# A Complete Analysis and Design Framework for Linear Impulsive and Related Hybrid Systems

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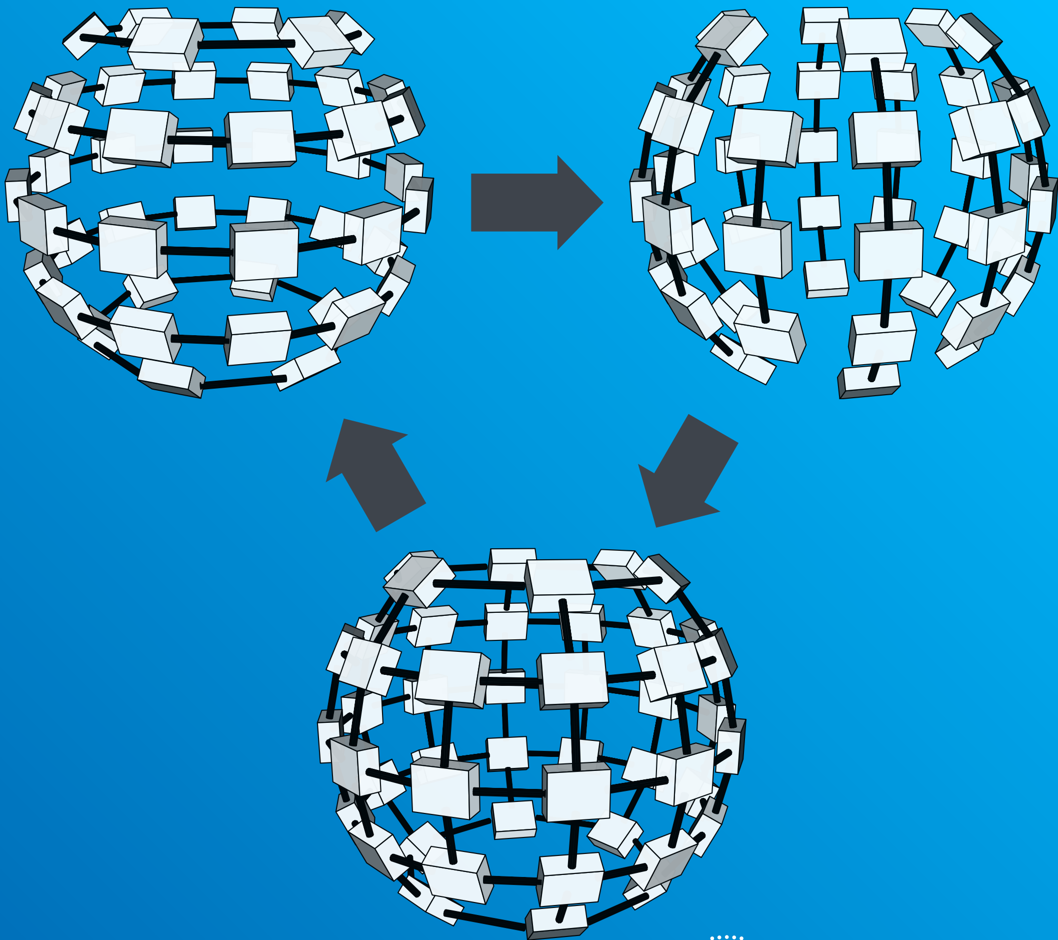

University of Stuttgart
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# A Complete Analysis and Design Framework for Linear Impulsive and Related Hybrid Systems

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Thank you

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# Abstract

We establish a framework for systematically analyzing and designing output-feedback controllers for linear impulsive and related hybrid systems that might even be affected by various types of uncertainties. In particular, the framework encompasses uncertain switched and sampled-data systems as well as networked systems with switching communication topologies.

The framework is based on recently developed convex criteria involving a so-called clock for analyzing impulsive systems under dwell-time constraints. We elaborate on the extension of those criteria for dynamic output-feedback controller synthesis by means of convex optimization and generalize the so-called dual iteration to impulsive systems. The latter originally and still constitutes a promising heuristic procedure for the challenging and non-convex design of static output-feedback controllers for standard linear time-invariant systems. Moreover, for uncertain impulsive systems as modeled in terms of linear fractional representations, we generalize the nominal analysis criteria by providing novel robust analysis conditions based on a novel time-domain and clock-dependent formulation of integral quadratic constraints. Finally, by combining the insights on nominal synthesis and robust analysis, we are able to tackle challenging output-feedback designs of practical relevance, such as the design of gain-scheduled, robust or robust gain-scheduled controllers for impulsive systems.

Most of the obtained analysis and synthesis conditions involve infinite-dimensional (differential) linear matrix inequalities which can be numerically solved by using relaxation methods based on, e.g., linear splines, B-splines or matrix sum-of-squares that we discuss as well.

## Keywords

Impulsive Systems, Output-Feedback Synthesis, Robust Analysis, Integral Quadratic Constraints, Robust Synthesis, Gain-Scheduling Control, Linear Matrix Inequalities

# Zusammenfassung

Ziel dieser Arbeit ist die Entwicklung von Werkzeugen zur systematischen Analyse von impulsiven linearen Systemen und dem Entwurf von Ausgangsrückführungsreglern für solche Systeme und verwandte Hybridsysteme selbst, wenn deren Modelle diverse Arten von Unsicherheiten aufweisen. Zu den verwandten Hybridsystemen gehören beispielsweise geschaltete Systeme, Systeme mit Datenabtastung und Netzwerksysteme mit geschalteten Kommunikationsstrukturen.

Die Basis unserer Überlegungen bilden die vor Kurzem entwickelten konvexen Analysekriterien für impulsive Systeme mit Verweilzeit, die eine Art von Taktgeber beinhalten. Wir diskutieren die Erweiterung dieser Kriterien zum Entwurf von dynamischen Ausgangsrückführungsreglern mit Hilfe von Techniken aus der konvexen Optimierung und verallgemeinern die sogenannte Dual-Iteration so, dass diese auf impulsive Systeme angewendet werden kann. Die Dual-Iteration ist ein vielversprechendes heuristisches Verfahren für den Entwurf von statischen Ausgangsrückführungsreglern für lineare zeitinvariante Systeme, der ein schwieriges und nichtkonvexes Problem darstellt. Außerdem etablieren wir neue Kriterien für die Robustheitsanalyse von impulsiven Systemen, die mit gebrochen linear eingehenden Unsicherheiten behaftet sind, basierend auf einer neuartigen Zeitbereichsformulierung von sogenannten quadratintegrablen Neben-

bedingungen. Schließlich kombinieren wir die gewonnenen Einsichten aus dem nominalen Reglerentwurf und der Robustheitsanalyse, um einige herausfordernde und praktisch höchst relevante Reglerentwurfsprobleme für impulsive Systeme zu lösen. Dazu gehören beispielsweise der Entwurf von robusten, gain-scheduled[1] oder robusten gain-scheduled Reglern.

Beinahe alle erarbeiteten Analyse- und Entwurfskriterien bestehen aus unendlichdimensionalen linearen Matrix(differential)ungleichungen, zu deren numerischen Lösung wir verschiedene Relaxationen verwenden. Konkret diskutieren wir Verfahren, die auf linearen Splines, B-Splines oder Matrixquadratsummen basieren.

[1] Dieser Begriff is nicht vernünftig übersetzbar. Er beschreibt eine spezielle Art der Adaptivität.

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# 1

# Introduction

## 1.1 Motivation

### Hybrid and Linear Impulsive Systems

Hybrid systems are dynamical systems that evolve continuously but also undergo instantaneous changes at certain events. They, hence, admit a combination of continuous-time and discrete-time dynamics which makes their study interesting and challenging. One of the most intuitive and classical examples of a hybrid system is given by a bouncing ball as depicted on the right and which can be modeled as follows. While the ball is in the air, it behaves continuously as described by the differential equations

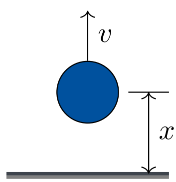


Figure 1.1: A bouncing ball.

$$\dot{v}(t) = -g \quad \text{and} \quad \dot{x}(t) = v(t)$$

where $g = 9.81\,\mathrm{m/s^2}$ denotes the acceleration

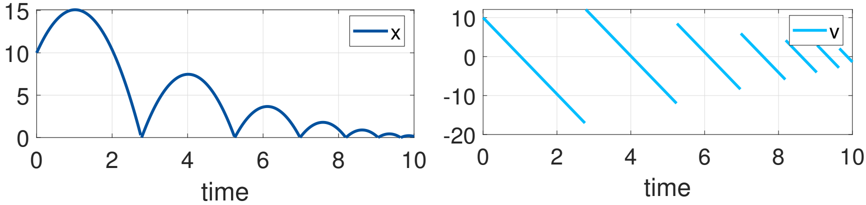



Figure 1.2: (Left) position $x$ and (right) velocity $v$ of the bouncing ball.

due to gravity, $x(t)$ is the ball's distance from the ground (in m) and $v(t)$ is the ball's velocity (in m/s) at time $t$. The hybrid nature of this example becomes apparent when modeling the collision of the ball with the ground. If one assumes a partially elastic collision with the ground, then the velocity before and after the collision (denoted by $v^-$ and $v^+$, respectively) can be related by

$$v^+ = -cv^-$$

where $c = 0.7$ is the ball's coefficient of restitution that is related to its deformation. Hence, the ball's velocity experiences a jump (or an impulse) whenever the ground is hit, i.e., whenever $x(t) = 0$ for some time instance $t$. This is confirmed in a simulation of the bouncing ball as shown in Fig. 1.2 where it is thrown up with a velocity of $v(0) = 10\,\mathrm{m/s}$ from a height of $x(0) = 10\,\mathrm{m}$.

Hybrid systems are practically highly relevant since they are encountered in numerous real-world applications related, e.g., to the fields of embedded control, fault tolerant control, traffic flow optimization, power electronics, robotics and system biology. Consequentially, considerable attention has been devoted to their investigation in the past years [63, 64, 88, 103]. Particularly in embedded control, one deals with interconnections of to-be-controlled systems and controllers modeled in continuous-time by differential equations with components for logical decision making that are

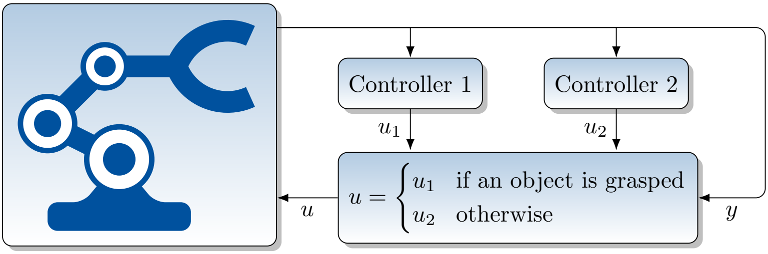



Figure 1.3: An embedded control system involving a manipulator.

intrinsically discrete. An example of such an interconnection involving a manipulator is schematically displayed in Fig. 1.3.

(Linear) impulsive systems constitute a rich subclass of hybrid systems which is considered on its own, e.g., in [76, 173, 68, 172]. We focus on this class because it even encompasses switched, sample-data and networked systems as shown for example in [22], [154] and [114]. Currently, networked systems composed of a great number of individual subsystems that share information enjoy increased attention due to, e.g., the recent interest in social and other digital networks as well as the developments on so-called smart cities. In such communication networks impulses and switches occur naturally, e.g., as a result of link failures or creations during operation of the system. An example of a switching network with three possible configurations or modes is illustrated on the cover of this thesis.

### Systems and Control Theory

Systems and control theory is a field with a long history and big impact on society due to its applications in various branches in science and engineering. Some of them are summarized in the survey [130]. In a nutshell, systems and control theory is essentially about

- the *analysis* of a given dynamic system's response to external excitations and

- the *design* of controllers that provide commands for actuatable systems based on measurements of the underlying system such that some desired objective is achieved.

In this thesis we conceptually follow the paradigm propagated in the monographs [149, 141, 179, 178] since it permits an efficient analysis and design even if the underlying system is affected by uncertainties or subcomponents that are difficult to grasp. This is highly relevant in practice since any employed mathematical model deviates from the underlying real system (e.g., caused by unknown system parameters or by neglected dynamics).

## 1.2 Main Goals

Despite the availability of many approaches to deal with impulsive and hybrid systems, there is still a lack of a common and flexible framework that permits a systematic analysis and design for such systems, particularly if they are affected by uncertainties. This is in contrast to standard linear time-invariant (LTI) systems where such a framework is available and elaborated on, e.g., in [160, 71, 78]. Its key ingredients are linear fractional representations (LFRs) for modeling uncertain or complex systems in a flexible fashion and integral quadratic constraints (IQCs) that allow for systematically analyzing such systems. These ingredients are accompanied by dedicated optimization tools based on linear matrix inequalities (LMIs) for the numerically efficient design of controllers. This motivates the following:

> The general theme of this thesis is the establishment of a systematic analysis and design framework for linear impulsive systems based on LFRs, IQCs and LMIs.

To this end, we have to achieve several (partly technical) subgoals that each constitute individual scientific contributions and which are discussed in the remaining chapters.

## 1.3 Outline and Contributions

In the remainder of this section, we briefly summarize the contents and individual contributions of the next four chapters that form the main part of this thesis. We also point the reader towards the already published contributions by the author that resulted from this work.

In Chapter 2 we elaborate in detail on analysis techniques for linear impulsive systems unaffected by uncertainties and with impulses satisfying dwell-time constraints based on the methodology introduced in [18] characterizing stability. In particular, we

- show that the proposed stability criteria generalize naturally to conditions for assuring dissipation based performance objectives and
- provide alternative analysis criteria in terms of so-called slack variables that permit the derivation of synthesis criteria with reduced conservatism.

In Chapter 3 we consider the systematic design of controllers for impulsive systems in the case that the full state is unavailable for control. We illustrate several variations and show that the provided approach seamlessly applies to switched and sampled-data systems. In particular, we provide

- several new LMI based criteria for designing dynamic output-feedback controllers for impulsive, sampled-data and switched systems and
- an extension of the so-called dual iteration for synthesizing static output-feedback controllers for impulsive systems by iteratively solving LMIs.

In Chapter 4 we provide tools for analyzing uncertain impulsive systems and demonstrate that these also permit the analysis of networked systems in a scalable fashion and even if the underlying communication topology is switching. Technically, this relies on

- a generalization of the IQC framework that applies to linear impulsive systems based on a novel notion of time-domain IQCs.

In Chapter 5 we combine the results from the previous chapters in order to tackle the problems of synthesizing so-called gain-scheduled and robust output-feedback controllers for linear impulsive systems. The contributions of this chapter are

- a gain-scheduled design approach for impulsive systems affected by piecewise constant parameters based on dynamic IQCs and

- an extension of the dual iteration for synthesizing robust output-feedback controllers for uncertain impulsive systems in an iterative fashion.

In Chapter 6 we provide several general concluding remarks as well as an outlook on potential future research. The appendix comprises an explanation of utilized symbols and abbreviations in Appendix A and Appendix B, respectively. Appendix C contains several auxiliary results on LMIs and tools for dealing with them. Most of these results are extracted from [149] and repeated here in order to turn this thesis self-contained. Thus, basic linear algebra and calculus should be sufficient for reading this work. Appendix D elaborates on three relaxations for turning differential LMIs, that frequently appear in this work, into standard semidefinite programs. These relaxations are based on sum-of-squares matrices, linear splines and B-splines. The B-spline relaxation is not standard and we

- demonstrate how to employ this relaxation for analyzing impulsive systems and for synthesizing impulsive output-feedback controllers and

- show for the first time that this relaxation is asymptotically exact.

A dependency graph including the main chapters and the relevant ones from the appendix is depicted in Fig. 1.4.

Finally, we stress that this work resulted in the peer-reviewed publications [79, 80, 81, 82, 83, 84, 85] and that, albeit most of the material has been streamlined and enhanced over the time, some portions of the text still overlap. In particular, there is some overlap between Chapter 2 and the reference [84], Chapter 3 and [84, 85], Chapter 4 and [80, 83], as well as Chapter 5 and [81, 83, 85]. We will more precisely link the presented results in these chapters with the publications by the author.

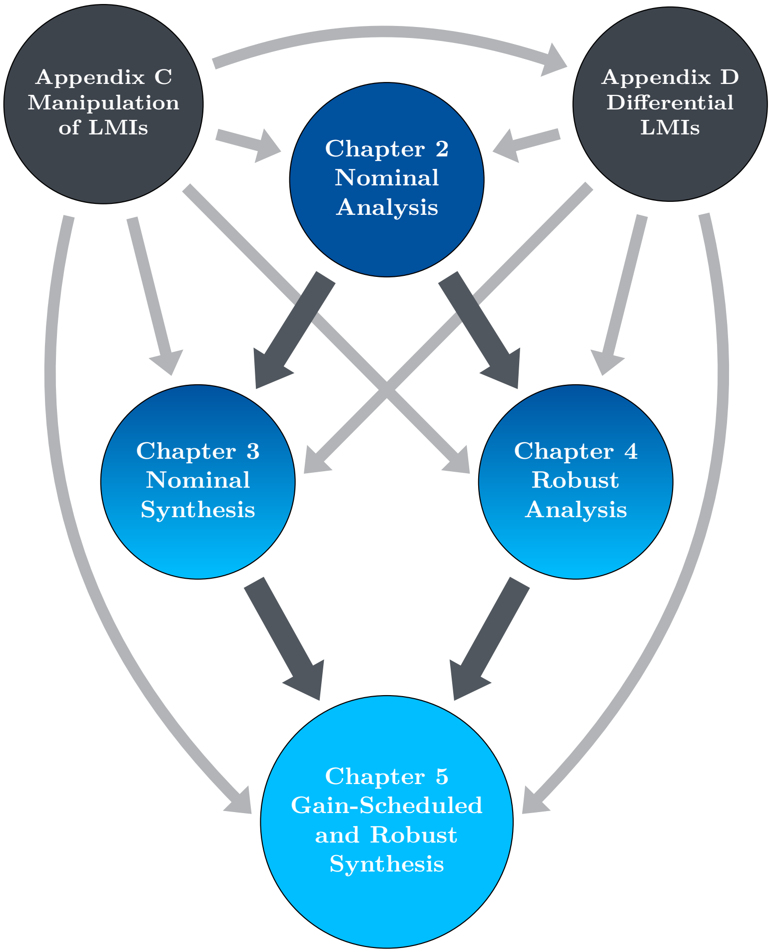



Figure 1.4: Dependency graph of covered topics.

# 2

# Nominal Analysis

A key task in control engineering is the investigation of the internal behavior of a dynamical system and the detailed analysis of its output in response to external excitations. Thereby, a fundamental and highly relevant property is *stability* which usually translates into a safe operation of the underlying system. Consequently, this property is of tremendous importance in numerous commercial and industrial applications involving, e.g., cars, trains, planes, rockets, power plants and robot manipulators.

Next to assuring stability, control engineers are also often asked to verify whether or not some desired *performance* objective is achieved. For example, airplanes and cars should be designed such that the passengers experience a comfortable flight or ride while being fuel efficient. Similarly, a solar power plant should generate as much electrical energy as possible and a robot manipulator could be tasked to grasp an object and to accurately place it at a specified target location.

In this chapter we focus on *nominal* stability and performance analysis for impulsive and related hybrid dynamical systems. This means that we provide tools and criteria for assuring stability and certain performance specifications for such dynamical systems in the case that they are unaffected by *uncertainties*. Of course, this is not very realistic because there are always discrepancies between the faced real system, which might be extremely complex in practice, and the employed mathematical model, which is limited, e.g., by the available computational resources and the knowledge about the real system. The theory of *robust control* [179, 141, 66] was precisely developed in order to provide means to deal with uncertainties as induced by the latter discrepancies. In Chapter 4 we will extend some of the most important analysis tools of robust control to uncertain interconnections involving impulsive systems. However, since the underlying techniques are based on those for nominal analysis, the latter are discussed here in detail.

## 2.1 Stability

### 2.1.1 Stability Analysis of Impulsive Systems

For matrices $A, A_J \in \mathbb{R}^{n\times n}$, some initial condition $x(0) \in \mathbb{R}^n$ and a sequence of impulse instants $0 = t_0 < t_1 < t_2 < \ldots$, let us consider an autonomous linear impulsive system with the description[1]

$$\dot{x}(t) = Ax(t), \tag{2.1a}$$

$$x(t_k) = A_J x(t_k^-) \tag{2.1b}$$

for $t \geq 0$ and $k \in \mathbb{N}$. Under the additional assumption that the monotone sequence $(t_k)_{k\in\mathbb{N}_0}$ does not admit an accumulation point, there exists a unique piecewise continuously differentiable and right continuous function

[1] For a normed vector space $X$, a function $f : [0,\infty) \to X$ and time $t > 0$ we let $f(t^-) := \lim_{s\nearrow t} f(s)$ denote the limit from below once it is well defined; for notational simplicity we set $f(0^-) := f(0)$.

$x : [0, \infty) \to \mathbb{R}^n$ satisfying (2.1a) and (2.1b). This function[2] is referred to as *state* or *state trajectory* of the system (2.1). The components (2.1a) and (2.1b) of the system (2.1) are usually called *flow* and *jump* component, respectively. If we assume that the jump component is absent or that it is rendered trivial by choosing $A_J = I$, then (2.1) describes again a standard linear time-invariant (LTI) dynamical system in continuous-time as modeled by a single ordinary differential equation.

As for such standard LTI systems, we intend to analyze the asymptotic behavior of the state trajectory of (2.1) for $t \to \infty$. The most important concept related to safe operation of the underlying system is the following.

**Definition 2.1** (Stability) *The system* (2.1) *is said to be (globally) (exponentially)* stable *if there exist constants* $M, \gamma > 0$ *such that* $\|x(t)\| \leq Me^{-\gamma t}\|x(0)\|$ *holds for all* $t \geq 0$ *and all initial conditions* $x(0) \in \mathbb{R}^n$.

For an impulsive system (2.1), stability is not only determined by properties of the describing matrices $A$ and $A_J$, but also greatly influenced by the sequence of impulse instants or, more precisely, by the differences $t_k - t_{k-1}$ for all $k$. These so-called *dwell-times* equal the duration of how long the flow component is active until a jump occurs and also determine the number of impulses within a given time period. Thus stability results are usually formulated under concrete assumptions on the dwell-times. The most typical and relevant ones are the following.

[2] The state of the system (2.1) is explicitly given by (2.1b) as well as $x(t) = e^{A(t-t_0)}x(0)$ for all $t \in [t_0, t_1)$ and

$$x(t) = e^{A(t-t_{k+1})}A_J e^{A(t_{k+1}-t_k)} \cdots A_J e^{A(t_1-t_0)}x(0)$$

for all $t \in [t_{k+1}, t_{k+2})$ and all $k \in \mathbb{N}_0$.

If (2.1b) is absent, the state is simply given by $x(t) = e^{At}x(0)$ for all $t \geq 0$.

**Definition 2.2** (Dwell-Time Conditions) *The strictly increasing sequence* $(t_k)_{k\in\mathbb{N}_0}$ *with* $t_0 = 0$ *is said to satisfy*

- *an* exact dwell-time *condition if there exists some* $T > 0$ *such that*

$$t_k - t_{k-1} = T \quad \textit{for all} \quad k \in \mathbb{N} \tag{EDT}$$

- *a* minimum dwell-time *condition if there exists some* $T_{\min} > 0$ *such that*

$$t_k - t_{k-1} \in [T_{\min}, \infty) \quad \textit{for all} \quad k \in \mathbb{N} \tag{MDT}$$

- *a* range dwell-time *condition if there exist* $0 < T_{\min} \leq T_{\max}$ *such that*

$$t_k - t_{k-1} \in [T_{\min}, T_{\max}] \quad \textit{for all} \quad k \in \mathbb{N}. \tag{RDT}$$

There are various more ways to constrain the dwell-times such as restricting the average number of jumps in given time intervals (average dwell-time) and it is also possible to constrain them not at all which is referred to as arbitrary dwell-time [74, 98, 97]. One could also allow the impulse instants $t_k$ to depend on the value of (parts of) the current state $x$ at time $t$ which leads to the field of event-triggered control [70, 153], but this is not pursued here. In the sequel, we focus on an analysis based on a range dwell-time condition and will comment on some of the alternative criteria stated in Definition 2.2.

Our first stability result is essentially taken from [18] and forms the basis of all upcoming results. It involves Lyapunov arguments [11] and also relies on the incorporation of a so-called clock. This clock is the function defined by

$$\theta(t) := t - t_k \quad \text{for all} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \mathbb{N}_0 \tag{2.2}$$

which depends on the sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ and which is illustrated in Fig. 2.1. This clock allows us, by its very nature, to define

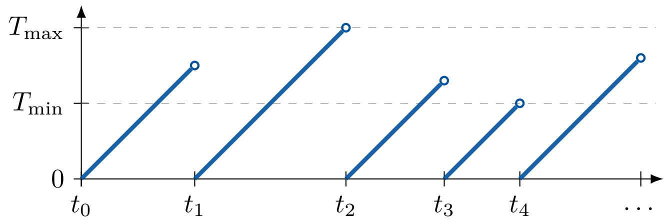



Figure 2.1: The clock (2.2) for a sequence $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT).

piecewise continuous Lyapunov functions capable of adequately dealing with the flow and jump component of the system (2.1) simultaneously.

We emphasize that, in contrast to, e.g., lifting or looped-functional based approaches [171, 45, 24], the resulting conditions are particularly well suited for deriving controller design criteria as the system matrices $A$ and $A_J$ enter in a convex and very convenient fashion.

**Theorem 2.3** (Clock-Based Stability Analysis Criteria) *The system* (2.1) *is stable for all sequences* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exists a function* $X \in C^1([0, T_{\max}], \mathbb{S}^n)$ *satisfying the (differential) linear matrix inequalities (LMIs)*[3]

$$X(\tau) \succ 0 \tag{2.3a}$$

*and*

$$\dot{X}(\tau) + A^\top X(\tau) + X(\tau)A = \begin{pmatrix} A \\ I \end{pmatrix}^\top \begin{pmatrix} 0 & X(\tau) \\ X(\tau) & \dot{X}(\tau) \end{pmatrix} \begin{pmatrix} A \\ I \end{pmatrix} \prec 0 \tag{2.3b}$$

[3] A brief introduction to standard LMIs and some useful tools are given in Appendix C; the employed (mostly standard) notation is recalled in Appendix A. How to numerically deal with non-standard LMIs as in (2.3) is discussed in Appendix D.

*for all* $\tau \in [0, T_{\max}]$ *as well as*

$$A_J^\top X(0) A_J - X(\tau) = \begin{pmatrix} A_J \\ I \end{pmatrix}^\top \begin{pmatrix} X(0) & 0 \\ 0 & -X(\tau) \end{pmatrix} \begin{pmatrix} A_J \\ I \end{pmatrix} \prec 0 \qquad (2.3c)$$

*for all* $\tau \in [T_{\min}, T_{\max}]$.

*Proof.* By continuity of $X$ and $\dot{X}$, compactness of $[0, T_{\max}]$, and strictness of the inequalities in (2.3a) and (2.3b), we infer the existence of positive constants $\alpha, \beta, \gamma$ satisfying

$$\alpha I \preccurlyeq X(\tau) \preccurlyeq \beta I \quad \text{and} \quad \dot{X}(\tau) + A^\top X(\tau) + X(\tau) A + \gamma X(\tau) \preccurlyeq 0$$

for all $\tau \in [0, T_{\max}]$. Let $x(0) \in \mathbb{R}^n$ and $(t_k)_{k \in \mathbb{N}_0}$ satisfying (RDT) be arbitrary and let $x$ be the corresponding state trajectory of the system (2.1). Then we define the function

$$\eta : t \mapsto x(t)^\top X(\theta(t)) x(t)$$

with $\theta$ being the clock as given in (2.2). From the previous inequalities we infer

$$\alpha \|x(t)\|^2 \leq \eta(t) \leq \beta \|x(t)\|^2 \quad \text{for all} \quad t \geq 0$$

and also

$$\dot{\eta}(t) + \gamma \eta(t) = x(t)^\top \left[ \dot{X} + A^\top X + XA + \gamma X \right] (\theta(t)) x(t) \leq 0$$

for all $t \in (t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. By continuity of $\eta$ on $(t_k, t_{k+1})$ and right continuity at $t_k$, this yields

$$\eta(t) \leq e^{-\gamma(t - t_k)} \eta(t_k) \quad \text{for all} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \mathbb{N}_0.$$

Moreover, by (2.3c) we have, for all $k \in \mathbb{N}$,

$$\begin{aligned}\eta(t_k) = x(t_k)^\top X(0)x(t_k) = x(t_k^-)^\top A_J^\top X(0)A_J x(t_k^-) \\ \leq x(t_k^-)^\top X(\theta(t_k^-))x(t_k^-) = \eta(t_k^-).\end{aligned}$$

A combination with the previous inequality leads to

$$\eta(t) \leq e^{-\gamma t}\eta(0) \quad \text{for all} \quad t \geq 0$$

and thus

$$\|x(t)\|^2 \leq \frac{1}{\alpha}\eta(t) \leq \frac{1}{\alpha}e^{-\gamma t}\eta(0) \leq \frac{\beta}{\alpha}e^{-\gamma t}\|x(0)\|^2 \quad \text{for all} \quad t \geq 0.$$

This yields the claim. ●

Intuitively, the piecewise continuous function $\eta$ plays a similar role as a Lyapunov function. In particular, (2.3b) ensures that $\eta$ is monotonically decreasing on each of the intervals $[t_k, t_{k+1})$, while (2.3c) assures a decrease from one of these intervals to the next; the condition (2.3a) allows us to link the values of $\eta$ with the norm of the state trajectory and, hence, to conclude an asymptotic decrease as desired.

Structurally, (2.3a) and (2.3b) constitute exactly the most commonly employed nominal stability analysis criteria for the continuous-time system $\dot{x}(t) = Ax(t)$ on the finite time horizon $[0, T_{\max}]$, and (2.3c) is closely related to the LMI based nominal analysis criteria for the discrete-time system $x(k+1) = A_J x(k)$. We will show that this is essentially the same for most of the results developed in this thesis, but with "nominal stability analysis" replaced by "nominal performance analysis", "nominal synthesis", "robust stability analysis", etc. and with the systems $\dot{x}(t) = Ax(t)$ and $x(k+1) = A_J x(k)$ replaced by systems with inputs and outputs corresponding to the considered problem.

**Remark 2.4** (a) We recover stability criteria for the system (2.1) involving sequences of impulse instants with arbitrary dwell-time by restricting the map $X$ to be constant. Moreover, in the case that the jump component is absent, i.e., (2.1) constitutes a standard continuous-time LTI system, we recover the well-known Lyapunov based stability criteria

$$X \succ 0 \quad \text{and} \quad A^\top X + XA \prec 0$$

by additionally omitting (2.3c). Feasibility of these LMIs is equivalent to the matrix $A$ being Hurwitz, i.e., all its eigenvalues are contained in the open left half-plane.

(b) Analogous stability conditions for sequences of impulse instants satisfying (EDT) are obtained by choosing $T_{\text{max}} := T_{\text{min}} := T$ in Theorem 2.3. In this case, it is shown in [18] that (2.3) is equivalent to the existence of some matrix $X$ satisfying

$$X \succ 0 \quad \text{and} \quad A_J^\top e^{A^\top T} X e^{AT} A_J - X \prec 0,$$

which is the same as saying that all eigenvalues of $e^{AT} A_J$ are located in the complex unit disk[4].

(c) Stability conditions for sequences $(t_k)_{k\in\mathbb{N}_0}$ satisfying (MDT) can be obtained by formally taking the limit $T_{\text{max}} \to \infty$. However, it is often more convenient to work with alternative ones obtained by choosing $T_{\text{max}} := T_{\text{min}}$ and by additionally enforcing $\dot{X}(T_{\text{min}}) = 0$. Their proof

[4] Note that if the sequence $(t_k)_{k\in\mathbb{N}_0}$ satisfies (EDT), then the state trajectory $x$ of (2.1) satisfies

$$x(t_{k+1}^-) = e^{AT} A_J x(t_k^-) \quad \text{for all} \quad k \in \mathbb{N}$$

and

$$\|x(t)\| \leq \max_{\tau\in[0,T]} \|e^{A\tau} A_J\| \cdot \|x(t_k^-)\| \quad \text{for all} \quad t \in [t_k, t_{k+1}).$$

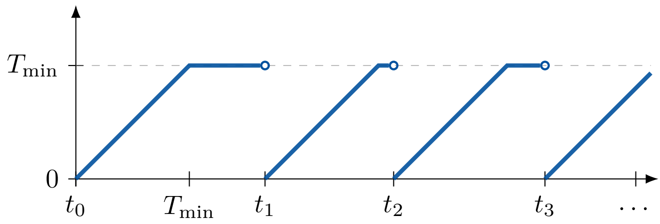



Figure 2.2: Modified clock (2.2) for a sequence $(t_k)_{k\in\mathbb{N}_0}$ with (MDT).

relies on a modification of the clock (2.2) as depicted in Fig. 2.2 and as given by $\theta(t) = t - t_k$ for $t \in [t_k, t_k + T_{\min}]$ and $\theta(t) = T_{\min}$ for $t \in [t_k + T_{\min}, t_{k+1})$ for all $k \in \mathbb{N}_0$.

(d) Note that Theorem 2.3 can also be viewed as a robust analysis result since the conditions (2.3) guarantee stability for *all* sequences of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT).

(e) Similarly as other approaches based on introducing a clock [4, 19, 21], Theorem 2.3 only yields sufficient conditions for stability. Necessary and sufficient stability criteria for hybrid systems are rather seldom and typically involve conditions that are numerically delicate; for example the approaches in [29, 169] are based on homogeneous polynomial Lyapunov functions and the resulting LMIs are very expensive to solve. However, in the case of exact dwell-time, it is shown in [18] that the above conditions are indeed necessary and sufficient.

(f) An inspection of the proof reveals that feasibility of the LMIs (2.3) also implies that, for any state trajectory of the system (2.1), the following inequality is satisfied

$$x(t)^\top X(\theta(t))x(t) \le x(0)^\top X(0)x(0) \quad \text{for all} \quad t \ge 0.$$

This inequality yields the following invariance property

$$x(t) \in \bigcup_{\tau\in[0,T_{\max}]} \left\{z \in \mathbb{R}^n \,:\, z^\top X(\tau)z \leq x(0)^\top X(0)x(0)\right\} \text{ for all } t \geq 0$$

involving the union of an infinite family of ellipsoids. Similarly as, e.g., in [12, 51], such inequalities and/or properties pave the way for a refined local stability analysis, but we will mostly stick to a global analysis in the sequel.

(g) The condition $X(\tau) \succ 0$ for all $\tau \in [0, T_{\max}]$ can be replaced without loss of generality by $X(0) \succ 0$, which usually results in a smaller computational burden; a similar complexity reduction is possible for some of our design results as well. Indeed, by denoting the left hand side of (2.3b) by $W(\tau)$, we can express $X(\tau)$ as

$$X(\tau) = e^{-A^\top \tau}\left(X(0) + \int_0^\tau e^{A^\top s}W(s)e^{As}ds\right)e^{-A\tau}$$

since both functions solve the same initial value problem and by the uniqueness of this problem's solution. Then we infer

$$\begin{aligned} X(\tau) &\succcurlyeq e^{-A^\top \tau}\left(X(0) + \int_0^{T_{\max}} e^{A^\top s}W(s)e^{As}ds\right)e^{-A\tau} \\ &= e^{A^\top (T_{\max}-\tau)}X(T_{\max})e^{A(T_{\max}-\tau)} \\ &\succ e^{A^\top (T_{\max}-\tau)}A_J^\top X(0)A_J e^{A(T_{\max}-\tau)} \succcurlyeq 0 \end{aligned}$$

for all $\tau \in [0, T_{\max}]$. Here, the three inequalities are consequences of (2.3b), (2.3c) and $X(0) \succ 0$, respectively.

(h) Note that finding a function $X$ satisfying (2.3) constitutes an infinite dimensional problem that cannot be solved directly in general. However, numerically tractable sufficient conditions can be obtained via each of the approaches discussed in Appendix D. They result in finite dimensional semidefinite programs that can be solved via semidefinite programming solvers such as SeDuMi [155], Mosek [113] or LMIlab [55].

### Example

As an illustration, let us consider a concrete linear impulsive system (2.1) described by the matrices

$$A := \begin{pmatrix} -1 & 0.1 \\ 0 & 1.2 \end{pmatrix} \quad \text{and} \quad A_J := \begin{pmatrix} 1.2 & 0 \\ 0 & 0.5 \end{pmatrix}, \tag{2.4}$$

which is also considered in [18]. Based on Theorem 2.3 and a simple bisection, we can, for example, determine the largest $T_{\max}$ such that this impulsive system is assured to be stable for all sequences of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) for $T_{\min} := 0.19$. In order to turn the analysis conditions (2.3) into a standard semidefinite program that can be solved numerically (e.g., with LMIlab [55]), we employ the B-spline relaxation as discussed in detail in Section D.3.

We obtain that stability is guaranteed for all $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) as long as $T_{\max} \leq 0.5776$ holds, which is the same upper bound as obtained in [18]. Note that this bound is essentially tight because the matrix $e^{AT}A_J$ has an eigenvalue $\lambda$ with $|\lambda| > 1$ if $T \geq 0.5777$. In particular, the system (2.1) with the (periodic) impulse instants $t_k = Tk$ is unstable for any $T \geq 0.5777$.

Three state trajectories $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ of the impulsive system (2.1) with describing matrices (2.4) and initial condition $x(0) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ are illustrated in Fig. 2.3; the corresponding sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ are given by $(0.5k)_{k\in\mathbb{N}_0}$, $(0.57k)_{k\in\mathbb{N}_0}$ and $(0.58k)_{k\in\mathbb{N}_0}$. The first two trajectories converge to zero as time goes to infinity and the last trajectory admits

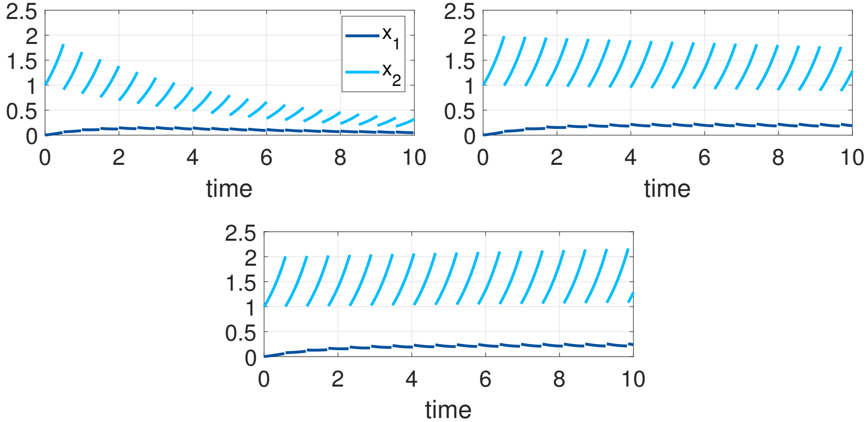



Figure 2.3: State trajectories of the system (2.1) with (2.4) and $(t_k)_{k\in\mathbb{N}_0}$ given by $(0.5k)_{k\in\mathbb{N}_0}$, $(0.57k)_{k\in\mathbb{N}_0}$ and $(0.58k)_{k\in\mathbb{N}_0}$, respectively.

unstable characteristics which is in accordance with our analysis results.

### 2.1.2 Stability Analysis of Switched Systems

The most popular class of hybrid systems consists of so-called switched systems as studied, e.g., in [4, 98, 97]. For a sequence of impulse instants $0 = t_0 < t_1 < t_2 < \dots$, matrices $A_1, \dots, A_N \in \mathbb{R}^{n\times n}$ and some initial condition $x(0) \in \mathbb{R}^n$, a switched linear system admits the description

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{2.5}$$

for $t \geq 0$; here, the *switching function* $\sigma : [0,\infty) \to \{1,\dots,N\}$ is constant on each of the intervals $[t_{k-1}, t_k)$. The switching function $\sigma$ determines which *mode* of the system (2.5) is currently active and a change of the value of $\sigma$ induces the transition from one mode to another. These transitions are instantaneous and, thus, introduce discrete-time behavior to the otherwise continuous dynamics. Hence, (2.5) constitutes indeed a hybrid system.

A benefit of studying impulsive systems is that numerous results can easily be converted to corresponding results for switched systems. In particular, we obtain the following criteria guaranteeing stability by a minor modification of the proof of Theorem 2.3; here, stability of the system (2.5) is analogously defined as in Definition 2.1. These criteria involve conditions of the form (2.3a) and (2.3b) for each of the modes of the system (2.5) as well as a jump condition of the form (2.3c) with $A_J = I$ for each pair of modes.

**Corollary 2.5** (Clock-Based Stability Analysis Criteria for Switched Systems) *The system* (2.5) *is stable for all switching functions* $\sigma$ *defined by sequences* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist* $X_1, \ldots, X_N \in C^1([0, T_{\max}], \mathbb{S}^n)$ *satisfying*

$$X_k(\tau) \succ 0 \quad \textit{and} \quad \begin{pmatrix} A_k \\ I \end{pmatrix}^\top \begin{pmatrix} 0 & X_k(\tau) \\ X_k(\tau) & \dot{X}_k(\tau) \end{pmatrix} \begin{pmatrix} A_k \\ I \end{pmatrix} \prec 0 \tag{2.6a,b}$$

*for all* $\tau \in [0, T_{\min}]$ *and* $k \in \{1, \ldots, N\}$ *as well as*

$$X_l(0) - X_k(\tau) = \begin{pmatrix} I \\ I \end{pmatrix}^\top \begin{pmatrix} X_l(0) & 0 \\ 0 & -X_k(\tau) \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix} \prec 0. \tag{2.6c}$$

*for all* $\tau \in [T_{\min}, T_{\max}]$ *and* $k, l \in \{1, \ldots, N\}$.

**Remark 2.6** (a) For switched systems one has even more possibilities to define constraints on the dwell-times by formulations that depend on the currently active mode and/or several past ones. This is for example done in [20].

(b) The above description allows the transition from each mode to each mode (including the current mode). In practice, it might be meaningful to consider or allow only some of those transitions. Formally, this can be achieved by introducing a directed (unweighted) graph

$G = (V, E)$ with vertices $V = \{1, \dots, N\}$ and edges $E \subset V^2$ and constraining the switching function $\sigma$ to satisfy $(\sigma(t_{k-1}), \sigma(t_k)) \in E$ for all $k \in \mathbb{N}$. For example, let us consider the system (2.5) with $N = 3$ modes and a switching function constrained by the graph $G$ with edges $E = \{(1,2), (2,3), (3,1), (3,3)\}$. Then the system can only switch from mode 1 to mode 2, from mode 2 to mode 3, from mode 3 to mode 1 and it can stay in mode 3. The corresponding criteria for stability are identical to the ones in Corollary 2.5, but (2.6c) must be satisfied only for all $(k, l) \in E$. It is also possible to constrain the values of $\sigma$ by utilizing other objects from computer science such as finite-state machines as done, e.g., in [92, 165].

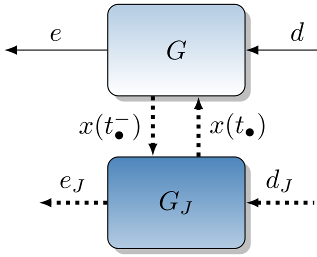



Figure 2.4: Block diagram corresponding to the impulsive system (2.7).

## 2.2 Performance

Next to analyzing stability of a dynamical system, it is important to investigate its behavior or performance with respect to exogenous inputs. To this end, for real matrices $A, B, C, D$, $A_J, B_J, C_J, D_J$ of appropriate dimensions, some initial condition $x(0) \in \mathbb{R}^n$ and a sequence of impulse instants $0 = t_0 < t_1 < t_2 < \dots$, let us consider now a linear impulsive system with the description

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix}, \tag{2.7a}$$

$$\begin{pmatrix} x(t_k) \\ e_J(k) \end{pmatrix} = \begin{pmatrix} A_J & B_J \\ C_J & D_J \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \end{pmatrix} \tag{2.7b}$$

for $t \geq 0$ and $k \in \mathbb{N}$. This system is also depicted in Fig. 2.4, where $G$ and $G_J$ stand for its flow and the jump component, respectively. The input signals $d \in L_2$ and $d_J \in \ell_2$ are generalized disturbances[5] and we wish to analyze the effect of those inputs on the (performance) output signals $e$, $e_J$

[5] The terminology *generalized* disturbance stems from the fact that this signal can comprise actual disturbances like wind hitting a truck on the highway and, e.g., for a controlled mechanical system, a reference signal provided by some user that the system is supposed to follow.

in some metric; in most situations the latter two signals are error indicators and the goal is to minimize their norm. There are various possible and interesting metrics to choose depending on the concrete application. For standard LTI systems, as obtained by omitting the impulsive component (2.7b), a summary is given, e.g., in [149, Section 3.3] and one can extend most of them to hybrid systems such as (2.7).

### 2.2.1 Quadratic Performance

We will mostly focus on so-called quadratic performance criteria that are defined as follows and involve two symmetric matrices $P$ and $P_J$ that are usually partitioned accordingly to the stacked signals $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e_J \\ d_J \end{smallmatrix}\right)$, respectively; we also make use of a notion of stability for (2.7) which is almost identical to the one in Definition 2.1. For reasons of space, we will in the sequel indicate objects that can be inferred by symmetry or are not relevant by the symbol "$\bullet$".

**Definition 2.7** (Stability and Quadratic Performance)

- *The system* (2.7) *is said to be stable if there exist constants* $M, \gamma > 0$ *such that* $\|x(t)\| \leq Me^{-\gamma t}\|x(0)\|$ *holds for all* $t \geq 0$ *and all initial conditions* $x(0) \in \mathbb{R}^n$ *and for vanishing disturbances* $d = 0$, $d_J = 0$.

- *The system* (2.7) *is said to* achieve quadratic performance with index $(P, P_J)$ *if there exists some* $\varepsilon > 0$ *such that*

$$\int_0^\infty (\bullet)^\top P \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} dt + \sum_{k=1}^\infty (\bullet)^\top P_J \begin{pmatrix} e_J(k) \\ d_J(k) \end{pmatrix} \leq -\varepsilon \|d\|_{L_2}^2 - \varepsilon \|d_J\|_{\ell_2}^2$$

*holds for the initial condition* $x(0) = 0$ *and for all* $(d, d_J) \in L_2 \times \ell_2$.

Note that Definition 2.7 is formulated only in terms of the state trajectory as well as the input and output signals. In particular, it allows for more general descriptions than (2.7) that involve, e.g., time-dependent describing matrices or nonlinear components.

For standard LTI systems, an important quantity with ample motivations found in the robust control literature (e.g., in [179, 66, 43]) is the *energy gain* which equals

$$\sup_{d\in L_2\setminus\{0\}} \frac{\|e\|_{L_2}}{\|d\|_{L_2}}$$

and is identical to the $H_\infty$-norm of the system if its transfer matrix is stable. An analogous quantity for the impulsive system (2.7), which we also refer to as energy gain, is given by

$$\sup_{(d,d_J)\in(L_2\times\ell_2)\setminus\{0\}} \frac{\sqrt{\|e\|_{L_2}^2+\|e_J\|_{\ell_2}^2}}{\sqrt{\|d\|_{L_2}^2+\|d_J\|_{\ell_2}^2}}.$$

It is not difficult to see that this gain is bounded by some $\gamma > 0$ if the system (2.7) achieves quadratic performance with index $\left(\begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I\end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I\end{pmatrix}\right)$, and that this gain equals the infimal $\gamma > 0$ such that (2.7) achieves quadratic performance with index $\left(\begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I\end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I\end{pmatrix}\right)$.

Let us provide our first performance analysis result which extends Theorem 2.3 and allows for more general performance indices than those corresponding to the energy gain.

**Theorem 2.8** (Quadratic Performance Analysis) *Let the symmetric matrices* $P = \begin{pmatrix} Q & S \\ S^\top & R\end{pmatrix}$, $P_J = \begin{pmatrix} Q_J & S_J \\ S_J^\top & R_J\end{pmatrix}$ *with* $Q \succcurlyeq 0$ *and* $Q_J \succcurlyeq 0$ *be given. Then the system* (2.7) *is stable and achieves quadratic performance with index* $(P, P_J)$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exists a function* $X \in C^1([0, T_{\max}], \mathbb{S}^n)$ *satisfying the LMIs*

$$X(\tau) \succ 0 \tag{2.8a}$$

*and*

$$(\bullet)^\top \begin{pmatrix} 0 & X(\tau) \\ X(\tau) & \dot{X}(\tau) \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \prec 0 \tag{2.8b}$$

*for all* $\tau \in [0, T_{\max}]$ *as well as*

$$(\bullet)^\top \begin{pmatrix} X(0) & 0 \\ 0 & -X(\tau) \end{pmatrix} \begin{pmatrix} A_J & B_J \\ I & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} C_J & D_J \\ 0 & I \end{pmatrix} \prec 0 \tag{2.8c}$$

*for all* $\tau \in [T_{\min}, T_{\max}]$.

*Proof. Stability:* The left upper blocks of (2.8b) and (2.8c) read as

$$\dot{X}(\tau) + A^\top X(\tau) + X(\tau)A + C^\top QC \prec 0$$

and

$$A_J^\top X(0)A_J - X(\tau) + C_J^\top Q_J C_J \prec 0,$$

respectively. By $Q \succcurlyeq 0$, $Q_J \succcurlyeq 0$ and (2.8a), we can then infer that the LMIs (2.3) hold and conclude stability from Theorem 2.3.

*Performance:* By continuity of $X$ and $\dot{X}$, compactness of $[0, T_{\max}]$ and $[T_{\min}, T_{\max}]$, and strictness of the inequalities in (2.8b) and (2.8c), we infer the existence of some positive constant $\varepsilon$ such that (2.8b) and (2.8c) hold for $R$ and $R_J$ replaced by $R + \varepsilon I$ and $R_J + \varepsilon I$, respectively.

Let now $d \in L_2$, $d_J \in \ell_2$ and $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) be arbitrary and let $x$ be the state trajectory of the system (2.7) corresponding to these inputs and sequence of impulse instants as well as to the initial condition $x(0) = 0$. With $\theta$ being the clock as given in (2.2), we then define the function

$$\eta : t \mapsto x(t)^\top X(\theta(t))x(t)$$

as in the proof of Theorem 2.3 which is nonnegative due to (2.8a). From the $\varepsilon$-modification of (2.8b), we infer, for all $t \in (t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$, that

$$\begin{aligned}
\dot{\eta}(t) &= (\bullet)^\top \begin{pmatrix} 0 & X(\theta(t)) \\ X(\theta(t)) & \dot{X}(\theta(t)) \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix} \\
&\leq -(\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix} - \varepsilon \|d(t)\|^2 \\
&= -(\bullet)^\top P \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} - \varepsilon \|d(t)\|^2
\end{aligned}$$

holds. Similarly, the $\varepsilon$-modification of (2.8c) leads to

$$\begin{aligned}
\eta(t_k) - \eta(t_k^-) &= (\bullet)^\top \begin{pmatrix} X(0) & 0 \\ 0 & -X(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} A_J & B_J \\ I & 0 \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \end{pmatrix} \\
&\leq -(\bullet)^\top P_J \begin{pmatrix} C_J & D_J \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \end{pmatrix} - \varepsilon \|d_J(k)\|^2 \\
&= -(\bullet)^\top P_J \begin{pmatrix} e_J(k) \\ d_J(k) \end{pmatrix} - \varepsilon \|d_J(k)\|^2
\end{aligned}$$

for all $k \in \mathbb{N}$. Moreover, continuity of $\eta$ on the intervals $[t_k, t_{k+1})$ yields

$$\begin{aligned}
\eta(t) - \eta(0) &= \eta(t) - \eta(t_k) + \sum_{l=1}^{k} \big(\eta(t_l) - \eta(t_{l-1})\big) \\
&= \eta(t) - \eta(t_k) + \sum_{l=1}^{k} \big(\eta(t_l^-) - \eta(t_{l-1})\big) + \sum_{l=1}^{k} \big(\eta(t_l) - \eta(t_l^-)\big) \\
&= \int_{t_k}^{t} \dot{\eta}(s)\, ds + \sum_{l=1}^{k} \int_{t_{l-1}}^{t_l} \dot{\eta}(s)\, ds + \sum_{l=1}^{k} \big(\eta(t_l) - \eta(t_l^-)\big)
\end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. Combining the latter identity and inequalities results in

$$\eta(t) - \eta(0) + \int_0^t (\bullet)^\top P \begin{pmatrix} e(s) \\ d(s) \end{pmatrix} ds + \sum_{l=1}^{k} (\bullet)^\top P_J \begin{pmatrix} e_J(l) \\ d_J(l) \end{pmatrix} \leq -\varepsilon \int_0^t \|d(s)\|^2 ds - \varepsilon \sum_{l=1}^{k} \|d_J(l)\|^2$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. Due to the nonnegativity of $\eta$ and $\eta(0) = x(0)^\top X(0) x(0) = 0$, this yields

$$\int_0^t (\bullet)^\top P \begin{pmatrix} e(s) \\ d(s) \end{pmatrix} ds + \sum_{l=1}^{k} (\bullet)^\top P_J \begin{pmatrix} e_J(l) \\ d_J(l) \end{pmatrix} \leq -\varepsilon \int_0^t \|d(s)\|^2 \, ds - \varepsilon \sum_{l=1}^{k} \|d_J(l)\|^2$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. The proof is finished by taking the limit $t \to \infty$. ●

**Remark 2.9** (Dissipativity) An inspection of the proof of Theorem 2.8 reveals that, even without positivity constraints on the matrices $Q$ and $Q_J$, we can also conclude from the LMIs (2.8) the existence of some $\varepsilon > 0$ such that

$$\begin{aligned} &x(t_b)^\top X(\theta(t_b)) x(t_b) - x(t_a)^\top X(\theta(t_a)) x(t_a) \\ &\leq -\int_{t_a}^{t_b} (\bullet)^\top P \begin{pmatrix} e(s) \\ d(s) \end{pmatrix} ds - \sum_{l=j+1}^{k} (\bullet)^\top P_J \begin{pmatrix} e_J(l) \\ d_J(l) \end{pmatrix} \\ &\quad - \varepsilon \int_{t_a}^{t_b} \|d(s)\|^2 \, ds - \varepsilon \sum_{l=j+1}^{k} \|d_J(l)\|^2 \end{aligned}$$

holds for all $t_a \in [t_j, t_{j+1})$, $t_b \in [t_k, t_{k+1})$ with $t_a \leq t_b$ and $j \leq k$ and all admissible system trajectories. This is essentially a (strict) *dissipation inequality* similarly as introduced in [167, 168] by Jan Willems, but for the impulsive system (2.7). Dissipation inequalities play a fundamental role in control and are strongly intertwined with quadratic performance. In the language of dissipation theory, the above inequality involves a (clock-dependent) *storage function*

$$V : \mathbb{R}^n \times [T_{\min}, T_{\max}] \to \mathbb{R}, \quad V(x,\tau) := x^\top X(\tau) x$$

as well as two *supply rates* $s : \mathbb{R}^{n_e} \times \mathbb{R}^{n_d} \to \mathbb{R}$ and $s_J : \mathbb{R}^{n_{e_J}} \times \mathbb{R}^{n_{d_J}} \to \mathbb{R}$ defined, respectively, by

$$s(e,d) := -(\bullet)^\top P \left( \begin{smallmatrix} e \\ d \end{smallmatrix} \right) \quad \text{ and } \quad s_J(e_J, d_J) := -(\bullet)^\top P_J \left( \begin{smallmatrix} e_J \\ d_J \end{smallmatrix} \right).$$

**Remark 2.10** (a) As for stability analysis in Theorem 2.3, we obtain corresponding performance analysis conditions for standard LTI systems in continuous-time by omitting (2.8c) and by constraining $X$ to be constant. Moreover, corresponding performance criteria for sequences of impulse instants satisfying (EDT) or (MDT) are obtained with the same modifications as mentioned in Remark 2.4.

(b) Note that, in contrast to the standard LTI case, the quadratic performance criteria in Theorem 2.8 are not necessary because the ones from the underlying stability result Theorem 2.3 are only sufficient in general. In particular, it is typically not possible to determine the energy gain of the system (2.7) exactly based on Theorem 2.8, but we can numerically determine good upper bounds on this value by computing the infimal $\gamma > 0$ such that there exists a function $X$ satisfying (2.8) for $(P, P_J) = \left( \left( \begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix} \right), \left( \begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix} \right) \right)$.

(c) Also note that the conditions (2.8) are related to the performance analysis criteria in [66, 156, 26] for (time-varying) linear systems that are defined on a finite time-horizon $[0, T]$. This is not surprising because one can view the impulsive system (2.7) as a family of linear systems (2.7a) defined on the finite time-horizons $[t_k, t_k + 1)$, $k \in \mathbb{N}$ with responses that are glued together according to (2.7b).

### Example

Let us consider an impulsive system (2.7) with describing matrices

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0.1 & 0 \\ 0 & 1.2 & 0.1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} A_J & B_J \\ \hline C_J & D_J \end{array}\right) = \left(\begin{array}{cc|c} 1.2 & 0 & 0 \\ 0 & 0.5 & 1 \\ \hline 1 & 0 & 0 \end{array}\right). \tag{2.9}$$

Since $A$ and $A_J$ coincide with the matrices (2.4) from the previous example, we already know that this system is stable for all $(t_k)$ satisfying (RDT) with $0.19 = T_{\min} \leq T_{\max} \leq 0.5776$. Theorem 2.8 now permits us to analyze this systems input-output behavior in more detail by determining upper bounds on its energy gain for various values of $T_{\max}$.

Two of these upper bounds are given by the full lines in Fig. 2.5. These curves are obtained by employing linear splines (Section D.2.1) and the B-spline relaxation (Section D.3), respectively. The parameters in both approaches are chosen such that the latter always admits smaller running times than the former. Since, in addition, uniformly smaller upper bounds are obtained, we conclude that the B-spline approach is superior than the one relying on linear splines; note that the sum-of-squares approach (Section D.1) yields the same upper bounds as B-splines, but tends to require larger computation times.

One observes that both curves are constant for $0.19 \leq T_{\max} \leq 0.44$ and grow dramatically as $T_{\max}$ moves towards 0.5776. The latter is expected since the system becomes unstable for $T_{\max} \geq 0.5777$ and the former can be partly explained by considering the upper bounds on the energy gain for sequences $(t_k)_{k\in\mathbb{N}_0}$ satisfying (EDT) with $T = T_{\max}$ (dashed lines). Since these upper bounds are relatively large for $T = 0.19$ and because $(Tk)_{k\in\mathbb{N}_0}$ satisfies (RDT) for $T_{\min} = T$ and any $T_{\max}$, the upper bounds obtained for sequences $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) are bounded from below by those

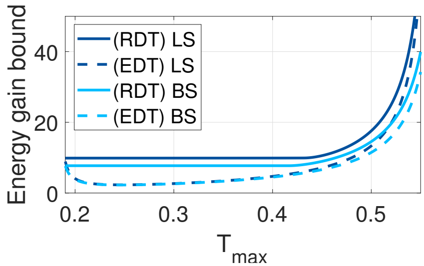



Figure 2.5: Upper bounds on the energy gain of the system (2.7) with (2.9) and sequences $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) / (EDT) resulting from applying the linear spline (LS) and the B-spline (BS) approach.

values, respectively.

### 2.2.2 Energy-to-Peak Performance

Next to the energy gain, another interesting quantity related to the performance of a given system is the so-called energy-to-peak gain, which is also referred to as generalized $H_2$-norm. A definition and discussion of this gain for standard LTI systems is found, e.g., in [127]. For an impulsive system (2.7), this gain can, e.g., be analogously defined as

$$\sup_{(d,d_J)\in(L_2\times\ell_2)\setminus\{0\}} \frac{\sup_{t\geq 0}\|e(t)\|}{\sqrt{\|d\|_{L_2}^2+\|d_J\|_{\ell_2}^2}}.$$

Similarly as before, we can compute good upper bounds on this value based on another extension of our stability result Theorem 2.3.

**Theorem 2.11** (Energy-to-Peak Analysis) *Let $Y$ be a positive definite matrix and suppose that $D = 0$. Then the system (2.7) is stable and its output $e$ satisfies*

$$e(t)^\top Y^{-1} e(t) \leq \int_0^t \|d(s)\|^2\, ds + \sum_{l=1}^{k} \|d_J(l)\|^2$$

*for all $t \in [t_k, t_{k+1})$, all $k \in \mathbb{N}_0$ and all disturbances $d, d_J$ all $(t_k)_{k\in\mathbb{N}_0}$ satisfying* (RDT) *and for the initial condition $x(0) = 0$ if there exists a function $X \in C^1([0, T_{\max}], \mathbb{S}^n)$ satisfying the LMIs*[6]

$$\begin{pmatrix} Y & C \\ C^\top & X \end{pmatrix} \succ 0,$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & -I \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline 0 & I \end{array}\right) \prec 0 \quad \text{and} \quad (\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & -I \end{array}\right) \left(\begin{array}{cc} A_J & B_J \\ I & 0 \\ \hline 0 & I \end{array}\right) \prec 0 \tag{2.10}$$

*on $[0, T_{\max}]$, $[0, T_{\max}]$ and $[T_{\min}, T_{\max}]$, respectively.*

Note that the energy-to-peak gain of (2.7) is smaller than $\gamma > 0$ if the LMIs (2.10) are feasible for $Y := \gamma^2 I$. Theorem 2.11 is formulated for a general matrix $Y$ because viewing this matrix as a decision variable and minimizing its trace subject to feasibility of (2.10) allows for determining a "smallest" ellipsoid with the invariance property

$$e(t) \in \left\{ x \;\middle|\; x^\top Y^{-1} x \leq 1 \right\}$$

for all $t \geq 0$ and all $(d, d_J)$ with $\|d\|_{L_2}^2 + \|d_J\|_{\ell_2}^2 \leq 1$.

[6] For a matrix-valued function $F$ with domain $X$, we will from now on frequently write

$$F \succ 0 \text{ on } X \quad \text{instead of} \quad F(x) \succ 0 \text{ for all } x \in X.$$

We proceed analogously for "$\succcurlyeq$", "$=$", "$\preccurlyeq$" and "$\prec$".

### Example

For the impulsive system (2.7) with (2.9) and for sequences $(t_k)_{k\in\mathbb{N}_0}$ with (RDT) and $[T_{\min}, T_{\max}] = [0.19, 0.5]$, we obtain the gray ellipsoid depicted in Fig. 2.6 by utilizing the B-spline relaxation. Restricting the sequence $(t_k)_{k\in\mathbb{N}_0}$ to satisfy (EDT) with $T = 0.5$ yields the black ellipsoid. The latter is tight as supported by two trajectories, which result from (worst-case) disturbances $(d, d_J)$ satisfying $\|d\|_{L_2}^2 + \|d_J\|_{\ell_2}^2 \leq 1$, hitting the boundary of the black ellipsoid at different points. The blue trajectory from Fig. 2.6 and its corresponding disturbances $(d, d_J)$ are also illustrated in Fig. 2.7. Note that the systematic construction of worst-case disturbances (especially in the case of impulse sequences satisfying (RDT)) is an open problem; the ones depicted in Fig. 2.7 are generated by a rather brute-force approach. In a nutshell, we did express $d = d(\alpha)$ and $d_J = d_J(\alpha_J)$ as a linear combination of few basis functions of $L_2$ and $\ell_2$ with coefficients $\alpha$ and $\alpha_J$, respectively. Then we applied a nonlinear program solver to the problem

$$\max_{\alpha,\alpha_J} \sup_{t\geq 0} \|e(t)\| \text{ subject to (2.7) and } \|d(\alpha)\|_{L_2}^2 + \|d_J(\alpha_J)\|_{\ell_2}^2 \leq 1$$

for several feasible initial values of $\alpha$ and $\alpha_J$.

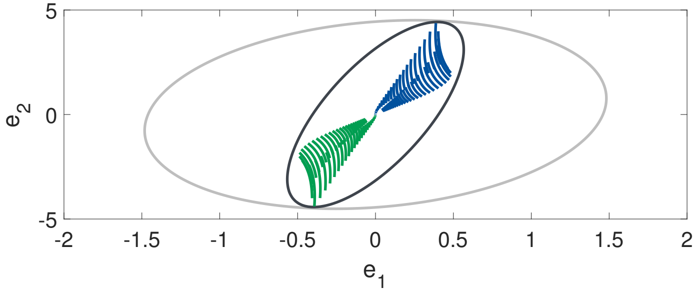



Figure 2.6: Invariant ellipsoids for the system (2.7) with (2.9) and sequences $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) (gray) and (EDT) (black) as well as two worst-case trajectories.

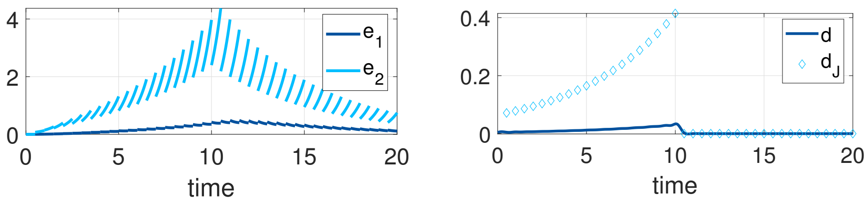



Figure 2.7: (Left) both components of the blue trajectory from Fig. 2.6 and (right) the corresponding input disturbances $(d, d_J)$.

### 2.2.3 Alternative Analysis Criteria with Slack-Variables

In order to render several interesting controller design approaches convex and thus numerically tractable, it is often mandatory to add constraints on some of the decision variables in the underlying performance analysis result which can introduce (severe) conservatism. The whole book [46] elaborates in detail on the idea to utilize equivalent alternative performance criteria involving so-called slack variables and to enforce the constraints required for the design on these new artificial variables instead of on the original ones. This procedure is usually referred to as *S-variable approach*, originates from [39] and is by now known to be beneficial in a multitude of situations.

Technically, the S-variable approach relies on the projection lemma C.12 as recalled in the appendix or variations thereof as, e.g., given in [38]. Our next result demonstrates how to appropriately add slack variables to the quadratic performance criteria from Theorem 2.8 by utilizing a non-standard variation of the projection lemma as stated in Lemma C.13. This variation is required for ensuring that the constructed slack variables are continuous functions.

**Theorem 2.12** (Quadratic Performance Analysis with Slack Variables) *Let $X$ be a fixed matrix-valued continuously differentiable map. Then the inequalities (2.8) are satisfied if and only if there exist some scalar $\rho > 0$ and continuous maps $G$, $G_J$ satisfying*

$$X \succ 0 \quad \text{and} \quad (\bullet)^\top \left(\begin{array}{ccc|c} 0 & \rho G^\top & G^\top & \\ \rho G & -\rho(G+G^\top) & X - G^\top & \\ G & X - G & \dot{X} & \\ \hline & & & P \end{array}\right) \left(\begin{array}{ccc} 0 & A & B \\ I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & C & D \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{2.11a,b}$$

*on $[0, T_{\max}]$ as well as*

$$(\bullet)^\top \left(\begin{array}{ccc|c} 0 & G_J & 0 & \\ G_J^\top & X(0) - G_J - G_J^\top & 0 & \\ 0 & 0 & -X & \\ \hline & & & P_J \end{array}\right) \left(\begin{array}{ccc} 0 & A_J & B_J \\ I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & C_J & D_J \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{2.11c}$$

*on $[T_{\min}, T_{\max}]$.*

*Proof.* We begin with some preparations. Note that the left hand side of

(2.11b) equals

$$\underbrace{\left(\begin{array}{c|c} 0 & \begin{pmatrix} X & 0 \end{pmatrix} \\ \hline (\bullet)^\top & \begin{pmatrix} \dot{X} & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \end{array}\right)}_{:=\, Q} + \mathrm{He}\left( \underbrace{\begin{pmatrix} \rho I \\ I \\ 0 \end{pmatrix}}_{:=\, U} G \underbrace{\begin{pmatrix} -I & A & B \end{pmatrix}}_{:=\, V} \right),$$

while the left hand side of (2.11c) reads as

$$\underbrace{\left(\begin{array}{c|c} X(0) & 0 \\ \hline 0 & \begin{pmatrix} -X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} C_J & D_J \\ 0 & I \end{pmatrix} \end{array}\right)}_{:=\, \hat{Q}} + \mathrm{He}\left( \underbrace{\begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}}_{:=\, \hat{U}} G_J \underbrace{\begin{pmatrix} -I & A_J & B_J \end{pmatrix}}_{:=\, \hat{V}} \right).$$

Basis matrices of the kernels of $U$, $V$, $\hat{U}$, $\hat{V}$ are given by

$$U_\perp := \begin{pmatrix} -\frac{1}{\rho} I & 0 \\ I & 0 \\ 0 & I \end{pmatrix}, \quad V_\perp := \begin{pmatrix} A & B \\ I & 0 \\ 0 & I \end{pmatrix}, \quad \hat{U}_\perp := \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix}, \quad \hat{V}_\perp := \begin{pmatrix} A_J & B_J \\ I & 0 \\ 0 & I \end{pmatrix},$$

respectively. Next, observe that we have the following identities

$$V_\perp^\top Q V_\perp = (\bullet)^\top \begin{pmatrix} 0 & X \\ X & \dot{X} \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix},$$

$$U_\perp^\top Q U_\perp = \begin{pmatrix} \dot{X} - \frac{2}{\rho} X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix},$$

$$\hat{V}_\perp^\top \hat{Q} \hat{V}_\perp = (\bullet)^\top \begin{pmatrix} X(0) & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A_J & B_J \\ I & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} C_J & D_J \\ 0 & I \end{pmatrix}$$

and

$$\hat{U}_\perp^\top \hat{Q} \hat{U}_\perp = \begin{pmatrix} -X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} C_J & D_J \\ 0 & I \end{pmatrix} = \hat{V}_\perp^\top \hat{Q} \hat{V}_\perp - (\bullet)^\top X(0) \begin{pmatrix} A_J & B_J \end{pmatrix}.$$

After these preparations, the proof is as follows.

*"If":* This statement follows from pointwise applying the standard projection lemma C.12, from the above identities as well as recalling that the inequalities $V_\perp^\top Q V_\perp \prec 0$ and $\hat{V}_\perp^\top \hat{Q} \hat{V}_\perp \prec 0$ are identical to (2.8b) and (2.8c), respectively.

*"Only if":* By assumption and the above computations, we have the inequalities $V_\perp^\top Q V_\perp \prec 0$ on $[0, T_{\max}]$ and $\hat{V}_\perp^\top \hat{Q} \hat{V}_\perp \prec 0$ on $[T_{\min}, T_{\max}]$. Due to $X \succ 0$ on $[0, T_{\max}]$, we can further conclude that

$$\hat{U}_\perp^\top \hat{Q} \hat{U}_\perp = \hat{V}_\perp^\top \hat{Q} \hat{V}_\perp - (\bullet)^\top X(0) \begin{pmatrix} A_J & B_J \end{pmatrix} \prec 0 \quad \text{on} \quad [T_{\min}, T_{\max}]$$

holds. Finally, by $(\bullet)^\top P \left( \begin{smallmatrix} D \\ I \end{smallmatrix} \right) \prec 0$ on $[0, T_{\max}]$, continuity of $\dot{X}$ and $X$, and compactness of $[0, T_{\max}]$, the Schur complement C.6 allows us to infer the existence of some small $\rho$ such that $U_\perp^\top Q U_\perp \prec 0$ on $[0, T_{\max}]$. In total we have

$$V_\perp^\top Q V_\perp \prec 0, \quad U_\perp^\top Q U_\perp \prec 0, \quad \hat{V}_\perp^\top \hat{Q} \hat{V}_\perp \prec 0 \quad \text{and} \quad \hat{U}_\perp^\top \hat{Q} \hat{U}_\perp \prec 0$$

on $[0, T_{\max}]$, $[0, T_{\max}]$, $[T_{\min}, T_{\max}]$ and $[T_{\min}, T_{\max}]$, respectively. Since $U$, $V$, $\hat{U}$ and $\hat{V}$ are constant, the latter inequalities permit us to apply the projection lemma C.13 in order to construct continuous functions $G$ and $G_J$ satisfying (2.11). ●

In the following chapters we will provide examples and comment on the benefits of including slack variables in various situations.

## 2.3 Summary

In this chapter we elaborate in detail on the clock dependent stability analysis criteria established in [18] for linear impulsive systems characterized by sequences of impulse instances satisfying dwell-time constraints. In particular, these criteria are formulated in Theorem 2.3 and in terms of differential LMIs that can be numerically solved based on, e.g., any of the relaxation schemes discussed in Appendix D. We show in Corollary 2.5 how to modify these criteria for analyzing switched systems, and demonstrate in Theorem 2.8 that they generalize naturally to tractable conditions for assuring dissipation based performance objectives for impulsive systems with inputs and outputs in their flow and jump component.

Moreover, we provide in Theorem 2.12 novel alternative analysis criteria involving slack variables. The benefit of introducing such variables in various concrete situations is illustrated, e.g., in the book [46]. We have published these criteria in [84] along with corresponding convex conditions for designing output-feedback controllers as discussed in the next chapter. In particular, we will employ these criteria for designing clock independent controllers for impulsive systems.

# 3

# Nominal Synthesis

Next to analyzing the behavior of a dynamical system, another key task in control engineering is to govern a system such that its operation is safe and some desired performance objective is achieved. This is usually accomplished by employing sensors that measure parts of the system's state (such as positions, velocities, angles, forces, temperatures, pressures, etc.), by processing the observed data, and by accordingly commanding the system's actuators (such as motors, heaters, pumps, etc.). It is instrumental to view the conversion from measurements to actuator commands again as a dynamical system that is *interconnected* to the original one and which is referred to as *controller*. Unfortunately, appropriately designing controllers is in general a challenging task that often requires a lot of tuning and physical insights on the underlying system.

In the first part of this chapter, we systematically design *dynamic* controllers for impulsive systems unaffected by uncertainties and where the full state is unavailable for control. In particular, we show that the synthesis of such a controller can be turned into a convex optimization problem that is efficiently solvable by means of standard algorithms. Alternative results have been proposed, e.g., in [7, 108, 95, 174]. These are based on separation principles and/or on suitable generalizations of geometric techniques. Moreover, apart from [95], all of the above mentioned papers rely on a specific structure of the underlying system. In contrast, our design results allow for general linear impulsive systems and the flexible nature of our approach permits us to illustrate several interesting variations such as designing controllers for switched and sampled-data systems.

In the second part, we provide a generalization of the so-called *dual iteration* to impulsive systems. The latter originally and still constitutes a promising heuristic procedure for the challenging and non-convex design of *static* output-feedback controllers for linear time-invariant systems. In contrast to dynamic ones, static controllers are more comfortable to implement, but much more difficult to construct due to the intrinsic non-convexity of the involved optimization problem. We will reconsider the dual iteration in Chapter 5 for the purpose of designing controllers for uncertain impulsive systems.

## 3.1 Dynamic Output-Feedback Controller Design

### 3.1.1 Controller Design for Impulsive Systems

For real matrices of appropriate dimensions, an initial condition $x(0) \in \mathbb{R}^n$, generalized disturbances $d \in L_2$, $d_J \in \ell_2$ and a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT), we consider now the impulsive open-loop plant with the generic description

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ u(t) \end{pmatrix}, \tag{3.1a}$$

$$\begin{pmatrix} x(t_k) \\ e_J(k) \\ y_J(k) \end{pmatrix} = \begin{pmatrix} A_J & B_J & B_{J2} \\ C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & 0 \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \\ u_J(k) \end{pmatrix} \tag{3.1b}$$

for $t \geq 0$ and $k \in \mathbb{N}$. In addition to the signals in (2.7), the system (3.1) involves measurement outputs $y$, $y_J$ and control inputs $u$, $u_J$. For given symmetric matrices $P$ and $P_J$, we aim in this section to design a dynamic output-feedback controller

$$\begin{pmatrix} \dot{x}_c(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} A^c(\theta(t)) & B^c(\theta(t)) \\ C^c(\theta(t)) & D^c(\theta(t)) \end{pmatrix} \begin{pmatrix} x_c(t) \\ y(t) \end{pmatrix}, \tag{3.2a}$$

$$\begin{pmatrix} x_c(t_k) \\ u_J(k) \end{pmatrix} = \begin{pmatrix} A_J^c(\theta(t_k^-)) & B_J^c(\theta(t_k^-)) \\ C_J^c(\theta(t_k^-)) & D_J^c(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} x_c(t_k^-) \\ y_J(k) \end{pmatrix} \tag{3.2b}$$

for $t \geq 0$ and $k \in \mathbb{N}$ such that the closed-loop interconnection of (3.1) and (3.2) is stable and achieves quadratic performance with index $(P, P_J)$. Here, $\theta$ denotes the clock as defined in (2.2) and $A^c$, $B^c$, $C^c$, $D^c$, $A_J^c$, $B_J^c$, $C_J^c$, $D_J^c$ are to-be-designed continuous matrix-valued maps[1]. The latter interconnection is illustrated in Fig. 3.1, where $G$, $K$, $G_J$ and $K_J$ denote the flow and jump components of the system (3.1) and the controller (3.2), respectively.

[1] In order to underline some of the progress made in this chapter, we highlight the describing maps of any controller in blue to emphasize that these enter the closed-loop analysis conditions as variables in a non-convex fashion. In contrast, our design results only involve variables that enter in a convex fashion and which we highlight in light blue.

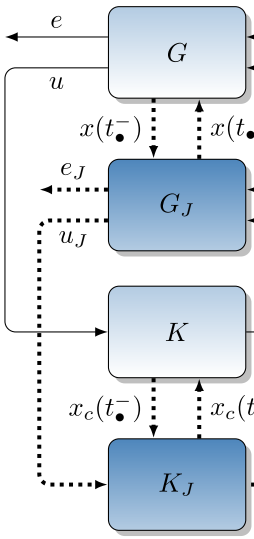



Figure 3.1: Block diagram of the closed-loop interconnection (3.3) of the system (3.1) with the controller (3.2).

Observe that this interconnection admits the equivalent description

$$\begin{pmatrix} \dot{x}_{cl}(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A}(\theta(t)) & \mathcal{B}(\theta(t)) \\ \mathcal{C}(\theta(t)) & \mathcal{D}(\theta(t)) \end{pmatrix} \begin{pmatrix} x_{cl}(t) \\ d(t) \end{pmatrix}, \tag{3.3a}$$

$$\begin{pmatrix} x_{cl}(t_k) \\ e_J(k) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_J(\theta(t_k^-)) & \mathcal{B}_J(\theta(t_k^-)) \\ \mathcal{C}_J(\theta(t_k^-)) & \mathcal{D}_J(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} x_{cl}(t_k^-) \\ d_J(k) \end{pmatrix} \tag{3.3b}$$

(for $t \geq 0$ and $k \in \mathbb{N}$) with state $x_{cl} := \left(\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right)$, initial condition $x_{cl}(0)$ and

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{cc|c} A & 0 & B \\ 0 & 0 & 0 \\ \hline C & 0 & D \end{array}\right) + \left(\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12} \end{array}\right) \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \left(\begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{array}\right) \tag{3.4}$$

as well as analogously defined maps $\mathcal{A}_J, \mathcal{B}_J, \mathcal{C}_J, \mathcal{D}_J$. Note that the closed-loop interconnection (3.3) is structurally of the same form as the impulsive system (2.7), but with the describing matrices replaced by clock-dependent ones. An inspection of the proof of Theorem 2.8 reveals that this analysis result also applies to systems with description (3.3). To this end, we proceed in this chapter under the following assumption on the performance index.

**Assumption 3.1** *The symmetric matrices* $P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ *and* $P_J = \begin{pmatrix} Q_J & S_J \\ S_J^\top & R_J \end{pmatrix}$ *are partitioned accordingly to the stacked signals* $\begin{pmatrix} e \\ d \end{pmatrix}$ *and* $\begin{pmatrix} e_J \\ d_J \end{pmatrix}$*, respectively. Moreover, the left upper blocks of* $P$ *and* $P_J$ *are positive semidefinite, i.e., there exist matrices* $T$*,* $T_J$*,* $U$*,* $U_J$ *satisfying* $U \succ 0$*,* $U_J \succ 0$*,* $Q = TU^{-1}T^\top$ *and* $Q_J = T_J U_J^{-1} T_J^\top$*.*

In particular, we have the following.

**Corollary 3.2** (Closed-Loop Analysis) *The closed-loop ystem* (3.3) *is stable and achieves quadratic performance with index* $(P, P_J)$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exists a function* $\mathcal{X} \in C^1([0, T_{\max}], \mathbb{S}^{2n})$ *satisfying the inequalities*

$$\mathcal{X} \succ 0 \quad \textit{and} \quad (\bullet)^\top \begin{pmatrix} 0 & \mathcal{X} \\ \mathcal{X} & \dot{\mathcal{X}} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix} \prec 0 \qquad \text{(3.5a,b)}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \begin{pmatrix} \mathcal{X}(0) & 0 \\ 0 & -\mathcal{X} \end{pmatrix} \begin{pmatrix} \mathcal{A}_J & \mathcal{B}_J \\ I & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} \mathcal{C}_J & \mathcal{D}_J \\ 0 & I \end{pmatrix} \prec 0 \qquad \text{(3.5c)}$$

*on* $[T_{\min}, T_{\max}]$*.*

One might wonder why we consider the design of controllers (3.2) with clock-dependent describing matrices instead of constant ones. As revealed by our next theorem, it is actually natural to search for controllers of the former type because we can provide necessary and sufficient LMI conditions for their existence. Of course, this dependence on the clock is also due to the employed closed-loop analysis criteria in Corollary 3.2. In a similar fashion, clock-dependent controllers have been designed, e.g., in [18] under the strong assumption that the full state $x$ is available for control. In particular, note that an implementation of the controller (3.2) requires the knowledge of the clock-value $\theta(t)$ and its left-limit $\theta(t^-)$ at time $t$, which is the same as knowing the last impulse instant $t_k$ with $t_k < t$. We comment on the design of controllers (3.2) with constant describing matrices in Remark 3.5. Finally, note that it is also possible to design controllers with impulses occurring asynchronously to the ones of the underlying open-loop system (3.1) based on the ideas from [170], but this is not elaborated on here. They are closely related to the ones on so-called inexact gain-scheduled controller design as considered, e.g., in [133].

In order to find a suitable controller (3.2) for the system (3.1) based on Corollary 3.2, we are required to simultaneously search for some $\mathcal{X}$ and $A^c$, $B^c$, $C^c$, $D^c$, $A_J^c$, $B_J^c$, $C_J^c$, $D_J^c$ satisfying (3.5). At the outset, this appears to be a difficult non-convex problem. A possibility to circumvent this issue is the application of a convexifying parameter transformation that is by now well-known in the LMI literature and has been proposed in [107, 137]. In our case, an extra issue results from the need to apply this transformation on the flow (3.3a) and jump component (3.3b) of the system (3.3) simultaneously. This leads to the following result which has also been published by the author in [84]. Similar conditions also appeared in [6] in the context of finite time stabilization, but they did consider neither a performance nor a control channel in the system's jump component and only block diagonal matrices $P$.

**Theorem 3.3** (Controller Design via Confexifying Parameter Transformation) *There exists a controller* (3.2) *for the system* (3.1) *such that the LMIs* (3.5) *are feasible if and only if there exist continuously differentiable* $X, Y$ *and continuous* $K, L, M, N$, $K_J, L_J, M_J, N_J$ *satisfying*

$$\mathbf{X} \succ 0 \quad \text{and} \quad (\bullet)^\top \begin{pmatrix} 0 & I \\ I & \mathbf{Z} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ I & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} \mathbf{C} & \mathbf{D} \\ 0 & I \end{pmatrix} \prec 0 \qquad (3.6a,b)$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \begin{pmatrix} \mathbf{X}(0)^{-1} & 0 \\ 0 & -\mathbf{X} \end{pmatrix} \begin{pmatrix} \mathbf{A}_J & \mathbf{B}_J \\ I & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} \mathbf{C}_J & \mathbf{D}_J \\ 0 & I \end{pmatrix} \prec 0 \qquad (3.6c)$$

*on* $[T_{\min}, T_{\max}]$. *Here, the matrix-valued maps* $\left(\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathbf{A}_J & \mathbf{B}_J \\ \mathbf{C}_J & \mathbf{D}_J \end{smallmatrix}\right)$, $\mathbf{X}$ *and* $\mathbf{Z}$ *are defined as*

$$\left(\begin{array}{cc|c} AY & A & B \\ 0 & XA & XB \\ \hline CY & C & D \end{array}\right) + \left(\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12} \end{array}\right) \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{array}\right),$$

$$\left(\begin{array}{cc|c} A_J Y & A_J & B_J \\ 0 & X(0)A_J & X(0)B_J \\ \hline C_J Y & C_J & D_J \end{array}\right) + \left(\begin{array}{cc} 0 & B_{J2} \\ I & 0 \\ \hline 0 & D_{J12} \end{array}\right) \begin{pmatrix} K_J & L_J \\ M_J & N_J \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_{J2} & D_{J21} \end{array}\right),$$

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\dot{Y} & 0 \\ 0 & \dot{X} \end{pmatrix},$$

*respectively.*

Note that the inequalities (3.6) can be turned into LMIs by our Assumption 3.1 on the quadratic performance index and by the linearization lemma C.8, i.e., by utilizing the Schur complement C.6. In particular, we can express the inequalities in (3.6b) and (3.6c) as

$$\begin{pmatrix} \mathbf{Z}+\mathbf{A}+\mathbf{A}^\top & \mathbf{B}+\mathbf{C}^\top S & \mathbf{C}^\top T \\ \mathbf{B}^\top + S^\top \mathbf{C} & \mathbf{D}^\top S + S^\top \mathbf{D} + R & \mathbf{D}^\top T \\ T^\top \mathbf{C} & T^\top D & -U \end{pmatrix}$$
$$= \begin{pmatrix} I & 0 \\ 0 & S^\top \\ 0 & T^\top \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} + (\bullet)^\top + \begin{pmatrix} \mathbf{Z} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & -U \end{pmatrix} \prec 0 \tag{3.7a}$$

on $[0, T_{\max}]$ and

$$\begin{pmatrix} -\mathbf{X} & \mathbf{C}_J^\top S_J & \mathbf{A}_J^\top & \mathbf{C}_J^\top T_J \\ S_J^\top \mathbf{C}_J & \mathbf{D}_J^\top S_J + S_J^\top \mathbf{D}_J + R_J & \mathbf{B}_J^\top & \mathbf{D}_J^\top T_J \\ \mathbf{A}_J & \mathbf{B}_J & -\mathbf{X}(0) & 0 \\ T_J^\top \mathbf{C}_J & T_J^\top \mathbf{D}_J & 0 & -U_J \end{pmatrix} =$$
$$\begin{pmatrix} 0 & 0 \\ 0 & S_J^\top \\ I & 0 \\ 0 & T_J^\top \end{pmatrix} \begin{pmatrix} \mathbf{A}_J & \mathbf{B}_J \\ \mathbf{C}_J & \mathbf{D}_J \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} + (\bullet)^\top - \begin{pmatrix} \mathbf{X} & 0 & 0 & 0 \\ 0 & -R_J & 0 & 0 \\ 0 & 0 & \mathbf{X}(0) & 0 \\ 0 & 0 & 0 & U_J \end{pmatrix} \prec 0 \tag{3.7b}$$

on $[T_{\min}, T_{\max}]$, respectively. Observe that all decision variables $X$, $Y$, $K$, $L$, $M$, $N$, $K_J$, $L_J$, $M_J$, $N_J$ enter those inequalities and the coupling condition (3.6a) in an affine fashion which permits us to solve them by employing, e.g., any of the methods suggested in Appendix D. For $(P, P_J) = \left( \left( \begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix} \right), \left( \begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix} \right) \right)$, the performance index corresponding to the energy gain, we can even simultaneously minimize the upper bound on the energy gain $\gamma$ since $\gamma^2$ also enters in an affine fashion. Finally note that an explicit formula for constructing the describing matrices $A^c$, $B^c$, $C^c$, $D^c$, $A_J^c$, $B_J^c$, $C_J^c$, $D_J^c$ of the controller (3.2) is given in the proof below.

*Proof.* We only prove sufficiency as necessity is essentially obtained by reversing the arguments. Whenever we take an inverse of a matrix valued map in the sequel, this is meant pointwise, i.e., for a map $F$ the function $F^{-1}$ satisfies $F^{-1}(\tau)F(\tau) = I$ for all $\tau$ in its domain.

*Step 1: Construction of $\mathcal{X}$:* Due to (3.6a), we can infer the existence of differentiable and pointwise nonsingular functions $U$ and $V$ satisfying $UV^\top = I - XY$; a possible choice is $U := X$ and $V := X^{-1} - Y$. By (3.6a), we can then additionally infer that $\mathcal{X} := \mathcal{Y}^{-T}\mathcal{Z} = \mathcal{Y}^{-T}\mathbf{X}\mathcal{Y}^{-1} \succ 0$ holds for

$$\mathcal{Y} := \begin{pmatrix} Y & I \\ V^\top & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Z} := \begin{pmatrix} I & 0 \\ X & U \end{pmatrix}.$$

*Step 2: Transformation of Parameters:* Let us define the continuous maps

$$\begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} := \begin{pmatrix} U & XB_2 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K - XAY - \dot{X}Y - \dot{U}V^\top & L \\ M & N \end{pmatrix} \begin{pmatrix} V^\top & 0 \\ C_2 Y & I \end{pmatrix}^{-1}$$

and

$$\begin{pmatrix} A_J^c & B_J^c \\ C_J^c & D_J^c \end{pmatrix} := \begin{pmatrix} U(0) & X(0)B_{J2} \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K_J - X(0)A_J Y & L_J \\ M_J & N_J \end{pmatrix} \begin{pmatrix} V^\top & 0 \\ C_{J2} Y & I \end{pmatrix}^{-1}.$$

The bijective mapping from $A^c, B^c, C^c, D^c, A_J^c, B_J^c, C_J^c, D_J^c$ to $K, L, M, N, K_J, L_J, M_J, N_J$ is referred to as convexifying parameter transformation. Its definition is motivated by the following observations. Note at first that $\mathcal{Y}^\top \dot{\mathcal{X}} \mathcal{Y}$ equals

$$\dot{\mathcal{Z}}\mathcal{Y} - \dot{\mathcal{Y}}^\top \mathcal{Z}^\top = \begin{pmatrix} -\dot{Y} & -\dot{Y}X - \dot{V}U^\top \\ \dot{X}Y + \dot{U}V^\top & \dot{X} \end{pmatrix} = \mathbf{Z} + \begin{pmatrix} 0 & (\bullet)^\top \\ \dot{X}Y + \dot{U}V^\top & 0 \end{pmatrix}$$

since $\mathcal{Z} = \mathcal{Y}^\top \mathcal{X}$ and $\dot{\mathcal{Z}} = \mathcal{Y}^\top \dot{\mathcal{X}} + \dot{\mathcal{Y}}^\top \mathcal{X}$. Moreover, we infer by routine computations that $\begin{pmatrix} \mathcal{Y}^\top \mathcal{X}\mathcal{A}\mathcal{Y} & \mathcal{Y}^\top \mathcal{X}\mathcal{B} \\ \mathcal{C}\mathcal{Y} & \mathcal{D} \end{pmatrix}$ equals

$$\begin{pmatrix} \mathcal{Z} & 0 \\ 0 & I \end{pmatrix} \left( \left( \begin{array}{cc|c} A & 0 & B \\ 0 & 0 & 0 \\ \hline C & 0 & D \end{array} \right) + \left( \begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12} \end{array} \right) \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \left( \begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{array} \right) \right) \begin{pmatrix} \mathcal{Y} & 0 \\ 0 & I \end{pmatrix}$$
$$= \left( \begin{array}{cc|c} AY & A & B \\ 0 & XA & XB \\ \hline CY & C & D \end{array} \right) + \left( \begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12} \end{array} \right)$$
$$\times \left( \begin{pmatrix} U & XB_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \begin{pmatrix} V^\top & 0 \\ C_2 Y & I \end{pmatrix} + \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix} \right) \left( \begin{array}{cc|c} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{array} \right).$$

This equals by the convexifying parameter transformation

$$\left( \begin{array}{cc|c} AY & A & B \\ 0 & XA & XB \\ \hline CY & C & D \end{array} \right) + \left( \begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12} \end{array} \right) \left( \begin{pmatrix} K & L \\ M & N \end{pmatrix} - \begin{pmatrix} \dot{X}Y + \dot{U}V^\top & 0 \\ 0 & 0 \end{pmatrix} \right) \left( \begin{array}{cc|c} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{array} \right)$$
$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} - \operatorname{diag}\left( \begin{pmatrix} 0 & 0 \\ \dot{X}Y + \dot{U}V^\top & 0 \end{pmatrix}, 0 \right).$$

In particular, a combination with the identity for $\mathcal{Y}^\top \dot{\mathcal{X}} \mathcal{Y}$ results in

$$\begin{pmatrix} \mathcal{Y}^\top (\dot{\mathcal{X}} + \mathcal{A}^\top \mathcal{X} + \mathcal{X}\mathcal{A})) \mathcal{Y} & \mathcal{Y}^\top \mathcal{X}\mathcal{B} \\ \mathcal{C}\mathcal{Y} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \mathbf{Z} + \mathbf{A}^\top + \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}.$$

Finally, we compute in a similar fashion, for any $\tau \in [T_{\min}, T_{\max}]$,

$$\begin{pmatrix} \mathcal{Y}(0)^\top \mathcal{X}(0) \mathcal{A}_J(\tau) \mathcal{Y}(\tau) & \mathcal{Y}(0)^\top \mathcal{X}(0) \mathcal{B}_J(\tau) \\ \mathcal{C}_J(\tau) \mathcal{Y}(\tau) & \mathcal{D}_J(\tau) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_J(\tau) & \mathbf{B}_J(\tau) \\ \mathbf{C}_J(\tau) & \mathbf{D}_J(\tau) \end{pmatrix}.$$

*Step 3: Transformation of LMIs:* Due to the identities from the previous step, the LMI (3.6a) and (3.6b) read, after congruence transformation with $\mathcal{Y}^{-1}$ and $\mathrm{diag}(\mathcal{Y}^{-1}, I)$, as (3.5a) and (3.5b), respectively. Similarly, a congruence transformation with $\mathrm{diag}(\mathcal{Y}^{-1}, I)$ leads from (3.6c) to (3.5c). ●

**Remark 3.4** (a) The corresponding controller design result for an impulsive system (3.1) involving a sequence of impulse instants with (MDT) is obtained by choosing $T_{\max} := T_{\min}$ and adding the constraints $\dot{X}(T_{\min}) = \dot{Y}(T_{\min}) = 0$. Indeed, this permits us to choose $U$ and $V$ such that they additionally satisfy $\dot{U}(T_{\min})V(T_{\min})^\top = 0$; a possible choice is still $U := X$ and $V := X^{-1} - Y$. This yields $\dot{\mathcal{X}}(T_{\min}) = 0$ as required by the modification of Corollary 3.2 for sequence of impulse instants with (MDT).

(b) For systems (3.1) involving impulse instants with (EDT), the domain of the maps in the jump component of the controller (3.2) is the singleton $\{T\}$ and, hence, they can be viewed as constant matrices. Moreover, note that in this case the impulses occur periodically which permits the use of lifting techniques as in [45, 28] in order to obtain alternative design criteria.

**Remark 3.5** (Design of Controllers with Constant Describing Matrices) In order to simplify the implementation of the impulsive controller (3.2), one might aim for controllers with constant describing matrices; note that their implementation still requires knowledge of the impulse instants $(t_k)_{k\in\mathbb{N}_0}$ due to the nature of the jump component. An inspection of the proof of Theorem 3.3 reveals that we can synthesize such controllers by enforcing all the matrices $X, Y, K, L, M, N, K_J, L_J, M_J, N_J$ to be constant. However, doing so renders the resulting synthesis inequalities (3.6) very conservative and, thus, rarely feasible. As a remedy, we can also design such controllers based on the alternative analysis criteria in Theorem 2.12 involving slack variables by carefully adapting the parameter transformation from [40].

This approach still requires to enforce several variables to be constant, but turns out to be much less conservative. The derivation of the resulting design criteria is similar to the one of the conditions in Theorem 3.11 and, thus, omitted here. These criteria are explicitly shown in [84] where we also provide a comparison to the ones in Theorem 3.3.

**Remark 3.6** (Static State-Feedback) As for standard LTI systems, dynamic output-feedback controller synthesis is conceptually more difficult than designing a static state-feedback controller based on the same analysis result. Such a static state-feedback controller admits the natural description

$$u(t) = D^c(\theta(t))x(t), \quad u_J(k) = D_J^c(\theta(t_k^-))x(t_k^-)$$

and relies on availability of measurements of the full state trajectory of the system (3.1), i.e., $y(t) = x(t)$ and $y_J(k) = x(t_k^-)$ as well as $(C_2, D_{21}) = (I, 0)$ and $(C_{J2}, D_{J21}) = (I, 0)$. Similarly as in [18], the corresponding synthesis LMIs are the same as in (3.6), but with simpler bold-face matrix-valued maps given by $\mathbf{X} = Y$, $\mathbf{Z} = -\dot{Y}$,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} AY + B_2N & B \\ CY + D_{12}N & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{A}_J & \mathbf{B}_J \\ \mathbf{C}_J & \mathbf{D}_J \end{pmatrix} = \begin{pmatrix} A_JY + B_{J2}N_J & B_J \\ C_JY + D_{J12}N_J & D_J \end{pmatrix}.$$

**Remark 3.7** (Possible Spectrum of Performance Index) Let $D \in \mathbb{R}^{n_e \times n_d}$ and $D_J \in \mathbb{R}^{n_{e_J} \times n_{d_J}}$ and suppose that the analysis inequalities (3.5) are satisfied for some controller (3.2) and a performance index $(P, P_J)$ satisfying Assumption 3.1. Then we have $(\bullet)^\top P \left(\begin{smallmatrix} I_{n_e} \\ 0 \end{smallmatrix}\right) \succcurlyeq 0$ and infer from the right lower block of (3.5b) that $(\bullet)^\top P \left(\begin{smallmatrix} \mathcal{D}(0) \\ I_{n_d} \end{smallmatrix}\right) \prec 0$ holds as well. By the strictness of the inequalities (3.5), by continuity and by compactness of $[0, T_{\max}]$, we can perturb $P$ such that it is nonsingular and the inequalities are still satisfied. This permits us to apply Lemma C.10 in order to infer that this matrix $P$ must have exactly $n_d$ negative and $n_e$ positive eigenvalues. Analogously, we obtain that $P_J$ must have exactly $n_{d_J}$ negative and $n_{e_J}$ positive eigenvalues. In the next results we will explicitly incorporate a variant of these necessary design conditions while taking Assumption 3.1 into account.

As an alternative to the convexifying parameter transformation, we can utilize in various situations the elimination lemma C.11 as recalled in the appendix. This lemma can either be applied directly to the closed-loop analysis LMIs (3.5) or to the inequalities in Theorem 3.3 in order to eliminate almost all of the appearing variables. The technical difficulty is to assure continuity when reconstructing the eliminated variables.

**Theorem 3.8** (Controller Design via Elimination) *Let $D \in \mathbb{R}^{\bullet \times n_d}$ and $D_J \in \mathbb{R}^{\bullet \times n_{d_J}}$, and suppose that $P$ and $P_J$ are nonsingular with exactly $n_d$ and $n_{d_J}$ negative eigenvalues, respectively. Moreover, let $U$, $V$, $U_J$ and $V_J$ be basis matrices of* $\ker((B_2^\top, D_{12}^\top))$, $\ker((C_2, D_{21}))$, $\ker((B_{J2}^\top, D_{J12}^\top))$ *and* $\ker((C_{J2}, D_{J21}))$, *respectively. Then there exists a controller* (3.2) *for the system* (3.1) *such that the LMIs* (3.5) *are feasible if and only if there exist continuously differentiable $X, Y$ satisfying*

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \tag{3.8a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) V \prec 0 \quad \text{and} \quad (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & P^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A^\top & -C^\top \\ \hline 0 & I \\ -B^\top & -D^\top \end{array}\right) U \succ 0$$
(3.8b,c)

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \left(\begin{array}{cc|c} \color{cyan}{X}(0) & 0 & \\ 0 & -\color{cyan}{X} & \\ \hline & & P_J \end{array}\right) \begin{pmatrix} A_J & B_J \\ I & 0 \\ \hline C_J & D_J \\ 0 & I \end{pmatrix} V_J \prec 0 \tag{3.8d}$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c} \color{cyan}{Y}(0) & 0 & \\ 0 & -\color{cyan}{Y} & \\ \hline & & P_J^{-1} \end{array}\right) \begin{pmatrix} I & 0 \\ -A_J^\top & -C_J^\top \\ \hline 0 & I \\ -B_J^\top & -D_J^\top \end{pmatrix} U_J \succ 0 \tag{3.8e}$$

*on* $[T_{\min}, T_{\max}]$.

*Proof. Only if:* Observe at first that we can express (3.5b) as

$$0 \succ (\bullet)^\top \begin{pmatrix} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & \dot{\mathcal{X}} & 0 & 0 \\ 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \\ \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix} = (\bullet)^\top \underbrace{\begin{pmatrix} \dot{\mathcal{X}} & 0 & \mathcal{X} & 0 \\ 0 & R & 0 & S^\top \\ \mathcal{X} & 0 & 0 & 0 \\ 0 & S & 0 & Q \end{pmatrix}}_{=:\tilde{P}} \begin{pmatrix} I & 0 \\ 0 & I \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

$$= (\bullet)^\top \tilde{P} \begin{pmatrix} I_{2n+n_d} \\ \tilde{W} + \tilde{U}^\top Z \tilde{V} \end{pmatrix}$$

on $[0, T_{\max}]$ with $Z := \left(\begin{smallmatrix} A^c & B^c \\ C^c & D^c \end{smallmatrix}\right)$ and matrices $\tilde{W}$, $\tilde{U}$, $\tilde{V}$ that can be read off from (3.4). Next, note that, for any $\tau \in [0, T_{\max}]$, the matrix $\tilde{P}(\tau)$ is nonsingular with exactly $2n + n_d$ negative eigenvalues by its structure, the assumptions on $P = \left(\begin{smallmatrix} Q & S \\ S^\top & R \end{smallmatrix}\right)$ and by (3.5a). This permits us to apply the elimination lemma C.11 pointwise for each $\tau \in [0, T_{\max}]$ and leads after few computations to the LMIs (3.8b) and (3.8c) with maps $\color{cyan}{X}$ and $\color{cyan}{Y}$ that are coupled as (3.8a) by (3.5a). Analogously, we can infer from (3.5c) that the LMIs (3.8d) and (3.8e) are satisfied which yields the claim.

Note that we did apply the elimination lemma pointwise on the closed-loop analysis inequalities (3.5) in order to infer feasibility of the LMIs (3.8). One can come to the same conclusion by pointwise applying elimination on the synthesis LMIs (3.6) in Theorem 3.3.

*If:* For the converse we can also apply the elimination lemma C.11 pointwise in order to construct maps $K, L, M, N, K_J, L_J, M_J, N_J$ satisfying the transformed synthesis LMIs (3.6). However, these maps might then be discontinuous. By combining the elimination lemma C.11 with the continuous selection theorem of [110] or the findings from [14], the existence of continuous maps satisfying the LMIs (3.6) is ensured. However, the latter two results do not provide means to construct these continuous maps and, hence, we need some extra work for their construction.

To this end let us now suppose that we have discontinuous maps $K$, $L$, $M$, $N$, $K_J$, $L_J$, $M_J$, $N_J$ satisfying (3.6). By a Schur complement, we then infer that the LMI (3.7) is satisfied. The major benefit of considering these LMIs is that we can apply a variant of the projection lemma C.13 to remove the discontinuous maps and then apply it once more in order to construct continuous ones. Once the latter continuous maps are obtained, it remains to apply the parameter transformation in the proof of Theorem 3.3 in order to construct the describing maps $A^c$, $B^c$, $C^c$, $D^c$, $A^c_J$, $B^c_J$, $C^c_J$, $D^c_J$ of the controller (3.2). ●

For practical implementations, it usually seems to be sufficient to take sufficiently fine grids of the intervals $[0, T_{\min}]$ and $[T_{\min}, T_{\max}]$, to build $\mathcal{X}$ as in the proof of Theorem 3.3, to apply the elimination lemma C.11 on the LMIs (3.5b) and (3.5c) for each of the knots in the grids, and to perform an interpolation of the resulting matrices.

**Remark 3.9** Due to the much smaller number of decision variables, it is typically preferable to work with Theorem 3.8 instead of Theorem 3.3. In general, it is, however, more difficult to generalize the former to structured design problems than the latter.

**Remark 3.10** (Numerical Reconstruction of Controller Matrices) Similarly as for standard $H_\infty$-controller design, the construction of the describing matrices of the controller (3.2), once a feasible solution of the synthesis criteria in Theorem 3.3 or 3.8 is available, can suffer from a number of numerical issues. As a remedy, one can try one or multiple of the following suggestions.

- Additionally enforce bounds on (all or some of) the decision variables and on (all or some of) their derivatives. For example by introducing the constraints $\begin{pmatrix} \gamma I & K \\ K^\top & \frac{1}{\gamma} I \end{pmatrix} \succ 0$ and/or $X \prec \beta I$ on $[0, T_{\max}]$ for some $\gamma, \beta > 0$.

- Include $\begin{pmatrix} Y & \beta I \\ \beta I & X \end{pmatrix} \succ 0$ on $[0, T_{\max}]$ for some $\beta > 1$. This aims at pushing the eigenvalues of $X - Y^{-1}$ away from zero and, hence, improves the quality of $\mathcal{X}$.

- Some SDP solvers have trouble to deal properly with strict inequalities. It can then be beneficial to replace "$\succ 0$" and "$\prec 0$" by "$\succ \varepsilon I$" and "$\prec -\varepsilon I$", respectively, for some small $\varepsilon > 0$.

- If one aims, for example, to determine a controller that achieves a small energy gain for the closed-loop by choosing the performance index as $\left( \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix} \right)$ in Theorem 3.3 or 3.8 and by minimizing $\gamma$, one should not try to construct a controller corresponding to the optimal gain $\gamma_{\mathrm{opt}}$. Instead, it is recommended to construct a *close-to-optimal* controller corresponding to $\gamma := (1 + \varepsilon)\gamma_{\mathrm{opt}}$ for some small $\varepsilon > 0$.

Note that for $(P, P_J) = \left( \left( \begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix} \right), \left( \begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix} \right) \right)$, the performance index corresponding to the energy gain, and if the orthogonality conditions

$$D_{12}^\top (C, D_{12}) = (0, I), \quad \left( \begin{smallmatrix} B \\ D_{21} \end{smallmatrix} \right) D_{21}^\top = \left( \begin{smallmatrix} 0 \\ I \end{smallmatrix} \right),$$

$$D_{J12}^\top (C_J, D_{J12}) = (0, I), \quad \left( \begin{smallmatrix} B_J \\ D_{J21} \end{smallmatrix} \right) D_{J21}^\top = \left( \begin{smallmatrix} 0 \\ I \end{smallmatrix} \right)$$

as well as $D = 0$ and $D_J = 0$ hold, then the inequalities (3.8b)-(3.8e) can be turned into

$$\dot{X} + A^\top X + XA + \gamma^{-2} XBB^\top X + C^\top C - \gamma^2 C_2^\top C_2 \prec 0,$$

$$\dot{Y} - AY - YA^\top - YC^\top CY - \gamma^{-2} BB^\top + B_2 B_2^\top \succ 0,$$

$$\begin{aligned} A_J^\top X(0) A_J - X - (A_J^\top X(0) B_J) \big(B_J^\top X(0) B_J - \gamma^2 I\big)^{-1} (\bullet)^\top & \\ + C_J^\top C_J - \gamma^2 C_{J2}^\top C_{J2} \prec 0 & \end{aligned}$$

and

$$Y(0) - A_J Y A_J^\top - (A_J Y C_J^\top) \big(I - C_J Y C_J^\top\big)^{-1} (\bullet)^\top - \gamma^{-2} B_J B_J^\top + B_{J2} B_{J2}^\top \succ 0,$$

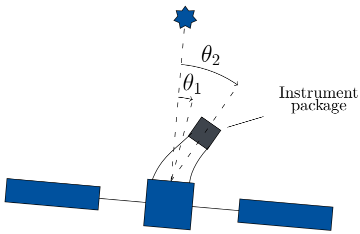
respectively. Orthogonality conditions of this type are frequently employed in $H_\infty$-control for non-hybrid systems (see, e.g., [178, Section 14.2]) and shown in [129] to be not restrictive in this context. The latter inequalities essentially admit the form of continuous- and discrete-time *Riccati inequalities* which also have a long history in control, but we will not work with them. Let us just note that the reformulation of (3.8b) relies on choosing

$$V := \begin{pmatrix} I & 0 \\ -D_{21}^\top C_2 & D_{21\perp} \end{pmatrix} \text{ with } D_{21\perp} \text{ being a basis matrix of } \ker(D_{21}),$$

observing that $B^\top = D_{21\perp} W$ holds for some matrix $W$, and on an application of the Schur complement C.6. The remaining inequalities (3.8c)-(3.8e) are modified analogously.

## Example

As an illustration let us consider a simple model for a flexible satellite which is explained in [53] and modeled as follows with state $\tilde{x} = \mathrm{col}(\theta_2, \dot\theta_2, \theta_1, \dot\theta_1)$ and with constants $J_1 = 1$, $J_2 = 0.1$, $k = 0.091$ and $b = 0.0036$:




$$\begin{pmatrix} \dot{\tilde{x}}(t) \\ \hline v(t) \end{pmatrix} = \left(\begin{array}{cccc|c:c} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k}{J_2} & -\frac{b}{J_2} & \frac{k}{J_2} & \frac{b}{J_2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{J_1} & \frac{b}{J_1} & -\frac{k}{J_1} & -\frac{b}{J_1} & 0 & \frac{1}{J_1} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} \tilde{x}(t) \\ \hline \tilde{d}(t) \\ u(t) \end{pmatrix}. \tag{3.9}$$

The standard $H_\infty$-design procedure allows us, for example, to synthesize a (non-impulsive) dynamic output-feedback controller $K$ such that the closed-loop interconnection with (3.9) is stable, the output $v$ nicely follows a given piecewise constant reference signal $r$ despite the presence of a disturbance $\tilde{d}$, and such that the control input $u$ is not too large.

To this end, we consider a standard weighted reference tracking configuration depicted in Fig. 3.2 with weights

$$W_r = 1, \quad W_d = 0.2, \quad W_u = 0.1 \quad \text{and} \quad W_{err}(s) = \frac{0.5s + 0.433}{s + 0.00433}$$

and where $G$ denotes the system (3.9); note that the dynamic weight $W_{err}$ can be equivalently expressed as a standard LTI system with state $\xi_{W_{err}}$ via the inverse Laplace transformation. Disconnecting the controller $K$ from

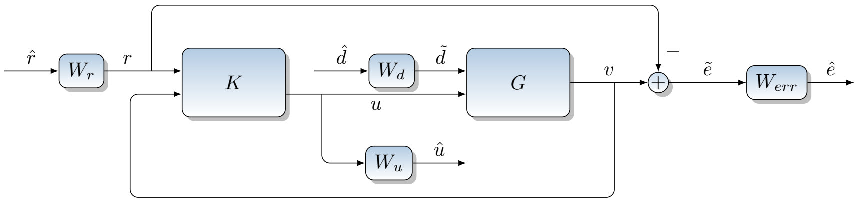



Figure 3.2: A standard weighted tracking configuration.

this configuration results in a weighted open-loop system which fits into the generic description (3.1a) with the signals

$$x := \begin{pmatrix} \tilde{x} \\ \xi_{W_{err}} \end{pmatrix}, \quad e := \begin{pmatrix} \hat{e} \\ \hat{u} \end{pmatrix}, \quad y := \begin{pmatrix} v \\ r \end{pmatrix}, \quad d := \begin{pmatrix} \hat{r} \\ \hat{d} \end{pmatrix} \quad \text{as well as} \quad u$$

and for easily computed describing matrices $A, B, B_2, C, D, D_{12}, C_2, D_{21}$. In particular, the latter description permits us to apply the specialization of Theorem 3.8 to non-impulsive systems, which is standard in the LMI literature and implemented in `hinfsyn` from Matlab. This results in a close-to-optimal controller $K$ which achieves a closed-loop energy gain of 0.877. A simulation of the interconnection of this controller and the system (3.9) for some reference $r$ and some small random disturbance $\tilde{d}$ is shown at the top of Fig. 3.3. In particular, we observe that the controller $K$ indeed admits the desired properties.

Let us now assume that, due to limited communication, the output $v$ of the system (3.9) can only be measured at times $t_0, t_1, \dots$ with $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) with $[T_{\min}, T_{\max}] = [0.4, 0.5]$. Naively driving the obtained controller $K$ without any modifications with the reference $r$ and with the piecewise constant signal

$$\tilde{v}(t) := v(t_k) \quad \text{for} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \mathbb{N}_0$$

results in the closed-loop response depicted on the middle of Fig. 3.3. We clearly observe that the tracking quality deteriorates and that the control input $u$ grows dramatically in size which is usually unacceptable.

As a remedy, we can explicitly take into account in the model of our (weighted) open-loop system that $r$ is available at all times and that we have access to $v$ only at time instances $t_k$. To this end, recall from (3.1a) that we have

$$\dot{x}(t) = Ax(t) + Bd(t) + B_2u(t) \quad \text{and} \quad e(t) = Cx(t) + Dd(t) + D_{12}u(t).$$

Since the reference $r$ is available at all times we define the measured output as

$$y(t) := r(t) = \left( \begin{smallmatrix} 1 & 0 \end{smallmatrix} \right) d(t) \quad \text{for} \quad t \geq 0.$$

Since $v$ is only available at the time instances $t_k$, we define the additional measured output

$$y_J(k) := \left( \begin{smallmatrix} v(t_k) \\ r(t_k) \end{smallmatrix} \right) = C_2x(t_k) + D_{21}d(t_k) \quad \text{for} \quad k \in \mathbb{N}_0,$$

where we incorporated the reference for simplifying the exposition. By defining the input disturbance $d_J(k) := d(t_k)$ and by recalling that the open-loop state $x$ is continuous in our configuration, i.e., $x(t_k) = x(t_k^-)$ holds for all $k \in \mathbb{N}$, this leads to the generic description of an impulsive system (3.1)

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ 0 & \left( \begin{smallmatrix} 1 & 0 \end{smallmatrix} \right) & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ u(t) \end{pmatrix}, \quad \begin{pmatrix} x(t_k) \\ e_J(k) \\ y_J(k) \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \\ u_J(k) \end{pmatrix},$$

where we incorporated the redundant signals $e_J$ and $u_J$ for the sake of compatibility. An application of Theorem 3.8 yields an impulsive controller (3.2) which achieves a closed-loop energy gain that is guaranteed to be

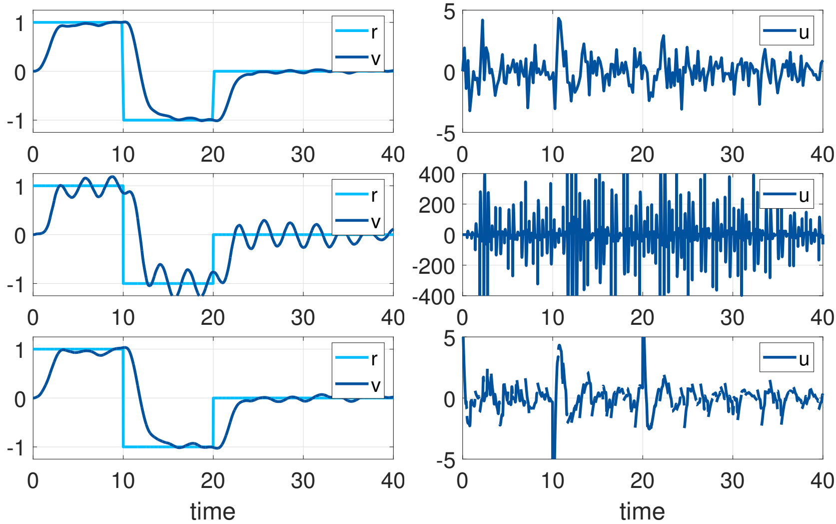



Figure 3.3: Some reference $r$ and closed-loop responses of the system (3.9) with controllers that have access to $r$ as well as to the output $v$ at all times (top) and only at fixed time instances $t_0, t_1, \ldots$ (middle and bottom). The one in the middle is naively designed and the one at the bottom based on Theorem 3.8.

smaller than 0.936. The response of the interconnection of this controller and (3.9) is shown at the bottom of Fig. 3.3. We observe a clear improvement in terms of the size of the control input $u$ and of the tracking behavior which is of almost identical quality as initially obtained by the controller $K$ when measuring the output $v$ at all times.

### Clock-Independent Controller Design

We have mentioned that the ideas from [170] permit the design of impulsive controllers with impulses occurring asynchronously to the ones of the underlying open-loop system (3.1). In a similar vein, one might raise the question whether it is possible by means of convex optimization to design a controller that is independent of the clock and which does not rely on knowledge of the impulse instants. Let us show that this is answered in the affirmative under some structural assumptions on the system (3.1) and by introducing slack variables similarly as in Theorem 2.12.

**Theorem 3.11** *Let* $\left(\begin{smallmatrix} B_2 \\ D_{12} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ -I \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} B_{J2} \\ D_{J12} \end{smallmatrix}\right) = 0$ *and* $(C_{J2}, D_{J21}) = 0$. *Then there exists a controller*

$$\begin{pmatrix} \dot{x}_c(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \begin{pmatrix} x_c(t) \\ y(t) \end{pmatrix}$$

*for the system* (3.1) *such that the LMIs* (3.5) *are feasible for the corresponding closed-loop interconnection if there exist some* $\rho > 0$, *a continuously differentiable* $\mathbf{X}$, *continuous* $G, H, S$ *with* $S - G$ *being constant, and matrices* $K, L, M, N$ *satisfying*

$$\mathbf{X} \succ 0 \quad \text{and} \quad (\bullet)^\top \left(\begin{array}{ccc|c} 0 & \rho I & I & \\ \rho I & -\rho(\mathbf{G} + \mathbf{G}^\top) & \mathbf{X} - \mathbf{G}^\top & \\ I & \mathbf{X} - \mathbf{G} & \dot{\mathbf{X}} & \\ \hline & & & P \end{array}\right) \left(\begin{array}{ccc} 0 & \mathbf{A} & \mathbf{B} \\ I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & \mathbf{C} & \mathbf{D} \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{3.10a,b}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \begin{pmatrix} \mathbf{X}(0) & 0 \\ 0 & -\mathbf{X} \end{pmatrix} \begin{pmatrix} \mathbf{A}_J & \mathbf{B}_J \\ I & 0 \end{pmatrix} + (\bullet)^\top P_J \begin{pmatrix} \mathbf{C}_J & \mathbf{D}_J \\ 0 & I \end{pmatrix} \prec 0 \tag{3.10c}$$

*on* $[T_{\min}, T_{\max}]$. *Here, the matrix-valued maps* $\left(\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathbf{A}_J & \mathbf{B}_J \\ \mathbf{C}_J & \mathbf{D}_J \end{smallmatrix}\right)$ *and* $\mathbf{G}$ *are defined as*

$$\left(\begin{array}{cc|c} HA & HA & HB \\ GA & GA & GB \\ \hline C & C & D \end{array}\right) + \left(\begin{array}{cc} 0 & 0 \\ I & 0 \\ \hline 0 & -I \end{array}\right) \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ C_2 & C_2 & D_{21} \end{array}\right), \quad \left(\begin{array}{cc|c} I & 0 & 0 \\ A_J - I & A_J & B_J \\ \hline C_J & C_J & D_J \end{array}\right)$$

*and*

$$\begin{pmatrix} H & H \\ S & G \end{pmatrix},$$

*respectively.*

The structural constraints on the describing matrices of (3.1) are such that the control channel in the jump component vanishes and such that the flow component corresponds to the open-loop plant for an *estimator synthesis problem*. A block diagram of the corresponding closed-loop interconnection is depicted in Fig. 3.4 where $K$ denotes the to-be-designed estimator, $G$ stands for the flow-component of the to-be-estimated system

$$\begin{pmatrix} \dot{x}(t) \\ v(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \\ C_2 & D_{21} \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix}$$

and $G_J$ is the underlying system's jump component. The estimation of non-measurable signals constitutes one of the most important problems in systems and control theory and is, hence, frequently considered in the literature and for various classes of dynamical systems [59, 60, 131]. In particular, the goal is to determine a controller (or estimator or filter) which takes the measured signal $y$ as input and generates an optimal approximation $u$ of the signal $v := e + u$; by choosing $P = \left(\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right)$ and neglecting the jump component, the estimation quality is measured in terms of the energy gain from the disturbance input $d$ to the estimation error $e$.

An explicit formula for constructing the controller matrices $A^c$, $B^c$, $C^c$, $D^c$ is given in the proof below which is inspired by [60]. They employ a convexifying parameter transformation in order to design robust estimators for uncertain systems based on an analysis result involving parameter dependent Lyapunov functions and slack variables. In contrast to [60], we merely require the difference $S - G$ to be constant instead of all the maps

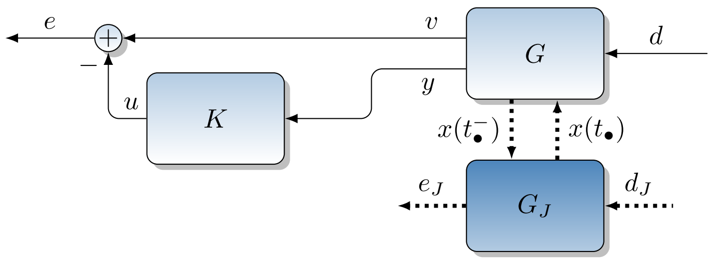



Figure 3.4: Block diagram corresponding to closed-loop interconnection in Theorem 3.11.

$H$, $S$, $G$. This leads to less conservatism. Note that deriving convex design criteria without constraining some of the maps $H$, $S$, $G$ and the describing matrices in the flow component (3.1a) seems not to be possible since those matrices appear in the parameter transformation.

*Proof. Step 1: Construction of Certificate $\mathcal{X}$ and Slack Variable $\mathcal{G}$:* Since $S - G$ is constant and by the left upper block of (3.10b), we infer the existence of pointwise nonsingular functions $U$ and $V$ satisfying $SH^{-1} = UV^\top + GH^{-1}$ and such that $U$ as well as $V^\top H$ are constant; a possible choice is $U := S - G$ and $V := H^{-1}$. Note that $\mathcal{G} := \mathcal{Y}^{-\top}\mathcal{Z} = \mathcal{Y}^{-\top}\mathbf{G}\mathcal{Y}^{-1}$ holds for

$$\mathcal{Y} := \begin{pmatrix} I & I \\ V^\top H & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Z} := \begin{pmatrix} H & 0 \\ G & U \end{pmatrix}.$$

Moreover, we have $\mathcal{X} := \mathcal{Y}^{-\top}\mathbf{X}\mathcal{Y}^{-1} \succ 0$ and $\dot{\mathcal{X}} = \mathcal{Y}^{-\top}\dot{\mathbf{X}}\mathcal{Y}^{-1}$ by (3.10a) and since $\dot{\mathcal{Y}} = 0$.

*Step 2: Transformation of Parameters:* Let us now define the matrices

$$\begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} := \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K & L \\ M & N \end{pmatrix} \begin{pmatrix} V^\top H & 0 \\ 0 & I \end{pmatrix}^{-1}.$$

Observe that $A^c, B^c, C^c, D^c$ are indeed independent of $\tau$ because $U$ and $V^\top H$ as well as $K, L, M, N$ are constant matrices. These choices are motivated by the fact that $\begin{pmatrix} \mathcal{Y}^\top \mathcal{G}\mathcal{A}\mathcal{Y} & \mathcal{Y}^\top \mathcal{G}\mathcal{B} \\ \mathcal{C}\mathcal{Y} & \mathcal{D} \end{pmatrix}$ equals

$$
\begin{aligned}
&\begin{pmatrix} \mathcal{Z} & 0 \\ 0 & I \end{pmatrix} \left( \left( \begin{array}{cc|c} A & 0 & B \\ 0 & 0 & 0 \\ \hline C & 0 & D \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ I & 0 \\ \hline 0 & -I \end{array} \right) \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \left( \begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{array} \right) \right) \begin{pmatrix} \mathcal{Y} & 0 \\ 0 & I \end{pmatrix} \\
&= \left( \begin{array}{cc|c} HA & HA & HB \\ GA & GA & GB \\ \hline C & C & D \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ I & 0 \\ \hline 0 & -I \end{array} \right) \left( \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \begin{pmatrix} V^\top H & 0 \\ 0 & I \end{pmatrix} \right) \left( \begin{array}{cc|c} I & 0 & 0 \\ C_2 & C_2 & D_{21} \end{array} \right) \\
&= \left( \begin{array}{cc|c} HA & HA & HB \\ GA & GA & GB \\ \hline C & C & D \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ I & 0 \\ \hline 0 & -I \end{array} \right) \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left( \begin{array}{cc|c} I & 0 & 0 \\ C_2 & C_2 & D_{21} \end{array} \right) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}.
\end{aligned}
$$

For the jump component, we infer via elementary computations

$$
\begin{aligned}
\begin{pmatrix} \mathcal{Y}^\top \mathcal{X}(0)\mathcal{A}_J\mathcal{Y} & \mathcal{Y}^\top \mathcal{X}\mathcal{B}_J \\ \mathcal{C}_J\mathcal{Y} & \mathcal{D}_J \end{pmatrix} &= \begin{pmatrix} \mathbf{X}(0)\mathcal{Y}^{-1}\mathcal{A}_J\mathcal{Y} & \mathbf{X}(0)\mathcal{Y}^{-1}\mathcal{B}_J \\ \mathcal{C}_J\mathcal{Y} & \mathcal{D}_J \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{X}(0)\mathbf{A}_J & \mathbf{X}(0)\mathbf{B}_J \\ \mathbf{C}_J & \mathbf{D}_J \end{pmatrix}.
\end{aligned}
$$

due to the particular choice of the transformation matrix $\mathcal{Y}$.

*Step 3: Transformation of LMIs:* Due to the identities from the previous step and after congruence transformation with $\mathcal{Y}^{-1}$ and $\operatorname{diag}(\mathcal{Y}^{-1}, I)$, the inequalities (3.10a) and (3.10c) read as (3.5a) and (3.5c), respectively. Finally, a congruence transformation of (3.10b) with $\operatorname{diag}(\mathcal{Y}^{-1}, \mathcal{Y}^{-1}, I)$ followed by an application of the projection lemma C.12, in order to eliminate the slack variable $\mathcal{G}$, yields (3.5b). ●

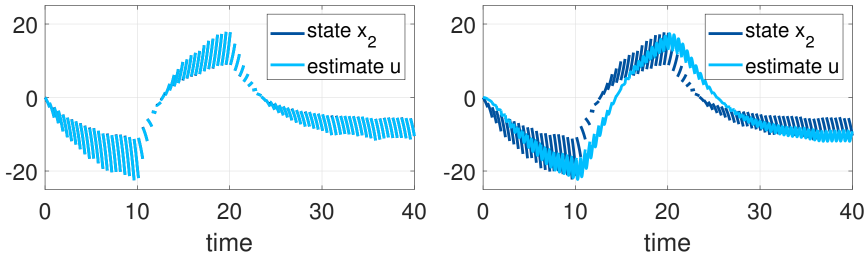



Figure 3.5: Second state of the system (3.1) with (3.11) and its estimate obtained by an impulsive controller (3.2) and by a non-impulsive LTI controller, respectively.

## Example

As an illustration let us consider the system (3.1) with matrices describing the flow and jump component given by

$$\left(\begin{array}{cc|c|c} -1 & 0.1 & 0 & 0 \\ 0 & 1.2 & 1 & 0 \\ \hline 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c|c} 1.2 & 0 & 0 & 0 \\ 0 & 0.5 & 0.1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \tag{3.11}$$

respectively, and for a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) with $[T_{\min}, T_{\max}] = [0.45, 0.5]$. In particular, this open-loop configuration is chosen with the intention to design an estimator which measures the first state $x_1$ and generates a good approximation $u$ of the second state $x_2$ despite the presence of the disturbances $d$ and $d_J$.

By employing Theorem 3.8 and choosing $(P, P_J)$ corresponding to the energy gain, we can design an impulsive (and in particular clock-dependent) estimator of the form (3.2). Depicted on the left of Fig. 3.5 is the state $x_2$ and the resulting estimate $u$ in response to a random disturbance $d_J$ and a step function $d(t) := -4\chi_{[0,10)}(t) + 4\chi_{[10,20)}(t) - 2\chi_{[20,50)}(t)$. Here, $\chi_{[a,b)}$

is a so-called indicator function which equals 1 on the interval $[a, b)$ and vanishes elsewhere. We essentially achieve a perfect estimation as both curves are on top of each other. This behavior is expected since Theorem 3.8 yields a very small upper bound of 0.05 on the closed-loop energy gain from disturbance to estimation error.

Naturally, the design of an estimator which does not rely on knowledge of the clock and the impulse instants $(t_k)_{k\in\mathbb{N}_0}$ is conceptually much more challenging, but still possible based on Theorem 3.11. Indeed, the estimate $u$ obtained from such an estimator is shown on the right of Fig. 3.5. The estimation error is of course larger if compared to the one obtained before, but we stress that the estimator resulting from Theorem 3.11 is merely a standard LTI system.

### 3.1.2 Controller Design for Sampled-Data Systems

Sampled-data systems constitute a highly relevant class of hybrid systems because in practice essentially any continuous-time open-loop plant is controlled with a digital device. Output-feedback design approaches for such systems are typically based on lifting techniques [45, 28] or on their interpretation as a delay system as, e.g., in [122]. In contrast, we follow [61] and rely on a representation as an impulsive system. This permits us to employ Theorem 3.3 or 3.8 for systematic output-feedback design with unprecedented ease.

Formally, for real matrices of appropriate dimensions, some initial condition $x(0) \in \mathbb{R}^n$, generalized disturbances $d \in L_2$, $d_J \in \ell_2$ and a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$, we consider an open-loop sampled-data system

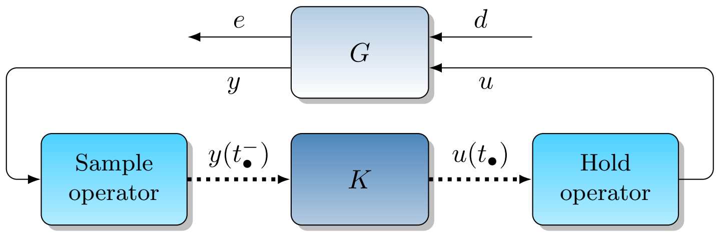



Figure 3.6: Standard sampled-data closed-loop interconnection involving a continuous-time system $G$, a discrete-controller $K$ a sample operator and a hold operator.

of the form

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ u(t) \end{pmatrix}, \quad \begin{pmatrix} e_J(k) \\ y_J(k) \end{pmatrix} = \begin{pmatrix} C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & 0_{\bullet \times n_u} \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \\ u(t_k^-) \end{pmatrix} \tag{3.12a}$$

for $t \geq 0$ and $k \in \mathbb{N}$ where the control input $u$ is piecewise constant, i.e.,

$$u(t) = u(t_k) \quad \text{for all} \quad t \in [t_k, t_{k+1}) \text{ and } k \in \mathbb{N}_0. \tag{3.12b}$$

In particular, only output samples are available for control and the control input is the result of a so-called zero-order-hold operation. The standard sampled-data closed-loop interconnection is illustrated in Fig. 3.6 and corresponds to choosing $(C_J, D_J, D_{J12}) = 0$ as well as $d_J = d(t_\bullet^-)$, $y(t) = C_{J2}x(t) + D_{J21}d(t)$ and $y_J = y(t_\bullet^-)$ in the description (3.12).

In order to reformulate (3.12) as an impulsive system with description (3.1), the property (3.12b) is handled by viewing $u$ as an additional state.

This leads to the description

$$
\begin{pmatrix} \dot{x}(t) \\ \dot{u}(t) \\ \hline e(t) \\ \hdashline y(t) \end{pmatrix} = \left(\begin{array}{cc|c:c} A & B_2 & B & 0 \\ 0 & 0 & 0 & 0 \\ \hline C & D_{12} & D & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} x(t) \\ u(t) \\ \hline d(t) \\ \hdashline \hat{u}(t) \end{pmatrix}, \quad \begin{pmatrix} x(t_k) \\ u(t_k) \\ \hline e_J(k) \\ \hdashline y_J(k) \end{pmatrix} = \left(\begin{array}{cc|c:c} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline C_J & D_{J12} & D_J & 0 \\ \hdashline C_{J2} & 0 & D_{J21} & 0 \end{array}\right) \begin{pmatrix} x(t_k^-) \\ u(t_k^-) \\ \hline d_J(k) \\ \hdashline u_J(k) \end{pmatrix} \tag{3.13}
$$

for $t \geq 0$ and $k \in \mathbb{N}$, which is indeed a special case of (3.1). As a consequence of Theorem 3.8, we obtain the first statement of the following result.

**Theorem 3.12** *Let $D \in \mathbb{R}^{\bullet \times n_d}$ and $D_J \in \mathbb{R}^{\bullet \times n_{d_J}}$, and suppose that $P$ and $P_J$ are nonsingular with exactly $n_d$ and $n_{d_J}$ negative eigenvalues, respectively. Then there exists a controller* (3.2) *for the system* (3.12) *such that the corresponding closed-loop analysis LMIs* (3.5) *are feasible if and only if the synthesis LMIs* (3.8) *are feasible for the system* (3.13). *Moreover, the latter synthesis LMIs are feasible if and only if there exist continuously differentiable $X_1$ and $Y = \left(\begin{smallmatrix} Y_1 & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)$ satisfying*

$$
\begin{pmatrix} Y & (\bullet)^\top \\ \begin{pmatrix} I_n & 0_{n \times n_u} \end{pmatrix} & X_1 \end{pmatrix} \succ 0, \tag{3.14a}
$$

$$
(\bullet)^\top \left(\begin{array}{cc|c} 0 & X_1 & \\ X_1 & \dot{X}_1 & \\ \hline & & P \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) \prec 0 \ \textit{and}\ (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & P^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -\hat{A}^\top & -\hat{C}^\top \\ \hline 0 & I \\ -\hat{B}^\top & -D^\top \end{array}\right) \succ 0 \tag{3.14b,c}
$$

*on $[0, T_{\max}]$ as well as*

$$
(\bullet)^\top \left(\begin{array}{cc|c} X_1(0) & 0 & \\ 0 & -X_1 & \\ \hline & & P_J \end{array}\right) \left(\begin{array}{cc} I & 0 \\ I & 0 \\ \hline C_J & D_J \\ 0 & I \end{array}\right) V_J \prec 0 \tag{3.14d}
$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c} Y_1(0) & 0 & \\ 0 & -Y & \\ \hline & & P_J^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -\hat{A}_J^\top & -\hat{C}_J^\top \\ \hline 0 & I \\ 0 & -D_J^\top \end{array}\right) \succ 0 \qquad (3.14e)$$

*on* $[T_{\min}, T_{\max}]$. *Here* $\hat{A} := \left(\begin{smallmatrix} A & B_2 \\ 0 & 0 \end{smallmatrix}\right)$, $\hat{B} := \left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)$, $\hat{C} := (C, D_{12})$, $\hat{A}_J := (I, 0)$, $\hat{C}_J := (C_J, D_{J12})$ *and* $V_J$ *is a basis matrix of* $\ker(C_{J2}, D_{J21})$.

Note that compared to the generic synthesis LMIs (3.8) for (3.13), the LMIs (3.14) are less expensive to solve since fewer decision variables are involved. Indeed, $X$ in (3.8) for (3.13) takes values in $\mathbb{S}^{n+n_u}$, while $X_1$ in (3.14) takes values in $\mathbb{S}^n$.

*Sketch of Proof.* We only have to show the second statement since the first one is a direct consequence of Theorem 3.8.

*Only if:* This follows from noting that, due to the particular structure of (3.13), the annihilators appearing in Theorem 3.8 can be chosen as

$$U = I, \quad V = I, \quad U_J = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad V_J = \begin{pmatrix} V_1 & 0 \\ 0 & I \\ V_2 & 0 \end{pmatrix},$$

where $\left(\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right)$ is a basis matrix of $\ker(C_{J2}, D_{J21})$. This leads immediately from (3.8c) and (3.8e) to (3.14c) and (3.14e), respectively. The remaining inequalities are obtained from (3.8a), (3.8b) and (3.8d) by canceling the block rows and columns corresponding to the right lower $n_u \times n_u$ block of $X$.

*If:* This follows from augmenting the given matrix-valued map $X_1$ as $X(\tau) := \operatorname{diag}(X_1(\tau), \frac{\alpha}{\tau+1} I_{n_u})$ for all $\tau \in [0, T_{\max}]$ and for some large enough $\alpha > 0$. Let us exemplary consider the inequality (3.8b) for the system (3.13) and denote its left hand side by $\Gamma$. This $\tau$-dependent matrix can naturally be partitioned into a $3 \times 3$ block matrix with its $(2, 2)$ block being

$$\Gamma_{22}(\tau) = (\bullet)^\top \dot{X}(\tau) \begin{pmatrix} 0 \\ I_{n_u} \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} D_{12} \\ 0 \end{pmatrix} = \frac{-\alpha}{(\tau+1)^2} I_{n_u} + (\bullet)^\top P \begin{pmatrix} D_{12} \\ 0 \end{pmatrix}.$$

By continuity of the involved functions, by compactness of $[0, T_{\max}]$, since all other blocks of $\Gamma$ do not depend on $\alpha$, and since $\left(\begin{smallmatrix} \Gamma_{11} & \Gamma_{13} \\ \Gamma_{31} & \Gamma_{33} \end{smallmatrix}\right)$ is exactly the left hand side of (3.14b), a Schur complement argument permits us indeed to conclude that (3.8b) holds for all sufficiently large $\alpha > 0$. Based on similar arguments we can find some $\alpha > 0$ such that the inequalities (3.8a) and (3.8d) also hold. The remaining LMIs (3.8c) and (3.8e) are directly obtained from (3.14c) and (3.14e). ●

Apparently, it does not seem to be possible to further reduce the computational burden by removing blocks of $Y$ without introducing conservatism. However, due to the particular block triangular structure of the closed-loop maps $\mathcal{A}$ and $\mathcal{A}_J$, further simplifications along the lines of the ones suggested in [18] for static-state feedback design are possible, but these are not discussed here.

Observe that if $(t_k)_{k \in \mathbb{N}_0}$ is a sequence satisfying (EDT), i.e., $t_{k+1} - t_k = T$ for all $k$, then the impulsive controller resulting from the corresponding modification of Theorem 3.12 can also be expressed as a discrete-time LTI controller of order $n + n_u$ due to the particular structure of (3.12). This follows from defining the state $\tilde{x}_c(k) := x_c(t_k^-)$ and the latter controller admits the form

$$\begin{pmatrix} \tilde{x}_c(k+1) \\ y_J(k) \end{pmatrix} = \begin{pmatrix} U(T) A_J^c & U(T) B_J^c \\ C_J^c & D_J^c \end{pmatrix} \begin{pmatrix} \tilde{x}_c(k) \\ u_J(k) \end{pmatrix}$$

for $k \in \mathbb{N}$ and where $U(t)$ is the so-called fundamental solution matrix that satisfies $\dot{U}(\tau) = A^c(\tau) U(\tau)$ on $[0, T]$ with initial condition $U(0) = I$.

Performing an analogous reformulation in the presence of more general sequences $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) yields a time-varying discrete-time controller instead.

Finally, we emphasize that the conditions (3.14) easily permit a seamless extension, e.g., to gain-scheduling controller synthesis or to the design of consensus protocols as discussed in Chapter 5.

## Example

As an illustration let us consider the LTI system

$$\begin{pmatrix} \dot{x}(t) \\ \hline e(t) \\ \hdashline y(t) \end{pmatrix} = \left(\begin{array}{c|c:c} A & B & B_2 \\ \hline C & D & D_{12} \\ \hdashline C_2 & 0 & 0 \end{array}\right) \begin{pmatrix} x(t) \\ \hline d(t) \\ \hdashline u(t) \end{pmatrix} = \left(\begin{array}{cc|c:c} 0 & 1 & 1 & 0 \\ -2 & 0.1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 \\ \hdashline 1 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} x(t) \\ \hline d(t) \\ \hdashline u(t) \end{pmatrix} \tag{3.15}$$

for $t \geq 0$ which is taken from [18] and augmented with a performance channel. The design goals are to render the fist state small despite the presence of a disturbance affecting this state and to bound the control input $u$ which acts on the second state. Synthesizing a standard $H_\infty$-controller for this system via `hinfsyn` in Matlab leads to an output of the closed-loop interconnection as depicted in the top row of Fig. 3.7; this output is the response to the input disturbance $d(t) := \chi_{[0,10)}(t) - \chi_{[10,20)}(t)$ and for zero initial conditions.

Next, we suppose that the measurements $y$ are only available at times $t_k$, where the sequence $(t_k)_{k\in\mathbb{N}_0}$ satisfies (RDT) with $[T_{\min}, T_{\max}] = [0.4, 0.5]$. Similarly as done earlier, we can still design a controller that achieves a small energy gain by viewing the resulting open-loop system as an impulsive system and by employing Theorem 3.8. Doing so results in a closed-loop response as depicted in the middle row of Fig. 3.7. As expected, we can

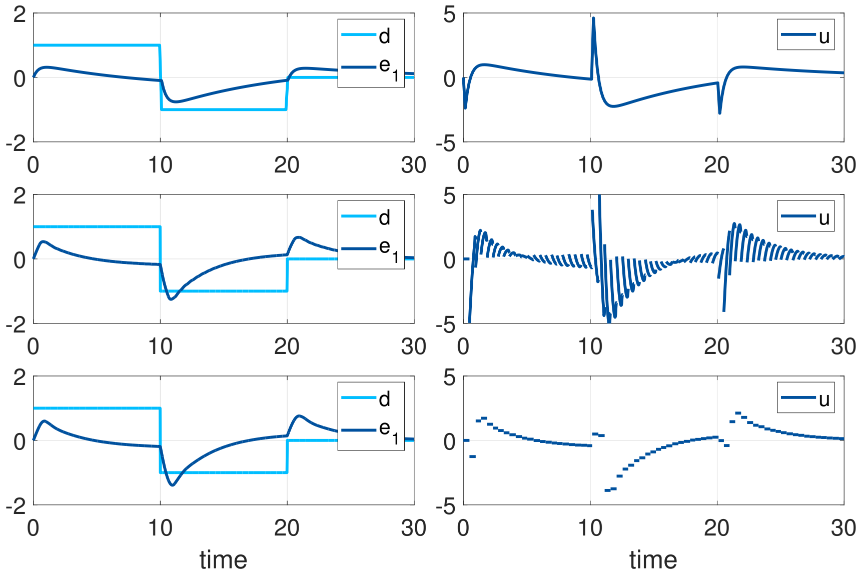



Figure 3.7: Some disturbance $d$ and closed-loop responses involving variations of the system (3.15) and correspondingly designed controllers.

observe that the disturbance rejection properties degrade and that more control effort is required because less information is available.

Finally, we can additionally require that the control input $u$ is piecewise constant on the intervals $[t_k, t_{k+1})$, $k \in \mathbb{N}_0$ which leads to a sampled-data system as described by (3.12) with $C_{J2} := C_2$ and vanishing matrices $C_J$, $D_J$, $D_{J12}$, $D_{J21}$. A suitable controller is now readily obtained by applying Theorem 3.12 and yields the closed-loop response as shown in the bottom row of Fig. 3.7. Here, we note that the disturbance is rejected almost as before, but the control signal $u$ is not as aggressive.

The computed upper bounds on the energy gain in the three design scenarios are 0.410, 0.764 and 0.847, respectively.

### 3.1.3 Controller Design for Switched Systems

Next, we demonstrate that the arguments underlying our synthesis results Theorem 3.3 and 3.8 even permit us to systematically design not one but several types of output-feedback controllers for switched systems. To this end, we consider, for real matrices of appropriate dimensions, some initial condition $x(0) \in \mathbb{R}^n$ and a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$, an open-loop switched system with the description

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_{\sigma(t)} & B_{\sigma(t)} & B_{2\sigma(t)} \\ C_{\sigma(t)} & D_{\sigma(t)} & D_{12\sigma(t)} \\ C_{2\sigma(t)} & D_{21\sigma(t)} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ u(t) \end{pmatrix} \tag{3.16}$$

for $t \geq 0$ and for a switching function $\sigma : [0, \infty) \to \{1, \ldots, N\}$ which is constant on each of the intervals $[t_{k-1}, t_k)$. The first controller we aim to design is of the form

$$\begin{aligned} \begin{pmatrix} \dot{x}_c(t) \\ u(t) \end{pmatrix} &= \begin{pmatrix} A^c_{\sigma(t)}(\theta(t)) & B^c_{\sigma(t)}(\theta(t)) \\ C^c_{\sigma(t)}(\theta(t)) & D^c_{\sigma(t)}(\theta(t)) \end{pmatrix} \begin{pmatrix} x_c(t) \\ y(t) \end{pmatrix}, \\ x_c(t_k) &= A^c_{J\sigma(t_k^-)\sigma(t_k)}(\theta(t_k^-))x_c(t_k^-) \end{aligned} \tag{3.17}$$

for $t \geq 0$ and $k \in \mathbb{N}$; recall that $\theta$ denotes the clock as defined in (2.2). Note that the latter controller is itself a switched system since it is defined by (time-varying) matrices that depend on the switching function $\sigma$, i.e., on the currently active mode. Its jump component additionally depends on the previous mode. Interconnecting (3.16) and (3.17) yields the closed-loop switched system

$$\begin{aligned} \begin{pmatrix} \dot{x}_{cl}(t) \\ e(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{A}_{\sigma(t)} & \mathcal{B}_{\sigma(t)} \\ \mathcal{C}_{\sigma(t)} & \mathcal{D}_{\sigma(t)} \end{pmatrix} \begin{pmatrix} x_{cl}(t) \\ d(t) \end{pmatrix}, \\ x_{cl}(t_k) &= \mathcal{A}_{J\sigma(t_k^-)\sigma(t_k)}(\theta(t_k^-))x_{cl}(t_k^-) \end{aligned} \tag{3.18}$$

with state $x_{cl} := \left(\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right)$ and jump map $\mathcal{A}_{Jkl}(\tau) := \operatorname{diag}(I, A^c_{Jkl}(\tau))$; the remaining calligraphic maps are analogously defined as the ones in (3.3a). By generalizing the stability criteria for switched systems in Corollary 2.5 to also account for quadratic performance with an index $P$ satisfying Assumption 3.1 and to systems with description (3.18), we obtain the following.

**Lemma 3.13** (Closed-Loop Analysis for Switched Systems) *The system* (3.18) *is stable and achieves quadratic performance with index $P$ for all $(t_k)_{k\in\mathbb{N}_0}$ satisfying* (RDT) *if there exists functions $\mathcal{X}_1, \dots, \mathcal{X}_N \in C^1([0, T_{\max}], \mathbb{S}^{2n})$ satisfying the inequalities*

$$\mathcal{X}_k \succ 0 \quad \text{and} \quad (\bullet)^\top \begin{pmatrix} 0 & \mathcal{X}_k \\ \mathcal{X}_k & \dot{\mathcal{X}}_k \end{pmatrix} \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ I & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} \mathcal{C}_k & \mathcal{D}_k \\ 0 & I \end{pmatrix} \prec 0 \tag{3.19a,b}$$

*on* $[0, T_{\max}]$ *for all* $k \in \{1, \dots, N\}$ *as well as*

$$\mathcal{A}_{Jkl}^\top \mathcal{X}_l(0) \mathcal{A}_{Jkl} - \mathcal{X}_k \prec 0 \tag{3.19c}$$

*on* $[T_{\min}, T_{\max}]$ *for all* $k, l \in \{1, \dots, N\}$*.*

Due to the intended similarities with the closed-loop analysis conditions for impulsive systems provided in Corollary 3.2, we can adjust the synthesis criteria in Theorem 3.3 or 3.8 in order to characterize the existence of a controller (3.17) by means of convex optimization. In particular, we have the following constructive result.

**Corollary 3.14** *Let $D_1 \in \mathbb{R}^{\bullet \times n_d}$ and suppose that $P$ is nonsingular with exactly $n_d$ negative eigenvalues. Moreover, let $U_k$ and $V_k$ be basis matrices of* $\ker((B_{2k}^\top, D_{12k}^\top))$ *and* $\ker((C_{2k}, D_{21k}))$*, respectively. Then there exists a controller* (3.17) *for the system* (3.16) *such that the analysis LMIs* (3.19) *are feasible if and only if there exist continuously differentiable $X_1, \dots, X_N$,*

$Y_1, \ldots, Y_N$ *satisfying*

$$\begin{pmatrix} Y_k & I \\ I & X_k \end{pmatrix} \succ 0, \tag{3.20a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X_k & \\ X_k & \dot{X}_k & \\ \hline & & P \end{array}\right) \begin{pmatrix} A_k & B_k \\ I & 0 \\ \hline C_k & D_k \\ 0 & I \end{pmatrix} V_k \prec 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y}_k & Y_k & \\ Y_k & 0 & \\ \hline & & P^{-1} \end{array}\right) \begin{pmatrix} I & 0 \\ -A_k^\top & -C_k^\top \\ \hline 0 & I \\ -B_k^\top & -D_k^\top \end{pmatrix} U_k \succ 0 \tag{3.20b,c}$$

*on* $[0, T_{\max}]$ *for all* $k \in \{1, \ldots, N\}$ *as well as*

$$X_l(0) - X_k \prec 0 \quad \text{and} \quad Y_l(0) - Y_k \succ 0 \tag{3.20d,e}$$

*on* $[T_{\min}, T_{\max}]$ *for all* $k, l \in \{1, \ldots, N\}$.

Since the conditions in Corollary 3.14 are necessary and sufficient, it is in a sense natural to consider the design of switched controllers (3.17) admitting a jump component. However, in the literature on switched systems the goal is almost always to design switched controllers without a jump component, i.e., with a trivial one. We can also design such controllers by employing suitable modifications of the proof of Theorem 3.3.

**Theorem 3.15** (Synthesizing Controllers without Jump Component) *There exists a controller* (3.17) *with* $A^c_{Jkl}(\tau) = I$ *for all* $k, l, \tau$ *such that the closed-loop system* (3.18) *is stable and achieves quadratic performance with index* $P$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist continuously differentiable* $X_1, \ldots, X_N$, $Y_1, \ldots, Y_N$ *satisfying* (3.20a)-(3.20d) *and* $Y_l(0) = Y_k(\tau)$ *for all* $\tau \in [T_{\min}, T_{\max}]$ *and all* $k, l \in \{1, \ldots, N\}$.

*Proof.* Recall that (3.20a) permits us to find differentiable and pointwise nonsingular functions $U_k$ and $V_k$ satisfying $U_k V_k^\top = I - X_k Y_k$. We can then define suitable certificates $\mathcal{X}_k \succ 0$ by $\mathcal{X}_k := \mathcal{Y}_k^{-T}\mathcal{Z}_k$ with

$$\mathcal{Y}_k := \begin{pmatrix} Y_k & I \\ V_k^\top & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Z}_k := \begin{pmatrix} I & 0 \\ X_k & U_k \end{pmatrix} = \begin{pmatrix} Y_k & I \\ I & X_k \end{pmatrix} \mathcal{Y}_k^{-1}.$$

It is now crucial to choose $V_k := Y_k$ and $U_k := Y_k^{-1} - X_k$ because this allows us to conclude that $\mathcal{Y}_l(0) = \mathcal{Y}_k(\tau)$ holds for all $\tau \in [T_{\min}, T_{\max}]$ and all $k, l \in \{1, \ldots, N\}$. In particular, for any $\tau \in [T_{\min}, T_{\max}]$ and $k, l \in \{1, \ldots, N\}$, we then infer by (3.20d)

$$\begin{aligned} \mathcal{Y}_l(0)^\top \big(\mathcal{X}_l(0) - \mathcal{X}_k(\tau)\big)\mathcal{Y}_l(0) &= \mathcal{Y}_l(0)^\top \mathcal{X}_l(0)\mathcal{Y}_l(0) - \mathcal{Y}_k(\tau)\mathcal{X}_k(\tau)\mathcal{Y}_k(\tau) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & X_l(0) - X_k(\tau) \end{pmatrix} \preccurlyeq 0. \end{aligned}$$

It remains then to recall that (3.20a)-(3.20c) imply the existence of maps $A_k^c, B_k^c, C_k^c, D_k^c$ such that (3.19b) is satisfied and that quadratic performance is still assured even if the inequality in (3.19c) is non-strict. ●

**Remark 3.16** Note that the maps $Y_1, \ldots, Y_N$ can vary on $(0, T_{\min})$ and restricting them to be identical and constant on $[0, T_{\max}]$ is not required. Our proposed additional constraint on $Y_1, \ldots, Y_N$ can still introduce some conservatism which can again be reduced by introducing slack variables as done in Theorem 2.12 if desired.

Even if the jump component of the controller (3.17) is rendered trivial, the control input $u$ is in general discontinuous and can involve large jumps due to the switching nature of the controller (3.17). Such large jumps in the control input might be not acceptable in a number of practical situations. Instead, one aims in such cases to design a controller such that the signal $u$ is as smooth as possible which is the so-called *bumpless transfer* controller design problem [58, 33] that was at first considered in the context of LTI systems, e.g., in [69]. In order to tackle this problem, the authors in [33] consider the design of switched controllers with vanishing direct feedthrough matrices $D_1^c, \ldots, D_N^c$ in order avoid the jumps from the measured output $y$. Moreover, they argue that bumps in the control input can be reduced by including a simple constraint which is, our situation, of the form

$$\sup_{\tau\in[T_{\min},T_{\max}]} \|\big(C_k^c(\tau) - C_l^c(0)\big)B_k^c(\tau)\|^2 \leq \beta^2 \quad \text{for all} \quad k, l \in \{1, \ldots, N\} \tag{3.21}$$

for some parameter $\beta > 0$. In order to provide convex criteria for designing a controller satisfying (3.21) next to achieving quadratic performance, we can no longer rely on the elimination lemma C.11 since multiple objectives are involved. Instead, we have to employ the convexifying parameter transformation as done in Theorem 3.3 which leads to the following synthesis criteria. Note that they could again be rendered less conservative but more expensive to solve by introducing slack variables.

**Theorem 3.17** (Synthesizing Controllers with Bump Limitation) *There exists a controller* (3.17) *satisfying* (3.21), $D_k^c = 0$ *and* $A_{Jkl}^c = I$ *for all* $k, l$ *such that the closed-loop system* (3.18) *is stable and achieves quadratic performance with index* $P$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist continuously differentiable* $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ *and continuous* $K_1, \ldots, K_N$, $L_1, \ldots, L_N$, $M_1, \ldots, M_N$ *satisfying*

$$\mathbf{X}_k \succ 0 \quad \text{and} \quad (\bullet)^\top \begin{pmatrix} 0 & I \\ I & \mathbf{Z}_k \end{pmatrix} \begin{pmatrix} \mathbf{A}_k & \mathbf{B}_k \\ I & 0 \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} \mathbf{C}_k & \mathbf{D}_k \\ 0 & I \end{pmatrix} \prec 0 \tag{3.22a,b}$$

*on* $[0, T_{\max}]$ *for all* $k \in \{1, \ldots, N\}$ *as well as*

$$X_l(0) - X_k \prec 0, \quad Y_l(0) = Y_k \tag{3.22c,d}$$

*and*

$$\left(\begin{array}{cccc|cc} I & 0 & 0 & 0 & M_k(\tau) & M_l(0) \\ 0 & \beta^2 I & L_k(\tau)^\top & L_k(\tau)^\top & 0 & 0 \\ 0 & L_k(\tau) & X_k(\tau) & 0 & I & 0 \\ 0 & L_k(\tau) & 0 & X_k(\tau) & 0 & -I \\ \hline M_k(\tau)^\top & 0 & I & 0 & Y_k(\tau) & 0 \\ M_l(0)^\top & 0 & 0 & -I & 0 & Y_l(0) \end{array}\right) \succ 0 \tag{3.22e}$$

*on* $[T_{\min}, T_{\max}]$ *for all* $k, l \in \{1, \ldots, N\}$*. Here, the maps* $\left(\begin{smallmatrix} \mathbf{A}_k & \mathbf{B}_k \\ \mathbf{C}_k & \mathbf{D}_k \end{smallmatrix}\right)$, $\mathbf{X}_k$ *and* $\mathbf{Z}_k$ *are defined as*

$$\left(\begin{array}{cc|c} A_k Y_k & A_k & B_k \\ 0 & X_k A_k & X_k B_k \\ \hline C_k Y_k & C_k & D_k \end{array}\right) + \left(\begin{array}{cc} 0 & B_{2k} \\ I & 0 \\ \hline 0 & D_{12k} \end{array}\right) \begin{pmatrix} K_k & L_k \\ M_k & 0 \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_{2k} & D_{21k} \end{array}\right),$$

$$\begin{pmatrix} Y_k & I \\ I & X_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\dot{Y}_k & 0 \\ 0 & \dot{X}_k \end{pmatrix},$$

*respectively.*

The argument that the inequality (3.22e) implies the bump limitation (3.21) is essentially from [33] and repeated here for convenience.

*Proof.* Due to (3.22a)-(3.22d), we can construct $A_k^c$, $B_k^c$, $C_k^c$ similarly as in the proof of Theorem 3.3 in such a fashion that the inequalities (3.19) are satisfied with $D_k^c = 0$ and $A_{Jkl}^c = I$ for all $k, l \in \{1, \ldots, N\}$. As in the proof of Theorem 3.15, the latter identity requires to choose $V_k := Y_k$ and $U_k := Y_k^{-1} - X_k$ in the construction of the certificate $\mathcal{X}_k$. In particular, for this choice we have more precisely

$$B_k^c = U_k^{-1} L_k = (Y_k^{-1} - X_k)^{-1} L_k \quad \text{and} \quad C_k^c = M_k V_k^{-T} = M_k Y_k^{-1}.$$

Let now $\tau \in [T_{\min}, T_{\max}]$ and $k, l \in \{1, \dots, N\}$ be arbitrary. Then applying the Schur complement C.6 on the inequality (3.22e) yields

$$\left(\begin{array}{cc|cc} I + \Omega_1 & 0 & -C_k^c(\tau) & C_l^c(0) \\ 0 & \beta^2 I & L_k(\tau)^\top & L_k(\tau)^\top \\ \hline -C_k^c(\tau)^\top & L_k(\tau) & X_k(\tau) - Y_k(\tau)^{-1} & 0 \\ C_l^c(0)^\top & L_k(\tau) & 0 & X_k(\tau) - Y_k(\tau)^{-1} \end{array}\right) \succ 0$$

with $\Omega_1 := -(\bullet)^\top Y_k(\tau)^{-1} M_k(\tau) - (\bullet)^\top Y_l(0)^{-1} M_l(0) \preccurlyeq 0$; here we did also make use of (3.22d). Another application of the Schur complement results in

$$\begin{pmatrix} I + \Omega_2 & (C_l^c(0) - C_k^c(\tau))B_k^c(\tau) \\ (\bullet)^\top & \beta^2 I + \Omega_3 \end{pmatrix} \succ 0$$

for some $\Omega_2, \Omega_3 \preccurlyeq 0$. Hence, we conclude

$$\begin{pmatrix} I & (C_l^c(0) - C_k^c(\tau))B_k^c(\tau) \\ (\bullet)^\top & \beta^2 I \end{pmatrix} \succ 0$$

which yields the desired constraint (3.21) by a final application of the Schur complement and since $\tau \in [T_{\min}, T_{\max}]$ as well as $k, l \in \{1, \dots, N\}$ were arbitrary. ●

**Remark 3.18** (Alternative Designs for Switched Systems) We stress that the multitude of possibilities to constrain the switching function $\sigma$ in (3.16) leads to various design approaches that each result in (structurally) different analysis criteria. Unfortunately, this makes it rather difficult to find appropriate literature on the concrete case one is working on and we, hence, only mention few alternatives. Probably one of the most well-cited paper dealing with dynamic output-feedback in the context of arbitrary dwell-time is [75] which relies on the Youla parametrization. For switching functions with (minimum) dwell-time constraints, even recent publications as, e.g., [4, 20, 89], that employ dedicated analysis results, merely consider state-feedback which is in contrast to our last three results. Output-feedback design for parametrically varying systems with a hysteresis switching function is considered in [102]; here, the switching function does depend on the time-varying parameter entering the underlying system.

Another relevant synthesis problem is referred to as *co-design* and deals with the issue of simultaneously finding a feedback controller and a switching function for the underlying system such some closed-loop objective is achieved [41, 99]; we stress that in this case the switching function is a degree of freedom. Output-feedback co-design is performed, e.g., in [41], and usually relies on so-called Riccati-Metzler inequalities and min-type piecewise quadratic Lyapunov functions for the underlying analysis; the considered switching functions are typically of the form $\sigma(y(t)) = \arg\min y(t)^\top X_i y(t)$ where $y$ is the system's measured output. In [23] a sampled-data co-design problem is considered involving clock-dependent LMIs, but assumes that the full state is available for control. It is expected the approach illustrated in this chapter permits removing the latter limitation.

## Example

As an example, let us consider a switched system (3.16) with

$$\begin{pmatrix} A_1 & B_1 \end{pmatrix} = \left(\begin{array}{cc|c} 0 & 1 & 1 \\ -5 & 1 & 0 \end{array}\right), \quad \begin{pmatrix} A_2 & B_2 \end{pmatrix} = \left(\begin{array}{cc|c} 5 & 5 & 5 \\ 1 & 1 & 1 \end{array}\right), \quad \begin{pmatrix} A_3 & B_3 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & -5 & 1 \\ 5 & 2 & 0 \end{array}\right), \tag{3.23a}$$

as well as

$$B_{2k} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} C_k & D_k & D_{12k} \\ C_{2k} & D_{21k} & 0 \end{pmatrix} = \left(\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.3 \\ \hline 1 & 0 & 0 & 0 \end{array}\right) \quad \text{for} \quad k \in \{1, 2, 3\} \tag{3.23b}$$

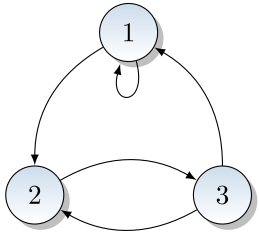



Figure 3.8: A graph $G = (V, E)$ with vertices $V = \{1, 2, 3\}$ and edges $E = \{(1, 1), (1, 2), (2, 3), (3, 2), (3, 1)\}$.

and a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (EDT) with $T = 2$. Moreover, let us assume that the switching function $\sigma$ is constrained as

$$(\sigma(t_k^-), \sigma(t_k)) \in E \quad \text{for all} \quad k \in \mathbb{N}_0 \tag{3.23c}$$

where $E$ denotes the edge set of the unweighted graph $G$ depicted in Fig. 3.8. Recall that designing an output-feedback controller for this system is possible based on small modifications of Corollary 3.14, Theorem 3.15 or of Theorem 3.17 along the lines of the ones mentioned in Remark 2.6. In the sequel, we choose the performance index $P$ accordingly to the energy gain and utilize the sum-of-squares approach D.1 with $\varepsilon = 0.01$, ansatz polynomials of degree $d_a = 4$ and multiplier polynomials of degree $d_m = 2$ in order to turn the involved differential LMIs into standard SDPs.

Note at first that an application of Corollary 3.14 for (3.23) yields an optimal upper bound of 3.698 on the achievable closed-loop energy gain. In contrast, omitting the constraint (3.23c), i.e., allowing for arbitrary switching, yields an upper bound of 4.143 which is partly because the system might stay longer in mode 2 which amplifies the input disturbance by a lot. The response of the switched system (3.23) interconnection with a close-to-optimal controller (3.17) obtained from Corollary 3.14 to the dis-

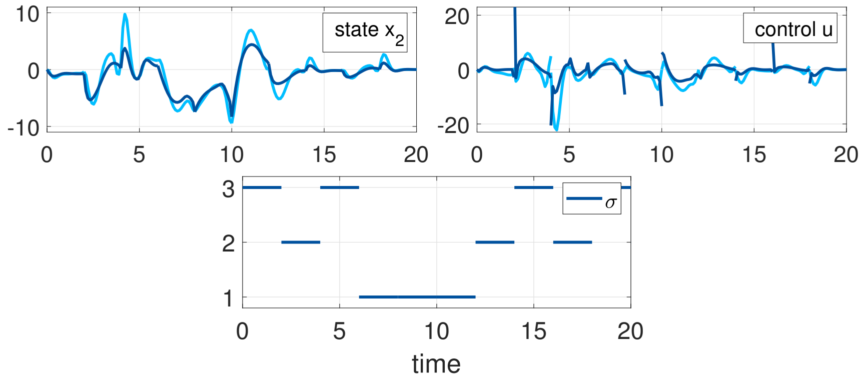



Figure 3.9: Second state and control input of the system (3.16) with (3.23) interconnected with a controller with jump component obtained from Corollary 3.14 (dark blue) and one with bump limitation from Theorem 3.17 (light blue) for some switching function $\sigma$.

turbance $d(t) = -4\chi_{[0,5)}(t) + 4\chi_{[5,10)}(t) - \chi_{[10,18)}(t)$ is depicted in Fig. 3.9 together with the switching function $\sigma$ in dark blue; recall that $\chi_{[a,b)}$ is the characteristic function of the interval $[a,b)$ and note that the performance output of the system (3.16) with (3.23) is given by $e = \left(\begin{smallmatrix} x_2 \\ 0.3u \end{smallmatrix}\right)$.

For this particular example, employing Theorem 3.17 instead of Corollary 3.14 in order to design a controller (3.17) with trivial jump component, yields visually not much of a difference in the closed-loop response and we merely observe a minor increase in the determined optimal upper bound which equals 3.758. We emphasize at this point that enforcing the jump component of the controller (3.17) to be trivial can actually lead to *larger* jumps in the control signal $u$ which might be counter intuitive. However, for several examples we observe that the controller's jump component plays a vital role in reducing the jumps induced by the switching of modes in the flow component of the controller.

Finally, let us employ Theorem 3.17 in order to design a controller satisfying the bump limitation constraint (3.21) with $\beta = \sqrt{10}$. The resulting closed-loop response illustrated in Fig. 3.9 in light blue shows that the control input $u$ indeed admits fewer jumps as desired. The price we have to pay for this improved behavior is that the disturbance attenuation properties of the controller degrade as reflected by the larger amplitude of the corresponding state response $x_2$.

### Summary

This concludes the first part of this chapter on the design of dynamic output-feedback controllers for impulsive and related hybrid systems. A summary of the presented results is given in Section 3.3. Next, we move on to the design of static output-feedback controllers for such systems.

## 3.2 Static Output-Feedback Controller Design

Even for standard LTI systems, the design of static output-feedback controllers constitutes a conceptually simple and yet theoretically very challenging problem. Such a design is also a popular approach of practical interest due to its straightforward implementation and the fact that, typically, only some (and not all) states of the underlying dynamical system are available for control. However, in contrast to, e.g., the design of static state-feedback or dynamic full-order controllers, the synthesis of static output-feedback controllers is intrinsically a challenging bilinear matrix inequality feasibility problem. Such problems are in general non-convex, non-smooth and NP-hard to solve [159]. These troublesome properties have led to the development of a multitude of (heuristic) design approaches, which only yield sufficient conditions for the existence of such static controllers. Next to providing only sufficient conditions, another downside of these approaches is that they might get stuck in a local minimum of the underlying optimization problem that can be far away from the global minimum of interest. Nevertheless, such approaches are employed and reported to work nicely on various practical examples. Two detailed surveys on static output-feedback design for standard LTI systems elaborating on several of such approaches are provided in [158, 128].

Static output-feedback controller design for hybrid systems, as studied for example in [37, 1], is even more difficult and is less frequently considered in the literature. This is partly due to vast amount of possibilities to describe hybrid systems and due to the typically more involved analysis conditions if compared to those for standard LTI systems. A consequence of the increased complexity is that not all of the approaches discussed in the surveys [158, 128] generalize nicely to hybrid systems such as the algorithm `hinfstruct` from [9] or `hifoo` from [27]. This is in contrast to approaches based on solving LMIs such as the classical D-K iteration as suggested, e.g., in [15, 47] or methods involving S-variables as, e.g., in Chapter 6.3 of [46]. These approaches are typically much more amenable for generalizations, but tend to be slower due to underlying complexity of solving LMI problems.

Throughout this section we restrict our attention to static controller synthesis for linear impulsive systems, but we emphasize that the provided results can be adapted without much effort to other interesting hybrid systems similarly as demonstrated in the previous section. We begin by concretely specifying the considered static design problem.

### 3.2.1 Problem Description

We consider again the open-loop impulsive system with generic description (3.1) and our main goal is the design of a *static* output-feedback controller for this system of the form

$$u(t) = K(\theta(t))y(t), \quad u_J(k) = K_J(\theta(t_k^-))y_J(k) \tag{3.24}$$

for $t \geq 0$ and $k \in \mathbb{N}$ such that the corresponding closed-loop energy gain is as small as possible. In contrast to the dynamic controller (3.2) considered in the previous section, the controller (3.24) does not involve an internal state variable which is particularly convenient for its implementation since its output $(u(t), u_J(k))$ for fixed $(t, k)$ is then readily obtained by two simple matrix-vector multiplications.

The interconnection of the system (3.1) and the controller (3.24) is given by

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ e(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{A}(\theta(t)) & \mathcal{B}(\theta(t)) \\ \mathcal{C}(\theta(t)) & \mathcal{D}(\theta(t)) \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix}, \\ \begin{pmatrix} x(t_k) \\ e_J(k) \end{pmatrix} &= \begin{pmatrix} \mathcal{A}_J(\theta(t_k^-)) & \mathcal{B}_J(\theta(t_k^-)) \\ \mathcal{C}_J(\theta(t_k^-)) & \mathcal{D}_J(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ d_J(k) \end{pmatrix} \end{aligned} \tag{3.25}$$

for $t \geq 0$ and $k \in \mathbb{N}$ as well as with describing matrix-valued maps

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} A+B_2KC_2 & B+B_2KD_{21} \\ C+D_{12}KC_2 & D+D_{12}KD_{21} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix} K \begin{pmatrix} C_2 & D_{21} \end{pmatrix}.$$

and analogously defined $\mathcal{A}_J, \mathcal{B}_J, \mathcal{C}_J, \mathcal{D}_J$. Since this closed-loop interconnection is of the same form as the impulsive system (2.7), we can easily determine (optimal) upper bounds on its energy gain based on our analysis result Theorem 2.8. The resulting LMI criteria are repeated here for convenience where $(P_\gamma, P_{J\gamma}) := \left( \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix} \right)$.

**Corollary 3.19** (Closed-Loop Analysis) *The system* (3.25) *is stable and its energy gain is bounded by* $\gamma$ *for all* $(t_k)_{k \in \mathbb{N}_0}$ *satisfying* (RDT) *if there exists a continuously differentiable function* $\mathcal{X}$ *satisfying the LMIs*

$$\mathcal{X} \succ 0 \quad \text{and} \quad (\bullet)^\top \begin{pmatrix} 0 & \mathcal{X} \\ \mathcal{X} & \dot{\mathcal{X}} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix} \prec 0 \quad \text{on} \quad [0, T_{\max}] \tag{3.26a,b}$$

*as well as*

$$(\bullet)^\top \begin{pmatrix} \mathcal{X}(0) & 0 \\ 0 & -\mathcal{X} \end{pmatrix} \begin{pmatrix} \mathcal{A}_J & \mathcal{B}_J \\ I & 0 \end{pmatrix} + (\bullet)^\top P_{J\gamma} \begin{pmatrix} \mathcal{C}_J & \mathcal{D}_J \\ 0 & I \end{pmatrix} \prec 0 \quad \text{on} \quad [T_{\min}, T_{\max}]. \tag{3.26c}$$

*We denote by* $\gamma_{\text{opt}}$ *the infimal* $\gamma > 0$ *such that there exists a static controller* (3.24) *that renders the closed-loop analysis LMIs* (3.26) *feasible.*

Note that $\gamma_{\text{opt}}$ is *not* the optimal energy gain achievable by controllers with description (3.24), but both values are often close to each other. The possible gap between both values is due to the conservatism in the employed analysis criteria in Theorem 2.8. Recall that we accept this gap since we require the structure of the latter criteria for applying controller design tools such as the convexifying parameter transformation or the elimination lemma.

As in the previous section, trouble arises through the simultaneous search for some certificate $\mathcal{X}$ and for the maps $K$, $K_J$ describing the controller (3.24), which constitutes a challenging non-convex problem. In contrast to the synthesis of dynamic controllers (3.2) and even if we would restrict our attention to standard LTI systems, this lack of convexity is *not* resolved by employing a convexifying parameter transformation similarly to the one suggested in [107, 137], by utilizing the elimination lemma C.11 or by relying on any other presently known technique. Exemplary, by directly using the elimination lemma on the closed-loop analysis LMIs, we obtain the following.

**Theorem 3.20** (Static Output-Feedback Controller Synthesis) *Let* $U$, $V$, $U_J$ *and* $V_J$ *be basis matrices of the subspaces* $\ker((B_2^\top, D_{12}^\top))$, $\ker((C_2, D_{21}))$, $\ker((B_{J2}^\top, D_{J12}^\top))$ *and* $\ker((C_{J2}, D_{J21}))$, *respectively. Then there exists a static controller* (3.24) *for the system* (3.1) *such that the closed-loop analysis LMIs* (3.26) *are feasible if and only if there exists a continuously differentiable* $X$ *satisfying*

$$X \succ 0, \tag{3.27a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_\gamma \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) V \prec 0 \quad \text{and} \quad (\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_\gamma \end{array}\right)^{-1} \left(\begin{array}{cc} I & 0 \\ -A^\top & -C^\top \\ \hline 0 & I \\ -B^\top & -D^\top \end{array}\right) U \succ 0 \tag{3.27b,c}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_{J\gamma} \end{array}\right) \left(\begin{array}{cc} A_J & B_J \\ I & 0 \\ \hline C_J & D_J \\ 0 & I \end{array}\right) V_J \prec 0 \tag{3.27d}$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_{J\gamma} \end{array}\right)^{-1} \left(\begin{array}{cc} I & 0 \\ -A_J^\top & -C_J^\top \\ \hline 0 & I \\ -B_J^\top & -D_J^\top \end{array}\right) U_J \succ 0 \tag{3.27e}$$

*on* $[T_{\min}, T_{\max}]$*. Moreover,* $\gamma_{\rm opt}$ *is equal to the infimal* $\gamma > 0$ *such that the above inequalities are feasible.*

The elimination lemma permits us to remove the describing maps $K$ and $K_J$ of the controller (3.24) from the closed-loop analysis LMIs (3.26). However, the variable $X$ now enters the above inequalities in a non-convex fashion. Therefore, determining $\gamma_{\rm opt}$ or computing a suitable static controller (3.24) remain difficult.

Since non-convexity seems to be an intrinsic feature of the static controller synthesis problem, heuristic approaches are usually employed and upper bounds on the optimal $\gamma_{\rm opt}$ are computed. In the sequel, we present a generalization to impulsive systems of the *dual iteration* originating from [90, 91] which is a heuristic procedure based on iteratively solving convex semidefinite programs. In [85] we elaborate on this procedure in detail for standard LTI systems and argue that it is especially useful if compared to alternative approaches for two reasons:

- It provides good upper bounds on the optimal achievable energy gain.
- It seamlessly generalizes, e.g., to robust and multi-objective design.

Its essential features are discussed next.

### 3.2.2 Dual Iteration

#### Initialization of the Iteration

In order to initialize the dual iteration, we propose a starting point that allows the computation of a lower bound on $\gamma_{\rm opt}$ as a valuable indicator of how conservative any later computed upper bound on $\gamma_{\mathrm{opt}}$ is. This lower bound is obtained by the following observation. If there exists a static controller (3.24) for the system (3.1) achieving a closed-loop energy gain of $\gamma$, then there also exists a dynamic controller (3.2) which achieves (at least) the same closed-loop energy gain. Indeed, by simply choosing

$$\begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} = \begin{pmatrix} -I_n & 0 \\ 0 & K \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_J^c & B_J^c \\ C_J^c & D_J^c \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & 0 \\ 0 & K_J \end{pmatrix},$$

we observe that the energy gain of (3.25) is identical to the one of the interconnection of the system (3.1) and the dynamic controller (3.2). Recall that $n$ denotes the number of columns of the matrix $A$ in (3.1).

We have already shown in Theorem 3.3 and 3.8 that finding such a dynamic controller (3.2) for the system (3.1) is possible by means of convex optimization. In particular, recall that we have the following.

**Corollary 3.21** (Dynamic Output-Feedback Controller Synthesis) *Let $U$, $V$, $U_J$ and $V_J$ be as in Theorem 3.20. Then there exists a controller* (3.2) *for the system* (3.1) *such that the corresponding closed-loop analysis LMIs* (3.5) *are feasible for $(P, P_J) = (P_\gamma, P_{J\gamma})$ if and only if there exist continuously differentiable $X, Y$ satisfying*

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \tag{3.28a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_\gamma \end{array}\right) \begin{pmatrix} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{pmatrix} V \prec 0 \quad \textit{and} \quad (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & P_\gamma^{-1} \end{array}\right) \begin{pmatrix} I & 0 \\ -A^\top & -C^\top \\ \hline 0 & I \\ -B^\top & -D^\top \end{pmatrix} U \succ 0$$

(3.28b,c)

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_{J\gamma} \end{array}\right) \left(\begin{array}{cc} A_J & B_J \\ I & 0 \\ \hline C_J & D_J \\ 0 & I \end{array}\right) V_J \prec 0 \tag{3.28d}$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c} Y(0) & 0 & \\ 0 & -Y & \\ \hline & & P_{J\gamma}^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A_J^\top & -C_J^\top \\ \hline 0 & I \\ -B_J^\top & -D_J^\top \end{array}\right) U_J \succ 0 \tag{3.28e}$$

*on* $[T_{\min}, T_{\max}]$. *In particular, we have* $\gamma_{\text{dof}} \leq \gamma_{\text{opt}}$ *for* $\gamma_{\text{dof}}$ *being the infimal* $\gamma > 0$ *such that the above LMIs are feasible.*

Note that by using the Schur complement C.6 on the inequalities (3.28c) and (3.28d), it is possible to solve the above synthesis LMIs (3.28) while simultaneously minimizing over $\gamma$ in order to compute $\gamma_{\text{dof}}$. In particular, since the latter is a lower bound on $\gamma_{\text{opt}}$, it is not possible to find a static output-feedback controller by relying on Corollary 3.19 that is guaranteed to achieve an energy gain smaller than $\gamma_{\text{dof}}$

As an intermediate step, let us consider the design of a *static full-information controller* for the system (3.1). This is a controller with description

$$u(t) = F(\theta(t))\tilde{y}(t), \quad u_J(k) = F_J(\theta(t_k^-))\tilde{y}_J(k)$$

for $t \geq 0$ and $k \in \mathbb{N}$. Here, the gains $F = (F_1, F_2)$ and $F_J = (F_{J1}, F_{J2})$ are continuous matrix-valued maps, while the input signals are given by $\tilde{y} := \left(\begin{smallmatrix} x \\ d \end{smallmatrix}\right)$ and $\tilde{y}_J(k) := \left(\begin{smallmatrix} x(t_k^-) \\ d_J(k) \end{smallmatrix}\right)$, respectively. Hence, this controller relies on access to the full state and the full input disturbances of the system (3.1). By replacing the measurements $y$, $y_J$ in (3.1) with the virtual measurements $\tilde{y}$, $\tilde{y}_J$, we can interconnect this controller with the system (3.1) which results in a closed-loop interconnection of the form (3.25), but with $\left(\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\mathcal{A}_J & \mathcal{B}_J\\ \mathcal{C}_J & \mathcal{D}_J\end{smallmatrix}\right)$ replaced by

$$\begin{pmatrix} A_F & B_F \\ C_F & D_F \end{pmatrix} := \begin{pmatrix} A + B_2F_1 & B + B_2F_2 \\ C + D_{12}F_1 & D + D_{12}F_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix} F \quad (3.29a)$$

and

$$\begin{pmatrix} A_{JF} & B_{JF} \\ C_{JF} & D_{JF} \end{pmatrix} := \begin{pmatrix} A_J + B_{J2}F_{J1} & B_J + B_{J2}F_{J2} \\ C_J + D_{J12}F_{J1} & D_J + D_{J12}F_{J2} \end{pmatrix} = \begin{pmatrix} A_J & B_J \\ C_J & D_J \end{pmatrix} + \begin{pmatrix} B_{J2} \\ D_{J12} \end{pmatrix} F_J, \quad (3.29b)$$

respectively. Consequently, we can characterize the existence of a suitable full-information controller for example by employing the elimination lemma C.11 in order to obtain the following.

**Lemma 3.22** (Full-Information Controller Synthesis) *There exist some full-information gains $F$ and $F_J$ such that the closed-loop analysis LMIs* (3.26) *with $\left(\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\mathcal{A}_J & \mathcal{B}_J\\ \mathcal{C}_J & \mathcal{D}_J\end{smallmatrix}\right)$ replaced by* (3.29) *are feasible if and only if there exists a continuously differentiable $Y$ satisfying $Y \succ 0$ on $[0, T_{\max}]$,* (3.28c) *and* (3.28e).

## Main Loop

We are now in the position to discuss the core of the dual iteration. The first key result provides LMI conditions that are sufficient for static output-feedback design based on the assumption that full-information gains $F = (F_1, F_2)$ and $F_J = (F_{J1}, F_{J2})$ are available.

**Theorem 3.23** (Primal Design Result) *Let $V$ and $V_J$ be as in Theorem 3.20. Then there exists a static controller* (3.24) *for the system* (3.1) *such that the closed-loop analysis LMIs* (3.26) *are feasible if there exists a continuously differentiable $X$ satisfying*

$$X \succ 0, \tag{3.30a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_\gamma \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) V \prec 0 \quad \textit{and} \quad (\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_\gamma \end{array}\right) \left(\begin{array}{cc} A_F & B_F \\ I & 0 \\ \hline C_F & D_F \\ 0 & I \end{array}\right) \prec 0 \tag{3.30b,c}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_{J\gamma} \end{array}\right) \left(\begin{array}{cc} A_J & B_J \\ I & 0 \\ \hline C_J & D_J \\ 0 & I \end{array}\right) V_J \prec 0 \tag{3.30d}$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_{J\gamma} \end{array}\right) \left(\begin{array}{cc} A_{JF} & B_{JF} \\ I & 0 \\ \hline C_{JF} & D_{JF} \\ 0 & I \end{array}\right) \prec 0 \tag{3.30e}$$

*on* $[T_{\min}, T_{\max}]$. *Moreover, we have $\gamma_{\mathrm{dof}} \leq \gamma_{\mathrm{opt}} \leq \gamma_F$ for $\gamma_F$ being the infimal $\gamma > 0$ such that the LMIs* (3.30) *are feasible.*

*Proof.* Applying the elimination lemma C.11 in order to remove the full-information controller gain $F$ from (3.30c) yields exactly the inequality (3.27c). Analogously, we obtain (3.27e) from (3.30e). Since the remaining inequalities are satisfied by assumption, we can construct the desired static controller via Theorem 3.20. ●

Note that we even have $\gamma_{\mathrm{opt}} = \gamma_F$ if we view the gains $F$ and $F_J$ as decision variables in (3.30). However, this would render the computation of $\gamma_F$ as troublesome as that of $\gamma_{\mathrm{opt}}$ itself.

Intuitively, Theorem 3.23 links the difficult static output-feedback and the manageable full-information design problem with a common quantity (here, the Lyapunov matrix $X$). This underlying idea is also employed in order to deal with many other non-convex and/or difficult problems such as the ones considered in [46, 10, 73].

While Theorem 3.23 is interesting on its own, the key idea of the dual iteration is that improved upper bounds on $\gamma_{\mathrm{opt}}$ are obtained by also considering a problem that is dual to full-information synthesis. This consists of finding *full-actuation* gains $E$ and $E_J$ such that the closed-loop analysis LMIs (3.26) are feasible if we replace $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathcal{A}_J & \mathcal{B}_J \\ \mathcal{C}_J & \mathcal{D}_J \end{smallmatrix}\right)$ by

$$\begin{pmatrix} A_E & B_E \\ C_E & D_E \end{pmatrix} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} + E \begin{pmatrix} C_2 & D_{21} \end{pmatrix} \tag{3.31a}$$

and

$$\begin{pmatrix} A_{JE} & B_{JE} \\ C_{JE} & D_{JE} \end{pmatrix} := \begin{pmatrix} A_J & B_J \\ C_J & D_J \end{pmatrix} + E_J \begin{pmatrix} C_{J2} & D_{J21} \end{pmatrix}, \tag{3.31b}$$

respectively. Determining such gains is again a convex problem and a solution is obtained for example by utilizing the elimination lemma C.11.

**Lemma 3.24** (Full-Actuation Controller Synthesis) *There exists some full-actuation gains $E$ and $E_J$ such that the closed-loop analysis LMIs* (3.26) *with* $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} \mathcal{A}_J & \mathcal{B}_J \\ \mathcal{C}_J & \mathcal{D}_J \end{smallmatrix}\right)$ *replaced by* (3.31) *are feasible if and only if there exists a continuously differentiable $X$ satisfying $X \succ 0$ on* $[0, T_{\max}]$, (3.28b) *and* (3.28d)

Given some full-actuation gains $E$ and $E_J$ we can formulate another set of LMI conditions that are sufficient for static output-feedback design. The proof is analogous to the one of Theorem 3.23 and is hence omitted.

**Theorem 3.25** (Dual Design Result) *Let $U$ and $U_J$ be as in Theorem 3.20. Then there exists a static controller* (3.24) *for the system* (3.1) *such that the closed-loop analysis LMIs* (3.26) *are feasible if there exists a continuously differentiable $Y$ satisfying*

$$Y \succ 0, \tag{3.32a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & P_\gamma^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A^\top & -C^\top \\ \hline 0 & I \\ -B^\top & -D^\top \end{array}\right) U \succ 0 \text{ and } (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & P_\gamma^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A_E^\top & -C_E^\top \\ \hline 0 & I \\ -B_E^\top & -D_E^\top \end{array}\right) \succ 0 \tag{3.32b,c}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \left(\begin{array}{cc|c} Y(0) & 0 & \\ 0 & -Y & \\ \hline & & P_{J\gamma}^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A_J^\top & -C_J^\top \\ \hline 0 & I \\ -B_J^\top & -D_J^\top \end{array}\right) U_J \succ 0 \tag{3.32d}$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c} Y(0) & 0 & \\ 0 & -Y & \\ \hline & & P_{J\gamma}^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A_{JE}^\top & -C_{JE}^\top \\ \hline 0 & I \\ -B_{JE}^\top & -D_{JE}^\top \end{array}\right) \succ 0 \tag{3.32e}$$

*on* $[T_{\min}, T_{\max}]$*. Moreover, we have $\gamma_{\text{dof}} \leq \gamma_{\text{opt}} \leq \gamma_E$ for $\gamma_E$ being the infimal $\gamma > 0$ such that the LMIs* (3.32) *are feasible.*

In the sequel, we refer to the LMIs (3.30) and (3.32) as primal and dual synthesis LMIs, respectively. Accordingly, we address Theorems 3.23 and 3.25 as primal and dual design results, respectively. Observe that the latter are nicely intertwined as follows.

**Theorem 3.26** *The following two statements hold.*

- *If the primal synthesis LMIs* (3.30) *are satisfied for some* $\gamma$*, some matrix* $X$ *and some full-information gains* $F$ *and* $F_J$*, then there exists some full-actuation gains* $E$ *and* $E_J$ *such that the dual synthesis LMIs* (3.32) *are satisfied for the same* $\gamma$ *and for* $Y = X^{-1}$*. In particular, we have* $\gamma_E < \gamma$*.*

- *If the dual synthesis LMIs* (3.32) *are satisfied for some* $\gamma$*, some matrix* $Y$ *and some full-actuation gains* $E$ *and* $E_J$*, then there exists some full-information gains* $F$ *and* $F_J$ *such that the primal synthesis LMIs* (3.30) *are satisfied for the same* $\gamma$ *and for* $X = Y^{-1}$*. In particular, we have* $\gamma_F < \gamma$*.*

*Proof.* We only show the first statement as the second one follows with analogous arguments. If the primal synthesis LMIs (3.30) are feasible, we have in particular $X \succ 0$ on $[0, T_{\max}]$, (3.28b) and (3.28d). This permits us to apply Lemma 3.24 and we can thus conclude the existence of full-actuation gains $E$ and $E_J$ satisfying

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_\gamma \end{array}\right) \left(\begin{array}{cc} A_E & B_E \\ I & 0 \\ \hline C_E & D_E \\ 0 & I \end{array}\right) \prec 0 \text{ and } (\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_{J\gamma} \end{array}\right) \left(\begin{array}{cc} A_{JE} & B_{JE} \\ I & 0 \\ \hline C_{JE} & D_{JE} \\ 0 & I \end{array}\right) \prec 0$$

on $[0, T_{\max}]$ and $[T_{\min}, T_{\max}]$, respectively. An application of the dualization lemma C.9 as given in the appendix allows us to infer that (3.32c) and (3.32e) are satisfied for $Y = X^{-1} \succ 0$. Finally, by using the elimination lemma C.11 on the LMIs (3.30c) and (3.30e) to remove the full-information gains $F$ and $F_J$, we conclude that (3.32b) and (3.32d) are satisfied as well. This finishes the proof. ●

The dual iteration now essentially amounts to alternately applying the

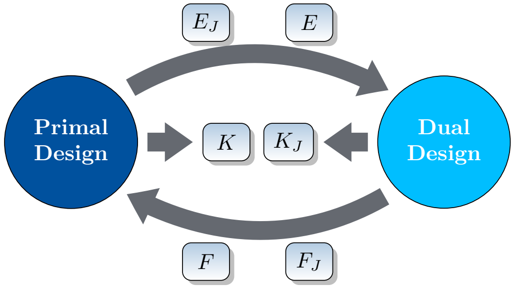



Figure 3.10: Schematic description of the main loop of the dual iteration.

two statements in Theorem 3.26 and is conceptually stated as follows. An illustration of its main loop is provided in Fig. 3.10.

**Algorithm 3.27** (Dual Iteration for Static Output-Feedback Design)

(a) *Initialization:* Compute the lower bound $\gamma_{\mathrm{dof}}$ based on solving the dynamic synthesis LMIs (3.28) and set $\gamma^0 := +\infty$ as well as $k = 1$. Design initial full-information gains $F$ and $F_J$ from Lemma 3.22.

(b) *Primal step:* Compute $\gamma_F$ by solving the primal synthesis LMIs (3.30) for the given gains $F$ and $F_J$ and choose some small $\varepsilon_k > 0$ such that $\gamma^k := \gamma_F(1 + \varepsilon_k) < \gamma^{k-1}$. For $\gamma = \gamma^k$, determine some $X$ satisfying the LMIs (3.30) and apply the elimination lemma C.11 on (3.30b) and (3.30d) in order to construct full-actuation gains $E$ and $E_J$ satisfying the dual synthesis LMIs (3.32) for $Y = X^{-1}$.

(c) *Dual step:* Compute $\gamma_E$ by solving the dual synthesis LMIs (3.32) for the given gains $E$ and $E_J$ and choose some small $\varepsilon_{k+1} > 0$ such that $\gamma^{k+1} := \gamma_E(1 + \varepsilon_{k+1}) < \gamma^k$. For $\gamma = \gamma^{k+1}$, determine a matrix $Y$ satisfying the LMIs (3.32) and apply the elimination lemma C.11 on (3.32b) and (3.32d) in order to construct full-information gains $F$ and $F_J$ satisfying the primal synthesis LMIs (3.30) for $X = Y^{-1}$.

(d) *Termination:* If $k$ is too large or $\gamma^k$ does not decrease any more, then stop and construct a static output-feedback controller according to Theorem 3.25.
Otherwise set $k = k + 2$ and go to the primal step.

**Remark 3.28** (a) Theorem 3.26 ensures that Algorithm 3.27 is recursively feasible, i.e., it will not get stuck due to infeasibility of some LMI, if the primal synthesis LMIs (3.30) are feasible when performing the primal step for the first time. Additionally, the proof of Theorem 3.26 demonstrates that we can even *warm start* the feasibility problems in the primal and dual steps by providing a feasible initial guess for the involved variables. This reduces the computational burden remarkably.

(b) The small numbers $\varepsilon_k > 0$ are introduced since, in general, it is not possible to determine optimal controllers or gains because these might not even exist; this is the reason for working with close-to-optimal solutions instead.

(c) We have $\gamma_{\text{dof}} \leq \gamma_{\text{opt}} \leq \gamma^k < \cdots < \gamma^2 < \gamma^1$ for all $k \in \mathbb{N}$ and thus the sequence $(\gamma^k)_{k\in\mathbb{N}}$ converges to some value $\gamma^* \geq \gamma_{\text{opt}}$. As for other approaches, there is no guarantee that $\gamma^* = \gamma_{\text{opt}}$. Nevertheless, the number of required iterations to obtain acceptable bounds on the optimal energy gain is rather low as will be demonstrated.

(d) As for any heuristic design, it can be beneficial to perform an a posteriori closed-loop analysis via Corollary 3.19. The resulting closed-loop energy gain is guaranteed to be not larger than the corresponding computed upper bound $\gamma^k$.

**Remark 3.29** (Initialization) (a) If static controller gains $K$ and $K_J$ are available that achieve a closed-loop energy gain bounded by $\gamma$, then the dual iteration can be initialized with $F = (KC_2, KD_{21})$ and $F_J = (K_JC_{J2}, K_JD_{J21})$. In particular, the primal synthesis LMIs (3.30) are then feasible and we have $\gamma_F \leq \gamma$.

(b) The selection of suitable gains $F$ and $F_J$ during the initialization of Algorithm 3.27 can be crucial, since feasibility of the primal synthesis LMIs (3.30) is *not* guaranteed from the feasibility of dynamic synthesis LMIs (3.28) and depends on the concrete choice of the gains $F$ and $F_J$. Similarly as in [91], we propose to compute the lower bound $\gamma_{\text{dof}}$ and then to reconsider the LMIs (3.28) for $\gamma = (1 + \varepsilon)\gamma_{\text{dof}}$ and some fixed $\varepsilon > 0$ while minimizing trace$(X + Y)$. Due to (3.28a), this is a common heuristic that aims to push $X$ towards $Y^{-1}$ and which promotes feasibility of the non-convex design matrix inequalities in Theorem 3.20. Constructing gains $F$ and $F_J$ based on Lemma 3.22 and these modified LMIs promotes feasibility of the primal synthesis LMIs (3.30) as well.

**Remark 3.30** (Control Theoretic Interpretation) The dual iteration as explained above solely relies on algebraic manipulations by heavily exploiting the elimination lemma C.11. This turns the derivation of the algorithm rather simple, but not that insightful. In [85] we additionally provide a control theoretic interpretation of the individual steps and argue that these can be related to the well-known *separation principle*[2]. More precisely, it is shown that the primal synthesis LMIs correspond to solving a particular robust design problem that structurally resembles robust estimation and which depends on a previously designed full-information controller.

[2] The classical separation principle for LTI systems states that one can synthesize a stabilizing dynamic output-feedback controller by combining a state observer with a state-feedback controller, which can be designed completely independently from each other.

We omit the details here for brevity since they are fully provided in [85]. Instead, we emphasize that the latter robust design problem is convex with a solution that is obtainable by a convexifying parameter transformation. As shown in [85], this permits the extension of the algorithm to much more challenging design problems such as those involving multiple objectives.

While the dual iteration, as illustrated above, theoretically extends nicely from standard LTI systems to impulsive ones due to the underlying analysis criteria in Corollary 3.19, there are also some issues that we have swept under the carpet so far.

### Compatibility Issues with DLMI Relaxations

Recall that all LMI problems appearing in Algorithm 3.27 are in fact infinite dimensional differential LMI problems which we can numerically solve only by relying on relaxations such as the ones discussed in Appendix D. Let us exemplary suppose that we intend to employ the sum-of-squares relaxation which relies on restricting all appearing decision variables to be polynomials and which is capable to deal with inequalities of the form $P(\tau) \prec 0$ for all $\tau \in [a, b]$ if $P$ is a polynomial (see Section D.1 for more details). Next, note that any full-information gains $F$ and $F_J$ as obtained from Lemma 3.22, i.e., by employing the elimination lemma C.11, will in general not be polynomials. Consequently, in order to apply the SOS relaxation for solving the primal synthesis LMIs (3.30), we have to interpolate the functions $F$ and $F_J$ with polynomials. Moreover, we note that once the latter LMIs are feasible for some polynomial certificate $X$, the dual synthesis LMIs (3.32) are still guaranteed to be feasible for $Y = X^{-1}$ if the generated gains $E$ and $E_J$ are approximated sufficiently well, but $X^{-1}$ is a rational function in general and not a polynomial. All of this leads to the following.

**Remark 3.31**
- The dual iteration as explained in Algorithm 3.27 and by solving the underlying DLMIs via the SOS relaxation can be implemented *if* we incorporate suitable polynomial approximations of the full-information and full-actuation gains (and, if warm starts are desired, of the inverse certificates $X^{-1}$ and $Y^{-1}$).

- In general, the generated sequence of upper bounds $(\gamma^k)_k$ will only be monotonically decreasing if large polynomial degrees are used in the interpolation *and* in the decision variables of the SOS relaxation. However, utilizing large polynomial degrees becomes quickly prohibitive from a numerical point of view.

- Analogously, the dual iteration as explained in Algorithm 3.27 and by solving the underlying DLMIs via the piecewise linear polynomial relaxation (see Section D.2.1) can be implemented *if* we incorporate suitable piecewise linear polynomial approximations of the full-information and full-actuation gains. Note that in this case the left hand sides of the inequalities (3.30c) and (3.32c) will in general be a piecewise quadratic polynomial and that these inequalities have to be relaxed accordingly. One can for example use that $f(t) = at^2 + bt + c > 0$ holds on $[x, y]$ if $f(x) > 0$, $f(y) > 0$ and $4f(\frac{x+y}{2}) - f(x) - f(y) = f(x) + f(y) - a(x - y)^2 > 0$ hold which is a consequence of expressing $f(t)$ as

$$\frac{(t-x)^2}{(y-x)^2}f(y) + \frac{(t-x)(y-t)}{(y-x)^2}[4f(\tfrac{x+y}{2}) - f(x) - f(y)] + \frac{(t-y)^2}{(y-x)^2}f(x).$$

Admittedly, the interpolation of the full-information and full-actuation gains and the loss of monotonicity in the sequence of upper bounds is quite unfavorable from a numerical point of view. At this point we leave it for future research to investigate possibilities to circumvent the discussed compatibility issues with DLMI relaxations. We expect such further investigations to be highly fruitful due to the effectiveness of the iteration for LTI systems as demonstrated in [85]. Moreover, we show in the next example that even the present status of the dual iteration is capable to outperform, e.g., the D-K iteration which was recently employed in [26] for the related design problem of synthesizing static controllers for time-varying systems on a finite-horizon.

### Example

As an illustration let us consider some modified examples from the collection in $\mathrm{COMPl_eib}$ [96] which consists of numerous continuous-time LTI systems with description

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B_1 & B \\ C_1 & D_{11} & D_{12} \\ C & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ u(t) \end{pmatrix} \tag{3.33}$$

for $t \geq 0$. The dual iterations permits us, e.g., to systematically design *static* sampled-data controllers for these systems. To this end, recall from Subsection 3.1.2 that we have to consider corresponding impulsive systems of the form

$$\left(\begin{array}{c} \dot{x}(t) \\ \dot{u}(t) \\ \hline e(t) \\ \hdashline y(t) \end{array}\right) = \left(\begin{array}{cc|c:c} A & B & B_1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline C_1 & D_{12} & D_{11} & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} x(t) \\ u(t) \\ \hline d(t) \\ \hdashline \hat{u}(t) \end{array}\right), \quad \left(\begin{array}{c} x(t_k) \\ u(t_k) \\ \hline e_J(k) \\ \hdashline y_J(k) \end{array}\right) = \left(\begin{array}{cc|c:c} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline 0 & 0 & 0 & 0 \\ \hdashline C & 0 & D_{21} & 0 \end{array}\right) \left(\begin{array}{c} x(t_k^-) \\ u(t_k^-) \\ \hline d_J(k) \\ \hdashline u_J(k) \end{array}\right)$$

involving some redundant signals for the sake of compatibility with (3.1). Here, we suppose that the involved impulse sequence $(t_k)_{k \in \mathbb{N}_0}$ satisfies (EDT) with $T = 0.5$. Note that considering such sampled-data systems conveniently removes almost all of the compatibility issues mentioned earlier.

In order to design a static sampled-data controller for (3.33), we can alternatively apply a D-K iteration scheme (also termed V-K iteration, e.g., in [15, 48] and similarly as recently employed in [26]). Here, this scheme relies on minimizing $\gamma$ subject to the closed-loop analysis inequalities (3.26) and with decision variables $(\gamma, K, K_J, \mathcal{X})$ while alternately fixing $(K, K_J)$ and $\mathcal{X}$. We emphasize that this approach requires an initialization with a static controller for which the inequalities (3.26) are satisfied and employ the static controller as obtained from computing $\gamma^1$ to this end. We denote the resulting upper bounds on $\gamma_{\mathrm{opt}}$ as $\gamma_{\mathrm{dk}}^k$, where the superscript $k$ indicates that the algorithm was stopped after $k$ iterations.

All computations are carried out with Matlab on a general purpose desktop computer (Intel Core i7, 4.0 GHz, 8 GB of ram) and we use LMIlab [55] for solving the LMIs resulting from relaxing all involved DLMIs via the linear spline relaxation D.2.1 with a grid of the interval $[0, T]$ consisting of 21 knots.

The numerically obtained results are illustrated in Table 3.1 and show that the dual iteration outperforms the D-K iteration in terms of the computed upper bounds which is analogous to observations in [85] for LTI systems. The dual iteration is slightly slower if compared to the D-K iteration in terms of required running time per iteration, but the latter converges very slowly. Note that the most time-consuming part of the dual iteration is its initialization since it involves twice as many variables; the actual iteration is relatively fast in comparison. The initialization is also numerically more delicate than the iteration which explains the phenomenon that the computed lower bounds are actually larger than the obtained upper bounds for few of the examples; this phenomenon is also promoted by the hard bounds on the entries of the decision variables introduced by the solver LMIlab.

For an additional comparison, we also show in Table 3.1 the computed optimal (upper bounds on the) energy gains achieved by dynamic and static

Table 3.1: Numerically determined lower and several upper bounds on $\gamma_{\text{opt}}$ resulting from the dual iteration and a D-K iteration for a sampled-data design together with corresponding bounds for a standard LTI design. All values are rounded to two decimals.

| | Sample-Data Design | | | | | | LTI Design | |
|---|---|---|---|---|---|---|---|---|
| | | Dual Iteration | | | D-K Iteration | | `hinfsyn` | `hinfstruct` |
| Name | $\gamma_{\text{dof}}$ | $\gamma^1$ | $\gamma^5$ | $\gamma^9$ | $\gamma_{\text{dk}}^5$ | $\gamma_{\text{dk}}^9$ | $\gamma_{\text{dof}}$ | $\gamma_{\text{his}}$ |
| AC3 | 5.42 | 6.54 | 5.45 | 5.42 | 6.30 | 6.22 | 1.61 | 3.64 |
| AC6 | 4.78 | 5.83 | 5.58 | 5.57 | 5.89 | 5.88 | 2.36 | 4.11 |
| HE2 | 3.43 | 6.13 | 5.01 | 5.00 | 5.39 | 5.39 | 2.44 | 4.25 |
| REA1 | 3.88 | 4.27 | 3.93 | 3.75 | 4.12 | 4.11 | 1.78 | 0.87 |
| DIS1 | 4.72 | 5.40 | 4.38 | 4.27 | 5.40 | 5.40 | 4.20 | 4.19 |
| DIS3 | 2.41 | 2.41 | 2.07 | 2.07 | 2.41 | 2.41 | 1.05 | 1.09 |
| DIS4 | 3.48 | 2.95 | 2.88 | 2.68 | 2.94 | 2.93 | 0.17 | 0.74 |
| PSM | 1.86 | 1.94 | 1.91 | 1.91 | 1.94 | 1.93 | 0.84 | 0.92 |
| NN2 | 2.06 | 2.45 | 2.03 | 2.03 | 2.26 | 2.25 | 1.78 | 2.22 |
| NN4 | 1.72 | 3.47 | 2.20 | 2.19 | 2.76 | 2.75 | 16.61 | 1.36 |
| NN15 | 0.15 | 0.24 | 0.21 | 0.21 | 0.22 | 0.22 | 0.10 | 0.10 |

output-feedback controllers for the system (3.33) as obtained via `hinfsyn` and `hinfstruct` [9], respectively.

Finally, note that based on the underlying closed-loop analysis result Corollary 3.19, one could also generalize other LMI-based static controller design approaches from LTI systems to impulsive ones such as the one suggested in Chapter 6.3 of [46] involving S-variables. This is in contrast to more specialized (and fast) algorithms such as `hinfstruct` [9] or `hifoo` [27] that are much less amenable for generalizations.

## 3.3 Summary

In the first part of this chapter, we show how to employ the convexifying parameter transformation introduced in [107, 137] and the elimination lemma C.11 from [72] for designing impulsive dynamic output-feedback controllers for impulsive open-loop systems. The corresponding convex design criteria are given in Theorem 3.3 and Theorem 3.8, respectively, and have been published by the author in [84]. Note that there are few alternative dynamic output-feedback design results for impulsive systems, e.g., in [6, 7, 174], but these often consider particularly structured underlying systems. More importantly, none of them provides design criteria based on elimination even though these are numerically much more favorable if compared to those relying on a convexifying parameter transformation.

Moreover, we demonstrate that the flexibility of our approach along with the richness of the class of impulsive systems permits us to provide new dynamic output-feedback synthesis criteria for designing

- LTI estimators for impulsive systems in Theorem 3.11;
- aperiodic sampled-data controllers in Theorem 3.12;
- two structurally different controllers for switched systems in Corollary 3.14 and Theorem 3.15, as well as a controller with bump limitation in Theorem 3.17.

Similar design criteria for sampled-data controllers have been obtained in [61], but only based on a convexifying parameter transformation. For switched systems there are output-feedback approaches, e.g., if considering switching functions with arbitrary dwell-time [75] or performing co-design [41, 99]. However, for switching functions with dwell-time constraints, even recent publications as, e.g., [4, 20, 89], that employ dedicated analysis results, rely on measurements of the full state which is in contrast to our results.

In the second part of this chapter, we consider the challenging design of static output-feedback controllers for impulsive systems. To this end and based on the developed design approach for designing dynamic controllers, we propose an extension to impulsive systems of the dual iteration that was established in [90, 91] for designing static stabilizing controllers for standard LTI systems. In [85] we revisit the dual iteration in the context of LTI systems, provide a novel control theoretic interpretation of its individual steps, extend it to multi-objective design problems, and generalize it to robust output-feedback design.

# 4

# Robust Analysis

Engineers are typically faced with discrepancies between the real system, which might be extremely complex in practice, and some employed mathematical model, which is limited, e.g., by the available computational resources and the knowledge about the real system. In the field of *robust control* [179, 141, 66] such discrepancies are called *uncertainties*. One of the most common sources of such uncertainties is the presence of several unknown parameters in the employed model that might even change over time. As an example, just think of the total mass of a car which varies by the weight of all passengers and their luggage. Such uncertainties are referred to as *parametric* uncertainties. Another common source is the approximation or the deliberate neglect of (difficult) dynamics in order to simplify the considered model of the real system which leads to so-called *dynamic* uncertainties.

In order to systematically analyze the real system despite the presence of numerous of such uncertainties, the essential strategy established in robust control is

- to identify (rough) descriptions of the uncertainties present in the real system,
- to directly include these descriptions into the considered mathematical model, and
- to verify stability and performance for all possible objects corresponding to the identified descriptions.

In particular, these steps guarantee stability and performance for the true dynamical system. Note that the identification in the first step is motivated by the fact that determining some bounds on an uncertain parameter is usually much easier compared to obtaining its true value. Think again of the mass of a car with some passengers which is surely bounded by its tare weight and by its admissible total weight, if it is loaded in a reasonable way. The last step might sound challenging since stability and performance have to be guaranteed for a whole family of models, but over the past years numerous techniques have been developed for exactly this purpose and for various classes of uncertainty descriptions. Still, even today it is of interest to refine those techniques and to expand their scope. To this end, we consider in this chapter robust analysis for linear impulsive and related hybrid systems.

## 4.1 Robust Analysis for Impulsive Systems

Let us at first confine the discussion to linear impulsive systems affected by arbitrarily time-varying parametric uncertainties. To this end, we rely on the framework of linear fractional representations (LFRs) and on so-called separation techniques which are summarized and briefly discussed in Section C.6 for the reader's convenience in the context of standard LTI systems. The LFR framework is discussed in detail, e.g., in [149, 44, 178] for LTI systems and is known to be a highly flexible modeling tool that

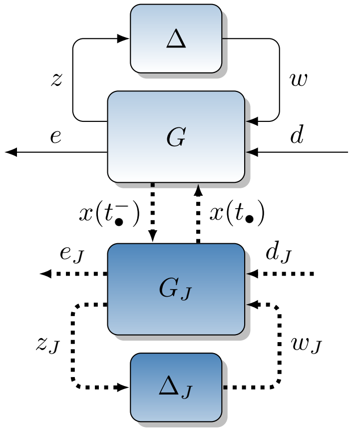



Figure 4.1: Block diagram of the uncertain impulsive system (4.1).

permits for effectively capturing structural dependencies of models on uncertain scalar parameters or on matrix sub-blocks. As we will see, another big advantage of this framework is the perfect fit to all our preparations provided in Chapter 2.

### 4.1.1 Arbitrarily Time-Varying Parametric Uncertainties

For real matrices of appropriate dimensions, an initial condition $x(0) \in \mathbb{R}^n$, generalized disturbances $d \in L_2$ and $d_J \in \ell_2$, a sequence of impulse instants $0 = t_0 < t_1 < t_2 < \dots$ as well as two sets $\mathbf{\Delta} \subset \mathbb{R}^{q\times p}$ and $\mathbf{\Delta}_J \subset \mathbb{R}^{q_J\times p_J}$, let us consider an uncertain linear impulsive system with the description

$$\begin{pmatrix} \dot{x}(t) \\ z(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \\ d(t) \end{pmatrix}, \quad \begin{pmatrix} x(t_k) \\ z_J(k) \\ e_J(k) \end{pmatrix} = \begin{pmatrix} A_J & B_J & B_{J2} \\ C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & D_{J22} \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ w_J(k) \\ d_J(k) \end{pmatrix},$$
$$w(t) = \Delta(t) z(t), \qquad\qquad w_J(k) = \Delta_J(k) z_J(k) \tag{4.1}$$

for $t \geq 0$ and $k \in \mathbb{N}$ and as schematically displayed in Fig. 4.1. Here, $w$, $w_J$, $z$ and $z_J$ are interconnection variables and the *uncertainties* $\Delta$ and $\Delta_J$ are piecewise continuous maps that are merely known to satisfy

$$\Delta(t) \in \mathbf{\Delta} \quad \text{for all} \quad t \geq 0 \qquad \text{and} \qquad \Delta_J(k) \in \mathbf{\Delta}_J \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.2}$$

In particular, note that we do not make any assumptions on the rate of variation of the uncertainties. Moreover, recall that the sets $\mathbf{\Delta}$ and $\mathbf{\Delta}_J$ are typically of the form

$$\left\{\mathrm{diag}(\delta_1 I, \ldots, \delta_{m_r} I, \Delta_1, \ldots, \Delta_{m_f}) \;\middle|\; |\delta_i| \leq 1 \text{ and } \|\Delta_i\| \leq 1\right\}$$

with (repeated) diagonal and full unstructured blocks on the diagonal, all bounded in norm by one. The purpose of these sets is to encode a priori available (crude) guesses on the ranges of all the involved uncertain parameters and to specify the structural dependencies of the underlying system on these uncertain parameters. Robust stability and performance of the uncertain system (4.1) are defined as follows.

**Definition 4.1** (Robust Stability and Robust Quadratic Performance)

- *The system* (4.1) *is said to be* well-posed *if $I - D\Delta$ and $I - D_J\Delta_J$ are nonsingular for all $\Delta \in \mathbf{\Delta}$ and all $\Delta_J \in \mathbf{\Delta}_J$.*
- *The system* (4.1) *is said to be* robustly stable *if it is well-posed and there exist constants $M, \gamma > 0$ such that $\|x(t)\| \leq Me^{-\gamma t}\|x(0)\|$ holds for all $t \geq 0$, all initial conditions $x(0) \in \mathbb{R}^n$ and all uncertainties $\Delta, \Delta_J$ with* (4.2) *and for vanishing disturbances $d = 0$ and $d_J = 0$.*
- *It* achieves robust quadratic performance with index $(P_p, P_{Jp})$ *if*

*there exists some $\varepsilon > 0$ such that*

$$\int_0^\infty \begin{pmatrix} e(t) \\ d(t) \end{pmatrix}^\top P_p \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} dt + \sum_{k=1}^\infty \begin{pmatrix} e_J(k) \\ d_J(k) \end{pmatrix}^\top P_{Jp} \begin{pmatrix} e_J(k) \\ d_J(k) \end{pmatrix} \leq -\varepsilon \|d\|_{L_2}^2 - \varepsilon \|d_J\|_{\ell_2}^2$$

*holds for the initial condition $x(0) = 0$, for all $d \in L_2$, all $d_J \in \ell_2$ and for all uncertainties $\Delta, \Delta_J$ satisfying* (4.2)*.*

As in the previous chapter, we assume throughout this section that the performance index $(P_p, P_{Jp})$ satisfies Assumption 3.1, i.e., that these matrices are partitioned accordingly to the signals $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e_J \\ d_J \end{smallmatrix}\right)$ and have positive semidefinite left upper blocks.

Note that well-posedness allows us to remove the interconnection variables $z, z_J, w, w_J$ and to equivalently express (4.1) as an impulsive system of the form (2.7) with time-varying describing matrices in the flow and jump component given by

$$\begin{pmatrix} A & B_2 \\ C_2 & D_{22} \end{pmatrix} + \begin{pmatrix} B \\ D_{21} \end{pmatrix} \Delta(t)(I - D\Delta(t))^{-1} \begin{pmatrix} C & D_{12} \end{pmatrix}$$

and

$$\begin{pmatrix} A_J & B_{J2} \\ C_{J2} & D_{J22} \end{pmatrix} + \begin{pmatrix} B_J \\ D_{J21} \end{pmatrix} \Delta_J(k)(I - D_J\Delta_J(k))^{-1} \begin{pmatrix} C_J & D_{J12} \end{pmatrix},$$

respectively. Note that these matrices depend rationally on the uncertainties; in particular, the description (4.1) involves an LFR for the systems' flow and for its jump component. As a consequence of this alternative representation, we can combine the arguments of the proofs of our nominal analysis result Theorem 2.8 and of Lemma C.22, which is an application of the so-called full block S-procedure, in order to obtain the following.

**Theorem 4.2** (Robust Analysis Criteria for Arbitrarily Time-Varying Uncertainties) *The system* (4.1) *is robustly stable and achieves robust quadratic performance with index* $(P_p, P_{Jp})$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist functions* $X \in C^1([0,T_{\max}],\mathbb{S}^n)$, $P \in C([0,T_{\max}],\mathbb{S}^{p+q})$ *and* $P_J \in C([T_{\min},T_{\max}],\mathbb{S}^{p_J+q_J})$ *satisfying the LMIs*

$$X \succ 0, \quad (\bullet)^\top \left(\begin{array}{cc|c:c} 0 & X & & \\ X & \dot{X} & & \\ \hline & & P & \\ \hdashline & & & P_p \end{array}\right) \left(\begin{array}{ccc} A & B & B_2 \\ I & 0 & 0 \\ \hline C & D & D_{12} \\ 0 & I & 0 \\ \hdashline C_2 & D_{21} & D_{22} \\ 0 & 0 & I \end{array}\right) \prec 0 \quad \textit{and} \quad (\bullet)^\top P \begin{pmatrix} I \\ \Delta \end{pmatrix} \succcurlyeq 0 \tag{4.3a,b,c}$$

*on* $[0,T_{\max}]$ *for all* $\Delta \in \boldsymbol{\Delta}$ *as well as*

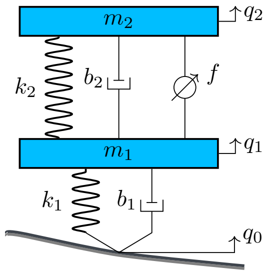
$$(\bullet)^\top \left(\begin{array}{cc|c:c} X(0) & 0 & & \\ 0 & -X & & \\ \hline & & P_J & \\ \hdashline & & & P_{Jp} \end{array}\right) \left(\begin{array}{ccc} A_J & B_J & B_{J2} \\ I & 0 & 0 \\ \hline C_J & D_J & D_{J12} \\ 0 & I & 0 \\ \hdashline C_{J2} & D_{J21} & D_{J22} \\ 0 & 0 & I \end{array}\right) \prec 0 \quad \textit{and} \quad (\bullet)^\top P_J \begin{pmatrix} I \\ \Delta_J \end{pmatrix} \succcurlyeq 0 \tag{4.3d,e}$$

*on* $[T_{\min},T_{\max}]$ *for all* $\Delta_J \in \boldsymbol{\Delta}_J$.

Analogously as discussed in Section C.6 or, e.g., in [149, 160], the functions $P$ and $P_J$ are usually referred to as *multipliers* and confined to take values in so-called *multiplier sets* as defined in Definition C.18 and corresponding to $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_J$, respectively. The latter restriction is imposed in order to ensure in a numerically tractable fashion that the inequalities (4.3c) and (4.3e) are satisfied.

### Example

As an illustration let us consider an active suspension system of one wheel of a transport vehicle as schematically depicted on the right and as also considered, e.g., in [13, 149]. This system is modeled by the differential equations




$$
\begin{aligned}
0 &= m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) - f \\
0 &= m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + b_1(\dot{q}_1 - \dot{q}_0) + k_2(q_1 - q_2) \\
&\quad + k_1(q_1 - q_0) + f.
\end{aligned}
\tag{4.4}
$$

Here, the force $f$ is a control input acting on the chassis mass $m_2$ and the axle mass $m_1$. Moreover, $q_2 - q_1$ is the distance between chassis and axle (and also called suspension deflection), $\ddot{q}_2$ denotes the acceleration of the chassis mass and $q_0$ denotes the road profile. The remaining parameters $b_i$, $k_1$ and $k_2$ are damping, tire and air spring coefficients, respectively. In the sequel we assume that $k_1$ and $m_2$ can vary over time and are only known within 10% of their nominal values $k_1^n$ and $m_2^n$. This means that

$$k_1(t) = k_1^n(1 + 0.1\delta_1(t)) \quad \text{and} \quad m_2(t) = m_2^n(1 + 0.1\delta_2(t))$$

with uncertainties $\delta_1(t), \delta_2(t) \in [-1, 1]$. The nominal values and remaining constants are assumed to be given by

$$
\begin{array}{lll}
b_1 = 5.0 \cdot 10^1 \text{ Ns/m}, & b_2 = 1.45 \cdot 10^3 \text{ Ns/m}, & k_1^n = 3.1 \cdot 10^5 \text{ N/m}, \\
k_2 = 3.0 \cdot 10^4 \text{ N/m}, & m_1 = 5.0 \cdot 10^1 \text{ kg}, & m_2^n = 4.0 \cdot 10^2 \text{ kg}.
\end{array}
$$

Let us say that we intend to design a controller that keeps $q_2$ constant and close to zero which is related to passenger comfort and such that the suspension deflection $q_2 - q_1$ is bounded to avoid damage to the mechanical system by measuring the signal $y := \left(\begin{smallmatrix} q_2 \\ q_2 - q_1 \end{smallmatrix}\right)$. Of course, the controller should achieve those objectives for all admissible uncertainties $\delta_1, \delta_2$. By introducing the state $x := \mathrm{col}\left(q_1, q_2, \dot q_1 - \frac{b_1}{m_1} q_0, \dot q_2\right)$, we arrive at the uncertain unweighted open-loop plant $P : \left(\begin{smallmatrix} d \\ u \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} e \\ y \end{smallmatrix}\right)$

$$\begin{pmatrix} \dot x(t) \\ \hline e(t) \\ \hdashline y(t) \end{pmatrix} = \left(\begin{array}{c|c:c} \tilde A & \tilde B & \tilde B_2 \\ \hline \tilde C & \tilde D & \tilde D_{12} \\ \hdashline \tilde C_2 & \tilde D_{21} & \tilde D_{22} \end{array}\right) \begin{pmatrix} x(t) \\ \hline d(t) \\ \hdashline u(t) \end{pmatrix}$$

$$= \left(\begin{array}{cccc|c:c} 0 & 0 & 1 & 0 & \frac{b_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{b_2}{m_1} & \frac{k_1}{m_1} - \frac{b_1}{m_1}\frac{b_1+b_2}{m_1} & -\frac{1}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2}{m_2} & \frac{b_1}{m_1}\frac{b_2}{m_2} & \frac{1}{m_2} \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} x(t) \\ \hline d(t) \\ \hdashline u(t) \end{pmatrix}$$

with signals $e := y$, $d := q_0$ and $u := f$ and where we omit the time-dependence of the uncertain coefficients.

In order to design a controller that satisfies the desired specifications by means of $H_\infty$-techniques, one usually augments the plant $P$ with weights to $P_w = W_o P W_i$ and designs an $H_\infty$-controller for the latter weighted plant. Here, we choose the weights (defined in the frequency domain) as

$$W_i(s) := \mathrm{diag}\left(\frac{0.01}{0.4s+1}, 1\right) \text{ and } W_o(s) := \mathrm{diag}\left(200, 0.6 \cdot \frac{0.125s+10}{0.05s+0.4}, 1, 1\right).$$

Such weights are typically chosen such that the specifications are as desired if the energy gain of the weighted closed-loop interconnection is bounded by one. Appropriately designing weights is essentially an art in itself, requires a lot of tuning or expertise and is not discussed here. The above choices are just picked for the purpose of illustration.

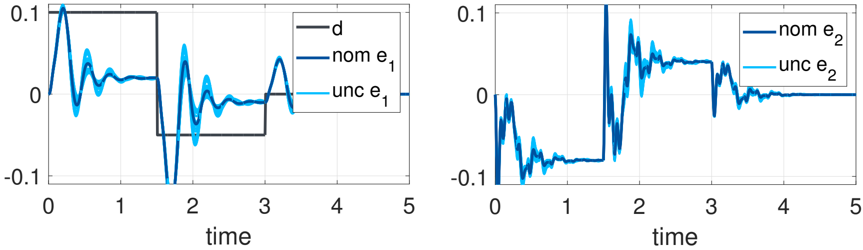



Figure 4.2: An input disturbance $d$ and closed-loop response for the nominal system (dark blue) and for the uncertain system for several samples of $\delta_i$ (light blue).

Now note that designing *robust* controllers is difficult in general and, thus, we design here a controller for the nominal weighted plant, i.e., for vanishing uncertainties $\delta_1$, $\delta_2$, and afterwards perform a robustness analysis of the uncertain closed-loop interconnection in order to decide whether this controller also does its job for the uncertain system.

Finally, we design here a sampled-data controller

$$\begin{pmatrix} \tilde{x}_c(k+1) \\ \tilde{u}(k) \end{pmatrix} = \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \begin{pmatrix} \tilde{x}_c(k) \\ y(t_k) \end{pmatrix}$$

with a sampling time of $t_{k+1} - t_k = 0.15$ seconds which can, e.g., be done in Matlab with the command `sdhinfsyn`. Note that the latter requires all $D$-matrices in the weighted open-loop interconnection to be zero which limits its applicability and is in contrast to our synthesis approach presented in Subsection 3.1.2. Anyhow, via `sdhinfsyn` we obtain a sampled-data controller for which the weighted nominal closed-loop's energy gain is guaranteed to be smaller than $0.455 < 1$. The corresponding (unweighted) closed-loop response to some disturbance $d$ for the nominal system and for the uncertain system for several samples of *constant* $\delta_1, \delta_2$ is depicted in Fig. 4.2. As expected, we observe that the closed-loop performance (slightly) deteriorates in the presence of uncertainties if compared to the response of the nominal closed-loop interconnection. Note that the response can be worse for uncertainties that are actually time-varying.

Let us now be more concrete by determining upper bounds on the *robust* energy gain achieved by the computed controller by means of our robust analysis result Theorem 4.2. To this end, we have to express the (weighted) closed-loop interconnection as impulsive system (4.1). This is achieved by *pulling out the uncertainties* as described, e.g., in Chapter 4.4 of [141]. To this end, observe that, by introducing the signals

$$z_1 := q_1 - q_0, \quad w_1 := \delta_1 z_1, \quad z_2 := \ddot{q}_2 \quad \text{and} \quad w_2 := \delta_2 z_2,$$

and by recalling that the state is $x = \operatorname{col}\Big(q_1, q_2, \dot{q}_1 - \frac{b_1}{m_1} q_0, \dot{q}_2\Big)$, the equations (4.4) read as

$$\begin{aligned}
0 &= m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) - f \\
&= m_2^n\ddot{q}_2 + m_2^n 0.1 w_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) - f \\
&= m_2^n\dot{x}_4 + m_2^n 0.1 w_2 + b_2(x_4 - x_3) + k_2(x_2 - x_1) - u - b_2\frac{b_1}{m_1}d
\end{aligned}$$

and

$$\begin{aligned}
0 &= m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + b_1(\dot{q}_1 - \dot{q}_0) + k_2(q_1 - q_2) + k_1(q_1 - q_0) + f \\
&= m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + b_1(\dot{q}_1 - \dot{q}_0) + k_2(q_1 - q_2) + k_1^n(q_1 - q_0) \\
&\quad + k_1^n 0.1 w_1 + f \\
&= m_1\dot{x}_3 - b_2x_4 + (b_2 + b_1)x_3 - k_2x_2 + (k_2 + k_1^n)x_1 + k_1^n 0.1 w_1 \\
&\quad + \Big((b_2 + b_1)\frac{b_1}{m_1} - k_1^n\Big)d + u.
\end{aligned}$$

This leads to the following linear fractional representation of the uncertain open-loop system

$$
\begin{pmatrix} \dot{x}(t) \\ \hline z(t) \\ \hline e(t) \\ \hdashline y(t) \end{pmatrix}
=
\left(\begin{array}{cccc|cc|c:c}
0 & 0 & 1 & 0 & 0 & 0 & \frac{b_1}{m_1} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{k_2+k_1^n}{-m_1} & \frac{k_2}{m_1} & \frac{b_2+b_1}{-m_1} & \frac{b_2}{m_1} & \frac{0.1k_1^n}{-m_1} & 0 & \frac{k_1^n}{m_1} - \frac{b_1}{m_1}\frac{b_2+b_1}{m_1} & \frac{-1}{m_1} \\
\frac{k_2}{m_2^n} & -\frac{k_2}{m_2^n} & \frac{b_2}{m_2^n} & -\frac{b_2}{m_2^n} & 0 & -0.1 & \frac{b_2 b_1}{m_1 m_2^n} & \frac{1}{m_2^n} \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\frac{k_2}{m_2^n} & -\frac{k_2}{m_2^n} & \frac{b_2}{m_2^n} & -\frac{b_2}{m_2^n} & 0 & -0.1 & \frac{b_2 b_1}{m_1 m_2^n} & \frac{1}{m_2^n} \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix} x(t) \\ \hline w(t) \\ \hline d(t) \\ \hdashline u(t) \end{pmatrix}
$$

$$
= \left(\begin{array}{c|c|c:c}
A & B & B_2 & B_3 \\ \hline
C & D & D_{12} & D_{13} \\ \hline
C_2 & 0 & 0 & 0 \\ \hdashline
C_3 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix} x(t) \\ \hline w(t) \\ \hline d(t) \\ \hdashline u(t) \end{pmatrix}, \quad
w(t) = \underbrace{\begin{pmatrix} \delta_1(t) & \\ & \delta_2(t) \end{pmatrix}}_{=:\Delta(t)} z(t).
$$

We stress that these representations are highly non-unique and that there are tools for their automatic generation available, e.g., in Matlab. By interconnecting the sampled-data controller we arrive at an impulsive closed-loop system with description

$$
\begin{pmatrix} \dot{x}(t) \\ \dot{u}(t) \\ \dot{x}_c(t) \\ \hline z(t) \\ \hdashline e(t) \end{pmatrix}
=
\left(\begin{array}{ccc|c:c}
A & B_3 & 0 & B & B_2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\ \hline
C & D_{13} & 0 & D & D_{12} \\ \hdashline
C_2 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix} x(t) \\ u(t) \\ x_c(t) \\ \hline w(t) \\ \hdashline d(t) \end{pmatrix}, \quad
\begin{pmatrix} x(t_k) \\ u(t_k) \\ x_c(t_k) \end{pmatrix}
=
\begin{pmatrix} I & 0 & 0 \\ D^c C_3 & 0 & C^c \\ B^c C_3 & 0 & A^c \end{pmatrix}
\begin{pmatrix} x(t_k^-) \\ u(t_k^-) \\ x_c(t_k^-) \end{pmatrix},
$$

$$
w(t) = \Delta(t) z(t)
$$

which is a special case of the generic description (4.1). Note that the weighted closed-loop can be expressed in a similar form, which is not shown here for reasons of space. Hence, we can apply Theorem 4.2 for its robustness analysis. Doing so with a multiplier set corresponding to D-G-scalings (see Remark C.19) and with the B-Spline relaxation (Section D.3) yields for example 1.176 as a guaranteed upper bound on the robust energy gain; note that this bound can be improved to some extend by adjusting the parameters in the relaxation and at the expense of a higher computational burden.

**Remark 4.3** (Lower Bounds and Worst-Case Uncertainties) Theorem (4.2) with quadratic performance index $(P_p, P_{Jp}) = \left(\begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}\right)$ permits us to systematically generate *upper bounds* on the robust energy gain. Unfortunately, determining good *lower bounds* on this number or finding *worst-case uncertainties* is much more challenging and not discussed here. Note that even for standard LTI systems, these are rather difficult tasks and there are only few methods for systematically coping with them. One of them is the $\mu$-analysis framework [44, 178] which, however, is limited in the variety of uncertainty classes that can be considered.

Let us finally demonstrate that restricting the multipliers $P$ and $P_J$ in Theorem 4.2 to be constant functions is conservative in general. Indeed, for our example, with this restriction and with exactly the same relaxation parameters, we merely obtain 1.569 as an upper bound on the robust energy gain

## 4.1.2 Integral Quadratic Constraints

In this subsection, we allow for vastly more general types of uncertainties compared to the previously considered parametric ones. We still rely on linear fractional representations, but, this time, adapt the framework of integral quadratic constraints (IQCs) to appropriately apply for impulsive systems. The latter robust analysis framework for standard LTI systems has been proposed in [109]. Since then it was established as a powerful and highly flexible tool, e.g., in aerospace applications precisely because it allows for a systematic robustness analysis for systems affected by various types of (challenging) uncertainties. We refer the reader to the recent theses [160] and [50] that provide a comprehensive in-depth discussion of system analysis via IQCs.

We consider, for real matrices of appropriate dimensions, some initial condition $x(0) \in \mathbb{R}^n$, generalized disturbances $d \in L_2$ and $d_J \in \ell_2$ and a sequence of impulse instants $0 = t_0 < t_1 < t_2 < \dots$, an uncertain linear impulsive system with the description

$$\begin{pmatrix} \dot{x}(t) \\ z(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \\ d(t) \end{pmatrix}, \qquad w(t) = \Delta(z)(t) \tag{4.5a}$$

$$\begin{pmatrix} x(t_k) \\ z_J(k) \\ e_J(k) \end{pmatrix} = \begin{pmatrix} A_J & B_J & B_{J2} \\ C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & D_{J22} \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ w_J(k) \\ d_J(k) \end{pmatrix}, \qquad w_J(k) = \Delta_J(z_J)(k) \tag{4.5b}$$

for $t \geq 0$ and $k \in \mathbb{N}$. The uncertainties $\Delta : L_{2e}^p \to L_{2e}^q$ and $\Delta_J : \ell_{2e}^{p_J} \to \ell_{2e}^{q_J}$ are potentially nonlinear functions that are merely known to be contained in given sets $\mathbf{\Delta}$ and $\mathbf{\Delta}_J$, respectively. Here, $L_{2e}^\bullet$ and $\ell_{2e}^\bullet$ denote the spaces of locally square integrable functions and summable sequences with values in $\mathbb{R}^\bullet$ and their corresponding standard norms, respectively.

In order to simplify the exposition, we assume that the uncertainty sets $\mathbf{\Delta}$ and $\mathbf{\Delta}_J$ are such that the description (4.5) is *well-posed* for any contained uncertainties; here well-posedness means that, for any $d \in L_2$ and $d_J \in \ell_2$, there exists a unique piecewise continuous and right continuous state trajectory $x$ satisfying (4.5). We also have to slightly weaken the definition of robust stability since an exponential decay rate of the state can usually no longer be assured for general non-parametric uncertainties.

**Definition 4.4** (Robust Stability and Robust Quadratic Performance)

- *The system* (4.5) *is said to be* robustly stable *if there exist a constant* $M > 0$ *such that*

$$\lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \|x(t)\| \leq M\|x(0)\| \text{ for all } t \geq 0$$

*hold for all* $(x(0), \Delta, \Delta_J) \in \mathbb{R}^n \times \mathbf{\Delta} \times \mathbf{\Delta}_J$ *and for vanishing disturbances* $d = 0$ *and* $d_J = 0$.

- *It is said to* achieve robust quadratic performance with index $(P, P_J)$ *if there exists some* $\varepsilon > 0$ *such that*

$$\int_0^\infty (\bullet)^\top P \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} dt + \sum_{k=1}^{\infty} (\bullet)^\top P_J \begin{pmatrix} e_J(k) \\ d_J(k) \end{pmatrix} \leq -\varepsilon\|d\|_{L_2}^2 - \varepsilon\|d_J\|_{\ell_2}^2$$

*holds for* $x(0) = 0$ *and for all* $(d, d_J, \Delta, \Delta_J) \in L_2 \times \ell_2 \times \mathbf{\Delta} \times \mathbf{\Delta}_J$.

Similarly as is the previous subsection, instead of directly considering the uncertain system (4.5) as a whole, the key idea behind IQCs is

- to identify constraints on interconnection variables $w, z, w_J, z_J$ that are enforced by the uncertainties $\Delta$ and $\Delta_J$, and afterwards
- to analyze the auxiliary system given by (4.5) with the uncertainties being removed, but with the identified constraints on the interconnection variables $w, z, w_J, z_J$.

Naturally, if we find that the latter auxiliary system is stable, then stability of the original system is guaranteed as well since the uncertain system is a particular instance of the auxiliary one. Note that the involved constraints are typically formulated in the frequency-domain such as in [109]. However, since we deal with impulsive systems with potentially aperiodic impulses,

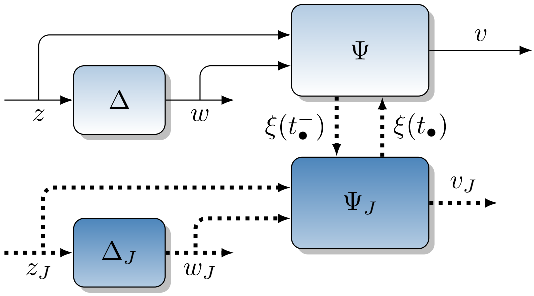



Figure 4.3: Graphical illustration of an integral quadratic constraint.

we cannot rely on frequency-domain techniques and will, instead, formulate constraints in the time-domain. Here, the definition of these constraints relies on an a priori chosen linear impulsive filter

$$\begin{pmatrix} \dot{\xi}(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} A_\Psi & B_\Psi & B_{\Psi 2} \\ C_\Psi & D_\Psi & D_{\Psi 2} \end{pmatrix} \begin{pmatrix} \xi(t) \\ z(t) \\ w(t) \end{pmatrix}, \quad \begin{pmatrix} \xi(t_k) \\ v_J(k) \end{pmatrix} = \begin{pmatrix} A_{\Psi_J} & B_{\Psi_J} & B_{\Psi_J 2} \\ C_{\Psi_J} & D_{\Psi_J} & D_{\Psi_J 2} \end{pmatrix} \begin{pmatrix} \xi(t_k^-) \\ z_J(k) \\ w_J(k) \end{pmatrix} \tag{4.6}$$

for $t \geq 0$ and $k \in \mathbb{N}$ with initial condition $\xi(0) = 0$ which captures the input-output behavior of the uncertainties $\Delta$ and $\Delta_J$; a graphical interpretation is shown in Fig. 4.3 where $\Psi$ and $\Psi_J$ stand for the flow and jump component of (4.6), respectively. Precisely, we introduce the following notion that involves again the clock $\theta$ as defined in (2.2). In order to specify dimensions, we suppose that $C_\Psi$ and $C_{\Psi_J}$ are elements of $\mathbb{R}^{n_v \times n_\Psi}$ and $\mathbb{R}^{n_{v_J} \times n_\Psi}$, respectively.

**Definition 4.5** (Finite-Horizon IQC with Terminal, Jump and Flow Cost) *The pair* $(\Delta, \Delta_J)$ *satisfies a* finite-horizon IQC with terminal, jump and flow cost *with respect to the impulsive filter* (4.6) *and the maps* $Z_T, Z_J, Z_F \in C^1([0, T_{\max}], \mathbb{S}^{n_\Psi})$, $M \in C([0, T_{\max}], \mathbb{S}^{n_v})$ *and* $M_J \in C([T_{\min}, T_{\max}], \mathbb{S}^{n_{v_J}})$ *if* *the inequality*

$$\int_0^t v(s)^\top M(\theta(s))v(s)\,ds + \sum_{l=1}^{k} v_J(l)^\top M_J(\theta(t_l^-))v_J(l)$$
$$+\,\mu_T(t) - \int_0^t \dot{\mu}_F(s)\,ds - \sum_{l=1}^{k}\big(\mu_J(t_l) - \mu_J(t_l^-)\big) \geq 0$$
$$\textit{for all} \quad t \in [t_k, t_{k+1}) \quad \textit{and all} \quad k \in \mathbb{N}_0$$

*holds for any trajectory of the impulsive filter* (4.6) *driven by* $\big(\begin{smallmatrix} z \\ \Delta(z) \end{smallmatrix}\big)$ *and* $\big(\begin{smallmatrix} z_J \\ \Delta_J(z_J) \end{smallmatrix}\big)$ *with any* $(z, z_J) \in L_{2e}^p \times \ell_{2e}^{p_J}$ *and any sequence of impulse instants* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT)*; here, we make use of the abbreviations* $\mu_\bullet(t) := \xi(t)^\top Z_\bullet(\theta(t))\xi(t)$ *for* $\bullet \in \{T, J, F\}$*. We denote the set of all such pairs* $(\Delta, \Delta_J)$ *by* $\mathrm{IQC}(Z_T, Z_J, Z_F, M, M_J)$*.*

Admittedly, this definition appears rather technical. However, for vanishing $Z_J$ and $Z_F$, it essentially states that the impulsive interconnection depicted in Fig. 4.3 satisfies a non-strict dissipation inequality similarly to the one that appeared in Remark 2.9. Recall that for deriving such dissipation inequalities for impulsive systems, we rely on expressing the function $\mu_T$ as

$$\begin{aligned}
&\mu_T(t) = \mu_T(t) - \mu_T(0)\\
&= \mu_T(t) - \mu_T(t_k) + \sum_{l=1}^{k}\big(\mu_T(t_l) - \mu_T(t_{l-1})\big)\\
&= \mu_T(t) - \mu_T(t_k) + \sum_{l=1}^{k}\big(\mu_T(t_l^-) - \mu_T(t_{l-1})\big) + \sum_{l=1}^{k}\big(\mu_T(t_l) - \mu_T(t_l^-)\big)\\
&= \int_0^t \dot{\mu}_T(s)\,ds + \sum_{l=1}^{k}\big(\mu_T(t_l) - \mu_T(t_l^-)\big)
\end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$, as well as on estimates for both the integral *and* the latter sum. The jump and flow costs, as characterized by the maps $Z_J$ and $Z_F$, are introduced to account for the situation that merely one of these estimates is available.

Note that the notion in Definition 4.5 generalizes the one proposed in [148] for standard LTI systems to impulsive ones. Indeed, we recover the dissipation inequality

$$\int_0^t v(s)^\top M v(s)\, ds + \xi(t)^\top Z \xi(t) \geq 0 \quad \text{for all} \quad t \geq 0$$

considered in [148] by dropping the jump component in Fig. 4.3 as well as the corresponding signals and maps, and by restricting the maps $Z$ and $M$ to be constant. Another related notion of IQCs was recently proposed in [26], where uncertain systems on a finite time horizon $[0, T]$ are considered. Their notion relies on an inequality of the form

$$\int_0^t v(s)^\top M(s) v(s)\, ds \geq 0 \quad \text{for} \quad t = T,$$

i.e., they do not consider a terminal cost and their inequality is not required to hold for all $t \in [0, T]$. Since we consider impulsive systems on the horizon $[0, \infty)$, the corresponding analogue is obtained by formally taking the limit $T \to \infty$ and accounting for the jumps:

$$\int_0^t v(s)^\top M(\theta(s)) v(s)\, ds + \sum_{l=1}^{k} v_J(l)^\top M_J(\theta(t_l^-)) v_J(l) \geq 0 \;\; \text{for} \;\; t = k = \infty.$$

This constitutes a so-called soft infinite-horizon IQC. It remains to be explored whether the arguments given in [148] generalize to impulsive systems and allow for linking the analysis criteria based on Definition 4.5 with those based on soft infinite-horizon IQCs.

Before giving several examples, we formulate the main result of this chapter, a genuine generalization of the IQC based analysis criteria in Theorem 4

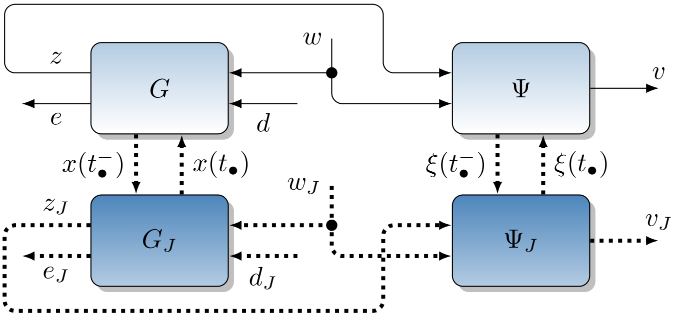



Figure 4.4: Block diagram of the augmented system (4.7) involving the system (4.5) and the filter (4.6).

from [148] to uncertain impulsive systems. To this end, we introduce the following augmented impulsive system

$$\begin{pmatrix}\dot\zeta(t)\\ v(t)\\ e(t)\end{pmatrix}=\begin{pmatrix}\mathcal{A} & \mathcal{B} & \mathcal{B}_2\\ \mathcal{C} & \mathcal{D} & \mathcal{D}_{12}\\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22}\end{pmatrix}\begin{pmatrix}\zeta(t)\\ w(t)\\ d(t)\end{pmatrix},\quad \begin{pmatrix}\zeta(t_k)\\ v_J(k)\\ e_J(k)\end{pmatrix}=\begin{pmatrix}\mathcal{A}_J & \mathcal{B}_J & \mathcal{B}_{J2}\\ \mathcal{C}_J & \mathcal{D}_J & \mathcal{D}_{J12}\\ \mathcal{C}_{J2} & \mathcal{D}_{J21} & \mathcal{D}_{J22}\end{pmatrix}\begin{pmatrix}\zeta(t_k^-)\\ w_J(k)\\ d_J(k)\end{pmatrix} \tag{4.7}$$

for $t \geq 0$ and $k \in \mathbb{N}$ with state $\zeta := \left(\begin{smallmatrix}\xi\\ x\end{smallmatrix}\right)$ as well as describing matrices

$$\left(\begin{array}{c|cc}\mathcal{A} & \mathcal{B} & \mathcal{B}_2\\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_{12}\\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22}\end{array}\right)=\left(\begin{array}{cc|cc}A_\Psi & B_\Psi C & B_\Psi D + B_{\Psi 2} & B_\Psi D_{12}\\ 0 & A & B & B_2\\ \hline C_\Psi & D_\Psi C & D_\Psi D + D_{\Psi 2} & D_\Psi D_{12}\\ C_2 & 0 & D_{21} & D_{22}\end{array}\right)$$

for the flow component; the matrices describing the jump component are given analogously. As illustrated in Fig. 4.4 in terms of a corresponding block diagram, this system results from the uncertain system (4.5) by removing the uncertainties and by an augmentation with the filter (4.6).

**Theorem 4.6** (IQC based Robust Analysis Criteria) *Let* $P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ *and* $P_J = \begin{pmatrix} Q_J & S_J \\ S_J^\top & R_J \end{pmatrix}$ *be symmetric matrices with* $Q \succcurlyeq 0$ *and* $Q_J \succcurlyeq 0$. *Then the system* (4.1) *is robustly stable and achieves robust quadratic performance with index* $(P, P_J)$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist functions* $X \in C^1([0, T_{\max}], \mathbb{S}^{n_\psi+n})$, $Z_T, Z_J, Z_F \in C^1([0, T_{\max}], \mathbb{S}^{n_\Psi})$, $M \in C([0, T_{\max}], \mathbb{S}^{n_v})$ *and* $M_J \in C([T_{\min}, T_{\max}], \mathbb{S}^{n_{vJ}})$ *satisfying*

$$X_T \succ 0 \quad \textit{and} \quad (\bullet)^\top \left(\begin{array}{cc|c:c} 0 & X_F & & \\ X_F & \dot{X}_F & & \\ \hline & & M & \\ \hdashline & & & P \end{array}\right) \left(\begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ I & 0 & 0 \\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \hdashline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{4.8a,b}$$

*on* $[0, T_{\max}]$,

$$(\bullet)^\top \left(\begin{array}{cc|c:c} X_J(0) & 0 & & \\ 0 & -X_J & & \\ \hline & & M_J & \\ \hdashline & & & P_J \end{array}\right) \left(\begin{array}{ccc} \mathcal{A}_J & \mathcal{B}_J & \mathcal{B}_{J2} \\ I & 0 & 0 \\ \hline \mathcal{C}_J & \mathcal{D}_J & \mathcal{D}_{J12} \\ \hdashline \mathcal{C}_{J2} & \mathcal{D}_{J21} & \mathcal{D}_{J22} \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{4.8c}$$

*on* $[T_{\min}, T_{\max}]$ *and*

$$\boldsymbol{\Delta} \times \boldsymbol{\Delta}_J \subset \mathrm{IQC}(Z_T, Z_J, Z_F, M, M_J) \tag{4.8d}$$

*where* $X_\bullet := X - \mathrm{diag}(Z_\bullet, 0)$ *for* $\bullet \in \{T, J, F\}$.

The full proof of Theorem 4.6 is given Subsection 4.1.3. Conceptually, it follows the lines of the one of Theorem 2.8 for nominal performance analysis, involves separation techniques similarly as in Lemma C.22, and relies on Lyapunov arguments involving a clock in order to capture the impulsive nature of the underlying augmented system (4.7). However, the individual steps are much more intricate due to the involved filter (4.6) that is itself an impulsive system.

In order make use of the IQC analysis conditions from Theorem 4.6, it is mandatory to employ numerically tractable criteria which imply the inclusion (4.8d). Analogously as discussed in Section C.6, these criteria should be not too conservative, not too costly to implement and, in particular, always be tailored to the concrete instance of the uncertainty sets $\mathbf{\Delta}$ and $\mathbf{\Delta}_J$. Next, we provide several of such choices as an illustration and begin by reconsidering arbitrarily time-varying parametric uncertainties as discussed in the previous subsection.

**Lemma 4.7** (Arbitrarily Time-Varying Parametric Uncertainties) *Suppose that $\tilde{\mathbf{\Delta}} \subset \mathbb{R}^{q\times p}$ and $\tilde{\mathbf{\Delta}}_J \subset \mathbb{R}^{q_J\times p_J}$ are given value sets and let the uncertainty sets are given by*

$$\mathbf{\Delta} := \left\{ \Delta(z)(t) := \tilde{\Delta}(t)z(t) \;\middle|\; \begin{array}{l} \tilde{\Delta} \text{ is piecewise continuous and} \\ \tilde{\Delta}(t) \in \tilde{\mathbf{\Delta}} \text{ for all } t \geq 0 \end{array} \right\},$$
$$\mathbf{\Delta}_J := \left\{ \Delta_J(z_J)(k) := \tilde{\Delta}_J(k) z_J(k) \;\middle|\; \tilde{\Delta}_J(k) \in \tilde{\mathbf{\Delta}}_J \text{ for all } k \in \mathbb{N} \right\}.$$

*Moreover, suppose that the filter* (4.6) *is static, i.e., $n_\Psi = 0$, and satisfies $(D_\Psi, D_{\Psi 2}) = I$, $(D_{\Psi_J}, D_{\Psi_J 2}) = I$. Then the inclusion* (4.8d) *holds for $Z_T := Z_J := Z_F := 0$ and all functions $M \in C([0, T_{\max}], \mathbb{S}^{p+q})$, $M_J \in C([T_{\min}, T_{\max}], \mathbb{S}^{p_J+q_J})$ satisfying*

$$(\bullet)^\top M \begin{pmatrix} I \\ \tilde{\Delta} \end{pmatrix} \succcurlyeq 0 \text{ for all } \tilde{\Delta} \in \tilde{\mathbf{\Delta}} \quad \text{and} \quad (\bullet)^\top M_J \begin{pmatrix} I \\ \tilde{\Delta}_J \end{pmatrix} \succcurlyeq 0 \text{ for all } \tilde{\Delta}_J \in \tilde{\mathbf{\Delta}}_J$$

*on $[0, T_{\max}]$ and on $[T_{\min}, T_{\max}]$, respectively.*

*Proof.* The proof is a consequence of Definition 4.5 and the particular choice of the filter since its outputs are given by $v = \left(\begin{smallmatrix} z \\ \tilde{\Delta} z \end{smallmatrix}\right) = \left(\begin{smallmatrix} I \\ \tilde{\Delta} \end{smallmatrix}\right) z$ and

$v_J = \left(\begin{smallmatrix} z_J \\ \tilde{\Delta}_J z_J \end{smallmatrix}\right) = \left(\begin{smallmatrix} I \\ \tilde{\Delta}_J \end{smallmatrix}\right) z_J$, respectively. Indeed, we have

$$\int_0^t v(s)^\top M(\theta(s))v(s)\,ds + \sum_{l=1}^k v_J(l)^\top M_J(\theta(t_l^-))v_J(l)$$
$$= \int_0^t (\bullet)^\top M(\theta(s))\begin{pmatrix} I \\ \tilde{\Delta}(s)\end{pmatrix} z(s)\,ds + \sum_{l=1}^k (\bullet)^\top M_J(\theta(t_l^-))\begin{pmatrix} I \\ \tilde{\Delta}_J(l)\end{pmatrix} z_J(l) \geq 0$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$ and any trajectory of the filter (4.6) and any sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT). ●

Note that the specific choice of the filter (4.6) in Lemma 4.7 renders the augmented system (4.7) identical to the known part of the original one (4.5). Thus, we recover the analysis criteria in Theorem 4.2 for impulsive systems affected by arbitrarily time-varying uncertainties by combining the IQC analysis criteria in Theorem 4.6 with Lemma 4.7. In particular, we have, e.g., the following result for a concrete choice of the value sets $\tilde{\mathbf{\Delta}}$ and $\tilde{\mathbf{\Delta}}_J$ which involves the abbreviation

$$P_p(S) := P_p^{a,b}(S) := \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix}^\top \begin{pmatrix} 0 & S^\top \\ S & 0 \end{pmatrix} \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix} \tag{4.9}$$

for any square matrix-valued map or square matrix $S$; many more robust analysis results for various instances of the value sets can be generated on the basis of Remark C.19.

**Corollary 4.8** (Arbitrarily Time-Varying Parametric Uncertainties for Concrete Value Sets) *Let the uncertainty sets $\mathbf{\Delta}$, $\mathbf{\Delta}_J$ and the filter* (4.6) *be as in Lemma 4.7 with value sets $\tilde{\mathbf{\Delta}}_J := \left\{\tilde{\Delta}_J \in \mathbb{R}^{q_J \times p_J} \mid \|\tilde{\Delta}_J\| \leq 1\right\}$ and $\tilde{\mathbf{\Delta}} := \left\{\tilde{\delta} I \mid \tilde{\delta} \in [a,b]\right\}$. Then the inclusion* (4.8d) *holds $Z_T := Z_J := Z_F := 0$, $M := P_p(N)$ and $M_J := \left(\begin{smallmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{smallmatrix}\right)$ and for any maps $N \in$*

$C([0,T_{\max}],\mathbb{R}^{q\times q})$ *and* $\alpha \in C([T_{\min},T_{\max}],\mathbb{R})$ *satisfying*

$$N + N^\top \succ 0 \quad \textit{on} \quad [0,T_{\max}] \qquad \textit{and} \qquad \alpha > 0 \quad \textit{on} \quad [T_{\min},T_{\max}].$$

*Proof.* This follows from Lemma 4.7 and from that

$$(\bullet)^\top M \begin{pmatrix} I \\ \tilde{\Delta} \end{pmatrix} = (\tilde{\delta} - a)(b - \tilde{\delta})(N + N^\top) \succcurlyeq 0$$

and

$$(\bullet)^T M_J \begin{pmatrix} I \\ \tilde{\Delta}_J \end{pmatrix} = \alpha(I - \tilde{\Delta}_J^\top \tilde{\Delta}_J) \succcurlyeq 0$$

hold for all $\tilde{\Delta} = \tilde{\delta} I \in \tilde{\mathbf{\Delta}}$ and all $\tilde{\Delta}_J \in \tilde{\mathbf{\Delta}}_J$, respectively. ●

Note that Lemma 4.7 and Corollary 4.8 rely on rendering the filter (4.6) trivial and only take information on the involved value sets into account; since the considered parametric uncertainties are assumed to be arbitrarily time-varying, there is also essentially nothing more we can do. However, in practice there frequently is additional information on the variation of the parametric uncertainties available or, e.g., it is known that they do not vary at all. One of the major benefits of Theorem 4.6 if compared to Theorem 4.2 is the possibility to readily incorporate such additional information via nontrivial filters (4.6) and on the basis of suitable IQCs. As an illustration, let us consider another class of parametric uncertainties that is relevant, e.g., in the context of switched systems.

**Lemma 4.9** (Piecewise Constant Parametric Uncertainty in Flow Component Only) *Suppose that the uncertainty channel in* (4.5b) *is absent, i.e.,* $p_J = q_J = 0$*, and that the uncertainty sets are given by* $\mathbf{\Delta}_J = \emptyset$ *and*

$$\mathbf{\Delta} := \left\{ \Delta(z)(t) := \delta(t) z(t) \;\middle|\; \begin{array}{l} \delta \textit{ is constant on each interval } [t_k, t_{k+1}) \\ \textit{with values in } [a,b] \end{array} \right\}.$$

*Moreover, let $P_p$ be as in* (4.9) *and, for some matrices $A_\psi$, $B_\psi$, $C_\psi$, $D_\psi$, let the describing matrices of the filter* (4.6) *be structured as*

$$\left(\begin{array}{c|c:c} A_\Psi & B_\Psi & B_{\Psi 2} \\ \hline C_\Psi & D_\Psi & D_{\Psi 2} \end{array}\right) = \left(\begin{array}{cc|c:c} A_\psi & 0 & B_\psi & 0 \\ 0 & A_\psi & 0 & B_\psi \\ \hline C_\psi & 0 & D_\psi & 0 \\ 0 & C_\psi & 0 & D_\psi \end{array}\right) \quad \textit{and} \quad A_{\Psi_J} = 0.$$

*Then the inclusion* (4.8d) *holds for $Z_T := Z_J := \frac{1}{2}P_p(R)$, $Z_F := 0$ and $M := P_p(N)$ with any functions $N \in C([0, T_{\max}], \mathbb{R}^{\frac{n_v}{2} \times \frac{n_v}{2}})$ and $R \in C^1([0, T_{\max}], \mathbb{S}^{\frac{n_\Psi}{2}})$ satisfying*

$$(\bullet)^\top \begin{pmatrix} 0 & R \\ R & \dot{R} \end{pmatrix} \begin{pmatrix} A_\psi & B_\psi \\ I & 0 \end{pmatrix} + (\bullet)^\top (N + N^\top) \begin{pmatrix} C_\psi & D_\psi \end{pmatrix} \succ 0 \quad \textit{on} \quad [0, T_{\max}]. \tag{4.10}$$

Note that by choosing the filter in Lemma 4.9 to be static, i.e., for $n_\Psi = 0$, we essentially treat the uncertainties in the same fashion as arbitrarily time-varying ones as in Corollary 4.8. Such a choice is by now well-known in the robust control literature to simplify the resulting analysis conditions and to reduce their computational burden at the prize of potentially being overly conservative. This is also natural from an optimization point of view, just because much less degrees of freedom are involved. Corresponding demonstrations are found, e.g., in [120] for standard LTI systems.

The detailed dissipation based proof of Lemma 4.9 is given in Subsection 4.1.3. Its key element is the state resetting property of the filter (4.6)

$$\xi(t_k) = 0 \quad \text{for all} \quad k \in \mathbb{N}_0$$

as induced by $A_{\Psi_J} = 0$. This property and the filter's block diagonal structure assure that the filter's output admits the commutation property

$$v = \Psi \begin{pmatrix} z \\ w \end{pmatrix} = \Psi \left( \begin{pmatrix} I \\ \delta I \end{pmatrix} z \right) = \begin{pmatrix} I & 0 \\ 0 & \delta I \end{pmatrix} \Psi \begin{pmatrix} z \\ z \end{pmatrix} = \begin{pmatrix} I \\ \delta I \end{pmatrix} v_1 \tag{4.11}$$

for the output $v_1$ of the (sub-)filter

$$\begin{pmatrix} \xi_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} A_\psi & B_\psi \\ C_\psi & D_\psi \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ z(t) \end{pmatrix}, \quad \xi_1(t_k) = 0$$

and for all admissible inputs and uncertainties; here, we denote by $\Psi$ the input-output map $\left( \begin{smallmatrix} z \\ w \end{smallmatrix} \right) \mapsto v$ corresponding to (4.6). The remaining ingredients for generating the desired estimates are the inequality

$$\int_0^t \dot{\tilde{\eta}}(s)\, ds \geq - \int_0^t v_1(s)^\top (N(\theta(s)) + N(\theta(s))^\top) v_1(s)\, ds \quad \text{for all} \quad t \geq 0$$

for the map $\tilde{\eta} : t \mapsto \xi_1(t)^\top R(\theta(t)) \xi_1(t)$ which is assured by (4.10), and the identity

$$(\bullet)^\top P_p(N) \begin{pmatrix} I \\ \delta I \end{pmatrix} = \underbrace{(\delta - a)(b - \delta)}_{\geq 0} (N + N^\top)$$

which holds for any $\delta \in [a, b]$ and any square matrix $N$.

By combining Lemma 4.9 with Theorem 4.6, we can employ IQCs with dynamic filters for the robustness analysis of systems affected by piecewise constant uncertainties for the first time. We have published the novel robust analysis criteria resulting from this combination in [83], where we additionally provide a detailed discussion of this particular result with several interesting applications, e.g., to consensus problems. In order to render the paper [83] self-contained, this paper provides an alternative motivation of these analysis criteria resulting in a more direct, but less modular proof.

If the underlying system (4.5) admits a trivial jump component, then we can recover a robust analysis result from [83] for standard LTI systems by combining Theorem 4.6 with the following lemma and by choosing all involved decision variables to be constant matrices. Its proof is similar to and simpler than the one of Lemma 4.9 and thus omitted.

**Lemma 4.10** (Constant Parametric Uncertainties in Flow Component Only) *Suppose that the uncertainty channel in* (4.5b) *is absent, i.e.,* $p_J = q_J = 0$ *and that the uncertainty sets are given by* $\mathbf{\Delta}_J = \emptyset$ *and*

$$\mathbf{\Delta} := \{\Delta(z)(t) := \delta z(t) \mid \delta \in [a,b]\}.$$

*Moreover, let the map* $P_p$ *and the filter* (4.6) *be as in Lemma 4.9, but with* $A_{\Psi_J} = I$*. Then the inclusion* (4.8d) *holds for* $Z_T := Z_J := \frac{1}{2}P_p(R)$*,* $Z_F := 0$*, and* $M := P_p(N)$ *with any maps* $N \in C([0,T_{\max}], \mathbb{R}^{\frac{n_v}{2}\times\frac{n_v}{2}})$ *and* $R \in C^1([0,T_{\max}], \mathbb{S}^{\frac{n_\Psi}{2}})$ *satisfying* (4.10)*.*

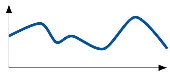
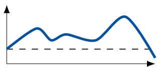
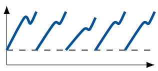
In particular, note that the employed filter in Lemma 4.10 is a standard LTI system since its jump component is trivial. It is not difficult to see that this filter also admits the essential commutation property (4.11) for the constant parametric uncertainties considered in Lemma 4.10.

In fact, for all of the so far considered (parametric) uncertainties, it is essential to choose filters (4.6) that are compatible with the encountered uncertainties in the sense that property (4.11) is assured. Some possible combinations are illustrated in Table 4.1 where

$$\begin{aligned}
\mathbf{\Delta}_c &:= \left\{\Delta(z)(t) := \delta z(t) \;\middle|\; \delta \in [a,b]\right\},\\
\mathbf{\Delta}_{pwc} &:= \left\{\Delta(z)(t) := \delta(t) z(t) \;\middle|\; \begin{array}{l}\delta \text{ is constant on intervals } [t_k, t_{k+1})\\ \text{with values in } [a,b]\end{array}\right\},\\
\mathbf{\Delta}_{atv} &:= \left\{\Delta(z)(t) := \delta(t) z(t) \;\middle|\; \begin{array}{l}\delta \text{ is piecewise continuous}\\ \text{with values in } [a,b]\end{array}\right\}
\end{aligned}$$

and where

Table 4.1: Illustration of the compatibility of filter types with classes of single repeated parametric uncertainties.

| | $\boldsymbol{\Delta}_c$  | $\boldsymbol{\Delta}_{pwc}$  | $\boldsymbol{\Delta}_{atv}$  |
|---|---|---|---|
| LTI $\Psi$  | ✓ | ✗ | ✗ |
| $\Psi$ with resets  | (✓) | ✓ | ✗ |
| Static $\Psi$  | (✓) | (✓) | ✓ |

- ✓ means that (4.11) is guaranteed for the particular combination of $\boldsymbol{\Delta}_\bullet$ and $\Psi$;
- (✓) means that (4.11) is guaranteed, but the resulting analysis criteria can be overly conservative for the particular instance of $\boldsymbol{\Delta}_\bullet$;
- ✗ means that (4.11) is not assured.

The plots on the top and on the left in Table 4.1 display the corresponding functions

$$t \mapsto \frac{\|\Delta(z)(t)\|}{\|z(t)\|} \quad \text{and} \quad t \mapsto \frac{\|\Psi(\left(\begin{smallmatrix} z \\ \Delta(z) \end{smallmatrix}\right))(t)\|}{\|\left(\begin{smallmatrix} z(t) \\ \Delta(z)(t) \end{smallmatrix}\right)\|},$$

respectively, for some pointwise nonzero function $z$ and illustrate that compatibility is related to the interplay of these functions over time (and on the intervals of interest). Based on such considerations it is, e.g., possible to develop novel filters that are compatible with and precisely capture piecewise constant and periodic parametric uncertainties, but this is not further elaborated on here.

Instead, let us formulate the jump component analogue of Lemma 4.10.

**Lemma 4.11** (Constant Parametric Uncertainties in Jump Component Only) *Suppose that the uncertainty channel in* (4.5a) *is absent, i.e., $p = q = 0$, and that the uncertainty sets are given by $\mathbf{\Delta} = \emptyset$ and*

$$\mathbf{\Delta}_J := \{\Delta_J(z_J)(k) := \delta_J z_J(k) \mid \delta_J \in [a, b]\}.$$

*Moreover, let $P_p$ be as in* (4.9) *and, for some matrices $A_{\psi_J}$, $B_{\psi_J}$, $C_{\psi_J}$, $D_{\psi_J}$, let the describing matrices of the filter* (4.6) *be structured as*

$$A_\Psi = 0 \quad \text{and} \quad \left(\begin{array}{c|c:c} A_{\Psi_J} & B_{\Psi_J} & B_{\Psi_{J2}} \\ \hline C_{\Psi_J} & D_{\Psi_J} & D_{\Psi_{J2}} \end{array}\right) = \left(\begin{array}{cc|c:c} A_{\psi_J} & 0 & B_{\psi_J} & 0 \\ 0 & A_{\psi_J} & 0 & B_{\psi_J} \\ \hline C_{\psi_J} & 0 & D_{\psi_J} & 0 \\ 0 & C_{\psi_J} & 0 & D_{\psi_J} \end{array}\right).$$

*Then the inclusion* (4.8d) *holds for $Z_T := Z_F := \frac{1}{2}P_p(R)$, $Z_J := 0$ and $M_J := P_p(N_J)$ with any maps $N_J \in C([T_{\min}, T_{\max}], \mathbb{R}^{\frac{n_{v_J}}{2} \times \frac{n_{v_J}}{2}})$ and $R \in C^1([0, T_{\max}], \mathbb{S}^{\frac{n_\Psi}{2}})$ satisfying*

$$(\bullet)^\top \begin{pmatrix} R(0) & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} A_{\psi_J} & B_{\psi_J} \\ I & 0 \end{pmatrix} + (\bullet)^\top (N_J + N_J^\top) \begin{pmatrix} C_{\psi_J} & D_{\psi_J} \end{pmatrix} \succ 0 \quad (4.12)$$

*on $[T_{\min}, T_{\max}]$.*

The proof of this novel result is also found in Subsection 4.1.3. We stress that such analogous results are not too difficult to obtain due to the flexibility and generality of Theorem 4.6 and the underlying notion of IQCs in Definition 4.5.

Note that all of the given examples so far involve parametric uncertainties. Naturally, IQCs as established in Definition 4.5 permit us to capture the input-output behavior of much more general uncertainties and Theorem 4.6 gives us dedicated robust analysis criteria for them. We give here two more examples involving nonparametric uncertainties and refer the reader to [109, 160, 50] for a multitude of further examples. In those publications they are formulated for standard LTI systems, but their extension to impulsive ones is not difficult based on the illustrations given next.

**Lemma 4.12** (Dynamic Repeated Uncertainty in Flow Component Only) *Suppose that the uncertainty channel in* (4.5b) *is absent, i.e.,* $p_J = q_J = 0$, *and that the uncertainty sets are given by* $\mathbf{\Delta}_J = \emptyset$ *and*

$$\mathbf{\Delta} := \Big\{ \Delta(z)(t) := \int_0^t C_\delta e^{A_\delta(t-s)} B_\delta z(s)\, ds + D_\delta z(t) \ \Big| \ \delta(s) := C_\delta(sI - A_\delta)^{-1}B_\delta + D_\delta \ \text{satisfies} \ \delta \in \mathrm{RH}_\infty^{1\times 1} \ \text{and} \ \|\delta\|_\infty \leq 1 \Big\}.$$

*Moreover, let the filter* (4.6) *be structured as in Lemma 4.10. Then the inclusion* (4.8d) *holds for* $Z_T := Z_J := \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right)$, $Z_F := 0$ *and* $M := \left(\begin{smallmatrix} N & 0 \\ 0 & -N \end{smallmatrix}\right)$ *with any matrices* $R \in \mathbb{S}^{\frac{n_\Psi}{2}}$ *and* $N \in \mathbb{S}^{\frac{n_v}{2}}$ *satisfying*

$$(\bullet)^\top \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} \begin{pmatrix} A_\psi & B_\psi \\ I & 0 \end{pmatrix} + (\bullet)^\top N \begin{pmatrix} C_\psi & D_\psi \end{pmatrix} \succ 0. \tag{4.13}$$

The proof is given in Subsection 4.1.3. Similarly as before, it revolves around the commutation property (4.11) which can be shown to be valid on the basis of Fubini's theorem.

Note that one of the arguments in the proof of Lemma 4.12 breaks down, if we replace the inequality (4.13) with

$$(\bullet)^\top \begin{pmatrix} \dot{R} & R \\ R & 0 \end{pmatrix} \begin{pmatrix} A_\psi & B_\psi \\ I & 0 \end{pmatrix} + (\bullet)^\top N \begin{pmatrix} C_\psi & D_\psi \end{pmatrix} \succ 0 \quad \text{on} \quad [0, T_{\max}] \tag{4.14}$$

for nonconstant maps $R \in C^1([0, T_{\max}], \mathbb{S}^{\frac{n_\Psi}{2}})$ and $N \in C([0, T_{\max}], \mathbb{S}^{\frac{n_v}{2}})$. This issue seems to be related to the fact that

$$N(t)\Delta(z)(t) = \Delta(Nz)(t) \quad \text{for all} \quad \Delta \in \mathbf{\Delta}$$

holds only for constant functions $N$. Nevertheless, one can show that the statement in Lemma 4.12 is true, if we replace (4.13) with (4.14) *and* a particular monotonicity criterion

$$(\bullet)^\top \begin{pmatrix} \ddot{R} & \dot{R} \\ \dot{R} & 0 \end{pmatrix} \begin{pmatrix} A_\psi & B_\psi \\ I & 0 \end{pmatrix} + (\bullet)^\top \dot{N} \begin{pmatrix} C_\psi & D_\psi \end{pmatrix} \preccurlyeq 0 \quad \text{on} \quad [0, T_{\max}]$$

for $R \in C^2([0, T_{\max}], \mathbb{S}^{\frac{n_\Psi}{2}})$ and $N \in C^1([0, T_{\max}], \mathbb{S}^{\frac{n_v}{2}})$. How these alternative criteria compare to the ones in Lemma 4.12 remains to be explored.

Finally, let us consider a class of nonlinear uncertainties and provide a suitable IQC for them that is essentially taken from [50]. The proof is found in Section 4.1.3.

**Lemma 4.13** (Slope-Restricted Nonlinear Repeated Uncertainty in Jump Component Only) *Suppose that the uncertainty channel in* (4.5a) *is absent, i.e.,* $p = q = 0$, *and that the uncertainty sets are given by* $\mathbf{\Delta} = \emptyset$ *and*

$$\mathbf{\Delta}_J := \Big\{ \Delta_J(z_J)(k) := (\phi(z_{J1}(k)), \ldots, \phi(z_{J2}(k)))^\top \;\Big|\; \phi : \mathbb{R} \to \mathbb{R} \textit{ satisfies } \phi(0) = 0 \textit{ and } 0 \leq \frac{\phi(x) - \phi(y)}{x - y} \textit{ for all } x \neq y \Big\}.$$

*Moreover, let the filter* (4.6) *be structured as*

$$A_\Psi = 0 \quad \textit{and} \quad \left(\begin{array}{c|c:c} A_{\Psi_J} & B_{\Psi_J} & B_{\Psi_{J2}} \\ \hline C_{\Psi_J} & D_{\Psi_J} & D_{\Psi_{J2}} \end{array}\right) = \left(\begin{array}{c|c:c} J_\nu \otimes I_{p_J} & e_\nu \otimes I_{p_J} & 0 \\ \hline C_\nu \otimes I_{p_J} & e_{\nu+1} \otimes I_{p_J} & 0 \\ 0 & 0 & I_{p_J} \end{array}\right)$$

*for some length* $\nu \in \mathbb{N}_0$*; here,* $C_\nu := \begin{pmatrix} I_\nu \\ 0_{1\times\nu} \end{pmatrix}$*,* $e_j$ *denotes the* $j$*th standard unit vector of appropriate dimension and* $J_\nu \in \mathbb{R}^{\nu\times\nu}$ *is a single Jordan block with ones in the entries located directly above the main diagonal and zeros elsewhere. Then the inclusion* (4.8d) *holds for* $Z_T := Z_J := Z_F := 0$ *and* $M_J := \begin{pmatrix} 0 & (\bullet)^\top \\ (\Lambda_1,\ldots,\Lambda_{\nu+1}) & 0_{p_J\times p_J} \end{pmatrix}$ *with any matrix* $\Lambda := (\Lambda_1, \ldots, \Lambda_{\nu+1}) \in \mathbb{R}^{p_J\times(\nu+1)p_J}$ *satisfying componentwise*

$$\sum_{j=1}^{\nu+1} \Lambda_j \mathbf{1} \geq 0, \quad \sum_{j=1}^{\nu+1} \mathbf{1}^\top \Lambda_j \geq 0, \quad \Lambda_1 \leq 0, \quad \ldots, \quad \Lambda_\nu \leq 0, \quad e_j^\top \Lambda_{\nu+1} e_i \leq 0 \text{ for } i \neq j, \tag{4.15}$$

*where* $\mathbf{1}$ *denotes the all-one vector.*

Note that all of the provided examples merely deal with a single or two uncertainties affecting the underlying impulsive system (4.1) or only one of its components. In view of the *modularity* of the IQC approach, this is more than sufficient because it is a simple exercise to combine Theorem 4.6 with, e.g., Lemmas 4.7, 4.12 and 4.13, in order to generate “novel” robust analysis criteria for an impulsive system affected by an arbitrarily time-varying parametric and a dynamic uncertainty in its flow component and a slope-restricted nonlinear uncertainty in its jump component. Completely analogous as illustrated, e.g., in Remark C.19 or in [160, 50], this is achieved by diagonally stacking the involved maps and the describing matrices of the filter (4.6). This modularity is one of most beautiful properties of the IQC framework and is also highly relevant for practical purposes when encountering a variety of different sources of uncertainties.

**Remark 4.14** (Generalizations of Definition 4.5 and Theorem 4.6)

(a) In principle we can replace the decoupled uncertainties in the system description (4.5) by a single more general one of the form $\Delta : \begin{pmatrix} z \\ z_J \end{pmatrix} \mapsto \begin{pmatrix} w \\ w_J \end{pmatrix}$. Such coupled uncertainties could emerge, e.g., from neglected or unknown impulsive dynamics in the underlying model or appear in descriptions of uncertain sampled-data systems. However, it seems that appropriately dealing with such uncertainties requires to merge the robustness analysis inequalities for the flow and jump component, (4.8b) and (4.8c), into a single, larger and more intricate inequality similarly as we did in [80] in a different context. There we considered systems affected by piecewise constant uncertainties with bounds on the involved jump heights.

(b) Analogously as for our initial nominal analysis result Theorem 2.3, the describing matrices of the uncertain system (4.1) can be allowed to depend continuously on the clock $\theta$ which is relevant, e.g., for closed-loop analysis involving controllers as designed in the next chapter. The describing matrices of the filter (4.6) are allowed to vary in the same fashion as long as this is compatible with the considered uncertainty set. While such filters are potentially superior to those with constant describing matrices, it is not yet clear how to systematically benefit from the additionally freedom that is involved.

**Remark 4.15** (Switched and Sampled-Data Systems) By recalling the previous chapters it is not much of a surprise that all our robust analysis results for linear impulsive systems also apply, e.g., to switched and sampled-data systems after minor modifications only.

### Example

Recall that we obtained via Theorem 4.2 the upper bound 1.176 on the robust energy gain in our previous example on page 111 involving an active suspension system affected by two scalar arbitrarily time-varying parametric uncertainties that take values in the interval $[-1, 1]$.

Let us now suppose that the latter uncertainties are known to be constant instead. This additional information permits us to generate better upper bounds based on combining Theorem 4.6 with a minor modification of

Lemma 4.10. To this end note that the matrices $(A_\psi, B_\psi, C_\psi, D_\psi)$ in the original version of Lemma 4.10 are typically chosen as

$$\left(-\begin{pmatrix}\alpha & 2\alpha & \dots & 2\alpha\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \dots & \alpha & 2\alpha\\ 0 & \dots & 0 & \alpha\end{pmatrix}\otimes I_p, \begin{pmatrix}-\sqrt{2\alpha}\\ \vdots\\ -\sqrt{2\alpha}\\ -\sqrt{2\alpha}\end{pmatrix}\otimes I_p, \begin{pmatrix}0 & \dots & 0\\ 0 & \dots & \sqrt{2\alpha}\\ \vdots & \iddots & \vdots\\ \sqrt{2\alpha} & \dots & \sqrt{2\alpha}\end{pmatrix}\otimes I_p, \begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix}\otimes I_p\right) \tag{4.16a}$$

or

$$\left(\begin{pmatrix}-\alpha & 0 & \dots & 0\\ 1 & \ddots & \ddots & \vdots\\ & \ddots & \ddots & 0\\ 0 & & 1 & -\alpha\end{pmatrix}\otimes I_p, \begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix}\otimes I_p, \begin{pmatrix}0_{1\times\nu}\\ I_\nu\end{pmatrix}\otimes I_p, \otimes\begin{pmatrix}1\\ 0_{\nu\times 1}\end{pmatrix}\otimes I_p\right) \tag{4.16b}$$

for some fixed $\alpha > 0$ and some length $\nu \in \mathbb{N}_0$. These choices correspond to the stable transfer matrices

$$\begin{pmatrix}I_p\\ \frac{s-\alpha}{s+\alpha}I_p\\ \vdots\\ \left(\frac{s-\alpha}{s+\alpha}\right)^\nu I_p\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}I_p\\ \frac{1}{s+\alpha}I_p\\ \vdots\\ \left(\frac{1}{s+\alpha}\right)^\nu I_p\end{pmatrix},$$

respectively, which are known to admit nice approximation properties [147, 119]. In order to analyze the concretely given system, we employ a diagonally stacked version of the first choice with $\alpha = 5$ and $\nu = 2$. This results in an upper bound of 0.708 on the robust energy gain which is much smaller than the previous one 1.176 (that is recovered by choosing $\nu = 0$). Note that this bound can be improved to some extend by increasing $\nu$ and/or by adjusting the underlying relaxation parameters at the prize of a higher computational burden. Of course, we can also reduce the computational burden, e.g., by restricting the functions in Lemma 4.10 to be constant, but this results in 0.993 as an upper bound which is close to the one obtained for arbitrarily time-varying uncertainties.

### 4.1.3 Technical Proofs and Auxiliary Results

#### Main IQC Based Robust Analysis Criteria

*Proof of Theorem 4.6.* We prove the result in four steps.

*Step 1: Preparations:* By continuity of the involved functions, compactness of $[0, T_{\max}]$ and $[T_{\min}, T_{\max}]$, and by strictness of the inequalities (4.8a)-(4.8c), we infer the existence of positive constants $\alpha, \beta, \varepsilon$ with

$$\alpha I \preccurlyeq X_T \preccurlyeq \beta I, \tag{$*1$}$$

such that the inequality (4.8b) also holds if $\varepsilon I$ is added to its left hand side and such that (4.8c) hold for $R_J$ replaced by $R_J + \varepsilon I$.

Now let $(d, d_J, \Delta, \Delta_J) \in L_2 \times \ell_2 \times \mathbf{\Delta} \times \mathbf{\Delta}_J$ be arbitrary and let $(t_k)_{k \in \mathbb{N}_0}$ be an arbitrary sequence of impulse instants satisfying (RDT). By our standing well-posedness assumption on $\mathbf{\Delta}$ and $\mathbf{\Delta}_J$, there exists a corresponding state trajectory $x$ of the system (4.5) and the remaining signals are also well-defined. Next, we let $\xi$ be the state of the filter (4.6) and infer that $\zeta := \left(\begin{smallmatrix} \xi \\ x \end{smallmatrix}\right)$ satisfies (4.7). With all those signals and the clock $\theta$ as given in (2.2), we define the functions

$$\nu : t \mapsto \zeta(t)^\top X(\theta(t))\zeta(t), \quad \mu_\bullet : t \mapsto \xi(t)^\top Z_\bullet(\theta(t))\xi(t) \quad \text{and} \quad \eta_\bullet := \nu - \mu_\bullet$$

for $\bullet \in \{T, J, F\}$. In particular, we have $\alpha \leq \eta_T(t) \leq \beta$ for all $t \geq 0$ due to ($*1$).

*Step 2: Individual Estimates:* From the $\varepsilon$-modification of (4.8b), we infer

$$
\begin{aligned}
\dot{\eta}_F(t) &= (\bullet)^\top \begin{pmatrix} 0 & X_F(\theta(t)) \\ X_F(\theta(t)) & \dot{X}_F(\theta(t)) \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta(t) \\ w(t) \\ d(t) \end{pmatrix} \\
&\leq -(\bullet)^\top \left(\begin{array}{c:c} M(\theta(t)) & \\ \hdashline & P \end{array}\right) \left(\begin{array}{ccc} \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \hdashline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \\ 0 & 0 & I \end{array}\right) \begin{pmatrix} \zeta(t) \\ w(t) \\ d(t) \end{pmatrix} - \varepsilon \left\| \begin{pmatrix} \zeta(t) \\ w(t) \\ d(t) \end{pmatrix} \right\|^2 \\
&= -(\bullet)^\top M(\theta(t)) v(t) - (\bullet)^\top P \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} - \varepsilon \left\| \begin{pmatrix} \zeta(t) \\ w(t) \\ d(t) \end{pmatrix} \right\|^2
\end{aligned}
$$

for all $t \in (t_k, t_{k+1})$ and $k \in \mathbb{N}_0$. Thus, continuity of $\eta_F$ on the intervals $[t_k, t_{k+1})$ implies

$$
\int_0^t \dot{\eta}_F(s)\, ds \leq - \int_0^t (\bullet)^\top M(\theta(s)) v(s) + (\bullet)^\top P \begin{pmatrix} e(s) \\ d(s) \end{pmatrix} + \varepsilon \left\| \begin{pmatrix} \zeta(s) \\ w(s) \\ d(s) \end{pmatrix} \right\|^2 ds
$$

for all $t \geq 0$. Similarly, the $\varepsilon$-modification of (4.8c) yields, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
&\eta_J(t_k) - \eta_J(t_k^-) \\
&= (\bullet)^\top \begin{pmatrix} X_J(0) & 0 \\ 0 & -X_J(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} \mathcal{A}_J & \mathcal{B}_J & \mathcal{B}_{J2} \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta(t_k^-) \\ w_J(k) \\ d_J(k) \end{pmatrix} \\
&\leq -(\bullet)^\top \left(\begin{array}{c:c} M_J(\theta(t_k^-)) & \\ \hdashline & P_J \end{array}\right) \left(\begin{array}{ccc} \mathcal{C}_J & \mathcal{D}_J & \mathcal{D}_{J12} \\ \hdashline \mathcal{C}_{J2} & \mathcal{D}_{J21} & \mathcal{D}_{J22} \\ 0 & 0 & I \end{array}\right) \begin{pmatrix} \zeta(t_k^-) \\ w_J(k) \\ d_J(k) \end{pmatrix} - \varepsilon \| d_J(k) \|^2 \\
&= -(\bullet)^\top M_J(\theta(t_k^-)) v_J(k) - (\bullet)^\top P_J \begin{pmatrix} e_J(k) \\ d_J(k) \end{pmatrix} - \varepsilon \| d_J(k) \|^2.
\end{aligned}
$$

*Step 3: Combined Estimates:* Recall that the state $\xi$ of the filter (4.6) is initialized in zero. Then the particular piecewise continuity of $\nu$ and the definition $\eta_\bullet = \nu - \mu_\bullet$ for $\bullet \in \{T, J, F\}$ yield

$$\begin{aligned}
&\eta_T(t) - \eta_T(0) + \mu_T(t) \\
&= \nu(t) - \nu(0) = \int_0^t \dot{\nu}(s)\, ds + \sum_{l=1}^{k} \big(\nu(t_l) - \nu(t_l^-)\big) \\
&= \int_0^t \dot{\eta}_F(s)\, ds + \int_0^t \dot{\mu}_F(s)\, ds + \sum_{l=1}^{k} \big(\eta_J(t_l) - \eta_J(t_l^-)\big) + \sum_{l=1}^{k} \big(\mu_J(t_l) - \mu_J(t_l^-)\big)
\end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. By the inclusion (4.8d), i.e., $(\Delta, \Delta_J) \in \mathrm{IQC}(Z_T, Z_J, Z_F, M, M_J)$, we then obtain

$$\begin{aligned}
\eta_T(t) - \eta_T(0) \leq \int_0^t \dot{\eta}_F(s)\, ds + \sum_{l=1}^{k} \big(\eta_J(t_l) - \eta_J(t_l^-)\big)& \\
+ \int_0^t (\bullet)^\top M(\theta(s)) v(s)\, ds + \sum_{l=1}^{k} (\bullet)^\top M_J(\theta(t_l^-)) v_J(l)&
\end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. Finally, we can conclude by the inequalities from the previous step that

$$\begin{aligned}
\eta_T(t) - \eta_T(0) \leq - \int_0^t (\bullet)^\top P \begin{pmatrix} e(s) \\ d(s) \end{pmatrix} ds - \sum_{l=1}^{k} (\bullet)^\top P_J \begin{pmatrix} e_J(l) \\ d_J(l) \end{pmatrix}& \\
- \varepsilon \int_0^t \left\| \begin{pmatrix} \zeta(s) \\ w(s) \\ d(s) \end{pmatrix} \right\|^2 ds - \varepsilon \sum_{l=1}^{k} \|d_J(l)\|^2 \quad (*2)&
\end{aligned}$$

holds for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$.

*Step 4: Robust Stability:* Let us now suppose that the input disturbances vanish, i.e., that $d = 0$ and $d_J = 0$ hold. Then $(*2)$ yields, in particular,

$$\eta_T(t)+\varepsilon\int_0^t \left\| \begin{pmatrix} \zeta(s) \\ w(s) \end{pmatrix} \right\|^2 ds \leq \eta_T(0) - \int_0^t (\bullet)^\top Q e(s) ds - \sum_{l=1}^{k} (\bullet)^\top Q_J e_J(l) \leq \eta_T(0)$$

for $t \geq 0$ because $Q \succcurlyeq 0$ and $Q_J \succcurlyeq 0$ hold by assumption. As a consequence of $(*1)$ and $\xi(0) = 0$, we get

$$\|x(t)\|^2 \leq \|\zeta(t)\|^2 \leq \frac{1}{\alpha}\eta_T(t) \leq \frac{1}{\alpha}\eta_T(0) \leq \frac{\beta}{\alpha}\|\zeta(0)\|^2 \leq \frac{\beta}{\alpha}\|x(0)\|^2$$

for all $t \geq 0$, i.e., uniform boundedness. Moreover, we can conclude $\zeta = \left(\begin{smallmatrix} \xi \\ x \end{smallmatrix}\right) \in L_2^{n_\Psi + n}$ and $w \in L_2^q$ which yields, in particular, $x \in L_2^n$ and $\dot{x} \in L_2^n$. Due to a variant of Barbalat's lemma [49], this finally yields $x(t) \to 0$ as $t \to \infty$.

*Step 5: Robust Performance:* Let us now suppose that $x(0) = 0$ holds which results in $\eta_T(0) = 0$ since $\xi(0) = 0$ holds as well. Together with the nonnegativity of $\eta_T$, this allows to conclude from $(*2)$ that

$$\int_0^t (\bullet)^\top P \begin{pmatrix} e(s) \\ d(s) \end{pmatrix} ds + \sum_{l=1}^{k} (\bullet)^\top P_J \begin{pmatrix} e_J(l) \\ d_J(l) \end{pmatrix} \leq -\varepsilon \int_0^t \|d(s)\|^2 ds - \varepsilon \sum_{l=1}^{k} \|d_J(l)\|^2$$

holds for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. The proof is finished by taking the limit $t \to \infty$. ●

## Assuring the IQC for Piecewise Constant Repeated Parametric Uncertainties in the Flow Component

*Proof of Lemma 4.9.* Let $(t_k)_{k\in\mathbb{N}_0}$ with (RDT), $\Delta \in \mathbf{\Delta}$ and $z \in L_{2e}^p$ be arbitrary. Then there exists a particularly piecewise constant function $\delta$ with $w := \Delta(z) = \delta z$. Let us partition the state $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and the output $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ of the filter (4.6) driven by $\left(\begin{smallmatrix} z \\ \Delta(z) \end{smallmatrix}\right)$ accordingly to the partition of $A_\Psi$ and $C_\Psi$, respectively. In order to proceed, it is crucial to note that $\xi(0) = 0$ and $A_{\Psi_J} = 0$ imply $\xi(t_k) = 0$ for all $k \in \mathbb{N}_0$. These resets of the filter's state lead to the following important commutation property which is due to the variation of constants formula and the piecewise constant nature of $\delta$:

$$\begin{aligned}\xi_2(t) = e^{A_\psi(t-t_k)}\xi_2(t_k) + \int_{t_k}^{t} e^{A_\psi(t-s)}B_\psi w(s)\, ds\\ = \delta(t_k)\int_{t_k}^{t} e^{A_\psi(t-s)}B_\psi z(s)\, ds = \delta(t)\xi_1(t)\end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. Similarly we have $v_2(t) = \delta(t)v_1(t)$ for all $t \geq 0$. Next, observe that

$$\begin{aligned}(\bullet)^\top P_p(S)\begin{pmatrix} I \\ rI \end{pmatrix} x &= (\bullet)^\top \begin{pmatrix} 0 & S^\top \\ S & 0 \end{pmatrix}\begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix}\begin{pmatrix} I \\ rI \end{pmatrix} x\\ &= (\bullet)^\top \begin{pmatrix} 0 & S^\top \\ S & 0 \end{pmatrix}\begin{pmatrix} (b-r)I \\ (r-a)I \end{pmatrix} x\\ &= (b-r)(r-a)x^\top(S+S^\top)x = 2(b-r)(r-a)x^\top Sx\end{aligned}$$

holds for any scalar $r$, vector $x$ and matrix $S$.

Let us now define the function $\mu_T : t \mapsto \xi(t)^\top Z_T(\theta(t))\xi(t)$ and observe that the preparations, $Z_T = \frac{1}{2}P_p(R)$ and $\xi(t) = \left(\begin{smallmatrix} I \\ \delta(t)I \end{smallmatrix}\right)\xi_1(t)$ imply

$$\mu_T(t) = \frac{1}{2}(\bullet)^\top P_p(R(\theta(t)))\xi(t) = (b-\delta(t))(\delta(t)-a)\cdot(\bullet)^\top R(\theta(t))\xi_1(t)$$

for all $t \geq 0$. Utilizing (4.10) and $a \leq \delta \leq b$ yields then

$$\begin{aligned}\dot{\mu}_T(t) &\geq -(b-\delta(t))(\delta(t)-a)\cdot(\bullet)^\top\Big(N(\theta(t)) + (\bullet)^\top\Big)\begin{pmatrix} C_\psi & D_\psi \end{pmatrix}\begin{pmatrix} \xi_1(t) \\ z(t) \end{pmatrix}\\ &= -(b-\delta(t))(\delta(t)-a)\cdot(\bullet)^\top\Big(N(\theta(t)) + (\bullet)^\top\Big)v_1(t)\end{aligned}$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Similarly as before, the last expression can be written as

$$-(\bullet)^\top P_p(N(\theta(t))) \begin{pmatrix} I \\ \delta(t)I \end{pmatrix} v_1(t) = -(\bullet)^\top M(\theta(t))v(t).$$

Consequently, we have

$$\begin{aligned} \mu_T(t) - \mu_T(0) &= \int_0^t \dot{\mu}_T(s)\, ds + \sum_{l=1}^{k} \big(\mu_T(t_l) - \mu_T(t_l^-)\big) \\ &\geq -\int_0^t (\bullet)^\top M(\theta(s))v(s)\, ds + \sum_{l=1}^{k} \big(\mu_T(t_l) - \mu_T(t_l^-)\big) \end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_0$. This yields the claim by $\mu_T(0) = 0$ as well as by defining $\mu_J := \mu_T$ and $\mu_F := 0$. ●

### Assuring the IQC for Constant Repeated Parametric Uncertainties in the Jump Component

*Proof of Lemma 4.11.* Let $(t_k)_{k\in\mathbb{N}_0}$ with (RDT), $\Delta_J \in \mathbf{\Delta}_J$ and $z_J \in \ell_{2e}^{p_J}$ be arbitrary. Then there exists some $\delta_J \in [a, b]$ with $w_J := \Delta_J(z_J) = \delta_J z_J$. Let us partition the state $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and the output $v_J = \left(\begin{smallmatrix} v_{J1} \\ v_{J2} \end{smallmatrix}\right)$ of the filter (4.6) driven by $\left(\begin{smallmatrix} z_J \\ \Delta_J(z_J) \end{smallmatrix}\right)$ accordingly to the partition of $A_{\Psi_J}$ and $C_{\Psi_J}$, respectively. In order to proceed, it is crucial to note that $A_\Psi = 0$ implies $\xi_i(t) = \xi_i(t_k)$ for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. Consequently, we obtain $\xi(t_k) = \left(\begin{smallmatrix} I \\ \delta_J I \end{smallmatrix}\right) \xi_1(t_k)$ and $v_J(k) = \left(\begin{smallmatrix} I \\ \delta_J I \end{smallmatrix}\right) v_{J1}(k)$ for all $k$ by the discrete-time variation of constants formula.

Let us now define the function $\mu_T : t \mapsto \xi(t)^\top Z_T(\theta(t))\xi(t)$ and observe that we have, similarly as in the proof of Lemma 4.9,

$$\mu_T(t) = \frac{1}{2}(\bullet)^\top P_p(R(\theta(t)))\xi(t_k) = (b - \delta_J)(\delta_J - a) \cdot (\bullet)^\top R(\theta(t))\xi_1(t)$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$. Utilizing (4.12) and $a \leq \delta_J \leq b$ yields then

$$\begin{aligned}
&\mu_T(t_k) - \mu_T(t_k^-) \\
&\geq -(b-\delta_J)(\delta_J - a)\cdot(\bullet)^\top \Big(N_J(\theta(t_k^-)) + (\bullet)^\top\Big)\Big(C_{\psi_J}\; D_{\psi_J}\Big)\begin{pmatrix}\xi_1(t_k^-)\\ z_J(k)\end{pmatrix} \\
&= -(b-\delta_J)(\delta_J - a)\cdot(\bullet)^\top \Big(N_J(\theta(t_k^-)) + (\bullet)^\top\Big) v_{J1}(k) \\
&= -(\bullet)^\top P_p(N_J(\theta(t_k^-)))\begin{pmatrix} I \\ \delta_J I\end{pmatrix} v_{J1}(k) = -(\bullet)^\top M_J(\theta(t_k^-)) v_J(k)
\end{aligned}$$

for all $k \in \mathbb{N}$ and consequently

$$\begin{aligned}
\mu_T(t) - \mu_T(0) &= \int_0^t \dot{\mu}_T(s)\,ds + \sum_{l=1}^k \big(\mu_T(t_l) - \mu_T(t_l^-)\big) \\
&\geq \int_0^t \dot{\mu}_T(s)\,ds - \sum_{l=1}^k (\bullet)^\top M_J(\theta(t_l^-)) v_J(l)
\end{aligned}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_0$. This yields the claim by $\mu_T(0) = 0$ as well as by defining $\mu_F := \mu_T$ and $\mu_J := 0$. ●

### Assuring the IQC for Dynamic Repeated Uncertainties in the Flow Component

In order to show Lemma 4.12, we require the following auxiliary result.

**Lemma 4.16** *Let $g : L_{2e}^1 \to L_{2e}^1$ and $H : L_{2e}^k \to L_{2e}^l$ be linear maps of the form*

$$g(w)(t) := \int_0^t C_g e^{A_g(t-s)} B_g w(s)\,ds + D_g w(t)$$

*and*

$$H(w)(t) := \int_0^t C e^{A(t-s)} B w(s)\,ds + D w(t),$$

*respectively. Then we have*

$$H \circ (gI_k) = (gI_l) \circ H,$$

*where* $gI_m : L_{2e}^m \to L_{2e}^m$, $w = \mathrm{col}(w_1, \ldots, w_m) \mapsto \mathrm{col}(g(w_1), \ldots, g(w_m))$.

*Proof.* Let us introduce the abbreviations $\tilde{g}(s) := C_g e^{A_g s} B_g$ and $\tilde{H}(s) := Ce^{As}B$. Since $\tilde{g}$ is scalar-valued, note that we have

$$\tilde{H}\tilde{g} = \tilde{g}\tilde{H}, \quad \tilde{g}D = D\tilde{g}, \quad \tilde{H}D_g = D_g\tilde{H} \text{ and } DD_g = D_gD. \tag{$*1$}$$

Let now $w \in L_{2e}^k$ and $t \geq 0$ be arbitrary. Then we have via integration by substitution

$$\begin{aligned}
&\int_0^t \tilde{H}(t-s) \left( \int_0^s \tilde{g}(s-r)w(r)\, dr \right) ds \\
&\qquad = \int_0^t \tilde{H}(s) \left( \int_0^{t-s} \tilde{g}(t-s-r)w(r)\, dr \right) ds \\
&\qquad = \int_0^t \tilde{H}(s) \left( \int_0^{t-s} \tilde{g}(r)w(t-s-r)\, dr \right) ds \\
&\qquad = \int_0^t \left( \int_0^t \tilde{H}(s)\tilde{g}(r)w(t-s-r)\chi_{[r \leq t-s]}\, dr \right) ds;
\end{aligned}$$

here $\chi_{[a \leq b]} := 1$ if $a \leq b$ and $\chi_{[a \leq b]} := 0$ otherwise. By using $\tilde{H}\tilde{g} = \tilde{g}\tilde{H}$ and Fubini's theorem, the last term equals

$$\begin{aligned}
&\int_0^t \left( \int_0^t \tilde{g}(r)\tilde{H}(s)w(t-s-r)\chi_{[r \leq t-s]}\, dr \right) ds \\
&\qquad = \int_0^t \left( \int_0^t \tilde{g}(r)\tilde{H}(s)w(t-s-r)\chi_{[r \leq t-s]}\, ds \right) dr \\
&\qquad = \int_0^t \left( \int_0^t \tilde{g}(r)\tilde{H}(s)w(t-s-r)\chi_{[s \leq t-r]}\, ds \right) dr.
\end{aligned}$$

Again via integration by substitution this is now the same as

$$\begin{aligned}\int_0^t \tilde{g}(r)\left(\int_0^{t-r} \tilde{H}(r)w(t-s-r)\,ds\right)\,dr \\ = \int_0^t \tilde{g}(r)\left(\int_0^{t-r} \tilde{H}(t-r-s)w(s)\,ds\right)\,dr \\ = \int_0^t \tilde{g}(t-r)\left(\int_0^{r} \tilde{H}(r-s)w(s)\,ds\right)\,dr\end{aligned}$$

which yields the statement for $D=0$ and $D_g=0$. The general case is obtained by using linearity and $(*1)$. ●

*Proof of Lemma 4.12.* Let $(t_k)_{k\in\mathbb{N}_0}$ with (RDT), $\Delta\in\mathbf{\Delta}$ and $z\in L_{2e}^p$ be arbitrary. Then there exists some $\delta\in\mathrm{RH}_\infty^{1\times 1}$ with $\|\delta\|_\infty\le 1$ characterizing the map $\Delta$. Without loss of generality, we can assume that $(A_\delta, B_\delta, C_\delta, D_\delta)$ is a minimal realization of $\delta$ and recall that $A_\delta$ is then Hurwitz

Next, let $\varepsilon>0$ be arbitrary and set $P_\varepsilon := \left(\begin{smallmatrix}1+\varepsilon & 0\\ 0 & -1\end{smallmatrix}\right)$. From the KYP lemma [123] and $\|\delta\|_\infty\le 1$, we can then infer the existence of a symmetric matrix $Y$ satisfying

$$(\bullet)^\top\begin{pmatrix}0 & Y\\ Y & 0\end{pmatrix}\begin{pmatrix}A_\delta & B_\delta\\ I & 0\end{pmatrix}+(\bullet)^\top P_\varepsilon\begin{pmatrix}0 & I\\ C_\delta & D_\delta\end{pmatrix}\succ 0. \tag{$*1$}$$

Since $A_\delta$ is Hurwitz and by negativity of the $(2,2)$ entry of $P_\varepsilon$, we even conclude $Y\succ 0$.

Let now $K\in\mathbb{S}^\bullet$ denote the whole left hand side of (4.13) and observe that we can incorporate this positive definite matrix into the LMI $(*1)$ in order to obtain

$$(\bullet)^\top\begin{pmatrix}0 & K\otimes Y\\ K\otimes Y & 0\end{pmatrix}\begin{pmatrix}I_\bullet\otimes A_\delta & I_\bullet\otimes B_\delta\\ I & 0\end{pmatrix}+(\bullet)^\top P_\varepsilon\otimes K\begin{pmatrix}0 & I\\ I_\bullet\otimes C_\delta & I_\bullet\otimes D_\delta\end{pmatrix}\succ 0. \tag{$*2$}$$

Next, we define $(\tilde{C}_\psi, \tilde{D}_\psi) := (\left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right))$ and the auxiliary signals

$$\tilde{v}_1 := \tilde{\psi}(z) := \tilde{C}_\psi \xi_1 + \tilde{D}_\psi z \quad \text{and} \quad \tilde{v}_2 := \tilde{\psi}(\Delta(z)) := \tilde{C}_\psi \xi_2 + \tilde{D}_\psi \Delta(z)$$

involving the state $\xi = \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right)$ of the filter (4.6) driven by $\left(\begin{smallmatrix} z \\ \Delta(w) \end{smallmatrix}\right)$ with a partition induced by the one of the describing matrices of the filter. Let us finally introduce the system

$$\begin{pmatrix} \dot{\rho}(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} I_\bullet \otimes A_\delta & I_\bullet \otimes B_\delta \\ I_\bullet \otimes C_\delta & I_\bullet \otimes D_\delta \end{pmatrix} \begin{pmatrix} \rho(t) \\ \tilde{v}_1(t) \end{pmatrix}, \quad \rho(0) = 0$$

as well as the function $\eta : t \mapsto \rho(t)^\top (K \otimes Y)\rho(t)$. Note that the latter is nonpositive by $K \succ 0$ and $Y \prec 0$ and that $(*2)$ implies

$$\dot{\eta}(t) \geq -(\bullet)^\top \big(P_\varepsilon \otimes K\big) \begin{pmatrix} 0 & I \\ I_\bullet \otimes C_\delta & I_\bullet \otimes D_\delta \end{pmatrix} \begin{pmatrix} \rho(t) \\ \tilde{v}_1(t) \end{pmatrix} = -(\bullet)^T (P_\varepsilon \otimes K\big) \begin{pmatrix} \tilde{v}_1(t) \\ y(t) \end{pmatrix}$$

for all $t \geq 0$. By integration, $\eta(0) = 0$ and nonpositivtiy of $\eta$ we obtain

$$0 \leq \int_0^t (\bullet)^\top \big(P_\varepsilon \otimes K\big) \begin{pmatrix} \tilde{v}_1(s) \\ y(s) \end{pmatrix} ds + \eta(t) - \eta(0) \leq \int_0^t (\bullet)^\top \big(P_\varepsilon \otimes K\big) \begin{pmatrix} \tilde{v}_1(s) \\ y(s) \end{pmatrix} ds$$

for $t \geq 0$. Next, observe that Lemma 4.16 yields

$$y = (\delta I_\bullet)(\tilde{v}_1) = \big((\delta I_\bullet) \circ \tilde{\psi}\big)(z) = \big(\tilde{\psi} \circ (\delta I_p)\big)(z) = \tilde{\psi}(\Delta(z)) = \tilde{v}_2.$$

It remains to untangle the matrix $K$, i.e., to observe that we have after few elementary computations

$$
\begin{aligned}
0 \leq \int_0^t (\bullet)^\top \big(P_\varepsilon \otimes K\big) \begin{pmatrix} \tilde{v}_1(s) \\ \tilde{v}_2(s) \end{pmatrix} ds \\
&= \int_0^t (\bullet)^\top \big(P_\varepsilon \otimes N\big) v(t) + \frac{d}{ds}\left(\xi(s)^\top \big(P_\varepsilon \otimes R\big)\xi(s)\right) ds \\
&= \int_0^t (\bullet)^\top \big(P_\varepsilon \otimes N\big) v(s) ds + \xi(t)^\top \big(P_\varepsilon \otimes R\big)\xi(t)
\end{aligned}
$$

for all $t \geq 0$. Finally, taking the limit $\varepsilon \to 0$ and defining $\mu_F := 0$ as well as $\mu_T := \mu_J : t \mapsto \xi(t)^\top Z_T \xi(t)$ for $Z_T := \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right)$ yields the claim. ●

### Assuring the IQC for Nonlinear Repeated Uncertainties in the Jump Component

We employ the following auxiliary result from [50].

**Lemma 4.17** *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ satisfies $\phi(0) = 0$ and $0 \leq \frac{\phi(x)-\phi(y)}{x-y}$ for all $x \neq y$ and let $L = (L_{ij}) \in \mathbb{R}^{k \times k}$ be a matrix satisfying*

$$
L_{ij} \leq 0 \text{ for } i \neq j, \quad L\mathbf{1} \geq 0 \quad \text{and} \quad \mathbf{1}^\top L \geq 0.
$$

*Then the repeated map $(\phi I_k) : \mathbb{R}^k \to \mathbb{R}^k$, $x \mapsto (\phi(x_1), \ldots, \phi(x_k))^\top$ satisfies*

$$
(\phi I_k)(x)^\top L x \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^k.
$$

*Proof of Lemma 4.13.* Let $(t_k)_{k \in \mathbb{N}_0}$ with (RDT), $\Delta_J \in \mathbf{\Delta}_J$ and $z_J \in \ell_{2e}^{p_J}$ be arbitrary. Then there exists some slope-restricted function $\phi$ with $w_J(k) := \Delta_J(z_J)(k) = (\phi I_{p_J}(z_J(k)))$.

Observe at first that $A_\Psi = 0$ implies $\xi(t) = \xi(t_k)$ for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_0$ and, thus, $\xi(t_k) = \xi(t_{k+1}^-)$ for all $k \in \mathbb{N}_0$. By induction and the particular choice of the filter, we infer

$$v_J(k) = \mathrm{col}(z_J(k-\nu),\ \ldots,\ z_J(k), w_J(k)) \quad \text{for all} \quad k \in \mathbb{N},$$

where we set $z_J(l) := 0$ for $l \leq 0$. This allows us to conclude

$$\begin{aligned}\sum_{l=1}^{k} (\bullet)^\top M_J v_J(l) &= 2 \sum_{l=1}^{k} w_J(l)^\top \sum_{j=1}^{\nu+1} \Lambda_j z_J(l-\nu+j+1) \\ &= 2 \left( (\phi I_{p_J k}) \begin{pmatrix} z_J(1) \\ \vdots \\ z_J(k) \end{pmatrix} \right)^\top L_k \begin{pmatrix} z_J(1) \\ \vdots \\ z_J(k) \end{pmatrix}\end{aligned}$$

for all $k \in \mathbb{N}$, where $L_k$ is a block Toeplitz matrix in $\mathbb{R}^{p_J k \times p_J k}$ with its $(l, 1)$ block given by $\Lambda_{\nu+2-l}$ if $l \leq \min\{\nu+1, k\}$ and $0_{p_J \times p_J}$ otherwise. From (4.15) we infer that $L_k$ is as required by Lemma 4.17 and can, hence, conclude that

$$\sum_{l=1}^{k} (\bullet)^\top M_J v_J(l) \geq 0 \quad \text{holds for all} \quad k \in \mathbb{N}.$$

This yields the claim by defining $\mu_F := \mu_T := \mu_J := 0$. ●

## 4.2 Analysis of Networked Systems

In this section we demonstrate the application of our robust analysis approach for analyzing (potentially large-scale) *networked systems* composed of $M \in \mathbb{N}$ *homogeneous* subsystems of possibly high order in a scalable[1] fashion. These subsystems are of the form

$$\begin{pmatrix} \dot{x}_i(t) \\ z_i(t) \\ e_i(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x_i(t) \\ w_i(t) \\ d_i(t) \end{pmatrix} \tag{4.17a}$$

for $t \geq 0$, $i \in \{1, \ldots, M\}$ and with initial conditions $x_1(0), \ldots, x_M(0) \in \mathbb{R}^n$ as well as generalized disturbances $d_1, \ldots, d_M \in L_2$. The interconnection structure of these subsystems is specified by constraining the interconnection variables $z_i$ and $w_i$ according to the *coupling condition*

$$w_i(t) = \sum_{j=1}^{M} a_{ij\sigma(t)} z_j(t) \tag{4.17b}$$

which involves coupling *weights* $a_{ijk} \in [0, \infty)$ and a switching function $\sigma : [0, \infty) \to \{1, \ldots, N\}$ corresponding to a given sequence of impulse instants $(t_k)_{k \in \mathbb{N}_0}$. This concrete coupling condition allows for modeling a possibly non-periodically switching communication topology with $N$ configurations similarly as also considered, e.g., in [100, 105, 57]. Such switching topologies occur in practice, e.g., as a result of link failures or creations during operation of the networked systems. Similarly, as we have already seen when considering constrained switching functions, the communication topology as defined by the coupling conditions (4.17b) is usually visualized by means of a weighted time-varying graph $G(t) = (V, E(t), d(t))$ with vertices $V = \{1, \ldots, M\}$, edges $E(t) \subset V^2$ and weight function

[1] Here, scalability is meant in the sense that one aims for analysis criteria with a resulting computational burden that grows slowly as $M$ increases.

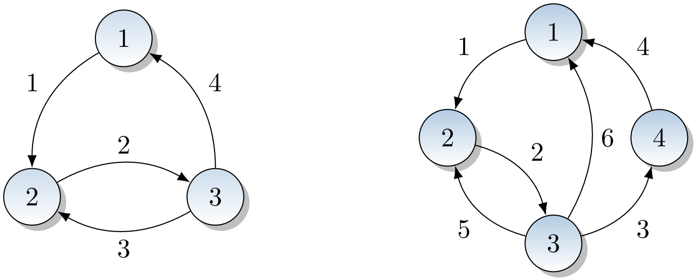



Figure 4.5: Two constant graphs corresponding to the adjacency matrices $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 4 & 3 & 0 \end{pmatrix}$ (left) and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 6 & 5 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix}$ (right), respectively.

$d(t) : E(t) \to \mathbb{R}$. The latter graph is conveniently characterized by its time-varying *adjacency matrix*

$$\mathscr{A}_{\sigma(t)} := \begin{pmatrix} a_{11\sigma(t)} & \cdots & a_{1M\sigma(t)} \\ \vdots & \ddots & \vdots \\ a_{M1\sigma(t)} & \cdots & a_{MM\sigma(t)} \end{pmatrix} \in \mathbb{R}^{M\times M}$$

through $E(t) := \{(i,j) \in V^2 \mid a_{ij\sigma(t)} \neq 0\}$ and $d(t) : (i,j) \mapsto a_{ij\sigma(t)}$ for all times $t \geq 0$. Two constant weighted graphs with corresponding adjacency matrices are illustrated in Fig. 4.5 and its caption.

In order to obtain scalable analysis criteria for the networked system (4.17), we assume in the sequel that the adjacency matrix corresponding to (4.17b) is symmetric, i.e., the underlying graph is undirected and information is shared bilaterally for all time instances, and that bounds on the location of the spectra of $\mathscr{A}_1, \ldots, \mathscr{A}_N$ are available. Precisely, we assume that

$$\mathscr{A}_l \in \mathbb{S}^M \quad \text{and} \quad \mathrm{eig}(\mathscr{A}_l) \subset [a_l, b_l] \qquad \text{for all} \qquad l \in \{1, \ldots, N\} \tag{4.18}$$

for some a priori given $a_l < b_l$. Such bounds are typically not difficult to obtain; for example one can take $b_l = \|\mathscr{A}_l\|_2$ for all $l \in \{1, \ldots, N\}$. Note that our analysis results will not require precise knowledge of the communication topology, but are robust in sense that they ensure stability and performance for all $\mathscr{A}_1, \ldots, \mathscr{A}_N$ satisfying (4.18).

Next, note that the homogeneity of the subsystems in (4.17) permits us to express their interconnection by using the Kronecker product equivalently and more compactly as

$$\begin{pmatrix} \dot{x}(t) \\ z(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes B & I \otimes B_2 \\ I \otimes C & I \otimes D & I \otimes D_{12} \\ I \otimes C_2 & I \otimes D_{21} & I \otimes D_{22} \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \\ d(t) \end{pmatrix}, \quad w(t) = (\mathscr{A}_{\sigma(t)} \otimes I) z(t) \tag{4.19}$$

for all $t \geq 0$ and with stacked signals $x := \mathrm{col}(x_1, \ldots, x_M)$, etc. Similarly as for example in [100, 105, 57], the assumed symmetry of the matrices $\mathscr{A}_1, \ldots, \mathscr{A}_N$ permits us to find orthogonal matrices $T_1, \ldots, T_N$ and scalars $\lambda_{11}, \ldots, \lambda_{M1}, \ldots, \lambda_{MN}$ such that

$$T_l \mathscr{A}_l T_l^\top = \begin{pmatrix} \lambda_{1l} & 0 & \\ & \ddots & \\ 0 & & \lambda_{Ml} \end{pmatrix} \quad \text{holds for all} \quad l \in \{1, \ldots, N\}.$$

Based on the latter representation and the rules of the Kronecker product, we can rewrite (4.19) equivalently as an impulsive system with description

$$\begin{aligned} &\begin{pmatrix} \dot{\hat{x}}(t) \\ \hat{z}(t) \\ \hat{e}(t) \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes B & I \otimes B_2 \\ I \otimes C & I \otimes D & I \otimes D_{12} \\ I \otimes C_2 & I \otimes D_{21} & I \otimes D_{22} \end{pmatrix} \begin{pmatrix} \hat{x}(t) \\ \hat{w}(t) \\ \hat{d}(t) \end{pmatrix}, \quad \hat{x}(t_k) = \left(T_{\sigma(t_k)} T_{\sigma(t_k^-)}^\top \otimes I\right) \hat{x}(t_k^-) \\ &\quad \hat{w}(t) = \mathrm{diag}\left(\lambda_{1\sigma(t)} I, \ \ldots, \ \lambda_{M\sigma(t)} I\right) \hat{z}(t) \end{aligned} \tag{4.20}$$

for all $t \geq 0$, $k \in \mathbb{N}$ and with the transformed signals $\hat{x} := (T_\sigma \otimes I)x$, etc.

We emphasize that this transformation leads from (4.19) to a system with *decoupled* flow component in the following sense: If we partition all signals as induced by the involved matrix blocks, then we can express the flow component of (4.20) as $M$ individual systems of the form

$$\begin{pmatrix} \dot{\hat{x}}_i(t) \\ \hat{z}_i(t) \\ \hat{e}_i(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \hat{x}_i(t) \\ \hat{w}_i(t) \\ \hat{d}_i(t) \end{pmatrix}, \qquad \hat{w}_i(t) = \lambda_{i\sigma(t)} \hat{z}_i(t)$$

affected by a single repeated parametric uncertainty $\lambda_{i\sigma(t)}$ satisfying $\lambda_{il} \in [a_l, b_l]$ for all $l$ by our assumption (4.18). However, the transformation reintroduces a coupling of the individual states $\hat{x}_i$ through the impulsive component of (4.20).

As a consequence of all these considerations, we are now in position to apply a modification of Theorem 4.6 and Lemma 4.9 in order to obtain the following novel analysis result for networked systems (4.17).

**Theorem 4.18** *Let $P = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ be a symmetric matrix with $Q \succcurlyeq 0$, let $P_p$ be as in (4.9) and let $A_\psi$, $B_\psi$, $C_\psi$, $D_\psi$ be matrices with $C_\psi \in \mathbb{R}^{n_v \times n_\psi}$. Then the interconnection (4.19) is stable and achieves quadratic performance with index $\begin{pmatrix} I\otimes Q & I\otimes S \\ I\otimes S^\top & I\otimes R \end{pmatrix}$ for all $\mathscr{A}_1, \ldots, \mathscr{A}_N$ satisfying (4.18) and all $(t_k)_{k\in\mathbb{N}_0}$ with (RDT) if there exist $X_1, \ldots, X_N \in C^1([0, T_{\max}], \mathbb{S}^{2n_\psi+n})$, $R_1, \ldots, R_N \in C([0, T_{\max}], \mathbb{S}^{n_\psi})$ and $N_1, \ldots, N_N \in C([0, T_{\max}], \mathbb{R}^{n_v \times n_v})$ satisfying*

$$\hat{X}_l \succ 0, \quad (\bullet)^\top \left(\begin{array}{cc|c|c} 0 & X_l & & \\ X_l & \dot{X}_l & & \\ \hline & & P_p^{a_l,b_l}(N_l) & \\ \hline & & & P \end{array}\right) \left(\begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ I & 0 & 0 \\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \hline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \\ 0 & 0 & I \end{array}\right) \prec 0 \qquad (4.21a,b)$$

*and*

$$(\bullet)^\top \begin{pmatrix} 0 & R_l \\ R_l & \dot{R}_l \end{pmatrix} \begin{pmatrix} A_\psi & B_\psi \\ I & 0 \end{pmatrix} + (\bullet)^\top (N_l + N_l^\top) \begin{pmatrix} C_\psi & D_\psi \end{pmatrix} \succ 0 \tag{4.21c}$$

*on* $[0, T_{\max}]$ *for all* $l \in \{1, \ldots, N\}$ *as well as*

$$(\bullet)^\top \hat{X}_l(0) \mathcal{A}_J - \hat{X}_k \prec 0 \tag{4.21d}$$

*on* $[T_{\min}, T_{\max}]$ *for all* $k, l \in \{1, \ldots, N\}$*; here, we employ the abbreviations* $\hat{X}_l := X_l - \operatorname{diag}(\frac{1}{2} P_p^{a_l, b_l}(R_l), 0)$ *as well as*

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix} := \left(\begin{array}{ccc|c:c} A_\psi & 0 & B_\psi C & B_\psi D & B_\psi D_{12} \\ 0 & A_\psi & 0 & B_\psi & 0 \\ 0 & 0 & A & B & B_2 \\ \hline C_\psi & 0 & D_\psi C & D_\psi D & D_\psi D_{12} \\ 0 & C_\psi & 0 & D_\psi & 0 \\ \hdashline 0 & 0 & C_2 & D_{21} & D_{22} \end{array}\right) \quad \text{and} \quad \mathcal{A}_J := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

Note that this result substantially generalizes the findings of, e.g., [100, 57], since they only consider constant Lyapunov matrices and because they merely employ static filters with $n_\psi = 0$.

We also emphasize that the size of the LMI problems corresponding to the conditions (4.21) is completely independent of the total number of subsystems $M$ and, thus, Theorem 4.18 is easily applicable even for large scale networks. This scalability is achieved by relying on a particularly structured Lyapunov matrix which is well-known to introduce some conservatism. However, this conservatism is typically accepted since scalability is considered to be more important in the context of networked systems.

*Proof.* For notational convenience, we drop the performance channel and we only consider the case of a static filter with $n_\psi = 0$ and $D_\psi = I$. The general case is shown with essentially the same arguments.

In order to apply a corresponding modification of Theorem 4.6 and Lemma 4.9 on the transformed system (4.20), it then suffices to show that the particularly structured maps

$$\check{X}_l := I \otimes X_l \quad \text{and} \quad \check{N}_l := I \otimes N_l \quad \text{for} \quad l \in \{1, \ldots, N\}$$

satisfy the inequalities

$$\check{X}_l \succ 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} 0 & \check{X}_l & \\ \check{X}_l & \dot{\check{X}}_l & \\ \hline & & P_p^{a_l, b_l}(\check{N}_l) \end{array}\right) \left(\begin{array}{cc} I \otimes A & I \otimes B \\ I & 0 \\ \hline I \otimes C & I \otimes D \\ 0 & I \end{array}\right) \prec 0 \;\; \text{and} \;\; \check{N}_l + \check{N}_l^\top \succ 0 \tag{$*$1a}$$

on $[0, T_{\max}]$ for all $l \in \{1, \ldots, N\}$ as well as

$$(\bullet)^\top \check{X}_l(0) \big(T_l T_k^\top \otimes I\big) - \check{X}_k \prec 0 \tag{$*$1b}$$

on $[T_{\min}, T_{\max}]$ for all $k, l \in \{1, \ldots, N\}$. Due to the structure of $\check{X}_l$ and $\check{N}_l$ and by some Kronecker algebra, we observe that ($*$1a) is equivalent to

$$X_l \succ 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} 0 & X_l & \\ X_l & \dot{X}_l & \\ \hline & & P_p^{a_l, b_l}(N_l) \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) \prec 0 \quad \text{and} \quad N_l + N_l^\top \succ 0. \tag{$*$2}$$

By making use of the orthogonality of $T_l$, ($*$1b) can be expressed as

$$0 \succ \big(T_k T_l^\top T_l T_k^\top\big) \otimes X_l(0) - \check{X}_k = I \otimes X_l(0) - \check{X}_k = \check{X}_l(0) - \check{X}_k$$

which is equivalent to

$$X_l(0) - X_k \prec 0.$$

It remains to note that the latter inequality and (∗2) correspond exactly to the inequalities (4.21) for the considered specialization. ●

Note that Theorem 4.18 allows for arbitrary switches between the communication topologies described by the adjacency matrices $\mathscr{A}_1, \ldots, \mathscr{A}_N$. Analogously as we have seen for switched systems, we can adjust Theorem 4.18 to incorporate additional knowledge on the switching sequence $\sigma$ with ease if such information is available.

Moreover, note that if the network (4.17) involves numerous individual communication topologies, i.e., if $N$ is large, then it can make sense to enforce several of the decision variables in Theorem 4.18 to be identical and to coordinate these restrictions with the available bounds in (4.18). We recover our analysis result for networked systems from [83] by considering the extreme case that all intervals in (4.18) are identical.

**Corollary 4.19** *Suppose that* $[a_1, b_1] = \cdots = [a_N, b_N] =: [a, b]$. *Then the interconnection* (4.19) *is stable and achieves quadratic performance with index* $\begin{pmatrix} I\otimes Q & I\otimes S \\ I\otimes S^\top & I\otimes R \end{pmatrix}$ *for all* $\mathscr{A}_1, \ldots, \mathscr{A}_N$ *satisfying* (4.18) *and all* $(t_k)_{k\in\mathbb{N}_0}$ *with* (RDT) *if there exist* $X \in C^1([0, T_{\max}], \mathbb{S}^{2n_\psi+n})$, $R \in C([0, T_{\max}], \mathbb{S}^{n_\psi})$ *and* $N \in C([0, T_{\max}], \mathbb{R}^{n_v\times n_v})$ *satisfying*

$$\hat{X} \succ 0, \quad (\bullet)^\top \left(\begin{array}{cc|c:c} 0 & X & & \\ X & \dot{X} & & \\ \hline & & P_p^{a,b}(N) & \\ \hdashline & & & P \end{array}\right) \left(\begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ I & 0 & 0 \\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \hdashline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \\ 0 & 0 & I \end{array}\right) \prec 0$$

*and*

$$(\bullet)^\top \begin{pmatrix} 0 & R \\ R & \dot{R} \end{pmatrix} \begin{pmatrix} A_\psi & B_\psi \\ I & 0 \end{pmatrix} + (\bullet)^\top (N + N^\top) \begin{pmatrix} C_\psi & D_\psi \end{pmatrix} \succ 0$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \hat{X}(0)\mathcal{A}_J - \hat{X} \prec 0$$

*on* $[T_{\min}, T_{\max}]$, *where* $\hat{X} := X - \mathrm{diag}(\frac{1}{2}P_p^{a,b}(R), 0)$.

**Remark 4.20** (Consensus) Theorem 4.18 and Corollary 4.19 can also be modified to yield criteria which guarantee that the subsystems in (4.17) asymptotically achieve *consensus*, i.e., they agree on a common value of their states asymptotically. Precisely, this means that

$$\lim_{t\to\infty} \|x_k(t) - x_l(t)\| = 0 \quad \text{for all} \quad k, l \in \{1, \ldots, M\}$$

for any initial conditions $x_1(0), \ldots, x_M(0) \in \mathbb{R}^n$. In this context the adjacency matrices $\mathscr{A}_l = (a_{ijl})$ are usually replaced by the Laplacian matrices of the graphs $\mathscr{L}_l := \mathscr{D}_l - \mathscr{A}_l$ where $\mathscr{D}_l := \sum_{\nu=1}^{M} \mathrm{diag}(a_{1\nu l}, \ldots, a_{M\nu l})$. One can show that in order to guarantee consensus, one has merely to replace the numbers $a_l$ in (4.18) by lower bounds on the algebraic connectivity, i.e., the second smallest eigenvalue, of the Laplacian matrices $\mathscr{L}_l$ for all $l \in \{1, \ldots, N\}$. Some lower bounds on this number can for example be found in [35]. For a general and detailed introduction to consensus problems see for example [124] or [166].

**Remark 4.21** (Including Uncertainties) Due to the modularity of the IQC approach, it is not difficult to derive scalable stability analysis criteria for networks (4.17) involving various types of uncertainties. In particular, by viewing a *heterogeneous* network as a homogeneous one subject to uncertainties, we even have means to treat such more general networks in a scalable fashion.

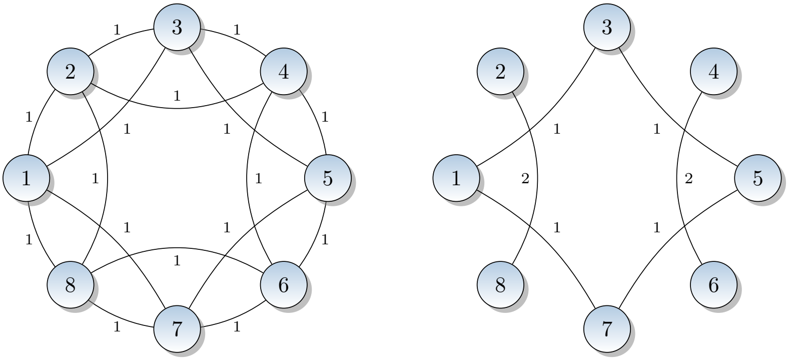



Figure 4.6: Two constant graphs with Laplacian matrices $\mathscr{L}_1$ and $\mathscr{L}_2$.

### Example

As an illustration let us perform a consensus analysis for a networked system composed of $M = 8$ subsystems and given by

$$\begin{pmatrix} \dot{x}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes B \\ I \otimes C & I \otimes D \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}, \quad w(t) = (\mathscr{L}_{\sigma(t)} \otimes I) z(t) \tag{4.23a}$$

with describing matrices

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{\beta}{7} \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 0 \end{pmatrix} \quad \text{and} \quad D = \frac{\beta}{7} \tag{4.23b}$$

involving some parameter $\beta \in (0, 2)$ and with Laplacian matrices $\mathscr{L}_1$ and $\mathscr{L}_2$ corresponding to the graphs depicted in Fig. 4.6. Further, we assume that the switching sequence $\sigma : [0, \infty) \to \{1, 2\}$ is constant on the intervals defined by the sequence of impulse instants $(t_k)_{k \in \mathbb{N}_0}$ with $t_{k-1} - t_k = 0.5$ for all $k \in \mathbb{N}$ and that the switches are constrained by

$$(\sigma(t_{k-1}), \sigma(t_k)) \in \{(1, 2), (2, 1)\} \quad \text{for all} \quad k \in \mathbb{N}.$$

In other words, the communication topology switches periodically between the two configurations depicted in Fig. 4.6.

We stress that we can allow for much more complex subsystem dynamics which is in contrast to, e.g., the underlying analysis result of [134] which requires that all eigenvalues of the matrix $A$ are located in the closed left half-plane. Moreover, note that only the first graph in Fig. 4.6 is connected[2] since the node 4 cannot be reached from the node 2 in the second graph. Naturally, connectedness is a crucial property in consensus problems and, for time-varying graphs, there are various notions thereof; several of them can be found, e.g., in [166]. The analysis criteria from [100, 57] or the one in Corollary 4.19 require (implicitly) that the underlying graph is connected for all time instances or that the matrix $A$ is Hurwitz. This means that none of them can be applied here since one of the eigenvalues of $A$ is larger than zero.

In order to apply the corresponding modification of Theorem 4.18 for consensus analysis, we observe that we can employ

- the lower bounds $(a_1, a_2) = (2.5858, 0)$ on the algebraic connectivity
- the upper bounds $(b_1, b_2) = (6, 4)$ on the maximal eigenvalue

of the Laplacian matrices $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Moreover, we choose $(A_\psi, B_\psi, C_\psi, D_\psi)$ as in (4.16a) with $\alpha = 5$ and length $\nu \in \{0, 1, 2, 3\}$. Combining Theorem 4.18 with a bisection permits us now for example to determine for each of the latter lengths $\nu$ the maximal value of the parameter $\beta$ for which consensus is guaranteed; we denote these maximal values by $\beta_\nu$. By employing the B-spline relaxation (with a fixed set of parameters) we obtain $\beta_1 = 1.011$, $\beta_2 = 1.013$, $\beta_3 = 1.013$ and that no

[2] A graph $G = (V, E)$ is called *connected*, if any node can be reached from any other node by moving along the edges of the graph. Formally, this means that for any pair $(t_0, t_f) \in V \times V$ with $t_0 \neq t_f$ there exists some $N \in \mathbb{N}$ and $t_1, \ldots, t_N \in V$ such that $(t_0, t_1), \ldots, (t_{N-1}, t_N), (t_N, t_f) \in E$. As shown, e.g., in [62], a graph $G$ is connected if and only if its algebraic connectivity nonzero.

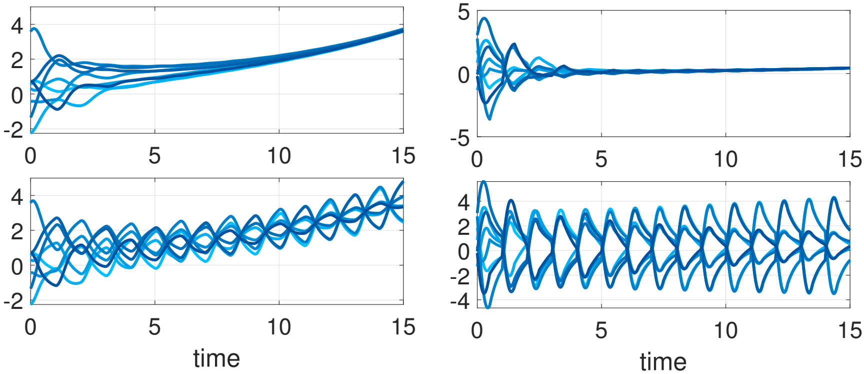



Figure 4.7: First (left) and second (right) component of the state trajectories of the subsystems of the network (4.23) for $\beta = 1$ (top) and for $\beta = 1.065$ (bottom).

such $\beta$ exists for $\nu = 0$. These findings demonstrate the benefit of utilizing dynamic filters as in (4.16a) with length $\nu > 0$ over static ones with $\nu = 0$ and show that even small values of $\nu$ can yield good results. Indeed, the bottom of Fig. 4.7 depicts the subsystem's state trajectories for $\beta = 1.065$, which is close to $\beta_3$, and illustrates that for this parameter value consensus is no longer achieved. Interestingly, we still observe some kind of agreement since the eight individual states follow four distinct trajectories. This behavior is termed cluster consensus which is studied in few papers only; one of them is [67].

Finally, note that there is a gap between the maximal $\beta \in (0, 2)$ for which consensus is achieved and our computed values $\beta_i$; the size of this gap is difficult to estimate in general. In this example this gap is mostly due to desired scalability in the employed analysis criteria and the fact that we did not take into account the periodicity of $\mathscr{L}_{\sigma(t)}$ in the utilized filters.

## 4.3 Summary

In this chapter's first part, we develop tools for analyzing uncertain impulsive system modeled in terms of LFRs. We begin by considering systems affected by arbitrarily time-varying parametric uncertainties and derive corresponding robust analysis criteria based on the full block S-procedure and multiplier separation techniques in Theorem 4.2.

Afterwards, we substantially generalize this result in Theorem 4.6 by relying on the dissipation based notion of finite-horizon IQCs with terminal, jump and flow cost as established in Definition 4.5. Theorem 4.6 constitutes a genuine generalization of Theorem 4 in [148] from non-impulsive systems to impulsive ones. Moreover, we provide several numerically verifiable criteria for assuring that IQCs with terminal, jump and flow costs are satisfied. These criteria are given in Lemmas 4.7 - 4.13 and tailored to concrete classes of uncertainties affecting the underlying system, with some of the involved dissipation inequalities appearing for the first time.

In particular, the novel robust analysis criteria resulting from combining Theorem 4.6 with Lemma 4.9 have been published by the author in [83] along with a detailed discussion and applications, e.g., to consensus problems for networked systems. This happened before we were able to derive Theorem 4.6 in its full generality and relies on a more direct, but much less modular proof.

In this chapter's second part, we demonstrate how our analysis results can be employed for analyzing networked systems in a scalable fashion and even if the underlying communication topology is switching. Our main result Theorem 4.18 is new and generalizes the findings of, e.g., [100, 57], since these authors only consider constant Lyapunov matrices and rely on the use of IQCs with static filters. Moreover, we illustrate that our criteria not only allow for guaranteeing stability and quadratic performance, but also permit us to assure that the considered network achieves consensus.

# 5

# Gain-Scheduled and Robust Synthesis

In Chapter 3, we elaborated on the design of feedback controllers for impulsive systems unaffected by uncertainties, and in Chapter 4, we developed new tools for systematically analyzing uncertain impulsive systems. In this chapter we benefit from the modularity of our employed approach which permits us to almost seamlessly combine those insights in order to synthesize output-feedback controllers for uncertain impulsive systems. Recall that the design of such *robust* controllers is of tremendous practical relevance since any designed controller is required to appropriately deal with the mismatch between the employed model and the real dynamical system to be controlled.

As an intermediate step, we consider the synthesis of so-called *gain-scheduled* controllers which can be viewed as a special case of the design of robust ones and which is also of independent interest. Roughly speaking and in contrast to a robust controller, a gain-scheduled controller aims to exploit that the parameters (or other objects such as delays) emerging in the encountered underlying system are only unknown at the outset but, in fact, measurable on-line. As an example think of the mass of a commercial airplane which is not known at the outset due to the unknown weight of the passengers and time-varying due to the consumption of fuel. However, by incorporating suitable sensors one can measure the weight on-line and should then incorporate those measurements in a controller design.

Usually one faces a mixture of genuine uncertainties and parameters that can be measured (or well approximated) on-line which leads to the challenging design of *robust gain-scheduled* controllers as elaborated on, e.g., in [160]. We will only briefly comment on this design for impulsive systems since it relies on another seamless combination of the approaches for robust and gain-scheduled synthesis.

## 5.1 Gain-Scheduled Synthesis

Recall that we adopt the framework of linear fractional representations (LFRs) [178] for robust analysis in Chapter 4 because it poses a well-established and flexible modeling tool in robust control [179]. In particular, this framework nicely permits us to separate known from unknown (or difficult) components. It is hence not surprising that the design of gain-scheduled controllers for non-impulsive systems modeled by LFRs has been considered in a number of works such as [116, 71, 140, 151, 147, 145]. In particular, the synthesis of gain-scheduled controllers for such system is by now well-established in various situations and successfully employed, e.g., in aerospace applications. Due to all of our preparations, we can essentially follow the conceptual design procedure even if the underlying open-loop system is impulsive and with few mandatory technical modifications only. In the sequel, we will briefly describe the conceptual procedure and highlight the required modifications.

Note that gain-scheduled controller synthesis is also possible without employing LFRs as for example done in [8, 132] for non-impulsive systems and in [19] for impulsive ones by relying on measurements of the full state. Without much difficulty, we could also design output-feedback gain-scheduled controllers based on these alternative approaches, but this is omitted here.

For brevity and in order to simplify the notation, we consider only the design of controllers assuring closed-loop stability. In view of the related works, e.g., [116, 71, 140, 145], this is no limitation as our synthesis results can be extended to the design of controllers achieving a desired quadratic performance criterion in a straightforward fashion. As in Chapter 4 we begin by considering arbitrarily time-varying parameters.

### 5.1.1 Arbitrarily Time-Varying Parameters

For real matrices of appropriate dimensions, an initial condition $x(0) \in \mathbb{R}^n$, a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT) and two sets $\mathbf{\Delta} \subset \mathbb{R}^{q\times p}$ and $\mathbf{\Delta}_J \subset \mathbb{R}^{q_J\times p_J}$, we consider now an impulsive open-loop plant with the description

$$\begin{pmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \\ u(t) \end{pmatrix}, \quad w(t) = \Delta(t)z(t), \tag{5.1a}$$

$$\begin{pmatrix} x(t_k) \\ z_J(k) \\ y_J(k) \end{pmatrix} = \begin{pmatrix} A_J & B_J & B_{J2} \\ C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & 0 \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ w_J(k) \\ u_J(k) \end{pmatrix}, \quad w_J(k) = \Delta_J(k)z_J(k) \tag{5.1b}$$

for $t \geq 0$ and $k \in \mathbb{N}$. Here, $y$, $y_J$ denote measured outputs, $u$, $u_J$ are control inputs and $w$, $w_J$, $z$, $z_J$ are interconnection variables. The involved time-varying parameters $\Delta$ and $\Delta_J$ are assumed to be piecewise continuous maps that *can be measured on-line*, i.e., they are available for control. However, at the outset, these parameters are merely known to satisfy

$$\Delta(t) \in \mathbf{\Delta} \quad \text{for all} \quad t \geq 0 \qquad \text{and} \qquad \Delta_J(k) \in \mathbf{\Delta}_J \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.2}$$

In this subsection our goal is the design of a dynamic *gain-scheduled* controller which ensures that the resulting closed-loop interconnection is robustly stable, i.e., stable for all admissible parameters $\Delta$ and $\Delta_J$. Concretely, we aim for controllers that admit the description

$$\begin{aligned}
\begin{pmatrix} \dot{x}_c(t) \\ z_c(t) \\ u(t) \end{pmatrix} &= \begin{pmatrix} A^c(\theta(t)) & B^c(\theta(t)) & B_2^c(\theta(t)) \\ C^c(\theta(t)) & D^c(\theta(t)) & D_{12}^c(\theta(t)) \\ C_2^c(\theta(t)) & D_{21}^c(\theta(t)) & D_{22}^c(\theta(t)) \end{pmatrix} \begin{pmatrix} x_c(t) \\ w_c(t) \\ y(t) \end{pmatrix}, \\
w_c(t) &= S(\theta(t), \Delta(t)) z_c(t) \\
\begin{pmatrix} x_c(t_k) \\ z_{Jc}(k) \\ u_J(k) \end{pmatrix} &= \begin{pmatrix} A_J^c(\theta(t_k^-)) & B_J^c(\theta(t_k^-)) & B_{J2}^c(\theta(t_k^-)) \\ C_J^c(\theta(t_k^-)) & D_J^c(\theta(t_k^-)) & D_{J12}^c(\theta(t_k^-)) \\ C_{J2}^c(\theta(t_k^-)) & D_{J21}^c(\theta(t_k^-)) & D_{J22}^c(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} x_c(t_k^-) \\ w_{Jc}(k) \\ y_J(k) \end{pmatrix}, \\
w_{Jc}(k) &= S_J(\theta(t_k^-), \Delta_J(k)) z_{Jc}(k)
\end{aligned} \tag{5.3}$$

for $t \geq 0$ and $k \in \mathbb{N}$ with initial condition $x_c(0) \in \mathbb{R}^{n_c}$ and with to-be-designed continuous describing matrices $A^c$, $B^c$, etc. and continuous *scheduling functions*

$$S : [0, T_{\max}] \times \mathbf{\Delta} \to \mathbb{R}^{r \times s} \quad \text{as well as} \quad S_J : [T_{\min}, T_{\max}] \times \mathbf{\Delta}_J \to \mathbb{R}^{r_J \times s_J}.$$

Due to these scheduling functions, the controller (5.3) is able to adjust its describing matrices according to the concrete instances of $\Delta$ and $\Delta_J$ and, thus, has the opportunity to benefit from their measurements. Also recall that $\theta$ denotes the clock (2.2) which appears naturally in (5.3) as discussed in the beginning of Chapter 3.

The closed-loop interconnection (5.1) and (5.3) reads as

$$\begin{pmatrix} \dot{x}_{cl}(t) \\ z_{cl}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A}(\theta(t)) & \mathcal{B}(\theta(t)) \\ \mathcal{C}(\theta(t)) & \mathcal{D}(\theta(t)) \end{pmatrix} \begin{pmatrix} x_{cl}(t) \\ w_{cl}(t) \end{pmatrix}, \quad w_{cl}(t) = \Delta_e(t) z_{cl}(t) \tag{5.4a}$$

$$\begin{pmatrix} x_{cl}(t_k) \\ z_{Jcl}(k) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_J(\theta(t_k^-)) & \mathcal{B}_J(\theta(t_k^-)) \\ \mathcal{C}_J(\theta(t_k^-)) & \mathcal{D}_J(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} x_{cl}(t_k^-) \\ w_{Jcl}(k) \end{pmatrix}, \quad w_{Jcl}(k) = \Delta_{Je}(k) z_{Jcl}(k) \tag{5.4b}$$

with stacked signals $x_{cl} := \mathrm{col}(x, x_c)$, $w_{cl} := \mathrm{col}(w, w_c)$, etc., parameters

$$\Delta_e(t) := \begin{pmatrix} \Delta(t) & 0 \\ 0 & S(\theta(t), \Delta(t)) \end{pmatrix} \text{ and } \Delta_{Je}(k) := \begin{pmatrix} \Delta_J(k) & 0 \\ 0 & S_J(\theta(t_k^-), \Delta_J(k)) \end{pmatrix}$$

as well as

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & 0 & 0 & 0 \\ \hline C & 0 & D & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{ccc} 0 & 0 & B_2 \\ I_{n_c} & 0 & 0 \\ \hline 0 & 0 & D_{12} \\ 0 & I_s & 0 \end{array}\right) \begin{pmatrix} A^c & B^c & B_2^c \\ C^c & D^c & D_{12}^c \\ C_2^c & D_{21}^c & D_{22}^c \end{pmatrix} \left(\begin{array}{cc|cc} 0 & I_{n_c} & 0 & 0 \\ 0 & 0 & 0 & I_r \\ C_2 & 0 & D_{21} & 0 \end{array}\right)$$

and analogously defined maps $\mathcal{A}_J, \mathcal{B}_J, \mathcal{C}_J, \mathcal{D}_J$. Two equivalent block diagrams of this closed-loop interconnection are depicted in Fig. 5.1 where $\mathcal{G}$, $\mathcal{G}_J$ $K$ and $K_J$ stand for the flow and jump component of (5.4) and the ones of (5.3), respectively. $G_e$ and $G_{Je}$ denote the flow and jump component of an augmented system whose flow component is

$$\left(\begin{array}{c} \dot{x}(t) \\ \hline z_{cl}(t) \\ \hdashline w_c(t) \\ y(t) \end{array}\right) = \left(\begin{array}{c|cc:cc} A & B & 0 & 0 & B_2 \\ \hline C & D & 0 & 0 & D_{12} \\ 0 & 0 & 0 & I_s & 0 \\ \hdashline 0 & 0 & I_r & 0 & 0 \\ C_2 & D_{21} & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} x(t) \\ \hline w_{cl}(t) \\ \hdashline z_c(t) \\ u(t) \end{array}\right);$$

its jump component is structured in the same fashion.

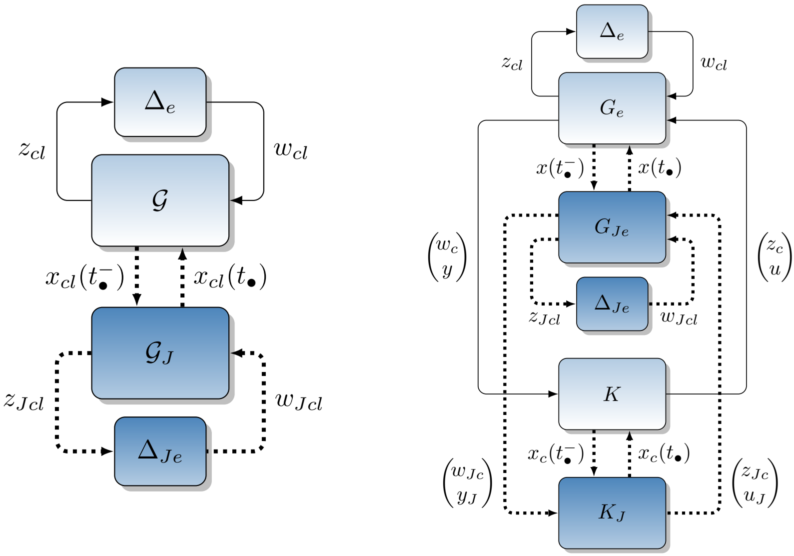



Figure 5.1: Two equivalent block diagrams of the closed-loop interconnection (5.4) of the system (5.1) with the controller (5.3).

Due to the particular structure of the closed-loop interconnection (5.4) and by the nature of the involved parameters, we can make use of a variation of Theorem 4.2 for its robust stability analysis. To this end, recall that a formal definition of robust stability was already given in Definition 4.1.

**Corollary 5.1** (Closed-Loop Robust Stability Analysis) *The interconnection* (5.4) *is robustly stable for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist functions* $\mathcal{X} \in C^1([0, T_{\max}], \mathbb{S}^{n+n_c})$, $\mathcal{P} \in C([0, T_{\max}], \mathbb{S}^{(p+s)+(q+r)})$ *and* $\mathcal{P}_J \in C([T_{\min}, T_{\max}], \mathbb{S}^{(p_J+s_J)+(q_J+r_J)})$ *satisfying, for all* $\Delta \in \mathbf{\Delta}$ *and all* $\Delta_J \in \mathbf{\Delta}_J$*, the inequalities*

$$\mathcal{X} \succ 0, \quad (\bullet)^\top \begin{pmatrix} 0 & \mathcal{X} \\ \mathcal{X} & \dot{\mathcal{X}} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{pmatrix} + (\bullet)^\top \mathcal{P} \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix} \prec 0, \qquad \text{(5.5a,b)}$$

$$(\bullet)^\top \begin{pmatrix} \mathcal{X}(0) & 0 \\ 0 & -\mathcal{X} \end{pmatrix} \begin{pmatrix} \mathcal{A}_J & \mathcal{B}_J \\ I & 0 \end{pmatrix} + (\bullet)^\top \mathcal{P}_J \begin{pmatrix} \mathcal{C}_J & \mathcal{D}_J \\ 0 & I \end{pmatrix} \prec 0, \tag{5.5c}$$

$$(\bullet)^\top \mathcal{P} \begin{pmatrix} I_p & 0 \\ 0 & I_s \\ \hdashline \Delta & 0 \\ 0 & S(\cdot, \Delta) \end{pmatrix} \succcurlyeq 0 \quad \textit{and} \quad (\bullet)^\top \mathcal{P}_J \begin{pmatrix} I_{p_J} & 0 \\ 0 & I_{s_J} \\ \hdashline \Delta_J & 0 \\ 0 & S_J(\cdot, \Delta_J) \end{pmatrix} \succcurlyeq 0 \tag{5.5d,e}$$

*on* $[0, T_{\max}]$, $[0, T_{\max}]$, $[T_{\min}, T_{\max}]$, $[0, T_{\max}]$ *and* $[T_{\min}, T_{\max}]$, *respectively.*

Similarly as for nominal controller synthesis as discussed in Chapter 3, attempting to solve the inequalities (5.5) and simultaneously searching a controller (5.3) (together with its corresponding scheduling functions) is numerically prohibitive due to the non-convex dependencies on all decision variables. However, we can render this simultaneous search convex if we add the (inertia) constraints

$$\mathcal{P} \ \text{ and } \ \mathcal{P}_J \ \text{ are constant maps,} \tag{5.6a}$$

$$(\bullet)^\top \mathcal{P} \begin{pmatrix} I \\ 0 \end{pmatrix} \succ 0 \ \text{ on } \ [0, T_{\max}] \quad \text{and} \quad (\bullet)^\top \mathcal{P}_J \begin{pmatrix} I \\ 0 \end{pmatrix} \succ 0 \ \text{ on } \ [T_{\min}, T_{\max}] \tag{5.6b}$$

as well as

$$(\bullet)^\top \mathcal{P} \begin{pmatrix} 0 \\ I \end{pmatrix} \prec 0 \ \text{ on } \ [0, T_{\max}] \quad \text{and} \quad (\bullet)^\top \mathcal{P}_J \begin{pmatrix} 0 \\ I \end{pmatrix} \prec 0 \ \text{ on } \ [T_{\min}, T_{\max}] \tag{5.6c}$$

to the closed-loop analysis criteria (5.5). With those additional constraints and due to the particular structure of the describing matrices in (5.4), we can employ the elimination lemma C.11 similarly is in Theorem 3.8 for nominal synthesis. This leads to the following theorem which is essentially an extension of the main result from [139] to impulsive systems.

**Theorem 5.2** (Gain-Scheduled Controller Design via Elimination) *Let $U$, $V$, $U_J$ and $V_J$ be basis matrices of the sets $\ker((B_2^\top, D_{12}^\top))$, $\ker((C_2, D_{21}))$, $\ker((B_{J2}^\top, D_{J12}^\top))$ and $\ker((C_{J2}, D_{J21}))$, respectively. Moreover, suppose that $0 \in \mathbf{\Delta}$ and $0 \in \mathbf{\Delta}_J$. Then there exist scheduling functions $S$, $S_J$ and a controller (5.3) for the system (5.1) such that the LMIs (5.5) and (5.6) are feasible if and only if there exist continuously differentiable maps $X, Y$ and matrices $P$, $\tilde{P}$, $P_J$, $\tilde{P}_J$ satisfying*

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \tag{5.7a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P \end{array}\right) \left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) V \prec 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & \tilde{P} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A^\top & -C^\top \\ \hline 0 & I \\ -B^\top & -D^\top \end{array}\right) U \succ 0, \tag{5.7b,c}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} X(0) & 0 & \\ 0 & -X & \\ \hline & & P_J \end{array}\right) \left(\begin{array}{cc} A_J & B_J \\ I & 0 \\ \hline C_J & D_J \\ 0 & I \end{array}\right) V_J \prec 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} Y(0) & 0 & \\ 0 & -Y & \\ \hline & & \tilde{P}_J \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A_J^\top & -C_J^\top \\ \hline 0 & I \\ -B_J^\top & -D_J^\top \end{array}\right) U_J \succ 0 \tag{5.7d,e}$$

*on $[0, T_{\max}]$, $[0, T_{\max}]$, $[0, T_{\max}]$, $[T_{\min}, T_{\max}]$, $[T_{\min}, T_{\max}]$, respectively, as well as*

$$(\bullet)^\top P \begin{pmatrix} 0 \\ I \end{pmatrix} \prec 0, \quad (\bullet)^\top P \begin{pmatrix} I \\ \Delta \end{pmatrix} \succcurlyeq 0, \quad (\bullet)^\top \tilde{P} \begin{pmatrix} I \\ 0 \end{pmatrix} \succ 0, \quad (\bullet)^\top \tilde{P} \begin{pmatrix} -\Delta^\top \\ I \end{pmatrix} \preccurlyeq 0 \tag{5.7f}$$

$$(\bullet)^\top P_J \begin{pmatrix} 0 \\ I \end{pmatrix} \prec 0, \quad (\bullet)^\top P_J \begin{pmatrix} I \\ \Delta_J \end{pmatrix} \succcurlyeq 0, \quad (\bullet)^\top \tilde{P}_J \begin{pmatrix} I \\ 0 \end{pmatrix} \succ 0, \quad (\bullet)^\top \tilde{P}_J \begin{pmatrix} -\Delta_J^\top \\ I \end{pmatrix} \preccurlyeq 0 \tag{5.7g}$$

*for all $(\Delta, \Delta_J) \in \mathbf{\Delta} \times \mathbf{\Delta}_J$.*

*Sketch of Proof.* *Only if:* This is the simple part and follows from pointwise applying the elimination lemma C.11 on each of the LMIs (5.5b), (5.5d), (5.5c) and (5.5e), as well as from defining $X$, $Y$, $P$, $\tilde{P}$, $P_J$ and $\tilde{P}_J$ by corresponding sub-blocks of $\mathcal{X}$, $\mathcal{X}^{-1}$, $\mathcal{P}$, $\mathcal{P}^{-1}$, etc. For example, one chooses

$$X := (\bullet)^\top \mathcal{X} \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad Y := (\bullet)^\top \mathcal{X}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \quad \text{and} \quad P := (\bullet)^\top \mathcal{P} \text{diag}\left( \begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} I \\ 0 \end{pmatrix} \right).$$

*If:* By continuity of $X, \dot{X}, Y, \dot{Y}$ on the compact interval $[0, T_{\max}]$ and strictness of all inequalities aside from the four in (5.7g) and (5.7f), we can perturb $P, \tilde{P}, P_J$ and $\tilde{P}_J$ as $P + \left(\begin{smallmatrix} \varepsilon I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\tilde{P} - \left(\begin{smallmatrix} 0 & 0 \\ 0 & \varepsilon I \end{smallmatrix}\right)$, $P_J + \left(\begin{smallmatrix} \varepsilon I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\tilde{P}_J - \left(\begin{smallmatrix} 0 & 0 \\ 0 & \varepsilon I \end{smallmatrix}\right)$, respectively, such that all inequalities in (5.7) are strict. Then Corollary C.10 allows us to infer from (5.7g) and (5.7f) that the perturbed $P$, $\tilde{P}$, $P_J$ and $\tilde{P}_J$ are nonsingular. Note that via another perturbation we can even ensure that $P - \tilde{P}^{-1}$ and $P_J - \tilde{P}_J^{-1}$ are nonsingular.

Due to (5.7g) and (5.7f) as well as $0 \in \mathbf{\Delta}$ and $0 \in \mathbf{\Delta}_J$, we can then apply Lemma C.16 in order to construct matrices $\mathcal{P}$ and $\mathcal{P}_J$ as well as continuous scheduling functions $S$ and $S_J$ such that (5.5d), (5.5e) and (5.6) are satisfied. Since $\mathcal{P}$ and $\mathcal{P}_J$ satisfy (5.6a) and (5.6b), we can follow the proof of Theorem 3.8 for nominal output-feedback design via elimination in order to construct describing maps of the controller (5.3) that are continuous. ●

As for the underlying analysis criteria in Corollary 5.1 or Theorem 4.2, in order to turn the gain-scheduled synthesis inequalities (5.7) into standard LMIs, we can apply one of the DLMI relaxations from Appendix D and confine the multipliers $P$, $P_J$, $\tilde{P}$ and $\tilde{P}_J$ to suitable choices of (dual) multiplier sets $\mathbf{P}(\mathbf{\Delta})$, $\mathbf{P}(\mathbf{\Delta}_J)$, $\tilde{\mathbf{P}}(\mathbf{\Delta})$ and $\tilde{\mathbf{P}}(\mathbf{\Delta}_J)$, respectively, similarly as explained in Section C.6 or, e.g., in [149, 160]. Here, the set $\mathbf{P}(\mathbf{\Delta})$ is required to admit a characterization in terms of LMIs and to satisfy

$$(\bullet)^\top P \begin{pmatrix} I \\ \Delta \end{pmatrix} \succcurlyeq 0 \quad \text{for all} \quad \Delta \in \mathbf{\Delta} \quad \text{and all} \quad P \in \mathbf{P}(\mathbf{\Delta});$$

the remaining sets are taken with analogous properties.

Recall that we summarized some strategies to improve the controller reconstruction in Remark 3.10 in Chapter 3 on nominal synthesis. These strategies can be applied here analogously.

Note that dropping (5.6a) and correspondingly searching for continuous functions $P, \tilde{P}, P_J$ and $\tilde{P}_J$ instead of constant matrices means that we have to construct continuous $\mathcal{P}$ and $\mathcal{P}_J$ in the proof of Theorem 5.2. Unfortunately, this is not easily possible based on Lemma C.16. One of the reasons is that the number of positive/negative eigenvalues of $P - \tilde{P}^{-1}$ and/or $P_J - \tilde{P}_J^{-1}$ might not be constant which breaks the whole construction. Another technical issue is that systematically constructing a continuous map $T$ that is pointwise nonsingular and satisfies $T(\tau)^\top M(\tau)T(\tau) \prec 0$ for all $\tau$ for some given continuous map $M$ with a constant number of negative eigenvalues is only easy if its eigenvalues are distinct for all $\tau$.

Further, note that we require (5.6b) in order to follow the proof of Theorem 3.8 for nominal output-feedback design via elimination and, in particular, to guarantee continuity of the reconstructed describing maps of the controller (5.3).

Finally, note that we employ (5.6c) for applying Lemma C.16. Similarly as shown in [140] for standard LTI systems, one might be able to drop this inequality along with the corresponding ones in (5.5c) and (5.5e). However, this will then lead to gain-scheduling controllers with a more general description than the one shown in (5.3).

**Remark 5.3** (Convexifying Parameter Transformation) In contrast to nominal synthesis, there seems to be no direct way to render the closed-loop analysis inequalities (5.5) convex by means of a convexifying parameter transformation. However, it is possible to perform such a transformation by relying on lifting techniques as an intermediate step. These techniques have been employed, e.g., in [126] for designing gain-scheduled controllers that (if desired) are particularly structured and in [163] for robust analysis via IQCs.

**Remark 5.4** (Simple Scheduling Functions) For some multiplier sets such as those based on D or D-G scalings (see Remark C.19), it is possible to choose the scheduling functions without loss of generality as

$$S(\tau, \Delta) := \Delta \quad \text{and} \quad S_J(\tau, \Delta_J) := \Delta_J$$

which simplifies implementations. This is partly due to availability of suitable results on matrix extensions as addressed in Section C.5. These dedicated results on matrix extensions even permit us to drop the constraint that all multipliers are constants maps which can be highly beneficial as illustrated in the examples of the last chapter.

### 5.1.2 Piecewise Constant Parameters

Next we consider the situation that more information on the involved parameters is available which permits us to employ Theorem 4.6 for a more dedicated underlying closed-loop analysis by means of IQCs with dynamic filters. Recall that utilizing static filters essentially corresponds to the criteria provided in Corollary 5.1 that capture only few properties of the involved parameters.

Unfortunately and even for standard LTI systems, it is still unknown how to perform gain-scheduled controller design based on IQCs with general dynamic multipliers and by means of convex optimization. One of the major technical stumbling stones is the lack of suitable dynamic version of Lemma C.16 on matrix extensions. So far, convex solutions to the gain-scheduled design problem have been found only for dynamic multipliers composed of D scalings and of D-G scalings [147, 145]. In [81, 83], we were able to extend the latter two approaches to impulsive systems and piecewise constant parameters based on special cases of our main analysis result Theorem 4.6. Here, we briefly repeat our main design result from [83].

For real matrices of appropriate dimensions, a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ and an initial condition $x(0)\in\mathbb{R}^n$ with (RDT), we consider now an impulsive open-loop plant with the description

$$\begin{pmatrix}\dot{x}(t)\\ z(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}A & B & B_2\\ C & D & D_{12}\\ C_2 & D_{21} & 0\end{pmatrix}\begin{pmatrix}x(t)\\ w(t)\\ u(t)\end{pmatrix},\quad \begin{pmatrix}x(t_k)\\ y_J(k)\end{pmatrix}=\begin{pmatrix}A_J & B_J\\ C_J & 0\end{pmatrix}\begin{pmatrix}x(t_k^-)\\ u_J(k)\end{pmatrix},$$
$$w(t)=\delta(t)z(t), \tag{5.8}$$

for $t\geq 0$ and $k\in\mathbb{N}$. Here, we assume that the parameter $\delta$ is measurable, piecewise constant and known to take values in a given interval, i.e.,

$$\delta(t)=\delta(t_k)\in[a,b]\ \text{ holds for all }\ t\in[t_k,t_{k+1})\text{ and all }k\in\mathbb{N}_0. \tag{5.9}$$

Our goal is the synthesis of a gain-scheduled controller of the form

$$\begin{pmatrix}\dot{x}_c(t)\\ z_c(t)\\ u(t)\end{pmatrix}=\begin{pmatrix}A^c(\theta(t)) & B^c(\theta(t)) & B_2^c(\theta(t))\\ C^c(\theta(t)) & D^c(\theta(t)) & D_{12}^c(\theta(t))\\ C_2^c(\theta(t)) & D_{21}^c(\theta(t)) & D_{22}^c(\theta(t))\end{pmatrix}\begin{pmatrix}x_c(t)\\ w_c(t)\\ y(t)\end{pmatrix},\quad w_c(t)=\delta(t)z_c(t),$$
$$\begin{pmatrix}x_c(t_k)\\ y_J(k)\end{pmatrix}=\begin{pmatrix}A_J^c(\theta(t_k^-)) & B_J^c(\theta(t_k^-))\\ C_J^c(\theta(t_k^-)) & D_J^c(\theta(t_k^-))\end{pmatrix}\begin{pmatrix}x_c(t_k^-)\\ u_J(k)\end{pmatrix} \tag{5.10}$$

for $t \geq 0$, $k \in \mathbb{N}$ and with continuous describing matrices such that the resulting closed-loop interconnection is stable for all parameters $\delta$ satisfying (5.9) and all $(t_k)_{k\in\mathbb{N}_0}$ with (RDT). Instead of simply applying a variant of Theorem 5.2, which would be possible in principle, we seek to provide another design approach based on the IQC analysis results in Theorem 4.6 and Lemma 4.9. We have already argued and shown in a numerical example that the combination of the latter results often permits a much more accurate analysis than the criteria that underlie the corresponding variant of Theorem 5.2. Recall that this stems from the possibility to employ dynamic filters (4.6) in Theorem 4.6 and Lemma 4.9 which are more flexible than static ones as implicitly used in Theorem 5.2. Hence, we intend to achieve analogous benefits for controller design.

The closed-loop interconnection of the system (5.8) and the controller (5.10) is essentially of the same form as the one in (5.4) and not repeated here. We stress that this interconnection's description is again structured in a way such that we can apply Theorem 4.6 together with Lemma 4.9 for its analysis. Recall that for this combination and in the present situation where the performance channel is absent, the analysis criteria in Theorem 4.6 are formulated in terms of an augmented system (4.7) with describing matrices

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} := \left(\begin{array}{ccc|c} A_\psi & 0 & B_\psi C & B_\psi D \\ 0 & A_\psi & 0 & B_\psi \\ 0 & 0 & A & B \\ \hline C_\psi & 0 & D_\psi C & D_\psi D \\ 0 & C_\psi & 0 & D_\psi \end{array}\right) \quad \text{and} \quad \mathcal{A}_J := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

for some matrices $A_\psi$, $B_\psi$, $C_\psi$ and $D_\psi$ with $C_\psi \in \mathbb{R}^{m_\psi \times n_\psi}$. Next to these describing matrices, we employ their dual[1] version

$$\begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} := \left(\begin{array}{ccc|c} -A_\phi^\top & 0 & 0 & C_\phi^\top \\ 0 & -A_\phi^\top & -C_\phi^\top B^\top & -C_\phi^\top D^\top \\ 0 & 0 & -A^\top & -C^\top \\ \hline -B_\phi^\top & 0 & 0 & D_\phi^\top \\ 0 & -B_\phi^\top & -D_\phi^\top B^\top & -D_\phi^\top D^\top \end{array}\right) \text{ and } \tilde{\mathcal{A}}_J := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_n^\top \end{pmatrix}$$

for some matrices $A_\phi, B_\phi, C_\phi$ and $D_\phi$ with $B_\phi \in \mathbb{R}^{n_\phi \times m_\phi}$ in order to state our design criteria. Finally, we introduce the abbreviation

$$P_d(S) := \frac{1}{(b-a)^2} \begin{pmatrix} I & I \\ aI & bI \end{pmatrix} \begin{pmatrix} 0 & S \\ S^\top & 0 \end{pmatrix} \begin{pmatrix} I & I \\ aI & bI \end{pmatrix}^\top$$

for any square valued map or square matrix $S$ which is also related to $P_p(S)$ as defined in (4.9) by duality. This permits us to formulate our gain-scheduled controller synthesis result based on combining Theorem 4.6 with Lemma 4.9 for the underlying closed-loop analysis.

**Theorem 5.5** *Let $\tilde{U}, \tilde{V}$, $\tilde{U}_J$ and $\tilde{V}_J$ be basis matrices of* $\ker(C_2, D_{21})$, $\ker(B_2^\top, D_{12}^\top)$, $\ker(C_J)$ *and* $\ker(B_J^\top)$, *respectively. Moreover, define*

$$U_J := \begin{pmatrix} I_{2n_\psi} & 0 \\ 0 & \tilde{U}_J \end{pmatrix}, \quad V_J := \begin{pmatrix} I_{2n_\phi} & 0 \\ 0 & \tilde{V}_J \end{pmatrix}, \quad U := \begin{pmatrix} I_{2n_\psi} & 0 \\ 0 & \tilde{U} \end{pmatrix}, \quad V := \begin{pmatrix} I_{2n_\phi} & 0 \\ 0 & \tilde{V} \end{pmatrix}.$$

*Then there exists a controller* (5.10) *for the system* (5.8) *such that their closed-loop is stable for all* $(t_k)_{k\in\mathbb{N}_0}$ *with* (RDT) *and all* $\delta$ *with* (5.9) *if there exist maps* $X \in C^1([0, T_{\max}], \mathbb{S}^{2n_\psi+n})$, $Y \in C^1([0, T_{\max}], \mathbb{S}^{2n_\phi+n})$, $\left(\begin{smallmatrix} R & S \\ S^\top & Q \end{smallmatrix}\right) \in C^1([0, T_{\max}], \mathbb{S}^{n_\psi+n_\phi})$, $M \in C([0, T_{\max}], \mathbb{R}^{m_\psi \times m_\psi})$ *as well as*

[1] A motivation of this notion is provided in [145, 147]. Essentially, the matrices $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ emerge after applying a variant of the dualization lemma C.9 on the underlying IQC based analysis criteria.

$N \in C([0, T_{\max}], \mathbb{R}^{m_\phi \times m_\phi})$ *satisfying*

$$\begin{pmatrix} \hat{Y} & \hat{S} \\ \hat{S}^\top & \hat{X} \end{pmatrix} \succ 0, \tag{5.11a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & \dot{X} & \\ \hline & & P_p(M) \end{array}\right) \left(\begin{array}{cc} \mathcal{A} & \mathcal{B} \\ I & 0 \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) V \prec 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} \dot{Y} & Y & \\ Y & 0 & \\ \hline & & P_d(N) \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \hline \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{array}\right) U \succ 0, \tag{5.11b,c}$$

$$(\bullet)^\top \left(\begin{array}{cc|c} 0 & R & \\ R & \dot{R} & \\ \hline & & M + M^\top \end{array}\right) \left(\begin{array}{cc} A_\psi & B_\psi \\ I & 0 \\ \hline C_\psi & D_\psi \end{array}\right) \succ 0, \quad (\bullet)^\top \left(\begin{array}{cc|c} \dot{Q} & Q & \\ Q & 0 & \\ \hline & & N + N^\top \end{array}\right) \left(\begin{array}{cc} I & 0 \\ -A_\phi^\top & C_\phi^\top \\ \hline -B_\phi^\top & D_\phi^\top \end{array}\right) \succ 0 \tag{5.11d,e}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \begin{pmatrix} \hat{X}(0) & 0 \\ 0 & -\hat{X} \end{pmatrix} \begin{pmatrix} \mathcal{A}_J \\ I \end{pmatrix} U_J \prec 0 \quad \text{and} \quad (\bullet)^\top \begin{pmatrix} \hat{Y}(0) & 0 \\ 0 & -\hat{Y} \end{pmatrix} \begin{pmatrix} I \\ \tilde{\mathcal{A}}_J \end{pmatrix} V_J \succ 0 \tag{5.11f,g}$$

*on* $[T_{\min}, T_{\max}]$ *where* $\hat{S} := \operatorname{diag}(S, S, I)$, $\hat{X} := X - \operatorname{diag}\left(\frac{1}{2} P_p(R), 0\right)$ *and* $\hat{Y} := Y - \operatorname{diag}\left(\frac{1}{2} P_d(Q), 0\right)$.

The constructive proof is admittedly, somewhat technical and not shown here. It is given in full detail in [83] together with several elaborating and technical comments. Further remarks can be extracted from the related results in [147, 145] for standard LTI systems.

At this point we only emphasize that for static filters (corresponding to $n_\psi = n_\phi = 0$ and $D_\psi = D_\phi = I$) the inequalities (5.11) simplify drastically and we recover a special case of Theorem 5.2 with multiplier sets corresponding to D-G-scalings. Moreover, the constructive proof of Theorem 5.5 leads usually to a controller (5.10) that has at most degree $n + 2n_\psi$ in the flow component and $q = p + 2n_\psi$ repetitions in the scheduling block.

Finally, next to the suggestions in Remark 3.10 for improving the numerical reconstruction of the controller matrices, it can be beneficial to enforce $S = I$ when solving the LMIs (5.11) in order to avoid the coordinate transformation in the proof and to reduce the number of required algebraic manipulations.

### 5.1.3 Distributed Control for Networked Systems

Recall that we have seen in Section 4.2 that our robust analysis tools can be employed for analyzing networked systems in a scalable fashion. Next, we show how to utilize those insights for designing controllers for such systems in the same vein.

To this end, we consider an homogeneous open-loop networked system composed of $M$ subsystems and with the description

$$\begin{pmatrix} \dot{x}_i(t) \\ z_i(t) \\ y_i(t) \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x_i(t) \\ w_i(t) \\ u_i(t) \end{pmatrix}, \quad w_i(t) = \sum_{j=1}^{M} a_{ij}(t) z_j(t) \tag{5.12}$$

for $t \geq 0$, $i \in \{1, \ldots, M\}$ and with initial conditions $x_1(0), \ldots, x_M(0) \in \mathbb{R}^n$. In order to simplify the formulation of our design criteria, we suppose that the time-varying communication topology of the network (5.12) is undirected and piecewise constant. Precisely, we assume that the corresponding adjacency matrix $\mathscr{A} = (a_{ij})$ satisfies

$$\mathscr{A}(t) = \mathscr{A}(t_k) \in \mathbb{S}^M \quad \text{and} \quad \operatorname{eig}(\mathscr{A}(t)) \subset [a, b] \quad \text{for all} \quad t \in [t_k, t_{k+1}), \;\; k \in \mathbb{N}_0 \tag{5.13}$$

for some sequence $(t_k)_{k \in \mathbb{N}_0}$ satisfying (RDT). We have seen in Section 4.2 how to incorporate additional knowledge on the communication topology and could apply those modifications here as well.

In the vast amount of literature on control for networked systems it

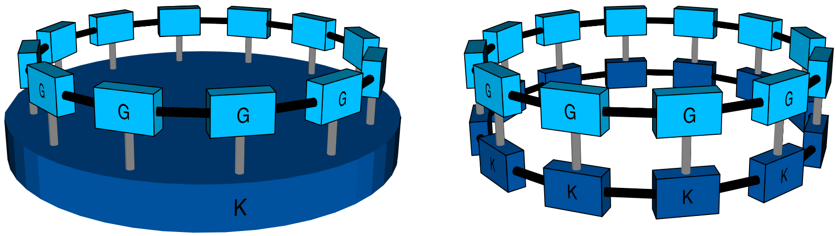



Figure 5.2: An undirected cyclic interconnected system with a centralized controller (left) and a distributed controller (right).

is generally suggested to design so-called *distributed* controllers (see, e.g., [100, 105, 57, 166, 34]). These are highly structured controllers that constitute themselves networked systems with a communication topology that is similar (or often even identical) to the one of the underlying open-loop network. The reason for synthesizing distributed controllers is that, for large networks, the design of a single controller which takes care of the full network becomes computationally prohibitive, e.g., due to the large number of required internal states; controllers of the latter type are referred to as *centralized* controllers in this context. Fig. 5.2 illustrates a closed-loop network involving a centralized and a distributed controller where the underlying open-loop network is characterized through a constant cyclic interconnection.

Hence, we follow this approach and consider the design of a distributed controller that admits, in our situation, the description

$$\begin{aligned}
\begin{pmatrix} \dot{x}_{ci}(t) \\ z_{ci}(t) \\ u_i(t) \end{pmatrix} &= \begin{pmatrix} A^c(\theta(t)) & B^c(\theta(t)) & B_2^c(\theta(t)) \\ C^c(\theta(t)) & D^c(\theta(t)) & D_{12}^c(\theta(t)) \\ C_2^c(\theta(t)) & D_{21}^c(\theta(t)) & D_{22}^c(\theta(t)) \end{pmatrix} \begin{pmatrix} x_{ci}(t) \\ w_{ci}(t) \\ y_i(t) \end{pmatrix}, \\
w_{ci}(t) &= \sum_{j=1}^{M} a_{ij}(t) z_{cj}(t), \quad x_{ci}(t_k) = A_J^c(\theta(t_k^-))x_{ci}(t_k^-)
\end{aligned} \tag{5.14}$$

for $t \geq 0$, $k \in \mathbb{N}$ and $i \in \{1, \ldots, M\}$. The key for developing scalable design criteria is to recall the transformation employed in Section 4.2 in order to diagonalize the adjacency matrix, as well as the resulting equivalent representations of the system (5.12) *and* the controller (5.14). In fact, this permits us to view the problem of constructing a distributed controller (5.14) as a gain-scheduled controller synthesis problem. This has the particularly nice benefit that we immediately obtain suitable design criteria from Theorem 5.5 as stated earlier.

**Corollary 5.6** *There exists a distributed controller* (5.14) *for the network* (5.12) *such that their closed-loop is stable for all* $(t_k)_{k\in\mathbb{N}_0}$ *with* (RDT) *and all switching communication topologies defined by* $\mathscr{A}$ *with* (5.13) *if there exist maps* $X \in C^1([0, T_{\max}], \mathbb{S}^{2n_\psi+n})$, $Y \in C^1([0, T_{\max}], \mathbb{S}^{2n_\phi+n})$, $\left(\begin{smallmatrix} R & S \\ S^\top & Q \end{smallmatrix}\right) \in C^1([0, T_{\max}], \mathbb{S}^{n_\psi+n_\phi})$, $M \in C([0, T_{\max}], \mathbb{R}^{m_\psi \times m_\psi})$ *as well as* $N \in C([0, T_{\max}], \mathbb{R}^{m_\phi \times m_\phi})$ *satisfying* (5.5) *with* $U_J = V_J = I$.

Let us stress that these design criteria do not depend on the number of subsystems in the network. Moreover, they involve for the first time (apart from [83]) IQCs with dynamic multipliers and a switching communication topology which generalizes, e.g., the findings of [100, 166, 57] that rely on static multipliers.

### Example

In view of our insights obtained in Subsection 3.1.2 on the synthesis of sampled-data controllers and due to the modularity of our approach, it is natural that our methodology also permits us to synthesize *distributed sampled-data* controllers in a scalable fashion. As an illustration, we consider a network of $M = 10$ simple subsystems with dynamics

$$\dot{x}_{i1}(t) = x_{i2}(t), \quad \dot{x}_{i2}(t) = -x_{i1}(t) + u_i(t) \quad \text{for all} \quad i \in \{1, \ldots, M\}$$

where the control inputs are restricted to be piecewise constant, i.e.,

$$u_i(t) = u_i(t_k) \ \text{ for all } \ t \in [t_k, t_{k+1}),\ k \in \mathbb{N}_0 \ \text{ and all } \ i \in \{1, \ldots, M\}$$

for some sequence $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT). Moreover, we assume that only the relative distances of the first states of neighbors at the time instances $t_0, t_1, t_2, \ldots$ are measurable, i.e., the outputs

$$y_{Ji}(k) := \sum_{j=1}^{M} a_{ij}\big(x_{j1}(t_k) - x_{i1}(t_k)\big) \quad \text{for} \quad i \in \{1, \ldots, M\}$$

are available for control. Here, we suppose that the coupling weights $a_{ij}$ are constant and describe an undirected cyclic communication graph as depicted in Fig. 5.2, i.e., they are given by

$$a_{ij} := 1 \text{ if } |i - j| = 1 \text{ or } |i - j| = M - 1 \quad \text{and} \quad a_{ij} := 0 \text{ otherwise.}$$

Our goal is now to find a distributed sampled-data controller such that consensus is asymptotically achieved. Recall that this means that

$$\lim_{t\to\infty} \|x_k(t) - x_l(t)\| = 0 \quad \text{holds for all} \quad k, l \in \{1, \ldots, M\}$$

and all initial conditions. To this end, we express the given network as

$$\begin{pmatrix} \dot{x}_i(t) \\ \dot{u}_i(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix}, \quad \left(\begin{array}{c} x_i(t_k) \\ u_i(t_k) \\ \hline z_{Ji}(t_k) \\ \hdashline y_{Ji}(t_k) \end{array}\right) = \left(\begin{array}{ccc|c:c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \end{array}\right) \left(\begin{array}{c} x_i(t_k^-) \\ u_i(t_k^-) \\ \hline w_{Ji}(k) \\ \hdashline u_{Ji}(k) \end{array}\right),$$

$$w_{Ji}(k) = \sum_{j=1}^{M} a_{ij}\big(z_{Jj}(k) - z_{Ji}(k)\big)$$

for $t \geq 0$, $k \in \mathbb{N}$ and $i \in \{1, \ldots, M\}$. Thus, we target at controllers with description

$$\begin{aligned}
\dot{x}_{ci}(t) &= A^c(\theta(t))x_{ci}(t),\\
\begin{pmatrix} x_{ci}(t_k) \\ z_{Jci}(k) \\ u_{Ji}(k) \end{pmatrix} &= \begin{pmatrix} A^c_J(\theta(t_k^-)) & B^c_J(\theta(t_k^-)) & B^c_{J2}(\theta(t_k^-)) \\ C^c_J(\theta(t_k^-)) & D^c_J(\theta(t_k^-)) & D^c_{J12}(\theta(t_k^-)) \\ C^c_{J2}(\theta(t_k^-)) & D^c_{J21}(\theta(t_k^-)) & D^c_{J22}(\theta(t_k^-)) \end{pmatrix} \begin{pmatrix} x_{ci}(t_k^-) \\ w_{Jci}(k) \\ y_{Ji}(k) \end{pmatrix}
\end{aligned}$$

that are coupled through

$$w_{Jci}(k) = \sum_{j=1}^{M} a_{ij}\big(z_{Jcj}(k) - z_{Jci}(k)\big).$$

For analogous reasons as stated above, we can design such a distributed controller, e.g., by employing a variant of Theorem 5.2 with multiplier sets corresponding to the set of D-G-scalings.

Fig. 5.3 displays the second states $x_{i2}$ of the closed-loop in response to random initial conditions for impulse sequences $(t_k)_{k \in \mathbb{N}_0}$ with (RDT) and $T_{\min} = 0.1$ and $T_{\max} \in \{0.5, 1, 2\}$ and correspondingly designed distributed controllers. As comparison, the analogous closed-loop response for a standard distributed controller that relies on permanently available measurements and unconstrained control inputs is shown as well. We observe that, in all cases, consensus is achieved even if information is only rarely exchanged. However, we note that reaching consensus takes longer in the latter cases which is expected intuitively.

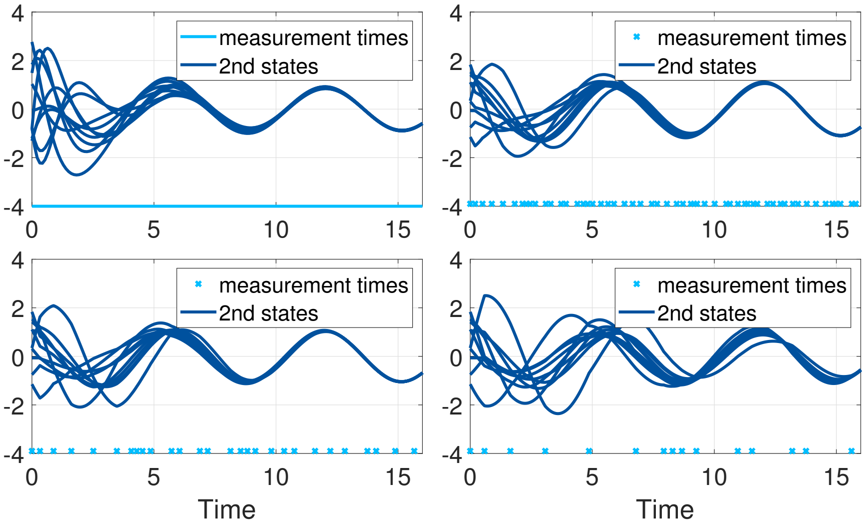



Figure 5.3: Second states $x_{i2}$ (dark blue) of the closed-loop interconnection involving a standard distributed controller (top left) and a sampled-data distributed controller obtained from Theorem 5.2 for $T_{\min} = 0.1$ as well as $T_{\max} = 0.5$ (top right), $T_{\max} = 1$ (bottom left) and $T_{\max} = 2$ (bottom right). The light blue markers denote the time instances at which measurements are taken.

## 5.2 Robust Synthesis

It is by now well-known in the literature on LMIs that the robust controller synthesis problem is amenable to techniques from convex optimization only for particular classes of uncertain dynamical systems. For example, if considering non-impulsive systems modeled by LFRs and employing multiplier theory or IQCs for robustness analysis of the closed-loop, the following classes have been identified.

- Systems for which the full-state is available for control if static multipliers are employed for the robustness analysis.
- Systems affected by a single unstructured uncertainty if using small-gain arguments.
- Systems emerging in estimation (or filter design) problems [157, 161] and feedforward control [152]. More generally, one can deal with systems having a control channel that is unaffected by uncertainties [143].
- Systems for which certain matrix pencils are left (or right) invertible [94].

For uncertain impulsive systems we essentially face the same challenges when tackling the general robust output-feedback design problem and are, hence, forced to employ heuristic procedures similarly as we did in Section 3.2 for (nominal) static output-feedback synthesis. In fact, the problem of finding such a static controller is closely related to the robust output-feedback problem in terms of reason for non-convexity, which can often be exploited.

In this section, we generalize the dual iteration to uncertain impulsive systems modeled by LFRs which is an extension of our results in Section 3.2 of tremendous practical relevance. Due to our particular (robust) analysis results and the flexibility of the LFR framework which is accompanied by corresponding design tools (such as the elimination lemma C.11 and the nonstandard variation thereof Lemma C.13), we can reuse several of the arguments already provided in Section 3.2 for static design and only require few additional ones. Isn't that nice?

Let us finally stress that, next to the dual iteration and based on our robust analysis criteria in Chapter 4, one could also extend several other static design approaches, e.g., of the ones suggested in [128], to uncertain impulsive systems. However, we focus on the dual iteration because it is, in our opinion, efficient with a high flexibility; due to the LFR framework it can deal with systems simultaneously affected by several uncertainties of different types. Unfortunately, so far we rely on IQCs with static filters for the underlying analysis which might be resolved in the future.

### 5.2.1 Problem Description

For real matrices of appropriate dimensions, a sequence of impulse instants $(t_k)_{k\in\mathbb{N}_0}$ satisfying (RDT), some initial condition $x(0) \in \mathbb{R}^n$ and two uncertainty (value) sets $\mathbf{\Delta} \subset \mathbb{R}^{q\times p}$ and $\mathbf{\Delta}_J \subset \mathbb{R}^{q_J\times p_J}$ that both contain the origin, we now consider an uncertain open-loop plant with the description

$$\begin{pmatrix} \dot{x}(t) \\ \hline z(t) \\ e(t) \\ y(t) \end{pmatrix} = \left(\begin{array}{c|ccc} A & B & B_2 & B_3 \\ \hline C & D & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & 0 \end{array}\right) \begin{pmatrix} x(t) \\ \hline w(t) \\ d(t) \\ u(t) \end{pmatrix}, \quad w(t) = \Delta(t) z(t), \tag{5.15a}$$

$$\begin{pmatrix} x(t_k) \\ \hline z_J(k) \\ e_J(k) \\ y_J(k) \end{pmatrix} = \left(\begin{array}{c|ccc} A_J & B_J & B_{J2} & B_{J3} \\ \hline C_J & D_J & D_{J12} & D_{J13} \\ C_{J2} & D_{J21} & D_{J22} & D_{J23} \\ C_{J3} & D_{J31} & D_{J32} & 0 \end{array}\right) \begin{pmatrix} x(t_k^-) \\ \hline w_J(k) \\ d_J(k) \\ u_J(k) \end{pmatrix}, \quad w_J(k) = \Delta_J(k) z_J(k) \tag{5.15b}$$

for $t \geq 0$ and $k \in \mathbb{N}$. As earlier in (5.1), the signals $y$, $y_J$ denote measured outputs, $u$, $u_J$ are control inputs and $w$, $w_J$, $z$, $z_J$ are interconnection variables; moreover, we include again a performance channel with generalized disturbances $d$, $d_J$ and error signals $e$, $e_J$.
The involved (arbitrarily) time-varying uncertainties $\Delta$ and $\Delta_J$ are assumed to be piecewise continuous maps satisfying

$$\Delta(t) \in \mathbf{\Delta} \quad \text{for all} \quad t \geq 0 \qquad \text{and} \qquad \Delta_J(k) \in \mathbf{\Delta}_J \quad \text{for all} \quad k \in \mathbb{N}.$$

In this section we aim to design a *robust* output-feedback controller for the system (5.15) such that the corresponding closed-loop energy gain is as small as possible for all admissible uncertainties; we refer to the worst of these gains as *robust* energy gain. We target at controllers of the form

$$\begin{aligned}\begin{pmatrix}\dot{x}_c(t)\\ u(t)\end{pmatrix} &= \begin{pmatrix}A^c(\theta(t)) & B^c(\theta(t))\\ C^c(\theta(t)) & D^c(\theta(t))\end{pmatrix}\begin{pmatrix}x_c(t)\\ y(t)\end{pmatrix},\\ \begin{pmatrix}x_c(t_k)\\ u_J(k)\end{pmatrix} &= \begin{pmatrix}A_J^c(\theta(t_k^-)) & B_J^c(\theta(t_k^-))\\ C_J^c(\theta(t_k^-)) & D_J^c(\theta(t_k^-))\end{pmatrix}\begin{pmatrix}x_c(t_k^-)\\ y_J(k)\end{pmatrix},\end{aligned} \tag{5.16}$$

for $t \geq 0$ and $k \in \mathbb{N}$ with initial condition $x_c(0) \in \mathbb{R}^n$ and with to-be-designed continuous describing matrices $A^c$, $B^c$, etc. The interconnection of the uncertain system (5.15) with the controller (5.16) is of the form

$$\begin{aligned}\begin{pmatrix}\dot{x}_{cl}(t)\\ z(t)\\ e(t)\end{pmatrix} &= \begin{pmatrix}\mathcal{A}(\theta(t)) & \mathcal{B}(\theta(t)) & \mathcal{B}(\theta(t))\\ \mathcal{C}(\theta(t)) & \mathcal{D}(\theta(t)) & \mathcal{D}_{12}(\theta(t))\\ \mathcal{C}_2(\theta(t)) & \mathcal{D}_{21}(\theta(t)) & \mathcal{D}_{22}(\theta(t))\end{pmatrix}\begin{pmatrix}x_{cl}(t)\\ w(t)\\ d(t)\end{pmatrix},\\ \begin{pmatrix}x_{cl}(t_k)\\ z_J(k)\\ e_J(k)\end{pmatrix} &= \begin{pmatrix}\mathcal{A}_J(\theta(t_k^-)) & \mathcal{B}_J(\theta(t_k^-)) & \mathcal{B}_{J2}(\theta(t_k^-))\\ \mathcal{C}_J(\theta(t_k^-)) & \mathcal{D}_J(\theta(t_k^-)) & \mathcal{D}_{J12}(\theta(t_k^-))\\ \mathcal{C}_{J2}(\theta(t_k^-)) & \mathcal{D}_{J21}(\theta(t_k^-)) & \mathcal{D}_{J22}(\theta(t_k^-))\end{pmatrix}\begin{pmatrix}x_{cl}(t_k^-)\\ w_J(k)\\ d_J(k)\end{pmatrix},\\ w(t) &= \Delta(t)z(t), \quad w_J(k) = \Delta_J(k)z_J(k)\end{aligned} \tag{5.17}$$

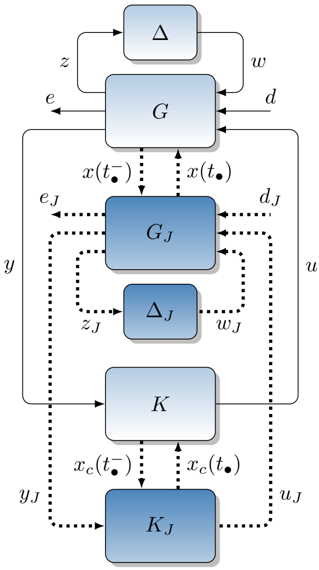



Figure 5.4: Block diagram of the closed-loop interconnection (5.17) of the uncertain system (5.15) with the robust controller (5.16).

with state $x_{cl} := \mathrm{col}(x, x_c)$,

$$\left(\begin{array}{c|cc} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}\right) := \left(\begin{array}{cc|cc} A & 0 & B & B_2 \\ 0 & 0 & 0 & 0 \\ \hline C & 0 & D & D_{12} \\ C_2 & 0 & D_{21} & D_{22} \end{array}\right) + \left(\begin{array}{cc} 0 & B_3 \\ I_{n_c} & 0 \\ \hline 0 & D_{13} \\ 0 & D_{23} \end{array}\right) \begin{pmatrix} A^c & B^c \\ C^c & D^c \end{pmatrix} \left(\begin{array}{cc|cc} 0 & I_{n_c} & 0 & 0 \\ C_3 & 0 & D_{31} & D_{32} \end{array}\right)$$

and analogously defined maps $\mathcal{A}_J, \mathcal{B}_J$, etc. A block diagram of this closed-loop interconnection is depicted in Fig. 5.4 where $G$, $G_J$ $K$ and $K_J$ refer to the flow and jump component of (5.15) and the ones of (5.16), respectively.

The particular structure of the closed-loop interconnection (5.17) permits us to apply Theorem 4.2 for its robustness analysis. Because we intend to measure performance in terms of the system's robust energy gain, recall that this gain is bounded by $\gamma > 0$ if the closed-loop system achieves robust quadratic performance with index $(P_\gamma, P_{J\gamma}) := \left(\begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}\right)$ as introduced in Definition 4.1.

In the sequel, all of our analysis and design criteria require only one of the relaxations in Appendix D to render them finite dimensional and thus numerically tractable. To this end, we suppose that we are given suitable multiplier sets $\mathbf{P}(\boldsymbol{\Delta})$ and $\mathbf{P}(\boldsymbol{\Delta}_J)$ as explained in Definition C.18 and corresponding to $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_J$. This leads to the following result.

**Corollary 5.7** (Closed-Loop Robust Analysis) *The interconnection* (5.17) *is robustly stable and achieves robust quadratic performance with index* $(P_\gamma, P_{J\gamma})$ *for all* $(t_k)_{k\in\mathbb{N}_0}$ *with* (RDT) *if there exist* $\mathcal{X} \in C^1([0, T_{\max}], \mathbb{S}^{2n})$, $P \in C([0, T_{\max}], \mathbf{P}(\boldsymbol{\Delta}))$ *and* $P_J \in C([T_{\min}, T_{\max}], \mathbf{P}(\boldsymbol{\Delta}_J))$ *satisfying the inequalities*

$$\mathcal{X} \succ 0, \tag{5.18a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c:c} 0 & \mathcal{X} & & \\ \mathcal{X} & \dot{\mathcal{X}} & & \\ \hline & & P & \\ \hdashline & & & P_\gamma \end{array}\right) \left(\begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ I & 0 & 0 \\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ 0 & I & 0 \\ \hdashline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{5.18b}$$

*and*

$$(\bullet)^\top \left(\begin{array}{cc|c:c} \mathcal{X}(0) & 0 & & \\ 0 & -\mathcal{X} & & \\ \hline & & P_J & \\ \hdashline & & & P_\gamma \end{array}\right) \left(\begin{array}{ccc} \mathcal{A}_J & \mathcal{B}_J & \mathcal{B}_{J2} \\ I & 0 & 0 \\ \hline \mathcal{C}_J & \mathcal{D}_J & \mathcal{D}_{J12} \\ 0 & I & 0 \\ \hdashline \mathcal{C}_{J2} & \mathcal{D}_{J21} & \mathcal{D}_{J22} \\ 0 & 0 & I \end{array}\right) \prec 0 \tag{5.18c}$$

*on* $[0, T_{\max}]$*,* $[0, T_{\max}]$ *and* $[T_{\min}, T_{\max}]$*, respectively. We denote by* $\gamma_{\mathrm{opt}}$ *the infimal* $\gamma > 0$ *such that there exists a robust controller* (5.16) *that renders the closed-loop analysis inequalities* (5.18) *feasible.*

Note that, as a consequence of the underlying conservatism in our analysis result, $\gamma_{\mathrm{opt}}$ is in general *not* the optimal robust energy gain achievable by robust controllers with description (5.16), but it often constitutes a good upper bound.

In order to slightly simplify the exposition, we proceed under the following assumption.

**Assumption 5.8** (Additional Properties of Multiplier Sets) *We assume that any multiplier* $P \in \mathbf{P}(\mathbf{\Delta})$ *is nonsingular and satisfies* $(\bullet)^\top P \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right) \prec 0$ *as well as that the* dual *multiplier set* $\tilde{\mathbf{P}}(\mathbf{\Delta}) := \{\tilde{P} \ : \ \tilde{P}^{-1} \in \mathbf{P}(\mathbf{\Delta})\}$ *admits an LMI representation. Moreover, we suppose that the multiplier set* $\mathbf{P}(\mathbf{\Delta}_J)$ *has analogous properties.*

Note that these assumptions are satisfied, e.g., for the concrete examples of multiplier sets given in Remark C.19. Moreover, note that the dual multiplier set of

$$\mathbf{P}(\mathbf{\Delta}) := \left\{ \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix}^\top \begin{pmatrix} 0 & H^\top \\ H & 0 \end{pmatrix} \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix} \;\middle|\; H + H^\top \succ 0 \right\}$$

for $\mathbf{\Delta} := \{\delta I \ : \ \delta \in [a, b]\}$ is given by

$$\tilde{\mathbf{P}}(\mathbf{\Delta}) := \left\{ \frac{1}{(b-a)^2} \begin{pmatrix} I & I \\ aI & bI \end{pmatrix} \begin{pmatrix} 0 & H \\ H^\top & 0 \end{pmatrix} \begin{pmatrix} I & I \\ aI & bI \end{pmatrix}^\top \;\middle|\; H + H^\top \succ 0 \right\}.$$

Since the origin is contained in $\mathbf{\Delta}$ and $\mathbf{\Delta}_J$ and by Assumption 5.8, we can apply the elimination lemma C.11 in all of the upcoming design scenarios. Otherwise, we would have to employ suitable perturbations ensuring nonsingularity and to explicitly enforce in some spots that the multipliers (in a pointwise fashion) have the correct amount of positive and negative eigenvalues. Note that the latter property is assured automatically at the most relevant spots.

In particular, we obtain the following intermediate non-convex design result by applying the elimination lemma C.11 on the analysis inequalities (5.18). The proof follows the lines of the one of Theorem 3.8 for nominal design via elimination even if the involved multipliers $P$ and $P_J$ not restricted to be constant matrices. Indeed, by $0 \in \mathbf{\Delta}$ and $0 \in \mathbf{\Delta}_J$, we can still apply the involved Schur complement argument and employ Lemma C.13 in order to construct describing maps of the controller (5.16) that are assured to be continuous.

**Theorem 5.9** (Robust Output-Feedback Controller Synthesis) *Let $U$, $V$, $U_J$ and $V_J$ be basis matrices of* $\ker((B_3^\top, D_{13}^\top, D_{23}^\top))$, $\ker((C_3, D_{31}, D_{32}))$, $\ker((B_{J3}^\top, D_{J13}^\top, D_{J23}^\top))$ *and* $\ker((C_{J3}, D_{J31}, D_{J32}))$, *respectively. Then there exists a controller* (5.16) *for the system* (5.15) *such that the analysis inequalities* (5.18) *are feasible if and only if there exist functions $X, Y \in C^1([0, T_{\max}], \mathbb{S}^n)$, $P \in C([0, T_{\max}], \mathbf{P}(\mathbf{\Delta}))$ and $P_J \in C([T_{\min}, T_{\max}], \mathbf{P}(\mathbf{\Delta}_J))$ satisfying the inequalities*

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \tag{5.19a}$$

$$(\bullet)^\top \left(\begin{array}{cc|c:c} 0 & X & & \\ X & \dot{X} & & \\ \hline & & P & \\ \hdashline & & & P_\gamma \end{array}\right) \left(\begin{array}{ccc} A & B & B_2 \\ I & 0 & 0 \\ \hline C & D & D_{12} \\ 0 & I & 0 \\ \hdashline C_2 & D_{21} & D_{22} \\ 0 & 0 & I \end{array}\right) V \prec 0, \tag{5.19b}$$

$$(\bullet)^\top \left(\begin{array}{cc|c:c} \dot{Y} & Y & & \\ Y & 0 & & \\ \hline & & \tilde{P} & \\ \hdashline & & & P_\gamma^{-1} \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ -A^\top & -C^\top & -C_2^\top \\ \hline 0 & I & 0 \\ -B^\top & -D^\top & -D_{21}^\top \\ \hdashline 0 & 0 & I \\ -B_2^\top & -D_{12}^\top & -D_{22}^\top \end{array}\right) U \succ 0 \tag{5.19c}$$

*on* $[0, T_{\max}]$ *as well as*

$$(\bullet)^\top \left(\begin{array}{cc|c:c} X(0) & 0 & & \\ 0 & -X & & \\ \hline & & P_J & \\ \hline & & & P_{J\gamma} \end{array}\right) \left(\begin{array}{ccc} A_J & B_J & B_{J2} \\ I & 0 & 0 \\ \hline C_J & D_J & D_{J12} \\ 0 & I & 0 \\ \hdashline C_{J2} & D_{J21} & D_{J22} \\ 0 & 0 & I \end{array}\right) V_J \prec 0, \tag{5.19d}$$

$$(\bullet)^\top \left(\begin{array}{cc|c:c} Y(0) & 0 & & \\ 0 & -Y & & \\ \hline & & \tilde{P}_J & \\ \hdashline & & & P_{J\gamma}^{-1} \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ -A_J^\top & -C_J^\top & -C_{J2}^\top \\ \hline 0 & I & 0 \\ -B_J^\top & -D_J^\top & -D_{J21}^\top \\ \hdashline 0 & 0 & I \\ -B_{J2}^\top & -D_{J12}^\top & -D_{J21}^\top \end{array}\right) U_J \succ 0 \tag{5.19e}$$

*on* $[T_{\min}, T_{\max}]$ *where*

$$\tilde{P} := P^{-1} \quad \textit{and} \quad \tilde{P}_J := P_J^{-1}. \tag{5.19f}$$

*Moreover,* $\gamma_{\text{opt}}$ *is equal to the infimal* $\gamma > 0$ *such that the above inequalities are feasible.*

In contrast to static output-feedback design as considered in Section 3.2, non-convexity emerges here through the multipliers $P$, $P_J$, $\tilde{P}$, $\tilde{P}_J$ and the coupling (5.19f) instead of the Lyapunov certificate $X$ and its inverse. Due to this non-convexity, computing $\gamma_{\mathrm{opt}}$ or a corresponding controller is difficult in general. Subsequently, we modify the dual iteration in order to compute upper bounds on $\gamma_{\mathrm{opt}}$ and, in particular, solve the robust output-feedback design problem for uncertain impulsive systems.

## 5.2.2 Dual Iteration

### Initialization of the Iteration

In order to initialize the dual iteration, we aim again to compute a meaningful lower bound on $\gamma_{\mathrm{opt}}$. Such a bound can be obtained by considering the design of a gain-scheduled controller as in Section 5.1. Indeed, if there exists a robust controller for the system (5.15) achieving a robust energy gain of $\gamma$, then there also exists a gain-scheduled controller (5.3) which achieves (at least) the same robust energy gain. This just follows from the observation that the robust controller (5.16) can be viewed as a gain-scheduled controller (5.3) with trivial scheduling functions $S = 0$ and $S_J = 0$.

Recall that the problem of finding a gain-scheduling controller (5.3) for the system (5.1) can be rendered convex with design criteria as given in Theorem 5.2. For our purposes we only need the following.

**Corollary 5.10** (Gain-Scheduled Design Criteria) *Suppose there exists a robust controller* (5.16) *such that the analysis inequalities* (5.18) *are feasible, then there exist continuously differentiable $X$, $Y$ as well as $P \in C([0,T_{\max}],\mathbf{P}(\mathbf{\Delta}))$, $P_J \in C([T_{\min},T_{\max}],\mathbf{P}(\mathbf{\Delta}_J))$, $\tilde{P} \in C([0,T_{\max}],\tilde{\mathbf{P}}(\mathbf{\Delta}))$ and $\tilde{P}_J \in C([T_{\min},T_{\max}],\tilde{\mathbf{P}}(\mathbf{\Delta}_J))$ satisfying the synthesis LMIs* (5.19a) - (5.19e). *Moreover, we have $\gamma_{\mathrm{gs}} \leq \gamma_{\mathrm{opt}}$ with $\gamma_{\mathrm{gs}}$ being the infimal $\gamma > 0$ such that the latter LMIs are feasible.*

Such lower bounds can be good indicators for measuring the conservatism of algorithms that generate upper bounds on some value of interest.

As for the static design in Section 3.2, the dual iteration is initialized by the design of a suitable full-information controller. For robust synthesis, such a controller admits the description

$$u(t) = F(\theta(t))\tilde{y}(t), \quad u_J(k) = F_J(\theta(t_k^-))\tilde{y}_J(k)$$

for $t \geq 0$, $k \in \mathbb{N}$. Here, the gains $F = (F_1, F_2, F_3)$ and $F_J = (F_{J1}, F_{J2}, F_{J3})$ are continuous matrix-valued maps and the input signals are given by $\tilde{y} := \mathrm{col}(x, w, d)$ and $\tilde{y}_J(k) := \mathrm{col}(x(t_k^-), w_J(k), d_J(k))$. Thus, these controllers are even able to measure both uncertain signals $w = \Delta z$ and $w_J = \Delta_J z_J$ in addition to the state $x$ and the generalized disturbances $d$, $d_J$. By replacing the measurements $y, y_J$ in (5.15) with the virtual measurements $\tilde{y}, \tilde{y}_J$, we can interconnect this controller with the system (5.15). This results in a closed-loop interconnection of the form (5.17), but with the maps

$$\mathcal{G} := \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B}_2 \\ \mathcal{C} & \mathcal{D} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{G}_J := \begin{pmatrix} \mathcal{A}_J & \mathcal{B}_J & \mathcal{B}_{J2} \\ \mathcal{C}_J & \mathcal{D}_J & \mathcal{D}_{J12} \\ \mathcal{C}_{J2} & \mathcal{D}_{J21} & \mathcal{D}_{J22} \end{pmatrix}$$

in the flow and jump component replaced by

$$G_F := \begin{pmatrix} A_F & B_F & B_{F2} \\ C_F & D_F & D_{F12} \\ C_{F2} & D_{F21} & D_{F22} \end{pmatrix} := \underbrace{\begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix}}_{=:G} + \begin{pmatrix} B_3 \\ D_{13} \\ D_{23} \end{pmatrix} F$$

and

$$G_{JF} := \begin{pmatrix} A_{JF} & B_{JF} & B_{JF2} \\ C_{JF} & D_{JF} & D_{JF12} \\ C_{JF2} & D_{JF21} & D_{JF22} \end{pmatrix} := \underbrace{\begin{pmatrix} A_J & B_J & B_{J2} \\ C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & D_{J22} \end{pmatrix}}_{=:G_J} + \begin{pmatrix} B_{J3} \\ D_{J13} \\ D_{J23} \end{pmatrix} F_J,$$

respectively; the abbreviations $\mathcal{G}$, $\mathcal{G}_J$, etc. are introduced here to save a lot of space in the upcoming results. By Assumption 5.8, the elimination lemma C.11 yields the following design criteria for full-information controllers as described above.

**Lemma 5.11** (Full-Information Controller Synthesis) *There exist some full-information gains $F$ and $F_J$ such that the closed-loop analysis LMIs* (5.18) *with the $(\mathcal{G}, \mathcal{G}_J)$ replaced by $(G_F, G_{JF})$ are feasible if and only if there exist functions $Y \in C^1([0, T_{\max}], \mathbb{S}^n)$, $\tilde{P} \in C([0, T_{\max}], \tilde{\mathbf{P}}(\boldsymbol{\Delta}))$ and $\tilde{P}_J \in C([T_{\min}, T_{\max}], \tilde{\mathbf{P}}(\boldsymbol{\Delta}_J))$ satisfying $Y \succ 0$ on $[0, T_{\max}]$,* (5.19c) *and* (5.19e).

## Main Loop

Once we have synthesized suitable initial full-information gains $F$, $F_J$ via Lemma 5.11, we can advance to the main loop of the dual iteration that begins with the following.

**Theorem 5.12** (Primal Design Result) *There exists a controller* (5.16) *for the system* (5.15) *such that the analysis LMIs* (5.18) *are feasible for the corresponding closed-loop system if there exist $X, Y \in C^1([0, T_{\max}], \mathbb{S}^n)$ as well as $P \in C([0, T_{\max}], \mathbf{P}(\boldsymbol{\Delta}))$, $P_J \in C([T_{\min}, T_{\max}], \mathbf{P}(\boldsymbol{\Delta}_J))$ satisfying*

$$\begin{pmatrix} X & Y \\ Y & Y \end{pmatrix} \succ 0, \quad (5.19\text{b}) \quad \textit{and} \quad (5.19\text{b}) \textit{ with } (X, G, V) \textit{ replaced by } (Y, G_F, I) \tag{5.20a,b,c}$$

*on $[0, T_{\max}]$ as well as*

$$(5.19\text{d}) \quad \textit{and} \quad (5.19\text{d}) \textit{ with } (X, G_J, V_J) \textit{ replaced by } (Y, G_{JF}, I) \tag{5.20d,e}$$

*on $[T_{\min}, T_{\max}]$. Moreover, we have $\gamma_{\mathrm{gs}} \leq \gamma_{\mathrm{opt}} \leq \gamma_F$ for $\gamma_F$ being the infimal $\gamma > 0$ such that the above LMIs are feasible.*

*Proof.* By strictness of the LMIs (5.20), continuity of $X, \dot{X}, P$ and $P_J$, and compactness of the intervals $[0, T_{\max}]$ and $[T_{\min}, T_{\max}]$, we can infer that the LMIs (5.20) remain satisfied if we replace $(P, P_J)$ by $(P_\varepsilon, P_{J\varepsilon}) := \left(P + \varepsilon \left(\begin{smallmatrix} I_p & 0 \\ 0 & 0 \end{smallmatrix}\right), P_J + \varepsilon \left(\begin{smallmatrix} I_{p_J} & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$ for some small $\varepsilon > 0$. Note that we then have the strict inequality

$$(\bullet)^\top P_\varepsilon \begin{pmatrix} I \\ \Delta \end{pmatrix} = (\bullet)^\top P \begin{pmatrix} I \\ \Delta \end{pmatrix} + \varepsilon I \succ 0 \quad \text{for all} \quad \Delta \in \mathbf{\Delta}$$

and similarly $(\bullet)^\top P_{J\varepsilon} \left(\begin{smallmatrix} I \\ \Delta_J \end{smallmatrix}\right) \succ 0$ for all $\Delta_J \in \mathbf{\Delta}_J$. In particular, we still have $P_\varepsilon \in C([0, T_{\max}], \mathbf{P}(\mathbf{\Delta}))$ and $P_{J\varepsilon} \in C([T_{\min}, T_{\max}], \mathbf{P}(\mathbf{\Delta}_J))$. Next observe that the $(2, 2)$ blocks of (5.20c) and (5.20e) then imply $(\bullet)^\top P_\varepsilon \left(\begin{smallmatrix} D_F \\ I_q \end{smallmatrix}\right) \prec 0$ on $[0, T_{\max}]$ and $(\bullet)^\top P_{J\varepsilon} \left(\begin{smallmatrix} D_{JF} \\ I_{q_J} \end{smallmatrix}\right) \prec 0$ on $[T_{\min}, T_{\max}]$, respectively. By Corollary C.10, we can then conclude that $P_\varepsilon$ has pointwise exactly $p$ positive and $q$ negative eigenvalues; we obtain analogously that $P_{J\varepsilon}$ has pointwise exactly $p_J$ positive and $q_J$ negative eigenvalues.

This permits us to eliminate the full-information gains $F$ and $F_J$ from the LMIs (5.20c) and (5.20e) which leads to (5.19c) and (5.19e) for $(Y, \tilde{P}, \tilde{P}_J)$ replaced by $(Y^{-1}, P_\varepsilon^{-1}, P_{J\varepsilon}^{-1})$. Finally, performing a congruence transformation of (5.20a) with $\operatorname{diag}(I, Y^{-1})$ yields (5.19a) for $Y$ replaced by $Y^{-1}$. Since we still have (5.19b), (5.19d), $P_\varepsilon \in C([0, T_{\max}], \mathbf{P}(\mathbf{\Delta}))$ and $P_{J\varepsilon} \in C([T_{\min}, T_{\max}], \mathbf{P}(\mathbf{\Delta}_J))$, we can apply Theorem 5.9 in order to construct the desired robust controller (5.16). ●

The employed dual versions of Lemma 5.11 and Theorem 5.9 are given next. They involve *full-actuation* gains $E = (E_1^\top, E_2^\top, E_3^\top)^\top$ and $E_J = (E_{J1}^\top, E_{J2}^\top, E_{J3}^\top)^\top$ as well as the maps

$$G_E := \begin{pmatrix} A_E & B_E & B_{E2} \\ C_E & D_E & D_{E12} \\ C_{E2} & D_{E21} & D_{E22} \end{pmatrix} := \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} + E \begin{pmatrix} C_3 & D_{31} & D_{31} \end{pmatrix}$$

and

$$G_{JE} := \begin{pmatrix} A_{JE} & B_{JE} & B_{JE2} \\ C_{JE} & D_{JE} & D_{JE12} \\ C_{JE2} & D_{JE21} & D_{JE22} \end{pmatrix} := \begin{pmatrix} A_J & B_J & B_{J2} \\ C_J & D_J & D_{J12} \\ C_{J2} & D_{J21} & D_{J22} \end{pmatrix} + E_J \begin{pmatrix} C_{J3} & D_{J31} & D_{J32} \end{pmatrix}.$$

The elimination lemma C.11 yields the following two results.

**Lemma 5.13** (Full-Actuation Controller Synthesis) *There exists some full-actuation gains $E$ and $E_J$ such that the closed-loop analysis LMIs* (5.18) *with $(\mathcal{G}, \mathcal{G}_J)$ replaced by $(G_E, G_{JE})$ are feasible if and only if there exist maps $X \in C^1([0, T_{\max}], \mathbb{S}^n)$, $P_J \in C([T_{\min}, T_{\max}], \mathbf{P}(\boldsymbol{\Delta}_J))$ and $P \in C([0, T_{\max}], \mathbf{P}(\boldsymbol{\Delta}))$ satisfying $X \succ 0$ on $[0, T_{\max}]$,* (5.19b) *and* (5.19d)*.*

**Theorem 5.14** (Dual Design Result) *There exists a controller* (5.16) *for the system* (5.15) *such that the analysis LMIs* (5.18) *are feasible for the corresponding closed-loop system if there exist $X, Y \in C^1([0, T_{\max}], \mathbb{S}^n)$ as well as $\tilde{P} \in C([0, T_{\max}], \tilde{\mathbf{P}}(\boldsymbol{\Delta}))$, $\tilde{P}_J \in C([T_{\min}, T_{\max}], \tilde{\mathbf{P}}(\boldsymbol{\Delta}_J))$ satisfying*

$$\begin{pmatrix} X & X \\ X & Y \end{pmatrix} \succ 0, \quad (5.19\text{c}) \quad \textit{and} \quad (5.19\text{c}) \textit{ with } (Y, G, U) \textit{ replaced by } (X, G_E, I) \tag{5.21a,b,c}$$

*on $[0, T_{\max}]$ as well as*

$$(5.19\text{e}) \quad \textit{and} \quad (5.19\text{e}) \textit{ with } (Y, G_J, U_J) \textit{ replaced by } (X, G_{JE}, I) \tag{5.21d,e}$$

*on $[T_{\min}, T_{\max}]$. Moreover, we have $\gamma_{\mathrm{gs}} \leq \gamma_{\mathrm{opt}} \leq \gamma_E$ for $\gamma_E$ being the infimal $\gamma > 0$ such that the above LMIs are feasible.*

Theorems 5.12 and 5.14 are again nicely intertwined, in analogy of what has been stated in Theorem 3.26 and illustrated in Fig. 3.10 in Section 3.2. In particular, the following conceptual algorithm generates a monotonically decreasing sequence $(\gamma^k)_{k \in \mathbb{N}}$ of upper bounds on $\gamma_{\mathrm{opt}}$ and we can essentially make the same statements as in Remark 3.28. As pointed out at the end of Section 3.2, we face again some compatibility issues with the employed DLMI relaxation for numerically solving the involved synthesis inequalities; we already proposed several initial suggestions, but leave a dedicated investigation for future research.

**Algorithm 5.15** (Dual Iteration for Robust Output-Feedback Design.)

(a) *Initialization:* Compute the lower bound $\gamma_{\mathrm{gs}}$ based on solving the gain-scheduling synthesis LMIs in Theorem 5.10 and set $\gamma^0 := +\infty$ as well as $k = 1$. Design initial full-information gains $F$ and $F_J$ from Lemma 5.11.

(b) *Primal step:* Compute $\gamma_F$ based on solving the primal synthesis LMIs (5.20) for the given gains $F$ and $F_J$ and choose some small $\varepsilon_k > 0$ such that $\gamma^k := \gamma_F(1 + \varepsilon_k) < \gamma^{k-1}$. For $\gamma = \gamma^k$, determine $X, Y$ and $P, P_J$ satisfying the LMIs (5.20) and apply Lemma 5.13 in order to design gains $E$ and $E_J$ satisfying the dual synthesis LMIs (5.21) for $(X, Y, \tilde{P}, \tilde{P}_J) = (X^{-1}, Y^{-1}, P^{-1}, P_J^{-1})$.

(c) *Dual step:* Compute $\gamma_E$ based on solving the dual synthesis LMIs (5.21) for the given gains $E$ and $E_J$ and choose some small $\varepsilon_{k+1} > 0$ such that $\gamma^{k+1} := \gamma_E(1 + \varepsilon_{k+1}) < \gamma^k$. For $\gamma = \gamma^{k+1}$, determine $X, Y$ and $\tilde{P}, \tilde{P}_J$ satisfying the LMIs (5.21) and apply Lemma 5.11 in order to design gains $F$ and $F_J$ satisfying the primal synthesis LMIs (5.20) for $(X, Y, P, P_J) = (X^{-1}, Y^{-1}, \tilde{P}^{-1}, \tilde{P}_J^{-1})$.

(d) *Termination:* If $k$ is too large or $\gamma^k$ does not decrease any more, then stop and construct a robust output-feedback controller (5.16) for the system (5.15) according to Theorem 5.14.
Otherwise set $k = k + 2$ and go to the primal step.

**Remark 5.16** (Robust Gain-Scheduled Output-Feedback Design) It is not difficult to extend Algorithm 5.15 to the more general and practically relevant design of robust gain-scheduling controllers as considered, e.g., in [162, 71]. For this problem, the uncertainties $\Delta$ and $\Delta_J$ in the description (5.15) are replaced by $\mathrm{diag}(\Delta_u, \Delta_s)$ and $\mathrm{diag}(\Delta_{Ju}, \Delta_{Js})$, respectively, with $\Delta_u$, $\Delta_{Ju}$ being unknown, while $\Delta_s$, $\Delta_{Js}$ are measurable on-line and taken into account by the to-be-designed controller. As for robust design, this synthesis problem is known to be convex only in very specific situations; for example if the control channel is unaffected by uncertainties [162].

An interesting special case of the general robust gain-scheduling design is sometimes referred to as inexact scheduling [133]. As for standard gain-scheduling it is assumed that a parameter dependent system (5.15) is given, but that the to-be-designed controller only receives noisy on-line measurements of the parameter instead of exact ones.

We emphasize that such modifications are all straightforward to handle, due to the flexibility of the design framework based on linear fractional representations and the employed multiplier separation techniques underlying Corollary 5.7. In a nutshell, these modifications amount to adding a scheduling channel to both of the components of the underlying system (5.15) and diagonally augmenting all synthesis LMIs with multipliers corresponding to the scheduling component. The augmentation is essentially the same as when moving from nominal controller design to robust synthesis.

### Dual Iteration: An Alternative Initialization

It can happen that the LMIs appearing in the primal step of algorithm 5.15 are infeasible for the initially designed full-information gains. In order to promote the feasibility of these LMIs, we propose an alternative initialization that relies on the following result.

**Lemma 5.17** *Suppose that the gain-scheduling synthesis LMIs in Theorem 5.10 are feasible and that some full-actuation gains $E$ and $E_J$ are designed from Lemma 5.13. Then there exist some $\alpha > 0$, continuously differentiable $X$, $Y$ as well as $P \in C([0, T_{\max}], \mathbf{P}(\mathbf{\Delta}))$, $\tilde{P} \in C([0, T_{\max}], \tilde{\mathbf{P}}(\mathbf{\Delta}))$, $P_J \in C([T_{\min}, T_{\max}], \mathbf{P}(\mathbf{\Delta}_J))$ and $\tilde{P}_J \in C([T_{\min}, T_{\max}], \tilde{\mathbf{P}}(\mathbf{\Delta}_J))$ satisfying the LMIs* (5.21) *with $\tilde{P}$ in* (5.21b) *replaced by $P$, with $\tilde{P}_J$ in* (5.21d) *replaced by $P_J$,*

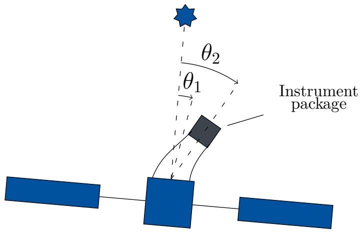
$$\begin{pmatrix} \alpha I & P - \tilde{P} \\ P - \tilde{P} & I \end{pmatrix} \succ 0 \quad \textit{and} \quad \begin{pmatrix} \alpha I & P_J - \tilde{P}_J \\ P_J - \tilde{P}_J & I \end{pmatrix} \succ 0 \tag{5.22}$$

Note that, with a Schur complement argument, (5.22) is equivalent to $\|P - \tilde{P}\|^2 < \alpha$ and $\|P_J - \tilde{P}_J\|^2 < \alpha$. Thus by minimizing $\alpha > 0$ subject to the above LMIs, we push the multipliers $(P, P_J)$ and $(\tilde{P}, \tilde{P}_J)$ as close together as possible. Due to the continuity of the map $M \mapsto M^{-1}$, this means that their inverses are close to each other as well. We can then design corresponding full-information gains $F$ and $F_J$ based on Lemma 5.11 for which the LMIs (5.20) are very likely to be feasible for $P^{-1} \approx \tilde{P}^{-1}$ and $P_J^{-1} \approx \tilde{P}_J^{-1}$.

**Remark 5.18** (a) In the case that the above procedure does not yield gains $F$ and $F_J$ for which the LMIs (5.20) are feasible, one can, e.g., iteratively double $\gamma$ and retry until a suitable gain is found. This practical approach works typically well in various situations.

(b) It would be nicer to directly employ additional constraints for the gain-scheduling synthesis LMIs in Theorem 5.10 which promote $P \approx \tilde{P}^{-1}$ and $P_J \approx \tilde{P}_J^{-1}$ and, thus, the feasibility of the primal synthesis LMIs (5.20) similarly as it was possible for static design in Remark 3.29. However, as far as we are aware of, this is only possible for specific multipliers and corresponding value sets.

### 5.2.3 Example




As an illustration let us consider a slight modification of the flexible satellite as considered in Chapter 3 which originates from [53]. Recall that we employed the following model with state $\tilde{x} = \mathrm{col}(\theta_2, \dot{\theta}_2, \theta_1, \dot{\theta}_1)$ and with exactly known constants $J_1 = 1$, $J_2 = 0.1$, $k = 0.091$ and $b = 0.0036$:

$$\begin{pmatrix} \dot{\tilde{x}}(t) \\ \hline v(t) \end{pmatrix} = \left(\begin{array}{cccc|c:c} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k}{J_2} & -\frac{b}{J_2} & \frac{k}{J_2} & \frac{b}{J_2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{J_1} & \frac{b}{J_1} & -\frac{k}{J_1} & -\frac{b}{J_1} & 0 & \frac{1}{J_1} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} \tilde{x}(t) \\ \hline \tilde{d}(t) \\ u(t) \end{pmatrix}. \tag{5.23}$$

This time, we assume instead that the constants $J_1$ and $b$ are merely known to be contained in the intervals $[0.8, 1.2]$ and $[0.0018, 0.0072]$, respectively. The goal now is to design a dynamic output-feedback controller $K$ such that, *for any* possible value of $J_1$ and $b$, the closed-loop interconnection with (5.23) is stable, the output $v$ nicely follows a given piecewise constant reference signal $r$ despite the presence of a disturbance $\tilde{d}$, and such that the control input $u$ is not too large.

To this end, we consider essentially the same reference tracking configuration as before and as shown again in Fig. 5.5 for convenience. It involves the weights

$$W_r = 1, \quad W_d = 0.2, \quad W_u = 0.1 \quad \text{and} \quad W_{err}(s) = \frac{0.5s + 0.433}{s + 0.00433}$$

and $G(\Delta)$ denotes the uncertain system (5.23). By disconnecting the controller $K$ from this configuration and by pulling out the uncertainties as for

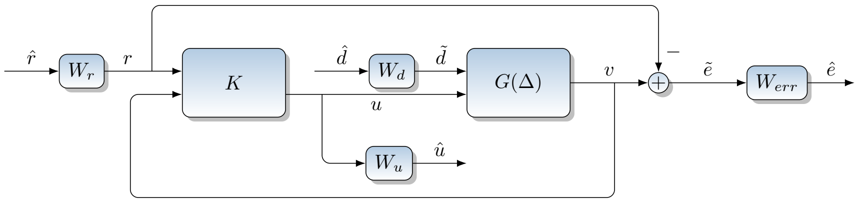



Figure 5.5: A standard weighted tracking configuration.

example explained in Section 9.2 of [178], we obtain a weighted open-loop system that fits into the description (5.15a) with the stacked signals

$$x := \begin{pmatrix} \tilde{x} \\ \xi_{W_{err}} \end{pmatrix}, \quad e := \begin{pmatrix} \hat{e} \\ \hat{u} \end{pmatrix}, \quad y := \begin{pmatrix} v \\ r \end{pmatrix}, \quad d := \begin{pmatrix} \hat{r} \\ \hat{d} \end{pmatrix},$$

where $\xi_{W_{err}}$ denotes the state corresponding to the weight $W_{err}$, with the static uncertainty

$$\Delta := \begin{pmatrix} J_1 & 0 \\ 0 & b \end{pmatrix} \in \mathbf{\Delta} := \left\{ \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \in \mathbb{R}^{2\times 2} \;\middle|\; \begin{array}{l} \delta_1 \in [0.8, 1.2] \text{ and} \\ \delta_2 \in [0.0018, 0.0072] \end{array} \right\},$$

and for some describing matrices $A, B$, etc. involving a vanishing matrix $D_{31}$. In Chapter 3 we have seen that once the system's output $v$ can only be measured at times $t_0, t_1, \ldots$ a standard (robust) $H_\infty$ design can easily lead to undesired closed-loop behavior. Here, we suppose that the sequence $(t_k)_{k\in\mathbb{N}_0}$ satisfying (EDT) with $T = 0.4$ and instead design a corresponding impulsive controller. To this end we express the uncertain open-loop system as an impulsive one of the form

$$\begin{pmatrix}\dot{x}(t)\\ z(t)\\ e(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}A & B & B_2 & B_3\\ C & D & D_{12} & D_{13}\\ C_2 & D_{21} & D_{22} & D_{23}\\ 0 & 0 & (\,1\;0\,) & 0\end{pmatrix}\begin{pmatrix}x(t)\\ w(t)\\ d(t)\\ u(t)\end{pmatrix}, \quad \begin{pmatrix}x(t_k)\\ z_J(k)\\ e_J(k)\\ y_J(k)\end{pmatrix} = \begin{pmatrix}I & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ C_3 & 0 & D_{32} & 0\end{pmatrix}\begin{pmatrix}x(t_k^-)\\ w_J(k)\\ d_J(k)\\ u_J(k)\end{pmatrix},$$

$$w(t) = \Delta z(t) \quad \text{and} \quad w_J = 0 \cdot z_J.$$

This description involves several redundant signals to emphasize that it constitutes a special case of the generic one in (5.15). In particular, we can apply the dual iteration as summarized in Algorithm 5.15 in order to determine an impulsive controller with the desired properties.

Before doing so let us demonstrate that neglecting the uncertainty $\Delta$ and designing a controller for the much simpler nominal system, i.e., for the unknown $\Delta$ replaced by $\left(\begin{smallmatrix}1 & 0\\ 0 & 0.0036\end{smallmatrix}\right)$, can lead to poor performance if the true parameters deviate from their guessed nominal values. In this case we can design a controller by applying Theorem 3.8 and, in particular, by means of solving a single convex optimization problem. Several simulations of the interconnection of this controller and the uncertain system (5.23) for some reference $r$, some small random disturbance $\tilde{d}$ and for several values of $\Delta \in \mathbf{\Delta}$ are shown at the top of Fig. 5.6; here the signals with subscript 'nom' denote the response of the nominal system. As expected, we observe that the tracking capabilities are fine for the nominal system, but deteriorate in the presence of uncertainties. In fact, if we were to increase the size of $\mathbf{\Delta}$ a bit, then there are uncertainties $\Delta \in \mathbf{\Delta}$ for which the closed-loop is not even stable anymore.

The same simulation is shown in the middle of Fig. 5.6, but, this time, the controller is designed via the dual iteration and explicitly takes the presence of uncertainties into account. We observe a much better tracking behavior and also a less aggressive control input for all of the samples $\Delta$.

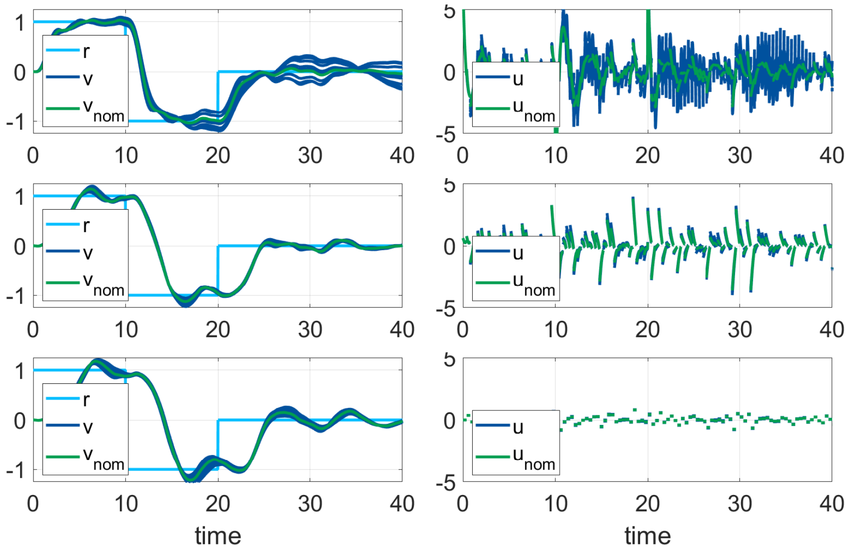



Figure 5.6: Some reference $r$ and closed-loop responses of the uncertain system (5.23) for several instances of $\Delta \in \boldsymbol{\Delta}$ with a controller designed for the nominal system (top) and two robust controller as obtained from the dual iteration (middle and bottom). The controller at the bottom has its output constrained to be piecewise constant.

Let us finally note that it is by now straightforward to include the additional constraint on the controller that its output is piecewise constant. Applying the dual iteration for this situation yields another robust impulsive controller and a simulation of the closed-loop response as depicted on the bottom of Fig. 5.6.

## 5.3 Summary

In the first part of this chapter, we consider the problem of convexifying the design of gain-scheduled output-feedback controllers for impulsive systems modeled in terms of LFRs and affected by on-line measurable parameters. We begin by considering arbitrarily time-varying parameters and derive Theorem 5.2, which constitutes an extension of the criteria in [139] from non-impulsive systems to impulsive ones. Afterwards, we consider piecewise constant parameters and establish Theorem 5.5 which relies on combining Theorem 4.6 and Lemma 4.9 for the underlying closed-loop robust analysis. In particular, it evolves around dynamic IQCs with impulsive filters admitting a state resetting property and D-G scalings along with a dedicated extension of those scalings. Theorem 5.5 has been published by the author in [83] and a related preliminary version involving D scalings is given in [81]. Related results that merely apply to non-impulsive systems are found in [145], but rely on an extension that is numerically much more intricate and susceptible to numerical errors.

In the second part of this chapter, we show how to employ the dual iteration for synthesizing robust output-feedback controllers for uncertain impulsive systems. This can be viewed as a generalization of the results in Section 3.2 on nominal static output-feedback design and relies on the use of static filters for the underlying closed-loop robustness analysis. We published the corresponding algorithm for non-impulsive uncertain systems in [85] since even this specialization is of tremendous practical relevance. Related (heuristic) approaches for uncertain impulsive systems usually rely on variations of the D-K iteration as, e.g., in [26], even though this method is known to be not very efficient. For non-impulsive systems there are efficient alternatives that completely avoid solving LMIs such as `hinfstruct` [9] or `hifoo` [27], but these merely apply to few classes of uncertainties and they are not amenable for generalizations to impulsive systems.

We also stress that the proposed variant of the dual iteration generalizes in a straightforward fashion to the interesting problem of synthesizing robust gain-scheduled output-feedback controllers. For non-impulsive systems, several specializations of this general problem that admit a convex solution are found, e.g., in [160].

# 6

# Conclusions

In this thesis we provide the essentials for a systematic analysis and design framework for linear impulsive and related hybrid systems with dwell-time constraints. Conveniently, this framework is in various ways analogous to the one for non-impulsive systems based on integral quadratic constraints (IQCs). The latter is capable to accurately handle such systems even in the presence of numerous and diverse uncertainties in an efficient fashion, and, therefore, is acknowledged by practitioners particularly from the aerospace industry. The most important ingredients of the proposed framework can be summarized as follows:

- Specifically tailored nominal analysis criteria as developed in [18] and as elaborated on in Chapter 2 which rely on Lyapunov arguments and the introduction of a clock to capture the impulse instants characterizing the considered hybrid systems. As illustrated in Chapter 3, the particular structure of these criteria permits us to design impulsive output-feedback controllers for impulsive open-loop systems in terms of convex optimization. This is achieved by carefully adjusting available design tools for non-impulsive systems and by employing suitable DLMI relaxations.

- A genuine generalization of the notion of finite-horizon IQCs with terminal cost as proposed in [148] for systematically analyzing uncertain non-impulsive systems which we introduce and discuss in Chapter 4. In particular, our time-domain formulation paves the way for extending various robust analysis results from the rich body of the IQC literature to uncertain impulsive systems.

- A convex solution to the gain-scheduled controller design problem for impulsive systems and an extension of the dual iteration for designing robust output-feedback controllers for uncertain impulsive systems in an iterative fashion as provided in Chapter 5. The former is expected to facilitate convexifying several of the related synthesis problems considered in [160] in the context of hybrid systems such as the design of robust gain-scheduled estimators.

We illustrate the flexibility of the proposed framework by demonstrating how to apply it, e.g., for switched, sampled-data and networked systems, and support most of the presented results by numerical examples.

Despite the title of this work, which is admittedly somewhat provocative since there is always room for improvements, there are several (technical) issues and challenges that are recommendable for future research. We highlight the ones that are expected to enhance the proposed framework the most.

- Most IQC based analysis and design criteria rely on numerically solving LMIs which is expensive or even prohibitive for large systems and if employing standard semidefinite programming solvers; this is the case for non-impulsive systems and is naturally even more delicate for impulsive ones since we are required to solve inherently larger LMIs resulting from DLMI relaxations. Hence, there is a strong need for numerically stable and fast LMI solvers that can be parallelized and which exploit the particular structure of optimization problems faced in control; currently, there are only few algorithms with the latter trait and most of them only apply to analysis problems. Moreover, there is a lack of suitable preconditioning techniques which can also help for dealing with large systems.

- As an alternative to the underlying clock based nominal analysis criteria, it could be interesting to analyze the feedback interconnection of linear systems with an impulsive component by means of suitable IQCs.

- As mentioned at the end of Section 3.2, we face some compatibility issues of the dual iteration with the employed DLMI relaxations. We have proposed several initial measures to produce relief, but we believe that a dedicated investigation would be fruitful.

- So far robust output-feedback design with the dual iteration relies on the use of static multipliers in the underlying IQC analysis criteria. It is well-known that dynamic multipliers can be much less conservative than static ones. Hence, incorporating those into the dual iteration is expected to be very beneficial.

# Appendix

# A

# Explanation of Symbols

## Basics and Matrices

Nonnegative integers are usually denoted by $i, j, k, l, m, n, p, q$ and $N$. Sometimes $\bullet$ is some unspecified but fixed nonnegative integer.

| | |
|---|---|
| $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ | Set of integers, of positive integers and of nonnegative integers. |
| $\mathbb{R}$, $\mathbb{C}$ | Set of real and complex numbers. |
| $\mathbb{K}$ | Stands for either $\mathbb{R}$ or $\mathbb{C}$. Its meaning does not change within theorems and other statements. |
| $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\overline{z}$ | Real part, imaginary part and complex conjugate of $z \in \mathbb{C}$. |
| $\mathbb{C}_\circ$, $\mathbb{C}_\circ^\infty$ | $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \circ 0\}$, $\mathbb{C}_\circ \cup \{\infty\}$ for $\circ \in \{<, \leq, =, \geq, >\}$. |
| $\mathbb{R}^n$, $\mathbb{C}^n$ | Vector space of real, complex $n$-tuples with standard Euclidian inner product $\langle \cdot, \cdot \rangle$, norm $\\| \cdot \\|$ and standard unit vectors $e_1, \dots, e_n$. |
| $\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$ | Vector space of real, complex $n \times m$ matrices with induced norm $\\|A\\| := \sup_{\\|x\\|=1} \\|Ax\\|$. |

| | |
|---|---|
| $\mathbb{S}^n$, $\mathbb{H}^n$ | Set of real symmetric and complex Hermitian $n \times n$ matrices. |

Let $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{p\times q}$ and $M, N \in \mathbb{C}^{n\times n}$.

| | |
|---|---|
| $I$ | Identity matrix. |
| $A^\top$, $A^*$ | Transpose and conjugate transpose of the matrix $A$. |
| $M^{-1}$ | Inverse of the square and nonsingular matrix $M$. |
| $\ker(A)$, $\mathrm{im}(A)$ | Kernel and image of the matrix $A$. |
| $\mathrm{trace}(M)$, $\det(M)$ | Trace and determinant of the matrix $M$. |
| $\mathrm{eig}(M)$ | Set of eigenvalues of the matrix $M$. |
| $M \succ N$ | $M = M^*$, $N = N^*$ and $M - N$ is positive definite. |
| $M \prec N$ | $M = M^*$, $N = N^*$ and $M - N$ is negative definite. |
| $M \succcurlyeq N$ | $M = M^*$, $N = N^*$ and $M - N$ is positive semidefinite. |
| $M \preccurlyeq N$ | $M = M^*$, $N = N^*$ and $M - N$ is negative semidefinite. |
| $A \otimes B$ | Kronecker product of $A$ and $B$ [see 87, P. 239 -287]. |
| $\mathrm{diag}(A_1, \ldots, A_N)$ | Block diagonal matrix with matrices $A_1, \ldots, A_N$ on its diagonal. |
| $\mathrm{col}(A_1, \ldots, A_N)$ | $:= (A_1^\top, \ldots, A_N^\top)^\top$ for matrices $A_1, \ldots, A_N$ with the same number of columns. |
| $\mathrm{He}(M)$ | $:= M + M^*$. |

## Function Spaces

| | |
|---|---|
| $L_{2e}^n$ | Set of locally square integrable functions from $[0, \infty)$ to $\mathbb{R}^n$. |
| $\ell_{2e}^n$ | Set of locally square summable sequences with elements in $\mathbb{R}^n$. |
| $L_2^n$ | $:= \left\{ x : [0,\infty) \to \mathbb{R}^n \;\middle|\; \lVert x \rVert := \left( \int_0^\infty x(t)^\top x(t)\, dt \right)^{\frac{1}{2}} < \infty \right\}$. |

| | |
|---|---|
| $\ell_2^n$ | $:= \left\{ (x_k)_{k\in\mathbb{N}} \;\middle|\; \|x\| := \left(\sum_{k=1}^{\infty} x(t)^\top x(t)\right)^{\frac{1}{2}} < \infty \right\}.$ |
| $\mathrm{RH}_\infty^{m\times n}, \mathrm{RL}_\infty^{m\times n}$ | Set of real rational proper $m \times n$ matrices without poles in the extended closed right half-plane (imaginary axis) equipped with the maximum norm $\|G\|_\infty = \max_{\omega\in\mathbb{R}\cup\{\infty\}} \|G(i\omega)\|$. |
| $C(X,Y)$, $C^1(X,Y)$ | Set of continous, continously differentiable functions from $X$ to $Y$ with normed spaces $X$ and $Y$. In this work $X$ is usually a closed interval and we consider one-sided derivatives at its boundaries. |

## Miscellaneous

- For a normed space $Y$, a function $f : [0,\infty) \to Y$ and $t > 0$ we let $f(t^-) := \lim_{s\nearrow t} f(s)$ denote the limit from below once it is well defined. For notational simplicity we further set $f(0^-) := f(0)$.

- Objects, that can be inferred by symmetry or are not relevant, are indicated by the symbol "$\bullet$". For example, we frequently abbreviate the expressions

$$A^\top M A \quad \text{and} \quad \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix} \qquad \text{as} \qquad (\bullet)^\top M A \quad \text{and} \quad \begin{pmatrix} A & B \\ (\bullet)^\top & D \end{pmatrix},$$

respectively.

B

# Abbreviations

| | |
|---|---|
| **ARE** | Algebraic Riccati Equation |
| **ARI** | Algebraic Riccati Inequality |
| **DLMI** | Differential Linear Matrix Inequality |
| **EDT** | Exact Dwell-Time |
| **FDI** | Frequency Domain Inequality |
| **IQC** | Integral Quadratic Constraint |
| **KYP** | Kalman Yakubovich Popov |
| **LFT** | Linear Fractional Transformation |
| **LFR** | Linear Fractional Representation |
| **LMI** | Linear Matrix Inequality |
| **LPV** | Linear Parameter Varying |
| **LTI** | Linear Time Invariant |
| **MDT** | Minimum Dwell-Time |
| **RDT** | Range Dwell-Time |
| **SOS** | (Matrix) Sum-of-Squares |

C

# Manipulation of Linear Matrix Inequalities

It is by now well-known in the control community that a multitude of difficult engineering optimization problems can be translated into or effectively approximated by linear matrix inequality (LMI) problems [16, 149]. However, such a transition, i.e., the required manipulation of the underlying problem, can be intricate and not obvious. In this chapter, we summarize several highly useful tools for (algebraically) manipulating matrix inequalities and, in particular, for generating LMIs problems.

## C.1 Linear Matrix Inequality Problems and Basics

Let us begin by providing the canonical description of LMIs and of the corresponding LMI problems as also given, e.g., in [16, 149]. To this end, we employ mostly standard notation from linear algebra[1] as recalled in Appendix A.

**Definition C.1** *Let* $F : \mathbb{R}^n \to \mathbb{S}^m$ *be an affine map and* $c : \mathbb{R}^n \to \mathbb{R}$ *be a linear function.*

- *An* LMI *is an expression of the form* $F(x) \preccurlyeq 0$.
- *An* LMI feasibility problem *amounts to testing whether there exists some* (decision) variable $x \in \mathbb{R}^n$ *such that* $F(x) \preccurlyeq 0$ *holds. The LMI* $F(x) \preccurlyeq 0$ *is said to be* feasible *if the result of the latter test is in the affirmative.*
- *An* LMI optimization problem *constitutes the minimization of the* cost function $c(x)$ *over all decision variables* $x \in \mathbb{R}^n$ *that satisfy* $F(x) \preccurlyeq 0$.

LMI optimization and feasibility problems are special cases of convex *semidefinite programs* (SDPs), which can be viewed as generalizations of *linear programs* (LPs); both, SDPs and LPs are extensively discusses in [17]. Thereby convexity plays a crucial role as it allows us for example to conclude that locally optimal solutions are also globally optimal. In particular, LMI problems can efficiently be solved if the problem size, as determined by the dimensions $n$ and $m$, is not too large [17, 16]. Some of the commonly used numerical solvers are LMIlab [55], SeDuMi [155] and Mosek [113], but nowadays there are many more available. As for any other optimization problem, it is recommended to pick a solver that takes as much properties of the underlying problem into account as possible.

In most control applications, one faces *strict* LMIs $F(x) \prec 0$ with particularly *structured* maps $F$ such as in the following well-known basic result.

[1] Working with LMIs requires some basic background in linear algebra. The books [86, 87] are particularly interesting in this context and provide much content beyond what is mandatory.

**Lemma C.2** *The matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz, i.e., all its eigenvalues are located in the open left half-plane, if and only if there exists a matrix $X \in \mathbb{S}^n$ satisfying $X \succ 0$ and $A^\top X + XA \prec 0$.*

This result involves an LMI described by the map $\tilde{F} : \mathbb{S}^n \to \mathbb{S}^{2n}$, $X \mapsto \left(\begin{smallmatrix} -X & 0 \\ 0 & A^\top X + XA \end{smallmatrix}\right)$. We recover the canonical LMI description by choosing a basis $(E_1, \ldots, E_r)$ of $\mathbb{S}^n$ and expressing any $X \in \mathbb{S}^n$ as linear combination of the basis elements. Indeed, for $X = \sum_{j=1}^r x_j E_j$, this yields

$$\tilde{F}(X) = \tilde{F}\left(\sum_{j=1}^r x_j E_j\right) = \tilde{F}(0) + \sum_{j=1}^r x_j \tilde{F}(E_j) =: F(x).$$

Naturally, it is not very efficient to utilize the canonical LMI description for such situations and most solvers try to exploit the available structured descriptions.

Note that even when facing non-strict LMIs $F(x) \preccurlyeq 0$, it is recommended for numerical reasons to render these inequalities strict by introducing some small $\varepsilon > 0$ and to consider the LMI $F_\varepsilon(x) := F(x) + \varepsilon I \preccurlyeq 0$ instead. This stems from the observation that a numerically determined optimizer $x^*$ of some cost subject to $F(x) \preccurlyeq 0$ might, due to numerical errors, merely satisfy $F(x^*) \preccurlyeq \hat{\varepsilon} I$ for some small $\hat{\varepsilon} > 0$ depending on the employed solver's accuracy. Exactly the same might happen if we replace $F$ by $F_\varepsilon$, but in the latter case we have $F(x^*) = F_\varepsilon(x^*) - \varepsilon I \preccurlyeq \hat{\varepsilon} I - \varepsilon I$ which is fine if $\varepsilon \geq \hat{\varepsilon}$.

Finally, we will in some situations deal with complex matrix inequalities, but these can equivalently be expressed as inequalities involving only real matrices based on the following immediate result.

**Lemma C.3** *Suppose that $A = X + iY \in \mathbb{C}^{n\times n}$ with $X$ and $Y \in \mathbb{R}^{n\times n}$. Then $A \preccurlyeq 0$ if and only if $\left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix}\right) \preccurlyeq 0$. Analogous statements hold for "$\succcurlyeq$", "$\prec$" and "$\succ$".*

*Proof.* We only show necessity as sufficiency follows from reversing the arguments. Since $A$ is Hermitian, we have $X+iY = (X+iY)^* = X^\top - iY^\top$ and hence $X = X^\top$ as well as $Y = -Y^\top$. This yields

$$\begin{pmatrix} Y & X \\ -X & Y \end{pmatrix}^\top = \begin{pmatrix} -Y & -X \\ X & -Y \end{pmatrix} = -\begin{pmatrix} Y & X \\ -X & Y \end{pmatrix}$$

and thus $x^\top \left(\begin{smallmatrix} Y & X \\ -X & Y \end{smallmatrix}\right) x = 0$ for all $x \in \mathbb{R}^{2n}$. Then we conclude

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}^\top \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix}^\top \left( \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} + i \begin{pmatrix} Y & X \\ -X & Y \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} u \\ v \end{pmatrix}^* \begin{pmatrix} I & iI \end{pmatrix}^* (X+iY) \begin{pmatrix} I & iI \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (\bullet)^* A(u+iv) \leq 0 \end{aligned}$$

for all $u, v \in \mathbb{R}^n$, i.e., negative definiteness of $\left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix}\right)$. ●

In order to directly cover the real and complex case, we denote by the set $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$.

## C.2 Schur Complement

The Schur complement is an elementary and very powerful tool in many practical and theoretical fields with a surprisingly large number of interesting applications [175, 17].

**Definition C.4** (Schur Complement) *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{K}^{(n+m)\times(n+m)}$. If $A$ is nonsingular then $D - CA^{-1}B$ is the* Schur complement *of $A$ in $M$, and if $D$ is nonsingular then $A - BD^{-1}C$ is the* Schur complement *of $D$ in $M$.*

Both Schur complements appear as a result from performing a single block Gaussian elimination. Indeed, we have

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

and

$$\begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix}$$

if $A$ and $D$ are nonsingular, respectively. Many interesting results can be inferred from this identity, but we only need and state two of them. The first is easily obtained by taking the inverses.

**Lemma C.5** (Block Inversion) *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{K}^{(n+m)\times(n+m)}$. Then the following statements hold.*

*(a) If $A$ is nonsingular, then $M$ is nonsingular if and only if $D - CA^{-1}B$ is nonsingular.*

*(b) If $A$ and $S_A := D - CA^{-1}B$ are nonsingular, then we have*

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} & -A^{-1}BS_A^{-1} \\ 0 & S_A^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} A^{-1} + A^{-1}BS_A^{-1}CA^{-1} & -A^{-1}BS_A^{-1} \\ -S_A^{-1}CA^{-1} & S_A^{-1} \end{pmatrix}. \end{aligned}$$

*(c) If $D$ is nonsingular, then $M$ is nonsingular if and only if $A - BD^{-1}C$ is nonsingular.*

(d) *If $D$ and $S_D := A - BD^{-1}C$ are nonsingular, then we have*

$$M^{-1} = \begin{pmatrix} S_D^{-1} & 0 \\ -D^{-1}CS_D^{-1} & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} S_D^{-1} & -S_D^{-1}BD^{-1} \\ -D^{-1}CS_D^{-1} & D^{-1}+D^{-1}CS_D^{-1}BD^{-1} \end{pmatrix}.$$

In the case that the matrix $M = \left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)$ is Hermitian and either $A$ or $D$ is nonsingular, then $M$ can even be rendered block diagonal based on a congruence transformation[2]. This leads to the following result.

**Lemma C.6** *Let $M = \left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right) \in \mathbb{K}^{(n+m)\times(n+m)}$ be Hermitian. Then the following statements are equivalent.*

(a) $M \succ 0$.

(b) $A \succ 0$ *and* $D - B^*A^{-1}B \succ 0$.

(c) $D \succ 0$ *and* $A - BD^{-1}B^* \succ 0$.

*Analogous statements hold for "$\succ$" replaced by "$\prec$".*

The proof of Lemma C.6 is an immediate consequence of the following result that is, e.g., found in [86] and also highly useful for our purposes.

**Lemma C.7** *Let $A$ be a Hermitian matrix and $T$ be nonsingular. Then the matrices $A$ and $T^*AT$ have exactly the same number of negative, zero and positive eigenvalues.*

A more general version of Lemma C.6 involving nonstrict inequalities and potentially singular matrices $A$ and $D$ can be found in [17, Appendix A], but the above version is sufficient for our purposes. An important consequence is the following lemma which is for example exploited in the design of $H_\infty$-controllers.

[2] Given a Hermitian matrix $A$ and some nonsingular matrix $T$, then the map $A \mapsto T^*AT$ is called a congruence transformation.

**Lemma C.8** (Linearization Lemma) *Let $V$, $S$, $T$ be real matrices of appropriate dimensions and let $Q$, $U$, $W$ be affine real matrix-valued functions. Then testing the existence of some $x$ such that*

$$U(x) \succ 0 \quad \text{and} \quad \begin{pmatrix} V \\ W(x) \end{pmatrix}^\top \begin{pmatrix} Q(x) & S \\ S^\top & TU(x)^{-1}T^\top \end{pmatrix} \begin{pmatrix} V \\ W(x) \end{pmatrix} \prec 0$$

*is an LMI feasibility problem.*

*Proof.* The second inequality reads as

$$V^\top Q(x)V + V^\top SW(x) + W(x)^\top S^\top V + W(x)^\top TU(x)^{-1}T^\top W(x) \prec 0.$$

By Lemma C.6, the first and second inequality are equivalent to

$$\begin{pmatrix} V^\top Q(x)V + V^\top SW(x) + W(x)^\top S^\top V & W(x)^\top T \\ T^\top W(x) & -U(x) \end{pmatrix} \prec 0.$$

It remains to observe that the term on the left hand side is affine in $x$. ●

## C.3 Dualization Lemma

The so-called *dualization lemma* plays a key role in many controller design approaches.

**Lemma C.9** (Dualization Lemma) *Let $A \in \mathbb{K}^{(p+q)\times q}$, $B \in \mathbb{K}^{(p+q)\times p}$, $P = P^* \in \mathbb{K}^{(p+q)\times(p+q)}$ and suppose that $(A, B)$ and $P$ are nonsingular. Further, let $U$ and $V$ be basis matrices of $\ker(A^*)$ and $\ker(B^*)$, respectively. Then the inequalities $A^*PA \prec 0$ and $B^*PB \succcurlyeq 0$ are equivalent to $U^*P^{-1}U \succ 0$ and $V^*P^{-1}V \preccurlyeq 0$.*

A proof of a more general version is provided, e.g, in [149]. The proof relies on a corollary of the so-called min-max theorem of Courant and

Fischer as found, e.g., in [86, Theorem 4.2.11]. Since this corollary is very useful for our purposes, it is repeated here.

**Corollary C.10** *Let $P$ be a Hermitian matrix and let $U$ be a subspace of dimension $k$ satisfying*

$$x^* P x > 0 \quad \textit{for all} \quad x \in U \setminus \{0\}.$$

*Then $P$ has at least $k$ positive eigenvalues. Analogous statements hold for the inequalities "$\geq$", "$\leq$" and "$<$".*

The dualization lemma is most typically applied in the case that $A = \left(\begin{smallmatrix} I_p \\ W \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 \\ I_q \end{smallmatrix}\right)$ for some matrix $W \in \mathbb{K}^{q \times p}$. Then Lemma C.9 states that

$$\begin{pmatrix} I_p \\ W \end{pmatrix}^* P \begin{pmatrix} I_p \\ W \end{pmatrix} \prec 0 \quad \text{and} \quad \begin{pmatrix} 0 \\ I_q \end{pmatrix}^* P \begin{pmatrix} 0 \\ I_q \end{pmatrix} \succcurlyeq 0 \tag{C.1}$$

are equivalent to

$$\begin{pmatrix} -W^* \\ I_q \end{pmatrix}^* P^{-1} \begin{pmatrix} -W^* \\ I_q \end{pmatrix} \succ 0 \quad \text{and} \quad \begin{pmatrix} I_p \\ 0 \end{pmatrix}^* P^{-1} \begin{pmatrix} I_p \\ 0 \end{pmatrix} \preccurlyeq 0$$

for any nonsingular Hermitian matrix $P$. Note that if both inequalities in (C.1) are strict, Corollary C.10 allows us to conclude that $P$ has exactly $p$ positive and $q$ negative eigenvalues, which implies that $P$ is nonsingular.

There are some variants of the dualization lemma that we do not need. We just mention the so-called partial dualization which is used, e.g., in [143] to show that robust controller synthesis is convex for systems without control channel uncertainties.

## C.4 Elimination Lemma

Very often one wishes to reduce the number of decision variables appearing in LMI problems or aims to eliminate variables that enter the appearing matrix inequalities in a non-convex fashion. This is possible based on the following *elimination lemma* which is a very powerful tool to turn several apparently non-convex controller design problems into convex LMI feasibility problems.

**Lemma C.11** (Elimination Lemma) *Let* $U \in \mathbb{K}^{r\times q}$, $V \in \mathbb{K}^{s\times p}$, $W \in \mathbb{K}^{q\times p}$, $P \in \mathbb{K}^{(p+q)\times(p+q)}$ *and suppose that* $P$ *is nonsingular and Hermitian with exactly* $p$ *negative eigenvalues. Further, let* $U_\perp$ *and* $V_\perp$ *be basis matrices of* $\ker(U)$ *and* $\ker(V)$*, respectively. Then there exists a matrix* $Z \in \mathbb{K}^{r\times s}$ *satisfying*

$$\begin{pmatrix} I_p \\ U^*ZV + W \end{pmatrix}^* P \begin{pmatrix} I_p \\ U^*ZV + W \end{pmatrix} \prec 0 \tag{C.2}$$

*if and only if*

$$V_\perp^* \begin{pmatrix} I_p \\ W \end{pmatrix}^* P \begin{pmatrix} I_p \\ W \end{pmatrix} V_\perp \prec 0 \quad \text{and} \quad U_\perp^* \begin{pmatrix} -W^* \\ I_q \end{pmatrix}^* P^{-1} \begin{pmatrix} -W^* \\ I_q \end{pmatrix} U_\perp \succ 0. \tag{C.3a,b}$$

Note that we make use of the standard convention that $U_\perp$ is an empty matrix if $\ker(U) = \{0\}$. In this case there exists a matrix $Z$ satisfying (C.2) if and only if (C.3a) holds. The case that $\ker(V) = \{0\}$ holds is treated analogously.

We give here a full proof of the elimination lemma since it provides a scheme for constructing a solution $Z \in \mathbb{K}^{r\times s}$ if it exists. The original proof is found in [72].

*Proof.* "Only if": Multiplying (C.2) with $V_\perp$ from the right and its conjugate transpose from the left leads immediately to (C.3a). By (C.2) and since $P$ is nonsingular with exactly $p$ negative eigenvalues, we also find a matrix $B$ such that $(A, B)$ is nonsingular for $A := \left(\begin{smallmatrix} I_p \\ U^*ZV+W \end{smallmatrix}\right)$ and such that $B^*PB \succcurlyeq 0$. Applying the dualization lemma C.9 yields then

$$\begin{pmatrix} -(U^*ZV+W)^* \\ I_q \end{pmatrix}^* P^{-1} \begin{pmatrix} -(U^*ZV+W)^* \\ I_q \end{pmatrix} \succ 0$$

and hence (C.3b) by multiplying $U_\perp$ from the right and its conjugate transpose from the left.

"If": By the singular value decomposition, we can find unitary matrices $W_u$, $W_v$ and nonsingular matrices $T_u$, $T_v$ such that

$$U = T_u \underbrace{\begin{pmatrix} I_{q_1} & 0 \\ 0 & 0_{\bullet\times q_2} \end{pmatrix}}_{=:\hat{U}} W_u^* \quad \text{and} \quad V = T_v \underbrace{\begin{pmatrix} I_{p_1} & 0 \\ 0 & 0_{\bullet\times p_2} \end{pmatrix}}_{=:\hat{V}} W_v^*.$$

With this decomposition we can express $U_\perp$ and $V_\perp$ more concretely as $W_u \left(\begin{smallmatrix} 0 \\ I_{q_2} \end{smallmatrix}\right) X_u$ and $W_v \left(\begin{smallmatrix} 0 \\ I_{p_2} \end{smallmatrix}\right) X_v$, respectively, for some nonsingular matrices $X_u$ and $X_v$. Let us now transform the remaining matrices accordingly as $\hat{P} := (\bullet)^* P \operatorname{diag}(W_v, W_u)$, $\hat{W} = W_u^* W W_v$ and $\hat{Z} := T_u^* Z T_v$ with a to-be-determined matrix $Z$. Further, we define the matrices

$$R := \left(\begin{pmatrix} I_p \\ \hat{W} \end{pmatrix}\begin{pmatrix} I_{p_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0_{p\times q_1} \\ I_{q_1} \\ 0_{q_2\times q_1} \end{pmatrix}\right), \quad S := \begin{pmatrix} I_p \\ \hat{W} \end{pmatrix}\begin{pmatrix} 0 \\ I_{p_2} \end{pmatrix} \text{ and } T := \begin{pmatrix} -\hat{W}^* \\ I_q \end{pmatrix}\begin{pmatrix} 0 \\ I_{q_2} \end{pmatrix}$$

that are elements from $\mathbb{K}^{(p+q)\times(p_1+q_1)}$, $\mathbb{K}^{(p+q)\times p_2}$ and $\mathbb{K}^{(p+q)\times q_2}$, respectively. Then we have by elementary computations

$$\begin{pmatrix} I_p \\ \hat{U}^*\hat{Z}\hat{V} + \hat{W} \end{pmatrix} = \left(\begin{array}{c:c} R \begin{pmatrix} I_{p_1} \\ \hat{Z}_{11} \end{pmatrix} & S \end{array}\right) \quad \text{for} \quad \hat{Z}_{11} := \begin{pmatrix} I_{q_1} \\ 0 \end{pmatrix}^* \hat{Z} \begin{pmatrix} I_{p_1} \\ 0 \end{pmatrix},$$

and (C.3) is equivalent to

$$S^*\hat{P}S \prec 0 \quad \text{and} \quad T^*\hat{P}^{-1}T \succ 0.$$

Moreover, (C.2) holds if and only if

$$0 \succ (\bullet)^*\hat{P}\begin{pmatrix} I_p \\ \hat{U}^*\hat{Z}\hat{V}+\hat{W}\end{pmatrix} = \begin{pmatrix} (\bullet)^*\hat{P}R\left(\begin{smallmatrix} I_{p_1} \\ \hat{Z}_{11}\end{smallmatrix}\right) & (\bullet)^* \\ S^*\hat{P}R\left(\begin{smallmatrix} I_{p_1} \\ \hat{Z}_{11}\end{smallmatrix}\right) & S^*\hat{P}S\end{pmatrix}.$$

By $S^*\hat{P}S \prec 0$ and the Schur complement, this inequality is equivalent to

$$\begin{pmatrix} I_{p_1} \\ \hat{Z}_{11}\end{pmatrix}^* \left(R^*\hat{P}R - R^*\hat{P}S(S^*\hat{P}S)^{-1}S^*\hat{P}R\right)\begin{pmatrix} I_{p_1} \\ \hat{Z}_{11}\end{pmatrix} \prec 0. \qquad (*1)$$

Let $\tilde{P}$ now be the inner matrix in $(*1)$ and let $\mathrm{in}_-(M)$ denote the number of negative eigenvalues of any Hermitian matrix $M$[3].

Next we show that $\mathrm{in}_-(\tilde{P}) = p_1$. If this holds, there exists $\left(\begin{smallmatrix} Z_1 \\ Z_2\end{smallmatrix}\right) \in \mathbb{K}^{\bullet\times p_1}$ with $(\bullet)^*\tilde{P}\left(\begin{smallmatrix} Z_1 \\ Z_2\end{smallmatrix}\right) \prec 0$. We can for example choose $\left(\begin{smallmatrix} Z_1 \\ Z_2\end{smallmatrix}\right) = (v_1,\ldots,v_{k_1})$ with $v_1,\ldots,v_{k_1}$ being orthonormal eigenvectors corresponding to the $p_1$ negative eigenvalues of $\tilde{P}$. Via a small perturbation of $Z_1$ if necessary, we can ensure that $Z_1$ is nonsingular and that $(\bullet)^*\tilde{P}\left(\begin{smallmatrix} Z_1 \\ Z_2\end{smallmatrix}\right) \prec 0$ remains valid. Then $(*1)$ holds for $\hat{Z}_{11} = Z_2Z_1^{-1}$ and $Z := T_u^{-*}\left(\begin{smallmatrix} \hat{Z}_{11} & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)T_v^{-1}$ is a solution of (C.2) for any choice of the $\bullet$ matrices.

Indeed, applying the Schur complement yields, for $Q := (R, S)$,

$$\mathrm{in}_-\left(Q^*\hat{P}Q\right) = \mathrm{in}_-\begin{pmatrix} R^*\hat{P}R & R^*\hat{P}S \\ S^*\hat{P}R & S^*\hat{P}S\end{pmatrix} = \mathrm{in}_-(\tilde{P}) + \mathrm{in}_-(S^*\hat{P}S).$$

[3] For a Hermitian matrix $M$, the ordered triple $\mathrm{in}(M) := (\mathrm{in}_+(M), \mathrm{in}_0(M), \mathrm{in}_-(M))$ where $\mathrm{in}_+(M)$, $\mathrm{in}_0(M)$ and $\mathrm{in}_-(M)$ denote the number of positive, zero and negative eigenvalues counting multiplicity, respectively, is referred to as *inertia* of $M$ [86, page 221]. We need the following two properties:
If $M$ is Hermitian and $T$ is nonsingular, then $\mathrm{in}_-(T^*MT) = \mathrm{in}_-(M)$.
If $M$ is Hermitian and nonsingular, then $\mathrm{in}_-(M) = \mathrm{in}_-(M^{-1})$.

By $S^* P S \prec 0$ and $S^* P S \in \mathbb{K}^{p_2 \times p_2}$, we get

$$\mathrm{in}_-(\tilde{P}) = \mathrm{in}_-\left(Q^* \hat{P} Q\right) - p_2.$$

Next, one can show, e.g., via a permutation and a block Gaussian elimination, that the matrix $(T, Q)$ is nonsingular. By Gram-Schmidt, we can then find an unitary matrix $(\hat{T}, \hat{Q})$ with $\mathrm{im}(\hat{T}) = \mathrm{im}(T)$ and $\mathrm{im}(\hat{Q}) = \mathrm{im}(Q)$. Let us define $\left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right) := (\bullet)^* \hat{P}^{-1} (\hat{T}, \hat{Q})$ and note that $A = \hat{T}^* \hat{P}^{-1} \hat{T}$ is positive definite by the choice of $\hat{T}$ and by $T^* \hat{P}^{-1} T \succ 0$. Then the Schur complement and the properties of $\mathrm{in}_-(\cdot)$ imply

$$\mathrm{in}_-(\hat{P}^{-1}) = \mathrm{in}_-(A) + \mathrm{in}_-(D - B^* A B) = \mathrm{in}_-\big((D - B^* A B)^{-1}\big).$$

Next, we can apply the block-inversion formula as $A$ is nonsingular:

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}^{-1} = (\bullet)^* \begin{pmatrix} A & 0 \\ 0 & D - B^* A^{-1} B \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ -B^* A^{-1} & I \end{pmatrix}.$$

Since $(\hat{T}, \hat{Q})$ is unitary, this leads to

$$\hat{Q}^* \hat{P} \hat{Q} = (\bullet)^* \hat{P} \begin{pmatrix} \hat{T} & \hat{Q} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = (\bullet)^* \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} = (D - B^* A B)^{-1}$$

and, thus,

$$\mathrm{in}_-(\hat{P}) = \mathrm{in}_-(\hat{P}^{-1}) = \mathrm{in}_-\big((D - B^* A B)^{-1}\big) = \mathrm{in}_-(\hat{Q}^* \hat{P} \hat{Q}) = \mathrm{in}_-(Q^* \hat{P} Q)$$

by the choice of $\hat{Q}$. Finally, we can conclude

$$\mathrm{in}_-(\tilde{P}) = \mathrm{in}_-(Q^* \hat{P} Q) - p_2 = \mathrm{in}_-(\hat{P}) - p_2 = \mathrm{in}_-(\mathbb{P}) - p_2 = p - p_2 = p_1$$

as claimed.

Unfortunately, it is in general not possible to enforce structural constraints on the matrix $Z$ in (C.2). This prevents the application of the elimination lemma, e.g., in multi-objective control problems. There are few exceptions such as the one in [144] involving a variant with a block triangular matrix $Z$.

By considering the special case $P = \left(\begin{smallmatrix} Q & I \\ I & 0 \end{smallmatrix}\right)$ and $W = 0$ for some Hermitian matrix $Q$, we recover a more common version of the elimination lemma which we refer to as *projection lemma* in order to distinguish both results. Another constructive proof of this result can also be found in [54].

**Lemma C.12** (Projection Lemma) *Let $Q = Q^* \in \mathbb{K}^{q\times q}$, $U \in \mathbb{K}^{r\times q}$ and $V \in \mathbb{K}^{s\times q}$ be given. Further, let $U_\perp$ and $V_\perp$ be basis matrices of* $\ker(U)$ *and* $\ker(V)$*, respectively. Then there exists a matrix $Z \in \mathbb{K}^{r\times s}$ satisfying*

$$Q + U^* Z V + V^* Z^* U \prec 0$$

*if and only if*

$$U_\perp^* Q U_\perp \prec 0 \quad \textit{and} \quad V_\perp^* Q V_\perp \prec 0.$$

The elimination and projection lemmas can not only be used to remove variables, but also to artificially add new variables. Such an application can be beneficial if one has to constrain some of the variables, e.g., in order to ensure convexity. Indeed, it is typically possible and much less conservative to constrain the artificially added variables instead of the original ones. The whole book [46] illustrates and elaborates on this idea.

Note that for $V = I$, the projection lemma can be shown to be equivalent to a variant of the so-called Finslers lemma [52]; the proof involves another Schur complement argument. Moreover, note that there are some robust versions of the projection lemma provided in [38] involving parameter dependent matrices $Q$, $U$ and/or $V$ and a parameter independent matrix $Z$. Unfortunately, they are not very practical for our purposes.

Instead, we will make use of the following nonstandard variant that involves continuous matrix-valued functions instead of matrices.

**Lemma C.13** (Projection Lemma) *Let* $Q \in C([a,b],\mathbb{S}^q)$, $U \in C([a,b],\mathbb{R}^{r\times q})$, $V \in C([a,b],\mathbb{R}^{s\times q})$ *and suppose that there exists a pointwise nonsingular continuous map* $S = (S_1, S_2, S_3, S_4)$ *with the property that, for all* $\tau \in [a,b]$,

$$S_3(\tau),\ (S_1(\tau), S_3(\tau)) \text{ and } (S_2(\tau), S_3(\tau)) \text{ are basis matrices of } \ker(U(\tau)) \cap \ker(V(\tau)),\quad \ker(U(\tau)) \quad \text{and} \quad \ker(V(\tau)), \tag{C.4}$$

*respectively. Then there exists a function* $Z \in C([a,b],\mathbb{R}^{r\times s})$ *satisfying*

$$Q + U^\top Z V + V^\top Z^\top U \prec 0 \quad \text{on} \quad [a,b] \tag{C.5}$$

*if and only if*

$$U_\perp^\top Q U_\perp \prec 0 \quad \text{and} \quad V_\perp^\top Q V_\perp \prec 0 \quad \text{hold on} \quad [a,b]; \tag{C.6}$$

*here,* $U_\perp = (S_1, S_3)$ *and* $V_\perp = (S_2, S_3)$.

Admittedly, the technical condition (C.4) is difficult to verify for general maps $U$ and $V$, but it is exactly what we need for the *construction* of a *continuous* function $Z$ satisfying (C.5). Fortunately, in most of our applications we are able to construct a map $S$ satisfying (C.4) due to the particular structure of the emerging maps $U$ and $V$. In particular, note that if $U$ and $V$ are constant functions, then it is a standard linear algebra problem to construct a suitable matrix $S$.

In order to prove Lemma C.13, we carefully modify the one of the standard projection lemma C.12 as given in [54].

*Proof.* We only have to show "if" since "only if" follows immediately. Note at first that by employing the standard projection lemma C.12 in a pointwise fashion and without additional care, we can construct a function $Z$ satisfying (C.5), but this functions might be discontinuous. An application of the standard projection lemma C.12 together with the continuous selection theorem from [110] or with the findings from [14] guarantees the existence of a continuous map $Z$ satisfying (C.5), but the latter do not provide means to construct this map. Hence, we need some extra work for its construction.

To this end, observe that a pointwise congruence transformation of (C.5) with $S$ leads to the equivalent LMI

$$S^\top QS + (US)^\top Z(VS) + (VS)^\top Z^\top (US) \prec 0 \quad \text{on} \quad [a,b]. \qquad (*1)$$

Here, $US$ and $VS$ have the structure $(0, U_2, 0, U_4)$ and $(V_1, 0, 0, V_4)$, respectively, where $(U_2, U_4)$ as well as $(V_1, V_4)$ have pointwise full column rank. In particular, we have

$$(US)^\top Z(VS) = \begin{pmatrix} 0 \\ U_2^\top \\ 0 \\ U_4^\top \end{pmatrix} Z \begin{pmatrix} V_1 & 0 & 0 & V_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ X_{21} & 0 & 0 & X_{24} \\ 0 & 0 & 0 & 0 \\ X_{41} & 0 & 0 & X_{44} \end{pmatrix}$$

for $X := \left(\begin{smallmatrix} X_{21} & X_{24} \\ X_{41} & X_{44} \end{smallmatrix}\right) := \begin{pmatrix} U_2 & U_4 \end{pmatrix}^\top Z \begin{pmatrix} V_1 & V_4 \end{pmatrix}$. With $P := S^\top QS$ partitioned accordingly, $(*1)$ reads then as

$$\left(\begin{array}{ccc|c} P_{11} & P_{12} + X_{21}^\top & P_{13} & P_{14} + X_{41}^\top \\ P_{21} + X_{21} & P_{22} & P_{23} & P_{24} + X_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ \hline P_{41} + X_{41} & P_{42} + X_{24}^\top & P_{43} & P_{44} + X_{44} + X_{44}^\top \end{array}\right) \prec 0 \quad \text{on} \quad [a,b]. \quad (*2)$$

Next, note that the hypothesis (C.6) read in terms of $P$ as

$$\begin{pmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{pmatrix} \prec 0 \quad \text{and} \quad \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix} \prec 0 \quad \text{on} \quad [a,b].$$

Thus we infer

$$P_{33} \prec 0, \quad P_{11} - P_{13}P_{33}^{-1}P_{31} \prec 0 \ \text{ and } \ P_{22} - P_{23}P_{33}^{-1}P_{32} \prec 0 \ \text{ on } \ [a,b]$$

by the Schur complement C.6. Let us now choose the continuous function

$$X_{21} : \tau \mapsto P_{32}(\tau)^\top P_{33}(\tau)^{-1}P_{31}(\tau) - P_{21}(\tau)$$

and observe that this choice renders the marked $3\times 3$ block in $(*2)$ negative definite. Indeed, by the Schur complement this is equivalent to

$$\begin{aligned} 0 \succ \begin{pmatrix} P_{11} & (\bullet)^\top \\ P_{21}+X_{21} & P_{22} \end{pmatrix} - \begin{pmatrix} P_{13} \\ P_{23} \end{pmatrix} P_{33}^{-1} \begin{pmatrix} P_{31} & P_{32} \end{pmatrix} \\ = \begin{pmatrix} P_{11} - P_{13}P_{33}^{-1}P_{31} & 0 \\ 0 & P_{22} - P_{23}P_{33}^{-1}P_{32} \end{pmatrix} \end{aligned}$$

on $[a,b]$ which is true by assumption. Next, we note that $(*2)$ is satisfied for the constant matrices

$$X_{41} := 0, \quad X_{24} := 0 \quad \text{and} \quad X_{44} := -\alpha I$$

if we choose $\alpha > 0$ sufficiently large. Indeed, this is a consequence of the Schur complement, continuity of all involved functions and compactness of $[a,b]$. It remains to recover a continuous map $Z$ from the constructed continuous map $X$. This is achieved by recalling that the ($\tau$-dependent) matrix equation $\begin{pmatrix} U_2 & U_4 \end{pmatrix}^\top Z \begin{pmatrix} V_1 & V_4 \end{pmatrix} = X$ can be expressed with the vectorization of matrices and the Kronecker product as

$$\left( \begin{pmatrix} V_1 & V_4 \end{pmatrix}^\top \otimes \begin{pmatrix} U_2 & U_4 \end{pmatrix}^\top \right) \mathrm{vec}(Z) = \mathrm{vec}(X) \quad \text{on} \quad [a,b].$$

Let us denote the matrix-valued map on the left hand side by $A$ and recall that $A(\tau)$ has full row rank for all $\tau \in [a, b]$ by construction. This permits us to construct the desired continuous matrix-valued function $Z$ by collecting its entries from the continuous function

$$\mathrm{vec}(Z)(\tau) := A(\tau)^\top (A(\tau)A(\tau)^\top)^{-1}\mathrm{vec}(X(\tau)) \quad \text{for} \quad \tau \in [a, b]$$

and finishes the proof. ●

## C.5 Matrix Extensions

After an application of the elimination lemma to remove decision variables, we often observe that several sub-blocks of the matrices $P$ and $P^{-1}$ in (C.3) are canceled. Since one usually aims to reconstruct the eliminated variables later on, it is required to extend these sub-blocks of $P$ and $P^{-1}$ to the original matrix $P$ that satisfies (C.2).

The most well-known result on matrix extensions in the control literature is the following because it plays an important role, e.g., in the design of dynamic output-feedback $H_\infty$-controllers. It is also not difficult to proof by applying the Schur complement as well as the block inversion formula.

**Lemma C.14** *Let $X$ and $Y$ be matrices satisfying $\left(\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right) \succ 0$ and choose nonsingular matrices $U$ and $V$ satisfying $I = YX + VU^*$. Note that a possible choice is $V = Y$ and $U = Y^{-1} - X$. Then the matrix*

$$\mathcal{X} := \begin{pmatrix} X & U \\ U^* & -V^{-1}YU \end{pmatrix}$$

*satisfies $\mathcal{X} \succ 0$, $\mathcal{X} = \left(\begin{smallmatrix} X & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)$ and $\mathcal{X}^{-1} = \left(\begin{smallmatrix} Y & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)$.*

In a similar fashion one obtains the following result.

**Lemma C.15** *Let $X$ and $Y$ be matrices with $X+X^*\succ 0$ and $Y+Y^*\succ 0$. Then the nonsingular matrix*

$$\mathcal{X} := \begin{pmatrix} X & X-Y^{-1} \\ X+Y^{-*} & X+Y^{-*} \end{pmatrix}$$

*satisfies $\mathcal{X}+\mathcal{X}^*\succ 0$, $\mathcal{X} = \left(\begin{smallmatrix} X & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)$ and $\mathcal{X}^{-1} = \left(\begin{smallmatrix} Y & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)$.*

More challenging is the follow result that is employed, e.g., for the design of gain-scheduling controllers and when working with so-called full-block multipliers.

**Lemma C.16** *Let $P = \left(\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right)$ and $\tilde{P} = \left(\begin{smallmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & R \end{smallmatrix}\right)$ be matrices in $\mathbb{H}^{p+q}$ with*

$$\begin{pmatrix} D \\ I_q \end{pmatrix}^* P \begin{pmatrix} D \\ I_q \end{pmatrix} \prec 0 \quad \text{and} \quad \begin{pmatrix} I_p \\ -D^* \end{pmatrix}^* \tilde{P} \begin{pmatrix} I_p \\ -D^* \end{pmatrix} \succ 0 \tag{C.7}$$

*for some matrix $D\in\mathbb{K}^{p\times q}$ and such that $\tilde{P}$ as well as $P-\tilde{P}^{-1}$ are non-singular. Moreover, let $\boldsymbol{\Delta}$ be a subset of $\mathbb{K}^{q\times p}$ such that additionally*

$$\begin{pmatrix} I_p \\ \Delta \end{pmatrix}^* P \begin{pmatrix} I_p \\ \Delta \end{pmatrix} \succ 0 \quad \text{and} \quad \begin{pmatrix} -\Delta^* \\ I_q \end{pmatrix}^* \tilde{P} \begin{pmatrix} -\Delta^* \\ I_q \end{pmatrix} \prec 0 \quad \text{hold for all} \quad \Delta\in\boldsymbol{\Delta}. \tag{C.8}$$

*Then there exist $p_2, q_2\in\mathbb{N}$ with $p_2+q_2 = p+q$, a nonsingular matrix $\mathcal{P}\in\mathbb{H}^{(p+p_2)+(q+q_2)}$ with exactly $q+q_2$ negative eigenvalues and a continuous function $\Delta_c:\boldsymbol{\Delta}\to\mathbb{K}^{q_2\times p_2}$ such that*

$$\mathcal{P} = \left(\begin{array}{cc|cc} Q & \bullet & S & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \hline S^* & \bullet & R & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}\right), \quad \mathcal{P}^{-1} = \left(\begin{array}{cc|cc} \tilde{Q} & \bullet & \tilde{S} & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \hline \tilde{S}^* & \bullet & \tilde{R} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}\right) \quad \text{and} \quad (\bullet)^*\mathcal{P}\left(\begin{array}{cc} I_p & 0 \\ 0 & I_{p_2} \\ \hline \Delta & 0 \\ 0 & \Delta_c(\Delta) \end{array}\right) \succ 0 \tag{C.9}$$

*holds for all $\Delta\in\boldsymbol{\Delta}$.*

This result is an extension of the one found in [139] since we neither require $0 \in \mathbf{\Delta}$ nor that (C.7) is satisfied for $D = 0$. The result from [140] is even more general since it does not require (C.7) at all, but the obtained inequality in (C.9) is structurally different in this case.

*Proof.* For some fixed $\Delta_0 \in \mathbf{\Delta}$, we define $Z := \left(\begin{smallmatrix} D \\ I_q \end{smallmatrix}\right)$, $\tilde{Z} := \left(\begin{smallmatrix} I_p \\ \Delta_0 \end{smallmatrix}\right)$ as well as

$$N := (P - \tilde{P}^{-1})^{-1},\; M := N - Z(Z^* P Z)^{-1} Z^* \text{ and } \tilde{M} := N - \tilde{Z}(\tilde{Z}^* P \tilde{Z})^{-1} \tilde{Z}^*.$$

Moreover, let $p_2 = \mathrm{in}_-(N)$ and $q_2 = \mathrm{in}_+(N)$ denote the number of negative and positive eigenvalues of $N$, respectively. Since $N$ is nonsingular with $p+q$ columns, these numbers indeed sum up to $p+q$. Next, one can pick a nonsingular matrix $T = (T_1, T_2) \in \mathbb{K}^{(p+q)\times(p_2+q_2)}$ such that the inequalities $T_2^* M T_2 \prec 0$ and $T_1^* \tilde{M} T_1 \succ 0$ hold. Indeed, we have

$$\mathrm{in}_-(M) + \mathrm{in}_-(Z^* P Z) = \mathrm{in}_- \begin{pmatrix} N & Z \\ Z^* & Z^* P Z \end{pmatrix} = \mathrm{in}_-(N) + \mathrm{in}_-(Z^* \tilde{P}^{-1} Z)$$

by the definition of $N$ and by the Schur complement C.6. By (C.7) and the dualization lemma C.9 we have $\mathrm{in}_-(Z^* P Z) = \mathrm{in}_-(Z^* \tilde{P}^{-1} Z)$ and thus $\mathrm{in}_-(M) = \mathrm{in}_-(N) = p_2$ which implies the existence of a matrix $T_2$ with the claimed properties. Analogously, we find a suitable matrix $T_1$ and can ensure nonsingularity of $T = (T_1, T_2)$ with a small perturbation if necessary.

Then we define the matrix

$$\tilde{\mathcal{P}} := \begin{pmatrix} P & T \\ T^* & T^* N T \end{pmatrix}$$

which satisfies

$$\tilde{\mathcal{P}}^{-1} = \begin{pmatrix} \tilde{P} & \bullet \\ \bullet & \bullet \end{pmatrix}, \quad (\bullet)^\top \tilde{\mathcal{P}} \begin{pmatrix} Z & 0 \\ 0 & Z_2 \end{pmatrix} \prec 0 \quad \text{and} \quad (\bullet)^\top \tilde{\mathcal{P}} \begin{pmatrix} \tilde{Z} & 0 \\ 0 & \tilde{Z}_2 \end{pmatrix} \succ 0$$

for $Z_2 = \left(\begin{smallmatrix} 0 \\ I_{q_2} \end{smallmatrix}\right)$ and $\tilde{Z}_2 = \left(\begin{smallmatrix} I_{p_2} \\ 0 \end{smallmatrix}\right)$ by another Schur complement, (C.7), (C.8), $T_2^* M T_2 \prec 0$ and by $T_1^* \tilde{M} T_1 \succ 0$. Finally, a permutation of $\tilde{\mathcal{P}}$ results in a matrix $\mathcal{P}$ with the desired properties.

Since $\mathcal{P}$ satisfies the requirements for pointwise applying the elimination lemma C.11, we can find a function $\Delta_c : \mathbf{\Delta} \to \mathbb{K}^{q_2 \times p_2}$ satisfying the inequality in (C.9) for all $\Delta \in \mathbf{\Delta}$. However, this function might be not continuous. Instead, we construct a continuous function $\Delta_c$ as follows. Let $\Delta \in \mathbf{\Delta}$ be fixed. By construction and by the elimination lemma we have

$$(\bullet)^* \mathcal{P} \begin{pmatrix} \mathcal{D} \\ I \end{pmatrix} \prec 0 \quad \text{and} \quad (\bullet)^* \mathcal{P} \begin{pmatrix} I \\ \Delta_e \end{pmatrix} \succ 0$$

for $\mathcal{D} = \operatorname{diag}(D, 0)$ and $\Delta_e = \operatorname{diag}(\Delta, \tilde{\Delta}_c)$ for some $\tilde{\Delta}_c \in \mathbb{K}^{q_2 \times p_2}$. Let us now partition the transformed matrix $\hat{\mathcal{P}} := (\bullet)^* \mathcal{P} \left(\begin{smallmatrix} I & \mathcal{D} \\ 0 & I \end{smallmatrix}\right) =: \left(\begin{smallmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}^* & \mathcal{R} \end{smallmatrix}\right)$ and observe that we have $\mathcal{R} \prec 0$ as well as

$$0 \prec (\bullet)^* \mathcal{P} \begin{pmatrix} I \\ \Delta_e \end{pmatrix} = (\bullet)^* \hat{\mathcal{P}} \begin{pmatrix} I & -\mathcal{D} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ \Delta_e \end{pmatrix} = (\bullet)^* \hat{\mathcal{P}} \begin{pmatrix} I - \mathcal{D}\Delta_e \\ \Delta_e \end{pmatrix}.$$

Since $I - \mathcal{D}\Delta_e = \operatorname{diag}(I - D\Delta, I)$ is nonsingular, the latter inequality is equivalent to

$$0 \prec (\bullet)^* \hat{\mathcal{P}} \begin{pmatrix} I \\ \Delta_e (I - \mathcal{D}\Delta_e)^{-1} \end{pmatrix} = (\bullet)^* \hat{\mathcal{P}} \begin{pmatrix} I \\ \tilde{\Delta}_e \end{pmatrix} = \mathcal{Q} + \mathcal{S}\tilde{\Delta}_e + (\bullet)^* + \tilde{\Delta}_e^* \mathcal{R} \tilde{\Delta}_e$$

with $\tilde{\Delta}_e := \operatorname{diag}(\tilde{\Delta}, \tilde{\Delta}_c)$ and $\tilde{\Delta} := \Delta(I - D\Delta)^{-1}$. It remains to note that the latter inequality is equivalent to

$$\begin{pmatrix} \mathcal{Q} - \mathcal{S}\mathcal{R}^{-1}\mathcal{S}^* & \mathcal{R}^{-1}\mathcal{S}^* + \tilde{\Delta}_e \\ (\bullet)^* & -\mathcal{R}^{-1} \end{pmatrix} \succ 0$$

by the Schur complement. By introducing the partitions

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} := \mathcal{Q} - \mathcal{S}\mathcal{R}^{-1}\mathcal{S}^*, \quad \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} := -\mathcal{R}^{-1}, \quad \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} := \mathcal{R}^{-1}\mathcal{S}^*,$$

a permutation as well as another Schur complement, we infer that the latter inequality is satisfied for

$$\Delta_c(\Delta) := \tilde{\Delta}_c := -W_{22} + \begin{pmatrix} W_{21} & V_{21} \end{pmatrix} \begin{pmatrix} U_{11} & (\bullet)^* \\ W_{11} + \tilde{\Delta} & V_{11} \end{pmatrix}^{-1} \begin{pmatrix} U_{12} & W_{21}^* \\ W_{12} & V_{12} \end{pmatrix}.$$

It remains to note that the function $\Delta_c$ obtained in this fashion is continuous and even smooth on $\mathbf{\Delta}$. ●

## C.6 Separation, LFRs and S-Procedure

A fundamental algebraic problem in robust control that is related to the well-posedness of feedback interconnections of systems with uncertain objects is to decide whether

$$I - D\Delta \text{ is nonsingular} \quad \text{for all} \quad \Delta \in \mathbf{\Delta}.$$

Here, $D$ is a matrix in $\mathbb{K}^{p\times q}$ and $\mathbf{\Delta}$ is a subset of $\mathbb{K}^{q\times p}$ that is usually referred to as *uncertainty set*. For the above problem it is typically rather difficult to come to a decision by numeric means and essentially impossible if $\Delta$ is a more general object than a matrix; e.g., we might then have to decide whether the map $x \mapsto x - D\Delta(x)$ is bijective for all $\Delta$ in a set of functions $\mathbf{\Delta}$.

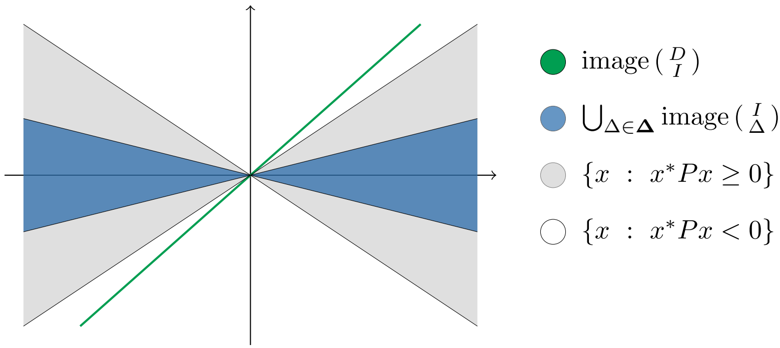



Figure C.1: Separation in terms of the positive and strictly negative cone of a Hermitian matrix $P$.

A key observation that leads to implementable criteria and nicely generalizes to more difficult settings is that, for any fixed $\Delta \in \boldsymbol{\Delta}$, we have

$$I - D\Delta \text{ is nonsingular} \quad \text{if and only if} \quad \text{image}\begin{pmatrix} D \\ I \end{pmatrix} \cap \text{image}\begin{pmatrix} I \\ \Delta \end{pmatrix} = \{0\}.$$

For nonsingularity of $I - D\Delta$, we hence need to make sure that the graph of $\Delta$ and the inverse graph of $D$ are *separated*, i.e., they only intersect at the origin. *Geometrically*, we can guarantee such a separation if these graphs are located in the positive cone $\{x \in \mathbb{K}^{p+q} : \ x^* P x \geq 0\}$ and strictly negative cone $\{x \in \mathbb{K}^{p+q} : \ x^* P x < 0\}$ of some Hermitian matrix $P$, respectively. This is illustrated in Fig. C.1. The involved matrix $P$ is usually referred to as *multiplier* and this observation leads to the following simple, but instrumental result.

**Lemma C.17** *Let $D \in \mathbb{K}^{p\times q}$ and $\mathbf{\Delta} \subset \mathbb{K}^{q\times p}$. Suppose that there exists a matrix $P \in \mathbb{H}^{p+q}$ with*

$$\begin{pmatrix} D \\ I \end{pmatrix}^* P \begin{pmatrix} D \\ I \end{pmatrix} \prec 0 \quad \text{and} \quad \begin{pmatrix} I \\ \Delta \end{pmatrix}^* P \begin{pmatrix} I \\ \Delta \end{pmatrix} \succcurlyeq 0 \quad \text{for all} \quad \Delta \in \mathbf{\Delta}. \tag{C.10}$$

*Then $I - D\Delta$ is nonsingular for all $\Delta \in \mathbf{\Delta}$. The converse holds if $\mathbf{\Delta}$ is compact.*

*Proof.* Sufficiency is obtained with elementary computations. For the converse one can choose $\lambda_1, \lambda_2 > 0$ such that $(I - D\Delta)^*(I - D\Delta) \succcurlyeq \lambda_1 I$ and $\Delta^*\Delta \preccurlyeq \lambda_2 I$ hold for all $\Delta \in \mathbf{\Delta}$ by compactness and continuity. Then a suitable multiplier is given by $P := \left(\begin{smallmatrix} 0 & 0 \\ 0 & -\lambda_1/\lambda_2 I \end{smallmatrix}\right) + \left(\begin{smallmatrix} I \\ -D^* \end{smallmatrix}\right)\left(\begin{smallmatrix} I & -D \end{smallmatrix}\right)$. ●

Note that the condition (C.10) is still numerically problematic as it involves infinitely many LMIs. In order to obtain numerical tractable criteria, one can employ so called *multiplier sets*.

**Definition C.18** (Multiplier Set) *The set $\mathbf{P}(\mathbf{\Delta}) \subset \mathbb{H}^{p+q}$ is called* multiplier set *for the set $\mathbf{\Delta}$ if it admits an LMI representation, i.e., there exist affine functions $F$ and $G$ such that $\mathbf{P}(\mathbf{\Delta}) = \{F(\nu) \mid \nu \in \mathbb{R}^\bullet \;\text{ and }\; G(\nu) \succ 0\}$, and if*

$$\begin{pmatrix} I \\ \Delta \end{pmatrix}^* P \begin{pmatrix} I \\ \Delta \end{pmatrix} \succcurlyeq 0 \quad \text{holds for all} \quad \Delta \in \mathbf{\Delta} \quad \text{and all} \quad P \in \mathbf{P}(\mathbf{\Delta}). \tag{C.11}$$

Given a fixed multiplier set $\mathbf{P}(\mathbf{\Delta})$, we can immediately conclude that

$I - D\Delta$ is nonsingular for all $\Delta \in \mathbf{\Delta}$

if there exists some $P \in \mathbf{P}(\mathbf{\Delta})$ satisfying $\left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right)^* P \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \prec 0$.

Due to the LMI representation of $\mathbf{P}(\mathbf{\Delta})$, finding a suitable multiplier $P$

constitutes a numerically tractable problem as desired. The price to pay for this tractability is that the resulting criterion is in general no longer necessary. To counteract the introduced conservatism while maintaining computational efficiency, one should aim for multiplier sets that describe the set $\mathbf{\Delta}$ as good as and as simple as possible in terms of quadratic inequalities. As a negative example consider the set $\mathbf{P}(\mathbf{\Delta}) := \{P \in \mathbb{H}^{p+q} \,:\, P \succ 0\}$ which always satisfies (C.11), but we will never find some $P \in \mathbf{P}(\mathbf{\Delta})$ satisfying $\left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right)^* P \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \prec 0$.

Next we give some examples of concrete sets $\mathbf{\Delta}$ and common choices for corresponding multiplier sets as well as some helpful insights for the construction of multiplier sets. Some additional examples can be found, e.g., in [149] and a detailed summary in the context of integral quadratic constraints [109] is available in [160].

**Remark C.19** (Examples and Properties of Multiplier Sets)

- Suppose that $\mathbf{\Delta} = \{\Delta \in \mathbb{K}^{q\times p} \;:\; \|\Delta\|_2 \leq r\}$ for some $r > 0$. Then the following set, that involves so-called D-scalings, is a suitable multiplier set for $\mathbf{\Delta}$:

$$\mathbf{P}(\mathbf{\Delta}) := \left\{ \begin{pmatrix} r^2 dI & 0 \\ 0 & -dI \end{pmatrix} \,\middle|\, d > 0 \right\}.$$

  Indeed, due to $\|\Delta\|_2 \leq r \;\Leftrightarrow\; \Delta^*\Delta \preccurlyeq r^2 I$, we can conclude that $(\bullet)^* d \left(\begin{smallmatrix} r^2 I & 0 \\ 0 & -I \end{smallmatrix}\right) \left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right) = d(r^2 I - \Delta^*\Delta) \succcurlyeq 0$ holds for all $\Delta \in \mathbf{\Delta}$ and all $d > 0$. Analogously, the set

$$\mathbf{P}(\mathbf{\Delta}) := \left\{ \begin{pmatrix} r^2 D & 0 \\ 0 & -D \end{pmatrix} \,\middle|\, D \succ 0 \right\}$$

  is a suitable multiplier set for $\mathbf{\Delta} = \{\delta I \;:\; |\delta| \leq r\}$.

- Suppose that $\mathbf{\Delta}$ is the convex hull of some generators $\Delta_1, \ldots, \Delta_N \in$

$\mathbb{K}^{q\times p}$, i.e., $\boldsymbol{\Delta} = \mathrm{co}\{\Delta_1,\ldots,\Delta_N\}$. Then

$$\mathbf{P}(\boldsymbol{\Delta}) := \left\{ P = P^* \;\middle|\; \begin{array}{l} (\bullet)^* P \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right) \prec 0 \;\text{ and }\; (\bullet)^* P \left(\begin{smallmatrix} I \\ \Delta_i \end{smallmatrix}\right) \succ 0 \\ \text{for all } \; i = 1,\ldots,N \end{array} \right\}$$

is a suitable multiplier set for $\boldsymbol{\Delta}$. Indeed, $\left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right)^* P \left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right) \prec 0$ implies that the mapping $M : \Delta \mapsto \left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right)^* P \left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right)$ is concave. Then convexity of $\boldsymbol{\Delta}$ and its concrete description in terms of the generators allows us to conclude that $M(\Delta) \succ 0$ holds for all $\Delta \in \boldsymbol{\Delta}$ if and only if $M(\Delta_i) \succ 0$ for all $i = 1,\ldots,N$.

- Suppose that $\boldsymbol{\Delta} = \{\delta I \;:\; \delta \in [a,b]\}$ for some $a < b$. Then it is possible to employ the above multiplier set for $\boldsymbol{\Delta}$ as well or the commonly used alternative involving the so-called D-G scalings

$$\begin{aligned} \mathbf{P}(\boldsymbol{\Delta}) &:= \left\{ \begin{pmatrix} -abD & \frac{b+a}{2}D + G^\top \\ \frac{b+a}{2}D + G & -D \end{pmatrix} \;\middle|\; D \succ 0 \;\text{ and }\; G + G^\top = 0 \right\} \\ &= \left\{ \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix}^\top \begin{pmatrix} 0 & H^\top \\ H & 0 \end{pmatrix} \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix} \;\middle|\; H + H^\top \succ 0 \right\}. \end{aligned}$$

  Indeed, observe that $(\bullet)^* P \left(\begin{smallmatrix} I \\ \delta I \end{smallmatrix}\right) = \delta(G + G^\top) + (b-\delta)(\delta - a)D \succcurlyeq 0$ holds for any $\delta \in [a,b]$ and any $P \in \mathbf{P}(\boldsymbol{\Delta})$.

- Suppose that $\mathbf{P}_1(\boldsymbol{\Delta})$ and $\mathbf{P}_2(\boldsymbol{\Delta})$ are two multiplier sets corresponding to $\boldsymbol{\Delta}$. Then

$$\mathbf{P}(\boldsymbol{\Delta}) := \left\{ \lambda_1 P_1 + \lambda_2 P_2 \;\middle|\; \begin{array}{l} \lambda_1 \ge 0,\; \lambda_2 \ge 0,\; P_1 \in \mathbf{P}_1(\boldsymbol{\Delta}) \\ \text{and } \; P_2 \in \mathbf{P}_2(\boldsymbol{\Delta}) \end{array} \right\}$$

  is another and even larger set of suitable multipliers for $\boldsymbol{\Delta}$.

- Suppose that $\boldsymbol{\Delta} := \{\mathrm{diag}(\Delta_1,\Delta_2) \;:\; \Delta_1 \in \boldsymbol{\Delta}_1,\; \Delta_2 \in \boldsymbol{\Delta}_2\}$ and that $\mathbf{P}(\boldsymbol{\Delta}_1)$ and $\mathbf{P}(\boldsymbol{\Delta}_2)$ are multiplier sets corresponding to $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$,

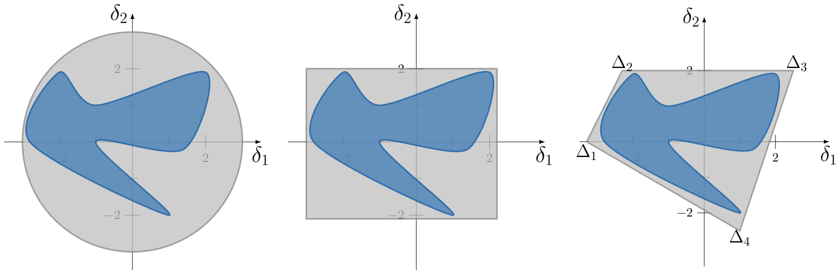



Figure C.2: Some uncertainty set $\mathbf{\Delta} \subset \mathbb{R}^2$ in blue and in gray the (basic) supersets $\{(\delta_1, \delta_2) \mid \delta_1^2 + \delta_2^2 \leq 3\}$ (left), $\{(\delta_1, \delta_2) \mid \delta_1 \in [-3, 2.2] \text{ and } \delta_2 \in [-2.1, 2]\}$ (middle) and $\mathrm{co}\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ (right).

respectively. Then

$$\mathbf{P}(\mathbf{\Delta}) := \left\{ \left(\begin{array}{cc|cc} Q_1 & 0 & S_1 & 0 \\ 0 & Q_2 & 0 & S_2 \\ \hline S_1^* & 0 & R_1 & 0 \\ 0 & S_2^* & 0 & R_2 \end{array}\right) \;\middle|\; \begin{array}{l} \begin{pmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{pmatrix} \in \mathbf{P}(\mathbf{\Delta}_1) \ \text{ and } \\ \begin{pmatrix} Q_2 & S_2 \\ S_2^* & R_2 \end{pmatrix} \in \mathbf{P}(\mathbf{\Delta}_2) \end{array} \right\}$$

is a suitable multiplier set for $\mathbf{\Delta}$. However, note that this set does not take any interaction between the components $\Delta_1$ and $\Delta_2$ into account.

- Suppose that $\mathbf{\Delta}_1 \subseteq \mathbf{\Delta}_2$ and that $\mathbf{P}(\mathbf{\Delta}_2)$ is a multiplier set corresponding to $\mathbf{\Delta}_2$. Then $\mathbf{P}(\mathbf{\Delta}_2)$ is also a multiplier set corresponding to $\mathbf{\Delta}_1$. In particular, even complex uncertainty sets can be captured by multiplier sets corresponding to rather basic uncertainty sets as long as the latter are supersets. This is also illustrated in Fig. C.2. However, such supersets are not always easy to detect in practice.

Next we introduce the so-called linear fractional representation frame-

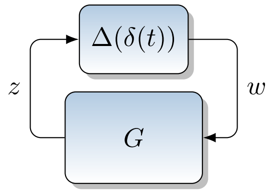



Figure C.3: Block diagram of the feedback interconnection (C.14).

work which is a powerful tool to systematically represent various types of uncertainties in dynamical systems and which permits a dedicated robustness analysis of the resulting models.

**Definition C.20** (Linear Fractional Representation (LFR)) *The map $F : \boldsymbol{\delta} \subset \mathbb{R}^r \to \mathbb{R}^{m\times n}$ admits a* linear fractional representation *if there exist matrices $A, B, C, D$ and a linear function $\Delta : \boldsymbol{\delta} \to \mathbb{R}^{q\times p}$ such that*

- *$I - D\Delta(\delta)$ is nonsingular for all $\delta \in \boldsymbol{\delta}$ and*
- *$F(\delta) = A + B\Delta(\delta)(I - D\Delta(\delta))^{-1}C =: \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \star \Delta(\delta)$ for all $\delta \in \boldsymbol{\delta}$.*

*The operation $\star$ is often called* lower linear fractional transformation *or* star-product*.*

A detailed discussion on the application of LFRs in the field of control can be found, e.g., in [149, 44, 178]. It is shown there, for example, that the sum, the product and the inverse of functions that admit an LFR does also admit an LFR. Moreover, we have the following result.

**Theorem C.21** *Let $F : \boldsymbol{\delta} \subset \mathbb{R}^r \to \mathbb{R}^{m\times n}$ be rational without a pole at zero. Then $F$ admits a linear fractional representation and we can construct matrices $A, B, C, D$ such that the corresponding linear function is given by $\Delta : \boldsymbol{\delta} \to \mathbb{R}^{q\times q},\ \delta \mapsto \mathrm{diag}(\delta_1 I_{\nu_1}, \ldots, \delta_r I_{\nu_r})$.*

Let us at first demonstrate that LFRs fit perfectly well to the ideas of separation presented so far. To this end, let us consider, for some initial condition $x(0) \in \mathbb{R}^n$, the autonomous system

$$\dot{x}(t) = F(\delta(t))x(t) \tag{C.12}$$

where $\delta$ is an unknown piecewise continuous function with values in some compact set $\boldsymbol{\delta} \subset \mathbb{R}^r$. It is not difficult to show that the system (C.12) is robustly stable, i.e., $\lim_{t\to\infty} x(t) = 0$ holds for all $\delta : [0, \infty) \to \boldsymbol{\delta}$ and all initial conditions $x(0) \in \mathbb{R}^n$, if there exists some matrix $X \succ 0$ satisfying

$$F(\delta)^\top X + XF(\delta) = \begin{pmatrix} I \\ F(\delta) \end{pmatrix}^\top \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \prec 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}. \tag{C.13}$$

Searching a suitable matrix $X$ does not constitute a practical robust stability test as infinitely many LMIs are involved and since $F$ is a rational function. However, if the function $F$ does admit an LFR, then we can express the system (C.12) equivalently as

$$\begin{pmatrix} \dot{x}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}, \quad w(t) = \Delta(\delta(t))z(t) \tag{C.14a,b}$$

by introducing two additional interconnection signals $w$ and $z$. In particular, by utilizing the LFR of $F$, we are able to express the uncertain system (C.12) as feedback interconnection of a known linear system (C.14a) and an unknown block (C.14b). This is illustrated in Fig. C.3 in terms of a block diagram where we identify the linear system (C.14a) with its transfer function $G(s) := C(sI - A)^{-1}B + D$. In this sense LFRs permit the *separation* of the unknown system components from the known ones and both components are only coupled through the interconnection signals $w$ and $z$. This leads to the following instrumental result which is taken from [140] and involves an application of the so-called full block S-procedure [138].

**Lemma C.22** (Concrete Full Block S-Procedure) *The LFR of $F$ is well-posed, i.e., the matrix $I - D\Delta(\delta)$ is nonsingular for all $\delta \in \boldsymbol{\delta}$, and the robust analysis LMI* (C.13) *holds if and only if there exists a symmetric matrix $P$ satisfying*

$$(\bullet)^\top \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + (\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \prec 0 \quad \textit{and} \quad (\bullet)^\top P \begin{pmatrix} I \\ \Delta(\delta) \end{pmatrix} \succcurlyeq 0 \tag{C.15}$$

*for all $\delta \in \boldsymbol{\delta}$.*

Similarly as before the LMI condition (C.15) is still numerically problematic, but one obtains tractable criteria by employing suitable multiplier sets as defined earlier and corresponding to the set $\{\Delta(\delta) \ : \ \delta \in \boldsymbol{\delta}\}$.

*Proof. Necessity:* See [138] or [140].
*Sufficiency:* The right lower block of the first LMI in (C.15) implies the inequality $\left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right)^\top P \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \prec 0$. By Lemma C.17 we can then conclude well-posedness of the LFR.

Next, we fix some $\delta \in \boldsymbol{\delta}$ and abbreviate $H := (I - D\Delta(\delta))^{-1}C$. Then we observe that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} I \\ H \end{pmatrix} = \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ H \end{pmatrix} = \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} H$$

hold. Consequently, we infer from (C.15)

$$\begin{aligned} (\bullet)^\top \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I \\ F(\delta) \end{pmatrix} &= (\bullet)^\top \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} I \\ H \end{pmatrix} \\ &\prec -(\bullet)^\top P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ H \end{pmatrix} = -(\bullet)^\top P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} H \preccurlyeq 0. \end{aligned}$$

This yields the claim since $\delta \in \boldsymbol{\delta}$ was arbitrary. ●

A major benefit of the formulation in (C.14) as depicted in Fig. C.3 over the one in (C.12) is that the former allows us to replace the uncertain time-varying matrix in (C.14b) with essentially any uncertain operator or (troublesome) object. In particular, in structured singular value theory [44, 178, 141] one studies the properties of feedback interconnections of linear systems as in (C.14a) with $w = \Delta z$ and for an uncertain $\Delta$ with a block diagonal structure

$$\Delta(s) = \operatorname{diag}\big(r_1 I, \ldots, r_{n_r} I, \delta_1(s) I, \ldots, \delta_{n_c} I, \Delta_1(s), \ldots, \Delta_{n_f}(s)\big);$$

here, the identity matrices can vary in size and $r_i \in \mathbb{R}$, $\delta_i \in \mathrm{RH}_\infty^{1\times 1}$ as well as $\Delta_i \in \mathrm{RH}_\infty^{\bullet\times\bullet}$; here, $\mathrm{RH}_\infty^{m\times n}$ denotes the set of real rational proper $m \times n$ matrices without poles in the extended closed right half-plane. It is shown that the resulting interconnections allow for efficiently modeling various types of uncertainties in dynamical systems such as parametric uncertainties, neglected dynamics and mixtures thereof. This high flexibility comes in tandem with dedicated extensions of Lemma C.22 and implementable derivations. Integral quadratic constraint theory [109] provides even more general separation based results and allows for additionally dealing, e.g., with uncertain time-delays and nonlinearities.

# D

# Differential Linear Matrix Inequalities

Most of the analysis and design criteria presented in this thesis are formulated in terms of so-called differential linear matrix inequalities (DLMIs). These appeared for the first time in the context of linear quadratic control for linear systems on a finite horizon where they are tightly linked to the Riccati differential equation and inequality. See, e.g., [25, 93, 111] for some of the early publications on this topic. They still emerge frequently when dealing with systems admitting finite horizon characteristics such as the hybrid systems considered in this work. Inequalities of this type also appear in analysis and design criteria for systems affected by constant or rate bounded uncertainties as, e.g., in [56, 5].

We base our discussion on the following canonical description of a DLMI and the corresponding DLMI problems.

**Definition D.1** *Let $F : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{S}^m$ be an affine map and $c : \mathbb{R}^n \to \mathbb{R}$ be linear. Further, let $\Omega$ be some set and suppose that $\tau_0 \in \Omega$.*

- *A* DLMI *is an expression of the form $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ for all $\tau \in \Omega$.*
- *A* DLMI feasibility problem *amounts to testing whether there exist some $x \in \mathbb{R}^n$ and some differentiable function $f : \Omega \to \mathbb{R}^r$ such that $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ holds for all $\tau \in \Omega$. The DLMI $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ is said to be* feasible *if the result of the latter test is in the affirmative.*
- *A* DLMI optimization problem *constitutes the minimization of the* cost $c(x)$ *over all vectors $x \in \mathbb{R}^n$ and all differentiable functions $f : \Omega \to \mathbb{R}^r$ that satisfy the DLMI $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ for all $\tau \in \Omega$.*

Note that this canonical description of a DLMI does not fit perfectly to the inequalities considered in this work, for example, since we sometimes face functions $F$ depending on multiple evaluations of $f$ (and $\dot{f}$) at fixed points $\tau_1, \ldots, \tau_N \in \Omega$. Nevertheless, this description covers the essential features.

Observe that DLMI feasibility and optimization problems are infinite dimensional which means that they cannot be numerically solved directly. A general recipe in order to reduce these problems to finite dimensional ones is to restrict the search to functions of the form

$$f(\tau) = \sum_{k=1}^{N} f_k b_k(\tau)$$

with scalar-valued differentiable basis functions $b_1, \ldots, b_N$, that span some finite dimensional space $S$, and free decision variables $f_1, \ldots, f_N \in \mathbb{R}^r$. The most common choices for $S$ are the space of polynomials of some fixed degree and finite dimensional subspaces of the space of piecewise polynomials; spaces containing rational or more involved function are rarely used. This approach allows us to express the DLMI $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ for all $\tau \in \Omega$ equivalently as

$$\tilde{F}(\tilde{x}, \tau) = \tilde{F}(x, f_1, \ldots, f_N, \tau) \preccurlyeq 0 \quad \text{for all} \quad \tau \in \Omega \tag{D.1}$$

for some function $\tilde{F}$ that is affine for any fixed $\tau$ and whose coefficients are affine combinations of $b_k(\tau)$ and $\dot{b}_k(\tau)$. An inequality of this form is usually referred to as parameter dependent LMI. Note that finding some decision variable $\tilde{x} = (x, f_1, \ldots, f_N)$ satisfying the latter inequality still constitutes a numerically intractable problem since this amounts to solving infinitely many LMIs. Moreover, we stress that introducing a discretization $\tau_0, \ldots, \tau_M$ of $\Omega$ and simply replacing (D.1) with the finite number of inequalities

$$\tilde{F}(\tilde{x}, \tau_k) \preccurlyeq 0 \quad \text{for all} \quad k \in \{0, \ldots, M\} \tag{D.2}$$

leads to a problem than can be handled numerically, but there is in general *no guarantee* that some $\tilde{x}$ with (D.2) also satisfies (D.1). Instead, one typically employs so-called inner approximations that assure (D.1) and usually rely on rather simple basis functions $b_1, \ldots, b_N$.

Next we illustrate some possibilities to (inner) approximate DLMI problems via finite dimensional SDPs and start by briefly recalling the matrix sum-of-squares (SOS) approach from [118, 146] that is also discussed, e.g., in [42, Chapter 2] and [142]. The latter two publications also elaborate on several alternative approaches such as those based on the KYP lemma [123] or on separation techniques similarly as briefly demonstrated in Section C.6. Further approximation strategies can be found, e.g., in [65]. As a demonstration, we will apply such approximations to the nominal stability result Theorem 2.3 and recapitulate here a variation thereof for convenience.

**Theorem D.2** (Clock-Based Stability Analysis) *The system* (2.1) *is stable for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exists a function* $X \in C^0([0, T_{\max}], \mathbb{S}^n)$ *that is piecewise continuously differentiable with finitely many pieces which satisfies the inequalities*

$$X(\tau) \succ 0 \quad \text{and} \quad \dot{X}(\tau) + A^\top X(\tau) + X(\tau)A \prec 0 \quad \text{for all} \quad \tau \in [0, T_{\max}] \tag{D.3a}$$

*as well as*

$$A_J^\top X(0) A_J - X(\tau) \prec 0 \quad \text{for all} \quad \tau \in [T_{\min}, T_{\max}]. \tag{D.3b}$$

## D.1 Sum-of-Squares Relaxation

Sum-of-squares polynomials constitute a topic with a rather long history in mathematics and a survey thereof is provided in [125]. Applications in the field of optimization and control appeared, e.g., in [118], and the extension to matrix valued polynomials[1] is due to [146].

**Definition D.3** (Sum-of-Squares Matrices) $P \in \mathbb{R}[x]^{p\times p}$ *is said to be a* sum-of-squares (SOS) matrix *if there exists* $Q \in \mathbb{R}[x]^{q\times p}$ *with* $q \in \mathbb{N}$ *such that* $P(x) = Q(x)^\top Q(x)$. *If* $p = 1$, *then* $P$ *is said to be an* SOS polynomial.

If $p = 1$, then $Q(x) = \mathrm{col}(Q_1(x), \ldots, Q_q(x))$ is a polynomial vector and we can express $P$ as

$$P(x) = \sum_{k=1}^{q} Q_k(x)^2$$

which motivates the terminology. The interest in SOS matrices in the control literature stems from the fact that SOS matrices constitute a nice

[1] In the sequel, $\mathbb{R}[x]^{q\times p}$ denotes the set of polynomials in the variable $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{R}^{q\times p}$. Any $P \in \mathbb{R}[x]^{q\times p}$ can be expressed as $P(x) = \sum_{\alpha\in\mathbb{N}_0^n,\, |\alpha|\le d} P_\alpha x^\alpha$ with coefficients $P_\alpha$ by recalling the multi-index notation $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. If $P_\alpha \neq 0$ for some $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = d$, then $d$ is the *degree* of $P$.

numerically tractable approximation to globally positive semidefinite polynomial matrices. Indeed, it is rather immediate that we have

$$P(x) \succcurlyeq 0 \text{ for all } x \in \mathbb{R}^n \quad \text{if} \quad P \text{ is an SOS matrix.}$$

Moreover, we have the following result which states that testing the SOS property amounts to solving an LMI feasibility problem.

**Theorem D.4** (Characterization of the SOS Property) *Let $P \in \mathbb{R}[x]^{p \times p}$ have degree $2d$ and let us abbreviate $u_d(x) := (1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^d)^\top$. Then $P$ is an SOS matrix if and only if there exists a symmetric matrix $X$ satisfying*

$$P(x) = (u_d(x) \otimes I_p)^\top X (u_d(x) \otimes I_p) \quad \text{and} \quad X \succcurlyeq 0.$$

Instead of global positivity properties one is typically much more interested in positivity on prescribed subsets of $\mathbb{R}^n$. In the context of SOS matrices, one usually employs subsets with the following general description.

**Definition D.5** (Semi-Algebraic Sets) *A set $\mathcal{G} \subset \mathbb{R}^n$ is called a (basic)* semi-algebraic set *if there exist polynomials $q_1, \ldots, g_q \in \mathbb{R}[x]$ such that $\mathcal{G} = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_q(x) \geq 0\}$.*

One of the possibilities to guarantee positivity of some $P \in \mathbb{R}[x]^{p \times p}$ on a semi-algebraic set $\mathcal{G}$ is motivated by introducing multipliers analogously as in Lagrange duality theory (see, e.g., [17, Section 5.2] and [115, Section 12.9]). More precisely, $P(x) \succcurlyeq 0$ holds for all $x \in \mathcal{G}$ if there exist positive semidefinite matrices $\Lambda_1, \ldots, \Lambda_q$ such that

$$P(x) - \sum_{k=1}^{q} \Lambda_k g_k(x) \succcurlyeq 0 \quad \text{holds for all} \quad x \in \mathbb{R}^n.$$

In general, the latter test is improved by considering polynomial matrices $\Lambda_1, \ldots, \Lambda_q$ and can be rendered computational by replacing global positive semidefiniteness with the SOS property. This gives the following.

**Lemma D.6** *Let $P \in \mathbb{R}[x]^{p\times p}$ be given. Then $P(x) \succcurlyeq 0$ holds for all $x \in \mathcal{G}$ if there exist SOS matrices $\Lambda_1, \ldots, \Lambda_q \in \mathbb{R}[x]^{p\times p}$ such that $P - \sum_{k=1}^{q} \Lambda_k g_k$ is an SOS matrix.*

This observation permits us to define a *hierarchy of relaxations* for a number of optimization problems involving infinitely many LMIs such as in (D.1) with a basis consisting of polynomials. The following theorem from [146] states that this hierarchy consists of improving relaxations and is even *asymptotically exact*. It relies on an extension of a famous result from [121] to matrix valued polynomials.

**Theorem D.7** (Asymptotically Exact SOS Relaxation Hierarchy) *Let $c \in \mathbb{R}^m$ and the polynomial matrix $P(x,y) = \sum_{\alpha\in\mathbb{N}_0^n,\, |\alpha|\leq d_a} L_\alpha(y)x^\alpha$ with degree $d_a \in \mathbb{N}_0$ and affine functions $L_\alpha : \mathbb{R}^m \to \mathbb{S}^p$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq d_a$ be given. Moreover, define*

$$\gamma_{\text{opt}} := \inf\left\{c^\top y \;\middle|\; P(x,y) \succ 0 \;\; \textit{for all} \;\; x \in \mathcal{G}\right\}$$

*and, for any $d \in \mathbb{N}_0$,*

$$\gamma_d := \inf\left\{ c^\top y \;\middle|\; \begin{array}{l} \textit{There exist SOS matrices } \Lambda_1, \ldots, \Lambda_n \in \mathbb{R}[x]^{p\times p} \\ \textit{of degree } d \textit{ and some } \varepsilon > 0 \textit{ such that} \\ P(\cdot, y) - \varepsilon I - \sum_{k=1}^{q} \Lambda_k g_k \textit{ is an SOS matrix} \end{array} \right\}.$$

*Then the following statements hold.*

*(a) $\gamma_d$ can be computed by solving a standard linear SDP.*

*(b) The sequence $(\gamma_d)_{d\in\mathbb{N}_0}$ is monotonically decreasing and bounded from below by $\gamma_{\text{opt}}$.*

(c) *The sequence* $(\gamma_d)_{d\in\mathbb{N}_0}$ *converges to* $\gamma_{\text{opt}}$ *for* $d \to \infty$ *if one of the polynomials* $g_k$ *characterizing the set* $\mathcal{G}$ *equals* $r^2 - \|x\|^2$ *for some fixed radius* $r > 0$.

Applying the matrix SOS relaxation to the DLMI stability criteria in Theorem D.2 yields the following set of conditions that also appeared in [18] and which can be turned into a standard SDP via Theorem D.4.

**Corollary D.8** (Stability Analysis via Sum-of-Squares) *The system* (2.1) *is stable for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist a polynomial matrix* $X \in \mathbb{R}[\tau]^{n\times n}$ *of degree* $d_a$*, SOS matrices* $\Lambda_1, \Lambda_2, \Lambda_J \in \mathbb{R}[\tau]^{n\times n}$ *of degree* $d_m$ *and some* $\varepsilon > 0$ *such that*

$$X - \varepsilon I - \Lambda_1 g, \quad -\big[\dot{X} + A^\top X + XA\big] - \varepsilon I - \Lambda_2 g$$

*and*

$$-[A_J^\top X(0) A_J - X] - \varepsilon I - \Lambda_J g_J$$

*are SOS matrices. Here, the polynomials* $g$ *and* $g_J$ *are given by* $g(\tau) := (T_{\max} - \tau)\tau$ *and* $g_J(\tau) := (T_{\max} - \tau)(\tau - T_{\min})$*, respectively.*

**Remark D.9** (a) By now several software packages are available that allow for introducing polynomial variables, their manipulation and for turning SOS constraints into SDP constraints which can dealt with by various SDP solvers such as LMIlab [55], SeDuMi [155] and Mosek [113]. One of these packages is SOSTOOLS [117] and the very versatile parser Yalmip [101] also supports optimization problems involving SOS matrices.

(b) SDPs resulting from SOS constraints are often costly to solve for reasonably large underlying systems. There are several techniques for improving the computational efficiency of such SDPs by relying on sparsity. For example one can exploit the inherent sparsity of the SDPs resulting from SOS constraints or make use of structure in the underlying system [3]; both approaches are implemented in the Matlab packages SOSADMM [176] and CDCS [177]. Another possibility to reduce the computational burden for solving DLMIs such as (D.3) is the application of the diagonally-dominant SOS or scaled-diagonally-dominant SOS approaches from [3, 2] instead of the SOS relaxation. While this typically leads to less accurate results, severe speed-ups can be observed as these approaches are based on solving linear and second order cone programs, respectively.

## D.2 Piecewise Convex/Concave Polynomial Relaxation

In contrast to the previous section, we now consider an approach that relies on searching a map $f$ satisfying the DLMI $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ for all $\tau \in \Omega := [a, b]$ in the space of piecewise polynomial functions defined on the grid $a = \lambda_1 < \cdots < \lambda_{M+1} = b$. It is then natural to consider the inequality $F(x, f(\tau), \dot{f}(\tau), f(\tau_0)) \preccurlyeq 0$ individually on each of the smaller intervals $[\lambda_k, \lambda_{k+1}]$. Note that, as explicitly required in Theorem D.2, we usually have to ensure continuity of the function $f$, but enforcing $f$ to be continuously differentiable everywhere is not necessary. Moreover, because the restriction of $F(x, f(\tau), \dot{f}(\tau), f(\tau_0))$ on the interval $[\lambda_k, \lambda_{k+1}]$ depends polynomially on $\tau$, one could apply an SOS approach for each of these intervals. However the resulting SDP would then be rather costly to solve even for small values $M > 2$. Instead, the approach presented in this sections relies on convexity and concavity and, more precisely, on the following result.

**Lemma D.10** *Let* $G : [\lambda_k, \lambda_{k+1}] \to \mathbb{S}^m$, $G(\tau) := F(x, f(\tau), \dot{f}(\tau), f(\tau_0))$. *Then the following two statements hold.*

*(a) Suppose that* $G$ *is convex*[2]. *Then* $G(\tau) \preccurlyeq 0$ *holds for all* $\tau \in [\lambda_k, \lambda_{k+1}]$ *if and only if* $G(\lambda_k) \preccurlyeq 0$ *and* $G(\lambda_{k+1}) \preccurlyeq 0$ *hold.*

*(b) Suppose that* $G$ *is two times continuously differentiable on* $(\lambda_k, \lambda_{k+1})$. *Then* $G$ *is convex if and only if* $\ddot{G}(\tau) \succcurlyeq 0$ *holds for all* $\tau \in (\lambda_k, \lambda_{k+1})$.

Of course, applying Lemma D.10 makes only sense if the inequality $\ddot{G}(\tau) \succcurlyeq 0$ for all $\tau \in (\lambda_k, \lambda_{k+1})$ is very easy to guarantee.

## D.2.1 Piecewise Linear Polynomials

The idea to employ piecewise linear polynomials for solving DLMIs appeared, e.g., in [4] in the context of switched systems and in [106] for solving general parameter dependent LMIs involving functions on more general domains. Linear polynomials are particularly convenient because they and their derivatives are automatically convex *and* concave. Hence, no additional constraints are required to enforce convexity.

Concretely, by applying this approach to the DLMI conditions in Theorem D.2, we obtain the following result involving only standard LMIs.

**Corollary D.11** (Stability Analysis via Piecewise Linear Polynomials) *Suppose that we are given, for some* $M, N \in \mathbb{N}$, *a grid defined by*

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{M+1} = T_{\min} < \lambda_{M+2} < \cdots < \lambda_{M+N+1} = T_{max}.$$

*Then the system* (2.1) *is stable for all* $(t_k)_{k \in \mathbb{N}_0}$ *satisfying* (RDT) *if there exist symmetric matrices* $X_1, \ldots, X_{M+N+1} \in \mathbb{R}^{n \times n}$ *satisfying*

$$X_k \succ 0 \quad \textit{for all} \quad k \in \{1, \ldots, M+N+1\},$$

[2] A function $G : \Omega \to \mathbb{S}^m$ is called *convex* if $G((1-t)x + ty) \preccurlyeq (1-t)G(x) + tG(y)$ for all $t \in [0,1]$ and all $x, y \in \Omega$. The function $G$ is called *concave* if $-G$ is convex.

$$\partial_k X_k + A^\top X_k + X_k A \prec 0 \quad \textit{and} \quad \partial_k X + A^\top X_{k+1} + X_{k+1} A \prec 0$$

*for all* $k \in \{1, \ldots, M+N\}$ *as well as*

$$A_J^\top X_1 A_J - X_k \prec 0$$

*for all* $k \in \{M+1, \ldots, M+N+1\}$. *Here, we employ the abbreviation* $\partial_k X := \frac{1}{\lambda_{k+1}-\lambda_k}(X_{k+1} - X_k)$ *for all* $k \in \{1, \ldots, M+N\}$.

If the above LMIs are feasible, we recover a function $X \in C^0([0, T_{\max}], \mathbb{S}^n)$ that is piecewise continuously differentiable satisfying (D.3) by defining

$$X(\tau) := X_k \frac{\tau - \lambda_k}{\lambda_{k+1} - \lambda_k} + X_{k+1} \frac{\lambda_{k+1} - \tau}{\lambda_{k+1} - \lambda_k}$$

for all $\tau \in [\lambda_k, \lambda_{k+1}]$ and all $k$. Indeed, this defines a continuous function which satisfies $X(\lambda_k) = X_k$ for all $k$ and $\dot{X}(\tau) = \partial_k X$ for all $\tau \in (\lambda_k, \lambda_{k+1})$ and all $k$.

**Remark D.12** Employing piecewise linear polynomials usually requires rather fine grids in order to achieve comparable results to those obtained via an SOS approach. This is illustrated in our numerical experiments in Section D.4 and similar observations have been made, e.g., in [22] in the context of stability and stabilization of impulsive and switched positive systems. Intuitively, the reason for this issue is that we essentially *simultaneously* approximate a function $f$ and its derivative $\dot{f}$. While $f$ is approximated with piecewise linear polynomials, its derivative is merely approximated with piecewise constant ones that are known to admit very limited approximation properties. Since such fine grids lead to a large number of decision variables that can be difficult to handle, the piecewise linear polynomial approach is generally considered inferior if compared to the SOS approach.

### D.2.2 Piecewise Quadratic Polynomials

Due to the downsides of piecewise linear polynomials, it is natural to consider piecewise quadratic ones in a next step. In general, these have better approximation properties and their second derivative is a piecewise constant function which is very convenient for ensuring convexity or concavity on sub-intervals. Unfortunately, we will argue that, for our purposes, it is still numerically challenging to enforce convexity or concavity in a suitable fashion. To this end, observe that employing piecewise quadratic polynomials and utilizing Lemma D.10 leads to the following extension of Corollary D.11.

**Corollary D.13** (Stability Analysis via Piecewise Quadratic Polynomials) *Let $M, N \in \mathbb{N}$ and suppose that we are given a grid defined by*

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{M+1} = T_{\min} < \lambda_{M+2} < \cdots < \lambda_{M+N+1} = T_{max}.$$

*Then the system* (2.1) *is stable for all $(t_k)_{k \in \mathbb{N}_0}$ satisfying* (RDT) *if there exist symmetric matrices $X_1, \ldots, X_{M+N+1}, \tilde{X}_1, \ldots, \tilde{X}_{M+N} \in \mathbb{R}^{n \times n}$ satisfying*

$$A^\top \tilde{X}_k + \tilde{X}_k A \preccurlyeq 0, \quad \tilde{X}_k \succcurlyeq 0,$$

$$\partial_k X_k + A^\top X_k + X_k A \prec 0, \;\; \partial_k X_{k+1} + A^\top X_{k+1} + X_{k+1} A \prec 0 \;\; \text{and} \;\; X_k \succ 0$$

*for all $k \in \{1, \ldots, M+N\}$ and $X_{M+N+1} \succ 0$ as well as*

$$X_1 - A_J^\top X_k A_J \prec 0$$

*for all $k \in \{M+1, \ldots, M+N+1\}$. Here, we employ, for any $k$, the abbreviations $\partial_k X_k := \frac{1}{\lambda_{k+1} - \lambda_k}(X_{k+1} - X_k) + (\lambda_{k+1} - \lambda_k)\tilde{X}_k$ and $\partial_k X_{k+1} := \frac{1}{\lambda_{k+1} - \lambda_k}(X_{k+1} - X_k) - (\lambda_{k+1} - \lambda_k)\tilde{X}_k$.*

If the above LMIs are feasible, we recover a function $X \in C^0([0, T_{\max}], \mathbb{S}^n)$

that is piecewise continuously differentiable satisfying (D.3) by defining

$$X(\tau) := \tilde{X}_k(\lambda_{k+1}-\tau)(\tau-\lambda_k) + X_k\frac{\lambda_{k+1}-\tau}{\lambda_{k+1}-\lambda_k} + X_{k+1}\frac{\tau-\lambda_k}{\lambda_{k+1}-\lambda_k}$$

for all $\tau \in [\lambda_k, \lambda_{k+1}]$ and all $k$. Indeed, this defines a continuous function which satisfies $\ddot{X}(\tau) = -2\tilde{X}_k$ for all $\tau \in (\lambda_k, \lambda_{k+1})$, $X(\lambda_k) = X_k$ as well as

$$\lim_{\tau \nearrow \lambda_{k+1}} \dot{X}(\tau) = -(\lambda_{k+1}-\lambda_k)\tilde{X}_k + \frac{1}{\lambda_{k+1}-\lambda_k}(X_{k+1}-X_k) = \partial_k X_{k+1}$$

and

$$\lim_{\tau \searrow \lambda_k} \dot{X}(\tau) = (\lambda_{k+1}-\lambda_k)\tilde{X}_k + \frac{1}{\lambda_{k+1}-\lambda_k}(X_{k+1}-X_k) = \partial_k X_k$$

for all $k$. Moreover, note that we recover the conditions in Corollary D.11 by setting $\tilde{X}_k = 0$ for all $k$.

**Remark D.14** While the inequalities in Corollary D.13 are all standard LMIs, the inequalities $A^\top \tilde{X}_k + \tilde{X}_k A \preccurlyeq 0$ and $\tilde{X}_k \succcurlyeq 0$ constitute numerically troublesome ones in general. This issue stems from the fact that, for impulsive systems as described by (2.1), the matrix $A$ is usually not Hurwitz stable and, consequently, the latter two LMIs can not be strictly satisfied. Typically, this is poison to any solver for semidefinite programs. In order to circumvent this issue, one should perform some preprocessing by factorizing the matrices $\tilde{X}_k = T^\top \left(\begin{smallmatrix} 0 & 0 \\ 0 & \tilde{X}_k^r \end{smallmatrix}\right) T$ according to the eigenspaces of the matrix $A$ similarly as done, e.g., in [135, 136], but we do not further pursue this approach here.

Considering piecewise polynomials of a higher degree is possible, but enforcing piecewise convexity/concavity suffers from the same deficiencies as mentioned in Remark D.14 for our applications. For our purposes, merely the piecewise linear polynomials work “well” in this regard because convexity/concavity is guaranteed at the outset.

# D.3 B-Spline Relaxation

In this section we consider another relaxation hierarchy which, this time, relies on different representations of splines that are composed of so-called B-splines. In particular, as in the previous section, we deal with piecewise polynomial functions, but we enjoy much more flexibility. This relaxation hierarchy is also very briefly discussed in [77] and included in the Linear Control Toolbox [164] in Matlab. Here, we will elaborate on it in full detail and, in particular, prove that this hierarchy is asymptotically exact which has, to the best of our knowledge, not been done yet. Thereby, we focus our attention on univariate splines and B-splines because multivariate splines and B-splines inherit most of their properties from univariate ones.

Finally, most of the following results from spline theory are taken from the textbooks [104] and [36]. At this point, my thanks also go to Jörg Hörner and Benjamin Ziegler for stimulating discussions and sharing some of their knowledge on splines.

## D.3.1 Definition and Basic Properties of B-Splines

We start by presenting the basic definition of B-splines and splines as well as some basic properties of them. B-splines are piecewise polynomials and their definition relies on knot sequences which induce a partition of their domain.

**Definition D.15** (Knot Sequence) *Let $a, b \in \mathbb{R}$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$.*

- *The finite sequence $\lambda = (\lambda_i) \in [a, b]^{k+n+1}$ is called a* knot sequence *if*
$$a = \lambda_1 \leq \cdots \leq \lambda_{k+n+1} = b.$$
- *The knot $t \in [a, b]$ is said to have* multiplicity $m \in \mathbb{N}_0$ *in $\lambda$ if $t$ appears exactly $m$ times in the knot sequence $\lambda$. In particular, if $t$ does not appear in $\lambda$, then it has multiplicity* $0$.

- *The knot sequence* $\lambda$ *is called* $k+1$ regular *if the knots* $a$ *and* $b$ *have multiplicity* $k+1$ *and no other knot has multiplicity larger than* $k+1$.
- *The* $k+1$*-regular knot sequence* $\lambda$ *is called* simple *if all knots* $\lambda_i$ *with* $\lambda_i \notin \{a, b\}$ *have multiplicity* $1$.

The basic definition of B-splines is now as follows.

**Definition D.16** (B-Splines) *Let* $a, b \in \mathbb{R}$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ *and let* $\lambda \in [a, b]^{k+n+1}$ *be a knot sequence. Then, for* $j \in \{1, \ldots, n\}$, *the* $j$*th* B-spline *of* degree $k$ *corresponding to the knot sequence* $\lambda$ *is defined by*

$$B_{j,k,\lambda}(x) := \omega_{j,k,\lambda}(x) B_{j,k-1,\lambda}(x) + (1 - \omega_{j+1,k,\lambda}(x)) B_{j+1,k-1,\lambda}(x)$$

*where* $B_{j,0,\lambda} = \chi_{[\lambda_j, \lambda_{j+1})}$ *is the indicator function of the interval* $[\lambda_j, \lambda_{j+1})$ *and where*

$$\omega_{j,k,\lambda}(x) := \begin{cases} \frac{x - \lambda_j}{\lambda_{j+k} - \lambda_j} & \text{if } \lambda_j \neq \lambda_{j+k}, \\ 0 & \text{otherwise} \end{cases}$$

*for all* $x \in [a, b)$. *If* $\lambda$ *is* $k+1$ *regular, we set* $B_{n,k,\lambda}(b) := 1$.

As an illustration, let us consider the $k+1$ regular knot sequences defined by

$$k = 1, \quad n = 5, \quad \lambda = \big(-1,\ -1,\ -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ 1,\ 1\big), \tag{D.6a}$$

$$k = 2, \quad n = 6, \quad \lambda = \big(-1,\ -1,\ -1,\ -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ 1,\ 1,\ 1\big), \tag{D.6b}$$

$$k = 2, \quad n = 8, \quad \lambda = \big(-1,\ -1,\ -1,\ -\tfrac{1}{2},\ 0,\ 0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ 1,\ 1,\ 1\big). \tag{D.6c}$$

The corresponding B-splines $B_{1,k,\lambda}, \ldots, B_{n,k,\lambda}$ are depicted in Fig. D.1. Note that the knot sequences in (D.6a) and (D.6b) are simple, but the one in (D.6c) is not simple, because the inner knot 0 has multiplicity 2.

Equipped with the definition of B-splines, we now state some of their basic and highly useful properties. These are all consequences of the re-

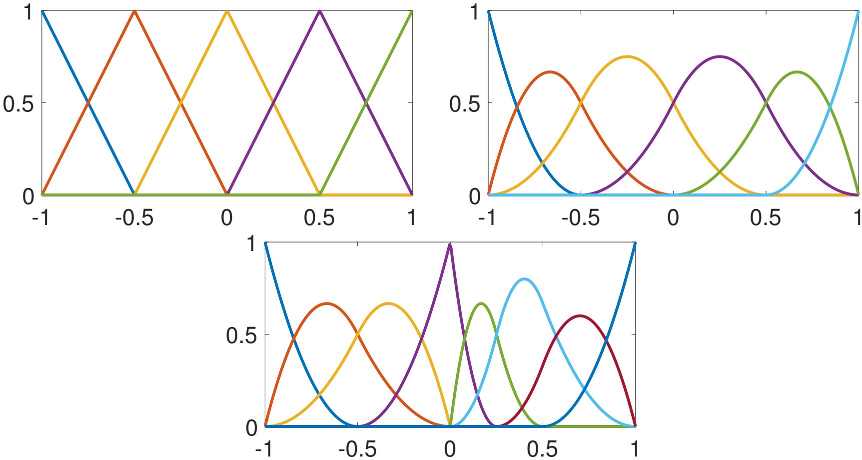



Figure D.1: The B-splines $B_{1,k,\lambda}, \dots, B_{n,k,\lambda}$ corresponding to the knot sequences defined in (D.6a) (top left), (D.6b) (top right) and (D.6c) (bottom).

cursion formula in Definition D.16, which is also referred to as *recurrence relation*, and can for example be found in [36, 150].

**Lemma D.17** (Basic Properties) *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and let $\lambda \in [a,b]^{k+n+1}$ be a knot sequence. Then the B-splines admit the following properties.*

*(a)* Positivity*: $B_{j,k,\lambda}(x) \geq 0$ for all $x \in [a,b]$.*

*(b)* Partition of unity*: $\sum_{j=1}^{n} B_{j,k,\lambda}(x) = 1$ for all $x \in [a,b]$.*

*(c)* Local support*: $B_{j,k,\lambda}(x) = 0$ for all $x \notin [\lambda_j, \lambda_{j+k+1}]$.*

The space of scalar-valued splines of degree $k$ corresponding to a knot sequence $\lambda$ is now naturally defined as the linear span of all B-splines of degree $k$ corresponding to the knot sequence $\lambda$. However, for our purposes, we consider matrix-valued splines.

**Definition D.18** (Space of Splines) *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $\lambda \in [a,b]^{k+n+1}$ be a knot sequence. The* space of (univariate polynomial matrix-valued) splines *of degree $k$ corresponding to the knot sequence $\lambda$ is defined as*

$$S_{k,\lambda}^{p\times q} := \left\{ \sum_{j=1}^{n} C_j B_{j,k,\lambda} \;\middle|\; C_j \in \mathbb{R}^{p\times q} \quad \text{for all} \quad j \in \{1,\dots,n\} \right\}.$$

*If the dimensions are not relevant we write $S_{k,\lambda}$ instead of $S_{k,\lambda}^{p\times q}$.*

As an example let us consider again the knot sequences (D.6b) and (D.6c) and the scalar-valued splines $S$ and $\hat{S}$ defined by their (stacked) coefficients

$$c := \left(\tfrac{1}{2},\ 1,\ 1,\ -\tfrac{1}{4},\ 1,\ 2\right)^\top \quad \text{and} \quad \hat{c} := \left(\tfrac{1}{2},\ 1,\ 1,\ -\tfrac{1}{2},\ 1,\ 2,\ -1,\ 2\right)^\top, \tag{D.7a,b}$$

respectively. The splines $S$ and $\hat{S}$ are depicted in Fig. D.2 together with their corresponding weighted B-splines $c_1 B_{1,k,\lambda}, \dots, c_n B_{n,k,\lambda}$ and $\hat{c}_1 B_{1,k,\lambda}, \dots, \hat{c}_n B_{n,k,\lambda}$, respectively. Note that both splines are continuous and observe that $\hat{S}$ is not continuously differentiable at the point 0. The latter stems from 0 being a knot with multiplicity 2 in the knot sequence (D.6c). Indeed, this is a consequence from a more general statement which is included in the following remark.

**Remark D.19**

- The B-splines $B_{1,k,\lambda}, \dots, B_{n,k,\lambda}$ are linearly independent and form a basis of the space of scalar-valued splines $S_{k,\lambda}^{1\times 1}$ if the knot sequence $\lambda$ is $k+1$ regular. Hence, the space $S_{k,\lambda}^{p\times q}$ has dimension $npq$ in this case.
- One can show that any $S \in S_{k,\lambda}^{p\times q}$ is a matrix-valued piecewise polynomial of degree $k$, which constitutes an alternative definition of the space of spline.
- A spline of degree $k$ corresponding to a $k+1$ regular knot sequence

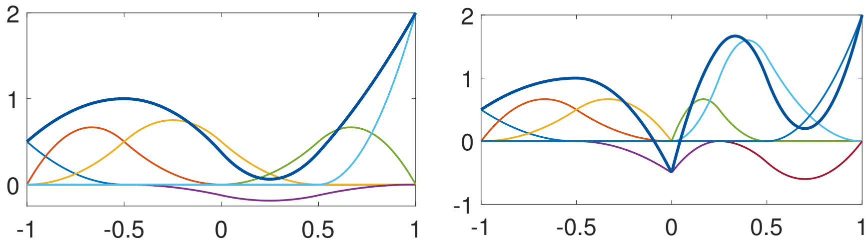



Figure D.2: (Left) the spline $S$ defined by the coefficients (D.7a) and the knot sequence (D.6b) as well as (right) the spline $\hat{S}$ defined by the coefficients (D.7b) and the knot sequence (D.6c). The corresponding scaled B-splines are illustrated as well.

is $d \leq k-1$ times continuously differentiable at an inner knot $\lambda_j \in (a,b)$ if $\lambda_j$ has multiplicity $k-d$. In particular, if $\lambda$ is a simple knot sequence, then $S_{k,\lambda} \subset C^{k-1}([a,b])$.

We are now in position to state an important intermediate idea underlying the B-spline relaxation for solving parameter dependent LMIs. It is a consequence of the positivity of the B-spline functions and since they do not vanish all at the same time.

**Lemma D.20** *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\lambda \in [a,b]^{k+n+1}$ be a knot sequence and suppose that $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda}^{p\times p}$ is given. Then $S(x) \succ 0$ holds for all $x \in [a,b]$ if $C_j \succ 0$ holds for all $j \in \{1,\ldots,n\}$.*

Note that, obviously, testing whether $C_j \succ 0$ holds for all $j \in \{1,\ldots,n\}$ is a standard LMI and hence amounts to a numerically tractable test. The converse of Lemma D.20 is not true in general as illustrated by the spline $S$ on the left of Fig. D.2. Indeed, this spline is positive on its domain, but one of its defining coefficients in (D.7a) is negative.

The B-spline relaxation hierarchy relies on finer representation of a given spline as we precisely discuss next.

## D.3.2 Spline Representations

In this subsection we present the main ingredients for the B-spline relaxation hierarchy. The key observation is that a spline $S$ in some spline space $S_{k,\lambda}$ is also contained in a certain ordered sequence of higher dimensional spline spaces $(S_{k_\nu,\lambda_\nu})_{\nu\in\mathbb{N}}$, i.e.,

$$S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda} \subset S_{k_1,\lambda_1} \subset S_{k_2,\lambda_2} \subset \dots$$

Thus, instead of considering the original coefficients $C_1, \dots, C_n$, we can also consider the coefficients of the same spline in the space $S_{k_\nu,\lambda_\nu}$ and apply Lemma D.20 there. In the sequel, we illustrate that this procedure is indeed beneficial.

The main operations for generating dedicated higher dimensional spline spaces are *knot insertion* and *degree elevation* which we both discuss next.

### Knot Insertion

Inserting knots into a given knot sequence will yield a refinement of this sequence. Refinements are naturally defined as follows.

**Definition D.21** (Refinements) *Let $\lambda \in [a,b]^m$ be a knot sequence. A knot sequence $\tilde{\lambda} \in [a,b]^{\tilde{m}}$ is said to be a* refinement *of $\lambda$ and we write $\lambda \subset \tilde{\lambda}$ if the multiplicity of any knot $t$ in the sequence $\lambda$ is not larger than the multiplicity of $t$ in the sequence $\tilde{\lambda}$.*

Note that the knot sequence (D.6c) is a refinement of (D.6b) and that the latter is a refinement of (D.6a). The following instrumental result involving refined knot sequences is from [31].

**Lemma D.22** (Knot Insertion) *For $k \in \mathbb{N}_0$ and $n, m \in \mathbb{N}$ let $\lambda \in [a,b]^{k+n+1}$ be a knot sequence with refinement $\tilde{\lambda} \in [a,b]^{k+m+1}$. Then the following statements hold.*

(a) $S^{p\times q}_{k,\lambda} \subset S^{p\times q}_{k,\tilde{\lambda}}$.

(b) $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} = \sum_{j=1}^{m} \tilde{C}_j B_{j,k,\tilde{\lambda}}$ *with* $\tilde{C}_l = \sum_{j=1}^{n} a^{(k)}_{lj} C_j$ *for all* $l \in \{1,\ldots,m\}$. *The numbers* $a^{(k)}_{lj}$ *are recursively given as*

$$a^{(k)}_{lj} = \omega_{j,k,\lambda}(\tilde{\lambda}_{l+k}) a^{(k-1)}_{lj} + (1-\omega_{j+1,k,\lambda}(\tilde{\lambda}_{l+k})) a^{(k-1)}_{l,j+1}$$

*and* $a^{(0)}_{lj} = B_{j,0,\lambda}(\tilde{\lambda}_l)$ *with* $\omega_{j,k,\lambda}$ *given in Definition D.16.*

(c) *The numbers* $a^{(k)}_{lj}$ *are sometimes called discrete B-splines and admit the following properties.*

- *If* $\tilde{\lambda}_l < \lambda_j$ *or* $\tilde{\lambda}_{l+k} \geq \lambda_{j+k+1}$ *then* $a^{(k)}_{lj} = 0$.
- $a^{(k)}_{lj} \geq 0$ *for all* $l \in \{1,\ldots,m\}$ *and all* $j \in \{1,\ldots,n\}$.
- $\sum_{j=1}^{n} a^{(k)}_{lj} = 1$ *for all* $l \in \{1,\ldots,m\}$.

*In particular, the* knot insertion matrix $M^{\mathrm{ki}}_{k,\tilde{\lambda},\lambda} := (a^{(k)}_{lj,\tilde{\lambda},\lambda})_{lj} \in \mathbb{R}^{m\times n}$ *is a sparse row stochastic matrix*[3].

Recall that we were unable to guarantee positivity of the scalar-valued spline $S$ depicted on the left hand side of Fig. D.2 corresponding knot sequence (D.6b) and with stacked coefficients $c = (\frac{1}{2}, 1, 1, -\frac{1}{4}, 1, 2)^\top$ by employing the test in Lemma D.20. Moreover, as stated before, the knot sequence (D.6c) is a refinement of (D.6b). By applying the above result, we can hence alternatively express $S$ as sum of B-splines corresponding to (D.6c) and with the new stacked coefficients

$$\tilde{c} := \left(\tfrac{1}{2},\ 1,\ 1,\ \tfrac{3}{8},\ \tfrac{1}{16},\ \tfrac{1}{16},\ 1,\ 2\right)^\top.$$

[3] A matrix $A = (a_{lj})_{lj} \in \mathbb{R}^{m\times n}$ is said to be *row stochastic* if its entries are nonnegative and the entries of each row sum up to one.

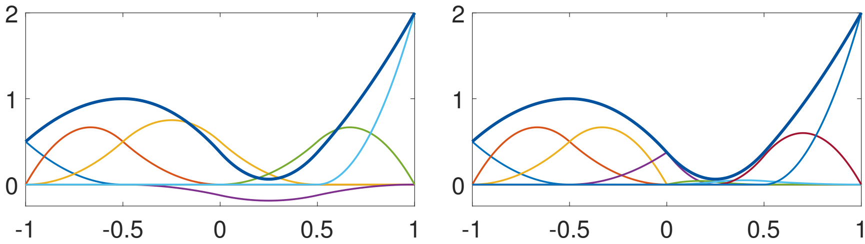



Figure D.3: (Left) the spline $S$ defined by the coefficients $c$ in (D.7) and the knot sequence (D.6b) as well as (right) the same spline represented as sum of B-splines corresponding to the knot sequence (D.6c). The corresponding scaled B-splines are illustrated as well.

Both representations are illustrated in Fig. D.3. For the new representation of $S$, the test in Lemma D.20 is in the affirmative and, hence, we have shown that $S(x) > 0$ holds for all $x \in [-1, 1]$. In total, this demonstrates the benefit utilizing knot refinements in tandem with Lemma D.20.

Let us stress that statement (b) implies, in particular, that the new coefficients are easily obtained by a *linear combination* of the original ones. This is of tremendous importance for our applications since the original coefficients will depend in an affine fashion on decision variables and because handling our optimization problems requires expressions that are affine in all decision variables! For later, note that we can also express statement (b) as follows. Suppose that $S \in S_{k,\lambda}^{p\times q}$ has the coefficients $C_1, \ldots, C_n$ and let us abbreviate $C := \mathrm{col}(C_1, \ldots, C_n)$. Then the coefficients of the refined representation of the spline $S$ in $S_{k,\tilde{\lambda}}^{p\times q}$ are given by the $p \times q$ blocks of the matrix

$$\tilde{C} = (M_{k,\tilde{\lambda},\lambda}^{\mathrm{ki}} \otimes I_p)C, \quad \text{i.e.,} \quad \tilde{C}_j = (e_j \otimes I_p)^\top (M_{k,\tilde{\lambda},\lambda}^{\mathrm{ki}} \otimes I_p)C$$

for $j \in \{1, \ldots, m\}$. Here, $e_1, \ldots, e_m$ denote the standard unit vectors in $\mathbb{R}^m$.

Moreover and as a consequence of statement (c), we have the following result that will ensure monotonicity of the B-spline hierarchy based on knot insertion.

**Corollary D.23** (Monotonicity of the Hierarchy) *Let $k \in \mathbb{N}_0$, $n, m \in \mathbb{N}$, let $\lambda \in [a,b]^{k+n+1}$ be a knot sequence with refinement $\tilde{\lambda} \in [a,b]^{k+m+1}$ and suppose that $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda}^{p\times p}$ is given. Then $C_j \succ 0$ for all $j \in \{1,\ldots,n\}$ implies that $\sum_{j=1}^{n} a_{lj}^{(k)} C_j \succ 0$ holds for all $j \in \{1,\ldots,m\}$.*

In other words, once the coefficients of a spline are positive definite, all alternative representations resulting from inserting additional knots also admit positive definite coefficients. Note that consecutive refinement via knot insertion are related as follows if the final knot sequence is $k+1$ regular and, hence, the corresponding B-splines are linearly independent.

**Lemma D.24** (Consecutive Refinements) *For $k \in \mathbb{N}_0$ let $\lambda \in [a,b]^{k+n+1}$, $\tilde{\lambda} \in [a,b]^{k+\tilde{n}+1}$ and $\hat{\lambda} \in [a,b]^{k+\hat{n}+1}$ be $k+1$ regular knot sequences satisfying $\hat{\lambda} \supset \tilde{\lambda} \supset \lambda$. Then the corresponding knot insertion matrices are related as $M_{k,\hat{\lambda},\lambda}^{\mathrm{ki}} = M_{k,\hat{\lambda},\tilde{\lambda}}^{\mathrm{ki}} M_{k,\tilde{\lambda},\lambda}^{\mathrm{ki}}$.*

**Remark D.25** If the refined knot sequence $\tilde{\lambda}$ results from $\lambda$ by inserting a single knot, then the knot insertion matrix is easily computed without a recursion and admits a particular triangular and band structure [36, page 136]. In particular, due to Lemma D.24, we can write any knot insertion matrix as the product of very specific rectangular matrices. However, we do not exploit this further.

Next we consider alternative spline representations resulting from degree elevation.

## Degree Elevation

It is clear that each spline of degree $k$ can be written as a spline of degree $k+1$ since any polynomial of degree $k$ can be viewed as a polynomial of degree $k+1$. The following result from [32] suggests a suitable corresponding knot sequence and yields a practical formula for the construction of the new coefficients. In particular, these new coefficients are again a linear combination of the original ones.

**Lemma D.26** (Degree Elevation) *For $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ let $\lambda \in [a,b]^{k+n+1}$ be a $k+1$ regular knot sequence composed of knots $\xi_1 < \cdots < \xi_\nu$ with multiplicities $m_1, \ldots, m_\nu$. Further, let $\tilde{\lambda}$ be the knot sequence composed of the same knots, but with multiplicities $m_1+1, \ldots, m_\nu+1$. Then the following statements hold.*

*(a)* $S_{k,\lambda}^{p\times q} \subset S_{k+1,\tilde{\lambda}}^{p\times q}$.

*(b)* $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} = \sum_{j=1}^{\tilde{n}} \tilde{C}_j B_{j,k+1,\tilde{\lambda}}$ *with* $\tilde{C}_l = \frac{1}{k+1}\sum_{j=1}^{n} \Lambda_{lj}^{(k)} C_j$ *for all* $l \in \{1,\ldots,\tilde{n}\}$ *and with* $\tilde{n} := n+\nu-1$. *The numbers* $\Lambda_{lj}^{(k)}$ *are recursively given as*

$$\Lambda_{lj}^{(k)} = \omega_{j,k,\lambda}(\tilde{\lambda}_{l+k+1})\Lambda_{lj}^{(k-1)} + (1-\omega_{j+1,k,\lambda}(\tilde{\lambda}_{l+k+1}))\Lambda_{l,j+1}^{(k-1)} + a_{lj,\tilde{\lambda},\lambda}^{(k)}$$

*and* $\Lambda_{lj}^{(0)} = a_{lj}^{(0)}$ *with* $\omega_{j,k,\lambda}$ *as in Definition D.16 and* $a_{lj}^{(k)}$ *as in Lemma D.22.*

*(c) The numbers* $\Lambda_{lj}^{k}$ *admit the following properties.*

- $\Lambda_{lj}^{(k)} \geq 0$ *for all* $l \in \{1,\ldots,\tilde{n}\}$ *and all* $j \in \{1,\ldots,n\}$.
- $\sum_{j=1}^{n} \Lambda_{lj}^{(k)} = k+1$ *for all* $l \in \{1,\ldots,\tilde{n}\}$.

*In particular, the* degree elevation matrix $M_{k,\lambda}^{\mathrm{de}} := \frac{1}{k+1}(\Lambda_{lj}^{(k)})_{lj}$ *is a row stochastic matrix.*

Let us again consider the scalar-valued spline $S$ depicted on the left of Fig. D.2 corresponding knot sequence (D.6b) and with stacked coefficients $c = (\frac{1}{2}, 1, 1, -\frac{1}{4}, 1, 2)^\top$. By applying the above result, we can equivalently express $S$ as sum of B-splines of degree $k+1 = 3$ corresponding to a

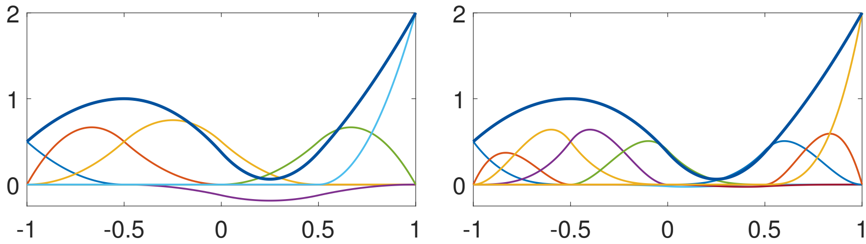



Figure D.4: (Left) the spline $S$ defined by the coefficients (D.7a) and the knot sequence (D.6b) as well as (right) the same spline represented as sum of B-splines of degree $k+1=3$. The corresponding scaled B-splines are illustrated as well.

modified knot sequence and with the new stacked coefficients

$$\tilde{c} := \left(\tfrac{1}{2},\ \tfrac{5}{6},\ 1,\ 1,\ \tfrac{19}{24},\ -\tfrac{1}{24},\ -\tfrac{1}{24},\ \tfrac{19}{24},\ \tfrac{4}{3},\ 2\right)^\top.$$

Both representations are illustrated in Fig. D.4. For the new representation the test in Lemma D.20 is still not in the affirmative, but the magnitude of the coefficients with negative sign is much smaller than before. For this example, after two further degree elevations the criteria in Lemma D.20 are satisfied which demonstrates the benefit utilizing degree elevation in tandem with Lemma D.20.

Due to the last statement of Lemma D.26, the complete analogue of Corollary D.23 holds and reads as follows.

**Corollary D.27** (Monotony of the Hierarchy) *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, let $\lambda \in [a,b]^{k+n+1}$ be a $k+1$ regular knot sequence and suppose that the spline $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda}^{p\times p}$ is given. Then $C_j \succ 0$ for all $j \in \{1,\dots,n\}$ implies that $\frac{1}{k+1}\sum_{j=1}^{n} \Lambda_{lj}^{(k)} C_j \succ 0$ holds for all $j \in \{1,\dots,\tilde{n}\}$.*

### Differentiation of Splines

Because we consider mainly differential LMIs in this work, it is natural that we need access to the derivative of a given spline or, more precisely, to its coefficients in a suitable representation. Moreover, these coefficients should be again a linear combination of the ones from the original spline. Fortunately, there is again such an explicit expression for the coefficients that can be found, e.g., in [36] and is repeated here. To this end, recall that we have already mentioned that splines in $S_{k,\lambda}$ are $d$ times continuously differentiable where $d \leq k-1$ depends on the multiplicity of the inner knots in the knot sequence $\lambda$.

**Lemma D.28** (Derivative of a Spline) *Let $k, n \in \mathbb{N}$ and let $\lambda \in [a,b]^{k+n+1}$ be a knot sequence such that $S_{k,\lambda} \subset C^1([a,b])$. Further, suppose that $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda}$. Then*

$$\dot{S} = \sum_{j=1}^{n-1} \tilde{C}_j B_{j,k-1,\tilde{\lambda}}$$

*with*

$$\tilde{C}_j = \alpha_{j+1,k,\lambda}(C_{j+1} - C_j) \quad \text{for all} \quad j \in \{1,\ldots,n-1\}.$$

*Here, the knot sequence $\tilde{\lambda}$ is obtained by discarding the first and last knot in $\lambda$ and $\alpha_{j,k,\lambda} := \frac{k}{\lambda_{j+k}-\lambda_j}$ if $\lambda_{j+k} \neq \lambda_j$ and $\alpha_{j,k,\lambda} := 0$ otherwise.*

For reasons of compatibility, we define the *derivative matrix* $M^{\mathrm{d}}_{k,\lambda} = (a_{lj})_{lj} \in \mathbb{R}^{(n-1)\times n}$ by

$$a_{lj} := \begin{cases} -\alpha_{j+1,k,\lambda} & \text{if } l = j, \\ \alpha_{j+1,k,\lambda} & \text{if } l = j-1, \\ 0 & \text{otherwise} \end{cases}$$

in order to infer that $\tilde{C}_l = \sum_{j=1}^{n} a_{lj} C_j$ holds for all $l \in \{1,\ldots,n-1\}$.

Note that we usually face conditions that simultaneously involve a variable $X$ and its derivative $\dot{X}$ such as in the inequality $\dot{X} + A^\top X + XA \prec 0$ on some interval $[a, b]$ and for some matrix $A \in \mathbb{R}^{n\times n}$. In order to apply Lemma D.20, it is mandatory to represent both $\dot{X}$ and $X$ as functions in a common spline space. This can be achieved as follows. Suppose that $X = \sum_{j=1}^n X_j B_{j,k,\lambda}$ with coefficients $X_1, \ldots, X_n \in \mathbb{R}^{n\times n}$ and for some given degree $k$ and knot sequence $\lambda$. Then, by Lemma D.28, the derivative of $X$ is given by

$$\dot{X} = \sum_{j=1}^{n-1} \tilde{X}_j B_{j,k-1,\tilde{\lambda}} \quad \text{with} \quad \tilde{X}_j := (e_j^\top M^{\mathrm{d}}_{k,\lambda} \otimes I_n)\mathrm{col}(X_1, \ldots, X_n)$$

and with $\tilde{\lambda}$ being the knot sequence obtained by discarding the first and last knot in $\lambda$. In order to achieve a common representation, the idea is to apply a degree elevation to $S_{k-1,\tilde{\lambda}} \subset S_{k,\hat{\lambda}}$ with yields another knot sequence $\hat{\lambda}$. By its construction, $\hat{\lambda}$ is a refinement of $\lambda$ and a knot insertion yields $S_{k,\lambda} \subset S_{k,\hat{\lambda}}$. In particular, we obtain

$$X = \sum_{j=1}^{\hat{n}} \hat{X}_j B_{j,k,\hat{\lambda}} \in S^{n\times n}_{k,\hat{\lambda}} \quad \text{with} \quad \hat{X}_j := (e_j^\top M^{\mathrm{ki}}_{k,\hat{\lambda},\lambda} \otimes I_n)\mathrm{col}(X_1, \ldots, X_n)$$

and

$$\dot{X} = \sum_{j=1}^{\hat{n}} \check{X}_j B_{j,k,\hat{\lambda}} \in S^{n\times n}_{k,\hat{\lambda}} \quad \text{with} \quad \check{X}_j := (e_j^\top M^{\mathrm{de}}_{k-1,\tilde{\lambda}} M^{\mathrm{d}}_{k,\lambda} \otimes I_n) \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

i.e., a common representation as desired. Thus, by applying Lemma D.20, we can ensure that the inequality $\dot{X} + A^\top X + XA \prec 0$ holds on $[a, b]$ if $\check{X}_j + A^\top \hat{X}_j + \hat{X}_j A \prec 0$ holds for all $j \in \{1, \ldots, \hat{n}\}$.

There are many more results on spline representations. We just mention two additional ones of them that can be useful for our purposes.

**Remark D.29** (a) *Representation of Polynomials:* Let $p$ be a polynomial of degree $\leq k$ and let $\lambda$ be a $k+1$ regular knot sequence. Then the so-called Marsden's identity states that the polynomial $p$ can be represented as linear combination of B-splines of degree $k$ corresponding to the knot sequence $\lambda$ and also provides an efficient way to compute the coefficients of the linear combination. The precise result can for example be found in [104].

(b) *Representation of Spline Products:* Intuitively, it is rather obvious that the product of two splines is again a spline. However, formulating and deriving the concrete representation as provided in [112] is tedious. In particular, the number of knots required for the representation is rather large in general which is partly due to the continuity requirements.

## D.3.3 Asymptotic Exactness

In this subsection, we precisely define the B-spline relaxation hierarchy and show that this hierarchy has the beautiful property of being asymptotically exact. The latter property is a consequence of characteristics of the so-called *control polygon* which is defined as follows.

**Definition D.30** (Control Polygon) *Let $k, n \in \mathbb{N}$, let $\lambda \in [a,b]^{k+n+1}$ be a knot sequence and suppose that $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda}$. Then the* control polygon *corresponding to $S \in S_{k,\lambda}$ is the piecewise linear interpolant to the points $(\lambda_1^*, C_1), \ldots, (\lambda_n^*, C_n)$, where $\lambda_j^* := (\lambda_{j+1}, \ldots, \lambda_{j+k})/k$ is the* $j$th knot average*.*

Let us once more consider the spline $S$ defined by (D.6b) and (D.7a). This spline and its control polygon are depicted on the left of Fig. D.5. The control polygon corresponding to the spline $S$ after a knot insertion and a degree elevation is illustrated in the middle and on the right of Fig. D.5, respectively. In both of the latter two cases, we observe that the control

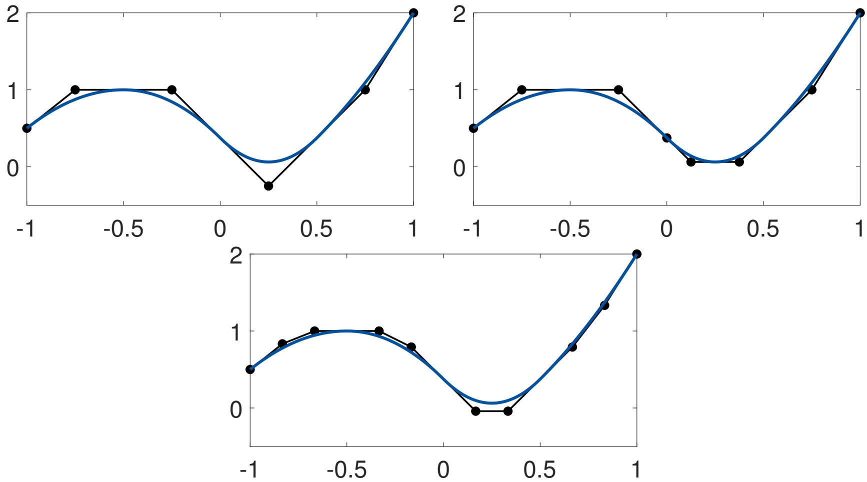



Figure D.5: (Top left) the spline $S$ defined by (D.6b) and (D.7a) as well as its control polygon. (Top right) the control polygon of $S$ after adding the knots 0 and $\frac{1}{4}$ to (D.6b). (Bottom) the control polygon of $S$ after performing a degree elevation.

polygon approaches the spline it is corresponding to. This observation is a provable fact that holds for splines and relies on the following two results from [104, Lemma 9.17] and [30], respectively.

**Lemma D.31** *Let $k, n \in \mathbb{N}$, let $\lambda \in [a, b]^{k+n+1}$ be a knot sequence and suppose that $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda}$. Then*

$$\|C_j - S(\lambda_j^*)\| \leq M(\lambda_{j+k} - \lambda_{j+1})^2 \quad \textit{holds for all} \quad j \in \{1, \ldots, n\}$$

*where $M$ is a constant that only depends on $S$ and $k$.*

**Lemma D.32** *Let $k, n \in \mathbb{N}$, let $\lambda \in [a,b]^{k+n+1}$ be a $k+1$ regular knot sequence and suppose that $S = \sum_{j=1}^{n} C_j B_{j,k,\lambda} \in S_{k,\lambda}$. Moreover, let $\Gamma_\nu$ denote the control polygon corresponding to $S$ after performing $\nu$ degree elevations. Then there exists a constant $M$ such that*

$$\max_{x \in [a,b]} \|S(x) - \Gamma_\nu(x)\| \leq M \frac{1}{\nu} \quad \textit{holds for all} \quad \nu \in \mathbb{N}.$$

A consequence of Lemma D.31 is that, loosely speaking, the control polygon obtained via suitable successive knot insertions converges quadratically to the spline it is corresponding to. This is in stark contrast to the linear convergence obtained via degree elevation in Lemma D.32. In the sequel we, thus, focus on the B-spline hierarchy based on knot insertions for brevity.

Combining Lemmas D.20 and D.31 yields the following instrumental result.

**Lemma D.33** *Let $k, n \in \mathbb{N}$, let $\lambda \in [a,b]^{k+n+1}$ be a $k+1$ regular knot sequence and suppose that $S := \sum_{j=1}^{n} C_j B_{j,k,\lambda}$. Moreover, let $(\lambda^\nu)_{\nu \in \mathbb{N}}$ be a sequence of refinements of $\lambda$ with $\lambda^\nu \in [a,b]^{k+n_\nu+1}$ satisfying $\lambda \subset \lambda^1 \subset \lambda^2 \subset \dots$ and $\lim_{\nu \to \infty} \max_{j \in \{1,\dots,n_\nu\}} |\lambda^\nu_{j+k} - \lambda^\nu_{j+1}| = 0$. Finally, let $C_1^\nu, \dots, C_{n_\nu}^\nu$ denote the coefficients of $S$ with respect to the knot sequence $\lambda^\nu$. Then $S(x) \succ 0$ holds for all $x \in [a,b]$ if and only if there exists some $\nu \in \mathbb{N}$ such that $C_j^\nu \succ 0$ holds for all $j \in \{1, \dots, n_\nu\}$.*

*Proof.* Sufficiency is a consequence of Lemma D.20. Necessity is obtained as follows. Due to the continuity of $S$ and compactness of $[a,b]$, we infer the existence of some $\varepsilon > 0$ such that $S(x) - \varepsilon I \succ 0$ holds for all $x \in [a,b]$. Let us define $\lambda_j^{\nu,*} := (\lambda^\nu_{j+1} + \dots + \lambda^\nu_{j+k})/k$ and $h_\nu := \max_{j \in \{1,\dots,n_\nu\}} |\lambda^\nu_{j+k} - \lambda^\nu_{j+1}|$ for all $\nu$ and $j$. From Lemma D.31 we infer the existence of some constant $M$ such that

$$\|C_j^\nu - S(\lambda_j^{\nu,*})\| \leq M(\lambda^\nu_{j+k} - \lambda^\nu_{j+1})^2 \leq M h_\nu^2$$

holds for all $j \in \{1, \ldots, n_\nu\}$ and all $\nu \in \mathbb{N}$. In particular, we have

$$C_j^\nu = C_j^\nu - S(\lambda_j^{\nu,*}) + S(\lambda_j^{\nu,*}) \succ C_j^\nu - S(\lambda_j^{\nu,*}) + \varepsilon I$$

for all $j \in \{1, \ldots, n_\nu\}$ and all $\nu \in \mathbb{N}$. Since $\|C_j^\nu - S(\lambda_j^{\nu,*})\|$ converges to 0 uniformly in $j$ and by continuity, we conclude the existence of some $\nu_0 \in \mathbb{N}$ such that $C_j^{\nu_0} - S(\lambda_j^{\nu_0,*}) + \varepsilon I \succ 0$ for all $j \in \{1, \ldots, n_{\nu_0}\}$. This yields $C_j^{\nu_0} \succ 0$ for all $j \in \{1, \ldots, n_{\nu_0}\}$. ●

We are now in position to precisely introduce the B-spline relaxation hierarchy based on knot insertion and refinements and state its properties. The latter are consequences of combining Lemma D.33 and Corollary D.23.

**Theorem D.34** (Asymptotically Exact B-Spline Relaxation Hierarchy via Knot Insertion) *Let $c \in \mathbb{R}^m$, $k, n \in \mathbb{N}$ and let $\lambda \in [a,b]^{k+n+1}$ be a $k+1$ regular knot sequence. Further, suppose that $C_1, \ldots, C_n : \mathbb{R}^m \to \mathbb{S}^p$ are affine functions and define*

$$\gamma_{\mathrm{opt}} := \inf \left\{ c^\top y \;\middle|\; \sum_{j=1}^{n} C_j(y) B_{j,k,\lambda}(x) \succ 0 \;\; \text{for all} \;\; x \in [a,b] \right\}.$$

*Moreover, let $(\lambda^\nu)_{\nu \in \mathbb{N}}$ be a sequence of knot sequences $\lambda^\nu \in [a,b]^{k+n_\nu+1}$ satisfying $\lambda \subset \lambda^1 \subset \lambda^2 \subset \ldots$ and $\lim_{\nu\to\infty} \max_{j\in\{1,\ldots,n_\nu\}} |\lambda_{j+k}^\nu - \lambda_{j+1}^\nu| = 0$. Finally, define*

$$\gamma_\nu := \inf \left\{ c^\top y \;\middle|\; \begin{array}{l} (e_j^\top M_{k,\lambda^\nu,\lambda}^{\mathrm{ki}} \otimes I_p)\mathrm{col}(C_1(y), \ldots, C_n(y)) \succ 0 \\ \text{for all} \quad j \in \{1, \ldots, n_\nu\} \end{array} \right\}.$$

*Then the following statements hold.*

*(a)* $\gamma_\nu$ *can be computed by solving a standard linear SDP.*

*(b)* $(\gamma_\nu)_{\nu \in \mathbb{N}_0}$ *is monotonically decreasing and bounded from below by* $\gamma_{\mathrm{opt}}$.

*(c)* $\lim_{\nu\to\infty} \gamma_\nu = \gamma_{\mathrm{opt}}$.

**Remark D.35** (a) A concrete strategy for constructing a suitable refinement is to insert the knots $\tilde{\xi}_j = \frac{1}{2}(\xi_j + \xi_{j+1})$ if the current knot sequence is composed of the knots $\xi_1 < \cdots < \xi_m$, i.e., to insert knots at the midpoints of the intervals defined by the current knot sequence. An adaptive alternative strategy could be as follows. Suppose that $\gamma_\nu$ is computed and pick some feasible $\tilde{y} \in \mathbb{R}^m$ such that $c^\top \tilde{y}$ is close to $\gamma_\nu$. Next, consider the corresponding spline $\tilde{S} = \sum_{j=1}^n C_j(\tilde{y}) B_{j,k,\lambda}$ and insert knots at locations where $S(x)$ has eigenvalues close to zero or at places that are interesting for other reasons in order to construct a suitable refined knot sequence.

(b) We stress that the relaxation hierarchy based on degree elevation is defined analogously and admits the same properties. Moreover, note that there can be situations where a combination of knot insertions and degree elevations is profitable.

As a final illustration, let us apply the B-spline relaxation to the DLMI stability criteria in Theorem D.2 which yields the following set of standard LMIs.

**Corollary D.36** *Let $k, n \in \mathbb{N}$ and let $\lambda \in [0, T_{\max}]^{k+n+1}$ be a $k+1$ regular knot sequence which contains the knot $\lambda_m = T_{\min}$ exactly once. Further, let*

- *$\lambda^d$ be the knot sequence obtained by discarding the first and last knot in $\lambda$,*
- *$\lambda^{de}$ the sequence obtained by increasing the multiplicity of each knot in $\lambda^d$ by one,*
- *$\lambda^{ki}$ be the sequence obtained by $k$ times inserting the knot $T_{\min}$ into the sequence $\lambda$ such that discarding the knots $\lambda_1, \ldots, \lambda_{m-1}$ yields a $k+1$ regular knot sequence $\lambda^{res} \in [T_{\min}, T_{\max}]^{k+n_{res}+1}$ with $n_{res} = n + k - m + 1$.*

*Moreover, let* $\lambda^D \in [0, T_{\max}]^{k+n^D+1}$ , $\lambda^F \in [0, T_{\max}]^{k+n^F+1}$ *and* $\lambda^J \in [T_{\min}, T_{\max}]^{k+n^J+1}$ *be refinements of* $\lambda$, $\lambda^{de}$ *and* $\lambda^{res}$, *respectively. Then the system* (2.1) *is stable for all* $(t_k)_{k\in\mathbb{N}_0}$ *satisfying* (RDT) *if there exist* $X_1, \ldots, X_n \in \mathbb{S}^n$ *satisfying*

$$X_j^D \succ 0 \quad \textit{for all} \quad j \in \{1, \ldots, n^D\},$$

$$\dot{X}_j^F + A^\top X_j^F + X_j^F A \prec 0 \quad \textit{for all} \quad j \in \{1, \ldots, n^F\}$$

*and*

$$A_J^\top X_0^J A_J - X_j^J \prec 0 \quad \textit{for all} \quad j \in \{1, \ldots, n^J\}.$$

*Here, the matrices with indices are given by* $X_0^J := \sum_{j=1}^n X_j B_{j,k,\lambda}(0)$ *as well as*

$$X_j^D := (e_j^\top M^{\mathrm{ki}}_{k,\lambda^D,\lambda} \otimes I_n)X,$$

$$\dot{X}_j^F := (e_j^\top M^{\mathrm{ki}}_{k,\lambda^F,\lambda^{de}} M^{\mathrm{de}}_{k-1,\lambda^d} M^{\mathrm{d}}_{k,\lambda} \otimes I_n)X,$$

$$X_j^F := (e_j^\top M^{\mathrm{ki}}_{k,\lambda^F,\lambda} \otimes I_n)X$$

*and*

$$X_j^J := (e_j^\top M^{\mathrm{ki}}_{k,\lambda^J,\lambda^{res}} \begin{pmatrix} 0_{n^{res}\times(m-1)} & I_{n^{res}} \end{pmatrix} M^{\mathrm{ki}}_{k,\lambda^{ki},\lambda} \otimes I_n)X$$

*where* $X := \mathrm{col}(X_1, \ldots, X_n)$.

In particular, if the above LMIs are feasible, then $X = \sum_{j=1}^n X_j B_{j,k,\lambda}$ is a continuous function satisfying the DLMIs (D.3).

Note that the knot sequence $\lambda^{ki}$ is introduced because it guarantees that the B-splines $B_{1,k,\lambda^{ki}}, \ldots, B_{m-1,k,\lambda^{ki}}$ and $B_{m,k,\lambda^{ki}}, \ldots, B_{n+k,k,\lambda^{ki}}$ are supported only on the compact interval $[0, T_{\min}]$ and $[T_{\min}, T_{\max}]$, respectively. This is important since otherwise one might enforce negative definiteness of $A_J^\top X(0) A_J - X$ on a larger interval than $[T_{\min}, T_{\max}]$ which can lead to conservatism.

## D.4 A Numerical Comparison

In order to provide a brief comparison of the presented relaxations for numerically dealing with differential LMIs, we consider the stability criteria in Theorem D.2 and compare Corollaries D.8, D.11 and D.36 on a variety of examples that are generated as follows. We take several continuous-time LTI systems with description

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B_1 & B \\ C_1 & D_{11} & D_{12} \\ C & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \\ u(t) \end{pmatrix} \tag{D.8}$$

for $t \geq 0$ from the COMPl$_\mathrm{e}$ib [96] collection and design, for each of these open-loop systems, a static output-feedback $H_\infty$-controller $K_F$ via the Matlab command `hinfstruct` [9]. Additionally, we convert the continuous-time model (D.8) into a discrete-time model by applying a zero-order-hold approach with $T = 0.5$ time units via the command `c2d` and design another static output-feedback $H_\infty$-controller $K_J$ for the resulting model. Finally, we interconnect both static controllers to the original system by defining the control input signal $u$ as the convex combination

$$u(t) := \frac{2}{5} u_F(t) + \frac{3}{5} u_J(t)$$

with

$$u_F(t) := K_F y(t) \quad \text{and} \quad u_J(t) := u_J(t_k) := K_J y(t_k)$$

for all $t \in [t_k, t_{k+1})$ and all $k \in \mathbb{N}_0$; here $(t_k)_{k \in \mathbb{N}_0}$ is some given sequence of impulse instants. Note that $u_J$ is a piecewise constant function with values determined by the controller $K_J$ and samples of the signal $y$ at the time instants $t_k$. If we now drop the performance channel $d \to e$ in the description (D.8), we can equivalently express the resulting closed-loop interconnection as a linear impulsive system of the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{u}_J(t) \end{pmatrix} = \begin{pmatrix} A + \frac{2}{5}BK_F C & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ u_J(t) \end{pmatrix}, \quad \begin{pmatrix} x(t_k) \\ u_J(t_k) \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{3}{5}K_J C & 0 \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ u_J(t_k^-) \end{pmatrix} \tag{D.9}$$

for $t \geq 0$ and $k \in \mathbb{N}$, which we can analyze via Theorem D.2. We are then able, for example, to numerically determine the largest number $T_{\max}$ such that the interconnection (D.9) is stable for all sequences of impulse instants $(t_k)_{k \in \mathbb{N}_0}$ satisfying a range dwell-time condition (RDT) on the interval $[0.1, T_{\max}]$ by employing Corollary D.8, D.11 or D.36 as well as a bisection. Note that it is impossible to simplify the analysis criteria in Theorem D.2 by restricting $X$ to be a constant matrix because the describing matrices of the flow and jump component of (D.9) are not Hurwitz and not Schur stable, respectively.

Table D.1 displays the obtained results, where we denote by $T_{SOS}^{d_a,d_m}$, $T_{LS}^{M,N}$ and $T_{BS}^{M,N,r}$ the numerically determined largest number $T_{\max} \in [0.1, 40]$ such that the inequalities in Corollary D.8, D.11 and D.36 are feasible, respectively. The superscripts in the numbers $T_{SOS}^{d_a,d_m}$ and $T_{LS}^{M,N}$ denote the parameters appearing in Corollary D.8 and D.11, respectively. For the application of Corollary D.36, we employ B-splines of degree $k = 3$ and use a $k+1$ regular knot sequence corresponding to the partition used in Corollary D.11 which is defined by the numbers $M$ and $N$. We refine the emerging knot sequences as mentioned in the beginning of Remark D.35 (a) $r$ times. All computations are carried out with Matlab on a general purpose desktop computer (Intel Core i7, 4.0 GHz, 8 GB of ram) and we use Yalmip [101] together with Mosek [113]. Note that we render all inequalities strict by adding or subtracting $\varepsilon I$ with $\varepsilon = 10^{-3}$; reducing the size of $\varepsilon$ often yields better results, but also promotes the occurrence of numerical issues.

Moreover, we denote the average running times within twenty runs in seconds for applying Corollary D.8, D.11 and D.36 with $T_{\max} = 1$ analogously as $A_{SOS}^{d_a,d_m}$, $A_{LS}^{M,N}$ and $A_{BS}^{M,N,r}$. These numbers are depicted in Table D.2 where $n_{xu}$ stands for the number of columns in $(A, B)$.

We observe that the linear spline approach often requires rather fine grids to get results compatible with the other two approaches. This leads to a large number of decision variables which might trouble some solvers. Mosek [113] is one of the solvers that seems to be good in exploiting the resulting underlying problem structure and can handle such fine grids. Moreover, note that the computation of $T_{SOS}^{6,2}$ seems to be problematic for several of the considered examples. The numerically determined values are actually smaller than the ones for $T_{SOS}^{4,2}$ and, hence, only a poor approximation of $T_{SOS}^{6,2}$. Nevertheless, the SOS approach is usually superior if compared to the one using linear splines as demonstrated, e.g., in [22, 82]. For most of the examples, the B-spline approach is the least conservative one, while, at same time, being faster than the SOS or the linear spline approach. Next to the nice approximation properties of splines, another reason for this might be that relatively few decision variables but many constraints are involved in the B-spline approach.

Table D.1: Some instances of the values $T_{SOS}^{d_a,d_m}$, $T_{LS}^{M,N}$ and $T_{BS}^{M,N,r}$ for several examples from [96]. All values are rounded to two decimals.

| | Sum-of-Squares | | | Linear Splines | | | B-Splines | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Name | $T_{SOS}^{2,1}$ | $T_{SOS}^{4,2}$ | $T_{SOS}^{6,2}$ | $T_{LS}^{10,10}$ | $T_{LS}^{30,30}$ | $T_{LS}^{20,70}$ | $T_{BS}^{3,3,0}$ | $T_{BS}^{3,3,1}$ | $T_{BS}^{3,10,0}$ | $T_{BS}^{5,20,1}$ |
| AC2 | 2.66 | 3.58 | 3.46 | 2.66 | 3.21 | 3.42 | 2.84 | 3.15 | 3.54 | 3.69 |
| AC6 | 3.11 | 3.60 | 3.07 | 3.25 | 3.60 | 3.68 | 3.42 | 3.56 | 3.73 | 3.79 |
| AC8 | 2.35 | 2.45 | 2.41 | 1.92 | 2.21 | 2.31 | 2.06 | 2.19 | 2.39 | 2.45 |
| HE2 | 2.17 | 3.19 | 2.92 | 2.29 | 2.84 | 3.03 | 2.39 | 2.76 | 3.17 | 3.32 |
| HE5 | 1.47 | 1.84 | 0.32 | 2.55 | 0.95 | 0.85 | 2.27 | 2.53 | 3.21 | 3.34 |
| DIS2 | 2.82 | 9.89 | 12.66 | 7.55 | 22.79 | 40.00 | 4.47 | 6.68 | 19.63 | 40.00 |
| MFP | 1.22 | 1.84 | 1.86 | 1.49 | 1.71 | 1.79 | 1.55 | 1.69 | 1.82 | 1.90 |
| EB1 | 1.96 | 2.66 | 2.45 | 2.60 | 2.90 | 3.15 | 2.66 | 2.80 | 5.97 | 19.07 |
| PSM | 8.90 | 13.98 | 35.00 | 4.61 | 14.31 | 28.53 | 3.25 | 4.84 | 14.12 | 40.00 |
| NN2 | 1.88 | 2.17 | 2.17 | 1.80 | 2.00 | 2.08 | 1.94 | 2.04 | 2.14 | 2.16 |
| NN13 | 1.80 | 3.54 | 3.23 | 2.41 | 6.97 | 13.65 | 2.06 | 2.78 | 8.55 | 25.51 |
| CSE1 | 2.33 | 2.33 | 2.33 | 1.79 | 2.08 | 2.19 | 1.96 | 2.14 | 2.29 | 2.33 |
| IH | 1.10 | 1.38 | 1.38 | 1.12 | 1.28 | 1.32 | 1.22 | 1.30 | 1.38 | 1.38 |

Table D.2: Average running times in seconds $A_{SOS}^{d_a,d_m}$, $A_{LS}^{M,N}$, $A_{BS}^{M,N,r}$ and $n_{xu}$, the number of columns of $(A, B)$, for examples from [96].

| | | Sum-of-Squares | | | Linear Splines | | | B-Splines | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Name | $n_{xu}$ | $A_{SOS}^{2,1}$ | $A_{SOS}^{4,2}$ | $A_{SOS}^{6,2}$ | $A_{LS}^{10,10}$ | $A_{LS}^{30,30}$ | $A_{LS}^{20,70}$ | $A_{BS}^{3,3,0}$ | $A_{BS}^{3,3,1}$ | $A_{BS}^{3,10,0}$ | $A_{BS}^{5,20,1}$ |
| AC2 | 8 | 0.65 | 0.80 | 1.42 | 0.31 | 0.88 | 1.66 | 0.19 | 0.24 | 0.39 | 1.53 |
| AC6 | 9 | 0.77 | 0.98 | 1.90 | 0.37 | 1.02 | 2.00 | 0.22 | 0.29 | 0.46 | 1.78 |
| AC8 | 10 | 0.89 | 1.23 | 2.37 | 0.42 | 1.18 | 2.32 | 0.25 | 0.33 | 0.52 | 1.96 |
| HE2 | 6 | 1.75 | 2.93 | 3.90 | 0.53 | 1.42 | 2.49 | 0.27 | 0.32 | 0.53 | 1.69 |
| HE5 | 12 | 1.29 | 2.15 | 4.10 | 0.74 | 2.38 | 4.42 | 0.41 | 0.63 | 0.90 | 3.74 |
| DIS2 | 5 | 0.39 | 0.40 | 0.54 | 0.17 | 0.41 | 0.73 | 0.11 | 0.14 | 0.24 | 1.09 |
| MFP | 7 | 0.84 | 1.36 | 1.99 | 0.33 | 0.91 | 1.67 | 0.19 | 0.24 | 0.40 | 1.47 |
| EB1 | 11 | 1.06 | 1.61 | 3.17 | 0.53 | 1.55 | 2.96 | 0.32 | 0.45 | 0.68 | 2.53 |
| PSM | 9 | 0.75 | 0.98 | 1.85 | 0.34 | 0.94 | 1.77 | 0.21 | 0.27 | 0.43 | 1.59 |
| NN2 | 3 | 0.28 | 0.26 | 0.32 | 0.13 | 0.31 | 0.53 | 0.09 | 0.11 | 0.20 | 0.93 |
| NN13 | 8 | 0.66 | 1.04 | 1.88 | 0.35 | 1.00 | 1.86 | 0.21 | 0.28 | 0.44 | 1.70 |
| CSE1 | 22 | 5.46 | 11.03 | 26.07 | 3.09 | 9.25 | 16.84 | 1.44 | 2.52 | 3.49 | 20.59 |
| IH | 32 | 17.86 | 45.15 | 123.36 | 12.83 | 42.35 | 71.42 | 6.35 | 12.95 | 14.55 | 80.55 |

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# Declaration

I hereby certify that this thesis has been composed by myself, and describes my own work, unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted, and all sources of information have been specifically acknowledged.

Stuttgart, April 2022

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## Abstract

We establish a framework for systematically analyzing and designing output-feedback controllers for linear impulsive and related hybrid systems that might even be affected by various types of uncertainties. In particular, the framework encompasses uncertain switched and sampled-data systems as well as networked systems with switching communication topologies.

The framework is based on recently developed convex criteria involving a so-called clock for analyzing impulsive systems under dwell-time constraints. We elaborate on the extension of those criteria for dynamic output-feedback controller synthesis by means of convex optimization and generalize the so-called dual iteration to impulsive systems. The latter originally and still constitutes a promising heuristic procedure for the challenging and non-convex design of static output-feedback controllers for standard linear time-invariant systems. Moreover, for uncertain impulsive systems as modeled in terms of linear fractional representations, we generalize the nominal analysis criteria by providing novel robust analysis conditions based on a novel time-domain and clock-dependent formulation of integral quadratic constraints. Finally, by combining the insights on nominal synthesis and robust analysis, we are able to tackle challenging output-feedback designs of practical relevance, such as the design of gain-scheduled, robust or robust gain-scheduled controllers for impulsive systems.

Most of the obtained analysis and synthesis conditions involve infinite-dimensional (differential) linear matrix inequalities which can be numerically solved by using relaxation methods based on, e.g., linear splines, B-splines or matrix sum-of-squares that we discuss as well.

## Keywords

Impulsive Systems, Output-Feedback Synthesis, Robust Analysis, Integral Quadratic Constraints, Robust Synthesis, Gain-Scheduling Control, Linear Matrix Inequalities