# From Short-Range to Contact Interactions in Many-Body Quantum Systems

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# Contents

# Basic Notation and Conventions

## Numbers, sets, basic things

| Notation | Explanation |
|---|---|
| $\mathbb{N}$, $\mathbb{N}_0$ | $\mathbb{N} := \{1, 2, 3, ...\}$ denotes the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. |
| $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ | $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of integers, real and complex numbers, respectively. |
| $\mathbb{R}^+$, $\mathbb{R}^-$ | $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}^- := (-\infty, 0)$. |
| $\delta_{ab}$, $\delta(\cdot)$ | The Kronecker delta is defined as $\delta_{ab} = 1$ if $a = b$ and $\delta_{ab} = 0$ otherwise. This should not be confused with the $\delta$-distribution, which is denoted by $\delta(\cdot)$ (without index). The letter $\delta$ is also used to denote (small) positive numbers. |
| $\overline{x}, \overline{M}$ | For $x \in \mathbb{C}^d$, $d \in \mathbb{N}$, we denote by $\overline{x} \in \mathbb{C}^d$ the complex conjugate of $x$, while for sets $M \subseteq \mathbb{R}^d$ we denote by $\overline{M}$ the closure of $M$. |
| $x \cdot y$, $x^2$, $\lvert x\rvert$ | For $x = (x_i)_{i=1}^d, y = (y_i)_{i=1}^d \in \mathbb{C}^d$, $d \in \mathbb{N}$, we define $x \cdot y := \sum_{i=1}^d x_i y_i$, $x^2 := x \cdot x$ and $\lvert x\rvert := (\overline{x} \cdot x)^{1/2}$. |
| $B_r(x)$ | Open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$: $B_r(x) := \{y \in \mathbb{R}^d \mid \lvert y - x\rvert < r\}$. |
| $O(\cdot)$, $o(\cdot)$ | Landau symbols (Big-O and little-o notation). |

## Hilbert spaces and operators

| Notation | Explanation |
|---|---|
| $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ | For two (complex) Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ with norms $\lVert \cdot \rVert_i$, $i = 1, 2$, the space of all bounded linear operators $S : \mathscr{H}_1 \to \mathscr{H}_2$ is denoted by $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$. This is a Banach space w.r.t. the operator norm $\lVert S\rVert := \sup_{\psi \in \mathscr{H}_1 \setminus \{0\}} \lVert S\psi\rVert_2 / \lVert\psi\rVert_1$. We also write $\mathscr{L}(\mathscr{H}_1) := \mathscr{L}(\mathscr{H}_1, \mathscr{H}_1)$ for short. |
| $\langle\psi\rvert, \lvert\psi\rangle$ | For a given vector $\psi$ in some Hilbert space, $\langle\psi\rvert$ and $\lvert\psi\rangle$ denote the bra, respectively, ket associated with $\psi$. |
| $\rho(S)$, $\sigma(S)$ | The resolvent set $\rho(S)$ of a densely defined closed operator $S$ is the set of all $z \in \mathbb{C}$ for which $S + z : D(S) \subseteq \mathscr{H} \to \mathscr{H}$ is a bijection. This differs by a minus sign from the conventional definition. The spectrum $\sigma(S)$ is defined as usual, which means that $\sigma(S)$ is the complement of $-\rho(S)$. |
| $\operatorname{Ker} S$ | The kernel of a given operator $S$: $\operatorname{Ker} S := \{\psi \in D(S) \mid S\psi = 0\}$. |
| $\operatorname{Ran} S$ | The range of a given operator $S$: $\operatorname{Ran} S := \{S\psi \mid \psi \in D(S)\}$. |
| $S^*$ | The adjoint of a given densely defined operator $S$. |
| $\psi_n \rightharpoonup \psi$ | Weak convergence (of $\psi_n$ towards $\psi$). |
| $\psi_n \to \psi$ | Norm convergence (of $\psi_n$ towards $\psi$). |
| $\bigoplus$ | Direct sum (of Hilbert spaces). |
| $\bigotimes$ | Tensor product (of Hilbert spaces or operators) or elementary tensor (of vectors). |
| $\lVert \cdot \rVert_{\mathrm{HS}}$ | Hilbert-Schmidt norm. |

### Function spaces and operations on functions

In the following, $\Omega$ and $M$ denote given open, respectively, measurable subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$.

| Notation | Explanation |
|---|---|
| $f \upharpoonright \widetilde{M}$ | For $f : M \to \mathbb{C}$ and $\widetilde{M} \subseteq M$, the restriction $(f \upharpoonright \widetilde{M}) : \widetilde{M} \to \mathbb{C}$ is defined by $(f \upharpoonright \widetilde{M})(x) := f(x)$. |
| $C(M)$ | The space of all bounded and continuous functions $f : M \to \mathbb{C}$, equipped with the uniform norm $\|f\|_\infty := \sup_{x\in M} \lvert f(x)\rvert$. |
| $C^n(\Omega)$ | For $n \in \mathbb{N}_0 \cup \{\infty\}$, $C^n(\Omega)$ denotes the space of all $n$-times continuously differentiable functions $f : \Omega \to \mathbb{C}$ for which the norm $\|f\|_{C^n(\Omega)} := \sum_{\lvert\alpha\rvert\le n} \|\partial^\alpha f\|_\infty$ is finite (thus $C^0(\Omega) = C(\Omega)$). |
| $C^{0,s}(\Omega)$ | The space of all $f \in C(\Omega)$ that are Hölder continuous with exponent $s \in (0, 1]$ with the norm $\|f\|_{C^{0,s}(\Omega)} := \|f\|_\infty + \sup_{x,y\in\Omega,x\neq y} \lvert f(x) - f(y)\rvert / \lvert x - y\rvert^s$. |
| $\operatorname{supp} f$ | Suppose that $f : M \to \mathbb{C}$ extends by zero to a function that is locally integrable over $\mathbb{R}^d$, and hence it defines a distribution $T_f \in \mathcal{D}'(\mathbb{R}^d)$. Then the support of $f$, $\operatorname{supp} f$, is the intersection of $M$ with the support of $T_f$ (cf. [50, Definition 10.23]). In particular, it follows that $f = 0$ a.e. in $M \setminus (\operatorname{supp} f)$, and for $f \in C(M)$ this is equivalent to the conventional definition $\operatorname{supp} f = \overline{\{x \in M \mid f(x) \neq 0\}} \cap M$. |
| $C_0^\infty(\mathbb{R}^d)$ | The set of all $f \in C^\infty(\mathbb{R}^d)$ for which $\operatorname{supp} f$ is a compact set in $\mathbb{R}^d$. |
| $C_0^\infty(\Omega)$ | The set of all $f \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subseteq \Omega$. |
| $L^p(M)$ | The usual $L^p$-space of all measurable functions $f : M \to \mathbb{C}$ for which the norm $\|f\|_{L^p(M)} := (\int_M \lvert f(x)\rvert^p \,\mathrm{d}x)^{1/p}$ if $1 \le p < \infty$, or $\|f\|_{L^\infty(M)} := \operatorname{ess\,sup}_{x\in M} \lvert f(x)\rvert$ if $p = \infty$, is finite (for $f : M \to \mathbb{C}^m$, $m \ge 2$, the norm $\|f\|_{L^p(M)}$ is defined analogously). We note that $L^2(M)$ is a Hilbert space with scalar product $\langle f \mid g\rangle := \int_M \overline{f(x)} g(x) \,\mathrm{d}x$. In the case $M = \mathbb{R}^d$ we also write $\|\cdot\|_{L^p}$, or just $\|\cdot\|$ if $p = 2$, instead of $\|\cdot\|_{L^p(\mathbb{R}^d)}$. |
| $\widehat{f}$ | For $f \in L^1(\mathbb{R}^d)$ the Fourier transform $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ is defined as $\widehat{f}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \exp(-ix\cdot y) \,\mathrm{d}y$. By the Plancherel theorem, the linear map $L^1 \cap L^2(\mathbb{R}^d) \ni f \to \widehat{f}$ extends to a unitary operator in $L^2(\mathbb{R}^d)$. We also write $\widehat{f}$ when the Fourier transform is only taken w.r.t. some coordinates. |
| $H^s(\mathbb{R}^d)$ $(s \ge 0)$ | The usual Sobolev space of order $s$: the Hilbert space of all $f \in L^2(\mathbb{R}^d)$ for which the norm $\|f\|_{H^s} := (\int_{\mathbb{R}^d} (1 + \lvert k\rvert^2)^s \lvert\widehat{f}(k)\rvert^2 \,\mathrm{d}k)^{1/2}$ is finite. |
| $H^n(\Omega)$ $(n \in \mathbb{N}_0)$ | The usual Sobolev space of order $n$ over $\Omega$: The set of all $f \in L^2(\Omega)$ for which all weak partial derivatives $\partial^\alpha f$, $\lvert\alpha\rvert \le n$, exist and belong to $L^2(\Omega)$ becomes a Hilbert space with the norm $\|f\|_{H^n(\Omega)} := (\sum_{\lvert\alpha\rvert\le n} \|\partial^\alpha f\|^2_{L^2(\Omega)})^{1/2}$ (in particular, $H^0(\Omega) = L^2(\Omega)$). We note that for $\Omega = \mathbb{R}^d$ this norm is equivalent to the previously defined one. |

# Abstract

This thesis is devoted to the approximation of contact interactions by means of short-range interactions in quantum mechanics. *Contact interactions* are also called *zero-range interactions* or *point interactions* in the literature, and as these terms already suggest, they describe idealized interactions that only appear when the particles involved are in direct contact with each other (i.e. their spatial coordinates coincide exactly). Formally, this corresponds, for example, to the case where the interaction potential $V$ is replaced by a $\delta$-potential. Contact interactions have a long tradition in physics, going back to the early days of quantum mechanics, and can be rigorously described with the help of mathematical methods. The mathematical construction of a Hamilton operator with non-vanishing contact interactions is a challenging issue and, in principle, only possible in $d \leq 3$ dimensions. While the case of a single particle interacting with an external "contact potential" has long been well-understood [6], there are still many open problems in the $N$-particle case ($N \geq 2$). In the case of $N \geq 2$ particles in $d = 2$ dimensions a physically reasonable Hamiltonian with two-body contact interactions (TMS Hamiltonian) was first constructed by Dell'Antonio, Figari and Teta [30].

In the first (and major) part of this thesis we mathematically justify well-established models with contact interactions in $d \in \{1, 2\}$ that are used to describe short-range two-body interactions among $N \geq 2$ particles. To this end, we consider a suitable class of Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$, with local rescaled two-body potentials. The rescaling in $\varepsilon > 0$ is chosen so that in the limit $\varepsilon \to 0$ the effective range of the two-body interaction tends to zero while at the same time the strength of the interaction at the collision planes diverges at a rate that depends on the dimension $d$. Our main goal is to prove, irrespective of the underlying symmetry, norm resolvent convergence of $H_\varepsilon$ in the limit $\varepsilon \to 0$ and to identify the limit operator $H$ with a TMS Hamiltonian from the literature. Besides the mathematical justification of the underlying physical model, this also provides an alternative way of constructing the Hamiltonian. In analogy to the simpler one-particle case [6], a non-trivial renormalization of the coupling constants in front of the two-body potentials is necessary in $d \geq 2$ dimensions. The restriction that $d \leq 2$ is necessary in a certain sense because in $d = 3$ dimensions two closely related effects, namely the Efimov effect and the Thomas effect, prohibit the norm resolvent convergence of $H_\varepsilon$ towards a "non-trivial" semibounded limit operator. First, we show in $d = 1$, and later also in $d = 2$, that $H_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint operator $H$, and in $d = 1$ we also obtain an asymptotic estimate for the rate of norm resolvent convergence depending on a mild decay condition for the two-body potentials. A comparable result in $d = 1$ was previously only known for $N \leq 3$ particles [10], and in $d = 2$ we thereby improve a result that was recently obtained in connection with a certain limit in the stochastic heat equation [46]. From the norm resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ it follows almost immediately that $H$ fulfills the physical minimum requirements of translational invariance, locality and boundedness from below. In $d = 1$, $H$ can be constructed directly from a closed, semibounded quadratic form, which is a small perturbation of the quadratic form of the free operator. We characterize the domain of $H$ by means of a rigorous version of a well-known jump condition for the derivative of the wave function. In $d = 2$, we show that $H$ can be identified with the TMS Hamiltonian. Both in one and two dimensions, our key to prove norm resolvent convergence is a generalized Konno-Kuroda formula for the resolvent of $H_\varepsilon$.

In the second part of this thesis we examine the weakness of short-range interactions among identical fermions of equal spin in all dimensions $d \leq 3$. In physics, short-range interactions in ultracold Fermi gases are conveniently entirely neglected, i.e. the Hamiltonian $H$ of the system is equated with the kinetic energy operator, which is given by the negative Laplacian $-\Delta$ in

appropriate units. Formally, the approximation $H \approx -\Delta$ is usually justified as follows: The Pauli principle implies that the wave function vanishes when the coordinates of two fermions coincide, and consequently very short-range two-body interactions should be negligible. In the case of pure contact interactions we give a mathematical justification for the identity $H = -\Delta$ in $d \geq 2$. In one dimension, however, we give a counterexample that shows that non-vanishing contact interactions among identical fermions of equal spin do exist. In order to obtain results for short-range interactions of positive range, we again consider suitable Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$, with rescaled two-body potentials, which are now restricted to the subspace of antisymmetric functions. Our second major result provides a criterion that specifies when, from the physical point of view, the approximation $H_\varepsilon \approx -\Delta$ is reasonable for small enough $\varepsilon > 0$ and we obtain an asymptotic estimate for the approximation error in terms of $\varepsilon > 0$. More precisely, it follows from our assumptions that $H_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to $-\Delta$ and we obtain an asymptotic estimate for the rate of norm resolvent convergence. Moreover, our result is optimal in a certain sense: If one condition is only slightly violated, then the spectrum of $H_\varepsilon$ fills the whole real line in the limit $\varepsilon \to 0$, which is not compatible with norm resolvent convergence towards the positive operator $-\Delta$.

# Zusammenfassung

Diese Dissertation beschäftigt sich mit der Approximation von Kontaktwechselwirkungen durch kurzreichweitige Wechselwirkungen in der Quantenmechanik. *Kontaktwechselwirkungen* werden in der Literatur auch als *nullreichweitige Wechselwirkungen* oder *Punktwechselwirkungen* bezeichnet, und wie diese Bezeichnungen bereits nahelegen, beschreiben sie idealisierte Wechselwirkungen, die nur dann in Erscheinung treten, wenn die beteiligten Teilchen in direktem Kontakt miteinander sind (d.h. ihre Ortskoordinaten sind exakt gleich). Formal entspricht dies zum Beispiel dem Fall, wo das Wechselwirkungspotential $V$ durch ein $\delta$-Potential ersetzt wird. Kontaktwechselwirkungen haben eine lange Tradition in der Physik, die bis zu den Anfängen der Quantenmechanik zurückreicht, und lassen sich rigoros mit Methoden der Mathematik beschreiben. Die mathematische Konstruktion eines Hamilton-Operators mit nichtverschwindenden Kontaktwechselwirkungen ist eine anspruchsvolle Angelegenheit und prinzipiell nur in $d \leq 3$ Dimensionen möglich. Während der Fall eines einzelnen Teilchens in einem externen „Kontaktpotential“ seit langem gut verstanden ist [6], gibt es im $N$-Teilchenfall ($N \geq 2$) noch viele offene Fragen. Im Fall von $N \geq 2$ Teilchen in $d = 2$ Dimensionen wurde ein physikalisch sinnvoller Hamiltonian mit Zweiteilchen-Kontaktwechselwirkungen (TMS Hamiltonian) zuerst von Dell'Antonio, Figari und Teta [30] konstruiert.

Im ersten (und größeren) Teil dieser Dissertation geben wir in $d \in \{1, 2\}$ Dimensionen eine mathematische Rechtfertigung für bewährte Modelle mit Kontaktwechselwirkungen, die zur Beschreibung von kurzreichweitigen Zweiteilchen-Wechselwirkungen zwischen $N \geq 2$ Teilchen verwendet werden. Hierzu betrachten wir eine geeignete Klasse von Schrödinger-Operatoren $H_\varepsilon$, $\varepsilon > 0$, mit lokalen reskalierten Zweiteilchen-Potentialen. Die Reskalierung in $\varepsilon > 0$ ist dabei so gewählt, dass im Limes $\varepsilon \to 0$ die effektive Reichweite der Zweiteilchen-Wechselwirkung gegen Null geht während gleichzeitig die Stärke der Zweiteilchen-Wechselwirkung an den Kollisionsebenen mit einer von der Dimension $d$ abhängigen Rate divergiert. Unser Hauptziel ist es, unabhängig von der zugrundeliegenden Symmetrie, Normresolventenkonvergenz von $H_\varepsilon$ im Limes $\varepsilon \to 0$ zu beweisen, und den Grenzoperator $H$ mit einem in der Literatur bereits bekannten TMS Hamiltonian zu identifizieren. Neben der mathematischen Rechtfertigung des zugrundeliegenden physikalischen Modells liefert dies auch einen alternativen Weg zur Konstruktion des Hamiltonians. In Analogie zum einfacheren Einteilchen-Fall [6] ist dabei in $d \geq 2$ Dimensionen eine nichttriviale Renormalisierung der Kopplungskonstanten vor den Zweiteilchen-Potentialen erforderlich. Die Einschränkung $d \leq 2$ ist in gewisser Weise notwendig, da in $d = 3$ Dimensionen zwei eng miteinander verwandte Effekte, nämlich der Efimov-Effekt und der Thomas-Effekt, die Normresolventenkonvergenz von $H_\varepsilon$ gegen einen „nichttrivialen“, nach unten beschränkten Grenzoperator verbieten. Zunächst zeigen wir in $d = 1$ und später auch in $d = 2$, dass $H_\varepsilon$ für $\varepsilon \to 0$ im Normresolventensinn gegen einen selbstadjungierten Operator $H$ konvergiert, wobei wir in $d = 1$ zusätzlich eine asymptotische Abschätzung für die Rate der Normresolventenkonvergenz in Abhängigkeit einer milden Abfallbedingung an die Zweiteilchen-Potentiale bekommen. Ein vergleichbares Resultat in $d = 1$ war vorher nur für $N \leq 3$ Teilchen bekannt [10], und in $d = 2$ verbessern wir damit ein Resultat, das kürzlich im Zusammenhang mit einem gewissen Limes in der stochastischen Wärmeleitungsgleichung erzielt wurde [46]. Aus der Normresolventenkonvergenz $H_\varepsilon \to H$ für $\varepsilon \to 0$ folgt fast unmittelbar, dass $H$ den physikalischen Mindestanforderungen der Translationsinvarianz, Lokalität und Beschränktheit nach unten genügt. In $d = 1$ kann $H$ direkt mittels einer abgeschlossenen, nach unten beschränkten quadratischen Form konstruiert werden, die eine kleine Störung der quadratischen Form des freien Operators ist. Den Definitionsbereich von $H$ charakterisieren wir mittels einer rigorosen Version einer wohlbekannten Sprungbedingung für die Ableitung der Wellenfunktion. In $d = 2$ zeigen wir,

dass $H$ mit dem TMS Hamiltonian identifiziert werden kann. Sowohl in einer als auch in zwei Dimensionen ist unser Schlüssel zum Beweis von Normresolventenkonvergenz eine verallgemeinerte Konno-Kuroda-Formel für die Resolvente von $H_\varepsilon$.

Im zweiten Teil dieser Arbeit untersuchen wir die Schwäche von kurzreichweitigen Wechselwirkungen zwischen identischen Fermionen gleichen Spins in allen Dimensionen $d \leq 3$. In der Physik werden kurzreichweitige Wechselwirkungen in ultrakalten Fermigasen meist komplett vernachlässigt, d.h. der Hamiltonian $H$ des Systems wird mit dem Operator der kinetischen Energie gleichgesetzt, welcher in geeigneten Einheiten durch den negativen Laplace-Operator $-\Delta$ gegeben ist. Formal wird die Approximation $H \approx -\Delta$ üblicherweise folgendermaßen gerechtfertigt: Das Pauli-Prinzip impliziert, dass die Wellenfunktion verschwindet sobald die Koordinaten zweier Fermionen übereinstimmen, und folglich sollten sehr kurzreichweitige Zweiteilchen-Wechselwirkungen dann vernachlässigbar sein. Im Falle reiner Kontaktwechselwirkungen geben wir in $d \geq 2$ eine mathematische Begründung für die Identität $H = -\Delta$. In einer Dimension zeigen wir dagegen mithilfe eines Gegenbeispiels, dass dann auch nichtverschwindende Kontaktwechselwirkungen zwischen identischen Fermionen mit gleichem Spin existieren. Um auch Aussagen über kurzreichweitige Wechselwirkungen mit positiver Reichweite treffen zu können, betrachten wir wieder geeignete Schrödinger-Operatoren $H_\varepsilon$, $\varepsilon > 0$, mit reskalierten Zweiteilchen-Potentialen, die nun auf den Unterraum der antisymmetrischen Funktionen eingeschränkt sind. Unser zweites Hauptresultat liefert ein Kriterium dafür, wann die Approximation $H_\varepsilon \approx -\Delta$ für hinreichend kleine $\varepsilon > 0$ physikalisch sinnvoll ist und wir erhalten eine asymptotische Abschätzung für den Approximationsfehler in Abhängigkeit von $\varepsilon > 0$. Genauer folgt aus unseren Annahmen, dass $H_\varepsilon$ für $\varepsilon \to 0$ im Normresolventensinn gegen $-\Delta$ konvergiert und wir bekommen eine asymptotische Abschätzung für die Rate der Normresolventenkonvergenz. Darüber hinaus ist unser Resultat in gewisser Weise optimal: Wenn eine Bedingung nur minimal verletzt ist, dann füllt das Spektrum von $H_\varepsilon$ im Limes $\varepsilon \to 0$ die ganze reelle Gerade aus, was nicht mit der Normresolventenkonvergenz gegen den positiven Operator $-\Delta$ vereinbar ist.

# 1 Introduction

Contact interactions, which are also known as zero-range interactions or point interactions, have been studied extensively in the literature, both in mathematics and physics. Due to the huge amount of literature, we can not cover all aspects of contact interactions in this introduction and we emphasize that the list of given references is also selective. First, we briefly discuss the simplest case of a single particle that interacts with an external "contact potential" located at the origin, which has long been well-understood and which is analyzed in great detail in the monograph [6]. Afterwards, in Section 1.2, we turn to the much less understood case of $N$-particle Hamiltonians with contact interactions. The main intention of Sections 1.1 and 1.2 is to put the new results that are presented in Section 1.3 into a broader perspective.

## 1.1 Contact interactions in the one-particle case

The interaction of a non-relativistic, spinless quantum particle with an external $\delta$-potential located at $0 \in \mathbb{R}^d$, $d \in \mathbb{N}$, can be described *formally* by the operator

$$-\Delta + \alpha\delta(\cdot), \qquad \alpha \in \mathbb{R}, \tag{1.1}$$

where, for the sake of simplicity, we have chosen $m = 1$ for the mass and $\hbar = \sqrt{2}$ for the reduced Planck constant, so that the kinetic energy operator is given by the negative Laplacian $-\Delta$. The parameter $\alpha \in \mathbb{R}$ determines the sign and the strength of the interaction. If, in $d = 1$ dimension, one replaces the interaction potential $\alpha\delta(\cdot)$ by a whole chain of external $\delta$-potentials $\alpha \sum_{y\in\mathbb{Z}} \delta(\,\cdot\, - y)$, $\alpha > 0$, then one obtains the model that has been investigated by Kronig and Penney in 1931 in order to describe the movement of a non-relativistic electron through a stationary crystal lattice [29]. Historically, the Kronig–Penney model was the first influential model of quantum mechanics, where short-range interactions are modelled by idealized interactions of zero range, and today it has become a standard model in solid-state physics due to its simplicity and explicit solvability. A few years after Kronig and Penney, Bethe and Peierls [14] and Thomas [78] considered operators of the type (1.1) in $d = 3$ dimensions to describe the short-range interactions among a proton and a neutron in the deuteron (after separation of the center of mass motion). Moreover, the work of Thomas motivated many further investigations on the model of three particles interacting via two-body contact interactions, which are described in more detail in Section 1.2.2, below.

However, from the mathematical point of view, the expression (1.1) is ill-defined because a $\delta$-distribution does not define an operator in $L^2(\mathbb{R}^d)$. The rigorous definition of an operator of the type (1.1) in $d = 3$ dimensions goes back to Berezin and Faddeev [13], who have constructed all self-adjoint extensions of the densely defined symmetric operator

$$h_0 = -\Delta \restriction C_0^\infty(\mathbb{R}^d \setminus \{0\}) \tag{1.2}$$

with the help of Krein's theory of self-adjoint extensions. Apart from the point $x = 0$, such self-adjoint extensions agree with the free operator $-\Delta$, and hence they describe an idealized interaction of zero range that is localized at $x = 0$. This explains why these interactions are called contact interactions, zero-range interactions or point interactions in the literature. Moreover, the so-called Fermi pseudopotentials, which have been introduced by Enrico Fermi in the context of his investigation of the neutron movement through hydrogen substances [35], were later also identified as contact interactions in $d \leq 3$ dimensions [45]. It is known that the deficiency indices $n_\pm := \dim \mathrm{Ker}(h_0^* \mp i)$ are given by $n_+ = n_- = 2$ if $d = 1$, $n_+ = n_- = 1$ if $d \in \{2, 3\}$ and $n_+ = n_- = 0$ if $d \geq 4$ [6]. By von Neumann's theory of self-adjoint extensions (see, e.g., [69, Chapter X.1]), this means that non-trivial self-adjoint extensions only exist in $d \leq 3$ dimensions, while $h_0$ is essentially self-adjoint in $d \geq 4$ and hence the only self-adjoint extension is the Friedrichs extension $-\Delta : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. More precisely, there exists a four-parameter family of self-adjoint extensions in $d = 1$ (apart from $\delta$-interactions, the so-called $\delta'$-interactions are also notable), while in $d \in \{2, 3\}$ there is only a one-parameter family of self-adjoint extensions [6]. In $d \in \{2, 3\}$ all contact interactions are attractive, while there also exist repulsive contact interactions in $d = 1$ [6]. A rigorous and particularly simple construction of attractive contact interactions in $d \leq 3$ uses a regularization in momentum space that is obtained by introducing an ultraviolet cutoff. That is, one considers a sequence of operators of the form $h_n = -\Delta - g_n |\delta_n\rangle \langle\delta_n|$, $n \in \mathbb{N}$, where $\delta_n \in L^2(\mathbb{R}^d)$ is defined by $\widehat{\delta}_n(p) := (2\pi)^{-d/2}$ if $|p| \leq n$ and $\widehat{\delta}_n(p) := 0$ if $|p| > n$. In particular, this implies that $\varphi(0) = \lim_{n\to\infty} \langle\delta_n \,|\, \varphi\rangle$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. The coupling constant $g_n > 0$ is chosen in such a way that $h_n$ has exactly one simple eigenvalue $E_B < 0$ (the *binding energy*), which does not depend on $n \in \mathbb{N}$. Then one can show that the sequence $h_n$, $n \in \mathbb{N}$, has a limit in the norm resolvent sense that describes an attractive contact interaction at $x = 0$ [44, 54].

Since the main subject of this thesis is the approximation of contact interactions by means of Schrödinger operators with local short-range potentials (especially in the $N$-particle case, $N \geq 2$), we describe this alternative and very natural approach in more detail in Section 1.1.1, below. Thereby, we already introduce methods and notation that are generalized in the more challenging $N$-particle case. In the literature this approach is discussed, e.g., in [7] ($d = 1$), [5] ($d = 2$), [4, 8, 19] ($d = 3$), as well as in [6] in all dimensions $d \leq 3$. For a treatment with methods of non-standard analysis we refer to the references in the introduction of [6]. The most important properties of the so defined contact interactions are summarized in Section 1.1.2.

### 1.1.1 Approximation by Schrödinger operators with short-range potentials

In the Hilbert space $L^2(\mathbb{R}^d)$, $d \leq 3$, we consider Schrödinger operators of the form

$$h_\varepsilon := -\Delta + g_\varepsilon V_\varepsilon, \qquad \varepsilon > 0, \tag{1.3}$$

where $V_\varepsilon(x) := \varepsilon^{-d} V(x/\varepsilon)$ for a fixed potential $V : \mathbb{R}^d \to \mathbb{R}$ and $g_\varepsilon \in \mathbb{R}$ plays the role of a coupling constant. For simplicity, we assume that $V \in L^1 \cap L^2(\mathbb{R}^d)$ has compact support. In particular, this implies that $h_\varepsilon$ defines a self-adjoint operator on $D(h_\varepsilon) = H^2(\mathbb{R}^d)$. For $h_\varepsilon$ to have a limit in the norm resolvent sense as $\varepsilon \to 0$, a dimension-dependent renormalization of $g_\varepsilon$ is necessary. To explain this, we first note that the unitary rescaling $U_\varepsilon \in \mathscr{L}(L^2(\mathbb{R}^d))$, $\varepsilon > 0$, with $(U_\varepsilon\psi)(x) := \varepsilon^{d/2}\psi(\varepsilon x)$ allows us to rewrite $h_\varepsilon$ as

$$h_\varepsilon = \varepsilon^{-2} U_\varepsilon^* (-\Delta + g_\varepsilon \varepsilon^{2-d} V) U_\varepsilon. \tag{1.4}$$

As norm resolvent convergence implies the convergence of the associated spectra (cf. Proposition 2.3, below), a negative eigenvalue $E_B < 0$ can only arise in the limit $\varepsilon \to 0$ if $-\Delta + g_\varepsilon\varepsilon^{2-d}V$ has an eigenvalue $E(\varepsilon)$ that behaves asymptotically, as $\varepsilon \to 0$, like $E(\varepsilon) = \varepsilon^2 E_B + o(\varepsilon^2)$. It is well-known in the spectral theory of Schrödinger operators what this means for $g_\varepsilon$ and $V$ [48, 74]. In the simplest case $d = 1$, it follows from [74, Theorem 2.5] that such an eigenvalue $\lambda_1(\varepsilon)$ exists if and only if $g = \lim_{\varepsilon\to 0} g_\varepsilon$ exists and $\alpha := g \int V(x)\,\mathrm{d}x < 0$. The resulting binding energy is then $E_B = -\alpha^2/4 < 0$. If $d = 2$ and $\int V(x)\,\mathrm{d}x \neq 0$, then it follows from [74, Theorem 3.4] that for the existence of such an eigenvalue $E(\varepsilon)$ it is necessary that, as $\varepsilon \to 0$, $1/g_\varepsilon = a\ln(\varepsilon) + o(|\ln(\varepsilon)|)$ with $a = \int V(x)\,\mathrm{d}x/(2\pi) \neq 0$. However, the more profound analysis in [5] shows that the resulting eigenvalue essentially depends on the next order $O(1)$, which means that we have to assume, more precisely, that $1/g_\varepsilon = a\ln(\varepsilon) + b + o(1)$ as $\varepsilon \to 0$, where $a = \int V(x)\,\mathrm{d}x/(2\pi) \neq 0$ and $b \in \mathbb{R}$. Without restriction, we may also assume that $a > 0$ because $h_\varepsilon$ depends on the product $g_\varepsilon V$

only. In the remaining case $d = 3$, the existence of the desired eigenvalue $E(\varepsilon)$ is the result of a zero-energy resonance of $-\Delta + V$ combined with the asymptotics $g_\varepsilon = \varepsilon + b\varepsilon^2 + o(\varepsilon^2)$ for some $b < 0$ [6, 48]. In summary, the asymptotics of $g_\varepsilon$ is determined by

$$g_\varepsilon = \begin{cases} g + o(1) & \text{if } d = 1, \\ (a \ln(\varepsilon) + b + o(1))^{-1} & \text{if } d = 2, \\ \varepsilon + b\varepsilon^2 + o(\varepsilon^2) & \text{if } d = 3, \end{cases} \qquad (\varepsilon \to 0) \tag{1.5}$$

where $a = \int V(x)\,\mathrm{d}x/(2\pi) > 0$ and $b, g \in \mathbb{R}$ (the choices $\alpha = g \int V(x)\,\mathrm{d}x > 0$ in $d = 1$, respectively, $b \geq 0$ in $d = 3$ lead to contact interactions with empty discrete spectrum [6]). While we may simply choose $g_\varepsilon = g = \text{const.}$ for $d = 1$, Eq. (1.5) shows that a non-trivial renormalization of $g_\varepsilon$ is required for $d \in \{2, 3\}$. Assuming that (1.5) holds, it is well-known that $h_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint operator that defines a contact interaction at the origin [6]. In the following, we sketch the proof for $d \in \{1, 2\}$ because this serves as a good preparation for the more sophisticated $N$-particle case that is described in Section 1.3. The proof for $d = 3$ is similar, though some additional notation and results concerning the zero-energy resonance of $-\Delta + V$ are needed [4, 6, 8].

Let $d \leq 2$ and let $u, v \in L^2(\mathbb{R}^d)$ be given by

$$\begin{aligned} v(r) &:= |V(r)|^{1/2}, \\ u(r) &:= J|V(r)|^{1/2}, \qquad J := \operatorname{sgn}(V), \end{aligned}$$

so that $V = vu$. Then, on $H^2(\mathbb{R}^d)$,

$$g_\varepsilon V_\varepsilon = g_\varepsilon \varepsilon^{-d} U_\varepsilon^* vu U_\varepsilon = A_\varepsilon^* B_\varepsilon,$$

where

$$A_\varepsilon = v\, \varepsilon^{-d/2} U_\varepsilon, \tag{1.6}$$

$$B_\varepsilon = g_\varepsilon u\, \varepsilon^{-d/2} U_\varepsilon = g_\varepsilon J A_\varepsilon \tag{1.7}$$

are densely defined and closed on $D(A_\varepsilon) = D(B_\varepsilon) := \{\psi \in L^2(\mathbb{R}^d) \,|\, vU_\varepsilon \psi \in L^2(\mathbb{R}^d)\}$. This allows us to rewrite $h_\varepsilon$ as

$$h_\varepsilon = -\Delta + A_\varepsilon^* B_\varepsilon,$$

where both $A_\varepsilon^* B_\varepsilon = g_\varepsilon V_\varepsilon$ and $A_\varepsilon^* A_\varepsilon = |V_\varepsilon|$ are infinitesimally $(-\Delta)$-bounded since $V \in L^2(\mathbb{R}^d)$ (see, e.g., [77, Theorem 10.2 and Lemma 6.22]). By Theorem B.1 in the appendix, this means that a point* $z \in \rho(-\Delta) = \mathbb{C} \setminus (-\infty, 0]$ belongs to $\rho(h_\varepsilon)$ if and only if the bounded operator

$$1 + g_\varepsilon \phi_\varepsilon(z) := 1 + B_\varepsilon(-\Delta + z)^{-1} A_\varepsilon^* \in \mathscr{L}(L^2(\mathbb{R}^d)) \tag{1.8}$$

is invertible. Moreover, for $z \in \rho(-\Delta) \cap \rho(h_\varepsilon)$, the resolvent $(h_\varepsilon + z)^{-1}$ can be expressed by the Konno-Kuroda formula (see also [49])

$$(h_\varepsilon + z)^{-1} = (-\Delta + z)^{-1} - \left(A_\varepsilon(-\Delta + \overline{z})^{-1}\right)^* (1 + g_\varepsilon \phi_\varepsilon(z))^{-1} B_\varepsilon(-\Delta + z)^{-1}. \tag{1.9}$$

We are going to show that the right side of Eq. (1.9) has a limit as $\varepsilon \to 0$, provided that $z \in \mathbb{C}$ is chosen suitably. With the help of appropriate Sobolev estimates, it is not hard to show that for all $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$A_\varepsilon(-\Delta + z)^{-1} \to |v\rangle \langle \overline{G_z^d}| \qquad (\varepsilon \to 0) \tag{1.10}$$

*Note our convention concerning the resolvent set $\rho(S)$ of a densely defined closed operator $S$. That is, $\rho(S)$ is the set of all $z \in \mathbb{C}$ for which $S + z : D(S) \subseteq \mathscr{H} \to \mathscr{H}$ is a bijection, which differs by a minus sign from the conventional definition. The spectrum $\sigma(S)$ is defined as usual, and thus $\sigma(S) = \mathbb{C} \setminus (-\rho(S))$.

in $\mathscr{L}(L^2(\mathbb{R}^d))$, where

$$G_z^d(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\exp(ip \cdot x)}{p^2 + z} \, \mathrm{d}p$$

denotes the Green's function of $-\Delta + z : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ (for details we refer to Lemmas 3.8 ($d = 1$) and 4.3 ($d = 2$), below). To prove convergence of $(1 + g_\varepsilon \phi_\varepsilon(z))^{-1}$, a careful distinction between the cases $d = 1$ and $d = 2$ is necessary. However, regardless of the space dimension $d \le 2$, Eqs. (1.6)-(1.8) imply that $\phi_\varepsilon(z)$ is a Hilbert-Schmidt operator with integral kernel

$$\phi_\varepsilon(z, x, x') := u(x)\, G_z^d(\varepsilon(x - x'))\, v(x'). \tag{1.11}$$

That is, $(\phi_\varepsilon(z)\psi)(x) = \int \phi_\varepsilon(z, x, x')\psi(x')\, \mathrm{d}x'$ for all $\psi \in L^2(\mathbb{R}^d)$.

For $d = 1$, it is well-known that

$$G_z^1(x) = \frac{i}{2k} \exp(ik|x|), \qquad\qquad z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.12}$$

where $k = k(z) \in \mathbb{C}$ is uniquely determined by $k^2 = -z$ and $\mathrm{Im}(k) > 0$. Now, a direct computation (see [6, Chapter I.3.2]) shows that

$$\phi_\varepsilon(z) \to \phi(z) := \frac{i}{2k} \left|u\right\rangle \left\langle v\right| \qquad\qquad (\varepsilon \to 0) \tag{1.13}$$

w.r.t. Hilbert-Schmidt norm, and hence in $\mathscr{L}(L^2(\mathbb{R}))$. Moreover, $(1 + g\phi(z))^{-1}$ exists if and only if $\alpha := g \int V(x)\, \mathrm{d}x$ satisfies either $\alpha \ge 0$ or $\alpha < 0$ and $z \ne \alpha^2/4$. If this is the case, then

$$(1 + g\phi(z))^{-1} = 1 - \frac{ig}{2k + i\alpha} \left|u\right\rangle \left\langle v\right|. \tag{1.14}$$

Combining (1.9), (1.10), (1.13) and (1.14), it follows that for all $z \in \mathbb{C} \setminus (-\infty, 0]$, $z \ne \alpha^2/4$,

$$(h_\varepsilon + z)^{-1} \to (-\Delta + z)^{-1} - \frac{2k\alpha}{2k + i\alpha} \left|G_z^1\right\rangle \left\langle \overline{G_z^1}\right| = (h(\alpha) + z)^{-1} \qquad (\varepsilon \to 0), \tag{1.15}$$

where the right-hand side agrees with the resolvent $(h(\alpha) + z)^{-1}$ of the self-adjoint operator $h(\alpha) = -\Delta_{\alpha,0}$ from [6, Chapter I, Theorem 3.1.1]. This shows that $h_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a well-known Hamiltonian $h(\alpha)$ that defines a $\delta$-interaction of strength $\alpha = g \int V(x)\, \mathrm{d}x$ at the origin. We remark that the case $\alpha = \infty$ in [6, Chapter I, Theorem 3.1.1] corresponds to a Hamiltonian $h(\infty) := -\Delta_{\infty,0}$ that realizes a Dirichlet boundary condition at $x = 0$. That is, it is given by the Dirichlet Laplacian on $(-\infty, 0) \cup (0, \infty)$. As this operator does not arise as a resolvent limit of the Schrödinger operators studied here, it will not be considered in the following.

In $d = 2$ dimensions, establishing convergence of $g_\varepsilon(1 + g_\varepsilon \phi_\varepsilon(z))^{-1}$ is more subtle due to the non-trivial renormalization of the coupling constant $g_\varepsilon$ and the fact that only a series expansion of the Green's function $G_z^2$ is known. We only state the result here and we refer to Section 4.4.1 and [6, Chapter I.5] for details. In the notation of [6, Chapter I.5] our operator $g_\varepsilon \phi_\varepsilon(z)$ agrees with the operator $B_\varepsilon(k)$, where $-k^2 = z$, $\mathrm{Im}(k) > 0$, $\lambda_1 = 1/a$ and $\lambda_2 = -b/a^2$. Hence, it follows from [6, Eq. (5.61), p. 104]* that, for all $z \in \mathbb{C} \setminus (-\infty, 0]$, $z \ne 4\exp(-4\pi\alpha - 2\gamma)$,

$$g_\varepsilon \left(1 + g_\varepsilon \phi_\varepsilon(z)\right)^{-1} \to -2\pi \left[\ln\left(\frac{k}{2i}\right) + \gamma + 2\pi\alpha\right]^{-1} \frac{\left|u\right\rangle \left\langle v\right|}{\left\langle u \,|\, v\right\rangle^2} \qquad (\varepsilon \to 0), \tag{1.16}$$

*There is a typo in [6, Eq. (5.61), p. 104]: The term in braces is to be inverted.

where $\gamma \approx 0.5772$ denotes the Euler–Mascheroni constant, and

$$\alpha := \frac{\langle v \,|\, Lu \rangle}{2\pi \, \langle u \,|\, v \rangle^2} - \frac{b}{\langle u \,|\, v \rangle} \tag{1.17}$$

with the Hilbert-Schmidt operator $L \in \mathscr{L}(L^2(\mathbb{R}^2))$ that is defined in terms of the kernel

$$u(x)\, \ln(|x - x'|)\, v(x'), \qquad\qquad x \neq x'.$$

From (1.7), (1.9), (1.10) and (1.16) it follows that, as $\varepsilon \to 0$, $(h_\varepsilon + z)^{-1} \to (h(\alpha) + z)^{-1}$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$, $z \neq 4 \exp(-4\pi\alpha - 2\gamma)$, where

$$(h(\alpha) + z)^{-1} = (-\Delta + z)^{-1} + \frac{2\pi}{\ln(-ik/2) + \gamma + 2\pi\alpha} |G_z^2\rangle\langle \overline{G_z^2}| \tag{1.18}$$

agrees with the resolvent of the self-adjoint operator $h(\alpha) = -\Delta_{\alpha,0}$ from [6, Chapter I, Theorem 5.2]. This shows that $h_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a well-known Hamiltonian $h(\alpha)$ that defines an attractive contact interaction at the origin. We emphasize that the asymptotics from (1.5) with $a = \int V(x)\,\mathrm{d}x/(2\pi) \neq 0$ is essential in $d = 2$ dimensions: If $\int V(x)\,\mathrm{d}x = 0$ or $a \neq \int V(x)\,\mathrm{d}x/(2\pi)$, then the limit operator (provided it exists) is just the Friedrichs extension $-\Delta \restriction H^2(\mathbb{R}^2)$, which corresponds to the case $\alpha = \infty$ in [6].

In $d = 3$ dimensions the situation is quite similar: By [6, Chapter I, Theorem 1.1.1], the operator $h_0 = -\Delta \restriction C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ has a one-parameter family of self-adjoint extensions $h(\alpha) = -\Delta_{\alpha,0}$, $\alpha \in (-\infty, \infty]$, where $\alpha = \infty$ corresponds to the Friedrichs extension $-\Delta \restriction H^2(\mathbb{R}^3)$, and [6, Chapter I, Theorem 1.2.5] shows that all these extensions $h(\alpha)$ arise as resolvent limits of suitable Schrödinger operators $h_\varepsilon$, $\varepsilon > 0$.

### 1.1.2 Summary of characteristic properties

In the previous section we have introduced a one-parameter family $h(\alpha)$, $\alpha \in (-\infty, \infty]$, of self-adjoint extensions of $-\Delta \restriction C_0^\infty(\mathbb{R}^d \setminus \{0\})$ in all dimensions $d \leq 3$. The operator $h(\alpha)$ describes an external contact interaction located at the origin whose sign and strength is determined by the parameter $\alpha$. The essential spectrum of $h(\alpha)$ agrees with the non-negative half-axis $[0, \infty)$, and in the case of an attractive interaction there is, in addition, one simple eigenvalue $E_B < 0$ [6], which we refer to as the binding energy.

In $d = 1$ dimension $h(\alpha)$, $\alpha \in \mathbb{R}$, can be viewed as a rigorous version of $-\Delta + \alpha\delta(\cdot)$. That is, $h(\alpha)$ is the self-adjoint operator that is associated with the closed semibounded quadratic form

$$q_\alpha(\psi) := \|\psi'\|^2 + \alpha|\psi(0)|^2, \qquad\qquad \psi \in D(q_\alpha) = H^1(\mathbb{R}). \tag{1.19}$$

Note that the right-hand side is well-defined because the embedding $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ exists and is continuous. The repulsive and attractive $\delta$-interactions correspond to the choices $\alpha > 0$ and $\alpha < 0$, respectively, while the choice $\alpha = 0$ just leads to the negative Laplacian $-\Delta \restriction H^2(\mathbb{R})$. In the attractive case $\alpha < 0$, the binding energy is given by $E_B = -\alpha^2/4$ [6]. For $d \in \{2, 3\}$, a direct interpretation of $h(\alpha)$, $\alpha \in \mathbb{R}$, as an operator of the type $-\Delta + \alpha\delta(\cdot)$ is not possible, and a suitable renormalization of $\alpha$ is indispensable [6]. This stems from the fact that a $\delta$-potential is *not* a small perturbation, in the form sense, of $-\Delta$ in $d \geq 2$. In particular, functions from $H^1(\mathbb{R}^d)$, $d \geq 2$, can have singularities and hence they do not embed into $C(\mathbb{R}^d)$. As already explained in the previous section, all contact interactions in $d \in \{2, 3\}$ are attractive. In $d = 2$, the binding energy $E_B = -4 \exp(-4\pi\alpha - 2\gamma)$ is the only (simple) eigenvalue of $h(\alpha)$, $\alpha \in \mathbb{R}$ [6]. In $d = 3$, $\alpha < 0$ corresponds to the case where the binding energy $E_B = -(4\pi\alpha)^2$ is the only (simple) eigenvalue of $h(\alpha)$, while $h(\alpha)$ has no discrete spectrum for $\alpha \geq 0$ [6].

The characteristic property of $h(\alpha)$, $\alpha \in \mathbb{R}$, in all dimensions $d \leq 3$ is that the resolvent can be expressed by a Krein formula of the form

$$(h(\alpha) + z)^{-1} = (-\Delta + z)^{-1} + C(z, \alpha, d) \, |G_z^d\rangle \, \langle \overline{G_z^d}| \,, \qquad z \in \rho(h(\alpha)), \tag{1.20}$$

where the constant $C(z, \alpha, d) \in \mathbb{C}$ is explicitly given by (1.15), respectively, (1.18) if $d \leq 2$, and for $d = 3$ it follows from [6, Chapter I, Theorem 1.1.2] that $C(z, \alpha, d) = (\alpha - ik/(4\pi))^{-1}$. From (1.20) we obtain the following characterization of the domain and the action of $h(\alpha)$: Each $\psi \in D(h(\alpha))$ can be written as $\psi = (h(\alpha) + z)^{-1}\varphi$ for some $\varphi \in L^2(\mathbb{R}^d)$, and with $\psi_0 := (-\Delta + z)^{-1}\varphi \in H^2(\mathbb{R}^d)$ and $\langle \overline{G_z^d}|\varphi\rangle = \psi_0(0)$ (this is well-defined because the embedding $H^2(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$ exists for $d \leq 3$, see, e.g., [69, Theorem IX.24]) it follows from (1.20) that

$$\psi = \psi_0 + C(z, \alpha, d)\psi_0(0)G_z^d, \qquad \psi_0 \in H^2(\mathbb{R}^d), z \in \rho(h(\alpha)) \tag{1.21}$$

and

$$(h(\alpha) + z)\psi = (-\Delta + z)\psi_0. \tag{1.22}$$

Conversely, it is not hard to show that Eq. (1.21) defines an element $\psi \in D(H)$ for any given $\psi_0 \in H^2(\mathbb{R}^d)$. Thus, Eqs. (1.21) and (1.22) determine the domain and the action of $h(\alpha)$. Moreover, Eq. (1.22) also shows that $\psi_0 \in H^2(\mathbb{R}^d)$ is uniquely determined by $\psi \in D(h(\alpha))$ and $z \in \rho(h(\alpha))$. For $\psi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, we may choose $\psi_0 = \psi$ in Eq. (1.21), which yields that $\psi \in D(h(\alpha))$ and $h(\alpha)\psi = -\Delta\psi$. This confirms that $h(\alpha)$ is a self-adjoint extension of $-\Delta \restriction C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Moreover, in $d = 1$, one can derive from (1.21) the well-known jump condition $\psi'(0+) - \psi'(0-) = \alpha\psi(0)$ for the derivative of the wave function, where $\psi'(0\pm) := \lim_{x\to 0\pm} \psi'(x)$. This is, in particular in physics, the common characterization of the domain of the operator $-\Delta + \alpha\delta(\cdot)$. More precisely, a direct computation using $C(z, \alpha, 1) = -2k\alpha/(2k + i\alpha)$ and the explicit formula (1.12) for $G_z^1$ leads to the following alternative description of the domain of $h(\alpha)$ (for details we refer to [6, Chapter I, Theorem 3.1.1]):

$$D(h(\alpha)) = \Big\{\psi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) \,|\, \psi'(0+) - \psi'(0-) = \alpha\psi(0)\Big\}. \tag{1.23}$$

Here, the left-hand and right-hand limits are to be understood in terms of suitable trace operators. For $d \in \{2, 3\}$, Eq. (1.21) shows that functions $\psi \in D(h(\alpha))$, $\alpha \neq 0$, with $\psi_0(0) \neq 0$ have a singularity at $x = 0$ that comes from the singularity of $G_z^d$ at $x = 0$. More precisely, as $|x| \to 0$,

$$\psi(x) = C(z, \alpha, d)\psi_0(0) \cdot \begin{cases} (2\pi)^{-1}\left(-\ln|x| - \dfrac{1}{a_\alpha}\right) + o(1) & \text{if } d = 2, \\ (4\pi)^{-1}\left(\dfrac{1}{|x|} - \dfrac{1}{a_\alpha}\right) + o(1) & \text{if } d = 3, \end{cases} \tag{1.24}$$

where $a_\alpha := -(2\pi\alpha)^{-1}$ if $d = 2$ and $a_\alpha := -(4\pi\alpha)^{-1}$ if $d = 3$ agrees with the s-wave scattering length of the operator $h(\alpha)$. For a comprehensive discussion of the scattering properties of $h(\alpha)$ in all dimensions $d \leq 3$ we refer to [6].

## 1.2 Contact interactions in the $N$-particle case ($N \geq 2$)

In this section we summarize results and facts about contact interactions in the $N$-particle case. To this end, we consider $N \geq 2$ non-relativistic, spinless quantum particles in $d \leq 3$ dimensions.

### 1.2.1 The mathematical point of view

By an $N$-particle Hamiltonian with contact interactions we understand any self-adjoint extension of the densely defined symmetric operator

$$\sum_{i=1}^{N} (-\Delta_{x_i}/m_i) \restriction C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \tag{1.25}$$

in the Hilbert space

$$\mathscr{H} := L^2\left(\mathbb{R}^{dN}, \mathrm{d}(x_1, \dots, x_N)\right), \tag{1.26}$$

where $\Delta_{x_i} := \sum_{k=1}^{d} \partial^2_{x_{i,k}}$ denotes the Laplacian w.r.t. $x_i = (x_{i,k})_{k=1}^d$, $m_i > 0$ denotes the mass of the $i$th particle and

$$\Gamma := \bigcup_{1 \le i < j \le N} \Gamma_{(i,j)} \tag{1.27}$$

denotes the union of all two-particle collision planes

$$\Gamma_{(i,j)} := \left\{ x = (x_1, ..., x_N) \in \mathbb{R}^{dN} \,\middle|\, x_i = x_j \right\}, \qquad 1 \le i < j \le N. \tag{1.28}$$

The idea behind this definition is that the Hamiltonian should agree with the free operator unless some of the particles are in direct contact with each other, i.e. $x_i = x_j$ for some $1 \le i < j \le N$. We note that the trivial extension is the kinetic energy operator

$$H_0 = \sum_{i=1}^{N} (-\Delta_{x_i}/m_i) \tag{1.29}$$

that is self-adjoint on $D(H_0) = H^2(\mathbb{R}^{dN})$. In fact, it follows from [75, Theorem 3.2] and our assumption $N \ge 2$ and $d \le 3$ that the operator from (1.25) has the deficiency indices $n_+ = n_- = +\infty$, so von Neumann's theory of self-adjoint extensions guarantees the existence of a huge family of self-adjoint extensions that depends on infinitely many parameters. A crucial example of such a self-adjoint extension would be a rigorous version of the formal operator

$$H(\alpha) = H_0 + \sum_{(i,j) \in \mathcal{I}} \alpha_{(i,j)} \delta(x_j - x_i), \tag{1.30}$$

where $\mathcal{I}$ is the set of all pairs $(i,j)$ with $1 \le i < j \le N$, which we denote by Greek letters $\sigma, \nu, ...$ in the sequel, and the component $\alpha_\sigma \in \mathbb{R}$ of the vector $\alpha := (\alpha_\sigma)_{\sigma \in \mathcal{I}}$ determines the sign and the strength of the two-body interaction among the particles in the pair $\sigma$.

A family of $N$-particle Hamiltonians with contact interactions, which involves operators of the type (1.30), can be constructed rigorously within the framework of Posilicano's theory. This theory was developed, inter alia, in [21, 65, 66, 67, 68] and it is closely related to von Neumann's theory of self-adjoint extensions and the theory of boundary triples [66, 67]. To apply Posilicano's theory, we have to introduce some auxiliary spaces and operators first.

Let $\mathfrak{X} := \bigoplus_{\sigma \in \mathcal{I}} \mathfrak{X}_\sigma$, where

$$\mathfrak{X}_\sigma := L^2\left(\mathbb{R}^{d(N-1)}, \mathrm{d}\left(R, x_1, ...\widehat{x}_i ... \widehat{x}_j ..., x_N\right)\right), \qquad \sigma = (i,j) \in \mathcal{I} \tag{1.31}$$

can be identified with the Hilbert space of all square-integrable functions over $\Gamma_\sigma$ after introducing the relative and center of mass coordinates

$$r = r_\sigma := x_j - x_i, \qquad R = R_\sigma := \frac{m_i x_i + m_j x_j}{m_i + m_j}, \qquad \sigma = (i,j) \in \mathcal{I}. \tag{1.32}$$

The hat, as in $\widehat{x}_i$, indicates omission of that variable. Furthermore, let the unbounded operator $T : D(T) \subseteq \mathscr{H} \to \mathfrak{X}$ be defined by $T\psi := (T_\sigma \psi)_{\sigma \in \mathcal{I}}$, where, for $\psi \in C_0^\infty(\mathbb{R}^{dN})$, the trace operators $T_\sigma : D(T_\sigma) \subseteq \mathscr{H} \to \mathfrak{X}_\sigma$ are defined by

$$(T_\sigma \psi)(R, x_1, ...\widehat{x}_i ... \widehat{x}_j ..., x_N) := \psi(x_1, \dots, x_N)\big|_{x_i = x_j = R}, \qquad \sigma = (i,j). \tag{1.33}$$

A rigorous definition of the trace operators $T_\sigma$ is given in Section 2.2, below. However, independent of the space dimension $d \leq 3$, all trace operators $T_\sigma$, $\sigma \in \mathcal{I}$, and hence $T$, are $H_0$-bounded. This means that*

$$G(z) := TR_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}), \qquad z \in \rho(H_0)$$

defines a bounded operator, where $R_0(z) := (H_0 + z)^{-1}$ for short. The operator $G(z)$ is closely related to the Green's function $G_{z,\underline{m}}$, $\underline{m} = (m_1, ..., m_N)$, of $H_0 + z$ (see, e.g., Eq. (4.72), below) and the letter $G$ serves to remind us of this connection. From (1.33) it follows that $T\psi = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ because $\psi$ vanishes in a neighborhood of all collision planes $\Gamma_\sigma$. In particular, this means that any self-adjoint extension of $H_0 \restriction \operatorname{Ker} T$ is also a self-adjoint extension of $H_0 \restriction C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$. To construct such self-adjoint extensions, we are going to apply [68, Theorem 2.2]. For this purpose, we fix $\lambda \in (0, \infty)$ and in analogy to [68] we introduce the bounded operators

$$M(z) := (z - \lambda) G(z) G(\lambda)^* \in \mathscr{L}(\mathfrak{X}), \qquad z \in \rho(H_0).$$

Now, let $\Theta : D(\Theta) \subseteq \mathfrak{X} \to \mathfrak{X}$ be an arbitrary self-adjoint operator, let $\Theta(z) := \Theta + M(z)$ for $z \in \rho(H_0)$, and suppose that $\Theta(z)$ is invertible for some $z \in \rho(H_0)$, i.e.

$$Z_\Theta := \{z \in \rho(H_0) \,|\, 0 \in \rho(\Theta(z))\} \neq \emptyset.$$

Then [68, Theorem 2.2]† shows that

$$(H_\Theta + z)^{-1} := R_0(z) + G(\overline{z})^* \Theta(z)^{-1} G(z), \qquad z \in \rho(H_0) \cap \rho(H_\Theta) \tag{1.34}$$

defines the resolvent of a self-adjoint operator $H_\Theta$ in $\mathscr{H}$ and that $\rho(H_0) \cap \rho(H_\Theta) = Z_\Theta$. The resolvent formula (1.34) can be interpreted as a generalization of the usual Krein formula that relates the resolvents of two self-adjoint extensions of a given symmetric operator [65], and it is the characteristic resolvent formula for $N$-particle Hamiltonians with contact interactions. In analogy to the one-particle case described in Section 1.1.2, Eq. (1.34) leads to the following characterization of the domain and the action of $H_\Theta$ (see also [68, Theorem 2.2]): A vector $\psi \in \mathscr{H}$ belongs to $D(H_\Theta)$ if and only if for some (and hence all) $z \in \rho(H_0) \cap \rho(H_\Theta)$ there exist $\psi_0 \in H^2(\mathbb{R}^{dN})$ and $w \in D(\Theta)$ such that

$$\psi = \psi_0 + G(\overline{z})^* w \tag{1.35}$$

and

$$T\psi_0 = \Theta(z) w. \tag{1.36}$$

If this is the case, then

$$(H_\Theta + z)\psi = (H_0 + z)\psi_0, \tag{1.37}$$

and the vectors $\psi_0 \in H^2(\mathbb{R}^{dN})$ and $w \in D(\Theta)$ are uniquely determined by $\psi \in D(H_\Theta)$ and $z \in \rho(H_0) \cap \rho(H_\Theta)$‡.

If $T : H^2(\mathbb{R}^{dN}) \to \mathfrak{X}$ were even surjective, then all self-adjoint extensions of $H_0 \restriction \operatorname{Ker} T$ could be characterized in a similar fashion with the help of [67, Theorem 3.1], which provides

*In the notation of [21, 65, 66, 67, 68] the operators $G(z)$, respectively, $G_z$ correspond to $(T(-H_0 + \overline{z})^{-1})^*$, which differs from our convention.

†By Lemma 2.13, below, and the subsequent remark, $\operatorname{Ker} G(\lambda)^* = \{0\}$ and $\operatorname{Ran} G(\lambda)^* \cap H^2(\mathbb{R}^{dN}) = \{0\}$, so the hypotheses of [68, Theorem 2.2] are satisfied.

‡The proof of [68, Theorem 2.2] even shows that $w \in D(\Theta)$ only depends on $\psi \in D(H_\Theta)$, but not on $z \in \rho(H_0) \cap \rho(H_\Theta)$.

a one-to-one correspondence to von Neumann's theory of self-adjoint extensions. Although $T$ is *not* surjective with the above choice of $\mathfrak{X}$, it is, in principle, possible to modify $\mathfrak{X}$ and $T$ appropriately and to characterize all self-adjoint extensions of $H_0 \upharpoonright C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$. One way to do this is described in the introduction of [67]. However, we shall see in the next section that only specific choices of the self-adjoint operator $\Theta : D(\Theta) \subseteq \mathfrak{X} \to \mathfrak{X}$ lead to physically reasonable extensions. For such choices of $\Theta$, one then has to verify that $\Theta(z) = \Theta + M(z)$ is invertible for at least one $z \in \rho(H_0)$. In general, finding a physically reasonable choice of $\Theta$ and proving invertibility of $\Theta(z)$ is a challenging task. This explains why historically mostly older methods (e.g. suitably chosen quadratic forms, introducing appropriate boundary conditions on $\Gamma$, or defining the Hamiltonian as a limit of certain regularized Hamiltonians, to name only a few of them) are still common methods to define physically reasonable contact interactions.

### 1.2.2 The physical point of view

From the physical point of view, only those self-adjoint extensions $H$ of $H_0 \upharpoonright C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ that meet certain minimum requirements are reasonable [30, 58, 59]. In particular, this involves

**(i)** $H$ is bounded from below. That is, $H \geq -C$ for some constant $C > 0$.

**(ii)** $H$ is translation-invariant. That is, $H$ commutes with the strongly continuous unitary group of operators $T_{\mathrm{tot},h} \in \mathscr{L}(\mathscr{H})$, $h \in \mathbb{R}^d$, defined by

$$(T_{\mathrm{tot},h}\psi)(x_1, x_2, ..., x_N) := \psi(x_1 + h, x_2 + h, ..., x_N + h).$$

More precisely, this means that $D(H) \subseteq D(HT_{\mathrm{tot},h})$ and $T_{\mathrm{tot},h}H\psi = HT_{\mathrm{tot},h}\psi$ for all $\psi \in D(H)$ and $h \in \mathbb{R}^d$.*

**(iii)** $H$ is local. That is, if $\psi \in D(H)$ vanishes a.e. in some non-empty open set $U \subseteq \mathbb{R}^{dN}$, then also $H\psi = 0$ a.e. in $U$.

Property (**ii**) is motivated by the fact that, due to the absence of an external potential, a translation of the whole system should not affect its physical behavior. Moreover, this implies that $H$ commutes with every component of the total momentum operator $P_{\mathrm{tot}}$ in the sense of [71, Theorem VIII.13]. Note that after passing to Fourier space, $P_{\mathrm{tot}}$ simply acts by multiplication with $P_{\mathrm{tot}}(p_1, ..., p_N) := \sum_{i=1}^N p_i$, where $p_i \in \mathbb{R}^d$ is conjugated to $x_i \in \mathbb{R}^d$. Property (**iii**) is motivated by the fact that $H$ agrees with the local operator $H_0$ on the set $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$, which is dense in $\mathscr{H}$. Indeed, one can show that *every* self-adjoint extension of $H_0 \upharpoonright C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ satisfies Property (**iii**) (see, e.g., Lemma C.2 in the appendix of [6]). From a historical point of view, Property (**i**), which guarantees the stability of the considered system, has caused the greatest difficulties and confusion. This is due to the so-called *Thomas effect* in $d = 3$ dimensions that we are going to explain in the following.

To describe two-body contact interactions among $N = 3$ identical spinless particles in $d = 3$ dimensions, Ter Martirosyan and Skornyakov assumed that the boundary condition that had been derived by Bethe and Peierls for $N = 2$ [14] (after separation of the center of mass motion this is essentially Eq. (1.24)) remains valid in all two-particle subsystems [55]. This leads to a boundary condition at $\Gamma$, which is now known as the *TMS boundary condition* (or simply *TMS condition*). One version of the TMS condition demands that every wave function in the domain of the Hamiltonian has the following asymptotic behavior for each pair $\sigma = (i, j) \in \mathcal{I}$:

$$\psi(x_1, ..., x_N) = \left( \frac{1}{|x_i - x_j|} - \frac{1}{a_{\alpha,\sigma}} \right) \xi_\sigma(x_1, ...\widehat{x_j}..., x_N) + o(1) \qquad (|x_i - x_j| \to 0),$$

*In fact, the Hamiltonians considered in this thesis even satisfy the stronger operator identity $T_{\mathrm{tot},h}H = HT_{\mathrm{tot},h}$ for each $h \in \mathbb{R}^d$ (see Propositions 3.17 and 4.16, below).

where $\xi_\sigma \in L^2(\mathbb{R}^{3(N-1)})$ can be interpreted as a function on the collision plane $\Gamma_\sigma$ and $a_{\alpha,\sigma}$ agrees with the scattering length in the $(i,j)$-channel [56]. A few years later the model of Ter Martirosyan and Skornyakov was further analyzed by Minlos and Faddeev [58, 59], where a one-parameter family of self-adjoint extensions of the symmetric (but *not* self-adjoint) TMS Hamiltonian was constructed. However, it turned out that each of these extensions has a sequence of negative eigenvalues $E_n$, $n \in \mathbb{N}$, that diverges exponentially to $-\infty$ as $n \to \infty$ [27, 58, 59]. Today, this phenomenon is known as the Thomas effect (or Thomas collapse) since Thomas has already encountered this effect in 1935 in the context of his formal calculation of the neutron-proton interaction in tritium [78]. Since then, the Thomas effect has been discussed extensively in the literature, both in mathematics and physics, but the construction of a semibounded Hamiltonian describing contact interactions among $N = 3$ particles in $d = 3$ dimensions remained an open problem for a long time. However, a regularized version of the TMS Hamiltonian for $N = 3$ bosons in $d = 3$ dimensions that is self-adjoint and bounded from below has recently been constructed in [11], see also [36] for some historical remarks. The Thomas effect, which comes from the extremely strong attractive forces when all three particles are very close to each other, could be compensated in [11] by introducing an effective three-body interaction.

In the presence of symmetry restrictions, especially antisymmetry, the impact of the Thomas effect can be weaker or even negligible. For example, in the case of the Fermi polaron model, which describes contact interactions among a gas of $N \geq 1$ identical fermions of mass one and a different particle of mass $m > 0$ (often called impurity), the TMS Hamiltonian has self-adjoint extensions that are bounded from below, provided that $m > m^*$ for some critical mass $m^* > 0$ [24, 60]. Moreover, $m^*$ is even independent of $N \geq 1$ [60]. A similar result also applies in the case of a $(2+2)$-system consisting of two identical fermions that interact via contact interactions with two fermions of a different species [57, 61].

The stability problem discussed above does not occur in $d \leq 2$ dimensions. In the simplest case $d = 1$, two-body $\delta$-interactions are small perturbations in the form sense of the free operator $H_0$. This means that a rigorous version of the formal operator $H(\alpha)$ from Eq. (1.30) can be constructed by means of a closed semibounded quadratic form (for details, see Section 3.2, below). The resolvent of this self-adjoint operator $H(\alpha)$ can be expressed by a generalized Krein formula of the form (1.34) (see Theorem 3.1, below), and hence Eqs. (1.35) - (1.37) characterize the domain and the action of $H(\alpha)$. In analogy to (1.23), the following jump condition at $\Gamma$ provides an alternative characterization of the wave functions $\psi \in D(H(\alpha))$:

$$\left(\frac{\partial_j}{m_j} - \frac{\partial_i}{m_i}\right)\psi|_{x_j=x_i+} - \left(\frac{\partial_j}{m_j} - \frac{\partial_i}{m_i}\right)\psi|_{x_j=x_i-} = \alpha_\sigma \psi|_{x_i=x_j}, \qquad \sigma = (i,j) \in \mathcal{I}, \quad (1.38)$$

where the left-hand and right-hand limits and equating the particle positions $x_i$ and $x_j$ are to be understood in terms of suitable trace operators in Sobolev spaces (for details, we refer to Proposition 3.16, below). A related model that is very common in physics is the Lieb-Liniger model [52, 51], which describes a one-dimensional Bose gas in a box $[0, l]$, $l > 0$, that interacts by repulsive $\delta$-interactions of strength $2c > 0$. The $\delta$-interactions are implemented by the jump condition (1.38) with $m_i = 1$, $i = 1, ..., N$, and $\alpha_\sigma = 2c$, $\sigma \in \mathcal{I}$, and periodic boundary conditions are imposed at the boundary points 0 and $l$. The Lieb-Liniger model is soluble in the sense that the Hamiltonian has a complete orthonormal system of eigenfunctions, which can be computed via the Bethe ansatz by solving a set of transcendental equations [32]. This allows for the explicit calculation of many physically relevant quantities, e.g. in thermodynamics [81]. Moreover, it is shown in [73] that the Lieb-Liniger model can be considered as a scaling limit of a dilute 3d Bose gas that interacts via repulsive two-body potentials. In this limit the scattering length of the interaction potential and the radius of the cylindrical trap tend to zero simultaneously.

In $d = 2$, a Hamiltonian $H_\beta$ describing non-trivial contact interactions among an arbitrary number $N \geq 2$ of particles was first constructed by Dell'Antonio, Figari and Teta in terms of an appropriate closed and semibounded quadratic form $\mathrm{F}_\beta$ [30]. They do not need any symmetry restrictions (i.e. the underlying Hilbert space is $L^2(\mathbb{R}^{2N})$) and for simplicity they choose $m_i = 1$, $i = 1, ..., N$, in our setting. The component $\beta_\sigma \in \mathbb{R}$ of the vector $\beta$ in $H_\beta$ determines the strength of the contact interaction among the particles of the pair $\sigma \in \mathcal{I}$. Moreover, $H_\beta$ satisfies our minimum requirements **(i)** – **(iii)** and its resolvent can be expressed by a generalized Krein formula of the form (1.34) (see [30, Eqs. (5.12) and (5.13)]). This allows for characterizing the domain and the action of $H_\beta$ in the spirit of Eqs. (1.35) - (1.37) (cf. [30, Eqs. (5.3) and (5.4)]). It is pointed out in [30] that this characterization of $D(H_\beta)$ can be interpreted as a two-dimensional generalization of the TMS boundary condition. Therefore, we refer to $H_\beta$ as the TMS Hamiltonian in the following. Beyond that, $H_\beta$ is the limit, in the strong resolvent sense, of a suitable sequence of regularized Hamiltonians, where the regularization is achieved by introducing an ultraviolet cutoff in Fourier space (see [30, Remark 4.4]). Similar results for $N \geq 2$ bosons are derived in [31] in the language of second quantization.

To conclude this section, we briefly discuss some recent developments in physics that have contributed to the fact that contact interactions are of topical interest in many areas of physics. In particular, we put emphasis on the field of ultra-cold quantum gases. The characteristic feature of such gases is that the effective range of the two-body interaction is much smaller than the thermal wavelength of the particles, which results in a universal behavior where the detailed form of the two-body interaction becomes irrelevant and only a few low-energy parameters (e.g. the scattering length) are sufficient to describe the interaction [18, 36]. Such a universal behavior can be observed, for example, during the BCS-BEC crossover in ultra-cold Fermi gases [82]. The BCS-BEC crossover describes the transition between a system of weakly bound pairs of fermions within the framework of the Bardeen-Cooper-Schrieffer (BCS) theory to a Bose Einstein condensate (BEC) consisting of tightly bound bosonic molecules. The middle of this crossover is described by the so-called *unitary limit*, where the scattering length of the two-body interaction becomes infinite, while at the same time the effective range of the interaction potential tends to zero [82]. In mathematical terms, such a limit defines a contact interaction. The BCS-BEC crossover has even been realized experimentally with the help of Feshbach resonances [25], which allow for a fine-tuning of the two-body scattering length [72, 83]. This experimental and theoretical progress has laid the foundation for various new applications of contact interactions. Today, contact interactions are useful, and in fact indispensable, tools in many areas of contemporary physics, from solid-state physics to atomic and nuclear physics, and even in string theory [38].

## 1.3 New results

This thesis is devoted to the approximation of contact interactions among $N \geq 2$ particles in $d \leq 3$ dimensions by means of Schrödinger operators with local rescaled two-body potentials. The results presented in this section are based on the following three publications:

**(1)** M. Griesemer, M. Hofacker, and U. Linden. From Short-Range to Contact Interactions in the 1d Bose Gas. *Math. Phys. Anal. Geom.*, 23(2):Paper No. 19, 28, 2020.

**(2)** M. Griesemer and M. Hofacker. From Short-Range to Contact Interactions in Two-dimensional Many-Body Quantum Systems. *Ann. Henri Poincaré*, 23(8):2769–2818, 2022.

**(3)** M. Griesemer and M. Hofacker. On the Weakness of Short-Range Interactions in Fermi Gases. arXiv:2201.04362 [math-ph], 2022.

Our focus lies on the case $d \leq 2$, where we show that the Hamiltonians described in Section 1.2.2 are limits, in the norm resolvent sense, of suitably rescaled Schrödinger operators. This is the main result of Sections 3 ($d = 1$) and 4 ($d = 2$). Finally, in Section 5, we analyze and quantify the weakness of short-range interactions among equal spin fermions in all dimensions $d \leq 3$.

### 1.3.1 From short-range to contact interactions in the $N$-particle case ($N \geq 2$)

In particular in physics, contact interactions are often used to model real short-range interactions whose details are very complicated or even unknown. For example, this was the motivation of Bethe and Peierls to approximate the two-body potential describing the short-range interactions among a proton and a neutron in the deuteron by an idealized interaction of zero range [14]. From the mathematical point of view, such an idealization is also favorable because we already know from Section 1.2 that Hamiltonians with contact interactions have many desirable properties: few relevant parameters (e.g. the scattering lengths, which, in the attractive case, are directly related to the binding energy), a simple characterization of the domain and the action of the Hamiltonian, an explicit Krein-like formula for the resolvent, and from this further physically relevant quantities, such as spectra and scattering quantities, can either be derived explicitly or with the help of numerical methods. Therefore, it would be very beneficial if one could consider contact interactions as limits of suitably rescaled local short-range interactions. For this purpose, we consider short-range interactions among $N \geq 2$ non-relativistic spinless quantum particles in $d \leq 3$ dimensions, which are described, in suitable units, by the rescaled Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$, from Equation (1.39), below. Our goal is to prove norm resolvent convergence of $H_\varepsilon$ in the limit $\varepsilon \to 0$ and to identify the limit operator with a Hamiltonian describing contact interactions. Because of the symmetry independent choice of coordinates, this implies norm resolvent convergence in arbitrary symmetry regimes (e.g. in the symmetric and antisymmetric subspace of $L^2(\mathbb{R}^{dN})$ and in every conceivable composed system) by simply projecting the resolvents onto the respective subspace of $L^2(\mathbb{R}^{dN})$. Compared with the weaker strong resolvent convergence, norm resolvent convergence has stronger consequences for the convergence of the associated spectra and unitary groups (see Section 2.1). For example, norm resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ implies the convergence of the associated spectra in the sense that $\lambda \in \sigma(H)$ if and only if there exists a sequence $\lambda_\varepsilon \in \sigma(H_\varepsilon)$ with $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda$, while strong resolvent convergence only implies a weaker result.

We now start motivating our main results, whereby we introduce all required spaces and operators. In the Hilbert space $\mathscr{H}$ from (1.26) we consider Schrödinger operators of the form

$$H_\varepsilon := H_0 + \sum_{\sigma=(i,j)\in\mathcal{I}} g_{\varepsilon,\sigma}\, V_{\sigma,\varepsilon}(x_j - x_i), \qquad \varepsilon > 0, \tag{1.39}$$

where $H_0$ is given by (1.29), $g_{\varepsilon,\sigma} \in \mathbb{R}$ denotes a coupling constant depending on $\varepsilon > 0$ and $\sigma$, and, by some abuse of notation, the multiplication operator $V_{\sigma,\varepsilon}(x_j - x_i)$ is defined in terms of

$$V_{\sigma,\varepsilon}(r) := \varepsilon^{-d}\, V_\sigma(r/\varepsilon), \qquad \varepsilon > 0, \tag{1.40}$$

for some fixed potential $V_\sigma : \mathbb{R}^d \to \mathbb{R}$ satisfying $V_\sigma(-r) = V_\sigma(r)$ a.e. We further assume that $V_\sigma \in L^1 \cap L^2(\mathbb{R}^d)$, which implies that $H_\varepsilon$ is self-adjoint on $D(H_\varepsilon) = D(H_0) = H^2(\mathbb{R}^{dN})$ (see, e.g., [77, Theorem 11.1]). In analogy to the one-particle case described in Section 1.1.1, the two-body interaction among the particles in the pair $\sigma$ has, at best, a vanishing limit as $\varepsilon \to 0$ unless $g_{\varepsilon,\sigma}$ has an asymptotic behavior of the form

$$g_{\varepsilon,\sigma} = \begin{cases} g_\sigma + o(1) & \text{if } d = 1, \\ \Big(\mu_\sigma(a_\sigma \ln(\varepsilon) + b_\sigma) + o(1)\Big)^{-1} & \text{if } d = 2, \\ (\mu_\sigma)^{-1}(\varepsilon + b_\sigma \varepsilon^2) + o(\varepsilon^2) & \text{if } d = 3, \end{cases} \qquad (\varepsilon \to 0) \tag{1.41}$$

where $a_\sigma, b_\sigma, g_\sigma \in \mathbb{R}$, $a_\sigma > 0$, and $\mu_\sigma := m_i m_j/(m_i + m_j)$ denotes the reduced mass of the pair $\sigma = (i,j)$. The prefactor $(\mu_\sigma)^{-1}$, which we have introduced in $d \in \{2,3\}$ dimensions, will allow us to compare our results directly with their counterparts known from the one-particle case [6]. In $d = 1$, however, we absorb this prefactor in the constant $g_\sigma$ for simplicity.

In $d = 3$ dimensions there is a close connection between the Thomas effect and the Efimov effect, which has important consequences for the approximation of contact interactions by means of Schrödinger operators [2, 9, 18]. The Efimov effect describes the phenomenon that a Hamiltonian describing short-range two-body interactions among three particles has an infinite number of negative eigenvalues in the center of mass frame if the Hamiltonians of all two-body subsystems have no negative eigenvalues and at least two of them have a zero-energy resonance. This effect was predicted by Efimov [33] and later it has been proved rigorously under various additional assumptions of technical nature (the first rigorous proof was given in [80] but we also refer to [76]). We now consider $N \geq 3$ particles in $d = 3$ dimensions and we assume that $H_\varepsilon$ defined by Eqs. (1.39)-(1.41) converges, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint operator $H$. Then, as elaborated in Appendix C, the Efimov effect leads to the following dichotomy: Either the limit operator $H$ is unbounded from below (that is, the Thomas effect occurs) or $H$ is trivial in the sense that each three-body subsystem contains at least one particle that does not interact with the other two particles. This means that further restrictions or modifications in the above setting are inevitable if one demands that the limit operator is bounded from below. This shall not be further investigated in the course of this thesis, and instead we now formulate our main results in the cases $d = 1$ and $d = 2$, where we are neither confronted with the Thomas effect nor with the Efimov effect.

Let $d \leq 2$ and let $H_\varepsilon$, $\varepsilon > 0$, be defined by Eqs. (1.39)-(1.41) for some $V_\sigma \in L^1 \cap L^2(\mathbb{R}^d)$ with $V_\sigma(r) = V_\sigma(-r)$ a.e. In the case $d = 2$, we further assume that (the importance of these assumptions shall be discussed below):

**(As)** There exists some $s > 0$ such that, for all $\sigma \in \mathcal{I}$, $\int |r|^{2s} |V_\sigma(r)| \,\mathrm{d}r < \infty$.

**(Ag)** For all $\sigma \in \mathcal{I}$, the asymptotics of $g_{\varepsilon,\sigma}$ is given by Eq. (1.41) with $a_\sigma \geq \frac{1}{2\pi} \int V_\sigma(r) \,\mathrm{d}r$.

Then our main results Theorem 3.1 ($d = 1$) and Theorem 4.1 ($d = 2$) show that $H_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint semibounded operator $H$ that describes contact interactions among $N$ particles. If $d = 1$ and, in addition, $|g_{\varepsilon,\sigma} - g_\sigma| = O(\varepsilon^s)$ as $\varepsilon \to 0$ and $\int |r|^{2s} |V_\sigma(r)| \,\mathrm{d}r < \infty$ for some $s \in (0,1)$ and all $\sigma \in \mathcal{I}$, then the rate of norm resolvent convergence is at least as good as $O(\varepsilon^s)$. More precisely, this means that $z \in \rho(H)$ implies that $z \in \rho(H_\varepsilon)$ for small enough $\varepsilon > 0$ and

$$\|(H + z)^{-1} - (H_\varepsilon + z)^{-1}\| = O(\varepsilon^s) \qquad (\varepsilon \to 0).$$

For $N = 2$ particles this problem reduces to the well-understood one-particle case [6] after removing the center of mass motion, but for $N > 2$ this is a non-trivial and fairly challenging problem, and except of publications **(1)** and **(2)** only few comparable results have been available so far. For $N = 3$ particles in $d = 1$ dimension such a result was first established in [10]. However, the proof in [10] relies on Faddeev's equations, which have no natural generalization for $N > 3$. The first result for arbitrary $N \geq 2$ in $d = 1$ was derived in publication **(1)** in the special case of a Bose gas. In this thesis we generalize and enlarge the results and methods from **(1)**, which are essentially based on the analysis of integral operators and their kernels. In $d = 2$ the weaker result that $H_\varepsilon$ has a limit in the *strong* resolvent sense as $\varepsilon \to 0$ was shown quite recently in [46] for an arbitrary number $N \geq 2$ of identical particles, provided that all two-body potentials $V_\sigma$ belong to a specific class of compactly supported smooth functions. The interesting point, however, is that this problem is studied in [46] in a completely different context, namely the investigation of a certain limit in the two-dimensional stochastic heat equation. In publication **(2)**, where Theorem 4.1 and most of the other results from Section 4 were first published, the result from [46] has been improved to *norm* resolvent convergence for the large class of two-body potentials $V_\sigma$ introduced above with general masses $m_i > 0$, $i = 1, ..., N$.

After this short digression into the literature, we now continue describing the results of this thesis. The norm resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ implies that $H$ has the Properties **(i)** – **(iii)** from Section 1.2.2 (see Propositions 3.17 ($d = 1$) and 4.16 ($d = 2$), below). In accordance with Eq. (1.34), the resolvent of $H$ is given by a Krein-like formula of the form

$$(H + z)^{-1} = R_0(z) + G(\overline{z})^*\Theta(z)^{-1}G(z), \qquad z \in \rho(H_0) \cap \rho(H), \tag{1.42}$$

with the only difference that only the pairs from a certain subset $\mathcal{J} \subseteq \mathcal{I}$ contribute to the auxiliary Hilbert space $\mathfrak{X} := \bigoplus_{\sigma\in\mathcal{J}} \mathfrak{X}_\sigma$, and hence to $H$. This means that $G(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X})$, $z \in \rho(H_0)$, has the components $G(z)_\sigma = T_\sigma R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$, $\sigma \in \mathcal{J}$, and $\Theta(z)$, $z \in \rho(H_0)$, is an operator defined in $\mathfrak{X}$ that has a bounded inverse if and only if $z \in \rho(H_0) \cap \rho(H)$. For $d = 1$, $\Theta(z)$ defines a bounded operator in $\mathscr{L}(\mathfrak{X})$, while for $d = 2$ the operator $\Theta(z)$ is unbounded with domain $D \subseteq \mathfrak{X}$ independent of $z \in \rho(H_0)$.

As explained at the end of Section 1.2.2, we expect that the limit operator $H$ only depends on the particular choices of the interaction potentials $g_{\varepsilon,\sigma}V_{\sigma,\varepsilon}$ via a few low-energy parameters. Indeed, for $d = 1$, $H$ is the self-adjoint operator that is associated with the closed semibounded quadratic form $q$ (with $C = 0$) from Eq. (3.7), below, which only depends on $g_{\varepsilon,\sigma}V_{\sigma,\varepsilon}$ via $\alpha_\sigma = \lim_{\varepsilon\to 0} \int g_{\varepsilon,\sigma}V_{\sigma,\varepsilon}(r)\,\mathrm{d}r = g_\sigma \int V_\sigma(r)\,\mathrm{d}r$. The parameter $\alpha_\sigma$ determines the strength of the $\delta$-interaction among the particles of the pair $\sigma$ and $H$ can be viewed as a rigorous version of the formal operator $H(\alpha)$ from Eq. (1.30). In particular, this means that $\mathcal{J} = \{\sigma \in \mathcal{I} \,|\, \alpha_\sigma \neq 0\}$ and that the domain of $H$ can be characterized by a rigorous version of the jump condition from Eq. (1.38) (see Proposition 3.16, below). For $d = 2$, the subset $\mathcal{J} \subseteq \mathcal{I}$ is the set of all pairs $\sigma \in \mathcal{I}$ for which the asymptotic expansion (1.41) holds with $a_\sigma = \int V_\sigma(r)\,\mathrm{d}r/(2\pi) > 0$. Similarly to the case $d = 1$, the limit operator $H$ only depends on the particular choices of $g_{\varepsilon,\sigma}V_{\sigma,\varepsilon}$ via certain parameters $\beta_\sigma \in \mathbb{R}$, $\sigma \in \mathcal{J}$ (see Eqs. (4.52)-(4.54), below). Here, the assumption **(As)** is needed to ensure that the integral operator $L_\sigma$, which is the analog of the operator $L$ from Section 1.1.1, is a Hilbert-Schmidt operator (in fact, such a condition is already needed in the one-particle case [6]). We are going to identify $H$ with the TMS Hamiltonian $H_\beta$, $\beta = (\beta_\sigma)_{\sigma\in\mathcal{J}}$, from [30]. To this end, we compute the quadratic form of $H$ explicitly in Section 4.6, below.

In the remainder of this section, we explain our strategy for proving Theorems 3.1 and 4.1. Regardless of the space dimension $d \in \{1, 2\}$, our starting point for proving norm resolvent convergence of $H_\varepsilon$ is the new expression (1.51), below, for $H_\varepsilon$, which allows us to express the resolvent of $H_\varepsilon$ by the generalized Konno-Kuroda formula from Eq. (1.53), below. For later convenience, we now derive Eqs. (1.51) and (1.53) rigorously, and afterwards we sketch the extensive proofs of Theorems 3.1 and 4.1 that are given in Sections 3 and 4, respectively.

Let the auxiliary Hilbert space $\widetilde{\mathfrak{X}}$ be given by

$$\widetilde{\mathfrak{X}} := \bigoplus_{\sigma\in\mathcal{I}} \widetilde{\mathfrak{X}}_\sigma, \tag{1.43}$$

where the integration variables $r$ and $R$ in

$$\widetilde{\mathfrak{X}}_\sigma := L^2\left(\mathbb{R}^{dN}, \mathrm{d}\left(r, R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N\right)\right), \qquad \sigma = (i, j) \tag{1.44}$$

correspond to the relative and center of mass coordinates from Eq. (1.32). This change of coordinates is implemented unitarily by the operator $\mathscr{K}_{(i,j)} : \mathscr{H} \to \widetilde{\mathfrak{X}}_{(i,j)}$ with

$$\begin{aligned}&\left(\mathscr{K}_{(i,j)}\psi\right)(r, R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N)\\ &\quad := \psi\Big(x_1, ..., x_{i-1}, R - \frac{m_j r}{m_i + m_j}, x_{i+1}, ..., x_{j-1}, R + \frac{m_i r}{m_i + m_j}, x_{j+1}, ..., x_N\Big).\end{aligned} \tag{1.45}$$

The adjoint thereof is the operator $\mathscr{K}^*_{(i,j)} : \widetilde{\mathfrak{X}}_{(i,j)} \to \mathscr{H}$ with

$$(\mathscr{K}^*_{(i,j)}\psi)(x_1, ..., x_N) = \psi\Big(x_j - x_i, \frac{m_i x_i + m_j x_j}{m_i + m_j}, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N\Big). \tag{1.46}$$

Furthermore, let $U_\varepsilon \in \mathscr{L}(L^2(\mathbb{R}^d))$ denote the unitary rescaling from Section 1.1.1, which we also consider as an operator in $\mathscr{L}(\widetilde{\mathfrak{X}}_\sigma)$ by setting

$$\left(U_\varepsilon\psi\right)(r, \underline{X}) := \varepsilon^{d/2}\, \psi(\varepsilon r, \underline{X}), \qquad \underline{X} = (R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N),\ \sigma = (i,j) \in \mathcal{I}, \tag{1.47}$$

and let

$$\begin{aligned} v_\sigma(r) &:= |V_\sigma(r)|^{1/2},\\ u_\sigma(r) &:= J_\sigma |V_\sigma(r)|^{1/2}, \qquad J_\sigma := \operatorname{sgn}(V_\sigma),\end{aligned}$$

so that $V_\sigma = u_\sigma v_\sigma$. In terms of the above operators, we now introduce for $\varepsilon > 0$ the new operators $A_{\varepsilon,\sigma}, B_{\varepsilon,\sigma} : D(A_{\varepsilon,\sigma}) \subseteq \mathscr{H} \to \widetilde{\mathfrak{X}}_\sigma$ with

$$A_{\varepsilon,\sigma} := (v_\sigma \otimes 1)\, \varepsilon^{-d/2} U_\varepsilon \mathscr{K}_\sigma, \tag{1.48}$$

$$B_{\varepsilon,\sigma} := (u_\sigma \otimes 1)\, \varepsilon^{-d/2} U_\varepsilon \mathscr{K}_\sigma = J_\sigma A_{\varepsilon,\sigma}, \tag{1.49}$$

where the domain $D(A_{\varepsilon,\sigma})$ is determined by the domain of the multiplication operator $v_\sigma \otimes 1$. Since our assumption that $V_\sigma \in L^1(\mathbb{R}^d)$ implies that $u_\sigma, v_\sigma \in L^2(\mathbb{R}^d)$, it follows that $A_{\varepsilon,\sigma}$ and $B_{\varepsilon,\sigma}$ are densely defined and closed on $D(A_{\varepsilon,\sigma}) \supset H^2(\mathbb{R}^{dN})$. These new operators allow us to rewrite the two-body interaction as

$$V_{\sigma,\varepsilon}(x_j - x_i)\psi = \mathscr{K}^*_\sigma (V_{\sigma,\varepsilon} \otimes 1)\mathscr{K}_\sigma \psi = (A_{\varepsilon,\sigma})^* B_{\varepsilon,\sigma}\psi, \qquad \psi \in H^2(\mathbb{R}^{dN}), \tag{1.50}$$

which means that the Definition (1.39) of $H_\varepsilon$ is equivalent to

$$H_\varepsilon = H_0 + \sum_{\sigma \in \mathcal{I}} g_{\varepsilon,\sigma} (A_{\varepsilon,\sigma})^* B_{\varepsilon,\sigma}, \qquad \varepsilon > 0. \tag{1.51}$$

From (1.50) and from our assumption that $V_\sigma \in L^2(\mathbb{R}^d)$ it follows that both $(A_{\varepsilon,\sigma})^* B_{\varepsilon,\sigma}$ and $(A_{\varepsilon,\sigma})^* A_{\varepsilon,\sigma} = |V_{\sigma,\varepsilon}(x_j - x_i)|$ are infinitesimally $H_0$-bounded (see, e.g., [77, Theorem 11.1]), so the hypotheses of Corollary B.2 are satisfied. This means that

$$\phi_\varepsilon(z)_{\sigma\nu} := B_{\varepsilon,\sigma} R_0(z) (A_{\varepsilon,\nu})^* \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma), \qquad z \in \rho(H_0),\ \sigma, \nu \in \mathcal{I} \tag{1.52}$$

defines the components of a bounded operator $\phi_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$. Moreover, with $g_\varepsilon \in \mathscr{L}(\widetilde{\mathfrak{X}})$ defined by $(g_\varepsilon)_{\sigma\nu} := g_{\varepsilon,\sigma}\delta_{\sigma\nu}$, $\sigma, \nu \in \mathcal{I}$, it follows that $1 + g_\varepsilon\phi_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ has a bounded inverse if and only if $z \in \rho(H_\varepsilon) \cap \rho(H_0)$. If this is the case, then

$$(H_\varepsilon + z)^{-1} = R_0(z) - \sum_{\sigma,\nu \in \mathcal{I}} (A_{\varepsilon,\sigma} R_0(\overline{z}))^* \Big[(1 + g_\varepsilon \phi_\varepsilon(z))^{-1}\Big]_{\sigma\nu} g_{\varepsilon,\nu} B_{\varepsilon,\nu} R_0(z). \tag{1.53}$$

We refer to Eq. (1.53) as a generalized Konno-Kuroda formula. This is our key to prove norm resolvent convergence of $H_\varepsilon$ in $d \in \{1, 2\}$. For sufficiently large $z_0 > 0$, we are going to show that $1 + g_\varepsilon\phi_\varepsilon(z)$ is invertible for all $z \in (z_0, \infty)$ and all small enough $\varepsilon > 0$, which implies that $z \in \rho(H_\varepsilon) \cap \rho(H_0)$, where $(H_\varepsilon + z)^{-1}$ is explicitly given by Eq. (1.53). We shall then show that both $(1+g_\varepsilon\phi_\varepsilon(z))^{-1}g_\varepsilon$ and $A_{\varepsilon,\sigma}R_0(z)$ (and hence also $B_{\varepsilon,\nu}R_0(z) = J_\nu A_{\varepsilon,\nu}R_0(z)$) have suitable limits as $\varepsilon \to 0$. In view of Eq. (1.53), this means that $R(z) := \lim_{\varepsilon\to 0}(H_\varepsilon + z)^{-1}$ exists for all $z \in (z_0, \infty)$, and after having integrated out the contributions of the two-body potentials $V_\sigma$, we shall see that $R(z)$ agrees with the right-hand side of Eq. (1.42) for a suitable operator $\Theta(z)$. The next step is then to show that $R(z)$ defines the resolvent of a self-adjoint semibounded operator $H$. To this end, we proceed differently depending on the space dimension $d$. In $d = 1$ we are going to use the fact that strong resolvent convergence is equivalent to strong and weak $\Gamma$-convergence of the associated closed and semibounded quadratic forms. Before we prove norm convergence $(H_\varepsilon + z)^{-1} \to R(z)$ as $\varepsilon \to 0$, this allows us to prove strong resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$, where $H$ denotes the self-adjoint operator that is associated with the quadratic form $q$ from Eq. (3.7). Altogether, this means that $R(z) = (H + z)^{-1}$ for all $z \in (z_0, \infty)$ and that $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$. In $d = 2$ a standard result (see, e.g., [31, Theorem 5]) allows us to show that there exists a self-adjoint semibounded operator $H$ such that $R(z) = (H + z)^{-1}$ for all $z \in (z_0, \infty)$. In fact, we shall see in Section 4.6 that $H$ agrees with the TMS Hamiltonian $H_\beta$ for suitable $\beta$ [30]. This proves Theorems 3.1 and 4.1 for $z \in (z_0, \infty)$ and to conclude the proofs, it remains to show that $\Theta(z)$ allows for an analytic continuation to all $z \in \rho(H_0) \cap \rho(H) \subseteq \mathbb{C}$ and that the resolvent formula (1.42) remains valid for all $z \in \rho(H_0) \cap \rho(H)$. The latter is achieved by verifying that the extended operators $\Theta(z)$ satisfy the hypotheses of [21, Theorem 2.19].

In the remainder of this section we explain our strategy for proving convergence, as $\varepsilon \to 0$, of $(1+g_\varepsilon\phi_\varepsilon(z))^{-1}g_\varepsilon$ and $A_{\varepsilon,\sigma}R_0(z)$ in more detail. From the Definition (1.48) of $A_{\varepsilon,\sigma}$, it follows that proving convergence of $A_{\varepsilon,\sigma}R_0(z)$ is effectively only a one-particle problem, namely proving convergence of $v_\sigma \varepsilon^{-d/2} U_\varepsilon$ in $\mathscr{L}(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$. Using the regularity of the $H^2(\mathbb{R}^d)$-functions, it is not hard to show that the latter is true in all dimensions $d \leq 3$, though the rate of convergence depends on the dimension and the decay of $|V_\sigma(r)|$ as $|r| \to \infty$. The hard part, which is even non-trivial in the one-particle case, is proving convergence of $(1 + g_\varepsilon\phi_\varepsilon(z))^{-1}g_\varepsilon$.

In $d = 1$ we first give a simplified proof that is based on the fact that even the operators $A_{\varepsilon,\sigma}R_0(z)^{1/2}$, $z > 0$, have suitable limits as $\varepsilon \to 0$ (which is *not* true for $d \geq 2$). In view of the Definition (1.52) of $\phi_\varepsilon(z)_{\sigma\nu}$ with $B_{\varepsilon,\sigma} = J_\sigma A_{\varepsilon,\sigma}$, this implies that $\phi(z) = \lim_{\varepsilon\to 0}\phi_\varepsilon(z)$ exists in $\mathscr{L}(\widetilde{\mathfrak{X}})$. Since we shall also prove that $\|\phi_\varepsilon(z)_{\sigma\nu}\| \leq C(\sigma,\nu)/\sqrt{z}$ for some constant $C(\sigma,\nu) > 0$ that is independent of $\varepsilon, z > 0$ and, by assumption, $g = \lim_{\varepsilon\to 0} g_\varepsilon$ exists, it follows that $(1+g\phi(z))^{-1}$ exists and that $(1 + g_\varepsilon\phi_\varepsilon(z))^{-1}g_\varepsilon \to (1 + g\phi(z))^{-1}g$ as $\varepsilon \to 0$, provided that $z > 0$ is large enough. However, this short and simplified proof comes at the expense of the convergence rate. To improve the rate of convergence, we are going to use that all components $\phi_\varepsilon(z)_{\sigma\nu}$ define integral operators whose kernels can be computed explicitly in terms of the Green's function $G_z^N$ of $-\Delta + z : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. We carefully distinguish between the three cases, where the pairs $\sigma$ and $\nu$ have two, one or no common particle, respectively. In each case we pass to Fourier space in an appropriate subset of the coordinates, which allows us to greatly reduce the dimension of the problem. In the end, the good properties of the respective Green's function allow us to prove the desired rate of convergence.

In $d = 2$ dimensions we are confronted with many additional difficulties when proving norm convergence of $(1+g_\varepsilon\phi_\varepsilon(z))^{-1}g_\varepsilon$, which makes the proof difficult and somewhat technical. First of all, it follows from the asymptotic expansion (1.41) with $a_\sigma > 0$ that $g_\varepsilon$ is invertible for all sufficiently small $\varepsilon > 0$. This means that the convergence of $(1+g_\varepsilon\phi_\varepsilon(z))^{-1}g_\varepsilon$ in the limit $\varepsilon \to 0$ is equivalent to the convergence of the inverse of the operator $\Lambda_\varepsilon(z) := (g_\varepsilon)^{-1} + \phi_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ in the limit $\varepsilon \to 0$. We then decompose $\Lambda_\varepsilon(z)$ into its diagonal and off-diagonal parts:

$$\Lambda_\varepsilon(z) = \Lambda_\varepsilon(z)_{\mathrm{diag}} + \Lambda_\varepsilon(z)_{\mathrm{off}}.$$

We are going to show that both $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}$ and $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}\Lambda_\varepsilon(z)_{\text{off}}$ have suitable limits as $\varepsilon \to 0$, provided that $z > 0$ is large enough. This, in turn, implies that

$$(1 + g_\varepsilon \phi_\varepsilon(z))^{-1} g_\varepsilon = \Lambda_\varepsilon(z)^{-1} = [1 + (\Lambda_\varepsilon(z)_{\text{diag}})^{-1}\Lambda_\varepsilon(z)_{\text{off}}]^{-1}(\Lambda_\varepsilon(z)_{\text{diag}})^{-1} \tag{1.54}$$

has a suitable limit as $\varepsilon \to 0$. To prove convergence of $(\Lambda_\varepsilon(z)_{\sigma\sigma})^{-1}$, and hence of $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}$, we pass to Fourier space in all coordinates except of the relative coordinate $r$. We shall see that $\Lambda_\varepsilon(z)_{\sigma\sigma}$ acts pointwise in the conjugate variables $\underline{P}_\sigma \in \mathbb{R}^{2N-2}$ by an operator that is very similar to the operator $(g_\varepsilon)^{-1} + \phi_\varepsilon(z)$ from the one-particle case. In analogy to Section 1.1.1, a careful analysis shows that the divergence of $(g_{\varepsilon,\sigma})^{-1}$ in the limit $\varepsilon \to 0$ is to be chosen in such a way that a divergent part in $\phi_\varepsilon(z)_{\sigma\sigma}$ is compensated. The additional difficulty that arises in the $N$-particle case is that all estimates have to be uniform in $\underline{P}_\sigma \in \mathbb{R}^{2N-2}$ in order to obtain norm convergence of $(\Lambda_\varepsilon(z)_{\sigma\sigma})^{-1}$. In the end, a lemma that essentially goes back to Barry Simon allows us to introduce an ultraviolet cutoff in $\underline{P}_\sigma$ and to reduce the problem to the well understood one-particle case. Thereby, it turns out that the above assumption **(Ag)** is indispensable: If **(Ag)** is not satisfied for some pair $\sigma$, then one may expect strong resolvent convergence of $H_\varepsilon$ at best. To prove convergence of the off-diagonal parts $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$, we introduce a suitable space cutoff depending on the respective pairs $\sigma$ and $\nu$ that eliminates certain singular contributions to $\phi_\varepsilon(z)_{\sigma\nu}$. This provides us with the regularity to proceed in a similar way as in the simpler case $d = 1$. The uniform boundedness in $\varepsilon, z > 0$ of the operator norm $\|\Lambda_\varepsilon(z)_{\text{off}}\|$ in combination with the regularizing properties of $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}$ eventually allow us to remove all cutoffs again when proving convergence of $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}\Lambda_\varepsilon(z)_{\text{off}}$. In the end, we shall see that $\Lambda_\varepsilon(z)^{-1}$ exists for sufficiently large $z > 0$ and sufficiently small $\varepsilon > 0$ and, in the limit $\varepsilon \to 0$, it converges to an operator that has exactly the right form so that, upon integrating out the contributions of the two-body potentials $V_\sigma$, the desired Krein-like formula (1.42) arises.

### 1.3.2 Weakness of short-range interactions in Fermi gases

In the physics literature it is a common practice that short-range interactions among equal spin fermions in ultracold quantum gases are neglected, while at the same time the interaction among particles of opposite spin is modelled by contact interactions [23, 37, 63]. For example, in the case of the Fermi polaron model only the contact interactions among the $N$ fermions and the impurity particle are present in the Hamiltonian, while the interaction among the $N$ fermions is completely neglected [43, 63]. The common justification for this simplification is as follows: The Pauli principle forces the wave function to be antisymmetric w.r.t. permutations of the spatial coordinates of the spin-aligned fermions, that is

$$\psi(x_1, ...x_i...x_j..., x_N) = -\psi(x_1, ...x_j...x_i..., x_N), \qquad 1 \leq i < j \leq N, \tag{1.55}$$

where the coordinates of the other particles have been omitted for brevity. This implies that $\psi(x_1, ..., x_N) = 0$ if $x_i = x_j$ for some pair $(i, j)$, which suggests that short-range interactions among equal spin fermions are very weak and, in particular, that zero-range interactions (contact interactions) vanish entirely. An alternative justification for neglecting such short-range interactions is that $s$-wave scattering should be suppressed by the Pauli principle [63].

In Section 5 we are going to justify this simplification mathematically. For this purpose, we consider $N \geq 2$ identical spinless fermions with mass one in $d \geq 1$ dimensions, which means that the underlying Hilbert space is the fermionic subspace

$$\mathscr{H}_{\text{f}} := \bigwedge_{i=1}^{N} L^2(\mathbb{R}^d, \mathrm{d}x_i) \tag{1.56}$$

of $L^2(\mathbb{R}^{dN})$ that contains all functions $\psi$ that satisfy (1.55) almost everywhere. In appropriate units the kinetic energy operator of the system is given by $H_0 = -\Delta \restriction H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$.

Our first result, Theorem 5.1, shows that contact interactions among equal spin fermions are indeed impossible in $d \geq 2$ dimensions. To this end, we prove that $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ is dense in $H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$ w.r.t. the norm of $H^2$. This implies that $-\Delta$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ (with $H_0$ being the only self-adjoint extension), and hence contact interactions, as introduced in Section 1.2.1, vanish on the fermionic subspace $\mathscr{H}_\mathrm{f}$ for $d \geq 2$. However, this is *not* true for $d = 1$. This manifests in the existence of $\delta'$-interactions in $d = 1$, which, in contrast to $\delta$-interactions, only vanish when the derivative (and not the wave function itsself) vanishes at the origin [6].

As elaborated in the previous section, contact interactions in $d \leq 3$ can be viewed as idealizations of short-range interactions that arise in a suitable zero-range limit. To estimate the weakness of short-range interactions in fully spin-polarized Fermi gases, we again consider the family of Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$, from Eq. (1.39), which we now restrict to $\mathscr{H}_\mathrm{f}$. More precisely, we choose $g_{\varepsilon,\sigma} = g_\varepsilon > 0$ and $V_\sigma = V$ for all pairs $\sigma \in \mathcal{I}$, where the measurable function $V : \mathbb{R}^d \to \mathbb{R}$ satisfies $V(r) = V(-r)$ a.e. Then,

$$H_\varepsilon = -\Delta + g_\varepsilon \sum_{\substack{i,j=1 \\ i<j}}^{N} V_\varepsilon(x_j - x_i), \qquad \varepsilon > 0, \tag{1.57}$$

where $V_\varepsilon(r) = \varepsilon^{-d} V(r/\varepsilon)$. To ensure that $H_\varepsilon$ defines a self-adjoint operator on $D(H_\varepsilon) = D(H_0) = H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$, further assumptions on $V$ are needed. Obviously, $V \in L^2(\mathbb{R}^d)$, $d \leq 3$, would be sufficient. However, the antisymmetry of the wave function combined with the Hölder continuity of the $H^2(\mathbb{R}^d)$-functions allow us to relax this assumption slightly (see Lemma 5.2).

Assuming $d \leq 2$, $V \in L^1 \cap L^2(\mathbb{R}^d)$, some further decay of $V$ in the case $d = 2$, and that $g_\varepsilon$ satisfies (1.41), Theorems 3.1 and 4.1 show that $H_\varepsilon \to H_0$ in the norm resolvent sense as $\varepsilon \to 0$. Although this only works for this special choice of $g_\varepsilon$ and only in $d \leq 2$ dimensions, we shall see that the norm resolvent convergence $H_\varepsilon \to H_0$ remains valid for a much larger class of short-range interactions. For $d \geq 2$, the weaker strong resolvent convergence $H_\varepsilon \to H_0$ as $\varepsilon \to 0$ can be easily derived from our previous result that $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ is dense in $H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$ for a large class of two-body potentials $g_\varepsilon V_\varepsilon$ (see Proposition 5.4). However, our major focus lies on proving norm resolvent convergence, which has stronger consequences for the associated spectra and unitary groups. For $d \leq 3$, this is achieved by Theorem 5.6, which we are going to explain in the following.

For the sake of this introduction, we suppose, for simplicity, that $V \in L^1 \cap L^2(\mathbb{R}^d)$, $V(r) = V(-r)$ a.e., $C_V := \operatorname{ess\,sup}_{r \in \mathbb{R}^d} |r|^2 |V(r)| < \infty$, and that $g_\varepsilon > 0$ satisfies

$$\limsup_{\varepsilon \to 0} g_\varepsilon \varepsilon^{2-d} < \frac{d^2}{C_V N}, \tag{1.58}$$

although Theorem 5.6 holds for a larger class of two-body potentials $g_\varepsilon V_\varepsilon$. In the case $d = 1$ we further suppose that $\int |V(r)||r|^{2s} \,\mathrm{d}r < \infty$ for some $s > 1/2$. Then Theorem 5.6 shows that $H_\varepsilon \to H_0$ in the norm resolvent sense as $\varepsilon \to 0$. Our estimate for the rate of resolvent convergence depends on $g_\varepsilon$ and, to some extent, on the decay of $|V(r)|$ as $|r| \to \infty$. When this decay is sufficient, then the space dimension $d$, or, more precisely, the highest Hölder exponent $s > 0$ for which the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d)$ exists, becomes the limiting factor.

Although the condition (1.58) can be further relaxed, we shall see that there exists some critical value $\lambda_{\max} = \lambda_{\max}(d, N, V) \geq 0$ such that $\limsup_{\varepsilon \to 0} g_\varepsilon \varepsilon^{2-d} < \lambda_{\max}$ is sufficient for norm resolvent convergence $H_\varepsilon \to H_0$, while one may expect strong resolvent convergence $H_\varepsilon \to H_0$ at best if $\limsup_{\varepsilon \to 0} g_\varepsilon \varepsilon^{2-d} > \lambda_{\max}$. For the special choice (1.41) of $g_\varepsilon$, the condition (1.58) is automatically satisfied for $d \leq 2$, while for $d = 3$ this is only possible if $N \geq 2$ is small enough. Therefore, it is even more remarkable that this choice of $g_\varepsilon$ conspires with the regularity of $H^2(\mathbb{R}^d)$-functions in such a way that for all $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(\varepsilon^2) \qquad (\varepsilon \to 0)$$

independent of the space dimension $d \in \{1, 2, 3\}$, provided that $V$ has sufficient decay and that $N \geq 2$ is small enough if $d = 3$. In contrast to that, we shall see that for suitably chosen $V$ the Hamiltonian $\widetilde{H}_\varepsilon$ that is defined by the right side of Eq. (1.57) in the enlarged Hilbert space $L^2(\mathbb{R}^{dN})$ has a significantly different behavior, even in $d = 3$ dimensions.

Another choice of $g_\varepsilon$, which is consistent with (1.58) as well, is the one where $g_\varepsilon \varepsilon^{2-d}$ is a positive constant smaller than $d^2/(C_V N)$. In this case, a rescaling shows that $\widetilde{H}_\varepsilon$ is unitarily equivalent to $\varepsilon^{-2}\widetilde{H}_{\varepsilon=1}$. Hence, if $d \leq 2$ and $V \leq 0$, $V \neq 0$, has sufficient decay, then it follows from [74, Theorems 2.5 and 3.4] that the two-body binding energy diverges to $-\infty$ in the limit $\varepsilon \to 0$, which amounts to a combined short-range and strong interaction limit. However, by Theorem 5.6, the restriction $H_\varepsilon = \widetilde{H}_\varepsilon \restriction \mathscr{H}_\mathrm{f}$ converges in the norm resolvent sense to $H_0$.

## 1.4 Outline

In Section 2 we collect basic results and definitions that are needed in the course of this thesis. Results concerning strong and norm resolvent convergence are summarized in Section 2.1. In particular, we demonstrate to what extent norm resolvent convergence has stronger consequences for the convergence of the associated spectra and unitary groups than the weaker strong resolvent convergence. In Section 2.2 we give the rigorous definition of the trace operators $T_\sigma$ that have been introduced in Eq. (1.33) and we establish basic properties of these trace operators that are needed in the sequel. In addition, we introduce two versions $T_\sigma^+$ and $T_\sigma^-$ of these trace operators that are needed in Section 3.6 in order to formulate a mathematically rigorous version of the jump condition from Eq. (1.38).

Section 3 is devoted to the analysis of short-range interactions and contact interactions in $d = 1$. The main result is Theorem 3.1, where Hamiltonians describing contact interactions among $N \geq 2$ particles are shown to arise as limits, in the norm resolvent sense, of Schrödinger operators with suitably rescaled two-body potentials. The major part of this section is devoted to both proving Theorem 3.1 and optimizing the rate of norm resolvent convergence. In the last part of this section we show that the limit operator $H$ from Theorem 3.1 describes physically reasonable contact interactions, i.e. it is bounded from below, local and translation-invariant. Moreover, the domain and the action of $H$ are characterized by means of a rigorous version of the jump condition from Eq. (1.38).

Section 4 is devoted to the analysis of short-range interactions and contact interactions in $d = 2$. The main result is Theorem 4.1, which can be viewed as the two-dimensional analog of Theorem 3.1. The major part of this section is concerned with preparing the proof of Theorem 4.1, which is given in Section 4.5. Moreover, we show that our limit operator $H$ describes physically reasonable contact interactions, and our estimates yield an explicit lower bound for the spectrum. In the last part of this section, we compute the quadratic form of $H$ explicitly, which allows us to identify $H$ with the TMS Hamiltonian of Dell'Antonio, Figari and Teta [30].

Finally, in Section 5, the weakness of short-range interactions among equal spin fermions is analyzed and quantified. First, we show that contact interactions among equal spin fermions are indeed impossible in $d \geq 2$, while this is *not* true in $d = 1$. Our second main result reveals that fully spin-polarized Fermi gases in $d \leq 2$ with short-range interactions - the spin-up-spin-down interaction strength being fixed - are asymptotically free in the limit of zero-range interaction. This remains true in $d = 3$ for suitable two-body potentials $V$ and $N \geq 2$ small, depending on $V$.

# 2 Preliminaries and basic results

## 2.1 Strong and norm resolvent convergence

In many situations, where one has to deal with unbounded operators, convergence in the resolvent sense is the appropriate concept of convergence. There are two important types of resolvent convergence: strong resolvent convergence and norm resolvent convergence. In this section, we collect basic results concerning these types of convergence in an arbitrary separable complex Hilbert space $\mathscr{H}$.

**Definition 2.1.** *Let $H$ and $H_\varepsilon$, $\varepsilon > 0$, be a family of self-adjoint operators in $\mathscr{H}$. Then we say that $H_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to $H$ if $\|(H_\varepsilon + z)^{-1} - (H + z)^{-1}\| \to 0$ in $\mathscr{L}(\mathscr{H})$ for some $\varepsilon_0 > 0$ and some $z \in \bigcap_{\varepsilon\in(0,\varepsilon_0)} \rho(H_\varepsilon) \cap \rho(H)$. Strong resolvent convergence is defined analogously, where the norm convergence is replaced by the weaker strong convergence. That is, for all $\psi \in \mathscr{H}$, $(H_\varepsilon + z)^{-1}\psi \to (H + z)^{-1}\psi$ as $\varepsilon \to 0$.*

A first consequence of strong, respectively, norm resolvent convergence is stated in Proposition 2.2, below. For the proof, we refer to [71, Theorem VIII.20] and [77, Theorem 6.31].

**Proposition 2.2.** *Let $H$ and $H_\varepsilon$, $\varepsilon > 0$, be self-adjoint operators in $\mathscr{H}$, let $\varepsilon_0 > 0$ and let $\Sigma = \bigcup_{\varepsilon\in(0,\varepsilon_0)} \sigma(H_\varepsilon) \cup \sigma(H)$.*

*(i) If $H_\varepsilon \to H$ in the strong resolvent sense as $\varepsilon \to 0$ and $f : \Sigma \to \mathbb{C}$ is a bounded and continuous function, then, for all $\psi \in \mathscr{H}$, $f(H_\varepsilon)\psi \to f(H)\psi$ as $\varepsilon \to 0$.*

*(ii) If $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$ and $f : \Sigma \to \mathbb{C}$ is a bounded and continuous function satisfying $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x)$, then $\|f(H_\varepsilon) - f(H)\| \to 0$ as $\varepsilon \to 0$.*

*Remark.* The operators $f(H), f(H_\varepsilon) \in \mathscr{L}(\mathscr{H})$ are defined by the spectral theorem, see, e.g., [71, Theorem VIII.5].

In quantum mechanics, the spectra of $H$ and $H_\varepsilon$ are of particular interest. By Proposition 2.3, below, norm resolvent convergence is superior to the weaker strong resolvent convergence with regard to the spectral consequences. For the proof, we refer to [71, Theorems VIII.23 and VIII.24] and [77, Theorem 6.38].

**Proposition 2.3.** *Let $H$ and $H_\varepsilon$, $\varepsilon > 0$, be self-adjoint operators in $\mathscr{H}$.*

*(i) If $H_\varepsilon \to H$ in the strong resolvent sense as $\varepsilon \to 0$, then $\sigma(H) \subseteq \lim_{\varepsilon\to 0} \sigma(H_\varepsilon)$ in the sense that for each $\lambda \in \sigma(H)$ there exists a sequence $\lambda_\varepsilon \in \sigma(H_\varepsilon)$, $\varepsilon > 0$, with $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda$.*

*(ii) If $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$, then $\sigma(H) = \lim_{\varepsilon\to 0} \sigma(H_\varepsilon)$ in the sense that $\lambda \in \sigma(H)$ if and only if there exists a sequence $\lambda_\varepsilon \in \sigma(H_\varepsilon)$, $\varepsilon > 0$, with $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda$.*

From Propositions 2.2 and 2.3 we obtain the following corollary:

**Corollary 2.4.** *If $H_\varepsilon \to H$ in the strong resolvent sense as $\varepsilon \to 0$, then $(H_\varepsilon + z)^{-1}\psi \to (H + z)^{-1}\psi$ for all $\psi \in \mathscr{H}$ and all $z$ from the interior of $\bigcap_{\varepsilon\in(0,\varepsilon_0)} \rho(H_\varepsilon) \cap \rho(H)$. If $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$, then $z \in \rho(H)$ implies that $z \in \rho(H_\varepsilon)$ for $\varepsilon > 0$ small enough and $\|(H_\varepsilon + z)^{-1} - (H + z)^{-1}\| \to 0$ as $\varepsilon \to 0$.*

It is well-known that strong resolvent convergence is equivalent to the strong convergence of the associated strongly continuous unitary groups. That is, $H_\varepsilon \to H$ in the strong resolvent sense as $\varepsilon \to 0$ if and only if, for all $t \in \mathbb{R}$ and all $\psi \in \mathscr{H}$, $\exp(-iH_\varepsilon t)\psi \to \exp(-iHt)\psi$ as $\varepsilon \to 0$, see, e.g., [71, Theorem VIII.21]. Indeed, one can even show that this convergence is uniform on compact time intervals, see [71, Problem 21, p. 314]. In the case of norm resolvent convergence, the following proposition further improves this result to norm convergence modulo the resolvent of the limit operator:

**Proposition 2.5.** *Let $H$ and $H_\varepsilon$, $\varepsilon > 0$, be a family of self-adjoint operators in $\mathscr{H}$ and suppose that $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$. Then, for any $T > 0$,*

$$\lim_{\varepsilon\to 0} \sup_{t\in[-T,T]} \left\| [\exp(-iHt) - \exp(-iH_\varepsilon t)] \, (H+i)^{-1} \right\| = 0. \tag{2.1}$$

*Proof.* We start with the estimate

$$\begin{aligned}\left\| [\exp(-iHt) - \exp(-iH_\varepsilon t)] \, (H+i)^{-1} \right\| &\le \left\| \exp(-iHt)(H+i)^{-1} - \exp(-iH_\varepsilon t)(H_\varepsilon + i)^{-1} \right\| \\ &\quad + \left\| \exp(-iH_\varepsilon t) \left[ (H_\varepsilon + i)^{-1} - (H+i)^{-1} \right] \right\|. \end{aligned} \tag{2.2}$$

The second summand on the right side of (2.2) is bounded by $\|(H_\varepsilon + i)^{-1} - (H+i)^{-1}\|$, which is independent of $t \in \mathbb{R}$ and vanishes as $\varepsilon \to 0$. Hence, to prove (2.1), it remains to show that

$$\lim_{\varepsilon\to 0} \sup_{t\in[-T,T]} \|f_t(H) - f_t(H_\varepsilon)\| = 0, \tag{2.3}$$

where $f_t(x) := \exp(-ixt)/(x+i)$ for short. For fixed $t \in \mathbb{R}$, it follows from Proposition 2.2 that $\|f_t(H) - f_t(H_\varepsilon)\| \to 0$ as $\varepsilon \to 0$. To prove that this convergence is uniform in $t \in [-T,T]$, let $\delta > 0$ be given. Let $-T = t_0 < t_1 < ... < t_n = T$ be a partition of $[-T,T]$ with $|t_{k+1} - t_k| < \delta/3$ for $k = 0, ..., n-1$. Then we choose $\varepsilon_0 > 0$ so small that, for all $k = 0, ..., n$ and all $\varepsilon \in (0, \varepsilon_0)$, $\|f_{t_k}(H) - f_{t_k}(H_\varepsilon)\| < \delta/3$. For given $t \in [-T,T]$, we now choose $t_k$ so that $t \in [t_k, t_{k+1})$ (for $t = T$ we choose $t_k = t_n = T$). Then, using that for all $x \in \mathbb{R}$,

$$|f_t(x) - f_{t_k}(x)| = \frac{1}{|x+i|} \left| \int_{t_k}^{t} ix \exp(-isx) \, \mathrm{d}s \right| \le |t - t_k|,$$

we obtain the Lipschitz property $\|f_t(A) - f_{t_k}(A)\| \le \|f_t - f_{t_k}\|_{L^\infty} \le |t - t_k| < \delta/3$ for $A \in \{H, H_\varepsilon\}$. Hence, it follows that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}\|f_t(H) - f_t(H_\varepsilon)\| &\le \|f_t(H) - f_{t_k}(H)\| + \|f_{t_k}(H) - f_{t_k}(H_\varepsilon)\| + \|f_{t_k}(H_\varepsilon) - f_t(H_\varepsilon)\| \\ &< \delta/3 + \delta/3 + \delta/3 = \delta.\end{aligned}$$

As $t \in [-T,T]$ was arbitrarily chosen, this proves (2.3), and hence (2.1). □

In the case of norm resolvent convergence, the following lemma, which is a variant of [28, Lemma 2.6.1], shows that even the rate of resolvent convergence is independent of the particular choice of $z \in \rho(H)$:

**Lemma 2.6.** *Suppose that $H_\varepsilon$, $\varepsilon > 0$, are self-adjoint on a common domain $D(H_\varepsilon) = D \subseteq \mathscr{H}$ and that $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$. For $z \in \rho(H_\varepsilon) \cap \rho(H)$, let $R_\varepsilon(z) := (H_\varepsilon + z)^{-1}$ and $R(z) := (H+z)^{-1}$. Then $z, z_0 \in \rho(H)$ implies that $z, z_0 \in \rho(H_\varepsilon) \cap \rho(H)$ for small enough $\varepsilon > 0$ and, for some constant $C_z \ge 0$,*

$$\|R_\varepsilon(z) - R(z)\| \le (1 + |z - z_0| C_z)^2 \|R_\varepsilon(z_0) - R(z_0)\|.$$

*Proof.* Let $z, z_0 \in \rho(H)$ be given. Then it follows from Proposition 2.3 that $z, z_0 \in \rho(H_\varepsilon) \cap \rho(H)$ for all $\varepsilon \in (0, \varepsilon_0)$ and $C_z := \sup_{\varepsilon\in(0,\varepsilon_0)} \|R_\varepsilon(z)\| < \infty$, provided that $\varepsilon_0 > 0$ is small enough. Let $\varepsilon, \delta \in (0, \varepsilon_0)$. Then, by the first resolvent identity,

$$R_\varepsilon(z) = R_\varepsilon(z_0)S_\varepsilon(z, z_0) = S_\varepsilon(z, z_0)R_\varepsilon(z_0), \tag{2.4}$$

where $S_\varepsilon(z, z_0) = 1+(z_0-z)R_\varepsilon(z)$ has the norm bound $\|S_\varepsilon(z, z_0)\| \le 1+|z-z_0|C_z$. The second resolvent identity $R_\varepsilon(z) - R_\delta(z) = R_\varepsilon(z)(H_\delta - H_\varepsilon)R_\delta(z)$ and (2.4) imply that

$$R_\varepsilon(z) - R_\delta(z) = S_\varepsilon(z, z_0)\big(R_\varepsilon(z_0) - R_\delta(z_0)\big)S_\delta(z, z_0).$$

After taking norms of both sides and the limit $\delta \to 0$, the desired estimate follows. □

A variant of Lemma 2.6, where the constant $C_z$ is explicitly given, is the following lemma:

**Lemma 2.7.** *Let $H$ and $H_0$ be self-adjoint on a common domain $D(H) = D(H_0) \subseteq \mathscr{H}$ and let $R(z) := (H+z)^{-1}$ and $R_0(z) := (H_0+z)^{-1}$ for $z \in \rho(H)\cap\rho(H_0)$. Then, for all $z \in \rho(H)\cap\rho(H_0)$ and all $w \in \mathbb{C} \setminus \mathbb{R}$,*

$$\|R(w) - R_0(w)\| \le \left(1 + \frac{|w-z|}{|\operatorname{Im}(w)|}\right)^2 \|R(z) - R_0(z)\|.$$

*Proof.* Using that $\|R_0(w)\| \le |\operatorname{Im}(w)|^{-1}$ and $\|R(w)\| \le |\operatorname{Im}(w)|^{-1}$, the estimates from the proof of Lemma 2.6 can be easily adapted to the present case. □

We conclude this section with an alternative characterization of strong resolvent convergence, namely weak and strong Γ-convergence of the associated quadratic forms, see Proposition 2.9, below. For a comprehensive overview of Γ-convergence in real Hilbert spaces, we refer to the monograph [26]. In [12] some of the key results are extended to the complex case. To define Γ-convergence, it is convenient to extend all quadratic forms to $\mathscr{H}$:

**Definition 2.8.** *Let $q \ge 0$ and $q_\varepsilon \ge 0$, $\varepsilon > 0$, be densely defined and closed quadratic forms, which are extended to the whole Hilbert space $\mathscr{H}$ by setting $q(\psi) := +\infty$ if $\psi \in \mathscr{H} \setminus D(q)$ and $q_\varepsilon(\psi) := +\infty$ if $\psi \in \mathscr{H} \setminus D(q_\varepsilon)$. Then we say that $q_\varepsilon$* strongly *Γ-converges to $q$ as $\varepsilon \to 0$ if the following two conditions are satisfied:*

*(i) If $\psi_\varepsilon \to \psi$ strongly in $\mathscr{H}$ as $\varepsilon \to 0$, then*

$$q(\psi) \le \liminf_{\varepsilon\to 0} q_\varepsilon(\psi_\varepsilon). \tag{2.5}$$

*(ii) For every $\psi \in \mathscr{H}$ there exists a sequence $\psi_\varepsilon \in \mathscr{H}$, $\varepsilon > 0$, with $\psi_\varepsilon \to \psi$ as $\varepsilon \to 0$ and*

$$q(\psi) = \lim_{\varepsilon\to 0} q_\varepsilon(\psi_\varepsilon). \tag{2.6}$$

*Weak Γ-convergence $q_\varepsilon \to q$ in the limit $\varepsilon \to 0$ is defined analogously, where the strong convergence in (i) and (ii) is replaced by the weak convergence $\psi_\varepsilon \rightharpoonup \psi$ as $\varepsilon \to 0$.*

The analog of the following result in real Hilbert spaces is proved in [26, Theorem 13.6], and in [12, Theorem 1] the proof is then adapted to the complex case.

**Proposition 2.9.** *Let $q \ge 0$ and $q_\varepsilon \ge 0$, $\varepsilon > 0$, be densely defined closed quadratic forms in $\mathscr{H}$ and let $H$ and $H_\varepsilon$ denote the self-adjoint operators that are associated with $q$ and $q_\varepsilon$, respectively. Then, as $\varepsilon \to 0$, $H_\varepsilon \to H$ in the strong resolvent sense if and only if $q_\varepsilon \to q$ in the sense of strong and weak Γ-convergence.*

## 2.2 Trace operators

In this section we define all trace operators that appear in this thesis and we prove the properties that are needed in the sequel.

Let $d \in \mathbb{N}$, $N \geq 2$ and $\sigma \in \mathcal{I}$. Our first goal is to define a suitable unbounded trace operator $T_\sigma$ that assigns to functions from the Hilbert space $\mathscr{H}$ defined by Eq. (1.26) an appropriate trace on the collision plane $\Gamma_\sigma$ from Eq. (1.28). For $\psi \in C_0^\infty(\mathbb{R}^{dN}) \subseteq \mathscr{H}$, we define

$$(T_\sigma\psi)(R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N) := \psi(x_1, \ldots, x_N)\big|_{x_i=x_j=R}, \qquad \sigma = (i,j), \tag{2.7}$$

which means that $T_\sigma\psi$ defines a vector in the $(N-1)$-particle Hilbert space

$$\mathfrak{X}_\sigma := L^2\left(\mathbb{R}^{d(N-1)}, \mathrm{d}\,(R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N)\right), \qquad \sigma = (i,j). \tag{2.8}$$

We interpret (2.7) in the following way: $T_\sigma$ equates the positions of the $i$th and the $j$th particle, which then agree with their common center of mass coordinate $R$. However, for later purposes, we need to define $T_\sigma$ on a larger space of functions. To this end, we note that the Definition (1.45) of $\mathscr{K}_\sigma$ implies that $T_\sigma\psi = \tau\mathscr{K}_\sigma\psi$ for all $\psi \in C_0^\infty(\mathbb{R}^{dN})$, where

$$(\tau\psi)\,(R, x_3, ..., x_N) := \psi(r, R, x_3, ..., x_N)|_{r=0}. \tag{2.9}$$

It is well-known that $\tau$ extends to a bounded operator from $H^s(\mathbb{R}^{dN})$ to $H^{s-d/2}(\mathbb{R}^{d(N-1)})$ for any $s \in (d/2, \infty)$ and, for the convenience of the reader, we prove this in Lemma 2.10, below. For this purpose, we define in accordance with (2.9),

$$\widehat{\tau\psi}(\underline{P}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{\psi}(p, \underline{P})\,\mathrm{d}p, \tag{2.10}$$

$p$ and $\underline{P}$ being conjugate to $r$ and $(R, x_3, ..., x_N)$, respectively, on the domain

$$D(\tau) = \left\{\psi \in L^2(\mathbb{R}^{dN}) \;\middle|\; \int\left(\int|\widehat{\psi}(p, \underline{P})|\,\mathrm{d}p\right)^2 \mathrm{d}\underline{P} < \infty\right\}. \tag{2.11}$$

It is easy to verify that $D(\tau)$ is a linear subspace of $L^2(\mathbb{R}^{dN})$ and, for $\psi \in D(\tau)$, (2.10) defines a vector $\tau\psi \in L^2(\mathbb{R}^{d(N-1)})$. The trace operator $T_\sigma : D(T_\sigma) \subseteq \mathscr{H} \to \mathfrak{X}_\sigma$ is then defined by

$$T_\sigma := \tau\mathscr{K}_\sigma, \tag{2.12}$$

which implies that the domain $D(T_\sigma) = \mathscr{K}_\sigma^* D(\tau) \subseteq \mathscr{H}$ depends on the pair $\sigma$. Indeed, in the case $d = 2$, we shall see in Section 4 that defining $T_\sigma$ simply on the $\sigma$-independent domain $\bigcup_{s\in(1,\infty)} H^s(\mathbb{R}^{2N})$ would not be sufficient for our purposes.

**Lemma 2.10.** *For all $s \in (d/2, \infty)$, $H^s(\mathbb{R}^{dN}) \subseteq D(\tau)$ and $\tau : H^s(\mathbb{R}^{dN}) \to H^{s-d/2}(\mathbb{R}^{d(N-1)})$ is a bounded operator.*

*Proof.* For $\psi \in H^s(\mathbb{R}^{dN})$, we have by the Cauchy-Schwarz inequality,

$$\left(\int|\widehat{\psi}(p, \underline{P})|\,\mathrm{d}p\right)^2 \leq \int|\widehat{\psi}(p, \underline{P})|^2(1 + |p|^2 + |\underline{P}|^2)^s\,\mathrm{d}p \cdot \int(1 + |p|^2 + |\underline{P}|^2)^{-s}\,\mathrm{d}p, \tag{2.13}$$

where

$$\int(1 + |p|^2 + |\underline{P}|^2)^{-s}\,\mathrm{d}p = C(d,s)\,(1 + |\underline{P}|^2)^{d/2-s}, \tag{2.14}$$

and $C(d,s) < \infty$ because $s > d/2$. From (2.13), (2.14) and $(1 + |\underline{P}|^2)^{d/2-s} \leq 1$ it follows that $\psi \in D(\tau)$, and hence with $\tau\psi$ defined by (2.10), $\|\tau\psi\|_{H^{s-d/2}} \leq (2\pi)^{-d/2}C(d,s)^{1/2}\|\psi\|_{H^s}$. $\square$

The Definition (2.12) of $T_\sigma$, the fact that $\mathscr{K}_\sigma$ defines a bounded operator in $H^s(\mathbb{R}^{dN})$ for all $s \in \mathbb{N}_0$, and Lemma 2.10 imply the following corollary:

**Corollary 2.11.** *Suppose that $d, s \in \mathbb{N}$, $d/2 < s$ and $\sigma \in \mathcal{I}$. Then $H^s(\mathbb{R}^{dN}) \subseteq D(T_\sigma)$ and $T_\sigma : H^s(\mathbb{R}^{dN}) \to \mathfrak{X}_\sigma$ defines a bounded operator.*

In terms of $T_\sigma$, we now define a one-parameter family of bounded operators in $d \leq 3$:

**Proposition 2.12.** *Let $d \leq 3$, $z \in \rho(H_0)$ and $\sigma \in \mathcal{I}$. Then $G(z)_\sigma := T_\sigma R_0(z)$ defines a bounded operator in $\mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$ that has the following properties:*

*(i) For all $w, z \in \rho(H_0)$,*

$$\begin{aligned} G(z)_\sigma &= G(w)_\sigma + (w - z)G(z)_\sigma R_0(w) \\ &= G(w)_\sigma + (w - z)G(w)_\sigma R_0(z). \end{aligned}$$

*(ii) For all $\varphi \in \mathfrak{X}_\sigma$, $(H_0 + z)G(z)^*_\sigma \varphi = 0$ in $\mathbb{R}^{dN} \setminus \Gamma_\sigma$ in the sense of distributions.*

*(iii) If $d = 1$, then $G(z)^*_\sigma : \mathfrak{X}_\sigma \to H^1(\mathbb{R}^N)$ defines a bounded operator.*

*Proof.* Since $R_0(z) : \mathscr{H} \to H^2(\mathbb{R}^{dN})$ is bounded and, by Corollary 2.11, $T_\sigma : H^2(\mathbb{R}^{dN}) \to \mathfrak{X}_\sigma$ also defines a bounded operator for $d \leq 3$, it follows that $G(z)_\sigma = T_\sigma R_0(z)$ defines a bounded operator in $\mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$. Now $(i)$ is a direct consequence of the first resolvent identity

$$\begin{aligned} R_0(z) &= R_0(w) + (w - z)R_0(z)R_0(w) \\ &= R_0(w) + (w - z)R_0(w)R_0(z), \qquad z, w \in \rho(H_0). \end{aligned} \tag{2.15}$$

To prove $(ii)$, let $\psi \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma_\sigma)$ and observe that

$$\langle (H_0 + z)\psi \,|\, G(z)^*_\sigma \varphi \rangle = \langle G(z)_\sigma (H_0 + z)\psi \,|\, \varphi \rangle = \langle T_\sigma \psi \,|\, \varphi \rangle = 0$$

because $T_\sigma \psi = 0$ by (2.7). This proves $(ii)$, so it only remains to prove $(iii)$, where $d = 1$ is fixed. For $z = 1$, this follows from the identity $G(1)^*_\sigma = R_0(1)^{1/2}(T_\sigma R_0(1)^{1/2})^*$ because, by Corollary 2.11, $T_\sigma : H^1(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ defines a bounded operator and hence $T_\sigma R_0(1)^{1/2} \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$. For general $z \in \rho(H_0)$, it follows from property $(i)$ that

$$G(z)^*_\sigma = G(1)^*_\sigma + (1 - \overline{z})R_0(1)G(z)^*_\sigma,$$

and hence $G(z)^*_\sigma : \mathfrak{X}_\sigma \to H^1(\mathbb{R}^N)$ also defines a bounded operator. □

In Section 1.2.1 we have used [68, Theorem 2.2]. This is justified by the following lemma and the subsequent remark:

**Lemma 2.13.** *Let $d, s \in \mathbb{N}$, $s > d/2$, let $\mathcal{J}$ be a non-empty subset of $\mathcal{I}$, and let $\mathfrak{X} = \bigoplus_{\sigma \in \mathcal{J}} \mathfrak{X}_\sigma$. Then $T\psi := (T_\sigma \psi)_{\sigma \in \mathcal{J}}$ defines a bounded operator $T \in \mathscr{L}(H^s(\mathbb{R}^{dN}), \mathfrak{X})$. Moreover,* $\operatorname{Ker} T$ *contains the set $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$, which is dense in $\mathscr{H}$, and* $\operatorname{Ran} T$ *is dense in $\mathfrak{X}$.*

*Remark.* For $d \leq 3$, it follows from Lemma 2.13 that $T \in \mathscr{L}(H^2(\mathbb{R}^{dN}), \mathfrak{X})$, and hence, for all $z \in \rho(H_0)$, $G(z) := TR_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X})$. An alternative description of $G(z)$ is given by $G(z)\psi = (G(z)_\sigma \psi)_{\sigma \in \mathcal{J}}$, where the operators $G(z)_\sigma$, $\sigma \in \mathcal{J}$, were introduced in Proposition 2.12. Moreover, Lemma 2.13 and [68, Lemma 2.5] imply that $\operatorname{Ker} G(\overline{z})^* = \{0\}$ and $\operatorname{Ran} G(\overline{z})^* \cap H^2(\mathbb{R}^{dN}) = \{0\}$, so $G(\overline{z})^*$ satisfies the hypotheses of [68, Theorem 2.2]*.

*In the notation of [68] the operator $G_z$ corresponds to the adjoint $(T(-H_0 + \overline{z})^{-1})^*$.

*Proof.* The fact that $T$ defines a bounded operator in $\mathscr{L}(H^s(\mathbb{R}^{dN}), \mathfrak{X})$ follows from Corollary 2.11. Moreover, for $\sigma \in \mathcal{J}$ and $\psi \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$, it follows immediately from (2.7) that $T_\sigma \psi = 0$ because $\psi = 0$ on $\Gamma$. Therefore, $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \subseteq \operatorname{Ker} T$. As $\Gamma$ is a closed set of measure zero in $\mathbb{R}^{dN}$, it is well-known that $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ is dense in $\mathscr{H}$ (see, e.g., [77, Theorem 0.40]). Thus it only remains to show that $\operatorname{Ran} T$ is dense in $\mathfrak{X}$. For this purpose, let $\widetilde{\Gamma}_\sigma$, $\sigma = (i,j) \in \mathcal{J}$, denote the set of all $(R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N) \in \mathbb{R}^{d(N-1)}$ with at least two common entries, i.e. $x_k = R$ or $x_k = x_l$ for some $k \neq l$. Then, as above, $C_0^\infty(\mathbb{R}^{d(N-1)} \setminus \widetilde{\Gamma}_\sigma)$ is dense in $\mathfrak{X}_\sigma$, so it suffices to show that for any given $w = (w_\sigma)_{\sigma \in \mathcal{J}} \in \mathfrak{X}$ with $w_\sigma \in C_0^\infty(\mathbb{R}^{d(N-1)} \setminus \widetilde{\Gamma}_\sigma)$ there exists $\psi \in C_0^\infty(\mathbb{R}^{dN})$ with $T\psi = w$. To this end, let $\delta := \min_{\sigma \in \mathcal{J}} \operatorname{dist}(\widetilde{\Gamma}_\sigma, \operatorname{supp} w_\sigma) > 0$ and let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a function with $\chi(0) = 1$ and $\chi(x) = 0$ if $|x| \geq 1$. Then we define $\psi \in C_0^\infty(\mathbb{R}^{dN})$ by

$$\psi(x_1, ..., x_N) := \sum_{\nu=(k,l) \in \mathcal{J}} w_\nu \left( \frac{x_k + x_l}{2}, x_1, ...\widehat{x}_k...\widehat{x}_l..., x_N \right) \chi\left( \frac{x_l - x_k}{\delta} \right).$$

Now, using the Definition (2.7) of $T_\sigma \psi$, it is straightforward to verify that, for all $\sigma \in \mathcal{J}$, $T_\sigma \psi = w_\sigma$, and hence $T\psi = w$. This concludes the proof. □

So far we have only considered trace operators for functions that are defined on the whole space $\mathbb{R}^{dN}$. However, in Section 3.6, below, where $d = 1$ is fixed, we also have to assign a suitable trace on $\Gamma_\sigma$ to functions that have jumps at $\Gamma = \bigcup_{\nu \in \mathcal{I}} \Gamma_\nu$. As $\mathbb{R}^N \setminus \Gamma_\sigma$ is not connected, such a trace is, in general, not unique and depends on the direction from which $\Gamma_\sigma$ is approached. As illustrated in Figure 2.1, $\mathbb{R}^N \setminus \Gamma_\sigma$ is the disjoint union of the two open half-spaces $\Omega_\sigma^+$ and $\Omega_\sigma^-$ defined by

$$\Omega_\sigma^\pm := \left\{ (x_1, ..., x_N) \in \mathbb{R}^N \mid \pm (x_j - x_i) > 0 \right\}, \qquad \sigma = (i,j). \tag{2.16}$$

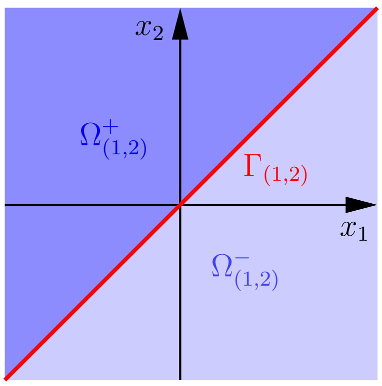


Figure 2.1: The open half-spaces $\Omega^+_{(1,2)}$ and $\Omega^-_{(1,2)}$ for $N = 2$.

In Proposition 2.14 below, we assign to $\Omega_\sigma^+ \setminus \Gamma$ and $\Omega_\sigma^- \setminus \Gamma$ suitable trace operators $T_\sigma^+$ and $T_\sigma^-$, respectively. The remark after Proposition 2.14 shows that these new trace operators agree with $T_\sigma$ on $H^1(\mathbb{R}^N)$.

**Proposition 2.14.** *Let $\sigma = (i,j) \in \mathcal{I}$ and let $D_\sigma^\pm$ denote the set of all $\psi \in C^\infty(\Omega_\sigma^\pm \setminus \Gamma)$ that extend to a function in $C(\overline{\Omega_\sigma^\pm} \setminus \bigcup_{\nu \neq \sigma} \Gamma_\nu)$ and for which $\overline{\operatorname{supp} \psi}$ is a compact set in $\mathbb{R}^N$. Then, for $\psi \in D_\sigma^\pm$, the limit*

$$(T_\sigma^\pm \psi)(R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N) := \lim_{x_j \to R\pm} \psi(x_1, ...x_{i-1}, R, x_{i+1}..., x_j, ..., x_N) \tag{2.17}$$

*exists for almost all $(R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N) \in \mathbb{R}^{N-1}$ and defines a vector $T_\sigma^\pm \psi \in \mathfrak{X}_\sigma$. The set $D_\sigma^\pm$ is dense in $H^1(\Omega_\sigma^\pm \setminus \Gamma)$, and $T_\sigma^\pm \restriction D_\sigma^\pm$ defined by (2.17) uniquely extends to a bounded operator $T_\sigma^\pm : H^1(\Omega_\sigma^\pm \setminus \Gamma) \to \mathfrak{X}_\sigma$. Furthermore, for all $\psi \in H^1(\Omega_\sigma^\pm \setminus \Gamma)$ and all $\varphi \in C_0^\infty(\mathbb{R}^N)$,*

$$T_\sigma^\pm(\psi\varphi) = (T_\sigma^\pm \psi)(T_\sigma \varphi). \tag{2.18}$$

*Remark.* By Corollary 2.11, $T_\sigma : H^1(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ defines a bounded operator. For $\psi \in C_0^\infty(\mathbb{R}^N)$, a comparison of (2.7) and (2.17) shows that $T_\sigma\psi = T_\sigma^\pm(\psi \restriction (\Omega_\sigma^\pm \setminus \Gamma))$, which, by an approximation argument, remains valid for general $\psi \in H^1(\mathbb{R}^N)$.

*Proof.* Since $\psi \in D_\sigma^\pm$ extends to a bounded and continuous function on $\overline{\Omega_\sigma^\pm} \setminus \bigcup_{\nu\neq\sigma} \Gamma_\nu$ and $\Gamma_\sigma \subseteq \overline{\Omega_\sigma^\pm}$, the limit on the right-hand side of Eq. (2.17) exists for all $(R, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N) \in \mathbb{R}^{N-1} \setminus \widetilde{\Gamma}_\sigma$, where $\widetilde{\Gamma}_\sigma \subseteq \mathbb{R}^{N-1}$ is defined as in the proof of Lemma 2.13. As $\widetilde{\Gamma}_\sigma$ has measure zero in $\mathbb{R}^{N-1}$ and $\overline{\operatorname{supp}\psi}$ is compact, it is clear that the limit $T_\sigma^\pm\psi$ defines a square-integrable function in $\mathfrak{X}_\sigma$. This proves the first part.

In the remainder of this proof, we assume, for the sake of notational simplicity, that $\sigma = (1, N)$ and we only consider the $(+)$-case, the $(-)$-case being similar. We are going to show that $D_\sigma^+$ is dense in $H^1(\Omega_\sigma^+ \setminus \Gamma)$. To this end, we first observe that $\Omega_\sigma^+ \setminus \Gamma$ is the disjoint union, over all permutations $\pi$ of $\{1, ..., N\}$ that satisfy $\pi^{-1}(1) < \pi^{-1}(N)$, of the open sets

$$\Omega_\pi := \left\{(x_1, ..., x_N) \in \mathbb{R}^N \,|\, x_{\pi(1)} < x_{\pi(2)} < ... < x_{\pi(N)}\right\}. \tag{2.19}$$

As each $\Omega_\pi$ is an open and connected set whose boundary is of class $C$ (cf. [50, Definition 9.57]), the restriction to $\Omega_\pi$ of functions in $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\Omega_\pi)$ (see, e.g., [50, Theorem 11.35]). Now, let $\psi \in H^1(\Omega_\sigma^+ \setminus \Gamma)$ be fixed. Then, using that $(\psi \restriction \Omega_\pi) \in H^1(\Omega_\pi)$ for any permutation $\pi$ with $\pi^{-1}(1) < \pi^{-1}(N)$, we may choose sequences $\psi_{\pi,n} \in C_0^\infty(\mathbb{R}^N)$, $n \in \mathbb{N}$, so that, in the limit $n \to \infty$, $(\psi_{\pi,n} \restriction \Omega_\pi) \to (\psi \restriction \Omega_\pi)$ in the norm of $H^1(\Omega_\pi)$. We now define a sequence $\psi_n \in D_\sigma^+$, $n \in \mathbb{N}$, by $\psi_n(x) := \psi_{\pi,n}(x)$ iff $x \in \Omega_\pi$. Then it follows that, as $n \to \infty$, $\psi_n \to \psi$ in the norm of $H^1(\Omega_\sigma^+ \setminus \Gamma)$. This proves that $D_\sigma^+$ is dense in $H^1(\Omega_\sigma^+ \setminus \Gamma)$.

We claim, and prove below, that there exists a constant $C(N) > 0$ such that, for all $\psi \in D_\sigma^+$,

$$\|T_\sigma^+\psi\|^2 \leq C(N)\|\psi\|^2_{H^1(\Omega_\sigma^+\setminus\Gamma)}. \tag{2.20}$$

As $D_\sigma^+$ is dense in $H^1(\Omega_\sigma^+ \setminus \Gamma)$, a standard argument then shows that (2.17) uniquely defines a bounded operator $T_\sigma^+ : H^1(\Omega_\sigma^+ \setminus \Gamma) \to \mathfrak{X}_\sigma$. Furthermore, for $\psi \in D_\sigma^+$, the identity (2.18) follows immediately from (2.17) because $\psi\varphi \in D_\sigma^+$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. An approximation argument then establishes (2.18) for general $\psi \in H^1(\Omega_\sigma^+ \setminus \Gamma)$.

It remains to prove (2.20). To this end, we introduce for any permutation $\pi$ satisfying $\pi^{-1}(1) < \pi^{-1}(N)$ the open set $O_\pi$ that is the interior of the closed set

$$\left\{(R, x_2, ..., x_{N-1}) \in \mathbb{R}^{N-1} \,|\, (R, x_2, ..., x_{N-1}, R) \in \overline{\Omega_\pi}\right\}.$$

Then it follows from the Definition (2.19) of $\Omega_\pi$ that $O_\pi = \emptyset$ unless $\pi \in S(\sigma)$, where $S(\sigma)$ denotes the set of all permutations $\pi$ that satisfy $\pi^{-1}(N) = \pi^{-1}(1) + 1$. Since $\Gamma_\sigma \subseteq \overline{\Omega_\sigma^+} = \bigcup_\pi \overline{\Omega_\pi}$, it is now immediate from the definition of $O_\pi$ that $\mathbb{R}^{N-1} = \bigcup_\pi \overline{O_\pi} = \bigcup_{\pi\in S(\sigma)} \overline{O_\pi}$. From this we conclude that, for all $\psi \in D_\sigma^+$,

$$\|T_\sigma^+\psi\|^2 = \sum_{\pi\in S(\sigma)} \|T_\sigma^+\psi\|^2_{L^2(O_\pi)}.$$

Hence, to prove (2.20), it suffices to show that for each $\pi \in S(\sigma)$ there exists a constant $C_{\pi,N} > 0$ such that, for all $\psi \in D_\sigma^+$, $\|T_\sigma^+\psi\|^2_{L^2(O_\pi)} \leq C_{\pi,N}\|\psi\|^2_{H^1(\Omega_\pi)}$. As $\Omega_\pi$ is an open set whose boundary is even uniformly Lipschitz continuous (cf. [50, Definition 13.11]), this follows from a standard result (see [50, Theorem 18.40]). However, for the convenience of the reader, we give a short proof here.

Let $\pi \in S(\sigma)$ be fixed. Then we define a vector $h = (h_i)_{i=1}^N \in \mathbb{R}^N$ by $h_i := 0$ if $\pi^{-1}(i) \leq \pi^{-1}(1)$ and $h_i := 1$ otherwise (in particular $h_1 = 0$ and $h_N = 1$). By construction, it follows that, for all $(R, x_2, ..., x_{N-1}) \in O_\pi$ and all $t > 0$, $x_t := (R, x_2, ..., x_{N-1}, R) + th \in \Omega_\pi$. Furthermore, the fundamental theorem of calculus shows that, for all $\psi \in D_\sigma^+$ and all $(R, x_2, ..., x_{N-1}) \in O_\pi$,

$$\left|(T_\sigma^+\psi)(R, x_2, ..., x_{N-1})\right|^2 = \left|\int_0^\infty \partial_t(|\psi(x_t)|^2)\, \mathrm{d}t\right| \leq 2N \int_0^\infty |\psi(x_t)||\nabla\psi(x_t)|\, \mathrm{d}t.$$

Integrating over $O_\pi$, we conclude that

$$\begin{aligned}\|T_\sigma^+\psi\|_{L^2(O_\pi)}^2 &\leq 2N \int_{O_\pi} \mathrm{d}(R, x_2, ..., x_{N-1}) \int_0^\infty \mathrm{d}t\, |\psi(x_t)||\nabla\psi(x_t)| \\ &= 2N \int_{\Omega_\pi} \mathrm{d}x\, |\psi(x)||\nabla\psi(x)| \leq 2N\|\psi\|_{H^1(\Omega_\pi)}^2,\end{aligned}$$

where the second line was obtained from the substitution $x_t \to x = (x_1, x_2, ..., x_{N-1}, x_N)$, and afterwards the Cauchy-Schwarz inequality was applied. This completes the proof of (2.20). □

In analogy to (2.12), an alternative description of the trace operators $T_\sigma^\pm$ can be obtained as follows. First, we note that Eq. (1.45) defines an isomorphism $\mathscr{K}_\sigma : H^1(\Omega_\sigma^\pm) \to H^1(\mathbb{R}^\pm \times \mathbb{R}^{N-1})$, where $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}^- := (-\infty, 0)$, and that we may restrict $T_\sigma^\pm$ to bounded operators $T_\sigma^\pm : H^1(\Omega_\sigma^\pm) \to \mathfrak{X}_\sigma$ by setting $T_\sigma^\pm\psi := T_\sigma^\pm(\psi \restriction (\Omega_\sigma^\pm \setminus \Gamma))$. Then, for $\psi \in C_0^\infty(\mathbb{R}^N)$, (2.17) is equivalent to $T_\sigma^\pm(\psi \restriction \Omega_\sigma^\pm) = \tau^\pm\varphi$, where $\varphi = \mathscr{K}_\sigma(\psi \restriction \Omega_\sigma^\pm) \in H^1(\mathbb{R}^\pm \times \mathbb{R}^{N-1})$ and

$$(\tau^\pm\varphi)(R, x_3, ..., x_N) := \lim_{r\to 0\pm} \varphi(r, R, x_3, ..., x_N). \tag{2.21}$$

Now, using that the restriction to $\Omega_\sigma^\pm$ of functions in $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\Omega_\sigma^\pm)$ (see, e.g., [50, Theorem 11.35]), an approximation argument shows that $\tau^\pm$ extend to bounded operators $\tau^\pm : H^1(\mathbb{R}^\pm \times \mathbb{R}^{N-1}) \to L^2(\mathbb{R}^{N-1})$ and that

$$T_\sigma^\pm\psi = \tau^\pm\mathscr{K}_\sigma\psi, \qquad \psi \in H^1(\Omega_\sigma^\pm),\ \sigma \in \mathcal{I}. \tag{2.22}$$

# 3 From short-range to contact interactions in $d = 1$ dimension

This section is devoted to the approximation of contact interactions among $N \geq 2$ particles in $d = 1$ space dimension by means of Schrödinger operators with suitably rescaled two-body potentials, the main result being Theorem 3.1, below. Thereby we generalize and enlarge the results from reference [41]. Throughout this section we work in the general framework of Section 1.3.1, where, if not stated otherwise, we assume that $d = 1$.

## 3.1 Main result and outline

We consider short-range interactions among $N \geq 2$ particles in $d = 1$ dimension in the Hilbert space $\mathscr{H}$ defined by Eq. (1.26). Recall from Eqs. (1.29), (1.39) and (1.40) that we assume that these short-range interactions can be described by Schrödinger operators of the form

$$H_\varepsilon = H_0 + \sum_{\sigma=(i,j)\in\mathcal{I}} g_{\varepsilon,\sigma}\, V_{\sigma,\varepsilon}(x_j - x_i), \qquad \varepsilon > 0, \tag{3.1}$$

where

$$H_0 = \sum_{i=1}^{N} (-\Delta_{x_i}/m_i) \tag{3.2}$$

denotes the kinetic energy operator, $g_{\varepsilon,\sigma} \in \mathbb{R}$ plays the role of a coupling constant and, for a given real-valued potential $V_\sigma \in L^1 \cap L^2(\mathbb{R})$ with $V_\sigma(-r) = V_\sigma(r)$ a.e.,

$$V_{\sigma,\varepsilon}(r) = \varepsilon^{-1}\, V_\sigma(r/\varepsilon), \qquad \sigma \in \mathcal{I},\ \varepsilon > 0. \tag{3.3}$$

In particular, $H_\varepsilon$ is self-adjoint on $D(H_\varepsilon) = D(H_0) = H^2(\mathbb{R}^N)$. As explained in Section 1.3.1, our goal is to prove norm resolvent convergence of $H_\varepsilon$ in the limit $\varepsilon \to 0$ and to identify the limit operator with a physically reasonable Hamiltonian describing contact interactions. Under the assumption that all coupling constants have suitable limits $g_\sigma = \lim_{\varepsilon\to 0} g_{\varepsilon,\sigma} \in \mathbb{R}$ (cf. Eq. (1.41)), Theorem 3.1, below, shows that $H_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint semibounded operator $H$. The quadratic form of $H$ is explicitly given by Eq. (3.7), below, where the parameters $\alpha_\sigma := g_\sigma \int V_\sigma(r)\,\mathrm{d}r$, $\sigma \in \mathcal{I}$, determine the strength of the resulting $\delta$-interaction among the particles of the pair $\sigma$. If $\alpha_\sigma = 0$, then the pair $\sigma$ does not contribute to $H$. Let $\mathcal{J}$ denote the set of all pairs $\sigma \in \mathcal{I}$ with $\alpha_\sigma \neq 0$.

From a heuristic point of view, the result from Theorem 3.1 is quite natural: from the Definition (3.3) of $V_{\sigma,\varepsilon}$ and from $\lim_{\varepsilon\to 0} g_{\varepsilon,\sigma} = g_\sigma$, it follows that for all $\varphi \in C_0^\infty(\mathbb{R})$,

$$\langle g_{\varepsilon,\sigma} V_{\sigma,\varepsilon} \,|\, \varphi\rangle \to \alpha_\sigma \varphi(0) = \alpha_\sigma \,\langle \delta \,|\, \varphi\rangle \qquad (\varepsilon \to 0),$$

where $\alpha_\sigma = g_\sigma \int V_\sigma(r)\,\mathrm{d}r$. This means that the two-body potential $g_{\varepsilon,\sigma} V_{\sigma,\varepsilon}$ converges, as $\varepsilon \to 0$, in the sense of distributions to a $\delta$-distribution of strength $\alpha_\sigma$. In particular, if $V_\sigma \geq 0$ and $\int V_\sigma(r)\,\mathrm{d}r = 1$, then $V_{\sigma,\varepsilon}$ defines a Dirac sequence in $\varepsilon > 0$ (see Figure 3.1).

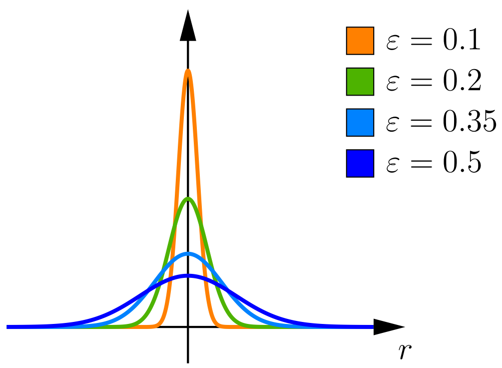


Figure 3.1: A Dirac sequence $V_{\sigma,\varepsilon}$ for some values of $\varepsilon > 0$.

Besides establishing norm resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ and estimating the rate of resolvent convergence, Theorem 3.1 also yields a Krein-like formula for the resolvent of $H$ in terms of the trace operators $T_\sigma$ from Section 2.2, see Eq. (3.6). We now define all operators appearing in Eq. (3.6). By Corollary 2.11, $T_\sigma : H^1(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ defines a bounded operator, where $\mathfrak{X}_\sigma$ is defined by Eq. (2.8), and hence $G(z)_\sigma = T_\sigma R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$ for $z \in \rho(H_0)$, where $R_0(z) = (H_0 + z)^{-1}$ for short. This means that $G(z)\psi := (G(z)_\sigma \psi)_{\sigma \in \mathcal{J}}$ defines a bounded operator in $\mathscr{L}(\mathscr{H}, \mathfrak{X})$ for any $z \in \rho(H_0)$, where the auxiliary Hilbert space

$$\mathfrak{X} := \bigoplus_{\sigma \in \mathcal{J}} \mathfrak{X}_\sigma \tag{3.4}$$

only contains contributions from pairs $\sigma \in \mathcal{J}$. It remains to define the operator $\Theta(z)$ from Eq. (3.6). By Property *(iii)* in Proposition 2.12, $G(\overline{z})^*_\nu : \mathfrak{X}_\nu \to H^1(\mathbb{R}^N)$ defines a bounded operator for any $z \in \rho(H_0)$, so it follows that $T_\sigma G(\overline{z})^*_\nu$, $\sigma, \nu \in \mathcal{I}$, define bounded operators in $\mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$. This means that

$$\Theta(z)_{\sigma\nu} := -(\alpha_\sigma)^{-1}\delta_{\sigma\nu} - T_\sigma G(\overline{z})^*_\nu \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma), \qquad z \in \rho(H_0), \sigma, \nu \in \mathcal{J}, \tag{3.5}$$

define the components of a bounded operator $\Theta(z) \in \mathscr{L}(\mathfrak{X})$. After these preparations, we now state the main result of this section:

**Theorem 3.1.** *Let $d = 1$, $N \geq 2$, and suppose, for all $\sigma \in \mathcal{I}$, that $V_\sigma \in L^1 \cap L^2(\mathbb{R})$, $V_\sigma(-r) = V_\sigma(r)$ a.e. and that the limit $g_\sigma = \lim_{\varepsilon \to 0} g_{\varepsilon,\sigma}$ exists. Let $H_\varepsilon$ be defined by Eqs.* (3.1)-(3.3)*. Then, as $\varepsilon \to 0$, $H_\varepsilon$ converges in the norm resolvent sense to the self-adjoint semibounded operator $H$ that is associated with the closed and semibounded quadratic form $q$ from Eq.* (3.7)*, where $C = 0$ and $\alpha_\sigma = g_\sigma \int V_\sigma(r)\, \mathrm{d}r$. If $z \in \rho(H_0) \cap \rho(H)$, then $\Theta(z) \in \mathscr{L}(\mathfrak{X})$ has a bounded inverse and*

$$(H + z)^{-1} = R_0(z) + G(\overline{z})^* \Theta(z)^{-1} G(z). \tag{3.6}$$

*If, in addition, $\int |r|^{2s} |V_\sigma(r)|\, \mathrm{d}r < \infty$ and $|g_{\varepsilon,\sigma} - g_\sigma| = O(\varepsilon^s)$ as $\varepsilon \to 0$, where $s \in (0, 1/2]$ is independent of the particular choice of $\sigma \in \mathcal{I}$, then $z \in \rho(H)$ implies that $z \in \rho(H_\varepsilon) \cap \rho(H)$ for small enough $\varepsilon > 0$ and $\|(H + z)^{-1} - (H_\varepsilon + z)^{-1}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

*Remarks.*

(i) For $N = 3$ particles and $g_{\varepsilon,\sigma} = 1$, Theorem 3.1 was first established in [10]. However, the proof in [10] relies on Faddeev's equations, which have no natural generalization to the case $N > 3$. The first result for an arbitrary number $N \geq 2$ of particles was derived in [41] in the case of a Bose gas. Theorem 3.1 generalizes the result from [41] to arbitrary symmetry regimes and general masses $m_i > 0$, $i = 1, ..., N$.

(ii) In Proposition 3.12, below, the restriction $s \in (0, 1/2]$ that limits the rate of resolvent convergence, namely $\|(H + z)^{-1} - (H_\varepsilon + z)^{-1}\| = O(\varepsilon^s)$, is improved to $s \in (0, 1)$. In the case $N = 3$, $g_{\varepsilon,\sigma} = 1$ and $\int |r|^2 |V_\sigma(r)|\, \mathrm{d}r < \infty$ for all $\sigma \in \mathcal{I}$, this improved rate of resolvent convergence, $O(\varepsilon^s)$ for any $s < 1$, agrees with the one established in [10]. However, if for some $\sigma \in \mathcal{I}$ and some $s \in (0, 1)$, $\int |r|^{2s} |V_\sigma(r)|\, \mathrm{d}r = \infty$, then [10] still asserts a higher rate of resolvent convergence. This might be a relict of the Konno-Kuroda formula that is the starting point of our proof.

The outline of this section is as follows: The proof of Theorem 3.1 is given in Section 3.4. In Section 3.2 we first establish strong resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$, which is easier to obtain than norm resolvent convergence due to its equivalence with strong and weak Γ-convergence of the associated quadratic forms. As explained in Section 1.3.1, our proof of norm resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ then relies on the generalized Konno-Kuroda formula (1.53) for $(H_\varepsilon + z)^{-1}$. More precisely, we are going to show that for sufficiently large $z > 0$ all contributions to the right side of Eq. (1.53) have suitable limits as $\varepsilon \to 0$. All required estimates and auxiliary results are derived in Section 3.3, which serves as a preparation for the subsequent proof of Theorem 3.1. However, the simplicity of this proof is gained at the expense of the rate of resolvent convergence, which is then improved in Section 3.5. Finally, the domain and the action of the limit operator $H$ from Theorem 3.1 are characterized in Section 3.6, where we shall also see that $H$ is a local and translation-invariant self-adjoint extension of $H_0 \restriction C_0^\infty(\mathbb{R}^N \setminus \Gamma)$.

## 3.2 Γ-Convergence

The goal of this section is to define the operators $H_\varepsilon$, $\varepsilon > 0$, and $H$ from Theorem 3.1 by appropriate closed semibounded quadratic forms $q_\varepsilon$ and $q$, respectively, on $H^1(\mathbb{R}^N)$ (see Eqs. (3.8) and (3.7), below), and to show that $q_\varepsilon \to q$ as $\varepsilon \to 0$ in the sense of weak and strong Γ-convergence. This implies Corollary 3.6, below, which is the main result of this section and which will allow us in Section 3.4 to conclude that the right-hand side of Eq. (3.6) defines the resolvent of the self-adjoint operator $H$ that is associated with the quadratic form $q$. All results of this section are valid under the assumption that, for all $\sigma \in \mathcal{I}$, $V_\sigma \in L^1(\mathbb{R})$ satisfies $V_\sigma(-r) = V_\sigma(r)$ a.e., the assumption that $V_\sigma \in L^2(\mathbb{R})$ is not needed here.

As in Theorem 3.1, let $g_\sigma = \lim_{\varepsilon\to 0} g_{\varepsilon,\sigma}$ and $\alpha_\sigma = g_\sigma \int V_\sigma(r)\,\mathrm{d}r$ for $\sigma \in \mathcal{I}$. Let $q$ and $q_\varepsilon$, $\varepsilon > 0$, denote the quadratic forms on $H^1(\mathbb{R}^N)$ defined by

$$q(\psi) := \sum_{i=1}^N \frac{\|\partial_i \psi\|^2}{m_i} + \sum_{\sigma\in\mathcal{I}} \alpha_\sigma \|T_\sigma \psi\|^2 + C\|\psi\|^2, \tag{3.7}$$

$$q_\varepsilon(\psi) := \sum_{i=1}^N \frac{\|\partial_i \psi\|^2}{m_i} + \sum_{\sigma=(i,j)\in\mathcal{I}} g_{\varepsilon,\sigma} \int V_{\sigma,\varepsilon}(x_j - x_i)|\psi(x)|^2\,\mathrm{d}x + C\|\psi\|^2, \tag{3.8}$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $C \in \mathbb{R}$ and the trace operator $T_\sigma$ is defined by Eq. (2.12), so Corollary 2.11 shows that $T_\sigma : H^1(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ defines a bounded operator. By Corollary 3.4, below, the quadratic forms $q$ and $q_\varepsilon$ are bounded from below and closed. More precisely, by choosing $C \geq 0$ large enough, we may assume that $q \geq 0$ and $q_\varepsilon \geq 0$ for all small enough $\varepsilon > 0$.

The main ingredients of this section are the following two inequalities:

**Proposition 3.2.** *For all $\psi \in H^1(\mathbb{R}^N)$,*

$$\sup_{r\in\mathbb{R}} \int_{\mathbb{R}^{N-1}} |\psi(r,x)|^2\,\mathrm{d}x \leq \|\partial_r \psi\|\|\psi\|, \tag{3.9}$$

$$\sup_{r\in\mathbb{R}\setminus\{0\}} \frac{1}{|r|^{1/2}} \left| \int_{\mathbb{R}^{N-1}} |\psi(r,x)|^2 - |\psi(0,x)|^2\,\mathrm{d}x \right| \leq 2\|\partial_r\psi\|^{3/2}\|\psi\|^{1/2}. \tag{3.10}$$

*The trace $\psi(r,\cdot) \in L^2(\mathbb{R}^{N-1})$, $r \in \mathbb{R}$, on the left sides of (3.9) and (3.10) is defined by $\psi(r,\cdot) := \tau\psi_r$, where $\tau : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^{N-1})$ denotes the trace operator from Eq. (2.10) and $\psi_r(x_1, x_2, ..., x_N) := \psi(x_1 + r, x_2, ..., x_N)$.*

*Proof.* First, we observe that $\|\psi_r\|_{H^1} = \|\psi\|_{H^1}$ for all $r \in \mathbb{R}$ and, by Lemma 2.10, the trace operator $\tau : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^{N-1})$ defines a bounded operator. Hence, since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, it suffices to prove (3.9) and (3.10) for $\psi \in C_0^\infty(\mathbb{R}^N)$ only (for general $\psi \in H^1(\mathbb{R}^N)$ an approximation argument then yields (3.9) and (3.10)).

To this end, let $\psi \in C_0^\infty(\mathbb{R}^N)$ be fixed. Then we apply to $\varphi(r) := \int_{\mathbb{R}^{N-1}} |\psi(r,x)|^2\,\mathrm{d}x$ the elementary Sobolev inequalities

$$|\varphi(r)| \le \frac{1}{2}\int_{\mathbb{R}} |\varphi'(s)|\,\mathrm{d}s, \tag{3.11}$$

$$|\varphi(r)-\varphi(0)| = \left|\int_0^r \varphi'(s)\,\mathrm{d}s\right| \le |r|^{1/2}\|\varphi'\|. \tag{3.12}$$

For fixed $r \in \mathbb{R}$, it follows from (3.11) and from the Cauchy-Schwarz inequality that

$$\int_{\mathbb{R}^{N-1}} |\psi(r,x)|^2\,\mathrm{d}x = |\varphi(r)| \le \frac{1}{2}\int_{\mathbb{R}} |\varphi'(s)|\,\mathrm{d}s \le \int_{\mathbb{R}}\int_{\mathbb{R}^{N-1}} |(\partial_r\psi)(s,x)||\psi(s,x)|\,\mathrm{d}x\,\mathrm{d}s \le \|\partial_r\psi\|\|\psi\|,$$

which proves (3.9) for $\psi \in C_0^\infty(\mathbb{R}^N)$, and hence for all $\psi \in H^1(\mathbb{R}^N)$. To prove (3.10) for $\psi \in C_0^\infty(\mathbb{R}^N)$, we apply (3.12) to the above function $\varphi$. We find that, for all $r \in \mathbb{R}\setminus\{0\}$,

$$\begin{aligned}\frac{|\varphi(r)-\varphi(0)|}{|r|^{1/2}} &\le \|\varphi'\| \le 2\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{N-1}} |(\partial_r\psi)(s,x)||\psi(s,x)|\,\mathrm{d}x\right)^2\mathrm{d}s\right)^{1/2}\\ &\le 2\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{N-1}} |(\partial_r\psi)(s,x)|^2\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^{N-1}} |\psi(s,x)|^2\,\mathrm{d}x\right)\mathrm{d}s\right)^{1/2}\\ &\le 2\|\partial_r\psi\|^{3/2}\|\psi\|^{1/2},\end{aligned}$$

where the Cauchy-Schwarz inequality was used for the second line, and the second factor in the second line was then estimated using (3.9). This establishes (3.10) for $\psi \in C_0^\infty(\mathbb{R}^N)$, and hence for all $\psi \in H^1(\mathbb{R}^N)$. □

The next lemma and the subsequent corollary serve as a preparation for the proof of Proposition 3.5, below.

**Lemma 3.3.** *For all $\delta > 0$, there exists $K_\delta > 0$ such that for all $\psi \in H^1(\mathbb{R}^N)$ and all $\sigma \in \mathcal{I}$,*

$$\|T_\sigma\psi\| \le \delta\|\nabla\psi\| + K_\delta\|\psi\|, \tag{3.13}$$

$$\left|\int_{\mathbb{R}^N} V_\sigma(x_j-x_i)|\psi(x)|^2\,\mathrm{d}x\right| \le \|V_\sigma\|_{L^1}\|\nabla\psi\|\|\psi\|. \tag{3.14}$$

*Proof.* Since $T_\sigma : H^1(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ defines a bounded operator, there exists a constant $K > 0$ such that for all $\psi \in H^1(\mathbb{R}^N)$ and all $\sigma \in \mathcal{I}$, $\|T_\sigma\psi\| \le K(\|\nabla\psi\| + \|\psi\|)$. Now, let $\psi_\lambda(x) := \lambda^{N/2}\psi(\lambda x)$ for $\lambda > 0$ and observe that $\|\psi_\lambda\| = \|\psi\|$, $\|\nabla\psi_\lambda\| = \lambda\|\nabla\psi\|$ and $\|T_\sigma\psi_\lambda\| = \lambda^{1/2}\|T_\sigma\psi\|$. Hence, it follows from $\|T_\sigma\psi_\lambda\| \le K(\|\nabla\psi_\lambda\| + \|\psi_\lambda\|)$ that, for all $\lambda > 0$, $\psi \in H^1(\mathbb{R}^N)$ and $\sigma \in \mathcal{I}$, $\|T_\sigma\psi\| \le K(\lambda^{1/2}\|\nabla\psi\| + \lambda^{-1/2}\|\psi\|)$, which proves (3.13).

To prove (3.14) for $\sigma = (i,j)$, we set

$$\widetilde{\psi}(r,R,x') := \psi\left(x_1,...,x_{i-1},R-\frac{r}{2},x_{i+1},...,x_{j-1},R+\frac{r}{2},x_{j+1},...,x_N\right), \tag{3.15}$$

where $x' := (x_1,...\widehat{x_i}...\widehat{x_j}...,x_N)$ for short. Then,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} V_\sigma(x_j-x_i)|\psi(x)|^2\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^N} V_\sigma(r)|\widetilde{\psi}(r,R,x')|^2\,\mathrm{d}(r,R,x')\right|\\ &\le \|V_\sigma\|_{L^1}\sup_{r\in\mathbb{R}}\int_{\mathbb{R}^{N-1}} |\widetilde{\psi}(r,R,x')|^2\,\mathrm{d}(R,x').\end{aligned} \tag{3.16}$$

To estimate the right side of (3.16), we apply (3.9) to $\widetilde{\psi} \in H^1(\mathbb{R}^N)$ and then we use that $\|\widetilde{\psi}\| = \|\psi\|$ and $\|\partial_r\widetilde{\psi}\| \le \|\nabla\psi\|$. This yields (3.14). □

Lemma 3.3 and $\|V_{\sigma,\varepsilon}\|_{L^1} = \|V_\sigma\|_{L^1}$ imply the following corollary:

**Corollary 3.4.** *Under the hypotheses of Proposition 3.5, below, for every $a > 0$ there exists $b > 0$ such that for all small enough $\varepsilon > 0$ and all $\psi \in H^1(\mathbb{R}^N)$,*

$$(1-a)\sum_{i=1}^{N}\frac{\|\partial_i\psi\|^2}{m_i} - b\|\psi\|^2 \le q_\varepsilon(\psi) \le (1+a)\sum_{i=1}^{N}\frac{\|\partial_i\psi\|^2}{m_i} + b\|\psi\|^2,$$

*and a similar estimate holds for $q$ in place of $q_\varepsilon$.*

**Proposition 3.5.** *Suppose, for all $\sigma \in \mathcal{I}$, that $V_\sigma \in L^1(\mathbb{R})$ satisfies $V_\sigma(-r) = V_\sigma(r)$ a.e. and that $g_\sigma = \lim_{\varepsilon\to 0} g_{\varepsilon,\sigma}$ exists. Let $q$ and $q_\varepsilon$, $\varepsilon > 0$, be defined on $H^1(\mathbb{R}^N)$ by Eqs. (3.7) and (3.8), respectively, where $\alpha_\sigma = g_\sigma \int V_\sigma(r)\,\mathrm{d}r$, and let $q(\psi) = q_\varepsilon(\psi) := +\infty$ for $\psi \in L^2(\mathbb{R}^N)\setminus H^1(\mathbb{R}^N)$. Then, as $\varepsilon \to 0$, $q_\varepsilon \to q$ in the sense of weak and strong $\Gamma$-convergence.*

*Proof.* We have to verify the conditions $(i)$ and $(ii)$ in Definition 2.8 and their analogs, where the strong convergence $\psi_\varepsilon \to \psi$ is replaced by the weak convergence $\psi_\varepsilon \rightharpoonup \psi$ as $\varepsilon \to 0$. Due to the fact that all form domains are equal, it suffices to show that, for all $\psi \in H^1(\mathbb{R}^N)$,

$$q(\psi) = \lim_{\varepsilon\to 0} q_\varepsilon(\psi) \tag{3.17}$$

and, for all $\psi_\varepsilon, \psi \in L^2(\mathbb{R}^N)$,

$$\psi_\varepsilon \rightharpoonup \psi \quad (\varepsilon \to 0) \quad \Rightarrow \quad q(\psi) \le \liminf_{\varepsilon\to 0} q_\varepsilon(\psi_\varepsilon). \tag{3.18}$$

We begin with the proof of (3.17). If $\psi \in C_0^\infty(\mathbb{R}^N)$, then it is a fairly straightforward application of Lebesgue dominated convergence to show that (3.17) holds. Now, let $\psi \in H^1(\mathbb{R}^N)$ and let $\psi_n \in C_0^\infty(\mathbb{R}^N)$, $n \in \mathbb{N}$, be a sequence with $\psi_n \to \psi$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Then, on the one hand,

$$|q(\psi) - q(\psi_n)| \to 0, \qquad (n \to \infty) \tag{3.19}$$

because $q$ is continuous w.r.t. its form norm, which is equivalent to the norm of $H^1(\mathbb{R}^N)$ by Corollary 3.4. On the other hand,

$$|q_\varepsilon(\psi) - q_\varepsilon(\psi_n)| \to 0, \qquad (n \to \infty) \tag{3.20}$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. This also follows from Corollary 3.4, which shows that the interaction is uniformly $H^1$-bounded in $\varepsilon \in (0, \varepsilon_0)$. Due to (3.19) and (3.20), the validity of (3.17) extends from $C_0^\infty(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$.

Now, we turn to the proof of (3.18). Let $\psi, \psi_\varepsilon \in L^2(\mathbb{R}^N)$ and suppose that, as $\varepsilon \to 0$, $\psi_\varepsilon \rightharpoonup \psi$ in $L^2(\mathbb{R}^N)$. To prove (3.18), we may assume that $\liminf_{\varepsilon\to 0} q_\varepsilon(\psi_\varepsilon) < \infty$ without loss of generality. We choose a zero sequence $\varepsilon_n > 0$, $n \in \mathbb{N}$, so that $\liminf_{\varepsilon\to 0} q_\varepsilon(\psi_\varepsilon) = \lim_{n\to\infty} q_{\varepsilon_n}(\psi_{\varepsilon_n})$. Then it follows from Corollary 3.4 that $\|\psi_{\varepsilon_n}\|_{H^1}$ is uniformly bounded in $n \in \mathbb{N}$. Therefore, after passing to a subsequence, we may assume that $\psi_{\varepsilon_n} \rightharpoonup \widetilde{\psi}$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Since $\psi_{\varepsilon_n} \rightharpoonup \psi$ in $L^2(\mathbb{R}^N)$, it follows that $\psi = \widetilde{\psi} \in H^1(\mathbb{R}^N)$.

From the weak lower semicontinuity of positive quadratic forms we know that

$$q(\psi) \le \liminf_{n\to\infty} q(\psi_{\varepsilon_n}).$$

On the right-hand side we may replace $q(\psi_{\varepsilon_n})$ by $q_{\varepsilon_n}(\psi_{\varepsilon_n})$ if we can show that

$$\sup_{0\neq\varphi\in H^1(\mathbb{R}^N)} \frac{|q_\varepsilon(\varphi) - q(\varphi)|}{\|\varphi\|_{H^1}^2} \to 0, \qquad (\varepsilon \to 0). \tag{3.21}$$

To prove this, we first assume that all $V_\sigma$ have compact support and we note that

$$|q_\varepsilon(\varphi) - q(\varphi)| \le \sum_{\sigma\in\mathcal{I}} \left| \int g_{\varepsilon,\sigma} V_{\sigma,\varepsilon}(x_j - x_i) |\varphi(x)|^2 \, \mathrm{d}x - \alpha_{\varepsilon,\sigma} \|T_\sigma \varphi\|^2 \right| + o(1) \cdot \|\varphi\|_{H^1}^2, \tag{3.22}$$

where $\alpha_{\varepsilon,\sigma} := g_{\varepsilon,\sigma} \int V_\sigma(r) \, \mathrm{d}r$, and the remainder $o(1)$ vanishes in the limit $\varepsilon \to 0$ because $\alpha_{\varepsilon,\sigma} \to \alpha_\sigma$. Now, let $\sigma = (i,j)$ be fixed and let $\widetilde{\varphi} := \mathscr{K}_\sigma \varphi$, where $\mathscr{K}_\sigma$ is defined by Eq. (1.45). Then the contribution of the pair $\sigma$ to (3.22) has the bound

$$\begin{aligned}
&\left| g_{\varepsilon,\sigma} \int V_{\sigma,\varepsilon}(r) |\widetilde{\varphi}(r,R,x')|^2 \, \mathrm{d}r \, \mathrm{d}R \, \mathrm{d}x' - \alpha_{\varepsilon,\sigma} \int |\widetilde{\varphi}(0,R,x')|^2 \, \mathrm{d}R \, \mathrm{d}x' \right| \\
&= \left| g_{\varepsilon,\sigma} \int \mathrm{d}r \, V_\sigma(r) \int \left( |\widetilde{\varphi}(\varepsilon r, R, x')|^2 - |\widetilde{\varphi}(0,R,x')|^2 \right) \mathrm{d}R \, \mathrm{d}x' \right| \\
&\le 2C_\sigma |g_{\varepsilon,\sigma}| \varepsilon^{1/2} \int |V_\sigma(r)| |r|^{1/2} \, \mathrm{d}r \cdot \|\varphi\|_{H^1}^2,
\end{aligned}$$

where the last line was obtained from (3.10) in combination with the estimate $\|\widetilde{\varphi}\|_{H^1}^2 \le C_\sigma \|\varphi\|_{H^1}^2$ for some constant $C_\sigma > 0$ that is independent of $\varphi \in H^1(\mathbb{R}^N)$. It follows that (3.21) is true in the case where all $V_\sigma$ have compact support, so the proposition is established in this case.

It remains to prove (3.21) in the case of general $V_\sigma \in L^1(\mathbb{R})$. To this end, we define for $\sigma \in \mathcal{I}$ and $k \in \mathbb{N}$ the cutoff potential

$$V_\sigma^k(r) := \begin{cases} V_\sigma(r) & \text{if } |r| \le k \\ 0 & \text{if } |r| > k \end{cases} \tag{3.23}$$

and we set $\alpha_\sigma^k := g_\sigma \int V_\sigma^k(r) \, \mathrm{d}r$. Then we define quadratic forms $q^k$ and $q_\varepsilon^k$ like $q$ and $q_\varepsilon$ with $\alpha_\sigma$ and $V_\sigma$ replaced by $\alpha_\sigma^k$ and $V_\sigma^k$, respectively. The constant $C$ in the defining expressions for $q$ and $q_\varepsilon$ is left unchanged. Since $V_\sigma^k$ has compact support, we know from the above proof that (3.21) holds for $V_\sigma^k$. That is, for each $k \in \mathbb{N}$,

$$\sup_{0\ne\varphi\in H^1(\mathbb{R}^N)} \frac{|q_\varepsilon^k(\varphi) - q^k(\varphi)|}{\|\varphi\|_{H^1}^2} \to 0, \qquad (\varepsilon \to 0). \tag{3.24}$$

From Lemma 3.3 and from the fact that $\|V_{\sigma,\varepsilon} - V_{\sigma,\varepsilon}^k\|_{L^1} = \|V_\sigma - V_\sigma^k\|_{L^1}$ we also know that

$$|q_\varepsilon(\varphi) - q_\varepsilon^k(\varphi)| + |q^k(\varphi) - q(\varphi)| \le \text{const.} \sum_{\sigma\in\mathcal{I}} \|V_\sigma - V_\sigma^k\|_{L^1} \|\varphi\|_{H^1}^2 \tag{3.25}$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. Choosing first $k$ large, then $\varepsilon$ small, we see that (3.24) and (3.25) combined imply (3.21) for general $V_\sigma \in L^1(\mathbb{R})$. □

In view of Proposition 2.9, Proposition 3.5 has the following corollary:

**Corollary 3.6.** *Under the hypotheses of Proposition 3.5, let $H_\varepsilon$, $\varepsilon > 0$, and $H$ denote the self-adjoint operators that are associated with the closed semibounded quadratic forms $q_\varepsilon$ and $q$, respectively, where $C = 0$. Then $H_\varepsilon \to H$ in the strong resolvent sense as $\varepsilon \to 0$.*

## 3.3 Preparation of the proof of Theorem 3.1

This section serves as a preparation for the next one, where the proof of Theorem 3.1 is given. As in the previous section, all results of this section are valid under the assumption that, for all $\sigma \in \mathcal{I}$, $V_\sigma \in L^1(\mathbb{R})$ satisfies $V_\sigma(-r) = V_\sigma(r)$ a.e., the assumption that $V_\sigma \in L^2(\mathbb{R})$ is still not needed.

We are going to prove estimates and convergence results for the contributions $A_{\varepsilon,\sigma}R_0(z)$, $B_{\varepsilon,\sigma}R_0(z)$ and $\phi_\varepsilon(z)$ to the Konno-Kuroda formula (1.53). Recall from Eqs. (1.48), (1.49) and (1.52) that, for $z \in (0,\infty)$ and $\sigma,\nu \in \mathcal{I}$,

$$A_{\varepsilon,\sigma} = (v_\sigma \otimes 1)\,\varepsilon^{-1/2}U_\varepsilon \mathscr{K}_\sigma, \tag{3.26}$$

$$B_{\varepsilon,\sigma} = (u_\sigma \otimes 1)\,\varepsilon^{-1/2}U_\varepsilon \mathscr{K}_\sigma = J_\sigma A_{\varepsilon,\sigma}, \tag{3.27}$$

$$\phi_\varepsilon(z)_{\sigma\nu} = B_{\varepsilon,\sigma}R_0(z)(A_{\varepsilon,\nu})^* \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma), \tag{3.28}$$

where $J_\sigma$ denotes multiplication with $\mathrm{sgn}(V_\sigma)$, $v_\sigma = |V_\sigma|^{1/2}$, $u_\sigma = J_\sigma v_\sigma$, and the unitary operators $\mathscr{K}_\sigma \in \mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$ and $U_\varepsilon \in \mathscr{L}(\widetilde{\mathfrak{X}}_\sigma)$ are defined by (1.45) and (1.47), respectively. The first result of this section is Proposition 3.7, below, which estimates the norm of $\phi_\varepsilon(z)_{\sigma\nu}$. In particular, the estimate (3.29) shows that $\|\phi_\varepsilon(z)_{\sigma\nu}\|$ vanishes uniformly in $\varepsilon > 0$ as $z \to \infty$.

**Proposition 3.7.** *Let $\sigma,\nu \in \mathcal{I}$ and suppose that $V_\sigma, V_\nu \in L^1(\mathbb{R})$. Then, for all $\varepsilon, z > 0$,*

$$\|\phi_\varepsilon(z)_{\sigma\nu}\| \le \frac{(\mu_\sigma\mu_\nu)^{1/4}}{z^{1/2}} \|V_\sigma\|_{L^1}^{1/2} \|V_\nu\|_{L^1}^{1/2}. \tag{3.29}$$

*Proof.* We claim, and prove below, that $A_{\varepsilon,\sigma}R_0(z)^{1/2} \in \mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$ and

$$\|A_{\varepsilon,\sigma}R_0(z)^{1/2}\| \le \left(\frac{\mu_\sigma}{z}\right)^{1/4} \|V_\sigma\|_{L^1}^{1/2}. \tag{3.30}$$

Since all assumptions are symmetric in $\sigma$ and $\nu$, a similar estimate also holds for $\|A_{\varepsilon,\nu}R_0(z)^{1/2}\|$. This means that Eq. (3.28) can be rewritten as

$$\phi_\varepsilon(z)_{\sigma\nu} = J_\sigma A_{\varepsilon,\sigma}R_0(z)^{1/2} \left(A_{\varepsilon,\nu}R_0(z)^{1/2}\right)^*, \tag{3.31}$$

and (3.29) then follows by taking norms of both sides.

It remains to prove (3.30) for given $\sigma = (i,j)$ and all $\varepsilon, z > 0$. To this end, we first consider $\psi \in H^1(\mathbb{R}^N)$. Then it follows from the Definition (3.26) of $A_{\varepsilon,\sigma}$ that $R_0(z)^{1/2}\psi \in H^2(\mathbb{R}^N) \subseteq D(A_{\varepsilon,\sigma})$ and

$$\begin{aligned} \left\|A_{\varepsilon,\sigma}R_0(z)^{1/2}\psi\right\|^2 &= \int \mathrm{d}r\,\mathrm{d}\underline{X}\ |V_\sigma(r)| \left|\left(\mathscr{K}_\sigma R_0(z)^{1/2}\psi\right)(\varepsilon r, \underline{X})\right|^2 \\ &\le \|V_\sigma\|_{L^1} \cdot \sup_{r\in\mathbb{R}} \int_{\mathbb{R}^{N-1}} \mathrm{d}\underline{X} \left|\left(\mathscr{K}_\sigma R_0(z)^{1/2}\psi\right)(r, \underline{X})\right|^2 \\ &\le \|V_\sigma\|_{L^1} \|\partial_r \mathscr{K}_\sigma R_0(z)^{1/2}\psi\| \|\mathscr{K}_\sigma R_0(z)^{1/2}\psi\|, \end{aligned} \tag{3.32}$$

where $\underline{X} := (R, x_1, ...\widehat{x_i}...\widehat{x_j}..., x_N)$ for short and the estimate from (3.9) was used for the last line. Next, it follows from the Definition (1.45) of $\mathscr{K}_\sigma$ that, in the sense of operators,

$$\mathscr{K}_\sigma^* \left(-\frac{\partial_r^2}{\mu_\sigma}\right) \mathscr{K}_\sigma \le (H_0 + z), \qquad z > 0,$$

and hence $\|\partial_r \mathscr{K}_\sigma R_0(z)^{1/2}\psi\| \le \mu_\sigma^{1/2}\|\psi\|$. Using this together with $\|\mathscr{K}_\sigma R_0(z)^{1/2}\psi\| \le \|\psi\|/z^{1/2}$ to estimate the right side of (3.32), we obtain that, for all $\psi \in H^1(\mathbb{R}^N)$,

$$\|A_{\varepsilon,\sigma}R_0(z)^{1/2}\psi\| \le \left(\frac{\mu_\sigma}{z}\right)^{1/4} \|V_\sigma\|_{L^1}^{1/2} \|\psi\|.$$

Since $H^1(\mathbb{R}^N)$ is dense in $\mathscr{H}$, we conclude that $A_{\varepsilon,\sigma}R_0(z)^{1/2}$ defines an operator in $\mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$ that satisfies the desired norm estimate (3.30). This concludes the proof. □

For the rest of this section, we are concerned with proving convergence, as $\varepsilon \to 0$, of $A_{\varepsilon,\sigma}R_0(z)$ and $\phi_\varepsilon(z)_{\sigma\nu}$, the main results being Corollary 3.9 and Proposition 3.10, below. In view of (3.26), proving convergence of $A_{\varepsilon,\sigma}$ is a problem in $L^2(\mathbb{R})$, which we solve in the next lemma. For a given pair $\sigma \in \mathcal{I}$, let $V = V_\sigma$, $v = v_\sigma$ and let the rank-one operator $|v\rangle\langle\delta| : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$|v\rangle\langle\delta|\,\psi := \psi(0)v, \qquad\qquad \psi \in H^1(\mathbb{R}),$$

where $\psi(0) := (2\pi)^{-1/2}\int \widehat{\psi}(p)\,\mathrm{d}p$ (note that $\widehat{\psi} \in L^1(\mathbb{R})$). Then the following holds:

**Lemma 3.8.** *For $n \in \{1,2\}$, $V \in L^1(\mathbb{R})$ and $v = |V|^{1/2}$,*

$$v\varepsilon^{-1/2}U_\varepsilon \to |v\rangle\langle\delta| \qquad\qquad (\varepsilon \to 0)$$

*in the norm of $\mathscr{L}(H^n(\mathbb{R}), L^2(\mathbb{R}))$. If, in addition, $\int |r|^{2s}\,|V(r)|\,\mathrm{d}r < \infty$ for some $s \in (0, n/2]$, then the rate of convergence is at least as good as $O(\varepsilon^s)$.*

*Proof.* We first assume that $\int |r|^{2s}\,|V(r)|\,\mathrm{d}r < \infty$ for some $s \in (0, n/2]$. As the Sobolev embedding $H^n(\mathbb{R}) \hookrightarrow C^{0,s}(\mathbb{R})$ exists and is continuous for $s \in (0, n/2]$ (see, e.g., [50, Theorems 12.48 and 12.55]), there exists a constant $c_{s,n} > 0$ such that, for all $\psi \in H^n(\mathbb{R})$ and almost all $r \in \mathbb{R}$,

$$|v(r)|\,|\psi(\varepsilon r) - \psi(0)| \le |v(r)|\,c_{s,n}|\varepsilon r|^s\|\psi\|_{H^n}.$$

This implies that, for all $\psi \in H^n(\mathbb{R})$,

$$\|(v\varepsilon^{-1/2}U_\varepsilon - |v\rangle\langle\delta|)\psi\| \le c_{s,n}\varepsilon^s\left(\int |r|^{2s}\,|V(r)|\,\mathrm{d}r\right)^{1/2}\|\psi\|_{H^n},$$

which proves the lemma under the assumption that $\int |r|^{2s}\,|V(r)|\,\mathrm{d}r < \infty$ for some $s \in (0, n/2]$.

For general $V \in L^1(\mathbb{R})$, we introduce for $k > 0$ the cutoff potential

$$V^k(r) := \begin{cases} V(r) & \text{if } |r| \le k \\ 0 & \text{if } |r| > k \end{cases} \tag{3.33}$$

and we set $v^k(r) := |V^k(r)|^{1/2}$. Then it follows that $\int |r|^{2s}|V^k(r)|\,\mathrm{d}r < \infty$ for all $k > 0$, so the above estimates show that, for all $k > 0$ and as $\varepsilon \to 0$, $v^k\varepsilon^{-1/2}U_\varepsilon \to |v^k\rangle\langle\delta|$ in the norm of $\mathscr{L}(H^n(\mathbb{R}), L^2(\mathbb{R}))$. Now, $v\varepsilon^{-1/2}U_\varepsilon \to |v\rangle\langle\delta|$ in the norm of $\mathscr{L}(H^n(\mathbb{R}), L^2(\mathbb{R}))$ follows from a simple $\delta/3$-argument because, for all $\psi \in H^n(\mathbb{R})$,

$$\|(v - v^k)\varepsilon^{-1/2}U_\varepsilon\psi\| \le \|v - v^k\|\|\psi\|_{L^\infty} \le \|V - V^k\|_{L^1}^{1/2}\|\psi\|_{H^1},$$

which vanishes uniformly in $\varepsilon > 0$ as $k \to \infty$, and a similar estimate holds for $\|\,|v - v^k\rangle\langle\delta|\,\|$. □

Lemma 3.8 implies the following convergence result:

**Corollary 3.9.** *Let $n \in \{1,2\}$, $z > 0$, $\sigma \in \mathcal{I}$ and suppose that $V_\sigma \in L^1(\mathbb{R})$. Then, as $\varepsilon \to 0$,*

$$A_{\varepsilon,\sigma}R_0(z)^{n/2} \to A_\sigma R_0(z)^{n/2}$$

*in $\mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$, where $A_\sigma \in \mathscr{L}(H^n(\mathbb{R}^N), \widetilde{\mathfrak{X}}_\sigma)$ is defined by*

$$A_\sigma\psi = v_\sigma \otimes (T_\sigma\psi). \tag{3.34}$$

*If, in addition, $\int |r|^{2s}\,|V_\sigma(r)|\,\mathrm{d}r < \infty$ for some $s \in (0, n/2]$, then $\left\|(A_{\varepsilon,\sigma} - A_\sigma)R_0(z)^{n/2}\right\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

*Proof.* By Corollary 2.11, $T_\sigma : H^n(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ defines a bounded operator, so it is clear that (3.34) defines an operator $A_\sigma \in \mathscr{L}(H^n(\mathbb{R}^N), \widetilde{\mathfrak{X}}_\sigma)$. Moreover, inserting the Definition (2.12) of $T_\sigma$, it follows that, for all $\psi \in H^n(\mathbb{R}^N)$, $A_\sigma\psi = v_\sigma \otimes (\tau \mathscr{K}_\sigma \psi) = (|v_\sigma\rangle \langle\delta| \otimes 1)\mathscr{K}_\sigma\psi$, where $\tau$ is defined by Eq. (2.10). Comparing this with the Definition (3.26) of $A_{\varepsilon,\sigma}$, we see that the corollary follows from Lemma 3.8 because $\mathscr{K}_\sigma$ defines a bounded operator in $\mathscr{L}(H^n(\mathbb{R}^N))$ and $R_0(z)^{n/2} \in \mathscr{L}(\mathscr{H}, H^n(\mathbb{R}^N))$. □

We conclude this section with the proof that $\phi_\varepsilon(z)$ has a suitable limit as $\varepsilon \to 0$:

**Proposition 3.10.** *Let $z > 0$, $\sigma, \nu \in \mathcal{I}$ and suppose that $V_\sigma, V_\nu \in L^1(\mathbb{R})$. Then the limit*

$$\phi(z)_{\sigma\nu} = \lim_{\varepsilon\to 0} \phi_\varepsilon(z)_{\sigma\nu} \tag{3.35}$$

*exists in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$, and*

$$\phi(z)_{\sigma\nu} = |u_\sigma\rangle \langle v_\nu| \otimes (T_\sigma G(z)^*_\nu) \tag{3.36}$$

*with $T_\sigma G(z)^*_\nu \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$. If, in addition, $\int |r|^{2s}(|V_\sigma(r)|+|V_\nu(r)|)\,\mathrm{d}r < \infty$ for some $s \in (0, 1/2]$, then $\|\phi_\varepsilon(z)_{\sigma\nu} - \phi(z)_{\sigma\nu}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

*Proof.* From Eq. (3.31) and Corollary 3.9 it follows that $\lim_{\varepsilon\to 0} \phi_\varepsilon(z)_{\sigma\nu}$ exists in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$, where the limit operator is explicitly given by

$$\phi(z)_{\sigma\nu} = J_\sigma A_\sigma R_0(z)^{1/2} \left(A_\nu R_0(z)^{1/2}\right)^* = J_\sigma A_\sigma \left(A_\nu R_0(z)\right)^*. \tag{3.37}$$

If, in addition, $\int |r|^{2s}(|V_\sigma(r)| + |V_\nu(r)|)\,\mathrm{d}r < \infty$ for some $s \in (0, 1/2]$, then Corollary 3.9 also shows that the rate of convergence is at least as good as $O(\varepsilon^s)$, so it only remains to show that the limit operator $\phi(z)_{\sigma\nu}$ from Eq. (3.37) agrees with the right side of Eq. (3.36). To this end, we first observe that Eq. (3.34) implies that, for all $\psi \in \mathscr{H}$,

$$A_\nu R_0(z)\psi = v_\nu \otimes (G(z)_\nu \psi),$$

where $G(z)_\nu = T_\nu R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\nu)$. The adjoint thereof is the operator

$$(A_\nu R_0(z))^* = G(z)^*_\nu \langle v_\nu| \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \mathscr{H}), \tag{3.38}$$

where $\langle v_\nu| \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \mathfrak{X}_\nu)$ denotes the adjoint of the operator $|v_\nu\rangle \in \mathscr{L}(\mathfrak{X}_\nu, \widetilde{\mathfrak{X}}_\nu)$ that is defined by $|v_\nu\rangle\, \psi := v_\nu \otimes \psi$. Inserting the identity (3.38) and the Definition (3.34) of $A_\sigma$ on the right side of Eq. (3.37), it is straightforward to verify that the right sides of Eqs. (3.37) and (3.36) coincide. The fact that $T_\sigma G(z)^*_\nu$ defines a bounded operator in $\mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$ follows from Property (*iii*) in Proposition 2.12 and from Corollary 2.11. □

## 3.4 Proof of Theorem 3.1

*Proof of Theorem 3.1.* Recall from Eqs. (1.52), (1.53) and the sentence in between that for any $z \in (0, \infty) \subseteq \rho(H_0)$ we have that $z \in \rho(H_\varepsilon) \cap \rho(H_0)$ if and only if $1 + g_\varepsilon\phi_\varepsilon(z)$ is invertible in $\mathscr{L}(\widetilde{\mathfrak{X}})$, and then $(H_\varepsilon + z)^{-1}$ is given by the Konno-Kuroda formula

$$(H_\varepsilon + z)^{-1} = R_0(z) - \sum_{\sigma,\nu\in\mathcal{I}} (A_{\varepsilon,\sigma} R_0(z))^* \left[(1 + g_\varepsilon\phi_\varepsilon(z))^{-1}\right]_{\sigma\nu} g_{\varepsilon,\nu} J_\nu A_{\varepsilon,\nu} R_0(z). \tag{3.39}$$

First, we are going to show that $1 + g_\varepsilon\phi_\varepsilon(z)$ is invertible for $z > 0$ large enough and $\varepsilon > 0$ small enough and that $\lim_{\varepsilon\to 0} (1 + g_\varepsilon\phi_\varepsilon(z))^{-1}$ exists in $\mathscr{L}(\widetilde{\mathfrak{X}})$. From Proposition 3.7 we know that

$$\|\phi_\varepsilon(z)\| \le \frac{N(N-1)}{2} \max_{\sigma,\nu\in\mathcal{I}} \|\phi_\varepsilon(z)_{\sigma\nu}\| \le \frac{\text{const.}}{\sqrt{z}}, \qquad z > 0 \tag{3.40}$$

uniformly in $\varepsilon > 0$. By assumption, $g_\sigma = \lim_{\varepsilon\to 0} g_{\varepsilon,\sigma} \in \mathbb{R}$ exists for all $\sigma \in \mathcal{I}$, and hence $\lim_{\varepsilon\to 0} g_\varepsilon = g$ in $\mathscr{L}(\widetilde{\mathfrak{X}})$, where $g$ is defined in terms of the components $g_{\sigma\nu} := g_\sigma \delta_{\sigma\nu}$, $\sigma, \nu \in \mathcal{I}$. Consequently, it follows from (3.40) that there exist $\varepsilon_0 > 0$ and $z_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $z \in (z_0, \infty)$, $\|g_\varepsilon \phi_\varepsilon(z)\| \leq 1/2$ and thus $(1 + g_\varepsilon\phi_\varepsilon(z))^{-1}$ exists. Moreover, Proposition 3.10 shows that $\lim_{\varepsilon\to 0} g_\varepsilon \phi_\varepsilon(z) = g\phi(z)$ in $\mathscr{L}(\widetilde{\mathfrak{X}})$, where $\phi(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ is defined in terms of the components $\phi(z)_{\sigma\nu} \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$, $\sigma, \nu \in \mathcal{I}$. We conclude that $(1 + g\phi(z))^{-1}$ exists and

$$\lim_{\varepsilon\to 0} (1 + g_\varepsilon\phi_\varepsilon(z))^{-1} = (1 + g\phi(z))^{-1}, \qquad z > z_0. \tag{3.41}$$

Since we also know from Corollary 3.9 that $\lim_{\varepsilon\to 0} A_{\varepsilon,\sigma} R_0(z) = A_\sigma R_0(z)$ in $\mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$ for all $\sigma \in \mathcal{I}$ and all $z > 0$, (3.41) now allows us to take the limit $\varepsilon \to 0$ on the right side of (3.39). We find that, for all $z \in (z_0, \infty)$, $\lim_{\varepsilon\to 0}(H_\varepsilon + z)^{-1} = R(z)$ in $\mathscr{L}(\mathscr{H})$, where

$$R(z) := R_0(z) - \sum_{\sigma,\nu\in\mathcal{I}} (A_\sigma R_0(z))^* \left[(1 + g\phi(z))^{-1}\right]_{\sigma\nu} g_\nu J_\nu A_\nu R_0(z). \tag{3.42}$$

However, as we already know from Corollary 3.6 that $H_\varepsilon \to H$ in the strong resolvent sense as $\varepsilon \to 0$, it follows that $R(z) = (H + z)^{-1}$ for all $z \in (z_0, \infty)$ and that $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$. If, in addition, $\int |r|^{2s} |V_\sigma(r)| \,\mathrm{d}r < \infty$ for some $s \in (0, 1/2]$ and all $\sigma \in \mathcal{I}$, then Corollary 3.9 and Proposition 3.10 also yield an estimate for the rate of convergence:

$$\|A_{\varepsilon,\sigma} R_0(z) - A_\sigma R_0(z)\| = O(\varepsilon^s), \tag{3.43}$$

$$\|\phi_\varepsilon(z) - \phi(z)\| = O(\varepsilon^s). \tag{3.44}$$

Hence, if $|g_{\varepsilon,\sigma} - g_\sigma| = O(\varepsilon^s)$ for all $\sigma \in \mathcal{I}$, then the above proof shows that, for each $z \in (z_0, \infty)$, $\|(H_\varepsilon + z)^{-1} - (H + z)^{-1}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$. For general $z \in \rho(H)$, Lemma 2.6 now reveals that $z \in \rho(H_\varepsilon) \cap \rho(H)$ for small enough $\varepsilon > 0$ and $\|(H_\varepsilon + z)^{-1} - (H + z)^{-1}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.

It remains to show that the expression (3.42) for $R(z)$ agrees with the right side of Eq. (3.6) and that Eq. (3.6) defines the resolvent $(H + z)^{-1}$ for all $z \in \rho(H_0) \cap \rho(H)$. To this end, we first integrate out the two-body potentials $V_\sigma$ in $R(z)$. In view of Eq. (3.36), $1 + g\phi(z)$ can be rewritten as

$$1 + g\phi(z) = 1 + g\,[W \circ S(z)]\,, \tag{3.45}$$

where $W$ and $S(z)$ are defined in terms of the components $W_{\sigma\nu} := |u_\sigma\rangle\,\langle v_\nu| \in \mathscr{L}(L^2(\mathbb{R}))$ and $S(z)_{\sigma\nu} := T_\sigma G(z)^*_\nu \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$, respectively, for $\sigma, \nu \in \mathcal{I}$. In Eq. (3.45), the operator product $Y \circ Z$ is defined by $(Y \circ Z)_{\sigma\nu} := Y_{\sigma\nu} \otimes Z_{\sigma\nu}$, a notation inspired by the Hadamard-Schur product of matrices. Now, using Eq. (3.45) and

$$W_{\sigma\eta}\, g_\eta W_{\eta\nu} = \alpha_\eta W_{\sigma\nu}, \qquad \sigma, \eta, \nu \in \mathcal{I},$$

where $\alpha_\eta = g_\eta \int V_\eta(r)\,\mathrm{d}r$, it is straightforward to check that

$$(1 + g\phi(z))^{-1} = 1 - g\left(W \circ \left[S(z)(1 + \alpha S(z))^{-1}\right]\right), \tag{3.46}$$

where $\alpha$ is a matrix operator with entries $\alpha_{\sigma\nu} := \alpha_\sigma \delta_{\sigma\nu}$, $\sigma, \nu \in \mathcal{I}$. Indeed, $(1 + \alpha S(z))^{-1}$ exists for large enough $z > 0$: Eq. (3.36), $\phi(z)_{\sigma\nu} = \lim_{\varepsilon\to 0} \phi_\varepsilon(z)_{\sigma\nu}$ and Proposition 3.7 imply that

$$\|S(z)_{\sigma\nu}\| \|V_\sigma\|_{L^1}^{1/2} \|V_\nu\|_{L^1}^{1/2} = \|\phi(z)_{\sigma\nu}\| \leq \frac{(\mu_\sigma\mu_\nu)^{1/4}}{z^{1/2}} \|V_\sigma\|_{L^1}^{1/2} \|V_\nu\|_{L^1}^{1/2}, \qquad \sigma, \nu \in \mathcal{I},$$

and hence, as we may assume that $V_\sigma \neq 0 \neq V_\nu$ without loss of generality, $\alpha S(z) \to 0$ as $z \to \infty$. Moreover, Eq. (3.34) shows that $A_\sigma R_0(z)\psi = v_\sigma \otimes (G(z)_\sigma \psi)$ with $G(z)_\sigma = T_\sigma R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$

and, similarly, $J_\nu A_\nu R_0(z)\psi = u_\nu \otimes (G(z)_\nu \psi)$. Inserting this together with (3.46) in (3.42) and using that $g_\sigma \langle v_\sigma \mid u_\sigma \rangle = \alpha_\sigma$, we find that

$$\begin{aligned} R(z) &= R_0(z) + \sum_{\sigma,\nu\in\mathcal{I}} G(z)_\sigma^* \left(-\alpha_\sigma \delta_{\sigma\nu} + \alpha_\sigma \left[S(z)(1+\alpha S(z))^{-1}\right]_{\sigma\nu} \alpha_\nu\right) G(z)_\nu \\ &= R_0(z) + \sum_{\sigma,\nu\in\mathcal{J}} G(z)_\sigma^* \left[-\alpha + \alpha S(z)(1+\alpha S(z))^{-1}\alpha\right]_{\sigma\nu} G(z)_\nu, \end{aligned} \tag{3.47}$$

where, from the first to the second line, the summation was restricted to the subset $\mathcal{J} \subseteq \mathcal{I}$, which contains all $\sigma \in \mathcal{I}$ with $\alpha_\sigma \neq 0$. Now, observe that on the subspace $\mathfrak{X} = \bigoplus_{\sigma\in\mathcal{J}} \mathfrak{X}_\sigma \subseteq \bigoplus_{\sigma\in\mathcal{I}} \mathfrak{X}_\sigma$,

$$-\alpha + \alpha S(z)(1+\alpha S(z))^{-1}\alpha = -(1+\alpha S(z))^{-1}\alpha = \Theta(z)^{-1},$$

where $\Theta(z) \in \mathscr{L}(\mathfrak{X})$ was introduced in Eq. (3.5) for general $z \in \rho(H_0) = \mathbb{C} \setminus (-\infty, 0]$, that is

$$\Theta(z)_{\sigma\nu} = -(\alpha_\sigma)^{-1}\delta_{\sigma\nu} - T_\sigma G(\overline{z})_\nu^* \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma), \qquad z \in \rho(H_0), \sigma, \nu \in \mathcal{J}. \tag{3.48}$$

In view of Eq. (3.47), this means that $R(z) = (H+z)^{-1}$ agrees with the right side of Eq. (3.6) for all $z \in (z_0, \infty)$, where (recall) $G(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X})$ is given by $G(z)\psi = (G(z)_\sigma \psi)_{\sigma\in\mathcal{J}}$.

To show that Eq. (3.6) defines $(H+z)^{-1}$ for all $z \in \rho(H) \cap \rho(H_0)$, it suffices to verify the hypotheses of [21, Theorems 2.4 and 2.19]. To this end, we first observe that $G(z) = TR_0(z)$, $z \in \rho(H_0)$, where the trace $T : H^1(\mathbb{R}^N) \to \mathfrak{X}$ is defined by $T\psi := (T_\sigma \psi)_{\sigma\in\mathcal{J}}$. The trace $T$ defines the operator $\tau$ in the notation of [21] and, by Lemma 2.13, $\operatorname{Ker} T$ is dense in $\mathscr{H}$. Therefore, it only remains to check that $\Theta(z)$, $z \in \rho(H_0)$, satisfies the hypotheses of [21, Theorems 2.4 and 2.19]. This follows from Proposition 3.11, below, so the proof of Theorem 3.1 is complete. □

**Proposition 3.11.** *Let $\emptyset \neq \mathcal{J} \subseteq \mathcal{I}$, let $\alpha_\sigma \in \mathbb{R} \setminus \{0\}$ for all $\sigma \in \mathcal{J}$, and let $w, z \in \rho(H_0)$. Then Eq.* (3.48) *defines an operator $\Theta(z) \in \mathscr{L}(\mathfrak{X})$ that has the following properties:*

*(i)* $\Theta(z)^* = \Theta(\overline{z})$.

*(ii)* $\Theta(z) = \Theta(w) + (z-w)G(z)G(\overline{w})^*$.

*(iii)* $0 \in \rho(\Theta(z))$ *for some* $z \in \rho(H_0)$.

*Proof.* Property $(iii)$ has already been verified in the proof of Theorem 3.1. By the Definition (3.48) of $\Theta(z)$ and by $G(z)\psi = (G(z)_\sigma\psi)_{\sigma\in\mathcal{J}}$, Property $(ii)$ is equivalent to

$$-T_\sigma G(\overline{z})_\nu^* = -T_\sigma G(\overline{w})_\nu^* + (z-w)G(z)_\sigma G(\overline{w})_\nu^*, \qquad \sigma, \nu \in \mathcal{J}. \tag{3.49}$$

From Proposition 2.12 $(i)$ we know that $G(\overline{z})_\nu = G(\overline{w})_\nu + (\overline{w} - \overline{z})G(\overline{w})_\nu R_0(\overline{z})$, and hence

$$G(\overline{z})_\nu^* = G(\overline{w})_\nu^* + (w-z)R_0(z)G(\overline{w})_\nu^*, \qquad w, z \in \rho(H_0), \nu \in \mathcal{J},$$

which yields Eq. (3.49) after applying $-T_\sigma$ to both sides. This proves Property $(ii)$.

To prove Property $(i)$ for $z \in (0,\infty)$, we first observe that Corollary 2.11 implies that $T_\nu R_0(z)^{1/2} \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\nu)$ for all $\nu \in \mathcal{J}$, and hence

$$T_\sigma G(z)_\nu^* = T_\sigma R_0(z)^{1/2} \left(T_\nu R_0(z)^{1/2}\right)^* = \left(T_\nu G(z)_\sigma^*\right)^*.$$

This means that $\Theta(z)$ is self-adjoint for $z \in (0,\infty)$. For general $z \in \rho(H_0)$, $(ii)$ now implies that

$$\begin{aligned} \Theta(\overline{z}) = \Theta(1) + (\overline{z}-1)\,G(\overline{z})G(1)^* &= [\Theta(1) + (z-1)\,G(1)G(\overline{z})^*]^* \\ &= [\Theta(1) + (z-1)\,G(z)G(1)^*]^* = \Theta(z)^*. \end{aligned}$$

□

## 3.5 Improving the rate of resolvent convergence in Theorem 3.1

The goal of this section is to improve the rate of resolvent convergence in Theorem 3.1, which is limited to $O(\varepsilon^s)$ with $s \in (0, 1/2]$. More precisely, we are going to prove the following:

**Proposition 3.12.** *Let the hypotheses of Theorem 3.1 be satisfied and suppose that the additional assumptions that $\int |r|^{2s}|V_\sigma(r)|\,\mathrm{d}r < \infty$ and $|g_{\varepsilon,\sigma} - g_\sigma| = O(\varepsilon^s)$ as $\varepsilon \to 0$ are satisfied for some $s \in (0, 1)$ that is independent of the particular choice of $\sigma \in \mathcal{I}$. Then $z \in \rho(H)$ implies that $z \in \rho(H_\varepsilon) \cap \rho(H)$ for small enough $\varepsilon > 0$ and $\|(H + z)^{-1} - (H_\varepsilon + z)^{-1}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

To prove Proposition 3.12, we have to show that the estimates (3.43) and (3.44) from the proof of Theorem 3.1 are still valid for general $s \in (0, 1)$, provided that $\int |r|^{2s}|V_\sigma(r)|\,\mathrm{d}r < \infty$ for all $\sigma \in \mathcal{I}$. In the case of (3.43), this follows immediately from Corollary 3.9. However, in the case of (3.44), our previous estimates are only valid for $s \in (0, 1/2]$, so we need more refined estimates for $\|\phi_\varepsilon(z) - \phi(z)\|$.

In Sections 3.5.1 and 3.5.2, below, we shall see that all components $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma, \nu \in \mathcal{I}$, define integral operators whose kernels can be explicitly computed in terms of the Green's function of $H_0 + z$. This is independent of the space dimension $d \in \{1, 2\}$, and in $d = 1$ this will be the key that allows us to prove the desired rate of convergence in Propositions 3.13, 3.14 and 3.15, below. However, these integral kernels also play a major role in our analysis of $\phi_\varepsilon(z)_{\sigma\nu}$ in $d = 2$ dimensions, see Sections 4.4.1 and 4.4.2, below. Hence, for the sake of later reference, we are going to compute these integral kernels in all dimensions $d \in \{1, 2\}$ simultaneously.

Let $d \in \{1, 2\}$ and $\varepsilon, z > 0$ be given and assume, for all pairs $\sigma \in \mathcal{I}$, that $V_\sigma \in L^1 \cap L^2(\mathbb{R}^d)$ satisfies $V_\sigma(r) = V_\sigma(-r)$ a.e. Recall from Eq. (1.52) that the components of $\phi_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ are given by $\phi_\varepsilon(z)_{\sigma\nu} = B_{\varepsilon,\sigma} R_0(z)(A_{\varepsilon,\nu})^* \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$, $\sigma, \nu \in \mathcal{I}$. Inserting the Definitions (1.48) and (1.49) of $A_{\varepsilon,\nu}$ and $B_{\varepsilon,\sigma}$, respectively, leads to the explicit expression

$$\phi_\varepsilon(z)_{\sigma\nu} = \varepsilon^{-d}\,(u_\sigma \otimes 1)\,U_\varepsilon \mathscr{K}_\sigma R_0(z) \mathscr{K}_\nu^* U_\varepsilon^* (v_\nu \otimes 1), \qquad \sigma, \nu \in \mathcal{I}. \tag{3.50}$$

In Section 3.5.1, below, we first consider the diagonal contributions $\phi_\varepsilon(z)_{\sigma\sigma}$, and Section 3.5.2 is then devoted to the off-diagonal contributions $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$. Finally, the proof of Proposition 3.12 is given at the end of Section 3.5.2.

### 3.5.1 The diagonal contributions $\phi_\varepsilon(z)_{\sigma\sigma}$

Let $d \in \{1, 2\}$ and $\varepsilon, z > 0$ be given. As the particular choice of the pair $\sigma \in \mathcal{I}$ is immaterial for the analysis of $\phi_\varepsilon(z)_{\sigma\sigma}$, we may assume that $\sigma = (1, 2)$ without restriction, and we drop the index $(1, 2)$ in the following: $V = V_{(1,2)} \in L^1 \cap L^2(\mathbb{R}^d)$, $v = v_{(1,2)}$, $\mu = \mu_{(1,2)}$, $\phi_\varepsilon(z) = \phi_\varepsilon(z)_{(1,2)(1,2)}$ etc. In particular, this means that $\phi_\varepsilon(z)$ is used as a shorthand notation for the $(1, 2)(1, 2)$-component of the operator $\phi_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$.

With this said, we now compute the integral kernel of $\phi_\varepsilon(z)$ in terms of the Green's function $G^d_\lambda$, $\lambda > 0$, of $-\Delta + \lambda : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. First, we note that the Definition (1.45) of $\mathscr{K} = \mathscr{K}_{(1,2)}$ implies that

$$\mathscr{K} R_0(z) = (\widetilde{H}_0 + z)^{-1} \mathscr{K}, \qquad z > 0, \tag{3.51}$$

where $\widetilde{H}_0$ is the free Hamiltonian $H_0$ expressed in the relative and center of mass coordinates of the pair $(1, 2)$. That is

$$\widetilde{H}_0 = -\frac{\Delta_r}{\mu} - \frac{\Delta_R}{m_1 + m_2} + \sum_{i=3}^{N} \left( -\frac{\Delta_{x_i}}{m_i} \right). \tag{3.52}$$

Now, starting from Eq. (3.50), the identities (3.51) and $\mathscr{K}\mathscr{K}^* = 1$ ($\mathscr{K}$ is unitary) show that

$$\phi_\varepsilon(z) = \varepsilon^{-d}\,(u \otimes 1)\, U_\varepsilon(\widetilde{H}_0 + z)^{-1} U_\varepsilon^*(v \otimes 1).$$

Hence, after a Fourier transform in $(R, x_3, ..., x_N)$, $\phi_\varepsilon(z)$ acts pointwise in the conjugate variable $\underline{P} = (P, p_3, ..., p_N)$ by the operator

$$\phi_\varepsilon(z, \underline{P}) = \varepsilon^{-d} \mu u\, U_\varepsilon\, (-\Delta_r + \mu(Q+z))^{-1}\, U_\varepsilon^* v \in \mathscr{L}(L^2(\mathbb{R}^d)), \tag{3.53}$$

where

$$Q := \frac{P^2}{m_1 + m_2} + \sum_{i=3}^{N} \frac{p_i^2}{m_i}. \tag{3.54}$$

From Eq. (3.53) we conclude that $\phi_\varepsilon(z, \underline{P})$ defines an integral operator in $\mathscr{L}(L^2(\mathbb{R}^d))$, where after the scaling $r'/\varepsilon \to r'$ the associated integral kernel is given by

$$K_\varepsilon(r, r', z, \underline{P}) := \mu\, u(r)\, G^d_{\mu(Q+z)}(\varepsilon(r - r'))\, v(r'). \tag{3.55}$$

This means that, for all $\psi \in L^2(\mathbb{R}^d)$, $(\phi_\varepsilon(z, \underline{P})\psi)(r) = \int K_\varepsilon(r, r', z, \underline{P})\psi(r')\, \mathrm{d}r'$.

In the remainder of this section, we now restrict ourselves to the case $d = 1$, where the Green's function is explicitly given by

$$G^1_\lambda(x) = \frac{\exp(-\sqrt{\lambda}\,|x|)}{2\sqrt{\lambda}}, \qquad \lambda > 0. \tag{3.56}$$

Due to the facts that $u, v \in L^2(\mathbb{R})$ and $G^1_\lambda \in L^\infty(\mathbb{R})$, we see that $\phi_\varepsilon(z, \underline{P})$ is a Hilbert-Schmidt operator and we expect, and prove below, that $\lim_{\varepsilon \to 0} \phi_\varepsilon(z) = \phi_0(z)$, where $\phi_0(z, \underline{P})$ is defined in terms of the integral kernel

$$K_0(r, r', z, \underline{P}) = \frac{1}{2}\sqrt{\frac{\mu}{Q+z}}\, u(r)v(r'). \tag{3.57}$$

**Proposition 3.13.** *Let $d = 1$, $\sigma = (1, 2)$, $z > 0$ and suppose that $V \in L^1(\mathbb{R})$ satisfies $\int |r|^{2s}\, |V(r)|\, \mathrm{d}r < \infty$ for some $s \in (0, 1]$. Then $\|\phi_\varepsilon(z) - \phi_0(z)\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

*Proof.* From Eqs. (3.55)-(3.57) it follows that for fixed $\varepsilon, z > 0$ and $\underline{P} \in \mathbb{R}^{N-1}$ the integral kernel of $\phi_\varepsilon(z, \underline{P}) - \phi_0(z, \underline{P})$ is given by

$$\mu\, u(r) \left( G^1_{\mu(Q+z)}(\varepsilon(r - r')) - G^1_{\mu(Q+z)}(0) \right) v(r'). \tag{3.58}$$

With the help of the elementary inequality $|\exp(-x) - 1| \leq x^s$, valid for all $x \geq 0$, and the explicit formula (3.56) for $G^1_\lambda$ we estimate

$$\begin{aligned} \left| G^1_{\mu(Q+z)}(\varepsilon(r - r')) - G^1_{\mu(Q+z)}(0) \right|^2 &\leq \frac{\varepsilon^{2s}}{4}\, (\mu(Q+z))^{s-1}\, |r - r'|^{2s} \\ &\leq \frac{\varepsilon^{2s}}{2} (\mu z)^{s-1} (|r|^{2s} + |r'|^{2s}). \end{aligned} \tag{3.59}$$

Using this to estimate the Hilbert-Schmidt norm of $\phi_\varepsilon(z, \underline{P}) - \phi_0(z, \underline{P})$, we find that

$$\|\phi_\varepsilon(z, \underline{P}) - \phi_0(z, \underline{P})\|^2_{\mathrm{HS}} \leq \varepsilon^{2s} \mu^{1+s} z^{s-1} \|V\|_{L^1} \int |r|^{2s}\, |V(r)|\, \mathrm{d}r, \tag{3.60}$$

which proves the proposition because the right side is independent of $\underline{P}$ and the operator norm is bounded from above by the Hilbert-Schmidt norm (cf. [71, Theorem VI.22 (d)]). ☐

### 3.5.2 The off-diagonal contributions $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$, and proof of Proposition 3.12

We now show that the off-diagonal contributions $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$, define integral operators whose kernels can be expressed in terms of the Green's function $G_{z,\underline{m}}$ of $H_0 + z$ that is defined by Eq. (3.61), below. As explained at the beginning of Section 3.5, we compute these integral kernels in all dimensions $d \in \{1, 2\}$ simultaneously and, without loss of generality, we may assume that $\sigma = (1, 2)$ and $\nu = (k, l) \neq (1, 2)$.

Let $\varepsilon, z > 0$ be fixed and let $G_{z,\underline{m}}$ denote the Green's function of $H_0 + z$, where the vector $\underline{m} := (m_1, ..., m_N)$ collects the masses of the $N$ particles. By a simple scaling argument,

$$G_{z,\underline{m}}(x_1, ..., x_N) = \left(\prod_{i=1}^{N} m_i^{d/2}\right) G_z^{dN}(\sqrt{m_1}x_1, ..., \sqrt{m_N}x_N), \tag{3.61}$$

where $G_z^{dN}$ denotes the usual Green's function of $-\Delta + z : H^2(\mathbb{R}^{dN}) \to L^2(\mathbb{R}^{dN})$ (we refer to Appendix A for details concerning $G_z^{dN}$). Now, with $\mathscr{K}_{(1,2)}$, $\mathscr{K}^*_{(k,l)}$ and $U_\varepsilon$ defined by Eqs. (1.45), (1.46) and (1.47), respectively, we find that Eq. (3.50) defines an integral operator, and for $\psi \in \widetilde{\mathfrak{X}}_{(k,l)}$,

$$\begin{aligned}
&\left(\phi_\varepsilon(z)_{(1,2)(k,l)}\,\psi\right)(r, R, x_3, ..., x_N) \\
&= \varepsilon^{-d} u_{(1,2)}(r) \int \mathrm{d}x'_1 \cdots \mathrm{d}x'_N\; G_{z,\underline{m}}\Big(R - \frac{\varepsilon m_2 r}{m_1 + m_2} - x'_1, R + \frac{\varepsilon m_1 r}{m_1 + m_2} - x'_2, x_3 - x'_3, ..., x_N - x'_N\Big) \\
&\quad \cdot v_{(k,l)}\left(\frac{x'_l - x'_k}{\varepsilon}\right) \psi\left(\frac{x'_l - x'_k}{\varepsilon}, \frac{m_k x'_k + m_l x'_l}{m_k + m_l}, x'_1, ...\widehat{x'_k}...\widehat{x'_l}..., x'_N\right) \\
&= u_{(1,2)}(r) \int \mathrm{d}x'_1 \cdots \mathrm{d}x'_N\, \mathrm{d}r'\, \mathrm{d}R'\; G_{z,\underline{m}}(R - \varepsilon c_{21} r - x'_1, R + \varepsilon c_{12} r - x'_2, x_3 - x'_3, ..., x_N - x'_N) \\
&\quad \cdot v_{(k,l)}(r')\, \psi\left(r', R', x'_1, ...\widehat{x'_k}...\widehat{x'_l}..., x'_N\right) \delta(x'_k - R' + \varepsilon c_{lk} r')\, \delta(x'_l - R' - \varepsilon c_{kl} r'),
\end{aligned} \tag{3.62}$$

where

$$c_{ij} := \frac{m_i}{m_i + m_j}, \qquad i, j = 1, ..., N. \tag{3.63}$$

The second equation of (3.62) was obtained by the substitution

$$r' := \frac{x'_l - x'_k}{\varepsilon}, \qquad R' := \frac{m_k x'_k + m_l x'_l}{m_k + m_l},$$

and two more integrations were introduced that are compensated by $\delta$-distributions. In the following, we distinguish between two cases: In the first case $\sigma = (1, 2)$ and $\nu = (k, l)$ have one particle in common, so $k \in \{1, 2\}$ and $l \geq 3$, and in the second case $\sigma$ and $\nu$ are composed of distinct particles, which means that $3 \leq k < l \leq N$.

Let us first consider the case $\sigma = (1, 2)$ and $\nu = (1, l)$ for some $l \geq 3$. Then it follows from Eq. (3.62) that, after the evaluation of the $\delta$-distributions in $x'_1$ and $x'_l$, the operator $\phi_\varepsilon(z)_{\sigma\nu}$ simply acts by convolution in $(x_3, ...\widehat{x}_l..., x_N)$. Consequently, it follows from the explicit formula (3.61) for $G_{z,\underline{m}}$ and from Lemma A.1 (*vi*) that $\phi_\varepsilon(z)_{\sigma\nu}$ acts pointwise in the conjugate variables $\underline{p}_{1l} := (p_3, ...\widehat{p}_l..., p_N)$ by the integral operator $\phi_\varepsilon(z, \underline{p}_{1l})_{\sigma\nu}$ that has the kernel

$$(m_1 m_2 m_l)^{d/2}\, u_\sigma(r)\, G^{3d}_{z+Q_\nu}(X_{\varepsilon,\sigma\nu})\, v_\nu(r'), \qquad \sigma = (1, 2), \nu = (1, l), l \geq 3, \tag{3.64}$$

where

$$X_{\varepsilon,(1,2)(1,l)} := \begin{pmatrix} \sqrt{m_1}(R - R' - \varepsilon(c_{21} r - c_{l1} r')) \\ \sqrt{m_2}(R - x'_2 + \varepsilon c_{12} r) \\ \sqrt{m_l}(x_l - R' - \varepsilon c_{1l} r') \end{pmatrix} \in \mathbb{R}^{3d} \tag{3.65}$$

and

$$Q_{(k,l)} := \sum_{\substack{n=3 \\ n\neq k,l}}^{N} \frac{p_n^2}{m_n}, \qquad 1 \le k < l \le N. \tag{3.66}$$

Similar considerations for $\sigma = (1,2)$ and $\nu = (2,l)$ with $l \ge 3$ show that $\phi_\varepsilon(z)_{\sigma\nu}$ acts pointwise in $\underline{p}_{2l} := (p_3, ...\widehat{p_l}..., p_N)$ by the integral operator $\phi_\varepsilon(z, \underline{p}_{2l})_{\sigma\nu}$ with kernel

$$(m_1 m_2 m_l)^{d/2}\, u_\sigma(r) G^{3d}_{z+Q_\nu}(X_{\varepsilon,\sigma\nu})\, v_\nu(r')\,, \qquad \sigma = (1,2), \nu = (2,l), l \ge 3, \tag{3.67}$$

where

$$X_{\varepsilon,(1,2)(2,l)} := \begin{pmatrix} \sqrt{m_1}(R - x_1' - \varepsilon c_{21} r) \\ \sqrt{m_2}(R - R' + \varepsilon(c_{12} r + c_{l2} r')) \\ \sqrt{m_l}(x_l - R' - \varepsilon c_{2l} r') \end{pmatrix} \in \mathbb{R}^{3d}. \tag{3.68}$$

By inspection, the integral kernels defined by (3.64) and (3.67) only differ by the permutations $x_1' \leftrightarrow x_2'$, $m_1 \leftrightarrow m_2$, $v_{(1,l)} \leftrightarrow v_{(2,l)}$ and the reflection $r \to -r$, which allows us to analyze them simultaneously.

So far we have considered all operators that occur in the case of $N \le 3$ particles. Let now $N > 3$ and let $\sigma$ and $\nu$ be composed of distinct particles, i.e. $\sigma = (1,2)$ and $\nu = (k,l)$ with $3 \le k < l \le N$. Then, after the evaluation of the $\delta$-distributions in $x_k'$ and $x_l'$, it follows from Eq. (3.62) that $\phi_\varepsilon(z)_{\sigma\nu}$ acts by convolution in $(x_3, ...\widehat{x_k}...\widehat{x_l}..., x_N)$. Hence, $\phi_\varepsilon(z)_{\sigma\nu}$ acts pointwise in the conjugate variables $\underline{p}_{kl} := (p_3, ...\widehat{p_k}...\widehat{p_l}..., p_N)$ by the integral operator $\phi_\varepsilon(z, \underline{p}_{kl})_{\sigma\nu}$ with kernel

$$(m_1 m_2 m_k m_l)^{d/2}\, u_\sigma(r) G^{4d}_{z+Q_\nu}(X_{\varepsilon,\sigma\nu})\, v_\nu(r')\,, \qquad \sigma = (1,2), \nu = (k,l), 3 \le k < l \le N, \tag{3.69}$$

where

$$X_{\varepsilon,(1,2)(k,l)} := \begin{pmatrix} \sqrt{m_1}(R - x_1' - \varepsilon c_{21} r) \\ \sqrt{m_2}(R - x_2' + \varepsilon c_{12} r) \\ \sqrt{m_k}(x_k - R' + \varepsilon c_{lk} r') \\ \sqrt{m_l}(x_l - R' - \varepsilon c_{kl} r') \end{pmatrix} \in \mathbb{R}^{4d}. \tag{3.70}$$

Now, we have computed the integral kernel of $\phi_\varepsilon(z)_{\sigma\nu}$ for all combinations of $d \in \{1,2\}$, $\varepsilon, z > 0$, $\sigma = (1,2)$ and $\nu = (k,l) \ne (1,2)$. In the rest of this section, we now restrict ourselves to the case $d = 1$, and we define, a priori, new operators $\phi_0(z)_{\sigma\nu}$ by the corresponding integral kernels with $\varepsilon = 0$. Thus we expect that $\phi_\varepsilon(z)_{\sigma\nu} \to \phi_0(z)_{\sigma\nu}$ as $\varepsilon \to 0$, which we confirm in Propositions 3.14 and 3.15, below. Since we know from Proposition 3.10 that $\phi_\varepsilon(z)_{\sigma\nu} \to \phi(z)_{\sigma\nu}$ as $\varepsilon \to 0$, it is then clear that $\phi_0(z)_{\sigma\nu} = \phi(z)_{\sigma\nu}$, where $\phi(z)_{\sigma\nu}$ has been introduced in Eq. (3.36). The essential point, however, is that Propositions 3.14 and 3.15 yield the desired rate of convergence that is needed for the proof of Proposition 3.12.

**Proposition 3.14.** *Let $d = 1$, $z > 0$, $\sigma = (1,2)$, $\nu = (k,l)$ with $k \in \{1,2\}$ and $l \ge 3$, and suppose that $V_\sigma, V_\nu \in L^1(\mathbb{R})$ satisfy $\int |r|^{2s}(|V_\sigma(r)| + |V_\nu(r)|)\,\mathrm{d}r < \infty$ for some $s \in (0,1)$. Then $\|\phi_\varepsilon(z)_{\sigma\nu} - \phi_0(z)_{\sigma\nu}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

*Proof.* As explained above, the operators $\phi_\varepsilon(z)_{(1,2)(1,l)}$ and $\phi_\varepsilon(z)_{(1,2)(2,l)}$ coincide up to the unitary reflection $r \to -r$ (and some obvious changes of indices that are immaterial for the estimates below). Hence, it suffices to consider the case $\nu = (1,l)$ with $l \ge 3$ only. Then $\phi_\varepsilon(z)_{\sigma\nu} - \phi_0(z)_{\sigma\nu}$ acts pointwise in $\underline{p}_{1l}$ by the integral operator with kernel

$$u_\sigma(r)\, \left(G^3_{z+Q_\nu,\underline{m}}(X_\varepsilon) - G^3_{z+Q_\nu,\underline{m}}(X_0)\right) v_\nu(r'), \tag{3.71}$$

where we have introduced the shorthand notations

$$G^3_{\lambda,\underline{m}}(x_1,x_2,x_l) := (m_1 m_2 m_l)^{1/2}\, G^3_\lambda(\sqrt{m_1}x_1,\sqrt{m_2}x_2,\sqrt{m_l}x_l), \qquad \lambda>0, \tag{3.72}$$

and

$$X_\varepsilon := \begin{pmatrix} R-R'-\varepsilon(c_{21}r-c_{l1}r') \\ R-x_2'+\varepsilon c_{12}r \\ x_l-R'-\varepsilon c_{1l}r' \end{pmatrix} \in \mathbb{R}^3, \qquad \varepsilon\ge 0.$$

To estimate the norm of $\phi_\varepsilon(z)_{\sigma\nu}-\phi_0(z)_{\sigma\nu}$, we first note that our assumptions on $V_\sigma$ and $V_\nu$ imply that $I_s(V_\sigma)<\infty$ and $I_s(V_\nu)<\infty$, where

$$I_s(V) := \int (1+|r|^{2s})\,|V(r)|\,\mathrm{d}r. \tag{3.73}$$

Now, using the Cauchy-Schwarz inequality in the $r'$-integration, we find that, for all $\psi\in L^2(\mathbb{R}^3)$,

$$\begin{aligned}
&\left\|\Big(\phi_\varepsilon(z,\underline{p}_{1l})_{\sigma\nu}-\phi_0(z,\underline{p}_{1l})_{\sigma\nu}\Big)\,\psi\right\|^2 \\
&= \int \mathrm{d}r\,|V_\sigma(r)| \int \mathrm{d}R\,\mathrm{d}x_l \left|\int \mathrm{d}r'\,v_\nu(r')\int \mathrm{d}R'\,\mathrm{d}x_2' \Big(G^3_{z+Q_\nu,\underline{m}}(X_\varepsilon)-G^3_{z+Q_\nu,\underline{m}}(X_0)\Big)\psi(X')\right|^2 \\
&\le I_s(V_\nu)\int \mathrm{d}r\,\mathrm{d}r' \frac{|V_\sigma(r)|}{1+|r'|^{2s}} \int \mathrm{d}R\,\mathrm{d}x_l \left|\int \mathrm{d}R'\,\mathrm{d}x_2' \Big(G^3_{z+Q_\nu,\underline{m}}(X_\varepsilon)-G^3_{z+Q_\nu,\underline{m}}(X_0)\Big)\psi(X')\right|^2, 
\end{aligned} \tag{3.74}$$

where $X' := (r',R',x_2')$ for brevity.

For a further estimate of (3.74), we consider for fixed $r,r'\in\mathbb{R}$, $Q_\nu\ge 0$ and $\varepsilon>0$ the integral operator $B_{r,r',Q_\nu,\varepsilon} : L^2(\mathbb{R}^2,\mathrm{d}(R,x_2)) \to L^2(\mathbb{R}^2,\mathrm{d}(R,x_l))$ that is defined in terms of the kernel $G^3_{z+Q_\nu,\underline{m}}(X_\varepsilon)-G^3_{z+Q_\nu,\underline{m}}(X_0)$. We are going to estimate $\|B_{r,r',Q_\nu,\varepsilon}\|$ with the help of the Schur test. To this end, we first introduce the intermediate point

$$X_{\varepsilon,0} := \begin{pmatrix} R-R'-\varepsilon(c_{21}r-c_{l1}r') \\ R-x_2' \\ x_l-R' \end{pmatrix} \in \mathbb{R}^3, \qquad \varepsilon>0.$$

Now, using (3.72) and the properties of the Green's function $G^3_\lambda$ from Appendix A, we estimate

$$\begin{aligned}
&\sup_{R,x_l\in\mathbb{R}}\left(\int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z+Q_\nu,\underline{m}}(X_\varepsilon)-G^3_{z+Q_\nu,\underline{m}}(X_0)\right|\right) \\
&\le \sup_{R,x_l\in\mathbb{R}}\left(\int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z,\underline{m}}(X_\varepsilon)-G^3_{z,\underline{m}}(X_{\varepsilon,0})\right|\right) + \sup_{R,x_l\in\mathbb{R}}\left(\int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z,\underline{m}}(X_{\varepsilon,0})-G^3_{z,\underline{m}}(X_0)\right|\right) \\
&\le \sup_{R,x_l\in\mathbb{R}}\left(\int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z,\underline{m}}\left(0,R-x_2'+\varepsilon c_{12}r,\,x_l-R'-\varepsilon c_{1l}r'\right)-G^3_{z,\underline{m}}\left(0,R-x_2',x_l-R'\right)\right|\right) \\
&\quad+\sup_{R,x_l\in\mathbb{R}}\left(\int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z,\underline{m}}(R-R'-\varepsilon(c_{21}r-c_{l1}r'),R-x_2',0)-G^3_{z,\underline{m}}(R-R',R-x_2',0)\right|\right) \\
&= \int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z,\underline{m}}\left(0,x_2'+\varepsilon c_{12}r,R'-\varepsilon c_{1l}r'\right)-G^3_{z,\underline{m}}\left(0,x_2',R'\right)\right| \\
&\quad+\int \mathrm{d}R'\,\mathrm{d}x_2' \left|G^3_{z,\underline{m}}(R'-\varepsilon(c_{21}r-c_{l1}r'),x_2',0)-G^3_{z,\underline{m}}(R',x_2',0)\right| \\
&\le (\sqrt{m_1}+\sqrt{m_l}) \sup_{|y|\le c(|r|+|r'|)} \left(\int_{\mathbb{R}^2} \left|G^3_z(x+\varepsilon y,0)-G^3_z(x,0)\right|\mathrm{d}x\right), \qquad c=\max_{i,j}(\sqrt{m_j}c_{ij}),
\end{aligned} \tag{3.75}$$

where the first inequality used the fact that, by the estimate (A.4) from Lemma A.2, the integrand in the first line attains its maximum for $Q_\nu=0$, the second inequality made use of the estimate (A.5) from Lemma A.2, the subsequent equality was obtained by substituting

$R-x_2'\to x_2'$, $x_l-R'\to R'$ in the first and $R-R'\to R'$, $R-x_2'\to x_2'$ in the second integral, and the last inequality simply used the Definition (3.72) of $G^3_{z,\underline{m}}$ and a scaling in $R'$ and $x_2'$. From (3.75) and a similar estimate with the roles of $(R,x_l)$ and $(R',x_2')$ interchanged, we conclude with the help of the Schur test that $B_{r,r',Q_\nu,\varepsilon}$ defines a bounded operator with

$$\|B_{r,r',Q_\nu,\varepsilon}\| \le (\sqrt{m_1}+\sqrt{m_2}+\sqrt{m_l}) \sup_{|y|\le c(|r|+|r'|)} \left( \int_{\mathbb{R}^2} \left| G_z^3(x+\varepsilon y,0) - G_z^3(x,0)\right| \mathrm{d}x \right). \tag{3.76}$$

From (3.76) and from Lemma A.3 it follows that $\|B_{r,r',Q_\nu,\varepsilon}\|^2 \le C\varepsilon^{2s}(|r|^{2s}+|r'|^{2s})$ for some constant $C=C(s,z,m_1,...,m_N)>0$ that does not depend on $r,r',Q_\nu$ and $\varepsilon$. Using this to estimate the right side of (3.74), we see that

$$\begin{aligned}\left\| \left(\phi_\varepsilon(z,\underline{p}_{1l})_{\sigma\nu} - \phi_0(z,\underline{p}_{1l})_{\sigma\nu}\right)\psi\right\|^2 &\le I_s(V_\nu)\int \mathrm{d}r\,\mathrm{d}r'\,\frac{|V_\sigma(r)|}{1+|r'|^{2s}}\left\|B_{r,r',Q_\nu,\varepsilon}\psi(r',\,\cdot\,)\right\|^2 \\ &\le C\varepsilon^{2s} I_s(V_\sigma)I_s(V_\nu)\,\|\psi\|^2.\end{aligned}$$

As the right side is independent of $\underline{p}_{1l}$, this proves that $\|\phi_\varepsilon(z)_{\sigma\nu}-\phi_0(z)_{\sigma\nu}\| = O(\varepsilon^s)$ as $\varepsilon\to 0$. □

**Proposition 3.15.** *Let $d=1$, $z>0$, $\sigma=(1,2)$, $\nu=(k,l)$ with $3\le k<l\le N$, and suppose that $V_\sigma, V_\nu\in L^1(\mathbb{R})$ satisfy $\int |r|^{2s}(|V_\sigma(r)|+|V_\nu(r)|)\,\mathrm{d}r<\infty$ for some $s\in(0,1)$. Then $\|\phi_\varepsilon(z)_{\sigma\nu}-\phi_0(z)_{\sigma\nu}\| = O(\varepsilon^s)$ as $\varepsilon\to 0$.*

*Proof.* From (3.69) and (3.70) we know that $\phi_\varepsilon(z)_{\sigma\nu}-\phi_0(z)_{\sigma\nu}$ acts pointwise in $\underline{p}_{kl}$ by the integral operator with kernel

$$u_\sigma(r)\,\left(G^4_{z+Q_\nu,\underline{m}}(X_\varepsilon) - G^4_{z+Q_\nu,\underline{m}}(X_0)\right) v_\nu(r'), \tag{3.77}$$

where we have introduced the shorthand notations

$$G^4_{\lambda,\underline{m}}(x_1,x_2,x_k,x_l) := (m_1m_2m_km_l)^{1/2}\,G^4_\lambda(\sqrt{m_1}x_1,\sqrt{m_2}x_2,\sqrt{m_k}x_k,\sqrt{m_l}x_l), \qquad \lambda>0, \tag{3.78}$$

and

$$X_\varepsilon := \begin{pmatrix} R-x_1'-\varepsilon c_{21}r \\ R-x_2'+\varepsilon c_{12}r \\ x_k-R'+\varepsilon c_{lk}r' \\ x_l-R'-\varepsilon c_{kl}r'\end{pmatrix} \in\mathbb{R}^4, \qquad \varepsilon\ge 0.$$

To estimate the norm of $\phi_\varepsilon(z)_{\sigma\nu}-\phi_0(z)_{\sigma\nu}$, we start our estimates similarly to (3.74). Using the Cauchy-Schwarz inequality in the $r'$-integration, we find that, for all $\psi\in L^2(\mathbb{R}^4)$,

$$\begin{aligned}&\left\| \left(\phi_\varepsilon(z,\underline{p}_{kl})_{\sigma\nu} - \phi_0(z,\underline{p}_{kl})_{\sigma\nu}\right)\psi\right\|^2 \\ &\le I_s(V_\nu)\int \mathrm{d}r\,\mathrm{d}r'\frac{|V_\sigma(r)|}{1+|r'|^{2s}}\int \mathrm{d}R\,\mathrm{d}x_k\,\mathrm{d}x_l\left|\int \mathrm{d}R'\,\mathrm{d}x_1'\,\mathrm{d}x_2'\left(G^4_{z+Q_\nu,\underline{m}}(X_\varepsilon)-G^4_{z+Q_\nu,\underline{m}}(X_0)\right)\psi(X')\right|^2 \\ &= I_s(V_\nu)\int \mathrm{d}r\,\mathrm{d}r'\,\frac{|V_\sigma(r)|}{1+|r'|^{2s}}\left\|F_{r,r',Q_\nu,\varepsilon}\psi(r',\,\cdot\,)\right\|^2,\end{aligned} \tag{3.79}$$

where $X':=(r',R',x_1',x_2')$ for short and for fixed $r,r'\in\mathbb{R}$, $Q_\nu\ge 0$ and $\varepsilon>0$ the integral operator $F_{r,r',Q_\nu,\varepsilon}: L^2\left(\mathbb{R}^3,\mathrm{d}(R,x_1,x_2)\right)\to L^2\left(\mathbb{R}^3,\mathrm{d}(R,x_k,x_l)\right)$ is defined in terms of the kernel $G^4_{z+Q_\nu,\underline{m}}(X_\varepsilon)-G^4_{z+Q_\nu,\underline{m}}(X_0)$. To show that $F_{r,r',Q_\nu,\varepsilon}$ defines a bounded operator and to estimate $\|F_{r,r',Q_\nu,\varepsilon}\|$, we are going to apply the Schur test again. Let

$$X_{\varepsilon,0} := \begin{pmatrix} R-x_1' \\ R-x_2' \\ x_k-R' \\ x_l-R'-\varepsilon c_{kl}r'\end{pmatrix} \in\mathbb{R}^4, \qquad \varepsilon>0.$$

Then, similarly to (3.75), we find

$$
\begin{aligned}
&\sup_{R,x_k,x_l\in\mathbb{R}} \left(\int \mathrm{d}R'\,\mathrm{d}x_1'\,\mathrm{d}x_2' \left|G^4_{z+Q_\nu,\underline{m}}(X_\varepsilon) - G^4_{z+Q_\nu,\underline{m}}(X_0)\right|\right) \\
&\quad\le \sup_{R,x_k,x_l\in\mathbb{R}} \left(\int \mathrm{d}R'\,\mathrm{d}x_1'\,\mathrm{d}x_2' \left|G^4_{z,\underline{m}}(X_\varepsilon) - G^4_{z,\underline{m}}(X_{\varepsilon,0})\right|\right) \\
&\qquad + \sup_{R,x_k,x_l\in\mathbb{R}} \left(\int \mathrm{d}R'\,\mathrm{d}x_1'\,\mathrm{d}x_2' \left|G^4_{z,\underline{m}}(X_{\varepsilon,0}) - G^4_{z,\underline{m}}(X_0)\right|\right) \\
&\quad\le \int \mathrm{d}R'\,\mathrm{d}x_1'\,\mathrm{d}x_2' \left|G^4_{z,\underline{m}}\left(x_1' - \varepsilon c_{21} r, x_2' + \varepsilon c_{12} r,\, R' + \varepsilon c_{lk} r', 0\right) - G^4_{z,\underline{m}}\left(x_1', x_2', R', 0\right)\right| \\
&\qquad + \int \mathrm{d}R'\,\mathrm{d}x_1'\,\mathrm{d}x_2' \left|G^4_{z,\underline{m}}\left(x_1', x_2', 0, R' - \varepsilon c_{kl} r'\right) - G^4_{z,\underline{m}}\left(x_1', x_2', 0, R'\right)\right| \\
&\quad\le (\sqrt{m_k} + \sqrt{m_l}) \sup_{|y|\le 2c(|r|+|r'|)} \left(\int_{\mathbb{R}^3} \left|G^4_z(x+\varepsilon y, 0) - G^4_z(x,0)\right| \mathrm{d}x\right), \qquad c = \max_{i,j}(\sqrt{m_j}c_{ij}), \quad (3.80)
\end{aligned}
$$

where the second inequality first used the estimate (A.5) from Lemma A.2, and afterwards the substitutions $R - x_1' \to x_1'$, $R - x_2' \to x_2'$ , $x_k - R' \to R'$ in the first and $R - x_1' \to x_1'$, $R - x_2' \to x_2'$, $x_l - R' \to R'$ in the second integral were carried out. By the Schur test and by Lemma A.3, the estimate (3.80) and a similar estimate with the roles of $(R, x_k, x_l)$ and $(R', x_1', x_2')$ interchanged imply that $\|F_{r,r',Q_\nu,\varepsilon}\|^2 \le C\varepsilon^{2s}(|r|^{2s} + |r'|^{2s})$ for some constant $C = C(s, z, m_1, ..., m_N) > 0$. Using this to estimate the right side of (3.79), it follows that

$$\left\|\left(\phi_\varepsilon(z, \underline{p}_{kl})_{\sigma\nu} - \phi_0(z, \underline{p}_{kl})_{\sigma\nu}\right)\psi\right\|^2 \le C\varepsilon^{2s} I_s(V_\sigma) I_s(V_\nu)\, \|\psi\|^2,$$

where $I_s(V_\sigma)$ and $I_s(V_\nu)$ are defined by Eq. (3.73). As the right side is independent of $\underline{p}_{kl}$, this proves that $\|\phi_\varepsilon(z)_{\sigma\nu} - \phi_0(z)_{\sigma\nu}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$. □

*Proof of Proposition 3.12.* Suppose that for some given $s \in (0,1)$ and all $\sigma \in \mathcal{I}$, $|g_{\varepsilon,\sigma} - g_\sigma| = O(\varepsilon^s)$ as $\varepsilon \to 0$ and $\int(1 + |r|^{2s})|V_\sigma(r)|\,\mathrm{d}r < \infty$. Then, for fixed $z > 0$, Corollary 3.9 shows that $\|A_{\varepsilon,\sigma}R_0(z) - A_\sigma R_0(z)\| = O(\varepsilon^s)$ as $\varepsilon \to 0$, and from Propositions 3.13, 3.14, 3.15 and their analogs for pairs $\sigma \neq (1,2)$, it follows that

$$\|\phi_\varepsilon(z) - \phi_0(z)\| = O(\varepsilon^s) \qquad\qquad (\varepsilon \to 0),$$

where $\phi_0(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ is defined in terms of the components $\phi_0(z)_{\sigma\nu}$, $\sigma, \nu \in \mathcal{I}$. Since we know from Proposition 3.10 that $\phi_\varepsilon(z) \to \phi(z)$ as $\varepsilon \to 0$, it follows that $\phi_0(z) = \phi(z)$ and $\|\phi_\varepsilon(z) - \phi(z)\| = O(\varepsilon^s)$ as $\varepsilon \to 0$. This means that the estimates (3.43) and (3.44) are valid, and hence the proof of Theorem 3.1 still works for general $s \in (0,1)$. We conclude that $z \in \rho(H)$ implies that $z \in \rho(H_\varepsilon) \cap \rho(H)$ for small enough $\varepsilon > 0$ and $\|(H+z)^{-1} - (H_\varepsilon + z)^{-1}\| = O(\varepsilon^s)$ as $\varepsilon \to 0$. □

## 3.6 Domain and action of the Hamiltonian

In this section we derive an explicit description of the domain and the action of the Hamiltonian $H$ from Theorem 3.1, see Proposition 3.16, below. Furthermore, we show that $H$ is a local and translation-invariant self-adjoint extension of $H_0 \restriction C_0^\infty(\mathbb{R}^N \setminus \Gamma)$ (see Remark (i) after Proposition 3.16 and Proposition 3.17, below), so the physical requirements **(i)** – **(iii)** from Section 1.2.2 are satisfied. In addition, we draw some conclusions concerning the spectrum of $H$.

Recall from Eq. (1.38) that in one space dimension a two-body $\delta$-interaction of strength $\alpha_\sigma \in \mathbb{R}$, supported on the collision plane $\Gamma_\sigma$, can be characterized by the following jump condition for the derivative of the wave function:

$$\left(\frac{\partial_j}{m_j} - \frac{\partial_i}{m_i}\right)\psi|_{x_j=x_i+} - \left(\frac{\partial_j}{m_j} - \frac{\partial_i}{m_i}\right)\psi|_{x_j=x_i-} = \alpha_\sigma \psi|_{x_i=x_j}, \qquad \sigma = (i,j) \in \mathcal{I}. \quad (3.81)$$

However, the jump condition (3.81) only makes sense on a formal level because it involves the evaluation of functions that are only defined almost everywhere in $\mathbb{R}^N$ on the hyper plane $\Gamma_\sigma$ that has measure zero in $\mathbb{R}^N$. To obtain a rigorous version of (3.81), these evaluations on $\Gamma_\sigma$ have to be defined in the sense of appropriate trace operators in Sobolev spaces. For $N = 3$ particles this is achieved by [10, Theorem 2]. For the sake of completeness, we establish a similar result for general $N \geq 2$ in Proposition 3.16, below, and we also give an explicit description of all required trace operators. To this end, we define for functions $\psi$ that belong to $H^1(O)$ for some open set $O \supset \Omega_\sigma^+ \setminus \Gamma$ the trace $T_\sigma^+\psi := T_\sigma^+(\psi \restriction (\Omega_\sigma^+ \setminus \Gamma))$, where the trace operator $T_\sigma^+ : H^1(\Omega_\sigma^+ \setminus \Gamma) \to \mathfrak{X}_\sigma$ on the right side has been introduced in Proposition 2.14. For open sets $O \supset \Omega_\sigma^- \setminus \Gamma$ and $\psi \in H^1(O)$, we define analogously $T_\sigma^-\psi := T_\sigma^-(\psi \restriction (\Omega_\sigma^- \setminus \Gamma))$.

**Proposition 3.16.** *Under the hypotheses of Theorem 3.1, $D(H)$ is the set of all functions $\psi \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma) \subseteq \mathscr{H}$ that satisfy, for all $\sigma \in \mathcal{I}$,*

$$(T_\sigma^+ - T_\sigma^-)\partial_\sigma\psi = \alpha_\sigma T_\sigma\psi, \tag{3.82}$$

*where, for $\sigma = (i,j)$, $\partial_\sigma := \partial_j/m_j - \partial_i/m_i$. If $\psi \in D(H)$, then, in the sense of distributions,*

$$H\psi = H_0\psi \qquad\qquad \text{in } \mathbb{R}^N \setminus \Gamma. \tag{3.83}$$

*Remarks.*

(i) For $\psi \in H^2(\mathbb{R}^N)$, the left side of Eq. (3.82) vanishes for all $\sigma \in \mathcal{I}$ because, by the remark after Proposition (2.14), $T_\sigma^+\partial_\sigma\psi = T_\sigma\partial_\sigma\psi = T_\sigma^-\partial_\sigma\psi$. This means that a function $\psi \in H^2(\mathbb{R}^N)$ belongs to $D(H)$ if and only if $\psi \in \operatorname{Ker} T_\sigma$ for all $\sigma \in \mathcal{J}$, and then $H\psi = H_0\psi$. In particular, since $C_0^\infty(\mathbb{R}^N \setminus \Gamma) \subseteq \operatorname{Ker} T_\sigma$ for all $\sigma \in \mathcal{J}$, this shows that $H$ is a self-adjoint extension of $H_0 \restriction C_0^\infty(\mathbb{R}^N \setminus \Gamma)$.

(ii) The identity (3.83) allows us to compute $H\psi$ explicitly for any given $\psi \in D(H)$.

The proof of Proposition 3.16 is given at the end of this section. Before we start preparing the proof, we state and prove the following:

**Proposition 3.17.** *The Hamiltonian $H$ from Theorem 3.1 is local in the following sense: If $\psi \in D(H)$ and $\psi = 0$ a.e. in some non-empty open set $U \subseteq \mathbb{R}^N$, then $H\psi = 0$ a.e. in $U$. Moreover, the operator identity $HT_{\mathrm{tot},h} = T_{\mathrm{tot},h}H$ holds for all $h \in \mathbb{R}$, where $T_{\mathrm{tot},h} \in \mathscr{L}(\mathscr{H})$ is given by*

$$(T_{\mathrm{tot},h}\psi)(x_1, x_2, ..., x_N) = \psi(x_1 + h, x_2 + h, ..., x_N + h).$$

*Remark.* The first part of Proposition 3.17 agrees with Property **(iii)** from Section 1.2.2 and the second part is a stronger version of Property **(ii)** from Section 1.2.2.

*Proof.* For the first part, it suffices to consider open sets $U \subseteq \mathbb{R}^N \setminus \Gamma$ because $\Gamma$ is a closed set of measure zero in $\mathbb{R}^N$. Then $H\psi = 0$ a.e. in $U$ follows immediately from Eq. (3.83) since $H_0$ is a local operator. For the second part, we first observe that the Definition (3.1) of $H_\varepsilon$ implies that, for all $\varepsilon > 0$, $H_\varepsilon T_{\mathrm{tot},h} = T_{\mathrm{tot},h}H_\varepsilon$. This is equivalent to, for all $\varepsilon > 0$, $T_{\mathrm{tot},h}(H_\varepsilon + i)^{-1} = (H_\varepsilon + i)^{-1}T_{\mathrm{tot},h}$ in $\mathscr{L}(\mathscr{H})$. Using Theorem 3.1 to take the limit $\varepsilon \to 0$, we obtain that $T_{\mathrm{tot},h}(H + i)^{-1} = (H + i)^{-1}T_{\mathrm{tot},h}$, which is equivalent to $HT_{\mathrm{tot},h} = T_{\mathrm{tot},h}H$. □

The translation invariance of $H_\varepsilon$, $\varepsilon > 0$, implies that $\sigma(H_\varepsilon) = \sigma_{\mathrm{ess}}(H_\varepsilon) = [\Sigma_\varepsilon, \infty)$ for some $\Sigma_\varepsilon \leq 0$. Hence, if $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$, then, with the help of Proposition 2.3, it is straightforward to verify that $\Sigma = \lim_{\varepsilon\to 0}\Sigma_\varepsilon$ exists and that $\sigma(H) = [\Sigma, \infty)$. In the special case of an attractive Bose gas with $m_i = 1$, $i = 1, ..., N$, and $\alpha_\sigma = \alpha$, $\sigma \in \mathcal{I}$, for some fixed $\alpha < 0$, the exact value of $\Sigma$ and the ground state are explicitly known, see [22] and the references therein. After removing the center of mass motion, the symmetric and translation-invariant ground state of the system is given by

$$\psi_{N,\alpha}(x_1, ..., x_N) = C_N \exp\left(\frac{\alpha}{4} \sum_{1\le i<j\le N} |x_j - x_i|\right), \tag{3.84}$$

where $C_N > 0$ denotes an appropriate normalization factor and the corresponding ground state energy is given by $\Sigma = -\alpha^2 N(N^2-1)/48$. Due to the purely attractive interaction, such a system collapses, for large $N$, to an interval whose length is of order $(|\alpha|N)^{-1}$ and the binding energy per particle diverges like $-(\alpha N)^2$ as $N \to \infty$ [22]. We shall not discuss the spectral properties of $H$ further, and instead we will be concerned with the proof of Proposition 3.16 in the remainder of this section.

As in the previous sections, let $T : H^1(\mathbb{R}^N) \to \mathfrak{X}$ and $G(z) : \mathscr{H} \to \mathfrak{X}$, $z \in \rho(H_0)$, be defined in terms of the components $T_\sigma$ and $G(z)_\sigma$, respectively, for $\sigma \in \mathcal{J}$. Then the following proposition, which is essentially due to [42, Proposition 4.4], gives an abstract, operator theoretic characterization of $D(H)$:

**Proposition 3.18.** *Under the hypotheses of Theorem 3.1, a vector $\psi \in \mathscr{H}$ belongs to $D(H)$ if and only if the following holds: For some (and hence all) $z \in \rho(H_0) \cap \rho(H)$ there exist $\psi_0 \in H^2(\mathbb{R}^N)$ and $w \in \mathfrak{X}$ such that*

$$\psi = \psi_0 + G(\overline{z})^* w \tag{3.85}$$

*and*

$$T\psi_0 = \Theta(z)w. \tag{3.86}$$

*The vectors $\psi_0$ and $w$ are uniquely determined by $\psi \in D(H)$ and $z \in \rho(H_0) \cap \rho(H)$, and*

$$(H+z)\psi = (H_0+z)\psi_0. \tag{3.87}$$

*Proof.* From Theorem 3.1 we know that $\Theta(z) \in \mathscr{L}(\mathfrak{X})$ has a bounded inverse for $z \in \rho(H_0)\cap\rho(H)$.

Suppose $\psi \in D(H)$ and $z \in \rho(H_0) \cap \rho(H)$. Let $\varphi := (H+z)\psi$, $\psi_0 := R_0(z)\varphi \in H^2(\mathbb{R}^N)$, and $w := \Theta(z)^{-1}T\psi_0$, so that (3.86) is trivially satisfied. Moreover, using (3.6) and $G(z)\varphi = TR_0(z)\varphi = T\psi_0$, (3.85) follows from

$$\begin{aligned}\psi = (H+z)^{-1}\varphi &= R_0(z)\varphi + G(\overline{z})^*\Theta(z)^{-1}G(z)\varphi \\ &= \psi_0 + G(\overline{z})^* w.\end{aligned}$$

Conversely, if (3.85) and (3.86) hold for some $z \in \rho(H_0) \cap \rho(H)$, let $\varphi := (H_0+z)\psi_0$. Then $T\psi_0 = G(z)\varphi$ and thus $w = \Theta(z)^{-1}T\psi_0 = \Theta(z)^{-1}G(z)\varphi$. Hence, by (3.85) and (3.6),

$$\begin{aligned}\psi = \psi_0 + G(\overline{z})^* w = R_0(z)\varphi + G(\overline{z})^*\Theta(z)^{-1}G(z)\varphi \\ = (H+z)^{-1}\varphi \in D(H),\end{aligned}$$

and from this it follows that $(H+z)\psi = \varphi = (H_0+z)\psi_0$. In particular, $\psi_0 = (H_0+z)^{-1}(H+z)\psi$ is uniquely determined by $\psi \in D(H)$ and $z \in \rho(H_0)\cap\rho(H)$, and since $\Theta(z)$ is invertible, it follows from (3.86) that $w$ is unique as well. □

The result from Proposition 3.18 is difficult to apply in concrete calculations since it is not clear how to compute the vectors $\psi_0 \in H^2(\mathbb{R}^N)$ and $w \in \mathfrak{X}$ for a given $\psi \in D(H)$. This is clarified by the following proposition:

**Proposition 3.19.** *Suppose that $\psi \in D(H)$ and $z \in \rho(H_0) \cap \rho(H)$. Then $\psi \in H^1(\mathbb{R}^N)$ and the vectors $w = (w_\sigma)_{\sigma\in\mathcal{J}} \in \mathfrak{X}$ and $\psi_0 \in H^2(\mathbb{R}^N)$ from Proposition 3.18 are explicitly given by*

$$w_\sigma = -\alpha_\sigma T_\sigma \psi \in H^{1/2}(\mathbb{R}^{N-1}), \qquad \sigma \in \mathcal{J}, \tag{3.88}$$

$$\psi_0 = \psi + \sum_{\nu\in\mathcal{J}} \alpha_\nu G(\overline{z})^*_\nu T_\nu \psi. \tag{3.89}$$

*Proof.* Let $w = (w_\sigma)_{\sigma\in\mathcal{J}} \in \mathfrak{X}$ and $\psi_0 \in H^2(\mathbb{R}^N)$ denote the vectors from Proposition 3.18. From Proposition 2.12 (*iii*) we know that $G(\overline{z})^*_\nu w_\nu \in H^1(\mathbb{R}^N)$ for all $\nu \in \mathcal{J}$, so it follows from (3.85) that $\psi \in H^1(\mathbb{R}^N)$. Now, using first (3.85) and then the identity $T_\sigma G(\overline{z})^*_\nu = -(\alpha_\sigma)^{-1}\delta_{\sigma\nu} - \Theta(z)_{\sigma\nu}$, $\sigma, \nu \in \mathcal{J}$, from (3.5) in combination with (3.86), we conclude that

$$T_\sigma \psi = T_\sigma \psi_0 + \sum_{\nu\in\mathcal{J}} T_\sigma G(\overline{z})^*_\nu w_\nu = -(\alpha_\sigma)^{-1} w_\sigma, \qquad \sigma \in \mathcal{J}. \tag{3.90}$$

This means that $w_\sigma = -\alpha_\sigma T_\sigma \psi$, where, by Eq. (2.12), $T_\sigma \psi = \tau \mathscr{K}_\sigma \psi \in H^{1/2}(\mathbb{R}^{N-1})$ because $\mathscr{K}_\sigma$ leaves the space $H^1(\mathbb{R}^N)$ invariant and, by Lemma 2.10, $\tau : H^1(\mathbb{R}^N) \to H^{1/2}(\mathbb{R}^{N-1})$ defines a bounded operator. This proves (3.88) and (3.89) is now immediate from (3.85). □

Before we turn to the proof of Proposition 3.16, we need some preparation for the computation of $(T^+_\sigma - T^-_\sigma)\partial_\sigma \psi$, $\sigma \in \mathcal{I}$. To this end, we want to use the decomposition (3.85) of $\psi \in D(H)$ for fixed $z \in (0,\infty) \cap \rho(H)$. We will see that the contribution of $\psi_0 \in H^2(\mathbb{R}^N)$ vanishes, so it remains to consider the contributions of the vectors $G(z)^*_\nu w_\nu$.

**Lemma 3.20.** *Let $z \in (0,\infty)$ and $\nu \in \mathcal{I}$ be given and suppose that $w_\nu \in \mathfrak{X}_\nu \cap H^{1/2}(\mathbb{R}^{N-1})$. Then $G(z)^*_\nu w_\nu \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ and, for all $\sigma \in \mathcal{I}$,*

$$(T^+_\sigma - T^-_\sigma)\partial_\sigma G(z)^*_\nu w_\nu = -\delta_{\sigma\nu} w_\nu. \tag{3.91}$$

*Remark.* $H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ is the set of all $\varphi \in H^1(\mathbb{R}^N)$ for which the restriction $\varphi \restriction \mathbb{R}^N \setminus \Gamma$ belongs to $H^2(\mathbb{R}^N \setminus \Gamma)$.

*Proof.* Without restriction, we may assume that $\nu = (1,2)$. From Proposition 2.12 (*iii*) we already know that $G(z)^*_\nu w_\nu \in H^1(\mathbb{R}^N)$. To show that $G(z)^*_\nu w_\nu \restriction \mathbb{R}^N \setminus \Gamma$ belongs to $H^2(\mathbb{R}^N \setminus \Gamma)$, it suffices to prove the stronger result that $G(z)^*_\nu w_\nu \restriction \mathbb{R}^N \setminus \Gamma_\nu$ belongs to $H^2(\mathbb{R}^N \setminus \Gamma_\nu)$. To this end, we note that Eq. (1.45) defines an isomorphism $\mathscr{K}_\nu : H^2(\mathbb{R}^N \setminus \Gamma_\nu) \to H^2((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{N-1})$, so it suffices to show that $\mathscr{K}_\nu G(z)^*_\nu w_\nu$ defines a function in $H^2((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{N-1})$. We now derive an explicit expression for $\mathscr{K}_\nu G(z)^*_\nu w_\nu$ in Fourier space. First, we observe that the identities (2.12), (3.51) and $\mathscr{K}_\nu \mathscr{K}^*_\nu = 1$ imply that

$$\mathscr{K}_\nu G(z)^*_\nu = \mathscr{K}_\nu (\tau \mathscr{K}_\nu R_0(z))^* = (\tau(\widetilde{H}_0 + z)^{-1})^*, \tag{3.92}$$

where $\widetilde{H}_0$ is defined by Eq. (3.52). Now, with $p$ and $\underline{P} = (P, p_3, ..., p_N)$ being conjugate to $r$ and $(R, x_3, ..., x_N)$, respectively, and $Q \geq 0$ defined by Eq. (3.54), we find that, for all $\varphi \in \widetilde{\mathfrak{X}}_\nu$,

$$\begin{aligned}\left\langle \varphi \,\middle|\, \left(\tau(\widetilde{H}_0 + z)^{-1}\right)^* w_\nu \right\rangle &= \left\langle \tau(\widetilde{H}_0 + z)^{-1}\varphi \,\middle|\, w_\nu \right\rangle \\ &= \frac{\mu_\nu}{\sqrt{2\pi}} \int \mathrm{d}\underline{P}\, \mathrm{d}p \left(p^2 + \mu_\nu(Q+z)\right)^{-1} \overline{\widehat{\varphi}(p,\underline{P})}\, \widehat{w_\nu}(\underline{P}).\end{aligned} \tag{3.93}$$

From (3.92) and (3.93) we conclude that

$$(\widehat{\mathscr{K}_\nu G(z)^*_\nu w_\nu})(p,\underline{P}) = \frac{\mu_\nu}{\sqrt{2\pi}} \left(p^2 + \mu_\nu(Q+z)\right)^{-1} \widehat{w_\nu}(\underline{P}), \tag{3.94}$$

and after applying the inverse Fourier transform in $r$, it follows from Lemma A.1 $(ii)$ together with the explicit formula (3.56) for the Green's function $G^1_z$, $z > 0$, that

$$(\mathscr{K}_\nu \widehat{G(z)^*_\nu} w_\nu)(r, \underline{P}) = \frac{\sqrt{\mu_\nu}}{2\sqrt{Q+z}} \exp\left(-|r|\sqrt{\mu_\nu(Q+z)}\right) \widehat{w_\nu}(\underline{P}). \tag{3.95}$$

Using the explicit expressions (3.94) and (3.95), it is now straightforward to verify that all (weak) partial derivatives $\partial^\beta \mathscr{K}_\nu G(z)^*_\nu w_\nu$, $|\beta| \le 2$, exist in $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{N-1}$ and define square-integrable functions. We demonstrate this in the case of the partial derivative $\partial_r \partial_{x_i}$, $i \in \{3, ..., N\}$:

$$\begin{aligned}\int \mathrm{d}\underline{P}\, \mathrm{d}p\, |p|^2 |p_i|^2 |(\mathscr{K}_\nu \widehat{G(z)^*_\nu} w_\nu)(p, \underline{P})|^2 &\le \frac{\mu_\nu^2}{2\pi} \int \mathrm{d}\underline{P} \left(\int \mathrm{d}p \left(p^2 + \mu_\nu(Q+z)\right)^{-1}\right) |p_i|^2 |\widehat{w_\nu}(\underline{P})|^2 \\ &= \frac{\mu_\nu^{3/2}}{2} \int \mathrm{d}\underline{P}\, \frac{|p_i|^2 |\widehat{w_\nu}(\underline{P})|^2}{(Q+z)^{1/2}} \\ &\le \frac{(m_i \mu_\nu^3)^{1/2}}{2} \int \mathrm{d}\underline{P}\, |p_i| |\widehat{w_\nu}(\underline{P})|^2 < \infty,\end{aligned}$$

where the last line was obtained from the estimate $|p_i| \le (m_i(Q+z))^{1/2}$ and the integral in the last line is finite because $w_\nu \in H^{1/2}(\mathbb{R}^{N-1})$ by assumption. Using similar estimates for the other partial derivatives (for $\partial_r^2$ one should use (3.95) instead of (3.94)), we conclude that $\mathscr{K}_\nu G(z)^*_\nu w_\nu$ defines a function in $H^2((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{N-1})$. This means that $G(z)^*_\nu w_\nu \restriction \mathbb{R}^N \setminus \Gamma_\nu$ belongs to $H^2(\mathbb{R}^N \setminus \Gamma_\nu)$, and hence $G(z)^*_\nu w_\nu \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$.

It remains to prove Eq. (3.91), and to this end we first assume that $\sigma = (1,2) = \nu$. Then the identity (2.22) for $T^+_\nu \restriction H^1(\Omega^+_\nu)$ and the Definition (1.45) of $\mathscr{K}_\nu$ imply that on $H^2(\Omega^+_\nu)$, $T^+_\nu \partial_\nu = \tau^+ \mathscr{K}_\nu \partial_\nu = (\mu_\nu)^{-1} \tau^+ \partial_r \mathscr{K}_\nu$. Since we already know that $G(z)^*_\nu w_\nu \restriction \mathbb{R}^N \setminus \Gamma_\nu$ belongs to $H^2(\mathbb{R}^N \setminus \Gamma_\nu)$, it follows that $(G(z)^*_\nu w_\nu \restriction \Omega^+_\nu) \in H^2(\Omega^+_\nu)$ and

$$T^+_\nu \partial_\nu G(z)^*_\nu w_\nu = (\mu_\nu)^{-1} \tau^+ \partial_r \mathscr{K}_\nu G(z)^*_\nu w_\nu.$$

To evaluate the trace operator $\tau^+$ on the right side, we use (3.95) to conclude that, for all $r > 0$,

$$\| - w_\nu/2 - (\mu_\nu)^{-1} (\partial_r \mathscr{K}_\nu G(z)^*_\nu w_\nu)(r, \cdot)\|^2 = \frac{1}{4} \int \mathrm{d}\underline{P} \left|1 - \exp\left(-r\sqrt{\mu_\nu(Q+z)}\right)\right|^2 |\widehat{w_\nu}(\underline{P})|^2,$$

where the right side vanishes as $r \to 0+$ by the Lebesgue dominated convergence theorem. This implies that $T^+_\nu \partial_\nu G(z)^*_\nu w_\nu = -w_\nu/2$. Similarly, $T^-_\nu \partial_\nu G(z)^*_\nu w_\nu = w_\nu/2$, so Eq. (3.91) is established for $\sigma = \nu$.

To prove Eq. (3.91) for given $\sigma \neq \nu = (1,2)$, we set $\psi := \partial_\sigma G(z)^*_\nu w_\nu$ and we note that $\psi \restriction \mathbb{R}^N \setminus \Gamma_\nu$ belongs to $H^1(\mathbb{R}^N \setminus \Gamma_\nu)$ because $G(z)^*_\nu w_\nu \restriction \mathbb{R}^N \setminus \Gamma_\nu$ belongs to $H^2(\mathbb{R}^N \setminus \Gamma_\nu)$. This implies that $\varphi\psi \in H^1(\mathbb{R}^N)$ for all $\varphi \in C^\infty_0(\mathbb{R}^N \setminus \Gamma_\nu)$, so the identity (2.18) and the remark after Proposition 2.14 show that

$$(T_\sigma \varphi)(T^+_\sigma \psi) = T^+_\sigma(\varphi\psi) = T_\sigma(\varphi\psi) = T^-_\sigma(\varphi\psi) = (T_\sigma\varphi)(T^-_\sigma \psi). \tag{3.96}$$

Comparing both sides, we conclude that $T^+_\sigma \psi = T^-_\sigma \psi$ in the interior of $\mathrm{supp}(T_\sigma \varphi)$. Next, we note that $\Gamma_\sigma \cap \Gamma_\nu$ is a hyperplane of codimension one in $\Gamma_\sigma$, and it is not hard to show that for almost all $y \in \mathbb{R}^{N-1}$ there exists a function $\varphi \in C^\infty_0(\mathbb{R}^N \setminus \Gamma_\nu)$ such that $T_\sigma \varphi$ does not vanish in some neighborhood of $y$. In view of Eq. (3.96), this means that $T^+_\sigma \psi = T^-_\sigma \psi$ a.e. in $\mathbb{R}^{N-1}$, and hence Eq. (3.91) also holds for $\sigma \neq \nu$. □

*Proof of Proposition 3.16.* Suppose first that $\psi \in D(H)$, so we have to verify that $\psi \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ and that Eq. (3.82) holds for all $\sigma \in \mathcal{I}$. To this end, let $z \in (0, \infty) \cap \rho(H)$ be fixed. Then we know from Proposition 3.19 that

$$\psi = \psi_0 + \sum_{\nu \in \mathcal{J}} G(z)^*_\nu w_\nu, \tag{3.97}$$

where $w_\nu = -\alpha_\nu T_\nu \psi \in H^{1/2}(\mathbb{R}^{N-1})$, $\nu \in \mathcal{J}$, and $\psi_0 \in H^2(\mathbb{R}^N)$. Hence, Lemma 3.20 shows that $G(z)^*_\nu w_\nu \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$, $\nu \in \mathcal{J}$, so Eq. (3.97) implies that $\psi \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$. To prove Eq. (3.82) for all $\sigma \in \mathcal{I}$, we again use the decomposition (3.97). In view of Lemma 3.20, we find that

$$(T^+_\sigma - T^-_\sigma)\partial_\sigma \psi = (T^+_\sigma - T^-_\sigma)\partial_\sigma \psi_0 - \sum_{\nu \in \mathcal{J}} \delta_{\sigma\nu} w_\nu = \begin{cases} -w_\sigma & \text{if } \sigma \in \mathcal{J} \\ 0 & \text{if } \sigma \in \mathcal{I} \setminus \mathcal{J}, \end{cases} \tag{3.98}$$

where we have used that $T^+_\sigma \partial_\sigma \psi_0 = T_\sigma \partial_\sigma \psi_0 = T^-_\sigma \partial_\sigma \psi_0$ because $\partial_\sigma \psi_0 \in H^1(\mathbb{R}^N)$ (see the remark after Proposition 2.14). Now, the identity $w_\sigma = -\alpha_\sigma T_\sigma \psi$ if $\sigma \in \mathcal{J}$ and the fact that $\alpha_\sigma = 0$ if $\sigma \in \mathcal{I} \setminus \mathcal{J}$ show that the right-hand sides of Eqs. (3.98) and (3.82) coincide, so Eq. (3.82) is established for all $\sigma \in \mathcal{I}$. To prove Eq. (3.83), we recall from Propositions 3.18 and 2.12 that $(H+z)\psi = (H_0+z)\psi_0$ and, in the sense of distributions, $(H_0+z)G(z)^*_\nu w_\nu = 0$ in $\mathbb{R}^N \setminus \Gamma_\nu$. Hence, Eq. (3.97) implies that, in the sense of distributions, $(H+z)\psi = (H_0+z)\psi_0 = (H_0+z)\psi$ in $\mathbb{R}^N \setminus \Gamma$, which proves Eq. (3.83).

Conversely, suppose that $\psi \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ satisfies (3.82) for all $\sigma \in \mathcal{I}$. Then we have to show that $\psi \in D(H)$. Let $z \in (0, \infty) \cap \rho(H)$ be fixed and, in accordance with Eqs. (3.88) and (3.89), let $w_\sigma := -\alpha_\sigma T_\sigma \psi \in \mathfrak{X}_\sigma \cap H^{1/2}(\mathbb{R}^{N-1})$, $\sigma \in \mathcal{I}$, and $\psi_0 := \psi - \sum_{\nu \in \mathcal{J}} G(z)^*_\nu w_\nu$. We are going to show that $\psi_0 \in H^2(\mathbb{R}^N)$ and that the conditions (3.85) and (3.86) are satisfied, so Proposition 3.18 then shows that $\psi \in D(H)$. (3.85) is obvious from the definition of $\psi_0$ and, moreover, it follows that $\psi_0 \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ because $\psi \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ by assumption and, by Lemma 3.20, $G(z)^*_\nu w_\nu \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$, $\nu \in \mathcal{J}$. Now, the definitions of $w_\sigma$ and $\psi_0$ imply that

$$T_\sigma \psi_0 = -(\alpha_\sigma)^{-1} w_\sigma - \sum_{\nu \in \mathcal{J}} T_\sigma G(z)^*_\nu w_\nu = (\Theta(z)w)_\sigma, \qquad \sigma \in \mathcal{J},$$

where the second equality used the Definition (3.5) of $\Theta(z)$. This proves (3.86), so it only remains to show that $\psi_0 \in H^2(\mathbb{R}^N)$. For this purpose, we first observe that $\mathbb{R}^N \setminus \Gamma$ is the disjoint union of the sets $\Omega_\pi$ that have been introduced in Eq. (2.19), where $\pi$ runs through all permutations of $\{1, ..., N\}$. Recall from the proof of Proposition 2.14 that each $\Omega_\pi$ is an open and connected set whose boundary is of class $C$, so it follows from [50, Theorem 11.35] that the restriction to $\Omega_\pi$ of functions in $C_0^\infty(\mathbb{R}^N)$ is dense in $H^2(\Omega_\pi)$. Hence, for any permutation $\pi$ we may choose a sequence $\psi_{\pi,n} \in C_0^\infty(\mathbb{R}^N)$, $n \in \mathbb{N}$, that converges, as $n \to \infty$, to $\psi_0 \restriction \Omega_\pi$ in the norm of $H^2(\Omega_\pi)$. We now define a sequence $\psi_n \in H^2(\mathbb{R}^N \setminus \Gamma) \cap C^\infty(\mathbb{R}^N \setminus \Gamma)$, $n \in \mathbb{N}$, by $\psi_n(x) := \psi_{\pi,n}(x)$ iff $x \in \Omega_\pi$, and it follows that, as $n \to \infty$, $\psi_n \to \psi_0$ in the norm of $H^2(\mathbb{R}^N \setminus \Gamma)$. Moreover, we define $H_0' \eta := \sum_{i=1}^N (-\partial_i^2 \eta)/m_i \in L^2(\mathbb{R}^N \setminus \Gamma)$ for $\eta \in D(H_0') = H^2(\mathbb{R}^N \setminus \Gamma)$. Then a version of Green's second identity shows that, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \Gamma} & \overline{\varphi(x)}(H_0'\psi_n)(x) - \overline{(H_0\varphi)(x)}\psi_n(x)\,\mathrm{d}x \\ &= \sum_{\sigma \in \mathcal{I}} \left( \left\langle T_\sigma \varphi \,\middle|\, (T^+_\sigma - T^-_\sigma)\partial_\sigma \psi_n \right\rangle - \left\langle T_\sigma \partial_\sigma \varphi \,\middle|\, (T^+_\sigma - T^-_\sigma)\psi_n \right\rangle \right).\end{aligned}$$

Taking the limit $n \to \infty$, we arrive at

$$\int_{\mathbb{R}^N \setminus \Gamma} \overline{\varphi(x)}(H_0'\psi_0)(x) - \overline{(H_0\varphi)(x)}\psi_0(x)\,\mathrm{d}x = \sum_{\sigma \in \mathcal{I}} \left\langle T_\sigma \varphi \,\middle|\, (T^+_\sigma - T^-_\sigma)\partial_\sigma \psi_0 \right\rangle, \tag{3.99}$$

where we used that $T^+_\sigma \psi_0 = T_\sigma \psi_0 = T^-_\sigma \psi_0$ because $\psi_0 \in H^1(\mathbb{R}^N)$ (see the remark after Proposition 2.14). Next, we use our assumption that $\psi \in H^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N \setminus \Gamma)$ satisfies (3.82) for all

$\sigma \in \mathcal{I}$ in combination with Lemma 3.20 to conclude that, for all $\sigma \in \mathcal{I}$,

$$\begin{aligned}(T_\sigma^+ - T_\sigma^-)\partial_\sigma \psi_0 &= (T_\sigma^+ - T_\sigma^-)\partial_\sigma \psi - \sum_{\nu \in \mathcal{J}} (T_\sigma^+ - T_\sigma^-)\partial_\sigma G(z)_\nu^* w_\nu \\ &= \alpha_\sigma T_\sigma \psi + w_\sigma = 0.\end{aligned}$$

Inserting this in (3.99), we obtain that, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\langle H_0 \varphi \,|\, \psi_0 \rangle = \langle \varphi \,|\, H_0' \psi_0 \rangle$, where $\langle \cdot \,|\, \cdot \rangle$ denotes the scalar product of $L^2(\mathbb{R}^N)$ and $H_0' \psi_0 \in L^2(\mathbb{R}^N \setminus \Gamma)$ is extended by zero to a function in $L^2(\mathbb{R}^N)$. From this we conclude that $\psi_0 \in H^2(\mathbb{R}^N)$ and that $H_0 \psi_0 = H_0' \psi_0$, which completes the proof of Proposition 3.16. □

# 4 From short-range to contact interactions in $d = 2$ dimensions

This section is devoted to the proof of the two-dimensional analog of Theorem 3.1, see Theorem 4.1, below. Moreover, we show that the limit operator $H$ from Theorem 4.1 agrees with the TMS Hamiltonian of Dell'Antonio, Figari and Teta [30]. This section is based on reference [39].

## 4.1 Main result and outline

Unlike the simpler case $d = 1$, a $\delta$-interaction is not a small perturbation, in the form sense, of the free operator $-\Delta : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ in $d \geq 2$ dimensions. Accordingly, the analog of the quadratic form $q$ from Eq. (3.7) is not bounded from below and closed on $H^1(\mathbb{R}^{dN})$ for $d \geq 2$ and $N \geq 2$. This already follows from the fact that the trace operators $T_\sigma : D(T_\sigma) \to \mathfrak{X}_\sigma$ in Corollary 2.11 do not define bounded operators on $H^1(\mathbb{R}^{dN})$ if $d \geq 2$. Therefore, the standard method from Section 3.2, which has allowed us to construct $N$-body Hamiltonians with non-trivial two-body contact interactions in $d = 1$, fails in $d \geq 2$. In $d = 2$ dimensions, a local, translation-invariant and semibounded $N$-body Hamiltonian with non-trivial two-body contact interactions was first constructed by Dell'Antonio, Figari and Teta in 1994 [30]. Their starting point is a suitable regularized quadratic form, where the regularization is achieved by introducing an ultraviolet cutoff in Fourier space. As the cutoff is removed, this regularized quadratic form converges, in the sense of strong and weak Γ-convergence, to a closed and semibounded quadratic form, which is denoted by $\mathrm{F}_\beta$ in [30]. The self-adjoint operator $H_\beta$ that is associated with $\mathrm{F}_\beta$ then describes non-trivial two-body contact interactions among $N \geq 2$ particles. The component $\beta_\sigma \in \mathbb{R}$ of the vector $\beta$ in $H_\beta$ determines the strength of the contact interaction among the particles of the respective pair $\sigma \in \mathcal{I}$. The resolvent of $H_\beta$ can be expressed by a Krein-like formula (see [30, Eqs. (5.12) and (5.13)]), and the domain and the action of $H_\beta$ can be characterized explicitly (see [30, Eqs. (5.3) and (5.4)]). It is pointed out in [30] that this characterization of $D(H_\beta)$ generalizes the condition derived by Ter Martirosyan and Skornyakov in the case $N = 3$, $d = 3$ [55], which was further analyzed in [58, 59]. Therefore, we refer to $H_\beta$ as the TMS Hamiltonian.

The goal of this section is to show that $H_\beta$ is the limit, in the norm resolvent sense, of suitably rescaled Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$. Recall from Eqs. (1.39), (1.40) and (1.29) that in $d = 2$ dimensions we consider Schrödinger operators of the form

$$H_\varepsilon = H_0 + \sum_{\sigma=(i,j)\in\mathcal{I}} g_{\varepsilon,\sigma}\, V_{\sigma,\varepsilon}(x_j - x_i), \qquad \varepsilon > 0, \tag{4.1}$$

where

$$H_0 = \sum_{i=1}^{N} (-\Delta_{x_i}/m_i) \tag{4.2}$$

denotes the kinetic energy operator, $g_{\varepsilon,\sigma} \in \mathbb{R}$ is a coupling constant and

$$V_{\sigma,\varepsilon}(r) = \varepsilon^{-2}\, V_\sigma(r/\varepsilon), \qquad \sigma \in \mathcal{I},\ \varepsilon > 0, \tag{4.3}$$

for some fixed real-valued potential $V_\sigma \in L^1 \cap L^2(\mathbb{R}^2)$ with $V_\sigma(-r) = V_\sigma(r)$ a.e. In particular, $H_\varepsilon$ is self-adjoint on $D(H_\varepsilon) = D(H_0) = H^2(\mathbb{R}^{2N})$. Recall from Eq. (1.41) that for $H_\varepsilon$ to have a non-trivial limit as $\varepsilon \to 0$, the coupling constants $g_{\varepsilon,\sigma}$ must have an asymptotic behavior of the form

$$\frac{1}{g_{\varepsilon,\sigma}} = \mu_\sigma\Big(a_\sigma \ln(\varepsilon) + b_\sigma\Big) + o(1) \qquad (\varepsilon \to 0), \tag{4.4}$$

where $a_\sigma, b_\sigma \in \mathbb{R}$, $a_\sigma > 0$ and the reduced mass $\mu_\sigma$ of the pair $\sigma$ has been factored out. We further assume that all two-body potentials $V_\sigma$ satisfy $\int |r|^{2s} |V_\sigma(r)| \, \mathrm{d}r < \infty$ for some $s > 0$ that is independent of the particular choice of $\sigma$, and that the coefficient $a_\sigma > 0$ in Eq. (4.4) satisfies

$$a_\sigma \geq \frac{1}{2\pi} \int V_\sigma(r) \, \mathrm{d}r. \tag{4.5}$$

While in the one-particle case a condition of the type (4.5) is not needed [6], this condition is necessary for $N \geq 2$: if (4.5) is not satisfied for some pair $\sigma$, then one may expect strong resolvent convergence of $H_\varepsilon$ at best. To see this, we consider a two-particle subsystem with pair potential $V_\sigma$ and reduced mass $\mu_\sigma = 1$. If $0 < a_\sigma < (2\pi)^{-1} \int V_\sigma(r) \, \mathrm{d}r$ and $(g_{\varepsilon,\sigma})^{-1} = a_\sigma \ln(\varepsilon) + b_\sigma$, then the Hamiltonian in the center of mass frame has a negative eigenvalue running towards $-\infty$ as $\varepsilon \to 0$. This follows from a rescaling in $\varepsilon > 0$ in combination with [74, Theorem 3.4]. Due to the HVZ theorem (see, e.g., [70, Theorem XIII.17]) and the center of mass motion, the Hamiltonian $H_\varepsilon$ then has essential spectrum filling the entire real axis in the limit $\varepsilon \to 0$. In view of Proposition 2.3, this is not compatible with norm resolvent convergence towards a semibounded Hamiltonian. If equality holds in (4.5), then, in analogy to the one-particle case described in Section 1.1.1, we shall see that the two-body potential $V_\sigma$ gives rise to a non-trivial contact interaction whose strength is determined by the parameter $b_\sigma$. In the case of inequality there is no contribution from $V_\sigma$. We use $\mathcal{J} \subseteq \mathcal{I}$ to denote the subset of pairs for which equality holds in (4.5).

Under the above hypotheses, Theorem 4.1, below, asserts norm resolvent convergence of $H_\varepsilon$ in the limit $\varepsilon \to 0$. Moreover, Eq. (4.8) yields a Krein-like formula for the resolvent of the limit operator $H$, which is our analog of [30, Eqs. (5.12) and (5.13)]. Before we state this result in detail, we introduce all required spaces and operators. Let the auxiliary Hilbert space $\mathfrak{X}$ be defined by

$$\mathfrak{X} := \bigoplus_{\sigma \in \mathcal{J}} \mathfrak{X}_\sigma, \tag{4.6}$$

where $\mathfrak{X}_\sigma$ has been introduced in Eq. (2.8). For $z \in \rho(H_0)$, let $R_0(z) = (H_0 + z)^{-1}$ for short and let $G(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X})$ be defined in terms of the components $G(z)_\sigma = T_\sigma R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$, $\sigma \in \mathcal{J}$, that have been introduced in Proposition 2.12. An alternative description of $G(z)$ is given by $G(z) = T R_0(z)$, where $T \in \mathscr{L}(H^2(\mathbb{R}^{2N}), \mathfrak{X})$ is defined as in Lemma 2.13. It remains to define the unbounded operator matrix $\Theta(z) : D \subseteq \mathfrak{X} \to \mathfrak{X}$, $z \in \rho(H_0)$, in Eq. (4.8). The off-diagonal contributions are given by

$$\Theta(z)_{\sigma\nu} := -T_\sigma G(\overline{z})^*_\nu, \qquad z \in \rho(H_0), \sigma, \nu \in \mathcal{J}, \sigma \neq \nu, \tag{4.7}$$

where the trace operator $T_\sigma : D(T_\sigma) \subseteq \mathscr{H} \to \mathfrak{X}_\sigma$ is defined by Eq. (2.12) on its maximal, $\sigma$-dependent domain $D(T_\sigma) = \mathscr{K}_\sigma^* D(\tau)$. Here, $\mathscr{K}_\sigma^*$, $\tau$ and $D(\tau)$ are given by Eqs. (1.46), (2.10) and (2.11), respectively. We will see that $\operatorname{Ran} G(\overline{z})^*_\nu \subseteq D(T_\sigma)$ and that $\Theta(z)_{\sigma\nu}$ defines a bounded operator in $\mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$ for $\sigma \neq \nu$. The diagonal parts $\Theta(z)_{\sigma\sigma}$, $\sigma \in \mathcal{J}$, are unbounded operators that are explicitly given by Eqs. (4.51)-(4.54), below, where $\ln(\cdot)$ denotes the principal branch of the logarithm. With these preparations at hand, we now state the main result of this section:

**Theorem 4.1.** *Let $d = 2$, $N \geq 2$ and suppose, for all $\sigma \in \mathcal{I}$, that $V_\sigma \in L^1 \cap L^2(\mathbb{R}^2)$, $V_\sigma(-r) = V_\sigma(r)$ a.e. and that there exists some $s > 0$ such that $\int |r|^{2s} |V_\sigma(r)| \, \mathrm{d}r < \infty$. Let $H_\varepsilon$, $\varepsilon > 0$, be defined by Eqs. (4.1)-(4.5). Then, as $\varepsilon \to 0$, $H_\varepsilon$ converges in the norm resolvent sense to a self-adjoint semibounded operator $H$. For $z \in \rho(H_0) \cap \rho(H)$, the resolvent of $H$ is given by*

$$(H + z)^{-1} = R_0(z) + G(\overline{z})^* \Theta(z)^{-1} G(z), \tag{4.8}$$

*where $\Theta(z) : D \subseteq \mathfrak{X} \to \mathfrak{X}$ is a densely defined and invertible operator, whose domain $D$ is independent of $z$.*

*Remark.* Theorem 4.1 shows that $0 \in \rho(\Theta(z))$ for all $z \in \rho(H_0) \cap \rho(H)$. In fact, it follows from [21, Theorem 2.19] combined with the properties of $\Theta(\cdot)$ from Proposition 4.14, below, that

$$\rho(H_0) \cap \rho(H) = \{z \in \rho(H_0) \,|\, 0 \in \rho(\Theta(z))\}.$$

This is closely related to the variational principle that has been used to prove stability of the Fermi polaron in two [42, 54] and in three dimensions [60], and in our setting this allows us to derive an explicit lower bound for $\sigma(H)$ in Proposition 4.15, below.

The main novelty in Theorem 4.1, compared to previous results of similar type, is that convergence is established in norm resolvent sense. We know from Proposition 2.3 that norm resolvent convergence implies convergence of the spectra, while this is not true for the weaker strong resolvent convergence. In the present case, where $\sigma(H_\varepsilon) = [\Sigma_\varepsilon, \infty)$ for some $\Sigma_\varepsilon \leq 0$, the norm resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ implies that $\Sigma = \lim_{\varepsilon\to 0} \Sigma_\varepsilon$ exists and that $\sigma(H) = [\Sigma, \infty)$. Recently, in a study of the 2d stochastic heat equation, Gu, Quastel and Tsai have derived a result very similar to Theorem 4.1 for $N$ identical particles [46]. In [46] the two-body potentials are compactly supported smooth functions and convergence in strong resolvent sense is established. In the case of bosons, Gu et al. give a formula for the resolvent that is similar to Eq. (4.8). As already indicated at the beginning of this section, TMS Hamiltonians in $d = 2$, like $H$ in Theorem 4.1, have also been described as resolvent limits of $N$-body Hamiltonians, where the regularized two-body contact interaction is an integral operator, rather than a potential, and the regularization is achieved by an ultraviolet cutoff [30, 31, 43] or a reversed heat flow [34]. In these cases the convergence is easier to establish than in the case studied here. Nevertheless, all previous approximation results of this kind in $d = 2$ with $N \geq 2$ particles establish *strong* resolvent convergence only.

Following the line of argument from the proof of Proposition 3.18, the resolvent formula from Eq. (4.8) implies the following characterization of the domain and the action of our Hamiltonian:

**Corollary 4.2.** *Under the hypotheses of Theorem 4.1, a vector $\psi \in \mathscr{H}$ belongs to $D(H)$ if and only if the following holds: For some (and hence all) $z \in \rho(H_0) \cap \rho(H)$ there exist $\psi_0 \in H^2(\mathbb{R}^{2N})$ and $w \in D$ such that*

$$\psi = \psi_0 + G(\overline{z})^* w \tag{4.9}$$

*and*

$$T\psi_0 = \Theta(z)w. \tag{4.10}$$

*The vectors $\psi_0$ and $w$ are uniquely determined by $\psi \in D(H)$ and $z \in \rho(H_0) \cap \rho(H)$, and*

$$(H + z)\psi = (H_0 + z)\psi_0. \tag{4.11}$$

The conditions (4.9) and (4.10) can be considered as an abstract, operator theoretic version of the TMS condition [30, 42] and Corollary 4.2 is our analog of [30, Eqs. (5.3) and (5.4)]. For $\psi \in \operatorname{Ker} T$, (4.9) and (4.10) are satisfied with $\psi_0 = \psi \in H^2(\mathbb{R}^{2N})$ and $w = 0$, and since $C_0^\infty(\mathbb{R}^{2N} \setminus \Gamma) \subseteq \operatorname{Ker} T$ by Lemma 2.13, this also shows that $H$ is a self-adjoint extension of $H_0 \restriction C_0^\infty(\mathbb{R}^{2N} \setminus \Gamma)$. Moreover, by Proposition 4.16, below, $H$ is a local operator that is invariant under all Euclidean isometries of $\mathbb{R}^2$. In particular, this confirms that $H$ is an $N$-particle Hamiltonian with contact interactions that has the Properties **(i)** – **(iii)** from Section 1.2.2.

The outline of this section is as follows: The auxiliary operators that are needed for the proof of Theorem 4.1 are introduced in Section 4.2, where the strategy of the proof is also briefly sketched. Sections 4.3 and 4.4 provide all preparations needed for the proof of Theorem 4.1, which is given in Section 4.5. In addition, a lower bound for $\sigma(H)$ and Proposition 4.16 are established in Section 4.5. Finally, in Section 4.6, we explicitly compute the quadratic form of $H$ and we show that it agrees, in the case where all masses are equal to one, with the quadratic form of the TMS Hamiltonian $H_\beta$ for suitable $\beta$ [30].

## 4.2 Auxiliary operators and strategy of the proof

In this section we define the most important auxiliary operators that are needed for the proof of Theorem 4.1 and we explain our general strategy.

The proof of Theorem 4.1 is based on the generalized Konno-Kuroda formula (1.53) for the resolvent of $H_\varepsilon$. In $d = 2$ dimensions the logarithmic divergence of $(g_{\varepsilon,\sigma})^{-1}$ in the limit $\varepsilon \to 0$, which is prescribed by Eq. (4.4), is indispensable for the convergence of the contributions $[(1 + g_\varepsilon \phi_\varepsilon(z))^{-1}]_{\sigma\nu}\, g_{\varepsilon,\nu}$. Recall from the sentence between Eqs. (1.52) and (1.53) that $g_\varepsilon \in \mathscr{L}(\widetilde{\mathfrak{X}})$ is given by the diagonal matrix with entries $(g_\varepsilon)_{\sigma\nu} = g_{\varepsilon,\sigma}\delta_{\sigma\nu}$, $\sigma, \nu \in \mathcal{I}$, where $\widetilde{\mathfrak{X}}$ denotes the auxiliary Hilbert space from Eqs. (1.43) and (1.44). We now rewrite the Konno-Kuroda formula in such a way that the dependence on $(g_{\varepsilon,\sigma})^{-1}$ becomes more evident. Since Eq. (4.4) with $a_\sigma > 0$ implies that $g_{\varepsilon,\sigma} < 0$ for small enough $\varepsilon > 0$, we may assume, without loss of generality, that $g_{\varepsilon,\sigma} \neq 0$, and hence that $(g_\varepsilon)^{-1} \in \mathscr{L}(\widetilde{\mathfrak{X}})$ exists. Then the identity

$$\left[(1 + g_\varepsilon \phi_\varepsilon(z))^{-1}\right]_{\sigma\nu} g_{\varepsilon,\nu} = \left[\Big((g_\varepsilon)^{-1} + \phi_\varepsilon(z)\Big)^{-1}\right]_{\sigma\nu}, \qquad \varepsilon > 0,\ \sigma, \nu \in \mathcal{I}$$

allows us to rewrite the Konno-Kuroda formula (1.53) in the form

$$(H_\varepsilon + z)^{-1} = R_0(z) - \sum_{\sigma,\nu \in \mathcal{I}} (A_{\varepsilon,\sigma} R_0(\overline{z}))^* \left[\Lambda_\varepsilon(z)^{-1}\right]_{\sigma\nu} B_{\varepsilon,\nu} R_0(z), \qquad z \in \rho(H_\varepsilon) \cap \rho(H_0), \tag{4.12}$$

where the bounded operator $\Lambda_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ is defined in terms of the components

$$\Lambda_\varepsilon(z)_{\sigma\nu} := (g_{\varepsilon,\sigma})^{-1}\delta_{\sigma\nu} + \phi_\varepsilon(z)_{\sigma\nu} \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma), \qquad z \in \rho(H_0), \sigma, \nu \in \mathcal{I}, \tag{4.13}$$

and $\Lambda_\varepsilon(z)$ is invertible if and only if $z \in \rho(H_\varepsilon) \cap \rho(H_0)$.

To prove norm resolvent convergence of $H_\varepsilon$ in the limit $\varepsilon \to 0$, we prove that all contributions to the right side of Eq. (4.12) have suitable limits as $\varepsilon \to 0$. In Section 4.3 we prove that

$$\lim_{\varepsilon \to 0} A_{\varepsilon,\sigma} R_0(z) = A_\sigma R_0(z), \qquad z \in \rho(H_0), \sigma \in \mathcal{I} \tag{4.14}$$

in $\mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$, where

$$A_\sigma R_0(z)\psi = v_\sigma \otimes (G(z)_\sigma \psi), \qquad \psi \in \mathscr{H}. \tag{4.15}$$

This also implies that $B_{\varepsilon,\nu} R_0(z) \to J_\nu A_\nu R_0(z)$ as $\varepsilon \to 0$.

The hard part, which is even non-trivial in the one-particle case, is the convergence of the middle part $\Lambda_\varepsilon(z)^{-1}$. Our analysis is based on the decomposition

$$\Lambda_\varepsilon(z) = \Lambda_\varepsilon(z)_{\text{diag}} + \Lambda_\varepsilon(z)_{\text{off}}$$

into diagonal and off-diagonal parts of the operator matrix from Eq. (4.13). In Section 4.4 we show that both $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}$ and $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}\Lambda_\varepsilon(z)_{\text{off}}$ have limits as $\varepsilon \to 0$, provided that $z > 0$ is large enough. The diagonal parts $\phi_\varepsilon(z)_{\sigma\sigma}$ contain a divergent contribution that must be canceled by the logarithmic divergence of $(g_{\varepsilon,\sigma})^{-1}$ in the limit $\varepsilon \to 0$. It turns out that $((g_{\varepsilon,\sigma})^{-1} + \phi_\varepsilon(z)_{\sigma\sigma})^{-1}$ has a vanishing limit unless $\sigma \in \mathcal{J}$, i.e. $\int V_\sigma(r)\,\mathrm{d}r = 2\pi a_\sigma$. In the end, we arrive at

$$\lim_{\varepsilon \to 0} \left[\Lambda_\varepsilon(z)^{-1}\right]_{\sigma\nu} = \begin{cases} -\dfrac{|u_\sigma\rangle\langle v_\nu|}{\langle u_\sigma \,|\, v_\sigma\rangle \langle u_\nu \,|\, v_\nu\rangle} \otimes (\Theta(z)^{-1})_{\sigma\nu} & \text{if } \sigma, \nu \in \mathcal{J} \\ 0 & \text{else} \end{cases} \tag{4.16}$$

for large enough $z > 0$, where $\Theta(z)$ is a densely defined and invertible operator in $\mathfrak{X}$. Combining (4.16) with (4.14) and (4.15), it follows that the expression on the right-hand side of Eq. (4.12) has the limit from Eq. (4.8). In analogy to the one-dimensional case, a standard argument then shows that Eq. (4.8) defines the resolvent of a self-adjoint operator $H$ for all $z \in \rho(H_0) \cap \rho(H)$.

## 4.3 The limit of $A_{\varepsilon,\sigma}R_0(z)$

In this section we prove in $d = 2$ that the operators $A_{\varepsilon,\sigma}R_0(z) \in \mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$, $z \in \rho(H_0)$, have suitable limits as $\varepsilon \to 0$. Although the proofs of Lemma 4.3 and Corollary 4.4, below, are very similar to their one-dimensional counterparts from Section 3.3, we recall some of the identities and estimates needed for the proof.

Recall from Eq. (1.48) that

$$A_{\varepsilon,\sigma} = (v_\sigma \otimes 1)\varepsilon^{-1}U_\varepsilon \mathscr{K}_\sigma, \qquad \varepsilon > 0,\ \sigma \in \mathcal{I}, \tag{4.17}$$

where $U_\varepsilon$ and $\mathscr{K}_\sigma$ are defined by Eqs. (1.47) and (1.45), respectively. This means that proving convergence of $A_{\varepsilon,\sigma}$ reduces to a problem in $L^2(\mathbb{R}^2)$, which we solve in the next lemma. For a given pair $\sigma \in \mathcal{I}$, let $V = V_\sigma$, $v = v_\sigma$ and let $|v\rangle\langle\delta| : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ denote the rank-one operator defined by

$$|v\rangle\langle\delta|\,\psi := \psi(0)v, \qquad \psi \in H^2(\mathbb{R}^2),$$

where $\psi(0) := (2\pi)^{-1}\int \widehat{\psi}(p)\,\mathrm{d}p$ (note that $\widehat{\psi} \in L^1(\mathbb{R}^2)$). Then the following holds:

**Lemma 4.3.** *If $V \in L^1(\mathbb{R}^2)$ and $v = |V|^{1/2}$, then*

$$v\varepsilon^{-1}U_\varepsilon \to |v\rangle\langle\delta| \qquad (\varepsilon \to 0)$$

*in the norm of $\mathscr{L}(H^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$. If, in addition, $\int |r|^{2s}\,|V(r)|\,\mathrm{d}r < \infty$ for some $s \in (0,1)$, then the rate of convergence is at least as good as $O(\varepsilon^s)$.*

*Proof.* It is well-known that the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow C^{0,s}(\mathbb{R}^2)$ exists and is continuous for $s \in (0,1)$ (see, e.g., [50, Theorem 12.55]). Hence, for any $s \in (0,1)$ there exists $c_s > 0$ such that, for all $\psi \in H^2(\mathbb{R}^2)$ and almost all $r \in \mathbb{R}^2$,

$$|v(r)|\,|\psi(\varepsilon r) - \psi(0)| \le |v(r)|\,c_s|\varepsilon r|^s\|\psi\|_{H^2}.$$

Squaring and integrating both sides proves the lemma in the case of $\int |r|^{2s}\,|V(r)|\,\mathrm{d}r < \infty$. For general $V \in L^1(\mathbb{R}^2)$, a $\delta/3$-argument, which is very similar to the one from the proof of Lemma 3.8, then proves the lemma. □

Observe that the coordinate transformation $\mathscr{K}_\sigma$ from Eq. (1.45) defines a bounded operator in $H^2(\mathbb{R}^{2N})$. Hence, the Definitions (2.12) and (2.10) of $T_\sigma$ and $\tau$, respectively, imply that $v_\sigma \otimes (T_\sigma\psi) = (|v_\sigma\rangle\langle\delta| \otimes 1)\mathscr{K}_\sigma\psi$ for all $\psi \in H^2(\mathbb{R}^{2N})$. Comparing this with the Definition (4.17) of $A_{\varepsilon,\sigma}$, we see that Lemma 4.3 implies that $\lim_{\varepsilon\to 0} A_{\varepsilon,\sigma} = A_\sigma$ in $\mathscr{L}(H^2(\mathbb{R}^{2N}), \widetilde{\mathfrak{X}}_\sigma)$, where $A_\sigma\psi := v_\sigma \otimes (T_\sigma\psi)$ for $\psi \in H^2(\mathbb{R}^{2N})$. If, in addition, $\int |r|^{2s}\,|V_\sigma(r)|\,\mathrm{d}r < \infty$ for some $s \in (0,1)$, then the rate of convergence is at least as good as $O(\varepsilon^s)$. This proves the following corollary:

**Corollary 4.4.** *Let $z \in \rho(H_0)$, $\sigma \in \mathcal{I}$ and suppose that $V_\sigma \in L^1(\mathbb{R}^2)$. Then, as $\varepsilon \to 0$,*

$$A_{\varepsilon,\sigma}R_0(z) \to A_\sigma R_0(z)$$

*in $\mathscr{L}(\mathscr{H}, \widetilde{\mathfrak{X}}_\sigma)$, where*

$$A_\sigma R_0(z)\psi = v_\sigma \otimes (G(z)_\sigma\psi), \qquad \psi \in \mathscr{H}, \tag{4.18}$$

*and $G(z)_\sigma = T_\sigma R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$. If, in addition, $\int |r|^{2s}\,|V_\sigma(r)|\,\mathrm{d}r < \infty$ for some $s \in (0,1)$, then $\|(A_{\varepsilon,\sigma} - A_\sigma)R_0(z)\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

## 4.4 Convergence of $\Lambda_\varepsilon(z)^{-1}$

The goal of this section is to show that the limit $\lim_{\varepsilon\to 0}\Lambda_\varepsilon(z)^{-1}$ exists for all large enough $z > 0$, provided that, for all $\sigma \in \mathcal{I}$, $V_\sigma \in L^1 \cap L^2(\mathbb{R}^2)$ and $\int |r|^{2s} |V_\sigma(r)| \,\mathrm{d}r < \infty$ for some $s > 0$. As explained at the end of Section 4.2, our proof is based on a decomposition of $\Lambda_\varepsilon(z)$ into diagonal and off-diagonal parts that we now define.

Recall from Eq. (4.13) that, for given $\varepsilon, z > 0$, the operator matrix $\Lambda_\varepsilon(z) \in \mathscr{L}(\widetilde{\mathfrak{X}})$ has the components

$$\Lambda_\varepsilon(z)_{\sigma\nu} = (g_{\varepsilon,\sigma})^{-1}\delta_{\sigma\nu} + \phi_\varepsilon(z)_{\sigma\nu}, \qquad \sigma,\nu \in \mathcal{I},$$

where, by Eq. (1.52),

$$\phi_\varepsilon(z)_{\sigma\nu} = B_{\varepsilon,\sigma} R_0(z)(A_{\varepsilon,\nu})^* \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma). \tag{4.19}$$

Using the Definitions (1.48) and (1.49) of $A_{\varepsilon,\nu}$ and $B_{\varepsilon,\sigma}$, respectively, leads to the explicit formula

$$\phi_\varepsilon(z)_{\sigma\nu} = \varepsilon^{-2}\,(u_\sigma \otimes 1)\, U_\varepsilon \mathscr{K}_\sigma R_0(z) \mathscr{K}_\nu^* U_\varepsilon^* (v_\nu \otimes 1). \tag{4.20}$$

We shall see that for fixed $z > 0$ the norm of the off-diagonal parts $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$, is uniformly bounded in $\varepsilon > 0$, whereas the asymptotics of the coupling constant $g_{\varepsilon,\sigma}$ from Eq. (4.4) cancels the singular part in the diagonal contributions $\phi_\varepsilon(z)_{\sigma\sigma}$ if and only if $a_\sigma = \int V_\sigma(r)\,\mathrm{d}r/(2\pi) > 0$. Therefore, we write the operator matrix $\Lambda_\varepsilon(z)$ in the form

$$\Lambda_\varepsilon(z) = \Lambda_\varepsilon(z)_{\mathrm{diag}} + \Lambda_\varepsilon(z)_{\mathrm{off}}, \tag{4.21}$$

where the diagonal part $\Lambda_\varepsilon(z)_{\mathrm{diag}}$ and the off-diagonal part $\Lambda_\varepsilon(z)_{\mathrm{off}}$ are defined in terms of the components $\Lambda_\varepsilon(z)_{\sigma\sigma}$ and $\Lambda_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$, respectively. That is

$$\big(\Lambda_\varepsilon(z)_{\mathrm{diag}}\big)_{\sigma\nu} = \Big((g_{\varepsilon,\sigma})^{-1} + \phi_\varepsilon(z)_{\sigma\sigma}\Big)\,\delta_{\sigma\nu}, \qquad \sigma,\nu \in \mathcal{I}, \tag{4.22}$$

and

$$\big(\Lambda_\varepsilon(z)_{\mathrm{off}}\big)_{\sigma\nu} = \phi_\varepsilon(z)_{\sigma\nu}(1 - \delta_{\sigma\nu}), \qquad \sigma,\nu \in \mathcal{I}. \tag{4.23}$$

Section 4.4.1 is devoted to $\Lambda_\varepsilon(z)_{\mathrm{diag}}$ and Section 4.4.2 is devoted to $\Lambda_\varepsilon(z)_{\mathrm{off}}$.

### 4.4.1 The limit of $(\Lambda_\varepsilon(z)_{\mathbf{diag}})^{-1}$

We first show that $\Lambda_\varepsilon(z)_{\mathrm{diag}}$ is invertible for small enough $\varepsilon > 0$ and large enough $z > 0$ and then we compute the limit $\lim_{\varepsilon\to 0}(\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1}$. This can be done for each component $\Lambda_\varepsilon(z)_{\sigma\sigma} = (g_{\varepsilon,\sigma})^{-1} + \phi_\varepsilon(z)_{\sigma\sigma}$ separately and without loss of generality we may choose $\sigma = (1,2)$. As in Section 3.5.1, we now drop the index $\sigma$ for brevity: $V = V_{(1,2)}$, $v = v_{(1,2)}$, $\mu = \mu_{(1,2)}$, $g_\varepsilon = g_{\varepsilon,(1,2)}$, $\phi_\varepsilon(z) = \phi_\varepsilon(z)_{(1,2)(1,2)}$ etc. In particular, this means that $g_\varepsilon$ and $\phi_\varepsilon(z)$ are used as shorthand notations for the $(1,2)(1,2)$-components of the corresponding operators in $\mathscr{L}(\widetilde{\mathfrak{X}})$.

First, we recall from Eqs. (3.53) and (3.54) that in $d = 2$ dimensions $\phi_\varepsilon(z)$ acts pointwise in $\underline{P} = (P, p_3, ..., p_N)$ by the operator

$$\begin{aligned}\phi_\varepsilon(z,\underline{P}) &= \varepsilon^{-2}\mu\, u\, U_\varepsilon(-\Delta_r + \mu(Q+z))^{-1}U_\varepsilon^* v\\ &= \mu\, u\,(-\Delta_r + \varepsilon^2\mu(Q+z))^{-1}\, v \;\in \mathscr{L}(L^2(\mathbb{R}^2)),\end{aligned} \tag{4.24}$$

where

$$Q = \frac{P^2}{m_1+m_2} + \sum_{i=3}^{N}\frac{p_i^2}{m_i} \tag{4.25}$$

and the scaling properties of the Laplace operator w.r.t. the unitary scaling $U_\varepsilon$ have been used to obtain the second line of Eq. (4.24). As the resolvent in Eq. (4.24) is divergent in the limit $\varepsilon \to 0$, the first step in the analysis of $\lim_{\varepsilon\to 0}(\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1}$ is to study the asymptotic behavior of $\phi_\varepsilon(z, \underline{P})$ for small $\varepsilon > 0$. A similar but less specific version of the following lemma can be found in [74, Proposition 3.2].

**Lemma 4.5.** *Suppose that $V \in L^1 \cap L^2(\mathbb{R}^2)$ satisfies $\int |r|^{2s} |V(r)| \,\mathrm{d}r < \infty$ for some $s \in (0, 2)$. Then there exists a constant $C = C(s, V) > 0$ such that*

$$\forall \alpha > 0: \qquad \left\| u\,(-\Delta + \alpha)^{-1}\, v + (2\pi)^{-1}\left[ (\ln(\sqrt{\alpha}/2) + \gamma)\, |u\rangle\langle v| + L \right] \right\|_{\mathrm{HS}} \le C\alpha^{s/2}, \tag{4.26}$$

$$\forall \alpha \ge 1: \qquad \left\| u\,(-\Delta + \alpha)^{-1}\, v \right\|_{\mathrm{HS}} \le C, \tag{4.27}$$

*where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm in $L^2(\mathbb{R}^2)$, $\gamma$ is the Euler–Mascheroni constant and $L$ is a Hilbert-Schmidt operator in $L^2(\mathbb{R}^2)$ that is defined in terms of the kernel*

$$u(r)\, \ln(|r - r'|)\, v(r'), \qquad r \ne r'.$$

*Proof.* First, we observe that $u(-\Delta + \alpha)^{-1} v$, $\alpha > 0$, has the integral kernel

$$u(r)\, G_\alpha(r - r')\, v(r'), \tag{4.28}$$

where $G_\alpha = G^2_\alpha$ denotes the Green's function of $-\Delta + \alpha : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. By [6, Equation (5.15), p. 99], we have the explicit description

$$G_\alpha(x) = G_1(\sqrt{\alpha}x) = \frac{i}{4} H_0^{(1)}(i\sqrt{\alpha}|x|), \qquad 0 \ne x \in \mathbb{R}^2, \tag{4.29}$$

where $H_0^{(1)}(\cdot)$ denotes the Hankel function of first kind and order zero. The asymptotics of $H_0^{(1)}(iy)$ for small $y > 0$ is well-known (see, e.g., [1, Chapter 9.1]):

$$iH_0^{(1)}(iy) = -2\pi^{-1}\left(\ln(y/2) + \gamma\right) + O(y^2|\ln y|) \qquad (y \to 0). \tag{4.30}$$

As $G_1$ is smooth in $\mathbb{R}^2 \setminus \{0\}$ and exponentially decaying as $|x| \to \infty$ (see Lemma A.4), we conclude from (4.29) and (4.30) that for some constant $\lambda = \lambda(s) > 0$,

$$\left| G_1(x) + (2\pi)^{-1}\left(\ln(|x|/2) + \gamma\right) \right| \le \lambda |x|^s, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \tag{4.31}$$

Using again Eq. (4.29), it follows that

$$\left| G_\alpha(x) + (2\pi)^{-1}\left(\ln(\sqrt{\alpha}|x|/2) + \gamma\right) \right| \le \lambda \alpha^{s/2} |x|^s$$

uniformly in $\alpha > 0$ and $x \in \mathbb{R}^2 \setminus \{0\}$. Therefore, the estimate (4.26) follows from

$$\begin{aligned}
&\left\| u\,(-\Delta + \alpha)^{-1}\, v + (2\pi)^{-1}\left[ (\ln(\sqrt{\alpha}/2) + \gamma)\, |u\rangle\langle v| + L \right] \right\|^2_{\mathrm{HS}} \\
&\quad = \int \mathrm{d}r\, \mathrm{d}r'\; |u(r)|^2 \left| G_\alpha(r - r') + (2\pi)^{-1}\left[ \ln(\sqrt{\alpha}|r - r'|/2) + \gamma \right] \right|^2 |v(r')|^2 \\
&\quad \le \lambda^2 \alpha^s \int \mathrm{d}r\, \mathrm{d}r'\; |V(r)|\, |r - r'|^{2s}\, |V(r')| \\
&\quad \le 2^{2s+1} \lambda^2 \alpha^s \|V\|_{L^1} \int |r|^{2s} |V(r)|\, \mathrm{d}r,
\end{aligned}$$

where the elementary inequality $(a+b)^{2s} \le 2^{2s}(a^{2s} + b^{2s})$, $a, b \ge 0$, was used for the last inequality. To show that $L$ indeed defines a Hilbert-Schmidt operator, we note that $\|L\|^2_{\mathrm{HS}} = I_1 + I_2$, where

$$I_1 = \int\limits_{|r|\le 1} \mathrm{d}r\; (\ln|r|)^2 \int \mathrm{d}r'\; |V(r + r')||V(r')| \le 2\pi \|V\|^2 \int_0^1 t\,(\ln t)^2\, \mathrm{d}t < \infty$$

by the Cauchy-Schwarz inequality, and

$$
\begin{aligned}
I_2 &= \int\limits_{|r|\geq 1} \mathrm{d}r\, (\ln|r|)^2 \int \mathrm{d}r'\, |V(r+r')||V(r')| \\
&\leq \sup_{t\geq 1} \left(t^{-2s}(\ln t)^2\right) \int \mathrm{d}r\, \mathrm{d}r'\, |r|^{2s}|V(r+r')||V(r')| \\
&= \sup_{t\geq 1} \left(t^{-2s}(\ln t)^2\right) \int \mathrm{d}r\, \mathrm{d}r'\, |r-r'|^{2s}|V(r)||V(r')| \\
&\leq 2^{2s+1} \sup_{t\geq 1} \left(t^{-2s}(\ln t)^2\right) \|V\|_{L^1} \int |r|^{2s}\, |V(r)|\, \mathrm{d}r < \infty.
\end{aligned}
$$

It remains to prove (4.27). By Lemma A.1 $(v)$, the function $\alpha \to G_\alpha(x)$ is strictly monotonically decreasing in $\alpha > 0$ for fixed $x \in \mathbb{R}^2 \setminus \{0\}$. Therefore, the Cauchy-Schwarz inequality and the fact that $G_1 \in L^2(\mathbb{R}^2)$ (cf. Lemma A.1 $(iv)$) yield the estimate

$$
\left\|u\,(-\Delta+\alpha)^{-1}\,v\right\|_{\mathrm{HS}}^2 \leq \int \mathrm{d}r\,\mathrm{d}r'\, |V(r)|\, |G_1(r-r')|^2\, |V(r')| \leq \|V\|^2\|G_1\|^2, \qquad \alpha \geq 1,
$$

where the right-hand side is finite and independent of $\alpha$. □

In Lemma 4.6, below, we show that $(g_\varepsilon)^{-1} + \phi_\varepsilon(z)$ is invertible for large enough $z > 0$ and small enough $\varepsilon > 0$. It is here, where the asymptotics of $g_\varepsilon$ from Eq. (4.4), namely

$$
\frac{1}{g_\varepsilon} = \mu\,(a\ln(\varepsilon) + b) + o(1) \qquad (\varepsilon \to 0) \tag{4.32}
$$

for some $a > 0$, plays a crucial role.

**Lemma 4.6.** *Let the hypotheses of Lemma 4.5 be satisfied and suppose that* (4.32) *holds for some* $a, b \in \mathbb{R}$, $a > 0$. *Then there exist* $\varepsilon_0 > 0$ *and* $z_0 \geq \mu^{-1}$ *such that* $(g_\varepsilon)^{-1} + \phi_\varepsilon(z, \underline{P})$ *is invertible for all* $\underline{P} \in \mathbb{R}^{2(N-1)}$, $z \in (z_0, \infty)$ *and* $\varepsilon \in (0, \varepsilon_0)$. *Moreover,*

$$
\left\|\left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z, \underline{P})\right)^{-1}\right\| \leq \widetilde{C} \begin{cases} |\ln\varepsilon|^{-1} & \text{if } \int V(r)\,\mathrm{d}r/(2\pi) < a, \\ \max\left\{|\ln\varepsilon|^{-1}, \ln(\mu(Q+z))^{-1}\right\} & \text{if } \int V(r)\,\mathrm{d}r/(2\pi) = a, \end{cases} \tag{4.33}
$$

*where the constant* $\widetilde{C} > 0$ *is independent of* $\underline{P} \in \mathbb{R}^{2(N-1)}$, $z \in (z_0, \infty)$ *and* $\varepsilon \in (0, \varepsilon_0)$.

*Proof.* Let $C > 0$ denote the constant from Lemma 4.5. In view of (4.32), there exists $\varepsilon_0 \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $|g_\varepsilon \mu C| \leq 1/2$ and $|g_\varepsilon| \leq \lambda |\ln\varepsilon|^{-1}$, where the constant $\lambda > 0$ is independent of $\varepsilon$.

If $\alpha_\varepsilon := \varepsilon^2\mu(Q+z) \geq 1$, then it follows from (4.24) and (4.27) that

$$
\|g_\varepsilon\phi_\varepsilon(z, \underline{P})\| \leq |g_\varepsilon|\mu \left\|u\,(-\Delta+\alpha_\varepsilon)^{-1}\,v\right\| \leq |g_\varepsilon\mu C| \leq \frac{1}{2}.
$$

Hence, $(g_\varepsilon)^{-1} + \phi_\varepsilon(z, \underline{P})$ is invertible and, for all $\varepsilon \in (0, \varepsilon_0)$,

$$
\left\|\left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z, \underline{P})\right)^{-1}\right\| = \left\|g_\varepsilon\,(1 + g_\varepsilon\phi_\varepsilon(z, \underline{P}))^{-1}\right\| \leq 2\lambda|\ln\varepsilon|^{-1}.
$$

This establishes (4.33) in the case of $\alpha_\varepsilon \geq 1$.

In all the following, $\alpha_\varepsilon = \varepsilon^2\mu(Q+z) < 1$. Then it follows from (4.24) and (4.26) that

$$
\phi_\varepsilon(z, \underline{P}) = -\frac{\mu}{4\pi}\ln(\alpha_\varepsilon)\,|u\rangle\,\langle v| + O(1), \tag{4.34}
$$

where the norm of the operator $O(1)$ is uniformly bounded in $\underline{P} \in \mathbb{R}^{2(N-1)}$, $z > 0$ and $\varepsilon > 0$ as long as $\alpha_\varepsilon < 1$. To compute the inverse of $(g_\varepsilon)^{-1} + \phi_\varepsilon(z, \underline{P})$, we apply the identity (cf. [6, Eq. (1.3.47), p. 33])

$$(1 + S + \beta\, |\widetilde{\eta}\rangle\, \langle\eta|)^{-1} = (1 + S)^{-1} - \frac{(1 + S)^{-1}\, |\widetilde{\eta}\rangle\, \langle\eta|\, (1 + S)^{-1}}{\beta^{-1} + \langle\eta\, |\, (1 + S)^{-1}\widetilde{\eta}\rangle}, \tag{4.35}$$

which is valid under the assumption that $\beta \in \mathbb{C}$, $\widetilde{\eta}, \eta \in \mathcal{H}$, $S, (1 + S)^{-1} \in \mathscr{L}(\mathcal{H})$ and $\beta^{-1} + \langle\eta\, |\, (1 + S)^{-1}\widetilde{\eta}\rangle \neq 0$ in some separable Hilbert space $\mathcal{H}$. This identity (with $S = 0$) and the fact that $|g_\varepsilon| = O(|\ln\varepsilon|^{-1})$ yield

$$\begin{aligned} \left(\frac{1}{g_\varepsilon} - \frac{\mu}{4\pi} \ln(\alpha_\varepsilon)\, |u\rangle\, \langle v|\right)^{-1} &= g_\varepsilon \left(1 - \frac{g_\varepsilon \mu}{4\pi} \ln(\alpha_\varepsilon)\, |u\rangle\, \langle v|\right)^{-1} \\ &= -\frac{g_\varepsilon^2}{f(\underline{P}, z, \varepsilon)}\, |u\rangle\, \langle v| + O(|\ln\varepsilon|^{-1}), \end{aligned} \tag{4.36}$$

provided that

$$f(\underline{P}, z, \varepsilon) := -\frac{4\pi}{\mu \ln(\alpha_\varepsilon)} + g_\varepsilon \int V(r)\, \mathrm{d}r \neq 0.$$

To derive a lower bound for $f(\underline{P}, z, \varepsilon)$, we consider the two cases $\int V(r)\, \mathrm{d}r/(2\pi) < a$ and $\int V(r)\, \mathrm{d}r/(2\pi) = a$ separately. In both cases we assume $z \in (z_0, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$ for some $z_0 \geq \mu^{-1}$ and $\varepsilon_0 \in (0, 1)$, which we will fix below. This implies that $\mu(Q + z) > 1$ and hence

$$0 < -\ln(\alpha_\varepsilon) \leq 2|\ln\varepsilon|. \tag{4.37}$$

If $\int V(r)\, \mathrm{d}r/(2\pi) < a$, then we use (4.32) and (4.37) to derive the lower bound

$$f(\underline{P}, z, \varepsilon) \geq (\mu|\ln\varepsilon|)^{-1} \left(2\pi - \frac{1}{a} \int V(r)\, \mathrm{d}r\right) + O((\ln\varepsilon)^{-2}) \qquad (\varepsilon \to 0). \tag{4.38}$$

Hence, there exists $\varepsilon_0 \in (0, 1)$ such that $f(\underline{P}, z, \varepsilon) > 0$ for all $\underline{P} \in \mathbb{R}^{2(N-1)}$, $z \in (\mu^{-1}, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$ with $\alpha_\varepsilon < 1$. Inserting (4.38) in (4.36) and using $g_\varepsilon^2 = O((\ln\varepsilon)^{-2})$ shows that

$$\left\| \left(\frac{1}{g_\varepsilon} - \frac{\mu}{4\pi} \ln(\alpha_\varepsilon)\, |u\rangle\, \langle v|\right)^{-1} \right\| = O(|\ln\varepsilon|^{-1}) \qquad (\varepsilon \to 0).$$

In view of (4.34), this proves (4.33) for $\int V(r)\, \mathrm{d}r/(2\pi) < a$ and $\alpha_\varepsilon < 1$.

If $\int V(r)\, \mathrm{d}r/(2\pi) = a$, then we again use (4.32) and (4.37) to derive a more refined version of (4.38):

$$\begin{aligned} |\ln\varepsilon|^2 |f(\underline{P}, z, \varepsilon)| &= \frac{|\ln\varepsilon|^2}{|\ln\alpha_\varepsilon|} \left|4\pi\mu^{-1} - g_\varepsilon \ln(\alpha_\varepsilon) \int V(r)\, \mathrm{d}r\right| \\ &\geq \frac{|\ln\varepsilon|}{2\mu} \left|4\pi - g_\varepsilon \mu \ln(\alpha_\varepsilon) \int V(r)\, \mathrm{d}r\right| \\ &= \frac{|\ln\varepsilon|}{2\mu} \left|4\pi - \frac{\int V(r)\, \mathrm{d}r}{a \ln(\varepsilon)} \Big(2\ln(\varepsilon) + \ln(\mu(Q + z))\Big) + O\left(\frac{|\ln\alpha_\varepsilon|}{(\ln\varepsilon)^2}\right)\right| \\ &= \pi\mu^{-1} \ln(\mu(Q + z)) + O(1), \end{aligned} \tag{4.39}$$

where for sufficiently small $\varepsilon_0 \in (0, 1)$ the remainder $O(1)$ is uniformly bounded in $\underline{P} \in \mathbb{R}^{2(N-1)}$, $z \in (\mu^{-1}, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$ as long as $\alpha_\varepsilon < 1$. By choosing $z_0 \geq \mu^{-1}$ large enough, we conclude from (4.39) that $|f(\underline{P}, z, \varepsilon)| \geq \pi \ln(\mu(Q+z))/(2\mu(\ln\varepsilon)^2) > 0$ for all $\underline{P}$, $z \in (z_0, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$ satisfying $\alpha_\varepsilon < 1$. With the help of this bound, the desired estimate for the operator norm of $\big((g_\varepsilon)^{-1} + \phi_\varepsilon(z, \underline{P})\big)^{-1}$ can be obtained similarly to the case $\int V(r)\, \mathrm{d}r/(2\pi) < a$. $\square$

An immediate consequence of Lemma 4.6 is that $(g_\varepsilon)^{-1} + \phi_\varepsilon(z)$ is invertible for all large enough $z > 0$ and small enough $\varepsilon > 0$. Yet, we shall see that the limit of the inverse operator essentially depends on the leading order of the coupling constant $g_\varepsilon$ or, more precisely, on $a$. If $\int V(r)\,\mathrm{d}r/(2\pi) < a$, then it follows from (4.33) that $\lim_{\varepsilon\to 0}\left((g_\varepsilon)^{-1} + \phi_\varepsilon(z)\right)^{-1}$ vanishes for $z > z_0$, while for $\int V(r)\,\mathrm{d}r/(2\pi) = a$ this limit will turn out to be non-trivial. To prove this, we first note that Eq. (4.24) implies that $g_\varepsilon \phi_\varepsilon(z, \underline{P})$ is an integral operator with kernel

$$g_\varepsilon \mu\, u(r)\, G_{\mu(Q+z)}(\varepsilon(r-r'))\, v(r'),$$

where $G_{\mu(Q+z)}(\varepsilon(r-r')) = G_{\varepsilon^2\mu(Q+z)}(r-r')$. Comparing this with Eq. (1.11), which defines the integral kernel of $g_\varepsilon\phi_\varepsilon(z)$ in the one-particle case, we see that both integral kernels coincide up to the substitution $z \to \mu(Q+z)$ because the respective coupling constants $g_\varepsilon$ differ by a factor of $\mu$. In the one-particle case the limit of $((g_\varepsilon)^{-1} + \phi_\varepsilon(z))^{-1}$ is given by (1.16) and (1.17), and hence we expect that

$$\lim_{\varepsilon\to 0}\left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z)\right)^{-1} = -\frac{|u\rangle\,\langle v|}{\langle u\,|\,v\rangle^2} \otimes D(z)^{-1}, \tag{4.40}$$

where $D(z)$ is an unbounded operator, which, after passing to Fourier space, acts as the multiplication operator

$$D(z, \underline{P}) := \frac{\mu}{2\pi}\left(\ln\left(\frac{\sqrt{\mu(Q+z)}}{2}\right) + \gamma + 2\pi\alpha\right), \tag{4.41}$$

$$\alpha := \frac{\langle v\,|\,Lu\rangle}{2\pi\,\langle u\,|\,v\rangle^2} - \frac{b}{\langle u\,|\,v\rangle}. \tag{4.42}$$

Here, $\gamma$ denotes the Euler–Mascheroni constant and $L$ is the Hilbert-Schmidt operator defined in Lemma 4.5. We now give a rigorous proof of (4.40):

**Proposition 4.7.** *Let the hypotheses of Lemma 4.5 be satisfied and suppose that the asymptotics of $g_\varepsilon$ is given by* (4.32) *with $a = \int V(r)\,\mathrm{d}r/(2\pi) > 0$ and $b \in \mathbb{R}$. Then $D(z)$ is invertible and* (4.40) *holds true for all large enough $z > 0$.*

*Proof.* Clearly, $D(z)$ is invertible if $\ln\left(\sqrt{\mu z}/2\right) + \gamma + 2\pi\alpha > 0$, i.e. $z > z_1 := 4\mu^{-1}\exp(-4\pi\alpha - 2\gamma)$. Let $\varepsilon_0, z_0 > 0$ be chosen as in Lemma 4.6 and, for the rest of the proof, let $z > \max(z_0, z_1)$ be fixed. Then (4.40) is equivalent to

$$\lim_{\varepsilon\to 0}\sup_{\underline{P}}\left\|\left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z,\underline{P})\right)^{-1} + D(z,\underline{P})^{-1}\frac{|u\rangle\,\langle v|}{\langle u\,|\,v\rangle^2}\right\| = 0 \tag{4.43}$$

with $\|\cdot\|$ denoting the operator norm in $\mathscr{L}(L^2(\mathbb{R}^2))$. The idea of the following proof is that the norm of both operators vanishes as $|\underline{P}| \to \infty$, while for $|\underline{P}| \leq$ const. the estimates from the one-particle case still work. To make this more explicit, we fix $\delta > 0$. Then it follows from Lemma 4.6 and Eq. (4.41) that

$$\left\|\left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z,\underline{P})\right)^{-1}\right\| + \left\|D(z,\underline{P})^{-1}\frac{|u\rangle\,\langle v|}{\langle u\,|\,v\rangle^2}\right\| < \delta,$$

provided that $\varepsilon > 0$ is sufficiently small and $|\underline{P}|$, and hence $Q$, are sufficiently large. To prove (4.43), it thus suffices to show that for any $K > 0$ there exists $\varepsilon_K > 0$ such that for all $\varepsilon \in (0, \varepsilon_K)$,

$$\sup_{\underline{P}\,:\,Q\leq K}\left\|\left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z,\underline{P})\right)^{-1} + D(z,\underline{P})^{-1}\frac{|u\rangle\,\langle v|}{\langle u\,|\,v\rangle^2}\right\| < \delta. \tag{4.44}$$

Let $K > 0$ be fixed. Then (4.24) and (4.26) imply that, for some constant $C = C(s, V) > 0$,

$$\left\| \phi_\varepsilon(z, \underline{P}) + \frac{\mu}{2\pi} \left[ \left( \ln \left( \frac{\varepsilon}{2} \sqrt{\mu(Q+z)} \right) + \gamma \right) |u\rangle\langle v| + L \right] \right\|_{\mathrm{HS}} \leq C\mu^{1+s/2}(K+z)^{s/2}\varepsilon^s$$

uniformly in $\underline{P}$ with $Q = Q(\underline{P}) \leq K$. Combining this with the asymptotic behavior of $g_\varepsilon$ in the limit $\varepsilon \to 0$ (see Eq. (4.32)), we see that

$$\begin{aligned} &g_\varepsilon \phi_\varepsilon(z, \underline{P}) \\ &= -\frac{1}{2\pi a} |u\rangle \langle v| - \frac{1}{2\pi a \ln(\varepsilon)} \left[ \left\{ \ln\left( \frac{\sqrt{\mu(Q+z)}}{2} \right) + \gamma - \frac{b}{a} \right\} |u\rangle\langle v| + L \right] + o\left( |\ln \varepsilon|^{-1} \right) \end{aligned} \tag{4.45}$$

is valid in Hilbert-Schmidt norm uniformly in $\underline{P}$ with $Q = Q(\underline{P}) \leq K$. As the right side coincides with the expansion of $B_\varepsilon(k)$ with $k = i\sqrt{\mu(Q+z)}$, $\lambda_1 = 1/a$ and $\lambda_2 = -b/a^2$ in the one-particle case (see [6, Eq. (5.56), p. 103]), the proof of (4.44) now follows the line of argument from that case. For the convenience of the reader, we spell out the details in the following.

To start with, we derive from (4.45) the equation

$$\left( \frac{1}{g_\varepsilon} + \phi_\varepsilon(z, \underline{P}) \right)^{-1} = g_\varepsilon \Big( 1 + \beta(Q) \, |u\rangle \, \langle v| + S_\varepsilon \Big)^{-1}, \tag{4.46}$$

where

$$\begin{aligned} \beta(Q) &:= -\frac{1}{2\pi a} \left( 1 + \frac{\widetilde{\beta}(Q)}{\ln(\varepsilon)} \right), \\ \widetilde{\beta}(Q) &:= \ln\left( \frac{\sqrt{\mu(Q+z)}}{2} \right) + \gamma - \frac{b}{a}, \\ S_\varepsilon &:= -\frac{1}{2\pi a \ln(\varepsilon)} L + o\left( |\ln \varepsilon|^{-1} \right). \end{aligned}$$

The expansion of $S_\varepsilon$ holds uniformly in $\underline{P}$ with $Q = Q(\underline{P}) \leq K$. To obtain an expression for the inverse on the right side of Eq. (4.46), we now apply the identity (4.35). This yields that

$$\Big( 1 + \beta(Q) \, |u\rangle \, \langle v| + S_\varepsilon \Big)^{-1} = (1 + S_\varepsilon)^{-1} - \frac{(1+S_\varepsilon)^{-1} \, |u\rangle \, \langle v| \, (1+S_\varepsilon)^{-1}}{\beta(Q)^{-1} + \langle v \,|\, (1+S_\varepsilon)^{-1} u \rangle} \tag{4.47}$$

as operators in $\mathscr{L}(L^2(\mathbb{R}^2))$. By the definitions of $\beta(Q)$ and $S_\varepsilon$, and by $2\pi a = \langle u \,|\, v \rangle$,

$$\beta(Q)^{-1} = -\langle u \,|\, v \rangle \left( 1 - \frac{\widetilde{\beta}(Q)}{\ln(\varepsilon)} \right) + O\left( (\ln \varepsilon)^{-2} \right), \qquad Q \leq K, \tag{4.48}$$

and

$$\left\langle v \,\middle|\, (1+S_\varepsilon)^{-1} u \right\rangle = \langle u \,|\, v \rangle + \frac{\langle v \,|\, L u \rangle}{2\pi a \ln(\varepsilon)} + o\left( |\ln \varepsilon|^{-1} \right), \tag{4.49}$$

where $\|S_\varepsilon\| = O(|\ln \varepsilon|^{-1})$ was used. Note that the sum of (4.48) and (4.49) is of order $|\ln \varepsilon|^{-1}$ because $\langle u \,|\, v \rangle$ cancels. It follows that the second summand in Eq. (4.47) is of order $|\ln \varepsilon|$. Hence, we may ignore the first summand, $(1+S_\varepsilon)^{-1}$, in Eq. (4.47) and we may replace the numerator in the second summand by $|u\rangle \, \langle v|$. It follows that, uniformly in $\underline{P}$ with $Q = Q(\underline{P}) \leq K$,

$$\Big( 1 + \beta(Q) \, |u\rangle \langle v| + S_\varepsilon \Big)^{-1} = -\frac{\ln(\varepsilon)}{\langle u \,|\, v \rangle} \left[ \ln\left( \frac{\sqrt{\mu(Q+z)}}{2} \right) + \gamma + 2\pi\alpha \right]^{-1} |u\rangle \langle v| + o\left( |\ln \varepsilon| \right)$$

with $\alpha$ defined by Eq. (4.42). Hence, it follows from (4.46) and from the asymptotics of $g_\varepsilon$ with $a = \langle u \,|\, v \rangle / (2\pi)$ that (4.44) with $D(z, \underline{P})$ defined by (4.41) holds true for all sufficiently small $\varepsilon > 0$. This concludes the proof. □

For later convenience, we now collect the conclusions of this section. To this end, we need to reinstall the index $\sigma$, which we dropped at the beginning of this section. Let $\sigma \in \mathcal{I}$ be a fixed pair. If $V_\sigma \in L^1 \cap L^2(\mathbb{R}^2)$ and $\int |r|^{2s} |V_\sigma(r)| \, \mathrm{d}r < \infty$ for some $s > 0$, then the analogs of Lemma 4.6 and Proposition 4.7 show that $\Lambda_\varepsilon(z)_{\sigma\sigma} = (g_{\varepsilon,\sigma})^{-1} + \phi_\varepsilon(z)_{\sigma\sigma}$ is invertible for all sufficiently small $\varepsilon > 0$ and sufficiently large $z > 0$ and

$$\lim_{\varepsilon \to 0} (\Lambda_\varepsilon(z)_{\sigma\sigma})^{-1} = \begin{cases} 0 & \text{if } \int V_\sigma(r) \, \mathrm{d}r/(2\pi) < a_\sigma, \\ -\dfrac{|u_\sigma\rangle \langle v_\sigma|}{\langle u_\sigma \,|\, v_\sigma \rangle^2} \otimes (\Theta(z)_{\sigma\sigma})^{-1} & \text{if } \int V_\sigma(r) \, \mathrm{d}r/(2\pi) = a_\sigma, \end{cases} \tag{4.50}$$

where $\Theta(z)_{\sigma\sigma}$ is a densely defined and invertible operator in $\mathfrak{X}_\sigma$. For $\sigma = (1,2)$, $\Theta(z)_{\sigma\sigma} = D(z)$ with $D(z)$ defined by Eqs. (4.41) and (4.42). For general pairs $\sigma = (i,j)$, the operator $\Theta(z)_{\sigma\sigma}$ acts pointwise in $\underline{P}_\sigma = (P, p_1, ...\widehat{p}_i...\widehat{p}_j..., p_N)$ by multiplication with

$$\Theta(z, \underline{P}_\sigma)_{\sigma\sigma} = \frac{\mu_\sigma}{4\pi} \left[ \ln\left( z + \frac{P^2}{m_i + m_j} + \sum_{\substack{n=1 \\ n \neq i,j}}^{N} \frac{p_n^2}{m_n} \right) + \frac{\beta_\sigma}{\pi} \right], \qquad \sigma = (i,j), \tag{4.51}$$

where

$$\beta_\sigma = 2\pi \left( \ln(\sqrt{\mu_\sigma}/2) + \gamma + 2\pi\alpha_\sigma \right), \tag{4.52}$$

$$\alpha_\sigma = \frac{\langle v_\sigma \,|\, L_\sigma u_\sigma \rangle}{2\pi \langle u_\sigma \,|\, v_\sigma \rangle^2} - \frac{b_\sigma}{\langle u_\sigma \,|\, v_\sigma \rangle}, \tag{4.53}$$

with the Hilbert-Schmidt operator $L_\sigma$ that has the integral kernel

$$u_\sigma(r) \, \ln(|r - r'|) \, v_\sigma(r'), \qquad r \neq r'. \tag{4.54}$$

### 4.4.2 Analysis of $\Lambda_\varepsilon(z)_{\mathbf{off}}$

If $N > 2$, then $\Lambda_\varepsilon(z)$ has an off-diagonal part $\Lambda_\varepsilon(z)_{\text{off}}$ defined by Eq. (4.23). We will see in this section that the norm $\|\Lambda_\varepsilon(z)_{\text{off}}\|$ is uniformly bounded in $\varepsilon > 0$ and $z > 0$ and that a regularized version of $\Lambda_\varepsilon(z)_{\text{off}}$ has a limit as $\varepsilon \to 0$. These results allow us in Section 4.4.3 to prove existence of $\lim_{\varepsilon \to 0} \Lambda_\varepsilon(z)^{-1}$ for sufficiently large $z > 0$. With this goal in mind, the results of this section are formulated for $z > 0$ only, although most of them are still valid in a slightly modified form for general $z \in \rho(H_0)$.

#### 4.4.2.1 Uniform boundedness of $\|\Lambda_\varepsilon(z)_{\mathbf{off}}\|$ in $\varepsilon > 0$ and $z > 0$

Recall from Eq. (4.23) that the non-vanishing components of $\Lambda_\varepsilon(z)_{\text{off}}$, $\varepsilon, z > 0$, are given by

$$(\Lambda_\varepsilon(z)_{\text{off}})_{\sigma\nu} = \phi_\varepsilon(z)_{\sigma\nu}, \qquad \sigma, \nu \in \mathcal{I}, \sigma \neq \nu. \tag{4.55}$$

We prove in this section, among other things, that for $\sigma \neq \nu$ the norm $\|\phi_\varepsilon(z)_{\sigma\nu}\|$ is uniformly bounded in $\varepsilon > 0$ and $z > 0$, the main result being Proposition 4.8, below. The presence of the distinct coordinate transformations $\mathscr{K}_\sigma$ and $\mathscr{K}_\nu$ in the Definition (4.20) of $\phi_\varepsilon(z)_{\sigma\nu}$ makes the proof very technical and somewhat tedious. Since the tools used in this proof are not needed anymore in the sequel, it is possible and advisable to take Proposition 4.8 for granted and to skip the proof at first reading.

With this said, we now start developing the tools for proving Proposition 4.8. As the choice of the pair $\sigma$ is immaterial for our estimates, we can assume that $\sigma = (1,2)$ and $\nu = (k,l) \neq (1,2)$ without restriction. In this case the integral kernel of the integral operator $\phi_\varepsilon(z)_{\sigma\nu}$ has been computed at the beginning of Section 3.5.2, and to prove Proposition 4.8 we are going to estimate all these integral kernels with the help of the Schur test. To this end, we first recall the required integral kernels for given $\varepsilon, z > 0$ and $d = 2$. Recall from the sentence including Eqs. (3.64)-(3.66) that for $\nu = (1,l)$, $l \geq 3$, the operator $\phi_\varepsilon(z)_{\sigma\nu}$ acts pointwise in $\underline{p}_{1l} = (p_3, ...\widehat{p_l}..., p_N)$ by the integral operator $\phi_\varepsilon(z, \underline{p}_{1l})_{\sigma\nu}$ that has the kernel

$$m_1 m_2 m_l \, u_\sigma(r) \, G^6_{z+Q_\nu}(X_{\varepsilon,\sigma\nu}) \, v_\nu(r') \,, \qquad \sigma = (1,2), \nu = (1,l), l \geq 3. \tag{4.56}$$

Here, $G^6_z \in L^1(\mathbb{R}^6)$ denotes the Green's function of $-\Delta + z : H^2(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$,

$$Q_{(k,l)} = \sum_{\substack{n=3\\ n\neq k,l}}^{N} \frac{p_n^2}{m_n}, \qquad (k,l) \in \mathcal{I}, \tag{4.57}$$

and

$$X_{\varepsilon,(1,2)(1,l)} = \begin{pmatrix} \sqrt{m_1}(R - R' - \varepsilon(c_{21}r - c_{l1}r')) \\ \sqrt{m_2}(R - x_2' + \varepsilon c_{12} r) \\ \sqrt{m_l}(x_l - R' - \varepsilon c_{1l} r') \end{pmatrix} \in \mathbb{R}^6, \tag{4.58}$$

where the constants $c_{ij} = m_i/(m_i + m_j)$, $1 \leq i < j \leq N$, were introduced in Eq. (3.63). Moreover, Eqs. (3.67) and (3.68) and the subsequent sentence show that, up to the permutations $x_1' \leftrightarrow x_2'$, $m_1 \leftrightarrow m_2$, $v_{(1,l)} \leftrightarrow v_{(2,l)}$ and the reflection $r \to -r$, the integral operators $\phi_\varepsilon(z)_{\sigma\nu}$ with $\nu = (1,l)$ and $\nu = (2,l)$, respectively, have the same kernels. If $N > 3$, then it remains to consider the case $3 \leq k < l \leq N$, where $\sigma = (1,2)$ and $\nu = (k,l)$ have no common particle. In this case it follows from the sentence including Eqs. (3.69) and (3.70) that $\phi_\varepsilon(z)_{\sigma\nu}$ acts pointwise in $\underline{p}_{kl} = (p_3, ...\widehat{p_k}...\widehat{p_l}..., p_N)$ by the integral operator $\phi_\varepsilon(z, \underline{p}_{kl})_{\sigma\nu}$ with kernel

$$m_1 m_2 m_k m_l \, u_\sigma(r) G^8_{z+Q_\nu}(X_{\varepsilon,\sigma\nu}) \, v_\nu(r') \,, \qquad \sigma = (1,2), \nu = (k,l), 3 \leq k < l \leq N, \tag{4.59}$$

where

$$X_{\varepsilon,(1,2)(k,l)} = \begin{pmatrix} \sqrt{m_1}(R - x_1' - \varepsilon c_{21} r) \\ \sqrt{m_2}(R - x_2' + \varepsilon c_{12} r) \\ \sqrt{m_k}(x_k - R' + \varepsilon c_{lk} r') \\ \sqrt{m_l}(x_l - R' - \varepsilon c_{kl} r') \end{pmatrix} \in \mathbb{R}^8. \tag{4.60}$$

Besides estimating the norm of $\phi_\varepsilon(z)_{\sigma\nu}$, we shall also estimate the error caused by cutting off the potential outside some ball of radius $h > 0$ in Proposition 4.8. This, in turn, will reduce the proof of our convergence result (see Proposition 4.12) to the case of compactly supported potentials. For a given pair $\nu \in \mathcal{I}$ and $h > 0$, let

$$V_\nu^h(x) := \begin{cases} V_\nu(x) & \text{if } |x| \leq h \\ 0 & \text{if } |x| > h \end{cases}$$

and let $\phi_\varepsilon^h(z)_{\sigma\nu}$ denote the operator $\phi_\varepsilon(z)_{\sigma\nu}$, where the potentials $V_\sigma$ and $V_\nu$ are replaced by $V_\sigma^h$ and $V_\nu^h$, respectively. This means that the kernel of $\phi_\varepsilon^h(z)_{\sigma\nu}$ emerges from the kernel of $\phi_\varepsilon(z)_{\sigma\nu}$ by replacing $u_\sigma$ and $v_\nu$ with $u_\sigma^h := \operatorname{sgn}(V_\sigma)|V_\sigma^h|^{1/2}$ and $v_\nu^h := |V_\nu^h|^{1/2}$, respectively.

**Proposition 4.8.** *For all pairs $\sigma = (i,j), \nu = (k,l)$ with $\sigma \neq \nu$ the norm of the operator $\phi_\varepsilon(z)_{\sigma\nu}$ is uniformly bounded in $\varepsilon > 0$ and $z > 0$, provided that $V_\sigma, V_\nu \in L^1(\mathbb{R}^2)$. Explicitly, it holds that*

$$\|\phi_\varepsilon(z)_{\sigma\nu}\| \leq C(\sigma,\nu) \, \|V_\sigma\|_{L^1}^{1/2} \|V_\nu\|_{L^1}^{1/2}, \tag{4.61}$$

*where $C(\sigma,\nu) := (4\sqrt{2})^{-1} m_i m_j m_k m_l / \min(m_i, m_j, m_k, m_l)^3$. Furthermore, for all $h > 0$,*

$$\|\phi_\varepsilon(z)_{\sigma\nu} - \phi_\varepsilon^h(z)_{\sigma\nu}\| \leq C(\sigma,\nu) \left( \|V_\sigma\|_{L^1} \|V_\nu\|_{L^1} - \|V_\sigma^h\|_{L^1} \|V_\nu^h\|_{L^1} \right)^{1/2}. \tag{4.62}$$

*Proof.* Without loss of generality, we may assume that $\sigma = (1,2)$ and then we have to establish (4.61) and (4.62) for all pairs $\nu \neq (1,2)$. We first consider the case $\nu = (1,l)$, $l \geq 3$.

The proofs of (4.61) and (4.62) are similar: It follows from (4.56) that in both cases we have to estimate the norm of an operator that, for fixed $\underline{p}_{1l}$, is given by a kernel of the form

$$m_1 m_2 m_l \, W(r,r') \, G^6_{z+Q}\,(X_\varepsilon)\,,$$

where $X_\varepsilon := X_{\varepsilon,(1,2)(1,l)}$ and $Q := Q_{(1,l)}$ for short. Explicitly, we have $W(r,r') = u_\sigma(r) v_\nu(r')$ in the case of (4.61) and $W(r,r') = u_\sigma(r)\, v_\nu(r') - u^h_\sigma(r)\, v^h_\nu(r')$ in the case of (4.62). Therefore, we only demonstrate the desired estimate in the case of (4.61).

For $\psi \in L^2(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)$, the Cauchy-Schwarz inequality in the $r'$-integration yields

$$\begin{aligned}
&(m_1 m_2 m_l)^{-2} \|\phi_\varepsilon(z, \underline{p}_{1l})_{\sigma\nu}\psi\|^2 \\
&\quad = \int \mathrm{d}r\, \mathrm{d}R\, \mathrm{d}x_l \left| \int \mathrm{d}r'\, \mathrm{d}R'\, \mathrm{d}x_2'\; W(r,r')\, G^6_{z+Q}(X_\varepsilon)\, \psi(r', R', x_2') \right|^2 \\
&\quad = \int \mathrm{d}r\, \mathrm{d}R\, \mathrm{d}x_l \left| \int \mathrm{d}r'\, W(r,r') \int \mathrm{d}R'\, \mathrm{d}x_2'\; G^6_{z+Q}(X_\varepsilon)\, \psi(r', R', x_2') \right|^2 \\
&\quad \leq \int \mathrm{d}r\, \mathrm{d}R\, \mathrm{d}x_l \left\{ \int \mathrm{d}r'\, W(r,r')^2 \right\} \int \mathrm{d}r' \left| \int \mathrm{d}R'\, \mathrm{d}x_2'\; G^6_{z+Q}(X_\varepsilon)\, \psi(r', R', x_2') \right|^2 \\
&\quad \leq \left\{ \int \mathrm{d}r\, \mathrm{d}r'\, W(r,r')^2 \right\} \cdot \sup_{r \in \mathbb{R}^2} \left( \int \mathrm{d}r'\, \mathrm{d}R\, \mathrm{d}x_l \left| \int \mathrm{d}R'\, \mathrm{d}x_2'\; G^6_{z+Q}(X_\varepsilon)\, \psi(r', R', x_2') \right|^2 \right),
\end{aligned} \tag{4.63}$$

where

$$\int \mathrm{d}r\, \mathrm{d}r'\; W(r,r')^2 = \|V_\sigma\|_{L^1} \|V_\nu\|_{L^1}. \tag{4.64}$$

For a further estimate of (4.63), we fix $r \in \mathbb{R}^2$. Then triangle inequality and the sequence of substitutions $R' + \varepsilon(c_{21} r - c_{l1} r') \to R'$, $x_l + \varepsilon c_{21} r - \varepsilon r' \to x_l$, $x_2' - \varepsilon c_{12} r \to x_2'$ in the first step and the monotonicity of the Green's function w.r.t. $z$ and $m_1, m_2, m_l$ (see Lemma A.1 ($v$)) in the second step yield that

$$\begin{aligned}
&\int \mathrm{d}r'\, \mathrm{d}R\, \mathrm{d}x_l \left| \int \mathrm{d}R'\, \mathrm{d}x_2'\, G^6_{z+Q}\,(X_\varepsilon)\;\; \psi(r', R', x_2') \right|^2 \\
&\quad \leq \int \mathrm{d}r'\, \mathrm{d}R\, \mathrm{d}x_l \left( \int \mathrm{d}R'\, \mathrm{d}x_2'\, G^6_{z+Q}\left(\sqrt{m_1}(R - R'), \sqrt{m_2}(R - x_2'), \sqrt{m_l}(x_l - R')\right) \widetilde{\psi}(r', R', x_2') \right)^2 \\
&\quad \leq \int \mathrm{d}r' \left\| F_1 \widetilde{\psi}(r', \cdot) \right\|^2,
\end{aligned} \tag{4.65}$$

where $\widetilde{\psi} \in L^2(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)$ is given by

$$\widetilde{\psi}(r', R', x_2') := \left| \psi\left(r', R' - \varepsilon(c_{21} r - c_{l1} r'), x_2' + \varepsilon c_{12} r\right) \right|$$

and $F_1 : L^2\left(\mathbb{R}^2 \times \mathbb{R}^2, \mathrm{d}\,(R, x_2)\right) \to L^2\left(\mathbb{R}^2 \times \mathbb{R}^2, \mathrm{d}\,(R, x_l)\right)$ is defined by the integral kernel

$$G^6_z\left(\sqrt{m}(R - R'), \sqrt{m}(R - x_2'), \sqrt{m}(x_l - R')\right), \qquad m = \min(m_1, m_2, m_l).$$

By Lemma A.5, $F_1$ is bounded with $\|F_1\| \leq (4\sqrt{2} m^2)^{-1}$. Using this together with the fact that $\|\psi\| = \|\widetilde{\psi}\|$, we obtain from (4.65) that

$$\int \mathrm{d}r'\, \mathrm{d}R\, \mathrm{d}x_l \left| \int \mathrm{d}R'\, \mathrm{d}x_2'\, G^6_{z+Q}\,(X_\varepsilon)\;\; \psi(r', R', x_2') \right|^2 \leq \left(32 \min(m_1, m_2, m_l)^4\right)^{-1} \|\psi\|^2, \tag{4.66}$$

where the right-hand side is independent of $r \in \mathbb{R}^2$ and $\underline{p}_{1l} \in \mathbb{R}^{2N-6}$. Hence, (4.61) for $\sigma = (1,2)$ and $\nu = (1,l)$ now follows by combining (4.63), (4.64) and (4.66).

As already mentioned above, the norm estimate (4.62) can be obtained in almost the same manner, with the only difference that $W(r,r') = u_\sigma(r)\, v_\nu(r') - u^h_\sigma(r)\, v^h_\nu(r')$ in this case. Then Eq. (4.64) has to be replaced by the identity

$$\int \mathrm{d}r\, \mathrm{d}r'\, W(r,r')^2 = \|V_\sigma\|_{L^1}\|V_\nu\|_{L^1} - \|V^h_\sigma\|_{L^1}\|V^h_\nu\|_{L^1},$$

which, in turn, is a consequence of the identities $u_\sigma \cdot u^h_\sigma = |V^h_\sigma|$ and $v_\nu \cdot v^h_\nu = |V^h_\nu|$. This completes the proof of (4.61) and (4.62) for $\sigma = (1,2)$ and $\nu = (1,l)$, $l \geq 3$.

The above proof also works in the case $\sigma = (1,2)$ and $\nu = (2,l)$, $l \geq 3$, because the integral kernels of $\phi_\varepsilon(z)_{\sigma\nu}$ with $\nu = (1,l)$ and $\nu = (2,l)$, respectively, only differ by the permutations $x'_1 \leftrightarrow x'_2$, $m_1 \leftrightarrow m_2$, $v_{(1,l)} \leftrightarrow v_{(2,l)}$ and the unitary reflection $r \to -r$.

It remains to consider the case $N > 3$, $\sigma = (1,2)$ and $\nu = (k,l)$ with $3 \leq k < l \leq N$, where $\phi_\varepsilon(z)_{\sigma\nu}$ acts pointwise in $\underline{p}_{kl} = (p_3, ...\widehat{p_k}...\widehat{p_l}..., p_N)$ by the integral operator $\phi_\varepsilon(z, \underline{p}_{kl})_{\sigma\nu}$ that is defined in terms of the kernel from Eq. (4.59). In this case the above estimates have to be slightly adjusted. The role of the integral operator $F_1$ is now played by the integral operator $F_2 \in \mathscr{L}\left(L^2(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)\right)$ from Lemma A.5. As the bounds for $\|F_1\|$ and for $\|F_2\|$ in Lemma A.5 differ by a factor of $m^{-1}$, while there is one mass factor more in front of the Green's function for $\nu = (k,l)$, $3 \leq k < l \leq N$, than for $\nu = (1,l)$, $l \geq 3$, we again obtain (4.61) and (4.62) with $C(\sigma,\nu)$ given in the statement of the proposition. $\square$

#### 4.4.2.2 The off-diagonal limit operators

This section is a preparation for the next one, where we shall be concerned with the convergence, as $\varepsilon \to 0$, of

$$\phi_\varepsilon(z)_{\sigma\nu} = B_{\varepsilon,\sigma} R_0(z) (A_{\varepsilon,\nu})^*, \qquad z > 0, \sigma \neq \nu. \tag{4.67}$$

From Corollary 4.4 it follows that for all $z > 0$,

$$R_0(z)(A_{\varepsilon,\nu})^* \to (A_\nu R_0(z))^* = G(z)^*_\nu \,\langle v_\nu| \qquad (\varepsilon \to 0) \tag{4.68}$$

in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \mathscr{H})$, and

$$B_{\varepsilon,\sigma} = J_\sigma A_{\varepsilon,\sigma} \to |u_\sigma\rangle\, T_\sigma \qquad (\varepsilon \to 0) \tag{4.69}$$

in $\mathscr{L}(H^2(\mathbb{R}^{2N}), \widetilde{\mathfrak{X}}_\sigma)$. Here, $|u_\sigma\rangle : \mathfrak{X}_\sigma \to \widetilde{\mathfrak{X}}_\sigma$ is defined by $|u_\sigma\rangle\, \psi = u_\sigma \otimes \psi$ and $\langle v_\nu| : \widetilde{\mathfrak{X}}_\nu \to \mathfrak{X}_\nu$ is the adjoint of $|v_\nu\rangle$. The formal composition of the limits in (4.68) and (4.69) is the operator

$$-\,|u_\sigma\rangle\,\langle v_\nu| \otimes \Theta(z)_{\sigma\nu}$$

with

$$\Theta(z)_{\sigma\nu} = -T_\sigma G(z)^*_\nu, \qquad z > 0, \sigma \neq \nu. \tag{4.70}$$

In the remainder of this section, we show that (4.70) defines an element $\Theta(z)_{\sigma\nu} \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$. As we shall see below, this requires that the trace operator $T_\sigma$ from Eq. (2.12) is defined on its maximal, $\sigma$-dependent domain $D(T_\sigma) = \mathscr{K}^*_\sigma D(\tau)$, where $D(\tau)$ is given by Eq. (2.11). We begin by computing representations of $T_\sigma$ and $G(z)^*_\nu$ in Fourier space. Using the Definitions (2.12), (2.10) and (1.45) of $T_\sigma$, $\tau$ and $\mathscr{K}_\sigma$, respectively, we find for $\sigma = (i,j)$ and $\psi \in D(T_\sigma)$ that

$$\begin{aligned}(\widehat{T_{(i,j)}\psi})\,(P, p_1, ...\widehat{p_i}...\widehat{p_j}..., p_N) &= \frac{1}{2\pi}\int \mathrm{d}p\; \widehat{\mathscr{K}_{(i,j)}\psi}\,(p, P, p_1, ...\widehat{p_i}...\widehat{p_j}..., p_N)\\ &= \frac{1}{2\pi}\int \mathrm{d}p\; \widehat{\psi}\left(p_1, ..., p_{i-1}, \frac{m_i P}{m_i + m_j} - p, p_{i+1}, ..., p_{j-1}, \frac{m_j P}{m_i + m_j} + p, p_{j+1}, ..., p_N\right).\end{aligned} \tag{4.71}$$

To compute the Fourier transform of $G(z)^*_\nu w$ for $z > 0$, $\nu = (k,l)$ and $w \in \mathfrak{X}_\nu$, we use $G(z)_\nu = T_\nu R_0(z)$ in combination with Eq. (4.71) and the substitution

$$P := p_k + p_l, \quad p := \frac{m_k p_l - m_l p_k}{m_k + m_l}.$$

After a straightforward computation, we find that $\langle w \,|\, G(z)_\nu \psi\rangle = \langle G(z)^*_\nu w \,|\, \psi\rangle$ for all $\psi \in \mathscr{H}$, where

$$(\widehat{G(z)^*_\nu w})\,(p_1, ..., p_N) = \frac{1}{2\pi}\left(z + \sum_{n=1}^{N} \frac{p_n^2}{m_n}\right)^{-1} \cdot \widehat{w}\,(p_k + p_l, p_1, ...\widehat{p}_k ... \widehat{p}_l ..., p_N)\,. \tag{4.72}$$

We now come to the main result of this section:

**Proposition 4.9.** *Let $\sigma = (i,j) \neq (k,l) = \nu$ and $z > 0$. Then $\Theta(z)_{\sigma\nu} = -T_\sigma G(z)^*_\nu$ defines a bounded operator in $\mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$ and $\|\Theta(z)_{\sigma\nu}\| \leq \max(m_i, m_j, m_k, m_l)/4$.*

In the proof of Proposition 4.9 we need the following result taken from [30, Lemma 3.1]. For the convenience of the reader, we give a short proof with a worse constant here.

**Lemma 4.10.** *For all $f, g \in L^2(\mathbb{R}^2)$,*

$$\int \frac{|f(x)||g(x')|}{|x|^2 + |x'|^2}\,\mathrm{d}x\,\mathrm{d}x' \leq \pi^2 \|f\|\|g\|. \tag{4.73}$$

*Proof.* Let $K$ denote the integral operator in $L^2(\mathbb{R}^2)$ that is defined by the kernel $K(x,x') = (|x|^2 + |x'|^2)^{-1}$. Using the Schur test with $h(x) = |x|^{-1}$, it is straightforward to verify that $K$ defines a bounded operator and that

$$\|K\| \leq \operatorname*{ess\,sup}_{x' \in \mathbb{R}^2} \left( |x'| \int_{\mathbb{R}^2} \frac{1}{|x|^2 + |x'|^2} \frac{1}{|x|}\,\mathrm{d}x \right) = \pi^2.$$

Hence, $\langle\, |\,f\,|\,, K\,|\,g\,|\,\rangle \leq \pi^2\|f\|\|g\|$, which establishes (4.73). □

*Proof of Proposition 4.9.* Without loss of generality, we may assume that $\sigma = (1,2)$. Moreover, it suffices to consider the two cases $\nu = (1,3)$ and $\nu = (3,4)$ corresponding to pairs with one common particle and no common particle, respectively (all other cases can be traced back to one of these cases by renaming some variables and masses).

To show that $\operatorname{Ran} G(z)^*_\nu \subseteq D(T_\sigma)$ or, equivalently, that $\operatorname{Ran} \mathscr{K}_\sigma G(z)^*_\nu \subseteq D(\tau)$, we have to verify that, for all $w \in \mathfrak{X}_\nu$,

$$\widehat{\varphi}(P, p_3, ..., p_N) := \frac{1}{2\pi}\int \mathrm{d}p \left| \widehat{G(z)^*_\nu w}\left( \frac{m_1 P}{m_1 + m_2} - p, \frac{m_2 P}{m_1 + m_2} + p, p_3, ..., p_N \right) \right|$$

defines an $L^2$-function. To this end, by the Riesz lemma, it suffices to show that

$$\int \mathrm{d}P\,\mathrm{d}p_3 \cdots \mathrm{d}p_N\,|(\widehat{\psi}\widehat{\varphi})(P, p_3, ..., p_N)| \leq \text{const.} \cdot \|\psi\|$$

for all $\psi \in \mathfrak{X}_{(1,2)}$. The substitution

$$P := p_1 + p_2, \quad p := \frac{m_1 p_2 - m_2 p_1}{m_1 + m_2}$$

and the identity (4.72) for $G(z)^*_\nu w$, $\nu = (k,l)$, show that

$$\begin{aligned} &\int \mathrm{d}P\,\mathrm{d}p_3 \cdots \mathrm{d}p_N\,|\widehat{\psi}(P, p_3, ..., p_N)||\widehat{\varphi}(P, p_3, ..., p_N)| \\ &= \frac{1}{2\pi}\int \mathrm{d}p_1 \cdots \mathrm{d}p_N\,|\widehat{\psi}(p_1 + p_2, p_3, ..., p_N)|\left|\widehat{G(z)^*_\nu w}(p_1, ..., p_N)\right| \\ &\leq \frac{\max(m_2, m_l)}{4\pi^2}\int \mathrm{d}p_1 \cdots \mathrm{d}p_N \frac{|\widehat{\psi}(p_1 + p_2, p_3, ..., p_N)||\widehat{w}\,(p_k + p_l, p_1, ...\widehat{p}_k ... \widehat{p}_l ..., p_N)|}{p_2^2 + p_l^2}. \end{aligned} \tag{4.74}$$

For $\nu = (k,l) = (1,3)$, let $\underline{p} := (p_4, ..., p_N)$ and let

$$f(p_3, \underline{p}) := \left(\int \mathrm{d}p_1 \, |\widehat{\psi}(p_1, p_3, \underline{p})|^2\right)^{1/2},$$
$$g(p_2, \underline{p}) := \left(\int \mathrm{d}p_1 \, |\widehat{w}(p_1, p_2, \underline{p})|^2\right)^{1/2}.$$

Then $f(\cdot, \underline{p}), g(\cdot, \underline{p}) \in L^2(\mathbb{R}^2)$ for almost all $\underline{p} \in \mathbb{R}^{2N-6}$. Hence, after applying the Cauchy-Schwarz inequality in the $p_1$-integration, it follows from Lemma 4.10 that

$$\begin{aligned}
&\int \mathrm{d}\underline{p} \, \mathrm{d}p_2 \, \mathrm{d}p_3 \, \mathrm{d}p_1 \, \frac{|\widehat{\psi}(p_1 + p_2, p_3, \underline{p})||\widehat{w}(p_1 + p_3, p_2, \underline{p})|}{p_2^2 + p_3^2} \\
&\quad \leq \int \mathrm{d}\underline{p} \, \mathrm{d}p_2 \, \mathrm{d}p_3 \, \frac{|f(p_3, \underline{p})||g(p_2, \underline{p})|}{p_2^2 + p_3^2} \leq \pi^2 \|\psi\| \|w\|.
\end{aligned} \tag{4.75}$$

For $\nu = (k,l) = (3,4)$, we set $\underline{p} := (p_5, ..., p_N)$ and

$$f(p_4, \underline{p}) := \left(\int \mathrm{d}p_1 \, \mathrm{d}p_3 \, |\widehat{\psi}(p_1, p_3, p_4, \underline{p})|^2\right)^{1/2},$$
$$g(p_2, \underline{p}) := \left(\int \mathrm{d}p_1 \, \mathrm{d}p_3 \, |\widehat{w}(p_3, p_1, p_2, \underline{p})|^2\right)^{1/2}.$$

Using the Cauchy-Schwarz inequality in the $(p_1, p_3)$-integration and Lemma 4.10, we estimate

$$\begin{aligned}
&\int \mathrm{d}\underline{p} \, \mathrm{d}p_2 \, \mathrm{d}p_4 \, \mathrm{d}p_1 \, \mathrm{d}p_3 \, \frac{|\widehat{\psi}(p_1 + p_2, p_3, p_4, \underline{p})||\widehat{w}(p_3 + p_4, p_1, p_2, \underline{p})|}{p_2^2 + p_4^2} \\
&\quad \leq \int \mathrm{d}\underline{p} \, \mathrm{d}p_2 \, \mathrm{d}p_4 \, \frac{|f(p_4, \underline{p})||g(p_2, \underline{p})|}{p_2^2 + p_4^2} \leq \pi^2 \|\psi\| \|w\|.
\end{aligned} \tag{4.76}$$

From (4.74), (4.75) and (4.76) it follows that $\widehat{\varphi} \in L^2(\mathbb{R}^{2N-2})$ and hence $\operatorname{Ran} G(z)^*_\nu \subseteq D(T_\sigma)$. Moreover, using that $\widehat{\varphi} \geq |\widehat{T_\sigma G(z)^*_\nu w}|$ a.e., the above estimates imply that for all $\psi \in \mathfrak{X}_{(1,2)}$,

$$|\langle \psi \,|\, T_\sigma G(z)^*_\nu w \rangle| \leq \frac{\max(m_1, m_2, m_3, m_4)}{4} \|\psi\| \|w\|.$$

Therefore, $\Theta(z)_{\sigma\nu} = -T_\sigma G(z)^*_\nu$ is a bounded operator, whose norm satisfies the desired estimate. □

#### 4.4.2.3 Cutoff functions and convergence

The goal of this section is to prove Proposition 4.12 on the convergence, as $\varepsilon \to 0$, of $\phi_\varepsilon(z)_{\sigma\nu}$, $\sigma \neq \nu$, with a suitable space cutoff $\chi_{\sigma\nu,c}$ defined below. In the analysis of $\lim_{\varepsilon\to 0} \Lambda_\varepsilon(z)^{-1}$ in Section 4.4.3 this cutoff will be removed again.

We begin by motivating the cutoff: By (4.69), $\lim_{\varepsilon\to 0} B_{\varepsilon,\sigma}$ exists in $\mathscr{L}(H^2(\mathbb{R}^{2N}), \widetilde{\mathfrak{X}}_\sigma)$. Here, the Sobolev index 2 could be reduced to $1 + \delta$ for some $\delta > 0$ but not further because of the presence of the trace operator $T_\sigma = \tau \mathscr{K}_\sigma$ in (4.69) (cf. Lemma 2.10). Moreover, by (4.68), $\lim_{\varepsilon\to 0} R_0(z)(A_{\varepsilon,\nu})^*$ exists in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \mathscr{H})$, or perhaps in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, H^{1-\delta}(\mathbb{R}^{2N}))$, which is not enough to prove that $\phi_\varepsilon(z)_{\sigma\nu}$ given by Eq. (4.67) has a limit as $\varepsilon \to 0$. The space cutoff to be introduced below removes the singularity due to $T^*_\nu$. More explicitly, for $\nu = (1,2)$, it follows from Eq. (4.20) and the identity $R_0(z)\mathscr{K}^*_\nu = \mathscr{K}^*_\nu (\widetilde{H}_0 + z)^{-1}$, where

$$\widetilde{H}_0 = -\frac{\Delta_r}{\mu_{(1,2)}} - \frac{\Delta_R}{m_1 + m_2} + \sum_{i=3}^{N} \left(-\frac{\Delta_{x_i}}{m_i}\right), \tag{4.77}$$

that existence of $\lim_{\varepsilon\to 0}(\widetilde{H}_0+z)^{-1}\varepsilon^{-1}U_\varepsilon^*(v_\nu\otimes 1)$ in $\mathscr{L}(L^2(\mathbb{R}^{2N}), H^2(\mathbb{R}^{2N}))$ would imply existence of $\lim_{\varepsilon\to 0}\phi_\varepsilon(z)_{\sigma\nu}$ in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$. Let $v = v_{(1,2)}$ and $\widetilde{R}_0(z) = (\widetilde{H}_0+z)^{-1}$, $z > 0$, for short. Then a straightforward computation shows that $\widetilde{R}_0(z)\varepsilon^{-1}U_\varepsilon^*(v\otimes 1)$ has the integral kernel

$$G_{z,\underline{\widetilde{m}}}(r-\varepsilon r', R-R', x_3-x_3', ..., x_N-x_N')\, v(r'),$$

where $\underline{\widetilde{m}} := (\mu_{(1,2)}, m_1+m_2, m_3, ..., m_N)$ and $G_{z,\underline{\widetilde{m}}}$ is the Green's function of $\widetilde{H}_0+z$, which is defined by Eq. (3.61). Hence, if $v$ has compact support and $|r| \geq c > 0$ is enforced by a space cutoff $\chi_c$, then, for sufficiently small $\varepsilon > 0$, the singularity of $G_{z,\underline{\widetilde{m}}}$ at the origin is avoided and, by Lemma A.4, this means that the above integral kernel defines a smooth function in $(r, R, x_3, ..., x_N)$.

Let $\chi \in C^\infty(\mathbb{R})$ be a function with $0 \leq \chi(x) \leq 1$, $\chi(x) = 0$ for $x \leq 1$ and $\chi(x) = 1$ for $x \geq 2$ (see Figure 4.1), and let $\chi_c(r) := \chi(|r|/c)$ for $r \in \mathbb{R}^2$ and $c > 0$.

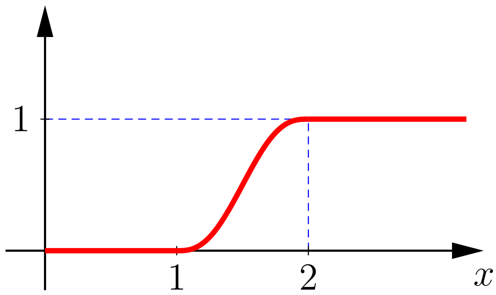


Figure 4.1: The cutoff function $\chi$

Then we have the following result:

**Lemma 4.11.** *Assume that* $v = v_{(1,2)} \in L^2(\mathbb{R}^2)$ *has compact support* $\operatorname{supp}(v) \subseteq B_h(0)$ *for some* $h > 0$ *and let* $c, z > 0$. *Then, for all* $n \in \mathbb{N}_0$, *the limit*

$$\lim_{\varepsilon\to 0}(\chi_c\otimes 1)\widetilde{R}_0(z)\varepsilon^{-1}U_\varepsilon^*(v\otimes 1) = (\chi_c\otimes 1)\mathscr{K}_{(1,2)}G(z)^*_{(1,2)}\langle v|$$

*exists in* $\mathscr{L}(L^2(\mathbb{R}^{2N}), H^n(\mathbb{R}^{2N}))$.

*Proof.* Clearly, the lemma will follow if we show that

$$\lim_{\varepsilon\to 0}(\partial^\alpha\chi_c)\widetilde{R}_0(z)\varepsilon^{-1}U_\varepsilon^*(v\otimes 1) = (\partial^\alpha\chi_c)\mathscr{K}G(z)^*\langle v| \qquad \text{in } \mathscr{L}(L^2(\mathbb{R}^{2N}), H^n(\mathbb{R}^{2N})) \quad (4.78)$$

for all $n \in \mathbb{N}_0$ and all multi-indices $\alpha \in \mathbb{N}_0^2$, where $\mathscr{K} = \mathscr{K}_{(1,2)}$, $G(z) = G(z)_{(1,2)}$ and $\partial^\alpha\chi_c = (\partial^\alpha\chi_c)\otimes 1$ for short. Using $\widetilde{R}_0(z) = \mathscr{K}R_0(z)\mathscr{K}^*$ in combination with the adjoint of (4.17) and (4.68), it follows that

$$\lim_{\varepsilon\to 0}\widetilde{R}_0(z)\varepsilon^{-1}U_\varepsilon^*(v\otimes 1) = \mathscr{K}G(z)^*\langle v| \qquad \text{in } \mathscr{L}(L^2(\mathbb{R}^{2N})),$$

so (4.78) holds for $n = 0$.

We now proceed by induction and assume that (4.78) holds for some $n \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^2$. To show that (4.78) also holds for $n+1$, we are going to use that

$$\eta\widetilde{R}_0(z) = \widetilde{R}_0(z)\left[\widetilde{H}_0, \eta\right]\widetilde{R}_0(z) + \widetilde{R}_0(z)\eta, \qquad \eta = \partial^\alpha\chi_c, \quad (4.79)$$

where $[\cdot,\cdot]$ denotes the commutator and

$$\lim_{\varepsilon\to 0}\widetilde{R}_0(z)\eta\,\varepsilon^{-1}U_\varepsilon^*(v\otimes 1) = 0 \qquad \text{in } \mathscr{L}(L^2(\mathbb{R}^{2N}), H^{n+1}(\mathbb{R}^{2N})) \quad (4.80)$$

because $\eta\, U_\varepsilon^*(v\otimes 1)=0$ for $\varepsilon h<c$. Let $D_k : H^1(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ denote the differential operator $D_k\varphi := \partial_k\varphi$ for $k\in\{1,2\}$. From the Definition (4.77) of $\widetilde{H}_0$ and from $\Delta_r = D_1D_1 + D_2D_2$ it follows that

$$\mu\left[\widetilde{H}_0,\eta\right] = [-\Delta_r,\eta] = -\sum_{k=1}^{2}\left(D_k(\partial_k\eta)+(\partial_k\eta)D_k\right) = (\Delta_r\eta) - 2\sum_{k=1}^{2} D_k(\partial_k\eta), \tag{4.81}$$

where $\mu=\mu_{(1,2)}$ is a reduced mass. Equations (4.79) and (4.81) imply that

$$\eta\widetilde{R}_0(z) = \widetilde{R}_0(z)\eta + \mu^{-1}\widetilde{R}_0(z)(\Delta_r\eta)\widetilde{R}_0(z) - 2\mu^{-1}\sum_{k=1}^{2} D_k\widetilde{R}_0(z)(\partial_k\eta)\widetilde{R}_0(z). \tag{4.82}$$

After multiplying (4.82) with $\varepsilon^{-1}U_\varepsilon^*(v\otimes 1)$ from the right, we get from (4.80) combined with the induction hypothesis and the fact that $\widetilde{R}_0(z)$ and $D_k\widetilde{R}_0(z)$ belong to $\mathscr{L}(H^n(\mathbb{R}^{2N}), H^{n+1}(\mathbb{R}^{2N}))$ that $\lim_{\varepsilon\to 0}\eta\widetilde{R}_0(z)\varepsilon^{-1}U_\varepsilon^*(v\otimes 1)$ exists in $\mathscr{L}(L^2(\mathbb{R}^{2N}), H^{n+1}(\mathbb{R}^{2N}))$. Furthermore, the limit operator has to be the same as in (4.78) because convergence in $\mathscr{L}(L^2(\mathbb{R}^{2N}), H^{n+1}(\mathbb{R}^{2N}))$ implies convergence in $\mathscr{L}(L^2(\mathbb{R}^{2N}), H^{n}(\mathbb{R}^{2N}))$. Hence, (4.78) holds with $n+1$ in place of $n$ and the proof is complete. □

With the help of Lemma 4.11 we can now prove convergence of the regularized operators $(1\otimes\chi_{\sigma\nu,c})\phi_\varepsilon(z)_{\sigma\nu}$ for pairs $\sigma=(i,j)\neq(k,l)=\nu$ and a cutoff function $\chi_{\sigma\nu,c}$ defined by

$$\chi_{(i,j)(k,l),c}\left(R,x_1,...\widehat{x_i}...\widehat{x_j}...,x_N\right) := \begin{cases} \chi_c\left(x_l - R\right) & \text{if } k\in\{i,j\}, l\notin\{i,j\},\\ \chi_c\left(x_k - R\right) & \text{if } k\notin\{i,j\}, l\in\{i,j\},\\ \chi_c\left(x_l - x_k\right) & \text{if } k,l\notin\{i,j\}.\end{cases} \tag{4.83}$$

**Proposition 4.12.** *Let $c,z>0$, $\sigma,\nu\in\mathcal{I}$, $\sigma\neq\nu$, and assume that $V_\sigma,V_\nu\in L^1(\mathbb{R}^2)$. Then*

$$\lim_{\varepsilon\to 0}\left(1\otimes\chi_{\sigma\nu,c}\right)\phi_\varepsilon(z)_{\sigma\nu} = \left(1\otimes\chi_{\sigma\nu,c}\right)\phi(z)_{\sigma\nu} \tag{4.84}$$

*in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu,\widetilde{\mathfrak{X}}_\sigma)$, where*

$$\phi(z)_{\sigma\nu} := -\left|u_\sigma\right\rangle\left\langle v_\nu\right|\otimes\Theta(z)_{\sigma\nu} \tag{4.85}$$

*and $\Theta(z)_{\sigma\nu} = -T_\sigma G(z)_\nu^* \in \mathscr{L}(\mathfrak{X}_\nu,\mathfrak{X}_\sigma)$.*

*Proof.* Assume, for the moment, that for all $h>0$,

$$\lim_{\varepsilon\to 0}(1\otimes\chi_{\sigma\nu,c})\,\phi_\varepsilon^h(z)_{\sigma\nu} = (1\otimes\chi_{\sigma\nu,c})\,\phi^h(z)_{\sigma\nu}, \tag{4.86}$$

where $\phi_\varepsilon^h(z)_{\sigma\nu}$ was introduced in Section 4.4.2.1 and $\phi^h(z)_{\sigma\nu} := -\left|u_\sigma^h\right\rangle\left\langle v_\nu^h\right|\otimes\Theta(z)_{\sigma\nu}$. By Proposition 4.9, $\Theta(z)_{\sigma\nu} = -T_\sigma G(z)_\nu^*$ defines an operator in $\mathscr{L}(\mathfrak{X}_\nu,\mathfrak{X}_\sigma)$ with $\|\Theta(z)_{\sigma\nu}\|\leq\widetilde{C}(\sigma,\nu)$, so the definitions of $\phi(z)_{\sigma\nu}$ and $\phi^h(z)_{\sigma\nu}$ imply that

$$\begin{aligned}\|\phi(z)_{\sigma\nu}-\phi^h(z)_{\sigma\nu}\| &\leq \left\|\left|u_\sigma\right\rangle\left\langle v_\nu\right| - \left|u_\sigma^h\right\rangle\left\langle v_\nu^h\right|\right\|\,\|\Theta(z)_{\sigma\nu}\|\\ &\leq \widetilde{C}(\sigma,\nu)\left(\|V_\sigma\|_{L^1}\|V_\nu\|_{L^1} - \|V_\sigma^h\|_{L^1}\|V_\nu^h\|_{L^1}\right)^{1/2},\end{aligned} \tag{4.87}$$

which is the equivalent of (4.62) for $\varepsilon=0$. Now, the general case of (4.84), where $u_\sigma$ and $v_\nu$ do not have compact support, follows from (4.62), (4.86) and (4.87) by a simple $\delta/3$-argument because $\|V_\sigma^h\|_{L^1}\|V_\nu^h\|_{L^1}\to\|V_\sigma\|_{L^1}\|V_\nu\|_{L^1}$ as $h\to\infty$.

It remains to prove (4.86). As the choice of the pair $\nu$ is immaterial for the following estimates, it suffices to consider the case $\nu=(1,2)\neq(i,j)=\sigma$ only. Moreover, we may assume that $\operatorname{supp}(u_\sigma)\cup\operatorname{supp}(v_\nu)\subseteq B_h(0)$, i.e. $u_\sigma^h = u_\sigma$ and $v_\nu^h = v_\nu$. Then Eq. (4.20) becomes

$$\phi_\varepsilon^h(z)_{\sigma\nu} = \varepsilon^{-2}\left(u_\sigma\otimes 1\right)U_\varepsilon\mathscr{K}_\sigma R_0(z)\mathscr{K}_\nu^* U_\varepsilon^*(v_\nu\otimes 1). \tag{4.88}$$

From the Definition (1.49) of $B_{\varepsilon,\sigma}$ and from (4.69) it follows that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} (u_\sigma \otimes 1) U_\varepsilon \mathscr{K}_\sigma = \lim_{\varepsilon\to 0} B_{\varepsilon,\sigma} = |u_\sigma\rangle T_\sigma \qquad \text{in } \mathscr{L}(H^2(\mathbb{R}^{2N}), \widetilde{\mathfrak{X}}_\sigma). \tag{4.89}$$

Next, defining

$$\chi_{\nu,c}(x_1, ..., x_N) := \chi_c(x_2 - x_1), \qquad \nu = (1,2),$$

the identities $R_0(z)\mathscr{K}_\nu^* = \mathscr{K}_\nu^* \widetilde{R}_0(z)$, $\chi_{\nu,c}\mathscr{K}_\nu^* = \mathscr{K}_\nu^*(\chi_c \otimes 1)$ and Lemma 4.11 imply that

$$\lim_{\varepsilon\to 0} \chi_{\nu,c} R_0(z) \mathscr{K}_\nu^* \varepsilon^{-1} U_\varepsilon^* (v_\nu \otimes 1) = \chi_{\nu,c} G(z)_\nu^* \langle v_\nu| \qquad \text{in } \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, H^n(\mathbb{R}^{2N})) \tag{4.90}$$

for all $n \in \mathbb{N}_0$. From (4.89) and (4.90) we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} (u_\sigma \otimes 1) U_\varepsilon \mathscr{K}_\sigma \chi_{\nu,c} R_0(z) \mathscr{K}_\nu^* U_\varepsilon^* (v_\nu \otimes 1) = |u_\sigma\rangle T_\sigma \chi_{\nu,c} G(z)_\nu^* \langle v_\nu| \tag{4.91}$$

in $\mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$. To complete the proof of (4.86), we have to move the cutoff to the left on both sides of (4.91). We start by considering the limit operators. For given $\psi \in \widetilde{\mathfrak{X}}_\nu$, let $\widetilde{\psi} := G(z)_\nu^* \langle v_\nu| \psi$. Then (4.90) shows that $\chi_{\nu,c}\widetilde{\psi} \in H^n(\mathbb{R}^{2N})$ for all $n \in \mathbb{N}$, so it follows from a standard Sobolev embedding theorem (see, e.g., [53, Theorem 8.8]) that $\chi_{\nu,c}\widetilde{\psi} \in C^\infty(\mathbb{R}^{2N})$. In particular, this implies that $\chi_{\nu,c}\widetilde{\psi} \in D(T_\sigma)$, where $T_\sigma(\chi_{\nu,c}\widetilde{\psi})$ is explicitly given by Eq. (2.7). Now, a direct calculation, using the defining relations for $\chi_{\nu,c}$ and $\chi_{\sigma\nu,c}$, shows that

$$T_\sigma(\chi_{\nu,c}\widetilde{\psi}) = \chi_{\sigma\nu,c} T_\sigma \widetilde{\psi},$$

where, by Proposition 4.9, $\widetilde{\psi} = G(z)_\nu^* \langle v_\nu| \psi \in D(T_\sigma)$. Therefore,

$$|u_\sigma\rangle T_\sigma \Big(\chi_{\nu,c} G(z)_\nu^* \langle v_\nu| \psi\Big) = |u_\sigma\rangle \chi_{\sigma\nu,c} T_\sigma \widetilde{\psi} = -\Big(|u_\sigma\rangle\langle v_\nu| \otimes [\chi_{\sigma\nu,c}\Theta(z)_{\sigma\nu}]\Big)\psi,$$

so the limit operator in (4.91) agrees with the desired limit operator in (4.86).
It remains to show that the left side of (4.91) agrees with the left side of (4.86), where $\phi_\varepsilon^h(z)_{\sigma\nu}$ is given by Eq. (4.88). For $\sigma = (i,j)$ with $3 \le i < j \le N$, this follows immediately from $U_\varepsilon \mathscr{K}_\sigma \chi_{\nu,c} = (1 \otimes \chi_{\sigma\nu,c}) U_\varepsilon \mathscr{K}_\sigma$. For $\sigma = (1,j)$ with $j \ge 3$, it holds, by some abuse of notation,

$$U_\varepsilon \mathscr{K}_\sigma \chi_{\nu,c} = \chi_c \left(x_2 - R + \frac{\varepsilon m_j r}{m_1 + m_j}\right) U_\varepsilon \mathscr{K}_\sigma. \tag{4.92}$$

For the purpose of computing the limit in (4.91), the right side of (4.92) may be replaced by $\chi_c(x_2 - R) U_\varepsilon \mathscr{K}_\sigma$ because $\chi_c$ is Lipschitz, $u_\sigma$ has compact support and, by Proposition 4.8, $\|\phi_\varepsilon^h(z)_{\sigma\nu}\|$ is uniformly bounded in $\varepsilon > 0$. Again, (4.86) follows. The remaining case $\sigma = (2,j)$ with $j \ge 3$ is treated similarly. □

### 4.4.3 Convergence of $\Lambda_\varepsilon(z)^{-1}$

In Sections 4.4.1 and 4.4.2.3 we have seen that $(\Lambda_\varepsilon(z)_{\text{diag}})^{-1}$ and a regularized version of $\Lambda_\varepsilon(z)_{\text{off}}$ have limits as $\varepsilon \to 0$. This will now allow us to prove invertibility of $\Lambda_\varepsilon(z)$ for small enough $\varepsilon > 0$ and large enough $z > 0$, and existence of $\lim_{\varepsilon\to 0} \Lambda_\varepsilon(z)^{-1}$. We claim that

$$\lim_{\varepsilon\to 0} \Big[\Lambda_\varepsilon(z)^{-1}\Big]_{\sigma\nu} = \begin{cases} -\dfrac{|u_\sigma\rangle\langle v_\nu|}{\langle u_\sigma \,|\, v_\sigma\rangle \langle u_\nu \,|\, v_\nu\rangle} \otimes \big[\Theta(z)^{-1}\big]_{\sigma\nu} & \text{if } \sigma, \nu \in \mathcal{J}, \\ 0 & \text{else,} \end{cases} \tag{4.93}$$

where $\Theta(z) = (\Theta(z)_{\sigma\nu})_{\sigma,\nu\in\mathcal{J}}$ is invertible in the reduced Hilbert space $\mathfrak{X}$ from Eq. (4.6).

To prove (4.93), we need to introduce some auxiliary operators. Let $\Pi : \widetilde{\mathfrak{X}} \to L^2(\mathbb{R}^2, \mathrm{d}r) \otimes \mathfrak{X}$ denote the matrix operator defined by

$$\Pi_{\sigma\nu} := \begin{cases} \delta_{\sigma\nu} & \text{if } \sigma, \nu \in \mathcal{J} \\ 0 & \text{if } \sigma \in \mathcal{J}, \nu \in \mathcal{I} \backslash \mathcal{J} \end{cases}$$

and let $U = (U_{\sigma\nu})_{\sigma,\nu\in\mathcal{J}}$, where $U_{\sigma\nu} \in \mathscr{L}(L^2(\mathbb{R}^2))$ is given by

$$U_{\sigma\nu} := \frac{|u_\sigma\rangle\langle v_\nu|}{\langle u_\sigma \,|\, v_\sigma\rangle\langle u_\nu \,|\, v_\nu\rangle}.$$

Let $\Theta(z)_{\mathrm{diag}}$ and $\Theta(z)_{\mathrm{off}}$ denote the operators in $\mathfrak{X}$ defined in terms of the components $\Theta(z)_{\sigma\sigma}$ and $\Theta(z)_{\sigma\nu}$, $\sigma \neq \nu$, that we have introduced in Sections 4.4.1 and 4.4.2.2, respectively, and let

$$\Theta(z) := \Theta(z)_{\mathrm{diag}} + \Theta(z)_{\mathrm{off}}, \qquad z > 0.$$

Proposition 4.9 shows that $\Theta(z)_{\mathrm{off}} \in \mathscr{L}(\mathfrak{X})$, while, by Eq. (4.51), $\Theta(z)_{\mathrm{diag}}$ is an unbounded operator. With the help of $\Pi$ and $U$, Equation (4.50) can now be written as

$$\lim_{\varepsilon\to 0} (\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1} = -\Pi^* \left( U \circ (\Theta(z)_{\mathrm{diag}})^{-1} \right) \Pi \tag{4.94}$$

for large enough $z > 0$. Here, as in Section 3.4, the operator product $Y \circ Z$ is defined by $(Y \circ Z)_{\sigma\nu} = Y_{\sigma\nu} \otimes Z_{\sigma\nu}$. The following proposition proves (4.93) in these new notations:

**Proposition 4.13.** *Under the hypotheses of Theorem 4.1, there exist $\varepsilon_0, z_0 > 0$ such that $\Lambda_\varepsilon(z)^{-1} \in \mathscr{L}(\widetilde{\mathfrak{X}})$ exists for all $\varepsilon \in (0, \varepsilon_0)$ and $z \in (z_0, \infty)$, and*

$$\lim_{\varepsilon\to 0} \Lambda_\varepsilon(z)^{-1} = -\Pi^* \left( U \circ \Theta(z)^{-1} \right) \Pi, \tag{4.95}$$

*where $\Theta(z)$ is a closed and invertible operator defined in $\mathfrak{X}$.*

*Proof.* Recall from Eqs. (4.21)-(4.23) that $\Lambda_\varepsilon(z) = \Lambda_\varepsilon(z)_{\mathrm{diag}} + \Lambda_\varepsilon(z)_{\mathrm{off}}$, where $\Lambda_\varepsilon(z)_{\mathrm{diag}}$ and $\Lambda_\varepsilon(z)_{\mathrm{off}}$ is the diagonal and off-diagonal part of the operator matrix $\Lambda_\varepsilon(z) = ((g_{\varepsilon,\sigma})^{-1}\delta_{\sigma\nu} + \phi_\varepsilon(z)_{\sigma\nu})_{\sigma,\nu\in\mathcal{I}}$, respectively. The Definition (4.23) of $\Lambda_\varepsilon(z)_{\mathrm{off}}$ and Proposition 4.8 imply that $\|\Lambda_\varepsilon(z)_{\mathrm{off}}\| \le C_{\mathrm{off}} < \infty$ uniformly in $\varepsilon, z > 0$. Moreover, Lemma 4.6 (and the analog for pairs $\sigma \neq (1,2)$) shows that $\Lambda_\varepsilon(z)_{\mathrm{diag}}$ is invertible and $\|(\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1}\| \le (2C_{\mathrm{off}})^{-1}$, provided that $z > z_0$ and $\varepsilon \in (0, \varepsilon_0)$ for sufficiently large $z_0 > 0$ and sufficiently small $\varepsilon_0 > 0$. We conclude that $\Lambda_\varepsilon(z)$ is invertible and

$$\Lambda_\varepsilon(z)^{-1} = \left(1 + (\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1}\Lambda_\varepsilon(z)_{\mathrm{off}}\right)^{-1} (\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1}, \qquad z > z_0,\ \varepsilon \in (0, \varepsilon_0). \tag{4.96}$$

Next, we claim that $\lim_{\varepsilon\to 0}(\Lambda_\varepsilon(z)_{\mathrm{diag}})^{-1}\Lambda_\varepsilon(z)_{\mathrm{off}}$ exists for large enough $z > 0$, which, by (4.22) and (4.23), is equivalent to the assertion that $\lim_{\varepsilon\to 0}((g_{\varepsilon,\sigma})^{-1} + \phi_\varepsilon(z)_{\sigma\sigma})^{-1}\phi_\varepsilon(z)_{\sigma\nu}$ exists for all pairs $\sigma, \nu \in \mathcal{I}, \sigma \neq \nu$. Without loss of generality, we may assume that $\sigma = (1,2) \neq (k,l) = \nu$ and in the following we use the shorthand notation of Section 4.4.1, which means that the index $\sigma$ is dropped in the diagonal contributions: $\phi_\varepsilon(z) := \phi_\varepsilon(z)_{\sigma\sigma}$, $V := V_\sigma$, $g_\varepsilon := g_{\varepsilon,\sigma}$, $a := a_\sigma$ etc. If $\int V(r)\,\mathrm{d}r/(2\pi) < a$, then it follows from Lemma 4.6 and from Proposition 4.8 that for large enough $z > 0$,

$$\lim_{\varepsilon\to 0} \left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z)\right)^{-1} \phi_\varepsilon(z)_{\sigma\nu} = 0, \qquad \nu \neq (1,2).$$

If $\int V(r)\,\mathrm{d}r/(2\pi) = a$, then it is more subtle to prove existence of the desired limit. We claim that, for all large enough $z > 0$,

$$\lim_{\varepsilon\to 0} \left(\frac{1}{g_\varepsilon} + \phi_\varepsilon(z)\right)^{-1} \phi_\varepsilon(z)_{\sigma\nu} = -\left(\frac{|u\rangle\langle v|}{\langle u \,|\, v\rangle^2} \otimes D(z)^{-1}\right) \phi(z)_{\sigma\nu}, \qquad \nu \neq (1,2), \tag{4.97}$$

where $D(z, \underline{P})$ is defined by Eqs. (4.41) and (4.42) and $\phi(z)_{\sigma\nu} \in \mathscr{L}(\widetilde{\mathfrak{X}}_\nu, \widetilde{\mathfrak{X}}_\sigma)$ is defined by Eq. (4.85). To prove this, we first show that large momenta may be neglected: As in Section 4.4.1, let $\underline{P} = (P, p_3, ..., p_N)$ be conjugate to $(R, x_3, ..., x_N)$ and let $\eta_K$, $K > 0$, denote multiplication with the characteristic function of the ball $\{\underline{P} \in \mathbb{R}^{2N-2} \,|\, |\underline{P}| \leq K\}$ in Fourier space. Now, observe that the Definition (4.25) of $Q$ implies that $Q \geq \lambda |\underline{P}|^2$ with $\lambda := 1/\max(m_1 + m_2, m_3, ..., m_N) > 0$, so for fixed $\delta > 0$ and $z > z_0$ Lemma 4.6 and Proposition 4.8 show that

$$\begin{aligned} &\left\| \left( \frac{1}{g_\varepsilon} + \phi_\varepsilon(z) \right)^{-1} (1 \otimes (1 - \eta_K))\, \phi_\varepsilon(z)_{\sigma\nu} \right\| \\ &\qquad \leq \widetilde{C} \max \left\{ |\ln \varepsilon|^{-1}, \ln(\mu(z + \lambda K^2))^{-1} \right\} C(\sigma, \nu)\, \|V_\sigma\|_{L^1}^{1/2} \|V_\nu\|_{L^1}^{1/2} < \delta/3, \end{aligned} \tag{4.98}$$

provided that $K > 0$ is large enough and $\varepsilon > 0$ is small enough. Using that

$$\lim_{\varepsilon \to 0} \left( \frac{1}{g_\varepsilon} + \phi_\varepsilon(z) \right)^{-1} = -\frac{|u\rangle \langle v|}{\langle u \,|\, v \rangle^2} \otimes D(z)^{-1} \tag{4.99}$$

by Proposition 4.7, a similar estimate also shows that

$$\left\| \left( \frac{|u\rangle \langle v|}{\langle u \,|\, v \rangle^2} \otimes D(z)^{-1} \right) (1 \otimes (1 - \eta_K))\, \phi(z)_{\sigma\nu} \right\| < \delta/3 \tag{4.100}$$

for large enough $K > 0$. In view of (4.98) and (4.100), it is clear that (4.97) will follow when we show that

$$\lim_{\varepsilon \to 0} \left( \frac{1}{g_\varepsilon} + \phi_\varepsilon(z) \right)^{-1} (1 \otimes \eta_K)\, \phi_\varepsilon(z)_{\sigma\nu} = -\left( \frac{|u\rangle \langle v|}{\langle u \,|\, v \rangle^2} \otimes D(z)^{-1} \right) (1 \otimes \eta_K)\, \phi(z)_{\sigma\nu} \tag{4.101}$$

for any fixed $K > 0$, $\nu \neq (1, 2)$ and large enough $z > 0$. To further simplify (4.101), we decompose

$$\eta_K = \eta_K\, \chi_{\sigma\nu,c} + \eta_K\, (1 - \chi_{\sigma\nu,c}), \qquad c > 0, \tag{4.102}$$

where $\chi_{\sigma\nu,c}$ is defined by Eq. (4.83). Furthermore, for $\nu = (k, l) \neq (1, 2) = \sigma$ we set

$$\gamma_{\sigma\nu}(R, x_3, ..., x_N) := \begin{cases} |R - x_l|^{1/2} & \text{if } k \in \{1, 2\}, l \geq 3, \\ |x_k - x_l|^{1/2} & \text{if } 3 \leq k < l \leq N, \end{cases}$$

and we note that the definition of $\chi_{\sigma\nu,c}$ implies that

$$\begin{aligned} \|\eta_K(1 - \chi_{\sigma\nu,c})\| = \|(1 - \chi_{\sigma\nu,c})\eta_K\| &\leq \|(1 - \chi_{\sigma\nu,c})\gamma_{\sigma\nu}\| \left\| (\gamma_{\sigma\nu})^{-1}(-\Delta' + 1)^{-1}(-\Delta' + 1)\eta_K \right\| \\ &\leq (2c)^{1/2} \left\| (\gamma_{\sigma\nu})^{-1}(-\Delta' + 1)^{-1} \right\| (K^2 + 1), \end{aligned} \tag{4.103}$$

where

$$-\Delta' := -\Delta_R + \sum_{i=3}^{N} (-\Delta_{x_i}).$$

As $|\cdot|^{-1/2} \in L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, a standard result (see, e.g., [77, Theorem 11.1]) shows that $\|(\gamma_{\sigma\nu})^{-1}(-\Delta' + 1)^{-1}\| < \infty$, so the right side of (4.103) vanishes as $c \to 0$. Hence, with the help of (4.102) and another $\delta/3$-argument, we see that the proof of (4.101) can be reduced to the assertion that for any fixed $c, K > 0$, $\nu \neq (1, 2)$ and large enough $z > 0$,

$$\lim_{\varepsilon \to 0} \left( \frac{1}{g_\varepsilon} + \phi_\varepsilon(z) \right)^{-1} (1 \otimes (\eta_K \chi_{\sigma\nu,c}))\, \phi_\varepsilon(z)_{\sigma\nu} = -\left( \frac{|u\rangle \langle v|}{\langle u \,|\, v \rangle^2} \otimes D(z)^{-1} \right) (1 \otimes (\eta_K \chi_{\sigma\nu,c}))\, \phi(z)_{\sigma\nu}.$$

However, this is immediate from (4.99) and Proposition 4.12, so (4.101) and thus (4.97) are established after all.

Since the choice $\sigma = (1,2)$ in the analysis above was immaterial, we conclude that for all pairs $\sigma, \nu \in \mathcal{I}, \sigma \neq \nu$, and large enough $z > 0$,

$$\lim_{\varepsilon \to 0} \left( (g_{\varepsilon,\sigma})^{-1} + \phi_\varepsilon(z)_{\sigma\sigma} \right)^{-1} \phi_\varepsilon(z)_{\sigma\nu} = \begin{cases} -\left( \dfrac{|u_\sigma\rangle \langle v_\sigma|}{\langle u_\sigma \,|\, v_\sigma \rangle^2} \otimes (\Theta(z)_{\sigma\sigma})^{-1} \right) \phi(z)_{\sigma\nu} & \text{if } \sigma \in \mathcal{J}, \\ 0 & \text{else.} \end{cases} \tag{4.104}$$

Let the operator $\Lambda(z)_{\text{off}} \in \mathscr{L}(\widetilde{\mathfrak{X}})$ be defined by $(\Lambda(z)_{\text{off}})_{\sigma\nu} := \phi(z)_{\sigma\nu}(1 - \delta_{\sigma\nu})$, $\sigma, \nu \in \mathcal{I}$. Then, in the notation of (4.94), (4.104) takes the form

$$\lim_{\varepsilon \to 0} (\Lambda_\varepsilon(z)_{\text{diag}})^{-1} \Lambda_\varepsilon(z)_{\text{off}} = -\Pi^* \left[ U \circ (\Theta(z)_{\text{diag}})^{-1} \right] \Pi \, \Lambda(z)_{\text{off}} =: L(z). \tag{4.105}$$

From (4.96), (4.94) and (4.105) it follows that

$$\lim_{\varepsilon \to 0} \Lambda_\varepsilon(z)^{-1} = -\left(1 + L(z)\right)^{-1} \Pi^* \left( U \circ (\Theta(z)_{\text{diag}})^{-1} \right) \Pi, \tag{4.106}$$

provided that $z \in (z_0, \infty)$ with $z_0 > 0$ large enough.

To simplify the right side of (4.106), we first note that the inverse $(1+L(z))^{-1}$ is only needed on $\operatorname{Ran} \Pi^*$, which, in view of (4.105), is left invariant by $L(z)$. Explicitly, we have that

$$(1 + L(z))^{-1} \Pi^* = \Pi^* \left( 1 - \left[ U \circ (\Theta(z)_{\text{diag}})^{-1} \right] \Pi \, \Lambda(z)_{\text{off}} \Pi^* \right)^{-1}. \tag{4.107}$$

Secondly, by Proposition 4.12, we have the factorization property

$$(\Lambda(z)_{\text{off}})_{\sigma\nu} = \phi(z)_{\sigma\nu} = -\,|u_\sigma\rangle \langle v_\nu| \otimes \Theta(z)_{\sigma\nu}, \qquad \sigma \neq \nu, \tag{4.108}$$

where $\Theta(z)_{\sigma\nu} \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$. From (4.108) it follows that

$$\left[ U \circ (\Theta(z)_{\text{diag}})^{-1} \right] \Pi \, \Lambda(z)_{\text{off}} \Pi^* = -\widetilde{U} \circ \left[ (\Theta(z)_{\text{diag}})^{-1} \Theta(z)_{\text{off}} \right], \tag{4.109}$$

where the components of $\widetilde{U} = (\widetilde{U}_{\sigma\nu})_{\sigma,\nu \in \mathcal{J}}$ are defined by

$$\widetilde{U}_{\sigma\nu} := \frac{|u_\sigma\rangle \langle v_\nu|}{\langle u_\sigma \,|\, v_\sigma \rangle}$$

and the identity $\widetilde{U}_{\sigma\nu} = U_{\sigma\eta} |u_\eta\rangle \langle v_\nu|$ was used. Furthermore, a direct computation, using the identity $(\widetilde{U} \circ Y)(\widetilde{U} \circ Z) = (\widetilde{U} \circ (YZ))$ for $Y, Z \in \mathscr{L}(\mathfrak{X})$, shows that on $\operatorname{Ran} \Pi$,

$$\left( 1 + \widetilde{U} \circ \left[ (\Theta(z)_{\text{diag}})^{-1} \Theta(z)_{\text{off}} \right] \right)^{-1} = 1 - \widetilde{U} \circ \left[ \Theta(z)^{-1} \Theta(z)_{\text{off}} \right]. \tag{4.110}$$

Indeed, $\Theta(z)^{-1} = [1 + (\Theta(z)_{\text{diag}})^{-1}\Theta(z)_{\text{off}}]^{-1} (\Theta(z)_{\text{diag}})^{-1}$ exists for large enough $z > 0$ because the Definition (4.51) of $\Theta(z, \underline{P}_\sigma)_{\sigma\sigma}$ implies that $\lim_{z \to \infty} (\Theta(z)_{\text{diag}})^{-1} = 0$ and, by Proposition 4.9, $\|\Theta(z)_{\sigma\nu}\|$, $\sigma \neq \nu$, and hence $\|\Theta(z)_{\text{off}}\|$ are uniformly bounded in $z > 0$. Now, (4.106) combined with (4.107), (4.109) and (4.110) yield

$$\begin{aligned} \lim_{\varepsilon \to 0} \Lambda_\varepsilon(z)^{-1} &= -\Pi^* \left( 1 + \widetilde{U} \circ \left[ (\Theta(z)_{\text{diag}})^{-1} \Theta(z)_{\text{off}} \right] \right)^{-1} \left( U \circ (\Theta(z)_{\text{diag}})^{-1} \right) \Pi \\ &= -\Pi^* \left( 1 - \widetilde{U} \circ \left[ \Theta(z)^{-1} \Theta(z)_{\text{off}} \right] \right) \left( U \circ (\Theta(z)_{\text{diag}})^{-1} \right) \Pi \\ &= -\Pi^* \left( U \circ \left[ (\Theta(z)_{\text{diag}})^{-1} - \Theta(z)^{-1} \Theta(z)_{\text{off}} (\Theta(z)_{\text{diag}})^{-1} \right] \right) \Pi \\ &= -\Pi^* \left( U \circ \Theta(z)^{-1} \right) \Pi, \end{aligned}$$

where the last equation follows from the second resolvent identity and the second to last equation used $\widetilde{U}_{\sigma\eta} U_{\eta\nu} = U_{\sigma\nu}$. □

## 4.5 Proof of Theorem 4.1 and properties of the Hamiltonian

*Proof of Theorem 4.1.* Recall from Section 4.2 that a point $z \in \rho(H_0)$ belongs to $\rho(H_\varepsilon) \cap \rho(H_0)$ if and only if $\Lambda_\varepsilon(z)$ is invertible in $\widetilde{\mathfrak{X}}$, and in this case $(H_\varepsilon + z)^{-1}$ can be expressed by the generalized Konno-Kuroda formula

$$(H_\varepsilon + z)^{-1} = R_0(z) - \sum_{\sigma,\nu\in\mathcal{I}} (A_{\varepsilon,\sigma} R_0(\overline{z}))^* \left[\Lambda_\varepsilon(z)^{-1}\right]_{\sigma\nu} J_\nu A_{\varepsilon,\nu} R_0(z). \tag{4.111}$$

In virtue of Proposition 4.13, there exist $\varepsilon_0, z_0 > 0$ such that $\Lambda_\varepsilon(z)$ is invertible for all $z \in (z_0, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$. This implies that $(z_0, \infty) \subseteq \rho(H_\varepsilon) \cap \rho(H_0)$ for $\varepsilon \in (0, \varepsilon_0)$ and $(H_\varepsilon + z)^{-1}$ is given by Eq. (4.111). Moreover, Proposition 4.13 also shows that $\lim_{\varepsilon\to 0} \Lambda_\varepsilon(z)^{-1}$ exists for all $z \in (z_0, \infty)$ and, by the componentwise version (4.93) of (4.95),

$$\lim_{\varepsilon\to 0} \left[\Lambda_\varepsilon(z)^{-1}\right]_{\sigma\nu} = \begin{cases} -\dfrac{|u_\sigma\rangle\langle v_\nu|}{\langle u_\sigma \,|\, v_\sigma\rangle \langle u_\nu \,|\, v_\nu\rangle} \otimes \left[\Theta(z)^{-1}\right]_{\sigma\nu} & \text{if } \sigma, \nu \in \mathcal{J}, \\ 0 & \text{else,} \end{cases}$$

where $\Theta(z)$ is a closed and invertible operator in the reduced Hilbert space $\mathfrak{X}$ from Eq. (4.6). Since $J_\nu = \operatorname{sgn}(V_\nu)$ is bounded and since, by Corollary 4.4, $\lim_{\varepsilon\to 0} A_{\varepsilon,\nu} R_0(z) = A_\nu R_0(z)$ for all $\nu \in \mathcal{I}$ and $z > 0$, we conclude that we can take the limit $\varepsilon \to 0$ on the right side of Eq. (4.111). We find that $\lim_{\varepsilon\to 0}(H_\varepsilon + z)^{-1} = R(z)$ for all $z \in (z_0, \infty)$, where

$$R(z) := R_0(z) + \sum_{\sigma,\nu\in\mathcal{J}} (A_\sigma R_0(\overline{z}))^* \left( \frac{|u_\sigma\rangle\langle v_\nu|}{\langle u_\sigma \,|\, v_\sigma\rangle \langle u_\nu \,|\, v_\nu\rangle} \otimes \left[\Theta(z)^{-1}\right]_{\sigma\nu} \right) J_\nu A_\nu R_0(z). \tag{4.112}$$

Expression (4.112) can be simplified as follows: By Corollary 4.4, we have that $A_\sigma R_0(z)\psi = v_\sigma \otimes (G(z)_\sigma \psi)$ with $G(z)_\sigma \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$ and, similarly, $J_\nu A_\nu R_0(z)\psi = u_\nu \otimes (G(z)_\nu \psi)$. Hence, (4.112) takes the form

$$R(z) = R_0(z) + \sum_{\sigma,\nu\in\mathcal{J}} G(\overline{z})^*_\sigma \left[\Theta(z)^{-1}\right]_{\sigma\nu} G(z)_\nu = R_0(z) + G(\overline{z})^* \,\Theta(z)^{-1} \,G(z), \tag{4.113}$$

where $G(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X})$ is given by $G(z)\psi = (G(z)_\sigma \psi)_{\sigma\in\mathcal{J}}$. This expression for $R(z)$ agrees with the right side of Eq. (4.8). Moreover, it also shows that $R(z)$ only depends on $V_\sigma$, $\sigma \in \mathcal{J}$, via the parameter $\beta_\sigma$ that is defined by Eqs. (4.52)-(4.54).

To prove Theorem 4.1 for $z \in (z_0, \infty)$, it remains to show that $R(z)$ indeed defines the resolvent $(H + z)^{-1}$ of a self-adjoint operator $H$. This follows from the Trotter-Kato theorem (see, e.g., [31, Theorem 5] and [71, Theorem VIII.22]*), provided that we can show that $\operatorname{Ran} R(z)$ is dense in $\mathscr{H}$. Let $T : H^2(\mathbb{R}^{2N}) \to \mathfrak{X}$ be defined as in Lemma 2.13, i.e. $T\psi = (T_\sigma \psi)_{\sigma\in\mathcal{J}}$. Let $\varphi \in (\operatorname{Ran} R(z))^\perp$ and observe that for any $\psi \in \operatorname{Ker} T$ (4.113) and $G(z) = T R_0(z)$ imply that

$$0 = \langle \varphi \,|\, R(z)(H_0 + z)\psi\rangle = \langle \varphi \,|\, \psi\rangle + \left\langle \varphi \,\middle|\, G(\overline{z})^* \,\Theta(z)^{-1}\, T\psi \right\rangle = \langle \varphi \,|\, \psi\rangle\,.$$

By Lemma 2.13, $\operatorname{Ker} T$ is dense in $\mathscr{H}$, so it follows that $\varphi = 0$. This shows that $\operatorname{Ran} R(z)$ is dense in $\mathscr{H}$, and hence, by the Trotter-Kato theorem, there exists a self-adjoint operator $H$ such that $R(z) = (H + z)^{-1}$ for all $z \in (z_0, \infty)$. We conclude that $H_\varepsilon \to H$ in the norm resolvent sense as $\varepsilon \to 0$, which completes the proof of Theorem 4.1 for $z \in (z_0, \infty)$.

To prove Theorem 4.1 for all $z \in \rho(H) \cap \rho(H_0) \subseteq \mathbb{C}$, it suffices to verify the hypotheses of [21, Theorem 2.19]. To this end, we first need to define $\Theta(z)$ for all $z \in \rho(H_0) = \mathbb{C} \setminus (-\infty, 0]$. For the diagonal parts $\Theta(z)_{\sigma\sigma}$, this is achieved by (4.51) with $\ln(\cdot)$ denoting the principal branch

*In [71] existence of $\lim_{\varepsilon\to 0}(H_\varepsilon + z)^{-1}$ in two points $z$ with $\pm \operatorname{Im}(z) > 0$ is assumed, but the proof can be adapted to the case where the limit exists for all $z$ from a non-empty open interval $I \subseteq \mathbb{R}$.

of the logarithm. To define the off-diagonal parts for $z \in \rho(H_0)$, we use $\Theta(z)_{\sigma\nu} := -T_\sigma G(\overline{z})^*_\nu$, $\sigma \neq \nu$, which agrees with (4.70) for $z > 0$, but, a priori, may be an unbounded operator for other values of $z \in \rho(H_0)$. From Proposition 2.12 $(i)$ it follows that

$$G(\overline{z})^*_\nu = G(\overline{w})^*_\nu + (w - z)R_0(z)G(\overline{w})^*_\nu, \qquad z, w \in \rho(H_0). \tag{4.114}$$

Choosing $w \in (0, \infty)$, we know from Proposition 4.9 that $\operatorname{Ran} G(\overline{w})^*_\nu \subseteq D(T_\sigma)$ and that $T_\sigma G(\overline{w})^*_\nu$ is bounded. Likewise, $T_\sigma R_0(z)$ is a bounded operator and hence, by Eq. (4.114), $\Theta(z)_{\sigma\nu} = -T_\sigma G(\overline{z})^*_\nu$ is a bounded operator in $\mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$ for all $z \in \rho(H_0)$. Furthermore, Eq. (4.114) now implies that

$$\Theta(z)_{\sigma\nu} = \Theta(w)_{\sigma\nu} + (z - w)G(z)_\sigma G(\overline{w})^*_\nu, \qquad \sigma \neq \nu, \quad z, w \in \rho(H_0). \tag{4.115}$$

By Proposition 4.14, below, the extended operator $\Theta(z)$ satisfies the hypotheses of [21, Theorem 2.19], so the proof of Theorem 4.1 is complete. □

**Proposition 4.14.** *Let $w, z \in \rho(H_0)$. Then the operator $\Theta(z)$ has the following properties:*

*(i)* $\Theta(z)^* = \Theta(\overline{z})$.

*(ii)* $\Theta(z) = \Theta(w) + (z - w)G(z)G(\overline{w})^*$.

*(iii)* $0 \in \rho(\Theta(z))$ *for some* $z \in \rho(H_0)$.

*Remark.* Property $(ii)$ implies that $D = D(\Theta(z))$ is independent of $z \in \rho(H_0)$.

*Proof.* Property $(iii)$ has already been verified in Proposition 4.13. To prove Property $(ii)$, we first recall from the proof of Theorem 4.1 that $\Theta(z)$ and $G(z)$ have the components $\Theta(z)_{\sigma\nu}$, $\sigma, \nu \in \mathcal{J}$, and $G(z)_\sigma, \sigma \in \mathcal{J}$, respectively. Hence, we have to verify that

$$\Theta(z)_{\sigma\nu} = \Theta(w)_{\sigma\nu} + (z - w)G(z)_\sigma G(\overline{w})^*_\nu, \qquad \sigma, \nu \in \mathcal{J}. \tag{4.116}$$

For $\sigma \neq \nu$, this has been shown in Eq. (4.115). In the case $\sigma = \nu$, we assume $\sigma = (1, 2) = \nu$ for notational simplicity. The Definition (2.12) of $T_\sigma$ and $\mathscr{K}_\sigma R_0(z) = (\widetilde{H}_0 + z)^{-1}\mathscr{K}_\sigma$, where $\widetilde{H}_0$ is defined by Eq. (4.77), imply that

$$G(z)_\sigma G(\overline{w})^*_\sigma = \tau \mathscr{K}_\sigma R_0(z) \left(\tau \mathscr{K}_\sigma R_0(\overline{w})\right)^* = \tau(\widetilde{H}_0 + z)^{-1} \left(\tau(\widetilde{H}_0 + \overline{w})^{-1}\right)^*.$$

Using now the Definition (2.10) of $\tau$ together with a Fourier transform in $(R, x_3, ..., x_N)$, we find that $(z - w)G(z)_\sigma G(\overline{w})^*_\sigma$ acts pointwise in $\underline{P}_\sigma = (P, p_3, ..., p_N)$ by multiplication with

$$\frac{z - w}{4\pi^2} \int_{\mathbb{R}^2} \left(\frac{p^2}{\mu_\sigma} + Q + z\right)^{-1} \left(\frac{p^2}{\mu_\sigma} + Q + w\right)^{-1} \mathrm{d}p = \frac{\mu_\sigma}{4\pi} \left[\ln(z + Q) - \ln(w + Q)\right],$$

where $Q \geq 0$ is defined by Eq. (4.25). Now, (4.116) for $\sigma = \nu = (1, 2)$ follows from the Definition (4.51) of $\Theta(z, \underline{P}_\sigma)_{\sigma\sigma}$.

To prove Property $(i)$, we use the fact that $\Theta(z)$ does not depend on the particular choices of $V_\sigma$, $a_\sigma$ and $b_\sigma$ as long as the set $\mathcal{J}$ and the parameters $\beta_\sigma$ remain unchanged. For $\sigma \in \mathcal{J}$ we choose $V_\sigma > 0$, $a_\sigma = \int V_\sigma(r)\,\mathrm{d}r/(2\pi)$ and $b_\sigma$ to solve (4.52) and (4.53) for the given values of $\beta_\sigma$. For $\sigma \in \mathcal{I} \setminus \mathcal{J}$ we choose $V_\sigma = 0$. Moreover, we fix $z_0 > 0$ so that (4.95) holds for all $z \in (z_0, \infty)$. Then $u_\sigma = v_\sigma$ for all pairs $\sigma \in \mathcal{I}$ and hence Eqs. (4.13) and (4.20) show that $\Lambda_\varepsilon(z)$ is self-adjoint for all $\varepsilon > 0$ and all $z \in (z_0, \infty)$. Now, it follows from (4.95), or more directly from (4.93), that $\Theta(z)^{-1}$ is self-adjoint, too. Therefore, $\Theta(z)$ is self-adjoint for $z \in (z_0, \infty)$, and Property $(i)$ for general $z \in \rho(H_0)$ now follows from Property $(ii)$ by choosing $w \in (z_0, \infty)$. □

Our results on $\Theta(z)$ imply the following lower bound for $\sigma(H)$ (see also [30, 31]).

**Proposition 4.15.** *Let $H$ denote the Hamiltonian from Theorem 4.1. Then, with $N_{\mathcal{J}} := |\mathcal{J}|$, $\mu^- := \min_{\sigma\in\mathcal{J}} \mu_\sigma$, $\beta^- := \min_{\sigma\in\mathcal{J}} \beta_\sigma$ and $m := \max_{i=1,\dots,N} m_i$, it holds that*

$$\Sigma = \inf \sigma(H) \geq -\exp\left(\frac{\pi m}{\mu^-}(N_{\mathcal{J}} - 1) - \frac{\beta^-}{\pi}\right). \tag{4.117}$$

*Proof.* Without restriction, we can assume that $N_{\mathcal{J}} \geq 1$ because for $H = H_0$ the lower bound from (4.117) is obvious. From [21, Theorem 2.19] and from Proposition 4.14 we know that a point $z \in \rho(H_0)$ belongs to $\rho(H)$ if and only if $\Theta(z)$ is invertible in $\mathfrak{X}$. Since $\Theta(z)$ is self-adjoint for $z > 0$, it suffices to show that $\Theta(z)$ is bounded from below by a positive constant for $z > z_0 := \exp(\pi m(N_{\mathcal{J}} - 1)/\mu^- - \beta^-/\pi)$. To this end, we use the Definition (4.51) of $\Theta(z, \underline{P_\sigma})_{\sigma\sigma}$ and for $\sigma \neq \nu$ we use the bound $\|\Theta(z)_{\sigma\nu}\| \leq m/4$ from Proposition 4.9. Let $z \in (z_0, \infty)$ and observe that $\ln(z) + \beta^-/\pi > 0$, so we conclude that, for all $w = (w_\sigma)_{\sigma\in\mathcal{J}} \in D(\Theta(z))$,

$$\begin{aligned}
\langle w \,|\, \Theta(z) w\rangle &= \sum_{\sigma\in\mathcal{J}} \langle w_\sigma \,|\, \Theta(z)_{\sigma\sigma} w_\sigma\rangle + \sum_{\substack{\sigma,\nu\in\mathcal{J}\\ \sigma\neq\nu}} \langle w_\sigma \,|\, \Theta(z)_{\sigma\nu} w_\nu\rangle \\
&\geq \frac{\mu^-}{4\pi}\left(\ln(z) + \frac{\beta^-}{\pi}\right) \sum_{\sigma\in\mathcal{J}} \|w_\sigma\|^2 - \frac{m}{4} \sum_{\substack{\sigma,\nu\in\mathcal{J}\\ \sigma\neq\nu}} \|w_\sigma\|\|w_\nu\| \\
&\geq \frac{\mu^-}{4\pi}\left(\ln(z) + \frac{\beta^-}{\pi}\right) \|w\|^2 - \frac{m}{8} \sum_{\substack{\sigma,\nu\in\mathcal{J}\\ \sigma\neq\nu}} (\|w_\sigma\|^2 + \|w_\nu\|^2) \\
&= \left[\frac{\mu^-}{4\pi}\left(\ln(z) + \frac{\beta^-}{\pi}\right) - \frac{m}{4}(N_{\mathcal{J}} - 1)\right] \|w\|^2.
\end{aligned}$$

The expression in brackets is positive for $z > z_0 = \exp(\pi m(N_{\mathcal{J}} - 1)/\mu^- - \beta^-/\pi)$, which proves (4.117). □

We conclude this section with the proof that $H$ satisfies a stronger version of the Properties **(i)** – **(iii)** from Section 1.2.2:

**Proposition 4.16.** *The Hamiltonian $H$ from Theorem 4.1 is local in the following sense: If $\psi \in D(H)$ and $\psi = 0$ a.e. in some non-empty open set $U \subseteq \mathbb{R}^{2N}$, then $H\psi = 0$ a.e. in $U$. Moreover, $H$ is invariant under all Euclidean isometries of $\mathbb{R}^2$. That is, for any orthogonal matrix $O \in \mathbb{R}^{2\times 2}$ and any $h \in \mathbb{R}^2$, $HT(O,h)_{\text{tot}} = T(O,h)_{\text{tot}} H$, where $T(O,h)_{\text{tot}} \in \mathscr{L}(\mathscr{H})$ is given by*

$$(T(O,h)_{\text{tot}}\psi)(x_1, x_2, \dots, x_N) := \psi(Ox_1 + h, Ox_2 + h, \dots, Ox_N + h).$$

*Proof.* The first part follows from Lemma C.2 in the appendix of [6], but for the convenience of the reader we spell out the details here. First, we suppose that $U \subseteq \mathbb{R}^{2N} \setminus \Gamma$ and we recall from Section 4.1 that Corollary 4.2 implies that $H$ is a self-adjoint extension of $H_0 \restriction C_0^\infty(\mathbb{R}^{2N} \setminus \Gamma)$. Therefore, all $\varphi \in C_0^\infty(U)$ belong to $C_0^\infty(\mathbb{R}^{2N} \setminus \Gamma) \subseteq D(H)$ and

$$\langle H\psi \,|\, \varphi\rangle = \langle \psi \,|\, H_0 \varphi\rangle = 0$$

because $\operatorname{supp}(\psi) \cap \operatorname{supp}(\varphi) = \emptyset$. This implies that $H\psi = 0$ a.e. in $U$. If $U \cap \Gamma \neq \emptyset$, then we apply the above argument to the open set $U \setminus \Gamma$. It follows that $H\psi = 0$ a.e. in $U \setminus \Gamma$ and hence $H\psi = 0$ a.e. in $U$ because $\Gamma$ is a set of measure zero in $\mathbb{R}^{2N}$.

For the second part we again use that $\Theta(z)$, and hence $H$, do not depend on the particular choices of $V_\sigma$, $a_\sigma$ and $b_\sigma$ as long as the set $\mathcal{J}$ and the parameters $\beta_\sigma$ remain unchanged. Hence, by adjusting $b_\sigma$ accordingly, we may choose $V_\sigma(r) = \exp(-|r|)$ and $a_\sigma = \int V_\sigma(r)\,\mathrm{d}r/(2\pi)$ for $\sigma \in \mathcal{J}$. For $\sigma \in \mathcal{I} \setminus \mathcal{J}$ we choose $V_\sigma(r) = 0$. Then it is immediate from the Definition (4.1) of $H_\varepsilon$, $\varepsilon > 0$, that $H_\varepsilon T(O,h)_{\mathrm{tot}} = T(O,h)_{\mathrm{tot}} H_\varepsilon$. It follows that $T(O,h)_{\mathrm{tot}}(H_\varepsilon + i)^{-1} = (H_\varepsilon + i)^{-1} T(O,h)_{\mathrm{tot}}$, and in the limit $\varepsilon \to 0$ we arrive at $T(O,h)_{\mathrm{tot}}(H + i)^{-1} = (H + i)^{-1} T(O,h)_{\mathrm{tot}}$. This proves the second part of the proposition. □

## 4.6 The quadratic form of the Hamiltonian

In this section we determine the quadratic form of the Hamiltonian $H$ from Theorem 4.1 and we show, in the case of $N$ particles of mass one, that it agrees with a quadratic form $\mathrm{F}_\beta$ introduced in [30]. This proves that $H$ agrees with the TMS Hamiltonian $H_\beta$ defined by [30, Eqs. (5.3)-(5.4)].

We start by deriving an explicit formula for the quadratic form of $H$ restricted to $D(H)$:

**Lemma 4.17.** *Let $H$ denote the Hamiltonian from Theorem 4.1 and let $z \in \rho(H_0) \cap \rho(H)$. Then, for any $\psi \in D(H)$, it holds that*

$$\langle \psi \,|\, H\psi \rangle = \langle \psi - G(z)^* w \,|\, (H_0 + z)(\psi - G(\overline{z})^* w) \rangle + \langle w \,|\, \Theta(z) w \rangle - z\|\psi\|^2, \tag{4.118}$$

*where $w = (w_\sigma)_{\sigma \in \mathcal{J}} \in D(\Theta(z))$ is uniquely determined by* (4.9) *and* (4.10)*.*

*Proof.* By Eq. (4.11), we have that $(H + z)\psi = (H_0 + z)\psi_0$, which yields

$$\begin{aligned} \langle \psi \,|\, (H + z)\psi \rangle &= \langle \psi \,|\, (H_0 + z)\psi_0 \rangle \\ &= \langle \psi - G(z)^* w \,|\, (H_0 + z)\psi_0 \rangle + \langle w \,|\, G(z)(H_0 + z)\psi_0 \rangle \,. \end{aligned} \tag{4.119}$$

By the definition of $G(z)$ and by (4.10), $G(z)(H_0 + z)\psi_0 = T\psi_0 = \Theta(z) w$, so it follows from (4.9) and (4.119) that

$$\langle \psi \,|\, (H + z)\psi \rangle = \langle \psi - G(z)^* w \,|\, (H_0 + z)\,(\psi - G(\overline{z})^* w) \rangle + \langle w \,|\, \Theta(z) w \rangle \,,$$

which proves (4.118). □

Since $H$ is self-adjoint, the quadratic form from Lemma 4.17 is closable and we are now going to determine an explicit description of its closure. From Eq. (4.51), it follows that $\Theta(z)_{\sigma\sigma} \geq c_1 \ln(z) + c_2$ in operator sense, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ depend on the pair $\sigma \in \mathcal{J}$ but not on $z \in (0, \infty)$. Combining this with the fact that, by Proposition 4.9, $\|\Theta(z)_{\mathrm{off}}\|$ is uniformly bounded in $z \in (0, \infty)$, we see that $\Theta(z) \geq c > 0$ for sufficiently large $z \in \rho(H) \cap (0, \infty)$. For such $z$, we introduce a quadratic form $q$ with domain

$$D(q) := \left\{ \psi \in \mathscr{H} \,\middle|\, \exists w \in D(\Theta(z)^{1/2}) : \psi - G(z)^* w \in H^1(\mathbb{R}^{2N}) \right\}$$

by

$$q(\psi) := \|(H_0 + z)^{1/2}(\psi - G(z)^* w)\|^2 + \|\Theta(z)^{1/2} w\|^2 - z\|\psi\|^2, \tag{4.120}$$

which agrees with the right side of (4.118) if $\psi \in D(H) \subseteq D(q)$. In virtue of the following lemma, which is our analog of [30, Lemma 3.2], it is clear that $w = w^\psi$ is uniquely determined by $\psi \in D(q)$ and, in particular, that $q$ is well-defined:

**Lemma 4.18.** *If $z > 0$ is so large that $\Theta(z) \geq c > 0$ and $w = (w_\sigma)_{\sigma \in \mathcal{J}} \in D(\Theta(z)^{1/2}) \setminus \{0\}$, then $G(z)^* w \notin H^1(\mathbb{R}^{2N})$.*

*Proof.* Recall from the sentence including Eq. (4.113) that $G(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X})$ is given by $G(z)\psi = (G(z)_\sigma \psi)_{\sigma\in\mathcal{J}}$ with $G(z)_\sigma = T_\sigma R_0(z) \in \mathscr{L}(\mathscr{H}, \mathfrak{X}_\sigma)$. Hence, for all $w = (w_\sigma)_{\sigma\in\mathcal{J}} \in \mathfrak{X}$,

$$G(z)^* w = \sum_{\sigma\in\mathcal{J}} G(z)^*_\sigma w_\sigma, \tag{4.121}$$

where, by Eq. (4.72) for $\sigma = (i,j)$,

$$\left(\widehat{G(z)^*_\sigma w_\sigma}\right)(p_1, ..., p_N) = \frac{1}{2\pi}\left(z + \sum_{n=1}^N \frac{p_n^2}{m_n}\right)^{-1} \cdot \widehat{w_\sigma}\left(p_i + p_j, p_1, ...\widehat{p_i}...\widehat{p_j}..., p_N\right). \tag{4.122}$$

Now, assume that $w \in D(\Theta(z)^{1/2}) \setminus \{0\}$ satisfies $G(z)^* w \in H^1(\mathbb{R}^{2N})$. To show that this leads to a contradiction, we first observe that for all $h > 0$,

$$\begin{aligned}
&\left\langle (H_0 + z)^{1/2} G(z)^* w \,\middle|\, (H_0 + z)^{1/2} G(z)^* w \right\rangle \\
&\geq (2\pi)^{-2} \int \mathrm{d}p_1 \cdots \mathrm{d}p_N \sum_{\sigma=(i,j)\in\mathcal{J}} \left( \frac{|\widehat{w_\sigma}(p_i + p_j, p_1, ...\widehat{p_i}...\widehat{p_j}..., p_N)|^2}{z + \sum_{n=1}^N p_n^2/m_n} \chi_{[0,2h]}\left(\sum_{n=1}^N \frac{p_n^2}{m_n}\right) \right. \\
&\qquad \left. - \sum_{\substack{\nu=(k,l)\in\mathcal{J} \\ \nu\neq\sigma}} \frac{|\widehat{w_\sigma}(p_i + p_j, p_1, ...\widehat{p_i}...\widehat{p_j}..., p_N)|\,|\widehat{w_\nu}(p_k + p_l, p_1, ...\widehat{p_k}...\widehat{p_l}..., p_N)|}{z + \sum_{n=1}^N p_n^2/m_n} \right),
\end{aligned} \tag{4.123}$$

where $\chi_{[0,2h]}$ denotes the characteristic function of the interval $[0, 2h]$. For $\sigma = (1,2)$ and $\nu = (k,l) \in \{(1,3), (3,4)\}$, it follows from (4.75) and (4.76) that

$$\begin{aligned}
&\int \mathrm{d}p_1 \cdots \mathrm{d}p_N \, \frac{|\widehat{w_\sigma}(p_1 + p_2, p_3, ..., p_N)|\,|\widehat{w_\nu}(p_k + p_l, p_1, ...\widehat{p_k}...\widehat{p_l}..., p_N)|}{z + \sum_{n=1}^N p_n^2/m_n} \\
&\leq \pi^2 \max(m_2, m_l) \|w_\sigma\| \|w_\nu\| < \infty.
\end{aligned}$$

Using similar estimates for the other pairs $\sigma, \nu \in \mathcal{J}$, $\sigma \neq \nu$, we find that all summands in the last line of (4.123) define integrable functions. We now show that the integral over the diagonal contributions to (4.123) diverges as $h \to \infty$. Since $w \neq 0$, there exists a pair $\sigma \in \mathcal{J}$ with $\|w_\sigma\| > 0$ and without restriction we may assume that $\sigma = (1,2)$. With the help of the substitution

$$P := p_1 + p_2, \qquad p := \frac{m_1 p_2 - m_2 p_1}{m_1 + m_2},$$

we find that the integral over the $(1,2)$-contribution has the lower bound

$$\begin{aligned}
&\int \mathrm{d}p_1 \cdots \mathrm{d}p_N \, \chi_{[0,2h]}\left(\sum_{n=1}^N \frac{p_n^2}{m_n}\right) \frac{|\widehat{w_\sigma}(p_1 + p_2, p_3, ..., p_N)|^2}{z + \sum_{n=1}^N p_n^2/m_n} \\
&\geq \int \mathrm{d}\underline{P}\, \chi_{[0,h]}(Q)\, |\widehat{w_\sigma}(\underline{P})|^2 \int \mathrm{d}p\, \chi_{[0,h]}(p^2/\mu) \left(z + \frac{p^2}{\mu} + Q\right)^{-1} \\
&= \pi\mu \int \mathrm{d}\underline{P}\, \chi_{[0,h]}(Q)\, |\widehat{w_\sigma}(\underline{P})|^2 \left(\ln(z + h + Q) - \ln(z + Q)\right) \\
&\geq \mu(\pi \ln(h) + \beta_\sigma) \int \chi_{[0,h]}(Q)\, |\widehat{w_\sigma}(\underline{P})|^2 \, \mathrm{d}\underline{P} - 4\pi^2 \int \chi_{[0,h]}(Q)\, |\widehat{w_\sigma}(\underline{P})|^2\, \Theta(z, \underline{P})_{\sigma\sigma} \, \mathrm{d}\underline{P},
\end{aligned} \tag{4.124}$$

where $\underline{P} = (P, p_3, ..., p_N)$, $\mu = \mu_{(1,2)}$ is a reduced mass, $Q \geq 0$ is defined by Eq. (4.25), and $\Theta(z, \underline{P})_{\sigma\sigma}$ is defined by Eq. (4.51). Since $w \in D(\Theta(z)^{1/2})$ by assumption, it follows that in the limit $h \to \infty$, the second term in the last line of (4.124) converges to $4\pi^2 \|(\Theta(z)_{\sigma\sigma})^{1/2} w_\sigma\|^2 < \infty$, while the first term diverges to $\infty$. From this we conclude that the right side of (4.123) diverges to $\infty$ in the limit $h \to \infty$, which contradicts our assumption that $G(z)^* w \in H^1(\mathbb{R}^{2N})$. $\square$

It is obvious from (4.120) that $q$ is bounded from below and a standard argument also shows that $q$ is closed. Hence, there exists a unique self-adjoint operator $H_q$ associated with $q$. We are going to show that $H_q = H$. Let $\psi \in D(H_q)$ be fixed. Then the definitions of $q$ and $H_q$ imply that, for all $\varphi \in D(q)$,

$$\begin{aligned}\langle (H_q+z)\psi \,|\, \varphi\rangle =& \left\langle (H_0+z)^{1/2}(\psi - G(z)^* w^\psi) \,\middle|\, (H_0+z)^{1/2}(\varphi - G(z)^* w^\varphi)\right\rangle \\ &+ \left\langle \Theta(z)^{1/2} w^\psi \,\middle|\, \Theta(z)^{1/2} w^\varphi \right\rangle. \end{aligned} \tag{4.125}$$

Choosing $w^\varphi = 0$, we see that each $\varphi \in H^1(\mathbb{R}^{2N})$ belongs to $D(q)$, and hence, for $\varphi \in H^1(\mathbb{R}^{2N})$, Eq. (4.125) becomes

$$\langle (H_q+z)\psi \,|\, \varphi\rangle = \left\langle (H_0+z)^{1/2}(\psi - G(z)^* w^\psi) \,\middle|\, (H_0+z)^{1/2}\varphi\right\rangle. \tag{4.126}$$

This equation implies that $\psi_0 := \psi - G(z)^* w^\psi \in D(H_0)$, and that

$$(H_q+z)\psi = (H_0+z)\psi_0. \tag{4.127}$$

We have thus verified condition (4.9) of Corollary 4.2. It remains to check condition (4.10).

Given $w \in D(\Theta(z)^{1/2})$, we choose $\varphi := G(z)^* w$, so that $\varphi - G(z)^* w = 0 \in H^1(\mathbb{R}^{2N})$ and hence $\varphi \in D(q)$. From Eqs. (4.125) and (4.127), we now see that, for all $w \in D(\Theta(z)^{1/2})$,

$$\begin{aligned}\left\langle \Theta(z)^{1/2} w^\psi \,\middle|\, \Theta(z)^{1/2} w\right\rangle &= \langle (H_q+z)\psi \,|\, \varphi\rangle \\ &= \langle (H_0+z)\psi_0 \,|\, G(z)^* w\rangle \\ &= \langle G(z)(H_0+z)\psi_0 \,|\, w\rangle = \langle T\psi_0 \,|\, w\rangle\,. \end{aligned}$$

This equation implies that $w^\psi \in D(\Theta(z))$ and that $\Theta(z) w^\psi = T\psi_0$, which is condition (4.10). Corollary 4.2 now shows that $\psi \in D(H)$ and, in view of Eq. (4.127), that $H_q \subseteq H$. From the self-adjointness of $H_q$ and $H$, we conclude that $H_q = H$.

In the case of $m_i = 1$ for $i = 1, ..., N$, we are going to show that $q$ agrees with a quadratic form introduced in [30, Eqs. (2.13)-(2.16)]. For given $\beta = (\beta_\sigma)_{\sigma\in\mathcal{J}}$, the quadratic form in [30] is denoted by $\mathrm{F}_\beta$ and, in our notation, it is defined by

$$\mathrm{F}_\beta(\psi) := \|\nabla(\psi - G_z^{2N} * \xi^\psi)\|^2 + z\|\psi - G_z^{2N} * \xi^\psi\|^2 + \Phi_\beta^{z,1}(\xi^\psi) + \Phi^{z,2}(\xi^\psi) - z\|\psi\|^2 \tag{4.128}$$

on the domain

$$D(\mathrm{F}_\beta) = \left\{\psi \in \mathscr{H} \,\middle|\, \exists \xi^\psi \in D(\Phi_\beta^{z,1}) : \psi - G_z^{2N} * \xi^\psi \in H^1(\mathbb{R}^{2N})\right\}.$$

The right side of (4.128) is independent of the particular choice of $z \in (0,\infty)$ and $\xi^\psi = (\xi_\sigma)_{\sigma\in\mathcal{J}}$ is a collection of "charges" $\xi_\sigma \in L^2(\mathbb{R}^{2(N-1)})$, which are uniquely determined by $\psi \in D(\mathrm{F}_\beta)$. For $\sigma = (i,j)$, $\xi_\sigma$ may be interpreted as a function on the hyperplane $x_i = x_j$ and the convolution $G_z^{2N} * \xi^\psi$ is to be understood in the sense that (see the proof of [30, Lemma 3.2 (*a*)])

$$\widehat{G_z^{2N} * \xi^\psi}(p_1, ..., p_N) = \sum_{\sigma=(i,j)\in\mathcal{J}} \left(z + \sum_{n=1}^N p_n^2\right)^{-1} \cdot \widehat{\xi_\sigma}\left(p_1, ..., p_{i-1}, \frac{p_i+p_j}{\sqrt{2}}, p_{i+1}, ...\widehat{p_j}..., p_N\right). \tag{4.129}$$

Moreover, by [30, Theorem 3.3], $\mathrm{F}_\beta$ is bounded from below and closed on $D(\mathrm{F}_\beta)$.

Assuming that $m_i = 1$, $i = 1, ..., N$, and that $z \in \rho(H) \cap (0,\infty)$ is so large that $\Theta(z) \geq c > 0$, we now write the various contributions to the right side of (4.120) in a form that will allow us to compare them to their counterparts in (4.128). We first infer from Eqs. (4.121), (4.122) and (4.129) that

$$G(z)^* w = G_z^{2N} * \xi, \qquad w = (w_\sigma)_{\sigma\in\mathcal{J}}, \xi = (\xi_\sigma)_{\sigma\in\mathcal{J}}, \tag{4.130}$$

where $\xi_\sigma$ agrees with $w_\sigma$ up to a permutation of the arguments and rescaling:

$$\xi_\sigma(x_1,...\widehat{x}_j...,x_N) = \frac{1}{4\pi}\, w_\sigma\left(\frac{x_i}{\sqrt{2}}, x_1, ...\widehat{x}_i...\widehat{x}_j..., x_N\right), \qquad \sigma=(i,j), \tag{4.131}$$

which, after Fourier transform, is equivalent to

$$\widehat{\xi_\sigma}(p_1,...\widehat{p}_j...,p_N) = \frac{1}{2\pi}\, \widehat{w_\sigma}\left(\sqrt{2}p_i, p_1, ...\widehat{p}_i...\widehat{p}_j..., p_N\right), \qquad \sigma=(i,j). \tag{4.132}$$

Conversely, given $\xi_\sigma \in L^2(\mathbb{R}^{2(N-1)})$ for all $\sigma \in \mathcal{J}$, the identity (4.131) uniquely determines $w = (w_\sigma)_{\sigma\in\mathcal{J}} \in \mathfrak{X}$ and again (4.130) holds true. In particular, it follows that

$$\|(H_0+z)^{1/2}(\psi - G(z)^* w)\|^2 = \|\nabla(\psi - G_z^{2N} * \xi)\|^2 + z\|\psi - G_z^{2N} * \xi\|^2,$$

provided that $\psi - G(z)^* w \in H^1(\mathbb{R}^{2N})$. The third and the fourth terms in (4.128) correspond to the diagonal and off-diagonal contributions, respectively, in

$$\|\Theta(z)^{1/2} w\|^2 = \sum_{\sigma\in\mathcal{J}} \|(\Theta(z)_{\sigma\sigma})^{1/2} w_\sigma\|^2 + \sum_{\substack{\sigma,\nu\in\mathcal{J}\\ \sigma\neq\nu}} \langle w_\sigma \,|\, \Theta(z)_{\sigma\nu} w_\nu\rangle\,.$$

We begin with the diagonal ones. The operator $\Theta(z)_{\sigma\sigma}$, $\sigma = (i,j)$, acts pointwise in $\underline{P}_\sigma = (P, p_1, ...\widehat{p}_i...\widehat{p}_j..., p_N)$ by multiplication with $\Theta(z, \underline{P}_\sigma)_{\sigma\sigma}$ defined by Eq. (4.51). For a given vector $w = (w_\sigma)_{\sigma\in\mathcal{J}} \in D(\Theta(z)^{1/2})$, it follows that

$$\sum_{\sigma\in\mathcal{J}} \|(\Theta(z)_{\sigma\sigma})^{1/2} w_\sigma\|^2 = \sum_{\sigma\in\mathcal{J}} \int \mathrm{d}\underline{P}_\sigma\, \Theta(z,\underline{P}_\sigma)_{\sigma\sigma}\, |\widehat{w_\sigma}\,(\underline{P}_\sigma)|^2\,,$$

which agrees with $\Phi_\beta^{z,1}(\xi)$ defined by [30, Eq. (2.15)], where $\beta = (\beta_\sigma)_{\sigma\in\mathcal{J}}$ and $\xi \in D(\Phi_\beta^{z,1})$ are given by Eqs. (4.52)-(4.54) and (4.132), respectively. Conversely, given $\xi \in D(\Phi_\beta^{z,1})$, a straightforward computation shows that Eq. (4.132) defines some $w \in D(\Theta(z)^{1/2})$ and that $\Phi_\beta^{z,1}(\xi) = \sum_{\sigma\in\mathcal{J}} \|(\Theta(z)_{\sigma\sigma})^{1/2} w_\sigma\|^2$. In particular, we see that $w \in D(\Theta(z)^{1/2})$ if and only if Eq. (4.132) defines a vector $\xi \in D(\Phi_\beta^{z,1})$, and hence, using Eq. (4.130), it follows that $D(q) = D(\mathrm{F}_\beta)$. It remains to examine the contributions of the off-diagonal operators $\Theta(z)_{\sigma\nu} \in \mathscr{L}(\mathfrak{X}_\nu, \mathfrak{X}_\sigma)$. Here, the key is the identity (cf. Eq. (4.70))

$$\Theta(z)_{\sigma\nu} = -T_\sigma G(z)_\nu^*, \qquad \sigma \neq \nu. \tag{4.133}$$

Recall from Eq. (4.71) that for $\sigma = (i,j)$ and $\psi \in D(T_\sigma)$,

$$\begin{aligned}&(\widehat{T_\sigma \psi})\,(P, p_1, ...\widehat{p}_i...\widehat{p}_j..., p_N)\\ &\quad = \frac{1}{2\pi}\int \mathrm{d}p\; \widehat{\psi}\left(p_1, ..., p_{i-1}, \frac{P}{2} - p, p_{i+1}, ..., p_{j-1}, \frac{P}{2} + p, p_{j+1}, ..., p_N\right).\end{aligned} \tag{4.134}$$

Now, using first Eqs. (4.133), (4.134) and the substitution

$$P := p_i + p_j, \quad p := \frac{p_j - p_i}{2},$$

and inserting the identity (4.122) for $G(z)_\nu^* w_\nu$ thereafter, we obtain that

$$\begin{aligned}&\sum_{\substack{\sigma,\nu\in\mathcal{J}\\ \sigma\neq\nu}} \langle w_\sigma \,|\, \Theta(z)_{\sigma\nu} w_\nu\rangle\\ &\quad = -\sum_{\substack{\sigma,\nu\in\mathcal{J}\\ \sigma\neq\nu}} \int \mathrm{d}p_1 \cdots \mathrm{d}p_N\, \frac{\overline{\widehat{w_\sigma}}\,(p_i+p_j, p_1, ...\widehat{p}_i...\widehat{p}_j..., p_N)\, \widehat{w_\nu}\,(p_k+p_l, p_1, ...\widehat{p}_k...\widehat{p}_l..., p_N)}{4\pi^2(z + \sum_{n=1}^N p_n^2)},\end{aligned}$$

where the sum runs over all $\sigma = (i,j) \neq (k,l) = \nu$ in $\mathcal{J}$. In view of Eq. (4.132), this coincides with the expression for $\Phi^{z,2}(\xi)$ from [30, Eq. (2.16)]. This completes the proof that $q = \mathrm{F}_\beta$.

# 5 Weakness of short-range interactions in Fermi gases

In this section we analyze and quantify the weakness of short-range interactions among identical spinless fermions, the main results being Theorems 5.1 and 5.6. This is based on reference [40].

## 5.1 Main results

As explained in Section 1.3.2, short-range interactions among equal spin fermions in ultracold quantum gases are usually neglected, while at the same time the interaction among particles of opposite spin is modeled by contact interactions. In the physics literature this is often justified by arguing that the Pauli principle forces the wave function to be antisymmetric w.r.t. permutations of the fermion positions, which, in particular, implies that the wave function vanishes when the positions of two fermions coincide. The goal of this section is to justify this simplification mathematically and, in the case of interactions of very small but positive range, to derive an (asymptotic) estimate for the approximation error. To this end, we consider an ultracold Fermi gas consisting of $N \geq 2$ identical spinless fermions in $d \geq 1$ dimensions. The underlying Hilbert space is the fermionic subspace

$$\mathscr{H}_{\mathrm{f}} = \bigwedge_{i=1}^{N} L^2(\mathbb{R}^d, \mathrm{d}x_i) \tag{5.1}$$

of $L^2(\mathbb{R}^{dN})$ that contains all antisymmetric functions $\psi$, that is

$$\psi(x_1, ...x_i...x_j..., x_N) = -\psi(x_1, ...x_j...x_i..., x_N), \qquad 1 \leq i < j \leq N.$$

In appropriate units ($\hbar = \sqrt{2}$ and $m_i = 1$ for $1, ..., N$), we may assume that the kinetic energy operator of the system is given by $H_0 = -\Delta$ on $D(H_0) = H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_{\mathrm{f}}$. For $d \geq 2$, it follows from Theorem 5.1, below, that $-\Delta$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_{\mathrm{f}}$, where $\Gamma$ denotes the union of the collision planes defined by Eqs. (1.27) and (1.28). This means that $H_0$ is the only self-adjoint extension and it proves that two-body contact interactions, as introduced in Section 1.2.1, are suppressed on the fermionic subspace $\mathscr{H}_{\mathrm{f}}$ if $d \geq 2$.

**Theorem 5.1.** *If $d \geq 2$ and $N \geq 2$, then $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_{\mathrm{f}}$ is dense in $H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_{\mathrm{f}}$ with respect to the norm of $H^2$. This means that*

$$H_0^2(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_{\mathrm{f}} = H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_{\mathrm{f}}, \tag{5.2}$$

*and it implies that $-\Delta \restriction C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_{\mathrm{f}}$ is essentially self-adjoint in $\mathscr{H}_{\mathrm{f}}$.*

*Remarks.*

(i) The main point of Theorem 5.1 is that elements of $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ vanish in an entire neighborhood of $\Gamma$. Elements of $C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_{\mathrm{f}}$ vanish on $\Gamma$ too. But the weaker statement that $C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_{\mathrm{f}}$ is dense in $H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_{\mathrm{f}}$ is true for all $d \geq 1$, which easily follows from the fact that $C_0^\infty(\mathbb{R}^{dN})$ is dense in $H^2(\mathbb{R}^{dN})$ (see, e.g., [50, Theorem 11.35]).

(ii) In $d = 1$ dimension the assertion of Theorem 5.1 is wrong. This manifests in the existence of $\delta'$-interactions, which only vanish when the derivative (and not the wave function itsself) vanishes at the origin (see [6, Chapter I.4] for details). To see this, we consider a sequence $\psi_n \in C_0^\infty(\mathbb{R}^N \setminus \Gamma)$, $n \in \mathbb{N}$, with $\psi_n \to \psi$ in the norm of $H^2(\mathbb{R}^N)$. Then $\partial_1 \psi_n \to \partial_1 \psi$ in the norm of $H^1(\mathbb{R}^N)$. Since, by Corollary 2.11, the trace operators $T_\sigma : H^1(\mathbb{R}^N) \to \mathfrak{X}_\sigma$ are continuous and since, clearly, $T_\sigma \partial_1 \psi_n = 0$, it follows that

$$T_\sigma \partial_1 \psi = 0, \qquad \sigma \in \mathcal{I}. \tag{5.3}$$

We now give an example of a wave function $\psi \in H^2(\mathbb{R}^N) \cap \mathscr{H}_\mathrm{f}$ without Property (5.3), which proves that $C_0^\infty(\mathbb{R}^N \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ is *not* dense in $H^2(\mathbb{R}^N) \cap \mathscr{H}_\mathrm{f}$. Let $x = (x_1, ..., x_N)$ and let

$$\psi(x) := e^{-|x|^2} \prod_{1 \le i < j \le N} (x_j - x_i).$$

Apart from the Gaussian, this is a Vandermonde determinant, so $\psi$ is antisymmetric. On the hyperplane $\Gamma_{(1,2)}$ we have

$$-\frac{\partial \psi}{\partial x_1}(x)\Big|_{x_1 = x_2} = e^{-|x|^2} \cdot \prod_{\substack{1 \le i < j \le N \\ (i,j) \neq (1,2)}} (x_j - x_i),$$

which shows that $T_{(1,2)} \partial_1 \psi \neq 0$.

So far, we know that contact interactions among identical fermions of equal spin vanish in $d \ge 2$, while this is *not* true in $d = 1$ due to the existence of non-trivial $\delta'$-interactions (but $\delta$-interactions vanish either way). From the previous sections we also know that (at least some) physically reasonable contact interactions in $d \le 3$ can be considered as limits $\varepsilon \to 0$ of suitable short-range two-body potentials $g_\varepsilon V_\varepsilon$, $\varepsilon > 0$. This suggests the following question: When can such short-range interactions be considered as weak on the fermionic subspace $\mathscr{H}_\mathrm{f}$ (in the sense that they vanish, in an appropriate sense, as $\varepsilon \to 0$) and, moreover, how large is the approximation error in terms of $\varepsilon > 0$? To address this question, we again consider the Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$, from Eq. (1.39). To ensure that $H_\varepsilon$ leaves the space $\mathscr{H}_\mathrm{f}$ invariant, we assume that $m_i = 1$ for $i = 1, ..., N$, $V_\sigma = V$ for all $\sigma \in \mathcal{I}$, where $V : \mathbb{R}^d \to \mathbb{R}$ is a fixed potential satisfying $V(r) = V(-r)$ a.e., and $g_{\varepsilon,\sigma} = g_\varepsilon > 0$ for all $\sigma \in \mathcal{I}$. Then

$$H_\varepsilon = -\Delta + g_\varepsilon \sum_{1 \le i < j \le N} V_\varepsilon(x_j - x_i), \qquad \varepsilon > 0, \tag{5.4}$$

where $V_\varepsilon(r) = \varepsilon^{-d} V(r/\varepsilon)$. For $H_\varepsilon$ to define a self-adjoint operator on $D(H_\varepsilon) = D(H_0)$, further assumptions on $V$ are needed. Clearly, the assumption $V \in L^2(\mathbb{R}^d)$, $d \le 3$, would be sufficient. However, the antisymmetry of the wave function and the fact that the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d)$ exists and is continuous for all $s \in I_d$, where $I_1 = (0, 1]$, $I_2 = (0, 1)$ and $I_3 = [0, 1/2]$ (see, e.g., [50, Theorem 12.55]), allow us to weaken the assumption $V \in L^2(\mathbb{R}^d)$ a bit. To this end, let $L^2_{\mathrm{odd}}(\mathbb{R}^d)$ denote the subspace of $L^2(\mathbb{R}^d)$ that contains all odd functions $\psi$ (that is $\psi(-r) = -\psi(r)$ a.e.). Then we have the following criterion for the self-adjointness of $H_\varepsilon$:

**Lemma 5.2.** *If $d \le 3$ and $V : \mathbb{R}^d \to \mathbb{R}$ is a measurable function that satisfies $V(r) = V(-r)$ a.e. and $\int \min(1, |r|^{2s}) |V(r)|^2 \, \mathrm{d}r < \infty$ for some $s \in I_d \cup \{0\}$, then $V$ is infinitesimally $(-\Delta)$-bounded in $L^2_{\mathrm{odd}}(\mathbb{R}^d)$. This, in turn, implies that $\sum_{i<j} |V_\varepsilon(x_j - x_i)|$ and hence $\sum_{i<j} V_\varepsilon(x_j - x_i)$ are infinitesimally $H_0$-bounded for all $\varepsilon > 0$ and, in particular, that Eq. (5.4) defines a self-adjoint operator $H_\varepsilon$ on $D(H_\varepsilon) = D(H_0)$.*

*Remark.* The assumption on $V$ in Lemma 5.2 essentially means that $V$ defines an $L^2$-function outside every neighborhood of the origin and $|V(r)| = O(|r|^{-\alpha})$ as $r \to 0$, where $\alpha < 3/2$ if $d = 1$ and $\alpha < 2$ if $d \in \{2, 3\}$.

As explained above, a heuristic argument suggests that the approximation $H_\varepsilon \approx H_0$ is reasonable for small $\varepsilon > 0$. Our goal is to justify this approximation mathematically by estimating the approximation error with regard to quantities that are of particular interest in physics. In particular, we consider spectra, expectation values, resolvents and unitary groups.

For expectation values we use the many-particle Hardy inequality for fermions from [47, Theorem 2.8]: For all $d \geq 1$, $N \geq 2$ and $\psi \in H^1(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$,

$$\sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^{dN}} \frac{|\psi(x_1, ..., x_N)|^2}{|x_i - x_j|^2} \,\mathrm{d}x_1 \cdots \mathrm{d}x_N \leq \frac{N}{d^2} \|\nabla\psi\|^2. \tag{5.5}$$

This implies the following result on the convergence of expectation values:

**Proposition 5.3.** *Let $d \leq 3$ and let $V : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that satisfies $V(r) = V(-r)$ a.e. and $C_V := \operatorname{ess\,sup}_{r \in \mathbb{R}^d} |r|^2 |V(r)| < \infty$. Suppose that Eq. (5.4) defines a self-adjoint operator $H_\varepsilon$ on $D(H_\varepsilon) = D(H_0)$ for all $\varepsilon > 0$. Then, for all $\varepsilon > 0$ and $\psi \in D(H_0)$,*

$$|\langle \psi \mid H_\varepsilon \psi \rangle - \langle \psi \mid H_0 \psi \rangle| \leq g_\varepsilon \varepsilon^{2-d} C_V N d^{-2} \|\nabla\psi\|^2. \tag{5.6}$$

If $\lambda_0 := \limsup_{\varepsilon \to 0} g_\varepsilon \varepsilon^{2-d} = 0$, then it follows from (5.6) and the variational principle that $\inf \sigma(H_\varepsilon) \to \inf \sigma(H_0) = 0$ as $\varepsilon \to 0$. To obtain more profound results, we recall from Section 2.1 that strong or norm resolvent convergence, respectively, implies convergence, in an appropriate sense, of the associated spectra and unitary groups. The following proposition establishes strong resolvent convergence $H_\varepsilon \to H_0$ as $\varepsilon \to 0$ under rather weak assumptions on $V$ and $g_\varepsilon$.

**Proposition 5.4.** *Let $d \geq 2$ and let $V : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that satisfies $V(r) = V(-r)$ a.e. and $\int_{|r| \geq r_0} |r|^{2s} |V(r)|^2 \,\mathrm{d}r < \infty$ for some $r_0 > 0$ and some $s \geq 0$. Suppose that $g_\varepsilon > 0$ satisfies $\limsup_{\varepsilon \to 0} g_\varepsilon \varepsilon^{s-d/2} < \infty$ and that Eq. (5.4) defines a self-adjoint operator $H_\varepsilon$ on $D(H_\varepsilon) = D(H_0)$ for all $\varepsilon > 0$. Then $H_\varepsilon \to H_0$ in the strong resolvent sense as $\varepsilon \to 0$.*

*Remark.* Proposition 5.4 is only formulated for $d \geq 2$ since the proof is based on Theorem 5.1. However, a similar result in $d = 1$ follows from Theorem 5.6, below.

As explained in Section 2.1, norm resolvent convergence has stronger consequences for the convergence of the associated spectra and unitary groups than the weaker strong resolvent convergence. For $d \leq 2$, norm resolvent convergence $H_\varepsilon \to H_0$ as $\varepsilon \to 0$ is an immediate consequence of Theorems 3.1 and 4.1. Indeed, for $\psi \in C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$, it follows from Definition (2.7) of $T_\sigma \psi$ and the antisymmetry of $\psi$ that $T_\sigma \psi = 0$ and, by an approximation argument, this extends to all $\psi \in D(H_0)$ because $T_\sigma : H^2(\mathbb{R}^{dN}) \to \mathfrak{X}_\sigma$ is continuous. Hence, $G(z)_\sigma = T_\sigma(-\Delta + z)^{-1}$ vanishes on $\mathscr{H}_\mathrm{f}$, so Theorems 3.1 and 4.1 show that $H_\varepsilon \to H_0$ in the norm resolvent sense as $\varepsilon \to 0$. However, Theorems 3.1 and 4.1 prescribe the asymptotics of $g_\varepsilon$ way too much as we shall see now.

Let $d \leq 3$ and let $U_{\mathrm{tot},\varepsilon} \in \mathscr{L}(\mathscr{H}_\mathrm{f})$, $\varepsilon > 0$, denote the unitary rescaling

$$(U_{\mathrm{tot},\varepsilon}\psi)(x_1, ..., x_N) := \varepsilon^{dN/2} \psi(\varepsilon x_1, ..., \varepsilon x_N). \tag{5.7}$$

Then a straightforward computation shows that

$$U_{\mathrm{tot},\varepsilon} H_\varepsilon (U_{\mathrm{tot},\varepsilon})^* = \varepsilon^{-2} \Bigg( -\Delta + g_\varepsilon \varepsilon^{2-d} \sum_{\substack{i,j=1 \\ i<j}}^{N} V(x_j - x_i) \Bigg). \tag{5.8}$$

This means that $\varepsilon^2 H_\varepsilon$ is unitarily equivalent to $H^{\mathrm{scal}}_{\lambda = \lambda_\varepsilon}$, where

$$H^{\mathrm{scal}}_\lambda := -\Delta + \lambda \sum_{\substack{i,j=1 \\ i<j}}^{N} V(x_j - x_i) \tag{5.9}$$

is self-adjoint on $D(H_\lambda^{\mathrm{scal}}) = D(H_0)$ and $\lambda_\varepsilon = g_\varepsilon \varepsilon^{2-d}$. In view of Lemma 5.2, $H_\lambda^{\mathrm{scal}}$ satisfies the hypotheses of Lemma 5.5, below, and accordingly we now define

$$\lambda_{\max} := \sup\left\{\lambda \geq 0 \,\middle|\, H_\lambda^{\mathrm{scal}} \geq 0\right\} \in [0, \infty]. \tag{5.10}$$

The following is well-known in a generic setting:

**Lemma 5.5.** *Let $A$ be self-adjoint and let $B$ be symmetric in some (complex) Hilbert space and suppose that $\sigma(A) = [0, \infty)$ and that $B$ is infinitesimally $A$-bounded. Then $f(\lambda) := \inf \sigma(A+\lambda B)$ defines a concave (and hence continuous) function $f : [0, \infty) \to \mathbb{R}$. In particular, with $\lambda_{\max} := \sup\{\lambda \geq 0 \,|\, f(\lambda) \geq 0\} \in [0, \infty]$, it holds*

*(i)* $f(\lambda) \geq 0$ *for all* $\lambda \in [0, \lambda_{\max}]$, $\lambda < \infty$.

*(ii) If* $\lambda_{\max} < \infty$, *then* $\lambda_{\max} < \lambda < \lambda'$ *implies that* $f(\lambda') < f(\lambda) < 0 = f(\lambda_{\max})$.

*Proof.* By the Kato-Rellich theorem (see, e.g., [69, Theorem X.12]), $A + \lambda B$ is self-adjoint and bounded from below on $D(A + \lambda B) = D(A)$ for all $\lambda \in [0, \infty)$. In particular, $f$ is well-defined, and by the variational principle,

$$f(\lambda) = \inf_{\psi \in D(A)\setminus\{0\}} \frac{\langle \psi \,|\, (A + \lambda B)\psi \rangle}{\|\psi\|^2}.$$

Since $A + \lambda B$ depends linearly on $\lambda$, this implies that $f$ is concave and hence continuous. It remains to prove $(i)$ and $(ii)$. By the definition of $\lambda_{\max}$, there exists a sequence $\lambda_n \in [0, \lambda_{\max}]$, $n \in \mathbb{N}$, with $\lambda_n < \infty$, $f(\lambda_n) \geq 0$ and $\lambda_n \to \lambda_{\max}$ as $n \to \infty$. This implies that $f(\lambda) \geq 0$ for all $\lambda \in [0, \lambda_n]$ because $f$ is concave and $f(0) = 0$ by assumption. Since $f$ is continuous and $\lambda_n \to \lambda_{\max}$ as $n \to \infty$, this proves $(i)$. In particular, if $\lambda_{\max} < \infty$, then $f(\lambda_{\max}) = 0$ because $f(\lambda_{\max}) \geq 0$ by $(i)$ and, clearly, $f(\lambda) < 0$ for all $\lambda \in (\lambda_{\max}, \infty)$. $(ii)$ is now immediate from the concavity of $f$. ☐

Obviously, $V \geq 0$ implies that $\lambda_{\max} = \infty$. If $V$ is not purely repulsive, then some decay of $V$ is needed to ensure that $\lambda_{\max} > 0$. For example, if

$$C_{V_-} := \operatorname*{ess\,sup}_{r \in \mathbb{R}^d} |r|^2 \max(-V(r), 0) < \infty \tag{5.11}$$

and $C_{V_-} \neq 0$, then it follows from the fermionic Hardy inequality (5.5) that

$$\lambda_{\max} \geq d^2/(C_{V_-} N) > 0. \tag{5.12}$$

If $\lambda_{\max} \in (0, \infty)$ and $\lambda_0 = \limsup_{\varepsilon \to 0} g_\varepsilon \varepsilon^{2-d} > \lambda_{\max}$, then Eq. (5.8) and Lemma 5.5 imply that a subsequence of $\inf \sigma(H_\varepsilon)$, $\varepsilon > 0$, diverges like $-\varepsilon^{-2}$ to $-\infty$. Since the translational invariance of $H_\varepsilon$ implies that $\sigma(H_\varepsilon) = [\inf \sigma(H_\varepsilon), \infty)$, the spectrum then fills the whole real line in the limit $\varepsilon \to 0$, which is not compatible with norm resolvent convergence to the positive operator $H_0$ (cf. Proposition 2.3). We conclude that norm resolvent convergence is only possible for $\lambda_0 \leq \lambda_{\max}$, and in the case of strict inequality the following theorem asserts norm resolvent convergence $H_\varepsilon \to H_0$ as $\varepsilon \to 0$ under some additional decay conditions on $V$.

**Theorem 5.6.** *Let $d \leq 3$ and assume that $V \in L^1(\mathbb{R}^d)$, $V(r) = V(-r)$ a.e. and that $V$ is infinitesimally $(-\Delta)$-bounded in $L^2_{\mathrm{odd}}(\mathbb{R}^d)$. Suppose that $g_\varepsilon > 0$, $\lambda_0 = \limsup_{\varepsilon\to 0} g_\varepsilon \varepsilon^{2-d} < \lambda_{\max}$ and that (at least) one of the following three cases occurs: (a) $V \geq 0$, (b) $V \leq 0$ and $\lambda_{\max} > 0$ or (c) $\operatorname{ess\,sup}_{r\in\mathbb{R}^d} |r|^2|V(r)| < \infty$. Then $H_\varepsilon \geq 0$ for $\varepsilon > 0$ small enough, and for all $z \in \mathbb{C}\setminus(-\infty, 0]$,*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = o(g_\varepsilon), \qquad (\varepsilon \to 0). \tag{5.13}$$

*Moreover, the following is true:*

*(i) If* $\int |r|^{2s}|V(r)|\,\mathrm{d}r < \infty$ *for some* $s \in I_d$*, then*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(g_\varepsilon \varepsilon^{2s}) \qquad (\varepsilon \to 0). \tag{5.14}$$

*(ii) If* $d = 2$ *and* $\int |r|^2 |\ln|r|||V(r)|\,\mathrm{d}r < \infty$*, then*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(g_\varepsilon \varepsilon^2 |\ln \varepsilon|) \qquad (\varepsilon \to 0). \tag{5.15}$$

*Remarks.*

(i) If $\limsup_{\varepsilon\to 0} g_\varepsilon < \infty$ or $\int |r|^{2s}|V(r)|\,\mathrm{d}r < \infty$ for some $s \in I_d$ and $\lim_{\varepsilon\to 0} g_\varepsilon \varepsilon^{2s} = 0$, then Theorem 5.6 implies that $H_\varepsilon \to H_0$ in the norm resolvent sense as $\varepsilon \to 0$.

(ii) As explained above, the condition $\lambda_0 < \lambda_{\max}$ is necessary: If $\lambda_0 > \lambda_{\max}$, then one can expect strong resolvent convergence at best. Indeed, if $d \geq 2$, $\lambda_0 < \infty$ and $V$ decays sufficiently fast as $|r| \to \infty$, then Proposition 5.4 yields strong resolvent convergence $H_\varepsilon \to H_0$ as $\varepsilon \to 0$.

As the regime where non-trivial contact interactions arise in the limit $\varepsilon \to 0$ is of particular interest, we state the estimates resulting from Theorem 5.6 for such choices of $g_\varepsilon$ separately in Corollary 5.7, below. While in $d \leq 2$ this asymptotics of $g_\varepsilon$ (see Eq. (1.41)) implies immediately that $\lambda_0 = \limsup_{\varepsilon\to 0} g_\varepsilon \varepsilon^{2-d} = 0$, a positive value of $\lambda_0$ is the result in $d = 3$. Therefore, for $d = 3$ it depends heavily on $V$ and $N$ whether the condition $\lambda_0 < \lambda_{\max}$ is satisfied or not. This shall be discussed in more detail in Section 5.5, below, where we shall see that, for appropriate $g_\varepsilon$ and $V$, the operator $\widetilde{H}_\varepsilon$ defined by the right side of Eq. (5.4) on the enlarged Hilbert space $L^2(\mathbb{R}^{3N})$ behaves essentially different than its restriction $H_\varepsilon = \widetilde{H}_\varepsilon \restriction \mathscr{H}_{\mathrm{f}}$, which converges in the norm resolvent sense to $H_0$, provided that $N$ is small enough. Either way, it is interesting to note that this asymptotics of $g_\varepsilon$ results in the same rate of resolvent convergence in all dimensions $d \leq 3$:

**Corollary 5.7.** *Suppose that the hypotheses of Theorem 5.6 are satisfied and that*

$$g_\varepsilon = \begin{cases} O(1) & \text{if } d = 1, \\ O(|\ln \varepsilon|^{-1}) & \text{if } d = 2, \\ O(\varepsilon) & \text{if } d = 3. \end{cases} \qquad (\varepsilon \to 0)$$

*Then, for all* $z \in \mathbb{C} \setminus (-\infty, 0]$*,*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(\varepsilon^2) \qquad (\varepsilon \to 0),$$

*provided that* $d \in \{1, 3\}$ *and* $\int |r|^{(5-d)/2}|V(r)|\,\mathrm{d}r < \infty$ *or* $d = 2$ *and* $\int |r|^2 |\ln|r|||V(r)|\,\mathrm{d}r < \infty$*.*

The outline of this section is as follows: The proof of Theorem 5.1 is given in Section 5.2, the proofs of Lemma 5.2 and Proposition 5.4 are given in Section 5.3 and the proof of Theorem 5.6 is given in Section 5.4. Finally, in Section 5.5, we put Theorem 5.6 into a broader perspective.

## 5.2 Proof of Theorem 5.1

The proof of Theorem 5.1 is based on the following two lemmas, Lemma 5.8 being its heart.

**Lemma 5.8.** *If* $d \geq 2$*, then there exists a sequence* $u_n \in C_0^\infty(\mathbb{R}^d)$*,* $n \in \mathbb{N}$*, with* $0 \leq u_n(x) \leq 1$*,* $u_n(x) = 1$ *if* $|x| \leq 1/n$ *and, as* $n \to \infty$*,* $\mathrm{diam}(\mathrm{supp}\, u_n) := \sup\{|x - y| \mid x, y \in \mathrm{supp}\, u_n\} \to 0$ *and*

$$\int |\nabla u_n(x)|^2\,\mathrm{d}x \to 0, \tag{5.16}$$

$$\int |x|^2 |\Delta u_n(x)|^2\,\mathrm{d}x \to 0. \tag{5.17}$$

*Proof.* In the case $d \geq 3$ we may choose any function $u \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq u(x) \leq 1$ and $u(x) = 1$ if $|x| \leq 1$. Then $u_n(x) := u(nx)$ has the desired properties because the substitution $y = nx$ shows that, in the limit $n \to \infty$,

$$\int |\nabla u_n(x)|^2 \, \mathrm{d}x = n^{2-d} \int |\nabla u(y)|^2 \, \mathrm{d}y \to 0,$$
$$\int |x|^2 |\Delta u_n(x)|^2 \, \mathrm{d}x = n^{2-d} \int |y|^2 |\Delta u(y)|^2 \, \mathrm{d}y \to 0.$$

In the remaining case $d = 2$ we first choose $u_1 = u_2 = f$, where $f \in C_0^\infty(\mathbb{R}^2)$, $0 \leq f(x) \leq 1$ and $f(x) = 1$ if $|x| \leq 1$. For $n \geq 3$ (and hence $\ln(\ln n) > 0$) we define $u_n(0) := 1$ and

$$u_n(x) := g\left(\frac{\ln(n|x|)}{\ln(\ln n)}\right), \qquad x \neq 0,$$

where $g \in C^\infty(\mathbb{R})$ denotes a fixed function with $0 \leq g(s) \leq 1$, $g(s) = 1$ if $s \leq 0$, and $g(s) = 0$ if $s \geq 1$. Then $u_n(x) = 1$ if $|x| \leq 1/n$, $u_n(x) = 0$ if $|x| \geq (\ln n)/n$, and hence $u_n \in C_0^\infty(\mathbb{R}^2)$ and $\operatorname{diam}(\operatorname{supp} u_n) \to 0$ as $n \to \infty$. Next, the substitution $s = \ln(nr)/\ln(\ln n)$ yields

$$\begin{aligned}
\frac{1}{2\pi} \int |\nabla u_n(x)|^2 \, \mathrm{d}x &= \frac{1}{(\ln(\ln n))^2} \int_{1/n}^{(\ln n)/n} g'\left(\frac{\ln(nr)}{\ln(\ln n)}\right)^2 \frac{\mathrm{d}r}{r} \\
&= \frac{1}{\ln(\ln n)} \int_0^1 g'(s)^2 \, \mathrm{d}s,
\end{aligned}$$

which vanishes in the limit $n \to \infty$. Furthermore, using that on radially symmetric functions

$$|x|^2 \Delta = \left(r \frac{\partial}{\partial r}\right)^2,$$

we find that

$$\begin{aligned}
\frac{1}{2\pi} \int |x|^2 |\Delta u_n(x)|^2 \, \mathrm{d}x &= \int_{1/n}^{(\ln n)/n} \left| \left(r \frac{\partial}{\partial r}\right)^2 g\left(\frac{\ln(nr)}{\ln(\ln n)}\right) \right|^2 \frac{\mathrm{d}r}{r} \\
&= \frac{1}{(\ln(\ln n))^4} \int_{1/n}^{(\ln n)/n} g''\left(\frac{\ln(nr)}{\ln(\ln n)}\right)^2 \frac{\mathrm{d}r}{r} \\
&= \frac{1}{(\ln(\ln n))^3} \int_0^1 g''(s)^2 \, \mathrm{d}s,
\end{aligned}$$

which also vanishes in the limit $n \to \infty$. This concludes the proof. ☐

**Lemma 5.9.** *Let $d \geq 2$ and let $\psi \in C_0^\infty(\mathbb{R}^{dN})$ with $\psi = 0$ on $\Gamma$. Then, for each pair $\sigma \in \mathcal{I}$ and for each $\varepsilon > 0$, there exists $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma_\sigma)$ with $\psi_\varepsilon = 0$ on $\Gamma$, $\operatorname{supp} \psi_\varepsilon \subseteq \operatorname{supp} \psi$ and*

$$\|(-\Delta + 1)(\psi - \psi_\varepsilon)\| < \varepsilon.$$

*Proof.* Without restriction, we may assume that $\sigma = (1, 2)$. Let $\psi \in C_0^\infty(\mathbb{R}^{dN})$ with $\psi = 0$ on $\Gamma$ and let

$$\psi_n(x_1, \ldots, x_N) := \psi(x_1, \ldots, x_N) \cdot (1 - u_n(x_1 - x_2)), \qquad n \in \mathbb{N},$$

where $u_n$ is given by Lemma 5.8. Then $\operatorname{supp} \psi_n \subseteq \operatorname{supp} \psi$, $\psi_n = 0 = \psi$ on $\Gamma$ and the properties of $u_n$ imply that $\psi_n \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma_\sigma)$. Therefore, it suffices to show that $\|(-\Delta+1)(\psi-\psi_n)\| \to 0$ as $n \to \infty$. In the following $u_n$ also denotes the function $(x_1, \ldots, x_N) \mapsto u_n(x_1 - x_2)$. Then $\psi - \psi_n = \psi u_n$, and hence

$$\|(-\Delta + 1)(\psi - \psi_n)\| \leq \|\Delta(\psi u_n)\| + \|\psi u_n\|.$$

Clearly, $\|\psi u_n\| \to 0$ as $n \to \infty$ by Lebesgue dominated convergence because $|\psi u_n| \leq |\psi|$ and $\psi u_n \to 0$ pointwise as $n \to \infty$. To estimate the first term $\|\Delta(\psi u_n)\|$, we introduce the relative and center of mass coordinates

$$r := x_2 - x_1, \qquad R := \frac{x_1 + x_2}{2}. \tag{5.18}$$

Then

$$\Delta = 2\Delta_r + \frac{1}{2}\Delta_R + \Delta_{x'}, \tag{5.19}$$

where $x' := (x_3, \dots, x_N)$. Since $u_n$ depends on $r$ only, it follows that

$$\Delta(\psi u_n) = 2\Delta_r(\psi u_n) + \frac{1}{2}(\Delta_R \psi) u_n + (\Delta_{x'}\psi) u_n,$$

where the first term equals

$$2\Delta_r(\psi u_n) = 2(\Delta_r \psi) u_n + 4(\nabla_r \psi) \cdot (\nabla_r u_n) + 2\psi \Delta_r u_n.$$

By Lebesgue dominated convergence, as explained above, $(\Delta_r\psi)u_n, (\Delta_R\psi)u_n$ and $(\Delta_{x'}\psi)u_n$ have vanishing $L^2$-norm in the limit $n \to \infty$. Hence, it remains to show that, as $n \to \infty$, $\|(\nabla_r\psi) \cdot (\nabla_r u_n)\| \to 0$ and $\|\psi\Delta_r u_n\| \to 0$. For $\|(\nabla_r\psi) \cdot (\nabla_r u_n)\|$ this follows from

$$\begin{aligned}\|(\nabla_r\psi) \cdot (\nabla_r u_n)\|^2 &\leq \int |\nabla_r \psi(r, R, x')|^2 |\nabla u_n(r)|^2 \, \mathrm{d}r \, \mathrm{d}R \, \mathrm{d}x' \\ &\leq \sup_{r \in \mathbb{R}^d} \int |\nabla_r \psi(r, R, x')|^2 \, \mathrm{d}R \, \mathrm{d}x' \cdot \|\nabla u_n\|^2\end{aligned}$$

because, by Lemma 5.8, $\|\nabla u_n\| \to 0$ as $n \to \infty$. For $\|\psi \Delta_r u_n\|$ we use that $\psi = 0$ on $\Gamma$ implies that $\psi(0, R, x') = 0$, and hence

$$\psi(r, R, x') = \int_0^1 (\nabla_r \psi)(tr, R, x') \cdot r \, \mathrm{d}t.$$

It follows that

$$\begin{aligned}\int \mathrm{d}r \, \mathrm{d}R \, dx' \, |(\psi\Delta_r u_n)(r, R, x')|^2 &\leq \int \mathrm{d}r \, \mathrm{d}R \, \mathrm{d}x' \left( \int_0^1 |\nabla_r \psi(tr, R, x')|^2 \, \mathrm{d}t \right) |r|^2 |\Delta u_n(r)|^2 \\ &\leq C \int |r|^2 |\Delta u_n(r)|^2 \, \mathrm{d}r,\end{aligned}$$

where

$$C := \sup_{r \in \mathbb{R}^d} \int \mathrm{d}R \, \mathrm{d}x' \int_0^1 |\nabla_r \psi(tr, R, x')|^2 \, \mathrm{d}t < \infty$$

and, by Lemma 5.8, $\int |r|^2 |\Delta u_n(r)|^2 \, \mathrm{d}r \to 0$ as $n \to \infty$. This shows that $\|\psi \Delta_r u_n\| \to 0$ as $n \to \infty$, so the proof is complete. □

*Proof of Theorem 5.1.* For given $\psi \in H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$ and $\varepsilon > 0$ it suffices to find a function $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ with $\|(-\Delta + 1)(\psi - \varphi_\varepsilon)\| < \varepsilon$. Then $\psi_\varepsilon := P_\mathrm{f} \varphi_\varepsilon$, where $P_\mathrm{f} \in \mathscr{L}(L^2(\mathbb{R}^{dN}))$ denotes the orthogonal projection onto the subspace $\mathscr{H}_\mathrm{f} \subseteq L^2(\mathbb{R}^{dN})$, defines an element in $C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ and, as $\varepsilon \to 0$, $\psi_\varepsilon \to \psi$ w.r.t. the norm of $H^2(\mathbb{R}^{dN})$ because

$$\|(-\Delta + 1)(\psi - \psi_\varepsilon)\| = \|P_\mathrm{f}(-\Delta + 1)(\psi - \varphi_\varepsilon)\| \leq \|(-\Delta + 1)(\psi - \varphi_\varepsilon)\| < \varepsilon.$$

To construct $\varphi_\varepsilon$, we may assume that $\psi$ belongs to $C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$, which is dense in $H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_\mathrm{f}$, and we use Lemma 5.9 repeatedly. That is, we use $\sigma(1), ..., \sigma(n)$ to denote the $n = N(N-1)/2$ pairs in $\mathcal{I}$, and we define $\Omega_0 := \mathbb{R}^{dN}$ and

$$\Omega_k := \mathbb{R}^{dN} \setminus \bigcup_{j=1}^{k} \Gamma_{\sigma(j)}, \qquad k = 1, \dots, n.$$

Then we set $\eta_0 := \psi$ and we construct smooth functions $\eta_k$, $k = 1, ..., n$, recursively with $\eta_k = 0$ on $\Gamma$, $\mathrm{supp}(\eta_k) \subseteq \mathrm{supp}(\eta_{k-1}) \cap \Omega_k$ and $\|\eta_k - \eta_{k-1}\| < \varepsilon/n$. This is achieved with the help of Lemma 5.9. The function $\varphi_\varepsilon := \eta_n$ has the desired properties. □

## 5.3 Proofs of Lemma 5.2 and Proposition 5.4

*Proof of Lemma 5.2.* For the first part, it suffices to show that for given $\delta > 0$ there exists $C(\delta) > 0$ so that $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d)$ implies that $V\psi \in L^2(\mathbb{R}^d)$ with

$$\|V\psi\|^2 \leq \delta^2 \|\Delta\psi\|^2 + C(\delta)^2 \|\psi\|^2. \tag{5.20}$$

If the hypotheses of Lemma 5.2 are satisfied with $s = 0$, then $V \in L^2(\mathbb{R}^d)$ and hence (5.20) follows from a standard result (see, e.g., [77, Theorem 11.1]). In the remaining case where $\int \min(1, |r|^{2s})|V(r)|^2 \,\mathrm{d}r < \infty$ for some $s \in I_d$, the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d)$ exists and is continuous, so it follows that, for some constant $c_s > 0$ and all $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d)$,

$$|\psi(r)| = |\psi(r) - \psi(0)| \leq c_s |r|^s (\|\Delta\psi\|^2 + \|\psi\|^2)^{1/2}, \qquad r \in \mathbb{R}^d, \tag{5.21}$$

where we used that $\psi(0) = 0$ since $\psi$ defines a continuous odd function. Now, let the cutoff potential $V^k$, $k > 0$, be defined as in Eq. (3.33). Then (5.21) implies that $V^k\psi \in L^2(\mathbb{R}^d)$ and

$$\|V^k\psi\|^2 \leq c_s^2 (\|\Delta\psi\|^2 + \|\psi\|^2) \int\limits_{|r|\leq k} |r|^{2s} |V(r)|^2 \,\mathrm{d}r,$$

where the integral on the right side vanishes as $k \to 0$. Hence, by choosing $k > 0$ small enough, we obtain that, for all $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d)$,

$$\|V^k\psi\|^2 \leq \frac{\delta^2}{2} \left( \|\Delta\psi\|^2 + \|\psi\|^2 \right). \tag{5.22}$$

Furthermore, since $V - V^k \in L^2(\mathbb{R}^d)$, there exists a constant $C(\delta, k) > 0$ such that, for all $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d)$,

$$\|(V - V^k)\psi\|^2 \leq \frac{\delta^2}{2} \|\Delta\psi\|^2 + C(\delta, k)^2 \|\psi\|^2. \tag{5.23}$$

Adding (5.22) and (5.23) yields (5.20), which proves the first part of the lemma.

For the second part we use that $C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_{\text{f}}$ is dense in $D(H_0) = H^2(\mathbb{R}^{dN}) \cap \mathscr{H}_{\text{f}}$ w.r.t. the $H^2$-norm, and that for all $\varphi \in C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_{\text{f}}$,

$$\int |V_\varepsilon(x_2 - x_1)\varphi(x)|^2 \,\mathrm{d}x = \int |V_\varepsilon(x_j - x_i)\varphi(x)|^2 \,\mathrm{d}x, \qquad 1 \leq i < j \leq N,$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^{dN}$. Therefore, to show that $\sum_{i<j} |V_\varepsilon(x_j - x_i)|$ is infinitesimally $H_0$-bounded for each $\varepsilon > 0$, it suffices to show that (5.20) implies that for all $\varepsilon > 0$ and all $\varphi \in C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_{\text{f}}$,

$$\left( \int |V_\varepsilon(x_2 - x_1)\varphi(x)|^2 \,\mathrm{d}x \right)^{1/2} \leq c_1(\varepsilon)\delta \|H_0\varphi\| + c_2(\varepsilon) C(\delta) \|\varphi\| \tag{5.24}$$

for some constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ that are independent of $\delta > 0$. In particular, it then follows that $\sum_{i<j} V_\varepsilon(x_j - x_i)$ is infinitesimally $H_0$-bounded, so $H_\varepsilon$ is self-adjoint on $D(H_\varepsilon) = D(H_0)$ by the Kato-Rellich theorem.

To prove (5.24), we introduce the relative and center of mass coordinates from (5.18), so the wave function $\varphi$ in the new coordinates is given by $\widetilde{\varphi}(r, R, x') := \varphi(R - r/2, R + r/2, x')$ with $x' = (x_3, ..., x_N)$, and afterwards we substitute $r/\varepsilon \to r$. We find that for all $\varphi \in C_0^\infty(\mathbb{R}^{dN}) \cap \mathscr{H}_{\text{f}}$,

$$\begin{aligned} \int |V_\varepsilon(x_2 - x_1)\varphi(x)|^2 \,\mathrm{d}x &= \varepsilon^{-d} \int \mathrm{d}R \,\mathrm{d}x' \int \mathrm{d}r \, |V(r)|^2 |\widetilde{\varphi}(\varepsilon r, R, x')|^2 \\ &\leq \varepsilon^{4-2d} \delta^2 \|\Delta_r \widetilde{\varphi}\|^2 + \varepsilon^{-2d} C(\delta)^2 \|\widetilde{\varphi}\|^2, \end{aligned}$$

where the second line was obtained from (5.20) and $\widetilde{\varphi}(-r, R, x') = -\widetilde{\varphi}(r, R, x')$. Now, observe that the definition of $\widetilde{\varphi}$ implies that $\|\widetilde{\varphi}\| = \|\varphi\|$ and, by Eq. (5.19), $\|\Delta_r \widetilde{\varphi}\| \leq \|H_0\varphi\|/2$, so (5.24) is established with $c_1(\varepsilon) = \varepsilon^{2-d}/2$ and $c_2(\varepsilon) = \varepsilon^{-d}$. This concludes the proof. $\square$

*Proof of Proposition 5.4.* In view of $\|(H_\varepsilon + i)^{-1}\| = 1 = \|R_0(i)\|$, where $R_0(i) = (H_0 + i)^{-1}$ for short, it suffices to prove that, as $\varepsilon \to 0$, $(H_\varepsilon + i)^{-1}\psi \to R_0(i)\psi$ for $\psi$ from a dense subset of $\mathscr{H}_\mathrm{f}$. By Theorem 5.1, the set of all $\psi = (H_0 + i)\varphi$ with $\varphi \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ is dense in $\mathscr{H}_\mathrm{f}$, and for such $\psi$,

$$\begin{aligned} \|(H_\varepsilon + i)^{-1}\psi - R_0(i)\psi\| &= \left\|(H_\varepsilon + i)^{-1}\Big(g_\varepsilon \sum_{i<j} V_\varepsilon(x_j - x_i)\Big)R_0(i)\psi\right\| \\ &\le g_\varepsilon \sum_{i<j} \|V_\varepsilon(x_j - x_i)\varphi\| = \frac{g_\varepsilon N(N-1)}{2}\|V_\varepsilon(x_2 - x_1)\varphi\|, \end{aligned} \tag{5.25}$$

where the last equality used the antisymmetry of $\varphi$. For a further estimate of (5.25), we again introduce the relative and center of mass coordinates from (5.18), so $\varphi$ is then given by $\widetilde{\varphi}(r, R, x') = \varphi(R - r/2, R + r/2, x')$. Like $\varphi$, $\widetilde{\varphi}$ is a compactly supported smooth function, and hence $\operatorname{supp}\widetilde{\varphi} \subseteq \mathbb{R}^d \times B_{N-1}$ for some ball $B_{N-1} \subseteq \mathbb{R}^{d(N-1)}$. It follows that for any $c > 0$,

$$\begin{aligned} \|V_\varepsilon(x_2 - x_1)\varphi\|^2 = &\,\varepsilon^{-d} \int_{B_{N-1}} \mathrm{d}R\,\mathrm{d}x' \int_{|r|\le c} \mathrm{d}r\,|V(r)|^2\,|\widetilde{\varphi}(\varepsilon r, R, x')|^2 \\ &+ \varepsilon^{-d} \int_{B_{N-1}} \mathrm{d}R\,\mathrm{d}x' \int_{|r|>c} \mathrm{d}r\,|V(r)|^2\,|\widetilde{\varphi}(\varepsilon r, R, x')|^2 . \end{aligned} \tag{5.26}$$

Since $\varphi \in C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma) \cap \mathscr{H}_\mathrm{f}$ and thus $\widetilde{\varphi}(r, R, x') = 0$ for $r < \operatorname{dist}(\operatorname{supp}\varphi, \Gamma)$, the first summand vanishes for $\varepsilon < c^{-1}\operatorname{dist}(\operatorname{supp}\varphi, \Gamma)$ and, with the given value of $s \ge 0$,

$$|\widetilde{\varphi}(r, R, x')|^2 \le C(\widetilde{\varphi}, s)^2 |r|^{2s}, \qquad (R, x') \in \mathbb{R}^{d(N-1)}$$

for some constant $C(\widetilde{\varphi}, s) > 0$. Hence, it follows from (5.26) that for all $\varepsilon < c^{-1}\operatorname{dist}(\operatorname{supp}\varphi, \Gamma)$,

$$g_\varepsilon \|V_\varepsilon(x_2 - x_1)\varphi\| \le C(\widetilde{\varphi}, s) g_\varepsilon \varepsilon^{s-d/2}\left(|B_{N-1}| \int_{|r|>c} |r|^{2s}\,|V(r)|^2\,\mathrm{d}r\right)^{1/2},$$

where the integral on the right side vanishes as $c \to \infty$ because $\int_{|r|\ge r_0} |r|^{2s}|V(r)|^2\,\mathrm{d}r < \infty$ by assumption. Since $\limsup_{\varepsilon\to 0} g_\varepsilon\,\varepsilon^{s-d/2} < \infty$ by assumption, by choosing first $c > r_0$ large and then $\varepsilon > 0$ small, we see that $\lim_{\varepsilon\to 0} g_\varepsilon\|V_\varepsilon(x_2 - x_1)\varphi\| = 0$. In view of (5.25), this means that $(H_\varepsilon + i)^{-1}\psi \to R_0(i)\psi$ as $\varepsilon \to 0$ and the proof is complete. □

## 5.4 Proof of Theorem 5.6

We first note that our assumptions on $V$ and Lemma 5.2 imply that $H_\varepsilon$ is self-adjoint on $D(H_\varepsilon) = D(H_0)$ for all $\varepsilon > 0$. The proof of Theorem 5.6 is based on the new expression (5.32), below, for $H_\varepsilon$ that allows us to express the resolvent by the Konno-Kuroda formula (5.34). To derive Eq. (5.32), we first need to introduce some auxiliary spaces and operators.

Let

$$\mathfrak{X}_\mathrm{f} := L^2_\mathrm{odd}(\mathbb{R}^d, \mathrm{d}r) \otimes L^2(\mathbb{R}^d, \mathrm{d}R) \otimes \bigwedge_{i=3}^{N} L^2(\mathbb{R}^d, \mathrm{d}x_i), \tag{5.27}$$

where the integration variables $r$ and $R$ in (5.27) correspond to the relative and center of mass coordinates of the fermion positions $x_1$ and $x_2$ (see Eq. (5.18)). This change of coordinates is implemented by the isometric operator $\mathscr{K} : \mathscr{H}_\mathrm{f} \to \mathfrak{X}_\mathrm{f}$ with

$$(\mathscr{K}\psi)(r, R, x_3, ..., x_N) := \psi\left(R - \frac{r}{2}, R + \frac{r}{2}, x_3, ..., x_N\right). \tag{5.28}$$

It follows that for all $\varphi, \psi \in D(H_0)$,

$$\begin{aligned}\sum_{i<j} \int \overline{\varphi(x)}\, V_\varepsilon(x_j - x_i)\psi(x)\,\mathrm{d}x &= \frac{(N-1)N}{2} \int \overline{\varphi(x)}\, V_\varepsilon(x_2 - x_1)\psi(x)\,\mathrm{d}x \\ &= \frac{(N-1)N}{2} \int \overline{(\mathscr{K}\varphi)(r,R,x')}\, V_\varepsilon(r)\,(\mathscr{K}\psi)(r,R,x')\,\mathrm{d}r\,\mathrm{d}R\,\mathrm{d}x',\end{aligned}$$

where the antisymmetry of $\varphi$ and $\psi$ was used in the first line. This means that

$$\sum_{i<j} V_\varepsilon(x_j - x_i)\psi = \frac{(N-1)N}{2} \mathscr{K}^*(V_\varepsilon \otimes 1)\mathscr{K}\psi, \qquad \psi \in D(H_0). \tag{5.29}$$

Next, let the unitary rescaling $U_\varepsilon \in \mathscr{L}(L^2(\mathbb{R}^d))$, $\varepsilon > 0$, be extended to $\mathscr{L}(\mathfrak{X}_\mathrm{f})$ via

$$(U_\varepsilon\psi)\,(r,R,x') := \varepsilon^{d/2}\,\psi(\varepsilon r, R, x'),$$

and let

$$\begin{aligned} v(r) &:= |V(r)|^{1/2}, \\ u(r) &:= J|V(r)|^{1/2}, \qquad J := \operatorname{sgn}(V),\end{aligned}$$

so that $V = uv$. Note that $u, v \in L^2(\mathbb{R}^d)$ because $V \in L^1(\mathbb{R}^d)$ by assumption. In terms of the above operators, we now define for $\varepsilon > 0$ the new operators $A_\varepsilon, B_\varepsilon : D(A_\varepsilon) \subseteq \mathscr{H}_\mathrm{f} \to \mathfrak{X}_\mathrm{f}$, where

$$A_\varepsilon := \sqrt{\frac{(N-1)N}{2}} (v \otimes 1)\, \varepsilon^{-d/2} U_\varepsilon \mathscr{K}, \tag{5.30}$$

$$B_\varepsilon := g_\varepsilon J A_\varepsilon = g_\varepsilon \sqrt{\frac{(N-1)N}{2}} (u \otimes 1)\, \varepsilon^{-d/2} U_\varepsilon \mathscr{K}. \tag{5.31}$$

The domain $D(A_\varepsilon)$ is determined by the domain of the multiplication operator $v\otimes 1$, so it follows that $A_\varepsilon$ and $B_\varepsilon$ are densely defined and closed on $D(A_\varepsilon) \supseteq D(H_0)$. In view of Eq. (5.29), these new operators allow us to express the Schrödinger operator $H_\varepsilon$ from Eq. (5.4) in the form

$$H_\varepsilon = -\Delta + A_\varepsilon^* B_\varepsilon. \tag{5.32}$$

This means that $A_\varepsilon^* B_\varepsilon$ is a sum of two-body potentials, and similarly,

$$A_\varepsilon^* A_\varepsilon \psi = \sum_{i<j} |V_\varepsilon(x_j - x_i)|\psi, \qquad \psi \in D(H_0). \tag{5.33}$$

In particular, it follows from Lemma 5.2 that both $A_\varepsilon^* B_\varepsilon$ and $A_\varepsilon^* A_\varepsilon$ are infinitesimally $H_0$-bounded, so the hypotheses of Theorem B.1 are satisfied. This means that

$$\phi_\varepsilon(z) = B_\varepsilon R_0(z) A_\varepsilon^*, \qquad z \in \rho(H_0)$$

defines a bounded operator in $\mathscr{L}(\mathfrak{X}_\mathrm{f})$ and that $1 + \phi_\varepsilon(z)$ has a bounded inverse if and only if $z \in \rho(H_\varepsilon) \cap \rho(H_0)$. If $z \in \rho(H_\varepsilon) \cap \rho(H_0)$, then

$$(H_\varepsilon + z)^{-1} = R_0(z) - (A_\varepsilon R_0(\overline{z}))^*(1 + \phi_\varepsilon(z))^{-1} B_\varepsilon R_0(z). \tag{5.34}$$

Taking for granted Lemmas 5.10 and 5.12, below, we now give the proof of Theorem 5.6:

*Proof of Theorem 5.6.* Lemma 5.12 shows that $(0,\infty) \subseteq \rho(H_\varepsilon)$ for all sufficiently small $\varepsilon > 0$, which means that $(H_\varepsilon + z)^{-1}$ is given by Eq. (5.34) for all $z > 0$. For fixed $z > 0$, it follows that

$$\|(H_\varepsilon + z)^{-1} - R_0(z)\| \leq g_\varepsilon \|A_\varepsilon R_0(z)\|^2 \|(1 + \phi_\varepsilon(z))^{-1}\|, \tag{5.35}$$

where we used that $B_\varepsilon = g_\varepsilon J A_\varepsilon$. Depending on the decay of $V$, Lemmas 5.10 and 5.12 yield an asymptotic estimate for the right side of (5.35) that proves Theorem 5.6 for $z \in (0,\infty)$. For general $z \in \rho(H_0) = \mathbb{C} \setminus (-\infty, 0]$, Lemma 2.7 now proves Theorem 5.6. ☐

**Lemma 5.10.** *If $d \le 3$, $z > 0$ and $V \in L^1(\mathbb{R}^d)$ satisfies $V(r) = V(-r)$ a.e., then $A_\varepsilon R_0(z) \to 0$ as $\varepsilon \to 0$ in $\mathscr{L}(\mathscr{H}_\mathrm{f}, \mathfrak{X}_\mathrm{f})$. Furthermore, the following is true:*

*(i) If $\int |r|^{2s} |V(r)| \,\mathrm{d}r < \infty$ for some $s \in I_d$, then $\|A_\varepsilon R_0(z)\| = O(\varepsilon^s)$ as $\varepsilon \to 0$.*

*(ii) If $d = 2$ and $\int |r|^2 |\ln|r|| |V(r)| \,\mathrm{d}r < \infty$, then $\|A_\varepsilon R_0(z)\| = O(\varepsilon|\ln\varepsilon|^{1/2})$ as $\varepsilon \to 0$.*

*Proof.* From Eqs. (5.28) and (5.19), it follows that $\|(-2\Delta_r + z)\mathscr{K}(H_0 + z)^{-1}\| \le 1$. Hence, Definition (5.30) of $A_\varepsilon$ implies that $\|A_\varepsilon R_0(z)\| \le \sqrt{N(N-1)/2}\|v\varepsilon^{-d/2}U_\varepsilon(-2\Delta + z)^{-1}\|_{\mathrm{odd}}$, where $\|\cdot\|_{\mathrm{odd}}$ denotes the operator norm in $L^2_{\mathrm{odd}}(\mathbb{R}^d)$. Now, recall from the proof of Lemma 5.2 that the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d)$ exists and is continuous for all $s \in I_d$. This means that for some constant $c_s > 0$ and all $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\mathrm{odd}}(\mathbb{R}^d)$,

$$\left|(\varepsilon^{-d/2}U_\varepsilon\psi)(r)\right| = |\psi(\varepsilon r) - \psi(0)| \le c_s|\varepsilon r|^s\|\psi\|_{H^2}, \qquad r \in \mathbb{R}^d. \tag{5.36}$$

If $\int |r|^{2s} |V(r)| \,\mathrm{d}r < \infty$, then it follows that

$$\|v\varepsilon^{-d/2}U_\varepsilon(-2\Delta + z)^{-1}\|_{\mathrm{odd}} \le C(s,z)\varepsilon^s \left(\int |r|^{2s} |V(r)| \,\mathrm{d}r\right)^{1/2} \tag{5.37}$$

for some constant $C(s,z) > 0$. This proves $(i)$, and a similar estimate, where (5.36) is replaced by the improved Sobolev estimate from Lemma 5.11, below, also proves $(ii)$.

It remains to show that $A_\varepsilon R_0(z) \to 0$ as $\varepsilon \to 0$ for general $V \in L^1(\mathbb{R}^d)$. To this end, we again consider the cutoff potential $V^k$, $k > 0$, from the proof of Lemma 5.2 and we set $v^k(r) := |V^k(r)|^{1/2}$. Then (5.37) holds for $v^k$ in place of $v$ because $\int |r|^{2s}|V^k(r)| \,\mathrm{d}r < \infty$ for all $s > 0$. Moreover, using that the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ exists and is continuous, we find that, for all $\varphi \in L^2_{\mathrm{odd}}(\mathbb{R}^d)$,

$$\begin{aligned}\|(v - v^k)\varepsilon^{-d/2}U_\varepsilon(-2\Delta + z)^{-1}\varphi\| &\le \|v - v^k\|\|(-2\Delta + z)^{-1}\varphi\|_{L^\infty} \\ &\le C(z)\|V - V^k\|_{L^1}^{1/2}\|\varphi\|.\end{aligned}$$

This implies that $\|(v - v^k)\varepsilon^{-d/2}U_\varepsilon(-2\Delta + z)^{-1}\|_{\mathrm{odd}} \to 0$ as $k \to \infty$ uniformly in $\varepsilon > 0$, and a simple $\delta/2$-argument now shows that $\|v\varepsilon^{-d/2}U_\varepsilon(-2\Delta + z)^{-1}\|_{\mathrm{odd}} \to 0$ as $\varepsilon \to 0$, which concludes the proof. □

The next lemma shows that the usual Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow C^{0,s}(\mathbb{R}^2)$, $s \in (0,1)$, can be slightly improved. This has been used in part $(ii)$ of the previous lemma.

**Lemma 5.11.** *For all $\psi \in H^2(\mathbb{R}^2)$ and all $x, y \in \mathbb{R}^2, y \ne 0$, we have*

$$|\psi(x+y) - \psi(x)| \le \frac{1}{2\sqrt{\pi}}|y|\,(2 + |\ln|y||)^{1/2}\left(\|\Delta\psi\|^2 + \|\nabla\psi\|^2\right)^{1/2}.$$

*Remark.* In fact, our proof yields a slightly better factor than $(2 + |\ln|y||)^{1/2}$.

*Proof.* We first note that $\psi \in H^2(\mathbb{R}^2)$ implies that $\widehat{\psi} \in L^1(\mathbb{R}^2)$, and hence for all $x \in \mathbb{R}^2$,

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \widehat{\psi}(p)\exp(ip\cdot x)\,\mathrm{d}p.$$

Therefore, by the Cauchy-Schwarz inequality,

$$2\pi\frac{|\psi(x+y) - \psi(x)|}{|y|} \le \int_{\mathbb{R}^2} \frac{|\exp(ip\cdot y) - 1|}{|y|}|\widehat{\psi}(p)|\,\mathrm{d}p \le I(y)^{1/2}\left(\|\Delta\psi\|^2 + \|\nabla\psi\|^2\right)^{1/2}, \tag{5.38}$$

where

$$I(y) := \int_{\mathbb{R}^2} \frac{|\exp(ip\cdot y) - 1|^2}{|y|^2(|p|^4 + |p|^2)}\,\mathrm{d}p.$$

To estimate the integral $I(y)$, we may assume that $y = (|y|, 0)$ because $I(y)$ is invariant under rotations. Then, after the substitution $q = (q_1, q_2) := p|y|$, we find that for any $Q > 0$,

$$I(y) = \int_{\mathbb{R}^2} \frac{|\exp(iq_1) - 1|^2}{|q|^4 + |y|^2|q|^2}\,\mathrm{d}q \leq \int_{|q|\leq Q} \frac{q_1^2}{|q|^4 + |y|^2|q|^2}\,\mathrm{d}q + \int_{|q|>Q} \frac{4}{|q|^4}\,\mathrm{d}q, \tag{5.39}$$

where the first and the second integral on the right side were obtained from $|\exp(iq_1) - 1| \leq |q_1|$ and $|\exp(iq_1) - 1| \leq 2$, respectively. Both integrals can be computed explicitly. For the first one we obtain

$$\begin{aligned}\int_{|q|\leq Q} \frac{q_1^2}{|q|^4 + |y|^2|q|^2}\,\mathrm{d}q &= \frac{1}{2}\int_{|q|\leq Q} \frac{1}{|q|^2 + |y|^2}\,\mathrm{d}q \\ &= \frac{\pi}{2}\ln\left(1 + \frac{Q^2}{|y|^2}\right) \leq \pi\left(|\ln|y|| + |\ln Q| + \frac{1}{2}\ln 2\right),\end{aligned} \tag{5.40}$$

where the inequality used $\ln(1+t) \leq |\ln t| + \ln 2$, valid for all $t > 0$. The second integral on the right side of (5.39) equals $4\pi/Q^2$, so it follows from (5.39) and (5.40) that for any $Q > 0$,

$$I(y) \leq \pi\left(|\ln|y|| + |\ln Q| + \frac{1}{2}\ln 2\right) + \frac{4\pi}{Q^2}.$$

The optimal value of $Q$ is $Q = 2\sqrt{2}$, which yields $I(y) \leq \pi(|\ln|y|| + c)$ with $c = \frac{1}{2} + \ln 4 < 2$. Using this to estimate the right side of (5.38), the desired estimate follows. □

**Lemma 5.12.** *If the hypotheses of Theorem 5.6 are satisfied and $\varepsilon > 0$ is small enough, then $(0,\infty) \subseteq \rho(H_\varepsilon)$, and hence $(1 + \phi_\varepsilon(z))^{-1}$ exists for all $z \in (0,\infty)$. Moreover, for all $z \in (0,\infty)$,*

$$\limsup_{\varepsilon\to 0} \|(1 + \phi_\varepsilon(z))^{-1}\| < \infty.$$

*Proof.* We claim, and prove below, that in each case (a), (b) and (c) from the statement of Theorem 5.6 there exist constants $C = C(d,N,V,\lambda_0), \varepsilon_0 = \varepsilon_0(d,N,V,\lambda_0) > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and all $\psi \in D(H_\varepsilon) = D(H_0)$,

$$g_\varepsilon \sum_{i<j} \int_{\mathbb{R}^{dN}} |V_\varepsilon(x_j - x_i)||\psi(x)|^2\,\mathrm{d}x \leq C\,\langle\psi\,|\,H_\varepsilon\psi\rangle\,. \tag{5.41}$$

Since $g_\varepsilon > 0$ by assumption, (5.41) shows that $H_\varepsilon \geq 0$ and hence $(0,\infty) \subseteq \rho(H_\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_0)$. By Theorem B.1, this means that $(1 + \phi_\varepsilon(z))^{-1}$ exists for all $z \in (0,\infty)$ and

$$(1 + \phi_\varepsilon(z))^{-1} = 1 - g_\varepsilon\, JA_\varepsilon(H_\varepsilon + z)^{-1}A_\varepsilon^*. \tag{5.42}$$

To estimate the norm of $(1 + \phi_\varepsilon(z))^{-1}$, we first note that (5.33) and (5.41) imply that for all $\varepsilon \in (0,\varepsilon_0)$, $z \in (0,\infty)$ and $\psi \in D(H_0)$,

$$g_\varepsilon\|A_\varepsilon\psi\|^2 \leq C\|(H_\varepsilon + z)^{1/2}\psi\|^2. \tag{5.43}$$

Using that $D(H_\varepsilon) = D(H_0)$ is dense in $D(H_\varepsilon^{1/2})$ w.r.t. the graph norm of $H_\varepsilon^{1/2}$ and that $A_\varepsilon$ is closed, an approximation argument shows that $D(H_\varepsilon^{1/2}) \subseteq D(A_\varepsilon)$ and that (5.43) is valid for all $\psi \in D(H_\varepsilon^{1/2})$. We conclude that $A_\varepsilon(H_\varepsilon + z)^{-1/2}$ defines a bounded operator and that

$g_\varepsilon \|A_\varepsilon(H_\varepsilon + z)^{-1/2}\|^2 \leq C$ for all $\varepsilon \in (0, \varepsilon_0)$ and $z \in (0, \infty)$. Using this to estimate the norm of the right side of Eq. (5.42), we find that for all $z \in (0, \infty)$,

$$\limsup_{\varepsilon \to 0} \|(1 + \phi_\varepsilon(z))^{-1}\| \leq 1 + C, \tag{5.44}$$

which proves the lemma given the validity of (5.41).

It remains to prove (5.41) in all cases (a), (b) and (c). In the case of (a), where $V \geq 0$, (5.41) with $C = 1$ is immediate from the definition of $H_\varepsilon$. In the other cases, the scaling $x \to \varepsilon x$ shows that (5.41) is equivalent to the assertion that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in D(H_0)$,

$$\lambda_\varepsilon \sum_{i<j} \int_{\mathbb{R}^{dN}} |V(x_j - x_i)||\psi(x)|^2 \, \mathrm{d}x \leq C \left\langle \psi | H^{\mathrm{scal}}_{\lambda_\varepsilon} \psi \right\rangle, \tag{5.45}$$

where $H^{\mathrm{scal}}_{\lambda_\varepsilon}$ is defined by (5.9) with $\lambda = \lambda_\varepsilon = g_\varepsilon \varepsilon^{2-d}$. From Lemma 5.5 we know that $H^{\mathrm{scal}}_\lambda \geq 0$ for all $\lambda \in [0, \lambda_{\max}]$, $\lambda < \infty$. In the case of (b), where $V \leq 0$ and $\lambda_{\max} > 0$ by assumption, we now choose $C > 0$ so large that $(1 + 1/C)\lambda_0 < \lambda_{\max}$ (here we use the assumption $\lambda_0 = \limsup_{\varepsilon \to 0} \lambda_\varepsilon < \lambda_{\max}$) and we choose $\varepsilon_0 > 0$ so small that $(1 + 1/C)\lambda_\varepsilon < \lambda_{\max}$ for all $\varepsilon \in (0, \varepsilon_0)$. Then (5.45) follows immediately. Finally, in the remaining case (c), where $V$ does not necessarily have a definite sign but $C_V = \operatorname{ess\,sup}_{r \in \mathbb{R}^d} |r|^2 |V(r)| \in (0, \infty)$, we first conclude from (5.12) that $\lambda_{\max} \geq d^2/(C_V N) > 0$. Let $\eta \in (\lambda_0, \lambda_{\max})$ be fixed, let $\delta \in (0, 1 - \lambda_0/\eta)$ be fixed, and let $\varepsilon_0 > 0$ be so small that $1 - \lambda_\varepsilon/\eta \geq \delta$ for all $\varepsilon \in (0, \varepsilon_0)$. Then, on the one hand, it follows that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in D(H_0)$,

$$\delta \|\nabla \psi\|^2 \leq \left(1 - \frac{\lambda_\varepsilon}{\eta}\right) \|\nabla \psi\|^2 \leq \left\langle \psi \,\middle|\, H^{\mathrm{scal}}_{\lambda_\varepsilon} \psi \right\rangle, \tag{5.46}$$

where the second inequality used the Definition (5.9) of $H^{\mathrm{scal}}_{\lambda_\varepsilon}$ together with $H^{\mathrm{scal}}_\eta \geq 0$ (because $\eta < \lambda_{\max}$). On the other hand, the fermionic Hardy inequality (5.5) yields for all $\psi \in D(H_0)$,

$$\sum_{i<j} \int_{\mathbb{R}^{dN}} |V(x_j - x_i)||\psi(x)|^2 \, \mathrm{d}x \leq \frac{C_V N}{d^2} \|\nabla \psi\|^2. \tag{5.47}$$

Combining (5.46) and (5.47) proves (5.45) for any particular choice of $C > \lambda_0 C_V N/(d^2 \delta)$. This completes the proof of Lemma 5.12, so Theorem 5.6 is established after all. □

## 5.5 Examples and discussion

To compare Theorem 5.6 with our previous results from Sections 3 and 4 and to show that the antisymmetry of the wave function is indispensable for Theorem 5.6, we now consider $H_\varepsilon$ as the restriction

$$H_\varepsilon = \widetilde{H}_\varepsilon \restriction \mathscr{H}_{\mathrm{f}},$$

where $\widetilde{H}_\varepsilon$ denotes the Schrödinger operator defined by expression (5.4) on the enlarged Hilbert space $\mathscr{H}$ from Eq. (1.26). We shall give choices for $g_\varepsilon$ and $V$, where $\widetilde{H}_\varepsilon$ has a limit describing non-trivial contact interactions, or even no limit at all, while $H_\varepsilon$ converges in the norm resolvent sense to the free operator $H_0$.

For $d \leq 2$, we choose, for simplicity, a two-body potential $V \in C_0^\infty(\mathbb{R}^d)$ with $V(r) = V(-r)$ a.e. and $\int V(r) \, \mathrm{d}r = -1$. Moreover, similarly to Eq. (1.41), we choose

$$\begin{cases} g_\varepsilon = g & \text{if } d = 1, \\ 2/g_\varepsilon = (2\pi)^{-1} |\ln \varepsilon| + b & \text{if } d = 2, \end{cases}$$

for some $g, b \in \mathbb{R}$, $g > 0$. Then $\lambda_0 = \limsup_{\varepsilon\to 0} g_\varepsilon \varepsilon^{2-d} = 0$ and it follows from (5.12) that $\lambda_{\max} > 0$. On the one hand, this means that the hypotheses of Theorem 5.6 are satisfied and hence $H_\varepsilon \to H_0$ in the norm resolvent sense as $\varepsilon \to 0$. On the other hand, it follows from Theorems 3.1 and 4.1, respectively, that $\widetilde{H}_\varepsilon \to \widetilde{H}$ in the norm resolvent sense as $\varepsilon \to 0$, where $\widetilde{H}$ describes non-trivial contact interactions among $N$ particles. That is, $\widetilde{H}$ is a self-adjoint extension of $-\Delta \restriction C_0^\infty(\mathbb{R}^{dN} \setminus \Gamma)$ that is distinct from the free Laplacian.

In $d = 3$ dimensions this is more complicated. In this case we choose, in accordance with Eq. (1.41),

$$g_\varepsilon = 2(\varepsilon + b\varepsilon^2) \tag{5.48}$$

for some $b \in \mathbb{R}$, which yields $\lambda_0 = \limsup_{\varepsilon\to 0} g_\varepsilon \varepsilon^{-1} = 2$. Comparing this with the lower bound

$$\lambda_{\max} \geq 9/(C_{V_-} N) \tag{5.49}$$

from (5.12), this suggests that the condition $\lambda_0 < \lambda_{\max}$ is only satisfied if both $V$ and $N$ are "sufficiently small". For this reason, the dependence on $N$ is exhibited in the notation in the following: We write $H_{N,\varepsilon}$ and $\widetilde{H}_{N,\varepsilon}$ for $H_\varepsilon$ and $\widetilde{H}_\varepsilon$, respectively. To construct an appropriate two-body potential, we first consider the case $N = 2$ and we introduce the relative and center of mass coordinates from Eq. (5.18). Then,

$$\widetilde{H}_{2,\varepsilon} = h_\varepsilon \otimes 1 + 1 \otimes (-\Delta_R/2), \tag{5.50}$$

where the one-particle operator

$$h_\varepsilon = -2\Delta_r + g_\varepsilon V_\varepsilon$$

is self-adjoint on $D(h_\varepsilon) = H^2(\mathbb{R}^3)$, provided that $V \in L^2(\mathbb{R}^3)$. The prefactor 2 in the Definition (5.48) of $g_\varepsilon$ allows us to compare $h_\varepsilon$ directly with the operator from [6, Chapter I, Eq. (1.2.11)]. Thus, we know from [6, Chapter I, Theorem 1.2.5.] that $h_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to $-2\Delta \restriction H^2(\mathbb{R}^3)$ unless

$$h := -\Delta + V$$

has a zero-energy resonance. We claim that an appropriate potential with a zero-energy resonance is given by

$$V(r) := \begin{cases} -\dfrac{2}{|r|} + 1 & \text{if } |r| \leq 1, \\ 0 & \text{if } |r| > 1. \end{cases}$$

To see this, we first observe that $V \in L^1 \cap L^2(\mathbb{R}^3)$, so the integral operator $uG_0v$ that is defined in terms of the kernel

$$(uG_0v)(x, x') := \frac{u(x)v(x')}{4\pi|x - x'|}$$

is a Hilbert-Schmidt operator [6]. Now, it is straightforward to verify that $(uG_0v)\varphi = -\varphi$, where $\varphi \in L^2(\mathbb{R}^3)$ is given by

$$\varphi(r) := \begin{cases} \left(\dfrac{2}{|r|} - 1\right)^{1/2} \exp(-|r|) & \text{if } |r| \leq 1, \\ 0 & \text{if } |r| > 1. \end{cases}$$

In analogy to [6, Chapter I, Eq. (1.2.25)] we now define

$$\psi(r) := (G_0v\varphi)(r) = \begin{cases} \exp(-|r|) & \text{if } |r| \leq 1, \\ \dfrac{1}{e|r|} & \text{if } |r| > 1. \end{cases}$$

Then it follows from [6, Chapter I, Lemma 1.2.3] that $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $(-\Delta + V)\psi = 0$ in the distributional sense, but, clearly, $\psi \notin L^2(\mathbb{R}^3)$. This proves that $h = -\Delta + V$ has a zero-energy resonance in the sense of [6]. Furthermore, one can verify that $h$ has no discrete spectrum in $(-\infty, 0]$, which, in particular, implies that $h \geq 0$.

Now, on the one hand, it follows immediately from the definition of $V$ that $-|r|^{-2} \leq V(r) \leq 0$ and hence $C_{V_-} = C_V = \operatorname{ess\,sup}_{r\in\mathbb{R}^3} |r|^2 |V(r)| \leq 1$. Thus, for $N \leq 4$, it follows from (5.49) that $\lambda_0 = 2 < \lambda_{\max}$ and hence, by Theorem 5.6, $H_{N,\varepsilon} \to H_0$ in the norm resolvent sense as $\varepsilon \to 0$. On the other hand, concerning $\widetilde{H}_{N,\varepsilon}$, we can say the following:

**Proposition 5.13.** *With the above notations, in the case $d = 3$ we have*

*(i)* *For $N = 2$, $\widetilde{H}_{2,\varepsilon} \to \widetilde{H}_2$ in the norm resolvent sense as $\varepsilon \to 0$, where $\widetilde{H}_2$ is a non-trivial self-adjoint extension of $-\Delta \restriction C_0^\infty(\mathbb{R}^6 \setminus \Gamma)$.*

*(ii)* *For each $N \geq 3$ there exists a constant $C_N < 0$ such that*

$$\sigma(\widetilde{H}_{N,\varepsilon}) = [C_N \varepsilon^{-2}, \infty),$$

*provided that $b = 0$ in* (5.48).

*Remark.* Proposition 5.13 and the above considerations for $d \leq 2$ show that the antisymmetry of the wave function is indispensable for Theorem 5.6 in all dimensions $d \leq 3$: The analog of Theorem 5.6 in $L^2(\mathbb{R}^{dN})$ is wrong.

*Proof.* To prove $(i)$, we use that the one-particle operator $h = -\Delta + V$ has a zero-energy resonance but no discrete spectrum in $(-\infty, 0]$. This means that case II of [6, Chapter I, Theorem 1.2.5] applies to $h_\varepsilon$, and it follows that $h_\varepsilon \to h(\alpha)$ in the norm resolvent sense as $\varepsilon \to 0$, where $h(\alpha) = -2\Delta_{\alpha,0}$ is a self-adjoint extension of $-2\Delta \restriction C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ that is distinct from the free Laplacian. From [6, Chapter I, Theorem 1.3.1] it now follows that $h(\alpha) \geq -C$, $h_\varepsilon \geq -C$, and hence, by (5.50), $\widetilde{H}_{2,\varepsilon} \geq -C$ for some constant $C > 0$ that is independent of $\varepsilon \in (0, 1)$. To prove norm resolvent convergence of $\widetilde{H}_{2,\varepsilon}$, we now pass to Fourier space in $R$ and we note that $\widetilde{H}_{2,\varepsilon}$ acts pointwise in the conjugate variable $P$ by the operator $\widetilde{H}_{2,\varepsilon}(P) = h_\varepsilon + P^2/2$. Therefore, it suffices to show that, uniformly in $z \geq C + 1$, $\|(h_\varepsilon + z)^{-1} - (h(\alpha) + z)^{-1}\| \to 0$ as $\varepsilon \to 0$. To this end, let $\delta \in (0, 1)$ be fixed. Then $z > C + 2\delta^{-1}$ and $\varepsilon < 1$ imply that $\|(h_\varepsilon + z)^{-1}\| < \delta/2$ and $\|(h(\alpha) + z)^{-1}\| < \delta/2$, and hence

$$\|(h_\varepsilon + z)^{-1} - (h(\alpha) + z)^{-1}\| < \delta. \tag{5.51}$$

For $z, w \in [C + 1, C + 2\delta^{-1}]$ we use that

$$\|(h_\varepsilon + z)^{-1} - (h_\varepsilon + w)^{-1}\| = |z - w| \|(h_\varepsilon + z)^{-1}(h_\varepsilon + w)^{-1}\| \leq |z - w|,$$

and a similar estimate for $h(\alpha)$ in place of $h_\varepsilon$. Hence, it suffices to consider finitely many points $z_1 = C + 1 < z_2 < ... < z_n = C + 2\delta^{-1}$ with $|z_{i+1} - z_i| < \delta/3$ and to choose $\varepsilon_0 > 0$ so small that $\|(h_\varepsilon + z_i)^{-1} - (h(\alpha) + z_i)^{-1}\| < \delta/3$ for all $\varepsilon \in (0, \varepsilon_0)$ and $i = 1, ..., n$. Then (5.51) is true for all $z \in [C + 1, C + 2\delta^{-1}]$ and all $\varepsilon \in (0, \varepsilon_0)$, and hence $\|(h_\varepsilon + z)^{-1} - (h(\alpha) + z)^{-1}\| \to 0$ as $\varepsilon \to 0$ uniformly in $z \geq C + 1$. This implies that $\widetilde{H}_{2,\varepsilon} \to \widetilde{H}_2$ in the norm resolvent sense as $\varepsilon \to 0$, where $\widetilde{H}_2$ is the closure of the operator $h(\alpha) \otimes 1 + 1 \otimes (-\Delta_R/2)$ defined on $D(h(\alpha)) \otimes H^2(\mathbb{R}^3)$. By construction, $\widetilde{H}_2$ is a self-adjoint extension of $-\Delta \restriction C_0^\infty(\mathbb{R}^6 \setminus \Gamma)$, so the proof of $(i)$ is complete.

To prove $(ii)$, we first note that, similarly to Eq. (5.8), $b = 0$ implies that $\widetilde{H}_{N,\varepsilon}$ is unitarily equivalent to $\varepsilon^{-2}\widetilde{H}_{N,\varepsilon=1}$. In the case $N = 3$ the zero-energy resonance of $h = -\Delta + V$ and the fact that $h \geq 0$ lead to the Efimov effect. For our particular choice of $V$, this follows, e.g., from [62, Theorem 1.1]. This means that $\widetilde{H}_{3,\varepsilon=1}$ with the center of mass motion removed has infinitely many negative eigenvalues and, in particular, that $C_3 := \inf \sigma(\widetilde{H}_{3,\varepsilon=1}) < 0$. For general $N \geq 3$, $C_N := \inf \sigma(\widetilde{H}_{N,\varepsilon=1}) \leq C_3$ now follows from the HVZ theorem, which proves $(ii)$. ☐

# Open problems and related works

Some of our results give rise to new questions that have not (or only partially) been answered in the course of this thesis. We collect such problems and suggestions for future research here:

- In the one-particle case the convergence, as $\varepsilon \to 0$, of the Schrödinger operators $h_\varepsilon = -\Delta + g_\varepsilon V_\varepsilon$ from Section 1.1.1 towards the respective Hamiltonian $h(\alpha)$, $\alpha \in (-\infty, \infty]$, defining a contact interaction at $x = 0$ also holds w.r.t. quantities that are fundamental in scattering theory. This involves, for example, convergence of the associated on-shell scattering amplitudes, on shell scattering operators, and hence of the associated scattering lengths in the limit $\varepsilon \to 0$ [6]. See also [17, 64] ($d = 1$), [15, 16] ($d = 2$) and [3, 4] ($d = 3$) for a more comprehensive discussion of low-energy scattering and similar results, as well as [20] for general results relating (strong) resolvent convergence to (strong) continuity of wave and scattering operators.

- If $d = 1$ and $\int V_\sigma(r)\,\mathrm{d}r = 0$ for some pair $\sigma$, then $\alpha_\sigma = g_\sigma \int V_\sigma(r)\,\mathrm{d}r = 0$ by Theorem 3.1 and the pair $\sigma$ has a vanishing contribution to the limit operator $H$. In the one-particle case this is not true anymore if the coupling constant $g_\varepsilon$ has a certain divergence as $\varepsilon \to 0$ even if $V \in C_0^\infty(\mathbb{R})$ obeys $\int V(r)\,\mathrm{d}r = 0$ but $V \neq 0$ (see [79, Theorem 4]). If $g_\varepsilon = \varepsilon^{-1/2}$, then $h_\varepsilon = -\Delta + g_\varepsilon V_\varepsilon$ converges in the norm resolvent sense to the operator $h(\alpha)$ from Section 1.1.1, which defines a contact interaction of strength $\alpha = \int \mathrm{d}x\,\mathrm{d}y\, V(x)|x-y|V(y)/2$ at the origin. If $g_\varepsilon = \varepsilon^{-\lambda}$ for some $\lambda \in (1/2, 1)$, then $h_\varepsilon \to h(\infty)$ in the norm resolvent sense as $\varepsilon \to 0$, where $h(\infty)$ is given by the Dirichlet Laplacian on $(-\infty, 0) \cup (0, \infty)$. We expect that similar results also hold for $N \geq 2$ particles and that our proof of Theorem 3.1 can be largely adjusted to these cases.

- As explained in Section 4.1, the *norm* resolvent convergence $H_\varepsilon \to H$ established by our two-dimensional main result Theorem 4.1 can only be valid under the assumption that the leading coefficient $a_\sigma > 0$ in Eq. (4.4) is not smaller than $(2\pi)^{-1} \int V_\sigma(r)\,\mathrm{d}r$ for all pairs $\sigma \in \mathcal{I}$. In the one-particle case, the analog of this condition is not necessary: Then, by [6, Chapter I, Theorem 5.5], one still has norm resolvent convergence towards the negative Laplacian $-\Delta$. Hence, it would be interesting to know whether *strong* resolvent convergence $H_\varepsilon \to H$ as $\varepsilon \to 0$ still holds in the case where $0 < a_\sigma < (2\pi)^{-1} \int V_\sigma(r)\,\mathrm{d}r$ for some pairs $\sigma$.

- The Fermi polaron model describes two-body contact interactions among $N \geq 1$ identical fermions and a particle of different type (called impurity). It has been shown in [42, 54] that the Hamiltonian of the 2d Fermi polaron has an $N$-independent lower bound, provided that the ratio $m$ of the impurity mass and the fermion mass is larger than 1.225, and it has been conjectured that this remains true for *arbitrary* $m > 0$. In view of Theorem 4.1, it would be sufficient to show that $H_\varepsilon \geq -C(m)$ for some constant $C(m) > 0$ that is independent of $\varepsilon \in (0, \varepsilon_0)$ and $N \geq 1$, where $H_\varepsilon$, $\varepsilon > 0$, are suitably rescaled Schrödinger operators that are defined as in Eqs. (4.1)-(4.5) in the respective subspace of $L^2$. This provides a new approach to the problem, since the Hamiltonian has so far only been constructed as a strong resolvent limit of operators, where the regularized two-body interaction is an integral operator rather than a potential.

- As explained in Section 1.2.2, the Thomas effect that arises for $N = 3$ particles in $d = 3$ dimensions is much weaker or even absent in the presence of some fermionic antisymmetry. Indeed, it has been recently shown that the Hamiltonian of the 3d Fermi polaron is bounded from below if $m > m^*$ for some critical mass ratio $m^* > 0$ [24, 60]. A similar result also holds for a $(2+2)$-system of two identical fermions that interact via two-body contact interactions with two fermions of a different species [57, 61]. Therefore, it is a natural and physically relevant question to ask whether these Hamiltonians in $d = 3$ can be also obtained as norm resolvent limits of suitably rescaled Schrödinger operators $H_\varepsilon$, $\varepsilon > 0$.

In view of the one-particle case described in [6, Chapter I.1.2], we expect that a suitable ansatz for $H_\varepsilon$ is given by Eqs. (1.39)-(1.41), where the two-body potentials $V_\sigma$ with a non-vanishing contribution in the limit $\varepsilon \to 0$ must have a zero-energy resonance. Making use of the antisymmetry, the resolvent of $H_\varepsilon$ can then be expressed by a simplified version of the generalized Konno-Kuroda formula from Eq. (4.12). This allows one to transfer the methods of this thesis. Proving convergence, as $\varepsilon \to 0$, of the inverse $\Lambda_\varepsilon(z)^{-1}$ will be the key point of the analysis, where the zero-energy resonances of the two-body potentials, the antisymmetry of the wave function, and the assumption $m > m^*$ on the mass ratio will play a crucial role. Here, the methods and results from [24, 60] and [57, 61], respectively, may be helpful.

# Appendix

# A Properties of the Green's function

This section collects facts and estimates on the Green's function of $-\Delta + z : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

For $d \in \mathbb{N}$ and $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$, let the function $G_z^d : \mathbb{R}^d \to \mathbb{C}$ be defined by

$$G_z^d(x) := \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{x^2}{4t} - zt\right) \mathrm{d}t. \tag{A.1}$$

Notice that $G_z^d$ has a singularity at $x = 0$ unless $d = 1$. The following lemma identifies $G_z^d$ as the Green's function of $-\Delta + z : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and establishes its well-known properties.

**Lemma A.1.** *Let $d \in \mathbb{N}$ and $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$. Then $G_z^d$ defined by (A.1) has the following properties:*

*(i)* $G_z^d \in L^1(\mathbb{R}^d)$ *and* $\|G_z^d\|_{L^1} \leq \operatorname{Re}(z)^{-1}$.

*(ii) The Fourier transform of $G_z^d$ is given by $\widehat{G_z^d}(p) = (2\pi)^{-d/2}(p^2 + z)^{-1}$.*

*(iii) $G_z^d$ is the Green's function of $-\Delta + z$. That is $(-\Delta + z)^{-1} f = G_z^d * f$ for all $f \in L^2(\mathbb{R}^d)$, where the convolution $G_z^d * f$ is defined by $(G_z^d * f)(x) := \int G_z^d(x - y) f(y)\, \mathrm{d}y$.*

*(iv) $G_z^d \in L^2(\mathbb{R}^d)$ if and only if $d \leq 3$.*

*(v) $G_z^d$ is spherically symmetric, i.e. $G_z^d(x)$ only depends on $|x|$. For $z \in (0, \infty)$, $G_z^d(x)$ is positive and strictly monotonically decreasing both as a function of $|x|$ and $z$.*

*(vi) Let $d_1, d_2 \in \mathbb{N}$ with $d_1 + d_2 = d$ and let $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. If $x_1 \neq 0$ or $d_1 = 1$, then $G_z^d(x_1, \cdot) \in L^1(\mathbb{R}^{d_2})$ and the Fourier transform is*

$$\widehat{G_z^d}(x_1, p_2) = (2\pi)^{-d_2/2}\, G_{z+p_2^2}^{d_1}(x_1). \tag{A.2}$$

*In particular, this means that*

$$\int_{\mathbb{R}^{d_2}} G_z^d(x_1, x_2)\, \mathrm{d}x_2 = G_z^{d_1}(x_1). \tag{A.3}$$

*Proof.* Applying first Tonelli's theorem to change the order of integration and using the substitution $\widetilde{x} = x/\sqrt{4t}$ afterwards, we see that

$$\begin{aligned}\int_{\mathbb{R}^d} |G_z^d(x)|\, \mathrm{d}x &\leq \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}x\, (4\pi t)^{-d/2} \exp\left(-\frac{x^2}{4t} - \operatorname{Re}(z)t\right) \\ &= \pi^{-d/2} \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}\widetilde{x}\, \exp\left(-\widetilde{x}^2 - \operatorname{Re}(z)t\right) = \operatorname{Re}(z)^{-1},\end{aligned}$$

which proves $(i)$. This, in turn, allows us to compute the Fourier transform $\widehat{G_z^d}$ with the help of Fubini's theorem and the substitution $\widetilde{x} = x/\sqrt{2t}$:

$$\begin{aligned}\widehat{G_z^d}(p) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}x \int_0^\infty \mathrm{d}t\, (4\pi t)^{-d/2} \exp\left(-\frac{x^2}{4t} - zt - ip \cdot x\right) \\ &= (2\pi)^{-d} \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}\widetilde{x}\, \exp\left(-\frac{\widetilde{x}^2}{2} - zt - i\sqrt{2t}p \cdot \widetilde{x}\right) \\ &= (2\pi)^{-d/2} \int_0^\infty \mathrm{d}t\, \exp\left(-(p^2 + z)t\right) = (2\pi)^{-d/2}(p^2 + z)^{-1},\end{aligned}$$

where the third line was obtained from the fact that $x \mapsto \exp(-x^2/2)$ is the fixed point of the Fourier transform. This proves (*ii*) and the proof of (*vi*) is similar: It is straightforward to verify that the assumption that $x_1 \neq 0$ or $d_1 = 1$ ensures that $G_z^d(x_1, \cdot) \in L^1(\mathbb{R}^{d_2})$ and that Fubini's theorem allows us to change the order of the $x_2$-integration and the $t$-integration in the computation of $\widehat{G_z^d}(x_1, \cdot)$ (see also Lemma A.4, below). Hence, upon the substitution $\widetilde{x}_2 = x_2/\sqrt{2t}$, (*vi*) follows from

$$\begin{aligned}
\widehat{G_z^d}(x_1, p_2) &= (2\pi)^{-d_2/2} \int\limits_{\mathbb{R}^{d_2}} \mathrm{d}x_2 \int_0^\infty \mathrm{d}t\ (4\pi t)^{-d/2} \exp\left(-\frac{x_1^2 + x_2^2}{4t} - zt - ip_2 \cdot x_2\right) \\
&= (2\pi)^{-d_2/2} \int_0^\infty \mathrm{d}t\, (4\pi t)^{-d_1/2} \exp\left(-\frac{x_1^2}{4t} - zt\right) (2\pi)^{-d_2/2} \int\limits_{\mathbb{R}^{d_2}} \mathrm{d}\widetilde{x}_2 \exp\left(-\frac{\widetilde{x}_2^2}{2} - i\sqrt{2t}p_2 \cdot \widetilde{x}_2\right) \\
&= (2\pi)^{-d_2/2} \int_0^\infty \mathrm{d}t\ (4\pi t)^{-d_1/2} \exp\left(-\frac{x_1^2}{4t} - (z + p_2^2)t\right) = (2\pi)^{-d_2/2}\, G_{z+p_2^2}^{d_1}(x_1).
\end{aligned}$$

Finally, (*iv*) is a direct consequence of (*ii*) and (*v*) is immediate from (A.1). □

The following lemma allows us to estimate differences of integral operators depending on $G_z^d$:

**Lemma A.2.** *Let $d \in \mathbb{N}$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Then for all $x, \widetilde{x} \in \mathbb{R}^d$ and all $Q \geq 0$,*

$$\left|G_{z+Q}^d(x) - G_{z+Q}^d(\widetilde{x})\right| \leq \left|G_{\mathrm{Re}(z)}^d(x) - G_{\mathrm{Re}(z)}^d(\widetilde{x})\right|. \tag{A.4}$$

*Similarly, if $d = d_1 + d_2$ for some $d_1, d_2 \in \mathbb{N}$, then for all $x_1, y_1 \in \mathbb{R}^{d_1}$ and all $x_2 \in \mathbb{R}^{d_2}$,*

$$\left|G_z^d(x_1, x_2) - G_z^d(y_1, x_2)\right| \leq \left|G_{\mathrm{Re}(z)}^d(x_1, 0) - G_{\mathrm{Re}(z)}^d(y_1, 0)\right|. \tag{A.5}$$

*Proof.* To prove (A.5) we may assume, without loss of generality, that $|x_1| \leq |y_1|$. Then,

$$\begin{aligned}
\left|G_z^d(x_1, x_2) - G_z^d(y_1, x_2)\right| &= \left|\int_0^\infty \mathrm{d}t\ (4\pi t)^{-d/2} \left(\exp\left(-\frac{x_1^2}{4t}\right) - \exp\left(-\frac{y_1^2}{4t}\right)\right) \exp\left(-\frac{x_2^2}{4t} - zt\right)\right| \\
&\leq \int_0^\infty \mathrm{d}t\ (4\pi t)^{-d/2} \left(\exp\left(-\frac{x_1^2}{4t}\right) - \exp\left(-\frac{y_1^2}{4t}\right)\right) \exp\left(-\mathrm{Re}(z)t\right) \\
&= \left|G_{\mathrm{Re}(z)}^d(x_1, 0) - G_{\mathrm{Re}(z)}^d(y_1, 0)\right|.
\end{aligned}$$

The proof of (A.4) is very similar and therefore omitted. □

**Lemma A.3.** *Let $d \geq 2$, $s \in (0, 1)$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Then, for some constant $C(s, z) > 0$ and all $y \in \mathbb{R}^{d-1}$,*

$$\int\limits_{\mathbb{R}^{d-1}} \mathrm{d}x \left|G_z^d(x + y, 0) - G_z^d(x, 0)\right| \leq C(s, z)|y|^s. \tag{A.6}$$

*Proof.* Since $G_z^d(\cdot, 0) \in L^1(\mathbb{R}^{d-1})$ by Lemma A.1 (*vi*), the left side of (A.6) is bounded, uniformly in $y \in \mathbb{R}^{d-1}$. So it remains to prove (A.6) for $|y| \leq 1$, and to this end it suffices to show that there exists a constant $C(z) > 0$ such that, for all $y \in \mathbb{R}^{d-1} \setminus \{0\}$,

$$\int\limits_{\mathbb{R}^{d-1}} \mathrm{d}x \left|G_z^d(x + y, 0) - G_z^d(x, 0)\right| \leq C(z)(1 + |\ln|y||)\, |y|. \tag{A.7}$$

Using the integral representation (A.1) for $G_z^d$ and making the substitution $x/\sqrt{4t} \to x$, we find

$$\int\limits_{\mathbb{R}^{d-1}} \mathrm{d}x \left|G_z^d(x + y, 0) - G_z^d(x, 0)\right| \leq \int_0^\infty \mathrm{d}t\, \frac{e^{-\lambda t}}{2\pi^{d/2} t^{1/2}} \int\limits_{\mathbb{R}^{d-1}} \mathrm{d}x \left|\exp\left(-\left(x + \frac{y}{\sqrt{4t}}\right)^2\right) - \exp\left(-x^2\right)\right|,$$

where $\lambda = \mathrm{Re}(z)$. By applications of triangle inequality and the fundamental theorem of calculus, respectively,

$$\int\limits_{\mathbb{R}^{d-1}} \mathrm{d}x \left| \exp\left( -\left( x + \frac{y}{\sqrt{4t}} \right)^2 \right) - \exp\left( -x^2 \right) \right| \leq C \min\left( 1, \frac{|y|}{\sqrt{t}} \right).$$

Since

$$\begin{aligned} \int_0^\infty \mathrm{d}t \, \frac{e^{-\lambda t}}{t^{1/2}} \min\left(1, \frac{|y|}{\sqrt{t}}\right) &\leq \int_0^{|y|^2} t^{-1/2} \, \mathrm{d}t + |y| \int_{|y|^2}^\infty \frac{e^{-\lambda t}}{t} \, \mathrm{d}t \\ &\leq 2|y| + |y| \left( \lambda^{-1} + 2 \ln |y| \right), \end{aligned}$$

the desired estimate follows. □

For $z \in (0, \infty)$ and $d \geq 2$, it follows from (A.1) that $G_z^d(x) \to \infty$ as $|x| \to 0$. However, the next lemma shows that, apart from $x = 0$, $G_z^d$ is surprisingly regular:

**Lemma A.4.** *Let $d \in \mathbb{N}$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Then $G_z^d$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and, for any multi-index $\alpha \in \mathbb{N}_0^d$, $\partial^\alpha G_z^d$ is exponentially decaying as $|x| \to \infty$.*

*Proof.* We claim, and prove below, that for all $c > 0$, $k \in \mathbb{R}$, $l \geq 0$ and $\lambda \in (0, \sqrt{\mathrm{Re}(z)})$,

$$\sup_{|x| \geq c} \left( \int_0^\infty t^k |x|^l \exp\left( -\frac{x^2}{4t} - \mathrm{Re}(z)t + \lambda |x| \right) \mathrm{d}t \right) < \infty. \tag{A.8}$$

This allows us to change the order of differentiation and integration in (A.1). We find that, for all $x \in \mathbb{R}^d \setminus \{0\}$, $\partial^\alpha G_z^d(x)$ is a sum of terms of the form

$$\int_0^\infty (4\pi t)^{-d/2} \frac{x^{\alpha'}}{(2t)^n} \exp\left( -\frac{x^2}{4t} - zt \right) \mathrm{d}t,$$

where $0 \leq n \leq |\alpha|$ and $\alpha' \in \mathbb{N}_0^d$ is a multi-index with $|\alpha'| \leq |\alpha|$. In view of (A.8), all of these terms are exponentially decaying as $|x| \to \infty$, which proves the lemma.

It remains to prove (A.8) for given $c > 0$, $k \in \mathbb{R}$, $l \geq 0$ and $\lambda \in (0, \sqrt{\mathrm{Re}(z)})$. To this end, we first choose $\delta > 0$ so small that $\lambda \leq (1 - \delta)\sqrt{\mathrm{Re}(z)}$. Then the estimate

$$\begin{aligned} \exp\left( -\frac{x^2}{4t} - \mathrm{Re}(z)t + \lambda|x| \right) &\leq \exp\left( -\frac{x^2}{4t} - \mathrm{Re}(z)t + (1 - \delta)\sqrt{\mathrm{Re}(z)}|x| \right) \\ &= \exp\left( -(1 - \delta) \left( \frac{|x|}{2\sqrt{t}} - \sqrt{\mathrm{Re}(z)t} \right)^2 - \delta \left( \frac{x^2}{4t} + \mathrm{Re}(z)t \right) \right) \\ &\leq \exp\left( -\frac{\delta x^2}{4t} - \delta \, \mathrm{Re}(z)t \right) \end{aligned}$$

holds for all $t > 0$ and all $x \in \mathbb{R}^d$. With $C_{l,\delta} := \sup_{s \geq 0} \left( s^{l/2} \exp\left( -\delta s \right) \right) < \infty$, it follows that

$$\begin{aligned} &\sup_{|x| \geq c} \left( \int_0^\infty t^k |x|^l \exp\left( -\frac{x^2}{4t} - \mathrm{Re}(z)t + \lambda|x| \right) \mathrm{d}t \right) \\ &\leq 8^{l/2} \sup_{|x| \geq c} \left( \int_0^\infty t^{k+l/2} \left( \frac{x^2}{8t} \right)^{l/2} \exp\left( -\frac{\delta x^2}{8t} - \frac{\delta c^2}{8t} - \delta \, \mathrm{Re}(z)t \right) \mathrm{d}t \right) \\ &\leq 8^{l/2} C_{l,\delta} \int_0^\infty t^{k+l/2} \exp\left( -\frac{\delta c^2}{8t} - \delta \, \mathrm{Re}(z)t \right) \mathrm{d}t, \end{aligned}$$

where the last integral is finite because the term $\exp\left( -\delta c^2/(8t) \right)$ cancels a possible divergence at $t = 0$. This proves (A.8) and hence the lemma. □

In the proof of Proposition 4.8 two integral operators $F_1$ and $F_2$ depending on $G_z^d$ were introduced. We conclude this section with the bypassed proof of their boundedness:

**Lemma A.5.** *Let $z, m \in (0,\infty)$. Then the integral operator $F_1 : L^2(\mathbb{R}^2 \times \mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ that is defined in terms of the kernel*

$$K_1(x,y,x',y') := G_z^6\left(\sqrt{m}(x-x'), \sqrt{m}(x-y'), \sqrt{m}(y-x')\right)$$

*is bounded with $\|F_1\| \le (4\sqrt{2}m^2)^{-1}$. Similarly, the kernel*

$$K_2(x,y,w,x',y',w') := G_z^8\left(\sqrt{m}(x-y'), \sqrt{m}(x-w'), \sqrt{m}(y-x'), \sqrt{m}(w-x')\right)$$

*defines an integral operator $F_2 \in \mathscr{L}(L^2(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2))$ with $\|F_2\| \le (4\sqrt{2}m^3)^{-1}$.*

*Proof.* Observe that the kernels $K_1$ and $K_2$ are both symmetric. The bounds for $\|F_1\|$ and $\|F_2\|$ are based on the Schur test. In the case of $F_1$, a general version of the Schur test shows that $F_1$ is a bounded operator with

$$\|F_1\| \le \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^2} \left( h(x,y)^{-1} \int \mathrm{d}x'\,\mathrm{d}y'\; h(x',y') K_1(x,y,x',y') \right), \tag{A.9}$$

provided that the right side is finite for some measurable test function $h : \mathbb{R}^2 \times \mathbb{R}^2 \to (0,\infty)$. We choose the test function $h(x,y) := |x-y|^{-1}$, so after a scaling we may assume that $m = 1$. To evaluate the right side of (A.9), we insert the integral representation (A.1) of $G_z^6$ and we substitute $x - y' \to y'$. This results in the identity

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^2} \left( |x-y| \int \mathrm{d}x'\,\mathrm{d}y'\; |x'-y'|^{-1}\, G_z^6\left(x-x', x-y', y-x'\right) \right) \\
&= \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^2} \left( |x-y| \int \mathrm{d}x' \int_0^\infty \mathrm{d}t\; (4\pi t)^{-3} \exp\left(-\frac{1}{4t}\left[(x-x')^2 + (y-x')^2\right]\right) \right. \\
&\quad \left. \cdot \int \mathrm{d}y'\; |y'+x'-x|^{-1} \exp\left(-\frac{|y'|^2}{4t} - zt\right) \right).
\end{aligned} \tag{A.10}$$

Using a rearrangement inequality (see [53, Theorem 3.4]), we see that the last integral has the bound

$$\begin{aligned}
\int \mathrm{d}y'\; |y'+x'-x|^{-1} \exp\left(-\frac{|y'|^2}{4t} - zt\right) &\le \int \mathrm{d}y'\; |y'|^{-1} \exp\left(-\frac{|y'|^2}{4t}\right) \\
&= 2\pi \int_0^\infty \mathrm{d}r\; \exp\left(-\frac{r^2}{4t}\right) = \sqrt{4\pi^3 t}.
\end{aligned} \tag{A.11}$$

For the remaining integral in (A.10), we use $(x-x')^2 + (y-x')^2 = \frac{1}{2}\left((2x'-x-y)^2 + (x-y)^2\right)$ together with the substitution $2x'-x-y \to 2x'$. This yields that

$$\begin{aligned}
&\pi \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^2} \left( |x-y| \int \mathrm{d}x' \int_0^\infty \mathrm{d}t\; (4\pi t)^{-5/2} \exp\left(-\frac{1}{4t}\left[(x-x')^2 + (y-x')^2\right]\right) \right) \\
&= \pi \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^2} \left( |x-y| \int_0^\infty \mathrm{d}t \int \mathrm{d}x'\; (4\pi t)^{-5/2} \exp\left(-\frac{|x'|^2}{2t} - \frac{(x-y)^2}{8t}\right) \right) \\
&= (16\sqrt{\pi})^{-1} \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^2} \left( |x-y| \int_0^\infty \mathrm{d}t\; t^{-3/2} \exp\left(-\frac{(x-y)^2}{8t}\right) \right) \\
&= (4\sqrt{2\pi})^{-1} \int_0^\infty \mathrm{d}t\; t^{-3/2} \exp\left(-t^{-1}\right),
\end{aligned} \tag{A.12}$$

where the essential supremum was canceled by the scaling $t \to (x-y)^2 t/8$. The last integral can be explicitly computed with the help of the substitution $\tilde{t} = t^{-1}$. We find that

$$\int_0^\infty t^{-3/2} \exp\left(-t^{-1}\right)\,\mathrm{d}t = \int_0^\infty \tilde{t}^{-1/2} \exp\left(-\tilde{t}\right)\,\mathrm{d}\tilde{t} = \Gamma(1/2) = \sqrt{\pi}, \tag{A.13}$$

where $\Gamma(\cdot)$ denotes the gamma function. Using (A.11), (A.12) and (A.13) to bound the right side of (A.10), the Schur test reveals that $F_1$ is bounded with $\|F_1\| \leq (4\sqrt{2})^{-1}$ (for $m = 1$).

The proof for the operator $F_2$ is similar, but for the convenience of the reader we provide the details here. First, the Schur test with the test function $h(y, w) = |y - w|^{-1}$ yields

$$\|F_2\| \leq \operatorname*{ess\,sup}_{x,y,w \in \mathbb{R}^2} \left( |y - w| \int \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}w' \, |y' - w'|^{-1} K_2(x, y, w, x', y', w') \right),$$

provided that the right side is finite. After a scaling, we may again assume that $m = 1$. Then the integral representation (A.1) of $G_z^8$ combined with the substitution $x - y' \to y'$ leads to

$$\begin{aligned}\|F_2\| \leq \operatorname*{ess\,sup}_{x,y,w \in \mathbb{R}^2} \Bigg( |y - w| \int \mathrm{d}x' \, \mathrm{d}w' \int_0^\infty \mathrm{d}t \, (4\pi t)^{-4} \exp\left( -\frac{1}{4t} \left[ (x - w')^2 + (y - x')^2 + (w - x')^2 \right] \right) \\ \cdot \int \mathrm{d}y' \, |x - y' - w'|^{-1} \exp\left( -\frac{|y'|^2}{4t} - zt \right) \Bigg). \qquad \text{(A.14)}\end{aligned}$$

Similarly to (A.11), a rearrangement inequality shows that

$$\int \mathrm{d}y' \, |x - y' - w'|^{-1} \exp\left( -\frac{|y'|^2}{4t} - zt \right) \leq \sqrt{4\pi^3 t}. \qquad \text{(A.15)}$$

In the remaining integral, we use the identity $(y - x')^2 + (w - x')^2 = \frac{1}{2}\left( (2x' - w - y)^2 + (y - w)^2 \right)$ in combination with the substitutions $2x' - w - y \to 2x'$ and $x - w' \to w'$. We find that

$$\begin{aligned}&\pi \operatorname*{ess\,sup}_{x,y,w \in \mathbb{R}^2} \left( |y - w| \int \mathrm{d}x' \, \mathrm{d}w' \int_0^\infty \mathrm{d}t \, (4\pi t)^{-7/2} \exp\left( -\frac{1}{4t} \left[ (x - w')^2 + (y - x')^2 + (w - x')^2 \right] \right) \right) \\ &= \pi \operatorname*{ess\,sup}_{x,y,w \in \mathbb{R}^2} \left( |y - w| \int_0^\infty \mathrm{d}t \int \mathrm{d}x' \, \mathrm{d}w' \, (4\pi t)^{-7/2} \exp\left( -\frac{1}{4t} \left( |w'|^2 + 2|x'|^2 \right) - \frac{(y - w)^2}{8t} \right) \right) \\ &= (16\sqrt{\pi})^{-1} \operatorname*{ess\,sup}_{y,w \in \mathbb{R}^2} \left( |y - w| \int_0^\infty \mathrm{d}t \, t^{-3/2} \exp\left( -\frac{(y - w)^2}{8t} \right) \right) \\ &= (4\sqrt{2\pi})^{-1} \int_0^\infty \mathrm{d}t \, t^{-3/2} \exp\left( -t^{-1} \right), \qquad \text{(A.16)}\end{aligned}$$

where the essential supremum was canceled by the scaling $t \to (y - w)^2 t/8$. By (A.13), the last integral is equal to $\sqrt{\pi}$, so the desired estimate for $\|F_2\|$ follows from (A.14), (A.15) and (A.16). □

# B Konno-Kuroda formula

In this section we sketch the proof of the Konno-Kuroda resolvent identity, see [49, Eq. (2.3)], for operators of the type (B.1), below. The main difference between Theorem B.1, below, and [49] is that we do not assume that $\phi(z)$ defined by Eq. (B.2) extends to a compact operator for some (and hence all) $z \in \rho(H_0)$.

Let $\mathscr{H}$ and $\widetilde{\mathfrak{X}}$ be arbitrary complex Hilbert spaces, let $H_0 \geq 0$ be a self-adjoint operator in $\mathscr{H}$ and let $A : D(A) \subseteq \mathscr{H} \to \widetilde{\mathfrak{X}}$ be densely defined and closed with $D(A) \supseteq D(H_0)$. Let $J \in \mathscr{L}(\widetilde{\mathfrak{X}})$ be self-adjoint and let $B = JA$. Suppose that $BD(H_0) \subseteq D(A^*)$ and that $A^*A$ and $A^*B$ are $H_0$-bounded with relative bound less than one. Then

$$H = H_0 + A^*B \tag{B.1}$$

is self-adjoint on $D(H_0)$ by the Kato-Rellich theorem (see [69, Theorem X.12]). For $z \in \rho(H_0)$, let $R_0(z) := (H_0 + z)^{-1}$ and let $\phi(z) : D(A^*) \subseteq \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{X}}$ be defined by

$$\phi(z) := BR_0(z)A^*. \tag{B.2}$$

Note that $D(A^*) \subseteq \widetilde{\mathfrak{X}}$ is dense because $A$ is closed. The resolvent $(H + z)^{-1}$ and the operator $\phi(z)$ are related by the following theorem:

**Theorem B.1.** *Let the above hypotheses be satisfied and let $z \in \rho(H_0)$. Then $\phi(z)$ defines a bounded operator. The operator $1 + \phi(z)$ is invertible if and only if $z \in \rho(H_0) \cap \rho(H)$, and then*

$$(H + z)^{-1} = R_0(z) - R_0(z)A^*(1 + \phi(z))^{-1}BR_0(z), \tag{B.3}$$

$$(1 + \phi(z))^{-1} = 1 - B(H + z)^{-1}A^*. \tag{B.4}$$

*Remark.* Note that $(1 + \phi(z))^{-1}$ leaves $D(A^*)$ invariant. This follows from (B.4) and from the assumption $BD(H_0) \subseteq D(A^*)$.

*Proof.*

*Step 1.* $AR_0(z)^{1/2}$ is bounded for $z > 0$, and $A(H + z)^{-1/2}$ is bounded for $z > 0$ large enough.

As $A^*A$ is $H_0$-bounded with relative bound less than one, the Kato-Rellich theorem shows that $H_0 - A^*A$ is bounded from below. This implies that, for all $\psi \in D(H_0)$ and all $z > 0$,

$$\|A\psi\|^2 \leq \|(H_0 + z)^{1/2}\psi\|^2 + C\|\psi\|^2$$

for some constant $C > 0$. Since $D(H_0)$ is dense in $D(H_0^{1/2})$ w.r.t. the graph norm of $H_0^{1/2}$ (see e.g. [71, Chapter VIII.6]) and $A$ is closed, this bound extends to all $\psi \in D(H_0^{1/2})$ by an approximation argument. In particular, $D(A) \supseteq D(H_0^{1/2})$ and the first statement of Step 1 follows. The second statement is a consequence of the first and the fact that $H$ and $H_0$ have equivalent form norms (in particular, the form domain of $H$ agrees with $D(H_0^{1/2})$).

*Step 2.* If $z \in \rho(H_0)$, then $\phi(z)$ is a bounded operator, and if $z \in \rho(H)$, then

$$S(z) := B(H + z)^{-1}A^*$$

is a bounded operator too.

This easily follows from Step 1 and from the first resolvent identity.

*Step 3.* If $z \in \rho(H_0) \cap \rho(H)$, then $1 + \phi(z)$ is invertible and $1 - S(z) = (1 + \phi(z))^{-1}$.

Both $\phi(z)$ and $S(z)$ leave $D(A^*)$ invariant and on this subspace, by straightforward computations using the second resolvent identity, $(1 + \phi(z))(1 - S(z)) = 1 = (1 - S(z))(1 + \phi(z))$.

*Step 4.* If $z \in \rho(H_0)$ and $1 + \phi(z)$ is invertible, then $z \in \rho(H)$, $(1 + \phi(z))^{-1}$ leaves $D(A^*)$ invariant and (B.3) holds.

By Step 3, $(1+\phi(i))^{-1} = 1-S(i)$, which leaves $D(A^*)$ invariant. Now suppose that $z \in \rho(H_0)$ and that $1 + \phi(z)$ has a bounded inverse. Then

$$\begin{aligned}(1+\phi(z))^{-1} &= (1+\phi(i))^{-1} + (1+\phi(i))^{-1}(\phi(i) - \phi(z))(1+\phi(z))^{-1} \\ &= (1+\phi(i))^{-1} + (z-i)(1+\phi(i))^{-1}BR_0(i)(AR_0(\bar{z}))^*(1+\phi(z))^{-1}.\end{aligned}$$

Since $BR_0(i) : \mathscr{H} \to D(A^*)$, it follows that $(1 + \phi(z))^{-1}$ leaves $D(A^*)$ invariant as well, and

$$R(z) := R_0(z) - R_0(z)A^*(1+\phi(z))^{-1}BR_0(z)$$

is well-defined. Now it is a matter of straightforward computations to show that $(H+z)R(z) = 1$ on $\mathscr{H}$ and that $R(z)(H + z) = 1$ on $D(H)$. □

Suppose that $\widetilde{\mathfrak{X}} = \bigoplus_{i=1}^{I} \widetilde{\mathfrak{X}}_i$, where $I \in \mathbb{N}$ and $\widetilde{\mathfrak{X}}_i$ are complex Hilbert spaces for $i = 1, ..., I$. Then we consider operators of the more general form

$$H = H_0 + \sum_{i=1}^{I} g_i\, A_i^* J_i A_i, \tag{B.5}$$

where $g_i \in \mathbb{R}$, $A_i : D(A_i) \subseteq \mathscr{H} \to \widetilde{\mathfrak{X}}_i$ are densely defined and closed with $D(A_i) \supseteq D(H_0)$ and $J_i = J_i^* \in \mathscr{L}(\widetilde{\mathfrak{X}}_i)$ for $i = 1, ..., I$. Suppose that $J_iA_iD(H_0) \subseteq D(A_i^*)$ and that $A_i^*A_i$ and $A_i^*J_iA_i$ are infinitesimally $H_0$-bounded for $i = 1, ..., I$. Then $H$ is self-adjoint on $D(H_0)$ and we observe that operators of the form (B.5) can be reduced to the form (B.1), where $A : D(A) \subseteq \mathscr{H} \to \widetilde{\mathfrak{X}}$ is the closure of the operator $A_0 : D(H_0) \to \widetilde{\mathfrak{X}}$ defined by $A_0\psi := (A_i\psi)_{i=1}^{I}$ and $J \in \mathscr{L}(\widetilde{\mathfrak{X}})$ is defined by $J(\psi_i)_{i=1}^{I} := (g_iJ_i\psi_i)_{i=1}^{I}$. Hence, Theorem B.1 implies the following corollary:

**Corollary B.2.** *Let the above hypotheses be satisfied and let $z \in \rho(H_0)$. Then*

$$\phi(z)_{ij} := J_iA_iR_0(z)A_j^*, \qquad i, j = 1, ..., N$$

*defines a bounded operator $\phi(z) = (\phi(z)_{ij})_{i,j=1}^{I} \in \mathscr{L}(\widetilde{\mathfrak{X}})$. Moreover, with the matrix operator $g \in \mathscr{L}(\widetilde{\mathfrak{X}})$ defined by $g_{ij} := g_i\delta_{ij}$, $i, j = 1, ..., I$, it follows that $1 + g\phi(z)$ is invertible if and only if $z \in \rho(H_0) \cap \rho(H)$. If this is the case, then*

$$\begin{aligned}(H+z)^{-1} &= R_0(z) - \sum_{i,j=1}^{I} (A_iR_0(\overline{z}))^* \left[(1+g\phi(z))^{-1}\right]_{ij} g_jJ_jA_jR_0(z), \\ (1+g\phi(z))^{-1} &= \left(\delta_{ij} - g_iJ_iA_i(H+z)^{-1}A_j^*\right)_{i,j=1}^{I}.\end{aligned}$$

# C Connection between the Efimov effect and the Thomas effect

We consider the Schrödinger operator $H_\varepsilon$, $\varepsilon > 0$, from Eqs. (1.39)-(1.41) for $N \geq 3$ particles in $d = 3$ dimensions. That is

$$H_\varepsilon = \sum_{i=1}^{N} (-\Delta_{x_i}/m_i) + \sum_{\sigma=(i,j)\in\mathcal{I}} g_{\varepsilon,\sigma}\, V_{\sigma,\varepsilon}(x_j - x_i), \qquad \varepsilon > 0, \tag{C.1}$$

where $V_{\sigma,\varepsilon}(r) = \varepsilon^{-3} V_\sigma(r/\varepsilon)$ for some fixed real-valued potential $V_\sigma \in L^1 \cap L^2(\mathbb{R}^3)$ with $V_\sigma(r) = V_\sigma(-r)$ a.e., and the asymptotics of $g_{\varepsilon,\sigma} \in \mathbb{R}$ is determined by

$$g_{\varepsilon,\sigma} = (\mu_\sigma)^{-1}(\varepsilon + b_\sigma \varepsilon^2) + o(\varepsilon^2) \qquad (\varepsilon \to 0) \tag{C.2}$$

for some constant $b_\sigma \in \mathbb{R}$. Thus $H_\varepsilon$ is self-adjoint on $D(H_\varepsilon) = H^2(\mathbb{R}^{3N})$ and the translation invariance of $H_\varepsilon$ implies that $\sigma(H_\varepsilon) = [\Sigma_\varepsilon, \infty)$ for some $\Sigma_\varepsilon \leq 0$. The asymptotics of $g_{\varepsilon,\sigma}$ is chosen so that the one-particle operators $h_{\sigma,\varepsilon} := -(\mu_\sigma)^{-1}\Delta + g_{\varepsilon,\sigma} V_{\sigma,\varepsilon}$ with $D(h_{\sigma,\varepsilon}) = H^2(\mathbb{R}^3)$ converge, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint operator $-(\mu_\sigma)^{-1}\Delta_\alpha$ that defines a non-trivial contact interaction at the origin, provided that $h_\sigma = -\Delta + V_\sigma$ has a zero-energy resonance (cf. [6, Chapter I, Theorem 1.2.5]). Since $h_{\sigma,\varepsilon}$ is nothing but the two-body Hamiltonian

$$-\Delta_{x_i}/m_i - \Delta_{x_j}/m_j + g_{\varepsilon,\sigma} V_{\sigma,\varepsilon}(x_j - x_i), \qquad \sigma = (i,j) \tag{C.3}$$

with the center of mass motion removed, we thus expect that $H_\varepsilon$ defines a non-trivial contact interaction among the $i$th and the $j$th particle in the limit $\varepsilon \to 0$.

Let us now assume that $H_\varepsilon$ converges, as $\varepsilon \to 0$, in the norm resolvent sense to a self-adjoint operator $H$. We further suppose that there exist two pairs $\sigma_1, \sigma_2 \in \mathcal{I}$ with exactly one common particle so that $h_{\sigma_1}$ and $h_{\sigma_2}$ both have a zero-energy resonance (and hence the associated two-body Hamiltonians have a non-trivial limit as $\varepsilon \to 0$). Without restriction, we may thereby assume that $\sigma_1 = (1,2)$ and $\sigma_2 = (1,3)$. We are going to show that $\sigma(H_\varepsilon)$ then fills the whole real line $(-\infty, \infty)$ in the limit $\varepsilon \to 0$, which, in view of Proposition 2.3, implies that $\sigma(H) = (-\infty, \infty)$. This is the Thomas effect. For this purpose, we consider the unitary rescaling $U_{\mathrm{tot},\varepsilon} \in \mathscr{L}(L^2(\mathbb{R}^{3N}))$, $\varepsilon > 0$, defined by $(U_{\mathrm{tot},\varepsilon}\psi)(x_1, ..., x_N) := \varepsilon^{3N/2}\psi(\varepsilon x_1, ..., \varepsilon x_N)$. Then a straightforward computation shows that

$$U_{\mathrm{tot},\varepsilon} H_\varepsilon (U_{\mathrm{tot},\varepsilon})^* = \varepsilon^{-2}\left(\sum_{i=1}^{N}(-\Delta_{x_i}/m_i) + \sum_{\sigma=(i,j)\in\mathcal{I}} \varepsilon^{-1} g_{\varepsilon,\sigma} V_\sigma(x_j - x_i)\right).$$

Since $\lim_{\varepsilon\to 0} \varepsilon^{-1} g_{\varepsilon,\sigma} = (\mu_\sigma)^{-1}$ by (C.2), we conclude that $\lim_{\varepsilon\to 0} \varepsilon^2 \Sigma_\varepsilon = \inf \sigma(H_{\mathrm{scal}})$, where

$$H_{\mathrm{scal}} := \sum_{i=1}^{N}(-\Delta_{x_i}/m_i) + \sum_{\sigma=(i,j)\in\mathcal{I}} (\mu_\sigma)^{-1}\, V_\sigma(x_j - x_i). \tag{C.4}$$

Hence, to show that $\sigma(H_\varepsilon) = [\Sigma_\varepsilon, \infty)$ fills the whole real line in the limit $\varepsilon \to 0$, we only have to show that $\inf \sigma(H_{\mathrm{scal}}) < 0$. On the one hand, if $h_\sigma$ has a negative eigenvalue for some pair $\sigma \in \mathcal{I}$, then $\inf \sigma(H_{\mathrm{scal}}) < 0$ by the HVZ theorem (see e.g. [70, Theorem XIII.17]). On the other hand, if $h_\sigma \geq 0$ for all $\sigma \in \mathcal{I}$, then the zero-energy resonances of $h_{(1,2)}$ and $h_{(1,3)}$ entail that the Hamiltonian $H_{\mathrm{scal}}$ for $N = 3$ with the center of mass motion removed has an infinite number of negative eigenvalues. This is the Efimov effect. For general $N \geq 3$, it again follows from the HVZ theorem that $\inf \sigma(H_{\mathrm{scal}}) < 0$, so the Thomas effect is inevitable. However, we emphasize that a rigorous proof of the Efimov effect requires further technical assumptions on the two-body potentials $V_\sigma$ (see, e.g., [80, Theorems 5 and 5'] and [76, Theorem 5.1]). For example, if $h_{(2,3)}$ also has a zero-energy resonance, then [76, Theorem 5.1] shows that it is sufficient to assume that, for all $1 \leq i < j \leq 3$, $|V_{(i,j)}(r)| \leq C(1+|r|)^{-\rho}$ for some $\rho > 2$ and some constant $C > 0$.

In summary, this means that either the limit operator $H$ is trivial in the sense that each three-body subsystem contains at least one particle that does not interact with the other two particles or, as an immediate consequence of the Efimov effect, $H$ is unbounded from below and the Thomas effect occurs.

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