial()`.

4 Lattice Parameter Estimation Tool

We will now proceed to describe our tool in more detail. The core part of the Python library is a function that generically searches for secure parameters for both LWE and SIS instances, as well as for ring and module variants. Moreover, some schemes depend on the statistical security of either LWE or SIS. We hence include classes to find parameters satisfying this criteria as well as basic variants of LWE and SIS. Furthermore, we provide a set of utility classes and methods for the most used distributions and norms. A configuration class allows for a substantial customization of the estimation process. The tool can either estimate the bit security level of fixed parameter sets or generically search for parameter sets that satisfy a certain bit security level.

4.1 Supported Distributions

The secret distribution used in encryption schemes based on LWE or SIS can be either distributed according to a Gaussian or a uniform distribution. The error distribution in LWE must be a Gaussian distribution with parameter α. In general, if both error and secret in LWE are sampled according to Gaussian distributions, their parameters may differ. Currently, however, the *estimator* only supports secrets that follow a uniform distribution or a Gaussian distribution with the same parameter as the error distribution.

4.1.1 Gaussian Distribution

In some applications, we receive a Gaussian distribution as input but require a bound in some norm to estimate the hardness of an SIS instance. Hence, we need to transform a Gaussian with parameter $s = \sqrt{2\pi}\sigma$ into a bound β given some security parameter sec. Note that a n-dimensional Gaussian $D_{\mathbb{Z}^n,s}$ can be sampled by combining samples from n independent one-dimensional Gaussians $D_{\mathbb{Z},s}$ [44].

For a Gaussian distribution and a random variable X with $X \sim D_{\mathbb{Z},s}$, the following inequality holds for any $k > 0$ with $\beta = k\sigma$ [59, Lemma 4.4]:

$$\Pr\left[|X| > k\sigma\right] \leq 2e^{\frac{-k^2}{2}} \iff \Pr\left[|X| > \beta\right] \leq 2e^{\frac{-\beta^2}{2\sigma^2}} \tag{4.1}$$

$$\iff \Pr\left[|X| > \beta\right] \leq 2e^{\frac{-\pi\beta^2}{s^2}} \tag{4.2}$$

$$\tag{4.3}$$

We demand $2e^{-\pi\beta^2/s^2} \leq 2^{-\mathsf{sec}}$ and obtain

$$
\begin{aligned}
2e^{-\pi\beta^2/s^2} \le 2^{-\mathsf{sec}} &\iff -\pi\frac{\beta^2}{s^2} \le (-\mathsf{sec} - 1)\ln(2) \\
&\iff \beta \ge s\sqrt{\frac{(\mathsf{sec}+1)\ln(2)}{\pi}}.
\end{aligned}
$$

Theorem 5 (Gaussian to Bound)
Given a Gaussian distribution $D_{\mathbb{Z}^n,s}$ with width parameter $s = \sqrt{2\pi}\sigma$ and a security parameter sec*, we can compute a bound β such that a sample $\mathbf{v}$ drawn from $D_{\mathbb{Z}^n,s}$ satisfies Pr $[\|\mathbf{v}\|_\infty \ge \beta] \le 2^{-\mathsf{sec}}$ as follows:*

$$
\beta = s\sqrt{\frac{(\mathsf{sec}+1)\ln(2)}{\pi}}. \tag{4.4}
$$

Analogously, if we require a bound in the ℓ_2-norm, we have that $\Pr\left[\|X\|_2 > k\sigma\sqrt{n}\right] \le k^n e^{\frac{n}{2}(1-k^2)}$, for an n-dimensional Gaussian $D_{\mathbb{Z}^n,s}$ and a random variable X with $X \sim D_{\mathbb{Z}^n,s}$, for any $k > 1$ [59, Lemma 4.4]. We set $k = \sqrt{2}$ and obtain

$$
\begin{aligned}
\Pr\left[\|X\|_2 > \sigma\sqrt{2n}\right] \le 2^{\frac{n}{2}} e^{\frac{n}{2}(1-2)} &= 2^{\frac{n}{2}} 2^{-\log e \frac{n}{2}} \\
&= 2^{\frac{n}{2}(1-\log e)}
\end{aligned}
$$

If $2^{\frac{n}{2}(1-\log e)} \le 2^{-\mathsf{sec}}$, we take $\sigma\sqrt{2n}$ as our bound β. Otherwise, we bound the ℓ_2-norm of β by the ℓ_∞-norm bound from Theorem 5 as described in Section 4.2.

We provide the classes `GaussianAlpha`, `GaussianSigma` and `GaussianS` to allow for the most flexibility in specifying a Gaussian distribution.

4.1.2 Uniform Distribution

For uniform distributions, we support all distributions that are supported by the *estimator*, namely, uniform modulo q, uniform in the interval $[a, \dots, b]$ and a sparse uniform distribution with parameters $((a, b), h)$, where exactly h components are in the interval $[a, \dots, b] \setminus \{0\}$ and all other components are zero. Note that we only consider discrete distributions are over $\mathbb{Z}$.

We can estimate the corresponding standard deviation for a Gaussian distribution by computing the variance of the uniform distribution. Given a lower and upper bound (a, b), the variance for a discrete uniform distribution is defined as $\sigma^2 = \frac{(b-a+1)^2-1}{12}$. For a uniform distribution modulo q we set $a = -\left\lfloor \frac{q}{2} \right\rfloor, b = \left\lfloor \frac{q}{2} \right\rfloor$.

For a sparse discrete n-dimensional uniform distribution $\mathcal{U}_h$ with an additional sparseness parameter h and a random variabel $X \sim \mathcal{U}_h$, we can compute the expected values $\mathbb{E}(X^2)$ and $\mathbb{E}(X)^2$ as follows [13]:

$$\mathbb{E}(X^2) = \frac{h}{n} \cdot \frac{2b^3 + 3b^2 + b - 2a^3 + 3a^2 - a}{6(b-a)} \tag{4.5}$$

$$\mathbb{E}(X)^2 = \frac{h}{n} \cdot \frac{b(b+1) - a(a-1)}{2(b-a)} \tag{4.6}$$

and obtain the variance $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.

4.2 Norm Bound Inequalities

In practical scenarios, oftentimes we require that the bound of a result does not exceed a certain value to guarantee correctness. In addition, we have seen that estimation algorithms rely on bounds in different ℓ_p-norms. For example, the combinatorial attack in Section 3.3.2 needs a bound in ℓ_∞ norm, whereas the dual attack in Section 3.3.1 uses bounds in the Euclidean norm. We thus need a way to bound a value of a bound in one norm by a value in a different norm. In our tool, we define a norm class with parameter p and provide a method `to_Lp()` for conversions into all other norms mentioned here. We begin by the following Theorem.

Theorem 6
Let $f \in \mathcal{R}_q$ with $f = \sum_i f_i X^i$ as in in [21] and $p, q \in \mathbb{N}$ with $\infty \geq p \geq q \geq 1$. Then the following inequation holds:

$$\|f\|_p \leq \|f\|_q. \tag{4.7}$$

Let $p, q \in \mathbb{N}$ with $1 \leq p \leq q \leq \infty$. Then the following inequation holds:

$$\lim_{q' \to q} \|f\|_p \leq \lim_{q' \to q} n^{\frac{1}{p} - \frac{1}{q'}} \|f\|_{q'}. \tag{4.8}$$

It is easy to see from the definition of the norms in Section 2.2 that Equation (4.7) is true. Equation (4.8) follows from Hölder's inequality. We proof the statement in Appendix B.1.

In addition, oftentimes we consider norms in the canonical embedding. Let $\mathcal{O}_K$ be the ring of integers of a number field $K = \mathbb{Q}(\theta)$, where θ is an algebraic number and σ denote the canonical embedding as defined in [34]. Then, according to [34, Theorem 7] for $f \in \mathcal{O}_K$ the following inequations hold:

$$\|f\|_\infty \leq \|\sigma(f)\|_\infty, \tag{4.9}$$

$$\|\sigma(f)\|_\infty \leq \|f\|_1. \tag{4.10}$$

(We assume that the constant C_m used in the original Theorem is 1 for m a power of 2 [34, Lemma 3].) If we combine Equations (4.9) and (4.10) respectively with Equation (4.8) we obtain the following theorem.

Theorem 7
Let $f \in \mathcal{O}_K$ as in in [34]. Then, the following two inequations hold:

$$\|\sigma(f)\|_\infty \leq \|f\|_1 \leq n^{1-\frac{1}{p}} \|f\|_p, \text{ and} \tag{4.11}$$

$$\|f\|_p \leq n^{\frac{1}{p}} \|f\|_\infty \leq n^{\frac{1}{p}} \|\sigma(f)\|_\infty \tag{4.12}$$

We can thus bound a given bound in the norm ℓ_q for some q by a bound in some ℓ_p norm using the above inequalities and analogously in the embedding norm, which we call C_p. We combine both results in the method `to_Lp()` and `to_Cp()` in the norm classes `Lp` and `Cp` respectively. Note that the embedding norm C_p works internally like the ℓ_p norm, as we define both for vector spaces of dimension n. We merely need the results in Theorem 7 for the conversion between both norms.

We also want to be able to estimate bounds on the result of addition and multiplication of vectors of a bounded length in different norms. Note that the degree of the polynomial of the underlying ring to which we apply the norms must match. For addition, we simply bound the second addend by the used norm of the first as described in Theorems 6 and 7 and add both bounds to obtain a bound on the result in the norm of the first addend. It is slightly more complicated for multiplication, and we state the results in the next theorem.

Theorem 8 (Multiplication of Norm Bounds [21, 34])
Let f be defined as above and let $g \in \mathcal{R}_q$ where $g = \sum_i \overline{g}_i X^i$ where $g_i \in [-(q-1)/2, (q-1)/2]$ and $\overline{g}_i = g_i \mod q$ as in [21]. We then can define the following inequations for multiplication according to [21]:

$$\begin{aligned} \|f \cdot g\|_\infty &\leq \|f\|_\infty \cdot \|g\|_1, \\ \|f \cdot g\|_\infty &\leq \|f\|_2 \cdot \|g\|_2. \end{aligned} \tag{4.13}$$

Let $x, y \in \mathcal{O}_K$. Then, the following inequation holds according to [34]:

$$\|\sigma(x \cdot y)\|_p \leq \|\sigma(x)\|_\infty \cdot \|\sigma(y)\|_p. \tag{4.14}$$

(Again, we assume that $C_m = 1$ in the original statement.)

If Equation (4.13) does not apply and the bounds for both vectors that we want to multiply is given in some ℓ_p-norm, we simply convert both bounds to the C_∞-norm as described in Equation (4.11) and apply Equation (4.14) with $p = \infty$. In the case that we have an ℓ_p-norm and a C_q-norm for some p, q, we similarly convert the ℓ_p-norm to the C_∞-norm and apply Equation (4.14). If both bounds are C_p-norm bounds with $p < \infty$, we apply Equation (4.14) twice, once with the first bound converted to the C_∞-norm and once with the second norm converted to the C_∞-norm, and take the best value as our result bound.

4.3 Problem Classes

We now present the problem classes in `lattice_parameter_estimation/problem` (see Figure 4.1). `LWE` and `SIS` inherit from the base class `BaseProblem`. All instances provide a method `get_estimate_algorithms()` that returns a list of algorithm instances that can be executed by the function `estimate()`. Furthermore, any instance of `BaseProblem` can be compared to a bit security level (for more details, we refer the reader to the documentation). The `LWE` class is initialized by the secret dimension n, a modulus q, the number of samples m, a `secret_distribution` and a `error_distribution`. Both `secret_distribution` and `error_distribution` must be instances of the class `distributions.Distribution`. Instead of `secret_distribution` and `error_distribution`,

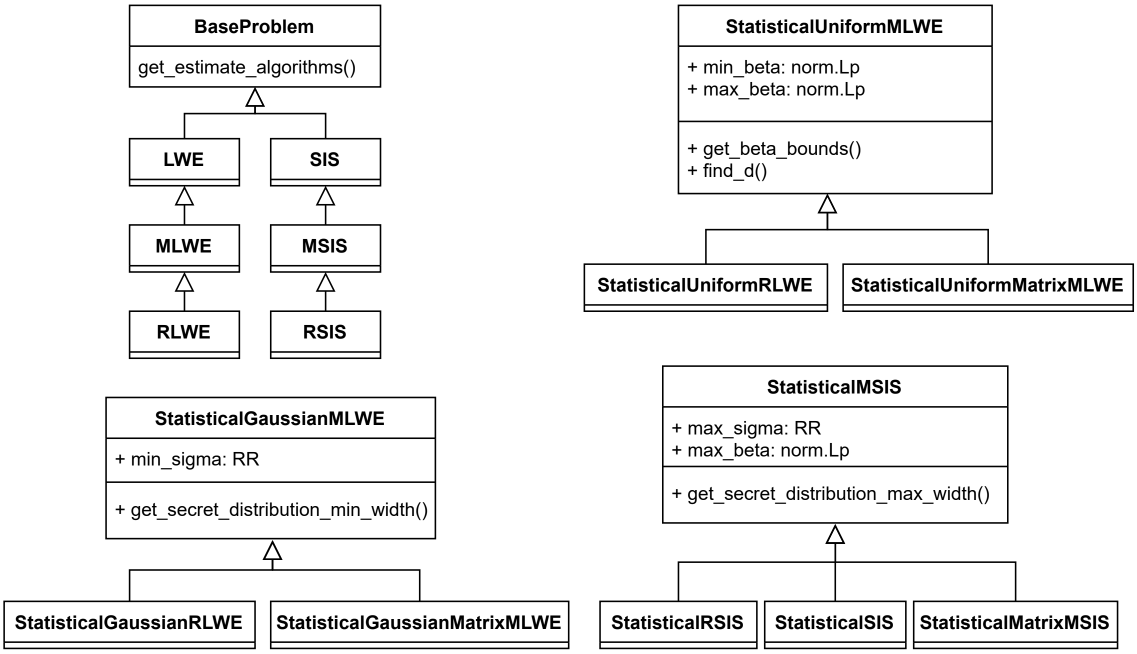


Figure 4.1: Problem classes

a bound of type `norm.BaseNorm` must be set for `SIS`. Note that both `distributions.Uniform` and `distributions.Gaussian` are instances of `norm.BaseNorm` and can thus be used as a bound. We compute the bound for a given distribution instance as described in Section 4.1.

For ring and module variants `RLWE`, `RSIS` and `MLWE`, `MSIS` respectively, n denotes the degree of the polynomial of the underlying ring $\mathcal{R}_q$. The module variants `MLWE` and `MSIS` take an addition parameter d for the rank of the module.

While there exist special cases where the ring structure of problem instances can be exploited in an attack on LWE or SIS, in general, the hardness of ring and Module variants is estimated by interpreting the coefficients of elements of $\mathcal{R}_q$ as vectors in $\mathbb{Z}_q^n$ [8]. If we take into account the considerations we presented in Section 2.5.3, we can thus reduce ring and Module instances, when calling `get_estimate_algorithms()` on the ring and module variant of `LWE` and `SIS`, as follows:

- $\text{RLWE}_{n,q,m,\chi} \longrightarrow \text{LWE}_{n,q,m\cdot n,\chi}$
- $\text{MLWE}_{n,d,q,m,\chi} \longrightarrow \text{LWE}_{n\cdot d,q,m\cdot n,\chi}$
- $\text{RSIS}_{n,q,m,\beta} \longrightarrow \text{SIS}_{n,q,m\cdot n,\beta}$
- $\text{MSIS}_{n,d,q,m,\beta} \longrightarrow \text{SIS}_{n\cdot d,q,m\cdot n,\beta}$

In addition to the base problem variants, we define some statistically secure variants for both LWE and SIS, since some schemes depend on the unconditional hardness of either LWE or SIS. More precisely, we ask for parameters such that an arbitrary powerful attacker can only break the scheme with probability less than $2^{-\mathsf{sec}}$.

StatisticalGaussianMLWE. For LWE, we define a statistically secure variant over a Gaussian distribution and over a uniform distribution. `StatisticalGaussianMLWE` follows Corollary 7.5 and Theorem 7.4 in [63]. The mapping of the parameters in [63] to the usage in this work can be found in Table C.1. We obtain the following theorem:

Theorem 9 (Statistically Secure MLWE Over a Gaussian Distribution [63])
Let $\mathcal{R}$ be the ring of integers in the m'th cyclotomic number field K of degree n, and $q \geq 2$ an integer. For positive integers $m \leq m + d \leq poly(n)$, let $\mathbf{A} = [\mathbf{I}_{[m]} \mid \bar{\mathbf{A}}] \in (\mathcal{R}_q)^{[m]\times[m+d]}$, where $\mathbf{I}_{[m]} \in (\mathcal{R}_q)^{[m]\times[m]}$ is the identity matrix and $\bar{\mathbf{A}} \in (\mathcal{R}_q)^{[m]\times[d]}$ is uniformly random. Then with probability $1 - 2^{-\Omega(n)}$ over the choice of $\bar{\mathbf{A}}$, the distribution of $\mathbf{Ax} \in (\mathcal{R}_q)^{[m]}$ where each coordinate of $\mathbf{x} \in (\mathcal{R}_q)^{[m+d]}$ is chosen from a discrete Gaussian distribution of parameter $s > 2n \cdot q^{m/(m+d)+2/(n(m+d))}$ over $\mathcal{R}$, satisfies that the probability of each of the q^{nm} possible outcomes is in the interval $(1 \pm 2^{-\Omega(n)})q^{-n}$ (and in particular is within statistical distance $2^{-\Omega(n)}$ of the uniform distribution over $(\mathcal{R}_q)^{[m]}$).

If a security parameter is passed and `sec` $> n$, we raise an exception. The resulting minimal standard deviation is stored in the instance variable `min_sigma` and the corresponding distribution can be obtained by calling `get_secret_distribution_min_width()` on the class instance.

StatisticalUniformMLWE. The authors of [21] describe statistically secure MLWE instances over a Uniform distribution with invertible elements. The samples $(\mathbf{A}', h_{\mathbf{A}'}(y))$ of the resulting MLWE instance are within statistical distance $2^{-\mathsf{sec}}$ of $(\mathbf{A}', \mathbf{u})$ for uniformly distributed $\mathbf{u}$.

We obtain the following theorem (for a mapping of the parameters, see Table C.2):

Theorem 10 (Statistically Secure MLWE Over a Uniform Distribution [21, Lemma 4])
Let $1 < d_2 < n$ be a power of 2. If q is a prime congruent to $2d_2 + 1 \pmod{4d_2}$ and

$$q^{m/(m+d)} \cdot 2^{2\mathsf{sec}/((m+d)\cdot n)} \leq 2\beta < \frac{1}{\sqrt{d_2}} \cdot q^{1/d_2} \tag{4.15}$$

then any (all-powerful) algorithm $\mathcal{A}$ has advantage at most $2^{-\mathsf{sec}}$ in solving $DKS^{\infty}_{m,m+d,\beta}$, where DKS^{∞} is the decisional knapsack problem in ℓ_∞-norm.

Note that we replaced the fixed value of 128 in the original Lemma with `sec`. The proof is essentially the same as in [21] and we will therefore refrain from stating it again here.

Hence, we have:

$$\beta_{min} = \frac{q^{m/(m+d)} \cdot 2^{2\mathsf{sec}/((m+d)\cdot n)}}{2} \tag{4.16}$$

$$\beta_{max} = \frac{1}{2\sqrt{d_2}} \cdot q^{1/d_2} - 1 \tag{4.17}$$

The variable d_2 can be passed as an argument. If it is not passed, we try to find d_2 by iterating over all powers of 2 that are smaller than n. The resulting bounds are converted to ℓ_∞ and stored in the instance variables `min_beta` and `max_beta`. We also provide an instance method `get_beta_bounds()` to obtain a tuple of both.

For both statistically secure MLWE variants, we include the corresponding ring versions `StatisticalGaussianRLWE` and `StatisticalUniformRLWE` for $d = 1$ and matrix versions `StatisticalGaussianMatrixMLWE` and `StatisticalUniformMatrixMLWE` for which the width and height of the matrix $\mathbf{A}$ in [63] can be passed instead of m and d.

`StatisticalMSIS`. We can find parameters for a statistically secure MSIS instance by following Section 4.1 of [33]. The mapping of the parameters in [33] is shown in Table C.3. More specifically, we ask to find a MLWE instance where the probability that non zero elements $\mathbf{r}$ in the Euclidean ball $B_m(0, 2B)$ satisfy $\hat{\mathbf{A}}_1 \cdot \mathbf{r} = \mathbf{0}$ is smaller than $2^{-\text{sec}}$.

The number of elements in $B_{m+d}(0, 2B)$ can be estimated from above as $|B_{m+d}(0, 2B)| \ll (2\pi e/((m+d)n))^{(m+d)n/2} \cdot (2B)^{(m+d)n}$. The scheme is statistically binding if the probability that non zero elements in $B_{m+d}(0, 2B)$ of radius $2B$ in $\mathcal{R}_q^{m+d}$ map to $\mathbf{0}$ in $\mathcal{R}_q^m$ is negligible. Hence, it must hold that $|B_{m+d}(0, 2B)|/q^{mn} \leq 2^{-\text{sec}}$ and we get:

$$\left(\sqrt{\frac{2\pi e}{(m+d)\cdot n}} \cdot 2B\right)^{(m+d)\cdot n} \leq 2^{-\text{sec}} \cdot q^{m \cdot n} \tag{4.18}$$

$$B \leq 2^{\frac{-\text{sec}}{(m+d)\cdot n} - 1} \cdot q^{\frac{m}{m+d}} \cdot \sqrt{\frac{(m+d)\cdot n}{2\pi e}} \tag{4.19}$$

We convert the bound B to a Gaussian over ℓ_2-norm by following the procedure described in Section 4.2:

$$s \approx x \sqrt{\frac{\pi}{(\text{sec} + 1)\ln(2)}} \tag{4.20}$$

The resulting parameters B and s can be accessed by the instance variables `max_sigma` and `max_beta` or by calling `get_secret_distribution_max_width()` on the class instance.

As for statistically secure MLWE, we again include a matrix version `StatisticalMatrixMSIS` and a ring `StatisticalRSIS` by setting $d = 1$. In addition, the proof also applies to the base SIS variant and hence we include `StatisticalSIS`. Here, the height of the matrix n becomes the rank of the modulus in the MSIS instance, i.e., $d = n$, and the degree of the polynomial is 1.

4.4 Parameter Search

The main functionality of our tool is to search for secure parameter sets given a set of problem instances and is encapsulated in the function `param_search.generic_search()`. It is also possible to directly estimate the cost of a list of parameter problems by calling the function `problem.estimate()`. For more details, we refer to the documentation.

The high-level idea of the search is presented in Algorithm 6. We begin with an initial parameter set. We then create a list of problem instances generated by a `parameter_problem` function and estimate the cost of all instances in the list by calling `problem.estimate()`. If the list contains multiple SIS instances or multiple LWE instances, we first attempt to reduce the instances to the easiest problem instance respectively. The reduction is not exhaustive and is realized by a simple comparison of the parameters n, q, m and α or bound β according to the following intuitive hardness results. For LWE, we have that the larger n and α are and the smaller q and m are, the harder the problem becomes. For SIS, the problem becomes harder for increasing n and q and decreasing β and m. The function then generates a list of estimation algorithm instances for each remaining problem instance and the resulting list is executed by an algorithm executor. The list is ordered according to expected runtime and output quality of the specified algorithms (see Section 4.5). If the estimation is configured to be parallel, the algorithms are split up into sublist and executed concurrently on multiple processors. Once we find an insecure problem instance (i.e., the estimated attack cost in clock cycles is smaller than 2^{sec}), we terminate the estimation procedure and `estimate()` returns a result that is labeled insecure. We then use the `next_parameters` function to generate a list L of (multiple) new parameter sets from our current parameter set and sort each of the new parameter sets into an ordered list with duplicate detection. The order is defined by a `parameter_cost` function. In the next step, we retrieve the parameter set with the lowest cost from L and repeat the procedure until the cost estimation step returns a secure result. The result includes the estimates for all cost models and algorithms.

Algorithm 6: Generic Search

```
Input: sec, initial_params, next_parameters, parameter_cost, problem_instance
L = OrderedList(initial_params)
while $L \neq \emptyset$ do
    current_params = L.pop()
    instances = parameter_problem(current_params)
    result = estimate(instances, sec)
    if result is secure then
        return (result, current_params)
    else
        next_param_sets = next_parameters(current_params)
        forall param_set in next_param_sets do
            sort param_set into L according to parameter_cost function
```

4.5 Configuration Options

The configuration of the estimation can be customized via the class `algorithms.Configuration`, which is passed as an optional argument of `param_search.generic_search()`.

We noted that the `estimate()` function can be configured to run in parallel, which may speed up the search, in particular if many cost models need to be tested (e.g., with configuration setting `conservative=False` and long running algorithms like `ARORA_GB`, `CODED_BKW` and `PRIMAL_DECODE` are used (see Section 4.5). The list of used algorithms can be changed in the configuration. We recommend to include at least `PRIMAL_USVP` for LWE instances and `LATTICE_REDUCTION` for SIS instances (as in the default configuration) to make full use of early termination, since the estimate algorithms for these attacks have a short runtime and yield relatively low cost estimates. A user can also specify a security strategy. We included three strategies, namely, `ALL_SECURE`, `SOME_SECURE` and

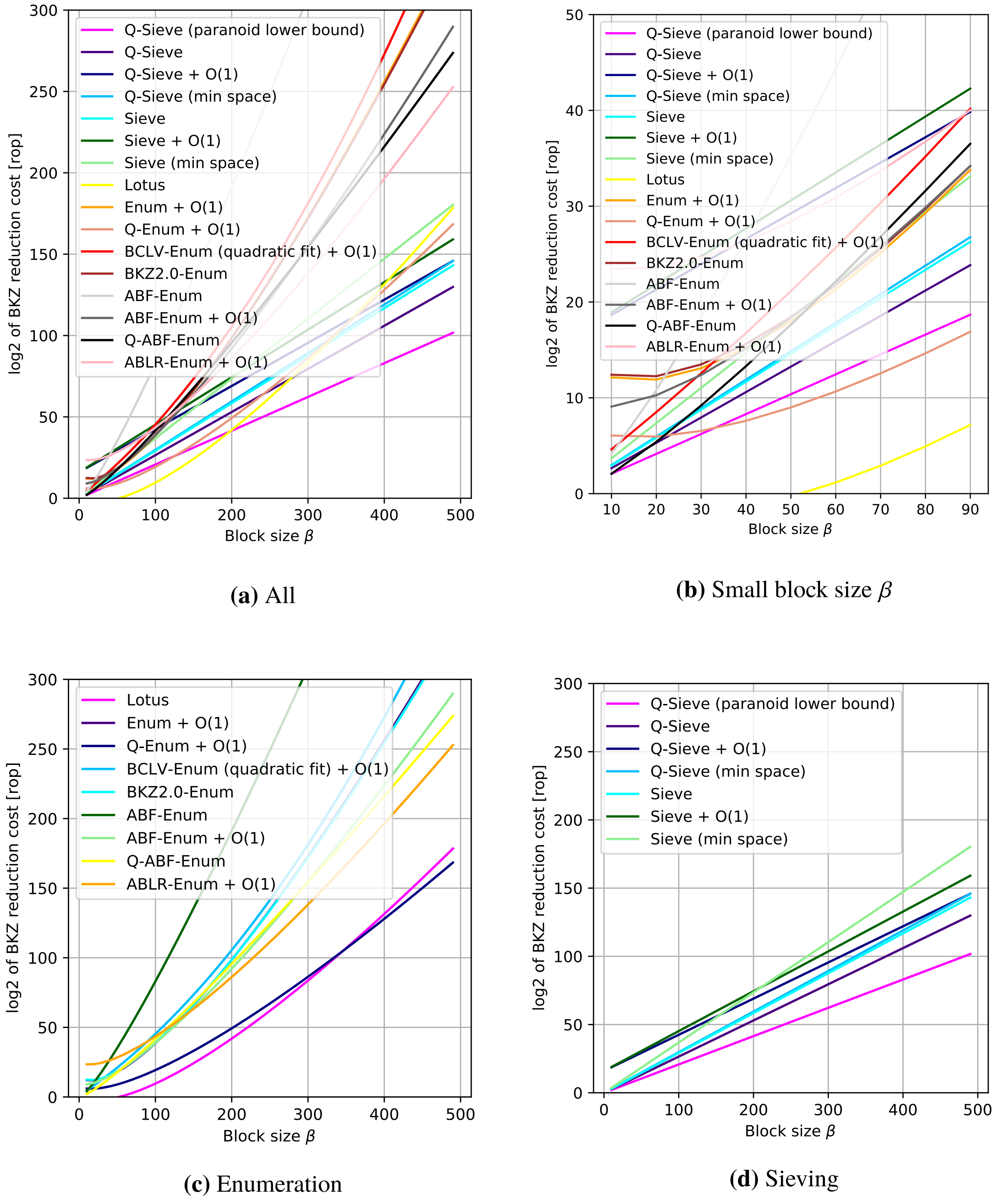
NOT_INSECURE. For ALL_SECURE, the search only terminates if all specified algorithms return a cost that satisfies the security level. Depending on the choice of algorithms, the search may not terminate since, for example, Arora-GB for larger n takes too long and is therefore aborted once the timeout is reached and cannot return a cost estimate. SOME_SECURE demands that at least one algorithm returns a secure estimate for each problem instance for a given parameter set. NOT_INSECURE only demands that no algorithm returns an insecure cost. We recommend using SOME_SECURE (as in the default configuration).

Cost Models. Attacks that use BKZ for lattice reduction require a cost model to estimate the number of CPU cycles in the SVP subroutine. Default cost models are shown in Table 3.2. We distinguish between estimates for classical, quantum, sieving and enumeration and each of these categories can be deselected by setting the respective parameter to False. At least one of classical and quantum and of sieving and enumeration respectively must be selected to make use of the default cost models.

Note that classical and quantum cost models cannot be directly compared with each other. The number of operations per second that can be executed by a quantum computer may be significantly smaller than for classical computers.

If all are unselected, custom cost models must be specified and passed as an argument. We included an option of taking the most conservative estimate for each category combination for a more efficient parameter search or estimation. Furthermore, we assigned a priority value on an ordinal scale to each cost model, which enables us to first run cost models that yield a lower cost and thus terminate the estimation process earlier for an insecure parameter set. The priority values of the default cost models are derived from Figure 4.2. The number of BKZ rounds can be configured according to the more conservative Core-SVP model algorithms.BKZ_SVP_repeat_core [14] in which the polynomial factor of the runtime complexity of BKZ is completely ignored. In addition, we provide a more realistic model algorithms.BKZ_SVP_repeat_8d for a BKZ cost of $8 \cdot d \cdot t_k$ BKZ rounds, where d again referes to the lattice dimension and t_k is the number of clock cycles required for the SVP subroutine (see Line 17).

Algorithms. Figures 4.3 to 4.5 show the plots of runtime and performance tests for the various algorithms that can be used in our tool. The parameters of SIS in Figure 4.3 are derived from [66], the parameters of LWE in Figure 4.5 are based on [82]. We chose the parameters of Figure 4.4, such that most algorithms yield a result. Note that Arora-GB in Figure 4.4 and Meet-in-the-Middle and Coded-BKW for higher dimensions in Figure 4.5 return a cost of ∞ and therefore do not appear in the bit security plot. In Figure 4.5, computing the cost for Arora-GB for $n > 512$ takes longer than the configured timeout of 200s and does therefore also not show up in the plot. In accordance with the results, we assigned priority values on an ordinal scale to the estimation algorithms. In Tables 4.1 and 4.2, we present the list of algorithms and their corresponding priority values and justify our choice. Algorithms with a smaller priority value are expected to yield relatively good results quickly and can therefore be executed first. If the estimate result does not satisfy the specified security requirement, we can terminate the estimation process early in order to maximize the efficiency of our search. Note that, while convenient, directly comparing the results for different algorithms is not always admissible, as the compared algorithms may rely on different assumptions. Some algorithms may compute a more realistic cost, while others may return a more conservative or even paranoid



(a) All

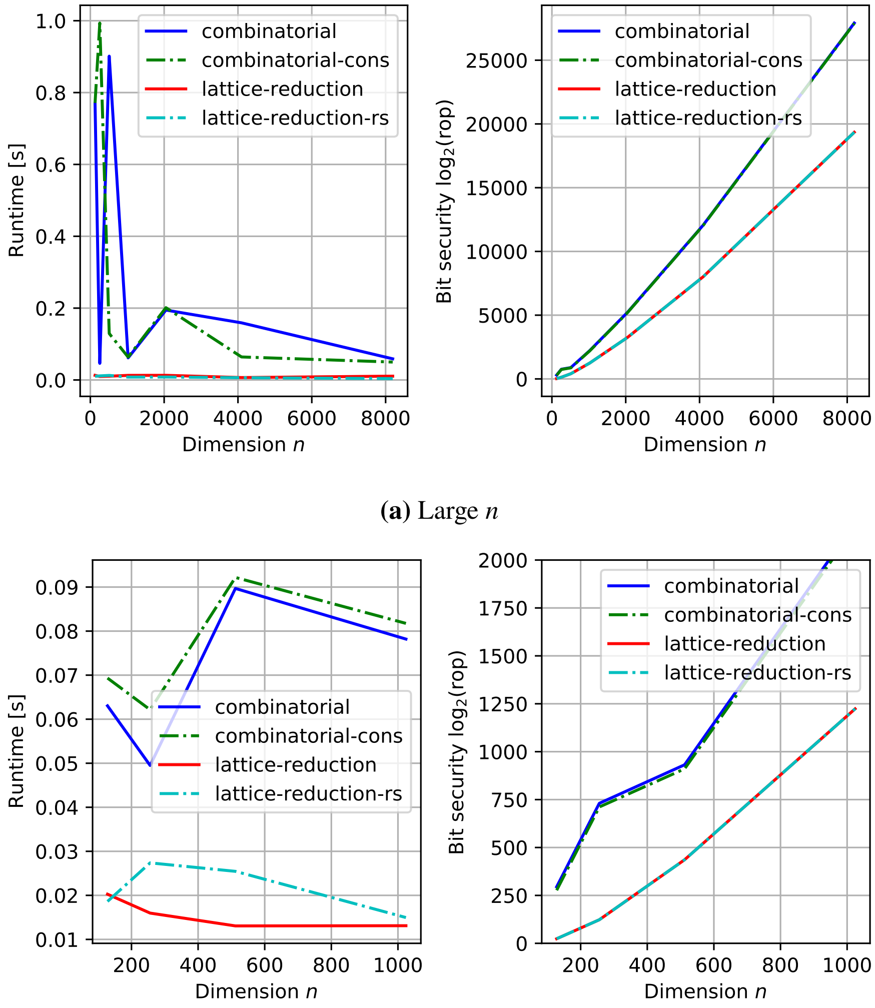
(b) Small block size β

(c) Enumeration

(d) Sieving

Figure 4.2: Cost models

cost estimate (e.g., BKZ is used with the Core-SVP paranoid lower bound). The results must be weighed carefully in order to guarantee the security of a given scheme in the forseeable future, while trying to keep the cost of using the scheme as low as possible. In the default configuration, we included `PRIMAL_USVP` for LWE and `LATTICE_REDUCTION` for SIS.



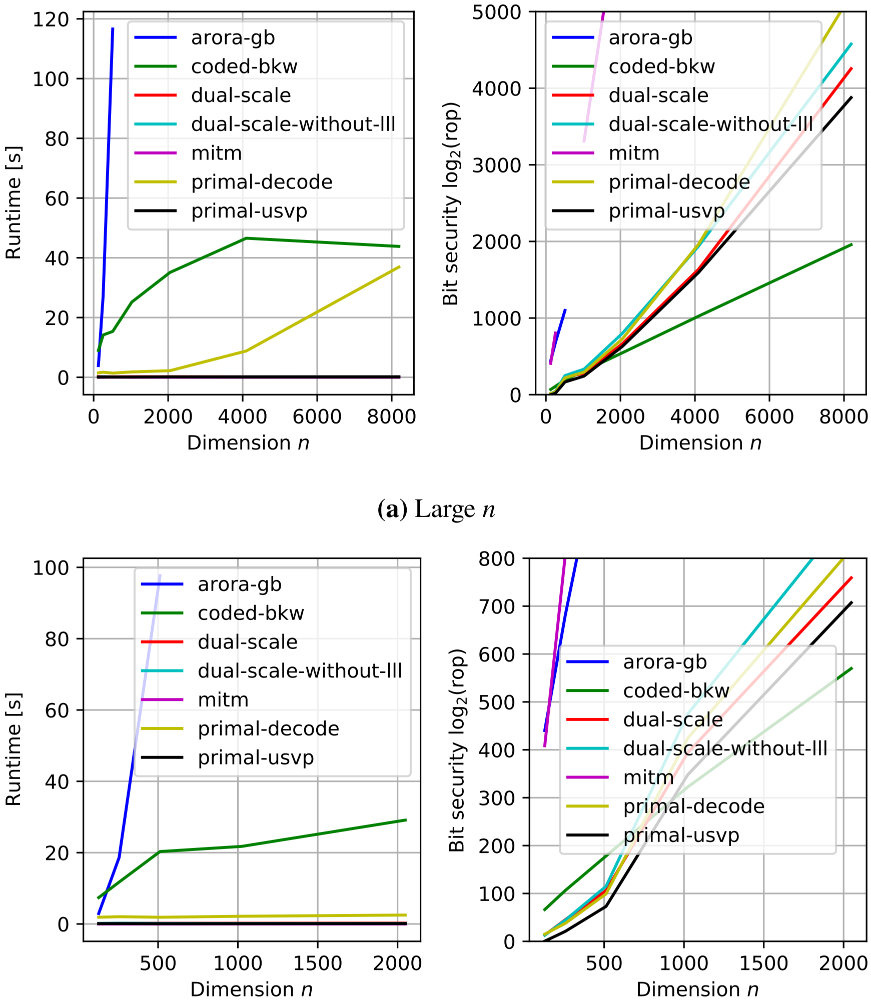
(a) Large n

(b) Small n

Figure 4.3: SIS with $n^2 < q < 2n^2,\ m = 2n\sqrt{n \log q},\ s = 2\sqrt{n \log q}$

4.6 Usage Examples

We provide several usage examples in the Python script `usage_example.py`. A simple estimation example for LWE and SIS show the application of the function `problem.estimate()` to estimate the security of problem instances for a fixed parameter set. Two simple parameter searches, one for LWE and one for SIS, explain the functionality of the function `param_search.generic_search()`. In both cases, as for the algorithm performance analysis in Figures 4.3 and 4.4, we derive the parameter choices from [82] and [66] for LWE and SIS respectively. In addition, our code includes two more advanced examples for schemes that are currently under research at the Institute of Information Security (SEC), University of Stuttgart. The first is a parameter search for variation of the BGV scheme based on [29, 34] that ensures accountability. The second finds parameters for a commitment scheme based on [21] in an advanced setting and combines LWE and SIS in one search. The examples demonstrate the usefulness of our tool. On the one hand, our tool provides



(a) Large n

(b) Small n

Figure 4.4: LWE with $\sigma = 2.828,\ m = \infty,\ n < q < 2n$

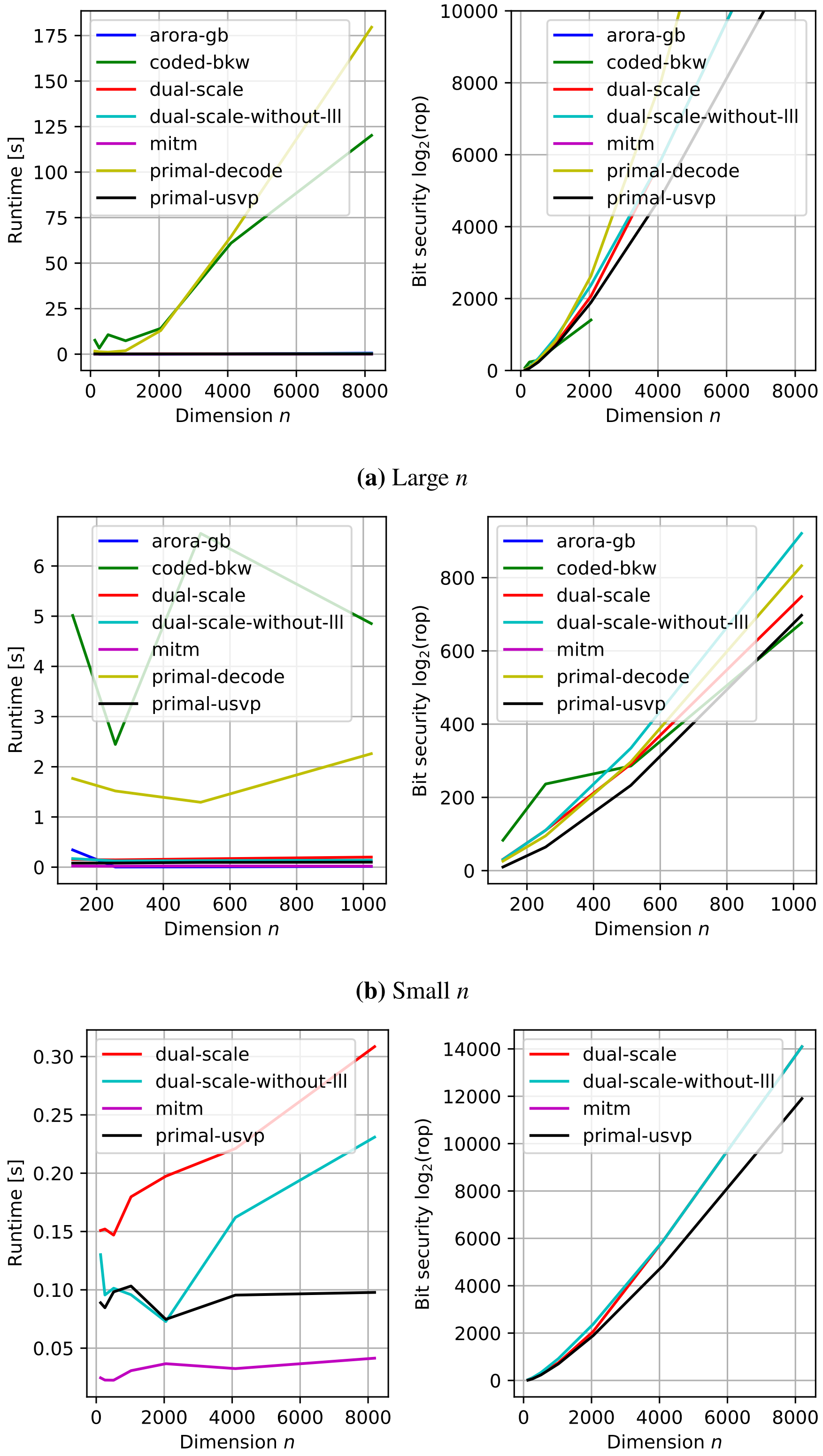
interfaces and default settings that are easy use without much background knowledge. On the other hand, the flexibility of the generic search allows for the combination of multiple primitive building blocks of a complex application, while maintining a simple and clear structure.

Algorithm	Priority	Justification
Meet-in-the-Middle	5	fastest, high cost estimate, as prefilter
Primal uSVP	10	fast, low cost estimatate estimates
Dual Attack	20	fast, often higher estimates than Primal uSVP
Dual Attack (without LLL)	30	fast, often higher estimates than Dual
Coded-BKW	90	slow, somtimes very low cost estimate (for small stddev), does not always yield results
Decoding Attack	100	slow, often higher estimates than faster algorithms
Arora-Ge	200	extremely slow, often higher estimates, does not always yield results

Table 4.1: Prioritization of LWE Algorithms

Algorithm	Priority	Justification
Lattice Reduction MR	1	fastest, low cost estimates
Lattice Reduction RS	2	same results as lattice-reduction, not always applicable
Combinatorial Attack	10	fast, often higher cost results
Combinatorial Conservative	9	fast, slighly better than Combinatorial

Table 4.2: Prioritization of SIS Algorithms



(a) Large *n*

(b) Small *n*

(c) Only algorithms with short runtime

Figure 4.5: LWE with parameters chosen as in [82]

5 Conclusion

In summary, we examined two important problems in modern cryptography, namely, LWE and SIS, and discussed various strategies and algorithms for solving both problems from a perspective of practical hardness estimation. Furthermore, we compiled a list of up-to-date reduction cost models from the literature. These cost models play a crucial role in predicting the behavior of attack algorithms that rely on a lattice reduction subroutine. We introduced a new unified and user-friendly tool that can be used to find secure parameters for a given cryptographic scheme. The tool includes all popular variants of LWE and SIS as well as respective cost estimation procedures, provides several utility classes and functions, and allows for detailed customization. A good choice of algorithms for initial parameter searches are Primal uSVP for LWE and Lattice Reduction for SIS, as they yield low cost estimates in sort time. The initial parameter set can then be adapted and the search run again with all algorithms for the final result.

Future Work. In the future, our tool could be further extended with more estimation algorithms, particularly, for SIS. We included a very basic version of a combinatorial attack on SIS in the dual lattice [67]. It should be possible to apply improvements to the BKW algorithm to the combinatorial attack and obtain more realistic estimates. Furthermore, the LWE Estimator [13] has not been updated over the last two years. Recently, more accurate estimates of the Primal uSVP attack have been published in [79]. In addition, we may see a major refactoring of the *estimator* in terms of how distributions are handled. In our tool, we introduced wrapper classes for this purpose. In the case of an update, our tool may not be compatible with the new version of the estimator anymore and will need to be adapted accordingly. Lastly, as of now, our tool does not provide a convenient way to specify custom estimate algorithms. This functionality could be included as part of the configuration class.

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All links were last followed on October 21, 2021.

A Additional Math

A.1 Linear Codes

Let $\mathbb{F}_q^n$ be the n-dimensional vector space over the field $\mathbb{F}_q$. A q-ary linear code C or $[n, k]$-code [92] is a k-dimensional linear subspace of $\mathbb{F}_q^n$ such that $\mathbf{0} \in C$, $\mathbf{x} + \mathbf{y} \in C$ for all $\mathbf{x}, \mathbf{y} \in C$ and $\gamma \mathbf{x} \in C$ for $\mathbf{x} \in C, \gamma \in \mathbb{F}_q$. There are q^k different codewords in C.

Let C be a q-ary linear $[n, k]$-code. The lattice over C [44] is defined as

$$\Lambda(C) = \{\mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{y} \in C : \mathbf{x} = \mathbf{y} \mod q\} . \tag{A.1}$$

Similarly, for a lattice $\Lambda(\mathbf{B})$ a lattice code C defined by $\Lambda(\mathbf{B})$ and a shaping region $\mathcal{V} \subset \mathbb{R}^n$ (e.g., the Voronoi region, see Equation (2.5)) is a subspace of $\mathbb{R}^n$ such that all codewords are lattice vectors in $\Lambda(\mathbf{B})$ within the region $\mathcal{V}$ [91]:

$$C = \{x \in \Lambda(\mathbf{B}) \mid x \in \mathcal{V}\} . \tag{A.2}$$

B Proofs

B.1 Proof of Norm Inequalities

Let $p, q \in \mathbb{N}$ with $1 \le p \le q \le \infty$. Then the following inequation holds:

$$\lim_{q' \to q} \|f\|_p \le \lim_{q' \to q} n^{\frac{1}{p} - \frac{1}{q'}} \|f\|_{q'}. \tag{B.1}$$

Proof. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\frac{1}{u} + \frac{1}{v} = 1$, Hölder's inequality states that

$$\begin{aligned}\sum_{k=1}^{n} |\mathbf{x}_k||\mathbf{y}_k| &\le \left(\sum_{k=1}^{n} |\mathbf{x}_k|^u\right)^{\frac{1}{u}} \left(\sum_{k=1}^{n} |\mathbf{y}_k|^v\right)^{\frac{1}{v}} \\ &= \left(\sum_{k=1}^{n} |\mathbf{x}_k|^u\right)^{\frac{1}{u}} \left(\sum_{k=1}^{n} |\mathbf{y}_k|^{\frac{u}{u-1}}\right)^{1-\frac{1}{u}}\end{aligned}$$

Following [49], we set $|\mathbf{x}_i| = |\mathbf{z}_i|^p$, $\mathbf{y}_i = 1$ and $u = \frac{q}{p} > 1$ and get

$$\begin{aligned}\sum_{k=1}^{n} |\mathbf{z}_i|^p = \sum_{k=1}^{n} |\mathbf{z}_i|^p |\mathbf{y} &\le \left(\sum_{k=1}^{n} |\mathbf{z}_i|^{p\frac{q}{p}}\right)^{\frac{p}{q}} \left(\sum_{k=1}^{n} 1^{\frac{q}{q-p}}\right)^{1-\frac{p}{q}} \\ &\le n^{1-\frac{p}{q}} \left(\sum_{k=1}^{n} |\mathbf{z}_i|^q\right)^{\frac{p}{q}}\end{aligned}$$

We now apply the definition of the ℓ_p-norm (see Section 2.2) and obtain

$$\begin{aligned}\|\mathbf{z}\|_p = \left(\sum_{k=1}^{n} |\mathbf{z}_i|^p\right)^{\frac{1}{p}} &\le \left(n^{1-\frac{p}{q}} \left(\sum_{k=1}^{n} |\mathbf{z}_i|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\ &= n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^{n} |\mathbf{z}_i|^q\right)^{\frac{1}{q}} \\ &= n^{\frac{1}{p}-\frac{1}{q}} \|\mathbf{z}\|_q\end{aligned}$$

□

C Tables

C.1 Mapping of Parameters in Referenced Papers

Parameters in [63]	Noation here	Represents
l	$m + d$	width of matrix $\mathbf{A}$
k	m	height of matrix $\mathbf{A}$

Table C.1: Parameter Mapping from [63]

Parameters in [21]	Noation Here	Represents
k	$m + d$	width of matrix $[\mathbf{I}_n\ \mathbf{A}']$
n	m	height of matrix $[\mathbf{I}_n\ \mathbf{A}']$
d	d_2	variable
N	n	degree of Ring polynomial

Table C.2: Parameter Mapping from [21]

Parameters in [33]	Noation Here	Represents
m'	$m + d$	width of matrix $\hat{\mathbf{A}}_1$
m	m	height of matrix $\hat{\mathbf{A}}_1$
B	B	norm-bound of secret
s	s	Gaussian width (not stddev)
N	n	degree of polynomial

Table C.3: Parameter Mapping from [33]

Declaration

I hereby declare that the work presented in this thesis is entirely my own and that I did not use any other sources and references than the listed ones. I have marked all direct or indirect statements from other sources contained therein as quotations. Neither this work nor significant parts of it were part of another examination procedure. I have not published this work in whole or in part before. The electronic copy is consistent with all submitted copies.

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