Characters and Character Sheaves of Finite Groups of Lie Type

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Abstract

A crucial problem in the representation theory of finite groups is the determination of the character tables of said groups. As the classification of finite simple groups shows, the main difficulties arise for the finite groups of Lie type: These are defined as the fixed-point sets $G = \mathbf{G}^F$ of a connected reductive algebraic group $\mathbf{G}$ over an algebraic closure of the finite field $\mathbb{F}_p$ with p elements (for a prime p) under so-called Frobenius maps (or, more generally, Steinberg maps) $F \colon \mathbf{G} \to \mathbf{G}$. In this way, there is an infinite family of finite groups associated to a given connected reductive group $\mathbf{G}$ as above, so one seeks to find 'generic' methods which should allow a 'uniform' treatment of the groups $\mathbf{G}^F$ with varying F. To that end, Lusztig developed the theory of character sheaves on $\mathbf{G}$ in the 1980s. This theory yields a basis of the space of class functions $\mathrm{CF}(G)$ of $G = \mathbf{G}^F$, consisting of characteristic functions of F-stable character sheaves on $\mathbf{G}$, and these characteristic functions are 'in principle' computable. After introducing the almost characters as certain explicitly defined linear combinations of the ordinary irreducible characters of G and showing that these almost characters form a basis of $\mathrm{CF}(G)$ as well, Lusztig conjectured that any almost character coincides with the characteristic function of a suitable character sheaf up to multiplication with a root of unity.

In the case where the centre of $\mathbf{G}$ is connected, Lusztig's Conjecture was proven in a theorem of Shoji in 1995. In this framework, specifying the roots of unity which appear in Lusztig's Conjecture/Shoji's Theorem is a major step towards determining the character table of $G = \mathbf{G}^F$. With regard to the unipotent characters and unipotent character sheaves and as far as classical groups $\mathbf{G}$ with a connected centre are concerned, these roots of unity have been determined by Shoji (1997, 2009); it thus remains to consider the *exceptional groups* $\mathbf{G}$. On the other hand, already in 1986, Lusztig developed methods to address the *explicit* computation of irreducible characters of G at unipotent elements: Lusztig's arguments in this context are ultimately based on exploiting certain congruence conditions, but these are only valid under some restrictions on the characteristic p — in particular, the *bad primes* p for $\mathbf{G}$ are excluded.

The main results of this thesis concern the exceptional groups and their bad primes p. We determine the roots of unity appearing in Shoji's Theorem with respect to the

so-called *cuspidal* unipotent character sheaves for groups of type E_6 in characteristic $p = 3$ and for groups of type E_7 in characteristic $p = 2$, and we resolve several cases for groups of type E_8 in the characteristics $p = 2, 3, 5$. Based on those results (and combined with earlier works), we obtain the complete tables containing the values of unipotent characters at unipotent elements for the groups $\mathsf{E}_6(q)$ and $\mathsf{E}_7(q)$ where q is a power of a prime p; these tables were previously not known for $p = 2, 3$. (For groups of type F_4, these tables have been determined by Marcelo–Shinoda in 1995; for groups of type G_2, the full character tables are known already since the 1970s–1980s anyway, due to Chang–Ree, Enomoto and Enomoto–Yamada.) Because of the sheer size of groups of type E_8, the picture will not be complete for these groups, but we are able to obtain new information with regard to character values at unipotent elements here as well — as one of the main outcomes during the work on this thesis, the author recently managed to resolve the last open cases in the generalised Springer correspondence, which occur for groups of type E_8 in characteristic $p = 3$; the generalised Springer correspondence is thus known in full generality now.

We begin with a short section recalling the definitions and main properties of connected reductive groups and finite groups of Lie type. We then explain the classification of irreducible characters of those finite groups of Lie type whose underlying connected reductive group has a connected centre, due to Deligne–Lusztig (1976) and Lusztig (1984). We further present the most important notions and results concerning Lusztig's theory of character sheaves, with a particular emphasis on those outcomes which have a direct influence on the computation of character values. This includes a detailed discussion on how one can single out 'good' normalisations of the characteristic functions of character sheaves.

The core of this thesis is the investigation of the simple groups of exceptional type with regard to determining the roots of unity involved in Shoji's Theorem for the unipotent character sheaves: We explain how this task can largely be reduced to considering cuspidal unipotent character sheaves on simple groups. We then go through the simple groups of exceptional type one by one and tackle this (reduced) problem or provide appropriate references in the cases where it has already been solved. At the same time, we consider the problem of computing the values of unipotent characters at unipotent elements for any of these groups. The above two problems are then completely resolved except for the groups of type E_8, but we obtain partial results here as well. Our argumentation mostly relies on exploiting a formula due to Ree: This formula relates the unipotent principal series characters of a finite group of Lie type G with characters of the associated Hecke algebra in terms of intersections of conjugacy classes of G with Bruhat cells.

Zusammenfassung

Ein zentrales Problem in der Darstellungstheorie endlicher Gruppen ist die Bestimmung der Charaktertafeln besagter Gruppen. Wie die Klassifikation der endlichen einfachen Gruppen zeigt, treten die Hauptschwierigkeiten bei den endlichen Gruppen vom Lie-Typ auf: Nach Definition sind diese durch die Fixpunktmengen $G = \mathbf{G}^F$ einer zusammenhängenden reduktiven algebraischen Gruppe $\mathbf{G}$ über einem algebraischen Abschluss des endlichen Körpers $\mathbb{F}_p$ mit p Elementen (für eine Primzahl p) unter sogenannten Frobenius-Abbildungen (oder allgemeiner Steinberg-Abbildungen) $F\colon \mathbf{G} \to \mathbf{G}$ beschrieben. Demnach ist einer gegebenen zusammenhängenden reduktiven Gruppe $\mathbf{G}$ wie oben eine unendliche Familie endlicher Gruppen zugeordnet, für die man somit nach „generischen" Methoden Ausschau hält, welche eine „uniforme" Behandlung der Gruppen $\mathbf{G}^F$ mit variierendem F erlauben sollen. Zu diesem Zweck entwickelte Lusztig die Theorie der Charaktergarben auf $\mathbf{G}$ in den 1980er Jahren. Diese Theorie liefert eine Basis des Raumes der Klassenfunktionen $\mathrm{CF}(G)$ von $G = \mathbf{G}^F$, bestehend aus den charakteristischen Funktionen F-stabiler Charaktergarben auf $\mathbf{G}$, und diese charakteristischen Funktionen sind „im Prinzip" berechenbar. Nach Einführung der Fast-Charaktere als gewisse explizit definierte Linearkombinationen der gewöhnlichen irreduziblen Charaktere von G sowie dem Nachweis, dass diese Fast-Charaktere ebenfalls eine Basis von $\mathrm{CF}(G)$ bilden, vermutete Lusztig, dass jeder Fast-Charakter bis auf Multiplikation mit einer Einheitswurzel mit der charakteristischen Funktion einer geeigneten Charaktergarbe übereinstimmt.

Falls $\mathbf{G}$ ein zusammenhängendes Zentrum besitzt, wurde Lusztigs Vermutung in einem Theorem von Shoji aus dem Jahre 1995 bewiesen. In diesem Bezugsrahmen stellt die Festlegung der in Lusztigs Vermutung/Shojis Theorem auftretenden Einheitswurzeln einen großen Schritt in Richtung der Bestimmung der Charaktertafel von $G = \mathbf{G}^F$ dar. Mit Blick auf die unipotenten Charaktere und unipotenten Charaktergarben und für klassische Gruppen $\mathbf{G}$ mit zusammenhängendem Zentrum wurden diese Einheitswurzeln von Shoji bestimmt (1997, 2009); es verbleibt somit, sich den *exzeptionellen Gruppen* $\mathbf{G}$ zu widmen. Andererseits entwickelte Lusztig bereits im Jahre 1986 Methoden, um die *explizite* Berechnung der irreduziblen Charaktere von G auf unipotenten Elementen in

Angriff zu nehmen: Lusztigs Argumente in diesem Zusammenhang basieren letztlich auf der Ausnutzung gewisser Kongruenzbedingungen, die allerdings nur unter bestimmten Einschränkungen an die Charakteristik p gültig sind — insbesondere sind die *schlechten Primzahlen* p für $\mathbf{G}$ hierbei nicht mit eingeschlossen.

Die Hauptergebnisse dieser Arbeit betreffen die exzeptionellen Gruppen und deren schlechte Primzahlen p. Wir erreichen die Bestimmung der in Shojis Theorem auftretenden Einheitswurzeln in Bezug auf die sogenannten *kuspidalen* unipotenten Charaktergarben für Gruppen vom Typ E_6 in Charakteristik $p = 3$ und für Gruppen vom Typ E_7 in Charakteristik $p = 2$ sowie die Lösung mehrerer Fälle für Gruppen vom Typ E_8 in den Charakteristiken $p = 2, 3, 5$. Basierend auf diesen Resultaten (und kombiniert mit früheren Arbeiten) erhalten wir die kompletten Tafeln der Werte unipotenter Charaktere auf unipotenten Elementen für die Gruppen $\mathsf{E}_6(q)$ und $\mathsf{E}_7(q)$, wobei q eine Potenz einer Primzahl p ist; für $p = 2, 3$ waren diese Tafeln zuvor nicht bekannt. (Für Gruppen vom Typ F_4 wurden diese Tafeln von Marcelo–Shinoda im Jahre 1995 bestimmt; für Gruppen vom Typ G_2 sind die kompletten Charaktertafeln ohnehin bereits seit den 1970er–1980er Jahren bekannt, nach Chang–Ree, Enomoto und Enomoto–Yamada.) Aufgrund der schieren Größe der Gruppen vom Typ E_8 ist das Bild bezüglich dieser Gruppen unvollständig, aber wir sind auch hier in der Lage, neue Informationen über die Charakterwerte auf unipotenten Elementen zu gewinnen — als eines der Hauptergebnisse während der Arbeit an dieser Dissertation ist es dem Autor kürzlich gelungen, die letzten offenen Fälle in der verallgemeinerten Springer-Korrespondenz zu lösen, welche für Gruppen vom Typ E_8 in Charakteristik $p = 3$ auftreten; damit ist die verallgemeinerte Springer-Korrespondenz nun in voller Allgemeinheit bekannt.

Wir beginnen mit einem kurzen Abschnitt, in welchem wir die Definitionen und wesentlichen Eigenschaften der zusammenhängenden reduktiven Gruppen sowie der endlichen Gruppen vom Lie-Typ in Erinnerung rufen. Dann beschreiben wir die Klassifikation der irreduziblen Charaktere all solcher endlichen Gruppen vom Lie-Typ, deren zugrunde liegende zusammenhängende reduktive Gruppe ein zusammenhängendes Zentrum besitzt, nach Deligne–Lusztig (1976) und Lusztig (1984). Ferner stellen wir die wichtigsten Begriffe und Ergebnisse hinsichtlich Lusztigs Theorie der Charaktergarben vor; besonderes Augenmerk legen wir dabei auf solche Resultate, die direkten Einfluss auf die Berechnung von Charakterwerten haben. Dies beinhaltet eine detaillierte Diskussion über die Wahl von „guten“ Normalisierungen der charakteristischen Funktionen von Charaktergarben.

Das Herzstück dieser Arbeit bildet die Untersuchung der einfachen Gruppen von exzeptionellem Typ im Hinblick auf die Bestimmung der in Shojis Theorem auftauchenden Einheitswurzeln für die unipotenten Charaktergarben: Wir zeigen auf, wie diese Fragestellung weitestgehend auf die Betrachtung kuspidal unipotenter Charaktergarben auf einfachen Gruppen zurückgeführt werden kann. Danach gehen wir nacheinander die einfachen Gruppen von exzeptionellem Typ durch und nehmen dieses (reduzierte) Problem in Angriff oder verweisen auf geeignete Quellen für die Fälle, in denen dies bereits bewerkstelligt wurde. Gleichzeitig betrachten wir das Problem der Berechnung der Werte unipotenter Charaktere auf unipotenten Elementen für jede dieser Gruppen. Die beiden oben genannten Probleme sind damit vollständig gelöst mit Ausnahme von Gruppen vom Typ E_8, für die wir jedoch ebenfalls Teilergebnisse erhalten. Unsere Argumentation beruht größtenteils auf der Auswertung einer Formel von Ree: Diese Formel setzt die unipotenten Charaktere in der Hauptserie einer endlichen Gruppe vom Lie-Typ G mit Charakteren der zugehörigen Hecke-Algebra bezüglich Schnitten von Konjugiertenklassen von G mit Bruhat-Zellen in Verbindung.

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Contents

1. Introduction

Let G be a finite group, $\mathbb{K}$ a field and V a finite-dimensional $\mathbb{K}$-vector space. A group homomorphism

$$\rho\colon G \to \mathrm{GL}(V)$$

is called a $\mathbb{K}$-representation of G. The $\mathbb{K}$-representation ρ is called irreducible if $V \neq \{0\}$ and if there exists no proper subspace $\{0\} \neq V' \subsetneq V$ which is invariant under the endomorphisms in $\rho(G) \subseteq \mathrm{GL}(V)$. Associated to ρ is its ($\mathbb{K}$-)character, defined as the function

$$\chi_\rho\colon G \to \mathbb{K}, \quad g \mapsto \mathrm{Trace}(\rho(g)).$$

The character χ_ρ is called irreducible if ρ is an irreducible representation. The study of $\mathbb{K}$-representations and $\mathbb{K}$-characters for various G and $\mathbb{K}$ is referred to as the representation theory and character theory of finite groups. This constitutes a very rich and intense subject matter of current research with numerous applications, both in mathematics but also in related natural sciences. By 'linearising' the structure of G (or, more precisely, the G-actions on finite sets) via such representations, one hopes to obtain a better understanding of the group G itself.

In the important special case where $\mathbb{K} = \mathbb{C}$ is the field of complex numbers (or any other algebraically closed field of characteristic zero), one speaks of ordinary representations and ordinary characters rather than of $\mathbb{C}$-representations and $\mathbb{C}$-characters and writes $\mathrm{Irr}(G)$ for the (finite) set consisting of all ordinary irreducible characters of G. The significance of the ordinary (irreducible) characters is highlighted by the following basic facts: First of all, any ordinary representation is (up to isomorphism) completely determined by its character. Secondly, any ordinary representation $\rho\colon G \to \mathrm{GL}(V)$ can be decomposed as a direct sum of ordinary irreducible representations in the sense that there exist $\rho(G)$-invariant subspaces $V_1, V_2, \ldots, V_m$ of V ($m \in \mathbb{N}$) such that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$ and such that for any $i \in \{1, 2, \ldots, m\}$, the group homomorphism $\rho_i\colon G \to \mathrm{GL}(V_i)$ induced by ρ defines an ordinary irreducible representation of G. In this case, we have

$$\chi_\rho = \chi_{\rho_1} + \ldots + \chi_{\rho_m}\colon G \to \mathbb{C}, \quad g \mapsto \chi_{\rho_1}(g) + \ldots + \chi_{\rho_m}(g),$$

where $\chi_{\rho_1}, \ldots, \chi_{\rho_m} \in \mathrm{Irr}(G)$. Therefore, studying the ordinary representations (respectively, characters) of a given finite group G is equivalent to studying the ordinary *irreducible* representations (respectively, characters) of G.

As for the ordinary irreducible characters themselves, let us denote by $\mathrm{CF}(G)$ the $\mathbb{C}$-vector space consisting of all functions $G \to \mathbb{C}$ which are constant on the conjugacy classes of G. It is well known that we have $\mathrm{Irr}(G) \subseteq \mathrm{CF}(G)$ and that $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathrm{CF}(G)$ with respect to the scalar product

$$\langle f, f' \rangle_G := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)} \quad (\text{for } f, f' \in \mathrm{CF}(G)).$$

In particular, the number of ordinary irreducible characters of G coincides with the number of conjugacy classes of G. Hence, if $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ and if $\{g_1, \ldots, g_r\}$ is a set of representatives for the conjugacy classes of G, we obtain a square matrix

$$X(G) = \big(\chi_i(g_j)\big)_{1 \leqslant i,j \leqslant r}.$$

This matrix is called the character table of G; it is unique up to reordering $\chi_1, \ldots, \chi_r$ and $g_1, \ldots, g_r$, which would lead to a corresponding swapping of its rows and columns, respectively. Moreover, $X(G)$ does not depend on the choice of the set of representatives $\{g_1, \ldots, g_r\}$ since characters are constant on conjugacy classes. Thus, $X(G)$ contains all the information on ordinary representations and characters of G in a concise form. As we are exclusively dealing with *ordinary* representations and characters of finite groups in this work, we will henceforth often omit the word ‘ordinary’ and just speak of representations and characters of G, thus tacitly assuming that the ground field of the involved vector space V is an algebraically closed field of characteristic zero.

In this thesis, we are concerned with the (ordinary) representations and characters of a very special and large class of finite groups G, the one consisting of finite groups of Lie type. These are defined as the fixed-point sets $G = \mathbf{G}^F$ of a connected reductive algebraic group $\mathbf{G}$ over an algebraic closure k of the finite field $\mathbb{F}_p$ with p elements (for a prime p) under certain bijective endomorphisms $F \colon \mathbf{G} \to \mathbf{G}$ called Frobenius maps or, more generally, Steinberg maps. By means of varying F, any connected reductive algebraic group $\mathbf{G}$ over k gives rise to an infinite family of finite groups of Lie type $\mathbf{G}^F$ and, as one should expect, these groups $\mathbf{G}^F$ inherit fundamental properties from the algebraic group $\mathbf{G}$ coming from a geometric or topological origin. Given a connected reductive group $\mathbf{G}$ over k, this often allows a ‘uniform’ description of the finite groups $\mathbf{G}^F$ as well as of their representations and characters.

The standard example is the general linear group $\mathbf{G} = \mathrm{GL}_n(k)$ ($n \in \mathbb{N}$): For any power q of p, the map

$$F\colon\ \mathrm{GL}_n(k) \to \mathrm{GL}_n(k), \quad (a_{ij}) \mapsto (a_{ij}^q),$$

defines a Frobenius map, and we obtain a finite group $\mathrm{GL}_n(k)^F = \mathrm{GL}_n(q)$ consisting of all invertible $n \times n$ matrices with entries in the finite field $\mathbb{F}_q \subseteq k$ with q elements. The character tables of these groups have been determined already in 1955 by Green [Gre55].

In general, determining the character table of a finite group of Lie type appears to be a very difficult problem. In their landmark paper [DL76] from 1976, Deligne and Lusztig utilised the concept of ℓ-adic cohomology groups with compact support (where ℓ is a prime different from p) to construct virtual representations (that is, $\mathbb{Z}$-linear combinations of irreducible representations) of finite groups of Lie type. Among other things, they showed that any irreducible representation of $\mathbf{G}^F$ appears as a constituent of at least one of these Deligne–Lusztig virtual representations. Based on this and other results of [DL76], Lusztig [Lus84a], [Lus88] established a classification of $\mathrm{Irr}(\mathbf{G}^F)$ in the 1980s. In this picture, an essential role is played by the so-called unipotent characters of finite groups of Lie type, which leads to a 'Jordan decomposition' of irreducible characters of $\mathbf{G}^F$ in terms of semisimple elements in the Langlands dual group $\mathbf{G}^*$ of $\mathbf{G}$ and unipotent characters of suitable Frobenius-fixed point subgroups of centralisers in $\mathbf{G}^*$ of such semisimple elements. With this approach, Lusztig also obtained explicitly computable formulae for the values of irreducible characters at semisimple elements of $\mathbf{G}^F$, which in particular includes the degrees of the irreducible characters.

However, as far as character values at non-semisimple elements are concerned, other methods seem to be required. In order to tackle this problem, Lusztig [LuCS1]–[LuCS5] developed the theory of character sheaves on connected reductive groups $\mathbf{G}$ over k, certain simple perverse sheaves in the bounded derived category $\mathscr{D}\mathbf{G}$ of constructible $\overline{\mathbb{Q}}_\ell$-sheaves which are equivariant for the conjugation action of $\mathbf{G}$ on itself. The Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$ naturally acts on $\mathscr{D}\mathbf{G}$, and one obtains the notion of F-stable character sheaves on $\mathbf{G}$, which provide an analogue to the irreducible characters of $\mathbf{G}^F$: To any F-stable character sheaf A on $\mathbf{G}$ (up to isomorphism), one can associate a class function of $\mathbf{G}^F$ (which is a priori only defined up to multiplication with a root of unity), called the characteristic function of A. In [LuCS1]–[LuCS5] and [Lus12b], Lusztig proved that the set of all these characteristic functions forms an orthonormal basis of the space of class functions $\mathrm{CF}(\mathbf{G}^F)$, which *in principle* is computable. Hence, finding the transformation between this basis and the one consisting of the irreducible characters of $\mathbf{G}^F$ would be a big step towards determining the character table of $\mathbf{G}^F$.

After introducing a further orthonormal basis of $\mathrm{CF}(\mathbf{G}^F)$ consisting of the so-called almost characters of $\mathbf{G}^F$, which are defined as explicit linear combinations of $\mathrm{Irr}(\mathbf{G}^F)$, Lusztig conjectured that any almost character coincides with the characteristic function of a suitable F-stable character sheaf on $\mathbf{G}$ (up to multiplication with a root of unity). In 1995, Shoji [Sho95a], [Sho95b] proved Lusztig's Conjecture in the case where $\mathbf{G}$ has a connected centre; there are also partial results for groups with a non-connected centre, due to Bonnafé [Bon06] and Waldspurger [Wal04] — in general, Lusztig's Conjecture is still open to this day.

But even in the cases where Lusztig's Conjecture is known to hold, one needs to specify the above-mentioned roots of unity (after having fixed normalisations of the characteristic functions of F-stable character sheaves) before hoping to obtain explicit character values in this framework. The determination of these roots of unity turns out to be a very subtle problem and is the main object of study in this thesis. More precisely, we will focus on the aforementioned unipotent characters of $\mathbf{G}^F$ and their analogues on the level of character sheaves, which are accordingly called the unipotent character sheaves. In this situation, our problem can largely be reduced to considering the so-called *cuspidal* unipotent character sheaves on *simple* groups $\mathbf{G}$. For classical groups, this has been completely settled by Shoji in [Sho97], [Sho09]. We will thus examine the various simple groups $\mathbf{G}$ of exceptional type and consider the following problem for such $\mathbf{G}$:

$$\begin{array}{l}\text{Determine the roots of unity in Lusztig's Conjecture/Shoji's Theorem}\\ \text{with respect to any } F\text{-stable cuspidal unipotent character sheaf on } \mathbf{G}.\end{array} \tag{$*$}$$

This problem has been addressed by various authors before, and we will provide appropriate references in the cases where it is already solved. Our main focus in this thesis with regard to $(*)$ lies on those groups $\mathbf{G}$ whose underlying field $k = \overline{\mathbb{F}}_p$ is of very small (so-called 'bad') characteristic p with respect to $\mathbf{G}$. As is well known, if p is not too small, the essential structural properties of $\mathbf{G}$ and its related objects tend to behave rather uniformly as p varies — however, this does typically not include the bad primes p for $\mathbf{G}$, and as it turns out, problem $(*)$ is no exception. Examples are the groups of type E_6 in characteristic $p = 3$ and the groups of type E_7 in characteristic $p = 2$, for which $(*)$ has been solved by the author in [Het19] and [Het22a], respectively; we present the full proofs of said results during the course of this thesis. Combined with the above-mentioned earlier solutions to $(*)$, the picture will then be complete for all exceptional groups other than the ones of type E_8 — for the latter, a complete solution to $(*)$ is not yet known, but we are able to resolve several previously unknown cases regarding cuspidal unipotent character sheaves whose support is contained in the unipotent variety, in particular with respect to the bad primes $p = 2, 3, 5$ for groups of type E_8.

On the other hand, we will also look at the following task:

$$\text{For any simple group } \mathbf{G} \text{ of exceptional type with Frobenius map } F\colon \mathbf{G} \to \mathbf{G}, \text{ compute the values of the unipotent characters of } \mathbf{G}^F \text{ at unipotent elements.} \tag{†}$$

This has been considered already in 1986 by Lusztig [Lus86], who established (†) with some rather mild restrictions on p and F — but again, the bad primes p for $\mathbf{G}$ are excluded there. For groups of type G_2, the complete character tables are known, due to Chang–Ree [CR74] for $p \geqslant 5$, Enomoto [Eno76] for $p = 3$, and Enomoto–Yamada [EY86] for $p = 2$, which in particular includes (†). As far as groups of type F_4 are concerned, (†) has been settled by Marcelo and Shinoda in [MS95]. Combined with earlier results on the so-called Green functions (see [Gec20b] and the further references there) and our solutions to (∗), we conclude (†) for the groups $\mathsf{E}_6(q)$ and $\mathsf{E}_7(q)$ where q is a power of p, which was previously not established in the bad characteristics $p = 2, 3$ for said groups. Again, (†) will not be complete for groups of type E_8, but we obtain partial results here as well. As one of the main results during the work on this thesis, the author settled the last open cases in the generalised Springer correspondence defined in [Lus84b], which occur for groups of type E_8 in characteristic $p = 3$ (see [Het22b]). This result is a contribution to (†); in fact, we will explicitly apply it to solve (∗) for two cuspidal unipotent character sheaves when dealing with the groups $\mathsf{E}_8(q)$ where q is a power of 3.

Our main method to tackle both (∗) and (†) relies on a reinterpretation of a formula due to Ree (see [CR81, §11D]), which appears in [Lus11b] and [Gec11]: It relates the unipotent principal series characters of a finite group of Lie type $\mathbf{G}^F$ with characters of the associated Hecke algebra in terms of intersections of $\mathbf{G}^F$-conjugacy classes with Bruhat cells. While these intersections (or even their sizes) are very difficult to compute in general, it will in many cases turn out to be sufficient to consider only one particular of these intersections and show that it is non-empty.

This thesis is organised as follows. We begin by recalling a rather detailed account of the theory of finite groups of Lie type and their ordinary representation theory in Chapter 2, with the aim of describing Lusztig's 'Main Theorem 4.23' in [Lus84a] on the classification of irreducible characters of $\mathbf{G}^F$ for connected reductive groups $\mathbf{G}$ with a connected centre in a precise way; we conclude this chapter with a section on Hecke algebras associated to finite groups of Lie type, which form a central ingredient in numerous arguments in this thesis. — All of this is of course well known and covered in standard surveys such as [Car85], [DM20], [GM20], but the reader may find it convenient to have the most important notions and properties all in one place with consistent notational conventions. In Chapter 3, we start by introducing Lusztig's theory of character sheaves. Apart from providing the fundamental definitions and stating some of the most important results, our focus always lies on those aspects which we will later explicitly refer to when dealing with finite groups of Lie type and their character values. In Section 3.3, we are then prepared to state Lusztig's Conjecture (3.3.8) and Shoji's Theorem (3.3.9) in a precise way. We proceed by describing Shoji's result with respect to the unipotent character sheaves on simple non-twisted groups with a trivial centre in Section 3.4. This section forms the foundation of our discussion in Chapter 4, where we go through the various simple groups $\mathbf{G}$ of exceptional type one by one and focus on the problems $(*)$ and $(\dagger)$. In the appendix, we include a short chapter recalling some basic properties of finite Coxeter groups and generic Iwahori–Hecke algebras (Appendix A), which play an important role throughout this thesis. We conclude by collecting some of the bigger tables which appear in this work (Appendix B).

1.1. Notation and conventions

Let us start by stating some conventions and introducing the basic notation that we use in this thesis.

Some conventions

Unless otherwise stated, $p \in \mathbb{N}$ always denotes a prime number and $k = \overline{\mathbb{F}}_p$ an algebraic closure of the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ with p elements. As soon as p is prescribed, q will always be a power of p of the form $q = p^e$, $e \in \mathbb{N}$. Then we denote by

$$\mathbb{F}_q = \{x \in k \mid x^q = x\} \subseteq k$$

the finite subfield of k with q elements, so that

$$k = \bigcup_{e\in\mathbb{N}} \mathbb{F}_{p^e}.$$

Let us fix a prime $\ell \in \mathbb{N}$ different from p and denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers (see, e.g., [Kob84]). It will be convenient to assume the existence of an isomorphism

$$\overline{\mathbb{Q}}_\ell \simeq \mathbb{C} \tag{1.1.0.1}$$

and to fix such an isomorphism once and for all[1] — in the representation theory of finite groups of Lie type one usually works with $\overline{\mathbb{Q}}_\ell$ (see [DL76]), but we sometimes want to be able to speak of complex conjugation, absolute values or the like. So we will also often identify the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$ with subsets of $\overline{\mathbb{Q}}_\ell$ and write $\mathbb{Q} \subseteq \mathbb{R} \subseteq \overline{\mathbb{Q}}_\ell$, thus tacitly referring to the isomorphism (1.1.0.1). Another choice that we need to make in several places is the one of a square root of p in $\overline{\mathbb{Q}}_\ell$, so it is useful to do this right away:

$$\begin{array}{l}\text{From now on we fix, once and for all, a square root } \sqrt{p} \text{ of } p \text{ in } \overline{\mathbb{Q}}_\ell.\\ \text{Whenever } q = p^e \ (e \in \mathbb{N}), \text{ we then set } \sqrt{q} := \sqrt{p}^{\,e}.\end{array} \tag{1.1.0.2}$$

We will still sometimes explicitly refer to (1.1.0.2) in order to emphasise that we are dependent on such a choice in a given situation. Next, let

$$\mu_{p'} := \{x \in \overline{\mathbb{Q}}_\ell^\times \mid x^n = 1 \text{ for some } n \in \mathbb{N} \text{ which is prime to } p\}.$$

We also consider the subring of $\mathbb{Q}$ defined by

$$\mathbb{Z}_{(p)} := \left\{ \frac{m}{n} \in \mathbb{Q} \;\middle|\; m, n \in \mathbb{Z},\ p \text{ does not divide } n \right\} \tag{1.1.0.3}$$

(i.e., the localization of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$). Identifying $\mathbb{Z}$ with an additive subgroup of $\mathbb{Z}_{(p)}$, we thus obtain an additive group $\mathbb{Z}_{(p)}/\mathbb{Z}$. Now the abelian groups $(k^\times, \cdot)$, $(\mu_{p'}, \cdot)$, $(\mathbb{Z}_{(p)}/\mathbb{Z}, +)$ are all isomorphic, although there is no canonical choice of such isomorphisms ([DL76, §5], cf. [Car85, 3.1.3]). We thus take any two such isomorphisms, denoted by

$$\imath\colon k^\times \xrightarrow{\sim} \mu_{p'} \tag{1.1.0.4}$$

and

$$\jmath\colon \mathbb{Z}_{(p)}/\mathbb{Z} \xrightarrow{\sim} k^\times, \tag{1.1.0.5}$$

and keep them fixed once and for all.

[1]Strictly speaking, the existence of such an isomorphism can only be guaranteed by referring to the axiom of choice. For the reader who is not willing to accept that axiom at this point, what we really need is actually an isomorphism between algebraic closures of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$, and such an isomorphism is known to exist without reference to the axiom of choice, cf. [Del80, 1.2.11].

General notation

Here we give an overview of some of the most frequently used notation in this thesis. Except for the very basic or standard notions, we usually introduce everything in the text as well.

Numbers

In the following table, p, ℓ are primes with $p \neq \ell$, and q is a power of p.

Symbol	Meaning
$\mathbb{N}$	The natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
Φ_n, $n \in \mathbb{N}$	The nth cyclotomic polynomial, or its evaluation at q
$\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$	The localization of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$, see (1.1.0.3)
$\mathbb{F}_p$	The finite field with p elements
$k = \overline{\mathbb{F}}_p$	An algebraic closure of $\mathbb{F}_p$; using the letter k thus tacitly assumes that p is fixed in a given setting
$\mathbb{F}_q$	The unique finite subfield of k with q elements
$\mathbb{Q}_\ell$	The field of ℓ-adic numbers
$\overline{\mathbb{Q}}_\ell$	An algebraic closure of $\mathbb{Q}_\ell$
$\sqrt{p}$	A fixed square root of p, usually assumed to be in $\overline{\mathbb{Q}}_\ell$, see (1.1.0.2)
$\mathcal{R}$	The set of all roots of unity in $\overline{\mathbb{Q}}_\ell$
$\mathcal{R}_n$, $n \in \mathbb{N}$	The set of all nth roots of unity in $\overline{\mathbb{Q}}_\ell$
$\mu_{p'}$	The set of all roots of unity in $\overline{\mathbb{Q}}_\ell$ of order prime to p
ζ_n, $n \in \mathbb{N}$	A (not necessarily fixed) primitive nth root of unity in $\mathcal{R}_n$; for $n \leqslant 4$, we typically write 1, -1, ω, i instead of ζ_1, ζ_2, ζ_3, ζ_4, respectively
$\mu_n(k^\times)$, $n \in \mathbb{N}$	The set of all nth roots of unity in $k^\times$
$\imath$	A fixed isomorphism $k^\times \xrightarrow{\sim} \mu_{p'}$, see (1.1.0.4)
$\jmath$	A fixed isomorphism $\mathbb{Z}_{(p)}/\mathbb{Z} \xrightarrow{\sim} k^\times$, see (1.1.0.5)

Abstract groups and modules

Given a commutative ring $\mathbb{K}$ with identity, whenever we speak of a '$\mathbb{K}$-algebra $\mathcal{A}$', we assume that $\mathcal{A}$ is associative and with identity element $1_{\mathcal{A}}$; furthermore, if we just speak of an '$\mathcal{A}$-module', we mean a left $\mathcal{A}$-module, usually assumed to be of finite $\mathbb{K}$-dimension.

In the following table, $\mathbb{K}$ is a field, $\mathcal{A}$ is a $\mathbb{K}$-algebra, G is any group, H is a subgroup of G, g, g' are elements of G, V and W are (left) $\mathcal{A}$-modules of finite $\mathbb{K}$-dimension, Γ is a finite group, and $n \in \mathbb{N}$.

Symbol	Meaning
C_n	The cyclic group of order n, written multiplicatively
$\mathbb{Z}/n\mathbb{Z}$	The cyclic group of order n, written additively, whose elements are denoted by $\overline{a} := a + n\mathbb{Z}$ for $a \in \mathbb{Z}$
$\mathfrak{S}_n$	The symmetric group on n letters
D_{2n}	The dihedral group of order $2n$
$N_G(H)$	The normaliser of H in G
$C_G(g)$	The centraliser of g in G
$C_G(H)$	The centraliser of H in G, i.e., the set $\bigcap_{h \in H} C_G(h)$
O_g	The conjugacy class of g in G (if G is clear from the context)
$g \sim_G g'$	g is G-conjugate to g', i.e., there exists $x \in G$ such that $xgx^{-1} = g'$
$\mathrm{Hom}_{\mathcal{A}}(V, W)$	The set of all $\mathcal{A}$-linear maps $V \to W$
$\mathrm{End}_{\mathcal{A}}(V)$	The endomorphism algebra of the $\mathcal{A}$-module V
Trace or Tr	Written with one argument, this refers to the trace of a matrix; if $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$, $\mathrm{Trace}(\varphi, V)$ or $\mathrm{Tr}(\varphi, V)$ is the trace of φ on V
$\mathbb{K}\Gamma$ or $\mathbb{K}[\Gamma]$	The group algebra of Γ over $\mathbb{K}$
$\mathrm{CF}(\Gamma)$	The finite-dimensional $\mathbb{K}$-vector space consisting of the class functions $\Gamma \to \mathbb{K}$ where $\mathbb{K}$ is either $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$, depending on the context
$\mathrm{Irr}(\Gamma)$	The subset of $\mathrm{CF}(\Gamma)$ consisting of the irreducible characters of Γ

Algebraic groups

In the following table, $\mathbf{H}$ denotes an algebraic group over $k = \overline{\mathbb{F}}_p$, h an element of $\mathbf{H}$ and $F\colon \mathbf{H} \to \mathbf{H}$ an endomorphism of algebraic groups.

Symbol	Meaning
$\mathbf{G}_{\mathrm{m}}$	The multiplicative algebraic group $(k^{\times}, \cdot)$
$\mathbf{G}_{\mathrm{a}}$	The additive algebraic group $(k, +)$
$\mathbf{Z}(\mathbf{H})$	The centre of $\mathbf{H}$
$\mathbf{H}^{\circ}$	The identity component of $\mathbf{H}$
$\mathbf{H}_{\mathrm{der}}$	The derived group $[\mathbf{H}, \mathbf{H}]$
$\mathbf{H}_{\mathrm{ss}}$	The group $\mathbf{H}/\mathbf{Z}(\mathbf{H})^{\circ}$
$\mathbf{H}_{\mathrm{uni}}$	The subvariety of $\mathbf{H}$ consisting of all unipotent elements in $\mathbf{H}$
$R_{\mathrm{u}}(\mathbf{H})$	The unipotent radical of $\mathbf{H}$
$C^{\circ}_{\mathbf{H}}(h)$	$C_{\mathbf{H}}(h)^{\circ}$
$A_{\mathbf{H}}(h)$	The (finite) component group $C_{\mathbf{H}}(h)/C^{\circ}_{\mathbf{H}}(h)$, whose elements are typically denoted by $\overline{x} := xC^{\circ}_{\mathbf{H}}(h)$ for $x \in C_{\mathbf{H}}(h)$
$\mathbf{H}^{F}$	The set of fixed points of $\mathbf{H}$ under F

Operators

In the following table, we collect a few of the somewhat less standard symbols for operators.

Symbol	Meaning
$\coprod$, $\amalg$	External disjoint union of sets, and its binary symbol
$\biguplus$, $\uplus$	Internal disjoint union of sets, and its binary symbol
$\boxtimes$	External tensor product

2. Finite groups of Lie type

In this chapter, we give an overview of some of the most important properties of finite groups of Lie type, with a special emphasis on their representation and character theory due to Deligne–Lusztig [DL76] and Lusztig [Lus84a]. In particular, under the assumption that the underlying connected reductive group has a connected centre, we state Lusztig's 'Main Theorem 4.23' of [Lus84a], which provides a classification of the irreducible characters of finite groups of Lie type.

In Section 2.1, we recall the essential notions of finite groups of Lie type, mostly following [GM20]. This involves the definition of the underlying connected reductive algebraic groups over $k = \overline{\mathbb{F}}_p$ and the root data attached to them (2.1.1–2.1.6). We also include a brief discussion on structural features of connected reductive groups over k (most notably the Bruhat decomposition) and how they transfer to the finite groups of Lie type (see 2.1.9–2.1.10 and 2.1.14–2.1.15, 2.1.20, respectively).

In Section 2.2, we present the definition of the virtual Deligne–Lusztig characters [DL76] (see Definition 2.2.3), and we give a detailed explanation of Lusztig's classification of the ordinary irreducible characters of those finite groups of Lie type whose underlying connected reductive group has a connected centre (Theorem 2.2.21), due to [Lus84a]. This requires a discussion on families in Weyl groups and the associated Fourier matrices (2.2.8–2.2.12), as well as the definition of almost characters (2.2.23). We briefly address the concepts of Harish-Chandra induction (2.2.6) and Lusztig induction (Definition 2.2.28). Furthermore, we give a summary of the state of knowledge concerning the Green functions in 2.2.5, whose computation will be of special relevance in several places later.

Section 2.3 is dedicated to Hecke algebras (cf. Appendix A) associated to finite groups of Lie type. We explain how these Hecke algebras give rise to (a reinterpretation of) a formula due to Ree (see 2.3.9). Said formula plays a pivotal role throughout this thesis; see especially Chapter 4 below, where we will exploit it numerous times in order to obtain explicit information on character values of finite groups of Lie type.

2.1. Connected reductive and finite reductive groups

We assume some familiarity with the basic definitions and properties concerning algebraic groups over $k = \overline{\mathbb{F}}_p$ (which we always assume to be affine) and refer to the literature (e.g., the standard textbooks [Bor91], [Hum75], [Spr09], see also [Gec03a]) for the details.

In this section, we introduce such notions which are of particular relevance in this thesis, for instance the connected reductive and (semi)simple algebraic groups over k or the associated finite groups of Lie type. For a much more elaborate discussion regarding the general theory of these groups and their representations, the reader is referred to, e.g., [Ste16], [DL76], [Car85], [DM20], [MT11], [GM20].

Definition 2.1.1. Let $\mathbf{G}$ be a connected algebraic group over k. Recall that the *unipotent radical* $R_{\mathrm{u}}(\mathbf{G})$ of $\mathbf{G}$ is the (unique) maximal closed connected unipotent normal subgroup of $\mathbf{G}$.

(a) $\mathbf{G}$ is called *reductive* if $R_{\mathrm{u}}(\mathbf{G}) = \{1\}$, and in this case we will often just refer to $\mathbf{G}$ as a *connected reductive group* (*over* k).

(b) $\mathbf{G}$ is called *semisimple* if $\mathbf{G}$ is reductive and the centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$ is finite (or equivalently, if its identity component $\mathbf{Z}(\mathbf{G})^\circ$ is the trivial subgroup $\{1\} \subseteq \mathbf{G}$).

(c) $\mathbf{G}$ is called *simple* if it is non-abelian and $\{1\}$, $\mathbf{G}$ are the only closed connected normal subgroups of $\mathbf{G}$.

Root data

The concept of root data, originally introduced in [DG70, Exposé XXI], provides a powerful combinatorial tool for classifying connected reductive groups. Here we give the definition of root data, following [GM20, 1.2.1], and state some basic properties of them. (For the proofs we refer to [GM20, §1.2].)

2.1.2. Let X, Y be free abelian groups of the same finite rank, and assume that there exists a bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$ such that we have induced isomorphisms of abelian groups

$$X \xrightarrow{\sim} \mathrm{Hom}(Y, \mathbb{Z}), \quad \lambda \mapsto (\nu \mapsto \langle \lambda, \nu \rangle),$$

and

$$Y \xrightarrow{\sim} \mathrm{Hom}(X, \mathbb{Z}), \quad \nu \mapsto (\lambda \mapsto \langle \lambda, \nu \rangle).$$

(Thus, the bilinear pairing $\langle\ ,\ \rangle$ is perfect.) Assume that there are finite subsets $R \subseteq X$, $R^\vee \subseteq Y$, together with a bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$, such that for any $\alpha \in R$, we have

$2\alpha \notin R$ and $\langle \alpha, \alpha^\vee \rangle = 2$. So for $\alpha \in R$, we obtain automorphisms of abelian groups

$$w_\alpha \colon X \xrightarrow{\sim} X, \quad \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha,$$

and

$$w_\alpha^\vee \colon Y \xrightarrow{\sim} Y, \quad \nu \mapsto \nu - \langle \alpha, \nu \rangle \alpha^\vee.$$

We require that $w_\alpha(R) = R$ and $w_\alpha^\vee(R^\vee) = R^\vee$ for any $\alpha \in R$. If all of the above conditions are satisfied, the quadruple $\mathscr{R} = (X, R, Y, R^\vee)$ is called a *root datum*. The groups

$$\mathbf{W} := \langle w_\alpha \mid \alpha \in R \rangle \subseteq \mathrm{Aut}(X) \quad \text{and} \quad \mathbf{W}^\vee := \langle w_\alpha^\vee \mid \alpha \in R \rangle \subseteq \mathrm{Aut}(Y)$$

are called the *Weyl groups* of R and $R^\vee$, respectively. We will refer to R as the *roots* and to $R^\vee$ as the *co-roots of the root datum* $\mathscr{R}$. Note that R is indeed a reduced crystallographic root system in the subspace $\mathbb{Q}R$ of $\mathbb{Q} \otimes_\mathbb{Z} X$ in the sense of [Bou68, Chap. VI, §1, Déf. 1], with Weyl group $\mathbf{W}$; see [GM20, 1.2.5]. Similarly, $R^\vee$ is a reduced crystallographic root system in $\mathbb{Q}R^\vee \subseteq \mathbb{Q} \otimes_\mathbb{Z} Y$, with Weyl group $\mathbf{W}^\vee$.

2.1.3. A special example of a connected reductive group is a *torus*, that is, an algebraic group which is isomorphic to the direct product of a finite number of copies of the multiplicative group $\mathbf{G}_\mathrm{m} := (k^\times, \cdot)$. Let $\mathbf{T}$ be a torus over k. The *rank of* $\mathbf{T}$ is defined to be the number $r \in \mathbb{N}_0$ such that $\mathbf{T}$ is isomorphic to $(\mathbf{G}_\mathrm{m})^r$, and is denoted by $r = \operatorname{rank} \mathbf{T}$. The *character group* $X(\mathbf{T})$ of $\mathbf{T}$ is the set consisting of all homomorphisms of algebraic groups $\mathbf{T} \to \mathbf{G}_\mathrm{m}$; similarly, the *co-character group* $Y(\mathbf{T})$ of $\mathbf{T}$ is the set of all homomorphisms of algebraic groups $\mathbf{G}_\mathrm{m} \to \mathbf{T}$. Both $X(\mathbf{T})$ and $Y(\mathbf{T})$ are indeed (abelian) groups, with group operation written additively and defined by (for $\lambda, \lambda' \in X(\mathbf{T})$ and $\nu, \nu' \in Y(\mathbf{T})$):

$$\lambda + \lambda' \colon \mathbf{T} \to \mathbf{G}_\mathrm{m}, \quad t \mapsto \lambda(t)\lambda'(t),$$

and

$$\nu + \nu' \colon \mathbf{G}_\mathrm{m} \to \mathbf{T}, \quad \xi \mapsto \nu(\xi)\nu'(\xi).$$

In particular, the multiplicative group $\mathbf{G}_\mathrm{m}$ itself is a torus, and it is easy to see that

$$X(\mathbf{G}_\mathrm{m}) = Y(\mathbf{G}_\mathrm{m}) = \{\mathbf{G}_\mathrm{m} \to \mathbf{G}_\mathrm{m},\ \xi \mapsto \xi^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Thus, if $\mathbf{T}$ is a torus of rank r, we obtain isomorphisms

$$X(\mathbf{T}) \cong Y(\mathbf{T}) \cong \mathbb{Z}^r.$$

We also see that, for $\lambda \in X(\mathbf{T})$, $\nu \in Y(\mathbf{T})$, we have $\lambda \circ \nu \in X(\mathbf{G}_\mathrm{m}) \cong \mathbb{Z}$, and we denote the corresponding integer by $\langle \lambda, \nu \rangle$. This gives rise to a perfect bilinear pairing

$$X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}, \quad (\lambda, \nu) \mapsto \langle \lambda, \nu \rangle.$$

The torus $\mathbf{T}$ can be recovered from both its character group and its co-character group, by means of the isomorphisms

$$\mathbf{T} \xrightarrow{\sim} \mathrm{Hom}(X(\mathbf{T}), k^\times), \quad t \mapsto (\chi \mapsto \chi(t)), \tag{2.1.3.1}$$

and

$$Y(\mathbf{T}) \otimes_\mathbb{Z} k^\times \xrightarrow{\sim} \mathbf{T}, \quad \text{determined by } \nu \otimes \xi \mapsto \nu(\xi) \text{ for } \nu \in Y(\mathbf{T}),\ \xi \in k^\times, \tag{2.1.3.2}$$

respectively (see [Car85, §3.1]).

2.1.4. Assume that we are given a connected reductive group $\mathbf{G}$ over k. A closed subgroup $\mathbf{T} \subseteq \mathbf{G}$ is called a *maximal torus* (of $\mathbf{G}$) if $\mathbf{T}$ is a torus and if $\mathbf{T}$ is maximal among the closed subgroups of $\mathbf{G}$ which are tori. We denote by $\mathscr{T}$ the set of all maximal tori of $\mathbf{G}$. The group $\mathbf{G}$ acts by conjugation on $\mathscr{T}$, and this action is transitive, that is, any two maximal tori of $\mathbf{G}$ are conjugate by some element of $\mathbf{G}$ (see [Hum75, §21.3]). Let us fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$. The (finite) group

$$\mathbf{W} = W_\mathbf{G}(\mathbf{T}) := N_\mathbf{G}(\mathbf{T})/\mathbf{T}$$

is called the *Weyl group of* $\mathbf{G}$ *with respect to* $\mathbf{T}$. It is clear that a different choice of $\mathbf{T}$ gives rise to a Weyl group isomorphic to $\mathbf{W}$ since any two maximal tori are conjugate in $\mathbf{G}$. For the same reason, all the maximal tori of $\mathbf{G}$ have the same rank, so it makes sense to define the *rank of* $\mathbf{G}$ as

$$\mathrm{rank}\,\mathbf{G} := \mathrm{rank}\,\mathbf{T}.$$

Following [GM20, 1.3.1], one can attach a root datum to the connected reductive group $\mathbf{G}$. Here, the set of roots $R \subseteq X(\mathbf{T})$ of $\mathbf{G}$ (with respect to $\mathbf{T}$) consists of all those $\alpha \in X(\mathbf{T})$ for which there exists a homomorphism of algebraic groups $u_\alpha \colon \mathbf{G}_\mathrm{a} \to \mathbf{G}$ (where $\mathbf{G}_\mathrm{a}$ denotes the additive group $(k, +)$) such that u_α is an isomorphism onto its image and

$$tu_\alpha(\xi)t^{-1} = u_\alpha(\alpha(t)\xi) \quad \text{for all } t \in \mathbf{T},\ \xi \in k.$$

In this case,

$$\mathbf{U}_\alpha := \{u_\alpha(\xi) \mid \xi \in k\}$$

is a one-dimensional closed connected unipotent subgroup of $\mathbf{G}$ which is normalised by $\mathbf{T}$. The group $\mathbf{U}_\alpha$ is called the *root subgroup corresponding to* $\alpha \in R$, and the assignment $\alpha \mapsto \mathbf{U}_\alpha$ gives a bijection between R and the set of one-dimensional closed connected unipotent subgroups of $\mathbf{G}$ which are normalised by $\mathbf{T}$. We have

$$\mathbf{G} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in R \rangle.$$

This allows a characterisation of the centre of $\mathbf{G}$ in terms of the roots: We have

$$\mathbf{Z}(\mathbf{G}) = \bigcap_{\alpha \in R} \ker \alpha \cong \mathrm{Hom}\Big(X(\mathbf{T})/\mathbb{Z}R, k^\times \Big) \tag{2.1.4.1}$$

(see, e.g., [MT11, 8.17]); here, the second isomorphism is induced from the one in (2.1.3.1). (In particular, the centre of $\mathbf{G}$ is contained in any maximal torus of $\mathbf{G}$.) Next, the conjugation action of $N_{\mathbf{G}}(\mathbf{T})$ on $\mathbf{T}$ induces actions of $\mathbf{W}$ on $X(\mathbf{T})$ and on $Y(\mathbf{T})$. Namely, let $w \in \mathbf{W}$, $\lambda \in X(\mathbf{T})$, $\nu \in Y(\mathbf{T})$, and denote by $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ a representative of w. Then we set

$$w.\lambda\colon \mathbf{T} \to \mathbf{G}_{\mathrm{m}}, \quad t \mapsto \lambda(\dot{w}^{-1} t \dot{w}),$$

and

$$w.\nu\colon \mathbf{G}_{\mathrm{m}} \to \mathbf{T}, \quad \xi \mapsto \dot{w}\nu(\xi)\dot{w}^{-1}.$$

Note that this allows us to identify $\mathbf{W}$ with a subgroup of both $\mathrm{Aut}(X(\mathbf{T}))$ and $\mathrm{Aut}(Y(\mathbf{T}))$. Using these actions and the perfect bilinear pairing $\langle\ ,\ \rangle\colon X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ defined in 2.1.3, it is possible to define a subset $\{w_\alpha \mid \alpha \in R\} \subseteq \mathbf{W}$, as well as a set of co-roots $R^\vee \subseteq Y(\mathbf{T})$, such that the quadruple $\mathscr{R} = (X(\mathbf{T}), R, Y(\mathbf{T}), R^\vee)$ together with the pairing $\langle\ ,\ \rangle$ is indeed a root datum, with Weyl group $\mathbf{W}$ (viewed as a subgroup of $\mathrm{Aut}(X(\mathbf{T}))$, as above), see [GM20, 1.3.1–1.3.2]. Finally, if we choose another maximal torus $\mathbf{T}'$ in $\mathbf{G}$, the root datum of $\mathbf{G}$ with respect to $\mathbf{T}'$ is isomorphic to $\mathscr{R}$ in the sense of [GM20, 1.2.2], see [GM20, 1.3.3]. Thus, we do not have to refer to a specific maximal torus in $\mathbf{G}$ when speaking of (the isomorphism class of) a root datum attached to $\mathbf{G}$.

2.1.5. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be a root datum. One can show that there exists a connected reductive group $\mathbf{G}$ over k such that applying the procedure outlined in 2.1.4 to $\mathbf{G}$ gives rise to a root datum isomorphic to $\mathscr{R}$ (see [GM20, 1.3.14] and the references there); moreover, $\mathbf{G}$ is uniquely determined up to isomorphism, that is, any root datum isomorphic to $\mathscr{R}$ leads to a connected reductive group isomorphic to $\mathbf{G}$ [GM20, 1.3.13].

Conversely, isomorphic connected reductive groups $\mathbf{G}$ give rise to the same isomorphism class of root data. Hence, the concept of root data provides a combinatorial tool for classifying connected reductive groups. For later use we give some further notions and properties of root data here (which are in fact mostly properties of the root system involved), see [GM20, 1.2.6] and also the references there. So let $\mathscr{R} = (X, R, Y, R^\vee)$ be a root datum, and let $\mathbf{G}$ be a connected reductive group corresponding to $\mathscr{R}$. There exists a subset $\Pi \subseteq R$ which is linearly independent in the $\mathbb{Q}$-vector subspace $\mathbb{Q}R \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} X$ and has the property that any root $\alpha \in R$ can be written as

$$\alpha = \sum_{\beta \in \Pi} n_{\alpha,\beta} \cdot \beta \quad \text{where } n_{\alpha,\beta} \in \mathbb{Z}, \tag{$*$}$$

with either $n_{\alpha,\beta} \geqslant 0$ for all $\beta \in \Pi$, or else $n_{\alpha,\beta} \leqslant 0$ for all $\beta \in \Pi$. If these conditions are satisfied, Π is called a *base* for the root system R. Except when $\mathbf{G}$ is a torus, such a base Π is not unique; in fact, the set of bases for R is in bijective correspondence with the Weyl group $\mathbf{W}$ of the root system (where, for a fixed base Π, such a bijection may be chosen by requiring that $w \in \mathbf{W}$ corresponds to the base $w(\Pi) \subseteq R$). Let us fix a base Π of R. Whenever we have done so, we will refer to the elements of Π as the *simple roots* of $\mathscr{R}$, or of R, or of $\mathbf{G}$. The *positive roots* R^+ of R (with respect to Π) are defined to be those $\alpha \in R$ such that the integers $n_{\alpha,\beta}$ in $(*)$ are non-negative; the *negative roots* R^- of R (with respect to Π) are defined by $R^- := -R^+ \subseteq R$. Writing $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, the matrix

$$\mathfrak{C} := \left(\langle \alpha_j, \alpha_i^\vee \rangle\right)_{1 \leqslant i,j \leqslant r}$$

is called the *Cartan matrix* of the root datum $\mathscr{R}$ or of the connected reductive group $\mathbf{G}$. Up to simultaneously reordering the rows and columns, $\mathfrak{C}$ does not depend on the choice of Π, thus only on $\mathscr{R}$. Note that $\mathfrak{C}$ is a Cartan matrix in the sense of Appendix A.1; more precisely, $\mathfrak{C}$ is a Cartan matrix of both finite and crystallographic type, that is, it satisfies the conditions ($\mathfrak{C}$1), ($\mathfrak{C}$2), ($\mathfrak{C}$fin) and ($\mathfrak{C}$crys) in A.1. The other notions are compatible with the ones in A.1 as well: Indeed, $\mathbf{W} \cong W(\mathfrak{C})$ is the reflection group associated with $\mathfrak{C}$, $R = R(\mathfrak{C})$ is the root system associated with $\mathfrak{C}$ (an abstract reduced crystallographic root system in the $\mathbb{R}$-vector space with basis $\Pi = \Pi(\mathfrak{C})$), and $r = |\Pi|$ is the rank of R. So $\mathbf{W}$ is also a Weyl group in the sense of A.1.8 and, hence, a Coxeter group, with Coxeter generators $S = \{w_\alpha \mid \alpha \in \Pi\}$ and defining relations

$$(w_\alpha w_\beta)^{m_{\alpha\beta}} = 1 \quad \text{for all } \alpha, \beta \in \Pi,$$

where $m_{\alpha\beta}$ is the order of $w_\alpha w_\beta \in \mathbf{W}$. The Dynkin diagram $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{C}$ is then also said to be the Dynkin diagram of $\mathbf{G}$. Furthermore, if $\mathfrak{D}$ is the name of $\mathfrak{D}(\mathfrak{C})$ (see A.1.8 and Figure A.2), it will sometimes be convenient to say that $\mathbf{G}$ *has a root system of type* $\mathfrak{D}$. For instance, when talking about the connected centraliser $C^\circ_{\mathbf{G}}(s)$ of a semisimple element s (note that $C^\circ_{\mathbf{G}}(s)$ is itself a connected reductive group, see [Car85, 3.5.4]), a description as above for $C^\circ_{\mathbf{G}}(s)$ will often be sufficient to characterise the $\mathbf{G}$-conjugacy class of s, and we may not need any further information on the group $C^\circ_{\mathbf{G}}(s)$ itself.

Example 2.1.6. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be a root datum, with pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$. Then one easily checks that $\mathscr{R}^* := (Y, R^\vee, X, R)$ is a root datum as well, with pairing given by

$$\langle\ ,\ \rangle^*\colon Y \times X \to \mathbb{Z},\ \langle \nu, \lambda\rangle^* := \langle \lambda, \nu\rangle \quad \text{for } \nu \in Y,\ \lambda \in X$$

(see [GM20, 1.2.3]). The quadruple $\mathscr{R}^*$ is called the *root datum dual to* $\mathscr{R}$. As mentioned in 2.1.5, there exists a connected reductive group $\mathbf{G}^*$ whose root datum is isomorphic to $\mathscr{R}^*$, and $\mathbf{G}^*$ is uniquely determined up to isomorphism. The group $\mathbf{G}^*$ is called the *dual group of* $\mathbf{G}$. Note that, since $(\mathscr{R}^*)^* = \mathscr{R}$, we have $(\mathbf{G}^*)^* \cong \mathbf{G}$.

2.1.7. (a) Let $\mathbf{G}$ be a connected reductive group over k. We fix a maximal torus $\mathbf{T} \subseteq \mathbf{G}$, which thus gives rise to the root datum $\mathscr{R} = (X, R, Y, R^\vee)$ of $\mathbf{G}$ with respect to $\mathbf{T}$, where $X = X(\mathbf{T})$ and $Y = Y(\mathbf{T})$. Following [MT11, pp. 70–71] (see also [Car85, 1.11]), let us consider the group

$$\Omega := \operatorname{Hom}(\mathbb{Z}R^\vee, \mathbb{Z}).$$

The group $\mathbf{G}$ is semisimple if and only if $\mathbb{Z}R^\vee \subseteq Y$ is a subgroup of finite index (which is equivalent to $\mathbb{Z}R \subseteq X$ being a subgroup of finite index), and in this case we obtain an embedding

$$X \xrightarrow{\sim} \operatorname{Hom}(Y, \mathbb{Z}) \hookrightarrow \operatorname{Hom}(\mathbb{Z}R^\vee, \mathbb{Z}) = \Omega,$$

where the first map is the isomorphism defined in 2.1.2 and the second is given by restriction of maps from Y to $\mathbb{Z}R^\vee$. So we may canonically identify $\mathbb{Z}R \subseteq X \subseteq \Omega$. Here, $\mathbb{Z}R$ is a subgroup of finite index in Ω, so

$$\Lambda := \Lambda(R) := {}^{\Omega}/_{\mathbb{Z}R}$$

is a finite group, called the *fundamental group* of the root system R.

(b) Note that the definition of Λ in (a) only depends on R, $R^\vee$ and not on X or Y. Hence, as in [GM20, 1.2.8], we may as well start with a Cartan matrix $\mathfrak{C}$ of both finite and crystallographic type (see Appendix A.1), without referring to a semisimple group just yet: In this case, let $R = R(\mathfrak{C})$ be the root system associated with $\mathfrak{C}$. We then say that

$$\Lambda(\mathfrak{C}) := \Lambda(R)$$

is the *fundamental group of* $\mathfrak{C}$. By [GM20, 1.5.1, 1.5.2], any subgroup $X/\mathbb{Z}R$ of $\Lambda(R)$ uniquely determines a root datum $(X, R, Y, R^\vee)$ up to isomorphism, which thus determines a semisimple algebraic group up to isomorphism (since $X/\mathbb{Z}R$ is finite by construction). Moreover, every semisimple algebraic group with root system isomorphic to R is obtained in this way (although the above procedure is in general not quite a one-to-one correspondence between the subgroups of $\Lambda(R)$ and the isomorphism classes of semisimple algebraic groups whose root system is isomorphic to R, see [MT11, 9.16(3)]). The fundamental groups of indecomposable Cartan matrices of finite and crystallographic type are given by Table 2.1 (see [GM20, p. 20]).

Type of $\mathfrak{C}$	$\Lambda(\mathfrak{C})$	
A_{n-1}	$\mathbb{Z}/n\mathbb{Z}$	
$\mathsf{B}_n, \mathsf{C}_n$	$\mathbb{Z}/2\mathbb{Z}$	
D_n	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	if n is even
	$\mathbb{Z}/4\mathbb{Z}$	if n is odd
$\mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_8$	$\{0\}$	
E_6	$\mathbb{Z}/3\mathbb{Z}$	
E_7	$\mathbb{Z}/2\mathbb{Z}$	

Table 2.1.: Fundamental groups of indecomposable Cartan matrices $\mathfrak{C}$ of finite and crystallographic type

(c) Let us assume again that we are given a semisimple group $\mathbf{G}$ with root system R and Cartan matrix $\mathfrak{C}$, and let $\mathfrak{D}$ be the name of the Dynkin diagram $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{C}$, as in A.1.8. We will then say that $\mathbf{G}$ *is of type* $\mathfrak{D}$. If $X = \mathbb{Z}R$, we say that $\mathbf{G}$ is *the adjoint group* (*of type* $\mathfrak{D}$), or sometimes just that $\mathbf{G}$ *is adjoint* or *is of adjoint type* in case we do not want to specify R or $\mathfrak{C}$. If $X = \Omega$, we say that $\mathbf{G}$ is *the simply connected group* (*of type* $\mathfrak{D}$), or sometimes just that $\mathbf{G}$ *is simply connected* or *is of simply connected type* in case we do not want to specify R or $\mathfrak{C}$. If $\mathbf{G}$ is any semisimple group with root system R (not necessarily of adjoint or simply connected type), we denote by $\mathbf{G}_{\mathrm{ad}}$ the (semisimple) adjoint group with root system isomorphic to R; similarly, $\mathbf{G}_{\mathrm{sc}}$ denotes the (semisimple) simply connected group with root system isomorphic to R. (In particular, we write $\mathbf{G} = \mathbf{G}_{\mathrm{ad}}$ if $\mathbf{G}$ is semisimple of adjoint type and $\mathbf{G} = \mathbf{G}_{\mathrm{sc}}$ if $\mathbf{G}$ is semisimple of simply connected type.) In view of (2.1.4.1), we have

$$\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) \cong \mathrm{Hom}(\Lambda(R), k^{\times}) \quad \text{and} \quad \mathbf{Z}(\mathbf{G}_{\mathrm{ad}}) = \{1\}. \tag{2.1.7.1}$$

Now, let us recall that a surjective homomorphism $f\colon \mathbf{H}_1 \to \mathbf{H}_2$ of connected algebraic groups $\mathbf{H}_1, \mathbf{H}_2$ is called an *isogeny* if its kernel $\ker f \subseteq \mathbf{H}_1$ is finite; we then automatically have $\ker f \subseteq \mathbf{Z}(\mathbf{H}_1)$. There exist canonical isogenies (see [GM20, §1.5])

$$(\pi_{\mathbf{G}})_{\mathrm{sc}}\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G} \quad \text{and} \quad (\pi_{\mathbf{G}})_{\mathrm{ad}}\colon \mathbf{G} \to \mathbf{G}_{\mathrm{ad}},$$

called the *simply connected covering* and the *adjoint quotient* of the semisimple group $\mathbf{G}$, respectively. Two non-isomorphic semisimple groups of type $\mathfrak{D}$ are said to be *of different isogeny type*. Finally, we note that the semisimple group $\mathbf{G}$ is simple if and only if its Cartan matrix $\mathfrak{C}$ is indecomposable (or, equivalently, if the Weyl group $\mathbf{W}$ of $\mathbf{G}$ is an irreducible Coxeter group). We refer to [MT11, Chap. 9] and [GM20, Chap. 1] (and the further references there) for a much more detailed description.

While there are numerous properties of (connected reductive) algebraic groups $\mathbf{G}$ and the associated finite groups of Lie type $\mathbf{G}^F$ which can be proven rather uniformly as the characteristic p of the ground field $k = \overline{\mathbb{F}}_p$ varies, there are a few distinguished very small primes p (depending on $\mathbf{G}$) which require a different treatment from the others at several places, or even give rise to features of $\mathbf{G}$ which are not shared by the analogous groups over ground fields of larger characteristics (and vice versa). These exceptional primes are specified by the following definition.

Definition 2.1.8 ([BCCISS, E-I-§4])**.** Let $\mathbf{G}$ be a connected reductive group over k, and let $\mathscr{R} = (X, R, Y, R^{\vee})$ be the corresponding root datum. Let $\Pi \subseteq R$ be a base for R, with corresponding positive roots $R^+ \subseteq R$, and let $\alpha_0 \in R^+$ be the highest root of R with respect to Π (that is, we have $\alpha_0 + \beta \notin R$ for any $\beta \in R^+$; the root α_0 is uniquely determined by this property). A prime $p \in \mathbb{N}$ is called *good* for $\mathbf{G}$ (or for R) if in the decomposition $(*)$ in 2.1.5 of α_0 into simple roots, p does not divide any of the coefficients $n_{\alpha_0,\beta}$. (This does not depend on the choice of Π.) A prime $p \in \mathbb{N}$ is called *bad* for $\mathbf{G}$ (or for R) if it is not good. The bad primes for simple groups $\mathbf{G}$ are listed in Table 2.2.

Important subgroups and conjugacy classes

2.1.9. Let $\mathbf{G}$ be a connected reductive group over k. A *Borel subgroup* of $\mathbf{G}$ is, by definition, a maximal closed connected solvable subgroup $\mathbf{B} \subseteq \mathbf{G}$. Let $\mathscr{B}$ be the variety of all Borel subgroups of $\mathbf{G}$. The group $\mathbf{G}$ acts transitively on $\mathscr{B}$ by conjugation (see [Hum75,

Type of $\mathbf{G}$	Bad primes p
A_n	—
$\mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n$	2
$\mathsf{E}_6, \mathsf{E}_7, \mathsf{F}_4, \mathsf{G}_2$	$2, 3$
E_8	$2, 3, 5$

Table 2.2.: Bad primes for simple groups $\mathbf{G}$

§21.3]), and we obtain a transitive action of $\mathbf{G}$ on the set $\{(\mathbf{T}, \mathbf{B}) \in \mathscr{T} \times \mathscr{B} \mid \mathbf{T} \subseteq \mathbf{B}\}$ given by simultaneous conjugation. Thus, for each two pairs $(\mathbf{T}, \mathbf{B}), (\mathbf{T}', \mathbf{B}') \in \mathscr{T} \times \mathscr{B}$ such that $\mathbf{T} \subseteq \mathbf{B}$ and $\mathbf{T}' \subseteq \mathbf{B}'$, there exists some $x \in \mathbf{G}$ such that $\mathbf{T}' = x\mathbf{T}x^{-1}$ and $\mathbf{B}' = x\mathbf{B}x^{-1}$. Let us fix a maximal torus $\mathbf{T} \in \mathscr{T}$. Since $\mathbf{T}$ is connected and abelian, thus solvable, there exists some Borel subgroup of $\mathbf{G}$ which contains $\mathbf{T}$. More precisely, as described in [GM20, 1.3.4], if $\mathscr{R} = (X(\mathbf{T}), R, Y(\mathbf{T}), R^\vee)$ is the root datum of $\mathbf{G}$ with respect to $\mathbf{T}$ and if we fix a base $\Pi \subseteq R$ with corresponding positive roots $R^+ \subseteq R$, the group

$$\mathbf{B} := \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in R^+ \rangle$$

is a Borel subgroup of $\mathbf{G}$ which contains $\mathbf{T}$, and for any Borel subgroup $\mathbf{B}'$ of $\mathbf{G}$ containing $\mathbf{T}$, there exists a unique $w \in \mathbf{W} = W_{\mathbf{G}}(\mathbf{T})$ such that

$$\mathbf{B}' = \dot{w}\mathbf{B}\dot{w}^{-1} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in w(R^+) \rangle.$$

(Note that $w(R^+)$ is the set of positive roots with respect to the base $w(\Pi) \subseteq R$.) In particular, this defines a bijection between the set of Borel subgroups of $\mathbf{G}$ which contain $\mathbf{T}$ and the set of bases for R.

2.1.10. Let $\mathbf{G}$ be a connected reductive group over k, and let us fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$, so that we obtain the Weyl group $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ as well as the root datum $\mathscr{R} = (X, R, Y, R^\vee)$ (with $X = X(\mathbf{T})$, $Y = Y(\mathbf{T})$) of $\mathbf{G}$ with respect to $\mathbf{T}$. Let us choose a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that $\mathbf{T} \subseteq \mathbf{B}$ and denote by $\Pi \subseteq R$ the base for R corresponding to $\mathbf{B}$, as described in 2.1.9. Setting $B := \mathbf{B}$, $N := N_{\mathbf{G}}(\mathbf{T})$ and $S := \{w_\alpha \mid \alpha \in \Pi\} \subseteq \mathbf{W}$, the groups B and N form a BN-pair (or Tits system) in $\mathbf{G}$ in the sense of [Bou68, Chap. IV, §2], see [Car85, Chap. 2] or [DM20, §3.1]. For later use we will state some of the most important properties and definitions related to $\mathbf{G}$ here, based on the fact that it is a group with a BN-pair. For proofs we refer to [DM20, Chap. 3] (or [Car85, Chap. 2]) and the references there. Firstly, there is the *Bruhat decomposition*

$$\mathbf{G} = \biguplus_{w \in \mathbf{W}} \mathbf{B}w\mathbf{B}$$

where for $w \in \mathbf{W}$, we write $\mathbf{B}w\mathbf{B} := \mathbf{B}\dot{w}\mathbf{B}$ for any chosen representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ of $w \in \mathbf{W}$ (this is clearly independent of the concrete choice of $\dot{w}$). Now we consider a subset $J \subseteq S$ and denote by $\mathbf{W}_J = \langle J \rangle \subseteq \mathbf{W}$ the subgroup of $\mathbf{W}$ generated by J. Any such $\mathbf{W}_J$ is itself a Coxeter group, with Coxeter generators J. We set

$$\mathbf{P}_J := \mathbf{B}\mathbf{W}_J\mathbf{B} := \biguplus_{w \in \mathbf{W}_J} \mathbf{B}w\mathbf{B} \subseteq \mathbf{G}.$$

Then $\mathbf{P}_J$ is a subgroup of $\mathbf{G}$, which we will call a *standard parabolic subgroup* of $\mathbf{G}$. (It should be noted that the term 'standard' only makes sense once a Borel subgroup $\mathbf{B}$ is prescribed, and then it is meant as 'standard with respect to $\mathbf{B}$'.) The unipotent radical of $\mathbf{P}_J$ is given by

$$R_{\mathrm{u}}(\mathbf{P}_J) = \prod_{\alpha \in R^+ \setminus R_J} \mathbf{U}_\alpha =: \mathbf{U}_J.$$

Any closed subgroup of $\mathbf{G}$ containing $\mathbf{B}$ has the form $\mathbf{P}_J$ for some $J \subseteq S$. More generally, a *parabolic* subgroup of $\mathbf{G}$ is a closed subgroup $\mathbf{P} \subseteq \mathbf{G}$ which contains some Borel subgroup of $\mathbf{G}$. Every parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is conjugate to a standard parabolic subgroup $\mathbf{P}_J$ for a suitable $J \subseteq S$. Furthermore, any parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ has a *Levi decomposition*, that is, there exist a closed subgroup $\mathbf{L} \subseteq \mathbf{P}$ and a semi-direct product decomposition

$$\mathbf{P} = R_{\mathrm{u}}(\mathbf{P}) \rtimes \mathbf{L}.$$

In this case, we call $\mathbf{L}$ a *Levi subgroup* (or a *Levi complement*) of $\mathbf{P}$. In general, $\mathbf{L}$ is not uniquely determined by $\mathbf{P}$; however, if $\mathbf{T} \subseteq \mathbf{P}$, there is exactly one Levi complement $\mathbf{L}$ of $\mathbf{P}$ which contains $\mathbf{T}$. If $\mathbf{P} = \mathbf{P}_J$ for some $J \subseteq S$, we denote by $\mathbf{L}_J$ the unique Levi complement of $\mathbf{P}_J$ such that $\mathbf{T} \subseteq \mathbf{L}_J$, and we call $\mathbf{L}_J$ *the standard Levi subgroup* of the standard parabolic subgroup $\mathbf{P}_J$ (which assumes that $\mathbf{T}$ and $\mathbf{B}$ are prescribed). A more direct description of $\mathbf{L}_J$ is given as follows: Let us denote by $\Pi_J \subseteq \Pi$ the subset of the simple roots such that $J = \{w_\alpha \mid \alpha \in \Pi_J\}$ and by $R_J \subseteq R$ the set of those roots which are in the subspace $\mathbb{Q}\Pi_J$ of $\mathbb{Q} \otimes_{\mathbb{Z}} X(\mathbf{T})$. Then R_J is a root system in that subspace, and Π_J is a base for R_J. We have

$$\mathbf{L}_J = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in R_J \rangle \subseteq \mathbf{G},$$

and this is a connected reductive group whose root datum with respect to $\mathbf{T}$ is given by $\mathscr{R}_J = (X, R_J, Y, R_J^\vee)$ where $R_J^\vee = \{\alpha^\vee \mid \alpha \in R_J\}$. The Weyl group $N_{\mathbf{L}_J}(\mathbf{T})/\mathbf{T}$ of $\mathbf{L}_J$ is isomorphic to $\mathbf{W}_J$. Any Levi complement $\mathbf{L}$ of any parabolic subgroup is conjugate to $\mathbf{L}_J$ for some $J \subseteq S$, so $\mathbf{L}$ is a connected reductive group. Finally, we mention that the Levi complements of parabolic subgroups of $\mathbf{G}$ are precisely the subgroups of $\mathbf{G}$ of the form $C_{\mathbf{G}}(\mathbf{S})$ for closed subgroups $\mathbf{S} \subseteq \mathbf{G}$ which are tori (see [DM20, 3.4.6, 3.4.7]).

2.1.11 Conjugacy classes. Let $\mathbf{G}$ be a connected reductive group over k. For later use and to fix our notation, we collect some well-known definitions and results concerning conjugacy classes of $\mathbf{G}$.

(a) For the following we refer to [Gec03a, §2.5]. Let $\mathscr{C}$ be a conjugacy class of $\mathbf{G}$. For any $g \in \mathscr{C}$, we have

$$\dim \mathbf{G} = \dim \mathscr{C} + \dim C_{\mathbf{G}}(g).$$

Furthermore, the (Zariski) closure $\overline{\mathscr{C}}$ of $\mathscr{C}$ is a union of conjugacy classes of $\mathbf{G}$. Given two conjugacy classes $\mathscr{C}, \mathscr{C}'$ of $\mathbf{G}$, we write $\mathscr{C}' \preccurlyeq \mathscr{C}$ if $\mathscr{C}' \subseteq \overline{\mathscr{C}}$. This defines a partial order on the set of all conjugacy classes of $\mathbf{G}$.

(b) An element $g \in \mathbf{G}$ is called *regular* if the centraliser $C_{\mathbf{G}}(g)$ is of minimal dimension among all centralisers of elements in $\mathbf{G}$; this is equivalent to $\dim C_{\mathbf{G}}(g) = \operatorname{rank} \mathbf{G}$ (see [BCCISS, E-III-§1]). Clearly, this condition is invariant under $\mathbf{G}$-conjugacy, so it makes sense to say that a conjugacy class $\mathscr{C}$ of $\mathbf{G}$ is *regular* if one (any) of its elements is regular.

(c) For $g \in \mathbf{G}$, let us write $g = g_{\mathrm{s}} g_{\mathrm{u}} = g_{\mathrm{u}} g_{\mathrm{s}}$ for its Jordan decomposition (that is, $g_{\mathrm{s}} \in \mathbf{G}$ is semisimple and $g_{\mathrm{u}} \in \mathbf{G}$ is unipotent; the elements g_{s} and g_{u} are uniquely determined by this property). Let $\mathcal{S} \subseteq \mathbf{G}$ be a set of representatives for the semisimple conjugacy classes of $\mathbf{G}$ and, for each $s \in \mathcal{S}$, let $\mathcal{U}_s \subseteq C_{\mathbf{G}}(s)$ be a set of representatives for the unipotent conjugacy classes of $C_{\mathbf{G}}(s)$. Then it immediately follows from the existence and uniqueness of the Jordan decomposition that

$$\{su = us \mid s \in \mathcal{S},\ u \in \mathcal{U}_s\}$$

is a set of representatives for the conjugacy classes of $\mathbf{G}$. Thus, studying the conjugacy classes of $\mathbf{G}$ is reduced to studying the semisimple conjugacy classes of $\mathbf{G}$ and the unipotent conjugacy classes of centralisers of semisimple elements of $\mathbf{G}$. Note that, by [Car85, 3.5.4], $C_{\mathbf{G}}^\circ(s)$ is a connected reductive group for any semisimple element $s \in \mathbf{G}$. In many instances the group $C_{\mathbf{G}}(s)$ is connected (although not in general), for example, if the derived group $\mathbf{G}_{\mathrm{der}} = [\mathbf{G}, \mathbf{G}]$ of $\mathbf{G}$ is simply connected; see [Car85, 3.5.6]. This highlights the importance of studying unipotent conjugacy classes of connected reductive algebraic groups.

(d) Let us now focus on unipotent elements and unipotent (conjugacy) classes of $\mathbf{G}$. (A reference for the following is [Car85, Chap. 5].) First of all, the set $\mathbf{G}_{\text{uni}}$ consisting of all unipotent elements of $\mathbf{G}$ is a closed irreducible subset of $\mathbf{G}$ and is called the *unipotent variety* of $\mathbf{G}$. Thus, if $\mathscr{O} \subseteq \mathbf{G}$ is a unipotent class, the set $\overline{\mathscr{O}}$ is contained in $\mathbf{G}_{\text{uni}}$ and, hence, is a union of unipotent classes of $\mathbf{G}$. There exist unipotent elements in $\mathbf{G}$ which are regular, and the set $\mathscr{O}_{\text{reg}}$ of all these regular unipotent elements turns out to be a single (unipotent) conjugacy class of $\mathbf{G}$, which is dense open in $\mathbf{G}_{\text{uni}}$. In particular, with respect to the order relation $\preccurlyeq$ defined in (a), we have $\mathscr{O} \preccurlyeq \mathscr{O}_{\text{reg}}$ for any unipotent class $\mathscr{O}$ of $\mathbf{G}$. This also shows that $\overline{\mathscr{O}} \cap \mathscr{O}_{\text{reg}} = \varnothing$ for any unipotent class $\mathscr{O} \neq \mathscr{O}_{\text{reg}}$.

It is known that in any given connected reductive group $\mathbf{G}$, the number of unipotent classes is finite, although the proof of this result in complete generality is very deep, see [Lus76a]. It is however easily seen that the classification of unipotent classes of a connected reductive group $\mathbf{G}$ can be reduced to the case where $\mathbf{G}$ is a simple (adjoint) group, using the canonical map $\mathbf{G} \to \mathbf{G}/\mathbf{Z}(\mathbf{G})$ (see the introduction to [Car85, Chap. 5]).

If $\mathbf{G}$ is a simple group, and if the characteristic p of k is good for $\mathbf{G}$, there exists a natural bijection between the nilpotent orbits of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ (under the adjoint action) and the unipotent conjugacy classes in $\mathbf{G}$. This allows a classification of the unipotent classes of $\mathbf{G}$, due to Bala–Carter [BC76a], [BC76b]. (In loc. cit., there is some lower bound imposed on p; Pommerening [Pom77], [Pom80] showed that the results of Bala–Carter hold whenever p is a good prime for $\mathbf{G}$.) In bad characteristic, there is in general no such bijection between the nilpotent orbits on $\mathfrak{g}$ and the unipotent conjugacy classes of $\mathbf{G}$, but the classification of the unipotent classes has still been carried out in all cases, see the detailed overview in [Car85, §5.11] and the references there. A single reference for the complete classification of unipotent classes of simple algebraic groups $\mathbf{G}$ (and also the nilpotent orbits on $\mathfrak{g}$), both in good and bad characteristic, is the book [LS12], which also contains results on centralisers of unipotent elements in $\mathbf{G}$ (and nilpotent orbits on $\mathfrak{g}$) which were previously not known.

We will be particularly concerned with simple *exceptional* groups (not necessarily of adjoint type) later, that is, groups of type E_6, E_7, E_8, F_4 or G_2 (cf. A.1.8). So let us assume that $\mathbf{G}$ is such a group. As mentioned in [Car85, p. 183] (see also [LS12, Chap. 22]), the unipotent classes which occur in good characteristic may be parametrised in the same way in bad characteristic, but in some cases there are a few additional unipotent classes when p is bad for $\mathbf{G}$. So it makes sense to use the names in [Car85] (which in turn are due to Bala–Carter) uniformly for all characteristics, although for groups of type E_n ($n = 6, 7, 8$), we will often also provide the names of Mizuno [Miz77], [Miz80]. (Note that this does not depend on the isogeny type of $\mathbf{G}$, using again the fact that the canonical map $\mathbf{G} \to {}^{\mathbf{G}}/_{\mathbf{Z}(\mathbf{G})}$ induces a bijection between the unipotent classes of $\mathbf{G}$ and those of ${}^{\mathbf{G}}/_{\mathbf{Z}(\mathbf{G})}$.) An exception is the regular unipotent conjugacy class, which we will denote by $\mathscr{O}_{\mathrm{reg}}$ (or sometimes $\mathscr{O}_{\mathrm{reg}}^{\mathfrak{D}}$ where $\mathfrak{D}$ is the name of the Dynkin diagram of the Cartan matrix of $\mathbf{G}$, in case $\mathbf{G}$ is not clear from the context), as above.

Frobenius maps and Steinberg maps

In order to obtain *finite* groups from a given (connected reductive) algebraic group $\mathbf{G}$ over k, one considers fixed-point sets of $\mathbf{G}$ under suitable bijective endomorphisms of $\mathbf{G}$, so-called Frobenius maps or, more generally, Steinberg maps.

2.1.12. Let us first recall the notion of Frobenius maps for affine varieties and how such maps give rise to finite subsets of these varieties. So let Z be an affine variety over k, and let $A := A[Z] \subseteq \mathrm{Maps}(Z, k)$ be the associated k-algebra consisting of regular functions $Z \to k$, with addition and multiplication defined pointwise (see [Gec03a, 2.1.6]). Let $F\colon Z \to Z$ be a morphism of affine varieties. Any such F induces an algebra homomorphism

$$F^*\colon A \to A, \quad a \mapsto a \circ F.$$

Let $q = p^e$ for some $e \in \mathbb{N}$. Following [Gec03a, 4.1.1] (see also [GM20, 1.4.3]), we say that F is a *Frobenius map corresponding to an $\mathbb{F}_q$-rational structure on Z*, or that Z *is defined over $\mathbb{F}_q$ with corresponding Frobenius map F*, if the following two conditions hold for (F, q):

(i) F^* is injective and $F^*(A) = \{a^q \mid a \in A\}$;

(ii) For each $a \in A$, there exists some $m \geqslant 1$ such that $(F^*)^m(a) = a^{q^m}$.

The associated set of fixed points

$$Z^F := \{z \in Z \mid F(z) = z\}$$

is called *the set of $\mathbb{F}_q$-rational points in Z (with respect to the $\mathbb{F}_q$-rational structure defined by F)* and is sometimes also denoted by $Z(\mathbb{F}_q)$. This definition implies, in particular, that F is a bijective map and that $Z(\mathbb{F}_q) = Z^F$ is a finite set (see [Gec03a, 4.1.4]).

Definition 2.1.13. Let $\mathbf{G}$ be an algebraic group over k. A map $F\colon \mathbf{G} \to \mathbf{G}$ which is an endomorphism of the algebraic group $\mathbf{G}$ and at the same time a Frobenius map of the affine variety underlying $\mathbf{G}$ in the sense of 2.1.12 (with respect to some power q of p) is called a *Frobenius map* or a *Frobenius endomorphism* of $\mathbf{G}$. More generally, following [GM20, 1.4.7], we say that a homomorphism $F\colon \mathbf{G} \to \mathbf{G}$ of algebraic groups is a *Steinberg map* of $\mathbf{G}$ if some power of F is a Frobenius map of $\mathbf{G}$. Thus, any Steinberg map $F\colon \mathbf{G} \to \mathbf{G}$ is bijective, and

$$\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$$

is a finite subgroup of $\mathbf{G}$. If the algebraic group $\mathbf{G}$ is connected and reductive, the group $\mathbf{G}^F$ is called a *finite group of Lie type* or a *finite reductive group.*

While we refer to the literature (e.g., [Gec03a, Chap. 4], [GM20, §1.4]) for standard properties concerning Frobenius and Steinberg maps of algebraic groups, let us explicitly state the following fundamental result, which is an indispensable tool when seeking to transfer properties of a connected algebraic group $\mathbf{G}$ (endowed with a Steinberg map $F\colon \mathbf{G} \to \mathbf{G}$) to the finite group $\mathbf{G}^F$.

Theorem 2.1.14 (Lang–Steinberg [Lan56], [Ste68, 10.1]; see [GM20, 1.4.8])**.** *Let $\mathbf{G}$ be a connected algebraic group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. Then the Lang–Steinberg map*

$$\mathcal{L}\colon \mathbf{G} \to \mathbf{G}, \quad g \mapsto g^{-1}F(g),$$

is surjective.

Proof. See, e.g., [GM20, p. 43]. □

Here are some direct consequences of the Lang–Steinberg Theorem.

Corollary 2.1.15. *Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. There exists a pair $(\mathbf{T}_0, \mathbf{B}_0) \in \mathscr{T} \times \mathscr{B}$ such that $\mathbf{T}_0 \subseteq \mathbf{B}_0$, $F(\mathbf{T}_0) = \mathbf{T}_0$ and $F(\mathbf{B}_0) = \mathbf{B}_0$. Moreover, the $\mathbf{G}^F$-conjugacy class of the pair $(\mathbf{T}_0, \mathbf{B}_0)$ is uniquely determined by this property.*

Proof. This is a standard application of the Lang–Steinberg Theorem 2.1.14, see, e.g., [GM20, 1.4.9, 1.4.12]. □

Definition 2.1.16. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. A maximal torus $\mathbf{T}_0 \subseteq \mathbf{G}$ is called *maximally split* if $F(\mathbf{T}_0) = \mathbf{T}_0$ and if there exists some Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ such that $F(\mathbf{B}_0) = \mathbf{B}_0$ and $\mathbf{T}_0 \subseteq \mathbf{B}_0$.

By Corollary 2.1.15, every connected reductive group $\mathbf{G}$ over k with a given Steinberg map $F\colon \mathbf{G} \to \mathbf{G}$ contains a maximally split torus, and any two maximally split tori of $\mathbf{G}$ are conjugate by an element of $\mathbf{G}^F$.

2.1.17. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. We fix a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{G}$. Since $N_{\mathbf{G}}(\mathbf{T}_0)$ is also F-stable, F induces an automorphism of the Weyl group $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ of $\mathbf{G}$ with respect to $\mathbf{T}_0$, which we denote by $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$. Consider the action of $\mathbf{W}$ on itself defined by

$$\mathbf{W} \times \mathbf{W} \to \mathbf{W}, \quad (w, y) \mapsto wy\sigma(w)^{-1}.$$

The orbits of this action are called the *σ-conjugacy classes* of $\mathbf{W}$. For $w \in \mathbf{W}$, let us choose a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of w and some $g \in \mathbf{G}$ such that $g^{-1}F(g) = \dot{w}$ (Theorem 2.1.14). Then

$$\mathbf{T}_w := g\mathbf{T}_0 g^{-1}$$

is an F-stable maximal torus of $\mathbf{G}$, called a *torus of type w* with respect to $\mathbf{T}_0$. The $\mathbf{G}^F$-conjugacy class of $\mathbf{T}_w$ is independent of the choice of $\dot{w}$ and g above, and the assignment $w \mapsto \mathbf{T}_w$ induces a bijection between the σ-conjugacy classes of $\mathbf{W}$ and the $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori of $\mathbf{G}$ (see, e.g., [Gec03a, 4.3.7]).

For later use it will be convenient to introduce the following notation.

Definition 2.1.18 (cf. [Lus77, 7.2])**.** Let $\mathbf{G}$ be a connected reductive algebraic group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. A closed (connected) subgroup $\mathbf{L} \subseteq \mathbf{G}$ is called a *regular subgroup* of $\mathbf{G}$ if $F(\mathbf{L}) = \mathbf{L}$ and $\mathbf{L}$ is the Levi complement of some (not necessarily F-stable) parabolic subgroup of $\mathbf{G}$.

2.1.19. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. Let us fix a pair $(\mathbf{T}_0, \mathbf{B}_0) \in \mathscr{T} \times \mathscr{B}$ consisting of a maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$ and an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{B}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$. (In particular, we have $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$.) As in 2.1.17, F induces an automorphism $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ of the Weyl group $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ of $\mathbf{G}$ with respect to $\mathbf{T}_0$. As before, we identify $\mathbf{W}$ with the Weyl group of R, using the action of $\mathbf{W}$ on X defined in 2.1.4. Let $\Pi \subseteq R$ be the base determined by $\mathbf{B}_0$, with corresponding positive roots $R^+ \subseteq R$, and let $S = \{w_\alpha \mid \alpha \in \Pi\}$ be the associated set of simple reflections in $\mathbf{W}$. Let $\alpha \in R$, and let us consider the corresponding root subgroup $\mathbf{U}_\alpha \subseteq \mathbf{G}$, as in 2.1.4. Clearly, $F(\mathbf{U}_\alpha)$ is again a one-dimensional closed connected unipotent subgroup of $\mathbf{G}$ which is normalised by $F(\mathbf{T}_0) = \mathbf{T}_0$, so $F(\mathbf{U}_\alpha) = \mathbf{U}_{\alpha^\dagger}$ for some $\alpha^\dagger \in R$. We thus obtain a bijection

$$R \to R, \quad \alpha \mapsto \alpha^\dagger.$$

On the other hand, F induces a group homomorphism

$$\varphi\colon X \to X, \quad \lambda \mapsto \lambda \circ F|_{\mathbf{T}_0},$$

and we have

$$\varphi(\alpha^\dagger) = q_\alpha \alpha \quad \text{for any } \alpha \in R,$$

where $q_\alpha \in \mathbb{N}$ are positive integers, each of which is a power of p (see [GM20, 1.3.11]). The numbers q_α ($\alpha \in R$) are called the *root exponents* of F (and also of φ), and φ is an example of a *p-isogeny of root data* in the sense of [GM20, 1.2.9]. In the case where F is a Frobenius map (and not only a Steinberg map) with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{G}$, we have $q_\alpha = q$ for all $\alpha \in R$ [GM20, 1.4.27]. Now let $\alpha \in R^+$. Then $\mathbf{U}_\alpha \subseteq \mathbf{U}_0 := R_{\mathrm{u}}(\mathbf{B}_0)$ and, since $\mathbf{B}_0$ is F-stable, we also have $\mathbf{U}_{\alpha^\dagger} \subseteq \mathbf{U}_0$. Therefore, the assignment $\alpha \mapsto \alpha^\dagger$ defines a permutation of R^+ and also of Π. Identifying $\mathbf{W} = \langle w_\alpha \mid \alpha \in R\rangle$ (as a subgroup of $\mathrm{Aut}(X)$), we get

$$\sigma(w_\alpha) = w_{\alpha^\dagger} \quad \text{for all } \alpha \in R$$

and

$$\varphi \circ \sigma(w) = w \circ \varphi \quad \text{for all } w \in \mathbf{W},$$

see [GM20, 1.2.10, 1.6.1]. In particular, we have $\sigma(S) = S$. Following [Lus84a, 3.1] (see also [GM20, 1.6.2]), the automorphism σ is called *ordinary* if for any two different elements $s \neq t$ of S which are in the same σ-orbit on S, the product st is of order 2 or 3. As in [GM20, 1.6.2], we say that the pair $(\mathbf{G}, F)$ is

$$\begin{cases} \textit{non-twisted} & \text{if } \sigma = \mathrm{id}_{\mathbf{W}}; \\ \textit{twisted} & \text{if } \sigma \neq \mathrm{id}_{\mathbf{W}} \text{ but } \sigma \text{ is ordinary}; \\ \textit{very twisted} & \text{otherwise.} \end{cases}$$

Now let us assume that $\mathbf{G}$ is a simple group. Let $d \in \mathbb{N}$ be such that F^d is a Frobenius map with respect to an $\mathbb{F}_{q_0}$-rational structure on $\mathbf{G}$ (where q_0 is a power of p), and let $q \in \mathbb{R}_{>0}$ be such that $q^d = q_0$. (The number q does not depend on the choice of d or q_0, see [GM20, 1.4.19].) Let $\mathfrak{C}$ be the (indecomposable) Cartan matrix of $\mathbf{G}$, and let $\mathfrak{D}$ be the name of the Dynkin diagram $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{C}$. If $c \in \mathbb{N}$ is the order of $\sigma \in \mathrm{Aut}(\mathbf{W})$, we say that $(\mathbf{G}, F)$ *is of type* ${}^c\mathfrak{D}$ and write $\mathbf{G}^F = {}^c\mathfrak{D}(q)$. A complete list containing all the possible types of $(\mathbf{G}, F)$ (including the order $|\mathbf{G}^F|$) for a simple group $\mathbf{G}$ is provided in [GM20, Table 1.3 (p. 73)]. Among this list, the very twisted groups are the *Suzuki groups* ${}^2\mathsf{B}_2(q)$ with $q = \sqrt{2}^{2m+1}$, and the *Ree groups* ${}^2\mathsf{F}_4(q)$ with $q = \sqrt{2}^{2m+1}$, ${}^2\mathsf{G}_2(q)$

with $q = \sqrt{3}^{2m+1}$, respectively[1]. We refer to Steinberg's lecture notes [Ste16, Chap. 11] for further details.

2.1.20. We keep the setting of 2.1.19. Given a σ-orbit κ on S, we set

$$s_\kappa := w_0^\kappa \in (\mathbf{W}_\kappa)^\sigma \subseteq \mathbf{W}^\sigma$$

where, for any subset $J \subseteq S$, w_0^J denotes the longest element in the Coxeter group $\mathbf{W}_J = \langle J \rangle \subseteq \mathbf{W}$ with respect to the length function determined by the Coxeter generators J of $\mathbf{W}_J$. Following [Car72, Chap. 13], the group $\mathbf{W}^\sigma \cong N_{\mathbf{G}}(\mathbf{T}_0)^F / \mathbf{T}_0^F$ is a finite Coxeter group with Coxeter generators

$$S_\sigma := \{s_\kappa \mid \kappa \subseteq S \text{ a } \sigma\text{-orbit}\}.$$

Of course, we just have $\mathbf{W}^\sigma = \mathbf{W}$ if $\sigma = \mathrm{id}_{\mathbf{W}}$. In the case where $\mathbf{W}$ is irreducible (that is, $\mathbf{G}$ is simple) and $\sigma \neq \mathrm{id}_{\mathbf{W}}$, the root system underlying $\mathbf{W}^\sigma$ is explicitly described in [Car72, §13.3]; in particular, we see that $\mathbf{W}^\sigma$ is itself a Weyl group if σ is ordinary. Just as $\mathbf{B}_0$ and $N_{\mathbf{G}}(\mathbf{T}_0)$ form a BN-pair in $\mathbf{G}$, with Weyl group $\mathbf{W}$ and simple reflections $S \subseteq \mathbf{W}$ (2.1.10), the groups $\mathbf{B}_0^F$ and $N_{\mathbf{G}}(\mathbf{T}_0)^F$ form a BN-pair in $\mathbf{G}^F$, with Weyl group $\mathbf{W}^\sigma$ and simple reflections $S_\sigma \subseteq \mathbf{W}^\sigma$. In particular, the Bruhat decomposition for $\mathbf{G}^F$ reads

$$\mathbf{G}^F = \biguplus_{w \in \mathbf{W}^\sigma} \mathbf{B}_0^F w \mathbf{B}_0^F$$

where $\mathbf{B}_0^F w \mathbf{B}_0^F := \mathbf{B}_0^F \dot{w} \mathbf{B}_0^F$ for any chosen representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)^F$ of $w \in \mathbf{W}^\sigma$. As described in [Car85, p. 36], given $w \in \mathbf{W}^\sigma$, it is possible to uniquely specify a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)^F$ of w (using the generators S_σ for $\mathbf{W}^\sigma$). It will sometimes be convenient to refer to this explicit choice.

Example 2.1.21. We place ourselves in the setting of 2.1.19. By [DL76, 5.21] (see also [Lus84a, 8.4], [Car85, §4.3], [GM20, 1.5.17]), the quadruple $(\mathbf{G}, \mathbf{T}_0, \mathbf{B}_0, F)$ gives rise to a *dual* quadruple $(\mathbf{G}^*, \mathbf{T}_0^*, \mathbf{B}_0^*, F')$ (we have chosen the symbol F' instead of F^* in order to avoid confusion with the inverse image functor, see Chapter 3 below), with the following ingredients: $\mathbf{G}^*$ is a group dual to $\mathbf{G}$ (see Example 2.1.6) and $F' \colon \mathbf{G}^* \to \mathbf{G}^*$ is a Steinberg map. Furthermore, $\mathbf{T}_0^* \subseteq \mathbf{G}^*$ is an F'-stable maximal torus of $\mathbf{G}^*$ and $\mathbf{B}_0^* \subseteq \mathbf{G}^*$ is an F'-stable Borel subgroup of $\mathbf{G}^*$ containing $\mathbf{T}_0^*$, such that the pair $(\mathbf{T}_0^*, \mathbf{B}_0^*)$

[1] We decided to conform with the notation for algebraic groups. As far as the Suzuki and Ree groups are concerned, this differs from the conventions in terms of finite group theory, where one would rather write ${}^2\mathsf{B}_2(q^2)$ for the Suzuki groups and ${}^2\mathsf{F}_4(q^2)$, ${}^2\mathsf{G}_2(q^2)$ for the Ree groups.

satisfies the following property: There exists an isomorphism

$$\delta\colon X(\mathbf{T}_0) \xrightarrow{\sim} Y(\mathbf{T}_0^*) \tag{2.1.21.1}$$

which maps the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}_0$ onto the set of co-roots of $\mathbf{G}^*$ with respect to $\mathbf{T}_0^*$, and the set of simple roots of $\mathbf{G}$ determined by $(\mathbf{T}_0, \mathbf{B}_0)$ onto the set of simple co-roots of $\mathbf{G}^*$ determined by $(\mathbf{T}_0^*, \mathbf{B}_0^*)$. Finally, we require that

$$\delta(\lambda \circ F|_{\mathbf{T}_0}) = F'|_{\mathbf{T}_0^*} \circ \delta(\lambda) \quad \text{for all } \lambda \in X(\mathbf{T}_0).$$

Let $\mathbf{W}^* = {}^{N_{\mathbf{G}^*}(\mathbf{T}_0^*)}/_{\mathbf{T}_0^*}$ be the Weyl group of $\mathbf{G}^*$ with respect to $\mathbf{T}_0^*$. By [Car85, 4.2.3], the isomorphism (2.1.21.1) induces a group isomorphism

$$\mathbf{W} \xrightarrow{\sim} \mathbf{W}^*, \quad w \mapsto w^*, \tag{2.1.21.2}$$

uniquely determined by the condition that for any root α of $\mathbf{G}$ and its associated reflection $w_\alpha \in \mathbf{W}$, $w_\alpha^* \in \mathbf{W}^*$ is the reflection associated to the co-root $\delta(\alpha)$ of $\mathbf{G}^*$. This isomorphism satisfies

$$\delta(w.\lambda) = w^*.\delta(\lambda) \quad \text{for any } w \in \mathbf{W},\ \lambda \in X(\mathbf{T}_0),$$

where the actions of $\mathbf{W}$ on $X(\mathbf{T}_0)$ and of $\mathbf{W}^*$ on $Y(\mathbf{T}_0^*)$ are defined as in 2.1.4. Moreover, under the isomorphism (2.1.21.2), the automorphism of $\mathbf{W}$ induced by F corresponds to the inverse of the automorphism of $\mathbf{W}^*$ induced by F' (see [Car85, 4.3.2]), that is, we have

$$\sigma(w)^* = {\sigma'}^{-1}(w^*) \quad \text{for any } w \in \mathbf{W},$$

where $\sigma'\colon \mathbf{W}^* \xrightarrow{\sim} \mathbf{W}^*$ denotes the automorphism induced by $F'\colon \mathbf{G}^* \to \mathbf{G}^*$.

2.2. Lusztig's classification of irreducible characters (the connected centre case)

We will be concerned with the ordinary representation theory of the finite groups of Lie type $\mathbf{G}^F$, that is, with representations (and characters) of $\mathbf{G}^F$ over an algebraically closed field of characteristic zero. It is well known that it does not really matter which exact algebraically closed field of characteristic zero we take for this purpose. While in general the field $\mathbb{C}$ of complex numbers would be the canonical choice, when dealing with finite groups of Lie type, the standard field to work with is an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers $\mathbb{Q}_\ell$ (where, as soon as the field $k = \overline{\mathbb{F}}_p$ is prescribed, ℓ is tacitly assumed to be any fixed prime number different from p), so that the typical tools from algebraic geometry or topology can be applied (see, e.g., [DL76], [Lus78]). Recall from the introduction that, although not strictly necessary, we assume the existence of an isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ (1.1.0.1). Given a finite group Γ, let $\mathrm{CF}(\Gamma)$ be the set of class functions $\Gamma \to \overline{\mathbb{Q}}_\ell$, and let us denote by

$$\langle f, f' \rangle_\Gamma := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} f(g)\overline{f'(g)} \quad (\text{for } f, f' \in \mathrm{CF}(\Gamma)) \tag{2.2.0.1}$$

the standard scalar product on $\mathrm{CF}(\Gamma)$, where the bar denotes the field automorphism of $\overline{\mathbb{Q}}_\ell$ which corresponds to complex conjugation under the isomorphism (1.1.0.1). (The only property of this field automorphism that we actually need is that it maps any root of unity in $\overline{\mathbb{Q}}_\ell$ to its inverse.) Let $\mathrm{Irr}(\Gamma) \subseteq \mathrm{CF}(\Gamma)$ be the subset of irreducible characters of Γ. They form an orthonormal basis of $\mathrm{CF}(\Gamma)$ with respect to the scalar product (2.2.0.1).

Fundamental induction processes for characters

2.2.1. A vital role in the character theory of finite groups of Lie type is played by the ℓ-adic cohomology attached to suitable algebraic varieties (or, more generally, schemes), due to SGA 4 [AGV73], SGA $4\frac{1}{2}$ [Del77], SGA 5 [Gro77], see also [Sri79, Chap. V], the appendix of [Car85], and the further references there. So let us consider an algebraic variety X over $k = \overline{\mathbb{F}}_p$. As mentioned in [Lus78, 1.2] (see also [Car85, §7.1], [GM20, 2.2.1]), it is a very deep result that one can canonically attach to X a family of $\overline{\mathbb{Q}}_\ell$-vector spaces $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ (for $i \in \mathbb{Z}$), called ℓ-adic cohomology groups with compact support. Each of these vector spaces is finite-dimensional, and we have $H^i_c(X, \overline{\mathbb{Q}}_\ell) = \{0\}$ for $i < 0$ and for $i > 2\dim X$. They are functorial in the sense that any finite morphism $f\colon X \to X'$ of algebraic varieties induces a linear map $f^*\colon H^i_c(X', \overline{\mathbb{Q}}_\ell) \to H^i_c(X, \overline{\mathbb{Q}}_\ell)$, for each $i \in \mathbb{Z}$. Assume that we are given a finite group Γ which acts on X via the homomorphism

$$\Theta\colon \Gamma \to \mathrm{Aut}(X), \quad g \mapsto \Theta_g$$

(where $\mathrm{Aut}(X)$ denotes the group of all algebraic automorphisms of X). Then each $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ ($i \in \mathbb{Z}$) has the structure of a Γ-module, with corresponding representation given by

$$\Gamma \to \mathrm{GL}\big(H^i_c(X, \overline{\mathbb{Q}}_\ell)\big), \quad g \mapsto \big(\Theta_{g^{-1}}\big)^*.$$

The *Lefschetz number* of $g \in \Gamma$ with respect to such an action is defined by

$$\mathfrak{L}(g, X) := \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Trace}((\Theta_{g^{-1}})^*, H_c^i(X, \overline{\mathbb{Q}}_\ell)).$$

This number does not depend on the choice of $\ell \neq p$, and we have

$$\mathfrak{L}(g, X) = \mathfrak{L}(g^{-1}, X) \in \mathbb{Z} \quad \text{for all } g \in \Gamma.$$

For more properties related to the ℓ-adic cohomology groups with compact support, we refer to [Car85, §7.1], see also [Lus78, §1], [DM20, Chap. 8].

2.2.2. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. The following construction based on the ℓ-adic cohomology groups with compact support will give rise to the definition of the virtual characters of Deligne and Lusztig in [DL76] (see Definition 2.2.3 below), which in turn are a central ingredient for Lusztig's classification of irreducible characters of $\mathbf{G}^F$ [Lus84a], [Lus88]. It is very convenient to use the model in [GM20, 2.2.5], as it allows the definition of Lusztig induction in a similar way (see Definition 2.2.28 below). We thus follow [GM20, §2.2] here.

Recall the definition of the Lang–Steinberg map $\mathcal{L}\colon \mathbf{G} \to \mathbf{G}$ in Theorem 2.1.14. For a closed subset $\mathbf{Y} \subseteq \mathbf{G}$, the preimage $\mathcal{L}^{-1}(\mathbf{Y})$ is a closed subset of $\mathbf{G}$ which is stable under left multiplication by any element of $\mathbf{G}^F$. Assume that we are given a finite subgroup H of $\mathbf{G}$ which satisfies the property

$$h^{-1}\mathbf{Y}F(h) \subseteq \mathbf{Y} \quad \text{for all } h \in H.$$

Then $\mathcal{L}^{-1}(\mathbf{Y})$ is invariant under right multiplication by any element of H. Thus, we obtain an (algebraic) action of $\mathbf{G}^F \times H$ on $\mathcal{L}^{-1}(\mathbf{Y})$, defined by

$$(\mathbf{G}^F \times H) \times \mathcal{L}^{-1}(\mathbf{Y}) \to \mathcal{L}^{-1}(\mathbf{Y}), \quad ((g, h), x) \mapsto gxh^{-1}.$$

This gives rise to an induced action of $\mathbf{G}^F \times H$ on the $\overline{\mathbb{Q}}_\ell$-vector space $H_c^i(\mathcal{L}^{-1}(\mathbf{Y}), \overline{\mathbb{Q}}_\ell)$, which makes the latter a module for the finite group $\mathbf{G}^F \times H$. For $\theta \in \mathrm{CF}(H)$, we may thus define (see [GM20, 2.2.5])

$$R_{H,\mathbf{Y}}^{\mathbf{G}}(\theta)\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell, \quad g \mapsto \frac{1}{|H|} \sum_{h \in H} \mathfrak{L}((g, h), \mathcal{L}^{-1}(\mathbf{Y}))\theta(h).$$

We have $R_{H,\mathbf{Y}}^{\mathbf{G}}(\theta) \in \mathrm{CF}(\mathbf{G}^F)$. If θ is a *virtual character* of H (that is, a $\mathbb{Z}$-linear combination of irreducible characters of H), then $R_{H,\mathbf{Y}}^{\mathbf{G}}(\theta)$ is a virtual character of $\mathbf{G}^F$.

If $\mathbf{H}$ is an F-stable closed subgroup of $\mathbf{G}$ and $H := \mathbf{H}^F$, it will be convenient to set

$$R^{\mathbf{G}}_{\mathbf{H},\mathbf{Y}}(\theta) := R^{\mathbf{G}}_{H,\mathbf{Y}}(\theta) \quad \text{for } \theta \in \mathrm{CF}(H).$$

Definition 2.2.3 (Deligne–Lusztig [DL76], Lusztig [Lus78])**.** Let $\mathbf{G}$ be a connected reductive group over k and $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. Let us fix a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{G}$, and let $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$.

(a) Let $\mathbf{T} \subseteq \mathbf{G}$ be an F-stable maximal torus, and let $\mathbf{B} \subseteq \mathbf{G}$ be a (not necessarily F-stable) Borel subgroup such that $\mathbf{T} \subseteq \mathbf{B}$. We denote by $\mathbf{U} := R_{\mathrm{u}}(\mathbf{B})$ the unipotent radical of $\mathbf{B}$. Then $\mathbf{Y} := \mathbf{U}$ and $H := \mathbf{T}^F$ meet the requirements of 2.2.2. For $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, we set

$$R^{\mathbf{G}}_{\mathbf{T}}(\theta) := R^{\mathbf{G}}_{\mathbf{T},\mathbf{U}}(\theta),$$

which is referred to as the *virtual character of Deligne–Lusztig* with respect to $\mathbf{T}, \theta$. (It is justified to omit $\mathbf{U}$ from the notation since the definition turns out to be independent of the choice of $\mathbf{B}$, see [DL76, 4.3] or [Lus78, 2.4].) By a slight abuse of notation, we will often omit the word 'virtual' and just speak of *the Deligne–Lusztig characters*. If $w \in \mathbf{W}$ and $\mathbf{T}_w \subseteq \mathbf{G}$ is a torus of type w with respect to $\mathbf{T}_0$ (see 2.1.17), it will be convenient to write

$$R_w := R^{\mathbf{G}}_w := R^{\mathbf{G}}_{\mathbf{T}_w}(1_{\mathbf{T}_w^F}).$$

(We will omit $\mathbf{G}$ from the notation when it is clear from the context.)

(b) An irreducible character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is called *unipotent* if there exists some $w \in \mathbf{W}$ such that $\langle \rho, R_w \rangle_{\mathbf{G}^F} \neq 0$. We denote the set of unipotent characters of $\mathbf{G}^F$ by

$$\mathrm{Uch}(\mathbf{G}^F) \subseteq \mathrm{Irr}(\mathbf{G}^F).$$

2.2.4. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. We collect some of the most important properties of the Deligne–Lusztig characters with regard to the irreducible characters of $\mathbf{G}^F$ here. (They are all contained in [DL76] and [Lus78]; see also the textbooks [Car85, Chap. 7], [DM20, Chap. 9] or [GM20, §2.2].) First of all, for every $\rho \in \mathrm{Irr}(\mathbf{G}^F)$, there exists some F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and some $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ such that

$$\langle \rho, R^{\mathbf{G}}_{\mathbf{T}}(\theta) \rangle_{\mathbf{G}^F} \neq 0.$$

An irreducible character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is called *uniform* if it can be written as a linear combination of $R^{\mathbf{G}}_{\mathbf{T}_i}(\theta_i)$ for suitable F-stable maximal tori $\mathbf{T}_i \subseteq \mathbf{G}$ and $\theta_i \in \mathrm{Irr}(\mathbf{T}_i^F)$. The values of a Deligne–Lusztig character $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ at unipotent elements of $\mathbf{G}^F$ are shown to be independent of $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, that is, for any F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$, we have

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(u) = R_{\mathbf{T}}^{\mathbf{G}}(1)(u) \quad \text{for any } u \in \mathbf{G}_{\text{uni}}^F \text{ and any } \theta \in \operatorname{Irr}(\mathbf{T}^F).$$

(Here, 1 denotes the trivial character of $\mathbf{T}^F$.) The functions

$$Q_{\mathbf{T}}^{\mathbf{G}} \colon \mathbf{G}_{\text{uni}}^F \to \overline{\mathbb{Q}}_\ell, \quad u \mapsto R_{\mathbf{T}}^{\mathbf{G}}(1)(u) \tag{2.2.4.1}$$

(where $\mathbf{T}$ runs over the F-stable maximal tori of $\mathbf{G}$) are called the *Green functions* of $\mathbf{G}^F$. We have $Q_{\mathbf{T}}^{\mathbf{G}}(u) \in \mathbb{Z}$ for all $u \in \mathbf{G}_{\text{uni}}^F$. If $\mathbf{T} = \mathbf{T}_w$ is a torus of type w, one also sometimes writes $Q_w := Q_{\mathbf{T}_w}^{\mathbf{G}}$. The importance of the Green functions is highlighted by the following character formula. Let $g \in \mathbf{G}^F$, and let $g = su = us$ be the Jordan decomposition of g (with $s \in \mathbf{G}^F$ semisimple and $u \in \mathbf{G}^F$ unipotent). Setting $\mathbf{H} := C_{\mathbf{G}}^\circ(s)$, we have

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|\mathbf{H}^F|} \sum_{\substack{x \in \mathbf{G}^F \\ x^{-1}sx \in \mathbf{T}^F}} \theta(x^{-1}sx) Q_{x\mathbf{T}x^{-1}}^{\mathbf{H}}(u). \tag{2.2.4.2}$$

Next, the Deligne–Lusztig characters satisfy the following 'scalar product formula', see [GM20, 2.2.8]: For any two F-stable maximal tori $\mathbf{T}, \mathbf{T}' \subseteq \mathbf{G}$ and any two irreducible characters $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, $\theta' \in \operatorname{Irr}(\mathbf{T}'^F)$, we have

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta') \rangle_{\mathbf{G}^F} = \frac{1}{|\mathbf{T}^F|} \cdot \left| \left\{ g \in \mathbf{G}^F \;\middle|\; g\mathbf{T}g^{-1} = \mathbf{T}' \text{ and } {}^g\theta = \theta' \right\} \right|$$

where

$${}^g\theta \colon \mathbf{T}'^F \to \overline{\mathbb{Q}}_\ell, \quad t' \mapsto \theta(g^{-1}t'g).$$

The scalar product formula has several immediate consequences. Namely, one deduces that two Deligne–Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are equal if and only if there exists some $g \in \mathbf{G}^F$ such that $g\mathbf{T}g^{-1} = \mathbf{T}'$ and ${}^g\theta = \theta'$; in any other case, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ are orthogonal to each other. Furthermore, we see that if $\theta \in \operatorname{Irr}(\mathbf{T}^F)$ is *in general position*, that is, we have ${}^g\theta \neq \theta$ for all $g \in N_{\mathbf{G}}(\mathbf{T})^F \setminus \mathbf{T}^F$, either $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ or $-R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible character of $\mathbf{G}^F$.

2.2.5 Green functions. In view of the character formula (2.2.4.2), the computation of the Deligne–Lusztig characters at arbitrary elements of $\mathbf{G}^F$ is reduced to that of the Green functions of finite reductive groups contained in $\mathbf{G}^F$ (see [Gec21, §2, §3] for information on the technical task of evaluating the coefficients of the Green functions in (2.2.4.2)). It is thus no surprise that a lot of work has been dedicated to the computation of the Green functions of finite groups of Lie type.

In good characteristic, based on a method due to Shoji [Sho82], the determination of the Green functions has been completely realised already in the 1980s (see Shoji [Sho82] for $\mathbf{G}$ of type F_4, [Sho83] for $\mathbf{G}$ of classical type, and Beynon–Spaltenstein [BS84] for $\mathbf{G}$ of type E_6, E_7 and E_8). However, this approach does not cover the cases where p is a bad prime for $\mathbf{G}$.

The obvious advantage of dealing with small primes is that the size of the group $\mathbf{G}^F$ is comparatively small, so one can try to take a simple group $\mathbf{G}$ and perform ad hoc computations in order to get information on character values; in this way, the Green functions have been determined for groups of type F_4, ${}^2\mathsf{F}_4$ and E_6 in characteristic 2 by Malle [Mal90], [Mal93], and for groups of type E_6 in characteristic 3 by Porsch [Por93].

But in order to obtain a uniform result for the infinite series of classical groups (or also for the big exceptional groups of type E_7 or E_8), it seems that more general methods are required. To this end, Lusztig defined another kind of Green functions [LuCS5, §24], using his theory of character sheaves, which we will introduce in Chapter 3 below. Up to certain signs, these new Green functions are computable by a purely combinatorial algorithm, which modifies and simplifies Shoji's approach mentioned above; see the explanations in Shoji's survey [Sho87a]. In the case where $F\colon \mathbf{G} \to \mathbf{G}$ is the Frobenius map for an $\mathbb{F}_q$-rational structure on $\mathbf{G}$ (for a power q of p) and if q is large enough (but with no restriction on p), Lusztig showed that these new Green functions coincide with the original ones multiplied by $(-1)^{\dim \mathbf{T}_0}$ ([Lus90, Thm. 1.14], see also the remarks in [Sho95a, 1.12]); this result was then proven to hold for arbitrary p and q by Shoji, see [Sho95a, Thm. 2.2], [Sho95b, Thm. 5.5].

So it remains to determine the unknown signs appearing in Lusztig's above-mentioned algorithm. This has been accomplished in most cases (but still not in full generality): Shoji [Sho06b], [Sho07], [Sho22] completed the computation of Green functions for any classical group (in characteristic $p = 2$). As far as groups of type F_4 are concerned, it is noted in Marcelo–Shinoda [MS95] that Shoji's tables for the Green functions are valid for $p = 3$ as well. The groups ${}^2\mathsf{E}_6(p^n)$ and $\mathsf{E}_7(p^n)$ for $p \leqslant 3$ and $n \in \mathbb{N}$ have been settled recently by Geck [Gec20b], by showing how the results of Lusztig and Shoji can be reduced to the case where $p = q$ and then using computer algebra methods; [Gec20b] also provides an independent verification of the results for $\mathsf{E}_6(3^n)$ and $\mathsf{F}_4(3^n)$. At least one previously open case for $\mathsf{E}_8(2^n)$ is solved in [Gec20b, §9] as well, but the complete computation of the Green functions for the groups $\mathsf{E}_8(2^n)$, $\mathsf{E}_8(3^n)$ and $\mathsf{E}_8(5^n)$ has not yet been realised.

2.2.6 Harish-Chandra series. Before introducing the necessary set-up of [Lus84a] to be able to state Lusztig's 'Main Theorem 4.23' (see Theorem 2.2.21 below), let us briefly mention another approach towards classifying the irreducible characters of finite groups of Lie type, namely that of *Harish-Chandra theory*.

Let $\mathbf{G}$ be a connected reductive group over k and $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. Let us fix a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{G}$ as well as an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ which contains $\mathbf{T}_0$. Thus, we obtain the Weyl group $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ of $\mathbf{G}$ with respect to $\mathbf{T}_0$ and denote by $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ the automorphism induced by F. Let $S \subseteq \mathbf{W}$ be the simple reflections determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$.

- If $\mathbf{L} \subseteq \mathbf{G}$ is an F-stable closed subgroup which is the Levi complement of some F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, then $\mathbf{P}^F$ is the semi-direct product $R_{\mathrm{u}}(\mathbf{P})^F \rtimes \mathbf{L}^F$ (see 2.1.10), so the canonical projection map $\mathbf{P}^F \to \mathbf{L}^F$ gives rise to the inflation map
$$\mathrm{Infl}_{\mathbf{L}^F}^{\mathbf{P}^F}\colon\ \mathrm{CF}(\mathbf{L}^F) \to \mathrm{CF}(\mathbf{P}^F),$$
which sends (irreducible) characters of $\mathbf{L}^F$ to (irreducible) characters of $\mathbf{P}^F$ (cf. [GM20, 2.1.3]). The *Harish-Chandra induction* from $\mathbf{L}^F$ to $\mathbf{G}^F$ (on the level of characters, or class functions) is defined as
$$R_{\mathbf{L}}^{\mathbf{G}} := \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \circ \mathrm{Infl}_{\mathbf{L}^F}^{\mathbf{P}^F}\colon\ \mathrm{CF}(\mathbf{L}^F) \to \mathrm{CF}(\mathbf{G}^F).$$
It is justified to omit $\mathbf{P}$ from the notation since $R_{\mathbf{L}}^{\mathbf{G}}$ turns out to be independent of the chosen F-stable parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ having $\mathbf{L}$ as a Levi complement, see [DM20, 5.3.1]. Clearly, $R_{\mathbf{L}}^{\mathbf{G}}$ sends characters of $\mathbf{L}^F$ to characters of $\mathbf{G}^F$, being the composition of two maps with the analogous properties. There is a notational overlap with the Deligne–Lusztig characters if $\mathbf{L}$ happens to be a maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$, but this is no issue since these two induction concepts coincide in this case (see [Car85, 7.2.4]).

- An irreducible character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is called *cuspidal* if $\langle \rho, R_{\mathbf{L}}^{\mathbf{G}}(\rho') \rangle_{\mathbf{G}^F} = 0$ for any F-stable closed subgroup $\mathbf{L} \subseteq \mathbf{G}$ which is the Levi complement of some *proper* F-stable parabolic subgroup of $\mathbf{G}$, and any $\rho' \in \mathrm{Irr}(\mathbf{L}^F)$. We denote by $\mathrm{Irr}(\mathbf{G}^F)^\circ \subseteq \mathrm{Irr}(\mathbf{G}^F)$ the subset of cuspidal (irreducible) characters of $\mathbf{G}^F$.

- For any $\rho \in \mathrm{Irr}(\mathbf{G}^F)$, there exists some F-stable closed subgroup $\mathbf{L} \subseteq \mathbf{G}$ which is the Levi complement of some F-stable parabolic subgroup of $\mathbf{G}$, and a cuspidal character $\rho_0 \in \mathrm{Irr}(\mathbf{L}^F)^\circ$ such that $\langle \rho, R_{\mathbf{L}}^{\mathbf{G}}(\rho_0) \rangle_{\mathbf{G}^F} \neq 0$. In fact, $\mathbf{L}$ can always be chosen to be a standard Levi subgroup $\mathbf{L}_J$ for some σ-stable $J \subseteq S$, and then ρ determines $\mathbf{L}_J$ and ρ_0 uniquely up to simultaneous conjugation with an element of

$\mathbf{W}$, see [Car85, Chap. 9]. This gives rise to a partition of $\mathrm{Irr}(\mathbf{G}^F)$ into so-called *Harish-Chandra series*, one for each 'cuspidal pair' (cf. [DM20, §5.3]) $(\mathbf{L}_J, \rho_0)$ as above up to $\mathbf{W}$-conjugacy.

Hence, in order to obtain a parametrisation of $\mathrm{Irr}(\mathbf{G}^F)$ in terms of Harish-Chandra series, one needs to know the cuspidal characters for any finite reductive group which is of the form $\mathbf{L}_J^F$ as above and, for any $\rho_0 \in \mathrm{Irr}(\mathbf{L}_J^F)^\circ$, one has to find the decomposition of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ into irreducible characters of $\mathbf{G}^F$. The latter can be tackled using the *Howlett–Lehrer theory* [HL80], which is based on the fact that there is a bijection between the irreducible constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ (counted without multiplicity) and the irreducible representations (up to isomorphism) of the $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-endomorphism algebra of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$, for any 'cuspidal pair' $(\mathbf{L}_J, \rho_0)$ as above (see, e.g., [Car85, Chap. 10]). Now there is a natural action of $N_{\mathbf{G}}(\mathbf{L}_J)^F$ on $\mathrm{Irr}(\mathbf{L}_J^F)$, defined by

$$N_{\mathbf{G}}(\mathbf{L}_J)^F \times \mathrm{Irr}(\mathbf{L}_J^F) \to \mathrm{Irr}(\mathbf{L}_J^F), \quad (n, \chi) \mapsto {}^n\chi,$$

where

$${}^n\chi \colon \mathbf{L}_J^F \to \overline{\mathbb{Q}}_\ell, \quad x \mapsto \chi(n^{-1}xn),$$

and this action leaves the set $\mathrm{Irr}(\mathbf{L}_J^F)^\circ$ invariant. It is also clear that ${}^x\chi = \chi$ for any $x \in \mathbf{L}_J^F$ and any $\chi \in \mathrm{Irr}(\mathbf{L}_J^F)$, so we obtain an induced action

$$N_{\mathbf{G}}(\mathbf{L}_J)^F/\mathbf{L}_J^F \times \mathrm{Irr}(\mathbf{L}_J^F) \to \mathrm{Irr}(\mathbf{L}_J^F). \tag{2.2.6.1}$$

Due to results of [HL80], [Lus84a, Chap. 8], [Gec93], it is known that the $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-endomorphism algebra of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ is always isomorphic to the group algebra (over $\overline{\mathbb{Q}}_\ell$) of the stabiliser of ρ_0 in $N_{\mathbf{G}}(\mathbf{L}_J)^F/\mathbf{L}_J^F$, so the irreducible constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ (counted without multiplicity) are parametrised by the irreducible characters of this stabiliser. In the special case where $\mathbf{L} = \mathbf{T}_0$ and $\rho_0 = 1_{\mathbf{T}_0^F}$ is the trivial character of $\mathbf{T}_0^F$, this leads to the so-called *principal series of unipotent characters* of $\mathbf{G}^F$, that is, the set of all irreducible constituents of $\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F}(1_{\mathbf{B}_0^F})$. The unipotent characters of $\mathbf{G}^F$ in the principal series are thus parametrised by $\mathrm{Irr}(\mathbf{W}^\sigma)$. In this situation, we will give a precise description of said parametrisation in Section 2.3 below. For a detailed coverage of Harish-Chandra theory in general, we refer to the literature (e.g., [DM20, Chap. 5], [GM20, Chap. 3]). However, note that this approach does not give any information on the cuspidal characters of connected reductive groups themselves, so one needs another method to determine the cuspidal characters of a given connected reductive group $\mathbf{G}$ (with Steinberg map $F \colon \mathbf{G} \to \mathbf{G}$). Such a criterion is provided by Lusztig in [Lus78, 2.18], using the Deligne–Lusztig characters: An irreducible character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is cuspidal if and only if for any F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ which is contained in a *proper* F-stable parabolic subgroup $\mathbf{P} \subsetneq \mathbf{G}$, we have

$$\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F} = 0 \quad \text{for any } \theta \in \mathrm{Irr}(\mathbf{T}^F). \tag{$*$}$$

The multiplicities of the irreducible characters of $\mathbf{G}^F$ in a given Deligne–Lusztig character are explicitly known, thanks to results of Lusztig. (The case where the centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$ is connected is contained in [Lus84a, Main Theorem 4.23], which we will formulate in Theorem 2.2.21 below; this result is extended to the non-connected centre case in [Lus88].) In particular, the condition $(*)$ can be checked effectively, and this will provide the list of cuspidal characters of $\mathbf{G}^F$.

Lusztig's parameter sets associated to Weyl groups and the non-abelian Fourier transform

2.2.7. Let Γ be a finite group, and let us assume that we are given a group automorphism $\gamma\colon \Gamma \xrightarrow{\sim} \Gamma$. Let $d \in \mathbb{N}$ be the order of $\gamma \in \mathrm{Aut}(\Gamma)$. Thus, we may consider the semidirect product

$$\Gamma(\gamma) := \Gamma \rtimes \langle\gamma\rangle \quad \text{where } \gamma \cdot g \cdot \gamma^{-1} = \gamma(g) \quad \text{for any } g \in \Gamma. \tag{2.2.7.1}$$

This uniquely determines a group structure for $\Gamma(\gamma)$ (a finite group, the underlying set being the Cartesian product of Γ and the cyclic group $\langle\gamma\rangle \cong C_d$, such that the usual inclusion maps $\Gamma \hookrightarrow \Gamma(\gamma)$ and $\langle\gamma\rangle \hookrightarrow \Gamma(\gamma)$ are both group homomorphisms). Following [GM20, 2.1.6, 2.1.9], we introduce some basic notions concerning extensions of characters of Γ to $\Gamma(\gamma)$. Consider the map

$$\mathrm{Irr}(\Gamma) \to \mathrm{Irr}(\Gamma), \quad \chi \mapsto \chi^\gamma := \chi \circ \gamma, \tag{2.2.7.2}$$

and denote by

$$\mathrm{Irr}(\Gamma)^\gamma := \{\chi \in \mathrm{Irr}(\Gamma) \mid \chi^\gamma = \chi\}$$

the set of γ-fixed points. Let $\chi \in \mathrm{Irr}(\Gamma)^\gamma$, and let $\Theta\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ (where $n = \chi(1)$) be a matrix representation which affords the character χ. Then $\Theta \circ \gamma$ also affords χ, so there exists an invertible matrix $E \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that

$$\Theta(\gamma(g)) = E \cdot \Theta(g) \cdot E^{-1} \quad \text{for all } g \in \Gamma.$$

By Schur's Lemma, this condition determines E up to multiplication with a non-zero scalar. In fact, E can be chosen in such a way that E^d is the identity matrix in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ (one can argue similarly as in the proof of [Fei82, III.2.14]), and then E is unique up to multiplication with a dth root of unity of $\overline{\mathbb{Q}}_\ell$. Unless otherwise stated, we will always assume that E is chosen in this way in such a set-up. The map

$$\tilde{\chi}\colon \Gamma \to \overline{\mathbb{Q}}_\ell, \quad g \mapsto \mathrm{Trace}(\Theta(g)\cdot E) = \mathrm{Trace}(E\cdot\Theta(g)).$$

is called a *γ-extension* of χ. This notion is justified since the set $\mathrm{Irr}(\Gamma)^\gamma$ consists precisely of those irreducible characters of Γ which can be extended to a character of $\Gamma(\gamma)$, and any such extension will automatically be an irreducible character of $\Gamma(\gamma)$: Namely, if Θ, E are as above (coming from $\chi \in \mathrm{Irr}(\Gamma)^\gamma$), it is easy to check that

$$\Gamma(\gamma) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell), \quad (g, \gamma^i) \mapsto \Theta(g)\cdot E^i,$$

is a well-defined representation of $\Gamma(\gamma)$, whose restriction to Γ is Θ. The d possible choices for E thus lead to d different extensions of $\chi \in \mathrm{Irr}(\Gamma)^\gamma$ to $\Gamma(\gamma)$, and any extension of χ arises in this way. By a slight abuse of notation, we will sometimes tacitly identify the γ-extension $\tilde{\chi}$ with the corresponding extension of χ to $\Gamma(\gamma)$.

2.2.8. Let W be the Weyl group of a reduced crystallographic root system R in the finite-dimensional $\mathbb{R}$-vector space V spanned by R. We assume that a set of simple roots in R has been fixed and denote by $S \subseteq W$ the corresponding set of simple reflections in W. In [Lus84a, 4.4–4.13], Lusztig describes a partition of the irreducible characters of W into families, which relies on a case-by-case investigation of the different irreducible Coxeter groups (W, S). To each family $\mathcal{F} \subseteq \mathrm{Irr}(W)$ is associated a certain finite group $\mathcal{G} = \mathcal{G}_\mathcal{F}$ and a set $\mathfrak{M}(\mathcal{G})$, defined as

$$\mathfrak{M}(\mathcal{G}) := \{(g, \sigma) \mid g \in \mathcal{G},\, \sigma \in \mathrm{Irr}(C_\mathcal{G}(g))\}/_\sim \tag{2.2.8.1}$$

where $\sim$ denotes the equivalence relation given by

$$(g, \sigma) \sim (hgh^{-1}, {}^h\sigma) \quad \text{for } h \in \mathcal{G},\, \sigma \in \mathrm{Irr}(C_\mathcal{G}(g)),$$

with ${}^h\sigma$ being the irreducible character of $C_\mathcal{G}(hgh^{-1}) = hC_\mathcal{G}(g)h^{-1}$ defined by composing σ with conjugation by h^{-1}. The set $\mathfrak{M}(\mathcal{G})$ is equipped with a pairing

$$\{\ ,\ \}\colon \mathfrak{M}(\mathcal{G}) \times \mathfrak{M}(\mathcal{G}) \to \overline{\mathbb{Q}}_\ell,$$

defined by

$$\{(g,\sigma),(h,\tau)\} := \frac{1}{|C_{\mathcal{G}}(g)|}\frac{1}{|C_{\mathcal{G}}(h)|}\sum_{\substack{x\in\mathcal{G}\\ gxhx^{-1}=xhx^{-1}g}} \tau(x^{-1}g^{-1}x)\sigma(xhx^{-1}), \qquad (2.2.8.2)$$

for $(g,\sigma),(h,\tau)\in\mathfrak{M}(\mathcal{G})$. Then, for any family $\mathcal{F}\subseteq \mathrm{Irr}(W)$, Lusztig specifies an embedding $\mathcal{F}\hookrightarrow\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$. Setting

$$\mathfrak{X}(W) := \coprod_{\substack{\mathcal{F}\subseteq\mathrm{Irr}(W)\\ \text{family}}} \mathfrak{M}(\mathcal{G}_{\mathcal{F}}), \qquad (2.2.8.3)$$

we therefore obtain an embedding

$$\mathrm{Irr}(W)\hookrightarrow\mathfrak{X}(W), \quad \phi\mapsto x_\phi. \qquad (2.2.8.4)$$

When no confusion may arise, we will sometimes write $\mathrm{Irr}(W)\subseteq\mathfrak{X}(W)$ or even $\phi\in\mathfrak{X}(W)$, thus tacitly referring to the embedding (2.2.8.4). The pairing $\{\ ,\ \}$ is extended to $\mathfrak{X}(W)\times\mathfrak{X}(W)$ in such a way that $\{(g,\sigma),(h,\tau)\}:=0$ whenever (g,σ), $(h,\tau)\in\mathfrak{X}(W)$ are not in the same $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$. Note that $\mathfrak{X}(W)$ is a finite set (whose definition also depends on R, S, but we omit them from the notation since we will always keep them fixed in a given setting). When we are given a concrete irreducible Weyl group W (of exceptional type) later, we will sometimes denote a family of $\mathrm{Irr}(W)$ by $\mathcal{F}_a$ where $a=a_\phi\in\mathbb{N}_0$ is as defined in [Lus84a, 4.1] for $\phi\in\mathcal{F}_a$ (this does not depend on the choice of $\phi\in\mathcal{F}_a$), cf. [Lus80, 1.7]; the number a often uniquely specifies the family $\mathcal{F}_a$, although not in general.

Definition 2.2.9 (Lusztig [Lus79, §4])**.** In the setting and with the notation of 2.2.8, let $\mathcal{F}\subseteq\mathrm{Irr}(W)$ be a family of irreducible characters of W. Assuming that an order of the elements in $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ has been fixed, the matrix

$$\Upsilon_{\mathcal{F}} := (\{x,y\})_{x,y\in\mathfrak{M}(\mathcal{G}_{\mathcal{F}})}$$

is called 'the' *Fourier matrix of the family* $\mathcal{F}$ (or *of the group* $\mathcal{G}_{\mathcal{F}}$). ($\Upsilon_{\mathcal{F}}$ is thus only determined up to simultaneously permuting its rows and columns.) If we fix an order $\mathcal{F}^1,\mathcal{F}^2,\ldots,\mathcal{F}^s$ of the families in $\mathrm{Irr}(W)$ and, for each such $\mathcal{F}^i$, an order of the elements in $\mathfrak{M}(\mathcal{G}_{\mathcal{F}^i})$, we obtain a block diagonal matrix

$$\Upsilon_W := \begin{pmatrix} \Upsilon_{\mathcal{F}^1} & & 0 \\ & \ddots & \\ 0 & & \Upsilon_{\mathcal{F}^s} \end{pmatrix}$$

of size $|\mathfrak{X}(W)| \times |\mathfrak{X}(W)|$. But even without having specified an order of the $\mathcal{F}^i$ or of the elements inside each $\mathfrak{M}(\mathcal{G}_{\mathcal{F}^i})$, we will (by a slight abuse of notation) still speak of 'the' *Fourier matrix* Υ_W *of* W. (So Υ_W is only determined up to the order of the blocks $\Upsilon_{\mathcal{F}^i}$ and up to simultaneously permuting the rows and columns of each block $\Upsilon_{\mathcal{F}^i}$.)

Remark 2.2.10. Let $\mathcal{G}$ be a finite group, and let $\mathfrak{M}(\mathcal{G})$ be as defined in 2.2.8. Following [Lus79, p. 323] we may define, for any function $f\colon \mathfrak{M}(\mathcal{G}) \to \overline{\mathbb{Q}}_\ell$, a function

$$\widehat{f}\colon \mathfrak{M}(\mathcal{G}) \to \overline{\mathbb{Q}}_\ell, \quad m \mapsto \sum_{m' \in \mathfrak{M}(\mathcal{G})} \{m, m'\} f(m'),$$

which Lusztig calls the '(non-abelian) Fourier transform of f'. This is motivated by the fact that if $\mathcal{G}$ is abelian, we have $\mathfrak{M}(\mathcal{G}) = \mathcal{G} \times \mathrm{Hom}(\mathcal{G}, \overline{\mathbb{Q}}_\ell^\times)$ and, hence, the function $\widehat{f}\colon \mathfrak{M}(\mathcal{G}) \to \overline{\mathbb{Q}}_\ell$ is given by

$$\widehat{f}(g,\sigma) = \frac{1}{|\mathcal{G}|} \sum_{(h,\tau) \in \mathcal{G} \times \mathrm{Hom}(\mathcal{G}, \overline{\mathbb{Q}}_\ell^\times)} \overline{\tau(g)} \sigma(h) f(h,\tau) \quad \text{for } (g,\sigma) \in \mathcal{G} \times \mathrm{Hom}(\mathcal{G}, \overline{\mathbb{Q}}_\ell^\times).$$

2.2.11. By [Lus79, (4.1), (4.2)], the Fourier matrix Υ_W in Definition 2.2.9 is hermitian, and Υ_W^2 is the identity matrix. As we see from the analysis in [Lus84a, 4.4–4.13], the group $\mathcal{G}_\mathcal{F}$ associated to a family $\mathcal{F} \subseteq \mathrm{Irr}(W)$ is always one of

$$\{1\},\ C_2^e \text{ (direct product of } e \geqslant 1 \text{ copies of the cyclic group of order 2)},\ \mathfrak{S}_3,\ \mathfrak{S}_4,\ \mathfrak{S}_5.$$

A case-by-case inspection reveals that we actually have $\{m, m'\} \in \mathbb{R}$ for any family $\mathcal{F} \subseteq \mathrm{Irr}(W)$ and any $m, m' \in \mathfrak{M}(\mathcal{G}_\mathcal{F})$, that is, the Fourier matrix Υ_W is in fact a *symmetric* matrix. Apart from the main reference [Lus84a, 4.3–4.15] on the families of irreducible characters in a Weyl group and the associated Fourier matrices, one may consult [Car85, §13.6], where the full 21×21 matrix arising from the group $\mathfrak{S}_4$ is included (which occurs for W of type F_4), except that the entry corresponding to $\big((1,\sigma),(g_3,1)\big)$ should be $-\frac{1}{3}$ instead of $\frac{1}{3}$. However, in either of these references, the full 39×39 matrix associated to the group $\mathfrak{S}_5$ (which occurs for W of type E_8) is not explicitly printed. This matrix can be accessed electronically through Michel's development version of the

CHEVIE package of GAP3 (see [MiChv]), using the commands

```
W:=CoxeterGroup("E",8);
FS5:=Fourier(UnipotentCharacters(W).families[46]);
Display(UnipotentCharacters(W).families[46]);
```

The second line gives the Fourier matrix $\Upsilon_{\mathcal{F}_{16}}$ of size 39×39, where the chosen order of the labels inside $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})$ can be read off from the output of the command in the third line. One can also get the other Fourier matrices $\Upsilon_{\mathcal{F}}$ with CHEVIE [MiChv], using similar commands as above, but it is important to note that, in general, the entry corresponding to $(m, m') \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}}) \times \mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ is

$$\Delta(m)\Delta(m') \cdot \{m, m'\} \quad \text{where} \quad \Delta\colon \mathfrak{M}(\mathcal{G}_{\mathcal{F}}) \to \{\pm 1\} \text{ is as defined in [Lus84a, 4.14].}$$

(For most of the families $\mathcal{F} \subseteq \mathrm{Irr}(W)$ however, we have $\Delta(m) = +1$ for all $m \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}})$; this includes the case where W is of type E_8 and $\mathcal{F} = \mathcal{F}_{16}$ is the family considered above, so the matrix `FS5` does indeed coincide with $\Upsilon_{\mathcal{F}_{16}}$ if the order of the elements in $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})$ is chosen accordingly.) The complete 'CHEVIE Fourier matrix' (that is, including the Δ function, as described above) can be obtained by typing

```
Fourier(UnipotentCharacters(W));
```

(with `W` assumed to be a Coxeter group), although this is not a block diagonal matrix most of the time since the order of the elements of $\mathfrak{X}(\mathtt{W})$ is in accordance with the one in `UnipotentCharacters(W)`, which is in terms of Harish-Chandra series rather than with respect to families in $\mathrm{Irr}(\mathtt{W})$. Since the author could not find a non-electronic reference for the Fourier matrix of $\mathfrak{S}_5$ (of size 39×39), it is printed in Table B.1 in the appendix.

2.2.12. In the setting of 2.2.8, assume that we are given an automorphism $\gamma\colon W \xrightarrow{\sim} W$ which satisfies $\gamma(S) = S$. In [Lus84a, 4.19–4.21], Lusztig generalises the situation in 2.2.8 to the present case by taking the automorphism γ into account, at least under the additional assumption that γ is ordinary (see 2.1.19). We do not give these definitions in the most general set-up at this point since they rely on quite an elaborate technical machinery which we shall never need in full extent. (For instance, in all the cases that we will be concerned with, the Weyl group W is actually irreducible; moreover, in most of the cases γ will in fact be just the identity on W, which leads to a drastic simplification of the description, see below.) Instead, we only sketch the most important notation here, which we shall make more explicit when dealing with the various specific examples later.

There is a finite set $\overline{\mathfrak{X}}(W, \gamma)$ as well as an infinite set $\mathfrak{X}(W, \gamma)$ attached to (W, γ) (see [Lus84a, (4.21.11), (4.21.12)]). These are given by a disjoint union of certain sets which are defined similarly to $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ in (2.2.8.1), where $\mathcal{F}$ runs over all those families in $\mathrm{Irr}(W)$ which are γ-stable with respect to the action of γ on $\mathrm{Irr}(W)$ described in (2.2.7.2). (In fact, as mentioned in [Lus84a, 4.17], if γ is assumed to be an ordinary automorphism of the Weyl group W, it follows from a case-by-case analysis that any γ-stable family $\mathcal{F}$ in $\mathrm{Irr}(W)$ is actually pointwise fixed by γ, that is, we have $\mathcal{F} \subseteq \mathrm{Irr}(W)^{\gamma}$.) The sets $\overline{\mathfrak{X}}(W,\gamma)$, $\mathfrak{X}(W,\gamma)$ are related by a pairing (defined similarly to the pairing (2.2.8.2), see [Lus84a, (4.21.7), (4.21.13)])

$$\{\ ,\ \}\colon \overline{\mathfrak{X}}(W,\gamma) \times \mathfrak{X}(W,\gamma) \to \overline{\mathbb{Q}}_\ell. \tag{2.2.12.1}$$

Let $\mathcal{R} \subseteq \overline{\mathbb{Q}}_\ell^\times$ be the set consisting of all roots of unity in $\overline{\mathbb{Q}}_\ell$. There is a natural free action of $\mathcal{R}$ on $\mathfrak{X}(W,\gamma)$ and a canonical surjective map

$$\mathfrak{X}(W,\gamma) \to \overline{\mathfrak{X}}(W,\gamma), \quad x \mapsto \overline{x}, \tag{2.2.12.2}$$

which induces a bijection between the set $\mathfrak{X}(W,\gamma)/\mathcal{R}$ of $\mathcal{R}$-orbits on $\mathfrak{X}(W,\gamma)$ and the set $\overline{\mathfrak{X}}(W,\gamma)$. Now consider the infinite cyclic group $\langle\tilde{\gamma}\rangle \cong \mathbb{Z}$ with generator $\tilde{\gamma}$, and let

$$\widetilde{W} := W(\tilde{\gamma}) := W \rtimes \langle\tilde{\gamma}\rangle \quad \text{where } \tilde{\gamma}\cdot w\cdot\tilde{\gamma}^{-1} = \gamma(w) \quad \text{for any } w \in W$$

(cf. (2.2.7.1), but here $W(\tilde{\gamma})$ is an infinite group). As in [Lus84a, 4.21], let us denote by $(\widetilde{W})^\vee_{\mathrm{ex}}$ the set of all (isomorphism classes of) those irreducible representations of $\widetilde{W}$ which factor through a finite quotient (that is, such that $\tilde{\gamma}$ acts as a map of finite order) and whose restriction to W is an irreducible representation of W. There is a natural free $\mathcal{R}$-action on $(\widetilde{W})^\vee_{\mathrm{ex}}$, defined by

$$\mathcal{R} \times (\widetilde{W})^\vee_{\mathrm{ex}} \to (\widetilde{W})^\vee_{\mathrm{ex}}, \quad (\zeta, \widetilde{\Theta}) \mapsto \widetilde{\Theta} \otimes \widetilde{\Theta}_\zeta, \tag{2.2.12.3}$$

where $\widetilde{\Theta}_\zeta$ is the one-dimensional representation of $\widetilde{W}$ on which W acts trivially and $\tilde{\gamma}$ acts by multiplication with $\zeta \in \mathcal{R}$. Let $\mathrm{Irr}(\widetilde{W})_{\mathrm{ex}}$ be the set of all characters of the representations in $(\widetilde{W})^\vee_{\mathrm{ex}}$. Since all the representations in $(\widetilde{W})^\vee_{\mathrm{ex}}$ are assumed to factor through a finite quotient, it is clear that associating to a representation in $(\widetilde{W})^\vee_{\mathrm{ex}}$ its character in $\mathrm{Irr}(\widetilde{W})_{\mathrm{ex}}$ defines a bijection

$$(\widetilde{W})^\vee_{\mathrm{ex}} \xrightarrow{\sim} \mathrm{Irr}(\widetilde{W})_{\mathrm{ex}},$$

so we can transfer the $\mathcal{R}$-action (2.2.12.3) to $\mathrm{Irr}(\widetilde{W})_{\mathrm{ex}}$. Then there is an $\mathcal{R}$-equivariant embedding [Lus84a, (4.21.14)]

$$\mathrm{Irr}(\widetilde{W})_{\mathrm{ex}} \hookrightarrow \mathfrak{X}(W,\gamma), \quad \vartheta \mapsto x_\vartheta. \tag{2.2.12.4}$$

The canonical projection

$$\widetilde{W} \to W(\gamma), \quad (w,\tilde{\gamma}^i) \mapsto (w,\gamma^i) \quad (\text{for } w\in W,\ i\in\mathbb{Z}),$$

induces an embedding $\mathrm{Irr}(W(\gamma)) \hookrightarrow \mathrm{Irr}(\widetilde{W})_{\mathrm{ex}}$, and composition with the map (2.2.12.4) gives rise to an embedding

$$\mathrm{Irr}(W(\gamma)) \hookrightarrow \mathrm{Irr}(\widetilde{W})_{\mathrm{ex}} \hookrightarrow \mathfrak{X}(W,\gamma), \quad \vartheta \mapsto x_\vartheta. \tag{2.2.12.5}$$

Now let $\phi \in \mathrm{Irr}(W)^\gamma$, and denote by $d\in\mathbb{N}$ the order of $\gamma\in\mathrm{Aut}(W)$. Let $\tilde{\phi}\in\mathrm{Irr}(W(\gamma))$ be one of the d extensions of ϕ (see 2.2.7), so we may view $\tilde{\phi}$ as an element of $\mathrm{Irr}(\widetilde{W})_{\mathrm{ex}}$. While $\tilde{\phi}$ is not uniquely determined by $\phi\in\mathrm{Irr}(W)^\gamma$, the $\mathcal{R}$-orbit of $\tilde{\phi}$ in $\mathrm{Irr}(\widetilde{W})_{\mathrm{ex}}$ certainly is. Hence, using the $\mathcal{R}$-equivariant embedding (2.2.12.4) and the fact that $\overline{\mathfrak{X}}(W,\gamma)$ is in bijection with the set of $\mathcal{R}$-orbits on $\mathfrak{X}(W,\gamma)$, we obtain an embedding

$$\mathrm{Irr}(W)^\gamma \hookrightarrow \overline{\mathfrak{X}}(W,\gamma), \quad \phi \mapsto \overline{x}_\phi, \tag{2.2.12.6}$$

defined as follows: For any $\phi\in\mathrm{Irr}(W)^\gamma$, $\overline{x}_\phi\in\overline{\mathfrak{X}}(W,\gamma)$ is the element corresponding to the $\mathcal{R}$-orbit of $x_{\tilde{\phi}}$ in $\mathfrak{X}(W,\gamma)$, where $\tilde{\phi}\in\mathrm{Irr}(W(\gamma))$ is an extension of ϕ to $W(\gamma)$. In the important special case where $\gamma=\mathrm{id}_W$ is the identity on W, one can canonically identify

$$\overline{\mathfrak{X}}(W,\mathrm{id}_W)\cong\mathfrak{X}(W) \quad \text{and} \quad \mathfrak{X}(W,\mathrm{id}_W)\cong\mathfrak{X}(W)\times\mathcal{R},$$

with $\mathfrak{X}(W)$ given by (2.2.8.3). With these identifications, we have (cf. [Lus84a, (4.21.8)])

$$\{m,(m',\zeta)\} = \zeta^{-1}\cdot\{m,m'\} \quad \text{for } m,m'\in\mathfrak{X}(W) \text{ and } \zeta\in\mathcal{R},$$

where $\{m,m'\}$ is given by (2.2.8.2). Moreover, the embedding (2.2.12.6) then coincides with the embedding defined in (2.2.8.4).

A partition of $\mathrm{Irr}(\mathbf{G}^F)$

2.2.13. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G}\to\mathbf{G}$ be a Steinberg map. Let $\mathbf{T}_0\subseteq\mathbf{G}$ be a maximally split torus, $\mathbf{B}_0\subseteq\mathbf{G}$ an F-stable Borel subgroup containing $\mathbf{T}_0$, and let $\mathscr{R}=(X,R,Y,R^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$.

(In particular, we have $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$.) Let $\mathbf{W} = W_{\mathbf{G}}(\mathbf{T}_0)$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$, and let $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ be the automorphism induced by F. As usual, we identify $\mathbf{W}$ with the Weyl group of R, using the action of $\mathbf{W}$ on X defined in 2.1.4. Denote by $\Pi \subseteq R$ the simple roots determined by $\mathbf{B}_0$, with positive roots $R^+ \subseteq R$ determined by Π, and let $S := \{w_\alpha \in \mathbf{W} \mid \alpha \in \Pi\}$ be the set of simple reflections corresponding to Π. The subsequent notation is still based on [Lus84a] but, as in [GM20, §2.4], we find it more convenient to work with the group X instead of line bundles over the variety $\mathscr{B}$ of Borel subgroups of $\mathbf{G}$: By [Lus84a, (1.3.2)], these line bundles can be canonically identified with X. We will thus mostly refer to [GM20, §2.4] rather than [Lus84a] in the sequel.

For any natural number $n \in \mathbb{N}$ which is prime to p, and for any $\lambda \in X$, we define

$$\mathscr{Z}_{\lambda,n} := \{w \in \mathbf{W} \mid \text{ there exists some } \lambda_w \in X \text{ such that } n\lambda_w = \lambda \circ F - w.\lambda\}$$

(see [GM20, 2.4.5]); it may happen that $\mathscr{Z}_{\lambda,n} = \varnothing$. Assume that (λ, n) is such that $\mathscr{Z}_{\lambda,n} \neq \varnothing$, and let $w \in \mathscr{Z}_{\lambda,n}$. It is easy to see that the $\lambda_w \in X$ which occurs in the definition of $\mathscr{Z}_{\lambda,n}$ is uniquely determined by w [GM20, 2.4.5]. Let us choose $g \in \mathbf{G}$ such that $g^{-1}F(g) = \dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ is a representative of w (see Theorem 2.1.14). So $\mathbf{T}_w = g\mathbf{T}_0 g^{-1}$ is a torus of type w with respect to $\mathbf{T}_0$. Following [GM20, 2.4.5], we obtain a linear character $\theta_w^{(\lambda,n)}$ of $\mathbf{T}_w^F$ by setting

$$\theta_w^{(\lambda,n)}\colon \mathbf{T}_w^F \to \overline{\mathbb{Q}}_\ell^\times, \quad t \mapsto \imath(\lambda_w(g^{-1}tg)). \tag{2.2.13.1}$$

(Recall that we have fixed an isomorphism $\imath\colon k^\times \xrightarrow{\sim} \mu_{p'} \subseteq \overline{\mathbb{Q}}_\ell^\times$ in (1.1.0.4).) We then associate to (λ, n) and $w \in \mathscr{Z}_{\lambda,n}$ the virtual Deligne–Lusztig character $R_{\mathbf{T}_w}^{\mathbf{G}}(\theta_w^{(\lambda,n)})$; note that this does not depend on the choices of $\dot{w}$ and g above. With these notions, we define

$$\mathcal{E}_{\lambda,n} := \Big\{\rho \in \operatorname{Irr}(\mathbf{G}^F) \;\Big|\; \langle R_{\mathbf{T}_w}^{\mathbf{G}}(\theta_w^{(\lambda,n)}), \rho\rangle_{\mathbf{G}^F} \neq 0 \text{ for some } w \in \mathscr{Z}_{\lambda,n}\Big\}$$

(see [GM20, 2.4.6]). It immediately follows from the definition that, for any $n' \in \mathbb{N}$ which is prime to p, the sets $\mathscr{Z}_{\lambda,n}$ and $\mathscr{Z}_{n'\lambda,n'n}$ coincide, and for any w which lies in these sets, the $\lambda_w \in X$ involved in their definition is the same. Hence, we also have $\mathcal{E}_{\lambda,n} = \mathcal{E}_{n'\lambda,n'n}$. Therefore, it is enough to consider pairs (λ, n) as above which are minimal in the sense that it is not possible to write $\lambda = n'\mu$, $n = n'm$, with $\mu \in X$ and $n', m \in \mathbb{N}$, $n' \geqslant 2$. In this case, the pair (λ, n) is called *indivisible* (see [GM20, 2.4.9]; note that this definition coincides with Lusztig's in [Lus84a, 6.1]). As in [GM20, p. 148], we set

$$\Lambda(\mathbf{G}, F) := \{(\lambda, n) \in X \times \mathbb{N} \mid n \text{ is prime to } p,\ \mathscr{Z}_{\lambda,n} \neq \varnothing,\ (\lambda, n) \text{ is indivisible}\}.$$

Then one can show (see [GM20, 2.4.11], [Lus84a, 6.5]) that for any $\rho \in \mathrm{Irr}(\mathbf{G}^F)$, there exists a suitable pair $(\lambda, n) \in \Lambda(\mathbf{G}, F)$ such that $\rho \in \mathcal{E}_{\lambda,n}$. In order to obtain a partition of $\mathrm{Irr}(\mathbf{G}^F)$ in terms of the sets $\mathcal{E}_{\lambda,n}$, one needs to impose an equivalence relation $\sim$ on the set $\Lambda(\mathbf{G}, F)$, given as follows (see [GM20, p. 148]). For $(\lambda, n), (\lambda', n') \in \Lambda(\mathbf{G}, F)$, we set

$$(\lambda, n) \sim (\lambda', n') \ :\Leftrightarrow\ n = n' \text{ and } \lambda' = x.\lambda + n\mu \text{ for some } x \in \mathbf{W},\ \mu \in X. \tag{2.2.13.2}$$

It is easy to check that $\sim$ defines an equivalence relation on $\Lambda(\mathbf{G}, F)$. By [GM20, 2.4.27] (cf. [Lus84a, 6.5]), we have, for any (λ, n), $(\lambda', n') \in \Lambda(\mathbf{G}, F)$:

$$(\lambda, n) \sim (\lambda', n') \quad \Longrightarrow \quad \mathcal{E}_{\lambda,n} = \mathcal{E}_{\lambda',n'}.$$

Proposition 2.2.14 ([Lus84a, 6.5], see [GM20, 2.4.29])**.** *In the setting of* 2.2.13*, we have a partition*

$$\mathrm{Irr}(\mathbf{G}^F) = \biguplus_{(\lambda,n) \in \Lambda(\mathbf{G},F)/\sim} \mathcal{E}_{\lambda,n},$$

where $\Lambda(\mathbf{G}, F)/_\sim$ *denotes a set of representatives for the equivalence classes of* $\Lambda(\mathbf{G}, F)$ *under the relation* $\sim$ *defined in* (2.2.13.2)*.*

Proof. See [Lus84a, 6.5] or [GM20, 2.4.29]. □

Remark 2.2.15. It is also shown in [GM20, 2.4.29] that the pieces $\mathcal{E}_{\lambda,n}$ in the partition in Proposition 2.2.14 are precisely the *geometric conjugacy classes of characters* or *geometric series of characters* (as defined, e.g., in [GM20, 2.3.4]).

2.2.16. Let $\lambda \in X$, and let $n \in \mathbb{N}$ be prime to p. As in [GM20, 2.4.13], we define

$$\mathbf{W}_{\lambda,n} := \{w \in \mathbf{W} \mid w.\lambda - \lambda \in n\mathbb{Z}R\} \tag{2.2.16.1}$$

and

$$R_{\lambda,n} := \{\alpha \in R \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ is divisible by } n\}.$$

As mentioned in [GM20, 2.4.13] (see also the references there), $\mathbf{W}_{\lambda,n}$ is a Weyl group with root system $R_{\lambda,n}$. Moreover, $R^+ \cap R_{\lambda,n}$ is a system of positive roots in $R_{\lambda,n}$. Let

$\Pi_{\lambda,n}$ be the unique base for $R_{\lambda,n}$ which is contained in $R^+ \cap R_{\lambda,n}$, and denote by

$$S_{\lambda,n} := \{w_\alpha \mid \alpha \in \Pi_{\lambda,n}\} \subseteq \mathbf{W}_{\lambda,n}$$

the corresponding set of simple reflections. (There is also another group $\hat{\mathbf{W}}_{\lambda,n}$ defined in [GM20, 2.4.12], but this group turns out to coincide with $\mathbf{W}_{\lambda,n}$ in the case where $\mathbf{G}$ has a connected centre, see [GM20, 2.4.14]; since we will only be concerned with this situation, we do not have to deal with $\hat{\mathbf{W}}_{\lambda,n}$ here.) We want to apply the machinery in [Lus84a, §4] (outlined in 2.2.12) to $(\mathbf{W}_{\lambda,n}, S_{\lambda,n})$, so we need to specify an automorphism of $\mathbf{W}_{\lambda,n}$.

Lemma 2.2.17 ([Lus84a, 2.15, 2.19], see [GM20, 2.4.14])**.** *In the setting of* 2.2.13*, assume that the centre* $\mathbf{Z}(\mathbf{G})$ *of* $\mathbf{G}$ *is connected. Then for any* $(\lambda, n) \in \Lambda(\mathbf{G}, F)$*, the following hold.*

(a) There is a unique element $w_1 \in \mathscr{Z}_{\lambda,n}$ *of minimal length with respect to the usual length function of the Coxeter system* $(\mathbf{W}, S)$*. We have* $\mathscr{Z}_{\lambda,n} = w_1\mathbf{W}_{\lambda,n}$*.*

(b) There is a well-defined group automorphism

$$\sigma_{\lambda,n} \colon \mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_{\lambda,n}, \quad w \mapsto \sigma(w_1 w w_1^{-1}),$$

and this automorphism satisfies $\sigma_{\lambda,n}(S_{\lambda,n}) = S_{\lambda,n}$*.*

Proof. See [Lus84a, 2.15, 2.19] or [GM20, 2.4.14]. □

2.2.18. We place ourselves in the setting of 2.2.13 and assume in addition that $\mathbf{Z}(\mathbf{G})$ is connected. Let us fix a pair $(\lambda, n) \in \Lambda(\mathbf{G}, F)$. Lemma 2.2.17(b) provides the desired group automorphism $\sigma_{\lambda,n} \colon \mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_{\lambda,n}$ satisfying $\sigma_{\lambda,n}(S_{\lambda,n}) = S_{\lambda,n}$, so that the machinery in 2.2.12 can be applied to $W = \mathbf{W}_{\lambda,n}$, $S = S_{\lambda,n}$ and $\gamma = \sigma_{\lambda,n}$. Let w_1 be the unique element of minimal length in $\mathscr{Z}_{\lambda,n}$, so that $\mathscr{Z}_{\lambda,n} = w_1\mathbf{W}_{\lambda,n}$ (see Lemma 2.2.17(a)). Let $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})^{\sigma_{\lambda,n}}$, and let us choose a $\sigma_{\lambda,n}$-extension $\tilde{\phi} \colon \mathbf{W}_{\lambda,n} \to \overline{\mathbb{Q}}_\ell$, as in 2.2.7. Then we define a class function

$$R_{\tilde{\phi}} := \frac{1}{|\mathbf{W}_{\lambda,n}|} \sum_{w \in \mathbf{W}_{\lambda,n}} \tilde{\phi}(w) \cdot R^{\mathbf{G}}_{\mathbf{T}_{w_1 w}}(\theta^{(\lambda,n)}_{w_1 w}) \in \mathrm{CF}(\mathbf{G}^F) \qquad (2.2.18.1)$$

(see [Lus84a, 3.7, 6.13]), which is called a *uniform almost character* of $\mathbf{G}^F$ [GM20, 2.4.17]. Note that, in general, we only know that the values $\tilde{\phi}(w)$ (for $w \in \mathbf{W}_{\lambda,n}$) are certain algebraic integers in $\overline{\mathbb{Q}}_\ell$.

Lemma 2.2.19 ([Lus84a, 3.2, 14.2], see [GM20, 2.1.14])**.** *Assume that we are in the setting of* 2.2.13 *and that* $\mathbf{Z}(\mathbf{G})$ *is connected. Let* $(\lambda, n) \in \Lambda(\mathbf{G}, F)$ *and* $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})^{\sigma_{\lambda,n}}$*, with* $\sigma_{\lambda,n} \colon \mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_{\lambda,n}$ *as defined in Lemma* 2.2.17(b)*.*

(a) There is a $\sigma_{\lambda,n}$*-extension* $\tilde{\phi} \colon \mathbf{W}_{\lambda,n} \to \overline{\mathbb{Q}}_\ell$ *of* ϕ *(see* 2.2.7*) which satisfies* $\tilde{\phi}(w) \in \mathbb{R}$ *for all* $w \in \mathbf{W}_{\lambda,n}$*.*

(b) If $F \colon \mathbf{G} \to \mathbf{G}$ *is a Frobenius map, then any* $\sigma_{\lambda,n}$*-extension* $\tilde{\phi}$ *as in* (a) *even satisfies* $\tilde{\phi}(w) \in \mathbb{Z}$ *for all* $w \in \mathbf{W}_{\lambda,n}$*.*

Proof. This is essentially proven in [GM20, 2.1.14] (cf. [Lus84a, 3.2, 14.2]): For (a) we only have to note in addition that $\mathbf{W}_{\lambda,n}$ is itself the Weyl group of a connected reductive group $\mathbf{H}$ and that $\sigma_{\lambda,n}$ is induced from a suitable Steinberg map on $\mathbf{H}$, as explained in [GM20, 2.5.10]. In order to deduce (b) from [GM20, 2.1.14], we need to know that $\sigma_{\lambda,n}$ is ordinary if F is a Frobenius map; this is stated in [Lus84a, (3.4.1)]. (Alternatively, one may also once again refer to [GM20, 2.5.10] and note that in this case $\sigma_{\lambda,n}$ is in fact induced from a *Frobenius map* on $\mathbf{H}$.) □

Lusztig's Main Theorem 4.23

2.2.20. We are now in a position to formulate Lusztig's 'Main Theorem 4.23' in [Lus84a]. To this end, we place ourselves in the setting of 2.2.13. In addition, we assume that $\mathbf{Z}(\mathbf{G})$ is connected and that $F \colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map for an $\mathbb{F}_q$-rational structure on $\mathbf{G}$, for some power q of p. (One may extend the theorem to the case where F is a Steinberg map, but then the formulation would become either somewhat less uniform or more technical, due to the remarks in [Lus84a, 14.2]; see, e.g., [GM20, 2.4.15]. In any case, the assumption on $\mathbf{Z}(\mathbf{G})$ is substantial.)

There is one more necessary piece of notation that we did not introduce so far. Namely, for a given pair $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, Lusztig defines a map

$$\Delta \colon \overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}) \to \{\pm 1\} \tag{2.2.20.1}$$

in [Lus84a, 4.14, 4.21]. Since the definition of Δ in the most general set-up is closely related to the very technical description of the set $\overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ itself (cf. 2.2.12), we refrain from providing it here. We merely mention that, if $\sigma_{\lambda,n}$ is the identity on $\mathbf{W}_{\lambda,n}$ (which will cover most of the cases that we shall consider), Δ can take the value -1 only if $\mathbf{W}_{\lambda,n}$ has an irreducible factor of type E_7 or E_8, and this in turn only concerns very few labels arising from these groups. Let us for now just refer to [Lus84a, Chap. 4] for the exact definition of Δ.

Theorem 2.2.21 (Lusztig [Lus84a, 4.23])**.** *In the setting of* 2.2.20 *(in particular, under the assumption that* $\mathbf{Z}(\mathbf{G})$ *is connected), consider any pair* $(\lambda, n) \in \Lambda(\mathbf{G}, F)$*. Let* w_1 *be the unique element of minimal length in* $\mathscr{Z}_{\lambda,n}$ *(see Lemma* 2.2.17(a)*), and let* $\sigma_{\lambda,n} \colon \mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_{\lambda,n}$ *be the automorphism defined in Lemma* 2.2.17(b)*. Then there is a bijection*

$$\mathcal{E}_{\lambda,n} \xrightarrow{\sim} \overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}), \quad \rho \mapsto \overline{x}_\rho, \tag{2.2.21.1}$$

such that for any $\rho \in \mathcal{E}_{\lambda,n}$*, any* $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})^{\sigma_{\lambda,n}}$ *and any* $\sigma_{\lambda,n}$*-extension* $\tilde{\phi}$ *of* ϕ*, we have*

$$\langle \rho, R_{\tilde{\phi}} \rangle_{\mathbf{G}^F} = (-1)^{l(w_1)} \Delta(\overline{x}_\rho) \{\overline{x}_\rho, x_{\tilde{\phi}}\}, \tag{2.2.21.2}$$

where $\{\ ,\ \}$ *is the pairing* (2.2.12.1) *(see* [Lus84a, (4.21.13)]*) and* $x_{\tilde{\phi}} \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ *is the image of* $\tilde{\phi}$ *under the embedding* (2.2.12.5) *with* $W = \mathbf{W}_{\lambda,n}$*,* $\gamma = \sigma_{\lambda,n}$*.*

Proof. The proof of this theorem occupies a large part of the book [Lus84a]; see, in particular, [Lus84a, Chap. 10]. □

Remark 2.2.22. We place ourselves in the setting of Theorem 2.2.21. In particular, let $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, and let us adopt the further notation with respect to this pair (λ, n).

(a) Lusztig requires the $\sigma_{\lambda,n}$-extension $\tilde{\phi}$ of $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})^{\sigma_{\lambda,n}}$ in Theorem 2.2.21 to be as in Lemma 2.2.19(b). This is due to the fact that his underlying pairing on the class functions of $\mathbf{G}^F$ (or, more precisely, on the Grothendieck group of virtual $\mathbf{G}^F$-modules of finite dimension over $\overline{\mathbb{Q}}_\ell$) is actually $\overline{\mathbb{Q}}_\ell$-bilinear, see [Lus84a, 3.7]. Recall (2.2.0.1) that we defined $\langle\ ,\ \rangle_{\mathbf{G}^F}$ to be the standard scalar product on $\mathrm{CF}(\mathbf{G}^F)$, which is therefore antilinear in the second argument. Now note that, if we assume $\tilde{\phi}$ to be chosen as in Lemma 2.2.19(b), $R_{\tilde{\phi}}$ is a $\mathbb{Q}$-linear combination of virtual characters, so $\langle \rho, R_{\tilde{\phi}} \rangle_{\mathbf{G}^F}$ (and, hence, $\{\overline{x}_\rho, x_{\tilde{\phi}}\}$) is a rational number. So in this case, the formulation of (2.2.21.2) is completely analogous to the one in [Lus84a, 4.23]. If we replace such a $\tilde{\phi}$ by another $\sigma_{\lambda,n}$-extension $\zeta \cdot \tilde{\phi}$ (where $\zeta \in \mathcal{R}$), the left side of (2.2.21.2) reads

$$\langle \rho, R_{\zeta \cdot \tilde{\phi}} \rangle_{\mathbf{G}^F} = \langle \rho, \zeta \cdot R_{\tilde{\phi}} \rangle_{\mathbf{G}^F} = \zeta^{-1} \cdot \langle \rho, R_{\tilde{\phi}} \rangle_{\mathbf{G}^F}.$$

On the other hand, since the embedding (2.2.12.4) is $\mathcal{R}$-equivariant, we get

$$\{\overline{x}_\rho, x_{\zeta \cdot \tilde{\phi}}\} = \{\overline{x}_\rho, \zeta . x_{\tilde{\phi}}\} = \zeta^{-1} \cdot \{\overline{x}_\rho, x_{\tilde{\phi}}\}$$

(where the last equality follows from [Lus84a, (4.21.8)]). Thus, the right side of (2.2.21.2) is also multiplied with ζ^{-1}, so we do not need to make any assumption on the $\sigma_{\lambda,n}$-extension in the formulation of Theorem 2.2.21. Whenever we perform any explicit calculations, we will however choose an integer-valued $\sigma_{\lambda,n}$-extension $\tilde{\phi}$ as in Lemma 2.2.19(b), such as the 'preferred extension' defined in [LuCS4, 17.2].

(b) The bijection (2.2.21.1) is in general not uniquely determined by the condition (2.2.21.2). We will be concerned with the task of uniquely specifying it for simple non-twisted groups with a trivial centre in Section 3.4 in the case where $(\lambda, n) = (0, 1)$, when $\mathcal{E}_{0,1} = \mathrm{Uch}(\mathbf{G}^F)$ is the set of unipotent characters of $\mathbf{G}^F$ (see Example 2.2.24 below). In general, [DM90, §6, §7] deals with the problem of formulating conditions which guarantee the 'unicity of the parametrisation' (2.2.21.1).

(c) Once Theorem 2.2.21 is established, the sign function Δ in (2.2.20.1) can also be described purely in terms of irreducible characters of $\mathbf{G}^F$, see [Lus84a, 6.6, 6.20]. Namely, for $\rho \in \mathcal{E}_{\lambda,n}$, $\Delta(\overline{x}_\rho)$ is uniquely determined by requiring that

$$(-1)^{l(w_1)}\Delta(\overline{x}_\rho) \cdot D(\rho) \in \mathrm{Irr}(\mathbf{G}^F),$$

where D denotes the 'duality homomorphism', see [Lus84a, 6.8].

2.2.23. We still keep the setting of Theorem 2.2.21. In particular, let $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})^{\sigma_{\lambda,n}}$, and let $\tilde{\phi}$ be a $\sigma_{\lambda,n}$-extension of ϕ. By [Lus84a, 3.9], $R_{\tilde{\phi}}$ is a linear combination of irreducible characters in $\mathcal{E}_{\lambda,n}$. Hence, (2.2.21.2) implies that

$$R_{\tilde{\phi}} = (-1)^{l(w_1)} \sum_{\rho \in \mathcal{E}_{\lambda,n}} \Delta(\overline{x}_\rho)\overline{\{\overline{x}_\rho, x_{\tilde{\phi}}\}} \cdot \rho.$$

More generally, following [Lus84a, (4.24.1)], we define for any $x \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ the *almost character*

$$R_x := (-1)^{l(w_1)} \sum_{\rho \in \mathcal{E}_{\lambda,n}} \Delta(\overline{x}_\rho)\overline{\{\overline{x}_\rho, x\}} \cdot \rho \in \mathrm{CF}(\mathbf{G}^F). \qquad (2.2.23.1)$$

We thus have $R_{\tilde{\phi}} = R_{x_{\tilde{\phi}}}$. (Note that the expressions for $R_{\tilde{\phi}}$ and R_x look somewhat different than in [Lus84a, 4.24]. This is due to the fact that the definition of our pairing $\langle\ ,\ \rangle_{\mathbf{G}^F}$ differs from the one in [Lus84a, 3.7], see Remark 2.2.22(a).) Recall from 2.2.12 that there is a free action of $\mathcal{R}$ on $\mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$. If $\zeta \in \mathcal{R}$, $x \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ and $\zeta.x \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ is the image of (ζ, x) under this action, we have

$$R_{\zeta.x} = \zeta \cdot R_x$$

(as follows from [Lus84a, (4.21.8)]). Hence, up to multiplication with a root of unity, R_x is uniquely determined by the $\mathcal{R}$-orbit of x in $\mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$. Let us fix a set

$\mathfrak{X}_0(\lambda, n) \subseteq \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ of representatives for the $\mathcal{R}$-orbits in $\mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$. As stated in 2.2.12, $\mathfrak{X}_0(\lambda, n)$ is in natural bijection with $\overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$. Now the set

$$\{R_x \mid x \in \mathfrak{X}_0(\lambda, n)\}$$

is an orthonormal basis of the linear subspace of $\big(\mathrm{CF}(\mathbf{G}^F), \langle\ ,\ \rangle_{\mathbf{G}^F}\big)$ spanned by $\mathcal{E}_{\lambda,n}$, see [Lus84a, 4.25]. In particular, denoting by $\Lambda(\mathbf{G}, F)/_\sim$ a set of representatives for the equivalence classes of $\Lambda(\mathbf{G}, F)$ under the relation (2.2.13.2), we deduce from Proposition 2.2.14 that the set

$$\biguplus_{(\lambda,n)\in\Lambda(\mathbf{G},F)/\sim} \{R_x \mid x \in \mathfrak{X}_0(\lambda, n)\} \subseteq \mathrm{CF}(\mathbf{G}^F)$$

is an orthonormal basis of $\big(\mathrm{CF}(\mathbf{G}^F), \langle\ ,\ \rangle_{\mathbf{G}^F}\big)$. The above discussion also shows that, for $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, an irreducible character $\rho \in \mathcal{E}_{\lambda,n}$ is expressed as

$$\rho = (-1)^{l(w_1)}\Delta(\overline{x}_\rho) \sum_{x\in\mathfrak{X}_0(\lambda,n)} \{\overline{x}_\rho, x\} \cdot R_x, \tag{2.2.23.2}$$

cf. [Lus84a, 4.25]. In particular, for any $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, knowing the values of the irreducible characters in $\mathcal{E}_{\lambda,n}$ is equivalent to knowing the values of the R_x for $x \in \mathfrak{X}_0(\lambda, n)$.

Example 2.2.24. In the setting of Theorem 2.2.21, consider the pair $(0, 1) \in \Lambda(\mathbf{G}, F)$. We immediately see from the definition that $\mathscr{X}_{0,1} = \mathbf{W}_{0,1} = \mathbf{W}$. Furthermore, we have $R_{0,1} = R$, $S_{0,1} = S$, and the element w_1 of minimal length in $\mathbf{W}$ is certainly the identity element of $\mathbf{W}$. Thus,

$$\sigma_{0,1} = \sigma \colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$$

is the map induced by F on $\mathbf{W}$. Also note that for any $w \in \mathbf{W}$, the $\lambda_w \in X$ appearing in the definition of $\mathscr{X}_{0,1}$ is the neutral element 0 of X. Thus, $\theta_w^{(0,1)}$ (see (2.2.13.1)) is the trivial character of $\mathbf{T}_w^F$ for any $w \in \mathbf{W}$, so

$$\mathcal{E}_{0,1} = \mathrm{Uch}(\mathbf{G}^F)$$

is just the set of unipotent characters of $\mathbf{G}^F$. Hence, Theorem 2.2.21 gives a bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \overline{\mathfrak{X}}(\mathbf{W}, \sigma), \quad \rho \mapsto \overline{x}_\rho,$$

such that for any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, any $\phi \in \mathrm{Irr}(\mathbf{W})^\sigma$ and any σ-extension $\tilde{\phi}$ of ϕ, we have

$$\langle \rho, R_{\tilde{\phi}} \rangle_{\mathbf{G}^F} = \Delta(\overline{x}_\rho)\{\overline{x}_\rho, x_{\tilde{\phi}}\}. \tag{2.2.24.1}$$

The (uniform) almost characters are given as follows: For $\phi \in \mathrm{Irr}(\mathbf{W})^\sigma$ and a σ-extension $\tilde{\phi}$ of ϕ, we get

$$R_{\tilde{\phi}} = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\phi}(w) \cdot R_w \in \mathrm{CF}(\mathbf{G}^F). \tag{2.2.24.2}$$

We will refer to $R_{\tilde{\phi}}$ as a *unipotent uniform almost character* (cf. 2.2.18). Moreover, for $x \in \mathfrak{X}(\mathbf{W}, \sigma)$, the definition of R_x now reads

$$R_x = \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \Delta(\overline{x}_\rho) \overline{\{\overline{x}_\rho, x\}} \cdot \rho. \tag{2.2.24.3}$$

Accordingly, we call R_x a *unipotent almost character*. As in 2.2.23, let us fix a set of representatives $\mathfrak{X}_0(0,1) \subseteq \mathfrak{X}(\mathbf{W}, \sigma)$ for the $\mathcal{R}$-orbits in $\mathfrak{X}(\mathbf{W}, \sigma)$. Thus, $\mathfrak{X}_0(0,1)$ is in natural bijection with $\overline{\mathfrak{X}}(\mathbf{W}, \sigma)$, so (2.2.23.2) becomes

$$\rho = \Delta(\overline{x}_\rho) \sum_{x \in \overline{\mathfrak{X}}(\mathbf{W}, \sigma)} \{\overline{x}_\rho, x\} \cdot R_x \quad \text{for } \rho \in \mathrm{Uch}(\mathbf{G}^F). \tag{2.2.24.4}$$

We will sometimes refer to the following notion, cf. [Lus80, 1.2]: We say that two unipotent characters $\rho, \rho' \in \mathrm{Uch}(\mathbf{G}^F)$ are *in the same family* if there exists some $n \in \mathbb{N}_0$ and a sequence of unipotent characters $\rho = \rho_0, \rho_1, \ldots, \rho_n = \rho'$, as well as $\phi_1, \ldots, \phi_n \in \mathrm{Irr}(\mathbf{W})^\sigma$, with σ-extensions $\tilde{\phi}_1, \ldots, \tilde{\phi}_n$, such that $\langle \rho_{i-1}, R_{\tilde{\phi}_i} \rangle_{\mathbf{G}^F} \neq 0 \neq \langle \rho_i, R_{\tilde{\phi}_i} \rangle_{\mathbf{G}^F}$ for $1 \leqslant i \leqslant n$. It follows from (2.2.24.1) and the classification of families of Weyl groups provided in [Lus84a, §4] that $\rho, \rho' \in \mathrm{Uch}(\mathbf{G}^F)$ are in the same family if and only if there exists a family $\mathcal{F} \subseteq \mathrm{Irr}(\mathbf{W})$ such that $\overline{x}_\rho, \overline{x}_{\rho'} \in \mathfrak{M}(\mathcal{G}_\mathcal{F})$.

Remark 2.2.25. With the notation of Example 2.2.24, let us write $Q_{\tilde{\phi}} := R_{\tilde{\phi}}|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $\phi \in \mathrm{Irr}(\mathbf{W})^\sigma$; we also recall from 2.2.4 that $Q_w = R_w|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $w \in \mathbf{W}$. By (2.2.24.2), we have

$$Q_{\tilde{\phi}} = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\phi}(w) \cdot Q_w \in \mathrm{CF}(\mathbf{G}^F).$$

Inverting these relations, one obtains

$$Q_w = \sum_{\phi \in \mathrm{Irr}(\mathbf{W})^\sigma} \tilde{\phi}(w) Q_{\tilde{\phi}}$$

(see [GM20, 2.8.2]). Thus, the knowledge of the Green functions Q_w for $w \in \mathbf{W}$ (cf. 2.2.5) is equivalent to that of the functions $Q_{\tilde{\phi}}$ for $\phi \in \mathrm{Irr}(\mathbf{W})^\sigma$. Since it is often more convenient to work with unipotent uniform almost characters rather than with the R_w ($w \in \mathbf{W}$), we will tacitly refer to this fact in numerous places below.

Parametrisation of $\mathrm{Irr}(\mathbf{G}^F)$ in terms of semisimple classes in the dual group

2.2.26. Let $\mathbf{G}$ be a connected reductive group over k, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. Let us fix a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{G}$ as well as an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ containing $\mathbf{T}_0$, and let $\mathscr{R} = (X, R, Y, R^\vee) = (X(\mathbf{T}_0), R, Y(\mathbf{T}_0), R^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$. Let $(\mathbf{G}^*, \mathbf{T}_0^*, \mathbf{B}_0^*, F')$ be a quadruple dual to $(\mathbf{G}, \mathbf{T}_0, \mathbf{B}_0, F)$ and $\mathbf{W}^* = {}^{N_{\mathbf{G}^*}(\mathbf{T}_0^*)}/_{\mathbf{T}_0^*}$ (see Example 2.1.21), and let $\delta\colon X(\mathbf{T}_0) \xrightarrow{\sim} Y(\mathbf{T}_0^*)$ be as in (2.1.21.1). Consider a pair $(\lambda, n) \in X \times \mathbb{N}$ where n is prime to p. Using the notation of [GM20, §2.5], we define an element

$$t_{\lambda,n} := \delta(\lambda)\Big(\jmath\Big(\tfrac{1}{n} + \mathbb{Z}\Big)\Big) \in \mathbf{T}_0^*,$$

where $\jmath\colon {}^{\mathbb{Z}_{(p)}}/_{\mathbb{Z}} \xrightarrow{\sim} k^\times$ is the isomorphism (1.1.0.5). Note that $t_{\lambda,n}$ is the image of $\lambda \otimes (\frac{1}{n} + \mathbb{Z})$ under the isomorphism

$$X \otimes_{\mathbb{Z}} {}^{\mathbb{Z}_{(p)}}/_{\mathbb{Z}} \xrightarrow{\delta\otimes\jmath} Y(\mathbf{T}_0^*) \otimes_{\mathbb{Z}} k^\times \xrightarrow{\sim} \mathbf{T}_0^*,$$

the last map being defined by (2.1.3.2) applied to the torus $\mathbf{T}_0^*$. Here, as mentioned in [GM20, 2.5.3], any element of $X \otimes_{\mathbb{Z}} {}^{\mathbb{Z}_{(p)}}/_{\mathbb{Z}}$ can be written as $\lambda \otimes (\frac{1}{n} + \mathbb{Z})$ for some $\lambda \in X$ and some $n \in \mathbb{N}$ which is prime to p. Hence, as (λ, n) runs over all elements of $X \times (\mathbb{N} \setminus p\mathbb{N})$, $t_{\lambda,n}$ runs over all elements of $\mathbf{T}_0^*$.

Now assume that $(\lambda, n) \in \Lambda(\mathbf{G}, F)$. If we replace (λ, n) by another element of its equivalence class under the relation (2.2.13.2), say by $(x.\lambda + n\mu, n)$ where $x \in \mathbf{W}$, $\mu \in X$, then one quickly verifies that

$$t_{x.\lambda+n\mu,n} = t_{x.\lambda,n} = \dot{x}^* t_{\lambda,n} (\dot{x}^*)^{-1},$$

where $\mathbf{W} \xrightarrow{\sim} \mathbf{W}^*$, $w \mapsto w^*$, is defined in (2.1.21.2), and $\dot{x}^* \in N_{\mathbf{G}^*}(\mathbf{T}_0^*)$ denotes a representative of $x^* \in \mathbf{W}^*$. Therefore, the set

$$\{t_{\lambda',n} \mid (\lambda', n) \in \Lambda(\mathbf{G}, F),\ (\lambda', n) \sim (\lambda, n)\} \subseteq \mathbf{T}_0^*$$

is the $\mathbf{W}^*$-class of $t_{\lambda,n}$ (with respect to the natural action of $\mathbf{W}^*$ on $\mathbf{T}_0^*$ induced by the conjugation action of $N_{\mathbf{G}^*}(\mathbf{T}_0^*)$). The $\mathbf{W}^*$-classes inside $\mathbf{T}_0^*$ are in natural bijection with the semisimple conjugacy classes of $\mathbf{G}^*$ (by assigning to the $\mathbf{W}^*$-class of $s \in \mathbf{T}_0^*$ the $\mathbf{G}^*$-conjugacy class of s, see [Car85, 3.7.1]). By [GM20, 2.5.4], the assumption that $\mathscr{X}_{\lambda,n} \neq \varnothing$ is equivalent to the assumption that the $\mathbf{G}^*$-conjugacy class of $t_{\lambda,n}$ is F'-stable. Hence, mapping the equivalence class of (λ, n) to the $\mathbf{G}^*$-conjugacy class of $t_{\lambda,n}$ defines a bijection

$$\Lambda(\mathbf{G}, F)/_{\sim} \xrightarrow{1-1} \{F'\text{-stable semisimple conjugacy classes of } \mathbf{G}^*\},$$

where $\Lambda(\mathbf{G}, F)/_{\sim}$ is a set of representatives for the equivalence classes of $\Lambda(\mathbf{G}, F)$ under the relation (2.2.13.2).

2.2.27. In the setting of 2.2.26, assume that $\mathbf{Z}(\mathbf{G})$ is connected. Instead of working with the set $\Lambda(\mathbf{G}, F)$, it is sometimes convenient to be able to express everything in terms of semisimple elements in the dual group $\mathbf{G}^*$. So let us consider an element $s \in \mathbf{T}_0^*$ whose conjugacy class in $\mathbf{G}^*$ is F'-stable. We define a subgroup (cf. [Sho95a, §5])

$$\mathbf{W}_s^* := \{w \in \mathbf{W}^* \mid \dot{w}s\dot{w}^{-1} = s\} \subseteq \mathbf{W}^*$$

as well as a subset

$$\mathscr{Z}_s^* := \{w \in \mathbf{W}^* \mid F'(s) = \dot{w}s\dot{w}^{-1}\} \subseteq \mathbf{W}^*,$$

where $\dot{w} \in N_{\mathbf{G}^*}(\mathbf{T}_0^*)$ denotes a representative of $w \in \mathbf{W}^*$. In view of our assumption that $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\mathbf{G})^\circ$, we have $C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s)$ (see [Car85, 4.5.9]); by [Car85, 3.5.4], $C_{\mathbf{G}^*}(s)$ is thus a connected reductive group over k, whose Weyl group with respect to the maximal torus $\mathbf{T}_0^* \subseteq C_{\mathbf{G}^*}(s)$ is $\mathbf{W}_s^*$. (Indeed, by definition, the Weyl group of $C_{\mathbf{G}^*}(s)$ with respect to the maximal torus $\mathbf{T}_0^*$ is $N_{C_{\mathbf{G}^*}(s)}(\mathbf{T}_0^*)/\mathbf{T}_0^*$, which is nothing but $\mathbf{W}_s^*$ if we identify the Weyl group of $C_{\mathbf{G}^*}(s)$ with a subgroup of $\mathbf{W}^*$ via the canonical embedding induced by $N_{C_{\mathbf{G}^*}(s)}(\mathbf{T}_0^*) \hookrightarrow N_{\mathbf{G}^*}(\mathbf{T}_0^*)$.) The discussion in 2.2.26 shows that there exists some $(\lambda, n) \in \Lambda(\mathbf{G}, F)$ such that $s = t_{\lambda,n}$. Using [GM20, 2.5.3, 2.5.4], it is easy to see that the isomorphism $\mathbf{W} \xrightarrow{\sim} \mathbf{W}^*$, $w \mapsto w^*$, defined in (2.1.21.2), induces (by restriction) an isomorphism

$$\mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_s^*,$$

and it maps $\mathscr{Z}_{\lambda,n}$ bijectively onto $\mathscr{Z}_s^*$. Let $\Pi \subseteq R$ be the simple roots determined by $\mathbf{B}_0$, and let $S = \{w_\alpha \mid \alpha \in \Pi\} \subseteq \mathbf{W}$ be the corresponding set of simple reflections. Then $S^* := \{w_\alpha^* \mid \alpha \in \Pi\} \subseteq \mathbf{W}^*$ is the set of simple reflections in $\mathbf{W}^*$ determined by $\mathbf{B}_0^*$. So it is clear that, if $w_1 \in \mathscr{Z}_{\lambda,n}$ is the unique element of minimal length with respect to the length function of $(\mathbf{W}, S)$, as in Lemma 2.2.17(a), $w_1^* \in \mathscr{Z}_s^*$ is the unique element of minimal length with respect to the length function of $(\mathbf{W}^*, S^*)$, and we have $\mathscr{Z}_s^* = w_1^*\mathbf{W}_s^*$. The automorphism $\sigma_s^* \in \operatorname{Aut}(\mathbf{W}_s^*)$ corresponding to the automorphism

$\sigma_{\lambda,n} \in \mathrm{Aut}(\mathbf{W}_{\lambda,n})$ defined in Lemma 2.2.17(b) is given by

$$\sigma_s^*\colon \mathbf{W}_s^* \xrightarrow{\sim} \mathbf{W}_s^*, \quad y^* \mapsto \sigma'^{-1}(w_1^* y^* w_1^{*-1}) \quad \text{for any } y \in \mathbf{W}_{\lambda,n},$$

and, since $\sigma(S_{\lambda,n}) = S_{\lambda,n}$ (see Lemma 2.2.17(b)), the set $S_{\lambda,n}^* := \{w_\alpha^* \mid \alpha \in \Pi_{\lambda,n}\}$ is left invariant by σ_s^*. Hence, we can label everything in terms of F'-stable semisimple conjugacy classes in $\mathbf{G}^*$. For any $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, let us denote the ($F'$-stable semisimple) conjugacy class of $s = t_{\lambda,n}$ in $\mathbf{G}^*$ by $[s]$. We may thus set

$$\mathcal{E}(\mathbf{G}^F, [s]) := \mathcal{E}_{\lambda,n},\ \overline{\mathfrak{X}}(\mathbf{W}_s^*, \sigma_s^*) := \overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}),\ \mathfrak{X}(\mathbf{W}_s^*, \sigma_s^*) := \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}).$$

In particular, Proposition 2.2.14 then states that

$$\mathrm{Irr}(\mathbf{G}^F) = \biguplus_{[s]} \mathcal{E}(\mathbf{G}^F, [s])$$

(where $[s]$ runs over the F'-stable semisimple classes in $\mathbf{G}^*$), and for each $s = t_{\lambda,n}$ as above, the parametrisation (2.2.21.1) can be written as

$$\mathcal{E}(\mathbf{G}^F, [s]) \xrightarrow{\sim} \overline{\mathfrak{X}}(\mathbf{W}_s^*, \sigma_s^*), \quad \rho \mapsto \overline{x}_\rho,$$

such that the conditions in Theorem 2.2.21 are satisfied. For the unipotent characters, we thus have

$$\mathrm{Uch}(\mathbf{G}^F) = \mathcal{E}(\mathbf{G}^F, [1]) \xrightarrow{\sim} \overline{\mathfrak{X}}(\mathbf{W}^*, \sigma'^{-1}).$$

Lusztig induction

Definition 2.2.28 (Lusztig [Lus76a])**.** Let $\mathbf{G}$ be a connected reductive group over k and $F\colon \mathbf{G} \to \mathbf{G}$ be a Steinberg map. Let $\mathbf{L} \subseteq \mathbf{G}$ be an F-stable closed subgroup which is the Levi complement of some (not necessarily F-stable) parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$. (So $\mathbf{L}$ is a regular subgroup of $\mathbf{G}$, see Definition 2.1.18.) Denote by $\mathbf{U_P} := R_{\mathrm{u}}(\mathbf{P})$ the unipotent radical of $\mathbf{P}$. Then $\mathbf{Y} := F(\mathbf{U_P})$ and $H := \mathbf{L}^F$ meet the requirements of 2.2.2, so that we can apply the construction there. In particular, for any irreducible character $\pi \in \mathrm{Irr}(\mathbf{L}^F)$, we have

$$R^{\mathbf{G}}_{\mathbf{L},F(\mathbf{U_P})}(\pi)\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell, \quad g \mapsto \frac{1}{|\mathbf{L}^F|} \sum_{l \in \mathbf{L}^F} \mathfrak{L}((g,l), \mathcal{L}^{-1}(F(\mathbf{U_P})))\pi(l),$$

a virtual character of $\mathbf{G}^F$. This induction process was defined by Lusztig [Lus76a] and is usually called *Lusztig induction*, or sometimes *twisted induction* [Lus90, 1.7].

Remark 2.2.29. We keep the setting of Definition 2.2.28.

(a) Consider the special case where $\mathbf{L} = \mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, so that $\mathbf{P} = \mathbf{B}$ is a Borel subgroup of $\mathbf{G}$, and $\mathbf{U_B} = R_\mathrm{u}(\mathbf{B})$. As mentioned in Definition 2.2.3, the definition of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ (for $\theta \in \mathrm{Irr}(\mathbf{T}^F)$) is independent of the choice of the Borel subgroup containing $\mathbf{T}$, so we may as well apply the construction with $F(\mathbf{B})$ instead of $\mathbf{B}$, and we see that Lusztig induction generalises the concept of the Deligne–Lusztig characters. In particular, it is justified to use the same symbol for both the Deligne–Lusztig characters and the Lusztig induction in this case.

(b) If the regular subgroup $\mathbf{L}$ is the Levi complement of some *F-stable* parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, then $R_{\mathbf{L},F(\mathbf{U_P})}^{\mathbf{G}}$ is in fact just Harish-Chandra induction (cf. 2.2.6), as remarked in the introduction of [Lus76a] (see also the note after [DM20, 9.1.4]). Hence, Lusztig induction also generalises Harish-Chandra induction, and again we do not have to worry about using the same symbol for these two concepts.

(c) As mentioned in [Lus76a, p. 204] (see also [DM20, 9.1.8] for a detailed proof), Lusztig induction is transitive, in the sense that if $\mathbf{L}$ is a Levi complement of some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, and if $\mathbf{M}$ is a Levi complement of some parabolic subgroup $\mathbf{Q}$ of $\mathbf{L}$, such that $F(\mathbf{L}) = \mathbf{L}$ and $F(\mathbf{M}) = \mathbf{M}$, then

$$R_{\mathbf{L},F(\mathbf{U_P})}^{\mathbf{G}} \circ R_{\mathbf{M},F(\mathbf{U_Q})}^{\mathbf{L}} = R_{\mathbf{M},F(\mathbf{U_Q})}^{\mathbf{G}}$$

(with the notation of 2.2.2).

(d) If $g = u \in \mathbf{G}_{\mathrm{uni}}^F$ is a unipotent element and $\pi \in \mathrm{Irr}(\mathbf{L}^F)$, we have

$$(R_{\mathbf{L},F(\mathbf{U_P})}^{\mathbf{G}}(\pi))(u) = \frac{1}{|\mathbf{L}^F|} \sum_{l \in \mathbf{L}_{\mathrm{uni}}^F} \mathfrak{L}((u,l), \mathcal{L}^{-1}(F(\mathbf{U_P})))\pi(l)$$

by [Lus76a, p. 203]. (This follows at once from a property of the ℓ-adic cohomology groups with compact support, see [Car85, 7.1.10].)

(e) It has been conjectured by Lusztig [Lus76a] that the twisted induction with respect to a regular subgroup $\mathbf{L} \subseteq \mathbf{G}$ is independent of the choice of a parabolic subgroup with Levi complement $\mathbf{L}$ used in its construction. At least if F is a Frobenius map for an $\mathbb{F}_q$-rational structure on $\mathbf{G}$ and if $q > 2$ (or alternatively, if the centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$ is connected, in which case no assumption on q is necessary), this is known due to [Lus90] and [BM11] (and [Sho96, §4], combined with the results in [Lus12b] as far as certain small p are concerned); see also 3.2.7 below for some more details.

2.3. Hecke algebras associated to finite groups of Lie type

In this section, we consider the permutation module $\overline{\mathbb{Q}}_\ell[{}^{G}/_{B}]$ where G is a finite group of Lie type and $B \subseteq G$ is a subgroup occurring in a BN-pair (B, N) in G (see 2.1.20). The endomorphism algebra of $\overline{\mathbb{Q}}_\ell[{}^{G}/_{B}]$ is a deformation of the group algebra of the Weyl group associated to the BN-pair (B, N) in G (cf. 2.2.6), and this fact gives rise to a formula which relates the irreducible characters of said endomorphism algebra with the unipotent principal series characters of G in terms of intersections of G-conjugacy classes with Bruhat cells, see (2.3.9.2) below. This formula is well known and appeared explicitly in [Gec11, §3], [Lus11b], cf. Ree's formula in [CR81, §11D], but since it is a crucial ingredient of numerous arguments in this thesis, we will provide a detailed exhibition here.

Let us fix the following notation and assumptions, which remain in force throughout this section. We denote by $\mathbf{G}$ a connected reductive group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$; furthermore, we assume that the centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$ is connected. We also fix a maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$ and an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ which contains $\mathbf{T}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$. (In particular, we have $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$.) Let $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$, and let $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ be the automorphism induced by F. We identify $\mathbf{W}$ with the Weyl group of R, using the action of $\mathbf{W}$ on X defined in 2.1.4. Denote by $\Pi \subseteq R$ the simple roots determined by $\mathbf{B}_0$, with positive roots $R^+ \subseteq R$ determined by Π, and let $S := \{w_\alpha \in \mathbf{W} \mid \alpha \in \Pi\}$ be the set of simple reflections corresponding to Π; we have $\sigma(S) = S$ (see 2.1.19). Recall from 2.1.20 that $\mathbf{W}^\sigma$ is a finite Coxeter group with Coxeter generators S_σ. We denote by $l_\sigma\colon \mathbf{W}^\sigma \to \mathbb{N}_0$ the length function of $\mathbf{W}^\sigma$ with respect to S_σ.

2.3.1. Let us consider the set of left cosets ${}^{\mathbf{G}^F}/_{\mathbf{B}_0^F}$ of $\mathbf{B}_0^F$ in $\mathbf{G}^F$. The left multiplication action of the finite group $\mathbf{G}^F$ on itself induces an action of $\mathbf{G}^F$ on ${}^{\mathbf{G}^F}/_{\mathbf{B}_0^F}$. Extending this action by linearity thus gives rise to a (left) module $\overline{\mathbb{Q}}_\ell[{}^{\mathbf{G}^F}/_{\mathbf{B}_0^F}]$ for the group algebra $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$ of $\mathbf{G}^F$, called the *permutation module* of $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$ with respect to the subgroup $\mathbf{B}_0^F \subseteq \mathbf{G}^F$. The character of $\mathbf{G}^F$ afforded by the corresponding representation is given by

$$\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F}\left(1_{\mathbf{B}_0^F}\right) = R_{\mathbf{T}_0}^{\mathbf{G}}\left(1_{\mathbf{T}_0^F}\right)$$

(see 2.2.6). In particular, the characters of the irreducible representations which appear as direct summands of the module $\overline{\mathbb{Q}}_\ell[{}^{\mathbf{G}^F}/_{\mathbf{B}_0^F}]$ are unipotent. The irreducible representations (characters) thus obtained are called the *unipotent principal series representations* (*characters*) *of* $\mathbf{G}^F$ (cf. 2.2.6). We consider the endomorphism algebra

$$\mathcal{H}_{\sigma,q} := \mathrm{End}_{\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]}\Big(\overline{\mathbb{Q}}_\ell\Big[\mathbf{G}^F/\mathbf{B}_0^F\Big]\Big)^{\mathrm{opp}};$$

here, 'opp' stands for the opposite algebra, that is, the multiplication in $\mathcal{H}_{\sigma,q}$ is given by

$$f \cdot g = g \circ f \quad \text{for } f, g \in \mathcal{H}_{\sigma,q}.$$

We define a function

$$S_\sigma \to \overline{\mathbb{Q}}_\ell, \quad s' \mapsto q_{s'} := \frac{|\mathbf{B}_0^F s' \mathbf{B}_0^F|}{|\mathbf{B}_0^F|} = q^{l(s')}.$$

(Here, l denotes the length function for the Coxeter system $(\mathbf{W}, S)$; the last equality follows from the *sharp form* of the Bruhat decomposition in the sense of [Car85, 2.5.14] or [Gec03a, 1.7.2], combined with a theorem of Rosenlicht [Gec03a, §4.2]. We typically write s' instead of s for elements of S_σ in order to avoid confusion with elements of S, as S_σ is in general not necessarily a subset of S.) We call $\mathcal{H}_{\sigma,q}$ the *Hecke algebra of* $\mathbf{G}^F$ (*with respect to the* BN*-pair* $(\mathbf{B}_0^F, N_{\mathbf{G}}(\mathbf{T}_0)^F)$ *in* $\mathbf{G}^F$, see 2.1.20), and $q_{s'}$ $(s' \in S_\sigma)$ *the parameters of* $\mathcal{H}_{\sigma,q}$. The justification of this notion will be given in Proposition 2.3.2 below. Following [Gec11, 3.6] (cf. [GP00, 8.4.1]) we define, for any $w \in \mathbf{W}^\sigma$, a $\overline{\mathbb{Q}}_\ell$-linear map

$$T_w \colon \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F] \to \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F],$$

uniquely determined by

$$T_w\Big(x\mathbf{B}_0^F\Big) := \sum_{\substack{y\mathbf{B}_0^F \in \mathbf{G}^F/\mathbf{B}_0^F \\ x^{-1}y \in \mathbf{B}_0^F w \mathbf{B}_0^F}} y\mathbf{B}_0^F \qquad \text{for } x \in \mathbf{G}^F.$$

Proposition 2.3.2 (Lusztig [Lus76b, §5, (7.7)], see [Lus78, Thm. 3.26])**.** *In the setting and with the notation of 2.3.1, we have* $T_w \in \mathcal{H}_{\sigma,q}$ *for any* $w \in \mathbf{W}^\sigma$. *Furthermore, the following hold.*

(a) The set $\{T_w \mid w \in \mathbf{W}^\sigma\}$ *is a* $\overline{\mathbb{Q}}_\ell$*-basis of* $\mathcal{H}_{\sigma,q}$.

(b) With the notation of A.2.3, $\mathcal{H}_{\sigma,q}$ *arises from the generic Iwahori–Hecke algebra* $\mathcal{H}(\mathbf{W}^\sigma, \mathbf{v}_{s'}^2 \mid s' \in S_\sigma)$ *over* $\mathbf{A} = \mathbb{Z}[\mathbf{v}_{s'}^{\pm 1} \mid s' \in S_\sigma]$ *by specialisation along the ring*

homomorphism

$$\varphi\colon \mathbf{A} \to \overline{\mathbb{Q}}_\ell, \quad \text{uniquely determined by } \varphi(\mathbf{v}_{s'}) = \sqrt{q_{s'}} \quad \text{for any } s' \in S_\sigma.$$

(Note that, since each $q_{s'}$ is an integral power of p, the choice of the square roots $\sqrt{q_{s'}}$ is taken care of by convention (1.1.0.2)*, see also Remark* 2.3.3 *below.) Hence, for $w \in \mathbf{W}^\sigma$ and $s' \in S_\sigma$, we have*

$$T_w \cdot T_{s'} = \begin{cases} T_{ws'} & \text{if} \quad l_\sigma(ws') = l_\sigma(w) + 1, \\ q_{s'} T_{ws'} + (q_{s'} - 1) T_w & \text{if} \quad l_\sigma(ws') = l_\sigma(w) - 1. \end{cases}$$

Proof. This is very similar to [GP00, §8.4], the main difference is that in our setting, we have to deal with the left cosets of $\mathbf{G}^F$ modulo $\mathbf{B}_0^F$ instead of right cosets, and that we consider the opposite endomorphism algebra. We will thus only sketch the proof here and mostly refer to the proof in loc. cit.

To simplify the notation, we write $B := \mathbf{B}_0^F$, $G := \mathbf{G}^F$. For $\psi \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\overline{\mathbb{Q}}_\ell[{}^G/_B])$, we define a function (cf. [GP00, 8.4.1])

$$\dot\psi\colon {}^G/_B \times {}^G/_B \to \overline{\mathbb{Q}}_\ell$$

by the requirement that

$$\psi(xB) = \sum_{yB \in G/B} \dot\psi(xB, yB) yB \quad \text{for all } xB \in {}^G/_B.$$

For $w \in \mathbf{W}^\sigma$, we thus have

$$\dot T_w(xB, yB) = \begin{cases} 1 & \text{if } Bx^{-1}yB = BwB, \\ 0 & \text{otherwise.} \end{cases}$$

Showing that the T_w ($w \in \mathbf{W}^\sigma$) are actually in $\mathrm{End}_{\overline{\mathbb{Q}}_\ell[G]}(\overline{\mathbb{Q}}_\ell[{}^G/_B])$ and that they form a basis of this $\overline{\mathbb{Q}}_\ell$-vector space is completely analogous to the proof of [GP00, 8.4.2]. Now let $w, w' \in \mathbf{W}^\sigma$. Then

$$\begin{aligned}(T_w \cdot T_{w'})(B) = T_{w'}(T_w(B)) &= T_{w'}\Bigg(\sum_{yB\in G/B} \dot{T}_w(B, yB)yB\Bigg) \\ &= \sum_{y'B\in G/B}\Bigg(\sum_{yB\in G/B} \dot{T}_w(B, yB)\dot{T}_{w'}(yB, y'B)\Bigg)y'B.\end{aligned}$$

Hence, the coefficient of $y'B$ in the above expression is given by

$$\left|\{yB \in {}^{G}/_{B} \mid BwB = ByB \text{ and } Bw'B = By^{-1}y'B\}\right| = \frac{|BwB \cap y'Bw'^{-1}B|}{|B|}.$$

By the Bruhat decomposition (see 2.1.20), there is a unique $w'' \in \mathbf{W}^\sigma$ such that $By'B = Bw''B$, and this w'' satisfies

$$\frac{|BwB \cap y'Bw'^{-1}B|}{|B|} = \frac{|BwB \cap w''Bw'^{-1}B|}{|B|} =: a_{w,w',w''}.$$

On the other hand, we have

$$\sum_{w''\in\mathbf{W}^\sigma} a_{w,w',w''}T_{w''}(B) = \sum_{y'B\in G/B}\Bigg(\sum_{w''\in\mathbf{W}^\sigma} a_{w,w',w''}\dot{T}_{w''}(B, y'B)\Bigg)y'B.$$

So if $w'' \in \mathbf{W}^\sigma$ is the (unique) element such that $Bw''B = By'B$, the coefficient of $y'B$ in the above expression is $a_{w,w',w''}$. This proves that

$$T_w \cdot T_{w'} = \sum_{w''\in\mathbf{W}^\sigma} a_{w,w',w''}T_{w''}.$$

(Note that it is sufficient to compare the evaluation of either side at B, since on both sides we have a $\overline{\mathbb{Q}}_\ell[G]$-linear function and $\overline{\mathbb{Q}}_\ell[{}^{G}/_{B}]$ is a cyclic $\overline{\mathbb{Q}}_\ell[G]$-module generated by B.)

Verifying the multiplication rules for $T_w \cdot T_{s'}$ ($w \in \mathbf{W}^\sigma$, $s' \in S_\sigma$) thus amounts to evaluating the $a_{w,s',w''}$ for $w'' \in \mathbf{W}^\sigma$, which is a simple computation based on the BN-pair axioms, entirely analogous to the proof of [GP00, 8.4.6]. These multiplication rules completely determine the multiplication of the $\overline{\mathbb{Q}}_\ell$-algebra $\mathcal{H}_{\sigma,q}$. Hence, in order to show that $\mathcal{H}_{\sigma,q}$ is the specialisation of the generic Iwahori–Hecke algebra $\mathcal{H}(\mathbf{W}^\sigma, \mathbf{v}_{s'}^2 \mid s' \in S_\sigma)$ along the map φ in the proposition, we only need to know that φ is actually well-defined. This follows from the fact that $q_{s'} = q_{s''}$ whenever $s', s'' \in S_\sigma$ are conjugate in $\mathbf{W}^\sigma$ (see [Lus78, 3.26] and [Lus76b, (7.7)]). □

Remark 2.3.3. (a) Since Proposition 2.3.2 is a special case of [Lus78, Thm. 3.26], we can extract the description of the Coxeter group $(\mathbf{W}^\sigma, S_\sigma)$ as well as the numbers $q_{s'}$ from [Lus78], for any simple algebraic group $\mathbf{G}$ of adjoint type. (The relevant lines in [Lus78, Table II (p. 35)] are those where Γ' is empty; the function $S_\sigma \to \overline{\mathbb{Q}}_\ell,\ s' \mapsto q_{s'}$, is denoted by λ in loc. cit.) For example, if $\sigma = \mathrm{id}_{\mathbf{W}}$, we have $q_s = q$ for all $s \in S$.

(b) Let $e \in \mathbb{N}$ be such that $q = p^e$. Recall that we fixed a square root $\sqrt{p} \in \overline{\mathbb{Q}}_\ell$ of p in (1.1.0.2). Since $q_{s'} = q^{l(s')}$ is an integral power of q for any $s' \in S_\sigma$, our convention (1.1.0.2) tells us that

$$\sqrt{q_{s'}} = (\sqrt{p})^{e \cdot l(s')} \quad \text{for } s' \in S_\sigma,$$

so the map φ in Proposition 2.3.2 is defined unambiguously.

2.3.4. We place ourselves in the setting of 2.3.1. By Example A.2.4 in the appendix, we have a natural bijection

$$\mathrm{Irr}(\mathcal{H}_L(\mathbf{W}^\sigma, \mathbf{v}_{s'}^2 \mid s' \in S_\sigma)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{W}^\sigma), \quad \phi \mapsto \phi_1,$$

where $L = \mathbb{Q}(\mathbf{v}_{s'} \mid s' \in S_\sigma)$ is the field of fractions of $\mathbf{A} = \mathbb{Z}[\mathbf{v}_{s'}^{\pm 1} \mid s' \in S_\sigma]$. On the other hand, in view of Proposition 2.3.2, we can also apply the procedure in A.2.3 to the specialisation $\mathcal{H}_{\sigma,q} \cong \mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{W}^\sigma, q_{s'} \mid s' \in S_\sigma)$. (Clearly, $\mathcal{H}_{\sigma,q}$ is a split semisimple algebra since $\overline{\mathbb{Q}}_\ell$ is an algebraically closed field of characteristic zero, so condition (♠) in A.2.3 is satisfied.) We thus obtain a bijection

$$\mathrm{Irr}(\mathcal{H}_L(\mathbf{W}^\sigma, \mathbf{v}_{s'}^2 \mid s' \in S_\sigma)) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_{\sigma,q}), \quad \phi \mapsto \phi_{\sigma,q}.$$

Hence, once a square root of p is chosen as in (1.1.0.2), there is a natural bijection between the irreducible characters of $\mathbf{W}^\sigma$ and the irreducible characters of $\mathcal{H}_{\sigma,q}$.

2.3.5. We keep the setting of 2.3.1. The algebra $\mathcal{H}_{\sigma,q}$ acts from the right on ${\mathbf{G}^F}/{\mathbf{B}_0^F}$ by

$${\mathbf{G}^F}/{\mathbf{B}_0^F} \times \mathcal{H}_{\sigma,q} \to {\mathbf{G}^F}/{\mathbf{B}_0^F}, \quad (g\mathbf{B}_0^F, f) \mapsto f(g\mathbf{B}_0^F).$$

Extending this action by linearity thus provides $E := \overline{\mathbb{Q}}_\ell[{\mathbf{G}^F}/{\mathbf{B}_0^F}]$ with the structure of a right $\mathcal{H}_{\sigma,q}$-module. Hence, in view of the left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-module structure defined in 2.3.1, E is a $(\overline{\mathbb{Q}}_\ell[\mathbf{G}^F], \mathcal{H}_{\sigma,q})$-bimodule. For any left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-module M, the algebra $\mathcal{H}_{\sigma,q}$ naturally acts from the right on the $\overline{\mathbb{Q}}_\ell$-vector space

$$V_M := \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]}(M, E)$$

by composition of functions, and the Hom-functor

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]}(-,E)\colon M \mapsto V_M \quad \text{(for a left } \overline{\mathbb{Q}}_\ell[\mathbf{G}^F]\text{-module } M)$$

defines a bijection between the isomorphism classes of the simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-modules which appear as direct summands of E and the isomorphism classes of the simple right $\mathcal{H}_{\sigma,q}$-modules. (This is a special case of a general statement concerning the Hom-functor with respect to a semisimple object E in any locally finite $\overline{\mathbb{Q}}_\ell$-linear abelian category, see, e.g., [TT20, Appendix A].) For any simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-module M which is isomorphic to a direct summand of E, the left action of $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$ on M and the right action of $\mathcal{H}_{\sigma,q}$ on V_M make the tensor product $M \otimes_{\overline{\mathbb{Q}}_\ell} V_M$ into a $(\overline{\mathbb{Q}}_\ell[\mathbf{G}^F], \mathcal{H}_{\sigma,q})$-bimodule, whose structure is uniquely determined by

$$g.(v \otimes f).T = (g.v) \otimes (T \circ f) \quad \text{for any } g \in \mathbf{G}^F,\ T \in \mathcal{H}_{\sigma,q},\ v \in M,\ f \in V_M.$$

Lemma 2.3.6. *We place ourselves in the setting of* 2.3.5*. There is an isomorphism of* $(\overline{\mathbb{Q}}_\ell[\mathbf{G}^F], \mathcal{H}_{\sigma,q})$*-bimodules*

$$\bigoplus_{(M|E)/\simeq} (M \otimes_{\overline{\mathbb{Q}}_\ell} V_M) \xrightarrow{\sim} E \tag{2.3.6.1}$$

(where the index $(M|E)/_\simeq$ *means that the sum is taken over a set of representatives for the isomorphism classes of the simple left* $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$*-submodules of* E*), such that for any* M *occurring in the sum, any* $v \in M$ *and any* $f \in V_M$*,* $v \otimes f$ *corresponds to* $f(v)$ *under* (2.3.6.1)*.*

Proof. Let us decompose E into a direct sum of simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-submodules, and let $M_1, M_2, \ldots, M_n$ ($n \in \mathbb{N}$) be a set of representatives for the isomorphism classes of left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-modules which occur in this decomposition. We can thus write

$$E = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} M_{ij},$$

where for any $1 \leqslant i \leqslant n$ and for any $1 \leqslant j \leqslant m_i$, M_{ij} is a simple $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-submodule of E which is isomorphic to M_i. (Hence, $m_i \in \mathbb{N}$ is the number of $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-submodules of E which are isomorphic to M_i.) Choosing isomorphisms $M_i \xrightarrow{\sim} M_{ij}$ (for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$) thus determines embeddings

$$\iota_{ij}\colon M_i \xrightarrow{\sim} M_{ij} \hookrightarrow E \quad \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m_i.$$

Now for $1 \leqslant i \leqslant n$, consider the $\overline{\mathbb{Q}}_\ell$-bilinear map

$$M_i \times V_{M_i} \to E, \quad (v_i, f_i) \mapsto f_i(v_i).$$

Taking the direct sum and using the universal property of the tensor product, we obtain a unique $\overline{\mathbb{Q}}_\ell$-linear map

$$\bigoplus_{i=1}^{n} M_i \otimes_{\overline{\mathbb{Q}}_\ell} V_{M_i} \to E \tag{2.3.6.2}$$

with the property that $v_i \otimes f_i$ is sent to $f_i(v_i)$ for any $i \in \{1, \dots, n\}$, $v_i \in M_i$ and $f_i \in V_{M_i}$. Since $\iota_{ij} \in V_{M_i}$ and $\iota_{ij}(M_i) = M_{ij}$ for any $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$, the map (2.3.6.2) is surjective. On the other hand, for any $1 \leqslant i \leqslant n$, the ι_{ij} $(1 \leqslant j \leqslant m_i)$ form a basis of the $\overline{\mathbb{Q}}_\ell$-vector space V_{M_i}, so $\dim_{\overline{\mathbb{Q}}_\ell} V_{M_i} = m_i$, and it follows that both sides of (2.3.6.2) have the same dimension. Hence, the map (2.3.6.2) defines an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces. Finally, it is clear from the construction that this isomorphism respects the $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-structure as well as the $\mathcal{H}_{\sigma,q}$-structure. □

Remark 2.3.7. The correspondence $M \mapsto V_M$ in 2.3.5 between the isomorphism classes of simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-modules which appear in $E = \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F]$ and the isomorphism classes of simple right $\mathcal{H}_{\sigma,q}$-modules is the same as the one described in [GP00, 8.4.4] (and also the one in [CR81, §11D]). Indeed, since $\mathcal{H}_{\sigma,q}$ is a split semisimple $\overline{\mathbb{Q}}_\ell$-algebra, it has a Wedderburn decomposition of the form

$$\mathcal{H}_{\sigma,q} = \bigoplus_{V \in \mathrm{Irr}(\mathcal{H}_{\sigma,q})} \mathcal{H}_{\sigma,q}(V)$$

(cf. [GP00, Chap. 7]), where $\mathrm{Irr}(\mathcal{H}_{\sigma,q})$ denotes a set of representatives for the isomorphism classes of simple right $\mathcal{H}_{\sigma,q}$-modules, and where $\mathcal{H}_{\sigma,q}(V)$ is a two-sided ideal in $\mathcal{H}_{\sigma,q}$, which is itself a simple $\overline{\mathbb{Q}}_\ell$-algebra isomorphic to $M_{n_V}(\overline{\mathbb{Q}}_\ell)$ with $n_V = \dim_{\overline{\mathbb{Q}}_\ell} V$. Thus, we have a decomposition

$$1_{\mathcal{H}_{\sigma,q}} = \sum_{V \in \mathrm{Irr}(\mathcal{H}_{\sigma,q})} e_V$$

of the identity element of $\mathcal{H}_{\sigma,q}$ into pairwise orthogonal centrally primitive idempotents of $\mathcal{H}_{\sigma,q}$. For each $V \in \mathrm{Irr}(\mathcal{H}_{\sigma,q})$, the right $\mathcal{H}_{\sigma,q}$-module $e_V \cdot \mathcal{H}_{\sigma,q}$ is isomorphic to $V^{\oplus n_V}$. On the other hand, we may further decompose e_V as a sum

$$e_V = \sum_{i=1}^{n_V} e_{ii}^V$$

of orthogonally primitive idempotents e_{ii}^V in $\mathcal{H}_{\sigma,q}$, so that

$$1_{\mathcal{H}_{\sigma,q}} = \sum_{V\in\mathrm{Irr}(\mathcal{H}_{\sigma,q})} \sum_{i=1}^{n_V} e_{ii}^V.$$

Since $\mathcal{H}_{\sigma,q}$ is the (opposite) endomorphism algebra of E, this corresponds to a decomposition

$$E = \bigoplus_{V\in\mathrm{Irr}(\mathcal{H}_{\sigma,q})} \bigoplus_{i=1}^{n_V} e_{ii}^V(E)$$

of E into a direct sum of simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-submodules. For any fixed $V \in \mathrm{Irr}(\mathcal{H}_{\sigma,q})$, the simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-modules $e_{ii}^V(E)$ $(1 \leqslant i \leqslant n_V)$ are isomorphic to each other, and their isomorphism class is the one corresponding to the simple $\mathcal{H}_{\sigma,q}$-module V in the setting of [GP00, 8.4.4]. Thus, we have to show that $V^{\oplus n_V} \cong e_V \cdot \mathcal{H}_{\sigma,q}$ is isomorphic to $\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]}(e_V(E), E)$ as right $\mathcal{H}_{\sigma,q}$-modules. Such an isomorphism is given by

$$e_V \cdot \mathcal{H}_{\sigma,q} \xrightarrow{\sim} \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]}(e_V(E), E), \quad e_V \cdot h \mapsto h|_{e_V(E)}.$$

Proposition 2.3.8 (cf. [Gec11, 3.6(b)], [Lus11b, 1.5(a)]). *Let $w \in \mathbf{W}^\sigma$, $g \in \mathbf{G}^F$, and denote by $O_g \subseteq \mathbf{G}^F$ the $\mathbf{G}^F$-conjugacy class of g. Then, with the notation of* 2.3.5*, we have*

$$\sum_{(M|E)/\simeq} \mathrm{Trace}(g, M)\,\mathrm{Trace}(T_w, V_M) = \frac{|O_g \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(g)|}{|\mathbf{B}_0^F|},$$

where the sum is taken over a set of representatives for the isomorphism classes of the simple left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-submodules of $E = \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F]$.

Proof. We set $G := \mathbf{G}^F$, $B := \mathbf{B}_0^F$, $E := \overline{\mathbb{Q}}_\ell[G/B]$. Let $g, x \in G$ and $w \in \mathbf{W}^\sigma$. Then, using the notation of the proof of Proposition 2.3.2, we get

$$g^{-1}.(xB).T_w = T_w(g^{-1}xB) = \sum_{yB\in G/B} \dot{T}_w(g^{-1}xB, yB)yB,$$

so

$$\mathrm{Trace}((g^{-1}, T_w), E) = \sum_{xB\in G/B} \dot{T}_w(g^{-1}xB, xB) = \frac{1}{|B|} \sum_{x\in G} \dot{T}_w(g^{-1}xB, xB).$$

By the definition of $\dot{T}_w$, we have

$$\dot{T}_w(g^{-1}xB, xB) = \begin{cases} 1 & \text{if } x^{-1}gx \in BwB, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $O_g \cap BwB = \varnothing$, we get $\mathrm{Trace}((g^{-1}, T_w), E) = 0$. Now assume that $O_g \cap BwB \neq \varnothing$; let us fix some $z \in O_g \cap BwB$ and $x_0 \in G$ such that $x_0^{-1} g x_0 = z$. Then for $x \in G$, we have $x^{-1}gx = z$ if and only if $x \in C_G(g)x_0$. This shows that for any $z \in O_g \cap BwB$, there are exactly $|C_G(g)|$ different $x \in G$ with $x^{-1}gx = z$. We deduce that

$$\mathrm{Trace}((g^{-1}, T_w), E) = \frac{|O_g \cap BwB| \cdot |C_G(g)|}{|B|}$$

(regardless of the assumption on $O_g \cap BwB$). Using Lemma 2.3.6, we obtain

$$\sum_{(M|E)/\simeq} \mathrm{Trace}(g^{-1}, M)\,\mathrm{Trace}(T_w, V_M) = \frac{|O_g \cap BwB| \cdot |C_G(g)|}{|B|}.$$

Viewing this as an equation in $\mathbb{C}$ by means of the isomorphism (1.1.0.1), we certainly have $\mathrm{Trace}(T_w, V_M) \in \mathbb{R}$ (see [GP00, 9.3.5]), so applying 'complex conjugation' yields the result. □

2.3.9. As discussed in 2.3.4, once a square root of p in $\overline{\mathbb{Q}}_\ell$ is fixed as in (1.1.0.2), we have a natural bijection

$$\mathrm{Irr}(\mathbf{W}^\sigma) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_{\sigma,q}), \quad \phi \mapsto V_\phi.$$

(Here, we write $\mathrm{Irr}(\mathcal{H}_{\sigma,q})$ for a set of representatives for the isomorphism classes of simple right $\mathcal{H}_{\sigma,q}$-modules, while $\mathrm{Irr}(\mathbf{W}^\sigma)$ is the set of irreducible characters of $\mathbf{W}^\sigma$, as usual.) For any $\phi \in \mathrm{Irr}(\mathbf{W}^\sigma)$, we choose a left $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-submodule M of $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_0^F]$ which corresponds to $V_\phi \in \mathrm{Irr}(\mathcal{H}_{\sigma,q})$ (so that $V_\phi \cong V_M$, see 2.3.5) and denote by $[\phi] \in \mathrm{Uch}(\mathbf{G}^F) \subseteq \mathrm{Irr}(\mathbf{G}^F)$ the character of M. With this notation, we set

$$m(g,w) := \sum_{\phi \in \mathrm{Irr}(\mathbf{W}^\sigma)} [\phi](g)\,\mathrm{Trace}(T_w, V_\phi) \quad \text{for } g \in \mathbf{G}^F,\ w \in \mathbf{W}^\sigma. \tag{2.3.9.1}$$

By Proposition 2.3.8 we have, for any $g \in \mathbf{G}^F$ and any $w \in \mathbf{W}^\sigma$:

$$m(g,w) = \frac{|O_g \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(g)|}{|\mathbf{B}_0^F|}. \tag{2.3.9.2}$$

Note that the character tables [GP00, 8.2.9] of the (generic) Iwahori–Hecke algebras have been completely determined (see [GP00, Chapters 10, 11]), so the numbers $\mathrm{Trace}(T_w, V_\phi)$ are known for all $w \in \mathbf{W}^\sigma$ and all $\phi \in \mathrm{Irr}(\mathbf{W}^\sigma)$. They are readily available through Michel's CHEVIE [MiChv], using the function `HeckeCharValues`. As already indicated in the introduction to this section (and also in the global introduction), (2.3.9.2) will be of paramount importance for numerous arguments in this thesis related to the computation of character values of finite groups of Lie type.

Remark 2.3.10. Proposition 2.3.2 (and even [Lus78, Thm. 3.26]) is just a special case of [Lus84a, Thm. 8.6], which covers the general situation (under the assumption that $\mathbf{Z}(\mathbf{G})$ is connected). Namely, for any σ-stable subset $J \subseteq S$ and any cuspidal character $\rho_0 \in \mathrm{Irr}(\mathbf{L}_J^F)^\circ$, consider the Harish-Chandra induced character $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ of $\mathbf{G}^F$. (Note that both the standard Levi subgroup $\mathbf{L}_J$ and the standard parabolic subgroup $\mathbf{P}_J$ are F-stable.) Let $\mathbf{W}(J, \rho_0)$ be the stabiliser of ρ_0 in $N_{\mathbf{G}}(\mathbf{L}_J)^F/\mathbf{L}_J^F$ under the action defined in (2.2.6.1). From [Lus84a, 8.5] (cf. [Lus76b, §5]), it follows that $\mathbf{W}(J, \rho_0)$ is a Coxeter group in a natural way. Then [Lus84a, Thm. 8.6] provides an isomorphism between the $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-endomorphism algebra of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ and a suitable specialisation (along a map which generalises the definition of φ in Proposition 2.3.2) of the generic Iwahori–Hecke algebra associated to $\mathbf{W}(J, \rho_0)$. Let us denote by

$$\mathrm{Irr}(\mathbf{G}^F \mid J, \rho_0) \subseteq \mathrm{Irr}(\mathbf{G}^F)$$

the subset consisting of those irreducible characters of $\mathbf{G}^F$ which appear as constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$. By [Lus84a, 8.7], our choice (1.1.0.2) of a square root of p (cf. 2.3.4) gives rise to a natural bijection

$$\mathrm{Irr}(\mathbf{W}(J, \rho_0)) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{G}^F \mid J, \rho_0), \quad \phi \mapsto \rho_0[\phi], \tag{2.3.10.1}$$

and this is exactly the parametrisation that we already referred to in 2.2.6. If $J = \varnothing$ and ρ_0 is the trivial character of $\mathbf{L}_\varnothing^F = \mathbf{T}_0^F$, we have $W(\varnothing, 1_{\mathbf{T}_0^F}) \cong \mathbf{W}^\sigma$, and $\mathrm{Irr}(\mathbf{G}^F \mid \varnothing, 1_{\mathbf{T}_0^F})$ is the set of irreducible constituents of

$$R_{\mathbf{T}_0}^{\mathbf{G}}(1_{\mathbf{T}_0^F}) = \mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F}(1_{\mathbf{B}_0^F}),$$

so we recover our special case treated in this section. (In this situation, it is common to omit $\rho_0 = 1_{\mathbf{T}_0^F}$ from the notation in (2.3.10.1) and just write $\phi \mapsto [\phi]$, as we have done in 2.3.9.)

3. Character sheaves

In this chapter, we present some of the essential features concerning Lusztig's theory of character sheaves on connected reductive groups over $k = \overline{\mathbb{F}}_p$ [LuCS1]–[LuCS5] and their consequences for the finite groups of Lie type. Our main references are [Lus84b], [LuCS1]–[LuCS5], [Lus12b] and [Sho95a], [Sho95b]. We put a special emphasis on those properties and results which are directly linked to the character theory of finite groups of Lie type, most notably with regard to the computation of actual character values.

In Section 3.1, we state the fundamental definitions and structural properties of character sheaves on connected reductive groups over k, without yet taking any $\mathbb{F}_q$-structure into consideration; this includes the unipotent character sheaves (Definition 3.1.8), as well as the cuspidal character sheaves and their relation to the irreducible cuspidal perverse sheaves (3.1.10–3.1.13). In analogy to the significance of cuspidal characters to the classification of irreducible characters of finite groups of Lie type in terms of Harish-Chandra series, we can then indicate the relevance of cuspidal character sheaves by means of a certain induction process on the level of perverse sheaves. The cuspidal character sheaves on connected reductive groups have been completely classified by Lusztig (see Remark 3.1.14).

In Section 3.2, we explain how an $\mathbb{F}_q$-rational structure on a connected reductive group over k transfers to the perverse sheaves on such a group and how this gives rise to characteristic functions of Frobenius-invariant character sheaves (see 3.2.1). We can then state Lusztig's result (Theorem 3.2.2) that these characteristic functions form a basis of the space of class functions of the associated finite group of Lie type. We proceed with a detailed discussion on the computation of characteristic functions and how this is in principle reduced to the computation of the generalised Green functions on finite reductive groups (3.2.3–3.2.11). These generalised Green functions are defined on the set of $\mathbb{F}_q$-rational points of the unipotent variety of the connected reductive group in question. As for their computation, Lusztig developed a purely combinatorial algorithm (which is implemented in Michel's CHEVIE [MiChv]) based on the generalised Springer correspondence (see 3.2.13–3.2.14 and 3.2.16–3.2.18). Since the generalised Springer correspondence is now explicitly known in complete generality (due to Lusztig [Lus84b], [Lus19]; Lusztig–Spaltenstein [LS85]; Spaltenstein [Spa85]; and the author [Het22b], where the last open cases for groups of type E_8 are settled), Lusztig's algorithm allows the determination of the generalised Green functions up to certain roots of unity (see 3.2.19). One of the problems which remains to be addressed in this framework is that of specifying suitable normalisations of characteristic functions of Frobenius-invariant character sheaves; we discuss this thoroughly in 3.2.20–3.2.25 where, in particular, the choice of a 'good' representative in the set of Frobenius-fixed points of a conjugacy class of the underlying connected reductive group plays an important role.

In Section 3.3, we start by describing a parametrisation of character sheaves in terms of a multiplicity formula for character sheaves due to Lusztig (see Theorem 3.3.2, Proposition 3.3.6), which highlights the analogy to the classification of irreducible characters of finite groups of Lie type via Theorem 2.2.21. Having those 'parallel' parametrisations of Frobenius-invariant character sheaves and almost characters at hand, we are then prepared to formulate Lusztig's Conjecture, which states that any almost character should coincide with a *suitably normalised* characteristic function of the corresponding character sheaf. — Hence, as the characteristic functions of character sheaves are computable 'in principle' and since the almost characters are given by explicit linear combinations of irreducible characters, the proof of Lusztig's Conjecture would constitute a major step towards determining the generic character tables of finite groups of Lie type. In the case where the underlying connected reductive group over k has a connected centre, Lusztig's Conjecture was proven by Shoji (see Theorem 3.3.9), by means of showing that any almost character coincides with a chosen characteristic function of the corresponding character sheaf *up to multiplication with a non-zero scalar*. Thus, the problem of determining the scalars involved in Shoji's Theorem remains to be resolved, and the investigation of said problem forms the core of this thesis. More precisely, we will consider this problem as far as the unipotent character sheaves on simple groups are concerned. In this situation, Shoji [Sho97], [Sho09] has determined the scalars in question for any classical group (see 3.3.13).

In Section 3.4, we consider non-twisted finite groups of Lie type arising from simple groups with a trivial centre, and we focus on the unipotent character sheaves on those groups. In this case, one obtains explicit parametrisations of the unipotent characters and unipotent character sheaves, which are 'compatible' in the sense of Corollary 3.4.8. We then explain how the task of determining the scalars in Shoji's Theorem with respect to unipotent character sheaves on simple groups of exceptional type can be reduced to considering *cuspidal* unipotent character sheaves on such groups (Corollary 3.4.17). In 3.4.18–3.4.24, we conclude this chapter by presenting the main methods that we utilise in Chapter 4 below to determine the scalars in Shoji's Theorem with regard to cuspidal unipotent character sheaves on various simple groups of exceptional type and also to complete the computation of the values of unipotent characters at unipotent elements for the groups $\mathsf{E}_6(q)$ and $\mathsf{E}_7(q)$, where q is any power of any prime p. To this end, the Hecke algebras associated to finite groups of Lie type (see Section 2.3) and the generalised Springer correspondence are of crucial importance.

3.1. Definition and some properties

We begin by introducing some notions of the theory of character sheaves, following [LuCS1]–[LuCS5], [Lus87], [Lus12b], [Sho95a]. For the definitions and properties of the underlying theory of perverse sheaves, we refer to [BBD82] (whose notation we adopt here as well, mostly without explicitly mentioning it in the sequel; see also [Lus87]). As before and throughout this chapter, $k = \overline{\mathbb{F}}_p$ is an algebraic closure of the finite field with p elements (for a prime p) and, as soon as p is prescribed, we fix any prime ℓ different from p and denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers. Starting from 3.1.4, $\mathbf{G}$ will always denote a connected reductive group over k.

3.1.1. Given an algebraic variety X over k, we denote by $\mathscr{D}X := \mathscr{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on X, whose objects are typically called *complexes* (cf. [LuCS1, 1.2]). For a morphism $f\colon X \to Y$ of algebraic varieties X, Y over k, let

$$f^*\colon \mathscr{D}Y \to \mathscr{D}X$$

be the inverse image functor and

$$f_!\colon \mathscr{D}X \to \mathscr{D}Y$$

be the direct image functor with compact support. They admit adjoint functors, denoted by

$$f_*\colon \mathscr{D}X \to \mathscr{D}Y, \quad f^!\colon \mathscr{D}Y \to \mathscr{D}X,$$

respectively. For instance, if $X \subseteq Y$ is a subvariety, $K \in \mathscr{D}Y$, and $i\colon X \hookrightarrow Y$ is the inclusion, the *restriction* of K to X is defined as

$$K|_X := i^*K \in \mathscr{D}X.$$

Now let $K \in \mathscr{D}X$ and $i \in \mathbb{Z}$. Associated to K is the ith *cohomology sheaf* $\mathscr{H}^iK$, a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X. There exists a complex $K[i] \in \mathscr{D}X$, called the ith *shift* of

K, which satisfies

$$\mathscr{H}^j(K[i]) = \mathscr{H}^{i+j}(K) \quad \text{for all } j \in \mathbb{Z}.$$

For $x \in X$, we denote the stalk of $\mathscr{H}^i K$ at x by $\mathscr{H}^i_x K$, a $\overline{\mathbb{Q}}_\ell$-vector space of finite dimension. The *support* of $K \in \mathscr{D}X$ is defined as

$$\operatorname{supp} K := \overline{\{x \in X \mid \mathscr{H}^i_x K \neq 0 \text{ for some } i \in \mathbb{Z}\}} \subseteq X$$

(where the bar stands for the Zariski closure, here in X). For later use we set

$$\hat{\varepsilon}_K := (-1)^{\dim X - \dim \operatorname{supp} K} \quad \text{for } K \in \mathscr{D}X. \tag{3.1.1.1}$$

If $\mathscr{F}$ is itself a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X (or even a $\overline{\mathbb{Q}}_\ell$-local system on X), there exists a unique complex $K \in \mathscr{D}X$ such that

$$\mathscr{H}^0(K) = \mathscr{F} \quad \text{and} \quad \mathscr{H}^i(K) = 0 \quad \text{if } i \neq 0,$$

so we will tacitly identify $\mathscr{F}$ with this K and write $\mathscr{F} \in \mathscr{D}X$. To simplify the notation, we shall mostly just speak of a *local system* when we mean a $\overline{\mathbb{Q}}_\ell$-local system.

Let $\mathscr{M}X$ be the full subcategory of $\mathscr{D}X$ consisting of the *perverse sheaves* on X in the sense of [BBD82] (see also [LuCS1, 1.3]). The category $\mathscr{M}X$ is abelian, and every simple object of $\mathscr{M}X$ can be expressed in terms of an intersection cohomology complex due to Deligne–Goresky–MacPherson [GM83], [BBD82]. More precisely, for any locally closed, smooth irreducible subvariety Y of X and any irreducible local system $\mathscr{L}$ on Y, $\mathscr{L}[\dim Y]$ is an irreducible perverse sheaf on Y, and there is a unique irreducible perverse sheaf extending $\mathscr{L}[\dim Y]$ to the closure $\overline{Y}$, namely, $\mathrm{IC}(\overline{Y}, \mathscr{L})[\dim Y]$. Let $i\colon \overline{Y} \hookrightarrow X$ be the inclusion. It will be convenient to use the notation

$$\mathrm{IC}(\overline{Y}, \mathscr{L})[\dim Y]^{\#X} := i_*(\mathrm{IC}(\overline{Y}, \mathscr{L})[\dim Y]) \in \mathscr{D}X.$$

(In other words, $\mathrm{IC}(\overline{Y}, \mathscr{L})[\dim Y]^{\#X}$ extends the complex $\mathrm{IC}(\overline{Y}, \mathscr{L})[\dim Y]$ to X, by 0 on $X \setminus \overline{Y}$.) Then $\mathrm{IC}(\overline{Y}, \mathscr{L})[\dim Y]^{\#X}$ is an irreducible perverse sheaf on X, and any irreducible perverse sheaf on X is obtained in this way. We note that for any smooth morphism $f\colon X \to Y$ of algebraic varieties with connected fibres of dimension $d \in \mathbb{N}_0$, the shifted inverse image functor $f^*[d]\colon \mathscr{D}Y \to \mathscr{D}X$ restricts to a functor

$$f^*[d]\colon \mathscr{M}Y \to \mathscr{M}X \tag{3.1.1.2}$$

(see [BBD82, 4.2.5] or [LuCS1, 1.8]). We will also need the cohomological functor

$$ {}^pH^i \colon \mathscr{D}X \to \mathscr{M}X \qquad (i \in \mathbb{Z}) $$

(see [BBD82, §1.3] or [LuCS1, 1.4]), which associates to any complex in $\mathscr{D}X$ (and any $i \in \mathbb{Z}$) a perverse sheaf on X.

If $\mathbf{H}$ is a connected algebraic group acting on the algebraic variety X via the morphism $m \colon \mathbf{H} \times X \to X$, and if $\pi_X \colon \mathbf{H} \times X \to X$ is the projection onto X, a perverse sheaf $K \in \mathscr{M}X$ is said to be $\mathbf{H}$-*equivariant* (*for the action of* $\mathbf{H}$ *on* X) if

$$ \pi_X^*[\dim \mathbf{H}](K) \cong m^*[\dim \mathbf{H}](K) \quad \text{in } \mathscr{M}(\mathbf{H} \times X). $$

Now let $\mathscr{L}$ be a local system on X; we thus have $\mathscr{L}[\dim X] \in \mathscr{M}X$. We say that the local system $\mathscr{L}$ is $\mathbf{H}$-*equivariant* (*for the action of* $\mathbf{H}$ *on* X) if the perverse sheaf $\mathscr{L}[\dim X]$ is $\mathbf{H}$-equivariant (for the action of $\mathbf{H}$ on X). Let us assume in addition that the action of $\mathbf{H}$ on X is transitive, and let us fix any $x \in X$. Consider the stabiliser

$$ \operatorname{Stab}_{\mathbf{H}}(x) := \{h \in \mathbf{H} \mid m(h, x) = x\} $$

of x in $\mathbf{H}$. Following [Sho88, 3.5], for any $\mathbf{H}$-equivariant irreducible local system $\mathscr{L}$ on X, the group $\operatorname{Stab}_{\mathbf{H}}(x)$ naturally acts on the stalk $\mathscr{L}_x$ of $\mathscr{L}$ at x, and this action induces a linear action on $\mathscr{L}_x$ of the finite group

$$ A_{\mathbf{H}}(x) = {}^{\operatorname{Stab}_{\mathbf{H}}(x)}\!/_{\operatorname{Stab}^\circ_{\mathbf{H}}(x)}, $$

which gives $\mathscr{L}_x$ the structure of an irreducible $A_{\mathbf{H}}(x)$-module. In fact, the assignment $\mathscr{L} \mapsto \mathscr{L}_x$ as above defines a bijection between the isomorphism classes of $\mathbf{H}$-equivariant irreducible local systems on X and the isomorphism classes of irreducible $A_{\mathbf{H}}(x)$-modules. If $\varsigma \in \operatorname{Irr}(A_{\mathbf{H}}(x))$ is the character of the $A_{\mathbf{H}}(x)$-module $\mathscr{L}_x$, we often just write ς instead of $\mathscr{L}$. In several applications that we will be concerned with later (mostly occurring when $X = \mathscr{C} \subseteq \mathbf{H}$ is a conjugacy class of $\mathbf{H}$), the group $A_{\mathbf{H}}(x)$ turns out to be cyclic and generated by the image $\overline{x}$ of $x \in \operatorname{Stab}_{\mathbf{H}}(x)$ under the canonical map $\operatorname{Stab}_{\mathbf{H}}(x) \to A_{\mathbf{H}}(x)$. In this case, we typically denote the irreducible characters of $A_{\mathbf{H}}(x)$ by their values at $\overline{x}$; thus, if the character of the irreducible $A_{\mathbf{H}}(x)$-module $\mathscr{L}_x$ takes the value $\zeta \in \mathcal{R}$ at $\overline{x}$, we will often denote the pair $(X, \mathscr{L})$ by (X, ζ), or by (x, ζ) if we have fixed $x \in X$ in a given setting.

3.1.2. Assume that the algebraic variety X over k is defined over $\mathbb{F}_q$ (q a power of p), with Frobenius morphism $F\colon X \to X$. Thus, for $K \in \mathscr{D}X$, we may consider the inverse image $F^*K \in \mathscr{D}X$. If F^*K is isomorphic to K, choosing an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$ gives rise, for any $x \in X$ and any $i \in \mathbb{Z}$, to a linear map

$$\varphi_{i,x}\colon \mathscr{H}^i_{F(x)}(K) \to \mathscr{H}^i_x(K)$$

at the level of stalks. Since for a fixed $x \in X$ only finitely many of the $\mathscr{H}^i_x K$ are non-zero, it makes sense to define the *characteristic function* $\chi_{K,\varphi}\colon X^F \to \overline{\mathbb{Q}}_\ell$ of K with respect to φ by

$$\chi_{K,\varphi}(x) := \sum_{i\in\mathbb{Z}}(-1)^i \operatorname{Trace}(\varphi_{i,x}, \mathscr{H}^i_x(K)) \quad \text{for } x \in X^F, \tag{3.1.2.1}$$

see [LuCS2, 8.4]. Assume in addition that $\mathbf{H}$ is a connected algebraic group over k, defined over $\mathbb{F}_q$, which acts on X such that the action is defined over $\mathbb{F}_q$. Then if $K \in \mathscr{M}X$ is a perverse sheaf such that there exists an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$, and if K is $\mathbf{H}$-equivariant for the action of $\mathbf{H}$ on X, the characteristic function $\chi_{K,\varphi}$ is an $\mathbf{H}(\mathbb{F}_q)$-invariant function on X^F, see [Sho95a, 1.1].

3.1.3. Let $\mathbf{S}$ be a torus over k, and let $\mathscr{L}$ be a local system on $\mathbf{S}$. (Since $\mathbf{S}$ is abelian, it is a trivial fact that $\mathscr{L}$ is automatically $\mathbf{S}$-equivariant for the conjugation action of $\mathbf{S}$ on itself, as this action is just the second projection $\mathbf{S} \times \mathbf{S} \to \mathbf{S}$.) Following [Sho95a, 1.2], we call $\mathscr{L}$ a *tame local system* if it has dimension 1 and if $\mathscr{L}^{\otimes n} \cong \overline{\mathbb{Q}}_\ell$ for some $n \in \mathbb{N}$ which is prime to p. We denote by $\mathscr{S}(\mathbf{S})$ the set of isomorphism classes of tame local systems on $\mathbf{S}$, even though we will (by a slight abuse of notation) often just speak of $\mathscr{S}(\mathbf{S})$ as 'the' tame local systems on $\mathbf{S}$ and write $\mathscr{L} \in \mathscr{S}(\mathbf{S})$ instead of $[\mathscr{L}] \in \mathscr{S}(\mathbf{S})$ or the like.

Another description of $\mathscr{S}(\mathbf{S})$ is given as follows (see [LuCS1, 1.11, 2.2]). Recall that we have fixed an isomorphism $\imath\colon k^\times \xrightarrow{\sim} \mu_{p'}$ in (1.1.0.4). So if we denote by $\mu_n(k^\times)$ the group consisting of all nth roots of unity in $k^\times$ (for $n \in \mathbb{N}$ which is prime to p), we get an injective group homomorphism

$$\imath_n\colon \mu_n(k^\times) \hookrightarrow k^\times \xrightarrow{\imath} \mu_{p'} \hookrightarrow \overline{\mathbb{Q}}_\ell^\times.$$

Now consider the morphism

$$\varrho_n\colon \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}, \quad \xi \mapsto \xi^n.$$

The group $\mu_n(k^\times)$ acts naturally on the direct image local system $(\varrho_n)_*\overline{\mathbb{Q}}_\ell$ on $\mathbf{G}_{\mathrm{m}}$, and we denote by $\mathscr{E}_{n,\imath_n}$ the summand of $(\varrho_n)_*\overline{\mathbb{Q}}_\ell$ on which $\mu_n(k^\times)$ acts according to the character $\imath_n$. Then for $\lambda \in X(\mathbf{S})$, we have $\lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{S})$, and this construction gives rise to an isomorphism

$$X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}/\mathbb{Z} \xrightarrow{\sim} \mathscr{S}(\mathbf{S}), \quad \lambda \otimes \left(\tfrac{1}{n} + \mathbb{Z}\right) \mapsto \lambda^*(\mathscr{E}_{n,\imath_n}). \tag{3.1.3.1}$$

(Recall the definition of $\mathbb{Z}_{(p)}$ in (1.1.0.3).)

If the torus $\mathbf{S}$ is defined over $\mathbb{F}_q$ (q a power of p), with Frobenius map $F\colon \mathbf{S} \to \mathbf{S}$, let us consider the set

$$\mathscr{S}(\mathbf{S})^F := \{\mathscr{L} \in \mathscr{S}(\mathbf{S}) \mid F^*\mathscr{L} \cong \mathscr{L}\}$$

of F-stable local systems in $\mathscr{S}(\mathbf{S})$. Given $\mathscr{L} \in \mathscr{S}(\mathbf{S})^F$, we can make a canonical choice for an isomorphism $\varphi_0\colon F^*\mathscr{L} \xrightarrow{\sim} \mathscr{L}$ by requiring that φ_0 induces the identity on the stalk of $\mathscr{L}$ at the identity element of $\mathbf{S}$. In this case, we just write

$$\chi_{\mathscr{L}} := \chi_{\mathscr{L},\varphi_0}\colon \mathbf{S}^F \to \overline{\mathbb{Q}}_\ell.$$

In fact, $\chi_{\mathscr{L}}$ is a (linear) character of $\mathbf{S}^F$, and the assignment $\mathscr{L} \mapsto \chi_{\mathscr{L}}$ defines a bijection

$$\mathscr{S}(\mathbf{S})^F \xrightarrow{\sim} \operatorname{Hom}(\mathbf{S}^F, \overline{\mathbb{Q}}_\ell^\times)$$

(see [Sho95a, (1.2.1)]), so we may think of the F-stable tame local systems on $\mathbf{S}$ (up to isomorphism) just as the irreducible characters of $\mathbf{S}^F$ using this correspondence.

3.1.4. From now until the end of this chapter, $\mathbf{G}$ denotes a connected reductive group over k. Furthermore, we fix a Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ and a maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{B}_0$. Let $\mathbf{U}_0 := R_{\mathrm{u}}(\mathbf{B}_0)$ be the unipotent radical of $\mathbf{B}_0$ and $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$. As usual, for $w \in \mathbf{W}$, we denote by $\dot{w}$ a representative of w in $N_{\mathbf{G}}(\mathbf{T}_0)$. Thus, $\mathbf{W}$ acts on $\mathbf{T}_0$ by

$$\mathbf{W} \times \mathbf{T}_0 \to \mathbf{T}_0, \quad (w,t) \mapsto w(t) := \dot{w}t\dot{w}^{-1}, \tag{3.1.4.1}$$

which is clearly independent of the choice of the representative $\dot{w}$ of w. Using the inverse image functor, we obtain an induced action of $\mathbf{W}$ on $\mathscr{S}(\mathbf{T}_0)$, given by

$$\mathbf{W} \times \mathscr{S}(\mathbf{T}_0) \to \mathscr{S}(\mathbf{T}_0), \quad (w,\mathscr{L}) \mapsto (w^{-1})^*\mathscr{L}. \tag{3.1.4.2}$$

(Here, w^{-1} is viewed as a map $\mathbf{T}_0 \xrightarrow{\sim} \mathbf{T}_0$ via (3.1.4.1), and $(w^{-1})^*\colon \mathscr{S}(\mathbf{T}_0) \xrightarrow{\sim} \mathscr{S}(\mathbf{T}_0)$ is the corresponding inverse image functor, not to be confused with the map (2.1.21.2).)

Then for $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$, we define $\mathbf{W}'_{\mathscr{L}}$ to be the fixed-point set under this action, that is,

$$\mathbf{W}'_{\mathscr{L}} := \{w \in \mathbf{W} \mid (w^{-1})^*\mathscr{L} \cong \mathscr{L}\}. \tag{3.1.4.3}$$

By (3.1.3.1), the set $\mathscr{S}(\mathbf{T}_0)$ can be described in a purely combinatorial way in terms of $X(\mathbf{T}_0) \otimes_{\mathbb{Z}} {}^{\mathbb{Z}_{(p)}}/_{\mathbb{Z}}$. Thus, if $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n})$ corresponds to $\lambda \otimes \left(\frac{1}{n} + \mathbb{Z}\right)$ under (3.1.3.1) (where $\lambda \in X(\mathbf{T}_0)$ and $n \in \mathbb{N}$ is prime to p), it follows from standard properties of tensor products (see, e.g., [GM20, 2.5.3]) that

$$\mathbf{W}'_{\mathscr{L}} = \{w \in \mathbf{W} \mid w.\lambda - \lambda \in nX(\mathbf{T}_0)\}.$$

This characterisation shows that $\mathbf{W}'_{\mathscr{L}}$ is the group denoted by $\hat{\mathbf{W}}_{\lambda,n}$ in [GM20, 2.4.12]; if $\mathbf{G}$ has a connected centre, we have $\mathbf{W}'_{\mathscr{L}} = \mathbf{W}_{\lambda,n}$ (2.2.16.1), see [GM20, 2.4.14]. In general (when $\mathbf{Z}(\mathbf{G})$ is not necessarily connected), $\mathbf{W}_{\lambda,n}$ coincides with the group $\mathbf{W}_{\mathscr{L}} \subseteq \mathbf{W}'_{\mathscr{L}}$ defined in [LuCS1, 2.3], but most of the time we will only be concerned with the connected centre case.

3.1.5. By applying the machinery outlined in 3.1.1 to $X = \mathbf{G}$, Lusztig gives the definition of character sheaves on $\mathbf{G}$ [LuCS1, 2.10], which we shall describe below. We mainly follow [Sho95a, 1.3] (although we only give the 'non-twisted' version of character sheaves here), the original references are [LuCS1, §2], [LuCS3, 12.1]. For $w \in \mathbf{W}$ and a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of w, consider the diagram

$$\begin{array}{ccccccc}
\mathbf{T}_0 & \xleftarrow{\varrho_{\dot{w}}} & \hat{Y}_w & \xrightarrow{\alpha} & Y_w & \xrightarrow{\pi_w} & \mathbf{G} \\
 & & & & \big\downarrow & \nearrow_{\overline{\pi}_w} & \\
 & & & & \overline{Y}_w & &
\end{array}$$

where

$$\begin{aligned}
Y_w &= \{(g, h\mathbf{B}_0) \in \mathbf{G} \times {}^{\mathbf{G}}/_{\mathbf{B}_0} \mid h^{-1}gh \in \mathbf{B}_0 w \mathbf{B}_0\}, \\
\overline{Y}_w &= \{(g, h\mathbf{B}_0) \in \mathbf{G} \times {}^{\mathbf{G}}/_{\mathbf{B}_0} \mid h^{-1}gh \in \overline{\mathbf{B}_0 w \mathbf{B}_0}\}, \\
\hat{Y}_w &= \{(g, h) \in \mathbf{G} \times \mathbf{G} \mid h^{-1}gh \in \mathbf{B}_0 w \mathbf{B}_0\},
\end{aligned}$$

and

$$\begin{aligned}
\pi_w(g, h\mathbf{B}_0) &= g, & \overline{\pi}_w(g, h\mathbf{B}_0) &= g, \\
\alpha(g, h) &= (g, h\mathbf{B}_0), & \varrho_{\dot{w}}(g, h) &= \mathrm{pr}_{\dot{w}}(h^{-1}gh).
\end{aligned}$$

Here, $\mathrm{pr}_{\dot{w}} \colon \mathbf{B}_0 w \mathbf{B}_0 \to \mathbf{T}_0$ is defined by $\mathrm{pr}_{\dot{w}}(u\dot{w}tu') = t$ (for $u, u' \in \mathbf{U}_0$, $t \in \mathbf{T}_0$). Now let $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$ be a tame local system on $\mathbf{T}_0$, and let $w \in \mathbf{W}'_{\mathscr{L}}$. Consider the inverse image $\varrho_{\dot{w}}^*\mathscr{L}$ of $\mathscr{L}$ under $\varrho_{\dot{w}}$. Then there exists a canonical local system $\tilde{\mathscr{L}}_{\dot{w}}$ of dimension 1 on Y_w such that $\varrho_{\dot{w}}^*\mathscr{L} \cong \alpha^*\tilde{\mathscr{L}}_{\dot{w}}$ (see [Sho95a, 1.3] or [LuCS1, 2.4], [LuCS3, 12.1]). The isomorphism class of $\tilde{\mathscr{L}}_{\dot{w}}$ is independent of the choice of the representative $\dot{w}$ of w, so it makes sense to define

$$K_w^{\mathscr{L}} := (\pi_w)_!\tilde{\mathscr{L}}_{\dot{w}} \in \mathscr{D}\mathbf{G} \qquad (\text{for } \mathscr{L} \in \mathscr{S}(\mathbf{T}_0),\ w \in \mathbf{W}'_{\mathscr{L}}). \tag{3.1.5.1}$$

Similarly, using the intersection cohomology complex to extend $\tilde{\mathscr{L}}_{\dot{w}}$ to $\overline{Y}_w$, we set

$$\overline{K}_w^{\mathscr{L}} := (\overline{\pi}_w)_!(\mathrm{IC}(\overline{Y}_w, \tilde{\mathscr{L}}_{\dot{w}})) \in \mathscr{D}\mathbf{G} \qquad (\text{for } \mathscr{L} \in \mathscr{S}(\mathbf{T}_0),\ w \in \mathbf{W}'_{\mathscr{L}}).$$

We can now give Lusztig's definition of character sheaves.

Definition 3.1.6 (Lusztig [LuCS1, 2.10])**.** For $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$, we denote by $\hat{\mathbf{G}}_{\mathscr{L}}$ the set of isomorphism classes of irreducible perverse sheaves A on $\mathbf{G}$ such that A is a constituent of ${}^pH^i(K_w^{\mathscr{L}})$ for some $w \in \mathbf{W}'_{\mathscr{L}}$ and some $i \in \mathbb{Z}$. A *character sheaf* on $\mathbf{G}$ is an irreducible perverse sheaf $A \in \mathscr{M}\mathbf{G}$ whose isomorphism class is in $\hat{\mathbf{G}}_{\mathscr{L}}$ for some $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$. We denote by $\hat{\mathbf{G}}$ the set of isomorphism classes of character sheaves on $\mathbf{G}$. By a slight abuse of notation, we will also write $A \in \hat{\mathbf{G}}$ (or $A \in \hat{\mathbf{G}}_{\mathscr{L}}$) if A is a character sheaf on $\mathbf{G}$ whose isomorphism class is in $\hat{\mathbf{G}}$ (or $\hat{\mathbf{G}}_{\mathscr{L}}$) and speak of $\hat{\mathbf{G}}$ just as 'the' character sheaves on $\mathbf{G}$.

Remark 3.1.7. (a) Let $\mathscr{L}, \mathscr{L}' \in \mathscr{S}(\mathbf{T}_0)$. By [LuCS3, 11.2], the sets $\hat{\mathbf{G}}_{\mathscr{L}}$ and $\hat{\mathbf{G}}_{\mathscr{L}'}$ coincide provided $\mathscr{L}, \mathscr{L}'$ are in the same orbit of the action of $\mathbf{W}$ on $\mathscr{S}(\mathbf{T}_0)$ defined in (3.1.4.2), while they are disjoint otherwise. Thus, if we denote by ${\mathscr{S}(\mathbf{T}_0)}/_{\mathbf{W}}$ a set of representatives for the $\mathbf{W}$-orbits on $\mathscr{S}(\mathbf{T}_0)$ under (3.1.4.2), we obtain a partition

$$\hat{\mathbf{G}} = \biguplus_{\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)/\mathbf{W}} \hat{\mathbf{G}}_{\mathscr{L}}.$$

(b) Let $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ where $\lambda \in X(\mathbf{T}_0)$ and $n \in \mathbb{N}$ is prime to p, see (3.1.3.1). By [LuCS1, 2.18], any $A \in \hat{\mathbf{G}}_{\mathscr{L}}$ is $(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ)$-equivariant for the action of $\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ$ on $\mathbf{G}$ defined by

$$(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ) \times \mathbf{G} \to \mathbf{G}, \quad ((x, z), g) \mapsto z^n x g x^{-1}.$$

In particular, any character sheaf on $\mathbf{G}$ is $\mathbf{G}$-equivariant for the conjugation action of $\mathbf{G}$ on itself.

(c) Let $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$, and let A be an irreducible perverse sheaf on $\mathbf{G}$. By [LuCS3, 12.7], we have $A \in \hat{\mathbf{G}}_{\mathscr{L}}$ if and only if A is a constituent of ${}^pH^i(\overline{K}_w^{\mathscr{L}})$ for some $w \in \mathbf{W}'_{\mathscr{L}}$ and some $i \in \mathbb{Z}$. The advantage of using this characterisation is that the complex $\overline{K}_w^{\mathscr{L}} \in \mathscr{D}\mathbf{G}$ is *semisimple* [LuCS3, 12.8], that is, each ${}^pH^i(\overline{K}_w^{\mathscr{L}})$ is a semisimple object of the abelian category $\mathscr{M}\mathbf{G}$.

Definition 3.1.8. Consider the trivial local system $\mathscr{L}_0 = \overline{\mathbb{Q}}_\ell \in \mathscr{S}(\mathbf{T}_0)$ (thus, $\mathbf{W}'_{\mathscr{L}_0} = \mathbf{W}$). The *unipotent character sheaves* on $\mathbf{G}$ are defined as

$$\hat{\mathbf{G}}^{\mathrm{un}} := \hat{\mathbf{G}}_{\mathscr{L}_0} \subseteq \hat{\mathbf{G}}.$$

As we shall see later (Section 3.3, Section 3.4), in the case where $F\colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map, the set $\hat{\mathbf{G}}^{\mathrm{un}}$ is indeed closely related to the set of unipotent characters of $\mathbf{G}^F$.

3.1.9. In analogy to Harish-Chandra induction and restriction for finite groups of Lie type, there are induction and restriction functors for perverse sheaves, defined as follows ([LuCS1, §3, §4], see also [TT20, §7]). Let $\mathbf{L} \subseteq \mathbf{G}$ be the Levi complement of some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Denote by $\mathbf{U_P} = R_{\mathrm{u}}(\mathbf{P})$ the unipotent radical of $\mathbf{P}$. Let $\pi_{\mathbf{P}\supseteq\mathbf{L}}\colon \mathbf{P} \to \mathbf{L}$ be the canonical projection with kernel $\mathbf{U_P}$, and let $i\colon \mathbf{P} \hookrightarrow \mathbf{G}$ be the inclusion. Then the functor

$$\mathrm{res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}\colon \mathscr{D}\mathbf{G} \to \mathscr{D}\mathbf{L}$$

is given by

$$\mathrm{res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(A) := (\pi_{\mathbf{P}\supseteq\mathbf{L}})_! i^* A[\dim \mathbf{U_P}] \in \mathscr{D}\mathbf{L} \quad (\text{for } A \in \mathscr{D}\mathbf{G}).$$

Now consider the diagram

$$\mathbf{L} \xleftarrow{\ \pi\ } V_1 \xrightarrow{\ \pi'\ } V_2 \xrightarrow{\ \pi''\ } \mathbf{G}$$

where

$$V_1 = \{(g,x) \in \mathbf{G} \times \mathbf{G} \mid x^{-1}gx \in \mathbf{P}\},$$
$$V_2 = \{(g,x\mathbf{P}) \in \mathbf{G} \times \mathbf{G}/\mathbf{P} \mid x^{-1}gx \in \mathbf{P}\},$$

and

$$\pi(g,x) = \pi_{\mathbf{P}\supseteq\mathbf{L}}(x^{-1}gx), \quad \pi'(g,x) = (g,x\mathbf{P}), \quad \pi''(g,x\mathbf{P}) = g.$$

Let $K \in \mathscr{M}\mathbf{L}$ be a perverse sheaf on $\mathbf{L}$ which is $\mathbf{L}$-equivariant for the conjugation action of $\mathbf{L}$ on itself. Then one can show that there exists a unique perverse sheaf $K_2 \in \mathscr{M}V_2$

(up to isomorphism) such that $\pi^* K[\dim \mathbf{G} + \dim \mathbf{U_P}] \cong (\pi')^* K_2[\dim \mathbf{P}]$. Setting

$$\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(K) := (\pi'')_! K_2 \in \mathscr{D}\mathbf{G}$$

then gives a functor

$$\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} \colon \{\mathbf{L}\text{-equivariant perverse sheaves on } \mathbf{L}\} \to \mathscr{D}\mathbf{G}. \tag{3.1.9.1}$$

Definition 3.1.10 ([LuCS1, 3.10], [LuCS2, 7.1])**.** Let $A \in \mathscr{M}\mathbf{G}$.

(a) We say that A is a *cuspidal perverse sheaf* if it satisfies the following two conditions.

(i) There is an integer $n \in \mathbb{N}$ which is prime to p such that A is $(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ)$-equivariant for the action of $\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ$ on $\mathbf{G}$ defined by

$$(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ) \times \mathbf{G} \to \mathbf{G}, \quad ((x, z), g) \mapsto z^n x g x^{-1}.$$

(ii) For any proper parabolic subgroup $\mathbf{P} \subsetneq \mathbf{G}$ with Levi complement $\mathbf{L} \subseteq \mathbf{P}$, we have

$$\dim \operatorname{supp} \mathscr{H}^i(\mathrm{res}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A)) < -i \quad \text{for all } i \in \mathbb{Z}.$$

(b) If $A \in \hat{\mathbf{G}}$, we say that A is a *cuspidal character sheaf* on $\mathbf{G}$ if it is a cuspidal perverse sheaf in the sense of (a). We denote by $\hat{\mathbf{G}}^\circ \subseteq \hat{\mathbf{G}}$ the subset consisting of the cuspidal character sheaves on $\mathbf{G}$.

Definition 3.1.11 ([LuCS2, 7.7], [LuCS3, 13.9])**.** (a) A cuspidal perverse sheaf $A \in \mathscr{M}\mathbf{G}$ is called *clean* if there exists a subset $\Sigma \subseteq \mathbf{G}$ which is the preimage of a conjugacy class of $\mathbf{G}/\mathbf{Z}(\mathbf{G})^\circ$ under the canonical map $\mathbf{G} \to \mathbf{G}/\mathbf{Z}(\mathbf{G})^\circ$ such that

$$\operatorname{supp} A = \overline{\Sigma} \quad \text{and} \quad A|_{\overline{\Sigma}\setminus\Sigma} = 0.$$

(b) $\mathbf{G}$ is called *clean* if for any Levi complement $\mathbf{L}$ of any parabolic subgroup of $\mathbf{G}$, every cuspidal character sheaf on $\mathbf{L}$ is clean in the sense of (a) applied to $\mathbf{L}$ instead of $\mathbf{G}$.

Remark 3.1.12. A character sheaf $A \in \hat{\mathbf{G}}$ automatically satisfies condition (i) in Definition 3.1.10(a), as mentioned in Remark 3.1.7(b). If $A \in \hat{\mathbf{G}}^\circ$, it can be shown that we even have $\mathrm{res}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A) = 0$ in $\mathscr{D}\mathbf{L}$ for any proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi complement $\mathbf{L} \subseteq \mathbf{P}$, see [LuCS1, 6.9(b)].

In [LuCS1], Lusztig actually always assumes that $\mathbf{T}_0 \subseteq \mathbf{L}$ and $\mathbf{B}_0 \subseteq \mathbf{P}$ (that is, with the notions in 2.1.10, $\mathbf{L}$ is the standard Levi subgroup of the standard parabolic subgroup

$\mathbf{P}$ with respect to the pair $\mathbf{T}_0 \subseteq \mathbf{B}_0$), but as remarked in [LuCS2, 7.1], one can drop this assumption for condition (ii) in Definition 3.1.10(a). Using suitable transitivity and 'Frobenius reciprocity' properties of res and ind (see [LuCS1, 4.2 and 4.4(d)]), we obtain the following characterisation of cuspidal character sheaves: $A \in \hat{\mathbf{G}}$ is in $\hat{\mathbf{G}}^\circ$ if and only if for any (standard) Levi subgroup of any proper (standard) parabolic subgroup $\mathbf{P} \subsetneq \mathbf{G}$, and any $A_0 \in \hat{\mathbf{L}}^\circ$, A is not a summand of $\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)$. (We may or may not include the 'standard' in this characterisation, whatever is more convenient in a given situation.)

Theorem 3.1.13 (Lusztig, see [LuCS5, Thm. 23.1], [Lus12b]). *Let* $\mathbf{G}$ *be a connected reductive group over* k.

(a) Any irreducible cuspidal perverse sheaf on $\mathbf{G}$ *is a (cuspidal) character sheaf.*

(b) $\mathbf{G}$ *is clean.*

Proof. The proof of this theorem occupies almost all of [LuCS4] and [LuCS5, §22, §23], and is finally concluded in [Lus12b] (which deals with some small primes p that have been excluded in [LuCS5, (23.0.1)]). □

Remark 3.1.14. (a) The (isomorphism classes of) irreducible cuspidal perverse sheaves on $\mathbf{G}$ are classified by Lusztig in [Lus84b] (cf. Example 3.1.19 below), which yields the classification of $\hat{\mathbf{G}}^\circ$ in view of Theorem 3.1.13(a). It should be mentioned that the proof of Theorem 3.1.13 is essentially based on a case-by-case analysis and strongly relies upon the classification of irreducible cuspidal perverse sheaves on connected reductive groups; in particular, showing that a given irreducible cuspidal perverse sheaf is in fact a character sheaf does not seem to be a direct consequence of the conditions (i) and (ii) in Definition 3.1.10(a). We also note that parts of the proof in [Lus12b] (concerning some small primes p) rely on explicit computer calculations!

(b) As we will describe below, knowing the cuspidal character sheaves on the Levi complements of the parabolic subgroups of $\mathbf{G}$ and inducing them to $\mathbf{G}$ gives rise to a parametrisation of $\hat{\mathbf{G}}$, which is rather analogous to Harish-Chandra theory for finite groups of Lie type. On the other hand, [LuCS5, Thm. 23.1] contains a further main part (which we have not included in the formulation of the above theorem) that together with the results of [Lus12b] provides another classification of $\hat{\mathbf{G}}$; it consists of a parametrisation of the sets $\hat{\mathbf{G}}_{\mathscr{L}}$ for $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$ in terms of families of the group $\mathbf{W}'_{\mathscr{L}}$. We will discuss this in Section 3.3 in the case where $\mathbf{G}$ has a connected centre. If $\mathbf{G}$ is equipped with a Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$, F naturally acts on $\hat{\mathbf{G}}$ (see 3.2.1 below) and, assuming that $\mathbf{Z}(\mathbf{G})$ is connected, the F-stable character sheaves on $\mathbf{G}$ are described via a scheme analogous to that in Theorem 2.2.21 for $\mathrm{Irr}(\mathbf{G}^F)$. In this case, given $(\lambda, n) \in \Lambda(\mathbf{G}, F)$

and $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ (3.1.3.1), the F-stable character sheaves inside $\hat{\mathbf{G}}_{\mathscr{L}}$ are parametrised in terms of the group $\mathbf{W}'_{\mathscr{L}} = \mathbf{W}_{\lambda,n}$, subject to properties analogous to those for the set $\mathcal{E}_{\lambda,n}$ via Theorem 2.2.21. We will be particularly concerned with the unipotent character sheaves $\hat{\mathbf{G}}^{\mathrm{un}} = \hat{\mathbf{G}}_{\mathscr{L}_0}$ (see Definition 3.1.8).

3.1.15. Following [Lus84b, 3.1] (see also [LuCS1, 3.11]), we define a partition of $\mathbf{G}$ into finitely many locally closed, smooth irreducible subvarieties which are stable by conjugation. Given an element $g \in \mathbf{G}$, let us write $g = g_{\mathrm{s}}g_{\mathrm{u}} = g_{\mathrm{u}}g_{\mathrm{s}}$ for its Jordan decomposition (with $g_{\mathrm{s}} \in \mathbf{G}$ semisimple, $g_{\mathrm{u}} \in \mathbf{G}$ unipotent). We say that $g \in \mathbf{G}$ is *isolated* in $\mathbf{G}$ [Lus84b, 2.6] if $C^\circ_{\mathbf{G}}(g_{\mathrm{s}})$ is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. This condition is clearly invariant under $\mathbf{G}$-conjugacy, so it makes sense to say that a conjugacy class of $\mathbf{G}$ is *isolated* if one (any) of its elements is isolated in $\mathbf{G}$.

Let $\mathbf{L} \subseteq \mathbf{G}$ be a Levi complement of some parabolic subgroup of $\mathbf{G}$, and let $\Sigma \subseteq \mathbf{L}$ be the preimage of an isolated conjugacy class of $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ under the canonical map $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$. We set

$$\Sigma_{\mathrm{reg}} := \{g \in \Sigma \mid C^\circ_{\mathbf{G}}(g_{\mathrm{s}}) \subseteq \mathbf{L}\}$$

and

$$Y_{(\mathbf{L},\Sigma)} := \bigcup_{x \in \mathbf{G}} x\Sigma_{\mathrm{reg}}x^{-1}.$$

Then $Y_{(\mathbf{L},\Sigma)}$ is a locally closed, smooth irreducible subvariety of $\mathbf{G}$ of dimension

$$\dim Y_{(\mathbf{L},\Sigma)} = |R_{\mathbf{G}}| - |R_{\mathbf{L}}| + \dim \Sigma$$

(where $R_{\mathbf{G}}$, $R_{\mathbf{L}}$ denote the sets of roots of $\mathbf{G}$, $\mathbf{L}$, respectively). Now $\mathbf{G}$ acts by simultaneous conjugation on the set of all pairs $(\mathbf{L}, \Sigma)$ as above, and there are only finitely many equivalence classes for this action. Clearly, $Y_{(\mathbf{L},\Sigma)}$ only depends on the class of $(\mathbf{L}, \Sigma)$. As described in [Lus84b, 3.1], given $g \in \mathbf{G}$, it is easy to find a pair $(\mathbf{L}, \Sigma)$ as above such that $g \in Y_{(\mathbf{L},\Sigma)}$. Indeed, we may take

$$\mathbf{L} := H_{\mathbf{G}}(g) := C_{\mathbf{G}}(\mathbf{Z}(C_{\mathbf{G}}(g_{\mathrm{s}}))^\circ),$$

which is the smallest closed subgroup of $\mathbf{G}$ that contains $C_{\mathbf{G}}(g_{\mathrm{s}})$ and is the Levi complement of some parabolic subgroup of $\mathbf{G}$. We then have $g \in \mathbf{L}$, and we choose

$$\Sigma := \mathbf{Z}(\mathbf{L})^\circ \cdot (\mathbf{L}\text{-conjugacy class of } g).$$

This shows that there is a finite set of pairs $(\mathbf{L}_i, \Sigma_i)$ $(1 \leqslant i \leqslant m$, some $m \in \mathbb{N})$ which gives rise to the desired partition

$$\mathbf{G} = \biguplus_{i=1}^{m} Y_{(\mathbf{L}_i,\Sigma_i)}.$$

Since the sets $Y_{(\mathbf{L}_i,\Sigma_i)}$ do not depend on the $\mathbf{G}$-conjugacy class of $(\mathbf{L}_i, \Sigma_i)$, we may even require that for all $i \in \{1, \ldots, m\}$, $\mathbf{L}_i$ is the *standard* Levi subgroup of some *standard* parabolic subgroup of $\mathbf{G}$ with respect to $\mathbf{T}_0 \subseteq \mathbf{B}_0$, see 2.1.10.

Definition 3.1.16 ([Lus84b, 2.4, 2.5])**.** (a) Let $\mathbf{G}$ be a semisimple algebraic group over k. For any parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, let us set $\overline{\mathbf{P}} := \mathbf{P}/R_{\mathrm{u}}(\mathbf{P})$, and let $\pi_{\mathbf{P}}\colon \mathbf{P} \to \overline{\mathbf{P}}$ be the canonical map. Furthermore, for $x \in \overline{\mathbf{P}}$, we write $x^{\overline{\mathbf{P}}}$ for the $\overline{\mathbf{P}}$-conjugacy class of x. Now let $\mathscr{C} \subseteq \mathbf{G}$ be a conjugacy class, and let $\mathscr{E}$ be an irreducible local system on $\mathscr{C}$ which is $\mathbf{G}$-equivariant for the conjugation action of $\mathbf{G}$ on $\mathscr{C}$. The pair $(\mathscr{C}, \mathscr{E})$ is called a *cuspidal pair for* $\mathbf{G}$ (and $\mathscr{E}$ is called a *cuspidal local system on* $\mathscr{C}$) if, for any proper parabolic subgroup $\mathbf{P} \subsetneq \mathbf{G}$, the following condition is satisfied:

$$\text{For any } x \in \overline{\mathbf{P}}, \text{ we have } H_c^d\left(\pi_{\mathbf{P}}^{-1}(x) \cap \mathscr{C}, \mathscr{E}\right) = 0 \quad \text{where } d = \dim \mathscr{C} - \dim\left(x^{\overline{\mathbf{P}}}\right).$$

(Here, $H_c^d(\pi_{\mathbf{P}}^{-1}(x) \cap \mathscr{C}, \mathscr{E})$ is the dth cohomology group with compact support of $\pi_{\mathbf{P}}^{-1}(x) \cap \mathscr{C}$, with coefficients in $\mathscr{E}|_{\pi_{\mathbf{P}}^{-1}(x)\cap\mathscr{C}}$ rather than in the constant local system $\overline{\mathbb{Q}}_\ell$ as considered in 2.2.1.)

(b) Let $\mathbf{G}$ be a connected reductive group over k. Consider the semisimple groups $\mathbf{G}_{\mathrm{ss}} := \mathbf{G}/\mathbf{Z}(\mathbf{G})^\circ$ and $\mathbf{G}_{\mathrm{der}} := [\mathbf{G}, \mathbf{G}]$, as well as the canonical maps

$$\pi_{\mathrm{ss}}\colon \mathbf{G} \to \mathbf{G}_{\mathrm{ss}}, \quad \pi_{\mathrm{der}}\colon \mathbf{G} \to \mathbf{G}/\mathbf{G}_{\mathrm{der}} \quad \text{and} \quad \pi := (\pi_{\mathrm{der}}, \pi_{\mathrm{ss}})\colon \mathbf{G} \to \mathbf{G}/\mathbf{G}_{\mathrm{der}} \times \mathbf{G}_{\mathrm{ss}}.$$

Let $\mathscr{C}$ be a conjugacy class of $\mathbf{G}_{\mathrm{ss}}$, and let $\Sigma := \pi_{\mathrm{ss}}^{-1}(\mathscr{C}) \subseteq \mathbf{G}$. Assume that $\mathscr{E}$ is a $\mathbf{G}$-equivariant irreducible local system on Σ (for the conjugation action of $\mathbf{G}$). The pair $(\Sigma, \mathscr{E})$ is called a *cuspidal pair for* $\mathbf{G}$ (and $\mathscr{E}$ is called a *cuspidal local system on* Σ) if there exists some cuspidal local system $\mathscr{E}'$ on $\mathscr{C}$ (that is, $(\mathscr{C}, \mathscr{E}')$ is a cuspidal pair for $\mathbf{G}_{\mathrm{ss}}$ as defined in (a)) and a tame local system $\mathscr{L} \in \mathscr{S}\left(\mathbf{G}/\mathbf{G}_{\mathrm{der}}\right)$ such that

$$\mathscr{E} \cong (\pi|_\Sigma)^*(\mathscr{L} \boxtimes \mathscr{E}'),$$

where $\pi|_\Sigma\colon \Sigma \to \mathbf{G}/\mathbf{G}_{\mathrm{der}} \times \mathscr{C}$ is the restriction of π.

Proposition 3.1.17 (Lusztig [LuCS1, 3.12], [Lus84b, §2])**.** *Let* $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$

be as in 3.1.3*, and let* $A \in \hat{\mathbf{G}}_{\mathscr{L}} \cap \hat{\mathbf{G}}^{\circ}$*. Consider the action of* $\mathbf{G} \times \mathbf{Z}(\mathbf{G})^{\circ}$ *on* $\mathbf{G}$ *defined by*

$$(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^{\circ}) \times \mathbf{G} \to \mathbf{G}, \quad ((x,z),g) \mapsto z^n xgx^{-1}. \tag{3.1.17.1}$$

(a) There exists a unique $(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^{\circ})$*-orbit* $\Sigma \subseteq \mathbf{G}$ *and a unique* $(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^{\circ})$*-equivariant irreducible local system* $\mathscr{E}$ *on* Σ *(up to isomorphism) such that*

$$A \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}}.$$

(b) The pair $(\Sigma, \mathscr{E})$ *in* (a) *is a cuspidal pair for* $\mathbf{G}$*.*

(c) Let Σ *be as in* (a)*. Then* Σ *is the preimage of an isolated conjugacy class of* $\mathbf{G}/\mathbf{Z}(\mathbf{G})^{\circ}$ *under the canonical map* $\mathbf{G} \to \mathbf{G}/\mathbf{Z}(\mathbf{G})^{\circ}$*. Furthermore, for any* $g \in \Sigma$*, the group* $C^{\circ}_{\mathbf{G}}(g)/\mathbf{Z}(\mathbf{G})^{\circ}$ *is unipotent.*

Proof. See [LuCS1, 3.12]. □

Remark 3.1.18. (a) As a converse to Proposition 3.1.17, given any cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{G}$, the complex

$$\mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}} \in \mathscr{M}\mathbf{G}$$

is a cuspidal character sheaf on $\mathbf{G}$. Indeed, by [LuCS2, (7.1.4)], $\mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}}$ is an irreducible cuspidal perverse sheaf on $\mathbf{G}$, so it remains to refer to Theorem 3.1.13(a). The cuspidal pairs for $\mathbf{G}$ thus parametrise the cuspidal character sheaves on $\mathbf{G}$.

(b) Let us set $\mathbf{H} := \mathbf{G} \times \mathbf{Z}(\mathbf{G})^{\circ}$ and consider the action (3.1.17.1) of $\mathbf{H}$ on $\mathbf{G}$. Let $\Sigma \subseteq \mathbf{G}$ be an $\mathbf{H}$-orbit, and let us fix a representative $g \in \Sigma$ for this orbit. Then, as noted in 3.1.1, the isomorphism classes of $\mathbf{H}$-equivariant irreducible local systems on Σ are in natural correspondence with the irreducible characters of $A_{\mathbf{H}}(g)$. In this way, any cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{G}$ and, hence, any cuspidal character sheaf $A = \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}}$ on $\mathbf{G}$, may concisely be described by a pair (g, ς) where $g \in \Sigma$ and $\varsigma \in \mathrm{Irr}(A_{\mathbf{H}}(g))$ parametrises the local system $\mathscr{E}$ on Σ. It will sometimes be convenient to write this as

$$A \leftrightarrow (g, \varsigma).$$

Example 3.1.19. Let us assume that $\mathbf{G}$ is a simple group, so that $\mathbf{Z}(\mathbf{G})^{\circ} = \{1\}$. Thus, the cuspidal pairs for $\mathbf{G}$ are of the form $(\mathscr{C}, \mathscr{E})$ where $\mathscr{C} \subseteq \mathbf{G}$ is an isolated conjugacy class of $\mathbf{G}$. As mentioned in Remark 3.1.14, the classification of the cuspidal pairs for $\mathbf{G}$ has been established by Lusztig [Lus84b]. We briefly indicate how the cuspidal pairs for $\mathbf{G}$ of type E_8 in good characteristic (that is, $p \geqslant 7$) can be explicitly extracted from the results of [Lus84b]. So let $\mathscr{C} \subseteq \mathbf{G}$ be an isolated conjugacy class, and let us choose some $g \in \mathscr{C}$ with Jordan decomposition $g = su = us$ (s semisimple, u unipotent). Let $\mathscr{E}$ be (the isomorphism class of) a $\mathbf{G}$-equivariant irreducible local system on $\mathscr{C}$. Thus, the pair $(\mathscr{C}, \mathscr{E})$ may be described by (g, ς) where $\varsigma \in \mathrm{Irr}(A_{\mathbf{G}}(g))$. Now we note that $C_{\mathbf{G}}(su) = C_{C_{\mathbf{G}}(s)}(u)$, so

$$A_{\mathbf{G}}(g) = C_{C_{\mathbf{G}}(s)}(u)/C^{\circ}_{C_{\mathbf{G}}(s)}(u) = A_{C_{\mathbf{G}}(s)}(u).$$

Let us denote by $\mathscr{O}_1$ the (unipotent) conjugacy class of u in $C_{\mathbf{G}}(s)$ and by $\mathscr{E}_1$ the irreducible local system on $\mathscr{O}_1$ described by ς (viewed as an irreducible character of $A_{C_{\mathbf{G}}(s)}(u)$). By [Lus84b, (2.10.1)], we have

$$(\mathscr{C}, \mathscr{E}) \text{ is a cuspidal pair for } \mathbf{G} \iff (\mathscr{O}_1, \mathscr{E}_1) \text{ is a cuspidal pair for } C_{\mathbf{G}}(s).$$

Now [Lus84b, 15.6] provides the isogeny types of all the $C_{\mathbf{G}}(s)$ where s is the semisimple part of an element in a conjugacy class $\mathscr{C} \subseteq \mathbf{G}$ which affords a cuspidal local system $\mathscr{E}$. The cuspidal pairs $(\mathscr{O}_1, \mathscr{E}_1)$ for $C_{\mathbf{G}}(s)$ for which $\mathscr{O}_1 \subseteq \mathbf{G}$ is a unipotent class can then be read off from [Lus84b, §10, §14, §15], and this gives all the cuspidal pairs $(\mathscr{C}, \mathscr{E})$ for $\mathbf{G}$.

Example 3.1.20. Let $n \geqslant 1$, and let $\mathbf{G}$ be a simple algebraic group over k of type A_n.

(i) Assume first that $|\mathbf{Z}(\mathbf{G})| < n+1$. (If the characteristic p of k does not divide $n+1$, this is equivalent to requiring that $\mathbf{G}$ is not isomorphic to the simply connected group $\mathrm{SL}_{n+1}(k)$.) Then there are no cuspidal character sheaves on $\mathbf{G}$.

(ii) Now let us assume that $|\mathbf{Z}(\mathbf{G})| = n+1$. (In particular, $\mathbf{G}$ is isomorphic to $\mathrm{SL}_{n+1}(k)$.) Then there are precisely $(n+1)\cdot\phi(n+1)$ isomorphism classes of cuspidal character sheaves on $\mathbf{G}$. (Here, ϕ is the Euler function.)

Proof. (a) Let us first consider the group $\mathbf{G}_{\mathrm{sc}} := \mathrm{SL}_{n+1}(k)$, that is, the simply connected group of type A_n. By [Lus84b, 10.3, 2.10] and [LuCS4, 18.5], the cuspidal character sheaves on $\mathbf{G}_{\mathrm{sc}}$ are given as follows: First, if p divides $n+1$ (so that we are in case (i) of the example), there are no cuspidal character sheaves on $\mathbf{G}_{\mathrm{sc}}$. Now assume that p does not divide $n+1$, so we are in case (ii) of the example. Let $\mathscr{O}_{\mathrm{reg}}$ be the regular unipotent class of $\mathbf{G}_{\mathrm{sc}}$, and let us fix an element $u_0 \in \mathscr{O}_{\mathrm{reg}}$. Then the elements of $\hat{\mathbf{G}}^{\circ}_{\mathrm{sc}}$ are precisely the complexes of the form

$$A_{(z,\varsigma)} = \mathrm{IC}(z \cdot (\mathbf{G}_{\mathrm{sc}})_{\mathrm{uni}}, \mathscr{E}^{\varsigma})[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}_{\mathrm{sc}}}$$

where $z \in \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) \cong C_{n+1}$ and $\mathscr{E}^{\varsigma}$ is a $\mathbf{G}_{\mathrm{sc}}$-equivariant irreducible local system on $z\mathscr{O}_{\mathrm{reg}}$ corresponding to a faithful linear character ς of

$$A_{\mathbf{G}_{\mathrm{sc}}}(zu_0) = A_{\mathbf{G}_{\mathrm{sc}}}(u_0) \cong \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) \cong C_{n+1}$$

(see [DM20, 12.2.3, 12.2.7] for the middle isomorphism). In particular, we have

$$|\hat{\mathbf{G}}^{\circ}_{\mathrm{sc}}| = (n+1) \cdot \phi(n+1),$$

which proves (ii).

(b) Now let $\mathbf{G}$ be any group of type A_n, and let us consider its simply connected covering

$$\pi \colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G},$$

see 2.1.7; recall that $\ker \pi$ is finite and contained in $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$. Assume that there exists a cuspidal character sheaf A on $\mathbf{G}$. By [LuCS4, (17.16.3)] (see also [Lus84b, 2.10]), A is a direct summand of $\pi_*\pi^*A \in \mathscr{D}\mathbf{G}$, and $\pi^*A \in \mathscr{D}\mathbf{G}_{\mathrm{sc}}$ is a direct sum of cuspidal character sheaves on $\mathbf{G}_{\mathrm{sc}}$ with trivial action of $\ker \pi$. In particular, $\hat{\mathbf{G}}^{\circ}_{\mathrm{sc}} \neq \varnothing$, so p cannot divide $n+1$ in view of (a); thus, $|\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})| = n+1$. Now let us fix one of the simple summands $A_0 \in \hat{\mathbf{G}}^{\circ}_{\mathrm{sc}}$ of π^*A. By (a), we have $A_0 \cong A_{(z,\varsigma)}$ for some $z \in \mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$ and some faithful linear character $\varsigma \in \mathrm{Irr}(A_{\mathbf{G}_{\mathrm{sc}}}(zu_0))$. On the other hand, $\ker \pi \subseteq \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) \cong A_{\mathbf{G}_{\mathrm{sc}}}(zu_0)$, and $\ker \pi$ acts trivially on A_0, so it also acts trivially on the local system $\mathscr{E}^{\varsigma}$ corresponding to ς. Hence, $\ker \pi \subseteq \ker \varsigma = \{1\}$, so π is a bijective group homomorphism. We conclude that $|\mathbf{Z}(\mathbf{G})| = |\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})| = n+1$, which proves (i). □

3.1.21. Let $\mathbf{L} \subseteq \mathbf{G}$ be a Levi complement of some parabolic subgroup of $\mathbf{G}$. Following [Lus84b, 3.2] (see also [LuCS2, 8.1] or [TT20, 7.3]), we briefly mention a method of 'inducing' cuspidal pairs for $\mathbf{L}$ to $\mathbf{G}$ by utilising the concept of intersection cohomology complexes, which turns out to coincide with the induction of the corresponding cuspidal character sheaves from $\mathbf{L}$ to $\mathbf{G}$. So let $\Sigma \subseteq \mathbf{L}$ be the preimage of an isolated conjugacy class in $\mathbf{L}/\mathbf{Z}(\mathbf{L})^{\circ}$ under the natural projection map $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^{\circ}$. We consider the diagram

$$\Sigma \xleftarrow{\ \alpha\ } \hat{Y} \xrightarrow{\ \beta\ } \tilde{Y} \xrightarrow{\ \gamma\ } Y$$

where

$$Y = Y_{(\mathbf{L},\Sigma)} \text{ (see 3.1.15)},$$
$$\hat{Y} = \{(g,x) \in \mathbf{G} \times \mathbf{G} \mid x^{-1}gx \in \Sigma_{\mathrm{reg}}\},$$
$$\tilde{Y} = \{(g,x\mathbf{L}) \in \mathbf{G} \times \mathbf{G}/\mathbf{L} \mid x^{-1}gx \in \Sigma_{\mathrm{reg}}\},$$

and

$$\alpha(g,x) = x^{-1}gx,$$
$$\beta(g,x) = (g,x\mathbf{L}),$$
$$\gamma(g,x\mathbf{L}) = g.$$

Assume now that $\mathscr{E}$ is a local system on Σ which is isomorphic to a direct sum of irreducible local systems $\mathscr{E}_i$ on Σ such that each $(\Sigma, \mathscr{E}_i)$ is a cuspidal pair for $\mathbf{L}$. (In particular, each $\mathscr{E}_i$ is $\mathbf{L}$-equivariant for the conjugation action of $\mathbf{L}$ on Σ, hence so is $\mathscr{E}$.) Then the local system $\alpha^*\mathscr{E}$ on $\hat{Y}$ is known to be isomorphic to $\beta^*\tilde{\mathscr{E}}$ for a uniquely determined local system $\tilde{\mathscr{E}}$ on $\tilde{Y}$ (up to isomorphism), so we can define

$$K_{\mathbf{L},\Sigma}^{\mathscr{E}} := \mathrm{IC}(\overline{Y}, \gamma_*\tilde{\mathscr{E}})[\dim Y]^{\#\mathbf{G}}.$$

The complex $K_{\mathbf{L},\Sigma}^{\mathscr{E}}$ is a semisimple perverse sheaf on $\mathbf{G}$. If we assume in addition that $\mathscr{E}$ is irreducible (so that $(\Sigma, \mathscr{E})$ itself is a cuspidal pair for $\mathbf{L}$), then by [Lus84b, 4.5], there is a canonical isomorphism

$$K_{\mathbf{L},\Sigma}^{\mathscr{E}} \cong \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0), \tag{3.1.21.1}$$

where

$$A_0 = \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}} \in \hat{\mathbf{L}}^\circ,$$

and where $\mathbf{P} \subseteq \mathbf{G}$ is *any* parabolic subgroup having $\mathbf{L}$ as a Levi complement. In particular, (3.1.21.1) implies that $\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)$ is independent of the chosen $\mathbf{P}$, so we may (and will) write $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$ instead of $\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)$ from now on, without referring to $\mathbf{P}$ at all. Let us also mention that, by [LuCS1, 4.3], the support of any simple direct summand of $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$ is $\overline{Y}$.

3.1.22. Let $\mathbf{L}$ be the *standard* Levi subgroup of some *standard* parabolic subgroup of $\mathbf{G}$ with respect to $\mathbf{T}_0 \subseteq \mathbf{B}_0$ (see 2.1.10), so $\mathbf{T}_0 \subseteq \mathbf{L} = \mathbf{L}_J$ and $\mathbf{B}_0 \subseteq \mathbf{P} = \mathbf{P}_J$ for some $J \subseteq S$, where $S \subseteq \mathbf{W}$ are the simple reflections determined by $\mathbf{T}_0 \subseteq \mathbf{B}_0$. Assume that $A_0 \in \hat{\mathbf{L}}^\circ$, and let $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$ be such that $A_0 \in \hat{\mathbf{L}}_{\mathscr{L}}$. Let $(\Sigma, \mathscr{E})$ be the corresponding cuspidal pair for $\mathbf{L}$, so that

$$A_0 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}}.$$

By [LuCS1, 4.8], any simple direct summand of the semisimple perverse sheaf $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$ is in $\hat{\mathbf{G}}_{\mathscr{L}}$. Conversely, if $A \in \hat{\mathbf{G}}$, there exists a standard Levi subgroup $\mathbf{L}$ of some standard parabolic subgroup of $\mathbf{G}$, and a cuspidal character sheaf $A_0 \in \hat{\mathbf{L}}^\circ$, such that A is a (simple) direct summand of $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$, see [LuCS1, 4.4]. This provides a classification of $\hat{\mathbf{G}}$ in terms of 'Harish-Chandra series', cf. 2.2.6 (see also 3.2.7 below).

3.2. $\mathbb{F}_q$-rational structure and characteristic functions

Let us fix the following conventions (extending the ones introduced in 3.1.4), which remain in force until the end of this chapter:

> From now on, we always assume that the connected reductive group $\mathbf{G}$ over $k = \overline{\mathbb{F}}_p$ is defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. In case there is no ambiguity about F, we will also assume that both the maximal torus $\mathbf{T}_0$ and the Borel subgroup $\mathbf{B}_0 \supseteq \mathbf{T}_0$ of $\mathbf{G}$ (see 3.1.4) are F-stable, unless stated otherwise.

An orthonormal basis for the class functions

3.2.1. The constructions outlined in 3.1.1 and 3.1.2 allow the definition of a Frobenius action on the character sheaves on $\mathbf{G}$, as follows: For $A \in \mathscr{M}\mathbf{G}$, the inverse image F^*A under the Frobenius endomorphism is also in $\mathscr{M}\mathbf{G}$, see (3.1.1.2). Assume that F^*A is isomorphic to A (such an A will be called *F-stable*), and let us choose an isomorphism $\varphi\colon F^*A \xrightarrow{\sim} A$. In the case where A is $\mathbf{G}$-equivariant for the conjugation action of $\mathbf{G}$ on itself, the associated characteristic function $\chi_{A,\varphi}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ is in $\mathrm{CF}(\mathbf{G}^F)$, see 3.1.2. If, in addition, A is a simple object of $\mathscr{M}\mathbf{G}$, then φ (and, hence, $\chi_{A,\varphi}$) is unique up to multiplication with a non-zero scalar in $\overline{\mathbb{Q}}_\ell$. (Here, multiplication of φ with the scalar $\xi \in \overline{\mathbb{Q}}_\ell^\times$ refers to the isomorphism $\xi\varphi\colon F^*A \xrightarrow{\sim} A$ which induces the $\overline{\mathbb{Q}}_\ell$-linear map $\xi\varphi_{i,g}\colon \mathscr{H}^i_{F(g)}(A) \to \mathscr{H}^i_g(A)$ for any $i \in \mathbb{Z}$, $g \in \mathbf{G}$, with the notation of 3.1.2.) In particular, all of the above applies to F-stable character sheaves on $\mathbf{G}$. We denote by

$$\hat{\mathbf{G}}^F := \{A \in \hat{\mathbf{G}} \mid F^*A \cong A\}$$

the set of F-stable character sheaves in $\hat{\mathbf{G}}$. Let $A \in \hat{\mathbf{G}}^F$, and let $\mathbf{L}$ be the standard Levi subgroup of some standard parabolic subgroup (with respect to $\mathbf{T}_0 \subseteq \mathbf{B}_0$), $(\Sigma, \mathscr{E})$ a cuspidal pair for $\mathbf{L}$, such that A is a direct summand of $K^{\mathscr{E}}_{\mathbf{L},\Sigma}$; we have $\operatorname{supp} A = \overline{Y}_{(\mathbf{L},\Sigma)}$ (see 3.1.21, 3.1.22).

We set $Y := Y_{(\mathbf{L},\Sigma)}$ and $d := \dim Y$. By [LuCS5, 25.1], an isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$ can be chosen in such a way that the following holds:

$$\begin{array}{l}\text{For any } n \in \mathbb{N} \text{ and any } y \in Y \text{ satisfying } F^n(y) = y,\\ \text{the eigenvalues of } (\varphi_A)^n_{-d,y} \colon \mathscr{H}^{-d}_y(A) \to \mathscr{H}^{-d}_y(A) \text{ are}\\ \text{of the form } q^{n(\dim \mathbf{G}-d)/2} \text{ times a root of unity in } \overline{\mathbb{Q}}_\ell.\end{array} \qquad (*)$$

In particular, $(*)$ determines φ_A up to multiplication with a root of unity. Sometimes, for $A \in \hat{\mathbf{G}}^F$, it will be convenient to just write χ_A for a characteristic function associated to the character sheaf A, without referring to a specific isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$. Whenever we do this, we tacitly assume that $\chi_A = \chi_{A,\varphi_A}$ for a chosen isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$ which satisfies $(*)$.

The following is one of the main results of [LuCS5], which we can now state without any restriction on the characteristic in view of the remarks in [Lus12b, 3.10].

Theorem 3.2.2 (Lusztig [LuCS5, §25], [Lus12b, 3.10])**.** *For any $A \in \hat{\mathbf{G}}^F$, assume that $\varphi_A \colon F^*A \xrightarrow{\sim} A$ is chosen as in* 3.2.1$(*)$*. Then the following hold.*

(a) The values of the characteristic functions χ_{A,φ_A} are cyclotomic integers;

(b) $\{\chi_{A,\varphi_A} \mid A \in \hat{\mathbf{G}}^F\}$ is an orthonormal basis of $\left(\mathrm{CF}(\mathbf{G}^F), \langle\ ,\ \rangle_{\mathbf{G}^F}\right)$.

On the computation of characteristic functions

Lusztig also provides a strategy which in principle allows the computation of the characteristic functions χ_A for $A \in \hat{\mathbf{G}}^F$. We will describe this here, following [LuCS2, §8, §10] and [LuCS5, §24].

3.2.3. Let $\mathbf{L} \subseteq \mathbf{G}$ be a Levi complement of some parabolic subgroup of $\mathbf{G}$, and let $(\Sigma, \mathscr{E})$ be a cuspidal pair for $\mathbf{L}$. Thus, we obtain the semisimple perverse sheaf

$$K := K^{\mathscr{E}}_{\mathbf{L},\Sigma} \in \mathscr{M}\mathbf{G},$$

see 3.1.21. We assume that $F^*K \cong K$ and choose an isomorphism $\varphi \colon F^*K \xrightarrow{\sim} K$.

Consider the endomorphism algebra

$$\mathscr{A} := \mathscr{A}_{\mathbf{L},\Sigma}^{\mathscr{E}} := \operatorname{End}_{\mathscr{M}\mathbf{G}}(K).$$

For any $A \in \mathscr{M}\mathbf{G}$ which is a summand of K, we set

$$V_A := \operatorname{Hom}_{\mathscr{M}\mathbf{G}}(A, K),$$

a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space which becomes a left $\mathscr{A}$-module via the action

$$\mathscr{A} \times V_A \to V_A, \quad (\vartheta, v) \mapsto \vartheta \circ v.$$

By a general argument concerning locally finite $\overline{\mathbb{Q}}_\ell$-linear abelian categories (see, e.g., [TT20, Appendix A]; cf. 2.3.5), one shows that $\mathscr{A}$ is a finite-dimensional semisimple $\overline{\mathbb{Q}}_\ell$-algebra, and that $A \mapsto V_A$ defines a bijection between the set of isomorphism classes of simple direct summands of K and the set of isomorphism classes of irreducible left $\mathscr{A}$-modules. Note that, since $F^*K \cong K$, F^*A is (isomorphic to) a simple direct summand of K as well, so V_{F^*A} is also an irreducible left $\mathscr{A}$-module. Thus, if $v\colon A \to K$ is in V_A, the inverse image functor F^* of the Frobenius endomorphism gives rise to a morphism $F^*(v)\colon F^*A \to F^*K$, and we get an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces

$$\varrho\colon V_A \to V_{F^*A}, \quad v \mapsto \varphi \circ F^*(v).$$

For any A as above which is F-stable, let us fix an isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$. This gives rise to an isomorphism of $\mathscr{A}$-modules

$$V_{F^*A} \to V_A, \quad v \mapsto v \circ \varphi_A^{-1}.$$

Composing this isomorphism with ϱ thus yields an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces

$$\sigma_A\colon V_A \to V_A, \quad v \mapsto \varrho(v) \circ \varphi_A^{-1} = \varphi \circ F^*(v) \circ \varphi_A^{-1}. \tag{3.2.3.1}$$

There is a natural isomorphism

$$\bigoplus_{(A|K)/\simeq} (A \otimes V_A) \xrightarrow{\sim} K$$

(where the index $(A|K)/_{\simeq}$ means that the sum is taken over a set of representatives for the isomorphism classes of the simple constituents A of K, thus tensoring with V_A is needed to take care of the multiplicity of A in K). Passing to stalks of the cohomology sheaves, we obtain isomorphisms

$$\bigoplus_{(A|K)/\simeq} (\mathscr{H}^i_g(A) \otimes V_A) \xrightarrow{\sim} \mathscr{H}^i_g(K) \tag{3.2.3.2}$$

for any $g \in \mathbf{G}$, $i \in \mathbb{Z}$. The latter can be described explicitly, as follows: Given $v \in V_A$, denote by $v_{i,g}$ the induced map $\mathscr{H}^i_g(A) \to \mathscr{H}^i_g(K)$. Let $a \in \mathscr{H}^i_g(A)$, $v \in V_A$, then $a \otimes v$ corresponds to $v_{i,g}(a)$ under (3.2.3.2). For any $g \in \mathbf{G}$ and $i \in \mathbb{Z}$, $\varphi\colon F^*K \xrightarrow{\sim} K$ induces a $\overline{\mathbb{Q}}_\ell$-linear bijective map $\varphi_{i,g}\colon \mathscr{H}^i_{F(g)}(K) \xrightarrow{\sim} \mathscr{H}^i_g(K)$ (see 3.1.2). Similarly, if F^*A is isomorphic to A, any isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$ induces a $\overline{\mathbb{Q}}_\ell$-linear bijective map $(\varphi_A)_{i,g}\colon \mathscr{H}^i_{F(g)}(A) \xrightarrow{\sim} \mathscr{H}^i_g(A)$. If $g \in \mathbf{G}^F$, it immediately follows from the above definitions that the $\overline{\mathbb{Q}}_\ell$-linear bijective map $(\varphi_A)_{i,g} \otimes \sigma_A \in \mathrm{GL}(\mathscr{H}^i_g(A) \otimes V_A)$ corresponds to $\varphi_{i,g}$ under the identification (3.2.3.2). If, on the other hand, F^*A is not isomorphic to A, then φ maps the image of $\mathscr{H}^i_g(A) \otimes V_A$ in $\mathscr{H}^i_g(K)$ onto a summand corresponding to a component of K which is not isomorphic to A. We thus obtain

$$\mathrm{Trace}(\varphi_{i,g}, \mathscr{H}^i_g(K)) = \sum_{\substack{(A|K)/\simeq \\ F^*A \cong A}} \mathrm{Trace}((\varphi_A)_{i,g}, \mathscr{H}^i_g(A)) \cdot \mathrm{Trace}(\sigma_A, V_A).$$

Taking the alternating sum over $i \in \mathbb{Z}$ on either side, it follows that

$$\chi_{K,\varphi} = \sum_{\substack{(A|K)/\simeq \\ F^*A \cong A}} \mathrm{Trace}(\sigma_A, V_A) \cdot \chi_{A,\varphi_A}. \tag{3.2.3.3}$$

3.2.4. We keep the setting and notation of 3.2.3. The algebra $\mathscr{A}$ may also be interpreted as the endomorphism algebra of the local system $\gamma_*\tilde{\mathscr{E}}$ used to define K in 3.1.21 (see [Lus84b, 4.1]), which allows a description of the structure of $\mathscr{A}$, see [Lus84b, 3.4, 3.5] and also [LuCS2, 10.2]. Let

$$\mathscr{N} := \mathscr{N}^{\mathscr{E}}_{\mathbf{L},\Sigma} := \{n \in N_{\mathbf{G}}(\mathbf{L}) \mid n\Sigma n^{-1} = \Sigma \text{ and } \mathrm{Int}(n)^*\mathscr{E} \cong \mathscr{E}\},$$

where

$$\mathrm{Int}(n)\colon \mathbf{G} \to \mathbf{G}, \quad x \mapsto nxn^{-1}.$$

We have $\mathbf{L} \subseteq \mathscr{N}$, and $\mathbf{L}$ is normal in $\mathscr{N}$, so we obtain a finite group

$$\mathscr{W} := \mathscr{W}^{\mathscr{E}}_{\mathbf{L},\Sigma} := \mathscr{N}/\mathbf{L}.$$

It can be shown that we have a decomposition

$$\mathscr{A} = \bigoplus_{w \in \mathscr{W}} \mathscr{A}_w,$$

where each $\mathscr{A}_w$ is a one-dimensional $\overline{\mathbb{Q}}_\ell$-vector space and, under the multiplication of $\mathscr{A}$, we have $\mathscr{A}_w \cdot \mathscr{A}_{w'} = \mathscr{A}_{ww'}$ for any $w, w' \in \mathscr{W}$. The unit element of $\mathscr{A}$ is contained in $\mathscr{A}_1$. Choosing a basis element $\vartheta_w \in \mathscr{A}_w$ for each $w \in \mathscr{W}$ (and taking $\vartheta_1 := 1_{\mathscr{A}}$), we can write

$$\vartheta_w \cdot \vartheta_{w'} = \lambda(w, w') \cdot \vartheta_{ww'} \quad \text{for } w, w' \in \mathscr{W} \text{ and suitable } \lambda(w, w') \in \overline{\mathbb{Q}}_\ell^\times.$$

Thus, $\mathscr{A}$ is isomorphic to the group algebra of $\mathscr{W}$ twisted by a 2-cocycle. For $w \in \mathscr{W}$ and a simple direct summand A of K which is F-stable, the construction for the definition of σ_A may of course also be applied with φ replaced by $\vartheta_w \circ \varphi\colon F^*K \xrightarrow{\sim} K$ (while keeping φ_A unchanged), so we get isomorphisms

$$\sigma_A^w\colon V_A \to V_A, \quad v \mapsto \vartheta_w \circ \varphi \circ F^*(v) \circ \varphi_A^{-1} \quad (\text{for } w \in \mathscr{W}).$$

The discussion in 3.2.3 yields that

$$\operatorname{Trace}\big((\vartheta_w \circ \varphi)_{i,g}, \mathscr{H}_g^i(K)\big) = \sum_{\substack{(A|K)/\simeq \\ F^*A \cong A}} \operatorname{Trace}\big((\varphi_A)_{i,g}, \mathscr{H}_g^i(A)\big) \cdot \operatorname{Trace}(\sigma_A^w, V_A),$$

valid for any $w \in \mathscr{W}$, $i \in \mathbb{Z}$ and $g \in \mathbf{G}^F$. Taking the sum over all $w \in \mathscr{W}$ and using certain orthogonality relations for the algebra $\mathscr{A}$ [LuCS2, 10.3], we obtain, for any F-stable A which is isomorphic to a simple direct summand of K and any chosen isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$:

$$\operatorname{Trace}\big((\varphi_A)_{i,g}, \mathscr{H}_g^i(A)\big) = \frac{1}{|\mathscr{W}|} \sum_{w \in \mathscr{W}} \operatorname{Trace}\big((\vartheta_w \circ \varphi)_{i,g}, \mathscr{H}_g^i(K)\big) \cdot \operatorname{Trace}\big((\sigma_A^w)^{-1}, V_A\big).$$

Taking the alternating sum over all $i \in \mathbb{Z}$ we get, for any A, φ_A as above,

$$\chi_{A,\varphi_A} = \frac{1}{|\mathscr{W}|} \sum_{w \in \mathscr{W}} \operatorname{Trace}\big((\sigma_A^w)^{-1}, V_A\big) \cdot \chi_{K, \vartheta_w \circ \varphi}. \tag{3.2.4.1}$$

3.2.5. As Lusztig shows in [LuCS2, 10.5], formula (3.2.4.1) can be applied to any $A \in \hat{\mathbf{G}}^F$, for a suitable K. Namely, if $A \in \hat{\mathbf{G}}^F$, the discussion in 3.1.22 first of all shows that A is isomorphic to an irreducible direct summand of $\operatorname{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$, where $\mathbf{L}$ is the (standard) Levi subgroup of some (standard) parabolic subgroup of $\mathbf{G}$, and $A_0 \in \hat{\mathbf{L}}^\circ$ is a cuspidal character sheaf on $\mathbf{L}$. By Proposition 3.1.17, we have

$$A_0 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}}$$

for some cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{L}$. Setting $K := K^{\mathscr{E}}_{\mathbf{L},\Sigma}$, there is a canonical isomorphism $K \cong \mathrm{ind}^{\mathbf{G}}_{\mathbf{L}}(A_0)$ (see 3.1.21), so A is isomorphic to a simple direct summand of K. By a standard application of the Lang–Steinberg Theorem 2.1.14 (see [LuCS2, 10.5]), one proves the existence of some $g_1 \in \mathbf{G}$ such that

$$F(g_1\mathbf{L}g_1^{-1}) = g_1\mathbf{L}g_1^{-1},\ F(g_1\Sigma g_1^{-1}) = g_1\Sigma g_1^{-1},\ F^*(\mathrm{Int}(g_1^{-1})^*\mathscr{E}) \cong \mathrm{Int}(g_1^{-1})^*\mathscr{E}.$$

Then

$$\mathrm{Int}(g_1^{-1})^*(A_0) \cong \mathrm{IC}(g_1\overline{\Sigma}g_1^{-1}, \mathrm{Int}(g_1^{-1})^*\mathscr{E})[\dim \Sigma]^{\#g_1\mathbf{L}g_1^{-1}}$$

is an F-stable cuspidal character sheaf on $g_1\mathbf{L}g_1^{-1}$, and inducing it to $\mathbf{G}$ gives a complex which is isomorphic to K. Thus, we may assume that $\mathbf{L}$, Σ, $\mathscr{E}$, A_0 and K are all F-stable (although $\mathbf{L}$ will in general no longer be the standard Levi subgroup of some standard parabolic subgroup of $\mathbf{G}$). Let us choose an isomorphism $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ of local systems over Σ. Since all the varieties Y, $\hat{Y}$, $\tilde{Y}$ and morphisms α, β, γ used to define $K = K^{\mathscr{E}}_{\mathbf{L},\Sigma}$ in 3.1.21 are defined over $\mathbb{F}_q$ (with respect to the obvious Frobenius morphisms induced by F), $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ gives rise to an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$ in $\mathscr{M}\mathbf{G}$ [LuCS2, (8.1.3)]. On the other hand, φ_0 also induces an isomorphism $\varphi_{A_0}\colon F^*A_0 \xrightarrow{\sim} A_0$ over $\mathscr{M}\mathbf{G}$, hence an isomorphism

$$\mathrm{ind}^{\mathbf{G}}_{\mathbf{L}}(\varphi_{A_0})\colon F^*(\mathrm{ind}^{\mathbf{G}}_{\mathbf{L}}(A_0)) \xrightarrow{\sim} \mathrm{ind}^{\mathbf{G}}_{\mathbf{L}}(A_0).$$

By [LuCS2, (8.2.4)], $\mathrm{ind}^{\mathbf{G}}_{\mathbf{L}}(\varphi_{A_0})$ corresponds to $\varphi\colon F^*K \xrightarrow{\sim} K$ under the canonical isomorphism $K \cong \mathrm{ind}^{\mathbf{G}}_{\mathbf{L}}(A_0)$. Thus, we are in the setting of 3.2.3, 3.2.4, so in principle the computation of the characteristic functions of F-stable character sheaves on $\mathbf{G}$ is reduced to that of the characteristic functions of induced complexes K from F-stable cuspidal character sheaves on various regular subgroups of $\mathbf{G}$ (see Definition 2.1.18).

3.2.6. Let $A \in \hat{\mathbf{G}}^F$, and let $\varphi_A\colon F^*A \xrightarrow{\sim} A$ be any isomorphism. As described in 3.2.5, there is a regular subgroup $\mathbf{L} \subseteq \mathbf{G}$ and an F-stable cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{L}$ such that A is a direct summand of $K = K^{\mathscr{E}}_{\mathbf{L},\Sigma} \in \mathscr{M}\mathbf{G}$, and a chosen isomorphism $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ naturally determines an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$. Then the formula (3.2.4.1) holds with these A, K and φ, for a chosen basis $\vartheta_w \in \mathscr{A}_w$ ($w \in \mathscr{W} = \mathscr{W}^{\mathscr{E}}_{\mathbf{L},\Sigma}$) of $\mathscr{A} = \mathscr{A}^{\mathscr{E}}_{\mathbf{L},\Sigma}$ (with the notation of 3.2.3, 3.2.4). Let $w \in \mathscr{W}$, and let us fix a representative $n \in \mathscr{N}^{\mathscr{E}}_{\mathbf{L},\Sigma}$ of w. In order to be able to describe the characteristic function $\chi_{K,\vartheta_w\circ\varphi}$ more precisely, we slightly modify the complex K, so that the isomorphism $\vartheta_w \circ \varphi\colon F^*K \xrightarrow{\sim} K$ can be replaced by one which is naturally induced from the underlying local system $\mathscr{E}$, following [LuCS2, 10.6]. By the Lang–Steinberg Theorem 2.1.14, there exists some $z \in \mathbf{G}$ such that $z^{-1}F(z) = n^{-1}$. We set

$$\mathbf{L}_{w^{-1}} := z\mathbf{L}z^{-1}, \quad \Sigma_{w^{-1}} := z\Sigma z^{-1} \quad \text{and} \quad \mathscr{E}_{w^{-1}} := \mathrm{Int}(z^{-1})^*\mathscr{E},$$

so that $(\Sigma_{w^{-1}}, \mathscr{E}_{w^{-1}})$ is a cuspidal pair for $\mathbf{L}_{w^{-1}}$. Note that we still have

$$F(\mathbf{L}_{w^{-1}}) = \mathbf{L}_{w^{-1}}, \quad F(\Sigma_{w^{-1}}) = \Sigma_{w^{-1}} \quad \text{and} \quad F^*(\mathscr{E}_{w^{-1}}) \cong \mathscr{E}_{w^{-1}}.$$

Now consider the complex

$$K_{w^{-1}} := K^{\mathscr{E}_{w^{-1}}}_{\mathbf{L}_{w^{-1}}, \Sigma_{w^{-1}}}$$

(see 3.1.21), so that

$$K_{w^{-1}} \cong \mathrm{ind}^{\mathbf{G}}_{\mathbf{L}_{w^{-1}}}\big((A_0)_{w^{-1}}\big),$$

with

$$(A_0)_{w^{-1}} := \mathrm{IC}\big(\overline{\Sigma}_{w^{-1}}, \mathscr{E}_{w^{-1}}\big)\big[\dim \Sigma_{w^{-1}}\big]^{\#\mathbf{L}_{w^{-1}}}.$$

Lusztig defines an isomorphism $(\varphi_0)_{w^{-1}} \colon F^*\mathscr{E}_{w^{-1}} \xrightarrow{\sim} \mathscr{E}_{w^{-1}}$ in terms of $\varphi_0 \colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ and ϑ_w in such a way that, if we denote by $\varphi_{w^{-1}} \colon F^*K_{w^{-1}} \xrightarrow{\sim} K_{w^{-1}}$ the isomorphism induced by $(\varphi_0)_{w^{-1}}$, we have

$$\mathrm{Trace}((\vartheta_w \circ \varphi)_{i,g}, \mathscr{H}^i_g K) = \mathrm{Trace}((\varphi_{w^{-1}})_{i,g}, \mathscr{H}^i_g K_{w^{-1}}) \quad \text{for any } g \in \mathbf{G}^F,\ i \in \mathbb{Z}.$$

Hence, (3.2.4.1) becomes

$$\chi_{A,\varphi_A} = \frac{1}{|\mathscr{W}|} \sum_{w \in \mathscr{W}} \mathrm{Trace}\big((\sigma_A^w)^{-1}, V_A\big) \cdot \chi_{K_{w^{-1}}, \varphi_{w^{-1}}}. \tag{3.2.6.1}$$

3.2.7. Before explaining how the computation of the characteristic functions $\chi_{K_{w^{-1}}, \varphi_{w^{-1}}}$ (for $w \in \mathscr{W} = \mathscr{W}^{\mathscr{E}}_{\mathbf{L},\Sigma}$ and $K_{w^{-1}}$, $\varphi_{w^{-1}}$ as in 3.2.6) can be approached, we will show how (3.2.6.1) can be reformulated using Lusztig induction (see Definition 2.2.28), at least for groups with a connected centre. We thus place ourselves in the setting of 3.2.6 and assume in addition that the centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$ is connected. So for any $w \in \mathscr{W}$, there is a canonical isomorphism

$$K_{w^{-1}} \cong \mathrm{ind}^{\mathbf{G}}_{\mathbf{L}_{w^{-1}}}\big((A_0)_{w^{-1}}\big),$$

and under this identification, the isomorphism $\varphi_{w^{-1}}\colon F^*K_{w^{-1}} \xrightarrow{\sim} K_{w^{-1}}$ corresponds to

$$\mathrm{ind}_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}\Big(\varphi_{(A_0)_{w^{-1}}}\Big)\colon F^*\Big(\mathrm{ind}_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}((A_0)_{w^{-1}})\Big) \xrightarrow{\sim} \mathrm{ind}_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}((A_0)_{w^{-1}})$$

where $\varphi_{(A_0)_{w^{-1}}}\colon F^*((A_0)_{w^{-1}}) \xrightarrow{\sim} (A_0)_{w^{-1}}$ is the isomorphism induced by $(\varphi_0)_{w^{-1}}$, see 3.2.5.

Thanks to the results of Lusztig [Lus90] and Shoji [Sho96, §4], it is known that under the above circumstances, the following hold:

(i) Lusztig induction is independent of the chosen parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ which has $\mathbf{L}_{w^{-1}}$ as Levi complement, so we may just choose any such $\mathbf{P}$ and write $R_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}} = R_{\mathbf{L}_{w^{-1}},F(\mathbf{U}_\mathbf{P})}^{\mathbf{G}}$.

(ii) We have $\chi_{K_{w^{-1}},\varphi_{w^{-1}}} = R_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}\Big(\chi_{(A_0)_{w^{-1}},\varphi_{(A_0)_{w^{-1}}}}\Big)$.

(The assumption that p is 'almost good' in [Sho96, Thm. 4.2] can be dropped since the results of [Sho95a], [Sho95b] used in the proof are now known to hold in complete generality, due to [Lus12b].) Hence, (3.2.6.1) becomes

$$\chi_{A,\varphi_A} = \frac{1}{|\mathscr{W}|}\sum_{w\in\mathscr{W}} \mathrm{Trace}\big((\sigma_A^w)^{-1}, V_A\big)\cdot R_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}\Big(\chi_{(A_0)_{w^{-1}},\varphi_{(A_0)_{w^{-1}}}}\Big). \tag{3.2.7.1}$$

As discussed in 3.2.5, such a formula holds for any $A \in \hat{\mathbf{G}}^F$ (with an appropriately chosen K depending on A) and any isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$. (Recall from 3.2.4 that the definition of σ_A^w, $w \in \mathscr{W}$, takes the choice of φ_A into account.) Hence, the characteristic function of any given F-stable character sheaf can be expressed as a linear combination of various $R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})$, with $\mathbf{L} \subseteq \mathbf{G}$ a regular subgroup and $A_0 \in (\hat{\mathbf{L}}^\circ)^F$ an F-stable cuspidal character sheaf on $\mathbf{L}$, such that the different $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ (for $\mathbf{L}$ occurring in the decomposition) are isomorphic to each other. In particular, in view of Theorem 3.2.2, we have

$$\mathrm{CF}(\mathbf{G}^F) = \langle R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0}) \mid \mathbf{L} \subseteq \mathbf{G} \text{ regular and } A_0 \in (\hat{\mathbf{L}}^\circ)^F\rangle_{\overline{\mathbb{Q}}_\ell}. \tag{3.2.7.2}$$

This also allows us to characterise the F-stable *cuspidal* character sheaves $(\hat{\mathbf{G}}^\circ)^F$ among the F-stable character sheaves on $\mathbf{G}$ in terms of Lusztig induction, which highlights the analogy to the Harish-Chandra theory for characters (cf. 2.2.6). Namely, we have

$$(\hat{\mathbf{G}}^\circ)^F = \{A \in \hat{\mathbf{G}}^F \mid \langle \chi_A, R_{\mathbf{L}}^{\mathbf{G}}(f)\rangle_{\mathbf{G}^F} = 0 \text{ for any } \mathbf{L} \subsetneq \mathbf{G} \text{ regular, } f \in \mathrm{CF}(\mathbf{L}^F)\}.$$

Indeed, let $A \in (\hat{\mathbf{G}}^\circ)^F$, and let $\mathbf{L} \subsetneq \mathbf{G}$ be a proper regular subgroup of $\mathbf{G}$, $f \in \mathrm{CF}(\mathbf{L}^F)$. Applying (3.2.7.2) to $\mathbf{L}$ (note that $\mathbf{L}$ also has a connected centre, see [DM20, 11.2.1]), we can write f as a linear combination of certain $R_{\mathbf{M}}^{\mathbf{L}}(\chi_{A_0})$, where any $\mathbf{M} \subseteq \mathbf{L}$ is a regular subgroup and $A_0 \in (\hat{\mathbf{M}}^\circ)^F$. By the transitivity of Lusztig induction (see Remark 2.2.29), we are thus reduced to showing that $\langle \chi_A, R_{\mathbf{M}}^{\mathbf{G}}(\chi_{A_0})\rangle_{\mathbf{G}^F} = 0$ for any such $\mathbf{M}$, A_0. By what we have said above, $R_{\mathbf{M}}^{\mathbf{G}}(\chi_{A_0})$ is a characteristic function of the induced complex $\mathrm{ind}_{\mathbf{M}}^{\mathbf{G}}(A_0)$, which by (3.2.3.3) is a linear combination of characteristic functions of the character sheaves on $\mathbf{G}$ which appear as simple summands of $\mathrm{ind}_{\mathbf{M}}^{\mathbf{G}}(A_0)$. But these are certainly not cuspidal (since $\mathbf{M} \subsetneq \mathbf{G}$), so they are all orthogonal to χ_A by Theorem 3.2.2. Conversely, let $A \in \hat{\mathbf{G}}^F$ be non-cuspidal. Then by the above discussion, χ_A is a linear combination of certain $R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})$ (with $\mathbf{L} \subseteq \mathbf{G}$ a regular subgroup, $A_0 \in (\hat{\mathbf{L}}^\circ)^F$), such that for every $\mathbf{L}$ which appears, $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ has the same type. It follows that $\mathbf{L} \neq \mathbf{G}$ for each such $\mathbf{L}$ (since A is not cuspidal, see again Theorem 3.2.2). As $\langle \chi_A, \chi_A\rangle_{\mathbf{G}^F} = 1$, writing χ_A as a linear combination of the $R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})$ above yields that at least one of the $\langle \chi_A, R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})\rangle_{\mathbf{G}^F}$ must be non-zero, as claimed.

Corollary 3.2.8. *Let $\mathbf{Z}(\mathbf{G})$ be connected. Let $A \in \hat{\mathbf{G}}^F$, and let $\mathbf{L} \subseteq \mathbf{G}$ be a regular subgroup, $(\Sigma, \mathscr{E})$ an F-stable cuspidal pair for $\mathbf{L}$, such that A is isomorphic to a direct summand of $K = K_{\mathbf{L},\Sigma}^{\mathscr{E}} \in \mathscr{M}\mathbf{G}$ (see* 3.2.5*). Assume that Σ does not contain any unipotent elements. Then we have*

$$\chi_A|_{\mathbf{G}^F_{\mathrm{uni}}} = 0.$$

Proof. We set $A_0 := \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}}$, an F-stable cuspidal character sheaf on $\mathbf{L}$ (see Remark 3.1.18). From the discussion in 3.2.6 and 3.2.7 (and with the notation there), it follows that χ_A is a linear combination of various $R_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}(\chi_{(A_0)_{w^{-1}}})$, where w runs through $\mathscr{W} = \mathscr{W}_{\mathbf{L},\Sigma}^{\mathscr{E}}$. Thus, in order to prove the corollary, it is sufficient to show that any such $R_{\mathbf{L}_{w^{-1}}}^{\mathbf{G}}(\chi_{(A_0)_{w^{-1}}})$ vanishes on all unipotent elements of $\mathbf{G}^F$. Recall that

$$(A_0)_{w^{-1}} = \mathrm{IC}\big(\overline{\Sigma}_{w^{-1}}, \mathscr{E}_{w^{-1}}\big)\big[\dim \Sigma_{w^{-1}}\big]^{\#\mathbf{L}_{w^{-1}}},$$

and the F-stable triple $(\mathbf{L}_{w^{-1}}, \Sigma_{w^{-1}}, \mathscr{E}_{w^{-1}})$ is obtained from $(\mathbf{L}, \Sigma, \mathscr{E})$ by simultaneous conjugation ('twisting with w^{-1}') with an element of $\mathbf{G}$. In particular, the quadruple $(\mathbf{L}_{w^{-1}}, \Sigma_{w^{-1}}, \mathscr{E}_{w^{-1}}, K_{w^{-1}})$ meets the assumptions of the corollary, so without loss of generality it suffices to prove that $R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})$ vanishes on all unipotent elements of $\mathbf{G}^F$. Let $u \in \mathbf{G}^F_{\mathrm{uni}}$. By Remark 2.2.29(d) (and the $\overline{\mathbb{Q}}_\ell$-linearity of $R_{\mathbf{L}}^{\mathbf{G}}$), we have

$$\Big(R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})\Big)(u) = \frac{1}{|\mathbf{L}^F|} \sum_{l \in \mathbf{L}^F_{\mathrm{uni}}} \mathfrak{L}((u,l), \mathcal{L}^{-1}(F(\mathbf{U_P})))\chi_{A_0}(l).$$

Since $\mathbf{G}$ is clean (see Theorem 3.1.13(b)), χ_{A_0} vanishes on all elements outside of Σ^F, so we have $\chi_{A_0}|_{\mathbf{L}^F_{\text{uni}}} = 0$ by our assumption on Σ. We deduce that

$$\left(R^{\mathbf{G}}_{\mathbf{L}}(\chi_{A_0})\right)(u) = 0 \quad \text{for all } u \in \mathbf{G}^F_{\text{uni}},$$

as desired. □

Definition 3.2.9 ([LuCS2, 8.3])**.** Let $\mathbf{L} \subseteq \mathbf{G}$ be a regular subgroup. Consider an F-stable unipotent class $\mathscr{O}$ of $\mathbf{L}$ and set $\Sigma = \mathbf{Z}(\mathbf{L})^\circ\mathscr{O} \subseteq \mathbf{L}$. Let $\mathscr{F}$ be an $\mathbf{L}$-equivariant local system on $\mathscr{O}$ which is isomorphic to $F^*\mathscr{F}$, and let $\psi_0\colon F^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ be an isomorphism. Let $\mathscr{E}$ be an $\mathbf{L}$-equivariant F-stable local system on Σ such that $\mathscr{F} = \mathscr{E}|_{\mathscr{O}}$, and let $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ be an isomorphism which extends the isomorphism ψ_0. (One may take $(\mathscr{E}, \varphi_0)$ as the inverse image of $(\mathscr{F}, \psi_0)$ under the canonical map $\Sigma \to \mathscr{O}$.) We assume that $\mathscr{E}$ is a direct sum of irreducible local systems $\mathscr{E}_i$ on Σ such that each $(\Sigma, \mathscr{E}_i)$ is a cuspidal pair for $\mathbf{L}$. Let $K = K^{\mathscr{E}}_{\mathbf{L},\Sigma}$ be as in 3.1.21, and let $\varphi\colon F^*K \xrightarrow{\sim} K$ be the isomorphism induced by φ_0, as described in 3.2.5. The function

$$Q^{\mathbf{G}}_{\mathbf{L},\mathscr{O},\mathscr{F},\psi_0}\colon \mathbf{G}^F_{\text{uni}} \to \overline{\mathbb{Q}}_\ell, \quad u \mapsto \chi_{K,\varphi}(u),$$

is called a *generalised Green function.* (It is justified to omit $\mathscr{E}$ and φ_0 from the notation since it can be shown [LuCS2, (8.3.2)] that for any $u \in \mathbf{G}^F_{\text{uni}}$, $\chi_{K,\varphi}(u)$ is independent of the choice of $(\mathscr{E}, \varphi_0)$ extending $(\mathscr{F}, \psi_0)$.)

3.2.10. Let $\mathbf{L} \subseteq \mathbf{G}$ be a regular subgroup, and let $\Sigma \subseteq \mathbf{L}$ be the preimage of an isolated conjugacy class of $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ under the canonical map $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$. We assume that $F(\Sigma) = \Sigma$ and that there exists a $\mathbf{G}$-equivariant F-stable irreducible local system $\mathscr{E}$ on Σ such that $(\Sigma, \mathscr{E})$ is a cuspidal pair for $\mathbf{L}$. Let us fix an isomorphism $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$. As discussed in 3.2.5, φ_0 naturally induces an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$, where $K := K^{\mathscr{E}}_{\mathbf{L},\Sigma} \in \mathscr{M}\mathbf{G}$. Consider an element $g \in \mathbf{G}^F$, and let $g = g_{\text{s}}g_{\text{u}} = g_{\text{u}}g_{\text{s}}$ be its Jordan decomposition ($g_{\text{s}} \in \mathbf{G}^F$ semisimple, $g_{\text{u}} \in \mathbf{G}^F$ unipotent). We denote by $\Sigma_{\text{ss}} \subseteq \mathbf{L}$ the set of semisimple parts of the Jordan decompositions of the elements in Σ. Assume that there exists some $x \in \mathbf{G}^F$ for which $x^{-1}g_{\text{s}}x \in \Sigma_{\text{ss}}$, and let us fix such an $x \in \mathbf{G}^F$. Then $g_{\text{s}} \in x\mathbf{L}x^{-1}$, and so the group $\mathbf{L}_x := x\mathbf{L}x^{-1} \cap C^\circ_{\mathbf{G}}(g_{\text{s}})$ is a regular subgroup of $C^\circ_{\mathbf{G}}(g_{\text{s}})$. Let

$$\mathscr{O}_x = \{v \in C^\circ_{\mathbf{G}}(g_{\text{s}})_{\text{uni}} \mid g_{\text{s}}v \in x\Sigma x^{-1}\}.$$

Since $g_{\text{s}}, x \in \mathbf{G}^F$ and $F(\Sigma) = \Sigma$, we see that $\mathscr{O}_x$ is F-stable. Let $\mathscr{F}_x$ be the inverse image of $\mathscr{E}$ under the morphism

$$\mathscr{O}_x \to \Sigma, \quad v \mapsto x^{-1}g_{\text{s}}vx. \tag{3.2.10.1}$$

Since this map commutes with F, a choice of an isomorphism $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ induces an isomorphism

$$\psi_x\colon F^*\mathscr{F}_x \xrightarrow{\sim} \mathscr{F}_x$$

of local systems on $\mathscr{O}_x$. Then $\mathscr{O}_x$ is a unipotent conjugacy class of $\mathbf{L}_x$, as follows from [LuCS2, 7.10, 7.11] applied to the cuspidal pair $(x\Sigma x^{-1}, \mathrm{Int}(x^{-1})^*\mathscr{E})$ for the connected reductive group $x\mathbf{L}x^{-1}$. Furthermore, let $\mathscr{E}_x$ be the inverse image of $\mathrm{Int}(x^{-1})^*\mathscr{E}$ under the map

$$\mathbf{Z}(x\mathbf{L}x^{-1})^{\circ}.\mathscr{O}_x \to x\Sigma x^{-1}, \quad z.v \mapsto zg_{\mathrm{s}}v.$$

By [LuCS2, 7.11(a)], $(\mathbf{Z}(x\mathbf{L}x^{-1})^{\circ}.\mathscr{O}_x, \mathscr{E}_x)$ is a cuspidal pair for $x\mathbf{L}x^{-1}$. Now we note that the map (3.2.10.1) may be written as the composition

$$\mathscr{O}_x \hookrightarrow \mathbf{Z}(x\mathbf{L}x^{-1})^{\circ}.\mathscr{O}_x \to x\Sigma x^{-1} \to \Sigma,$$

where the second map is the one just defined and the third map is given by conjugation with x^{-1}. Thus, $\mathscr{F}_x$ is the restriction of $\mathscr{E}_x$ to $\mathscr{O}_x$. Hence, for any $x \in \mathbf{G}^F$ such that $x^{-1}g_s x \in \Sigma_{\mathrm{ss}}$, we obtain a generalised Green function

$$Q^{C^\circ_{\mathbf{G}}(g_{\mathrm{s}})}_{\mathbf{L}_x,\mathscr{O}_x,\mathscr{F}_x,\psi_x}\colon C^\circ_{\mathbf{G}}(g_{\mathrm{s}})^F_{\mathrm{uni}} \to \overline{\mathbb{Q}}_\ell,$$

see Definition 3.2.9.

Theorem 3.2.11 (Lusztig [LuCS2, Thm. 8.5])**.** *Assume that we are in the setting of 3.2.10. In particular, $K = K^{\mathscr{E}}_{\mathbf{L},\Sigma} \in \mathscr{M}\mathbf{G}$ is defined with respect to a regular subgroup $\mathbf{L}$ of $\mathbf{G}$ and an F-stable cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{L}$, and an isomorphism $\varphi_0\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ is chosen, which thus induces an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$. Let $g \in \mathbf{G}^F$ with Jordan decomposition $g = g_{\mathrm{s}}g_{\mathrm{u}} = g_{\mathrm{u}}g_{\mathrm{s}}$, and let $\mathbf{L}_x$, $\mathscr{O}_x$, $\mathscr{F}_x$ and ψ_x be defined with respect to this g, for any $x \in \mathbf{G}^F$ such that $x^{-1}g_{\mathrm{s}}x \in \Sigma_{\mathrm{ss}}$. Then*

$$\chi_{K,\varphi}(g) = \sum_{\substack{x\in\mathbf{G}^F \\ x^{-1}g_{\mathrm{s}}x\in\Sigma_{\mathrm{ss}}}} \frac{|\mathbf{L}_x^F|}{|C^\circ_{\mathbf{G}}(g_{\mathrm{s}})^F|\cdot|\mathbf{L}^F|} \cdot Q^{C^\circ_{\mathbf{G}}(g_{\mathrm{s}})}_{\mathbf{L}_x,\mathscr{O}_x,\mathscr{F}_x,\psi_x}(g_{\mathrm{u}}). \tag{3.2.11.1}$$

Proof. See [LuCS2, §8]. □

According to 3.2.3–3.2.6, 3.2.9–3.2.11, the computation of the characteristic functions of F-stable character sheaves on a connected reductive group which is defined over $\mathbb{F}_q$ is thus in principle reduced to the computation of generalised Green functions of such groups.

Remark 3.2.12. Let us consider the special case where $\mathbf{L} = \mathbf{T} \subseteq \mathbf{G}$ is an F-stable maximal torus. For any $\mathscr{L} \in \mathscr{S}(\mathbf{T})^F$ with the canonical isomorphism $\varphi_0\colon F^*\mathscr{L} \xrightarrow{\sim} \mathscr{L}$, let $\theta := \chi_{\mathscr{L}} \in \mathrm{Irr}(\mathbf{T}^F)$ (see 3.1.3). Let $K = K_{\mathbf{T},\mathbf{T}}^{\mathscr{L}} \in \mathscr{M}\mathbf{G}$, and let $\varphi\colon F^*K \xrightarrow{\sim} K$ be the isomorphism induced by φ_0. We then have

$$\chi_{K,\varphi} = (-1)^{\dim \mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta),$$

see [Sho95a, 2.3]. If the centre of $\mathbf{G}$ is connected (or alternatively, if q is large enough), this is a special case of an even stronger result concerning any regular subgroup $\mathbf{L} \subseteq \mathbf{G}$, see [Lus90] and [Sho96, §4], which we have referred to already in 3.2.7 above.

3.2.13. In order to tackle the problem of computing the generalised Green functions which appear in Theorem 3.2.11, we are reduced to considering the complexes $K = K_{\mathbf{L},\Sigma}^{\mathscr{E}} \in \mathscr{M}\mathbf{G}$ for which $\mathbf{L} \subseteq \mathbf{G}$ is a regular subgroup and $(\Sigma, \mathscr{E})$ is an F-stable cuspidal pair for $\mathbf{L}$, where Σ is the preimage of a *unipotent* isolated conjugacy class of $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ under the canonical map $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$. In [LuCS5, §24], Lusztig provides an algorithm for the computation of these generalised Green functions. Let us describe this now, following [LuCS5, §24] (which relies on the results of [Lus84b]); see also the overview given in [Sho06b, 1.1–1.3].

Let $\mathbf{L}$ be a Levi complement of some parabolic subgroup of $\mathbf{G}$ (we do not yet assume that $\mathbf{L}$ is F-stable), $\mathscr{O}_0$ a unipotent class of $\mathbf{L}$ and $\mathscr{E}_0$ an $\mathbf{L}$-equivariant local system on $\mathscr{O}_0$ (for the conjugation action of $\mathbf{L}$) such that $(\mathbf{Z}(\mathbf{L})^\circ.\mathscr{O}_0, 1 \boxtimes \mathscr{E}_0)$ is a cuspidal pair for $\mathbf{L}$. (Here, $1 \boxtimes \mathscr{E}_0$ is the inverse image of $\mathscr{E}_0$ under the projection map $\mathbf{Z}(\mathbf{L})^\circ.\mathscr{O}_0 \to \mathscr{O}_0$. Moreover, $\mathscr{E}_0$ is always assumed to be taken up to isomorphism in this set-up.) Then $\mathbf{G}$ naturally acts on the set of such triples $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ by conjugation, via

$${}^g(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) := (g\mathbf{L}g^{-1}, g\mathscr{O}_0 g^{-1}, \mathrm{Int}(g^{-1})^*\mathscr{E}_0) \qquad \text{for } g \in \mathbf{G}.$$

We denote by $[(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)]$ the $\mathbf{G}$-conjugacy class of the triple $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ and by $\mathcal{M}_{\mathbf{G}}$ the set of $\mathbf{G}$-conjugacy classes of all triples $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ as above. For $\mathfrak{j} = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}$, let us consider the semisimple perverse sheaf

$$K_{\mathfrak{j}} := K_{\mathbf{L},\mathbf{Z}(\mathbf{L})^\circ\cdot\mathscr{O}_0}^{1\boxtimes\mathscr{E}_0} \in \mathscr{M}\mathbf{G}.$$

Setting $Y_{\mathfrak{j}} := Y_{(\mathbf{L},\mathbf{Z}(\mathbf{L})^\circ\cdot\mathscr{O}_0)}$ (see 3.1.15), we have $\operatorname{supp} K_{\mathfrak{j}} = \overline{Y}_{\mathfrak{j}}$. Recall from 3.2.3 that $K_{\mathfrak{j}}$ is decomposed as

$$K_{\mathfrak{j}} \cong \bigoplus_{(A|K_{\mathfrak{j}})/\simeq} (A \otimes V_A),$$

where $V_A = \mathrm{Hom}_{\mathscr{M}\mathbf{G}}(A, K_\mathfrak{j})$. As we have also stated in 3.2.3, the assignment $A \mapsto V_A$ gives rise to a bijection between the isomorphism classes of simple constituents of $K_\mathfrak{j}$ and the isomorphism classes of irreducible left $\mathscr{A}_\mathfrak{j}$-modules up to isomorphism, where $\mathscr{A}_\mathfrak{j} = \mathrm{End}_{\mathscr{M}\mathbf{G}}(K_\mathfrak{j})$. In the particular case at hand, Lusztig shows in [Lus84b, Thm. 9.2] that $\mathscr{A}_\mathfrak{j}$ is isomorphic to the group algebra of

$$\mathscr{W}_\mathfrak{j} = W_\mathbf{G}(\mathbf{L}) = {}^{N_\mathbf{G}(\mathbf{L})}/_{\mathbf{L}},$$

that is, in the setting of 3.2.4, we have $\mathscr{N} = N_\mathbf{G}(\mathbf{L})$, and the basis elements ϑ_w $(w \in \mathscr{W}_\mathfrak{j})$ of the algebra $\mathscr{A}_\mathfrak{j}$ can be chosen in such a way that the 2-cocycle involved is trivial. In fact, there is a *canonical* choice for the basis elements ϑ_w, $w \in \mathscr{W}_\mathfrak{j}$ (subject to the condition [Lus84b, 9.2(d)]), which we shall fix from now on, so we obtain a *canonical* isomorphism

$$\overline{\mathbb{Q}}_\ell[\mathscr{W}_\mathfrak{j}] \xrightarrow{\sim} \mathscr{A}_\mathfrak{j}, \quad \text{determined by } w \mapsto \vartheta_w \text{ for all } w \in \mathscr{W}_\mathfrak{j}. \tag{3.2.13.1}$$

Thus, we can write

$$K_\mathfrak{j} \cong \bigoplus_{\phi \in \mathrm{Irr}(\mathscr{W}_\mathfrak{j})} (A_\phi \otimes V_\phi),$$

where A_ϕ is the simple direct summand of $K_\mathfrak{j}$ (up to isomorphism) which corresponds to $\phi \in \mathrm{Irr}(\mathscr{W}_\mathfrak{j})$ via the isomorphism (3.2.13.1), and $V_\phi = \mathrm{Hom}_{\mathscr{M}\mathbf{G}}(A_\phi, K_\mathfrak{j})$.

Now we define $\mathcal{N}_\mathbf{G}$ to be the set of all pairs $(\mathscr{O}, \mathscr{E})$, where $\mathscr{O}$ is a unipotent conjugacy class of $\mathbf{G}$ and $\mathscr{E}$ is an irreducible local system on $\mathscr{O}$ (up to isomorphism) which is $\mathbf{G}$-equivariant for the conjugation action of $\mathbf{G}$. The restriction of the complex $K_\mathfrak{j}$ to $\mathbf{G}_\mathrm{uni}$ is a direct sum of intersection cohomology complexes $\mathrm{IC}(\overline{\mathscr{O}}, \mathscr{E})[\dim \mathbf{Z}(\mathbf{L})^\circ + \dim \mathscr{O}]^{\#\mathbf{G}_\mathrm{uni}}$, for suitable $(\mathscr{O}, \mathscr{E}) \in \mathcal{N}_\mathbf{G}$. More precisely, for any $\phi \in \mathrm{Irr}(\mathscr{W}_\mathfrak{j})$, there is a unique $(\mathscr{O}, \mathscr{E}) \in \mathcal{N}_\mathbf{G}$ such that

$$A_\phi|_{\mathbf{G}_\mathrm{uni}} \cong \mathrm{IC}(\overline{\mathscr{O}}, \mathscr{E})[\dim \mathbf{Z}(\mathbf{L})^\circ + \dim \mathscr{O}]^{\#\mathbf{G}_\mathrm{uni}}, \tag{3.2.13.2}$$

and the isomorphism class of A_ϕ is uniquely determined by this property among the simple perverse sheaves which are constituents of $K_\mathfrak{j}$ [LuCS5, 24.1]. So for each $\mathfrak{j} \in \mathcal{M}_\mathbf{G}$, the above procedure gives rise to an injective map

$$\mathrm{Irr}(\mathscr{W}_\mathfrak{j}) \hookrightarrow \mathcal{N}_\mathbf{G}.$$

Conversely, given any $(\mathscr{O}, \mathscr{E}) \in \mathcal{N}_\mathbf{G}$, there exists a unique $\mathfrak{j} \in \mathcal{M}_\mathbf{G}$ such that $(\mathscr{O}, \mathscr{E})$ is in the image of the corresponding map $\mathrm{Irr}(\mathscr{W}_\mathfrak{j}) \hookrightarrow \mathcal{N}_\mathbf{G}$ just defined. Thus, we have an associated surjective map

$$\tau\colon \mathcal{N}_{\mathbf{G}} \to \mathcal{M}_{\mathbf{G}},$$

and for any $\mathfrak{j} = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}$, the elements in $\tau^{-1}(\mathfrak{j}) \subseteq \mathcal{N}_{\mathbf{G}}$ are parametrised by the irreducible characters of $\mathscr{W}_{\mathfrak{j}} = W_{\mathbf{G}}(\mathbf{L})$. Hence, we obtain a bijection

$$\coprod_{[(\mathbf{L},\mathscr{O}_0,\mathscr{E}_0)]\in\mathcal{M}_{\mathbf{G}}} \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})) \cong \biguplus_{\mathfrak{j}\in\mathcal{M}_{\mathbf{G}}} \tau^{-1}(\mathfrak{j}) = \mathcal{N}_{\mathbf{G}}, \tag{3.2.13.3}$$

which is called the *generalised Springer correspondence.* If $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}_{\mathbf{G}}$ and if $\phi \in \mathrm{Irr}(\mathscr{W}_{\tau(\mathfrak{i})})$ corresponds to $\mathfrak{i}$ under (3.2.13.3), we will also set $A_{\mathfrak{i}} := A_\phi$. If we only consider the element $[(\mathbf{T}_0, \{1\}, \overline{\mathbb{Q}}_\ell)] \in \mathcal{M}_{\mathbf{G}}$, the map (3.2.13.3) restricts to an injection

$$\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathcal{N}_{\mathbf{G}}, \tag{3.2.13.4}$$

which is called the (*ordinary*) *Springer correspondence.* Up to tensoring with the sign representation of $\mathbf{W}$, the map (3.2.13.4) coincides with the one originally considered by Springer in [Spr76] (which is only defined for p large enough; see [Lus81] for arbitrary p). The generalised Springer correspondence as above has been defined by Lusztig [Lus84b], and the problem of determining it can be reduced to considering simple algebraic groups $\mathbf{G}$ of simply connected type, thus can be approached by means of a case-by-case analysis. This has been carried out explicitly in all cases, due to the work of Lusztig [Lus84b], Lusztig–Spaltenstein [LS85], Spaltenstein [Spa85] (see also the references there for earlier results concerning the ordinary Springer correspondence), Lusztig [Lus19], and was finally concluded by the author in [Het22b], see Theorem 4.5.13 below. The generalised Springer correspondence is electronically available in CHEVIE [MiChv].

3.2.14. Following [LuCS5, 24.2], let us now bring the $\mathbb{F}_q$-rational structure of $\mathbf{G}$ into the picture. We keep the setting and notation of 3.2.13. If $\mathbf{L} \subseteq \mathbf{G}$ is a regular subgroup, the Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$ induces an automorphism of the finite group $W_{\mathbf{G}}(\mathbf{L})$, hence also a bijection of the set $\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))$, as in (2.2.7.2). Here, we denote the set of fixed points under this action by

$$\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))^F \subseteq \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})).$$

Furthermore, F gives rise to an action on $\mathcal{N}_{\mathbf{G}}$ via

$$\mathcal{N}_{\mathbf{G}} \to \mathcal{N}_{\mathbf{G}}, \quad (\mathscr{O}, \mathscr{E}) \mapsto (F^{-1}(\mathscr{O}), F^*\mathscr{E}).$$

We set
$$\mathcal{N}_{\mathbf{G}}^F := \{(\mathscr{O}, \mathscr{E}) \in \mathcal{N}_{\mathbf{G}} \mid F^{-1}(\mathscr{O}) = \mathscr{O} \text{ and } F^*\mathscr{E} \cong \mathscr{E}\}.$$
Similarly, we get an action on $\mathcal{M}_{\mathbf{G}}$ via
$$\mathcal{M}_{\mathbf{G}} \to \mathcal{M}_{\mathbf{G}}, \quad [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \mapsto [(F^{-1}(\mathbf{L}), F^{-1}(\mathscr{O}_0), F^*\mathscr{E}_0)].$$
(This is clearly well-defined.) We set
$$\mathcal{M}_{\mathbf{G}}^F := \{[(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}} \mid [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] = [(F^{-1}(\mathbf{L}), F^{-1}(\mathscr{O}_0), F^*\mathscr{E}_0)]\}$$
and refer to the elements of $\mathcal{M}_{\mathbf{G}}^F$ as the F-stable elements of $\mathcal{M}_{\mathbf{G}}$. Let $[(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}^F$. The classification [Lus84b, §10–§15] of cuspidal pairs which involve a unipotent class shows that we can choose the representative $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ in such a way that $\mathbf{L}$ is the *standard* Levi subgroup of a *standard* parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and that
$$F(\mathbf{P}) = \mathbf{P},\ F(\mathbf{L}) = \mathbf{L},\ F(\mathscr{O}_0) = \mathscr{O}_0,\ F^*\mathscr{E}_0 \cong \mathscr{E}_0$$
(see also [Tay14, 6.2]). This property uniquely determines the triple $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$; by a slight abuse of notation, we will in this case sometimes write $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}_{\mathbf{G}}^F$ instead of $[(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}^F$.

3.2.15. With the notation of 3.2.14, let us assume that $\mathfrak{i} = (\mathscr{O}, \mathscr{E})$ is in $\mathcal{N}_{\mathbf{G}}^F$, and let $\mathfrak{j} = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}$ be such that $A_{\mathfrak{i}}$ is a (simple) constituent of $K_{\mathfrak{j}}$, that is, we have $\mathfrak{j} = \tau(\mathfrak{i})$. Then $F^*A_{\mathfrak{i}}$ is a simple constituent of $F^*K_{\mathfrak{j}}$, which is the complex associated to $[(F^{-1}(\mathbf{L}), F^{-1}(\mathscr{O}_0), F^*\mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}$, so $F^*A_{\mathfrak{i}}$ certainly corresponds to *some* pair $(\mathscr{O}', \mathscr{E}') \in \mathcal{N}_{\mathbf{G}}$ via (3.2.13.2). Thus,
$$(F^*A_{\mathfrak{i}})|_{\mathbf{G}_{\text{uni}}} \cong \mathrm{IC}(\overline{\mathscr{O}'}, \mathscr{E}')[\dim \mathbf{Z}(F^{-1}(\mathbf{L}))^\circ + \dim \mathscr{O}']^{\#\mathbf{G}_{\text{uni}}}.$$
On the other hand, the complex $A_{\mathfrak{i}}$ satisfies the condition (3.2.13.2) with respect to the pair $(\mathscr{O}, \mathscr{E})$. Applying F^* on both sides and using the F-invariance of $(\mathscr{O}, \mathscr{E})$ yields that
$$(F^*A_{\mathfrak{i}})|_{\mathbf{G}_{\text{uni}}} \cong \mathrm{IC}(\overline{\mathscr{O}}, \mathscr{E})[\dim \mathbf{Z}(\mathbf{L})^\circ + \dim \mathscr{O}]^{\#\mathbf{G}_{\text{uni}}},$$
so we must have $\mathscr{O}' = \mathscr{O}$ and $\mathscr{E}' \cong \mathscr{E}$. We conclude that $F^*A_{\mathfrak{i}} \cong A_{\mathfrak{i}}$. The discussion in 3.2.13 then shows that $[(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] = [(F^{-1}(\mathbf{L}), F^{-1}(\mathscr{O}_0), F^*\mathscr{E}_0)]$, that is, we have

$\mathfrak{j} = \tau(\mathfrak{i}) \in \mathcal{M}_{\mathbf{G}}^F$, so the map $\tau\colon \mathcal{N}_{\mathbf{G}} \to \mathcal{M}_{\mathbf{G}}$ restricts to a map

$$\tau\colon \mathcal{N}_{\mathbf{G}}^F \to \mathcal{M}_{\mathbf{G}}^F.$$

Let $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}_{\mathbf{G}}^F$ and $\mathfrak{j} = \tau(\mathfrak{i}) = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}^F$ be as above, where we assume that the triple $(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)$ is such that $\mathbf{L}$ is the standard Levi subgroup of some standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, and both $\mathbf{L}$ and $\mathbf{P}$ are F-stable (see 3.2.14). Let us choose an isomorphism $\varphi_0\colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ of local systems over $\mathscr{O}_0$ such that φ_0 induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Such a choice gives rise to an isomorphism $1 \boxtimes F^*\mathscr{E}_0 \xrightarrow{\sim} 1 \boxtimes \mathscr{E}_0$ of local systems over $\mathbf{Z}(\mathbf{L})^\circ\mathscr{O}_0$, and as described in 3.2.5, this determines an isomorphism $\varphi_{\mathfrak{j}}\colon F^*K_{\mathfrak{j}} \xrightarrow{\sim} K_{\mathfrak{j}}$. Any choice of an isomorphism $\varphi_{A_{\mathfrak{i}}}\colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$ allows us to define a bijective linear map

$$\sigma_{A_{\mathfrak{i}}}\colon V_{A_{\mathfrak{i}}} \to V_{A_{\mathfrak{i}}}, \quad v \mapsto \varphi_{\mathfrak{j}} \circ F^*(v) \circ \varphi_{A_{\mathfrak{i}}}^{-1}, \tag{3.2.15.1}$$

see (3.2.3.1). Consider the automorphism

$$\iota_{\mathfrak{j}}\colon \mathscr{A}_{\mathfrak{j}} \xrightarrow{\sim} \mathscr{A}_{\mathfrak{j}}, \quad \vartheta \mapsto \varphi_{\mathfrak{j}} \circ F^*(\vartheta) \circ \varphi_{\mathfrak{j}}^{-1},$$

of the algebra $\mathscr{A}_{\mathfrak{j}}$. Since F^* defines a functor $\mathscr{M}\mathbf{G} \to \mathscr{M}\mathbf{G}$ (see (3.1.1.2)), we get

$$\sigma_{A_{\mathfrak{i}}}(\vartheta \circ v) = \iota_{\mathfrak{j}}(\vartheta) \circ \sigma_{A_{\mathfrak{i}}}(v) \quad \text{for any } \vartheta \in \mathscr{A}_{\mathfrak{j}},\ v \in V_{A_{\mathfrak{i}}}. \tag{3.2.15.2}$$

Now $\mathscr{W}_{\mathfrak{j}} = W_{\mathbf{G}}(\mathbf{L})$ is a Coxeter group by [Lus84b, Thm. 9.2], and from the definition of its Coxeter generators in loc. cit. and our assumption that $F(\mathbf{P}) = \mathbf{P}$, it follows that the automorphism $F_{\mathfrak{j}}\colon \mathscr{W}_{\mathfrak{j}} \xrightarrow{\sim} \mathscr{W}_{\mathfrak{j}}$ induced by F is an automorphism of Coxeter groups. Then, as remarked in [LuCS2, 10.9], under the canonical isomorphism (3.2.13.1), we have

$$\vartheta_w \circ \vartheta_{w'} = \vartheta_{ww'} \quad \text{and} \quad \iota_{\mathfrak{j}}(\vartheta_w) = \vartheta_{F_{\mathfrak{j}}^{-1}(w)} \quad \text{for all } w, w' \in \mathscr{W}_{\mathfrak{j}}.$$

Using this and (3.2.15.2), we see that

$$\vartheta_{F_{\mathfrak{j}}^{-1}(w)} \circ v = \sigma_{A_i}(\vartheta_w \circ \sigma_{A_i}^{-1}(v)) \quad \text{for all } w \in \mathscr{W}_{\mathfrak{j}},\ v \in V_{A_{\mathfrak{i}}}.$$

Hence, as described in 2.2.7, $\sigma_{A_{\mathfrak{i}}}$ determines an extension of the $\mathscr{W}_{\mathfrak{j}}$-module $V_{A_{\mathfrak{i}}}$ to an irreducible module for the semidirect product $\mathscr{W}_{\mathfrak{j}}(F_{\mathfrak{j}}^{-1}) = \mathscr{W}_{\mathfrak{j}} \rtimes \langle F_{\mathfrak{j}}^{-1} \rangle$. Now for any irreducible module of $\mathscr{W}_{\mathfrak{j}}$ which can be extended to $\mathscr{W}_{\mathfrak{j}}(F_{\mathfrak{j}}^{-1})$, Lusztig singles out one particular extension to $\mathscr{W}_{\mathfrak{j}}(F_{\mathfrak{j}}^{-1})$, called the *preferred extension*, see [LuCS4, 17.2]. Since multiplying $\varphi_{A_{\mathfrak{i}}}$ with a scalar $\mathfrak{z} \in \overline{\mathbb{Q}}_\ell^\times$ changes $\sigma_{A_{\mathfrak{i}}}$ to $\mathfrak{z}^{-1} \cdot \sigma_{A_{\mathfrak{i}}}$, there is a unique choice for the isomorphism $\varphi_{A_{\mathfrak{i}}} \colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$ such that the map $\sigma_{A_{\mathfrak{i}}}$ defined with respect to $\varphi_{A_{\mathfrak{i}}}$ in (3.2.15.1) determines the preferred extension of $V_{A_{\mathfrak{i}}}$ to $\mathscr{W}_{\mathfrak{j}}(F_{\mathfrak{j}}^{-1})$. Thus, the choice of $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ gives rise to a distinguished isomorphism $\varphi_{A_{\mathfrak{i}}} \colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$. This discussion also shows that, for $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ and $\mathfrak{j} \in \mathcal{M}_{\mathbf{G}}^F$ as above, the character of the $\overline{\mathbb{Q}}_\ell[\mathscr{W}_{\mathfrak{j}}]$-module $V_{A_{\mathfrak{i}}}$ is in $\operatorname{Irr}(\mathscr{W}_{\mathfrak{j}})^{F_{\mathfrak{j}}}$. The generalised Springer correspondence (3.2.13.3) therefore induces a bijection

$$\coprod_{\mathfrak{j}=[(\mathbf{L},\mathscr{O}_0,\mathscr{E}_0)]\in\mathcal{M}_{\mathbf{G}}^F} \operatorname{Irr}(W_{\mathbf{G}}(\mathbf{L}))^{F_{\mathfrak{j}}} \cong \mathcal{N}_{\mathbf{G}}^F. \tag{3.2.15.3}$$

Let $\mathfrak{j} = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}^F$ and $\phi \in \operatorname{Irr}(\mathscr{W}_{\mathfrak{j}})^{F_{\mathfrak{j}}}$, and assume that $\mathfrak{i} = (\mathscr{O}, \mathscr{E})$ is the corresponding element of $\mathcal{N}_{\mathbf{G}}^F$. As before, we fix an isomorphism $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Let $\varphi_{A_{\mathfrak{i}}} \colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$ be the isomorphism determined by φ_0 as described above. We set

$$\begin{aligned} a_{\mathfrak{i}} &:= -\dim \mathscr{O} - \dim \mathbf{Z}(\mathbf{L})^\circ, \\ b_{\mathfrak{i}} &:= \dim \operatorname{supp} A_{\mathfrak{i}} = \dim Y_{(\mathbf{L},\mathbf{Z}(\mathbf{L})^\circ\cdot\mathscr{O}_0)}, \\ d_{\mathfrak{i}} &:= \tfrac{1}{2}(a_{\mathfrak{i}} + b_{\mathfrak{i}}). \end{aligned}$$

(In view of [Spa82, II.2.8], $a_{\mathfrak{i}} + b_{\mathfrak{i}}$ is even, so that $d_{\mathfrak{i}} \in \mathbb{N}_0$.) Thus, by (3.2.13.2), we have

$$A_{\mathfrak{i}}|_{\mathbf{G}_{\mathrm{uni}}} \cong \mathrm{IC}(\overline{\mathscr{O}}, \mathscr{E})[-a_{\mathfrak{i}}]^{\#\mathbf{G}_{\mathrm{uni}}}.$$

Since the intersection cohomology complex involved extends the local system $\mathscr{E}$ on $\mathscr{O}$ to $\overline{\mathscr{O}}$, we deduce that

$$\mathscr{H}^a(A_{\mathfrak{i}})|_{\mathscr{O}} \cong \begin{cases} \mathscr{E} & \text{if } a = a_{\mathfrak{i}}, \\ 0 & \text{if } a \neq a_{\mathfrak{i}}. \end{cases}$$

We may thus define an isomorphism $\psi_{\mathfrak{i}} \colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ by the requirement that $q^{d_{\mathfrak{i}}}\psi_{\mathfrak{i}}$ coincides with the isomorphism $F^*\mathscr{H}^{a_{\mathfrak{i}}}(A_{\mathfrak{i}})|_{\mathscr{O}} \xrightarrow{\sim} \mathscr{H}^{a_{\mathfrak{i}}}(A_{\mathfrak{i}})|_{\mathscr{O}}$ induced by $\varphi_{A_{\mathfrak{i}}}$. It is proven in [LuCS5, (24.2.4)] that for any $g \in \mathscr{O}^F$, the induced map $\psi_{\mathfrak{i},g} \colon \mathscr{E}_g \to \mathscr{E}_g$ on the stalk of $\mathscr{E}$ at g is of finite order. Now consider the two functions

$$X_{\mathfrak{i}} \colon \mathbf{G}_{\mathrm{uni}}^F \to \overline{\mathbb{Q}}_\ell \qquad \text{and} \qquad Y_{\mathfrak{i}} \colon \mathbf{G}_{\mathrm{uni}}^F \to \overline{\mathbb{Q}}_\ell,$$

defined as follows: For $g \in \mathbf{G}_{\mathrm{uni}}^F$, let

$$X_{\mathfrak{i}}(g) := (-1)^{a_{\mathfrak{i}}} q^{-d_{\mathfrak{i}}} \chi_{A_{\mathfrak{i}}, \varphi_{A_{\mathfrak{i}}}}(g)$$

and

$$Y_{\mathfrak{i}}(g) := \begin{cases} \mathrm{Trace}(\psi_{\mathfrak{i},g}, \mathscr{E}_g) & \text{if } g \in \mathscr{O}^F, \\ 0 & \text{if } g \notin \mathscr{O}^F. \end{cases}$$

Both $X_{\mathfrak{i}}$ and $Y_{\mathfrak{i}}$ are invariant under the conjugation action of $\mathbf{G}^F$ on $\mathbf{G}_{\mathrm{uni}}^F$.

Theorem 3.2.16 (Lusztig [LuCS5, §24])**.** *In the setting of* 3.2.15*, the following hold.*

(a) The functions $Y_{\mathfrak{i}}$*,* $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$*, form a basis of the vector space consisting of all functions* $\mathbf{G}_{\mathrm{uni}}^F \to \overline{\mathbb{Q}}_\ell$ *which are invariant under the conjugation action of* $\mathbf{G}^F$ *on* $\mathbf{G}_{\mathrm{uni}}^F$*.*

(b) There is a system of equations

$$X_{\mathfrak{i}} = \sum_{\mathfrak{i}' \in \mathcal{N}_{\mathbf{G}}^F} p_{\mathfrak{i}',\mathfrak{i}} Y_{\mathfrak{i}'}, \quad \mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F,$$

for some uniquely determined $p_{\mathfrak{i}',\mathfrak{i}} \in \mathbb{Z}$*. These* $p_{\mathfrak{i}',\mathfrak{i}}$ *are explicitly known and can be obtained by means of an algorithm which entirely relies on combinatorial data.*

Proof. See [LuCS5, §24]. Note that the restrictions [LuCS5, (23.0.1)] on the characteristic p of k can be removed, due to the remarks in [Lus12b, 3.10]. □

Corollary 3.2.17. *In the setting of Theorem* 3.2.16*, the following hold.*

(i) We have $p_{\mathfrak{i},\mathfrak{i}} = 1$ *for all* $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$*.*

(ii) If $\mathfrak{i}' = (\mathscr{O}', \mathscr{E}') \neq \mathfrak{i} = (\mathscr{O}, \mathscr{E})$*, then* $p_{\mathfrak{i}',\mathfrak{i}} \neq 0$ *implies that* $\mathscr{O}' \neq \mathscr{O}$ *and* $\mathscr{O}' \subseteq \overline{\mathscr{O}}$*.*

(iii) If $\tau(\mathfrak{i}') \neq \tau(\mathfrak{i})$*, we have* $p_{\mathfrak{i}',\mathfrak{i}} = 0$*.*

Proof. This is an immediate consequence of (the proof of) Theorem 3.2.16, so we again refer to [LuCS5, §24] (and [Lus12b]). □

Remark 3.2.18. Hence, if we define a total order $\leqslant$ on $\mathcal{N}_{\mathbf{G}}^F$ in such a way that for $\mathfrak{i} = (\mathscr{O}, \mathscr{E})$, $\mathfrak{i}' = (\mathscr{O}', \mathscr{E}') \in \mathcal{N}_{\mathbf{G}}^F$, we have

$$\mathfrak{i}' \leqslant \mathfrak{i} \text{ whenever } \mathscr{O}' \subseteq \overline{\mathscr{O}}$$

(note that the latter defines a partial order on the set of unipotent classes of $\mathbf{G}$), the matrix $(p_{\mathfrak{i}',\mathfrak{i}})_{\mathfrak{i}',\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F}$ has upper unitriangular shape. Lusztig's algorithm to compute the

$p_{\mathfrak{i}',\mathfrak{i}}$ is provided in [LuCS5, 24.4]. This algorithm is implemented in CHEVIE [MiChv] and is accessible via the functions `UnipotentClasses` and `ICCTable`.

3.2.19. This completes our description of Lusztig's strategy to compute the characteristic functions of F-stable character sheaves on $\mathbf{G}$ in principle, so let us briefly recap it here.

Let $A \in \hat{\mathbf{G}}^F$. From the discussion in 3.2.5, we see that A is isomorphic to a simple direct summand of the complex $K = K_{\mathbf{L},\Sigma}^{\mathscr{E}}$ for a suitable regular subgroup $\mathbf{L} \subseteq \mathbf{G}$ and an F-stable cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{L}$. Thus, formula (3.2.4.1) shows that χ_A is a linear combination of characteristic functions of K. The values of these characteristic functions at a given $g = g_s g_u = g_u g_s \in \mathbf{G}^F$ can then be expressed as linear combinations of generalised Green functions via Theorem 3.2.11. Recall that the latter are defined as the restrictions to $C_{\mathbf{G}}^\circ(g_s)_{\mathrm{uni}}^F$ of characteristic functions of complexes associated to elements $\mathfrak{j} \in \mathcal{M}_{C_{\mathbf{G}}^\circ(g_s)}^F$. Hence, in view of the discussion in 3.2.3 (see, in particular, formula (3.2.3.3)) applied to $C_{\mathbf{G}}^\circ(g_s)$ and to the specific induced complexes $K_{\mathfrak{j}}$ considered in 3.2.13–3.2.15, these functions on $C_{\mathbf{G}}^\circ(g_s)_{\mathrm{uni}}^F$ are given as linear combinations of the functions $X_{\mathfrak{i}}$, $\mathfrak{i} \in \mathcal{N}_{C_{\mathbf{G}}^\circ(g_s)}^F$, whose computation is reduced to that of the $Y_{\mathfrak{i}}$, $\mathfrak{i} \in \mathcal{N}_{C_{\mathbf{G}}^\circ(g_s)}^F$, thanks to Theorem 3.2.16.

However, we do not want to hide the fact that there are still several issues to be overcome if one seeks to find an explicit formula for the values of the characteristic functions of F-stable character sheaves in general, for instance the following:

(a) While it is easy to compute the functions $Y_{\mathfrak{i}}$ above *up to scalar multiples*, exactly pinpointing these scalars is a non-trivial task since it is difficult to describe the isomorphisms $\varphi_{A_{\mathfrak{i}}}$ (and thus the $\psi_{\mathfrak{i}}$) concretely. The latter has been accomplished for classical groups (in any characteristic) by Shoji [Sho06b], [Sho07], [Sho22]; as far as exceptional groups are concerned, this problem is not yet solved in complete generality.

(b) We also recall that, for an arbitrary $A \in \hat{\mathbf{G}}^F$, the requirement $(*)$ in 3.2.1 on $\varphi_A \colon F^*A \xrightarrow{\sim} A$ determines φ_A only up to multiplication with a root of unity, so one needs to find a way to uniquely specify such an isomorphism in the first place. As before, let $\mathbf{L} \subseteq \mathbf{G}$ be a regular subgroup, and let $(\Sigma, \mathscr{E})$ be the F-stable cuspidal pair for $\mathbf{L}$ such that A is a constituent of the complex

$$K = K_{\mathbf{L},\Sigma}^{\mathscr{E}} \cong \mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0) \in \mathscr{M}\mathbf{G},$$

where $A_0 = \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}} \in (\hat{\mathbf{L}}^\circ)^F$. Let $\varphi_0 \colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ be an isomorphism. As described in 3.2.5, φ_0 naturally induces isomorphisms $\varphi_{A_0} \colon F^*A_0 \xrightarrow{\sim} A_0$ and $\varphi \colon F^*K \xrightarrow{\sim} K$. Now, in general, as mentioned in 3.2.4 (and using the notation there), the endomorphism algebra of K is isomorphic to the group algebra of $\mathscr{W}_{\mathbf{L},\Sigma}^{\mathscr{E}}$ twisted by a 2-cocycle. Let us make the following assumption on the given triple $(\mathbf{L}, \Sigma, \mathscr{E})$:

$$\mathrm{End}_{\mathscr{M}\mathbf{G}}(K_{\mathbf{L},\Sigma}^{\mathscr{E}}) \cong \overline{\mathbb{Q}}_\ell[W_{\mathbf{G}}(\mathbf{L})]. \qquad (\spadesuit)$$

(Thus, we require that the 2-cocycle involved is trivial and that the stabiliser of $(\Sigma, \mathscr{E})$ in $W_{\mathbf{G}}(\mathbf{L}) = {}^{N_{\mathbf{G}}(\mathbf{L})}/_{\mathbf{L}}$ is the full group $W_{\mathbf{G}}(\mathbf{L})$. This is satisfied in many important special cases, e.g., whenever $\Sigma = \mathbf{Z}(\mathbf{L})^\circ.\mathscr{O}_0$ for a unipotent conjugacy class $\mathscr{O}_0 \subseteq \mathbf{L}$ and $\mathscr{E} = 1 \boxtimes \mathscr{E}_0$ where $\mathscr{E}_0$ is a local system on $\mathscr{O}_0$, see 3.2.13.) Then we can perform the analogous argument as in 3.2.15 to see that (once φ_0 is fixed) choosing an explicit isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$ can be achieved by singling out an extension of the $\overline{\mathbb{Q}}_\ell[W_{\mathbf{G}}(\mathbf{L})]$-module $V_A = \mathrm{Hom}_{\mathscr{M}\mathbf{G}}(A, K)$ to $(W_{\mathbf{G}}(\mathbf{L}))(\gamma_{\mathbf{L}}^{-1})$ (with the notation of 2.2.7, where $\gamma_{\mathbf{L}}$ denotes the map induced by F on $W_{\mathbf{G}}(\mathbf{L})$). For instance, we can take the preferred extension [LuCS4, 17.2]. So if ($\spadesuit$) is satisfied for a given $(\mathbf{L}, \Sigma, \mathscr{E})$, we only need to make a choice for $\varphi_0 \colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ in order to obtain specific isomorphisms $\varphi_A \colon F^*A \xrightarrow{\sim} A$, for any $A \in \hat{\mathbf{G}}^F$ which is a constituent of $K_{\mathbf{L},\Sigma}^{\mathscr{E}}$. In 3.2.20 and 3.2.21 below, we will discuss how one can make a concrete choice for φ_0 (at least in the case where the local system $\mathscr{E}$ is one-dimensional).

For example, under the assumptions that the centre $\mathbf{Z}(\mathbf{G})$ is connected, ${}^{\mathbf{G}}/_{\mathbf{Z}(\mathbf{G})}$ is a simple algebraic group and p is a good prime for $\mathbf{G}$, explicitly computable formulae for the values of characteristic functions at unipotent elements of $\mathbf{G}^F$ have been found by Taylor [Tay14] (where in particular problems (a) and (b) above needed to be dealt with).

Parametrisation of F-stable local systems on conjugacy classes and normalisation of characteristic functions

We conclude this section by providing some tools and remarks which will help us in making explicit choices for certain isomorphisms $F^*A \xrightarrow{\sim} A$ (where A is an F-stable character sheaf on $\mathbf{G}$).

3.2.20. Let $\mathbf{H}$ be a connected algebraic group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ (q a power of p), and let $F \colon \mathbf{H} \to \mathbf{H}$ be the corresponding Frobenius map. Let X be a non-empty algebraic variety over k, and let us assume that there is a transitive action

$$\mathbf{H} \times X \to X, \quad (h, x) \mapsto h.x.$$

We also assume that X is defined over $\mathbb{F}_q$, with corresponding Frobenius morphism $F' \colon X \to X$, such that the following conditions are satisfied (cf. [Gec03a, §4.3]):

- We have $F'(h.x) = F(h).F'(x)$ for all $h \in \mathbf{H}$, $x \in X$.

- The stabiliser $C_{\mathbf{H}}(x)$ of any $x \in X$ is a closed subgroup of $\mathbf{H}$.

We thus get an action of $\mathbf{H}^F$ on $X^{F'}$ by restriction of the action of $\mathbf{H}$ on X.

(a) The following is well known (see [BCCISS, E-I.2.7]) and in fact does not even require any geometric properties of X (see [Gec03a, §4.3]): First of all, the set $X^{F'}$ of F'-fixed points in X is non-empty, so we may fix an element $x_0 \in X^{F'}$ and consider the finite group

$$A_{\mathbf{H}}(x_0) = {C_{\mathbf{H}}(x_0)}/{C^\circ_{\mathbf{H}}(x_0)}.$$

Clearly, F leaves both $C_{\mathbf{H}}(x_0)$ and $C^\circ_{\mathbf{H}}(x_0)$ invariant, so we obtain an induced automorphism $\gamma\colon A_{\mathbf{H}}(x_0) \xrightarrow{\sim} A_{\mathbf{H}}(x_0)$. Just as in 2.1.17, the γ-conjugacy classes of $A_{\mathbf{H}}(x_0)$ are defined to be the orbits of the action

$$A_{\mathbf{H}}(x_0) \times A_{\mathbf{H}}(x_0) \to A_{\mathbf{H}}(x_0), \quad (a, a') \mapsto aa'\gamma(a)^{-1}.$$

The γ-conjugacy classes of $A_{\mathbf{H}}(x_0)$ correspond to the $\mathbf{H}^F$-orbits contained in $X^{F'}$. Specifically, if $a \in A_{\mathbf{H}}(x_0)$ and $h \in \mathbf{H}$ is such that $h^{-1}F(h) \in C_{\mathbf{H}}(x_0)$ is sent to a under the canonical map $C_{\mathbf{H}}(x_0) \to A_{\mathbf{H}}(x_0)$, we have $(x_0)_a := h.x_0 \in X^{F'}$. Note that, for a given $a \in A_{\mathbf{H}}(x_0)$, $(x_0)_a$ is not uniquely determined by this procedure, but its orbit under the action of $\mathbf{H}^F$ is. More precisely, associating the γ-conjugacy class of a in $A_{\mathbf{H}}(x_0)$ with the $\mathbf{H}^F$-orbit of $(x_0)_a$ defines a bijection between the set of γ-conjugacy classes of $A_{\mathbf{H}}(x_0)$ and the set of $\mathbf{H}^F$-orbits in $X^{F'}$.

(b) Now let $\mathscr{E}$ be an $\mathbf{H}$-equivariant irreducible local system on X such that $(F')^*\mathscr{E} \cong \mathscr{E}$. Recall from 3.1.1 that $\mathscr{E}_{x_0}$ carries in a natural way the structure of an irreducible $A_{\mathbf{H}}(x_0)$-module. Following [Lus04, 19.7], the bijection $\mathscr{E} \mapsto \mathscr{E}_{x_0}$ between the isomorphism classes of $\mathbf{H}$-equivariant irreducible local systems on X and the isomorphism classes of the irreducible $A_{\mathbf{H}}(x_0)$-modules restricts to a bijection between the isomorphism classes of $\mathbf{H}$-equivariant *F'-stable* irreducible local systems on X and the isomorphism classes of those irreducible $A_{\mathbf{H}}(x_0)$-modules *which can be extended to* $(A_{\mathbf{H}}(x_0))(\gamma)$ (with the notation of 2.2.7). For $\mathscr{E}$ as above, let us choose an isomorphism $\varphi\colon (F')^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$. Thus, for any $x \in X^{F'}$, φ induces a linear map $\varphi_x\colon \mathscr{E}_x \xrightarrow{\sim} \mathscr{E}_x$, and we obtain an $\mathbf{H}^F$-invariant characteristic function

$$\chi_{\mathscr{E},\varphi}\colon X^{F'} \to \overline{\mathbb{Q}}_\ell, \quad x \mapsto \mathrm{Trace}(\varphi_x, \mathscr{E}_x).$$

Let us denote by $\Theta_{\mathscr{E}}\colon A_{\mathbf{H}}(x_0) \to \mathrm{GL}(\mathscr{E}_{x_0})$ the representation of the $A_{\mathbf{H}}(x_0)$-module $\mathscr{E}_{x_0}$. Taking $E := \varphi_{x_0}$ in the setting of 2.2.7 gives rise to an extension of $\Theta_{\mathscr{E}}$ to $(A_{\mathbf{H}}(x_0))(\gamma)$, and we have

$$\chi_{\mathscr{E},\varphi}((x_0)_a) = \operatorname{Trace}(\varphi_{x_0} \circ \Theta_{\mathscr{E}}(a), \mathscr{E}_{x_0}) \quad \text{for any } a \in A_{\mathbf{H}}(x_0).$$

Thus, if $\varsigma_{\mathscr{E}} \colon A_{\mathbf{H}}(x_0) \to \overline{\mathbb{Q}}_\ell$ is the character of $\Theta_{\mathscr{E}}$, and if we denote by $\widetilde{\varsigma}_{\mathscr{E}}$ its γ-extension defined with respect to $E := \varphi_{x_0}$ as in 2.2.7, we have

$$\chi_{\mathscr{E},\varphi}((x_0)_a) = \widetilde{\varsigma}_{\mathscr{E}}(a) \quad \text{for any } a \in A_{\mathbf{H}}(x_0).$$

3.2.21. Let $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ (see 3.1.3), and let $A \in \hat{\mathbf{G}}^F_{\mathscr{L}} \cap \hat{\mathbf{G}}^\circ$. Consider the action

$$(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ) \times \mathbf{G} \to \mathbf{G}, \quad ((x,z),g) \mapsto z^n xgx^{-1}.$$

By Proposition 3.1.17(a), there exists a unique $(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ)$-orbit $\Sigma \subseteq \mathbf{G}$ and a unique $(\mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ)$-equivariant irreducible local system $\mathscr{E}$ on Σ (up to isomorphism) such that

$$A \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}}.$$

Since $F^*A \cong A$, it follows from the uniqueness of $(\Sigma, \mathscr{E})$ that $F(\Sigma) = \Sigma$ and $F^*\mathscr{E} \cong \mathscr{E}$. Let us set $\mathbf{H} := \mathbf{G} \times \mathbf{Z}(\mathbf{G})^\circ$ and $X := \Sigma$. By a slight abuse of notation, we shall just write $F \colon \mathbf{H} \to \mathbf{H}$ and $F \colon X \to X$ for the maps $F \times F|_{\mathbf{Z}(\mathbf{G})^\circ}$ and $F|_X$, respectively. Then one easily checks that the requirements in 3.2.20 are met. Let $d = \dim \Sigma$. For $g \in \Sigma$ and $i \in \mathbb{Z}$, we have

$$\mathscr{H}^i_g(A) \cong \begin{cases} \mathscr{E}_g & \text{if } i = -d, \\ 0 & \text{if } i \neq -d. \end{cases}$$

We make the following assumption:

> The local system $\mathscr{E}$ is one-dimensional.

Hence, by the description in 3.2.1, an isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$ can be chosen in such a way that, (in particular) for any $g \in \Sigma^F$, the induced map $(\varphi_A)_{-d,g} \colon \mathscr{E}_g \to \mathscr{E}_g$ is given by multiplication with $q^{(\dim \mathbf{G} - d)/2}$ times a root of unity. We choose an element $g_0 \in \Sigma^F$ and consider its $\mathbf{H}^F$-orbit inside Σ^F, and we require that $(\varphi_A)_{-d,g_0} \colon \mathscr{E}_{g_0} \to \mathscr{E}_{g_0}$ is given by scalar multiplication with $q^{(\dim \mathbf{G} - d)/2}$. Note that this uniquely determines the isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$. For $g \in \Sigma^F$, we then have

$$\chi_{A,\varphi_A}(g) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Trace}((\varphi_A)_{i,g}, \mathscr{H}^i_g(A)) = (-1)^{-d} \operatorname{Trace}((\varphi_A)_{-d,g}, \mathscr{E}_g).$$

Let us now consider the finite group $A_{\mathbf{H}}(g_0) = {C_{\mathbf{H}}(g_0)}/{C^\circ_{\mathbf{H}}(g_0)}$ and the automorphism $\gamma\colon A_{\mathbf{H}}(g_0) \xrightarrow{\sim} A_{\mathbf{H}}(g_0)$ induced by F, as in 3.2.20. Since $\mathscr{E}$ is one-dimensional, the $A_{\mathbf{H}}(g_0)$-module $\mathscr{E}_{g_0}$ affords a linear character $\varsigma = \varsigma_{\mathscr{E}}\colon A_{\mathbf{H}}(g_0) \to \overline{\mathbb{Q}}_\ell^\times$. Applying the discussion in 3.2.20(b) with $\varphi := (\varphi_A)_{-d}|_\Sigma\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$, and denoting by $\tilde{\varsigma}\colon A_{\mathbf{H}}(g_0) \to \overline{\mathbb{Q}}_\ell^\times$ the γ-extension of ς with respect to $(\varphi_A)_{-d,g_0}\colon \mathscr{E}_{g_0} \xrightarrow{\sim} \mathscr{E}_{g_0}$, we obtain

$$\chi_{A,\varphi_A}((g_0)_a) = (-1)^d\tilde{\varsigma}(a) = (-1)^d\varsigma(a)q^{(\dim \mathbf{G}-d)/2} \quad \text{for any } a \in A_{\mathbf{H}}(g_0).$$

By Theorem 3.1.13, we also know that $A|_{\overline{\Sigma}\setminus\Sigma} = 0$, so we have

$$\chi_{A,\varphi_A}(g) = \begin{cases} (-1)^d\varsigma(a)q^{(\dim \mathbf{G}-d)/2} & \text{if } g \sim_{\mathbf{G}^F} (g_0)_a \text{ for some } a \in A_{\mathbf{H}}(g_0),\\ 0 & \text{if } g \notin \Sigma^F. \end{cases}$$

Hence, we see that for an F-stable cuspidal character sheaf $A \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}}$ on $\mathbf{G}$ as above, choosing an isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$ admissible for Theorem 3.2.2 is equivalent to singling out an $\mathbf{H}^F$-orbit inside Σ^F, and the characteristic function χ_{A,φ_A} is then completely determined by the choice of the $\mathbf{H}^F$-orbit of a representative $g_0 \in \Sigma^F$ along with the character $\varsigma \in \mathrm{Irr}(A_{\mathbf{H}}(g_0))$ corresponding to the isomorphism class of the local system $\mathscr{E}$ on Σ (as in 3.2.20).

3.2.22. In the setting of 3.2.21, it thus remains to consider the problem of choosing an $\mathbf{H}^F$-orbit contained in Σ^F (or a representative g_0 of such an orbit). We would like to find some g_0 with certain 'good' properties, which should ideally lead to a distinguished $\mathbf{H}^F$-orbit inside Σ^F. In order to shed some light on what such properties could be in this context, we first focus on a natural condition on unipotent conjugacy classes of a connected reductive group in good characteristic and then try to formulate similar requirements for the general case (at least under the assumption that $\mathbf{G}$ is a simple group).

(a) Recall ([Car85, Chap. 5], see also 2.1.11(d)) that the classification of the unipotent conjugacy classes of connected reductive groups is reduced to considering simple groups, so let us assume here that $\mathbf{G}$ is a simple algebraic group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q \subseteq k$ (q a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. We assume in addition that the characteristic p is good for $\mathbf{G}$. Let $u \in \mathbf{G}^F_{\mathrm{uni}}$, and let $\mathscr{B}_u$ be the variety consisting of all Borel subgroups of $\mathbf{G}$ which contain u. Then F naturally acts on the set of irreducible components of $\mathscr{B}_u$. Following [Sho87a, 5.1] (see [Sho83], [BS84]), we say that u is *split unipotent* if said action is the trivial one; this notion only depends on the $\mathbf{G}^F$-conjugacy class of u, so we will then also say that the $\mathbf{G}^F$-conjugacy class of u

is a *split unipotent class*. Now consider any F-stable unipotent conjugacy class $\mathscr{O} \subseteq \mathbf{G}$. Except when $\mathbf{G}$ is of type E_8, $q \equiv -1 \pmod 3$ and $\mathscr{O} = \mathsf{D}_8(a_3) \subseteq \mathbf{G}$, it is known (due to results of Shoji [Sho82], [Sho83] and Beynon–Spaltenstein [BS84]) that there is exactly one split unipotent class contained in $\mathscr{O}^F$. (In the case excluded above, no such class exists inside $\mathscr{O}^F$.) If $u_0 \in \mathbf{G}^F_{\mathrm{uni}}$ is split unipotent, we have:

$$F \text{ acts trivially on } A_{\mathbf{G}}(u_0). \tag{$\clubsuit$}$$

So by 3.2.20(a), the $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}^F$ are parametrised by the conjugacy classes of $A_{\mathbf{G}}(u_0)$. Moreover, we note that any split unipotent element $u_0 \in \mathbf{G}^F_{\mathrm{uni}}$ has the following property:

$$\text{For any } n \in \mathbb{N} \text{ which is prime to } p, u_0 \text{ is conjugate to } u_0^n \text{ in } \mathbf{G}^F. \tag{$\diamondsuit$}$$

Indeed, let $\mathscr{O} \subseteq \mathbf{G}$ be the (F-stable) unipotent class containing u_0. Then we also have $u_0^n \in \mathscr{O}$ (see [Lus09, 2.5]; in fact, this result even holds in bad characteristic and for arbitrary connected reductive groups). Let p^r ($r \in \mathbb{N}_0$) be the order of u_0, and let $n', m \in \mathbb{Z}$ be such that $nn' = mp^r + 1$. Then $u_0^{nn'} = u_0$, so we get

$$\mathscr{B}_{u_0} \subseteq \mathscr{B}_{u_0^n} \subseteq \mathscr{B}_{u_0^{nn'}} = \mathscr{B}_{u_0}.$$

Thus, we have $\mathscr{B}_{u_0} = \mathscr{B}_{u_0^n}$, so u_0^n is split unipotent as well; since $u_0^n \in \mathscr{O}^F$, u_0 and u_0^n must be $\mathbf{G}^F$-conjugate. In particular, any split unipotent element of $\mathbf{G}^F$ is $\mathbf{G}^F$-conjugate to its inverse.

(b) Let us return to the situation in 3.2.21, for simplicity still assuming that $\mathbf{G}$ is simple, so that Σ is an F-stable conjugacy class of $\mathbf{G}$ (since $\mathbf{Z}(\mathbf{G})^\circ = \{1\}$ in this case). We also assume that $\Sigma = \Sigma^{-1}$. Thus, if Σ happens to be a unipotent conjugacy class and if p is a good prime for $\mathbf{G}$, a natural choice for the representative g_0 is $g_0 := u_0$ with $u_0 \in \Sigma^F$ split unipotent. If, on the other hand, Σ is a non-unipotent class of $\mathbf{G}$ (or if the characteristic is bad for $\mathbf{G}$), we would also like to specify a 'good' $\mathbf{G}^F$-conjugacy class inside Σ^F by formulating suitable conditions which characterise such a class. In view of the brief discussion concerning split unipotent elements in (a), one obvious requirement that we could impose on a representative g_0 of such a $\mathbf{G}^F$-conjugacy class is that it satisfies the analogue to ($\diamondsuit$) (or at least that g_0, g_0^{-1} are $\mathbf{G}^F$-conjugate). However, note that such a $\mathbf{G}^F$-class may not even exist inside Σ^F, or if it exists, it may not be uniquely determined by this property. Another requirement on a 'good' representative $g_0 \in \Sigma^F$ is motivated by ($\clubsuit$): Recall from Proposition 3.1.17(c) that the connected centraliser of any element of the conjugacy class Σ above is a unipotent group. Thus, for any $g \in \Sigma^F$, a theorem of Rosenlicht (see [Gec03a, 4.2.4]) states that $|C^\circ_{\mathbf{G}}(g)^F| = q^d$ with $d = \dim C^\circ_{\mathbf{G}}(g)$. By a slight abuse of notation, let $F\colon A_{\mathbf{G}}(g) \xrightarrow{\sim} A_{\mathbf{G}}(g)$ be the automorphism induced by F. A simple application of the Lang–Steinberg Theorem 2.1.14 shows that we have a canonical isomorphism

$$A_{\mathbf{G}}(g)^F \cong C_{\mathbf{G}}(g)^F / C^\circ_{\mathbf{G}}(g)^F,$$

so

$$|A_{\mathbf{G}}(g)|q^d \geqslant |A_{\mathbf{G}}(g)^F|q^d = |C_{\mathbf{G}}(g)^F|,$$

with equality if and only if F acts trivially on $A_{\mathbf{G}}(g)$. Since we know that this holds in the case where Σ is a unipotent class and $g \in \Sigma^F$ is a split unipotent element, a natural requirement on a ‘good’ representative $g_0 \in \Sigma^F$ would be that its centraliser order is maximal among the elements of Σ^F (or that F acts trivially on $A_{\mathbf{G}}(g_0)$, if such g_0 exists). But again, this condition does in general not uniquely specify the $\mathbf{G}^F$-class inside Σ^F.

(c) As we shall see when dealing with the various examples of simple groups of exceptional type in Chapter 4, there are several situations where the methods above are not sufficient to uniquely specify a $\mathbf{G}^F$-class inside a given Σ^F. As has been observed by Geck [Gec21], the condition (♡) below turns out to be very useful for those matters; it is formulated under the following assumptions:

(i) $\mathbf{G}$ is simple and simply connected.

(ii) The (F-stable) conjugacy class $\Sigma \subseteq \mathbf{G}$ consists of regular elements.

So let $\mathbf{G}$, Σ be such that (i) and (ii) are satisfied. Let us consider the Weyl group $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ of $\mathbf{G}$ with respect to $\mathbf{T}_0$ and denote by $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ the automorphism induced by F. Viewing $\mathbf{W}$ as a Coxeter group (as in 2.1.5), assume that there exists a Coxeter element w_{c} of $\mathbf{W}$ such that $\sigma(w_{\mathrm{c}}) = w_{\mathrm{c}}$. By [Gec21, 4.7], the set $\Sigma \cap \mathbf{B}_0 w_{\mathrm{c}} \mathbf{B}_0$ is (non-empty and) a single $\mathbf{B}_0$-orbit for the conjugation action of $\mathbf{B}_0$. Thus, the set $\Sigma^F \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F$ is non-empty as well, so we may pick $g_0 \in \Sigma^F$ in such a way that the following holds:

$$\text{The } \mathbf{G}^F\text{-conjugacy class } C_0 \subseteq \Sigma^F \text{ of } g_0 \text{ satisfies } C_0 \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F \neq \varnothing. \qquad (\heartsuit)$$

Now in general, (♡) still does not always determine the $\mathbf{G}^F$-conjugacy class of g_0 completely, but it will be restrictive enough for our purposes. (In order to uniquely specify the $\mathbf{G}^F$-class of g_0, one may choose a representative $\dot{w}_{\mathrm{c}} \in N_{\mathbf{G}}(\mathbf{T}_0)^F$ of w_{c} and then require that the $\mathbf{G}^F$-class of g_0 has a non-empty intersection with $\mathbf{U}_0^F \dot{w}_{\mathrm{c}} \mathbf{U}_0^F$, see

[Gec21, 4.8].) As far as the simple groups of type G_2, F_4 or E_8 are concerned, the $\mathbf{G}^F$-class of g_0 is in fact uniquely determined by (♡). Indeed, the fundamental groups of the Cartan matrices of these $\mathbf{G}$ are trivial (see 2.1.7), so we necessarily have $\mathbf{Z}(\mathbf{G}) = \{1\}$. By [Gec21, 4.9], this implies that

$$(\Sigma \cap \mathbf{B}_0 w_c \mathbf{B}_0)^F = C_0 \cap \mathbf{B}_0^F w_c \mathbf{B}_0^F,$$

where $C_0 \subseteq \Sigma^F$ is a single $\mathbf{G}^F$-conjugacy class.

3.2.23 Lusztig's map. Condition (♡) in 3.2.22(c) can be adapted to non-regular unipotent classes of $\mathbf{G}$ in a suitable way (at least if $\mathbf{G}$ is assumed to be simple of exceptional type), as follows. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$, $r = \operatorname{rank} \mathbf{G}$, and let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subseteq R$ be the set of simple roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$ (as described in 2.1.9). Let $\sigma \colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ be the automorphism induced by F. Viewing $\mathbf{W}$ as a subgroup of $\operatorname{Aut}(X)$ (see 2.1.4) and denoting by $s_i := w_{\alpha_i} \in \mathbf{W}$ ($1 \leqslant i \leqslant r$) the reflection corresponding to the simple root $\alpha_i \in \Pi$, $\mathbf{W}$ is a Coxeter group with Coxeter generators $S = \{s_1, s_2, \ldots, s_r\}$. Let us denote by ${}^{\mathbf{W}}/_{\sim}$ the set of conjugacy classes of $\mathbf{W}$ and by ${}^{\mathbf{G}_{\mathrm{uni}}}/_{\sim}$ the set of unipotent conjugacy classes of $\mathbf{G}$. For $C \in {}^{\mathbf{W}}/_{\sim}$, let $C_{\min} \subseteq C$ be the subset consisting of the elements of minimal length in C with respect to the length function of $(\mathbf{W}, S)$. In [Lus11a, 4.5], Lusztig defines a surjective map

$$\Phi \colon {}^{\mathbf{W}}/_{\sim} \to {}^{\mathbf{G}_{\mathrm{uni}}}/_{\sim}, \quad C \mapsto \mathscr{O}_C.$$

If $\mathbf{G}$ is simple of exceptional type, it follows from [Lus11a, 0.4] and [Lus12a, 4.8] (see also [MiChv, §6]) that Φ has the following property, which in turn uniquely determines Φ:

$$\begin{array}{l}\text{Let } C \in {}^{\mathbf{W}}/_{\sim} \text{ and } w \in C_{\min}. \text{ Then } \mathscr{O}_C \cap \mathbf{B}_0 w \mathbf{B}_0 \neq \varnothing, \text{ and if} \\ \mathscr{O} \in {}^{\mathbf{G}_{\mathrm{uni}}}/_{\sim} \text{ is such that } \mathscr{O} \cap \mathbf{B}_0 w \mathbf{B}_0 \neq \varnothing, \text{ we have } \mathscr{O}_C \subseteq \overline{\mathscr{O}}.\end{array} \qquad (*)$$

(Note that the formulation of $(*)$ is equivalent to the one in [Lus12a, 4.8], see the remarks in [Gec11, §3]; thus, by the argument in [Lus11a, 0.2], $(*)$ is independent of the choice of $w \in C_{\min}$.) For our purposes, whenever we deal with simple groups of exceptional type, it will be convenient to take $(*)$ as the definition of Φ. For example, the trivial class of $\mathbf{W}$ is sent to the trivial class of $\mathbf{G}$; the class of $\mathbf{W}$ containing the Coxeter elements is sent to the regular unipotent class of $\mathbf{G}$. The map Φ can be explicitly obtained using CHEVIE, see [MiChv, §6]. If $w \in \mathbf{W}^\sigma$ is an element of minimal length in its conjugacy class $C_w \in {}^{\mathbf{W}}/_{\sim}$, with the associated ($F$-stable) unipotent class $\mathscr{O}_{C_w} = \Phi(C_w) \in {}^{\mathbf{G}_{\mathrm{uni}}}/_{\sim}$, we may thus formulate the following requirement on a 'good' representative $u_0 \in \mathscr{O}_{C_w}^F$:

$$\text{The } \mathbf{G}^F\text{-conjugacy class } O_0 \subseteq \mathscr{O}_{C_w}^F \text{ of } u_0 \text{ satisfies } O_0 \cap \mathbf{B}_0^F w \mathbf{B}_0^F \neq \varnothing. \qquad (\heartsuit')$$

In the special case where $w = w_c \in \mathbf{W}^\sigma$ is a Coxeter element of $\mathbf{W}$, $(\heartsuit')$ just becomes $(\heartsuit)$ in 3.2.22(c) with $\Sigma = \mathscr{O}_{\mathrm{reg}}$. Note that, for any $w \in \mathbf{W}^\sigma$ such that w is of minimal length in its conjugacy class $C_w \in \mathbf{W}/_\sim$, there always exists some $u_0 \in \mathscr{O}_{C_w}^F$ such that $(\heartsuit')$ is satisfied (see [Gec11, 3.5(a)]). As it turns out, the condition $(\heartsuit')$ does in many cases determine the $\mathbf{G}^F$-conjugacy class of u_0 uniquely.

In several instances below, we will apply the following lemma to obtain a representative u_0 such that the condition $(\heartsuit')$ in 3.2.23 is satisfied. Recall that, for any $\alpha \in R$, $u_\alpha \colon \mathbf{G}_\mathrm{a} \to \mathbf{G}$ is the homomorphism whose image is the root subgroup $\mathbf{U}_\alpha$ (see 2.1.4).

Lemma 3.2.24. *In the setting and with the notation of* 3.2.23*, let $w \in \mathbf{W}$ be an element of length $e \in \mathbb{N}_0$ in $(\mathbf{W}, S)$, and let $w = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_e}$ be a reduced expression for w (where $1 \leqslant i_1, i_2, \ldots, i_e \leqslant r$). Let $t_1, t_2, \ldots, t_e \in k^\times$, and let*

$$u_0 := u_{-w_0(\alpha_{i_1})}(t_1) \cdot u_{-w_0(\alpha_{i_2})}(t_2) \cdot \ldots \cdot u_{-w_0(\alpha_{i_e})}(t_e) \in \mathbf{U}_0.$$

Then, denoting by $\dot{w}_0 \in N_\mathbf{G}(\mathbf{T}_0)$ a representative of the longest element w_0 of $(\mathbf{W}, S)$, we have

$$\dot{w}_0 u_0 \dot{w}_0^{-1} \in (\mathbf{U}_{-\alpha_{i_1}} \mathbf{U}_{-\alpha_{i_2}} \cdots \mathbf{U}_{-\alpha_{i_e}}) \cap \mathbf{B}_0 w \mathbf{B}_0.$$

In particular, this holds for $u_0 = u_{-w_0(\alpha_{i_1})}(1) u_{-w_0(\alpha_{i_2})}(1) \cdots u_{-w_0(\alpha_{i_e})}(1)$.

Proof. This is mainly a collection of well-known facts about root systems and basic structural properties of groups with a BN-pair, see, e.g., [Ste16] or [Car85, Chap. 2]. The longest element $w_0 \in \mathbf{W}$ is characterised by the property $w_0(R^+) = R^-$, so u_0 is indeed an element of $\mathbf{U}_0$. For $\alpha \in R$, we have $\dot{w}_0 \mathbf{U}_\alpha \dot{w}_0^{-1} = \mathbf{U}_{w_0(\alpha)}$, so $\dot{w}_0 u_{-w_0(\alpha_{i_j})}(t_j) \dot{w}_0^{-1} \in \mathbf{U}_{-\alpha_{i_j}}$ for $1 \leqslant j \leqslant e$ (since $w_0^2 = 1$). Now

$$\mathbf{U}_{-\alpha_{i_j}} \subseteq \mathbf{L}_{\{s_{i_j}\}} \subseteq \mathbf{P}_{\{s_{i_j}\}} = \mathbf{B}_0 \cup \mathbf{B}_0 s_{i_j} \mathbf{B}_0$$

(see [Car85, §2.6]). As $\mathbf{U}_{-\alpha_{i_j}} \subseteq \dot{w}_0 \mathbf{U}_0 \dot{w}_0^{-1} \subseteq \dot{w}_0 \mathbf{B}_0 \dot{w}_0^{-1}$, we have $\mathbf{U}_{-\alpha_{i_j}} \cap \mathbf{B}_0 = \{1\}$, so $\dot{w}_0 u_{-w_0(\alpha_{i_j})}(t_j) \dot{w}_0^{-1} \in \mathbf{B}_0 s_{i_j} \mathbf{B}_0$. Since $w = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_e}$ is a reduced expression for w, we know that

$$\mathbf{B}_0 s_{i_1} \mathbf{B}_0 \cdot \mathbf{B}_0 s_{i_2} \mathbf{B}_0 \cdot \ldots \cdot \mathbf{B}_0 s_{i_e} \mathbf{B}_0 = \mathbf{B}_0 w \mathbf{B}_0$$

(see, e.g., [Gec03a, 1.6.3]). It follows that

$$\dot{w}_0 u_0 \dot{w}_0^{-1} = (\dot{w}_0 u_{-w_0(\alpha_{i_1})}(t_1)\dot{w}_0^{-1}) \cdot \ldots \cdot (\dot{w}_0 u_{-w_0(\alpha_{i_e})}(t_e)\dot{w}_0^{-1}) \in \mathbf{B}_0 w \mathbf{B}_0. \qquad \square$$

3.2.25 Relation between characteristic functions. We place ourselves in the setting of 3.2.15 and adopt the notation from there. Let us consider a pair $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}_{\mathbf{G}}^F$, and let $\mathfrak{j} = \tau(\mathfrak{i}) = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}^F$, so that $A_\mathfrak{i} \in \hat{\mathbf{G}}^F$ is a simple direct summand of $K_\mathfrak{j}$. As in 3.2.15, we assume that $\mathbf{L}$ is F-stable and the standard Levi subgroup of some F-stable standard parabolic subgroup of $\mathbf{G}$. It should be noted that the choice of the isomorphism $\varphi_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ in 3.2.15 does not satisfy the condition $(*)$ in 3.2.1! Since we will need it in a number of places throughout this thesis, let us describe the relation between the isomorphism $\varphi_{A_\mathfrak{i}}$ and an isomorphism $F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ as in 3.2.1$(*)$ (up to multiplication with a root of unity).

Recall that, in order to define $\varphi_{A_\mathfrak{i}}$ as in 3.2.15, we have to choose an isomorphism $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ of local systems on $\mathscr{O}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Such an isomorphism then naturally determines an isomorphism $\varphi_\mathfrak{j} \colon F^*K_\mathfrak{j} \xrightarrow{\sim} K_\mathfrak{j}$, and $\varphi_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ is uniquely defined by the requirement that the bijective linear map

$$\sigma_{A_\mathfrak{i}} \colon V_{A_\mathfrak{i}} \to V_{A_\mathfrak{i}}, \quad v \mapsto \varphi_\mathfrak{j} \circ F^*(v) \circ \varphi_{A_\mathfrak{i}}^{-1},$$

determines the preferred extension of $V_{A_\mathfrak{i}}$ to a $\mathscr{W}_\mathfrak{j}(F_\mathfrak{j}^{-1})$-module. In particular, $\sigma_{A_\mathfrak{i}}$ is a map of finite order.

On the other hand, let $\overline{\varphi}_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ be an isomorphism which satisfies the condition $(*)$ in 3.2.1. (We use the bar here only to have a distinction from the isomorphism $\varphi_{A_\mathfrak{i}}$.) We set $\Sigma_0 := \mathbf{Z}(\mathbf{L})^\circ.\mathscr{O}_0$, $Y := Y_{(\mathbf{L},\Sigma_0)}$ (so that $\operatorname{supp} A_\mathfrak{i} = \overline{Y}$) and $d := \dim Y$. So we obtain a corresponding bijective linear map

$$\overline{\sigma}_{A_\mathfrak{i}} \colon V_{A_\mathfrak{i}} \to V_{A_\mathfrak{i}}, \quad v \mapsto \varphi_\mathfrak{j} \circ F^*(v) \circ \overline{\varphi}_{A_\mathfrak{i}}^{-1}.$$

We know that there exists some scalar $\mathfrak{z}_\mathfrak{i} \in \overline{\mathbb{Q}}_\ell^\times$ such that

$$\overline{\varphi}_{A_\mathfrak{i}} = \mathfrak{z}_\mathfrak{i} \cdot \varphi_{A_\mathfrak{i}} \quad \text{and, hence,} \quad \overline{\sigma}_{A_\mathfrak{i}} = \mathfrak{z}_\mathfrak{i}^{-1} \cdot \sigma_{A_\mathfrak{i}}.$$

Now let us pick any element $y \in \Sigma_0^F \cap Y^F$. In view of (3.2.3.2), we may identify $\mathscr{H}_y^{-d}(A_\mathfrak{i}) \otimes V_{A_\mathfrak{i}}$ with a subspace of $\mathscr{H}_y^{-d}(K_\mathfrak{j})$, and as mentioned in 3.2.3, the endomorphism $(\overline{\varphi}_A)_{-d,y} \otimes \overline{\sigma}_{A_\mathfrak{i}} \in \operatorname{End}(\mathscr{H}_y^{-d}(A_\mathfrak{i}) \otimes V_{A_\mathfrak{i}})$ corresponds to $(\varphi_\mathfrak{j})_{-d,y}$ under this identification.

From our assumption on φ_0 and the definition of the complex $K_\mathfrak{j}$, it follows that $(\varphi_\mathfrak{j})_{-d,y}$ is a map of finite order. Since the eigenvalues of $(\overline{\varphi}_{A_\mathfrak{i}})_{-d,y}\colon \mathscr{H}_y^{-d}(A_\mathfrak{i}) \to \mathscr{H}_y^{-d}(A_\mathfrak{i})$ are of the form $q^{(\dim \mathbf{G}-d)/2}$ times a root of unity, we deduce that the eigenvalues of $\overline{\sigma}_{A_\mathfrak{i}}$ must be of the form $q^{-(\dim \mathbf{G}-d)/2}$ times a root of unity. Thus,

$$q^{(\dim \mathbf{G}-d)/2} \cdot \overline{\sigma}_{A_\mathfrak{i}} = q^{(\dim \mathbf{G}-d)/2} \cdot \mathfrak{z}_\mathfrak{i}^{-1} \cdot \sigma_{A_\mathfrak{i}}$$

is a map of finite order, as is $\sigma_{A_\mathfrak{i}}$. Hence, the scalar $\mathfrak{z}_\mathfrak{i}$ is equal to $q^{(\dim \mathbf{G}-d)/2}$ times a root of unity. In other words, if $\varphi_{A_\mathfrak{i}}\colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ is defined as in 3.2.15, then

$$\overline{\varphi}_{A_\mathfrak{i}} := q^{(\dim \mathbf{G}-\dim \operatorname{supp} A_\mathfrak{i})/2} \cdot \varphi_{A_\mathfrak{i}}\colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$$

satisfies the requirement $(*)$ in 3.2.1, so it is a valid choice for Theorem 3.2.2. With this $\overline{\varphi}_{A_\mathfrak{i}}$, it is also clear that

$$\chi_{A_\mathfrak{i},\overline{\varphi}_{A_\mathfrak{i}}} = q^{(\dim \mathbf{G}-\dim \operatorname{supp} A_\mathfrak{i})/2} \cdot \chi_{A_\mathfrak{i},\varphi_{A_\mathfrak{i}}}. \tag{3.2.25.1}$$

By the definition of $X_\mathfrak{i}$, we thus get

$$\chi_{A_\mathfrak{i},\overline{\varphi}_{A_\mathfrak{i}}}|_{\mathbf{G}^F_{\mathrm{uni}}} = (-1)^{a_\mathfrak{i}} q^{(\dim \mathbf{G}-\dim \mathscr{O}-\dim \mathbf{Z}(\mathbf{L})^\circ)/2} X_\mathfrak{i}. \tag{3.2.25.2}$$

Remark 3.2.26. As soon as a choice for the isomorphism $\varphi_0\colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ is made, the procedure in 3.2.25 *uniquely* determines the isomorphisms $\varphi_{A_\mathfrak{i}}\colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ and $\overline{\varphi}_{A_\mathfrak{i}}\colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$. While both $\varphi_{A_\mathfrak{i}}$ and $\overline{\varphi}_{A_\mathfrak{i}}$ depend upon the choice of φ_0, the interrelation between $\varphi_{A_\mathfrak{i}}$ and $\overline{\varphi}_{A_\mathfrak{i}}$ does not, so it makes sense to refer to (3.2.25.1) and (3.2.25.2) even without having specified φ_0.

3.3. Lusztig's Conjecture and Shoji's Theorem

By Theorem 3.2.2, the set of (suitably normalised) characteristic functions of F-stable character sheaves (up to isomorphism) on a connected reductive group $\mathbf{G}$ over $k = \overline{\mathbb{F}}_p$ with Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$ is an orthonormal basis of the space of class functions of $\mathbf{G}^F$. Lusztig conjectured that this basis coincides with the one consisting of almost characters, after a suitable normalisation of the characteristic functions and with an appropriate generalisation of the definition of almost characters if the centre of $\mathbf{G}$ is not connected. In this section, we will formulate Lusztig's Conjecture (see 3.3.8) and state Shoji's Theorem (3.3.9), which proves Lusztig's Conjecture in the case where $\mathbf{Z}(\mathbf{G})$ is connected. Note that this is a big step towards the determination of the generic character tables of finite groups of Lie type, as the characteristic functions of F-stable character sheaves can be computed in principle, although their explicit computation is still a difficult problem (see 3.2.19).

Recall our standing assumptions formulated in the beginning of Section 3.2: So $\mathbf{G}$ is a connected reductive group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$; furthermore, $\mathbf{T}_0$ is a maximally split torus of $\mathbf{G}$, and $\mathbf{B}_0 \subseteq \mathbf{G}$ is an F-stable Borel subgroup which contains $\mathbf{T}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$ (so that $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$ and $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ be the automorphism induced by F. Let $(\mathbf{G}^*, \mathbf{T}_0^*, \mathbf{B}_0^*, F')$ be the dual quadruple associated to $(\mathbf{G}, \mathbf{T}_0, \mathbf{B}_0, F)$, and let $\mathbf{W}^* = N_{\mathbf{G}^*}(\mathbf{T}_0^*)/\mathbf{T}_0^*$ (see Example 2.1.21). Unless explicitly stated otherwise, we make in addition the following assumption, which remains in force until the end of this chapter:

The centre $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$ is connected.

We begin by stating a central result of [LuCS5] (see Theorem 3.3.2 below), which provides a classification of the character sheaves on $\mathbf{G}$ in terms of tame local systems $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$ and of families of the group $\mathbf{W}'_{\mathscr{L}}$; here we will only formulate this result for those $\mathbf{G}$ which have a connected centre, although Lusztig proves it for any connected reductive group $\mathbf{G}$ over k in a more general framework.

3.3.1. Let $(\lambda, n) \in X \times (\mathbb{N} \setminus p\mathbb{N})$, and let $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ be the associated tame local system, see 3.1.3. Recall the definition of $\mathbf{W}'_{\mathscr{L}}$ in 3.1.4 and the one of $\mathbf{W}_{\lambda,n}$ in 2.2.16. As already remarked in 3.1.4, we have

$$\mathbf{W}'_{\mathscr{L}} = \{w \in \mathbf{W} \mid w.\lambda - \lambda \in nX\}.$$

Since we assume $\mathbf{Z}(\mathbf{G})$ to be connected, it follows from [GM20, 2.4.14] (see the proof of [DM20, 11.2.1]) that $\mathbf{W}'_{\mathscr{L}} = \mathbf{W}_{\lambda,n}$. This also shows that the group $\mathbf{W}_{\mathscr{L}}$ and its root system $R_{\mathscr{L}}$ defined in [LuCS1, 2.3] coincide with $\mathbf{W}'_{\mathscr{L}} = \mathbf{W}_{\lambda,n}$ and the root system $R_{\lambda,n}$ as described in 2.2.16, respectively. For any local system $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ as above, we may thus apply the machinery of 2.2.8 to $W = \mathbf{W}_{\lambda,n}$ (with roots $R_{\lambda,n}$ and simple reflections $S_{\lambda,n}$, see 2.2.16). In particular, we have a pairing

$$\{\ ,\ \}\colon \mathfrak{X}(\mathbf{W}_{\lambda,n}) \times \mathfrak{X}(\mathbf{W}_{\lambda,n}) \to \overline{\mathbb{Q}}_\ell$$

and an embedding

$$\mathrm{Irr}(\mathbf{W}_{\lambda,n}) \hookrightarrow \mathfrak{X}(\mathbf{W}_{\lambda,n}), \quad \phi \mapsto x_\phi.$$

Now recall the definition of $K_w^{\mathscr{L}}$ for $w \in \mathbf{W}_{\lambda,n}$ in 3.1.5. For $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})$, we set (see [LuCS3, 14.10])

$$R_\phi^{\mathscr{L}} := \frac{1}{|\mathbf{W}_{\lambda,n}|} \sum_{w \in \mathbf{W}_{\lambda,n}} \phi(w^{-1}) \sum_{i \in \mathbb{Z}} (-1)^{i+\dim \mathbf{G}}\, {}^p\!H^i(K_w^{\mathscr{L}}),$$

an element of the subgroup of the Grothendieck group of $\mathscr{M}\mathbf{G}$ spanned by the character sheaves. We denote by $(\ :\ )$ the symmetric $\overline{\mathbb{Q}}_\ell$-bilinear pairing on this subgroup such that for any two character sheaves A, A' on $\mathbf{G}$, we have

$$(A : A') = \begin{cases} 1 & \text{if} \quad A \cong A', \\ 0 & \text{if} \quad A \not\cong A'. \end{cases}$$

Theorem 3.3.2 (Lusztig [LuCS5, Thm. 23.1], [Lus12b])**.** *Let* $\mathbf{G}$ *be a connected reductive group over* k *and assume that* $\mathbf{Z}(\mathbf{G})$ *is connected. Let* $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$, *where* $(\lambda, n) \in X \times (\mathbb{N} \setminus p\mathbb{N})$. *Then, with the notation of* 3.3.1, *there is a bijection*

$$\hat{\mathbf{G}}_{\mathscr{L}} \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}_{\lambda,n}), \quad A \mapsto x_A,$$

such that for any $A \in \hat{\mathbf{G}}_{\mathscr{L}}$ *and any* $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})$, *we have*

$$(A : R_\phi^{\mathscr{L}}) = \hat{\varepsilon}_A \{x_A, x_\phi\}.$$

(Recall from (3.1.1.1) *that* $\hat{\varepsilon}_A = (-1)^{\dim \mathbf{G} - \dim \operatorname{supp} A}$*.)*

Remark 3.3.3. Our framework in this section is based on the standing assumption that the centre of $\mathbf{G}$ is connected. The formulation of Theorem 3.3.2 can be generalised so that it holds for any connected reductive group $\mathbf{G}$ over k, see [LuCS5, Thm. 23.1] (and the remarks in [Lus12b, 3.10]), but the necessary broadening of the setting becomes slightly more technical. Since we will only be concerned with the case where $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\mathbf{G})^\circ$, we merely refer to [LuCS5, §23] at this point. Also note that, regardless of the assumption on the centre of $\mathbf{G}$, Theorem 3.3.2 is still only a part of [LuCS5, Thm. 23.1], the other part is Theorem 3.1.13. Thus, just as for Theorem 3.1.13, while the main portion of the proof of Theorem 3.3.2 is contained in [LuCS4] and [LuCS5, §22, §23], it is only concluded in [Lus12b], where some small primes are considered which were previously excluded in the formulation of [LuCS5, Thm. 23.1], see [LuCS5, (23.0.1)].

3.3.4. Recall that we have fixed an isomorphism $\jmath\colon \mathbb{Z}_{(p)}/\mathbb{Z} \xrightarrow{\sim} k^\times$ in (1.1.0.5). As discussed in 2.2.26, there is an isomorphism

$$X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}/\mathbb{Z} \xrightarrow{\delta\otimes\jmath} Y(\mathbf{T}_0^*) \otimes_{\mathbb{Z}} k^\times \xrightarrow{\sim} \mathbf{T}_0^*,$$

where the last map is given by (2.1.3.2) (applied to the torus $\mathbf{T}_0^*$) and $\delta\colon X \xrightarrow{\sim} Y(\mathbf{T}_0^*)$ is the isomorphism (2.1.21.1) used to describe the duality between $(\mathbf{G}, \mathbf{T}_0, \mathbf{B}_0, F)$ and $(\mathbf{G}^*, \mathbf{T}_0^*, \mathbf{B}_0^*, F')$. Combined with the discussion in 3.1.3, we thus obtain a (non-canonical) isomorphism

$$\mathscr{S}(\mathbf{T}_0) \cong \mathbf{T}_0^*. \tag{3.3.4.1}$$

Specifically, for $s \in \mathbf{T}_0^*$, if $(\lambda, n) \in X \times (\mathbb{N} \setminus p\mathbb{N})$ is such that $s = t_{\lambda,n}$ (see 2.2.26), then with the notation of 3.1.3, $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\jmath_n}) \in \mathscr{S}(\mathbf{T}_0)$ is the local system corresponding to s under (3.3.4.1). The isomorphism $\mathbf{W} \xrightarrow{\sim} \mathbf{W}^*$, $w \mapsto w^*$ (see (2.1.21.2)) and the action of $\mathbf{W}^*$ on $\mathbf{T}_0^*$ (i.e., (3.1.4.1) applied to the dual groups) give rise to an action

$$\mathbf{W} \times \mathbf{T}_0^* \to \mathbf{T}_0^*, \quad (w, s) \mapsto \dot{w}^* s (\dot{w}^*)^{-1}, \tag{3.3.4.2}$$

where $\dot{w}^* \in N_{\mathbf{G}^*}(\mathbf{T}_0^*)$ denotes a representative of $w^* \in \mathbf{W}^*$. For $w \in \mathbf{W}$ and $s = t_{\lambda,n}$, $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\jmath_n})$ as above, we have $\dot{w}^* s (\dot{w}^*)^{-1} = t_{w.\lambda,n} \in \mathbf{T}_0^*$, and this element corresponds to the local system $(w.\lambda)^*(\mathscr{E}_{n,\jmath_n}) \in \mathscr{S}(\mathbf{T}_0)$ under the isomorphism (3.3.4.1). On the other hand, $(w.\lambda)^*(\mathscr{E}_{n,\jmath_n})$ is the image of $(w, \mathscr{L})$ under the action of $\mathbf{W}$ on $\mathscr{S}(\mathbf{T}_0)$ defined in (3.1.4.2), so the isomorphism (3.3.4.1) is compatible with the actions of $\mathbf{W}$ on $\mathscr{S}(\mathbf{T}_0)$ and $\mathbf{T}_0^*$. Hence, in the setting of Definition 3.1.6, we may set $\hat{\mathbf{G}}_s := \hat{\mathbf{G}}_{\mathscr{L}}$ provided $s \in \mathbf{T}_0^*$ corresponds to $\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)$ under (3.3.4.1). In view of Remark 3.1.7(a), we thus obtain a partition

$$\hat{\mathbf{G}} = \biguplus_{s \in \mathbf{T}_0^*/\mathbf{W}} \hat{\mathbf{G}}_s,$$

where $\mathbf{T}_0^*/\mathbf{W}$ denotes a set of representatives for the $\mathbf{W}$-orbits on $\mathbf{T}_0^*$ under (3.3.4.2). We also recall from the discussion in 2.2.26 that $\mathbf{T}_0^*/\mathbf{W}$ is at the same time a set of representatives for the semisimple conjugacy classes of $\mathbf{G}^*$.

3.3.5. We now take the $\mathbb{F}_q$-rational structure on $\mathbf{G}$ into account. Recall that the Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$ gives rise to the inverse image functor $F^*\colon \mathscr{D}\mathbf{G} \to \mathscr{D}\mathbf{G}$. Since $F(\mathbf{T}_0) = \mathbf{T}_0$, we may also view F^* as a functor $\mathscr{D}\mathbf{T}_0 \to \mathscr{D}\mathbf{T}_0$. Let $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\jmath_n}) \in \mathscr{S}(\mathbf{T}_0)$ be the tame local system associated to $(\lambda, n) \in X \times (\mathbb{N} \setminus p\mathbb{N})$. Then

$$F^*\mathscr{L} = F^*(\lambda^*(\mathscr{E}_{n,\jmath_n})) = (\lambda \circ F|_{\mathbf{T}_0})^*(\mathscr{E}_{n,\jmath_n}),$$

so $F^*\mathscr{L}$ arises from $(\lambda \circ F|_{\mathbf{T}_0}, n) \in X \times (\mathbb{N} \setminus p\mathbb{N})$; in particular, F^* induces a bijection $\mathscr{S}(\mathbf{T}_0) \xrightarrow{\sim} \mathscr{S}(\mathbf{T}_0)$. Now assume that $A \in \hat{\mathbf{G}}_{\mathscr{L}}$. By definition, A is a constituent of ${}^pH^i(K_w^{\mathscr{L}})$ for some $w \in \mathbf{W}'_{\mathscr{L}} = \mathbf{W}_{\lambda,n}$ and some $i \in \mathbb{Z}$, so the inverse image F^*A is a constituent of $F^*({}^pH^i(K_w^{\mathscr{L}}))$. It follows rather directly from the definition of $K_w^{\mathscr{L}}$ and the properties mentioned in [LuCS1, 1.7, 1.8] that we have

$$F^*({}^pH^i(K_w^{\mathscr{L}})) = {}^pH^i(F^*K_w^{\mathscr{L}}) = {}^pH^i\Big(K^{F^*\mathscr{L}}_{\sigma_{\lambda,n}^{-1}(w)}\Big),$$

where $\sigma_{\lambda,n}\colon \mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_{\lambda,n}$ is the automorphism defined in Lemma 2.2.17(b). We conclude that $F^*A \in \hat{\mathbf{G}}_{F^*\mathscr{L}}$, so

$$F^*\hat{\mathbf{G}}_{\mathscr{L}} = \hat{\mathbf{G}}_{F^*\mathscr{L}} \quad \text{for any} \quad \mathscr{L} \in \mathscr{S}(\mathbf{T}_0).$$

Recall from Remark 3.1.7(a) that we have $\hat{\mathbf{G}}_{\mathscr{L}} = \hat{\mathbf{G}}_{F^*\mathscr{L}}$ if and only if $\mathscr{L}$ and $F^*\mathscr{L}$ are in the same $\mathbf{W}$-orbit in $\mathscr{S}(\mathbf{T}_0)$ under the action (3.1.4.2). Thus, $F^*\hat{\mathbf{G}}_{\mathscr{L}} = \hat{\mathbf{G}}_{\mathscr{L}}$ holds if and only if there exists some $w \in \mathbf{W}$ such that $F^*\mathscr{L} \cong (w^{-1})^*\mathscr{L}$. Writing $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n})$ as above, we see from the combinatorial description of $\mathscr{S}(\mathbf{T}_0)$ via (3.1.3.1) and the definition of the set $\mathscr{Z}_{\lambda,n}$ in 2.2.13 that

$$F^*\mathscr{L} \cong (w^{-1})^*\mathscr{L} \iff w \in \mathscr{Z}_{\lambda,n}.$$

Hence, for $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$, we have

$$F^*\hat{\mathbf{G}}_{\mathscr{L}} = \hat{\mathbf{G}}_{\mathscr{L}} \iff \mathscr{Z}_{\lambda,n} \neq \varnothing.$$

The discussion in 3.3.4 shows that the $\mathbf{W}$-orbits on $\mathscr{S}(\mathbf{T}_0)$ with respect to the action (3.1.4.2) and the $\mathbf{W}$-orbits on $\mathbf{T}_0^*$ with respect to the action (3.3.4.2) correspond to one another under the isomorphism (3.3.4.1). On the other hand, as we have seen in 2.2.26, the $\mathbf{W}$-orbits of $\mathbf{T}_0^*$ are in natural bijection with the semisimple classes of $\mathbf{G}^*$. Moreover, for $(\lambda, n) \in X \times (\mathbb{N} \setminus p\mathbb{N})$, the $\mathbf{G}^*$-conjugacy class of $t_{\lambda,n} \in \mathbf{T}_0^*$ is F'-stable if and only if $\mathscr{Z}_{\lambda,n} \neq \varnothing$, and in this way the set $\Lambda(\mathbf{G}, F)/_\sim$ parametrises the F'-stable semisimple conjugacy classes of $\mathbf{G}^*$. Hence, associating to $(\lambda, n) \in \Lambda(\mathbf{G}, F)$ the $\mathbf{W}$-orbit of $\lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ induces a bijection

$$\Lambda(\mathbf{G}, F)/_\sim \xrightarrow{1-1} \{\mathscr{L} \in \mathscr{S}(\mathbf{T}_0)/_{\mathbf{W}} \mid F^*\hat{\mathbf{G}}_{\mathscr{L}} = \hat{\mathbf{G}}_{\mathscr{L}}\},$$

where $\mathscr{S}(\mathbf{T}_0)/_{\mathbf{W}}$ denotes a set of representatives for the $\mathbf{W}$-orbits on $\mathscr{S}(\mathbf{T}_0)$. In particular, we have

$$\hat{\mathbf{G}}^F = \biguplus_{(\lambda,n)\in\Lambda(\mathbf{G},F)/\sim} \hat{\mathbf{G}}^F_{\lambda^*(\mathscr{E}_{n,\imath_n})}.$$

In view of Theorem 3.3.2, we would like to have similar parametrisations of the sets $\hat{\mathbf{G}}^F_{\lambda^*(\mathscr{E}_{n,\imath_n})}$ for $(\lambda, n) \in \Lambda(\mathbf{G}, F)/_\sim$. These are provided by the following result.

Proposition 3.3.6 (Shoji [Sho95a, §5])**.** *(As before, we assume that* $\mathbf{Z}(\mathbf{G})$ *is connected.) Let* $(\lambda, n) \in \Lambda(\mathbf{G}, F)$*,* $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n})$ *and* $\sigma_{\lambda,n} \colon \mathbf{W}_{\lambda,n} \xrightarrow{\sim} \mathbf{W}_{\lambda,n}$ *be as in Lemma 2.2.17(b). Then there is a bijection*

$$\hat{\mathbf{G}}^F_{\mathscr{L}} \xrightarrow{\sim} \overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}), \quad A \mapsto \overline{x}_A, \tag{3.3.6.1}$$

such that for any $A \in \hat{\mathbf{G}}^F_{\mathscr{L}}$ *and any* $\phi \in \mathrm{Irr}(\mathbf{W}_{\lambda,n})^{\sigma_{\lambda,n}}$*, we have*

$$(A : R^{\mathscr{L}}_\phi) = \hat{\varepsilon}_A\{\overline{x}_A, x_\phi\}. \tag{3.3.6.2}$$

Here, $\{\ ,\ \}\colon \overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}) \times \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n}) \to \overline{\mathbb{Q}}_\ell$ *denotes the pairing* (2.2.12.1) *(with* $W = \mathbf{W}_{\lambda,n}$ *and* $\gamma = \sigma_{\lambda,n}$*).*

Proof. See [Sho95a, §5]. □

Remark 3.3.7. Similarly to the parametrisation of the irreducible characters of $\mathbf{G}^F$ (see Theorem 2.2.21 and Remark 2.2.22(b)), the bijection (3.3.6.1) is in general not uniquely determined by the condition (3.3.6.2). For the simple non-twisted groups with a trivial centre (which we consider in Section 3.4), we will make an explicit choice for the parametrisation of the F-stable *unipotent* character sheaves $(\hat{\mathbf{G}}^{\mathrm{un}})^F = \hat{\mathbf{G}}^F_{\mathscr{L}_0}$ (corresponding to the pair $(\lambda, n) = (0, 1) \in \Lambda(\mathbf{G}, F)$, see 3.3.12 below).

3.3.8 Lusztig's Conjecture. In view of Theorem 2.2.21 and Proposition 3.3.6, the set $\overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ (for $(\lambda, n) \in \Lambda(\mathbf{G}, F)$) thus parametrises both the irreducible characters of $\mathbf{G}^F$ in the series $\mathcal{E}_{\lambda,n}$ and the character sheaves in $\hat{\mathbf{G}}^F_{\lambda^*(\mathscr{E}_{n,\imath_n})}$. Already in 1984, (i.e., before he even defined the character sheaves), Lusztig conjectured in [Lus84a, 13.7] the existence of suitable F-invariant irreducible perverse sheaves on $\mathbf{G}$ whose characteristic functions should coincide with the almost characters of $\mathbf{G}^F$. This conjecture is formulated in a more precise way in the introduction of [LuCS5]. However, note that at this point there was not even a clear definition of what an 'almost character of $\mathbf{G}^F$' should be in the case where the centre of $\mathbf{G}$ is not connected. Such a definition was provided by Lusztig only much later in [Lus18], but in general his conjecture is still open for these $\mathbf{G}$; see 3.3.13 below for some more details.

On the other hand, we are now in a position to state the following theorem of Shoji, which verifies Lusztig's Conjecture under the assumption that $\mathbf{G}$ has a connected centre.

Theorem 3.3.9 (Shoji [Sho95a, 5.7], [Sho95b, 3.2, 4.1])**.** *Let $\mathbf{G}$ be a connected reductive group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$, and assume that $\mathbf{Z}(\mathbf{G})$ is connected. Let A be an F-stable character sheaf on $\mathbf{G}$, and let $(\lambda, n) \in \Lambda(\mathbf{G}, F)$, $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\imath_n}) \in \mathscr{S}(\mathbf{T}_0)$ be such that $A \in \hat{\mathbf{G}}^F_{\mathscr{L}}$; thus, A is parametrised by $\overline{x}_A \in \overline{\mathfrak{X}}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ under* (3.3.6.1)*. Let $\varphi_A\colon F^*A \xrightarrow{\sim} A$ be an isomorphism. Then, for any $x \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ whose $\mathcal{R}$-orbit corresponds to $\overline{x}_A$ under* (2.2.12.2)*, there exists some scalar $\xi_x = \xi_x(\varphi_A) \in \overline{\mathbb{Q}}_\ell^\times$ such that*

$$R_x = \xi_x \cdot \chi_{A,\varphi_A}. \tag{3.3.9.1}$$

(Here, R_x denotes the almost character associated to x, see (2.2.23.1)*.)*

Remark 3.3.10. (a) The proof of Theorem 3.3.9 is given in [Sho95a], [Sho95b]. In fact, Shoji assumes that p is 'almost good', as Lusztig does in [LuCS5, (23.0.1)] — however, since the relevant results of [LuCS5] (most notably [LuCS5, Thm. 23.1], see Theorem 3.1.13 and Theorem 3.3.2) are now known to hold in complete generality, thanks to [Lus12b], we do not need to impose any restriction on p in the formulation of Theorem 3.3.9.

(b) In [Sho95a, §5], Shoji works with semisimple conjugacy classes in the dual group $\mathbf{G}^*$ instead of the set $\Lambda(\mathbf{G}, F)$. Thus, in his setting, the parametrisation of the character sheaves in $\hat{\mathbf{G}}^F_s$ for $s \in \mathbf{T}_0^*/\mathbf{W}$ (see 3.3.4) is in terms of the group $\mathbf{W}_s^*$ defined in 2.2.27, but we see from the discussion there that our set-up leads to exactly the same.

(c) We have written $\xi_x = \xi_x(\varphi_A)$ in Theorem 3.3.9 in order to emphasise that the scalar ξ_x depends upon the choice of the isomorphism $\varphi_A\colon F^*A \xrightarrow{\sim} A$. However, when trying to work out the scalar $\xi_x(\varphi_A)$ in the explicit examples later, we will usually choose and fix a specific isomorphism φ_A beforehand and then only write ξ_x.

(d) The scalars $\xi_x(\varphi_A)$ can be defined directly in terms of (A, φ_A), due to Lusztig [LuCS3, §13, §14], [Lus86, §3], and this interpretation is an essential ingredient in the proof of Theorem 3.3.9 in [Sho95a], [Sho95b]. In the case where A is a unipotent character sheaf on a simple group $\mathbf{G}$ with a trivial centre and a non-twisted $\mathbb{F}_q$-rational structure, we will describe this characterisation of $\xi_x(\varphi_A)$ in 3.4.12 below.

3.3.11. The determination of the character table of $\mathbf{G}^F$ can thus be reformulated to solving the following two problems (see [Lus92]):

(a) Compute the values of the characteristic functions of F-stable character sheaves.

(b) For any $x \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ (where $(\lambda, n) \in \Lambda(\mathbf{G}, F)$), determine the scalar ξ_x in (3.3.9.1).

We have explained in Section 3.2 how (a) can be approached, at least in principle. From now on we will mostly be concerned with (b). More precisely, instead of investigating problem (b) 'separately' for each $\mathbf{G}$, it can be replaced by inductive conditions. Let us consider the following two tasks, formulated *simultaneously for all* connected reductive groups $\mathbf{G}$ which are defined over $\mathbb{F}_q$ (for one and the same q), with corresponding Frobenius map $F \colon \mathbf{G} \to \mathbf{G}$, and for which $\mathbf{Z}(\mathbf{G})$ is connected:

(b1) For any regular subgroup $\mathbf{L} \subseteq \mathbf{G}$ and any $\chi \in \mathrm{Irr}(\mathbf{L}^F)$, find the explicit decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$ into irreducible characters of $\mathbf{G}^F$ (where $R_{\mathbf{L}}^{\mathbf{G}}$ denotes Lusztig induction, see Definition 2.2.28).

(b2) For $x \in \mathfrak{X}(\mathbf{W}_{\lambda,n}, \sigma_{\lambda,n})$ (where $(\lambda, n) \in \Lambda(\mathbf{G}, F)$) which in the setting of Shoji's Theorem 3.3.9 corresponds to an F-stable *cuspidal* character sheaf on $\mathbf{G}$, specify the scalar ξ_x in (3.3.9.1).

If (b1) and (b2) were dealt with for all $\mathbf{G}$ as above, and under the assumption that (a) can be carried out explicitly, this would give rise to the character table of $\mathbf{G}^F$. Indeed, recall from 3.2.5–3.2.7 that the characteristic function χ_A of any $A \in \hat{\mathbf{G}}^F$ can be written as an explicit linear combination of class functions of the form $R_{\mathbf{L}}^{\mathbf{G}}(\chi_{A_0})$, indexed by regular subgroups $\mathbf{L} \subseteq \mathbf{G}$ which are all conjugate to one another and F-stable cuspidal character sheaves $A_0 \in (\hat{\mathbf{L}}^\circ)^F$ (such that for each occurring $(\mathbf{L}, A_0)$, A is isomorphic to a simple summand of the complex obtained by inducing A_0 to $\mathbf{G}$). Note that, since $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\mathbf{G})^\circ$, we also have $\mathbf{Z}(\mathbf{L}) = \mathbf{Z}(\mathbf{L})^\circ$ for any Levi complement $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$, see [DM20, 11.2.1]. So in view of (b2) applied to all the $(\mathbf{L}, A_0)$ above, our hypothesis allows us to replace χ_{A_0} by an explicitly known multiple of the corresponding almost character, thus by an explicit linear combination of $\mathrm{Irr}(\mathbf{L}^F)$. So we obtain an explicit linear combination of χ_A in terms of various $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$, indexed by the same regular subgroups $\mathbf{L} \subseteq \mathbf{G}$ as before and $\chi \in \mathrm{Irr}(\mathbf{L}^F)$. Hence, the solution to (b1) would yield the decomposition of any χ_A ($A \in \hat{\mathbf{G}}^F$) into $\mathrm{Irr}(\mathbf{G}^F)$, that is, the desired base change between the two bases $\{\chi_A \mid A \in \hat{\mathbf{G}}^F\}$ and $\mathrm{Irr}(\mathbf{G}^F)$ of $\mathrm{CF}(\mathbf{G}^F)$. It remains to refer to hypothesis (a). (Of course, one would like to formulate a similar program for those $\mathbf{G}$ with a non-connected centre, but recall that in general the equations of the form (3.3.9.1) are only conjectural in this case, see 3.3.8.)

As for (b1), the special case where $\mathbf{L} = \mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ is dealt with by Theorem 2.2.21 (and is even known if $\mathbf{Z}(\mathbf{G})$ is not connected, due to [Lus88]). If

$\mathbf{L}$ is not a maximal torus of $\mathbf{G}$, the explicit decomposition of the $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$ into $\mathrm{Irr}(\mathbf{G}^F)$ (for $\chi \in \mathrm{Irr}(\mathbf{L}^F)$) is not yet established in complete generality but is known in numerous cases, especially as far as unipotent characters of $\mathbf{L}^F$ are concerned (see, e.g., Asai [Asa84a], [Asa84b], [Asa84c], [Asa85] and Broué–Malle–Michel [BMM93], as well as Shoji [Sho85], [Sho87b], which includes the consideration of non-unipotent characters). We refer to [GM20, §4.6] (and the further references there) for a detailed overview of the current state of knowledge. Once these gaps are closed, the goal of determining the character table of $\mathbf{G}^F$ will thus essentially be reduced to solving problem (b2), although we want to emphasise that there are several difficulties involved in order to concretely work out (a), one of them being the specification of the isomorphisms $\varphi_A \colon F^*A \xrightarrow{\sim} A$ for $A \in \hat{\mathbf{G}}^F$ in general.

On the other hand, it would certainly be desirable to be able to formulate an inductive condition for problem (b) itself. Such an approach is adopted by Lusztig [Lus86, §3], if only for those $A \in \hat{\mathbf{G}}^F$ whose support contains a unipotent element and under the assumptions that p is good, q satisfies certain congruence conditions, and that for any regular subgroup $\mathbf{L} \subseteq \mathbf{G}$ in question, F induces the identity map on $W_{\mathbf{G}}(\mathbf{L}) = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$: For any such $A \in \hat{\mathbf{G}}^F$, let $\mathbf{L} \subseteq \mathbf{G}$ be a regular subgroup such that A is a simple direct summand of $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$, where $A_0 \in (\hat{\mathbf{L}}^\circ)^F$. Then, using the concept of split unipotent elements (see 3.2.22(a)), there is a canonical choice for an isomorphism $\varphi_{A_0} \colon F^*A_0 \xrightarrow{\sim} A_0$ (see 3.2.20, 3.2.21), and φ_{A_0} naturally determines an isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$, see [Lus86, 3.2]. By [Lus86, 3.5], the determination of the scalars ξ_x in the equations (3.3.9.1) with respect to those $A \in \hat{\mathbf{G}}^F$ whose support contains a unipotent element is then reduced to considering cuspidal character sheaves; cf. 3.4.15–3.4.17 below, where we will follow Lusztig's approach to explain this reduction for the unipotent character sheaves on simple groups with a trivial centre and a non-twisted $\mathbb{F}_q$-rational structure.

In any case, we see that for both of these strategies towards determining the character table of $\mathbf{G}^F$, a crucial part consists in solving problem (b2) above, and this is what we will henceforth focus on (at least as far as unipotent character sheaves are concerned).

3.3.12. Consider the pair $(0, 1) \in \Lambda(\mathbf{G}, F)$, corresponding to the trivial local system $\mathscr{L}_0 = \overline{\mathbb{Q}}_\ell \in \mathscr{S}(\mathbf{T}_0)$. Thus,

$$\hat{\mathbf{G}}^{\mathrm{un}} = \hat{\mathbf{G}}_{\mathscr{L}_0} \subseteq \hat{\mathbf{G}}$$

are the unipotent character sheaves on $\mathbf{G}$, see Definition 3.1.8. As already observed in 2.2.24 and 3.1.4, we have

$$\mathbf{W}'_{\mathscr{L}_0} = \mathbf{W}_{0,1} = \mathscr{Z}_{0,1} = \mathbf{W}.$$

In particular, the element $w_1 \in \mathscr{Z}_{0,1}$ of minimal length is the identity element of $\mathbf{W}$, and

$\sigma_{0,1} = \sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ is the map induced by F. So by Proposition 3.3.6, there is a bijection

$$(\hat{\mathbf{G}}^{\mathrm{un}})^F = \hat{\mathbf{G}}^F_{\mathscr{L}_0} \xrightarrow{\sim} \overline{\mathfrak{X}}(\mathbf{W},\sigma), \quad A \mapsto \overline{x}_A, \tag{3.3.12.1}$$

such that for any $A \in (\hat{\mathbf{G}}^{\mathrm{un}})^F$ and any $\phi \in \mathrm{Irr}(\mathbf{W})^\sigma$, we have

$$(A : R_\phi^{\mathscr{L}_0}) = \hat{\varepsilon}_A\{\overline{x}_A, x_\phi\}.$$

As in 2.2.23, let us fix a set of representatives $\mathfrak{X}_0(0,1) \subseteq \mathfrak{X}(\mathbf{W},\sigma)$ for the $\mathcal{R}$-orbits in $\mathfrak{X}(\mathbf{W},\sigma)$. Thus, $\mathfrak{X}_0(0,1)$ is in natural bijection with $\overline{\mathfrak{X}}(\mathbf{W},\sigma)$, so we may label the elements of $(\hat{\mathbf{G}}^{\mathrm{un}})^F$ in terms of $\mathfrak{X}_0(0,1)$ and write

$$(\hat{\mathbf{G}}^{\mathrm{un}})^F = \{A_x \mid x \in \mathfrak{X}_0(0,1)\}.$$

In this setting and as far as unipotent character sheaves are concerned, the statement of Shoji's Theorem 3.3.9 is as follows: For any $x \in \mathfrak{X}_0(0,1)$ and any chosen isomorphism $\varphi_{A_x}\colon F^*A_x \xrightarrow{\sim} A_x$, there exists some scalar $\xi_x = \xi_x(\varphi_{A_x}) \in \overline{\mathbb{Q}}_\ell^\times$ such that

$$R_x = \xi_x \cdot \chi_{A_x,\varphi_{A_x}}. \tag{3.3.12.2}$$

Let us require in addition that $\varphi_{A_x}\colon F^*A_x \xrightarrow{\sim} A_x$ is as in 3.2.1($*$): By Theorem 3.2.2, we then have $\langle \chi_{A_x,\varphi_{A_x}}, \chi_{A_x,\varphi_{A_x}}\rangle_{\mathbf{G}^F} = 1$; since also $\langle R_x, R_x\rangle_{\mathbf{G}^F} = 1$, we get

$$|\xi_x(\varphi_{A_x})| = 1 \quad \text{whenever} \quad \varphi_{A_x}\colon F^*A_x \xrightarrow{\sim} A_x \text{ is as in 3.2.1}(*). \tag{3.3.12.3}$$

In what follows, we will consider various simple algebraic groups and try to determine the scalars ξ_x in (3.3.12.2) for $x \in \mathfrak{X}_0(0,1)$, after having fixed isomorphisms $\varphi_{A_x}\colon F^*A_x \xrightarrow{\sim} A_x$ which satisfy the condition ($*$) in 3.2.1.

3.3.13 Classical groups. (a) For any classical group $\mathbf{G}$ with a connected centre and with a Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$ such that $(\mathbf{G}, F)$ is non-twisted, the problem of determining the scalars ξ_x for $x \in \mathfrak{X}_0(0,1)$ in (3.3.12.2) has been completely solved by Shoji, see [Sho97, 6.2] and [Sho09, §6]. In fact, the latter even covers some non-unipotent character sheaves, but not all of them; we refer to [Sho09, 6.3, 6.4] for the precise statement. As for the unipotent character sheaves, note that the first thing to address when trying to solve equations like (3.3.12.2) is the specification of the isomorphisms $\varphi_{A_x}\colon F^*A_x \xrightarrow{\sim} A_x$ for $x \in \mathfrak{X}_0(0,1)$. As is shown in [Sho97], [Sho09], the problem of fixing these isomorphisms and determining the scalars ξ_x is reduced to considering cuspidal unipotent character sheaves on simple groups of classical type. So let $\mathbf{G}$ be a simple algebraic group of classical type, and let $A \in \hat{\mathbf{G}}^\circ \cap \hat{\mathbf{G}}^{\mathrm{un}}$; by Proposition 3.1.17, A is of the form

$$A \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E})[\dim \mathscr{C}]^{\#\mathbf{G}},$$

where $\mathscr{C} \subseteq \mathbf{G}$ is an F-stable conjugacy class and $\mathscr{E}$ is an F-stable cuspidal local system on $\mathscr{C}$. The possibilities for $(\mathscr{C}, \mathscr{E})$ can be extracted from [Lus84b] and [LuCS5, §22, §23]; see also [Sho97, 6.6], [Sho09, 3.1], or the appendix of [DLM14]. In particular, with our assumptions on $\mathbf{G}$ and A, we see that $\mathscr{E}$ is always one-dimensional. Recall from 3.2.21 that choosing an isomorphism $F^*A \xrightarrow{\sim} A$ is then equivalent to singling out a representative $g_0 \in \mathscr{C}^F$ (or, rather, its $\mathbf{G}^F$-conjugacy class). Hence, one is seeking to make a 'good' choice for (the $\mathbf{G}^F$-conjugacy class of) g_0, see 3.2.22, 3.2.23. In the case where $\mathscr{C}$ is a unipotent class of $\mathbf{G}$, the natural requirement on g_0 is that it is split unipotent — recall the definition of this notion from 3.2.22(a) in good characteristic; for classical groups, Shoji has introduced split unipotent elements in bad characteristic as well, see [Sho06b], [Sho07], [Sho22]. Now assume that $\mathscr{C}$ is not necessarily unipotent, and let $g_0 \in \mathscr{C}^F$, with Jordan decomposition $g_0 = s_0u_0 = u_0s_0$ ($s_0 \in \mathbf{G}^F$ semisimple, $u_0 \in C_{\mathbf{G}}(s_0)^F$ unipotent); one may thus require that u_0 is split unipotent in $C_{\mathbf{G}}(s_0) = C^\circ_{\mathbf{G}}(s_0)$, so it remains to specify s_0. This is discussed in detail in [Sho97, 6.6] and [Sho09, §3]; the isomorphism $F^*A \xrightarrow{\sim} A$ with respect to such a choice of g_0 is then defined as in 3.2.21.

(b) For classical groups with *disconnected* centre and especially as far as non-unipotent character sheaves are concerned, Theorem 3.3.9 is not applicable, and in fact one first needs to think of an appropriate generalisation of the definition of almost characters (cf. 3.3.8). This, and the determination of the scalars involved, is achieved in several cases (but not in complete generality), due to results of Waldspurger [Wal04] (for $\mathrm{Sp}_{2n}(q)$, and also for the disconnected orthogonal groups $\mathrm{O}_{2n}(q)$ with respect to an appropriate generalisation of Lusztig's Conjecture, in odd characteristic p and for q large enough), Bonnafé [Bon06] (for $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$, with q large enough) and Shoji [Sho06a] (for $\mathrm{SL}_n(q)$, with p large enough); see also the further references there.

3.4. Simple non-twisted groups with a trivial centre

Throughout this section, $\mathbf{G}$ denotes a *simple* algebraic group over the field $k = \overline{\mathbb{F}}_p$, defined over the finite subfield $\mathbb{F}_q \subseteq k$ where q is a power of p, with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. Let us fix an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ as well as an F-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ which is contained in $\mathbf{B}_0$, so that $\mathbf{W} := N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ is the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$. Thus, F induces an automorphism $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$. In this entire section, we make the following assumptions:

We have $\mathbf{Z}(\mathbf{G}) = \{1\}$ and $\sigma = \mathrm{id}_{\mathbf{W}}$, so that $(\mathbf{G}, F)$ is non-twisted.

In particular, the results stated in Section 3.3 (most notably Shoji's Theorem 3.3.9) can be applied. Let $\mathbf{U}_0 = R_{\mathrm{u}}(\mathbf{B}_0)$ be the unipotent radical of $\mathbf{B}_0$. Then $\mathbf{B}_0$ is the semidirect product of $\mathbf{U}_0$ and $\mathbf{T}_0$ (with $\mathbf{U}_0$ being normal in $\mathbf{B}_0$). Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum attached to $\mathbf{G}$ and $\mathbf{T}_0$ (so $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), with underlying bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$, see 2.1.4. Let $r = \operatorname{rank} \mathbf{G}$, $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subseteq R$ be the set of simple roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$ (as described in 2.1.9), $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subseteq R^\vee$ be the corresponding set of simple co-roots and $R^+ \subseteq R$ be the positive roots with respect to Π. Let $\mathfrak{C} = (\langle \alpha_j, \alpha_i^\vee \rangle)_{1 \leqslant i,j \leqslant r}$ be the associated Cartan matrix (see 2.1.5). For $1 \leqslant i \leqslant r$, let us set $s_i := w_{\alpha_i}$, with w_{α_i} being defined as in 2.1.2. Thus, viewing $\mathbf{W}$ as a subgroup of $\mathrm{Aut}(X)$ (as in 2.1.4), $\mathbf{W}$ is a Coxeter group with Coxeter generators $S = \{s_1, \ldots, s_r\}$ (see again 2.1.5).

Hence, if $J \subseteq S$ is a non-empty subset whose elements form a connected subgraph of the Dynkin diagram of $\mathbf{G}$, $\mathbf{L}'_J := \mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is a simple group with a trivial centre (see [DM20, 2.3.4]) and with Weyl group $\mathbf{W}_J$; furthermore, F naturally determines a Frobenius map $F'\colon \mathbf{L}'_J \to \mathbf{L}'_J$ (with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{L}'_J$) which induces the identity on $\mathbf{W}_J$, so $(\mathbf{L}'_J, F')$ also satisfies the assumptions that we imposed on $(\mathbf{G}, F)$ above. At least as far as unipotent (almost) characters and unipotent character sheaves on simple groups with a trivial centre and a non-twisted $\mathbb{F}_q$-rational structure are concerned, this allows a reduction of the problem of determining the scalars ξ_x in (3.3.12.2) to considering *cuspidal* unipotent character sheaves, see Corollary 3.4.17 below. As mentioned in 3.3.13, said problem has been completely solved by Shoji [Sho97], [Sho09] for all classical groups. In this thesis, we will thus consider the cuspidal unipotent character sheaves on the simple groups of exceptional type; see Chapter 4 below.

Parametrisation of unipotent characters and unipotent character sheaves

3.4.1. Recall from 2.2.12 that we have natural identifications

$$\overline{\mathfrak{X}}(\mathbf{W}, \mathrm{id}_{\mathbf{W}}) \cong \mathfrak{X}(\mathbf{W}) \quad \text{and} \quad \mathfrak{X}(\mathbf{W}, \mathrm{id}_{\mathbf{W}}) \cong \mathfrak{X}(\mathbf{W}) \times \mathcal{R},$$

with $\mathfrak{X}(\mathbf{W})$ defined as in 2.2.8. In particular, in view of Theorem 3.3.2 and 3.3.12, we have $\hat{\mathbf{G}}^{\mathrm{un}} = (\hat{\mathbf{G}}^{\mathrm{un}})^F$, that is, every unipotent character sheaf on $\mathbf{G}$ is automatically F-stable. We may thus label the elements of $\hat{\mathbf{G}}^{\mathrm{un}}$ in terms of $\mathfrak{X}(\mathbf{W})$ and rewrite the bijection (3.3.12.1) as

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}, \quad x \mapsto A_x, \tag{3.4.1.1}$$

such that

$$(A_x : R_\phi^{\mathscr{L}_0}) = \hat{\varepsilon}_{A_x}\{x, x_\phi\} \quad \text{for any } x \in \mathfrak{X}(\mathbf{W}) \text{ and any } \phi \in \mathrm{Irr}(\mathbf{W}). \tag{3.4.1.2}$$

On the other hand, by Theorem 2.2.21 (see Example 2.2.24), there is a bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho, \tag{3.4.1.3}$$

such that

$$\langle \rho, R_\phi \rangle_{\mathbf{G}^F} = \Delta(x_\rho)\{x_\rho, x_\phi\} \quad \text{for any } \rho \in \mathrm{Uch}(\mathbf{G}^F) \text{ and any } \phi \in \mathrm{Irr}(\mathbf{W}). \tag{3.4.1.4}$$

(Note that, since we assumed $\sigma = \mathrm{id}_{\mathbf{W}}$, the unique σ-extension of ϕ to $\mathbf{W}(\sigma) = \mathbf{W}$ is ϕ itself, x_ϕ is the image of ϕ under (2.2.8.4) with $W = \mathbf{W}$, and $\{\ ,\ \}\colon \mathfrak{X}(\mathbf{W}) \times \mathfrak{X}(\mathbf{W}) \to \overline{\mathbb{Q}}_\ell$ is the pairing defined in 2.2.8.) In general, as mentioned in Remark 2.2.22 and Remark 3.3.7, the bijections (3.4.1.1) and (3.4.1.3) are not uniquely determined by the conditions (3.4.1.2) and (3.4.1.4), respectively. Our first aim in this section is to formulate additional requirements which uniquely specify the bijections (3.4.1.1) and (3.4.1.3).

3.4.2. We want to describe the parametrisations of $\mathrm{Uch}(\mathbf{G}^F)$ and $\hat{\mathbf{G}}^{\mathrm{un}}$ in terms of Harish-Chandra series (cf. 2.2.6 and 3.1.21, 3.1.22, respectively). To this end, we need suitable parameter sets associated to various Coxeter groups. Their definition is mostly due to Lusztig [Lus15, §3], although we find it more convenient to use the scheme in [Gec18, §4], which is very similar but avoids the 'doubling' of certain elements in the sets defined below. Let W be the Weyl group of a root system, and let (W, S) be the corresponding Coxeter system. Following [Gec18, 4.4] (cf. [Lus15, 3.1]), we attach a certain finite (possibly empty) set $\mathfrak{S}_W^\circ$ to (W, S). If $W = \{1\}$, we set $\mathfrak{S}_W^\circ := \{(1,1)\}$. Assume now that (W, S) is irreducible, and let $n = |S|$. Let us make the following specification, to which we will refer in numerous places later.

We fix primitive roots of unity $\omega, \mathrm{i}, \zeta_5 \in \mathcal{R}$ of order $3, 4, 5$, respectively.

- (W, S) of type A_n $(n \geqslant 1)$: $\mathfrak{S}_W^\circ := \varnothing$.

- (W,S) of type B_n or C_n $(n \geqslant 2)$:

$$\mathfrak{S}^\circ_W := \begin{cases} \{((-1)^{n/2}, 2^l)\} & \text{if } n = l^2 + l \text{ for some } l \in \mathbb{N}, \\ \varnothing & \text{otherwise.} \end{cases}$$

- (W,S) of type D_n $(n \geqslant 4)$:

$$\mathfrak{S}^\circ_W := \begin{cases} \{((-1)^{n/4}, 2^{2l-1})\} & \text{if } n = 4l^2 \text{ for some } l \in \mathbb{N}, \\ \varnothing & \text{otherwise.} \end{cases}$$

- (W,S) of type G_2: $\mathfrak{S}^\circ_W := \{(1,6), (-1,2), (\omega,3), (\omega^2,3)\}$.
- (W,S) of type F_4: $\mathfrak{S}^\circ_W := \{(1,8), (1,24), (-1,4), (\mathrm{i},4), (-\mathrm{i},4), (\omega,3), (\omega^2,3)\}$.
- (W,S) of type E_6: $\mathfrak{S}^\circ_W := \{(\omega,3), (\omega^2,3)\}$.
- (W,S) of type E_7: $\mathfrak{S}^\circ_W := \{(\mathrm{i},2), (-\mathrm{i},2)\}$.
- (W,S) of type E_8:

$$\begin{aligned}\mathfrak{S}^\circ_W := \{&(1,8), (1,120), (-1,12), (\mathrm{i},4), (-\mathrm{i},4), (\omega,6), (-\omega,6), (\omega^2,6), (-\omega^2,6), \\ &(\zeta_5,5), (\zeta_5^2,5), (\zeta_5^3,5), (\zeta_5^4,5)\}.\end{aligned}$$

If W is reducible, let us write $W = W_1 \times \ldots \times W_m$ where $m \geqslant 2$ and where each W_j is the Weyl group of an irreducible root system, so that if $S_j \subseteq W_j$ are the corresponding simple reflections, (W_j, S_j) is an irreducible Coxeter system for $1 \leqslant j \leqslant m$. We then set

$$\mathfrak{S}^\circ_W := \mathfrak{S}^\circ_{W_1} \times \ldots \times \mathfrak{S}^\circ_{W_m}.$$

Next, we define

$$\mathfrak{S}_W := \left\{(J, \epsilon, \mathfrak{s}) \mid J \subseteq S,\, \epsilon \in \mathrm{Irr}(W^{S/J}),\, \mathfrak{s} \in \mathfrak{S}^\circ_{W_J}\right\}$$

where $W_J := \langle J \rangle \subseteq W$ and $W^{S/J}$ is the subgroup of W generated by the involutions

$$\sigma_s := w_0^{J\cup\{s\}} w_0^J = w_0^J w_0^{J\cup\{s\}} \quad \text{for } s \in S \setminus J.$$

(Recall from 2.1.20 that for any subset $J' \subseteq S$, we denote by $w_0^{J'}$ the longest element in $W_{J'}$.) If (W,S) is irreducible or $\{1\}$ and if $J \subseteq S$ is such that $\mathfrak{S}^\circ_{W_J} \neq \varnothing$, then (W_J, J) is irreducible or $\{1\}$, and we have $w_0^{J'} \cdot J \cdot w_0^{J'} = J$ for all $J \subseteq J' \subseteq S$, so by [Lus76b, §5], $\left(W^{S/J}, \{\sigma_s \mid s \in S \setminus J\}\right)$ is a Coxeter system as well. We identify $\mathfrak{S}_W^\circ$ with a subset of $\mathfrak{S}_W$ via the embedding

$$\mathfrak{S}_W^\circ \hookrightarrow \mathfrak{S}_W, \quad \mathfrak{s} \mapsto (S, 1, \mathfrak{s}). \tag{3.4.2.1}$$

(Note that $W^{S/S} = \{1\}$.) At the other extreme, considering $J = \varnothing$, we have $W^{S/\varnothing} = W$ and $\mathfrak{S}_{W_\varnothing}^\circ = \mathfrak{S}_{\{1\}}^\circ = \{(1,1)\}$, so we obtain an embedding

$$\mathrm{Irr}(W) \hookrightarrow \mathfrak{S}_W, \quad \phi \mapsto (\varnothing, \phi, (1,1)). \tag{3.4.2.2}$$

There is one further notion which we shall need below (cf. [DM90, §4, §6]). Assume that the Weyl group W is irreducible, and let $\mathcal{F} \subseteq \mathrm{Irr}(W)$ be a family (see 2.2.8). We say that $\mathcal{F}$ is *exceptional* in either of the following cases:

- W is of type E_7, and $\mathcal{F}$ is the (unique) family which contains the two irreducible characters of degree 512;
- W is of type E_8, and $\mathcal{F}$ is one of the two families which contain (two) irreducible characters of degree 4096.

Any other family $\mathcal{F} \subseteq \mathrm{Irr}(W)$ is called *non-exceptional*. Now let $x \in \mathfrak{X}(W)$, and let $\mathcal{F} \subseteq \mathrm{Irr}(W)$ be the family such that $x = (g, \tau) \in \mathfrak{M}(\mathcal{G}_\mathcal{F})$. If $\mathcal{F}$ is exceptional and if x is not in the image of the embedding $\mathcal{F} \hookrightarrow \mathfrak{M}(\mathcal{G}_\mathcal{F})$, we set

$$\tilde{\lambda}_x := \mathrm{i} \cdot \tfrac{\tau(g)}{\tau(1)}$$

where $\mathrm{i} \in \mathcal{R}$ is the primitive 4th root of unity that we fixed above. (In fact, we always have $\tau(1) = 1$ in this case.) In any other case, we set

$$\tilde{\lambda}_x := \tfrac{\tau(g)}{\tau(1)}.$$

3.4.3. We will require some numerical invariants attached to unipotent characters of $\mathbf{G}^F$ and also to unipotent character sheaves on $\mathbf{G}$; since this section is the only place where we explicitly need to deal with them, we merely give a brief description here which fits for the present situation.

(a) Let $w \in \mathbf{W}$. Following [DL76, §1] (see also [Gec18, 4.3]), we consider the variety

$$X_w = \{g\mathbf{B}_0 \in {}^{\mathbf{G}}/_{\mathbf{B}_0} \mid g^{-1}F(g) \in \mathbf{B}_0 w \mathbf{B}_0\}.$$

Clearly, $\mathbf{G}^F$ acts on X_w by left multiplication, so we obtain an induced $\mathbf{G}^F$-module structure on $H^i_c(X_w)$ (see [Car85, 7.1.3]). Using further properties of the ℓ-adic cohomology groups with compact support, one shows that the virtual character $R_w \in \mathrm{CF}(\mathbf{G}^F)$ (see Definition 2.2.3) can also be written as

$$R_w \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell, \quad g \mapsto \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Trace}(g, H^i_c(X_w)),$$

see [Car85, 7.2.3, 7.7.11]. (This is actually the original definition of R_w in [DL76, §1].) By definition, any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ appears as a constituent of R_w for some $w \in \mathbf{W}$, and therefore of the character of the $\mathbf{G}^F$-module $H^i_c(X_w)$ for some $w \in \mathbf{W}$ and some $i \in \mathbb{Z}$. Now note that, due to our assumption on F in this section, we have $F(X_w) = X_w$, and so F induces a linear map on $H^i_c(X_w)$ which commutes with the $\mathbf{G}^F$-action. Hence, if $\mu \in \overline{\mathbb{Q}}_\ell$ is an eigenvalue of F on $H^i_c(X_w)$, the generalised eigenspace $H^i_c(X_w)_\mu$ with respect to (F, μ) is a $\mathbf{G}^F$-module. So for any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, there exist $w \in \mathbf{W}$, $i \in \mathbb{Z}$ and $\mu \in \overline{\mathbb{Q}}_\ell$ such that ρ is a constituent of the character of $H^i_c(X_w)_\mu$. While the triple (w, i, μ) is not uniquely determined by this property, there is a root of unity $\lambda_\rho \in \mathcal{R}$ such that $\mu = \lambda_\rho q^{m/2}$ for some $m \in \mathbb{Z}$, and λ_ρ is independent of (w, i, μ) above (it only depends on ρ and possibly the choice (1.1.0.2) of a square root of q), see [Lus78, 3.9] and [DM85, III.2.3]. This root of unity $\lambda_\rho \in \mathcal{R}$ is called the *Frobenius eigenvalue* of ρ.

(b) Let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. One can attach to ρ its *degree polynomial* $\mathbb{D}_\rho$. Since we will not need it explicitly here, we only refer to [Gec18, 3.3] (see also [GM20, 2.3.25]) for the definition of $\mathbb{D}_\rho$. This is a polynomial in $\mathbb{Q}[\mathbf{q}]$ (where $\mathbf{q}$ is an indeterminate over $\mathbb{Q}$) whose evaluation at q gives the degree $\rho(1)$ of ρ. Then we define $n_\rho \in \mathbb{N}$ to be the smallest natural number such that $n_\rho \mathbb{D}_\rho \in \mathbb{Z}[\mathbf{q}]$.

(c) Let $g \in \mathbf{G}^F$, and let us write $g = x^{-1}F(x)$ where $x \in \mathbf{G}$ (see Theorem 2.1.14). Then, clearly, we also have $F(x)x^{-1} \in \mathbf{G}^F$, and assigning to the $\mathbf{G}^F$-conjugacy class of $x^{-1}F(x)$ the $\mathbf{G}^F$-conjugacy class of $F(x)x^{-1}$ gives rise to a well-defined bijection

$$t_1 \colon \mathbf{G}^F/_\sim \xrightarrow{1-1} \mathbf{G}^F/_\sim$$

(where we denote by $\mathbf{G}^F/_\sim$ the set of conjugacy classes of $\mathbf{G}^F$), as in [Sho95a, 1.16]. Thus, we obtain the transposed map

$$t_1^* \colon \mathrm{CF}(\mathbf{G}^F) \xrightarrow{1-1} \mathrm{CF}(\mathbf{G}^F), \quad f \mapsto f \circ t_1,$$

which Shoji calls the 'twisting operator' on $\mathbf{G}^F$. Now let $A \in \hat{\mathbf{G}}^{\mathrm{un}} = (\hat{\mathbf{G}}^{\mathrm{un}})^F$, and let us choose an isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$. Then by [Sho95a, 3.3] (see also [Eft94]), there exists a root of unity $\lambda_A \in \mathcal{R}$ such that

$$t_1^*(\chi_{A,\varphi_A}) = \lambda_A \chi_{A,\varphi_A}.$$

Clearly, λ_A is independent of the choice of φ_A. Note that this definition of λ_A coincides with the one in [Lus15, 3.6], see [Eft94].

3.4.4. With the notation of 3.4.2 and 3.4.3, we can now describe the parametrisation of $\mathrm{Uch}(\mathbf{G}^F)$ via the set $\mathfrak{S}_{\mathbf{W}}$, following Lusztig [Lus15, §3] (see also [Gec18, §4]). Let us first focus on the *cuspidal* unipotent characters of $\mathbf{G}^F$. Similarly to the notation $\mathrm{Irr}(\mathbf{G}^F)^\circ \subseteq \mathrm{Irr}(\mathbf{G}^F)$, we set

$$\mathrm{Uch}(\mathbf{G}^F)^\circ := \mathrm{Irr}(\mathbf{G}^F)^\circ \cap \mathrm{Uch}(\mathbf{G}^F).$$

The definition of the set $\mathfrak{S}_{\mathbf{W}}^\circ$ is designed as to give rise to a bijection

$$\mathrm{Uch}(\mathbf{G}^F)^\circ \xrightarrow{\sim} \mathfrak{S}_{\mathbf{W}}^\circ, \quad \rho \mapsto (\lambda_\rho, n_\rho). \tag{3.4.4.1}$$

In fact, this as well as all other results in [Lus15, §3] hold whenever $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ is simple or $\{1\}$ (which is slightly more general than our assumption on $\mathbf{G}$ in this section, as it includes the case where $\mathbf{G}$ is a torus, for example), and we will use this below as far as subgroups of $\mathbf{G}$ corresponding to connected subgraphs of the Dynkin diagram of $\mathbf{G}$ are concerned.

Now let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. By [Lus78, 3.25] (cf. 2.2.6), there exist a (σ-stable) subset $J \subseteq S$ and a cuspidal unipotent character $\rho_0 \in \mathrm{Uch}(\mathbf{L}_J^F)^\circ$ such that $\langle \rho, R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)\rangle_{\mathbf{G}^F} \neq 0$. In general, this merely determines $(\mathbf{L}_J, \rho_0)$ up to conjugation with an element of $\mathbf{W}$, but in the present situation it follows from the case-by-case analysis in [Lus78, 3.25] that $(\mathbf{L}_J, \rho_0)$ is the only element in its $\mathbf{W}$-conjugacy class, so ρ uniquely determines the pair $(\mathbf{L}_J, \rho_0)$. This also shows that the stabiliser $\mathbf{W}(J, \rho_0)$ of ρ_0 in $N_{\mathbf{G}}(\mathbf{L}_J)^F/\mathbf{L}_J^F$ under the action (2.2.6.1) is the full group $N_{\mathbf{G}}(\mathbf{L}_J)^F/\mathbf{L}_J^F$. One can thus extract from the discussion in [Lus84a, 8.5] that $\mathbf{W}(J, \rho_0) = N_{\mathbf{G}}(\mathbf{L}_J)^F/\mathbf{L}_J^F$ can be canonically identified with the Coxeter group $\mathbf{W}^{S/J}$ defined above. Hence, in view of (2.3.10.1) (and again depending on our choice (1.1.0.2) of a square root of p), the irreducible characters of $\mathbf{W}^{S/J}$ parametrise the irreducible characters of $\mathbf{G}^F$ which appear as constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$. (The corresponding specialisation of the generic Iwahori–Hecke algebra associated to $\mathbf{W}^{S/J}$ can be read off from the last column of the table in [Lus78, p. 35], due to the remarks in [Lus84a, 8.2].) Using the notation of (2.3.10.1), we thus have $\rho = \rho_0[\epsilon]$ for some $\epsilon \in \mathrm{Irr}(\mathbf{W}^{S/J})$. Now $\mathbf{W}_J$ is the Weyl group of $\mathbf{L}_J$, and $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple or $\{1\}$

(this is also part of the discussion in [Lus78, 3.25]), so we can apply (3.4.4.1) to $(\mathbf{L}_J, \mathbf{W}_J)$ as well. Hence, the assignment $\rho \mapsto (J, \epsilon, (\lambda_{\rho_0}, n_{\rho_0}))$ gives a bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{S}_{\mathbf{W}}.$$

In particular, identifying $\mathfrak{S}^\circ_{\mathbf{W}} \subseteq \mathfrak{S}_{\mathbf{W}}$ via (3.4.2.1), the restriction of this bijection to $\mathrm{Uch}(\mathbf{G}^F)^\circ$ yields the bijection (3.4.4.1).

By collecting several results of [Lus84a], we can now make a definite choice for the bijection (3.4.1.3). (This is a special case of [DM90, 6.4], where no assumptions on σ and the centre of $\mathbf{G}$ are made.)

Proposition 3.4.5 (cf. [DM90, 6.4])**.** *There is a bijection*

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho,$$

which satisfies the following conditions:

(i) Property (3.4.1.4) *holds;*

(ii) $\lambda_\rho = \tilde{\lambda}_{x_\rho}$ *for all* $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ *(with* $\tilde{\lambda}_{x_\rho}$ *as defined in 3.4.2);*

(iii) $x_\phi = x_{[\phi]}$ *for any* $\phi \in \mathrm{Irr}(\mathbf{W})$ *(where* $[\phi] \in \mathrm{Uch}(\mathbf{G}^F)$ *is as in* 2.3.9*, see also Remark* 2.3.10*).*

Moreover, the bijection $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W})$ *above is uniquely determined by* (i)*,* (ii)*,* (iii)*.*

Proof. As noted in 3.4.1 (see Example 2.2.24), there exists a bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho,$$

which satisfies (i). By [Lus84a, 11.2], this bijection can in fact be chosen so that (ii) holds as well. (The exceptions mentioned in loc. cit. are taken care of by the slight modification of the definition of $\tilde{\lambda}_x$ compared to [Lus84a, 11.1], as in [DM90, p. 135].) Next, by [Lus84a, 12.6], we have $x_\phi = x_{[\phi]}$ for all $\phi \in \mathrm{Irr}(\mathbf{W})$ which do not lie in an exceptional family. So it remains to consider the cases where $\phi \in \mathrm{Irr}(\mathbf{W})$ is a character of degree 512 (if $\mathbf{W}$ is of type E_7) or of degree 4096 (if $\mathbf{W}$ is of type E_8). We first have to show that x_ϕ and $x_{[\phi]}$ are actually in the same $\mathfrak{M}(\mathcal{G}_\mathcal{F})$: If $\mathbf{W}$ is of type E_7, there are exactly two $\phi \in \mathrm{Irr}(\mathbf{W})$ of degree 512, and they lie in the same family $\mathcal{F}$. Since we already know that $x_{\phi'} = x_{[\phi']}$ for any other $\phi' \in \mathrm{Irr}(\mathbf{W})$, we necessarily have $x_\phi, x_{[\phi]} \in \mathfrak{M}(\mathcal{G}_\mathcal{F})$ for both $\phi \in \mathrm{Irr}(\mathbf{W})$ of degree 512. If $\mathbf{W}$ is of type E_8, there are two families in $\mathrm{Irr}(\mathbf{W})$

which contain characters of degree 4096. Comparing the tables in [Lus84a, p. 105, p. 368] (specifically the 'a-function', as defined in [Lus84a, (4.1.1)]), we see that x_ϕ and $x_{[\phi]}$ are in the same $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ in this case as well. So let us consider any $\phi \in \mathrm{Irr}(\mathbf{W})$ which is of degree 512 (in type E_7) or of degree 4096 (in type E_8). Thus, ϕ lies in a family $\mathcal{F} \subseteq \mathrm{Irr}(\mathbf{W})$ consisting of two characters with the same degree, $\mathcal{G}_\mathcal{F} = C_2$ is the cyclic group of order 2, and the image of the embedding $\mathcal{F} \hookrightarrow \mathfrak{M}(\mathfrak{S}_2)$ is $\{(1,1),(1,\varepsilon)\}$ (where ε denotes the non-trivial character of C_2). We see from the explicit case-by-case description in [Lus80] (or also once again from the tables in the appendix of [Lus84a]) that the other two elements of $\mathfrak{M}(\mathfrak{S}_2)$ parametrise unipotent characters of $\mathbf{G}^F$ which are not in the principal Harish-Chandra series, so $[\phi] \in \mathrm{Uch}(\mathbf{G}^F)$ must correspond to one of the labels $(1,1)$, $(1,\varepsilon)$. Hence, considering the two $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ parametrised by the above two x_ϕ in our bijection $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W})$, we may (if necessary) just switch the assignment

$$\rho \mapsto x_\rho \in \{(1,1),(1,\varepsilon)\} \subseteq \mathfrak{M}(\mathcal{G}_\mathcal{F})$$

in order to achieve $x_\phi = x_{[\phi]}$ — note that this does not violate condition (i), as we see from an inspection of the 2×4 submatrix corresponding to $\mathcal{F} \times \mathfrak{M}(\mathcal{G}_\mathcal{F})$ of the Fourier block $\Upsilon_\mathcal{F}$; it is also clear that (ii) remains true since $\lambda_{[\phi]} = 1$ for all of the ϕ considered above. This proves the existence of a bijection as in the proposition.

To show the uniqueness, we first recall that any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ appears as a constituent of the almost character R_ϕ for some $\phi \in \mathrm{Irr}(\mathbf{W})$, so the family $\mathcal{F} \subseteq \mathrm{Irr}(\mathbf{W})$ for which $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ contains x_ρ is certainly independent of a chosen bijection $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W})$ which satisfies (i). Then [DM90, 6.3] shows that for any non-exceptional family $\mathcal{F}$ for which $\mathcal{G}_\mathcal{F}$ is neither $\mathfrak{S}_3$ nor $\mathfrak{S}_4$, the map

$$\mathfrak{M}(\mathcal{G}_\mathcal{F}) \hookrightarrow \mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \mathrm{Uch}(\mathbf{G}^F)$$

is uniquely determined by condition (i) already. In particular, this covers all classical groups. As far as exceptional families $\mathcal{F}$ are concerned, the uniqueness with respect to the principal series unipotent characters is contained in the proof of the existence part above, in view of condition (iii), while the characters corresponding to the other two elements of $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ have different Frobenius eigenvalues, so they are distinguished by (ii). Finally, if $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ is $\mathfrak{S}_3$ or $\mathfrak{S}_4$, the uniqueness of the parametrisation with respect to labels of $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ is easily verified using (i) and (ii), see also the proof of [DM90, 6.4]. □

Remark 3.4.6. As mentioned in the proof of Proposition 3.4.5, if $\mathbf{G}$ is of classical type, the bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho,$$

is already uniquely determined by (3.4.1.4). In the case of an exceptional group $\mathbf{G}$, the unique bijection which satisfies conditions (i), (ii) and (iii) in Proposition 3.4.5 can be read off from the tables in the appendix of [Lus84a] (assuming that the choice of all the primitive roots of unity involved in loc. cit. is the same as the one in 3.4.2), and we will often just refer to these tables in what follows.

3.4.7. Now we turn to the unipotent character sheaves on $\mathbf{G}$. Let us denote by

$$\hat{\mathbf{G}}^{\circ,\mathrm{un}} := \hat{\mathbf{G}}^\circ \cap \hat{\mathbf{G}}^{\mathrm{un}}$$

the set of cuspidal unipotent character sheaves. By [Lus15, 3.7], there is a bijection

$$\mathfrak{S}_{\mathbf{W}}^\circ \xrightarrow{\sim} \hat{\mathbf{G}}^{\circ,\mathrm{un}}. \tag{3.4.7.1}$$

(Again, such a bijection exists whenever $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ is simple or $\{1\}$.) Comparing our definition of the sets $\mathfrak{S}_{\mathbf{W}}^\circ$ with Lusztig's [Lus15, 3.1], we see that if $A \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ corresponds to $\mathfrak{s} \in \mathfrak{S}_{\mathbf{W}}^\circ$ under (3.4.7.1), the first component of $\mathfrak{s}$ is equal to λ_A (as defined in 3.4.3(c)). However, we also see that if $\mathbf{W}$ is of type F_4 or E_8, there are (in either of these cases) two cuspidal character sheaves $A \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ for which $\lambda_A = 1$, so the definition of λ_A alone is not quite sufficient to determine the bijection (3.4.7.1). In [Lus15, §3], this issue is resolved by considering the multiplicities of a given $A \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ in ${}^pH^i(K_w^{\mathscr{L}_0})$ ($i \in \mathbb{Z}$) for a suitably defined $w \in \mathbf{W}$. We will implicitly use this in Corollary 3.4.8 below by referring to a consequence [Lus15, 3.10] of Lusztig's explicit choice for the bijection (3.4.7.1) — for now we just refer to [Lus15, 3.7].

Now let $A \in \hat{\mathbf{G}}^{\mathrm{un}}$. So there exist $J \subseteq S$ and $A_0 \in \hat{\mathbf{L}}_J^{\circ,\mathrm{un}}$ such that A is a simple direct summand of $\mathrm{ind}_{\mathbf{L}_J}^{\mathbf{G}}(A_0)$ (see 3.1.22). In particular, $\mathfrak{S}_{\mathbf{W}_J}^\circ$ is non-empty, so as noted in 3.4.2, $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ must be simple or $\{1\}$. Let $(\Sigma, \mathscr{E})$ be the cuspidal pair for $\mathbf{L}_J$ corresponding to A_0 (via Proposition 3.1.17), and recall the definition of $\mathscr{W}_{\mathbf{L}_J,\Sigma}^{\mathscr{E}}$ in 3.2.4. We claim that

$$\mathscr{W}_{\mathbf{L}_J,\Sigma}^{\mathscr{E}} \cong \mathbf{W}^{S/J}.$$

Indeed, this is clear if $J = S$, so we can assume that $J \subsetneq S$. In view of the discussion in 3.4.4, proving our assertion is then equivalent to showing that $\mathscr{W}_{\mathbf{L}_J,\Sigma}^{\mathscr{E}}$ is isomorphic to the stabiliser $\mathbf{W}(J, \rho_0)$ of some (any) $\rho_0 \in \mathrm{Uch}(\mathbf{L}_J^F)^\circ$. This follows from a general argument if $|\hat{\mathbf{L}}_J^{\circ,\mathrm{un}}| = 1$, see [Sho95a, (5.16.1)]. (Note that this includes the case where $J = \varnothing$, as the unique cuspidal unipotent character sheaf on $\mathbf{L}_\varnothing = \mathbf{T}_0$ is $\mathscr{L}_0[\dim \mathbf{T}_0]$.) Since $J \subsetneq S$ and $(\mathbf{W}_J, J)$ is irreducible or $\{1\}$, we can apply (3.4.7.1) to $(\mathbf{L}_J, \mathbf{W}_J)$, and we see from the possibilities for the set $\mathfrak{S}^\circ_{\mathbf{W}_J}$ that $|\hat{\mathbf{L}}_J^{\circ,\mathrm{un}}| > 1$ can only happen when $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type E_6 or E_7. Then it is shown in [Sho95b, 4.2] that in either of these cases, we have $\mathscr{W}^{\mathscr{E}}_{\mathbf{L}_J,\Sigma} \cong \mathbf{W}(J, \rho_0)$ for both of the cuspidal unipotent characters $\rho_0 \in \mathrm{Uch}(\mathbf{L}_J^F)^\circ$, which verifies our claim. So by [Sho95a, 5.9], there is an algebra isomorphism

$$\mathrm{End}_{\mathscr{M}\mathbf{G}}(\mathrm{ind}^{\mathbf{G}}_{\mathbf{L}_J}(A_0)) \cong \overline{\mathbb{Q}}_\ell[\mathbf{W}^{S/J}]. \tag{3.4.7.2}$$

Hence, among the simple direct summands of $\mathrm{ind}^{\mathbf{G}}_{\mathbf{L}_J}(A_0)$ up to isomorphism, A is parametrised by some $\epsilon \in \mathrm{Irr}(\mathbf{W}^{S/J})$. Let $\mathfrak{s}_{A_0} \in \mathfrak{S}^\circ_{\mathbf{W}_J}$ be the element corresponding to A_0 under (3.4.7.1) (again applied to $(\mathbf{L}_J, \mathbf{W}_J)$). Then the assignment $(J, \epsilon, \mathfrak{s}_{A_0}) \mapsto A$ gives a well-defined bijection

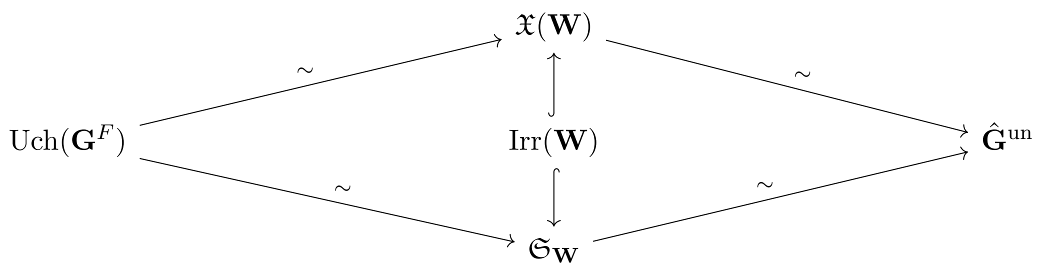
$$\mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}.$$

In particular, identifying $\mathfrak{S}^\circ_{\mathbf{W}} \subseteq \mathfrak{S}_{\mathbf{W}}$ via (3.4.2.1), the restriction of this bijection to $\mathfrak{S}^\circ_{\mathbf{W}}$ yields the bijection (3.4.7.1).

Corollary 3.4.8. *Let* $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W})$, $\rho \mapsto x_\rho$, *be the unique bijection specified through Proposition* 3.4.5*. Let* $\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{S}_{\mathbf{W}}$ *be the bijection defined in* 3.4.4*, and let* $\mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ *be the bijection defined in* 3.4.7*. Then there exists a unique bijection*

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$$

which satisfies the condition (3.4.1.2) *and makes the following diagram commutative (where* $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{X}(\mathbf{W})$ *is the embedding* (2.2.8.4) *and* $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_{\mathbf{W}}$ *is the embedding* (3.4.2.2)*, with* $W = \mathbf{W}$ *in both cases).*



Proof. Let $\phi \in \mathrm{Irr}(\mathbf{W})$. From 3.4.4, we see that the triple $(\varnothing, \phi, (1,1)) \in \mathfrak{S}_{\mathbf{W}}$ corresponds to the unipotent principal series character $[\phi]$ (see 2.3.9, Remark 2.3.10), which by condition (iii) in Proposition 3.4.5 is parametrised by $x_\phi \in \mathfrak{X}(\mathbf{W})$. Thus, the left triangle of the diagram in the corollary is commutative, so it is clear that there is a unique bijection $\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ which makes the full diagram commutative. Now let $A \in \hat{\mathbf{G}}^{\mathrm{un}}$, and let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ be the corresponding unipotent character under

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}.$$

Recall the definitions of $R_\phi^{\mathscr{L}_0}$ and the pairing $(:)$ in 3.3.1. By [Lus15, 3.10], we have

$$\langle \rho, R_w \rangle_{\mathbf{G}^F} = \sum_{i \in \mathbb{Z}} (-1)^{i+\dim \mathbf{G}} (A : {}^pH^i(K_w^{\mathscr{L}_0})).$$

Hence, we get

$$\begin{aligned}(A : R_\phi^{\mathscr{L}_0}) &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w^{-1}) \sum_{i \in \mathbb{Z}} (-1)^{i+\dim \mathbf{G}} (A : {}^pH^i(K_w^{\mathscr{L}_0})) \\ &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w^{-1}) \langle \rho, R_w \rangle_{\mathbf{G}^F} = \langle \rho, R_\phi \rangle_{\mathbf{G}^F} = \Delta(x_\rho)\{x_\rho, x_\phi\}.\end{aligned}$$

So in order to show that (3.4.1.2) holds for the bijection $\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$, we are reduced to proving that $\Delta(x_\rho) = \hat{\varepsilon}_A$ whenever $A \in \hat{\mathbf{G}}^{\mathrm{un}}$ corresponds to $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ as above. Let $(\mathbf{L}_J, \epsilon, \mathfrak{s}) \in \mathfrak{S}_{\mathbf{W}}$ be the triple which parametrises A (and ρ). Thus, A is a simple direct summand of $\mathrm{ind}_{\mathbf{L}_J}^{\mathbf{G}}(A_0)$, where $A_0 \in \hat{\mathbf{L}}_J^{\circ,\mathrm{un}}$. Let $(\Sigma, \mathscr{E})$ be the cuspidal pair for $\mathbf{L}_J$ corresponding to A_0 (via Proposition 3.1.17), so $\Sigma = \mathscr{C}.\mathbf{Z}(\mathbf{L}_J)$ for some conjugacy class $\mathscr{C}$ of $\mathbf{L}_J$ (note that $\mathbf{Z}(\mathbf{L}_J) = \mathbf{Z}(\mathbf{L}_J)^\circ$ here). By 3.1.15, 3.1.21, we have

$$\begin{aligned}\dim \operatorname{supp} A = \dim Y_{(\mathbf{L}_J, \Sigma)} &= |R_{\mathbf{G}}| - |R_{\mathbf{L}_J}| + \dim \Sigma \\ &= \dim \mathbf{G} - \dim \mathbf{L}_J + \dim \mathscr{C} + \dim \mathbf{Z}(\mathbf{L}_J),\end{aligned}$$

so

$$\dim \mathbf{G} - \dim \operatorname{supp} A = \dim \mathbf{L}_J - \dim \mathscr{C} - \dim \mathbf{Z}(\mathbf{L}_J).$$

Let us fix any element $x \in \mathscr{C}$. Then we get

$$\dim \mathbf{G} - \dim \operatorname{supp} A = \dim C_{\mathbf{L}_J}(x) - \dim \mathbf{Z}(\mathbf{L}_J)$$

and, hence,

$$\hat{\varepsilon}_A = (-1)^{\dim \mathbf{G} - \dim \operatorname{supp} A} = (-1)^{\dim C_{\mathbf{L}_J}(x) - \dim \mathbf{Z}(\mathbf{L}_J)}.$$

By [Spa82, II.2.8], we have

$$\dim C_{\mathbf{L}_J}(x) = \operatorname{rank} \mathbf{L}_J + 2 \dim \mathscr{B}_x^{\mathbf{L}_J},$$

where $\mathscr{B}_x^{\mathbf{L}_J}$ denotes the variety of Borel subgroups of $\mathbf{L}_J$ which contain x. We get

$$\hat{\varepsilon}_A = (-1)^{\operatorname{rank}\mathbf{L}_J - \dim \mathbf{Z}(\mathbf{L}_J)} = (-1)^{\operatorname{rank}(\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J))} = (-1)^{|J|}.$$

From (3.4.7.1) applied to $(\mathbf{L}_J, \mathbf{W}_J)$, we know that $\mathfrak{S}^\circ_{\mathbf{W}_J}$ must be non-empty. Now let us go through the definition of the sets $\mathfrak{S}^\circ_W$ for irreducible Weyl groups W: We see that the only case where $\mathfrak{S}^\circ_W$ is non-empty and the rank of W is odd occurs when W is of type E_7. Hence, we have $\hat{\varepsilon}_A = -1$ if and only if $(\mathbf{W}_J, J)$ is of type E_7. The same holds for $\Delta(x_\rho)$, see [Lus84a, 4.14]. The corollary is proved. □

Remark 3.4.9. From now on, whenever dealing with a simple group $\mathbf{G}$ with trivial centre and a non-twisted $\mathbb{F}_q$-rational structure, we always assume that the bijections $\operatorname{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W})$ and $\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ are chosen according to Corollary 3.4.8. In view of Remark 3.4.6, the parametrisation of $\hat{\mathbf{G}}^{\mathrm{un}}$ in terms of $\mathfrak{X}(\mathbf{W})$ can thus directly be read off from the tables provided in the appendix of [Lus84a] as well. It will often be convenient to also label the unipotent characters in terms of $\mathfrak{X}(\mathbf{W})$, that is, we write

$$\operatorname{Uch}(\mathbf{G}^F) = \{\rho_x \mid x \in \mathfrak{X}(\mathbf{W})\} \quad \text{and} \quad \hat{\mathbf{G}}^{\mathrm{un}} = \{A_x \mid x \in \mathfrak{X}(\mathbf{W})\}.$$

Example 3.4.10. (a) Consider an irreducible character $\phi \in \operatorname{Irr}(\mathbf{W})$ and the associated element $x_\phi \in \mathfrak{X}(\mathbf{W})$ as well as the corresponding character sheaf $A_{x_\phi} \in \hat{\mathbf{G}}^{\mathrm{un}}$. Thus, A_{x_ϕ} is parametrised by the triple $(\varnothing, \phi, (1,1)) \in \mathfrak{S}_{\mathbf{W}}$ under the bijection $\mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}$ defined in 3.4.7. Note that the unique (cuspidal) unipotent character sheaf on $\mathbf{T}_0$ is $A_0 = \mathscr{L}_0[\dim \mathbf{T}_0]$. Thus, among the (isomorphism classes of) simple direct summands of $\operatorname{ind}_{\mathbf{T}_0}^{\mathbf{G}}(A_0)$, A_{x_ϕ} is parametrised by ϕ under the isomorphism (3.4.7.2) (with $J = \varnothing$). On the other hand, since $\mathscr{L}_0[\dim \mathbf{T}_0] = \mathrm{IC}(\mathbf{T}_0, \mathscr{L}_0)[\dim \mathbf{T}_0]$, we know from (3.1.21.1) that

$$\operatorname{ind}_{\mathbf{T}_0}^{\mathbf{G}}(A_0) \cong K_{\mathbf{T}_0,\mathbf{T}_0}^{\mathscr{L}_0},$$

and $K_{\mathbf{T}_0,\mathbf{T}_0}^{\mathscr{L}_0}$ is denoted by $K_{\mathfrak{j}}$ with $\mathfrak{j} = (\mathbf{T}_0, \{1\}, \mathscr{L}_0) \in \mathcal{M}_{\mathbf{G}}^F$ in 3.2.13, 3.2.14. Hence, using the notation there, we have $A_{x_\phi} = A_\phi$, that is, A_{x_ϕ} is the character sheaf parametrised by the element of $\mathcal{N}_{\mathbf{G}}$ (which in fact lies in $\mathcal{N}_{\mathbf{G}}^F$) which is the image of ϕ under the ordinary Springer correspondence (3.2.13.4).

(b) More generally, assume that $A \in \hat{\mathbf{G}}^{\mathrm{un}}$ is parametrised by some $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}_{\mathbf{G}}^F$ in the setting of 3.2.13. Let $\mathfrak{j} = [(\mathbf{L}, \mathscr{O}_0, \mathscr{E}_0)] \in \mathcal{M}_{\mathbf{G}}^F$ and $\epsilon \in \operatorname{Irr}(W_{\mathbf{G}}(\mathbf{L}))^{F_{\mathfrak{j}}}$ be such that $\mathfrak{i}$ corresponds to ϵ under (3.2.15.3). As noted in 3.2.14, we may assume that $\mathbf{L} = \mathbf{L}_J$ is the standard Levi subgroup of the standard parabolic subgroup $\mathbf{P}_J$ of $\mathbf{G}$ (for some $J \subseteq S$). By 3.4.7, the groups $\mathscr{W}_{\mathfrak{j}} = W_{\mathbf{G}}(\mathbf{L}_J)$ and $\mathbf{W}^{S/J}$ are isomorphic, so the respective constructions in 3.2.13, 3.4.7 show that A corresponds to $(J, \epsilon, \mathfrak{s}_A) \in \mathfrak{S}_{\mathbf{W}}$ (for a certain $\mathfrak{s}_A \in \mathfrak{S}^\circ_{\mathbf{W}_J}$) under the parametrisation of $\hat{\mathbf{G}}^{\mathrm{un}}$ in terms of $\mathfrak{S}_{\mathbf{W}}$. In view of the commutativity of the diagram in Corollary 3.4.8, we can thus read off the corresponding element of $\mathfrak{X}(\mathbf{W})$ by comparing the parametrisations of $\mathrm{Uch}(\mathbf{G}^F)$ in terms of $\mathfrak{S}_{\mathbf{W}}$, $\mathfrak{X}(\mathbf{W})$, respectively (see the appendix of [Lus84a] and [Car85, §13.8]).

3.4.11. Below, it will be convenient to tacitly identify $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ (via (1.1.0.1)) and just use the terms such as 'complex conjugation' or 'real numbers' for elements of $\overline{\mathbb{Q}}_\ell$ in the obvious way. Given a character ρ of $\mathbf{G}^F$, recall that the contragredient character is defined by

$$\overline{\rho}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell, \quad g \mapsto \rho(g^{-1}) = \overline{\rho(g)}.$$

Clearly, if $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, we have $\overline{\rho} \in \mathrm{Uch}(\mathbf{G}^F)$ as well. For later use we show that the Frobenius eigenvalues associated to unipotent characters satisfy

$$\lambda_{\overline{\rho}} = \overline{\lambda}_\rho = \lambda_\rho^{-1} \quad \text{for all } \rho \in \mathrm{Uch}(\mathbf{G}^F) \tag{3.4.11.1}$$

(which is already contained in [DM85, III.3.4], see also [GM03, 4.3]). More precisely, we provide some references here which allow us to quickly determine $\overline{\rho}$ for a given $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. Let us begin by considering those $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ for which $\lambda_\rho \in \{\pm 1\}$. Then, unless ρ is one of the two characters of degree 512 in case $\mathbf{G}$ is of type E_7, or ρ is one of the four characters of degree 4096 in case $\mathbf{G}$ is of type E_8, we have $\rho(g) \in \mathbb{Q}$ for all $g \in \mathbf{G}^F$, see [Lus02, 1.8]. Now assume that ρ is one of the six characters excluded above. Then $\rho(g) \in \mathbb{Q}(\sqrt{q}) \subseteq \mathbb{R}$ by [Gec03b, 5.6]. This shows that

$$\overline{\rho} = \rho \quad \text{whenever } \lambda_\rho \in \{\pm 1\}, \tag{3.4.11.2}$$

so in particular $\lambda_{\overline{\rho}} = \overline{\lambda}_\rho = \lambda_\rho^{-1}$ holds for those ρ. For instance, this already covers all non-twisted classical groups, see [Lus02, 1.12]. We may thus assume from now on that $\mathbf{G}$ is of exceptional type and $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ is such that $\lambda_\rho \neq \pm 1$. The character field

$$\mathbb{Q}(\rho) = \mathbb{Q}(\rho(g) \mid g \in \mathbf{G}^F)$$

is provided by [Gec03b, 5.6, see Table 1]. Note that $\overline{\rho} = \rho$ if and only if $\mathbb{Q}(\rho) \subseteq \mathbb{R}$. By inspection of the character degrees (printed in the appendix of [Lus84a] and also in [Car85, §13.9]), we see that, except for the four cuspidal unipotent characters $\mathsf{E}_8[\zeta_5^i]$ $(1 \leqslant i \leqslant 4)$ in type E_8 (where $\zeta_5 \in \mathcal{R}$ is the 5th primitive root of unity that we fixed in 3.4.2), there are never more than two unipotent characters which have the same degree.

Hence, unless $\rho \in \{\mathsf{E}_8[\zeta_5^i] \mid 1 \leqslant i \leqslant 4\}$, the character field $\mathbb{Q}(\rho)$ of ρ uniquely determines the contragredient character $\overline{\rho}$, and one verifies that $\lambda_{\overline{\rho}} = \overline{\lambda}_\rho = \lambda_\rho^{-1}$. Thus, in order to complete the proof of (3.4.11.1), it only remains to deal with the four characters $\mathsf{E}_8[\zeta_5^i]$, $1 \leqslant i \leqslant 4$ (when $\mathbf{G}$ is of type E_8). To this end, let us consider the inversion map

$$\mathrm{inv}\colon \mathbf{G}^F \to \mathbf{G}^F, \quad g \mapsto g^{-1}.$$

Clearly, inv induces a bijection on the set of conjugacy classes of $\mathbf{G}^F$, which we again denote by inv, so we may compose it with t_1 (see 3.4.3(c)) in order to obtain the map defined in [DM85, I.7.2(i)]. By [Gec03b, 5.3] (see [DM85, III.2.3]), we have

$$[\phi] \circ \mathrm{inv} \circ t_1 = \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \langle \rho, R_\phi \rangle_{\mathbf{G}^F} \cdot \lambda_\rho \cdot \rho \quad \text{for any } \phi \in \mathrm{Irr}(\mathbf{W}). \tag{3.4.11.3}$$

Let us consider the character $\phi := 4480_y \in \mathrm{Irr}(\mathbf{W})$. (Thus, ϕ lies in the unique family $\mathcal{F}_{16} \subseteq \mathrm{Irr}(\mathbf{W})$ consisting of 17 characters, so $|\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})| = 39$.) Then

$$\langle \rho, R_{[4480_y]} \rangle_{\mathbf{G}^F} = \Delta(x_\rho)\{x_\rho, x_{4480_y}\} = \{x_\rho, x_{4480_y}\} \in \mathbb{Q} \quad \text{for all } \rho \in \mathrm{Uch}(\mathbf{G}^F)$$

(where the last equality follows from the fact that $\Delta(x) = 1$ for all $x \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})$ and $\{x, x_{4480_y}\} = 0$ if $x \notin \mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})$). We have $\lambda_{[4480_y]} = 1$, so $[4480_y]$ is real-valued in view of (3.4.11.2). Hence, the left side of (3.4.11.3) with $\phi = 4480_y$ is invariant under complex conjugation, so we get

$$\sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \{x_\rho, x_{4480_y}\}\lambda_\rho \cdot \rho = \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \{x_\rho, x_{4480_y}\}\overline{\lambda}_\rho \cdot \overline{\rho}. \tag{3.4.11.4}$$

Let us set

$$\mathcal{U}' := \mathrm{Uch}(\mathbf{G}^F) \setminus \{\mathsf{E}_8[\zeta_5^i] \mid 1 \leqslant i \leqslant 4\}.$$

By the above discussion, we have $\lambda_{\overline{\rho}} = \overline{\lambda}_\rho$ for all $\rho \in \mathcal{U}'$, and $\rho \mapsto \overline{\rho}$ defines a bijection on $\{\mathsf{E}_8[\zeta_5^i] \mid 1 \leqslant i \leqslant 4\}$, hence also on $\mathcal{U}'$. One checks that $\{x_\rho, x_{4480_y}\} = \{x_{\overline{\rho}}, x_{4480_y}\}$ for all $\rho \in \mathcal{U}'$, so we get

$$\sum_{\rho \in \mathcal{U}'} \{x_\rho, x_{4480_y}\}\lambda_\rho \cdot \rho = \sum_{\rho \in \mathcal{U}'} \{x_{\overline{\rho}}, x_{4480_y}\}\lambda_{\overline{\rho}} \cdot \overline{\rho} = \sum_{\rho \in \mathcal{U}'} \{x_\rho, x_{4480_y}\}\overline{\lambda}_\rho \cdot \overline{\rho}. \tag{3.4.11.5}$$

On the other hand, the $\mathsf{E}_8[\zeta_5^i]$ are named by their Frobenius eigenvalues, so

$$\lambda_{\mathsf{E}_8[\zeta_5^i]} = \zeta_5^i \quad \text{for } 1 \leqslant i \leqslant 4.$$

Furthermore, we have $\{x_{\mathsf{E}_8[\zeta_5^i]}, x_{4480_y}\} = \frac{1}{5}$ for all $i \in \{1,2,3,4\}$. Hence, subtracting (3.4.11.5) from (3.4.11.4), we deduce that

$$\tfrac{1}{5}\sum_{i=1}^{4} \zeta_5^i \cdot \mathsf{E}_8[\zeta_5^i] = \tfrac{1}{5}\sum_{i=1}^{4} \zeta_5^{-i} \cdot \overline{\mathsf{E}_8[\zeta_5^i]}.$$

Since the ζ_5^i $(1 \leqslant i \leqslant 4)$ are pairwise different, the only possibility is that $\overline{\mathsf{E}_8[\zeta_5^i]} = \mathsf{E}_8[\zeta_5^{-i}]$ for $1 \leqslant i \leqslant 4$. Thus, (3.4.11.1) is proved.

Some reductions

3.4.12. For $x \in \mathfrak{X}(\mathbf{W})$, let us for now pick any isomorphism $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ as in 3.2.1(∗). Lusztig has shown how one can attach a scalar $\zeta_x \in \overline{\mathbb{Q}}_\ell$ of absolute value 1 to (A_x, φ_x); we will sketch this only very briefly here and refer to [LuCS3, §13] and [Lus86, 3.5] for more details, see also [Gec21, 3.5]. By the definition of $\hat{\mathbf{G}}^{\mathrm{un}}$ (or, rather, by Remark 3.1.7(c)), there exist $w \in \mathbf{W}$ and $i \in \mathbb{Z}$ such that A_x is a constituent of ${}^pH^i(\overline{K}_w^{\mathscr{L}_0})$; in particular, we have ${}^pH^i(\overline{K}_w^{\mathscr{L}_0}) \neq \{0\}$. Let us fix any such w, i, and let $V_x := \mathrm{Hom}_{\mathscr{D}\mathbf{G}}(A_x, {}^pH^i(\overline{K}_w^{\mathscr{L}_0}))$. There is a natural isomorphism $\vartheta_{i,w} \colon F^*({}^pH^i(\overline{K}_w^{\mathscr{L}_0})) \xrightarrow{\sim} {}^pH^i(\overline{K}_w^{\mathscr{L}_0})$. (Writing w instead of $\dot{w}$ is justified here since all of the above is independent of the choice of the representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)^F$ of w in our situation.) Consider the $\overline{\mathbb{Q}}_\ell$-linear map

$$\Psi_x \colon V_x \to V_x, \quad v \mapsto \vartheta_{i,w} \circ F^*(v) \circ \varphi_x^{-1}.$$

By [LuCS3, 13.10], all the eigenvalues of Ψ_x are equal to $\zeta_x \cdot q^{(i-\dim \mathbf{G})/2}$, and we have $|\zeta_x| = 1$; in fact, ζ_x lies in $\mathcal{R}$ with our assumption on the isomorphism φ_x. This definition of $\zeta_x = \zeta_x(\varphi_x)$ does not depend on w, i (and $\dot{w}$), only on A_x, φ_x and the choice of $\sqrt{q}$ that we made in (1.1.0.2). On the other hand, let us consider the characteristic function

$$\chi_x := \chi_{A_x,\varphi_x} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell. \tag{3.4.12.1}$$

As noted in 2.2.11, we have $\{x, y\} \in \mathbb{R}$ for all $x, y \in \mathfrak{X}(\mathbf{W})$, so the definition (2.2.24.3) of the unipotent almost character $R_x \in \mathrm{CF}(\mathbf{G}^F)$ reads

$$R_x = \sum_{y \in \mathfrak{X}(\mathbf{W})} \Delta(y)\{y, x\}\rho_y.$$

By (3.3.12.2) and (3.3.12.3), we have

$$R_x = \xi_x \chi_x \quad \text{for } x \in \mathfrak{X}(\mathbf{W}), \text{ where } \xi_x = \xi_x(\varphi_x) \in \overline{\mathbb{Q}}_\ell^\times, \ |\xi_x| = 1. \tag{3.4.12.2}$$

Using [LuCS3, 14.14] (and [Lus86, 3.6(a)]), one finds that

$$R_x = (-1)^{\dim \mathbf{G}} \hat{\varepsilon}_{A_x} \zeta_x \chi_x \quad \text{for } x \in \mathfrak{X}(\mathbf{W}), \tag{3.4.12.3}$$

see [Gec19, §3]. Thus, we have

$$\xi_x = (-1)^{\dim \mathbf{G}} \hat{\varepsilon}_{A_x} \zeta_x \quad \text{for } x \in \mathfrak{X}(\mathbf{W}). \tag{3.4.12.4}$$

In particular, as $\zeta_x \in \mathcal{R}$, we also get $\xi_x \in \mathcal{R}$. Now let $m \in \mathbb{N}$. Then $F^m \colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map which provides $\mathbf{G}$ with an $\mathbb{F}_{q^m}$-rational structure, and F^m certainly induces the identity map on $\mathbf{W}$ (since F does, by our standing assumption in this section), so the above discussion may as well be applied to $(\mathbf{G}, F^m)$ instead of $(\mathbf{G}, F)$. We can thus write

$$\operatorname{Uch}(\mathbf{G}^{F^m}) = \{\rho_x^{(m)} \mid x \in \mathfrak{X}(\mathbf{W})\}$$

(still assuming that the corresponding bijection $\operatorname{Uch}(\mathbf{G}^{F^m}) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W})$ is the one defined through Proposition 3.4.5). Furthermore, for $x \in \mathfrak{X}(\mathbf{W})$, denote by $R_x^{(m)} \in \operatorname{CF}(\mathbf{G}^{F^m})$ the corresponding unipotent almost character, that is,

$$R_x^{(m)} = \sum_{y \in \mathfrak{X}(\mathbf{W})} \Delta(y)\{y, x\} \rho_y^{(m)}.$$

In order to define the corresponding characteristic function on $\mathbf{G}^{F^m}$, we need to specify an isomorphism $(F^m)^* A_x \xrightarrow{\sim} A_x$. This is done in [Gec19, 3.3]. Namely, $\varphi_x \colon F^* A_x \xrightarrow{\sim} A_x$ naturally induces isomorphisms

$$(F^*)^i(\varphi_x) \colon (F^*)^{i+1} A_x \xrightarrow{\sim} (F^*)^i A_x, \quad \text{for } 0 \leqslant i \leqslant m-1.$$

Setting

$$\tilde{\varphi}_x^{(m)} := \varphi_x \circ F^*(\varphi_x) \circ \ldots \circ (F^*)^{m-1}(\varphi_x) \colon (F^*)^m A_x \xrightarrow{\sim} A_x,$$

and composing $\tilde{\varphi}_x^{(m)}$ with the canonical isomorphism $(F^m)^* A_x \cong (F^*)^m A_x$, we thus obtain an isomorphism

$$\varphi_x^{(m)} \colon (F^m)^* A_x \xrightarrow{\sim} A_x.$$

Since φ_x satisfies the requirement $(*)$ in 3.2.1 with respect to F, it follows from this construction that $\varphi_x^{(m)}$ satisfies the requirement $(*)$ in 3.2.1 with respect to F^m. Let

$$\chi_x^{(m)} := \chi_{A_x, \varphi_x^{(m)}} \colon \mathbf{G}^{F^m} \to \overline{\mathbb{Q}}_\ell$$

be the associated characteristic function, and let $\zeta_x^{(m)} \in \overline{\mathbb{Q}}_\ell$ be defined by

$$R_x^{(m)} = (-1)^{\dim \mathbf{G}} \hat{\varepsilon}_{A_x} \zeta_x^{(m)} \chi_x^{(m)}.$$

Note that $R_x^{(1)} = R_x$, $\chi_x^{(1)} = \chi_x$, and so $\zeta_x^{(1)} = \zeta_x$. We can now formulate the following result due to Geck [Gec19].

Proposition 3.4.13 (Geck [Gec19, 3.4])**.** *In the setting of* 3.4.12*, we have*

$$\zeta_x^{(m)} = \zeta_x^m \quad \textit{for all } x \in \mathfrak{X}(\mathbf{W}),\ m \in \mathbb{N}.$$

Proof. See [Gec19, §3]. The proof is based on considering the linear map $\Psi_x \colon V_x \to V_x$ defined in 3.4.12 and comparing it with the linear maps $\Psi_x^{(m)} \colon V_x \to V_x$ defined with respect to $(\varphi_x^{(m)}, F^m)$ instead of (φ_x, F), for any $m \in \mathbb{N}$ and any $x \in \mathfrak{X}(\mathbf{W})$. It is shown in [Gec19, 3.7] that $\Psi_x^{(m)} = \Psi_x^m$ for any $m \in \mathbb{N}$, so we must have $\zeta_x^{(m)} = \zeta_x^m$. □

Remark 3.4.14. We place ourselves in the setting of 3.4.12.

(a) Let us assume that $A_x \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ is a cuspidal unipotent character sheaf on $\mathbf{G}$, corresponding to the (F-stable) cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{G}$. We also assume that $\mathscr{E}$ is one-dimensional and that there exists some $g_0 \in \Sigma^F$ such that F acts trivially on $A_{\mathbf{G}}(g_0)$. Of course, we then have $g_0 \in \Sigma^{F^m}$, and F^m acts trivially on $A_{\mathbf{G}}(g_0)$ for any $m \in \mathbb{N}$. Let $d = \dim \Sigma$, and let $\big(\varphi_x^{(m)}\big)_{-d,g_0} \colon \mathscr{E}_{g_0} \to \mathscr{E}_{g_0}$ be the linear map induced by $\varphi_x^{(m)} \colon (F^m)^* A_x \xrightarrow{\sim} A_x$. We have $\big(\varphi_x^{(m)}\big)_{-d,g_0} = \big(\varphi_x\big)_{-d,g_0}^m$ for any $m \in \mathbb{N}$ (see [Sho95a, 1.1]). Let $\varphi_x \colon F^* A_x \xrightarrow{\sim} A_x$ be the isomorphism corresponding to the choice of $g_0 \in \Sigma^F$, that is, the linear map $(\varphi_x)_{-d,g_0} \colon \mathscr{E}_{g_0} \to \mathscr{E}_{g_0}$ is given by scalar multiplication with $q^{(\dim \mathbf{G} - d)/2}$. Then $\big(\varphi_x^{(m)}\big)_{-d,g_0} = \big(\varphi_x\big)_{-d,g_0}^m$ is given by scalar multiplication with $\big(q^{(\dim \mathbf{G} - d)/2}\big)^m = (q^m)^{(\dim \mathbf{G} - d)/2}$. Thus, with respect to the $\mathbb{F}_{q^m}$-rational structure on $\mathbf{G}$ defined by F^m, the isomorphism $\varphi_x^{(m)} \colon (F^m)^* A_x \xrightarrow{\sim} A_x$ corresponds to the choice of $g_0 \in \Sigma^{F^m}$, for any $m \in \mathbb{N}$.

(b) Proposition 3.4.13 thus reduces the problem of determining the scalars ζ_x (and, hence, ξ_x) for $x \in \mathfrak{X}(\mathbf{W})$ to the base case $p = q$, for any simple $\mathbf{G}$ with trivial centre such that $(\mathbf{G}, F)$ is non-twisted. This reduction is especially advantageous as far as bad primes p for $\mathbf{G}$ are concerned, as in these cases many properties of the algebraic group $\mathbf{G}$ related to its geometric origin tend to be quite different compared to the cases where p is good. On the other hand, since the mere size of $\mathbf{G}(\mathbb{F}_p)$ is comparatively small for very small primes p, one might hope that there is enough information available for the character values of $\mathbf{G}(\mathbb{F}_p)$ by other means, possibly from an abstract group-theoretic nature, e.g., using a computer algebra system such as GAP [GAP4] and CHEVIE [GHLMP], [MiChv], or consulting the Cambridge $\mathbb{ATLAS}$ [CCNPW]. For example, Geck used such methods in [Gec19] for the groups $\mathsf{F}_4(q)$ and $\mathsf{E}_6(q)$ where q is a power of the prime $p = 2$.

3.4.15. From now on, we will mostly be concerned with the problem of determining the scalars ξ_x in (3.4.12.2) (or equivalently, the ζ_x in (3.4.12.3)), or provide appropriate references in the cases where they are already known. But recall that, for $x \in \mathfrak{X}(\mathbf{W})$, an isomorphism $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ as in 3.2.1(∗) is only defined up to multiplication with a root of unity, so the first task is to explain how one can go about fixing such φ_x. Let us start with elements of the form $x = x_\phi$ for $\phi \in \mathrm{Irr}(\mathbf{W})$; this is discussed in detail in [Sho95a, §1, §2]. By Corollary 3.4.8, A_{x_ϕ} is parametrised by the triple $(\varnothing, \phi, (1,1))$, so A_{x_ϕ} is isomorphic to a simple constituent of $\mathrm{ind}_{\mathbf{T}_0}^{\mathbf{G}}(A_0)$ where $A_0 = \mathscr{L}_0[\dim \mathbf{T}_0]$ is the unique (cuspidal) unipotent character sheaf on $\mathbf{T}_0$. The cuspidal pair for $\mathbf{T}_0$ corresponding to A_0 is thus $(\mathbf{T}_0, \mathscr{L}_0)$, and the neutral element 1 of $\mathbf{G}$ serves as a distinguished element of $\mathbf{T}_0$ (see 3.1.3). Following 3.2.21, we fix the isomorphism $\varphi_{A_0} \colon F^*A_0 \xrightarrow{\sim} A_0$ in such a way that the induced map $(\varphi_{A_0})_{-r,1} \colon (\mathscr{L}_0)_1 \to (\mathscr{L}_0)_1$ is scalar multiplication with $q^{(\dim \mathbf{G} - r)/2}$ (recall that $r = \operatorname{rank} \mathbf{G} = \dim \mathbf{T}_0$). The endomorphism algebra of the complex $K_{\mathbf{T}_0,\mathbf{T}_0}^{\mathscr{L}_0} \in \mathscr{M}\mathbf{G}$ is isomorphic to the group algebra $\overline{\mathbb{Q}}_\ell[\mathbf{W}]$ of $\mathbf{W}$, so the discussion in 3.2.19(b) shows how $(\varphi_{A_0})_{-r} \colon F^*\mathscr{L}_0 \xrightarrow{\sim} \mathscr{L}_0$ determines an isomorphism $\varphi_{x_\phi} \colon F^*A_{x_\phi} \xrightarrow{\sim} A_{x_\phi}$. The scalar $\xi_{x_\phi} = \xi_{x_\phi}(\varphi_{x_\phi}) \in \overline{\mathbb{Q}}_\ell$ which relates the almost character $R_{x_\phi} = R_\phi$ with the characteristic function $\chi_{A_{x_\phi}, \varphi_{x_\phi}}$ is explicitly known thanks to the results of Lusztig [Lus90] and Shoji, see [Sho95a, 2.18]: We have

$$R_\phi = (-1)^{\dim \mathbf{T}_0} \chi_{A_{x_\phi}, \varphi_{x_\phi}},$$

that is, $\xi_{x_\phi} = (-1)^{\dim \mathbf{T}_0}$ and $\zeta_{x_\phi} = 1$.

3.4.16. Continuing the discussion in 3.4.15, let us now assume that $x \in \mathfrak{X}(\mathbf{W})$ is not in the image of the embedding $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{X}(\mathbf{W})$. We want to describe how both specifying the isomorphism $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ and determining the scalar ξ_x (and ζ_x) are reduced to considering *cuspidal* unipotent character sheaves, under the acceptance that one has to simultaneously deal with all simple groups with a trivial centre and with a non-twisted $\mathbb{F}_q$-rational structure (for one and the same q), cf. 3.3.11.

So let $J \subseteq S$ and $A_0 \in \hat{\mathbf{L}}_J^{\circ,\mathrm{un}}$ be such that A_x is isomorphic to a simple direct summand of $\mathrm{ind}_{\mathbf{L}_J}^{\mathbf{G}}(A_0)$ (see 3.4.7). By our assumption on $(\mathbf{G}, F)$ in this section, $\mathbf{L}_J$ has a connected centre [DM20, 11.2.1], and the restriction of F to $\mathbf{L}_J$ defines a non-twisted $\mathbb{F}_q$-rational structure on $\mathbf{L}_J$. Moreover, by our assumption on x, we have $J \neq \varnothing$, so that $\mathbf{L}'_J := \mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J) \neq \{1\}$ is necessarily a simple group (see again 3.4.7); we also have $\mathbf{Z}(\mathbf{L}'_J) = \{1\}$. Let $\pi\colon \mathbf{L}_J \to \mathbf{L}'_J$ be the canonical map, and let $F'\colon \mathbf{L}'_J \to \mathbf{L}'_J$ be the Frobenius map on $\mathbf{L}'_J$ induced by F, so that $F' \circ \pi = \pi \circ F|_{\mathbf{L}_J}$; F' defines a non-twisted $\mathbb{F}_q$-rational structure on $\mathbf{L}'_J$. Thus, everything in this section can also be applied to $(\mathbf{L}'_J, F')$ instead of $(\mathbf{G}, F)$.

The shifted inverse image functor $\pi^*[\dim \mathbf{Z}(\mathbf{L}_J)]\colon \mathscr{D}\mathbf{L}'_J \to \mathscr{D}\mathbf{L}_J$ induces a bijection between the (cuspidal) unipotent character sheaves on $\mathbf{L}'_J$ and the (cuspidal) unipotent character sheaves on $\mathbf{L}_J$ (see [LuCS4, 17.10] and [Lus84b, 2.10]). Let $A'_0 \in (\hat{\mathbf{L}}'_J)^{\circ,\mathrm{un}}$ be such that $\pi^*(A'_0)[\dim \mathbf{Z}(\mathbf{L}_J)] = A_0$. As noted in 3.4.1, A'_0 is automatically F'-stable. Let $(\Sigma', \mathscr{E}')$ be the (F'-stable) cuspidal pair for $\mathbf{L}'_J$ such that $A'_0 \cong \mathrm{IC}(\overline{\Sigma'}, \mathscr{E}')[\dim \Sigma']^{\#\mathbf{L}'_J}$. We then have $A_0 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}_J}$ where $\Sigma = \pi^{-1}(\Sigma')$ and $\mathscr{E} = (\pi|_\Sigma)^*\mathscr{E}'$. Here, Σ' is a conjugacy class of $\mathbf{L}'_J$, and Σ is of the form $\mathscr{C} \cdot \mathbf{Z}(\mathbf{L}_J)$ for a conjugacy class $\mathscr{C}$ of $\mathbf{L}_J$ such that $\pi(\mathscr{C}) = \Sigma'$.

Let us for now fix any element $g'_0 \in \Sigma'^{F'}$, and let $g_0 \in \mathscr{C}$ be such that $\pi(g_0) = g'_0$; we also set $d' := \dim \Sigma'$ and $d := \dim \Sigma = d' + \dim \mathbf{Z}(\mathbf{L}_J)$. As described in 3.2.21, the choice of g'_0 uniquely determines an isomorphism $\varphi_{A'_0}\colon F'^*A'_0 \xrightarrow{\sim} A'_0$ by requiring that the induced map $(\varphi_{A'_0})_{-d',g'_0}\colon \mathscr{E}'_{g'_0} \xrightarrow{\sim} \mathscr{E}'_{g'_0}$ is given by scalar multiplication with $q^{(\dim \mathbf{L}'_J - d')/2}$. Let $\varphi_{A_0} := \pi^*[\dim \mathbf{Z}(\mathbf{L}_J)](\varphi_{A'_0})\colon F^*A_0 \xrightarrow{\sim} A_0$; then both $\varphi_{A'_0}$ and φ_{A_0} satisfy the requirement $(*)$ in 3.2.1. Now consider the complex $K = K_{\mathbf{L}_J,\Sigma}^{\mathscr{E}} \in \mathscr{M}\mathbf{G}$. We have $K \cong \mathrm{ind}_{\mathbf{L}_J}^{\mathbf{G}}(A_0)$, and the endomorphism algebra $\mathrm{End}_{\mathscr{M}\mathbf{G}}(K)$ is isomorphic to the group algebra $\overline{\mathbb{Q}}_\ell[\mathbf{W}^{S/J}]$, see (3.4.7.2). The discussion in 3.2.19(b) thus shows that $\varphi_0 := (\varphi_{A_0})_{-d}\colon F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ not only determines an isomorphism $\varphi\colon F^*K \xrightarrow{\sim} K$, but also an isomorphism $\varphi_x\colon F^*A_x \xrightarrow{\sim} A_x$. Hence, the choice of $g'_0 \in \Sigma'^{F'}$ (or, equivalently, the one of an isomorphism $F'^*A'_0 \xrightarrow{\sim} A'_0$) completely determines the isomorphisms $\varphi_x\colon F^*A_x \xrightarrow{\sim} A_x$ for any $x \in \mathfrak{X}(\mathbf{W})$ such that A_x is a simple direct summand of $\mathrm{ind}_{\mathbf{L}_J}^{\mathbf{G}}(A_0)$; see 3.2.22, 3.2.23 for comments on how one can make a 'good' choice for $g'_0 \in \Sigma'^{F'}$.

As for the scalars ξ_x, we can formulate the following result, which is implicitly contained in Lusztig's and Shoji's work, see [LuCS3, §13, §14], [Lus86], [Sho95a], [Sho95b]; for classical groups, it is also explicitly stated in [Sho09, 6.3].

Corollary 3.4.17 (cf. [Sho09, §6]). *Let $x \in \mathfrak{X}(\mathbf{W}) \setminus \mathrm{Irr}(\mathbf{W})$, with associated $\varnothing \neq J \subseteq S$ and $A'_0 \in (\hat{\mathbf{L}}'_J)^{\circ,\mathrm{un}}$ in the setting of* 3.4.16. *Let $\varphi_{A'_0}\colon F'^*A'_0 \xrightarrow{\sim} A'_0$ be any isomorphism as in* 3.2.1$(*)$ *(cf.* 3.2.21*) and $\varphi_x\colon F^*A_x \xrightarrow{\sim} A_x$ be the isomorphism determined by $\varphi_{A'_0}$, as explained in* 3.4.16. *Let $R'_0 \in \mathrm{CF}(\mathbf{L}'^{F'}_J)$ be the almost character corresponding to A'_0*

and $\xi_0' = \xi_0'(\varphi_{A_0'}) \in \overline{\mathbb{Q}}_\ell$ *be such that* $R_0' = \xi_0' \chi_{A_0', \varphi_{A_0'}}$. *Then we have*

$$\xi_x(\varphi_x) = (-1)^{\dim \mathbf{Z}(\mathbf{L}_J)} \xi_0',$$

that is,

$$R_x = (-1)^{\dim \mathbf{Z}(\mathbf{L}_J)} \xi_0' \cdot \chi_{A_x, \varphi_x}.$$

Proof. As explained in 3.4.16, we can apply the discussion in 3.4.12 to $(\mathbf{L}_J', F')$ as well. Thus, if $\zeta_0' = \zeta_{A_0'}(\varphi_{A_0'}) \in \overline{\mathbb{Q}}_\ell$ denotes the scalar attached to $(A_0', \varphi_{A_0'})$, we just need to compare (3.4.12.4) applied to $(\mathbf{G}, F)$ and $(\mathbf{L}_J', F')$, respectively. The advantage of arguing with $\zeta_x = \zeta_x(\varphi_x)$ and ζ_0' (instead of directly using $\xi_x = \xi_x(\varphi_x)$ and ξ_0') is that their definition immediately gives $\zeta_x = \zeta_0'$. (More precisely, it is shown in [Lus86, 3.5] that we have $\zeta_x = \zeta_{A_0}(\varphi_{A_0})$, with φ_{A_0} as defined in 3.4.16; the fact that $\zeta_{A_0}(\varphi_{A_0}) = \zeta_0'$ follows from the compatibility properties of $\pi^*[\dim \mathbf{Z}(\mathbf{L}_J)]$ mentioned in [LuCS4, 17.10].) So we only need to compare $(-1)^{\dim \mathbf{G}} \hat{\varepsilon}_{A_x} = (-1)^{\dim \operatorname{supp} A_x}$ with $(-1)^{\dim \mathbf{L}_J'} \hat{\varepsilon}_{A_0'} = (-1)^{\dim \operatorname{supp} A_0'}$. By 3.1.15 and 3.1.21, we have

$$\dim \operatorname{supp} A_x = \dim Y_{(\mathbf{L}_J, \Sigma)} \equiv \dim \Sigma \pmod 2.$$

On the other hand, the support of A_0' is (the closure of) Σ', which has dimension $\dim \Sigma - \dim \mathbf{Z}(\mathbf{L}_J)$ (see 3.4.16). So we get

$$\xi_x = (-1)^{\dim \operatorname{supp} A_x} \zeta_x = (-1)^{\dim \mathbf{Z}(\mathbf{L}_J)} (-1)^{\dim \operatorname{supp} A_0'} \zeta_0' = (-1)^{\dim \mathbf{Z}(\mathbf{L}_J)} \xi_0'. \qquad \square$$

The main methods

In view of Corollary 3.4.17 and as far as simple groups with a trivial centre and a non-twisted $\mathbb{F}_q$-rational structure are concerned, we are thus reduced to considering cuspidal unipotent character sheaves when seeking to determine the scalars ξ_x ($x \in \mathfrak{X}(\mathbf{W})$) in (3.4.12.2). Recall from 3.3.13 that this problem has been completely solved by Shoji for all classical groups. Hence, we will look at the cuspidal unipotent character sheaves on the various simple groups of exceptional type in what follows (see Chapter 4 below). In the remainder of this chapter, we present the main methods that we will use in order to get our hands on the scalars ξ_x in (3.4.12.2) for those $x \in \mathfrak{X}(\mathbf{W})$ which parametrise cuspidal unipotent character sheaves $A_x \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$.

3.4.18. Let us consider an F-stable cuspidal unipotent character sheaf

$$A = \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E})[\dim \mathscr{C}]^{\#\mathbf{G}} \in \hat{\mathbf{G}}^{\circ,\mathrm{un}},$$

where $\mathscr{C} \subseteq \mathbf{G}$ is an F-stable conjugacy class and $\mathscr{E}$ is an F-stable cuspidal local system on $\mathscr{C}$. As in 3.2.21, we assume that $\mathscr{E}$ is one-dimensional. (This is always the case when $\mathbf{G}$ is a simple group of exceptional type, see Chapter 4 below.) Thus, the first step consists in finding a 'good' representative $g_0 \in \mathscr{C}^F$; see 3.2.22, 3.2.23. It turns out that in many cases it is possible to choose $g_0 \in \mathscr{C}^F$ in such a way that (in particular) the following condition holds:

$$g_0 \text{ is conjugate to } g_0^{-1} \text{ in } \mathbf{G}^F. \qquad (\diamondsuit)$$

So let us assume that $g_0 \in \mathscr{C}^F$ satisfies $(\diamondsuit)$. By 3.2.21, g_0 determines an isomorphism $\varphi_A \colon F^*A \xrightarrow{\sim} A$. Let $x_A \in \mathfrak{X}(\mathbf{W})$ be such that $A = A_{x_A}$. As in 3.4.12, we write

$$\xi_{x_A} := \xi_{x_A}(\varphi_A), \quad \zeta_{x_A} := \zeta_{x_A}(\varphi_A), \quad \chi_{x_A} := \chi_{A,\varphi_A}.$$

We immediately get

$$\xi_{x_A} \in \{\pm 1\} \quad \text{(and thus } \zeta_{x_A} \in \{\pm 1\} \text{ as well)}.$$

Indeed, as noted in 3.4.12, the unipotent almost character R_{x_A} is not only a class function of $\mathbf{G}^F$ but at the same time an $\mathbb{R}$-linear combination of characters of $\mathbf{G}^F$. So, using $(\diamondsuit)$, we deduce that

$$\xi_{x_A}\chi_{x_A}(g_0) = R_{x_A}(g_0) = R_{x_A}(g_0^{-1}) = \overline{R_{x_A}(g_0)} = \overline{\xi}_{x_A}\chi_{x_A}(g_0),$$

where the last equality holds due to the fact that $\chi_{x_A}(g_0) \in \mathbb{R}$, see again 3.2.21. So we have $\xi_{x_A} = \overline{\xi}_{x_A}$ and, since $|\xi_{x_A}| = 1$, we conclude that $\xi_{x_A} \in \{\pm 1\}$, as claimed.

3.4.19. We keep the setting of 3.4.18. In particular,

$$A = \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E})[\dim \mathscr{C}]^{\#\mathbf{G}}$$

is an F-stable cuspidal unipotent character sheaf on $\mathbf{G}$, the local system $\mathscr{E}$ is one-dimensional, and we assume that we have found a 'good' representative $g_0 \in \mathscr{C}^F$ which is $\mathbf{G}^F$-conjugate to g_0^{-1}. In order to have a uniform notation, we will continue to write χ_{x_A} for the characteristic function of A as in (3.4.12.1), but with $\varphi_{x_A} \colon F^*A \xrightarrow{\sim} A$ explicitly defined with respect to the choice of g_0. (For any other $x \in \mathfrak{X}(\mathbf{W})$, let us for now just choose any $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ as in 3.2.1(∗), and let the further notation be as in 3.4.12.) We set

$$\mathfrak{X}^\circ(\mathbf{W}) := \{x \in \mathfrak{X}(\mathbf{W}) \mid A_x \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}\}$$

and

$$\mathfrak{X}'(\mathbf{W}) := \mathfrak{X}(\mathbf{W}) \setminus (\mathfrak{X}^\circ(\mathbf{W}) \cup \mathrm{Irr}(\mathbf{W})).$$

We also recall that $\{x, y\} \in \mathbb{R}$ for all $x, y \in \mathfrak{X}(\mathbf{W})$. Let us briefly describe the two main methods that we use in this thesis to obtain information on the sign $\xi_{x_A} \in \{\pm 1\}$.

(1) For $\rho \in \mathrm{Uch}(\mathbf{G}^F)$, we have

$$\begin{aligned}\Delta(x_\rho) \cdot \rho &= \sum_{x \in \mathfrak{X}(\mathbf{W})} \{x_\rho, x\} \cdot R_x \\ &= \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \{x_\rho, x_\phi\} R_\phi + \sum_{x \in \mathfrak{X}'(\mathbf{W})} \{x_\rho, x\} \xi_x \chi_x + \sum_{x \in \mathfrak{X}^\circ(\mathbf{W})} \{x_\rho, x\} \xi_x \chi_x.\end{aligned}$$

Recall from 2.2.5 that the values of the R_ϕ (for $\phi \in \mathrm{Irr}(\mathbf{W})$) at unipotent elements of $\mathbf{G}^F$ are known in most cases (the exceptions occurring only when $\mathbf{G}$ is of type E_8 and $p \in \{2, 3, 5\}$). But even for non-unipotent elements $g \in \mathbf{G}^F$, the values $R_\phi(g)$ can be computed in many cases; see [Gec21, §2, §3] for more details. So let us assume that we are able to compute the values $R_\phi(g_0)$ for all $\phi \in \mathrm{Irr}(\mathbf{W})$; as described in 3.2.21, the computation of $\chi_x(g_0)$ is easy for all $x \in \mathfrak{X}^\circ(\mathbf{W})$. Note that all those $R_\phi(g_0)$ and $\chi_x(g_0)$ are rational numbers. One idea to obtain information on the sign ξ_{x_A} is then to choose $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ in such a way that

$$\{x_\rho, x_A\} \neq 0 \quad \text{while} \quad \{x_\rho, x\} \cdot \chi_x(g_0) = 0 \text{ for all } x \in \mathfrak{X}'(\mathbf{W}).$$

Recall that it is sometimes possible to deduce that $\chi_x(g_0) = 0$ for certain fixed elements $g_0 \in \mathbf{G}^F$, especially in the case where g_0 is unipotent, see Corollary 3.2.8 or Theorem 3.2.16. If we manage to find such a ρ, we then get an equation for $\rho(g_0)$ in which the only unknowns are the signs $\xi_x \in \{\pm 1\}$ corresponding to the cuspidal unipotent character sheaves A_x which are supported by (the closure of) $\mathscr{C}$, one of them being A by construction. (It may be the only one which appears.) We know from basic character theory of finite groups that $\rho(g_0)$ is an algebraic integer, and if all the conditions mentioned above are satisfied, we know at the same time that it is a rational number, so it must be a rational integer. In several instances, this information is already sufficient to determine the sign ξ_{x_A}; see [Gec19, 6.6], [Gec21], and also the arguments in 4.5.5 and Proposition 4.5.7 below.

(2) Another approach is as follows, using the Hecke algebra associated to the permutation module of $\mathbf{G}^F$ with respect to $\mathbf{B}_0^F$; see Section 2.3. Recall the definition of $m(g, w)$ (for any $g \in \mathbf{G}^F$, $w \in \mathbf{W}$) in 2.3.9. In view of (2.2.24.4) and (3.4.12.2), we may express

this in terms of characteristic functions of character sheaves: We have

$$m(g,w) = \sum_{x\in\mathfrak{X}(\mathbf{W})} c_x(w)R_x(g) = \sum_{x\in\mathfrak{X}(\mathbf{W})} c_x(w)\xi_x\chi_x(g) \tag{3.4.19.1}$$

where

$$c_x(w) = \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_\phi, x\}\,\mathrm{Trace}(T_w, V_\phi).$$

(We have used that $\Delta(x_\phi) = +1$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$.) We thus get

$$m(g,w) = \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} c_{x_\phi}(w)R_\phi(g) + \sum_{x\in\mathfrak{X}'(\mathbf{W})} c_x(w)\xi_x\chi_x(g) + \sum_{x\in\mathfrak{X}^\circ(\mathbf{W})} c_x(w)\xi_x\chi_x(g).$$

Recall (2.2.11, 2.3.9) that the numbers $c_x(w)$ are explicitly known and electronically accessible through Michel's CHEVIE [MiChv] (for all $x \in \mathfrak{X}(\mathbf{W})$ and $w \in \mathbf{W}$). On the other hand, by (2.3.9.2), we have

$$m(g,w) = \frac{|O_g \cap \mathbf{B}_0^F w\mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(g)|}{|\mathbf{B}_0^F|}. \tag{3.4.19.2}$$

The idea is to evaluate this at (possibly various) suitable $w \in \mathbf{W}$ and $g \in \mathscr{C}^F$, in particular at $g = g_0$, to obtain information on the sign ξ_{x_A}. This works especially well if $c_x(w)R_x(g) = 0$ for 'many' of the $x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_A\}$. If $x \in \mathfrak{X}(\mathbf{W})$ is of the form $x = x_\phi$ for some irreducible character $\phi \in \mathrm{Irr}(\mathbf{W})$, we might also be able to compute $R_x(g) = R_\phi(g)$, as mentioned in (1) above. In general, it appears to be very difficult to explicitly compute the cardinalities $|O_{g_0} \cap \mathbf{B}_0^F w\mathbf{B}_0^F|$ on the right side of (3.4.19.2), but already knowing that this is certainly a non-negative number will turn out to solve the cases appearing in type E_6 with $p = 3$ (see Section 4.1) and several of the cases occurring in type E_8 with $p = 5$ (see Section 4.5). However, there are also many instances where more knowledge on both the set $O_{g_0} \cap \mathbf{B}_0^F w\mathbf{B}_0^F$ and on the values of almost characters R_x with $x \in \mathfrak{X}'(\mathbf{W})$ is required; an example is E_7 in characteristic $p = 2$ (see Section 4.2), but also numerous cases in type E_8 (see again Section 4.5). To this end, we will explain how one can compute the almost characters at *unipotent* elements up to certain roots of unity in 3.4.21–3.4.24 below.

As a matter of fact, the method (2) above combined with the discussion in 3.4.21–3.4.24 will enable us to directly obtain *explicit* values of unipotent almost characters R_x (where $x \in \mathfrak{X}'(\mathbf{W})$) at unipotent elements for groups of type E_6 and E_7 which were previously not known.

Remark 3.4.20. Note that the definition (2.3.9.1) of $m(g,w)$ only involves unipotent principal series characters of $\mathbf{G}^F$. An expression for $m(g,w)$ in terms of characteristic functions of unipotent character sheaves as in (3.4.19.1) can thus also directly be obtained from [LuCS3, 14.14] (see also [Lus86, 3.6]) using the definition of the scalars ζ_x ($x \in \mathfrak{X}(\mathbf{W})$) in 3.4.12, without reference to unipotent almost characters and Shoji's Theorem 3.3.9.

3.4.21. In several places below, we will need detailed information on the values of the unipotent almost characters R_x ($x \in \mathfrak{X}(\mathbf{W})$) at unipotent elements, so let us explain how one can go about computing those $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ up to some unknown roots of unity. In view of (3.4.12.2), this is (up to multiplication with roots of unity) equivalent to computing the functions $\chi_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $x \in \mathfrak{X}(\mathbf{W})$. Recall from 3.2.5 that A_x is a simple direct summand of some $K^{\mathscr{E}_1}_{\mathbf{L},\Sigma} \in \mathscr{M}\mathbf{G}$ where $\mathbf{L} \subseteq \mathbf{G}$ is a regular subgroup and $(\Sigma, \mathscr{E}_1)$ is an F-stable cuspidal pair for $\mathbf{L}$. By Corollary 3.2.8, we know that in the case where Σ does not contain any unipotent elements, we have $\chi_x|_{\mathbf{G}^F_{\mathrm{uni}}} = 0$. So we are reduced to considering those $x \in \mathfrak{X}(\mathbf{W})$ for which A_x is a direct summand of some $K^{\mathscr{E}_1}_{\mathbf{L},\Sigma}$ as above, and with $\Sigma \subseteq \mathbf{L}$ being the preimage of a *unipotent* conjugacy class of $\mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ under the canonical map $\mathbf{L} \to \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$. We may then assume that $\mathbf{L} = \mathbf{L}_J$ is the standard Levi subgroup of the standard parabolic subgroup $\mathbf{P}_J \subseteq \mathbf{G}$ for some $J \subseteq S$. Hence, we are in the setting of the generalised Springer correspondence (see 3.2.13, 3.2.15), that is, we have $A_x \cong A_{\mathfrak{i}}$ for some $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}^F_{\mathbf{G}}$, and $\mathfrak{j} := \tau(\mathfrak{i}) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}^F_{\mathbf{G}}$ is such that $\Sigma = \mathbf{Z}(\mathbf{L}_J)^\circ.\mathscr{O}_0$ and $\mathscr{E}_1 \cong 1 \boxtimes \mathscr{E}_0$. By 3.1.21, we have

$$K_{\mathfrak{j}} = K^{\mathscr{E}_1}_{\mathbf{L}_J,\Sigma} \cong \mathrm{ind}^{\mathbf{G}}_{\mathbf{L}_J}(A_0) \quad \text{where} \quad A_0 = \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}_J} \in \hat{\mathbf{L}}^\circ_J.$$

The parametrisation of the isomorphism classes of the simple direct summands of $K_{\mathfrak{j}}$ in terms of the irreducible characters of the group $\mathscr{W}_{\mathfrak{j}} = W_{\mathbf{G}}(\mathbf{L}_J)$ (see 3.2.13) thus corresponds to the parametrisation of the isomorphism classes of the simple direct summands of $\mathrm{ind}^{\mathbf{G}}_{\mathbf{L}_J}(A_0)$ in terms of $\mathrm{Irr}(\mathbf{W}^{S/J})$ via the isomorphism $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ in 3.4.7. In particular, given $\mathfrak{j} = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}^F_{\mathbf{G}}$, as soon as *one* $A_{\mathfrak{i}}$ with $\mathfrak{i} \in \tau^{-1}(\mathfrak{j})$ is in $\hat{\mathbf{G}}^{\mathrm{un}}$, we know that *every* $A_{\mathfrak{i}}$ with $\mathfrak{i} \in \tau^{-1}(\mathfrak{j})$ is in $\hat{\mathbf{G}}^{\mathrm{un}}$. In this case, the task of matching the $\mathfrak{i} \in \tau^{-1}(\mathfrak{j})$ with the $x \in \mathfrak{X}(\mathbf{W})$ so that $A_{\mathfrak{i}} \cong A_x$ is exactly the task of determining the generalised Springer correspondence with respect to the block $\tau^{-1}(\mathfrak{j}) \subseteq \mathcal{N}^F_{\mathbf{G}}$, and this is now established in complete generality (see the remarks in 3.2.13). So if $x \in \mathfrak{X}(\mathbf{W})$ is such that there exists some $\mathfrak{i} \in \mathcal{N}^F_{\mathbf{G}}$ for which $A_x \cong A_{\mathfrak{i}}$, it will often be convenient to say that *x corresponds to $\mathfrak{i}$ under the generalised Springer correspondence.*

3.4.22. Let $x \in \mathfrak{X}(\mathbf{W})$ and $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}^F_{\mathbf{G}}$ correspond to each other under the generalised Springer correspondence, and let $\tau(\mathfrak{i}) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0)$ (with $J \subseteq S$). In the case where $J = S$, $A_x \cong A_\mathfrak{i}$ is a cuspidal (unipotent) character sheaf on $\mathbf{G}$, and we refer to 3.2.21 for the computation of the characteristic function χ_x. Let us now assume that $J = \varnothing$, so that $\tau(\mathfrak{i}) = (\mathbf{T}_0, \{1\}, 1)$, and $\mathfrak{i}$ is in the image of the ordinary Springer correspondence $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathcal{N}_\mathbf{G}^F$, see (3.2.13.4). Recall (2.2.5) that in this situation, the computation of $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ is almost completely known, and it is explained in detail in [Gec20a, §2, §3] and [Gec20b, §2, §3] (cf. [Sho06b, 1.1–1.3]), so let us describe this here only briefly. First of all, we make the following assumption on a given pair $\mathfrak{i}' = (\mathscr{O}', \mathscr{E}') \in \mathcal{N}_\mathbf{G}^F$ (or, rather, on the unipotent class $\mathscr{O}' \subseteq \mathbf{G}$):

$$\text{There exists an element } u_0' \in \mathscr{O}'^F \text{ such that } F \text{ acts trivially on } A_\mathbf{G}(u_0'). \qquad (\clubsuit)$$

(This will be satisfied in every case that we are concerned with later.) If, under the ordinary Springer correspondence, $\phi', \phi \in \mathrm{Irr}(\mathbf{W})$ are mapped to $\mathfrak{i}' = (\mathscr{O}', \mathscr{E}')$, $\mathfrak{i} = (\mathscr{O}, \mathscr{E})$, respectively, we set

$$p_{\phi',\phi} := p_{\mathfrak{i}',\mathfrak{i}} \text{ (see Theorem 3.2.16)} \quad \text{and} \quad d_\phi := \tfrac{1}{2}(\dim \mathbf{G} - \dim \mathscr{O} - \dim \mathbf{T}_0).$$

Furthermore, if $u_0' \in \mathscr{O}'^F$ is as in ($\clubsuit$) and $\varsigma' \in \mathrm{Irr}(A_\mathbf{G}(u_0'))$ parametrises the local system $\mathscr{E}'$ on $\mathscr{O}'$, the function $Y_{\phi'} := Y_{\mathfrak{i}'} \colon \mathbf{G}^F_{\mathrm{uni}} \to \overline{\mathbb{Q}}_\ell$ (see 3.2.15) is given by

$$Y_{\phi'}((u_0')_a) = \delta_{\phi'}\varsigma'(a) \quad \text{for } a \in A_\mathbf{G}(u_0'), \text{ where } \delta_{\phi'} \in \{\pm 1\}$$

(and $Y_{\phi'}(u) = 0$ if $u \in \mathbf{G}^F_{\mathrm{uni}} \setminus \mathscr{O}'^F$). We have

$$R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}} = \sum_{\phi' \in \mathrm{Irr}(\mathbf{W})} q^{d_\phi} p_{\phi',\phi} Y_{\phi'},$$

so in order to compute $R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}}$, the only task is to determine the signs $\delta_{\phi'}$ for $\phi' \in \mathrm{Irr}(\mathbf{W})$, and we recall from 2.2.5 that this has been accomplished in almost all cases.

3.4.23. Let $x \in \mathfrak{X}(\mathbf{W})$, $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathcal{N}_\mathbf{G}^F$ and $\tau(\mathfrak{i}) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0)$ be as in 3.4.22 (so that $A_x \cong A_\mathfrak{i}$), but we no longer require $J \subseteq S$ to be the empty set. Let $(J, \epsilon, \mathfrak{s}) \in \mathfrak{S}_\mathbf{W}$ (where $\epsilon \in \mathrm{Irr}(\mathbf{W}^{S/J})$, $\mathfrak{s} \in \mathfrak{S}^\circ_{\mathbf{W}_J}$) be the triple corresponding to x via Corollary 3.4.8. Let us fix an isomorphism $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$; we then define the isomorphisms $\varphi_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$, $\overline{\varphi}_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ as in 3.2.25 and set $\varphi_x := \overline{\varphi}_{A_\mathfrak{i}}$, $\chi_x := \chi_{A_x,\varphi_x}$. Let $\xi_x := \xi_x(\varphi_x)$ be as in (3.4.12.2), so that

$$R_x|_{\mathbf{G}^F_{\mathrm{uni}}} = \xi_x\chi_x|_{\mathbf{G}^F_{\mathrm{uni}}} = (-1)^{a_\mathfrak{i}} q^{(\dim \mathbf{G} - \dim \mathscr{O} - \dim \mathbf{Z}(\mathbf{L}_J))/2} \xi_x X_\mathfrak{i}. \qquad (3.4.23.1)$$

Now recall from 3.2.15 that $a_\mathfrak{i} \equiv \dim \operatorname{supp} A_\mathfrak{i} \pmod 2$, so the definition of $\zeta_x = \zeta_x(\varphi_x)$ in 3.4.12 shows that $(-1)^{a_\mathfrak{i}}\xi_x = \zeta_x$. Moreover, with the same argument as in [Lus86, 3.5] (cf. the proof of Corollary 3.4.17), we see that the root of unity ζ_x is independent of which of the above x, $\mathfrak{i}$ we choose, as long as they give rise to the triple $\tau(\mathfrak{i}) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0)$. We may thus set $\zeta(J, \mathfrak{s}) := \zeta_x$ and obtain

$$R_x|_{\mathbf{G}^F_{\mathrm{uni}}} = q^{(\dim \mathbf{G} - \dim \mathscr{O} - \dim \mathbf{Z}(\mathbf{L}_J))/2}\zeta(J, \mathfrak{s})X_\mathfrak{i}. \tag{3.4.23.2}$$

(If $\mathfrak{S}^\circ_{\mathbf{W}_J} = \{\mathfrak{s}\}$ is a singleton, we will usually just write $\zeta_J := \zeta(J, \mathfrak{s})$.) As for the computation of the function $X_\mathfrak{i} \colon \mathbf{G}^F_{\mathrm{uni}} \to \overline{\mathbb{Q}}_\ell$, we recall from Theorem 3.2.16 and Corollary 3.2.17 that it is given by an explicitly known linear combination of the functions $Y_{\mathfrak{i}'}$, $\mathfrak{i}' = (\mathscr{O}', \mathscr{E}') \in \mathcal{N}^F_\mathbf{G}$, and the coefficients at these $Y_{\mathfrak{i}'}$ can be obtained via CHEVIE. Up to multiplication with a root of unity, the functions $Y_{\mathfrak{i}'} \colon \mathbf{G}^F_{\mathrm{uni}} \to \overline{\mathbb{Q}}_\ell$ are given as follows, see [Sho06b, 1.3]: For simplicity, we assume that there exists $u'_0 \in \mathscr{O}'^F$ such that F acts trivially on $A_\mathbf{G}(u'_0)$. Then there exists a root of unity $\gamma_{\mathfrak{i}'} \in \mathcal{R}$ such that, if $\varsigma' \in \operatorname{Irr}(A_\mathbf{G}(u'_0))$ parametrises the local system $\mathscr{E}'$ on $\mathscr{O}'$, we have

$$Y_{\mathfrak{i}'}|_{\mathscr{O}'^F} = \gamma_{\mathfrak{i}'} Y^0_{\mathfrak{i}'} \quad \text{where} \quad Y^0_{\mathfrak{i}'} \colon \mathscr{O}'^F \to \overline{\mathbb{Q}}_\ell, \quad (u'_0)_a \mapsto \varsigma'(a).$$

In particular, if x and $\mathfrak{i} = (\mathscr{O}, \mathscr{E})$ are as above and if $u \in \mathbf{G}^F_{\mathrm{uni}}$ is an element of the unipotent class $\mathscr{O}'$ of $\mathbf{G}$, we see from Corollary 3.2.17 that $R_x(u)$ can only be non-zero if $\mathscr{O}' \subseteq \overline{\mathscr{O}}$. The determination of the roots of unity $\zeta(J, \mathfrak{s})$ and $\gamma_{\mathfrak{i}'}$ is open in general (cf. 3.2.19(a)), but knowing a value $R_x(u)$ up to these roots of unity already provides powerful information, and combining this with the method described in 3.4.19(2), we will manage to specify the signs appearing in 3.4.18 and also determine some actual character values at unipotent elements which were previously not known.

Remark 3.4.24. To summarise the discussion in 3.4.21–3.4.23: If we are interested in the values of the unipotent almost characters R_x (for $x \in \mathfrak{X}(\mathbf{W})$) at *unipotent* elements of $\mathbf{G}^F$, we are reduced to considering those x which correspond to some $\mathfrak{i} \in \mathcal{N}^F_\mathbf{G}$ under the generalised Springer correspondence (as any other R_x is identically zero on $\mathbf{G}^F_{\mathrm{uni}}$). If $u \in \mathbf{G}^F_{\mathrm{uni}}$ lies in the unipotent class $\mathscr{O}' \subseteq \mathbf{G}$ and if $x \leftrightarrow \mathfrak{i} = (\mathscr{O}, \mathscr{E})$ as above, we have $R_x(u) = 0$ unless $\mathscr{O}' \subseteq \overline{\mathscr{O}}$, so with respect to values at elements of $\mathscr{O}'^F$, we only have to consider such R_x for which x corresponds to pairs involving unipotent classes which are 'bigger' than $\mathscr{O}'$. The values of these $R_x|_{\mathscr{O}'^F}$ can then be determined up to some roots of unity; in the case where x corresponds to some $\phi \in \operatorname{Irr}(\mathbf{W})$, the values of $R_\phi|_{\mathscr{O}'^F}$ are described up to some signs, and these signs are in fact known in almost all cases.

4. Simple groups of exceptional type

As explained in Section 3.4 (specifically due to 3.4.15 and Corollary 3.4.17), with regard to determining the scalars ξ_x in (3.4.12.2) for simple groups $\mathbf{G}$ with a trivial centre and with a non-twisted $\mathbb{F}_q$-rational structure defined by a Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$, it remains to consider this task for the *cuspidal* unipotent character sheaves on *exceptional* groups $\mathbf{G}$, that is, the groups of type E_6, E_7, E_8, F_4 and G_2. In this chapter, we thus go through these groups $\mathbf{G}$ one by one and try to solve the following problem, or provide appropriate references in the cases where its solution is already known:

$$\begin{array}{l}\text{For any } x \in \mathfrak{X}(\mathbf{W}) \text{ such that } A_x \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}, \text{ specify an isomorphism} \\ \varphi_x\colon F^*A_x \xrightarrow{\sim} A_x \text{ and determine the scalar } \xi_x = \xi_x(\varphi_x) \text{ in (3.4.12.2).}\end{array} \tag{$*$}$$

(Here, $\mathbf{W}$ denotes the Weyl group of the group $\mathbf{G}$ in question.)

In fact, recall from 2.1.7 that a simple exceptional group $\mathbf{G}$ can only have a non-trivial centre when it is of type E_6 with $p \neq 3$ or of type E_7 with $p \neq 2$; in both of these cases, $\mathbf{G}$ is then necessarily of simply connected type, and Geck [Gec21] has solved $(*)$ for these groups as well, with an obvious generalisation of Shoji's Theorem 3.3.9 with respect to the *unipotent* character sheaves on $\mathbf{G}$. Therefore, as far as the problem $(*)$ is concerned, we can actually drop the assumption on $\mathbf{Z}(\mathbf{G})$. As for the choice of an isomorphism $\varphi_x\colon F^*A_x \xrightarrow{\sim} A_x$ in $(*)$, recall from 3.2.21 that if $\mathscr{C} \subseteq \mathbf{G}$ is the (F-stable) conjugacy class whose closure supports A_x, this is equivalent to singling out a $\mathbf{G}^F$-conjugacy class contained in $\mathscr{C}^F$. We always aim to make a 'good' choice for such a $\mathbf{G}^F$-conjugacy class with respect to the guidelines described in 3.2.22–3.2.23.

In Section 4.1, we consider the simple groups of type E_6 over $k = \overline{\mathbb{F}}_p$. Apart from the non-twisted groups $\mathsf{E}_6(q)$, we also have to deal with the twisted groups ${}^2\mathsf{E}_6(q)$ here, so we need to slightly expand the setting of Section 3.4 (see 4.1.1–4.1.6). If $p \neq 3$, Geck solved $(*)$ in [Gec21, §5], and we provide a detailed summary of these results in 4.1.7–4.1.11. The case where $p = 3$ was solved by the author in [Het19]; we give the complete exposition and proof of $(*)$ in 4.1.12–4.1.22. All of this holds both for the non-twisted groups $\mathsf{E}_6(q)$ and for the twisted groups ${}^2\mathsf{E}_6(q)$ (and, as mentioned above, regardless of the isogeny type of E_6). At least for the non-twisted groups $\mathsf{E}_6(q)$ (and also for the twisted groups

${}^2\mathsf{E}_6(q)$ when $p \geqslant 3$), we are then in a position to conclude the computation of unipotent characters at unipotent elements (4.1.23–4.1.27), which was previously not completely known when $p \leqslant 3$.

In Section 4.2, we look at groups of type E_7 over $k = \overline{\mathbb{F}}_p$. If $p \neq 2$, Geck solved $(*)$ in [Gec21, §6]; we give a detailed summary of his results in 4.2.3–4.2.6. The case where $p = 2$ has been solved by the author in [Het22a], and we present the full proof in 4.2.7–4.2.12. We also complete the computation of unipotent characters at unipotent elements for the groups $\mathsf{E}_7(q)$ where q is any power of any prime p (see 4.2.13 and 4.2.16–4.2.28), which was previously only known for $p \geqslant 5$. Especially in the case where $p = 2$, this requires much more elaborate arguments than before.

We continue in Section 4.3 with the groups of type G_2. The solution to $(*)$ is rather easy here since the generic character tables of $\mathsf{G}_2(p^n)$ $(n \in \mathbb{N})$ are completely known, due to Chang–Ree [CR74] for $p \geqslant 5$, Enomoto [Eno76] for $p = 3$, and Enomoto–Yamada [EY86] for $p = 2$. But we still put some emphasis on making 'good' choices for the isomorphisms $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ in $(*)$, in accordance with the guidelines in 3.2.22–3.2.23, cf. Remark 4.3.5.

In Section 4.4, we report on the solution to $(*)$ for groups of type F_4, which has been accomplished by Marcelo–Shinoda [MS95] and Geck ([Gec19, §5], [Gec21, §7]). We thus summarise their results and partly sketch the methods used in the proofs.

Hence, the solution to $(*)$ is then complete for all exceptional groups other than those of type E_8, which we consider in Section 4.5. For these groups, we are rather far away from having a full answer to $(*)$, but we can solve several different cases. There are 13 cuspidal character sheaves on the group $\mathbf{G}$ of type E_8, and all of them are unipotent. Here, we will only focus on those cuspidal (unipotent) character sheaves whose support is contained in the unipotent variety; their number depends on the characteristic p of k and is given by the following table:

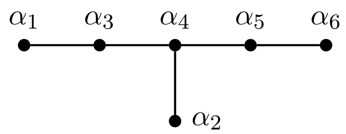
Characteristic	$\#\{A \in \hat{\mathbf{G}}^\circ \mid \operatorname{supp} A \subseteq \mathbf{G}_{\mathrm{uni}}\}$
$p \geqslant 7$	1
$p = 5$	5
$p = 3$	3
$p = 2$	5

There is always one cuspidal character sheaf on $\mathbf{G}$, denoted by A_1 below, which satisfies $\operatorname{supp} A_1 \subseteq \mathbf{G}_{\mathrm{uni}}$ and allows a uniform description regardless of the characteristic. If $p \geqslant 7$, we obtain the solution to $(*)$ with respect to A_1; if $p \leqslant 5$, we get congruence relations which yield the desired result as soon as the Green functions for groups of type E_8 in bad characteristic (cf. 2.2.5) are known, see 4.5.5, Proposition 4.5.7. As for the other cuspidal character sheaves occurring in the above table, we obtain the solution with respect to the four cuspidal character sheaves in characteristic $p = 5$ (Proposition 4.5.10), the two cuspidal character sheaves in characteristic $p = 3$ (Proposition 4.5.21) and two of the four cuspidal character sheaves in characteristic $p = 2$ (Proposition 4.5.29). Our arguments are mostly based on applying the strategy described in 3.4.19(2), in combination with the explicit knowledge of the generalised Springer correspondence to evaluate the unipotent almost characters at relevant unipotent elements up to certain roots of unity as described in 3.4.21–3.4.24. In particular, for the groups $\mathsf{E}_8(3^n)$ ($n \in \mathbb{N}$), this provides an application of the author's result in [Het22b] (see Theorem 4.5.13 below) completing the determination of the generalised Springer correspondence. As a by-product, we also obtain the values of some almost characters R_x at certain unipotent elements where $x \in \mathfrak{X}(\mathbf{W}) \setminus \mathrm{Irr}(\mathbf{W})$, see Corollary 4.5.22 and Corollary 4.5.30.

4.1. Groups of type E_6

In this section, we assume that $\mathbf{G}$ is the simple adjoint group of type E_6 over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ where q is a power of p, with Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. We fix an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ and an F-stable maximal torus $\mathbf{T}_0 \subseteq \mathbf{B}_0$. Thus, we get the associated Weyl group $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ and denote by $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ the automorphism induced by F. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum attached to $\mathbf{G}$ and $\mathbf{T}_0$ (where $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), with underlying bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$. Let $R^+ \subseteq R$ be the positive roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$, and let $\Pi = \{\alpha_1, \ldots, \alpha_6\} \subseteq R^+$ be the corresponding simple roots, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_6^\vee\}$ be the corresponding simple co-roots. We choose the order of $\alpha_1, \ldots, \alpha_6$ in such a way that the Dynkin diagram of $\mathbf{G}$ is as follows:

E_6



Let $\mathfrak{C} = (\langle \alpha_j, \alpha_i^\vee \rangle)_{1 \leqslant i,j \leqslant 6}$ be the associated Cartan matrix. Furthermore, let $\mathbf{U}_0 = R_{\mathrm{u}}(\mathbf{B}_0)$ be the unipotent radical of $\mathbf{B}_0$; then $\mathbf{B}_0$ is the semidirect product of $\mathbf{U}_0$ and $\mathbf{T}_0$ (with $\mathbf{U}_0$ being normal in $\mathbf{B}_0$). As described in 2.1.19, F induces a p-isogeny of root data

$$\varphi\colon X \to X, \quad \lambda \mapsto \lambda \circ F|_{\mathbf{T}_0},$$

as well as a bijection $R \to R$, $\alpha \mapsto \alpha^\dagger$, so that $\varphi(\alpha^\dagger) = q\alpha$ for all $\alpha \in R$ (since $F\colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{G}$). The assignment $\alpha \mapsto \alpha^\dagger$ restricts to a graph automorphism of the Dynkin diagram, so there are two possible cases: Either $\alpha \mapsto \alpha^\dagger$ is the identity (then $\mathbf{G}^F$ is the non-twisted group $\mathsf{E}_6(q)$, and $\sigma = \mathrm{id}_{\mathbf{W}}$), or else it is a map of order 2 (then $\mathbf{G}^F$ is the twisted group ${}^2\mathsf{E}_6(q)$, and $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$ is the inner automorphism given by conjugation with the longest element w_0 of $\mathbf{W}$). We identify $\mathbf{W}$ with a subgroup of $\mathrm{Aut}(X)$ (via 2.1.4), and for $1 \leqslant i \leqslant 6$, we set $s_i := w_{\alpha_i}$, with w_{α_i} being defined as in 2.1.2. Thus, $\mathbf{W}$ is a Coxeter group with Coxeter generators $S = \{s_1, \ldots, s_6\}$, which are arranged in the Coxeter diagram with the analogous numbering as in the Dynkin diagram of $\mathbf{G}$ printed above (see 2.1.5). We use the notation of Lusztig [Lus84a, 4.11] for the irreducible characters of $\mathbf{W}$, which is essentially due to Frame [Fra51], with one small difference: Given an irreducible character d_p (respectively, d_q) of $\mathbf{W}$ in [Fra51], we write $d_p' = d_p \otimes 1_p'$ (respectively, $d_q' = d_q \otimes 1_p'$), where $1_p'$ is the sign character of $\mathbf{W}$.

Parametrisations of unipotent characters and unipotent character sheaves in type E_6

4.1.1. Since σ is an inner automorphism of $\mathbf{W}$, we always have $\mathrm{Irr}(\mathbf{W})^\sigma = \mathrm{Irr}(\mathbf{W})$, and we can henceforth drop the superscript σ. As we have stated in 2.2.12, if $\sigma = \mathrm{id}_{\mathbf{W}}$, there is a canonical bijection between the sets $\overline{\mathfrak{X}}(\mathbf{W}, \sigma)$ and $\mathfrak{X}(\mathbf{W})$. But even in the case where σ is not the identity, the sets $\overline{\mathfrak{X}}(\mathbf{W}, \sigma)$ and $\mathfrak{X}(\mathbf{W})$ can be identified for groups of type E_6, see [Lus84a, 4.19]. Hence, the embedding (2.2.12.6) is just the embedding (2.2.8.4):

$$\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{X}(\mathbf{W}), \quad \phi \mapsto x_\phi. \tag{4.1.1.1}$$

Let us describe the set $\mathfrak{X}(\mathbf{W})$ and the families of $\mathrm{Irr}(\mathbf{W})$. We have $|\mathfrak{X}(\mathbf{W})| = 30$, $|\mathrm{Irr}(\mathbf{W})| = 25$, and $\mathrm{Irr}(\mathbf{W})$ is partitioned into 17 families: 14 of them consist of a single character; the characters which form these one-element families are

$$1_p,\ 6_p,\ 20_p,\ 64_p,\ 60_p,\ 81_p,\ 24_p,\ 1_p',\ 6_p',\ 20_p',\ 64_p',\ 60_p',\ 81_p',\ 24_p'.$$

Then there are two families consisting of 3 characters, namely,

$$\mathcal{F}_3 = \{30_p, 15_p, 15_q\} \quad \text{and} \quad \mathcal{F}_{15} = \{30_p', 15_p', 15_q'\}.$$

We have $\mathcal{G}_{\mathcal{F}_3} = \mathcal{G}_{\mathcal{F}_{15}} = C_2$, so that $|\mathfrak{M}(\mathcal{G}_{\mathcal{F}_3})| = |\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{15}})| = 4$. In order to conform with the notation that Lusztig uses in [Lus84a, 4.3], we denote the elements of $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_3})$ and

$\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{15}})$ by

$$(1,1), \quad (1,\varepsilon), \quad (g_2,1), \quad (g_2,\varepsilon),$$

where g_2 is the non-trivial element and ε is the non-trivial irreducible character of C_2. Finally, there is one family consisting of 5 characters, namely,

$$\mathcal{F}_7 = \{80_s, 20_s, 60_s, 10_s, 90_s\} \subseteq \mathrm{Irr}(\mathbf{W}).$$

We have $\mathcal{G}_{\mathcal{F}_7} = \mathfrak{S}_3$; as in [Lus84a, 4.3], we denote the 8 elements of $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_7}) \subseteq \mathfrak{X}(\mathbf{W})$ by

$$(1,1), \quad (1,\varepsilon), \quad (g_2,1), \quad (g_3,1), \quad (1,r), \quad (g_2,\varepsilon), \quad (g_3,\omega), \quad (g_3,\omega^2),$$

with the following conventions: For $i = 2, 3$, $g_i \in \mathfrak{S}_3$ is an i-cycle; the irreducible characters of $C_{\mathcal{G}_{\mathcal{F}_7}}(1) = \mathfrak{S}_3$ are named $1, \varepsilon, r$ (trivial, sign, reflection), the irreducible characters of $C_{\mathcal{G}_{\mathcal{F}_7}}(g_2) \cong C_2$ are denoted by $1, \varepsilon$ as above, and the irreducible characters of $C_{\mathcal{G}_{\mathcal{F}_7}}(g_3) = \langle g_3 \rangle \cong C_3$ are denoted by their values at g_3, where $\omega \in \mathcal{R}$ is a fixed primitive 3rd root of unity which, in the case where $\sigma = \mathrm{id}_{\mathbf{W}}$, we assume to be the same as the one in 3.4.2. For each of the three non-trivial families $\mathcal{F} \subseteq \mathrm{Irr}(\mathbf{W})$, the embedding $\mathcal{F} \hookrightarrow \mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ is obtained by matching the elements of $\mathcal{F}$ with the first $|\mathcal{F}|$ elements of $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ in the respective orders as listed above; it can also be read off from Table 4.1 below. This gives the embedding (4.1.1.1).

4.1.2. We want to describe the parametrisations of unipotent characters and unipotent character sheaves in terms of $\mathfrak{X}(\mathbf{W})$. Let $c \leqslant 2$ be the order of $\sigma \in \mathrm{Aut}(\mathbf{W})$. Instead of working with the infinite set $\mathfrak{X}(\mathbf{W}, \sigma)$ in the set-up of 2.2.12, it suffices to consider a finite subset of $\mathfrak{X}(\mathbf{W}, \sigma)$ which can be identified with $\mathfrak{X}(\mathbf{W}) \times \mathcal{R}_c$ where $\mathcal{R}_c \subseteq \mathcal{R}$ denotes the group of all cth roots of unity in $\overline{\mathbb{Q}}_\ell$. With these identifications, the pairing (2.2.12.1) restricts to a pairing

$$\{\ ,\ \}\colon \mathfrak{X}(\mathbf{W}) \times (\mathfrak{X}(\mathbf{W}) \times \mathcal{R}_c) \to \overline{\mathbb{Q}}_\ell, \ \{x, (y, a)\} = a^{-1}\{x, y\} \qquad (4.1.2.1)$$

for any $x, y \in \mathfrak{X}(\mathbf{W})$, $a \in \mathcal{R}_c$, where $\{x, y\}$ is defined as in 2.2.8. The action of all roots of unity $\mathcal{R}$ on $\mathfrak{X}(\mathbf{W}, \sigma)$ induces (by restriction) an action of $\mathcal{R}_c$ on $\mathfrak{X}(\mathbf{W}) \times \mathcal{R}_c$ given by multiplication on the second factor, and the embedding (2.2.12.5) gives rise (again by restriction) to an embedding

$$\mathrm{Irr}(\mathbf{W}(\sigma)) \hookrightarrow \mathfrak{X}(\mathbf{W}) \times \mathcal{R}_c, \qquad (4.1.2.2)$$

such that a σ-extension of $\phi \in \mathrm{Irr}(\mathbf{W})$ is mapped to (x_ϕ, a) for some $a \in \mathcal{R}_c$. For

$\phi \in \mathrm{Irr}(\mathbf{W})$, let $\tilde{\phi}\colon \mathbf{W} \to \overline{\mathbb{Q}}_\ell$ be the preferred σ-extension of ϕ, as defined in [LuCS4, 17.2]. Then $\tilde{\phi}$ is mapped to $(x_\phi, 1)$ under (4.1.2.2), see [Lus84a, 4.19]. In view of Example 2.2.24, we have

$$R_{\tilde{\phi}} = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \tilde{\phi}(w) R_w, \tag{4.1.2.3}$$

and there is a bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho, \tag{4.1.2.4}$$

such that

$$\langle \rho, R_{\tilde{\phi}} \rangle_{\mathbf{G}^F} = \Delta(x_\rho)\{x_\rho, x_\phi\} \quad \text{for any } \rho \in \mathrm{Uch}(\mathbf{G}^F),\ \phi \in \mathrm{Irr}(\mathbf{W}). \tag{4.1.2.5}$$

Now the entries $\{x, y\}$ $(x, y \in \mathfrak{X}(\mathbf{W}))$ of the Fourier matrix with respect to the Weyl group of type E_6 are all in $\mathbb{Q}$, hence so are the values of the pairing (4.1.2.1) as $c \leqslant 2$. So for any $(x, a) \in \mathfrak{X}(\mathbf{W}) \times \mathcal{R}_c$, the unipotent almost character $R_{(x,a)}$ (see (2.2.24.3)) is given by

$$R_{(x,a)} = \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \Delta(x_\rho)\{x_\rho, (x, a)\} \cdot \rho. \tag{4.1.2.6}$$

In particular, in case $c = 2$, we have $R_{(x,-1)} = -R_{(x,1)}$ for $x \in \mathfrak{X}(\mathbf{W})$. In any case, we have $R_{\tilde{\phi}} = R_{(x_\phi,1)}$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$. In the setting of 2.2.23, we take

$$\mathfrak{X}_0(0,1) := \{(x, 1) \mid x \in \mathfrak{X}(\mathbf{W})\} \subseteq \mathfrak{X}(\mathbf{W}) \times \mathcal{R}_c \hookrightarrow \mathfrak{X}(\mathbf{W}, \sigma)$$

as a set of representatives for the $\mathcal{R}$-orbits inside $\mathfrak{X}(\mathbf{W}, \sigma)$, so that we can (and will) identify the sets $\mathfrak{X}_0(0, 1)$ and $\mathfrak{X}(\mathbf{W})$. Thus,

$$\{R_{\tilde{\phi}} \mid \phi \in \mathrm{Irr}(\mathbf{W})\} \subseteq \{R_x \mid x \in \mathfrak{X}(\mathbf{W})\}.$$

By (2.2.24.4), we have

$$\rho = \Delta(x_\rho) \sum_{x \in \mathfrak{X}(\mathbf{W})} \{x_\rho, x\} \cdot R_x \quad \text{for } \rho \in \mathrm{Uch}(\mathbf{G}^F). \tag{4.1.2.7}$$

On the other hand, since $\overline{\mathfrak{X}}(\mathbf{W}, \sigma) \cong \mathfrak{X}(\mathbf{W})$, we have $\hat{\mathbf{G}}^{\mathrm{un}} = (\hat{\mathbf{G}}^{\mathrm{un}})^F$ by Theorem 3.3.2 and 3.3.12. So there is a bijection

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}} = (\hat{\mathbf{G}}^{\mathrm{un}})^F, \quad x \mapsto A_x, \tag{4.1.2.8}$$

such that

$$(A_x : R_\phi^{\mathscr{L}_0}) = \hat{\varepsilon}_{A_x}\{x, x_\phi\} \quad \text{for any } x \in \mathfrak{X}(\mathbf{W}),\ \phi \in \mathrm{Irr}(\mathbf{W}). \tag{4.1.2.9}$$

4.1.3. We keep the setting and notation of 4.1.2. By the proof of [LuCS4, 20.3(a)] (see also [Sho95b, 4.6]), there are exactly two cuspidal unipotent character sheaves A_1, A_2 on $\mathbf{G}$ (for any p); with the notation of 4.1.1, they are parametrised by

$$x_1 := (g_3, \omega) \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_7}) \quad \text{and} \quad x_2 := (g_3, \omega^2) \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_7})$$

under (4.1.2.8). Now the bijection (4.1.2.8) is not quite uniquely determined by the condition (4.1.2.9), but we see from an inspection of the Fourier matrix that the only ambiguity arises as far as the two cuspidal unipotent character sheaves are concerned. To fix this, note that 3.4.3(c) also applies to the case where $\sigma \neq \mathrm{id}_{\mathbf{W}}$, and in fact the root of unity $\lambda_A \in \mathcal{R}$ associated to $A \in \hat{\mathbf{G}}^{\mathrm{un}}$ only depends on A and not on the $\mathbb{F}_q$-structure on $\mathbf{G}$, see [Sho95a, 3.3]. So we first consider the non-twisted case $\sigma = \mathrm{id}_{\mathbf{W}}$ and specify the bijection (4.1.2.8) via Corollary 3.4.8. In particular, this implies that $\lambda_{A_{x_i}} = \tilde{\lambda}_{x_i}$ for $i = 1, 2$, so $\lambda_{A_{x_1}} = \omega$, $\lambda_{A_{x_2}} = \omega^2$. We then just impose the same condition for the twisted case $\sigma \neq \mathrm{id}_{\mathbf{W}}$, and this uniquely determines the bijection (4.1.2.8). We number the cuspidal unipotent character sheaves A_1, A_2 above in such a way that

$$A_1 = A_{x_1} \quad \text{and} \quad A_2 = A_{x_2}.$$

For $x \in \mathfrak{X}(\mathbf{W})$, let us for now choose any isomorphism $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ as in 3.2.1(∗). (For the cuspidal unipotent character sheaves A_1 and A_2 we will make an explicit choice below, depending on the characteristic p of k.) We denote by

$$\chi_x := \chi_{A_x,\varphi_x} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$$

the corresponding characteristic function. Thus, by (3.3.12.2) and (3.3.12.3), we have

$$R_x = \xi_x \chi_x \text{ for } x \in \mathfrak{X}(\mathbf{W}),\ \text{where } \xi_x = \xi_x(\varphi_x) \in \overline{\mathbb{Q}}_\ell^\times,\ |\xi_x| = 1. \tag{4.1.3.1}$$

Similarly to the parametrisation of the unipotent character sheaves, the bijection (4.1.2.4) which satisfies (4.1.2.5) will be uniquely determined as soon as we impose the additional requirement that $\lambda_{\rho_{x_1}} = \omega$, $\lambda_{\rho_{x_2}} = \omega^2$. (Cf. condition (ii) in Proposition 3.4.5; here we denote by $\rho_x \in \mathrm{Uch}(\mathbf{G}^F)$ the character corresponding to $x \in \mathfrak{X}(\mathbf{W})$; the fact that the two cuspidal unipotent characters in the twisted case also have Frobenius eigenvalues ω

and ω^2 follows from [Lus76b, (7.3)] and its proof.) So let us fix the bijection (4.1.2.4) in this way. Then, with the notation in the appendix of [Lus84a] or in [Car85, pp. 480–481], we have, for $i = 1, 2$:

$$\rho_{x_i} = \begin{cases} \mathsf{E}_6[\omega^i] & \text{if } \sigma = \mathrm{id}_{\mathbf{W}}, \\ {}^2\mathsf{E}_6[\omega^i] & \text{if } \sigma \neq \mathrm{id}_{\mathbf{W}}\,. \end{cases}$$

4.1.4. Let us describe the Harish-Chandra series of unipotent characters for groups of type E_6. We first assume that $\sigma = \mathrm{id}_{\mathbf{W}}$; thus, the parametrisations of $\mathrm{Uch}(\mathbf{G}^F)$ and $\hat{\mathbf{G}}^{\mathrm{un}}$ that we fixed in 4.1.3 are the ones given by Corollary 3.4.8. In view of the results of [Lus78, pp. 31–37] (see also the appendix of [Lus84a]), the 30 unipotent characters of $\mathbf{G}^F = \mathsf{E}_6(q)$ fall into the following Harish-Chandra series.

(a) There are 25 unipotent characters in the principal series, i.e., the irreducible constituents of $\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F}(1_{\mathbf{B}_0^F})$; these are parametrised by the irreducible characters of $\mathbf{W} = W(\mathsf{E}_6)$.

(b) Let $J = \{s_2, s_3, s_4, s_5\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)^\circ$ is simple of type D_4. We have $\mathfrak{S}_{\mathbf{W}_J}^\circ = \{(-1, 2)\}$ (so $\mathbf{L}_J^F$ has a unique cuspidal unipotent character, ρ_0, say), and the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J)$ is isomorphic to $W(\mathsf{A}_2) \cong \mathfrak{S}_3$. As before, we denote the three irreducible characters of this group by 1, ε, r (trivial, sign, reflection); accordingly, we denote the three irreducible constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ by $\mathsf{D}_4[1]$, $\mathsf{D}_4[\varepsilon]$, $\mathsf{D}_4[r]$, respectively.

(c) For $J = S$, the set $\mathfrak{S}_{\mathbf{W}_J}^\circ = \mathfrak{S}_{\mathbf{W}}^\circ$ consists of the two elements $(\omega, 3)$, $(\omega^2, 3)$ parametrising the two cuspidal unipotent characters $\mathsf{E}_6[\omega]$, $\mathsf{E}_6[\omega^2]$, respectively.

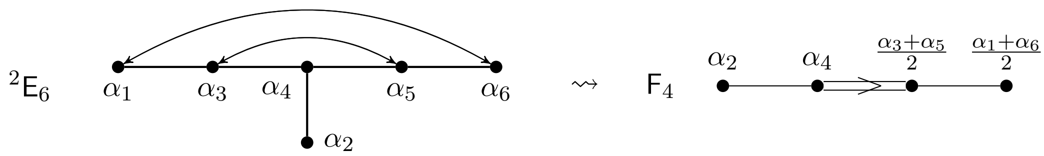
4.1.5. We now assume that $\sigma \neq \mathrm{id}_{\mathbf{W}}$ is the automorphism of $\mathbf{W}$ given by conjugation with the longest element $w_0 \in \mathbf{W}$. The 30 unipotent characters of $\mathbf{G}^F = {}^2\mathsf{E}_6(q)$ fall into the following Harish-Chandra series (see again [Lus78, pp. 31–37]).

(a) There are 25 unipotent characters in the principal series. These irreducible constituents of $\mathrm{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F}(1_{\mathbf{B}_0^F})$ are parametrised by the irreducible characters of $\mathbf{W}^\sigma$.

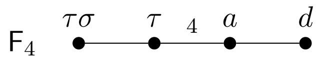
(b) Let $J = \{s_1, s_3, s_4, s_5, s_6\}$. There are two unipotent characters appearing as irreducible constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$, where ρ_0 is the unique cuspidal unipotent character of $\mathbf{L}_J^F$; note that F induces a non-split $\mathbb{F}_q$-rational structure on $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)^\circ$, which thus gives rise to the twisted group ${}^2\mathsf{A}_5(q)$. The irreducible constituents of $R_{\mathbf{L}_J}^{\mathbf{G}}(\rho_0)$ are parametrised by the irreducible characters ± 1 of the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong C_2$; accordingly, we denote these two unipotent characters by ${}^2\mathsf{A}_5[1]$ and ${}^2\mathsf{A}_5[-1]$.

(c) Finally, there are three cuspidal unipotent characters ${}^2\mathsf{E}_6[1]$, ${}^2\mathsf{E}_6[\omega]$ and ${}^2\mathsf{E}_6[\omega^2]$ (despite the fact that there are only two cuspidal character sheaves on $\mathbf{G}$).

Recall (2.1.20) that in (a), $\mathbf{W}^\sigma$ is a Weyl group of type F_4 with Coxeter generators $S_\sigma = \{s_2, s_4, s_3s_5, s_1s_6\}$; the simple roots of $\mathbf{W}^\sigma$ induced from the ones in $\mathbf{W}$ are α_2, α_4, $\frac{1}{2}(\alpha_3+\alpha_5)$, $\frac{1}{2}(\alpha_1+\alpha_6)$ (see [Car72, 13.3.3]). Thus, α_2 and α_4 are long roots, while $\frac{1}{2}(\alpha_3+\alpha_5)$, $\frac{1}{2}(\alpha_1+\alpha_6)$ are short roots; we may picture this as follows (see [GH22, §4]):



The character table of $W(\mathsf{F}_4)$ has been determined by Kondo [Kon65], who denotes the Coxeter generators by $\tau\sigma, \tau, a, d$ such that the associated Coxeter diagram is



(The σ appearing in $\tau\sigma$ is not to be confused with our automorphism $\sigma \in \mathrm{Aut}(\mathbf{W})$ here.) But note that there are two possible ways of matching s_2, s_4, s_3s_5, s_1s_6 with $\tau\sigma, \tau, a, d$. Following [GH22, §4], we identify

$$s_2 \leftrightarrow \tau\sigma, \quad s_4 \leftrightarrow \tau, \quad s_3s_5 \leftrightarrow a, \quad s_1s_6 \leftrightarrow d,$$

so that $\tau\sigma, \tau$ correspond to reflections in the long roots α_2, α_4, and a, d to reflections in the short roots $\frac{1}{2}(\alpha_3+\alpha_5)$, $\frac{1}{2}(\alpha_1+\alpha_6)$. With this identification, we then denote by d_j the jth irreducible character of degree d in Kondo's character table [Kon65] when referring to elements of $\mathrm{Irr}(\mathbf{W}^\sigma)$. (This is the same notation and convention that Lusztig uses in [Lus84a].)

4.1.6. The discussion in 4.1.3 uniquely specifies a bijection

$$\mathrm{Uch}(\mathsf{E}_6(q)) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \mathrm{Uch}({}^2\mathsf{E}_6(q)). \tag{4.1.6.1}$$

Let us explicitly describe this bijection, following Lusztig [Lus80, 1.10, 1.14–1.16] (but with the conventions in 4.1.5, which match with the ones in [Lus84a]). For $x \in \mathfrak{X}(\mathbf{W})$, let $\rho_x \in \mathrm{Uch}(\mathsf{E}_6(q))$ and $\rho'_x \in \mathrm{Uch}({}^2\mathsf{E}_6(q))$ be the associated unipotent characters under (4.1.6.1). Then the degree of ρ'_x is obtained from the one of ρ_x by replacing q by $-q$ and, if necessary, changing the sign to make the resulting number positive. As observed in

4.1.3, for $i = 1, 2$ and $x_i = (g_3, \omega^i) \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_7})$, we have $\rho_{x_i} = \mathsf{E}_6[\omega^i]$ and $\rho'_{x_i} = {}^2\mathsf{E}_6[\omega^i]$. The remaining 28 unipotent characters are uniquely determined by their degree. The bijection (4.1.6.1) is thus given by Table 4.1; in this table, the names of the unipotent characters are as in 4.1.4, 4.1.5; since each of the 14 lines on the left represents a family consisting of just one element, there is no need to provide the label in $\mathfrak{X}(\mathbf{W})$; any of the other three blocks is dedicated to a non-trivial family $\mathcal{F}$, and the column on the right gives the labels in $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ (with the notation of 4.1.1).

$\mathsf{E}_6(q)$	${}^2\mathsf{E}_6(q)$
$[1_p]$	$[1_1]$
$[6_p]$	$[2_1]$
$[20_p]$	$[4_2]$
$[64_p]$	${}^2\mathsf{A}_5[1]$
$[60_p]$	$[4_3]$
$[81_p]$	$[9_2]$
$[24_p]$	$[8_3]$
$[1'_p]$	$[1_4]$
$[6'_p]$	$[2_2]$
$[20'_p]$	$[4_5]$
$[64'_p]$	${}^2\mathsf{A}_5[-1]$
$[60'_p]$	$[4_4]$
$[81'_p]$	$[9_3]$
$[24'_p]$	$[8_4]$

$\mathsf{E}_6(q)$	${}^2\mathsf{E}_6(q)$	$\mathfrak{X}(\mathbf{W})$
$[30_p]$	$[2_3]$	$(1,1)$
$[15_p]$	$[9_1]$	$(1,\varepsilon)$
$[15_q]$	$[1_2]$	$(g_2,1)$
$\mathsf{D}_4[1]$	$[8_1]$	(g_2,ε)

$\mathsf{E}_6(q)$	${}^2\mathsf{E}_6(q)$	$\mathfrak{X}(\mathbf{W})$
$[30'_p]$	$[2_4]$	$(1,1)$
$[15'_p]$	$[9_4]$	$(1,\varepsilon)$
$[15'_q]$	$[1_3]$	$(g_2,1)$
$\mathsf{D}_4[\varepsilon]$	$[8_2]$	(g_2,ε)

$\mathsf{E}_6(q)$	${}^2\mathsf{E}_6(q)$	$\mathfrak{X}(\mathbf{W})$
$[80_s]$	${}^2\mathsf{E}_6[1]$	$(1,1)$
$[20_s]$	$[12_1]$	$(1,\varepsilon)$
$[60_s]$	$[4_1]$	$(g_2,1)$
$[10_s]$	$[6_1]$	$(g_3,1)$
$[90_s]$	$[6_2]$	$(1,r)$
$\mathsf{D}_4[r]$	$[16_1]$	(g_2,ε)
$\mathsf{E}_6[\omega]$	${}^2\mathsf{E}_6[\omega]$	(g_3,ω)
$\mathsf{E}_6[\omega^2]$	${}^2\mathsf{E}_6[\omega^2]$	(g_3,ω^2)

Table 4.1.: The bijection $\mathrm{Uch}(\mathsf{E}_6(q)) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \mathrm{Uch}({}^2\mathsf{E}_6(q))$ described in 4.1.6

Type E_6 in characteristic $p \neq 3$

In this subsection (that is, here and in 4.1.7–4.1.11 below), we assume that $p \neq 3$. In this case, the scalars ξ_x in (4.1.3.1) for the two $x \in \mathfrak{X}(\mathbf{W})$ parametrising cuspidal (unipotent) character sheaves have been determined by Geck [Gec21, §5]. To describe his results, we consider the simple, simply connected group $\mathbf{G}_{\mathrm{sc}}$ of type E_6 and the canonical map $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$. By [GM20, 1.5.9] (see [Ste68, 9.16]), there exists a unique isogeny $\tilde{F}\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$ such that $F \circ \pi = \pi \circ \tilde{F}$, and $\tilde{F}$ is a Frobenius map which provides $\mathbf{G}_{\mathrm{sc}}$

with an $\mathbb{F}_q$-rational structure.

4.1.7. There are exactly two cuspidal unipotent character sheaves on $\mathbf{G}_{\mathrm{sc}}$, and they are given by

$$\widetilde{A}_1 := \pi^*(A_1) \quad \text{and} \quad \widetilde{A}_2 := \pi^*(A_2), \tag{4.1.7.1}$$

where $A_1, A_2 \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ are as in 4.1.3. In particular, both $\widetilde{A}_1$ and $\widetilde{A}_2$ are $\tilde{F}$-stable. Let $\tilde{s} \in \mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$ be a semisimple element such that $C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})$ has a root system of type $\mathsf{A}_2 \times \mathsf{A}_2 \times \mathsf{A}_2$, and let $\tilde{u} \in C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})^{\tilde{F}}$ be a regular unipotent element in $C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})$. Denote by $\widetilde{\Sigma} \subseteq \mathbf{G}_{\mathrm{sc}}$ the ($\tilde{F}$-stable) conjugacy class containing the element $\tilde{g} := \tilde{s}\tilde{u} = \tilde{u}\tilde{s}$. (Up to $\mathbf{G}_{\mathrm{sc}}$-conjugacy, $\tilde{g}$ is uniquely determined by the above conditions; we refer to [Gec21, 5.1, 5.2] for a more detailed description of $\widetilde{\Sigma}$.) For $h \in C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})$, we denote by $\overline{h}$ the image of h in $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}) = C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})/C^{\circ}_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})$. Let $1 \neq z \in \mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$ be one of the two non-trivial elements of the centre of $\mathbf{G}_{\mathrm{sc}}$. It is shown in [Gec21, 5.2] that $\overline{\tilde{g}} = \overline{\tilde{s}}$ and that

$$A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}) = \langle \overline{\tilde{s}} \rangle \times \langle \overline{z} \rangle \cong C_3 \times C_3.$$

The action of $\tilde{F}$ on $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})$ depends on the congruence of q modulo 3. Namely, we have

$$\tilde{F}(\overline{z}) = \begin{cases} \overline{z} & \text{if} \quad q \equiv 1 \pmod 3, \\ \overline{z}^{-1} & \text{if} \quad q \equiv 2 \pmod 3, \end{cases}$$

so

$$A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})^{\tilde{F}} = \begin{cases} \langle \overline{\tilde{s}} \rangle \times \langle \overline{z} \rangle & \text{if} \quad q \equiv 1 \pmod 3, \\ \langle \overline{\tilde{s}} \rangle & \text{if} \quad q \equiv 2 \pmod 3. \end{cases}$$

But in any case, we see that $\widetilde{\Sigma}^{\tilde{F}}$ splits into an odd number (either 9 or 3) of $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy classes. Hence, there must be at least one $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-class inside $\widetilde{\Sigma}^{\tilde{F}}$ which is stable under taking inverses. Let us choose such a $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class, and let $\tilde{g}_0$ be a representative for this class; thus, $\tilde{g}_0$ is $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugate to $\tilde{g}_0^{-1}$. Now let $\Sigma := \pi(\widetilde{\Sigma})$, a conjugacy class of $\mathbf{G}$. Having fixed $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$, we set $g_0 := \pi(\tilde{g}_0) \in \Sigma^F$, so that g_0 is $\mathbf{G}^F$-conjugate to g_0^{-1}. Furthermore, let $\tilde{g}_0 = \tilde{s}_0\tilde{u}_0 = \tilde{u}_0\tilde{s}_0$ and $g_0 = s_0u_0 = u_0s_0$ be the Jordan decompositions of $\tilde{g}_0$ and g_0, respectively. By the proof of [LuCS4, 20.4] (see also [Sho95b, 4.6]), there exists a certain $a \in A_{\mathbf{G}}(g_0)$ such that

$$A_{\mathbf{G}}(g_0) \cong \langle \overline{g}_0 \rangle \times \langle a \rangle \cong C_3 \times C_3.$$

For $i = 1, 2$, let $\varsigma_i \in \mathrm{Irr}(A_{\mathbf{G}}(g_0))$ be the linear character of $A_{\mathbf{G}}(g_0)$ such that $\varsigma_i(a) = 1$ and $\varsigma_i(\overline{g}_0) = \omega^i$ (with $\omega \in \mathcal{R}$ the primitive 3rd root of unity that we fixed in 4.1.1), and let $\mathscr{E}_i$

be a one-dimensional $\mathbf{G}$-equivariant irreducible local system on Σ whose isomorphism class corresponds to ς_i (via 3.2.20). Then

$$A_1 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E}_1)[\dim\Sigma]^{\#\mathbf{G}} \quad \text{and} \quad A_2 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E}_2)[\dim\Sigma]^{\#\mathbf{G}}.$$

It is shown in [Gec21, 5.2] that $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})\widetilde{\Sigma} = \widetilde{\Sigma}$, so $\pi^{-1}(\Sigma) = \widetilde{\Sigma}$, and it follows from (4.1.7.1) that

$$\widetilde{A}_1 \cong \mathrm{IC}\Big(\overline{\widetilde{\Sigma}}, \pi^*\mathscr{E}_1\Big)[\dim\Sigma]^{\#\mathbf{G}_{\mathrm{sc}}} \quad \text{and} \quad \widetilde{A}_2 \cong \mathrm{IC}\Big(\overline{\widetilde{\Sigma}}, \pi^*\mathscr{E}_2\Big)[\dim\Sigma]^{\#\mathbf{G}_{\mathrm{sc}}}.$$

The map $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ canonically induces a map $\overline{\pi}\colon A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) \to A_{\mathbf{G}}(g_0)$, and the isomorphism classes of the (one-dimensional, $\mathbf{G}_{\mathrm{sc}}$-equivariant) irreducible local systems $\widetilde{\mathscr{E}}_i := \pi^*\mathscr{E}_i$ on $\widetilde{\Sigma}$ correspond to the linear characters

$$\widetilde{\varsigma}_i := \varsigma_i \circ \overline{\pi} \in \mathrm{Irr}(A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)), \quad i = 1, 2.$$

For $i = 1, 2$, let $\varphi_{A_i}\colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphisms corresponding to $g_0 \in \Sigma^F$ (via 3.2.21). Applying π^* on either side and using some standard properties of the inverse image functor, we thus obtain induced isomorphisms

$$\varphi_{\tilde{A}_i} := \pi^*(\varphi_{A_i})\colon \tilde{F}^*\widetilde{A}_i \xrightarrow{\sim} \widetilde{A}_i, \quad i = 1, 2.$$

It is clear from the above construction that $\varphi_{\tilde{A}_i}$ is the isomorphism corresponding to the choice of $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$ via 3.2.21. We will from now on fix the choice of g_0, $\tilde{g}_0$ and, hence, of the isomorphisms φ_{A_i}, $\varphi_{\tilde{A}_i}$ as just described. Furthermore, for $i = 1, 2$, when writing φ_{x_i}, χ_{x_i} or $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i})$, this is meant to be as in 4.1.3 but with respect to the isomorphisms $\varphi_{x_i} := \varphi_{A_i}$ that we just defined. The above discussion shows that we have the following identity concerning the characteristic functions:

$$\chi_{\tilde{A}_i, \varphi_{\tilde{A}_i}} = \chi_{A_i, \varphi_{A_i}} \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} = \chi_{x_i} \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} \quad \text{for } i = 1, 2. \tag{4.1.7.2}$$

4.1.8. As remarked in [Gec21, 5.4], at least with respect to the *unipotent* almost characters and character sheaves, Shoji's Theorem 3.3.9 can also be formulated for the simply connected group $\mathbf{G}_{\mathrm{sc}}$ of type E_6 in a natural way, as follows: Recall from 3.3.8 that, in general, almost characters are only defined provided the underlying connected reductive group has a connected centre, and extending this concept to the non-connected centre case appears to be a delicate problem (cf. [Lus18]). However, as far as unipotent almost characters are concerned, there is a natural definition: By [DL76, 7.10] (see also

[GM20, 2.3.15]), we have a bijection

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathrm{Uch}(\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}), \quad \rho \mapsto \rho \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}}. \tag{4.1.8.1}$$

Hence, in the definition of a unipotent almost character with respect to $\mathbf{G}$, we may just replace any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ by $\rho \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} \in \mathrm{Uch}(\mathbf{G}_{\mathrm{sc}}^{\tilde{F}})$ and set

$$R_x^{\mathbf{G}_{\mathrm{sc}}} := \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)} \Delta(x_\rho)\{x_\rho, x\} \cdot \left(\rho \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}}\right) \in \mathrm{CF}(\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}) \quad \text{for } x \in \mathfrak{X}(\mathbf{W}),$$

that is,

$$R_x^{\mathbf{G}_{\mathrm{sc}}} = R_x^{\mathbf{G}} \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} \quad \text{for } x \in \mathfrak{X}(\mathbf{W}).$$

(Here, in order to avoid any confusion, we write $R_x^{\mathbf{G}} \in \mathrm{CF}(\mathbf{G}^F)$ instead of R_x for the unipotent almost character defined in 4.1.2 with respect to the adjoint group $\mathbf{G}$.) Thus, composing either side of (4.1.3.1) with $\pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}}$ and using (4.1.7.2), we obtain

$$R_{x_i}^{\mathbf{G}_{\mathrm{sc}}} = \xi_{x_i} \chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} \quad \text{for } i = 1, 2, \text{ where } \quad \chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} := \chi_{\tilde{A}_i, \varphi_{\tilde{A}_i}}. \tag{4.1.8.2}$$

We briefly sketch the idea of [Gec21, §5] to determine the scalars ξ_{x_i} in (4.1.8.2), illustrated for the non-twisted simply connected group $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}} = (\mathsf{E}_6)_{\mathrm{sc}}(q)$. First of all, since $\tilde{g}_0 \in \tilde{\Sigma}^{\tilde{F}}$ is chosen so that condition ($\diamondsuit$) in 3.4.18 holds, we know that $\xi_{x_i} \in \{\pm 1\}$ for $i = 1, 2$, and a similar computation shows that $\xi_{x_1} = \xi_{x_2}$. (In fact, the necessary computation is entirely analogous to the one that we will carry out in detail in 4.1.17 below, as the only property of u_0 which we will exploit there is that it is $\mathbf{G}^F$-conjugate to u_0^{-1}.) So we have

$$R_{x_i}^{\mathbf{G}_{\mathrm{sc}}} = \xi \cdot \chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} \quad \text{for } i = 1, 2, \text{ where } \xi := \xi_{x_1} = \xi_{x_2} \in \{\pm 1\}.$$

Consider the cuspidal unipotent character $\mathsf{E}_6[\omega]^{\mathbf{G}_{\mathrm{sc}}} \in \mathrm{Uch}(\mathbf{G}_{\mathrm{sc}}^{\tilde{F}})$. Applying Lusztig's non-abelian Fourier transform, one gets

$$\mathsf{E}_6[\omega]^{\mathbf{G}_{\mathrm{sc}}} = \tfrac{1}{3}\big(R_{80_s}^{\mathbf{G}_{\mathrm{sc}}} - R_{90_s}^{\mathbf{G}_{\mathrm{sc}}} + R_{20_s}^{\mathbf{G}_{\mathrm{sc}}} - R_{10_s}^{\mathbf{G}_{\mathrm{sc}}} + \xi(2\chi_{x_1}^{\mathbf{G}_{\mathrm{sc}}} - \chi_{x_2}^{\mathbf{G}_{\mathrm{sc}}})\big).$$

The $R_\phi^{\mathbf{G}_{\mathrm{sc}}}$ ($\phi \in \mathrm{Irr}(\mathbf{W})$) have their values in $\mathbb{Q}$, and we see from 3.2.21 that $\chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) = q^3$ for $i = 1, 2$. Thus, $\mathsf{E}_6[\omega]^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$ is an algebraic integer (being the value of a character) and at the same time a rational number, so it must be a rational integer. Hence, as soon as the value

$$R_{80_s}^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) - R_{90_s}^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) + R_{20_s}^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) - R_{10_s}^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) \in \mathbb{Q}$$

is computed, we have

$$\mathsf{E}_6[\omega]^{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) = \tfrac{1}{3}\big(\text{known value in } \mathbb{Q} + \xi q^3\big) \in \mathbb{Z}. \tag{4.1.8.3}$$

Since 3 does not divide q, it is clear that this condition will then uniquely specify the sign $\xi \in \{\pm 1\}$. Now the computation of the unipotent uniform almost characters $R_\phi^{\mathbf{G}_{\mathrm{sc}}}$ for $\phi \in \mathrm{Irr}(\mathbf{W})$ is equivalent to that of the virtual Deligne–Lusztig characters $R_w^{\mathbf{G}_{\mathrm{sc}}}$ for $w \in \mathbf{W}$, which in turn is reduced to the computation of Green functions of groups smaller than $\mathbf{G}_{\mathrm{sc}}$ in view of the character formula (2.2.4.2). The technical problem of explicitly evaluating all the ingredients in this formula is worked out in [Gec21, §2, §3]. Once this is accomplished, one finds that $\xi = \xi_{x_1} = \xi_{x_2} = +1$. The twisted group ${}^2\mathsf{E}_6(q)$ is treated in a completely analogous way, by considering the cuspidal unipotent character ${}^2\mathsf{E}_6[\omega]$, and one also obtains $\xi = \xi_{x_1} = \xi_{x_2} = +1$. To summarise:

Proposition 4.1.9 (Geck [Gec21, 5.5])**.** *As in* 4.1.7 *(and with the further notation there), let $\tilde{g}_0 \in \tilde{\Sigma}^{\tilde{F}}$ be an element which is $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugate to $\tilde{g}_0^{-1}$, so that $g_0 = \pi(\tilde{g}_0) \in \Sigma^F$ is $\mathbf{G}^F$-conjugate to g_0^{-1}. For $i = 1, 2$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of g_0, and let $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i}) \in \overline{\mathbb{Q}}_\ell^\times$ be defined by* (4.1.3.1)*. Then*

$$\xi_{x_1} = \xi_{x_2} = 1,$$

that is, the characteristic function $\chi_{x_i} = \chi_{A_i,\varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ coincides with the unipotent almost character R_{x_i} for $i = 1, 2$, both for the non-twisted group $\mathbf{G}^F = \mathsf{E}_6(q)$ and for the twisted group $\mathbf{G}^F = {}^2\mathsf{E}_6(q)$, where q is any power of any prime $p \neq 3$. The analogous statement holds for the simply connected group $\mathbf{G}_{\mathrm{sc}}$ (using $\tilde{g}_0$ to define the characteristic functions $\chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} \colon \mathbf{G}_{\mathrm{sc}}^{\tilde{F}} \to \overline{\mathbb{Q}}_\ell$ for $i = 1, 2$).

4.1.10. Following [Gec21, p. 27], for groups of type E_6 in characteristic $p \neq 3$, we can now give the values of the cuspidal unipotent characters at elements of $\tilde{\Sigma}^{\tilde{F}}$, Σ^F. As in Proposition 4.1.9, let us fix an element $\tilde{g}_0 \in \tilde{\Sigma}^{\tilde{F}}$ which is $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugate to $\tilde{g}_0^{-1}$. In Table 4.2, the different elements of $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$ appearing in the top line represent different $\tilde{F}$-conjugacy classes of $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$, which in turn correspond to different $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy classes inside $\tilde{\Sigma}^{\tilde{F}}$ (respectively, different $\mathbf{G}^F$-conjugacy classes inside Σ^F), via 3.2.20(a). So if multiple elements appear in the top line of a column (which happens for the case $q \equiv 1 \pmod 3$), this means that the cuspidal unipotent characters listed in a given line take the same value on any of the corresponding $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy classes inside $\tilde{\Sigma}^{\tilde{F}}$ (respectively, on any of the corresponding $\mathbf{G}^F$-conjugacy classes inside Σ^F). In the case where $q \equiv 1 \pmod 3$, $\tilde{F}$ acts trivially on $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$, so the $\tilde{F}$-conjugacy classes of

$A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) \cong C_3 \times C_3$ are just singletons, and $\widetilde{\Sigma}^{\tilde{F}}$, Σ^F split into 9 different conjugacy classes of $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$, $\mathbf{G}^F$, respectively. If, on the other hand, $q \equiv 2 \pmod 3$, there are 3 different $\tilde{F}$-conjugacy classes in $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$ (consisting of 3 elements each), represented by the elements $\overline{1}$, $\overline{\tilde{s}}_0$, $\overline{\tilde{s}}_0^2$, and, correspondingly, $\widetilde{\Sigma}^{\tilde{F}}$, Σ^F split into 3 different conjugacy classes of $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$, $\mathbf{G}^F$, respectively. Finally note that Table 4.2 covers both the adjoint group and the simply connected group of type E_6, in view of (4.1.8.1).

$q \equiv 1 \pmod 3$	1, $\overline{z}$, $\overline{z}^2$	$\overline{\tilde{s}}_0$, $\overline{\tilde{s}}_0\overline{z}$, $\overline{\tilde{s}}_0\overline{z}^2$	$\overline{\tilde{s}}_0^2$, $\overline{\tilde{s}}_0^2\overline{z}$, $\overline{\tilde{s}}_0^2\overline{z}^2$
$\mathsf{E}_6[\omega]$, ${}^2\mathsf{E}_6[\omega]$	$\frac{1}{3}(q^3-1)$	$\frac{1}{3}(q^3-1)+\omega q^3$	$\frac{1}{3}(q^3-1)+\omega^2 q^3$
$\mathsf{E}_6[\omega^2]$, ${}^2\mathsf{E}_6[\omega^2]$	$\frac{1}{3}(q^3-1)$	$\frac{1}{3}(q^3-1)+\omega^2 q^3$	$\frac{1}{3}(q^3-1)+\omega q^3$

$q \equiv 2 \pmod 3$	1	$\overline{\tilde{s}}_0$	$\overline{\tilde{s}}_0^2$
$\mathsf{E}_6[\omega]$, ${}^2\mathsf{E}_6[\omega]$	$\frac{1}{3}(q^3+1)$	$\frac{1}{3}(q^3+1)+\omega q^3$	$\frac{1}{3}(q^3+1)+\omega^2 q^3$
$\mathsf{E}_6[\omega^2]$, ${}^2\mathsf{E}_6[\omega^2]$	$\frac{1}{3}(q^3+1)$	$\frac{1}{3}(q^3+1)+\omega^2 q^3$	$\frac{1}{3}(q^3+1)+\omega q^3$

Table 4.2.: Values of cuspidal unipotent characters on $\widetilde{\Sigma}^{\tilde{F}}, \Sigma^F$ in type E_6, $p \neq 3$

Remark 4.1.11. (a) Note that the only assumption which is made on the $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class of $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$ in Proposition 4.1.9 is that it is stable under taking inverses. In the case where $q \equiv 2 \pmod 3$, this requirement in fact uniquely determines the $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class of $\tilde{g}_0$ inside $\widetilde{\Sigma}^{\tilde{F}}$ (as well as the $\mathbf{G}^F$-conjugacy class of $g_0 = \pi(\tilde{g}_0)$ inside Σ^F), as we see at once by looking at Table 4.2 since the values of the cuspidal unipotent characters on the conjugacy classes parametrised by $\overline{\tilde{s}}_0$, $\overline{\tilde{s}}_0^2$ are non-real. On the other hand, if $q \equiv 1 \pmod 3$, there are 3 such $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-classes inside $\widetilde{\Sigma}^{\tilde{F}}$, parametrised by $\overline{1}, \overline{z}, \overline{z}^2$ with respect to a chosen $\tilde{g}_0$. (But, in view of how the proposition is stated, normalising the characteristic functions χ_{x_i} via another $\tilde{g}_0$ which is $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugate to $\tilde{g}_0^{-1}$ also leads to $\xi_{x_i} = 1$ for $i = 1, 2$.)

(b) Now let us compare this with 3.2.22(c): We set

$$\mathbf{B}_0^{\mathrm{sc}} := \pi^{-1}(\mathbf{B}_0),\ \mathbf{T}_0^{\mathrm{sc}} := \pi^{-1}(\mathbf{T}_0),\ \mathbf{W}^{\mathrm{sc}} := N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{T}_0^{\mathrm{sc}})/\mathbf{T}_0^{\mathrm{sc}}.$$

Thus, $\mathbf{B}_0^{\mathrm{sc}} \subseteq \mathbf{G}_{\mathrm{sc}}$ is an $\tilde{F}$-stable Borel subgroup, and $\mathbf{T}_0^{\mathrm{sc}}$ is an $\tilde{F}$-stable maximal torus of $\mathbf{G}_{\mathrm{sc}}$ contained in $\mathbf{B}_0^{\mathrm{sc}}$ (see [Bor91, 11.14]); furthermore, $\mathbf{W}^{\mathrm{sc}}$ is the Weyl group of $\mathbf{G}^{\mathrm{sc}}$ with respect to $\mathbf{T}_0^{\mathrm{sc}}$. So both $\mathbf{W}$ and $\mathbf{W}^{\mathrm{sc}}$ are Coxeter groups in a natural way (see 2.1.5). The map $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ induces an isomorphism $\mathbf{W}^{\mathrm{sc}} \xrightarrow{\sim} \mathbf{W}$, which by construction is compatible with the automorphisms $\sigma\colon \mathbf{W} \xrightarrow{\sim} \mathbf{W}$, $\tilde{\sigma}\colon \mathbf{W}^{\mathrm{sc}} \xrightarrow{\sim} \mathbf{W}^{\mathrm{sc}}$ induced by F, $\tilde{F}$, respectively. We set

$$w_{\mathrm{c}} := s_1 s_6 s_3 s_5 s_2 s_4 \in \mathbf{W},$$

a Coxeter element of $\mathbf{W}$ which satisfies $\sigma(w_{\mathrm{c}}) = w_{\mathrm{c}}$ (regardless of whether $\sigma = \mathrm{id}_{\mathbf{W}}$ or $\sigma \neq \mathrm{id}_{\mathbf{W}}$). Thus, the corresponding element $w_{\mathrm{c}}^{\mathrm{sc}} \in \mathbf{W}^{\mathrm{sc}}$ is a Coxeter element of $\mathbf{W}^{\mathrm{sc}}$, and we have $\tilde{\sigma}(w_{\mathrm{c}}^{\mathrm{sc}}) = w_{\mathrm{c}}^{\mathrm{sc}}$. Regardless of the congruence of q modulo 3, we conclude from [Gec21, 4.9, 4.10] that a $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class contained in $\widetilde{\Sigma}^{\tilde{F}}$ is stable under taking inverses if and only if it has a non-empty intersection with $(\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}} w_{\mathrm{c}}^{\mathrm{sc}} (\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}}$. So the admissible representatives $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$ for Proposition 4.1.9 are exactly those which satisfy the condition ($\heartsuit$) in 3.2.22(c) (with respect to the simply connected group $\mathbf{G}_{\mathrm{sc}}$).

Type E_6 in characteristic $p = 3$

In this subsection (that is, here and in 4.1.12–4.1.22 below), we assume that $p = 3$, so that $\mathbf{G}$ is the simple adjoint group of type E_6 over $k = \overline{\mathbb{F}}_3$. This prime requires a distinct treatment from the others: First of all, as we shall see below, the two cuspidal unipotent character sheaves (most notably their support) look quite different. But even if we could get a rationality condition such as (4.1.8.3) (for a suitable $\tilde{g}_0$ in the support of the cuspidal character sheaves), this would not provide any information on the sign ξ, since if the right side of (4.1.8.3) is an integer for $\xi = +1$, it will certainly also be an integer for $\xi = -1$ (and vice versa). We thus need to find another argument. Note that, since $p = 3$, we do not really have to distinguish between the simply connected group $\mathbf{G}_{\mathrm{sc}}$ and the adjoint group $\mathbf{G} = \mathbf{G}_{\mathrm{ad}}$ of type E_6. Indeed, consider the canonical isogeny $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ (see 2.1.7); its kernel is contained in $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$, which by (2.1.7.1) is isomorphic to $\mathrm{Hom}(\Lambda(\mathfrak{C}), k^\times)$. But for groups of type E_6, the fundamental group $\Lambda(\mathfrak{C})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, so the only homomorphism $\Lambda(\mathfrak{C}) \to k^\times$ is the trivial one. Thus, $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) = \{1\}$, so $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G} = \mathbf{G}_{\mathrm{ad}}$ is bijective and restricts to an isomorphism on the level of finite groups $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}} \xrightarrow{\sim} \mathbf{G}^F$ (see [GM20, 1.5.12]) where $\tilde{F}\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$ is the endomorphism commuting with π and induced by F; $\tilde{F}$ is a Frobenius map for an $\mathbb{F}_q$-rational structure on $\mathbf{G}_{\mathrm{sc}}$.

The following (4.1.12–4.1.19) is based on [Het19].

4.1.12. By [LuCS4, 20.3], the two cuspidal unipotent character sheaves A_1, A_2 on $\mathbf{G}$ are supported by the unipotent variety $\mathbf{G}_{\mathrm{uni}}$ of $\mathbf{G}$, that is, the closure of the regular unipotent class $\mathscr{O}_{\mathrm{reg}} \subseteq \mathbf{G}$ (see 2.1.11). So by Proposition 3.1.17 (see also 3.2.21), there exist $\mathbf{G}$-equivariant F-stable irreducible local systems $\mathscr{E}_1$, $\mathscr{E}_2$ on $\mathscr{O}_{\mathrm{reg}}$ such that

$$A_i \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_i)[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}} \quad \text{for } i = 1, 2.$$

Let $u \in \mathscr{O}_{\text{reg}}^F$. The group $A_{\mathbf{G}}(u) = {}^{C_{\mathbf{G}}(u)}/{}_{C^\circ_{\mathbf{G}}(u)}$ is cyclic of order 3 and generated by the image $\overline{u}$ of u in $A_{\mathbf{G}}(u)$ (see, e.g., [Miz77, §6] and [DM20, 12.2.3, 12.2.7]). Thus, the automorphism of $A_{\mathbf{G}}(u)$ induced by F is the identity, and the elements of $A_{\mathbf{G}}(u)$ correspond to the $\mathbf{G}^F$-conjugacy classes contained in $\mathscr{O}_{\text{reg}}^F$. In particular, there are three such classes, so it is clear that (at least) one of them is stable under taking inverses. Here we can make an explicit choice: For $1 \leqslant i \leqslant 6$, let us denote by $u_i := u_{\alpha_i}$ the homomorphism $\mathbf{G}_{\mathrm{a}} \to \mathbf{G}$ whose image is the root subgroup $\mathbf{U}_{\alpha_i} \subseteq \mathbf{U}_0$ (see 2.1.4). We then set

$$u_0 := \begin{cases} u_1(1) \cdot u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_6(1) & \text{if } \sigma = \mathrm{id}_{\mathbf{W}}, \\ u_1(1) \cdot u_6(1) \cdot u_3(1) \cdot u_5(1) \cdot u_2(1) \cdot u_4(1) & \text{if } \sigma \neq \mathrm{id}_{\mathbf{W}}\,. \end{cases} \tag{4.1.12.1}$$

Thus, we have $u_0 \in \mathbf{U}_0^F \cap \mathscr{O}_{\text{reg}}^F$ in either case (see [DM20, 12.2.2]).

Lemma 4.1.13. *The element* $u_0 \in \mathscr{O}_{\text{reg}}^F$ *defined in* (4.1.12.1) *is conjugate to* u_0^{-1} *in* $\mathbf{G}^F$.

Proof. Taking

$$t := \alpha_2^\vee(-1)\alpha_3^\vee(-1)\alpha_4^\vee(-1)\alpha_5^\vee(-1) \in \mathbf{T}_0^F,$$

we have

$$tu_0t^{-1} = \begin{cases} u_1(-1) \cdot u_2(-1) \cdot u_3(-1) \cdot u_4(-1) \cdot u_5(-1) \cdot u_6(-1) & \text{if } \sigma = \mathrm{id}_{\mathbf{W}}, \\ u_1(-1) \cdot u_6(-1) \cdot u_3(-1) \cdot u_5(-1) \cdot u_2(-1) \cdot u_4(-1) & \text{if } \sigma \neq \mathrm{id}_{\mathbf{W}}\,. \end{cases}$$

To get u_0^{-1}, we thus need to find a $\mathbf{G}^F$-conjugate of tu_0t^{-1} in which the $u_i(-1)$ appear in reversed order compared to the expression for tu_0t^{-1}. This can be achieved by mimicking a proof for the well-known fact that any two Coxeter elements in a given Coxeter group are conjugate (see, e.g., [Cas17, §1], to which we will refer in some more detail in the proof of Lemma 4.2.8 below). Specifically, setting

$$v := \begin{cases} u_6(-1)u_5(-1)u_6(-1)u_4(-1)u_5(-1)u_6(-1)u_1(1) & \text{if } \sigma = \mathrm{id}_{\mathbf{W}}, \\ u_4(-1)u_6(1)u_1(1) & \text{if } \sigma \neq \mathrm{id}_{\mathbf{W}}, \end{cases}$$

we have $(vt)u_0(vt)^{-1} = u_0^{-1}$ and $v \in \mathbf{U}_0^F$, so $vt \in \mathbf{B}_0^F \subseteq \mathbf{G}^F$. □

4.1.14. As noted in 4.1.12, the group $A_{\mathbf{G}}(u_0)$ is cyclic of order 3 and generated by $\overline{u}_0$, and F induces the identity map on $A_{\mathbf{G}}(u_0)$. So in view of 3.2.20, there are (up to isomorphism) three $\mathbf{G}$-equivariant irreducible local systems on $\mathscr{O}_{\text{reg}}$, each of which is one-dimensional and F-stable. By [Sho95b, 4.6], the isomorphism classes of the $\mathscr{E}_i$

($i = 1, 2$) correspond to the two non-trivial linear characters of $A_{\mathbf{G}}(u_0)$. For $i = 1, 2$, we denote by $\varsigma_i \in \mathrm{Irr}(A_{\mathbf{G}}(u_0))$ the character which describes the local system $\mathscr{E}_i$. Since we required $A_i = A_{x_i}$ in 4.1.3, it follows that

$$\varsigma_1(\overline{u}_0) = \omega \quad \text{and} \quad \varsigma_2(\overline{u}_0) = \omega^2.$$

Now that we have fixed the choice of u_0, we also obtain uniquely defined isomorphisms $\varphi_{A_i} \colon F^*A_i \xrightarrow{\sim} A_i$ for $i = 1, 2$, as described in 3.2.21. We thus also fix $\varphi_{x_i} := \varphi_{A_i}$ and assume that χ_{x_i} and $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i})$ are defined with respect to these choices of φ_{x_i} in the setting of 4.1.3, for $i = 1, 2$. By Theorem 3.1.13, the characteristic functions $\chi_{x_1}, \chi_{x_2} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ are identically zero on $\mathbf{G}^F \setminus \mathscr{O}_{\mathrm{reg}}^F$. Their values on $\mathscr{O}_{\mathrm{reg}}^F$ are given by the following table, where the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$ are described by giving the corresponding element of $A_{\mathbf{G}}(u_0)$ in the top line of each column.

	1	$\overline{u}_0$	$\overline{u}_0^2$
χ_{x_1}	q^3	ωq^3	$\omega^2 q^3$
χ_{x_2}	q^3	$\omega^2 q^3$	ωq^3

In particular, we see that $\overline{\chi}_{x_1} = \chi_{x_2}$. We want to prove the analogue to Proposition 4.1.9 for $p = 3$ with respect to the element u_0 defined in (4.1.12.1), so we have to show that the scalars ξ_{x_1}, ξ_{x_2} are both equal to 1 (see Proposition 4.1.19 below). We start with the following lemma; we may instead just directly refer to [Lus84b, §15] (which is what we will typically do later on), but since this is the first instance of this kind of argument, let us give a detailed proof (or, rather, detailed references) to indicate what the typical arguments look like in this context.

Lemma 4.1.15. *Let $\mathbf{L} \subseteq \mathbf{G}$ be a Levi complement of some parabolic subgroup of $\mathbf{G}$. Assume that $\mathbf{L}$ is neither a torus nor the full group $\mathbf{G}$, and denote by $\mathbf{L}_{\mathrm{ss}} := \mathbf{L}/\mathbf{Z}(\mathbf{L})^\circ$ the corresponding semisimple algebraic group. Then, unless $\mathbf{L}_{\mathrm{ss}}$ is simple of type D_4, there are no cuspidal character sheaves on $\mathbf{L}$.*

Proof. Assume that $\mathbf{L} \subseteq \mathbf{G}$ is neither a torus nor the full group $\mathbf{G}$ and that $\hat{\mathbf{L}}^\circ$ is non-empty. From the reduction arguments in [Lus84b, 2.10], we deduce that $\hat{\mathbf{L}}_{\mathrm{ss}}^\circ$ must be non-empty as well. Since $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\mathbf{G})^\circ$, we also have $\mathbf{Z}(\mathbf{L}) = \mathbf{Z}(\mathbf{L})^\circ$ [DM20, 11.2.1] and, hence, $\mathbf{Z}(\mathbf{L}_{\mathrm{ss}}) = \{1\}$. Thus, the adjoint quotient (see 2.1.7)

$$(\pi_{\mathbf{L}_{\mathrm{ss}}})_{\mathrm{ad}} \colon \mathbf{L}_{\mathrm{ss}} \to (\mathbf{L}_{\mathrm{ss}})_{\mathrm{ad}}$$

is in fact a bijective isogeny of algebraic groups. It follows that the inverse image functor $(\pi_{\mathbf{L}_{\mathrm{ss}}})^*_{\mathrm{ad}}\colon \mathscr{D}((\mathbf{L}_{\mathrm{ss}})_{\mathrm{ad}}) \to \mathscr{D}\mathbf{L}_{\mathrm{ss}}$ defines a bijection between the isomorphism classes of cuspidal character sheaves on $(\mathbf{L}_{\mathrm{ss}})_{\mathrm{ad}}$ and those on $\mathbf{L}_{\mathrm{ss}}$ (see, e.g., [Tay14, 3.9]), so we may assume without loss of generality that $\mathbf{L}_{\mathrm{ss}}$ is of adjoint type. Let $\mathbf{L}_{\mathrm{ss}}^1, \ldots, \mathbf{L}_{\mathrm{ss}}^r$ ($r \in \mathbb{N}$) be the simple (adjoint) factors of $\mathbf{L}_{\mathrm{ss}}$. Thus, the product map defines an isomorphism

$$\mathbf{L}_{\mathrm{ss}}^1 \times \ldots \times \mathbf{L}_{\mathrm{ss}}^r \xrightarrow{\sim} \mathbf{L}_{\mathrm{ss}}.$$

In view of [LuCS4, 17.11], the cuspidal character sheaves on $\mathbf{L}_{\mathrm{ss}}^1 \times \ldots \times \mathbf{L}_{\mathrm{ss}}^r$ are of the form $A_0^1 \boxtimes \ldots \boxtimes A_0^r$ where A_0^i is a cuspidal character sheaf on $\mathbf{L}_{\mathrm{ss}}^i$ for $1 \leqslant i \leqslant r$. By Example 3.1.20, there are no cuspidal character sheaves for adjoint groups of type A_n ($n \in \mathbb{N}$), so we know that none of the $\mathbf{L}_{\mathrm{ss}}^i$ can be of type A_n. Furthermore, by the proof of [LuCS4, 19.3], there are no cuspidal character sheaves for the adjoint simple group of type D_5 either. The only remaining possibility is that $r = 1$ and $\mathbf{L}_{\mathrm{ss}}$ is simple of type D_4, so the lemma is proved. □

Proposition 4.1.16. *Let $x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_1, x_2\}$ be an element which is not in the image of the map $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{X}(\mathbf{W})$ in* (4.1.1.1)*. Then the characteristic function χ_x of the character sheaf A_x satisfies*

$$\chi_x|_{\mathbf{G}^F_{\mathrm{uni}}} = 0.$$

Proof. (a) Let $x \in \mathfrak{X}(\mathbf{W})$ and consider the corresponding unipotent character sheaf $A_x \in \hat{\mathbf{G}}^{\mathrm{un}}$. By 3.2.5, there exist a regular subgroup $\mathbf{L} \subseteq \mathbf{G}$ and an F-stable cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{L}$ such that A_x is isomorphic to a direct summand of $K = K^{\mathscr{E}}_{\mathbf{L},\Sigma} \in \mathscr{M}\mathbf{G}$. Let

$$A_0 := \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{L}} \in \hat{\mathbf{L}}^{\circ,\mathrm{un}}$$

be the corresponding F-stable cuspidal unipotent character sheaf on $\mathbf{L}$. Then, according to 3.2.6 and 3.2.7 (and using the notation there), χ_x is a linear combination of various $R^{\mathbf{G}}_{\mathbf{L}_{w^{-1}}}(\chi_{(A_0)_{w^{-1}}})$, $w \in \mathscr{W}^{\mathscr{E}}_{\mathbf{L},\Sigma}$.

Let us first assume that $\mathbf{L}$ is a torus. Then $(A_0)_{w^{-1}}$ is of the form $(\mathscr{L}_0)_{w^{-1}}[\mathrm{rank}\,\mathbf{G}]$ (where $(\mathscr{L}_0)_{w^{-1}} \cong \overline{\mathbb{Q}}_\ell$ is the trivial local system on $\mathbf{T}_{w^{-1}}$), and its characteristic function is (up to scalar multiplication) just the trivial character of $\mathbf{T}^F_{w^{-1}}$ (see 3.1.3). Thus, χ_x is in fact a linear combination of the virtual Deligne–Lusztig characters $R_w = R^{\mathbf{G}}_{\mathbf{T}_w}(1)$, $w \in \mathbf{W}$. The number of different R_w equals the number of the σ-conjugacy classes in $\mathbf{W}$, which by [GKP00, 7.3] is equal to $|\mathrm{Irr}(\mathbf{W})^\sigma| = |\,\mathrm{Irr}(\mathbf{W})|$. But the R_{x_ϕ} ($\phi \in \mathrm{Irr}(\mathbf{W})$) already constitute $|\,\mathrm{Irr}(\mathbf{W})|$ many pairwise orthogonal class functions which are linear combinations of the R_w, so we must have $x = x_\phi$ for some $\phi \in \mathrm{Irr}(\mathbf{W})$.

On the other hand, $\mathbf{L} = \mathbf{G}$ holds precisely when A_x is cuspidal, that is, if and only if $x \in \{x_1, x_2\}$.

(b) Now assume that $x \in \mathfrak{X}(\mathbf{W})$ satisfies the requirements in the proposition. In view of Lemma 4.1.15 and by what we have shown in (a), the character sheaf A_x is a constituent of $K_{\mathbf{L},\Sigma}^{\mathscr{E}} \in \mathscr{M}\mathbf{G}$ where $\mathbf{L} \subseteq \mathbf{G}$ is a regular subgroup such that $\mathbf{L}_{\mathrm{ss}}$ is simple of type D_4, and $(\Sigma, \mathscr{E})$ is an F-stable cuspidal pair for $\mathbf{L}$. We want to apply Corollary 3.2.8, so we are reduced to showing that Σ does not contain any unipotent elements.

Let $\pi\colon \mathbf{L} \to \mathbf{L}_{\mathrm{ss}}$ be the canonical map. By [LuCS4, 17.10] and [Lus84b, 2.10], the (shifted) inverse image functor

$$\pi^*[\dim \mathbf{Z}(\mathbf{L})]\colon \mathscr{D}\mathbf{L}_{\mathrm{ss}} \to \mathscr{D}\mathbf{L}$$

defines a bijection $\hat{\mathbf{L}}_{\mathrm{ss}}^{\circ,\mathrm{un}} \xrightarrow{\sim} \hat{\mathbf{L}}^{\circ,\mathrm{un}}$. Since $\mathbf{Z}(\mathbf{L}_{\mathrm{ss}}) = \{1\}$ and since $p \neq 2$, $\mathbf{L}_{\mathrm{ss}}$ is necessarily the adjoint group $\mathrm{PSO}_8(k)$ of type D_4. By the proof of [LuCS4, 19.3], all the (four) cuspidal character sheaves on this group have the same support, namely, the closure of the conjugacy class $\mathscr{C} \subseteq \mathbf{L}_{\mathrm{ss}}$ of s_1u_1 where $s_1 \in \mathbf{L}_{\mathrm{ss}}$ is a semisimple element such that $C^{\circ}_{\mathbf{L}_{\mathrm{ss}}}(s_1)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$, and u_1 is a regular unipotent element in $C^{\circ}_{\mathbf{L}_{\mathrm{ss}}}(s_1)$. So we must have $\Sigma = \pi^{-1}(\mathscr{C})$, and there exists some $x \in \Sigma$ such that $\pi(x) = s_1u_1$. Let $x = su = us$ be the Jordan decomposition of x (with $s \in \mathbf{L}$ semisimple, $u \in \mathbf{L}$ unipotent). Then $\pi(s)\pi(u) = \pi(u)\pi(s)$ is the Jordan decomposition of $\pi(x) = s_1u_1$, so $\pi(s) = s_1$ and $\pi(u) = u_1$.

Let $y \in \Sigma$ be an arbitrary element. Then $\pi(y) = \pi(g)\pi(x)\pi(g)^{-1}$ for some $g \in \mathbf{L}$, so there exists $z \in \mathbf{Z}(\mathbf{L})$ such that $g^{-1}yg = xz = (zs)u$. Here, zs is semisimple, u is unipotent, and zs commutes with u, so $(zs)u = u(zs)$ is the Jordan decomposition of $g^{-1}yg$. By the condition on $C^{\circ}_{\mathbf{L}_{\mathrm{ss}}}(s_1)$, we certainly have $s_1 \neq 1$, so $s \notin \mathbf{Z}(\mathbf{L})$. This implies that $zs \neq 1$, so $g^{-1}yg$ and y are non-unipotent elements. We have thus shown that $\Sigma \cap \mathbf{L}_{\mathrm{uni}} = \varnothing$, as desired. □

4.1.17. For $x \in \mathfrak{X}(\mathbf{W})$, let us from now on denote by $\rho_x \in \mathrm{Uch}(\mathbf{G}^F)$ the corresponding unipotent character under the bijection (4.1.2.4) that we fixed in 4.1.3. Then the unipotent almost characters R_{x_1}, R_{x_2} are written as

$$R_{x_1} = \sum_{x \in \mathfrak{X}(\mathbf{W})} \{x, x_1\}\Delta(x)\rho_x = \tfrac{2}{3}\rho_{x_1} - \tfrac{1}{3}\rho_{x_2} + \sum_{x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_1, x_2\}} \{x, x_1\}\rho_x$$

and

$$R_{x_2} = \sum_{x \in \mathfrak{X}(\mathbf{W})} \{x, x_2\}\Delta(x)\rho_x = \tfrac{2}{3}\rho_{x_2} - \tfrac{1}{3}\rho_{x_1} + \sum_{x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_1, x_2\}} \{x, x_2\}\rho_x.$$

By [Gec03b, 5.6, see Table 1], we have $\mathbb{Q}(\rho_x) = \mathbb{Q}$ for any $x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_1, x_2\}$, while $\mathbb{Q}(\rho_{x_i})$ contains non-real numbers for $i = 1, 2$. We deduce that $\overline{\rho}_x = \rho_x$ for all $x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_1, x_2\}$, and $\overline{\rho}_{x_1} = \rho_{x_2}$, $\overline{\rho}_{x_2} = \rho_{x_1}$. (Cf. 3.4.11, but recall that we excluded the twisted groups there; we see however that this is completely analogous as far as ${}^2\mathsf{E}_6(q)$ is concerned.) Also note that $\{x, x_1\} = \{x, x_2\}$ for any $x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_1, x_2\}$. We thus get

$$\overline{R}_{x_1} = R_{x_2}.$$

Evaluating at the element $u_0 \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F$ defined in (4.1.12.1) and using Lemma 4.1.13, we obtain

$$R_{x_1}(u_0) = R_{x_1}(u_0^{-1}) = \overline{R}_{x_1}(u_0) = R_{x_2}(u_0).$$

Hence, we see that

$$\xi_{x_1} q^3 = \xi_{x_1} \chi_{x_1}(u_0) = R_{x_1}(u_0) = R_{x_2}(u_0) = \xi_{x_2} \chi_{x_2}(u_0) = \xi_{x_2} q^3,$$

and this also equals $\overline{R}_{x_1}(u_0) = \overline{\xi}_{x_1} q^3$. We deduce that $\xi_{x_1} = \xi_{x_2} = \overline{\xi}_{x_1}$ and then $\xi_{x_1} = \xi_{x_2} \in \{\pm 1\}$, since $|\xi_{x_1}| = 1$. Now let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. By (4.1.2.7), we have

$$\Delta(x_\rho) \cdot \rho = \sum_{x \in \mathfrak{X}(\mathbf{W})} \{x_\rho, x\} \cdot R_x.$$

In view of Proposition 4.1.16 and (4.1.3.1), we get

$$\Delta(x_\rho) \cdot \rho|_{\mathbf{G}^F_{\mathrm{uni}}} = \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \{x_\rho, x_\phi\} R_{\tilde{\phi}}|_{\mathbf{G}^F_{\mathrm{uni}}} + \xi \sum_{i=1}^{2} \{x_\rho, x_i\} \chi_{x_i}|_{\mathbf{G}^F_{\mathrm{uni}}} \tag{4.1.17.1}$$

where $\xi := \xi_{x_1} = \xi_{x_2} \in \{\pm 1\}$.

Lemma 4.1.18. *For* $\phi \in \mathrm{Irr}(\mathbf{W})$ *and any* $u \in \mathscr{O}_{\mathrm{reg}}^F$*, we have*

$$R_{\tilde{\phi}}(u) = \langle \phi, 1_{\mathbf{W}} \rangle_{\mathbf{W}} = \begin{cases} 1 & \text{if } \phi = 1_{\mathbf{W}}, \\ 0 & \text{if } \phi \neq 1_{\mathbf{W}}. \end{cases}$$

Proof. If $\sigma = \mathrm{id}_{\mathbf{W}}$, we have

$$R_{\tilde{\phi}}(u) = R_\phi(u) = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w) R_w(u) = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \phi(w),$$

where the last equality holds since $R_w(u) = 1$ for any $w \in \mathbf{W}$ [DL76, 9.16]. On the other hand, if $\sigma \neq \mathrm{id}_{\mathbf{W}}$, the preferred extension of ϕ is defined by the requirement that σ acts in the same way as the longest element w_0 of $\mathbf{W}$, see [LuCS4, 17.2(b)]. (Thus, in the setting of 2.2.7, if Θ is a representation of $\mathbf{W}$ affording the character ϕ, we can take $E := \Theta(w_0)$.) We obtain

$$R_{\tilde{\phi}}(u) = \frac{1}{|\mathbf{W}|}\sum_{w\in\mathbf{W}}\phi(ww_0)R_w(u) = \frac{1}{|\mathbf{W}|}\sum_{w\in\mathbf{W}}\phi(ww_0) = \frac{1}{|\mathbf{W}|}\sum_{w\in\mathbf{W}}\phi(w)$$

(again using [DL76, 9.16] and the fact that $w \mapsto ww_0$ defines a bijection $\mathbf{W} \to \mathbf{W}$). So for any σ, we have

$$R_{\tilde{\phi}}(u) = \frac{1}{|\mathbf{W}|}\sum_{w\in\mathbf{W}}\phi(w) = \langle \phi, 1_{\mathbf{W}}\rangle_{\mathbf{W}},$$

which proves the lemma. □

Proposition 4.1.19 (see [Het19])**.** *Let $u_0 \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F$ be as defined in* (4.1.12.1)*. For $i = 1, 2$, let $\varphi_{x_i}\colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of u_0 in $\mathscr{O}_{\mathrm{reg}}^F$, and let $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i}) \in \overline{\mathbb{Q}}_\ell^\times$ be defined by* (4.1.3.1)*. Then*

$$\xi_{x_1} = \xi_{x_2} = 1,$$

that is, the characteristic function $\chi_{x_i} = \chi_{A_i,\varphi_{x_i}}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ coincides with the unipotent almost character R_{x_i} for $i = 1, 2$, both for the non-twisted group $\mathbf{G}^F = \mathsf{E}_6(q)$ and for the twisted group $\mathbf{G}^F = {}^2\mathsf{E}_6(q)$, where q is any power of the prime $p = 3$.

Proof. We already know from 4.1.17 that $\xi = \xi_{x_1} = \xi_{x_2} \in \{\pm 1\}$. By Lemma 4.1.18 and since $\chi_{x_1}(u_0) = \chi_{x_2}(u_0) = q^3$ (see 4.1.14), the evaluation of (4.1.17.1) at u_0 gives

$$\Delta(x_\rho)\rho(u_0) = \{x_\rho, x_{1_\mathbf{W}}\} + \xi q^3 \sum_{i=1}^{2}\{x_\rho, x_i\} \quad \text{for any } \rho \in \mathrm{Uch}(\mathbf{G}^F). \tag{4.1.19.1}$$

To determine ξ, we consider the Hecke algebra $\mathcal{H}_{\sigma,q}$ of $\mathbf{G}^F$ with respect to the BN-pair $(\mathbf{B}_0^F, N_\mathbf{G}(\mathbf{T}_0)^F)$, see 2.3.1. Here, we have $S_{\mathrm{id}_\mathbf{W}} = S = \{s_1, s_2, \ldots, s_6\}$ (in particular, $(\mathbf{W}^{\mathrm{id_W}}, S_{\mathrm{id}_\mathbf{W}}) = (\mathbf{W}, S)$ is a Coxeter system of type E_6), and the parameters of $\mathcal{H}_{\mathrm{id}_\mathbf{W},q}$ are given by $q_{s_i} = q$ for $1 \leqslant i \leqslant 6$. In the case where $\sigma \neq \mathrm{id}_\mathbf{W}$, recall from 4.1.5 that $\mathbf{W}^\sigma$ is a Weyl group of type F_4 with Coxeter generators $S_\sigma = \{s_2, s_4, s_3s_5, s_1s_6\}$; the parameters of $\mathcal{H}_{\sigma,q}$ are given by

$$q_{s_2} = q_{s_4} = q, \quad q_{s_3s_5} = q_{s_1s_6} = q^2$$

(see [Lus78, Table II (p. 35)], [Lus76b, (7.7)] or [Lus84a, 8.2]). By 2.3.9 (and using the notation there), we have

$$m(g,w) = \sum_{\phi\in\mathrm{Irr}(\mathbf{W}^\sigma)} [\phi](g)\,\mathrm{Trace}(T_w, V_\phi) = \frac{|O_g \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(g)|}{|\mathbf{B}_0^F|},$$

valid for any $g \in \mathbf{G}^F$ and $w \in \mathbf{W}^\sigma$. We evaluate this equation with $g = u_0$ and with

$$w = w_{\mathrm{c}} := s_1 s_6 s_3 s_5 s_2 s_4 \in \mathbf{W}^\sigma,$$

a Coxeter element of $\mathbf{W}$:

(a) Let us first consider the non-twisted case where $\mathbf{G}^F = \mathsf{E}_6(q)$, so $\sigma = \mathrm{id}_{\mathbf{W}}$. As noted in 4.1.3, the bijection (4.1.2.4) that we fixed there coincides with the one provided by Proposition 3.4.5, so we have $x_{[\phi]} = x_\phi$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$. Hence, except for the sign ξ, we can compute (4.1.19.1) for any $\rho = [\phi]$, $\phi \in \mathrm{Irr}(\mathbf{W})$. Thus, the sum $\sum_{\phi\in\mathrm{Irr}(\mathbf{W}^\sigma)}[\phi](u_0)\,\mathrm{Trace}(T_w, V_\phi)$ evaluates to

$$\mathrm{Tr}(T_w, V_{1_p}) + \tfrac{2}{3}\xi q^3\big(\mathrm{Tr}(T_w, V_{80_s}) + \mathrm{Tr}(T_w, V_{20_s}) - \mathrm{Tr}(T_w, V_{10_s}) - \mathrm{Tr}(T_w, V_{90_s})\big).$$

Choosing $w = w_{\mathrm{c}}$ and using CHEVIE to get the character values of the Hecke algebra $\mathcal{H}_{\mathrm{id}_{\mathbf{W}},q}$, we obtain

$$0 \leqslant m(u_0, w_{\mathrm{c}}) = (2\xi + 1)\cdot q^6,$$

which would be false if $\xi = -1$, so we must have $\xi = +1$.

(b) Now assume that we are in the twisted case $\mathbf{G}^F = {}^2\mathsf{E}_6(q)$ (i.e., $\sigma \neq \mathrm{id}_{\mathbf{W}}$). Given an irreducible character $\phi \in \mathrm{Irr}(\mathbf{W}^\sigma)$, the label of $\mathfrak{X}(\mathbf{W})$ parametrising the unipotent character $[\phi]$ is obtained from Table 4.1. We get

$$\begin{aligned} m(u_0, w_{\mathrm{c}}) &= \mathrm{Tr}(T_{w_{\mathrm{c}}}, V_{1_1}) + \tfrac{2}{3}\xi q^3\big(\mathrm{Tr}(T_{w_{\mathrm{c}}}, V_{12_1}) - \mathrm{Tr}(T_{w_{\mathrm{c}}}, V_{6_1}) - \mathrm{Tr}(T_{w_{\mathrm{c}}}, V_{6_2})\big) \\ &= (2\xi+1)\cdot q^6, \end{aligned}$$

and this is non-negative; thus, $\xi = +1$. □

Remark 4.1.20. Recall from 3.2.23 that the conjugacy class of $\mathbf{W}$ containing the Coxeter elements is sent to the regular unipotent class $\mathscr{O}_{\mathrm{reg}} \subseteq \mathbf{G}$ under Lusztig's map. The characterisation (∗) in 3.2.23 of this map indicates why the argument in the proof of Proposition 4.1.19 works exactly when evaluating $m(u_0, w)$ with a Coxeter element w of $\mathbf{W}$. More generally, given a unipotent class $\mathscr{O}$ in a simple algebraic group $\mathbf{G}$ with Weyl group $\mathbf{W}$ and an element $u \in \mathscr{O}(\mathbb{F}_q)$, the standard choice for $w \in \mathbf{W}$ to evaluate $m(u, w)$

in order to obtain information on character values at u is such that the conjugacy class of w in $\mathbf{W}$ is sent to $\mathscr{O}$ under Lusztig's map, as we shall see in numerous places below.

Example 4.1.21. The previous discussion immediately yields the explicit values of any unipotent character at any regular unipotent element for $\mathbf{G}^F \in \{\mathsf{E}_6(q), {}^2\mathsf{E}_6(q)\}$ where q is a power of $p = 3$. Indeed, Proposition 4.1.16 and Lemma 4.1.18 show that

$$R_x|_{\mathscr{O}_{\mathrm{reg}}^F} = 0 \quad \text{unless} \quad x \in \{1_{\mathbf{W}}, x_1, x_2\}.$$

The trivial character $1_{\mathbf{W}}$ of $\mathbf{W}$ gives rise to the one-element family consisting of the trivial character $1_{\mathbf{G}^F}$ of $\mathbf{G}^F$, and of course we have $1_{\mathbf{G}^F}(u) = 1$ for any $u \in \mathscr{O}_{\mathrm{reg}}^F$. So we only need to consider the 8 unipotent characters parametrised by elements $x \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_7}) \subseteq \mathfrak{X}(\mathbf{W})$, and we only have to look up the Fourier coefficients $\{x, x_1\}$ and $\{x, x_2\}$, as well as the values of $R_{x_i} = \chi_{x_i}$ for $i = 1, 2$ given in 4.1.14. The values of the unipotent characters parametrised by elements of $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_7}) \subseteq \mathfrak{X}(\mathbf{W})$ are thus given in Table 4.3, where we denote the $\mathbf{G}^F$-conjugacy classes contained in $\mathscr{O}_{\mathrm{reg}}^F$ by giving the corresponding elements of $A_{\mathbf{G}}(u_0) = \langle \overline{u}_0 \rangle \cong C_3$ in the top line.

$\mathsf{E}_6(q)$	${}^2\mathsf{E}_6(q)$	1	$\overline{u}_0$	$\overline{u}_0^2$
$[80_s]$	${}^2\mathsf{E}_6[1]$	$\frac{2}{3}q^3$	$-\frac{1}{3}q^3$	$-\frac{1}{3}q^3$
$[20_s]$	$[12_1]$	$\frac{2}{3}q^3$	$-\frac{1}{3}q^3$	$-\frac{1}{3}q^3$
$[60_s]$	$[4_1]$	0	0	0
$[10_s]$	$[6_1]$	$-\frac{2}{3}q^3$	$\frac{1}{3}q^3$	$\frac{1}{3}q^3$
$[90_s]$	$[6_2]$	$-\frac{2}{3}q^3$	$\frac{1}{3}q^3$	$\frac{1}{3}q^3$
$\mathsf{D}_4[r]$	$[16_1]$	0	0	0
$\mathsf{E}_6[\omega]$	${}^2\mathsf{E}_6[\omega]$	$\frac{1}{3}q^3$	$\frac{1}{3}q^3(2\omega - \omega^2)$	$\frac{1}{3}q^3(2\omega^2 - \omega)$
$\mathsf{E}_6[\omega^2]$	${}^2\mathsf{E}_6[\omega^2]$	$\frac{1}{3}q^3$	$\frac{1}{3}q^3(2\omega^2 - \omega)$	$\frac{1}{3}q^3(2\omega - \omega^2)$

Table 4.3.: Values of unipotent characters on $\mathscr{O}_{\mathrm{reg}}^F$ in type E_6, $p = 3$

Remark 4.1.22. Table 4.3 shows that the only $\mathbf{G}^F$-conjugacy class inside $\mathscr{O}_{\mathrm{reg}}^F$ on which the cuspidal unipotent characters of groups of type E_6 in characteristic 3 take real values is the one which contains u_0. Hence, the $\mathbf{G}^F$-conjugacy class O_{u_0} of u_0 is the unique $\mathbf{G}^F$-class inside $\mathscr{O}_{\mathrm{reg}}^F$ which is stable under taking inverses. (The element $(u_0)_{\overline{u}_0}^{-1}$ is $\mathbf{G}^F$-conjugate to $(u_0)_{\overline{u}_0^2}$.) On the other hand, let us recall that, since $p = 3$, the canonical isogeny $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ is bijective, so we may as well have considered the simply connected group of type E_6 instead of the adjoint group. In particular, we may assume that $\mathbf{G}$ and $\Sigma := \mathscr{O}_{\mathrm{reg}}$ satisfy conditions (i) and (ii) in 3.2.22(c), so let us adopt the notation from there, except that we explicitly set $w_{\mathrm{c}} := s_1 s_6 s_3 s_5 s_2 s_4 \in \mathbf{W}^\sigma$ (as in the proof of Proposition 4.1.19). Since $\mathbf{G}$ has a trivial centre, [Gec21, 4.10] shows that

$$O_0 \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F = (\mathscr{O}_{\mathrm{reg}} \cap \mathbf{B}_0 w_{\mathrm{c}} \mathbf{B}_0)^F \neq \varnothing$$

where $O_0 \subseteq \mathscr{O}_{\mathrm{reg}}^F$ is a (unipotent) $\mathbf{G}^F$-conjugacy class, and we have $O_0 = O_0^{-1}$. So we must have $O_{u_0} = O_0$. Thus, our chosen representative u_0 satisfies the condition (♡) in 3.2.22(c), and its $\mathbf{G}^F$-class is uniquely determined by this property.

Values of unipotent characters at unipotent elements for $\mathsf{E}_6(q)$

As in the beginning of this section, let p be any prime and $\mathbf{G}$ be the simple adjoint group of type E_6 over $k = \overline{\mathbb{F}}_p$. Let us assume here that $F \colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map providing $\mathbf{G}$ with a split $\mathbb{F}_q$-rational structure for some power q of p, so that $\mathbf{G}^F = \mathsf{E}_6(q)$ is non-twisted. The further notation is as in the beginning of this section. Our goal is to explicitly determine the values of the unipotent characters of $\mathsf{E}_6(q)$ at unipotent elements. (Note that, for this purpose, the assumption that $\mathbf{G}$ is adjoint is not restrictive.) Recall from 2.2.23, 2.2.24 that this task is equivalent to the one of determining the unipotent almost characters at unipotent elements.

4.1.23. Recall the decomposition of $\mathrm{Uch}(\mathbf{G}^F)$ into Harish-Chandra series as described in 4.1.4. Thus, 25 of the 30 almost characters R_x ($x \in \mathfrak{X}(\mathbf{W})$) are of the form R_ϕ with $\phi \in \mathrm{Irr}(\mathbf{W})$. As mentioned in 2.2.5, the values of these 25 almost characters at unipotent elements are known in all characteristics. They are explicitly computed and printed by Malle [Mal93] (for $p = 2$), Porsch [Por93] (for $p = 3$); see Beynon–Spaltenstein [BS84] for good primes $p \geqslant 5$, for which we also refer to Lübeck's electronic library [Lüb]. Considering the elements $x_i \in \mathfrak{X}(\mathbf{W})$ ($i = 1, 2$) which parametrise the cuspidal unipotent characters (cuspidal unipotent character sheaves), we know the values of the $R_{x_i}|_{\mathbf{G}^F_{\mathrm{uni}}}$ from the previous results of this section:

- If $p \neq 3$, the support of the A_{x_i} is given by (the closure of) a non-unipotent class of $\mathbf{G}$, so the $\chi_{A_{x_i}}$ (and, hence, the R_{x_i}) are identically 0 on $\mathbf{G}^F_{\mathrm{uni}}$.
- If $p = 3$, the values of the R_{x_i} are obtained from Proposition 4.1.19 (as $R_{x_i}(g) = 0$ for any $g \in \mathbf{G}^F \setminus \mathscr{O}^F_{\mathrm{reg}}$).

It remains to consider the three elements $x \in \mathfrak{X}(\mathbf{W})$ corresponding to the ones of $\mathfrak{S}_{\mathbf{W}}$ described in 4.1.4(b). The associated unipotent character sheaves $A_x \in \hat{\mathbf{G}}^{\mathrm{un}}$ are thus the simple direct summands of the complex $K^{\mathscr{E}}_{\mathbf{L}_J,\Sigma} \in \mathscr{M}\mathbf{G}$ where $J = \{s_2, s_3, s_4, s_5\} \subseteq S$, and where the (unique) cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{L}_J$ is given as follows (see [Lus84b, 15.1] and the proof of [LuCS4, 19.3]):

- If $p \neq 2$, we have $\Sigma = \mathbf{Z}(\mathbf{L}_J).\mathscr{C}$ where $\mathscr{C} \subseteq \mathbf{L}_J$ is a conjugacy class containing elements of the form $g = g_{\mathrm{s}}g_{\mathrm{u}} = g_{\mathrm{u}}g_{\mathrm{s}}$ such that $C^\circ_{\mathbf{L}_J}(g_{\mathrm{s}})$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$. In particular, we have $g_{\mathrm{s}} \notin \mathbf{Z}(\mathbf{L}_J)$, so Σ does not contain any unipotent elements. It follows from Corollary 3.2.8 that χ_{A_x} (and, hence, R_x) vanishes identically on $\mathbf{G}^F_{\mathrm{uni}}$, for any of the three $x \in \mathfrak{X}(\mathbf{W})$ corresponding to the elements of $\mathfrak{S}_{\mathbf{W}}$ described in 4.1.4(b).

- If $p = 2$, we have $\Sigma = \mathbf{Z}(\mathbf{L}_J).\mathscr{O}_0$ where $\mathscr{O}_0 \subseteq \mathbf{L}_J$ is the regular unipotent class.

Summarising:

- If $p \geqslant 5$, the computation of the values of unipotent characters at unipotent elements is reduced to that of the Green functions (or, equivalently, the $R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $\phi \in \mathrm{Irr}(\mathbf{W})$), as $R_x|_{\mathbf{G}^F_{\mathrm{uni}}} = 0$ for any $x \in \mathfrak{X}(\mathbf{W}) \setminus \mathrm{Irr}(\mathbf{W})$. This is completely known, and it suffices to consult [Lüb].

- If $p = 3$, we have to consider 27 almost characters $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ $(x \in \mathfrak{X}(\mathbf{W}))$: 25 of them are of the form $R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}}$, and their values are provided by [Por93]; the other two $x \in \mathfrak{X}(\mathbf{W})$ parametrise cuspidal unipotent character sheaves, so the values of the associated $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ are obtained from Proposition 4.1.19.

- If $p = 2$, we have to consider 28 almost characters $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ $(x \in \mathfrak{X}(\mathbf{W}))$: 25 of them are of the form $R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}}$, and their values are provided by [Mal93]; the other three $x \in \mathfrak{X}(\mathbf{W})$ correspond to the elements of $\mathfrak{S}_{\mathbf{W}}$ as described in 4.1.4(b), and the values of those $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ are not yet known.

4.1.24. In view of the discussion in 4.1.23, it remains to consider the case where $p = 2$ and to compute the values of $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ for the three $x \in \mathfrak{X}(\mathbf{W})$ corresponding to the elements of $\mathfrak{S}_{\mathbf{W}}$ as described in 4.1.4(b).

So let $p = 2$, $J = \{s_2, s_3, s_4, s_5\} \subseteq S$, and let $x \in \mathfrak{X}(\mathbf{W})$ correspond to one of the three elements $(J, \phi, (-1, 2)) \in \mathfrak{S}_{\mathbf{W}}$ where $\phi \in \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}_J))$; in this case, we will also write $R_{\mathsf{D}_4[\phi]} := R_x$. In view of 4.1.23 and Remark 3.4.24, any such x corresponds to some $\mathfrak{i} \in \mathscr{N}^F_{\mathbf{G}}$ under the generalised Springer correspondence, and we have $\tau(\mathfrak{i}) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0)$ where $\mathscr{O}_0 \subseteq \mathbf{L}_J$ is the regular unipotent class (and up to isomorphism, $\mathscr{E}_0$ is uniquely determined by $\mathbf{L}_J$ and $\mathscr{O}_0$). These three $\mathfrak{i} \in \mathscr{N}^F_{\mathbf{G}}$ are thus in the image of the embedding

$$\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}_J)) \hookrightarrow \mathscr{N}^F_{\mathbf{G}}. \tag{4.1.24.1}$$

Recall that we denote the irreducible characters of $W_{\mathbf{G}}(\mathbf{L}_J) \cong W(\mathsf{A}_2) \cong \mathfrak{S}_3$ by 1, ε, r (trivial, sign, reflection). Then, following Spaltenstein [Spa85] (and using his notation for the unipotent classes of $\mathbf{G}$, except that we write $\mathscr{O}_{\mathrm{reg}} = \mathsf{E}_6$), the embedding (4.1.24.1) is given by

$$1 \mapsto (\mathscr{O}_{\mathrm{reg}}, -1), \quad r \mapsto (\mathsf{D}_5, -1), \quad \varepsilon \mapsto (\mathsf{D}_4, -1). \tag{4.1.24.2}$$

(Here, for each $\mathscr{O} \in \{\mathscr{O}_{\mathrm{reg}}, \mathsf{D}_5, \mathsf{D}_4\}$, we have $A_{\mathbf{G}}(u) \cong C_2$ for any $u \in \mathscr{O}$, and we write -1 for the non-trivial local system on $\mathscr{O}$.) Let us fix any isomorphism $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Let $\mathfrak{i} = (\mathscr{O}, -1) \in \mathcal{N}_{\mathbf{G}}^F$ be one of the three pairs considered above, and let $x \in \mathfrak{X}(\mathbf{W})$ be such that $A_x \cong A_{\mathfrak{i}}$; we define the isomorphisms $\varphi_{A_{\mathfrak{i}}} \colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$, $\overline{\varphi}_{A_{\mathfrak{i}}} \colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$ as in 3.2.25 and set $\varphi_x := \overline{\varphi}_{A_{\mathfrak{i}}}$, $\chi_x := \chi_{A_x, \varphi_x}$. In the setting of 3.4.23, let us put $\zeta := \zeta_J$; we then have

$$R_x|_{\mathbf{G}^F_{\mathrm{uni}}} = q^{(\dim \mathbf{G} - \dim \mathscr{O} - \dim \mathbf{Z}(\mathbf{L}_J))/2} \zeta X_{\mathfrak{i}}$$

and

$$X_{\mathfrak{i}} = p_{(\mathscr{O}_{\mathrm{reg}}, -1), \mathfrak{i}} Y_{(\mathscr{O}_{\mathrm{reg}}, -1)} + p_{(\mathsf{D}_5, -1), \mathfrak{i}} Y_{(\mathsf{D}_5, -1)} + p_{(\mathsf{D}_4, -1), \mathfrak{i}} Y_{(\mathsf{D}_4, -1)}.$$

In particular, $X_{\mathfrak{i}}$ (and, hence, $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$) vanishes outside of $\mathscr{O}^F_{\mathrm{reg}} \cup \mathsf{D}_5^F \cup \mathsf{D}_4^F$. The coefficients $p_{\mathfrak{i}', \mathfrak{i}}$ can be obtained, for example, via the function `ICCTable` in CHEVIE [MiChv]. This gives the following:

(i) For $u \in \mathscr{O}^F_{\mathrm{reg}}$, we have

$$\begin{aligned} R_{\mathsf{D}_4[1]}(u) &= q^2 \zeta X_{(\mathscr{O}_{\mathrm{reg}}, -1)}(u) = q^2 \zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(u), \\ R_{\mathsf{D}_4[r]}(u) &= q^4 \zeta X_{(\mathsf{D}_5, -1)}(u) = 0, \\ R_{\mathsf{D}_4[\varepsilon]}(u) &= q^8 \zeta X_{(\mathsf{D}_4, -1)}(u) = 0. \end{aligned}$$

(ii) For $u \in \mathsf{D}_5^F$, we have

$$\begin{aligned} R_{\mathsf{D}_4[1]}(u) &= q^2 \zeta X_{(\mathscr{O}_{\mathrm{reg}}, -1)}(u) = q^3 \zeta Y_{(\mathsf{D}_5, -1)}(u), \\ R_{\mathsf{D}_4[r]}(u) &= q^4 \zeta X_{(\mathsf{D}_5, -1)}(u) = q^4 \zeta Y_{(\mathsf{D}_5, -1)}(u), \\ R_{\mathsf{D}_4[\varepsilon]}(u) &= q^8 \zeta X_{(\mathsf{D}_4, -1)}(u) = 0. \end{aligned}$$

(iii) For $u \in \mathsf{D}_4^F$, we have

$$\begin{aligned} R_{\mathsf{D}_4[1]}(u) &= q^2\zeta X_{(\mathscr{O}_{\mathrm{reg}},-1)}(u) = q^5\zeta Y_{(\mathsf{D}_4,-1)}(u), \\ R_{\mathsf{D}_4[r]}(u) &= q^4\zeta X_{(\mathsf{D}_5,-1)}(u) = (q^7+q^6)\zeta Y_{(\mathsf{D}_4,-1)}(u), \\ R_{\mathsf{D}_4[\varepsilon]}(u) &= q^8\zeta X_{(\mathsf{D}_4,-1)}(u) = q^8\zeta Y_{(\mathsf{D}_4,-1)}(u). \end{aligned}$$

4.1.25. We keep the setting and notation of 4.1.24 and apply 3.4.19(2): For any $w \in \mathbf{W}$ and any $u \in \mathbf{G}_{\mathrm{uni}}^F$, we have

$$m(u,w) = \frac{|O_u \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(u)|}{|\mathbf{B}_0^F|} = \sum_{x \in \mathfrak{X}(\mathbf{W})} c_x(w) R_x(u), \tag{4.1.25.1}$$

where

$$c_x(w) = \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \{x_\phi, x\} \operatorname{Trace}(T_w, V_\phi).$$

We want to evaluate (4.1.25.1) to compute $R_x|_{\mathbf{G}_{\mathrm{uni}}^F}$ for the three $x \in \mathfrak{X}(\mathbf{W})$ considered in 4.1.24. As already observed there, it suffices to look at elements $u \in \mathscr{O}_{\mathrm{reg}}^F \cup \mathsf{D}_5^F \cup \mathsf{D}_4^F$, and we have $A_{\mathbf{G}}(u) \cong C_2$ for any such u; more precisely, the group $A_{\mathbf{G}}(u)$ is generated by $\overline{u}$, see [LS12, Table 17.4]. In particular, for any $u \in \mathscr{O}_{\mathrm{reg}}^F \cup \mathsf{D}_5^F \cup \mathsf{D}_4^F$, F acts trivially on $A_{\mathbf{G}}(u)$, so each of $\mathscr{O}_{\mathrm{reg}}^F$, D_5^F, D_4^F splits into two $\mathbf{G}^F$-conjugacy classes. As usual, for $1 \leqslant i \leqslant 6$, let us denote by $u_i := u_{\alpha_i}$ the homomorphism $\mathbf{G}_{\mathrm{a}} \to \mathbf{G}$ whose image is the root subgroup $\mathbf{U}_{\alpha_i} \subseteq \mathbf{U}_0$ (see 2.1.4). Following Mizuno [Miz77], we set

$$\begin{aligned} x_{20} &:= u_1(1) \cdot u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_6(1) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F, \\ x_{16} &:= u_6(1) \cdot u_5(1) \cdot u_4(1) \cdot u_3(1) \cdot u_2(1) \in \mathbf{U}_0^F \cap \mathsf{D}_5^F, \\ x_{14} &:= u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \in \mathbf{U}_0^F \cap \mathsf{D}_4^F. \end{aligned}$$

(Note that we do not have to refer to any convention for the choice of certain signs in a Chevalley basis in the Lie algebra underlying $\mathbf{G}$ since k is of characteristic 2.) We now apply Lemma 3.2.24: Note that $-w_0(\Pi) = \Pi$, so $-w_0$ induces a graph automorphism of the Dynkin diagram of E_6. This automorphism is the unique non-trivial one (see, e.g., [Bou68, Chap. VI]). We set

$$\begin{aligned} w_{20} &:= s_6 \cdot s_2 \cdot s_5 \cdot s_4 \cdot s_3 \cdot s_1 \in \mathbf{W} \text{ (a Coxeter element of } \mathbf{W}), \\ w_{16} &:= s_1 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_2 \in \mathbf{W}, \\ w_{14} &:= s_2 \cdot s_5 \cdot s_4 \cdot s_3 \in \mathbf{W}. \end{aligned}$$

Using CHEVIE (see [MiChv, §6]), one verifies that these are reduced expressions for the w_j ($j \in \{20, 16, 14\}$), each w_j is of minimal length in its conjugacy class $C_j \subseteq \mathbf{W}$, and C_j is sent to the unipotent conjugacy class of $\mathbf{G}$ containing x_j under Lusztig's map (see 3.2.23). Let $\dot{w}_0 \in N_{\mathbf{G}}(\mathbf{T}_0)^F \subseteq N_{\mathbf{G}}(\mathbf{T}_0)$ be a representative for the longest element w_0 of $(\mathbf{W}, S)$. By Lemma 3.2.24, we have

$$\dot{w}_0 x_j \dot{w}_0^{-1} \in (\mathbf{B}_0 w_j \mathbf{B}_0)^F = \mathbf{B}_0^F w_j \mathbf{B}_0^F \quad \text{for } j \in \{20, 16, 14\},$$

where the last equality follows from the uniqueness of expressions in the sharp form of the Bruhat decomposition [Car85, 2.5.14]. We have shown that $O_{x_j} \cap \mathbf{B}_0^F w_j \mathbf{B}_0^F \neq \varnothing$ for $j \in \{20, 16, 14\}$, so x_{20}, x_{16}, x_{14} satisfy condition ($\heartsuit'$) in 3.2.23 and are thus our choices for 'good' representatives in $\mathscr{O}_{\mathrm{reg}}^F, \mathsf{D}_5^F, \mathsf{D}_4^F$, respectively.

4.1.26. We keep the notation of 4.1.24, 4.1.25. We can now compute the values of the almost characters $R_{\mathsf{D}_4[1]}$, $R_{\mathsf{D}_4[r]}$ and $R_{\mathsf{D}_4[\varepsilon]}$ at unipotent elements.

(i) For $u \in \mathscr{O}_{\mathrm{reg}}^F$ and $w \in \mathbf{W}$, we have

$$m(u, w) = c_{1_{\mathbf{W}}}(w) R_{1_{\mathbf{W}}}(u) + c_{\mathsf{D}_4[1]}(w) R_{\mathsf{D}_4[1]}(u).$$

Taking $w = w_{20} = s_6 s_2 s_5 s_4 s_3 s_1$ and using CHEVIE [MiChv], we get $c_{1_{\mathbf{W}}}(w_{20}) = q^6$ and $c_{\mathsf{D}_4[1]}(w_{20}) = q^4$. Thus, 4.1.24(i) gives

$$m(u, w_{20}) = q^6 (1 + \zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(u)) \quad \text{for } u \in \mathscr{O}_{\mathrm{reg}}^F.$$

Since $\zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(u) \in \mathcal{R}$ is a root of unity and $m(u, w_{20})$ is (in particular) a real number, we must have $\zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(u) \in \{\pm 1\}$. As $O_{x_{20}} \cap \mathbf{B}_0^F w_{20} \mathbf{B}_0^F \neq \varnothing$, we obtain

$$0 < 1 + \zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(x_{20}) \quad \text{with} \quad \zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(x_{20}) \in \{\pm 1\},$$

which is only true if $\zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(x_{20}) = +1$. Now consider the other $\mathbf{G}^F$-conjugacy class inside $\mathscr{O}_{\mathrm{reg}}^F$, for which Mizuno [Miz77, p. 554] gives the representative

$$x_{45} := u_1(1) \cdot u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_6(1) \cdot u_{\alpha_2 + \alpha_3 + \alpha_4}(\eta) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F,$$

where $\eta \in \mathbb{F}_q$ is a fixed element such that the polynomial $\mathbf{x}^2 + \mathbf{x} + \eta$ is irreducible in $\mathbb{F}_q[\mathbf{x}]$ (with $\mathbf{x}$ an indeterminate over $\mathbb{F}_q$). Since $A_{\mathbf{G}}(x_{20}) \cong C_2$, the discussion in 3.4.23 shows that $Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(x_{45}) = -Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(x_{20})$, so we have $\zeta Y_{(\mathscr{O}_{\mathrm{reg}}, -1)}(x_{45}) = -1$. Thus, 4.1.24(i) gives

$$R_{\mathsf{D}_4[1]}(x_{20}) = q^2 \qquad \text{and} \qquad R_{\mathsf{D}_4[1]}(x_{45}) = -q^2.$$

(ii) For $u \in \mathsf{D}_5^F$ and $w \in \mathbf{W}$, we have

$$m(u,w) = c_{1_\mathbf{W}}(w)R_{1_\mathbf{W}}(u) + c_{\phi_{6,1}}(w)R_{\phi_{6,1}}(u) + c_{\phi_{20,2}}(w)R_{\phi_{20,2}}(u) \\ + c_{\mathsf{D}_4[1]}R_{\mathsf{D}_4[1]}(u) + c_{\mathsf{D}_4[r]}R_{\mathsf{D}_4[r]}(u).$$

We take $w = w_{16} = s_1 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_2$. Using the known values of the Green functions [Mal93] and 4.1.24(ii) (and CHEVIE for the coefficients $c_x(w_{16})$), we get, for any $u \in \mathsf{D}_5^F$:

$$m(u,w_{16}) = q^5 + (q^5 - q^4)q + (q^5 - q^4)q^2 + \Big(q^3 \cdot q^3\zeta + (q^3 - q^2)\cdot q^4\zeta\Big)Y_{(\mathsf{D}_5,-1)}(u) \\ = q^7(1 + \zeta Y_{(\mathsf{D}_5,-1)}(u)).$$

It follows that $\zeta Y_{(\mathsf{D}_5,-1)}(u) \in \{\pm 1\}$. Since $O_{x_{16}} \cap \mathbf{B}_0^F w_{16}\mathbf{B}_0^F \neq \varnothing$, we can argue exactly as in (i) to deduce that $\zeta Y_{(\mathsf{D}_5,-1)}(x_{16}) = +1$. For the other $\mathbf{G}^F$-class inside D_5^F, we may take the representative (see [Miz77, p. 554])

$$x_{43} := u_6(1)\cdot u_5(1)\cdot u_4(1)\cdot u_3(1)\cdot u_2(1)\cdot u_{\alpha_3+\alpha_4+\alpha_5}(\eta) \in \mathbf{U}_0^F \cap \mathsf{D}_5^F,$$

with $\eta \in \mathbb{F}_q$ as in the definition of x_{45}. We thus have $Y_{(\mathsf{D}_5,-1)}(x_{43}) = -Y_{(\mathsf{D}_5,-1)}(x_{16})$, so $\zeta Y_{(\mathsf{D}_5,-1)}(x_{43}) = -1$. In view of 4.1.24(ii), we obtain

$$R_{\mathsf{D}_4[1]}(x_{16}) = q^3 \quad \text{and} \quad R_{\mathsf{D}_4[1]}(x_{43}) = -q^3,$$
$$R_{\mathsf{D}_4[r]}(x_{16}) = q^4 \quad \text{and} \quad R_{\mathsf{D}_4[r]}(x_{43}) = -q^4.$$

(iii) For $u \in \mathsf{D}_4^F$ and $w \in \mathbf{W}$, we have

$$m(u,w) = \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} c_\phi(w)R_\phi(u) + \sum_{x \in \mathfrak{X}(\mathbf{W})\setminus \mathrm{Irr}(\mathbf{W})} c_x(w)R_x(u).$$

Taking $w = w_{14} = s_2 \cdot s_5 \cdot s_4 \cdot s_3$ and using the values of the Green functions in [Mal93], we get

$$\sum_{\phi \in \mathrm{Irr}(\mathbf{W})} c_\phi(w_{14})R_\phi(u) = q^{10} + 2q^9 + 2q^8 + q^7 \quad \text{for any } u \in \mathsf{D}_4^F.$$

Furthermore, we have $c_{\mathsf{D}_4[1]}(w_{14}) = c_{\mathsf{D}_4[\varepsilon]}(w_{14}) = q^2$ and $c_{\mathsf{D}_4[r]}(w_{14}) = 2q^2$, so in view of 4.1.24(iii), we get

$$\sum_{x \in \mathfrak{X}(\mathbf{W})\setminus \mathrm{Irr}(\mathbf{W})} c_x(w_{14})R_x(u) = q^2(q^5 + 2(q^7 + q^6) + q^8)\zeta Y_{(\mathsf{D}_4,-1)}(u) \quad \text{for } u \in \mathsf{D}_4^F.$$

We thus have

$$m(u, w_{14}) = (1 + \zeta Y_{(\mathsf{D}_4,-1)}(u))(q^{10} + 2q^9 + 2q^8 + q^7) \quad \text{for } u \in \mathsf{D}_4^F.$$

Since $O_{x_{14}} \cap \mathbf{B}_0^F w_{14} \mathbf{B}_0^F \neq \varnothing$, we can once again argue exactly as in (i) to see that $\zeta Y_{(\mathsf{D}_4,-1)}(x_{14}) = +1$. For the other $\mathbf{G}^F$-class inside D_4^F, we may take the representative (see [Miz77, p. 554])

$$x_{41} := u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_{\alpha_2+\alpha_3+\alpha_4}(\eta) \in \mathbf{U}_0^F \cap \mathsf{D}_4^F,$$

with $\eta \in \mathbb{F}_q$ as in the definitions of x_{45}, x_{43}. So we have $\zeta Y_{(\mathsf{D}_4,-1)}(x_{41}) = -1$. It follows from 4.1.24(iii) that

$$\begin{aligned} R_{\mathsf{D}_4[1]}(x_{14}) &= q^5 && \text{and} && R_{\mathsf{D}_4[1]}(x_{41}) = -q^5, \\ R_{\mathsf{D}_4[r]}(x_{14}) &= q^7 + q^6 && \text{and} && R_{\mathsf{D}_4[r]}(x_{41}) = -q^7 - q^6, \\ R_{\mathsf{D}_4[\varepsilon]}(x_{14}) &= q^8 && \text{and} && R_{\mathsf{D}_4[\varepsilon]}(x_{41}) = -q^8. \end{aligned}$$

Since R_x vanishes on $\mathbf{G}_{\text{uni}}^F \setminus (\mathscr{O}_{\text{reg}}^F \cup \mathsf{D}_5^F \cup \mathsf{D}_4^F)$ for any $x \in \mathfrak{X}(\mathbf{W}) \setminus \text{Irr}(\mathbf{W})$, Table 4.4 contains all the necessary values of these almost characters at unipotent elements. Thus, together with the results of Malle [Mal93], this yields the values of all unipotent (almost) characters at unipotent elements for the groups $\mathsf{E}_6(2^n)$, $n \in \mathbb{N}$.

$\mathbf{G}$-conjugacy class:	D_4		D_5		$\mathscr{O}_{\text{reg}}$	
$\mathbf{G}^F$-class (Mizuno [Miz77]):	x_{14}	x_{41}	x_{16}	x_{43}	x_{20}	x_{45}
$\mathbf{G}^F$-class (Malle [Mal93]):	u_{16}	u_{17}	u_{23}	u_{24}	u_{26}	u_{27}
$R_{\mathsf{D}_4[1]}$	q^5	$-q^5$	q^3	$-q^3$	q^2	$-q^2$
$R_{\mathsf{D}_4[\varepsilon]}$	q^8	$-q^8$	0	0	0	0
$R_{\mathsf{D}_4[r]}$	$q^7 + q^6$	$-q^7 - q^6$	q^4	$-q^4$	0	0
$R_{\mathsf{E}_6[\omega]}$	0	0	0	0	0	0
$R_{\mathsf{E}_6[\omega^2]}$	0	0	0	0	0	0

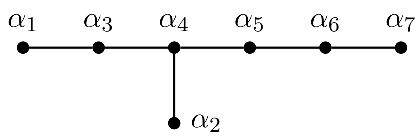
Table 4.4.: Values of $R_x|_{\mathbf{G}_{\text{uni}}^F}$ for $x \in \mathfrak{X}(\mathbf{W}) \setminus \text{Irr}(\mathbf{W})$, where $\mathbf{G}^F = \mathsf{E}_6(q)$, q a power of 2

Remark 4.1.27. In view of the discussion in 4.1.23, this completes the determination of the unipotent characters at unipotent elements for the groups $\mathbf{G}^F = \mathsf{E}_6(q)$ where q is any power of any prime p. In the case of $\mathsf{E}_6(2^n)$ ($n \in \mathbb{N}$), we have printed these values in the appendix, see Table B.2. If $p > 2$, the description in 4.1.23 easily transfers to the twisted groups ${}^2\mathsf{E}_6(q)$, so one can explicitly obtain the value of any unipotent character at any unipotent element — as far as the prime $p = 2$ is concerned, this problem is not yet solved for the groups ${}^2\mathsf{E}_6(2^n)$, but it should be possible to argue similarly as in the case of the non-twisted groups $\mathsf{E}_6(2^n)$.

4.2. Groups of type E_7

In this section, we consider the simple adjoint group $\mathbf{G}$ of type E_7 over $k = \overline{\mathbb{F}}_p$. Assume that $\mathbf{G}$ is defined over $\mathbb{F}_q \subseteq k$, where q is a power of p, and that $F\colon \mathbf{G} \to \mathbf{G}$ is the corresponding Frobenius map. We fix an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ and an F-stable maximal torus $\mathbf{T}_0 \subseteq \mathbf{B}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum attached to $\mathbf{G}$ and $\mathbf{T}_0$ (where $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), with underlying bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$. Let $R^+ \subseteq R$ be the positive roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$, and let $\Pi = \{\alpha_1, \ldots, \alpha_7\} \subseteq R^+$ be the corresponding simple roots, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_7^\vee\}$ be the corresponding simple co-roots. We choose the order of $\alpha_1, \ldots, \alpha_7$ in such a way that the Dynkin diagram of $\mathbf{G}$ is as follows:

E_7



Let $\mathfrak{C} = (\langle \alpha_j, \alpha_i^\vee \rangle)_{1 \leqslant i,j \leqslant 7}$ be the associated Cartan matrix, and let $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ be the Weyl group of $\mathbf{G}$ (with respect to $\mathbf{T}_0$). As usual, we identify $\mathbf{W}$ with a subgroup of $\operatorname{Aut}(X)$ (via 2.1.4), and for $1 \leqslant i \leqslant 7$, we set $s_i := w_{\alpha_i}$, with w_{α_i} being defined as in 2.1.2. Thus, $\mathbf{W}$ is a Coxeter group with Coxeter generators $S = \{s_1, \ldots, s_7\}$, arranged in the Coxeter diagram with the analogous numbering as in the Dynkin diagram of $\mathbf{G}$ printed above (see 2.1.5). We use the notation of [Lus84a, 4.12] for the irreducible characters of $\mathbf{W}$, which coincides with the one in [GP00, Table C.5 (p. 414)]. Let $\mathbf{U}_0 = R_{\mathrm{u}}(\mathbf{B}_0)$ be the unipotent radical of $\mathbf{B}_0$. As described in 2.1.19, F induces a p-isogeny of root data

$$\varphi\colon X \to X, \quad \lambda \mapsto \lambda \circ F|_{\mathbf{T}_0},$$

and a bijection $R \to R$, $\alpha \mapsto \alpha^\dagger$, so that $\varphi(\alpha^\dagger) = q\alpha$ for all $\alpha \in R$ (since $F\colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{G}$). The assignment $\alpha \mapsto \alpha^\dagger$ restricts to a graph automorphism of the Dynkin diagram, and since the only such automorphism in type E_7 is the identity, we must have $\alpha = \alpha^\dagger$ for all $\alpha \in R$, so $\mathbf{G}^F$ is necessarily the non-twisted group $\mathsf{E}_7(q)$ and $\sigma = \mathrm{id}_{\mathbf{W}}$. Hence, we are in the setting of Section 3.4 and adopt the further notation from there.

4.2.1. Consider the embedding

$$\operatorname{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{X}(\mathbf{W}), \quad \phi \mapsto x_\phi$$

(see 2.2.8). We have $|\mathfrak{X}(\mathbf{W})| = 76$ and $|\operatorname{Irr}(\mathbf{W})| = 60$. The irreducible characters $\operatorname{Irr}(\mathbf{W})$ are partitioned into 35 families, as follows: There are 24 families consisting of a single character, 8 families consisting of 3 characters (the associated sets $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ have 4 elements each), 2 families consisting of 5 characters (the associated sets $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ have 8 elements each), and then there is the exceptional family (see 3.4.2) consisting of 2 characters (the associated set $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ has 4 elements). As usual when dealing with simple non-twisted groups with a trivial centre, we fix the bijections

$$\operatorname{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho, \tag{4.2.1.1}$$

and

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}, \quad x \mapsto A_x, \tag{4.2.1.2}$$

through Corollary 3.4.8. As in Remark 3.4.9, we will then often write $\rho_x \in \operatorname{Uch}(\mathbf{G}^F)$ for the unipotent character corresponding to $x \in \mathfrak{X}(\mathbf{W})$ under (4.2.1.1).

By the proof of [LuCS4, 20.3(c)] (see also [Sho95b, 4.6]), there are exactly two cuspidal unipotent character sheaves A_1, A_2 on $\mathbf{G}$ (for any p), and both of them are F-stable. They are parametrised by the following elements of $\mathfrak{X}(\mathbf{W})$ under (4.2.1.2): Let $\mathcal{F}_{11} = \{512'_a, 512_a\} \subseteq \operatorname{Irr}(\mathbf{W})$ be the exceptional family, so that $\mathcal{G}_{\mathcal{F}_{11}} = \mathfrak{S}_2 = C_2$. We denote the non-trivial element of $\mathfrak{S}_2 = C_2$ by g_2 and the non-trivial irreducible character of this group by ε (thus conforming with the notation of [Lus84a, 4.3]). So we have

$$\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{11}}) = \{(1,1), (1,\varepsilon), (g_2,1), (g_2,\varepsilon)\}.$$

Then the two cuspidal unipotent character sheaves on $\mathbf{G}$ are labelled by

$$x_1 = (g_2, 1) \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_{11}}) \quad \text{and} \quad x_2 = (g_2, \varepsilon) \in \mathfrak{M}(\mathcal{G}_{\mathcal{F}_{11}}).$$

We number the $A_1, A_2 \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ above in such a way that

$$A_1 = A_{x_1} \quad \text{and} \quad A_2 = A_{x_2}.$$

Let us fix a square root i of -1 in $\overline{\mathbb{Q}}_\ell$ (which we always assume to be the same as the one in 3.4.2), and let $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$ be the square root of q that we fixed through (1.1.0.2).

Then, with the notation in the appendix of [Lus84a] or in [Car85, pp. 480–481], we have

$$\rho_{x_1} = \mathsf{E}_7[\mathrm{i}\sqrt{q}] \quad \text{and} \quad \rho_{x_2} = \mathsf{E}_7[-\mathrm{i}\sqrt{q}].$$

For $x \in \mathfrak{X}(\mathbf{W})$, let us for now pick any isomorphism $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ as in 3.2.1(∗). Just as we did for groups of type E_6, we will later make an explicit choice as far as the two cuspidal unipotent character sheaves $A_1 = A_{x_1}$ and $A_2 = A_{x_2}$ are concerned, depending on the characteristic p of k. In the case where $p = 3$, there is one further element of $\mathfrak{X}(\mathbf{W})$ (which will be denoted by x_0, see 4.2.11 below) for which we want to make an explicit choice for the isomorphism $\varphi_{x_0} \colon F^*A_{x_0} \xrightarrow{\sim} A_{x_0}$, but since our argumentation up to that point will not depend on such a concrete choice, we do not have to specify this right here. So we obtain the corresponding characteristic functions

$$\chi_x := \chi_{A_x,\varphi_x} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell \quad \text{for } x \in \mathfrak{X}(\mathbf{W}).$$

By (3.4.12.2), we have

$$R_x = \xi_x \chi_x \text{ for } x \in \mathfrak{X}(\mathbf{W}), \text{ where } \xi_x = \xi_x(\varphi_x) \in \overline{\mathbb{Q}}_\ell^\times,\ |\xi_x| = 1. \tag{4.2.1.3}$$

4.2.2. We fix a primitive 3rd root of unity $\omega \in \mathcal{R}_3$, which we always assume to be the same as the one in 3.4.2. The set $\mathfrak{S}^\circ_{\mathbf{W}_J}$ is non-empty for the following subsets $J \subseteq S$: $J = \varnothing$, $J = \{s_2, s_3, s_4, s_5\}$, $J = \{s_1, s_2, \ldots, s_6\}$ and $J = S$. In view of [Lus78, p. 36], the 76 elements of $\mathfrak{S}_{\mathbf{W}}$ fall into Harish-Chandra series as follows.

(a) The set $J = \varnothing$ gives rise to the 60 elements in the principal series, that is, the elements in the image of the embedding $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_{\mathbf{W}}$, $\phi \mapsto (\varnothing, \phi, (1,1))$.

(b) Let $J = \{s_2, s_3, s_4, s_5\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type D_4. We have $\mathfrak{S}^\circ_{\mathbf{W}_J} = \{(-1,2)\}$, and the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is isomorphic to $W(\mathsf{B}_3)$. So there are 10 elements in $\mathfrak{S}_{\mathbf{W}}$ of the form $(J, \epsilon, (-1,2))$, $\epsilon \in \mathrm{Irr}(\mathbf{W}^{S/J})$.

(c) Let $J = \{s_1, \ldots, s_6\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type E_6. We have $\mathfrak{S}^\circ_{\mathbf{W}_J} = \{(\omega,3), (\omega^2,3)\}$, and the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is isomorphic to $W(\mathsf{A}_1) \cong C_2$. This leads to the 4 elements $(J, \pm1, (\omega,3))$, $(J, \pm1, (\omega^2,3))$ of $\mathfrak{S}_{\mathbf{W}}$.

(d) For $J = S$, the set $\mathfrak{S}^\circ_{\mathbf{W}_J} = \mathfrak{S}^\circ_{\mathbf{W}}$ consists of the two elements $(\mathrm{i},2)$, $(-\mathrm{i},2)$ parametrising the two cuspidal unipotent characters of $\mathbf{G}^F$ (and the two cuspidal unipotent character sheaves on $\mathbf{G}$).

Type E_7 in characteristic $p \neq 2$

In this subsection (that is, here and in 4.2.3–4.2.6 below), we assume that $p \neq 2$. The scalars ξ_{x_1}, ξ_{x_2} in (4.2.1.3) (corresponding to the two cuspidal unipotent character sheaves on $\mathbf{G}$) have been determined by Geck, see [Gec21, §6]. To describe his results, we consider (similarly to the case E_6 with $p \neq 3$) the simple, simply connected group $\mathbf{G}_{\mathrm{sc}}$ of type E_7 and the canonical map $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$. By [GM20, 1.5.9] (see [Ste68, 9.16]), there exists a unique isogeny $\tilde{F}\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$ such that $F \circ \pi = \pi \circ \tilde{F}$, and $\tilde{F}$ is a Frobenius map which provides $\mathbf{G}_{\mathrm{sc}}$ with an $\mathbb{F}_q$-rational structure.

4.2.3. There are exactly two cuspidal unipotent character sheaves on $\mathbf{G}_{\mathrm{sc}}$, given by

$$\widetilde{A}_1 := \pi^*(A_1) \quad \text{and} \quad \widetilde{A}_2 := \pi^*(A_2), \tag{4.2.3.1}$$

where A_1, A_2 are the two cuspidal unipotent character sheaves on $\mathbf{G} = \mathbf{G}_{\mathrm{ad}}$ considered in 4.2.1. In particular, both $\widetilde{A}_1$ and $\widetilde{A}_2$ are $\tilde{F}$-stable. Let $\tilde{s} \in \mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$ be a semisimple element such that $C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})$ has a root system of type $\mathsf{A}_3 \times \mathsf{A}_3 \times \mathsf{A}_1$, and let $\tilde{u} \in C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})^{\tilde{F}}$ be a regular unipotent element in $C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})$. Let $\widetilde{\Sigma} \subseteq \mathbf{G}_{\mathrm{sc}}$ be the ($\tilde{F}$-stable) conjugacy class containing the element $\tilde{g} := \tilde{s}\tilde{u} = \tilde{u}\tilde{s}$. (These conditions determine $\tilde{g}$ up to $\mathbf{G}_{\mathrm{sc}}$-conjugacy; see [Gec21, 6.1] for a more detailed description of $\widetilde{\Sigma}$.) As usual, given $h \in C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})$, we denote by $\overline{h}$ the image of h in $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}) = C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})/C^\circ_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})$. Let $1 \neq z \in \mathbf{Z}(\mathbf{G}_{\mathrm{sc}})$ be the unique non-trivial element of the centre of $\mathbf{G}_{\mathrm{sc}}$. We have $\overline{\tilde{g}} = \overline{\tilde{s}}$ and

$$A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}) = \langle \overline{\tilde{s}} \rangle \times \langle \overline{z} \rangle \cong C_4 \times C_2.$$

Thus, $\tilde{F}$ acts trivially on $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g})$, so $\widetilde{\Sigma}^{\tilde{F}}$ splits into 8 conjugacy classes of $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$, each with centraliser order $8q^7$. Hence, in contrast to the case E_6 with $p \neq 3$ (see 4.1.7), we do not know at this point whether any of the $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-classes inside $\widetilde{\Sigma}^{\tilde{F}}$ is actually stable under taking inverses. Let $\mathbf{B}_0^{\mathrm{sc}} := \pi^{-1}(\mathbf{B}_0)$, $\mathbf{T}_0^{\mathrm{sc}} := \pi^{-1}(\mathbf{T}_0)$ be the Borel subgroup and the maximal torus of $\mathbf{G}_{\mathrm{sc}}$ corresponding to $\mathbf{B}_0$, $\mathbf{T}_0$, respectively (see [Bor91, 11.14]), and let $\mathbf{W}^{\mathrm{sc}} := N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{T}_0^{\mathrm{sc}})/\mathbf{T}_0^{\mathrm{sc}}$ be the Weyl group of $\mathbf{G}_{\mathrm{sc}}$ with respect to $\mathbf{T}_0^{\mathrm{sc}}$. Let $w_{\mathrm{c}}^{\mathrm{sc}} \in \mathbf{W}^{\mathrm{sc}}$ be a Coxeter element of $\mathbf{W}^{\mathrm{sc}}$. Following [Gec21, 6.4], we fix a representative $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$ whose $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class has a non-empty intersection with $(\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}} w_{\mathrm{c}}^{\mathrm{sc}} (\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}}$. In fact, due to the results of [Gec21, §4], there are two such $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-classes C, C' inside $\widetilde{\Sigma}^{\tilde{F}}$, but the relevant implications for us turn out to be independent of the exact choice of one of them. (As noted in 3.2.22(c), we could uniquely specify one of these two classes, C say, by choosing a representative $\dot{w}_{\mathrm{c}}^{\mathrm{sc}} \in N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{T}_0^{\mathrm{sc}})^{\tilde{F}}$ of $w_{\mathrm{c}}^{\mathrm{sc}}$ and then requiring that $C \cap (\mathbf{U}_0^{\mathrm{sc}})^{\tilde{F}} \dot{w}_{\mathrm{c}}^{\mathrm{sc}} (\mathbf{U}_0^{\mathrm{sc}})^{\tilde{F}} \neq \varnothing$ where $\mathbf{U}_0^{\mathrm{sc}} := R_{\mathrm{u}}(\mathbf{B}_0^{\mathrm{sc}})$.) In order to fix the notation, we assume from now on that $\tilde{g}_0 \in C$; then C' is parametrised by the element $\overline{z} \in A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$. Let us set $\Sigma := \pi(\widetilde{\Sigma})$ and $g_0 := \pi(\tilde{g}_0) \in \Sigma^F$. Let $\tilde{g}_0 = \tilde{s}_0\tilde{u}_0 = \tilde{u}_0\tilde{s}_0$, $g_0 = s_0u_0 = u_0s_0$ be the respective Jordan decompositions of $\tilde{g}_0$, g_0. By [Sho95b, 4.6] there exists a certain $a \in A_{\mathbf{G}}(g_0)$ such that

$$A_{\mathbf{G}}(g_0) \cong \langle \overline{g}_0 \rangle \times \langle a \rangle \cong C_4 \times C_2.$$

For $j = 1, 2$, we denote by ς_j the linear (irreducible) character of $A_{\mathbf{G}}(g_0)$ which takes the value i^{2j-1} at $\overline{g}_0$ and 1 at a (where i is as in 4.2.1; we use j instead of i here in order to avoid any notational confusion with the complex unit i). Let $\mathscr{E}_j$ be the one-dimensional $\mathbf{G}$-equivariant irreducible local system on Σ described by ς_j (see 3.2.20). Then

$$A_1 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E}_1)[\dim \Sigma]^{\#\mathbf{G}} \quad \text{and} \quad A_2 \cong \mathrm{IC}(\overline{\Sigma}, \mathscr{E}_2)[\dim \Sigma]^{\#\mathbf{G}}.$$

By [Gec21, 6.1], we have $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})\widetilde{\Sigma} = \widetilde{\Sigma}$, so $\pi^{-1}(\Sigma) = \widetilde{\Sigma}$, and it follows from (4.2.3.1) that

$$\widetilde{A}_1 \cong \mathrm{IC}\Big(\overline{\widetilde{\Sigma}}, \pi^*\mathscr{E}_1\Big)\big[\dim \Sigma\big]^{\#\mathbf{G}_{\mathrm{sc}}} \quad \text{and} \quad \widetilde{A}_2 \cong \mathrm{IC}\Big(\overline{\widetilde{\Sigma}}, \pi^*\mathscr{E}_2\Big)\big[\dim \Sigma\big]^{\#\mathbf{G}_{\mathrm{sc}}}.$$

The rest is entirely analogous to 4.1.7: The map $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ canonically induces a map $\overline{\pi}\colon A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0) \to A_{\mathbf{G}}(g_0)$, and the (one-dimensional, $\mathbf{G}_{\mathrm{sc}}$-equivariant) irreducible local systems $\widetilde{\mathscr{E}}_j := \pi^*\mathscr{E}_j$ on $\widetilde{\Sigma}$ are described by the linear characters

$$\widetilde{\varsigma}_j := \varsigma_j \circ \overline{\pi} \in \mathrm{Irr}(A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)), \quad j = 1, 2.$$

For $j = 1, 2$, let $\varphi_{A_j}\colon F^*A_j \xrightarrow{\sim} A_j$ be the isomorphisms corresponding to $g_0 \in \Sigma^F$ (via 3.2.21). We obtain induced isomorphisms

$$\varphi_{\tilde{A}_j} := \pi^*(\varphi_{A_j})\colon \tilde{F}^*\widetilde{A}_j \xrightarrow{\sim} \widetilde{A}_j, \quad j = 1, 2.$$

The $\varphi_{\tilde{A}_j}$ are the isomorphisms corresponding to the choice of $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$ via 3.2.21. We will henceforth fix the choice of g_0, $\tilde{g}_0$, and thus the one of the isomorphisms φ_{A_j}, $\varphi_{\tilde{A}_j}$ just described. Furthermore, for $j = 1, 2$, when writing φ_{x_j}, χ_{x_j} or $\xi_{x_j} := \xi_{x_j}(\varphi_{x_j})$, this is meant to be as in 4.2.1 but with respect to the isomorphisms $\varphi_{x_j} := \varphi_{A_j}$ just defined. The characteristic functions satisfy the following identity:

$$\chi_{\tilde{A}_j, \varphi_{\tilde{A}_j}} = \chi_{A_j, \varphi_{A_j}} \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} = \chi_{x_j} \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} \quad \text{for } j = 1, 2. \tag{4.2.3.2}$$

4.2.4. Similarly to 4.1.8, we can apply Shoji's Theorem 3.3.9 for the simply connected group $\mathbf{G}_{\mathrm{sc}}$ of type E_7 with regard to the unipotent character sheaves in a natural way.

Thus, with the analogous notation as in 4.1.8, we have

$$R_x^{\mathbf{G}_{\mathrm{sc}}} = R_x^{\mathbf{G}} \circ \pi|_{\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}} \quad \text{for } x \in \mathfrak{X}(\mathbf{W}),$$

and, using (4.2.3.2):

$$R_{x_i}^{\mathbf{G}_{\mathrm{sc}}} = \xi_{x_i} \chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} \quad \text{for } i = 1, 2, \text{ where } \quad \chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} := \chi_{\tilde{A}_i, \varphi_{\tilde{A}_i}}. \tag{4.2.4.1}$$

The values of the characteristic functions $\chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}}$ ($i = 1, 2$) are easily computed (see 3.2.21) and printed in [Gec21, p. 31]: First, we know from Theorem 3.1.13 that $\chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}}$ vanishes at any element of $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}} \setminus \tilde{\Sigma}^{\tilde{F}}$; the following table gives the values of these characteristic functions at elements of $\tilde{\Sigma}^{\tilde{F}}$, where the $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy classes inside $\tilde{\Sigma}^{\tilde{F}}$ are described by the corresponding elements of $A_{\mathbf{G}_{\mathrm{sc}}}(\tilde{g}_0)$ given at the top of each column, and where z, i, $\sqrt{q}$ are as before; we also set $q^{7/2} := \sqrt{q}^7$.

	$1, \overline{z}$	$\overline{\overline{s}}_0^2, \overline{\overline{s}}_0^2\overline{z}$	$\overline{\overline{s}}_0, \overline{\overline{s}}_0\overline{z}$	$\overline{\overline{s}}_0^{-1}, \overline{\overline{s}}_0^{-1}\overline{z}$
$\chi_{x_1}^{\mathbf{G}_{\mathrm{sc}}}$	$q^{7/2}$	$-q^{7/2}$	$\mathrm{i}q^{7/2}$	$-\mathrm{i}q^{7/2}$
$\chi_{x_2}^{\mathbf{G}_{\mathrm{sc}}}$	$q^{7/2}$	$-q^{7/2}$	$-\mathrm{i}q^{7/2}$	$\mathrm{i}q^{7/2}$

Recall that the $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-classes parametrised by $1, \overline{z}$ are denoted by C, C', respectively. In view of [Gec21, 4.10(a)], we have $\tilde{g}_0^{-1} \in C \cup C'$. Using this and the fact that the characteristic functions $\chi_{x_1}^{\mathbf{G}_{\mathrm{sc}}}$, $\chi_{x_2}^{\mathbf{G}_{\mathrm{sc}}}$ take the same value at any element of $C \cup C'$, a similar computation as the one for groups of type E_6 in characteristic $p \neq 3$ shows that $\xi_{x_1} = \xi_{x_2} \in \{\pm 1\}$. It is then finally shown in [Gec21, 6.5] that we have in fact $\xi_{x_1} = \xi_{x_2} = +1$, by considering the Hecke algebra associated to $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$ and its BN-pair $\big((\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}}, N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{T}_0^{\mathrm{sc}})^{\tilde{F}}\big)$ and explicitly evaluating the right side of the formula (3.4.19.2).

Proposition 4.2.5 (Geck [Gec21, 6.5])**.** *Let $\mathbf{G}^F = \mathsf{E}_7(q)$ where q is any power of any prime $p \neq 2$. As in 4.2.3 (and with the notation there), let $\tilde{g}_0 \in \tilde{\Sigma}^{\tilde{F}}$ be an element whose $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class has a non-empty intersection with $(\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}} w_{\mathrm{c}}^{\mathrm{sc}} (\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}}$, where $w_{\mathrm{c}}^{\mathrm{sc}}$ is a Coxeter element of $\mathbf{W}^{\mathrm{sc}}$. Let $g_0 = \pi(\tilde{g}_0) \in \Sigma^F$. For $i = 1, 2$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of g_0, and let $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i}) \in \overline{\mathbb{Q}}_\ell^\times$ be defined by (4.2.1.3). Then*

$$\xi_{x_1} = \xi_{x_2} = 1,$$

that is, the characteristic function $\chi_{x_i} = \chi_{A_i, \varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ coincides with the unipotent almost character R_{x_i} for $i = 1, 2$. The analogous statement holds for the simply connected group $\mathbf{G}_{\mathrm{sc}}$ (using $\tilde{g}_0$ to define the characteristic functions $\chi_{x_i}^{\mathbf{G}_{\mathrm{sc}}} \colon \mathbf{G}_{\mathrm{sc}}^{\tilde{F}} \to \overline{\mathbb{Q}}_\ell$ for $i = 1, 2$).

Remark 4.2.6. Recall from 4.2.3 that the only assumption which is made on the $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-conjugacy class of $\tilde{g}_0 \in \widetilde{\Sigma}^{\tilde{F}}$ in Proposition 4.2.5 is that it has a non-empty intersection with $(\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}} w_{\mathrm{c}}^{\mathrm{sc}} (\mathbf{B}_0^{\mathrm{sc}})^{\tilde{F}}$, and there are two $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}}$-classes inside $\widetilde{\Sigma}^{\tilde{F}}$ with this property. So here the proof is solely based on the fact that condition (♡) in 3.2.22(c) is satisfied. In view of [Gec21, p. 33], the values of the four unipotent characters parametrised by $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{11}})$ (where $\mathcal{F}_{11} = \{512_a', 512_a\} \subseteq \mathrm{Irr}(\mathbf{W})$ is the exceptional family) are given by Table 4.5 (with the same conventions as in 4.2.4).

	$1,\, \overline{z}$	$\overline{\tilde{s}}_0^{\,2},\, \overline{\tilde{s}}_0^{\,2}\overline{z}$	$\overline{\tilde{s}}_0,\, \overline{\tilde{s}}_0\overline{z}$	$\overline{\tilde{s}}_0^{\,-1},\, \overline{\tilde{s}}_0^{\,-1}\overline{z}$
$[512_a']$	$q^{7/2}$	$-q^{7/2}$	0	0
$[512_a]$	$-q^{7/2}$	$q^{7/2}$	0	0
$\mathsf{E}_7[\mathrm{i}\sqrt{q}]$	0	0	$-\mathrm{i}q^{7/2}$	$\mathrm{i}q^{7/2}$
$\mathsf{E}_7[-\mathrm{i}\sqrt{q}]$	0	0	$\mathrm{i}q^{7/2}$	$-\mathrm{i}q^{7/2}$

Table 4.5.: Values of unipotent characters in the exceptional family at elements of $\widetilde{\Sigma}^{\tilde{F}}$ (respectively, Σ^F) in type E_7, $p \neq 2$

Type E_7 in characteristic $p = 2$

In this subsection (that is, here and in 4.2.7–4.2.15 below), we assume that $p = 2$, so $\mathbf{G}$ is the simple adjoint group of type E_7 over $k = \overline{\mathbb{F}}_2$. Somewhat similarly to the case of E_6 with $p = 3$, the prime $p = 2$ for groups of type E_7 behaves distinctly from all the others, as the two cuspidal unipotent character sheaves look quite different. Note that, since $p = 2$, we do not really have to distinguish between the simply connected group $\mathbf{G}_{\mathrm{sc}}$ and the adjoint group $\mathbf{G} = \mathbf{G}_{\mathrm{ad}}$ of type E_7. Indeed, consider the canonical isogeny $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ (see 2.1.7); we have $\ker \pi \subseteq \mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) \cong \mathrm{Hom}(\Lambda(\mathfrak{C}), k^\times)$. Since $\Lambda(\mathfrak{C}) \cong \mathbb{Z}/2\mathbb{Z}$, the only homomorphism $\Lambda(\mathfrak{C}) \to k^\times$ is the trivial one, and we deduce that $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}}) = \{1\}$. Thus, $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G} = \mathbf{G}_{\mathrm{ad}}$ is bijective and gives rise to an isomorphism on the level of finite groups $\mathbf{G}_{\mathrm{sc}}^{\tilde{F}} \xrightarrow{\sim} \mathbf{G}^F$ (see [GM20, 1.5.12]) where $\tilde{F}\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$ is the endomorphism commuting with π and induced by F; $\tilde{F}$ is a Frobenius map for an $\mathbb{F}_q$-rational structure on $\mathbf{G}_{\mathrm{sc}}$.

The discussion in 4.2.7–4.2.14 below is due to [Het22a].

4.2.7. By [LuCS4, 20.3, 20.5], the support of the two cuspidal unipotent character sheaves A_1, A_2 on $\mathbf{G}$ is the unipotent variety $\mathbf{G}_{\mathrm{uni}} = \overline{\mathscr{O}}_{\mathrm{reg}} \subseteq \mathbf{G}$. By Proposition 3.1.17 (see 3.2.21), there are $\mathbf{G}$-equivariant F-stable irreducible local systems $\mathscr{E}_1$, $\mathscr{E}_2$ on $\mathscr{O}_{\mathrm{reg}}$

such that

$$A_j \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_j)[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}} \quad \text{for } j = 1, 2.$$

Let $u \in \mathscr{O}_{\mathrm{reg}}^F$. The group $A_{\mathbf{G}}(u) = C_{\mathbf{G}}(u)/C^\circ_{\mathbf{G}}(u)$ is cyclic of order 4 and generated by the image $\overline{u}$ of u in $A_{\mathbf{G}}(u)$ (see, e.g., [Miz80, Lm. 10] and [DM20, 12.2.3, 12.2.7]). Hence, the automorphism of $A_{\mathbf{G}}(u)$ induced by F is the identity, and the $\mathbf{G}^F$-conjugacy classes contained in $\mathscr{O}_{\mathrm{reg}}^F$ are parametrised by the elements of $A_{\mathbf{G}}(u)$; so there are four different $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}_{\mathrm{reg}}^F$. For $1 \leqslant i \leqslant 7$, let $u_i := u_{\alpha_i}$ be the homomorphism $\mathbf{G}_{\mathrm{a}} \to \mathbf{G}$ whose image is the root subgroup $\mathbf{U}_{\alpha_i} \subseteq \mathbf{U}_0$ (see 2.1.4). We set

$$y_1 := u_1(1) \cdot u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_6(1) \cdot u_7(1) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F. \tag{4.2.7.1}$$

(Mizuno [Miz80] defines y_1 in a slightly different way, but it is $\mathbf{G}^F$-conjugate to our chosen representative, as the following lemma shows.)

Lemma 4.2.8. *Consider the element $y_1 \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F$ defined in* (4.2.7.1)*. For any permutation ϱ of $\{1, 2, \ldots, 7\}$, there exists an element $v \in \mathbf{U}_0^F$ such that*

$$v \cdot y_1 \cdot v^{-1} = u_{\varrho(1)}(1) \cdot u_{\varrho(2)}(1) \cdot \ldots \cdot u_{\varrho(7)}(1).$$

In particular, y_1 is conjugate to y_1^{-1} in $\mathbf{U}_0^F \subseteq \mathbf{G}^F$.

Proof. Let us first consider the Weyl group $\mathbf{W}$ of $\mathbf{G}$, viewed as the irreducible finite Coxeter group of type E_7 with simple reflections $S = \{s_1, s_2, \ldots, s_7\}$. It is well known that any two Coxeter elements of $(\mathbf{W}, S)$ are conjugate in $\mathbf{W}$, that is, for any permutation ϱ of $\{1, 2, \ldots, 7\}$, there exists some $w \in \mathbf{W}$ such that

$$w \cdot (s_1 \cdot s_2 \cdot \ldots \cdot s_7) \cdot w^{-1} = s_{\varrho(1)} \cdot s_{\varrho(2)} \cdot \ldots \cdot s_{\varrho(7)}.$$

More precisely, [Cas17, §1] provides an algorithm to compute such an element w, which is only based on the facts that an element s_i ($1 \leqslant i \leqslant 7$) in the first (respectively, last) position of a given Coxeter word can be shifted to the last (respectively, first) position by means of conjugation with s_i, and that s_i, s_j ($1 \leqslant i, j \leqslant 7$, $i \neq j$) commute if and only if they are not linked in the Coxeter diagram of $(\mathbf{W}, S)$. But the analogous statements hold for the $u_i(1)$, $1 \leqslant i \leqslant 7$. (Note that $u_i(1) = u_i(-1)$ since k has characteristic 2.) So we can just mimic the proof of [Cas17, 1.4] to obtain an element $v \in \mathbf{U}_0^F$ (a product of certain $u_i(1)$, $1 \leqslant i \leqslant 7$) such that

$$v \cdot y_1 \cdot v^{-1} = u_{\varrho(1)}(1) \cdot u_{\varrho(2)}(1) \cdot \ldots \cdot u_{\varrho(7)}(1).$$

In particular, since $y_1^{-1} = u_7(1)\cdot u_6(1)\cdot\ldots\cdot u_1(1)$, y_1 is conjugate to y_1^{-1} in $\mathbf{U}_0^F \subseteq \mathbf{G}^F$. Specifically, setting

$$v := u_7(1)u_6(1)u_7(1)u_5(1)u_6(1)u_7(1)u_4(1)u_2(1)u_5(1)u_6(1)u_7(1)u_2(1)u_1(1) \in \mathbf{U}_0^F,$$

we have $vy_1v^{-1} = y_1^{-1}$. □

4.2.9. In the setting of 4.2.7, let us take $u := y_1 \in \mathscr{O}_{\mathrm{reg}}^F$. Thus, $A_{\mathbf{G}}(y_1)$ is a cyclic group of order 4 generated by $\overline{y}_1$, and F induces the identity map on $A_{\mathbf{G}}(y_1)$. So by 3.2.20, there are (up to isomorphism) four $\mathbf{G}$-equivariant irreducible local systems on $\mathscr{O}_{\mathrm{reg}}$, and each of them is one-dimensional and F-stable. For $j = 1, 2$, let $\varsigma_j \in \mathrm{Irr}(A_{\mathbf{G}}(y_1))$ be the faithful irreducible (linear) characters of $A_{\mathbf{G}}(y_1)$ with $\varsigma_1(\overline{y}_1) = \mathrm{i}$, $\varsigma_2(\overline{y}_1) = -\mathrm{i}$, where $\mathrm{i} \in \mathcal{R}_4$ is the primitive 4th root of unity fixed in 4.2.1 (and assumed to be the same as in 3.4.2). Then for $j = 1, 2$, ς_j corresponds to the isomorphism class of the local system $\mathscr{E}_j$ on $\mathscr{O}_{\mathrm{reg}}$ (via 3.2.20), see, e.g., [Spa85, p. 337]. The fixed choice of y_1 uniquely defines isomorphisms $\varphi_{A_j}\colon F^*A_j \xrightarrow{\sim} A_j$ for $j = 1, 2$, as described in 3.2.21. We will henceforth always assume that in the setting of 4.2.1, χ_{x_j} and $\xi_{x_j} := \xi_{x_j}(\varphi_{x_j})$ are defined with respect to $\varphi_{x_j} := \varphi_{A_j}$ for $j = 1, 2$. By Theorem 3.1.13, the characteristic functions χ_{x_1}, χ_{x_2} vanish at any element of $\mathbf{G}^F \setminus \mathscr{O}_{\mathrm{reg}}^F$. The values of χ_{x_1}, χ_{x_2} at elements of $\mathscr{O}_{\mathrm{reg}}^F$ are given by the following table where, as usual, we denote the $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}_{\mathrm{reg}}^F$ by the corresponding elements of $A_{\mathbf{G}}(y_1)$ and where $q^{7/2} := \sqrt{q}^{\,7}$, with $\sqrt{q}$ being the square root of q that we fixed in (1.1.0.2).

	1	$\overline{y}_1$	$\overline{y}_1^2$	$\overline{y}_1^3$
χ_{x_1}	$q^{7/2}$	$\mathrm{i}q^{7/2}$	$-q^{7/2}$	$-\mathrm{i}q^{7/2}$
χ_{x_2}	$q^{7/2}$	$-\mathrm{i}q^{7/2}$	$-q^{7/2}$	$\mathrm{i}q^{7/2}$

In particular, we see that $\overline{\chi}_{x_1} = \chi_{x_2}$.

4.2.10. Having described the characteristic functions χ_{x_1}, χ_{x_2}, let us now consider the almost characters R_{x_1}, R_{x_2}, as well as the scalars ξ_{x_1}, ξ_{x_2} defined through (4.2.1.3). We have

$$R_{x_1} = \sum_{x\in\mathfrak{X}(\mathbf{W})} \{x, x_1\}\Delta(x)\rho_x = -\tfrac{1}{2}\rho_{x_1} + \tfrac{1}{2}\rho_{x_2} + \sum_{x\in\mathfrak{X}(\mathbf{W})\setminus\{x_1,x_2\}} \{x, x_1\}\rho_x,$$

$$R_{x_2} = \sum_{x\in\mathfrak{X}(\mathbf{W})} \{x, x_2\}\Delta(x)\rho_x = -\tfrac{1}{2}\rho_{x_2} + \tfrac{1}{2}\rho_{x_1} + \sum_{x\in\mathfrak{X}(\mathbf{W})\setminus\{x_1,x_2\}} \{x, x_2\}\rho_x.$$

(Note that $\Delta(x_1) = \Delta(x_2) = -1$, see [Lus84a, 4.14].) Now, labelling the elements of $\mathfrak{X}(\mathbf{W})$ in terms of unipotent characters as in (4.2.1.1), we see from 3.4.11 and an inspection of the Fourier matrix that we have $\{x_\rho, x_1\}\Delta(x_\rho) = \{x_{\overline{\rho}}, x_2\}\Delta(x_{\overline{\rho}}) \in \mathbb{R}$ for any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. So we get

$$\overline{R}_{x_1} = \sum_{\rho\in\mathrm{Uch}(\mathbf{G}^F)} \{x_\rho, x_1\}\Delta(x_\rho)\overline{\rho} = \sum_{\rho\in\mathrm{Uch}(\mathbf{G}^F)} \{x_{\overline{\rho}}, x_2\}\Delta(x_{\overline{\rho}})\overline{\rho} = R_{x_2},$$

where the last equality follows from the fact that $\rho \mapsto \overline{\rho}$ defines a bijection on $\mathrm{Uch}(\mathbf{G}^F)$. Using Lemma 4.2.8, we deduce that

$$R_{x_1}(y_1) = \overline{R_{x_1}(y_1)} = R_{x_2}(y_1).$$

We thus obtain

$$\xi_{x_1} q^{7/2} = \xi_{x_1}\chi_{x_1}(y_1) = R_{x_1}(y_1) = R_{x_2}(y_1) = \xi_{x_2}\chi_{x_2}(y_1) = \xi_{x_2} q^{7/2},$$

which also equals $\overline{R_{x_1}(y_1)} = \overline{\xi}_{x_1} q^{7/2}$. So we have $\xi_{x_1} = \xi_{x_2} = \overline{\xi}_{x_1} \in \{\pm 1\}$ (since $|\xi_{x_1}| = 1$). Let us set

$$\xi := \xi_{x_1} = \xi_{x_2} \in \{\pm 1\}. \tag{4.2.10.1}$$

4.2.11. In order to determine the sign $\xi \in \{\pm 1\}$ in (4.2.10.1), we will use the method described in 3.4.19(2). With the notation there we have, for any $g \in \mathbf{G}^F$ and any $w \in \mathbf{W}$:

$$\frac{|O_g \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(g)|}{|\mathbf{B}_0^F|} = m(g, w) = \sum_{x\in\mathfrak{X}(\mathbf{W})} c_x(w) R_x(g), \tag{4.2.11.1}$$

where

$$c_x(w) = \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_\phi, x\}\,\mathrm{Trace}(T_w, V_\phi).$$

We want to evaluate (4.2.11.1) with $g = u \in \mathscr{O}_{\mathrm{reg}}^F$ (and a suitable $w \in \mathbf{W}$) to get hold of the sign ξ. For this purpose, we only need to consider such $x \in \mathfrak{X}(\mathbf{W})$ which correspond to some pair $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ of the form $(\mathscr{O}_{\mathrm{reg}}, \mathscr{E})$ under the generalised Springer correspondence, as $R_x|_{\mathscr{O}_{\mathrm{reg}}^F} = 0$ for any other $x \in \mathfrak{X}(\mathbf{W})$; see Remark 3.4.24. As already noted in 4.2.9, there are four isomorphism classes of (F-stable) $\mathbf{G}$-equivariant irreducible local systems on $\mathscr{O}_{\mathrm{reg}}$, and they are naturally parametrised by the irreducible characters of $A_{\mathbf{G}}(y_1) \cong \langle \overline{y}_1 \rangle \cong C_4$; accordingly, we denote any such local system on $\mathscr{O}_{\mathrm{reg}}$ by the value of the corresponding character of $A_{\mathbf{G}}(y_1)$ at $\overline{y}_1$. We thus have to consider the four pairs

$$(\mathscr{O}_{\mathrm{reg}}, 1),\ (\mathscr{O}_{\mathrm{reg}}, -1),\ (\mathscr{O}_{\mathrm{reg}}, \mathrm{i}),\ (\mathscr{O}_{\mathrm{reg}}, -\mathrm{i}) \in \mathcal{N}_{\mathbf{G}}^F.$$

Let $x_0 \in \mathfrak{X}(\mathbf{W})$ be the element corresponding to $(\mathsf{D}_4, 1, (-1,2)) \in \mathfrak{S}_{\mathbf{W}}$ under Corollary 3.4.8, where $\mathsf{D}_4 := \{s_2, s_3, s_4, s_5\} \subseteq S$, and where 1 denotes the trivial character of $W_{\mathbf{G}}(\mathbf{L}_{\mathsf{D}_4}) \cong \mathbf{W}^{S/\mathsf{D}_4} \cong W(\mathsf{B}_3)$. By the results of Spaltenstein [Spa85, p. 331] (and in view of the discussion in 4.2.9), the generalised Springer correspondence with respect to the above four elements of $\mathcal{N}_{\mathbf{G}}^F$ is then given by the following table.

Local system on $\mathscr{O}_{\mathrm{reg}}$	1	-1	i	$-$i
$x \in \mathfrak{X}(\mathbf{W}) \cong \mathfrak{S}_{\mathbf{W}}$	$1_{\mathbf{W}}$	$x_0 \leftrightarrow (\mathsf{D}_4, 1, (-1,2))$	x_1	x_2

We have $R_{1_{\mathbf{W}}} = 1_{\mathbf{G}^F}$ and $R_{x_j} = \xi\chi_{x_j}$ for $j = 1,2$. As for the almost character R_{x_0}, we follow 3.4.23 to compute its restriction to $\mathscr{O}_{\mathrm{reg}}^F$ (up to multiplication with a scalar). Let $\mathfrak{i}_0 := (\mathscr{O}_{\mathrm{reg}}, -1)$. We have $\tau(\mathfrak{i}_0) = (\mathbf{L}_{\mathsf{D}_4}, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}_{\mathbf{G}}^F$ where $(\mathscr{O}_0, \mathscr{E}_0)$ is uniquely determined by $\mathbf{L}_{\mathsf{D}_4}$, and where $\mathscr{O}_0 \subseteq \mathbf{L}_{\mathsf{D}_4}$ is the regular unipotent class. Let us fix an isomorphism $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_{\mathrm{reg}}^F$. Let $\varphi_{A_{\mathfrak{i}_0}} : F^*A_{\mathfrak{i}_0} \xrightarrow{\sim} A_{\mathfrak{i}_0}$, $\overline{\varphi}_{A_{\mathfrak{i}_0}} : F^*A_{\mathfrak{i}_0} \xrightarrow{\sim} A_{\mathfrak{i}_0}$ be the isomorphisms defined as in 3.2.25, and let $\varphi_{x_0} := \overline{\varphi}_{A_{\mathfrak{i}_0}}$, $\chi_{x_0} := \chi_{A_{x_0}, \varphi_{x_0}}$. As discussed in 3.4.23, we get

$$R_{x_0}|_{\mathbf{G}_{\mathrm{uni}}^F} = q^2 \zeta_{\mathsf{D}_4} X_{\mathfrak{i}_0} \quad \text{where } \zeta_{\mathsf{D}_4} \in \mathcal{R}.$$

We have $X_{\mathfrak{i}_0}|_{\mathscr{O}_{\mathrm{reg}}^F} = Y_{\mathfrak{i}_0}$ (see Corollary 3.2.17), and there exists a root of unity $\gamma_{\mathfrak{i}_0} \in \mathcal{R}$ such that

$$Y_{\mathfrak{i}_0}\Big((y_1)_{\overline{y}_1^{\,j}}\Big) = \gamma_{\mathfrak{i}_0}(-1)^j \quad \text{for } 0 \leqslant j \leqslant 3.$$

We thus have

$$R_{x_0}\Big((y_1)_{\overline{y}_1^{\,j}}\Big) = q^2 \zeta_{\mathsf{D}_4}\gamma_{\mathfrak{i}_0}(-1)^j \quad \text{for } 0 \leqslant j \leqslant 3.$$

Since R_{x_0} is an $\mathbb{R}$-linear combination of unipotent characters, Lemma 4.2.8 shows that

$$q^2 \cdot \zeta_{\mathsf{D}_4}\gamma_{\mathfrak{i}_0} = R_{x_0}(y_1) = R_{x_0}(y_1^{-1}) = \overline{R_{x_0}(y_1)} = q^2 \cdot \overline{\zeta_{\mathsf{D}_4}\gamma_{\mathfrak{i}_0}},$$

so we get $\zeta_{\mathsf{D}_4}\gamma_{\mathfrak{i}_0} \in \mathbb{R} \cap \mathcal{R}$, that is, we have $\zeta_{\mathsf{D}_4}\gamma_{\mathfrak{i}_0} \in \{\pm 1\}$.

As an analogue to Proposition 4.2.5, we obtain the following result.

Proposition 4.2.12 (see [Het22a])**.** *Let $\mathbf{G}^F = \mathsf{E}_7(q)$ where q is any power of the prime $p = 2$. Let $y_1 \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F$ be as defined in (4.2.7.1). For $i = 1,2$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$*

be the isomorphism corresponding to the $\mathbf{G}^F$*-conjugacy class of* y_1 *in* $\mathscr{O}_{\mathrm{reg}}^F$*, and let* $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i}) \in \overline{\mathbb{Q}}_\ell^\times$ *be defined by* (4.2.1.3)*. Then*

$$\xi_{x_1} = \xi_{x_2} = 1,$$

that is, the characteristic function $\chi_{x_i} = \chi_{A_i,\varphi_{x_i}} : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ *coincides with the unipotent almost character* R_{x_i} *for* $i = 1, 2$*.*

Proof. We set

$$w_{\mathrm{c}} := s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \in \mathbf{W},$$

a Coxeter element of $\mathbf{W}$. Thus, the conjugacy class of w_{c} in $\mathbf{W}$ is sent to $\mathscr{O}_{\mathrm{reg}}$ under Lusztig's map (see 3.2.23). With the notation of 4.2.11, let us now evaluate (4.2.11.1) with $g = y_1$ and $w = w_{\mathrm{c}}$. Using CHEVIE [MiChv], we get $c_{1_{\mathbf{W}}}(w_{\mathrm{c}}) = q^7$, $c_{x_0}(w_{\mathrm{c}}) = q^5$ and $c_{x_1}(w_{\mathrm{c}}) = c_{x_2}(w_{\mathrm{c}}) = q^{7/2}$, so we have

$$\frac{|O_{y_1} \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(y_1)|}{|\mathbf{B}_0^F|} = \sum_{x \in \mathfrak{X}(\mathbf{W})} c_x(w_{\mathrm{c}}) R_x(y_1) = q^7(1 + \zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0} + 2\xi), \quad (4.2.12.1)$$

where $\xi, \zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0} \in \{\pm 1\}$. We want to apply Lemma 3.2.24: Recall that the longest element w_0 of $\mathbf{W}$ (with respect to the length function on $\mathbf{W}$ determined by $\Pi = \{\alpha_1, \ldots, \alpha_7\} \subseteq R^+$) is characterised by the property $w_0(R^+) = -R^+$, so $-w_0(\Pi) = \Pi$, and $-w_0$ defines a graph automorphism of the Dynkin diagram of $\mathbf{G}$. But the only such automorphism is the identity, so $-w_0(\alpha_i) = \alpha_i$ for $1 \leqslant i \leqslant 7$. Let us choose a representative $\dot{w}_0 \in N_{\mathbf{G}}(\mathbf{T}_0)^F$ of $w_0 \in \mathbf{W}$ (cf. 2.1.20). It follows from Lemma 3.2.24 that $\dot{w}_0 y_1 \dot{w}_0^{-1} \in O_{y_1} \cap \mathbf{B}_0 w_{\mathrm{c}} \mathbf{B}_0$. Since $F(\dot{w}_0) = \dot{w}_0$ and $F(y_1) = y_1$, the uniqueness of expressions in the sharp form of the Bruhat decomposition [Car85, 2.5.14] implies that $\dot{w}_0 y_1 \dot{w}_0^{-1} \in O_{y_1} \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F$. Thus, the left side of (4.2.12.1) is strictly positive, so $\xi \neq -1$, and we must have $\xi = +1$. □

Corollary 4.2.13. *Let* $x_0 \in \mathfrak{X}(\mathbf{W})$ *be as in* 4.2.11 *and* y_1 *be as in* (4.2.7.1)*. The values of* R_{x_0} *at elements of* $\mathscr{O}_{\mathrm{reg}}^F$ *are given by the following table, where we denote the* $\mathbf{G}^F$*-classes inside* $\mathscr{O}_{\mathrm{reg}}^F$ *by giving the corresponding elements of* $A_{\mathbf{G}}(y_1)$ *in the top line of each column.*

	1	$\overline{y}_1$	$\overline{y}_1^2$	$\overline{y}_1^3$
R_{x_0}	q^2	$-q^2$	q^2	$-q^2$

Proof. We have seen in 4.2.11 that

$$R_{x_0}\Big((y_1)_{\overline{y}_1^j}\Big) = q^2\zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0}(-1)^j \quad \text{for } 0 \leqslant j \leqslant 3,$$

where $\zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0} \in \{\pm 1\}$, so we only have to show that $\zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0} = +1$. To see this, let us take $j = 2$ above and set $y_1' := (y_1)_{\overline{y}_1^2} \in \mathscr{O}_{\mathrm{reg}}^F$. We evaluate (4.2.11.1) with $g = y_1'$ and $w = w_{\mathrm{c}} = s_1 s_2 \cdots s_7 \in \mathbf{W}$. By Proposition 4.2.12 and in view of the values of the characteristic functions χ_{x_1} and χ_{x_2} given in 4.2.9, the almost characters R_{x_1} and R_{x_2} both take the value $-q^{7/2}$ at y_1', so we obtain

$$\frac{|O_{y_1'} \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(y_1')|}{|\mathbf{B}_0^F|} = m(y_1', w_{\mathrm{c}}) = q^7(1 + \zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0} - 2) = q^7(\zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0} - 1).$$

Hence, the sign $\zeta_{\mathsf{D}_4}\gamma_{\mathrm{i}_0}$ cannot be -1, so it must be $+1$. □

Remark 4.2.14. As described in 3.2.21, the isomorphisms $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ $(i = 1, 2)$ are determined by choosing one of the four $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}_{\mathrm{reg}}^F$. Now that we know the values of R_{x_0}, R_{x_1} and R_{x_2} at any regular unipotent element, we can explicitly compute $m(u, w_{\mathrm{c}})$ for any $u \in \mathscr{O}_{\mathrm{reg}}^F$ (where $w_{\mathrm{c}} = s_1 s_2 \cdots s_7$, as before). We get

$$m\Big((y_1)_{\overline{y}_1^j}, w_{\mathrm{c}}\Big) = \begin{cases} 4q^7 & \text{if } j = 0, \\ 0 & \text{if } j \in \{1, 2, 3\}. \end{cases}$$

Thus, among the $\mathbf{G}^F$-conjugacy classes contained in $\mathscr{O}_{\mathrm{reg}}^F$, the class of y_1 is the unique one whose elements satisfy the condition (♡) in 3.2.22(c), and we declare the $\mathbf{G}^F$-conjugacy class of y_1 as the *good* class among the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$.

But note that, in contrast to the case of E_6 with $p = 3$, there is another $\mathbf{G}^F$-conjugacy class inside $\mathscr{O}_{\mathrm{reg}}^F$ which is stable under taking inverses (namely, the one parametrised by $\overline{y}_1^2 \in A_{\mathbf{G}}(y_1)$), so this criterion alone does not specify the $\mathbf{G}^F$-class of y_1.

Example 4.2.15. We can now give the values of the unipotent characters in the family

$$\big\{[512_a'], [512_a], \mathsf{E}_7[\mathrm{i}\sqrt{q}], \mathsf{E}_7[-\mathrm{i}\sqrt{q}]\big\} \subseteq \mathrm{Uch}(\mathbf{G}^F)$$

at the regular unipotent elements. For $u \in \mathscr{O}_{\mathrm{reg}}^F$, we have

$$\begin{aligned} [512_a'](u) = -[512_a](u) &= \tfrac{1}{2}\big(R_{x_1}(u) + R_{x_2}(u)\big); \\ \mathsf{E}_7[\mathrm{i}\sqrt{q}](u) = -\mathsf{E}_7[-\mathrm{i}\sqrt{q}](u) &= -\tfrac{1}{2}\big(R_{x_1}(u) - R_{x_2}(u)\big). \end{aligned}$$

The values of R_{x_1} and R_{x_2} are obtained from Proposition 4.2.12 (and 4.2.9), so it is easy to compute the values of the four unipotent characters above at elements of $\mathscr{O}_{\mathrm{reg}}^F$. They are given in Table 4.6 where, as usual, we denote the $\mathbf{G}^F$-conjugacy classes contained in

$\mathscr{O}_{\mathrm{reg}}^F$ by writing the corresponding element of $A_{\mathbf{G}}(y_1) = \langle \overline{y}_1 \rangle \cong C_4$ in the top line of each column.

	1	$\overline{y}_1$	$\overline{y}_1^2$	$\overline{y}_1^3$
$[512'_a]$	$q^{7/2}$	0	$-q^{7/2}$	0
$[512_a]$	$-q^{7/2}$	0	$q^{7/2}$	0
$\mathsf{E}_7[\mathrm{i}\sqrt{q}]$	0	$-\mathrm{i}q^{7/2}$	0	$\mathrm{i}q^{7/2}$
$\mathsf{E}_7[-\mathrm{i}\sqrt{q}]$	0	$\mathrm{i}q^{7/2}$	0	$-\mathrm{i}q^{7/2}$

Table 4.6.: Values of unipotent characters in the exceptional family for $\mathsf{E}_7(q)$ at regular unipotent elements, where q is a power of $p = 2$

Values of unipotent characters at unipotent elements for $\mathsf{E}_7(q)$

Let us come back to the situation in the beginning of this section, where p is any prime (and $\mathbf{G}$ is the simple adjoint group of type E_7 over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$, $q = p^n$, with Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$). Similarly as for groups of type E_6, we consider the task of explicitly computing the values of the unipotent (almost) characters of $\mathbf{G}^F = \mathsf{E}_7(q)$ at unipotent elements.

4.2.16. Recall the decomposition of $\mathrm{Uch}(\mathbf{G}^F)$ into Harish-Chandra series described in 4.2.2. In particular, we see that 60 of the 76 almost characters R_x ($x \in \mathfrak{X}(\mathbf{W})$) are of the form R_ϕ with $\phi \in \mathrm{Irr}(\mathbf{W})$. Computing their values at unipotent elements of $\mathbf{G}^F$ is equivalent to determining the Green functions of $\mathbf{G}^F$, and this has been established in type E_7 in all characteristics, see 2.2.5. Considering the two $x \in \mathfrak{X}(\mathbf{W})$ which correspond to the elements of $\mathfrak{S}_{\mathbf{W}}^\circ$ in 4.2.2(d) parametrising cuspidal unipotent characters (cuspidal unipotent character sheaves), we know the values of $R_x|_{\mathbf{G}_{\mathrm{uni}}^F}$ from the previous results of this section:

- If $p \neq 2$, the support of A_x is given by (the closure of) a non-unipotent class of $\mathbf{G}$, so χ_{A_x} (and, hence, R_x) is identically 0 on $\mathbf{G}_{\mathrm{uni}}^F$.
- If $p = 2$, the values of R_x are obtained from Proposition 4.2.12 (as $R_x(g) = 0$ for any $g \in \mathbf{G}^F \setminus \mathscr{O}_{\mathrm{reg}}^F$).

The 14 remaining $x \in \mathfrak{X}(\mathbf{W})$ give rise to unipotent character sheaves $A_x \in \hat{\mathbf{G}}^{\mathrm{un}}$ which are simple constituents of a complex $K_{\mathbf{L}_J,\Sigma}^{\mathscr{E}} \in \mathscr{M}\mathbf{G}$, with J as in 4.2.2(b) or 4.2.2(c), and with $(\Sigma, \mathscr{E})$ a cuspidal pair for $\mathbf{L}_J$: If $J = \{s_2, s_3, s_4, s_5\}$, there is a unique such cuspidal pair for $\mathbf{L}_J$; if $J = \{s_1, s_2, \ldots, s_6\}$, there are two such cuspidal pairs, but the Σ involved is the same for both pairs. In this way, any of these 14 elements $x \in \mathfrak{X}(\mathbf{W})$ uniquely determines J, which in turn uniquely determines $\Sigma \subseteq \mathbf{L}_J$, and with the same argument as in 4.1.23, we see that $R_x|_{\mathbf{G}^F_{\mathrm{uni}}} = 0$ if Σ does not contain any unipotent elements; this happens for $J = \{s_2, s_3, s_4, s_5\}$ whenever $p \neq 2$, and for $J = \{s_1, s_2, \ldots, s_6\}$ whenever $p \neq 3$. Summarising:

- If $p \geqslant 5$, we only need to consider the 60 unipotent uniform almost characters $R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $\phi \in \mathrm{Irr}(\mathbf{W})$. Their values can be obtained via Lübeck's electronic library [Lüb].

- If $p = 3$, the values of the 60 almost characters R_ϕ ($\phi \in \mathrm{Irr}(\mathbf{W})$) at unipotent elements have been determined by Geck in [Gec20b]. In addition, there are four $x \in \mathfrak{X}(\mathbf{W})$ for which $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ is non-zero, associated to $J = \{s_1, s_2, \ldots, s_6\}$ as in 4.2.2(c); their values are not yet known.

- If $p = 2$, we have to consider 72 almost characters $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$: The $R_\phi|_{\mathbf{G}^F_{\mathrm{uni}}}$ ($\phi \in \mathrm{Irr}(\mathbf{W})$) are known, again due to [Gec20b]. The two $x \in \mathfrak{X}(\mathbf{W})$ parametrising cuspidal unipotent character sheaves give rise to non-zero functions $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$, whose values are obtained from Proposition 4.2.12. Finally, there are 10 additional elements of $\mathfrak{X}(\mathbf{W})$ for which $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ is non-zero; they arise from the subset $J = \{s_2, s_3, s_4, s_5\} \subseteq S$ as in 4.2.2(b), and their values are not yet known.

The case where $p = 3$

4.2.17. Let $p = 3$, $J = \{s_1, s_2, \ldots, s_6\}$, and let $\omega \in \mathcal{R}_3$ be the primitive 3rd root of unity that we fixed in 3.4.2, 4.2.2. In order to compute the four non-zero $R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $x \in \mathfrak{X}(\mathbf{W}) \setminus \mathrm{Irr}(\mathbf{W})$ corresponding to $(J, \pm 1, (\omega, 3)), (J, \pm 1, (\omega^2, 3)) \in \mathfrak{S}_{\mathbf{W}}$, we need to consider the corresponding four $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ under the generalised Springer correspondence (see Remark 3.4.24). By the results of [Lus84b] (see also [Spa85]), these $\mathfrak{i}$ are

$$(\mathscr{O}_{\mathrm{reg}}, \omega), \quad (\mathsf{E}_6, \omega), \quad (\mathscr{O}_{\mathrm{reg}}, \omega^2), \quad (\mathsf{E}_6, \omega^2),$$

where $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_3$ for any $u \in \mathscr{O}_{\mathrm{reg}} \cup \mathsf{E}_6$, so we denote the local systems by the values of the corresponding characters of $A_{\mathbf{G}}(u)$ at $\overline{u}$, as usual. We then have

$$\tau((\mathscr{O}_{\mathrm{reg}}, \omega)) = \tau((\mathsf{E}_6, \omega)) = (\mathbf{L}_J, \mathscr{O}_0, \omega), \quad \tau((\mathscr{O}_{\mathrm{reg}}, \omega^2)) = \tau((\mathsf{E}_6, \omega^2)) = (\mathbf{L}_J, \mathscr{O}_0, \omega^2),$$

where $\mathscr{O}_0 \subseteq \mathbf{L}_J$ is the regular unipotent class, and where $A_{\mathbf{L}_J}(u) = \langle \overline{u} \rangle \cong C_3$ for $u \in \mathscr{O}_0$. Identifying $\mathfrak{X}(\mathbf{W})$ with $\mathfrak{S}_{\mathbf{W}}$ (see Corollary 3.4.8), the correspondence $x \leftrightarrow \mathfrak{i}$ for the above

$x, \mathfrak{i}$ is then given by

$$(J,1,(\omega,3)) \leftrightarrow (\mathscr{O}_{\mathrm{reg}},\omega), \qquad (J,-1,(\omega,3)) \leftrightarrow (\mathsf{E}_6,\omega),$$
$$(J,1,(\omega^2,3)) \leftrightarrow (\mathscr{O}_{\mathrm{reg}},\omega^2), \qquad (J,-1,(\omega^2,3)) \leftrightarrow (\mathsf{E}_6,\omega^2).$$

Let us denote the almost characters R_x with x corresponding to

$$(J,1,(\omega,3)), \quad (J,-1,(\omega,3)), \quad (J,1,(\omega^2,3)), \quad (J,-1,(\omega^2,3)),$$

by

$$R_{\mathsf{E}_6[\omega,1]}, \quad R_{\mathsf{E}_6[\omega,-1]}, \quad R_{\mathsf{E}_6[\omega^2,1]}, \quad R_{\mathsf{E}_6[\omega^2,-1]},$$

respectively.

4.2.18. We keep the setting of 4.2.17. Let us fix isomorphisms $\varphi_0^{\omega}\colon F^*(\omega) \xrightarrow{\sim} \omega$ and $\varphi_0^{\omega^2}\colon F^*(\omega^2) \xrightarrow{\sim} \omega^2$ which induce maps of finite order at the stalks of ω, ω^2 at any element of $\mathscr{O}_0^F$. If x is one of the four elements considered in 4.2.17 and $x \leftrightarrow \mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$, we define the isomorphisms $\varphi_{A_\mathfrak{i}}\colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$, $\overline{\varphi}_{A_\mathfrak{i}}\colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ as in 3.2.25 and set $\varphi_x := \overline{\varphi}_{A_\mathfrak{i}}$, $\chi_x := \chi_{A_x,\varphi_x}$. We now apply 3.4.23 and use the notation there; setting $\zeta(\omega) := \zeta(J,(\omega,3))$ and $\zeta(\omega^2) := \zeta(J,(\omega^2,3))$, we thus obtain

$$R_{\mathsf{E}_6[\omega,1]}|_{\mathbf{G}^F_{\mathrm{uni}}} = q^3\zeta(\omega)X_{(\mathscr{O}_{\mathrm{reg}},\omega)} \quad \text{and} \quad R_{\mathsf{E}_6[\omega,-1]}|_{\mathbf{G}^F_{\mathrm{uni}}} = q^6\zeta(\omega)X_{(\mathsf{E}_6,\omega)}.$$

Using CHEVIE [MiChv] (see Remark 3.2.18), we get

$$X_{(\mathscr{O}_{\mathrm{reg}},\omega)} = Y_{(\mathscr{O}_{\mathrm{reg}},\omega)} + q^2Y_{(\mathsf{E}_6,\omega)} \quad \text{and} \quad X_{(\mathsf{E}_6,\omega)} = Y_{(\mathsf{E}_6,\omega)},$$

so we have

$$\begin{aligned} R_{\mathsf{E}_6[\omega,1]}|_{\mathbf{G}^F_{\mathrm{uni}}} &= q^3\zeta(\omega)Y_{(\mathscr{O}_{\mathrm{reg}},\omega)} + q^5\zeta(\omega)Y_{(\mathsf{E}_6,\omega)} \quad \text{and} \\ R_{\mathsf{E}_6[\omega,-1]}|_{\mathbf{G}^F_{\mathrm{uni}}} &= q^6\zeta(\omega)Y_{(\mathsf{E}_6,\omega)}. \end{aligned} \tag{4.2.18.1}$$

Similarly, replacing ω by ω^2, we obtain

$$\begin{aligned} R_{\mathsf{E}_6[\omega^2,1]}|_{\mathbf{G}^F_{\mathrm{uni}}} &= q^3\zeta(\omega^2)Y_{(\mathscr{O}_{\mathrm{reg}},\omega^2)} + q^5\zeta(\omega^2)Y_{(\mathsf{E}_6,\omega^2)} \quad \text{and} \\ R_{\mathsf{E}_6[\omega^2,-1]}|_{\mathbf{G}^F_{\mathrm{uni}}} &= q^6\zeta(\omega^2)Y_{(\mathsf{E}_6,\omega^2)}. \end{aligned} \tag{4.2.18.2}$$

Using 3.4.11 and arguing as in 4.2.10, we see that we have $\overline{R_{\mathsf{E}_6[\omega,1]}} = R_{\mathsf{E}_6[\omega^2,1]}$ and $\overline{R_{\mathsf{E}_6[\omega,-1]}} = R_{\mathsf{E}_6[\omega^2,-1]}$. Thus, applying 'complex conjugation' to the equations (4.2.18.1), we obtain

$$\begin{aligned} R_{\mathsf{E}_6[\omega^2,1]}|_{\mathbf{G}^F_{\text{uni}}} &= q^3 \cdot \overline{\zeta(\omega)} \cdot \overline{Y_{(\mathscr{O}_{\text{reg}},\omega)}} + q^5 \cdot \overline{\zeta(\omega)} \cdot \overline{Y_{(\mathsf{E}_6,\omega)}} \quad \text{and} \\ R_{\mathsf{E}_6[\omega^2,-1]}|_{\mathbf{G}^F_{\text{uni}}} &= q^6 \cdot \overline{\zeta(\omega)} \cdot \overline{Y_{(\mathsf{E}_6,\omega)}}. \end{aligned} \tag{4.2.18.3}$$

Hence, we must have

$$\overline{\zeta(\omega)} \cdot \overline{Y_{(\mathsf{E}_6,\omega)}} = \zeta(\omega^2) Y_{(\mathsf{E}_6,\omega^2)} \quad \text{and} \quad \overline{\zeta(\omega)} \cdot \overline{Y_{(\mathscr{O}_{\text{reg}},\omega)}} = \zeta(\omega^2) Y_{(\mathscr{O}_{\text{reg}},\omega^2)}. \tag{4.2.18.4}$$

We also see that $R_{\mathsf{E}_6[\omega,1]}|_{\mathbf{G}^F_{\text{uni}}}$ and $R_{\mathsf{E}_6[\omega^2,1]}|_{\mathbf{G}^F_{\text{uni}}}$ vanish outside of $\mathscr{O}^F_{\text{reg}} \cup \mathsf{E}_6^F$, while $R_{\mathsf{E}_6[\omega,-1]}|_{\mathbf{G}^F_{\text{uni}}}$ and $R_{\mathsf{E}_6[\omega^2,-1]}|_{\mathbf{G}^F_{\text{uni}}}$ vanish outside of E_6^F. Since $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_3$ for $u \in \mathscr{O}^F_{\text{reg}} \cup \mathsf{E}_6^F$, both $\mathscr{O}^F_{\text{reg}}$ and E_6^F split into three $\mathbf{G}^F$-conjugacy classes, so both of $\mathscr{O}^F_{\text{reg}}$ and E_6^F contain (at least) one $\mathbf{G}^F$-class which is stable under taking inverses. Let us fix representatives $y_1 \in \mathscr{O}^F_{\text{reg}}$ and $y_{21} \in \mathsf{E}_6^F$ which are $\mathbf{G}^F$-conjugate to their respective inverses. (As it will turn out, y_1 and y_{21} are in fact uniquely determined by this requirement up to $\mathbf{G}^F$-conjugacy, see Remark 4.2.20 below, but we do not know that at this point.)

4.2.19. In the setting and with the notation of 4.2.18, we now apply 3.4.19(2), as in 4.1.25. So for any $w \in \mathbf{W}$ and any $u \in \mathbf{G}^F_{\text{uni}}$, we have

$$m(u,w) = \frac{|O_u \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(u)|}{|\mathbf{B}_0^F|} = \sum_{x \in \mathfrak{X}(\mathbf{W})} c_x(w) R_x(u), \tag{4.2.19.1}$$

where

$$c_x(w) = \sum_{\phi \in \operatorname{Irr}(\mathbf{W})} \{x_\phi, x\} \operatorname{Trace}(T_w, V_\phi).$$

We define

$$w_1 := s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \in \mathbf{W} \qquad \text{and} \qquad w_{21} := s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \in \mathbf{W}.$$

These are reduced expressions for w_1, w_{21}, and w_1, w_{21} are of minimal length in their conjugacy classes in $\mathbf{W}$; furthermore, under Lusztig's map (see 3.2.23), the class of w_1 is sent to $\mathscr{O}_{\text{reg}}$, and the class of w_{21} is sent to E_6.

(i) For $u \in \mathscr{O}^F_{\text{reg}}$, we get

$$m(u,w_1) = c_{1_{\mathbf{W}}}(w_1) + c_{\mathsf{E}_6[\omega,1]}(w_1) R_{\mathsf{E}_6[\omega,1]}(u) + c_{\mathsf{E}_6[\omega^2,1]}(w_1) R_{\mathsf{E}_6[\omega^2,1]}(u).$$

Using CHEVIE [MiChv], we see that $c_{1_{\mathbf{W}}}(w_1) = q^7$ and $c_{\mathsf{E}_6[\omega,1]}(w_1) = c_{\mathsf{E}_6[\omega^2,1]}(w_1) = q^4$.

In view of (4.2.18.1) and (4.2.18.3), we thus obtain

$$m(u, w_1) = q^7\big(1 + \zeta(\omega)Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}(u) + \overline{\zeta(\omega)} \cdot \overline{Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}(u)}\big) \quad \text{for } u \in \mathscr{O}_{\mathrm{reg}}^F.$$

Since we have chosen $y_1 \in \mathscr{O}_{\mathrm{reg}}^F$ so that it is $\mathbf{G}^F$-conjugate to y_1^{-1} (see 4.2.18) and since $R_{\mathsf{E}_6[\omega,1]}$ is an $\mathbb{R}$-linear combination of unipotent characters, we have

$$R_{\mathsf{E}_6[\omega,1]}(y_1) = R_{\mathsf{E}_6[\omega,1]}(y_1^{-1}) = \overline{R_{\mathsf{E}_6[\omega,1]}(y_1)} \in \mathbb{R},$$

so (4.2.18.1) implies that $\zeta(\omega)Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}(y_1) \in \mathbb{R}$. The latter being at the same time a root of unity, we must have $\zeta(\omega)Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}(y_1) \in \{\pm 1\}$. We get

$$0 \leqslant m(y_1, w_1) = q^7\big(1 + 2\zeta(\omega)Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}(y_1)\big),$$

and it follows that $\zeta(\omega)Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}(y_1) = +1$. So we have

$$R_{\mathsf{E}_6[\omega,1]}(y_1) = R_{\mathsf{E}_6[\omega^2,1]}(y_1) = q^3. \tag{4.2.19.2}$$

Now recall from 3.4.23 that $Y_{(\mathscr{O}_{\mathrm{reg}},\omega)}$ and $Y_{(\mathscr{O}_{\mathrm{reg}},\omega^2)}$ are up to scalar multiplication determined by the linear characters of $A_{\mathbf{G}}(y_1) = \langle \overline{y}_1 \rangle$ which take the respective values ω and ω^2 at $\overline{y}_1$. Using (4.2.19.2), we can thus give the values of $R_{\mathsf{E}_6[\omega,1]}$ and $R_{\mathsf{E}_6[\omega^2,1]}$ at all elements of $\mathscr{O}_{\mathrm{reg}}^F$; see Table 4.7, where the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$ are described by giving the corresponding element of $A_{\mathbf{G}}(y_1) = \langle \overline{y}_1 \rangle \cong C_3$ in the top line of each column.

	1	$\overline{y}_1$	$\overline{y}_1^2$
$R_{\mathsf{E}_6[\omega,1]}$	q^3	ωq^3	$\omega^2 q^3$
$R_{\mathsf{E}_6[\omega^2,1]}$	q^3	$\omega^2 q^3$	ωq^3

Table 4.7.: Values of $R_{\mathsf{E}_6[\omega,1]}$ and $R_{\mathsf{E}_6[\omega^2,1]}$ on $\mathscr{O}_{\mathrm{reg}}^F$ for $\mathsf{E}_7(q)$, q a power of 3

(ii) For $u \in \mathsf{E}_6^F$, we have

$$\begin{aligned} m(u, w_{21}) = {} & c_{1_{\mathbf{W}}}(w_{21}) + c_{\phi_{7,1}}(w_{21})R_{\phi_{7,1}}(u) + c_{\phi_{27,2}}(w_{21})R_{\phi_{27,2}}(u) + c_{\phi_{21,3}}(w_{21})R_{\phi_{21,3}}(u) \\ & + c_{\mathsf{E}_6[\omega,1]}(w_{21})R_{\mathsf{E}_6[\omega,1]}(u) + c_{\mathsf{E}_6[\omega,-1]}(w_{21})R_{\mathsf{E}_6[\omega,-1]}(u) \\ & + c_{\mathsf{E}_6[\omega^2,1]}(w_{21})R_{\mathsf{E}_6[\omega^2,1]}(u) + c_{\mathsf{E}_6[\omega^2,-1]}(w_{21})R_{\mathsf{E}_6[\omega^2,-1]}(u). \end{aligned}$$

Using the known values of the $R_\phi(u)$ for $\phi \in \mathrm{Irr}(\mathbf{W})$ (see [Gec20b]) and (4.2.18.1), (4.2.18.3), this evaluates to

$$m(u, w_{21}) = (q^8 + q^9)\big(1 + \zeta(\omega)Y_{(\mathsf{E}_6,\omega)}(u) + \overline{\zeta(\omega)} \cdot \overline{Y_{(\mathsf{E}_6,\omega)}(u)}\big) \quad \text{for any } u \in \mathsf{E}_6^F.$$

Arguing exactly as in (i) (just with y_{21} instead of y_1), we see that

$$R_{\mathsf{E}_6[\omega,1]}(y_{21}) = R_{\mathsf{E}_6[\omega^2,1]}(y_{21}) = q^5 \quad \text{and} \quad R_{\mathsf{E}_6[\omega,-1]}(y_{21}) = R_{\mathsf{E}_6[\omega^2,-1]}(y_{21}) = q^6.$$

This yields all the non-zero values of $R_{\mathsf{E}_6[\omega,\pm1]}$, $R_{\mathsf{E}_6[\omega^2,\pm1]}$ at elements of E_6^F; they are given by Table 4.8 (with the analogous conventions as in (i), with respect to y_{21} instead of y_1). Thus, Tables 4.7 and 4.8 contain all the non-zero values of the four unipotent almost characters $R_{\mathsf{E}_6[\omega,\pm1]}$, $R_{\mathsf{E}_6[\omega^2,\pm1]}$ at elements of $\mathbf{G}^F_{\text{uni}}$.

	1	$\overline{y}_{21}$	$\overline{y}_{21}^2$
$R_{\mathsf{E}_6[\omega,1]}$	q^5	ωq^5	$\omega^2 q^5$
$R_{\mathsf{E}_6[\omega^2,1]}$	q^5	$\omega^2 q^5$	ωq^5
$R_{\mathsf{E}_6[\omega,-1]}$	q^6	ωq^6	$\omega^2 q^6$
$R_{\mathsf{E}_6[\omega^2,-1]}$	q^6	$\omega^2 q^6$	ωq^6

Table 4.8.: Values of $R_{\mathsf{E}_6[\omega,\pm1]}$ and $R_{\mathsf{E}_6[\omega^2,\pm1]}$ on E_6^F for $\mathsf{E}_7(q)$, q a power of 3

Remark 4.2.20. This completes the determination of the values of unipotent characters at unipotent elements for groups of type E_7 in characteristic $p = 3$. Note that the only assumption which we made on the representatives $y_1 \in \mathscr{O}_{\text{reg}}^F$ and $y_{21} \in \mathsf{E}_6^F$ is that they are $\mathbf{G}^F$-conjugate to their respective inverses. Since any almost character is an $\mathbb{R}$-linear combination of unipotent characters, we see from Tables 4.7 and 4.8 that both $\mathscr{O}_{\text{reg}}^F$ and E_6^F contain a unique $\mathbf{G}^F$-class which is stable under taking inverses: the one containing y_1, y_{21}, respectively. Furthermore, we have

$$m\Big((y_1)_{\overline{y}_1^i}, w_1\Big) = \begin{cases} 3q^7 & \text{if } i = 0, \\ 0 & \text{if } i \in \{1, 2\}, \end{cases}$$

and

$$m\Big((y_{21})_{\overline{y}_{21}^i}, w_{21}\Big) = \begin{cases} 3(q^8 + q^9) & \text{if } i = 0, \\ 0 & \text{if } i \in \{1, 2\}. \end{cases}$$

Thus, the $\mathbf{G}^F$-class of y_1 (respectively, y_{21}) is the unique one inside $\mathscr{O}_{\text{reg}}^F$ (respectively, E_6^F) which has a non-empty intersection with $\mathbf{B}_0^F w_1 \mathbf{B}_0^F$ (respectively, $\mathbf{B}_0^F w_{21} \mathbf{B}_0^F$), so y_1

and y_{21} satisfy the condition ($\heartsuit'$) in 3.2.23.

The case where $p = 2$

4.2.21. We now assume that $p = 2$. In view of 4.2.16, we thus have to consider the subset $J = \{s_2, s_3, s_4, s_5\} \subseteq S$ and the 10 unipotent almost characters $R_x|_{\mathbf{G}^F_{\text{uni}}}$ for $x \in \mathfrak{X}(\mathbf{W})$ corresponding to an element of $\mathfrak{S}_{\mathbf{W}}$ as in 4.2.2(b). Our argumentation will be similar (although more elaborate) as in the case of E_6 with $p = 2$ discussed in 4.1.24–4.1.26.

If $x \in \mathfrak{X}(\mathbf{W})$ corresponds to one of the 10 elements $(J, \epsilon, (-1, 2)) \in \mathfrak{S}_{\mathbf{W}}$ where $\epsilon \in \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}_J))$, we write $R_{\mathsf{D}_4[\epsilon]} := R_x$. The character sheaves $A_x \in \hat{\mathbf{G}}^{\text{un}}$ associated to these $x \in \mathfrak{X}(\mathbf{W})$ are the simple constituents of the complex $K^{\mathscr{E}}_{\mathbf{L}_J,\Sigma} \in \mathscr{M}\mathbf{G}$ where $\Sigma = \mathbf{Z}(\mathbf{L}_J).\mathscr{O}_0$, with $\mathscr{O}_0 \subseteq \mathbf{L}_J$ the regular unipotent class and $\mathscr{E} = 1 \boxtimes \mathscr{E}_0$ the unique cuspidal local system on Σ. Hence, each of the $x \in \mathfrak{X}(\mathbf{W})$ above corresponds to some $\mathfrak{i} \in \mathcal{N}^F_{\mathbf{G}}$ under the generalised Springer correspondence (see Remark 3.4.24); we have $\tau(\mathfrak{i}) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}^F_{\mathbf{G}}$, and $\mathfrak{i}$ is in the image of the embedding

$$\mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}_J)) \hookrightarrow \mathcal{N}^F_{\mathbf{G}}. \tag{4.2.21.1}$$

Following Spaltenstein [Spa85] (and using his notation for the irreducible characters of $W_{\mathbf{G}}(\mathbf{L}_J) \cong W(\mathsf{B}_3)$ and for the unipotent classes of $\mathbf{G}$, except that we write $\mathscr{O}_{\text{reg}} \subseteq \mathbf{G}$ for the regular unipotent class, as usual), the embedding (4.2.21.1) is given by

$$\begin{aligned}
&(3,0) \mapsto (\mathscr{O}_{\text{reg}}, -1), \quad (0,3) \mapsto (\mathsf{E}_7(a_1), -1), \quad (2,1) \mapsto (\mathsf{E}_7(a_2), -1), \quad (1,2) \mapsto (\mathsf{E}_6, -1)\\
&(21,0) \mapsto (\mathsf{D}_6, -1), \quad (0,21) \mapsto (\mathsf{D}_6(a_1), -1), \quad (1^2,1) \mapsto (\mathsf{D}_5{+}\mathsf{A}_1, -1)\\
&(1,1^2) \mapsto (\mathsf{D}_5, -1), \quad (1^3,0) \mapsto (\mathsf{D}_4{+}\mathsf{A}_1, -1), \quad (0,1^3) \mapsto (\mathsf{D}_4, -1).
\end{aligned}$$

We already know that any of the $R_x|_{\mathbf{G}^F_{\text{uni}}}$ above vanishes outside of the 10 unipotent classes $\mathscr{O}$ occurring in $(\mathscr{O}, -1)$ in this list (see 3.4.23, Corollary 3.2.17).

4.2.22. (We still assume that $p = 2$.) For any $u \in \mathscr{O}^F_{\text{reg}}$, we have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_4$, so F acts trivially on this group, and $\mathscr{O}^F_{\text{reg}}$ is the union of four $\mathbf{G}^F$-classes. Given any of the other 9 (F-stable) unipotent classes $\mathscr{O} \subseteq \mathbf{G}$ appearing in 4.2.21 and any $u \in \mathscr{O}^F$, we have $A_{\mathbf{G}}(u) \cong C_2$, so F again acts trivially on this group, and $\mathscr{O}^F$ is the union of two $\mathbf{G}^F$-conjugacy classes. We use Mizuno's notation [Miz80] for the representatives for the unipotent classes of $\mathbf{G}^F$, with one exception: In order to be consistent with 4.2.7, we still define y_1 as in (4.2.7.1); since we will only need properties of y_1 up to $\mathbf{G}^F$-conjugacy, this does not make any relevant difference in view of Lemma 4.2.8. Thus, representatives for the $\mathbf{G}^F$-conjugacy classes inside the above $\mathscr{O}^F$ are given as follows.

$$\mathscr{O}_{\mathrm{reg}}\colon y_1, y_7, y_8, y_9; \quad \mathsf{E}_7(a_1)\colon y_{10}, y_{12}; \quad \mathsf{E}_7(a_2)\colon y_{13}, y_{15}; \quad \mathsf{E}_6\colon y_{21}, y_{22}; \quad \mathsf{D}_6\colon y_{28}, y_{30};$$
$$\mathsf{D}_6(a_1)\colon y_{41}, y_{43}; \quad \mathsf{D}_5{+}\mathsf{A}_1\colon y_{38}, y_{40}; \quad \mathsf{D}_5\colon y_{52}, y_{53}; \quad \mathsf{D}_4{+}\mathsf{A}_1\colon y_{77}, y_{79}; \quad \mathsf{D}_4\colon y_{85}, y_{86}.$$

We would like to single out a representative y_j in each of these classes, following 3.2.23: So if $\mathscr{O} \subseteq \mathbf{G}$ is a unipotent class as above and if $C \subseteq \mathbf{W}$ is a conjugacy class of $\mathbf{W}$ which is sent to $\mathscr{O}$ under Lusztig's map, we want to choose $y_j \in \mathscr{O}^F$ in such a way that its $\mathbf{G}^F$-conjugacy class $O_{y_j} \subseteq \mathscr{O}^F$ has a non-empty intersection with $\mathbf{B}_0^F w \mathbf{B}_0^F$ for some (any) $w \in C$ of minimal length among the elements of C. For now we declare any such $y_j \in \mathscr{O}^F$ to be *good*. (It will turn out later that there is a unique good y_j in each $\mathscr{O}^F$ above, see Remark 4.2.27 below. Recall that we have already shown this for $y_1 \in \mathscr{O}_{\mathrm{reg}}^F$, see Remark 4.2.14.)

Lemma 4.2.23. *In the setting and with the notation of* 4.2.22*, the representatives*

$$y_1, \quad y_{10}, \quad y_{13}, \quad y_{21}, \quad y_{28}, \quad y_{41}, \quad y_{38}, \quad y_{52}, \quad y_{77}, \quad y_{85}$$

are good.

Proof. If $\alpha \in R^+$ is of the form $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_n}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \leqslant 7$, it will be convenient to write $u_{i_1 i_2 \ldots i_n} := u_\alpha(1) \in \mathbf{U}_\alpha \subseteq \mathbf{U}_0$. The strategy for the proof is as follows. Given any y_j as in the lemma, we construct an element $g_j \in \mathbf{G}^F$ such that $g_j y_j g_j^{-1} = u_{i_1} u_{i_2} \cdots u_{i_m}$ where $1 \leqslant i_1, \ldots, i_m \leqslant 7$ (with m and $i_1, \ldots, i_m$ depending on j). We then set $w_j := s_{i_1} s_{i_2} \cdots s_{i_m}$ and use CHEVIE to verify that this is a reduced expression for w_j, w_j is of minimal length in its conjugacy class C_{w_j}, and Lusztig's map (3.2.23) sends C_{w_j} to the unipotent class of $\mathbf{G}$ which contains y_j. (Recall that Lusztig's map can be obtained via CHEVIE as well, using the code provided in [MiChv, §6].) It remains to refer to Lemma 3.2.24 to deduce that $O_{y_j} \cap \mathbf{B}_0^F w_j \mathbf{B}_0^F \neq \varnothing$, thus proving that y_j is good. Whenever we write g_j and w_j below, we also mean that all of the above properties are satisfied, without explicitly mentioning it.

We have done this for the element

$$y_1 = u_1 u_2 u_3 u_4 u_5 u_6 u_7 \in \mathscr{O}_{\mathrm{reg}}^F$$

in the proof of Proposition 4.2.12: In this case, the expression for y_1 already has the desired form, so we can just take $g_1 := 1$ and $w_1 := s_1 s_2 s_3 s_4 s_5 s_6 s_7 \in \mathbf{W}$. Next, we consider the element

$$y_{10} = u_1u_3u_{34}u_{24}u_5u_6u_7 = u_1u_4u_3u_2u_4u_2u_5u_6u_7 \in \mathsf{E}_7(a_1)^F.$$

(The second equality follows from Chevalley's commutator relations [Che55]; note that we do not need to worry about the coefficients since $1 = -1$ in k.) Accordingly, we set $g_{10} := 1$ and $w_{10} := s_1s_4s_3s_2s_4s_2s_5s_6s_7 \in \mathbf{W}$. Similarly, we have

$$y_{13} = u_1u_2u_3u_{24}u_{45}u_{56}u_{67} = u_1u_3u_4u_2u_5u_4u_6u_5u_7u_6u_7 \in \mathsf{E}_7(a_2)^F,$$

and we set $g_{13} := 1$, $w_{13} := s_1s_3s_4s_2s_5s_4s_6s_5s_7s_6s_7 \in \mathbf{W}$. As for the remaining y_j, $j \in \{21, 28, 41, 38, 52, 77, 85\}$, we actually need to construct a non-trivial $g_j \in \mathbf{G}^F$. Let us first print Mizuno's definitions [Miz80] for these y_j:

$$\begin{aligned}
y_{21} &= u_1u_3u_{24}u_{45}u_{56}u_{67} \in \mathsf{E}_6^F,\\
y_{28} &= u_1u_{234}u_{345}u_{245}u_6u_7 \in \mathsf{D}_6^F,\\
y_{41} &= u_1u_{234}u_{245}u_{3456}u_{2456}u_7 \in \mathsf{D}_6(a_1)^F,\\
y_{38} &= u_1u_{234}u_{345}u_{245}u_{456}u_{567} \in (\mathsf{D}_5{+}\mathsf{A}_1)^F,\\
y_{52} &= u_1u_{234}u_{345}u_{456}u_{567} \in \mathsf{D}_5^F,\\
y_{77} &= u_1u_{23445}u_{23456}u_{34567}u_{24567} \in (\mathsf{D}_4{+}\mathsf{A}_1)^F,\\
y_{85} &= u_1u_{23445}u_{23456}u_{34567} \in \mathsf{D}_4^F.
\end{aligned}$$

(Note that we do not have to refer to any convention for the choice of certain signs in a Chevalley basis in the Lie algebra underlying $\mathbf{G}$ since k has characteristic 2.) Now let

$$\omega_i := u_{\alpha_i}(1)u_{-\alpha_i}(1)u_{\alpha_i}(1) \in \mathbf{G}^F \quad \text{for } 1 \leqslant i \leqslant 7.$$

We have $\omega_iu_\alpha(1)\omega_i^{-1} = u_{s_i(\alpha)}(1)$ for any $\alpha \in R^+$ and $1 \leqslant i \leqslant 7$ (again using the fact that $1 = -1$ in k; see [Ste16, Chap. 3]). The principal idea consists in conjugating a given y_j with a suitable product of the ω_i to obtain an expression of the form $u_{i_1}u_{i_2}\cdots u_{i_m}$ where $1 \leqslant i_1, \ldots, i_m \leqslant 7$. Let us execute this procedure in some detail for y_{21}. We start by conjugating with ω_6. Since

$$\begin{gathered}
s_6(\alpha_1) = \alpha_1, \quad s_6(\alpha_3) = \alpha_3, \quad s_6(\alpha_2+\alpha_4) = \alpha_2+\alpha_4,\\
s_6(\alpha_4+\alpha_5) = \alpha_4+\alpha_5+\alpha_6, \quad s_6(\alpha_5+\alpha_6) = \alpha_5, \quad s_6(\alpha_6+\alpha_7) = \alpha_7,
\end{gathered}$$

we get $\omega_6y_{21}\omega_6^{-1} = u_1u_3u_{24}u_{456}u_5u_7$, and we have reduced the sum of the heights of the roots appearing as indices of the u's. Continuing in this fashion, one possibility to arrive at an element of the desired form $u_{i_1}u_{i_2}\cdots u_{i_6}$ is illustrated by the following picture, where the ω_i on the left is the element that we conjugate with in the given step:

$$
\begin{array}{ccl}
 & & u_1u_3u_{24}u_{45}u_{56}u_{67} \\
\omega_6 & \rightsquigarrow & u_1u_3u_{24}u_{456}u_5u_7 \\
\omega_4 & \rightsquigarrow & u_1u_{34}u_2u_{56}u_{45}u_7 \\
\omega_5 & \rightsquigarrow & u_1u_{345}u_2u_6u_4u_7 \\
\omega_3 & \rightsquigarrow & u_{13}u_{45}u_2u_6u_{34}u_7 \\
\omega_4 & \rightsquigarrow & u_{134}u_5u_{24}u_6u_3u_7 \\
\omega_2 & \rightsquigarrow & u_{1234}u_5u_4u_6u_3u_7 \\
\omega_1 & \rightsquigarrow & u_{234}u_5u_4u_6u_{13}u_7 \\
\omega_3 & \rightsquigarrow & u_{24}u_5u_{34}u_6u_1u_7 \\
\omega_4 & \rightsquigarrow & u_2u_{45}u_3u_6u_1u_7 \\
\omega_5 & \rightsquigarrow & u_2u_4u_3u_{56}u_1u_7 \\
\omega_6 & \rightsquigarrow & u_2u_4u_3u_5u_1u_{67} \\
\omega_7 & \rightsquigarrow & u_2u_4u_3u_5u_1u_6
\end{array}
$$

So we set $g_{21} := \omega_7\omega_6\omega_5\omega_4\omega_3\omega_1\omega_2\omega_4\omega_3\omega_5\omega_4\omega_6$, $w_{21} := s_2s_4s_3s_5s_1s_6$ and verify that the conditions mentioned in the beginning of the proof are met. The argument regarding the other y_j is similar to that for y_{21}, although partly more tedious. So let us just give it via the following table, where the vector appearing in the second column describes the product of the ω_i. (For the representative y_{41}, we need to conjugate with u_{24} in the end.) The last column gives the element w_j in the form $w_j = s_{i_1}s_{i_2}\cdots s_{i_m}$, so that $g_jy_jg_j^{-1} = u_{i_1}u_{i_2}\cdots u_{i_m}$ and that y_j, g_j, w_j meet all the conditions stated in the beginning of this proof.

y_j	$g_j \in \mathbf{G}^F$	w_j
y_{21}	$(7,6,5,4,3,1,2,4,3,5,4,6)$	$s_2s_4s_3s_5s_1s_6$
y_{28}	$(1,3,4,5,6,7,2,4,3,5,4,1,6,5,2,3)$	$s_6s_7s_5s_3s_4s_2$
y_{41}	$u_{24}\cdot(2,1,3,4,5,6,7,2,4,5,3,1,4,6,5,3,4,7,6,4,2)$	$s_6s_7s_2s_4s_2s_3s_4s_5$
y_{38}	$(6,5,4,3,7,6,5,4,1,3,2,4,5,3,4,2,6,1,3,4,5,4,7,5)$	$s_3s_4s_1s_7s_5s_2$
y_{52}	$(1,3,4,5,2,7,6,4,3,5,4,2,7,3,4,1,2,5,3,6)$	$s_5s_4s_6s_3s_2$
y_{77}	$(6,7,5,6,1,3,4,5,2,4,6,5,3,4,2,1,3,4,5,6,4,3,2,4,6,7,4,5,6,7)$	$s_4s_5s_2s_3s_7$
y_{85}	$(6,5,1,4,3,4,1,2,7,6,5,4,3,1,2,4,6,7,5,1,6,4,1,3,6,5,7,6)$	$s_4s_2s_3s_5$

(The content of this table may be verified with CHEVIE [MiChv], using the function `UnipotentGroup`.) □

4.2.24. Let us now come back to the situation in 4.2.21. We fix an isomorphism $\varphi_0\colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$. Let $x \in \mathfrak{X}(\mathbf{W})$ be one of the elements corresponding to $(J, \epsilon, (-1,2)) \in \mathfrak{S}_{\mathbf{W}}$ where $J = \{s_2, s_3, s_4, s_5\}$ and $\epsilon \in \operatorname{Irr}(W_{\mathbf{G}}(\mathbf{L}_J))$, and let $\mathfrak{i} := (\mathscr{O}, -1) \in \mathcal{N}_{\mathbf{G}}^F$ be such that $A_{\mathfrak{i}} \cong A_x$. Thus, $\mathscr{O} \subseteq \mathbf{G}$ is one of the 10 unipotent classes considered in 4.2.22. We define the isomorphisms $\varphi_{A_{\mathfrak{i}}}\colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$, $\overline{\varphi}_{A_{\mathfrak{i}}}\colon F^*A_{\mathfrak{i}} \xrightarrow{\sim} A_{\mathfrak{i}}$ as in 3.2.25 and set $\varphi_x := \overline{\varphi}_{A_{\mathfrak{i}}}$, $\chi_x := \chi_{A_x,\varphi_x}$. Applying the discussion in 3.4.23 and setting $\zeta := \zeta_J$ there, we get

$$R_x|_{\mathbf{G}^F_{\mathrm{uni}}} = q^{(\dim \mathbf{G} - \dim \mathscr{O} - \dim \mathbf{Z}(\mathbf{L}_J))/2}\zeta X_{\mathfrak{i}}. \tag{4.2.24.1}$$

For any $x \in \mathfrak{X}(\mathbf{W})$ as above and any unipotent class $\mathscr{O}' \subseteq \mathbf{G}$ considered in 4.2.22, we use Theorem 3.2.16 and Remark 3.2.18 to obtain the coefficient of $Y_{(\mathscr{O}',-1)}$ in $\zeta^{-1}R_x|_{\mathbf{G}^F_{\mathrm{uni}}}$. These coefficients are given in Table 4.9, where $\mathscr{O}'$ is given in the top line of the column, and where the left column gives the irreducible character ϵ of $W_{\mathbf{G}}(\mathbf{L}_J) \cong W(\mathsf{B}_3)$ with the notation of Spaltenstein [Spa85]; furthermore, Φ_n ($n \in \mathbb{N}$) denotes the nth cyclotomic polynomial evaluated at q.

4.2.25. We keep the setting of 4.2.24 (and 4.2.21). We have already computed the values of $R_{\mathsf{D}_4[(3,0)]}$ at regular unipotent elements in Corollary 4.2.13, and Table 4.9 shows that the other $R_{\mathsf{D}_4[\epsilon]}$ vanish at regular unipotent elements. As for the remaining 9 unipotent classes $\mathscr{O} \subseteq \mathbf{G}$ considered in 4.2.22, we are reduced to determining the functions $\zeta Y_{(\mathscr{O},-1)}\colon \mathscr{O}^F \to \overline{\mathbb{Q}}_\ell$ (see again Table 4.9); note that the values of these functions are roots of unity. To achieve this, we once more apply the method described in 3.4.19(2):

	$\mathscr{O}_{\mathrm{reg}}$	$\mathsf{E}_7(a_1)$	$\mathsf{E}_7(a_2)$	E_6	D_6	$\mathsf{D}_6(a_1)$	$\mathsf{D}_5{+}\mathsf{A}_1$	D_5	$\mathsf{D}_4{+}\mathsf{A}_1$	D_4
$(3,0)$	q^2	0	q^3	q^3	q^4	0	q^5	q^5	q^8	q^8
$(0,3)$	0	q^3	0	q^4	0	q^6	0	q^6	0	q^{11}
$(2,1)$	0	0	q^4	q^4	q^5	0	q^6	$q^6\Phi_4$	$q^9\Phi_4$	$q^9\Phi_3\Phi_6$
$(1,2)$	0	0	0	q^5	0	q^7	q^7	$2q^7$	q^{10}	$q^{10}\Phi_3\Phi_6$
$(21,0)$	0	0	0	0	q^6	0	q^7	q^7	$q^{10}\Phi_4$	$q^{10}\Phi_4$
$(0,21)$	0	0	0	0	0	q^8	0	q^8	0	$q^{13}\Phi_4$
$(1^2,1)$	0	0	0	0	0	0	q^8	q^8	$q^{11}\Phi_4$	$q^{11}\Phi_3\Phi_6$
$(1,1^2)$	0	0	0	0	0	0	0	q^9	q^{12}	$q^{12}\Phi_3\Phi_6$
$(1^3,0)$	0	0	0	0	0	0	0	0	q^{14}	q^{14}
$(0,1^3)$	0	0	0	0	0	0	0	0	0	q^{17}

Table 4.9.: Coefficients of $Y_{(\mathscr{O}',-1)}$ in $\zeta^{-1}R_{\mathsf{D}_4[\epsilon]}|_{\mathbf{G}^F_{\mathrm{uni}}}$ for $\mathbf{G}^F = \mathsf{E}_7(q)$, $q=2^n$

For any $w \in \mathbf{W}$ and any $u \in \mathbf{G}^F_{\mathrm{uni}}$, we have

$$m(u,w) = \frac{|O_u \cap \mathbf{B}_0^F w \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(u)|}{|\mathbf{B}_0^F|} = \sum_{x\in\mathfrak{X}(\mathbf{W})} c_x(w)R_x(u), \qquad (4.2.25.1)$$

where

$$c_x(w) = \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_\phi, x\}\,\mathrm{Trace}(T_w, V_\phi).$$

We will evaluate (4.2.25.1) with the elements w_{10}, w_{13}, w_{21}, w_{28}, w_{41}, w_{38}, w_{52}, w_{77}, w_{85} of $\mathbf{W}$ defined in the proof of Lemma 4.2.23 and, given such a w_i, with $u \in \mathscr{O}^F$ where $\mathscr{O}$ is the image of the conjugacy class of w_i in $\mathbf{W}$ under Lusztig's map (3.2.23). We will thus also need the explicit values of $R_\phi|_{\mathscr{O}^F}$ for certain $\phi \in \mathrm{Irr}(\mathbf{W})$. So let us fix any of the 9 unipotent classes $\mathscr{O} \subseteq \mathbf{G}$ above. Then the only pair of the form $(\mathscr{O},\mathscr{E}) \in \mathscr{N}_{\mathbf{G}}^F$ which is in the image of the *ordinary* Springer correspondence $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathscr{N}_{\mathbf{G}}^F$ is $(\mathscr{O},1)$. In this situation, as explained in [Gec20b, 2.7], the values of $R_\phi|_{\mathscr{O}^F}$ (for any $\phi \in \mathrm{Irr}(\mathbf{W})$) are directly obtained from the ordinary Springer correspondence and Lusztig's algorithm (see Theorem 3.2.16, Remark 3.2.18), using the CHEVIE [MiChv] functions `UnipotentClasses` and `ICCTable`. Specifically, we have

$$7_a' \mapsto (\mathsf{E}_7(a_1),1),\ 27_a \mapsto (\mathsf{E}_7(a_2),1),\ 21_b' \mapsto (\mathsf{E}_6,1),\ 35_b \mapsto (\mathsf{D}_6,1),\ 210_a \mapsto (\mathsf{D}_6(a_1),1),$$
$$168_a \mapsto (\mathsf{D}_5{+}\mathsf{A}_1,1),\quad 189_c' \mapsto (\mathsf{D}_5,1),\quad 84_a \mapsto (\mathsf{D}_4{+}\mathsf{A}_1,1),\quad 105_c' \mapsto (\mathsf{D}_4,1)$$

under (3.2.13.4). The $R_\phi|_{\mathscr{O}^F}$ that we need below are given in Table B.4 in the appendix.

Proposition 4.2.26. *Let* $\mathscr{O} \in \{\mathsf{E}_7(a_1), \mathsf{E}_7(a_2), \mathsf{E}_6, \mathsf{D}_6, \mathsf{D}_6(a_1), \mathsf{D}_5{+}\mathsf{A}_1, \mathsf{D}_5, \mathsf{D}_4{+}\mathsf{A}_1, \mathsf{D}_4\}$*. Let* $y_j \in \mathscr{O}^F$ *be the good representative, and let* $y_{j'} \in \mathscr{O}^F$ *be the other representative (see* 4.2.22, 4.2.23*). Then, in the setting of* 4.2.24*, we have*

$$\zeta Y_{(\mathscr{O},-1)}(y_j) = 1 \quad \textit{and} \quad \zeta Y_{(\mathscr{O},-1)}(y_{j'}) = -1.$$

Proof. We keep the setting and notation of 4.2.24. Let us first note that the assertion on y_j implies the one on $y_{j'}$. Indeed, for any $\mathscr{O}$ and $y_j \in \mathscr{O}^F$ as in the proposition, we have $A_{\mathbf{G}}(y_j) \cong C_2$, so it follows from the discussion in 3.4.23 that $Y_{(\mathscr{O},-1)}(y_{j'}) = -Y_{(\mathscr{O},-1)}(y_j)$.

It therefore suffices to consider the elements y_j in Lemma 4.2.23. We will evaluate $m(y_j, w_j)$ for $j \in \{10, 13, 21, 28, 41, 38, 52, 77, 85\}$. Let us explain this in detail with respect to $w_{10} \in \mathbf{W}$ and $y_{10} \in \mathsf{E}_7(a_1)^F$. For $x \in \mathfrak{X}(\mathbf{W})$, we see from Theorem 3.2.16 and Corollary 3.2.17 that $R_x(y_{10})$ can only be non-zero if $A_x \cong A_{\mathfrak{i}}$ for $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathscr{N}_{\mathbf{G}}^F$ with $\mathsf{E}_7(a_1) \subseteq \overline{\mathscr{O}}$, so we only have to consider the four pairs $\mathfrak{i} = (\mathscr{O}_{\mathrm{reg}}, \pm 1)$ and $(\mathsf{E}_7(a_1), \pm 1)$ (since the two cuspidal character sheaves A_x corresponding to the pairs $(\mathscr{O}_{\mathrm{reg}}, \pm \mathrm{i})$ are identically zero outside of $\mathscr{O}_{\mathrm{reg}}$). We have

$$R_{1_a}(y_{10}) = 1 \quad \text{and} \quad R_{7'_a}(y_{10}) = q$$

(see Table B.4 in the appendix). Furthermore, Table 4.9 shows that

$$R_{\mathsf{D}_4[(3,0)]}(y_{10}) = 0 \quad \text{and} \quad R_{\mathsf{D}_4[(0,3)]}(y_{10}) = q^3 \zeta Y_{(\mathsf{E}_7(a_1),-1)}(y_{10}).$$

So we get

$$m(y_{10}, w_{10}) = c_{1_a}(w_{10}) + c_{7'_a}(w_{10}) q + c_{\mathsf{D}_4[(0,3)]}(w_{10}) q^3 \zeta Y_{(\mathsf{E}_7(a_1),-1)}(y_{10}).$$

Using CHEVIE, we see that $c_{1_a}(w_{10}) = q^9$, $c_{7'_a}(w_{10}) = 0$ and $c_{\mathsf{D}_4[(0,3)]} = q^6$, so we obtain

$$m(y_{10}, w_{10}) = \big(1 + \zeta Y_{(\mathsf{E}_7(a_1),-1)}(y_{10})\big) q^9.$$

On the other hand, we have

$$m(y_{10}, w_{10}) = \frac{|O_{y_{10}} \cap \mathbf{B}_0^F w_{10} \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(y_{10})|}{|\mathbf{B}_0^F|},$$

so the root of unity $\zeta Y_{(\mathsf{E}_7(a_1),-1)}(y_{10})$ can only be ± 1. Since $O_{y_{10}} \cap \mathbf{B}_0^F w_{10} \mathbf{B}_0^F \neq \varnothing$ by Lemma 4.2.23, we must have $\zeta Y_{(\mathsf{E}_7(a_1),-1)}(y_{10}) = +1$. Evaluating the other $m(y_j, w_j)$ is similar, but there are more $R_x(y_j)$ which can possibly be non-zero; we thus also need to compute more $c_x(w_j)$. The necessary information to get the $R_x(y_j)$ (depending on the unknown $\zeta Y_{(\mathscr{O},-1)}(y_j)$, where $\mathscr{O}$ is such that $y_j \in \mathscr{O}$) can be obtained from Table 4.9 and Table B.4. As before, the $c_x(w_j)$ are computed using CHEVIE; for the sake of completeness we have included their required values in the appendix, see Table B.3. So we just give the results for the remaining $m(y_j, w_j)$ below, where $\vartheta_{\mathscr{O}} := \zeta Y_{(\mathscr{O},-1)}(y_j)$.

$$
\begin{aligned}
m(y_{13}, w_{13}) &= (1 + \vartheta_{\mathsf{E}_7(a_2)}) \cdot q^{11}, \\
m(y_{21}, w_{21}) &= (1 + \vartheta_{\mathsf{E}_6}) \cdot q^{8}(q+1), \\
m(y_{28}, w_{28}) &= (1 + \vartheta_{\mathsf{D}_6}) \cdot q^{9}(q+1), \\
m(y_{41}, w_{41}) &= (1 + \vartheta_{\mathsf{D}_6(a_1)}) \cdot q^{12}(q+1), \\
m(y_{38}, w_{38}) &= (1 + \vartheta_{\mathsf{D}_5+\mathsf{A}_1}) \cdot q^{11}(q+1), \\
m(y_{52}, w_{52}) &= (1 + \vartheta_{\mathsf{D}_5}) \cdot q^{10}(q^2+2q+1), \\
m(y_{77}, w_{77}) &= (1 + \vartheta_{\mathsf{D}_4+\mathsf{A}_1}) \cdot q^{13}(q^4+2q^3+2q^2+2q+1), \\
m(y_{85}, w_{85}) &= (1 + \vartheta_{\mathsf{D}_4}) \cdot q^{10}(q^9+3q^8+5q^7+7q^6+8q^5+8q^4+7q^3+5q^2+3q+1).
\end{aligned}
$$

In each case, we conclude that $1+\vartheta_{\mathscr{O}} > 0$, which forces the root of unity $\vartheta_{\mathscr{O}} = \zeta Y_{(\mathscr{O},-1)}(y_j)$ to be $+1$. $\square$

Remark 4.2.27. For $\mathscr{O} \subseteq \mathbf{G}$ and $y_j, y_{j'} \in \mathscr{O}^F$ as in Proposition 4.2.26, we can of course also compute $m(y_{j'}, w_j)$ instead of $m(y_j, w_j)$ in the proof of the proposition; we just have to replace $\zeta Y_{(\mathscr{O},-1)}(y_j) = 1$ by $\zeta Y_{(\mathscr{O},-1)}(y_{j'}) = -1$. This shows that the $\mathbf{G}^F$-conjugacy class of $y_{j'}$ does in fact have an empty intersection with $\mathbf{B}_0^F w_j \mathbf{B}_0^F$, so the $\mathbf{G}^F$-conjugacy class of y_j is indeed the unique one in $\mathscr{O}^F$ consisting of good elements in the sense of 4.2.22.

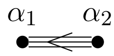
4.2.28. We are now in a position to explicitly compute the values of the 10 unipotent almost characters $R_{\mathsf{D}_4[\epsilon]}$ at unipotent elements for the groups $\mathbf{G}^F = \mathsf{E}_7(q)$ with $q = 2^n$, where $J = \{s_2, s_3, s_4, s_5\} \subseteq S$ and $\epsilon \in \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}_J))$. Indeed, recall that these functions have non-zero values only at elements of one of the 10 classes considered in 4.2.22. Their values at elements of these classes are given as follows. If $\mathscr{O} = \mathscr{O}_{\mathrm{reg}}$, we can refer to Corollary 4.2.13 (as $R_{\mathsf{D}_4[\epsilon]}|_{\mathscr{O}_{\mathrm{reg}}^F} = 0$ whenever $\epsilon \neq (3,0)$). If $\mathscr{O} \subseteq \mathbf{G}$ is one of the other 9 unipotent classes above and $y_j \in \mathscr{O}^F$ is the good representative (see Lemma 4.2.23), the values $R_{\mathsf{D}_4[\epsilon]}(y_j)$ are given by Table 4.9, while one has to multiply the entries of this table by -1 in order to obtain $R_{\mathsf{D}_4[\epsilon]}(y_{j'})$.

Combined with 4.2.19 (see also the overview given in 4.2.16), this completes the determination of the unipotent characters at unipotent elements for the groups $\mathbf{G}^F = \mathsf{E}_7(q)$

where q is any power of any prime p.

4.3. Groups of type G_2

In this section, we assume that $\mathbf{G}$ is the simple group of type G_2 over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. We fix an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ and an F-stable maximal torus $\mathbf{T}_0 \subseteq \mathbf{B}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum attached to $\mathbf{G}$ and $\mathbf{T}_0$ (where $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), with underlying bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$. Let $R^+ \subseteq R$ be the positive roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$, and let $\Pi = \{\alpha_1, \alpha_2\} \subseteq R^+$ be the corresponding simple roots, $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee\}$ be the corresponding simple co-roots. We choose the order of α_1, α_2 in such a way that the Dynkin diagram of $\mathbf{G}$ is as follows:

G_2 

(Thus, α_1 is the short simple root, α_2 is the long simple root. This differs from the convention in [Lus84a, 4.8], but we want to conform with the notation in [CR74], [Eno76], [EY86] and [His90] since these will be our references for the conjugacy classes and character tables for groups of type G_2.) Let $\mathfrak{C} = (\langle \alpha_j, \alpha_i^\vee \rangle)_{1 \leqslant i,j \leqslant 2}$ be the corresponding Cartan matrix and $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$. As in [His90, p. 96], we adopt the names for the irreducible characters from [Car85, p. 412]. Furthermore, let $\mathbf{U}_0 = R_\mathrm{u}(\mathbf{B}_0)$ be the unipotent radical of $\mathbf{B}_0$, so that $\mathbf{B}_0$ is the semidirect product of $\mathbf{U}_0$ and $\mathbf{T}_0$ (with $\mathbf{U}_0$ being normal in $\mathbf{B}_0$). As described in 2.1.19, F induces a p-isogeny of root data

$$\varphi\colon X \to X, \quad \lambda \mapsto \lambda \circ F|_{\mathbf{T}_0},$$

as well as a bijection $R \to R$, $\alpha \mapsto \alpha^\dagger$, so that $\varphi(\alpha^\dagger) = q\alpha$ for all $\alpha \in R$ (since $F\colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{G}$). The assignment $\alpha \mapsto \alpha^\dagger$ restricts to a graph automorphism of the Dynkin diagram, which is necessarily the identity since the Dynkin diagram does not have any non-trivial graph automorphisms (and again due to our assumption that F is a Frobenius map). Hence, $\mathbf{G}^F$ is the non-twisted group $\mathsf{G}_2(q)$ and $\sigma = \mathrm{id}_{\mathbf{W}}$. We are thus in the setting of Section 3.4 and adopt the further notation from there.

4.3.1. We have $|\mathfrak{X}(\mathbf{W})| = 10$ and $|\operatorname{Irr}(\mathbf{W})| = 6$. The set $\operatorname{Irr}(\mathbf{W})$ is partitioned into three families. Two of these families consist of a single character; the third family $\mathcal{F}_1$ consists of four characters, and we have $|\mathfrak{M}(\mathcal{G}_{\mathcal{F}_1})| = 8$. We fix the bijections

$$\mathrm{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho, \tag{4.3.1.1}$$

and

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}, \quad x \mapsto A_x, \tag{4.3.1.2}$$

according to Corollary 3.4.8. As usual, we will often write $\rho_x \in \mathrm{Uch}(\mathbf{G}^F)$ for the unipotent character corresponding to $x \in \mathfrak{X}(\mathbf{W})$ under (4.3.1.1).

There are exactly four cuspidal character sheaves A_1, A_2, A_3, A_4 in $\hat{\mathbf{G}}$, and all of them are in $\hat{\mathbf{G}}^{\mathrm{un}}$, see [LuCS4, 20.6] for $p \geqslant 5$ (that is, p is good for $\mathbf{G}$), and [Sho95a, 7.6] for $p \leqslant 3$; as noted in 3.4.1, the A_i ($1 \leqslant i \leqslant 4$) are thus automatically F-stable. Let $\mathcal{F}_1 = \{\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}\} \subseteq \mathrm{Irr}(\mathbf{W})$ be the unique family with four elements, so that $\mathcal{G}_{\mathcal{F}_1} = \mathfrak{S}_3$. In view of [LuCS4, 20.6] and [Sho95a, §7], all the elements $x_i \in \mathfrak{X}(\mathbf{W})$ for which A_{x_i} is a cuspidal (unipotent) character sheaf ($1 \leqslant i \leqslant 4$) are in $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_1}) = \mathfrak{M}(\mathfrak{S}_3)$. We have

$$x_1 = (1, \varepsilon),\ x_2 = (g_2, \varepsilon),\ x_3 = (g_3, \omega),\ x_4 = (g_3, \omega^2),$$

where for $j = 2, 3$, g_j denotes a j-cycle in $\mathfrak{S}_3$, ε is the sign character of $\mathfrak{S}_3$ or its restriction to $C_{\mathfrak{S}_3}(g_2)$, $\omega \in \mathcal{R}_3$ is a fixed primitive 3rd root of unity which we assume to be the same as the one in 3.4.2, and the irreducible characters of $C_{\mathfrak{S}_3}(g_3) = \langle g_3 \rangle$ are denoted by their values at g_3. We number the $A_1, A_2, A_3, A_4 \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ in such a way that

$$A_i = A_{x_i} \quad \text{for } 1 \leqslant i \leqslant 4. \tag{4.3.1.3}$$

The description of the (F-stable) cuspidal pairs $(\Sigma, \mathscr{E})$ corresponding to these four A_i (see 3.1.17, 3.1.18) depends on whether $p = 2$, $p = 3$ or $p \geqslant 5$. It is provided by Shoji in [Sho95a, §6, §7], where the root of unity $\lambda_{A_i} \in \mathcal{R}$ associated to A_i (see 3.4.3(c)) is also determined. From Corollary 3.4.8, it follows in particular that $\lambda_{A_i} = \tilde{\lambda}_{x_i}$ for $1 \leqslant i \leqslant 4$. We have

$$\tilde{\lambda}_{x_1} = 1,\ \tilde{\lambda}_{x_2} = -1,\ \tilde{\lambda}_{x_3} = \omega,\ \tilde{\lambda}_{x_4} = \omega^2.$$

Since these values are pairwise different, it is easy to compare our numbering of the A_i with Shoji's: It coincides in the case where $p \geqslant 5$ (see [Sho95a, 6.8]) but not for $p = 2$ or $p = 3$. (We prefer to have the uniform notation $A_i = A_{x_i}$, $1 \leqslant i \leqslant 4$, in any characteristic.)

On the other hand, denoting the cuspidal unipotent characters as in [Car85, p. 478], we have

$$\rho_{x_1} = \mathsf{G}_2[1],\ \rho_{x_2} = \mathsf{G}_2[-1],\ \rho_{x_3} = \mathsf{G}_2[\omega],\ \rho_{x_4} = \mathsf{G}_2[\omega^2].$$

Then the almost characters R_{x_i}, $1 \leqslant i \leqslant 4$, are given by

$$\begin{aligned}
R_{x_1} &= \tfrac{1}{6}\big([\phi_{2,1}] + 2[\phi'_{1,3}] + \mathsf{G}_2[1] - 3[\phi_{2,2}] - 3\mathsf{G}_2[-1] + 2[\phi''_{1,3}] + 2\mathsf{G}_2[\omega] + 2\mathsf{G}_2[\omega^2]\big),\\
R_{x_2} &= \tfrac{1}{2}\big([\phi_{2,1}] - \mathsf{G}_2[1] - [\phi_{2,2}] + \mathsf{G}_2[-1]\big),\\
R_{x_3} &= \tfrac{1}{3}\big([\phi_{2,1}] - [\phi'_{1,3}] + \mathsf{G}_2[1] - [\phi''_{1,3}] + 2\mathsf{G}_2[\omega] - \mathsf{G}_2[\omega^2]\big),\\
R_{x_4} &= \tfrac{1}{3}\big([\phi_{2,1}] - [\phi'_{1,3}] + \mathsf{G}_2[1] - [\phi''_{1,3}] - \mathsf{G}_2[\omega] + 2\mathsf{G}_2[\omega^2]\big).
\end{aligned}$$

The generic character tables of $\mathsf{G}_2(p^n)$ $(n \in \mathbb{N})$ are completely known, due to the work of Chang–Ree [CR74] for $p \geqslant 5$, Enomoto [Eno76] for $p = 3$, and Enomoto–Yamada [EY86] for $p = 2$. The character tables of $\mathsf{G}_2(p^n)$ for $p \geqslant 5$ are not explicitly printed in [CR74] but in [His90, Appendix B], so we will henceforth use the latter as our reference for the case where $p \geqslant 5$.

Hence, in order to evaluate the almost characters R_{x_i} $(1 \leqslant i \leqslant 4)$ at a given element of $\mathbf{G}^F$, we merely need to match the names of the unipotent characters in terms of Harish-Chandra series with the ones chosen in the above references. The relevant information to this end can be obtained almost entirely by just looking at the degrees of the irreducible characters. In fact, the only ambiguity arises from two pairs of unipotent characters, namely, $\{[\phi'_{1,3}], [\phi''_{1,3}]\}$ and $\{\mathsf{G}_2[\omega], \mathsf{G}_2[\omega^2]\}$. (The characters in either of these sets have the same degree.) Now note that $[\phi'_{1,3}], [\phi''_{1,3}]$ appear with the same multiplicity in any of the R_{x_i} $(1 \leqslant i \leqslant 4)$, so the computation of these almost characters does not even require a distinction between $[\phi'_{1,3}]$ and $[\phi''_{1,3}]$. Furthermore, it turns out that the characters $\mathsf{G}_2[\omega], \mathsf{G}_2[\omega^2]$ take the same value at any element of $\mathbf{G}^F$ that we will consider below, so distinguishing them will once again not be necessary for our purposes.

For any $x \in \mathfrak{X}(\mathbf{W})$, let $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ be an isomorphism as in 3.2.1(∗). (We will make an explicit choice for the cuspidal character sheaves below, depending on p.) We denote by

$$\chi_x := \chi_{A_x,\varphi_x} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$$

the corresponding characteristic function. We thus have

$$R_x = \xi_x\chi_x \text{ for } x \in \mathfrak{X}(\mathbf{W}),\ \text{where } \xi_x = \xi_x(\varphi_x) \in \overline{\mathbb{Q}}_\ell^\times,\ |\xi_x| = 1, \tag{4.3.1.4}$$

see (3.4.12.2). So for any of the four (F-stable) cuspidal pairs $(\Sigma, \mathscr{E})$ for $\mathbf{G}$ (corresponding to one of the four cuspidal character sheaves A_i), our main task in this section is to single out a 'good' $\mathbf{G}^F$-conjugacy class inside Σ^F (cf. 3.2.22, 3.2.23), which thus specifies the choice of φ_{x_i}. Once this is accomplished, the determination of the scalar $\xi_{x_i} = \xi_{x_i}(\varphi_{x_i})$ immediately follows from evaluating the almost character R_{x_i} at an element of the above $\mathbf{G}^F$-conjugacy class inside Σ^F.

Type G_2 in characteristic $p \geqslant 5$

4.3.2. Let us assume that $p \geqslant 5$, that is, p is a good prime for $\mathbf{G}$. The description of the cuspidal (unipotent) character sheaves on $\mathbf{G}$ in terms of the corresponding cuspidal pairs for $\mathbf{G}$ (see Proposition 3.1.17, Remark 3.1.18) is provided by [Sho95a, §6] and (the proof of) [LuCS4, 20.6]. It is given by the following list, where we use the notation of Remark 3.1.18 and the convention that 's' always denotes a semisimple element of $\mathbf{G}$ and 'u' a unipotent element of $\mathbf{G}$. The names for the unipotent classes of $\mathbf{G}$ are as in [Car85].

(a) $A_1 \leftrightarrow (u, \varsigma)$ where u is an element of the class $\mathsf{G}_2(a_1)$; we have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_3$, and ς is the sign character of $\mathfrak{S}_3$.

(b) $A_2 \leftrightarrow (su, \varsigma)$ where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1$, and u is regular unipotent in $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_2$ (where $\overline{su} = \overline{s}$ is the image in $A_{\mathbf{G}}(su)$ of su, s, respectively), and ς is the non-trivial linear character of $A_{\mathbf{G}}(su)$.

(c) $A_i \leftrightarrow (su, \varsigma_i)$, $i = 3, 4$, where $C_{\mathbf{G}}(s)$ has a root system of type A_2, and u is regular unipotent in $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_3$, and ς_3, ς_4 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_3(\overline{su}) = \omega$, $\varsigma_4(\overline{su}) = \omega^2$ (with $\omega \in \mathcal{R}_3$ as in 4.3.1).

(a) The character sheaf A_1 is supported by the closure of the (F-stable) unipotent class $\mathsf{G}_2(a_1) \subseteq \mathbf{G}$. By 3.2.22(a), there is a natural choice for a $\mathbf{G}^F$-conjugacy class inside $\mathsf{G}_2(a_1)^F$, the one consisting of split elements. If we fix such a split element $u_0 \in \mathsf{G}_2(a_1)^F$, we know that F acts trivially on $A_{\mathbf{G}}(u_0) \cong \mathfrak{S}_3$. In particular, $\mathsf{G}_2(a_1)^F$ splits into three $\mathbf{G}^F$-classes. Furthermore, since A_1 is a cuspidal character sheaf and the algebraic group $C^\circ_{\mathbf{G}}(u_0)$ is of dimension 4, we see from the discussion in 3.2.22 that

$$|C_{\mathbf{G}}(u_0)^F| = |A_{\mathbf{G}}(u_0)|q^4 = 6q^4.$$

The $\mathbf{G}^F$-class of u_0 inside $\mathsf{G}_2(a_1)^F$ is uniquely determined by this property according to [His90, Appendix A.4], and a representative for this $\mathbf{G}^F$-class is denoted by u_3 in loc.

cit., so let us also write u_3 instead of u_0 from now on. We obtain

$$R_{x_1}(u_3) = \begin{cases} -q^2 & \text{if} \quad q \equiv -1 \pmod 3, \\ q^2 & \text{if} \quad q \equiv +1 \pmod 3. \end{cases}$$

As described in 3.2.21, let $\varphi_{x_1} \colon F^*A_1 \xrightarrow{\sim} A_1$ be the isomorphism corresponding to the choice of u_3. Then the characteristic function $\chi_{x_1} := \chi_{A_1,\varphi_{x_1}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ is given by

$$\chi_{x_1}(g) = \begin{cases} \varepsilon(a)q^2 & \text{if } g \sim_{\mathbf{G}^F} (u_3)_a \text{ for some } a \in A_{\mathbf{G}}(u_3), \\ 0 & \text{if } g \notin \mathsf{G}_2(a_1)^F, \end{cases}$$

where ε is the sign character of $A_{\mathbf{G}}(u_3) \cong \mathfrak{S}_3$. Evaluating R_{x_1} and χ_{x_1} at u_3, we conclude that

$$\xi_{x_1}(\varphi_{x_1}) = \begin{cases} -1 & \text{if} \quad q \equiv -1 \pmod 3, \\ +1 & \text{if} \quad q \equiv +1 \pmod 3. \end{cases}$$

(b) The character sheaf A_2 is supported by the closure of the (F-stable) conjugacy class $\mathscr{C} \subseteq \mathbf{G}$ containing elements of the form $su = us$, where $s \in \mathbf{G}$ is a semisimple element such that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1$, and where u is a regular unipotent element of $C_{\mathbf{G}}(s)$. Let $su = us \in \mathscr{C}^F$. Since $A_{\mathbf{G}}(su) \cong C_2$ is generated by the image of su, F acts trivially on $A_{\mathbf{G}}(su)$, so $\mathscr{C}^F$ splits into two $\mathbf{G}^F$-conjugacy classes; moreover, as u is regular unipotent in $C_{\mathbf{G}}(s)$, we have

$$\dim C_{\mathbf{G}}(su) = \dim C_{C_{\mathbf{G}}(s)}(u) = \operatorname{rank} C_{\mathbf{G}}(s) = \operatorname{rank} \mathbf{G} = 2,$$

so the argument in 3.2.22(b) shows that

$$|C_{\mathbf{G}}(su)^F| = |A_{\mathbf{G}}(su)| \cdot |C_{\mathbf{G}}^\circ(su)^F| = 2q^2.$$

By inspection of the tables in [His90, pp. 147, 159], we see that there are only two $\mathbf{G}^F$-classes whose elements have this centraliser order, so we conclude that these must be precisely the two $\mathbf{G}^F$-classes into which $\mathscr{C}^F$ splits. The $\mathbf{G}^F$-class of s is uniquely determined by the property that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1$, so we only need to specify u. We take

$$u_0 := u_{\alpha_2}(1)u_{2\alpha_1+\alpha_2}(1) \in \mathbf{U}_0^F,$$

so that with the notation in [His90, p. 142], $k_{2,3} = su_0 = u_0 s$ is a representative for our chosen $\mathbf{G}^F$-class inside $\mathscr{C}^F$. Let $\varphi_{x_2}\colon F^*A_2 \xrightarrow{\sim} A_2$ be the corresponding isomorphism; the associated characteristic function $\chi_{x_2} := \chi_{A_2,\varphi_{x_2}}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ satisfies

$$\chi_{x_2}(k_{2,3}) = q.$$

On the other hand, evaluating R_{x_2} at $k_{2,3}$ gives

$$R_{x_2}(k_{2,3}) = q,$$

so we conclude that $\xi_{x_2}(\varphi_{x_2}) = +1$, independently of the congruence of q modulo 3.

(c) Finally, consider the cuspidal character sheaves A_3, A_4, supported by the closure of the (F-stable) conjugacy class $\mathscr{C}$ containing elements of the form $su = us$, where $s \in \mathbf{G}$ is a semisimple element such that $C_{\mathbf{G}}(s)$ has a root system of type A_2, and where u is a regular unipotent element of $C_{\mathbf{G}}(s)$. Let $su = us \in \mathscr{C}^F$. Since $A_{\mathbf{G}}(su) \cong C_3$ is generated by the image of su, we know that F acts trivially on $A_{\mathbf{G}}(su)$, so $\mathscr{C}^F$ splits into three $\mathbf{G}^F$-conjugacy classes; the same argument as in (b) yields that $C^\circ_{\mathbf{G}}(su)$ is a (unipotent) algebraic group of dimension 2 and

$$|C_{\mathbf{G}}(su)^F| = |A_{\mathbf{G}}(su)| \cdot |C^\circ_{\mathbf{G}}(su)^F| = 3q^2.$$

The tables in [His90, pp. 147, 159] show that there are exactly three $\mathbf{G}^F$-conjugacy classes whose elements have centraliser order $3q^2$, so these three classes must be the ones into which $\mathscr{C}^F$ splits. Following [His90, p. 142], exactly one of these classes is stable under taking inverses; a representative for said $\mathbf{G}^F$-class is denoted by $k_{3,2}$ in loc. cit. (Thus, $k_{3,2}$ is $\mathbf{G}^F$-conjugate to $k_{3,2}^{-1}$.) Let $\varphi_{x_i}\colon F^*A_i \xrightarrow{\sim} A_i$ ($i = 3, 4$) be the isomorphisms corresponding to the choice of $k_{3,2}$; the characteristic functions $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ thus satisfy

$$\chi_{x_i}(k_{3,2}) = q \quad \text{for } i = 3, 4.$$

Evaluating the R_{x_i} ($i = 3, 4$) at $k_{3,2}$ gives

$$R_{x_i}(k_{3,2}) = q \quad \text{for } i = 3, 4,$$

and we conclude that $\xi_{x_3}(\varphi_{x_3}) = \xi_{x_4}(\varphi_{x_4}) = +1$, independently of the congruence of q modulo 3.

Type G_2 in characteristic $p = 3$

4.3.3. Let us assume that $p = 3$. By [Sho95a, 7.2], the description of the character sheaves A_i ($1 \leqslant i \leqslant 4$) in terms of the corresponding cuspidal pairs for $\mathbf{G}$ is as follows (with the same conventions as in 4.3.2):

(a) $A_1 \leftrightarrow (u, \varsigma)$ where u is an element of the class $\mathsf{G}_2(a_1)$; we have $A_{\mathbf{G}}(u) \cong C_2$, and ς is the non-trivial linear character of $A_{\mathbf{G}}(u)$.

(b) $A_2 \leftrightarrow (su, \varsigma)$ where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1$, and u is regular unipotent in $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_2$ (where $\overline{su} = \overline{s}$ is the image in $A_{\mathbf{G}}(su)$ of su, s, respectively), and ς is the non-trivial linear character of $A_{\mathbf{G}}(su)$.

(c) $A_i \leftrightarrow (u, \varsigma_i)$, $i = 3, 4$, where u is a regular unipotent element in $\mathbf{G}$; we have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_3$, and ς_3, ς_4 are the linear characters of $A_{\mathbf{G}}(u)$ which satisfy $\varsigma_3(\overline{u}) = \omega$, $\varsigma_4(\overline{u}) = \omega^2$ (with $\omega \in \mathcal{R}_3$ as in 4.3.1).

(a) The character sheaf A_1 is supported by the closure of the (F-stable) unipotent class $\mathsf{G}_2(a_1) \subseteq \mathbf{G}$. For $u \in \mathsf{G}_2(a_1)^F$, we have $A_{\mathbf{G}}(u) \cong C_2$, so F induces the identity map on this group, and $\mathsf{G}_2(a_1)^F$ splits into two $\mathbf{G}^F$-conjugacy classes. By the argument in 3.2.22(b), $C_{\mathbf{G}}(u)^\circ$ is a unipotent algebraic group (of dimension 4), and we have

$$|C_{\mathbf{G}}(u)^F| = |A_{\mathbf{G}}(u)| \cdot |C^\circ_{\mathbf{G}}(u)^F| = 2q^4.$$

There are exactly two $\mathbf{G}^F$-conjugacy classes whose elements have this centraliser order (see [Eno76, Table VII-1]), so we know that these must be the two classes into which $\mathsf{G}_2(a_1)^F$ splits. We pick

$$u_0 := u_{\alpha_1+\alpha_2}(1)u_{3\alpha_1+\alpha_2}(-1) \in \mathsf{G}_2(a_1)^F$$

(an element of the $\mathbf{G}^F$-class denoted by 'A_{41}' in [Eno76]). Let $\varphi_{x_1} \colon F^*A_1 \xrightarrow{\sim} A_1$ be the isomorphism corresponding to this choice of u_0, so that the characteristic function $\chi_{x_1} := \chi_{A_1,\varphi_{x_1}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ satisfies

$$\chi_{x_1}(u_0) = q^2.$$

On the other hand, we have

$$R_{x_1}(u_0) = q^2,$$

and we conclude that $\xi_{x_1}(\varphi_{x_1}) = +1$.

(b) The character sheaf A_2 is supported by the closure of the (F-stable) conjugacy class $\mathscr{C} \subseteq \mathbf{G}$ containing elements of the form $su = us$, where $s \in \mathbf{G}$ is a semisimple element such that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_1$, and where u is a regular unipotent element of $C_{\mathbf{G}}(s)$. Let $su = us \in \mathscr{C}^F$. By checking the centraliser orders in [Eno76, Table VII-1], we see that this determines the $\mathbf{G}^F$-conjugacy class of s uniquely; moreover, since $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle \cong C_2$, F acts trivially on this group, so $\mathscr{C}^F$ splits into two $\mathbf{G}^F$-conjugacy classes. As u is regular unipotent in $C_{\mathbf{G}}(s)$, we have

$$\dim C_{\mathbf{G}}(su) = \dim C_{C_{\mathbf{G}}(s)}(u) = \operatorname{rank} C_{\mathbf{G}}(s) = \operatorname{rank} \mathbf{G} = 2,$$

so the argument in 3.2.22(b) shows that

$$|C_{\mathbf{G}}(su)^F| = |A_{\mathbf{G}}(su)|q^2 = 2q^2.$$

There are exactly two $\mathbf{G}^F$-classes whose elements have this centraliser order (denoted by 'B_4' and 'B_5' in [Eno76]). We set

$$u_0 := u_{\alpha_1+\alpha_2}(1)u_{3\alpha_1+\alpha_2}(-1) \in \mathbf{U}_0^F \quad \text{and} \quad s_0 := \alpha_1^\vee(-1)\alpha_2^\vee(-1) \in \mathbf{T}_0^F,$$

so that $g_0 := s_0u_0 = u_0s_0 \in \mathscr{C}^F$ is a representative for the class 'B_4' in [Eno76]. Let $\varphi_{x_2} \colon F^*A_2 \xrightarrow{\sim} A_2$ be the isomorphism corresponding to the choice of g_0, and let $\chi_{x_2} := \chi_{A_2,\varphi_{x_2}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$. We obtain

$$R_{x_2}(g_0) = q = \chi_{x_2}(g_0),$$

so we have $\xi_{x_2}(\varphi_{x_2}) = +1$.

(c) The character sheaves A_3, A_4 are supported by $\overline{\mathscr{O}}_{\mathrm{reg}} = \mathbf{G}_{\mathrm{uni}}$. We fix the representative

$$u_0 := u_{\alpha_1}(1) \cdot u_{\alpha_2}(1) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F.$$

(The $\mathbf{G}^F$-class of u_0 is named 'A_{51}' in [Eno76]; it is the unique one inside $\mathscr{O}_{\mathrm{reg}}^F$ which is stable under taking inverses.) Let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ ($i = 3, 4$) be the corresponding isomorphisms. Then the characteristic functions $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ ($i = 3, 4$) satisfy

$$\chi_{x_3}(u_0) = \chi_{x_4}(u_0) = q.$$

On the other hand, we have

$$R_{x_3}(u_0) = R_{x_4}(u_0) = q,$$

so we get $\xi_{x_3}(\varphi_{x_3}) = \xi_{x_4}(\varphi_{x_4}) = +1$.

Type G_2 in characteristic $p = 2$

4.3.4. Let us assume that $p = 2$. By [Sho95a, 7.2], the description of the character sheaves A_i $(1 \leqslant i \leqslant 4)$ in terms of the corresponding cuspidal pairs for $\mathbf{G}$ is as follows (with the same conventions as in 4.3.2):

(a) $A_1 \leftrightarrow (u, \varsigma)$ where u is an element of the class $\mathsf{G}_2(a_1)$; we have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_3$, and ς is the sign character of $\mathfrak{S}_3$.

(b) $A_2 \leftrightarrow (u, \varsigma)$ where u is a regular unipotent element in $\mathbf{G}$; we have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_2$ (where $\overline{u}$ denotes the image of u in $A_{\mathbf{G}}(u)$), and ς is the non-trivial linear character of $A_{\mathbf{G}}(u)$.

(c) $A_i \leftrightarrow (su, \varsigma_i)$, $i = 3, 4$, where $C_{\mathbf{G}}(s)$ has a root system of type A_2, and u is regular unipotent in $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_3$, and ς_3, ς_4 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_3(\overline{su}) = \omega$, $\varsigma_4(\overline{su}) = \omega^2$ (with $\omega \in \mathcal{R}_3$ as in 4.3.1).

(a) The character sheaf A_1 is supported by the closure of the (F-stable) unipotent class $\mathsf{G}_2(a_1)$. For $u \in \mathsf{G}_2(a_1)^F$, we have

$$\mathfrak{S}_3 \cong A_{\mathbf{G}}(u) \supseteq A_{\mathbf{G}}(u)^F \cong C_{\mathbf{G}}(u)^F / C^\circ_{\mathbf{G}}(u)^F.$$

Here, $C^\circ_{\mathbf{G}}(u)$ is a unipotent group of dimension 4, so

$$|C_{\mathbf{G}}(u)^F| = |A_{\mathbf{G}}(u)^F| \cdot |C^\circ_{\mathbf{G}}(u)^F| \leqslant |A_{\mathbf{G}}(u)| \cdot |C^\circ_{\mathbf{G}}(u)^F| = 6q^4.$$

From [EY86, Table IV-1], we see that there is a unique $\mathbf{G}^F$-conjugacy class whose elements have centraliser order $6q^4$ (named 'A_{31}' in loc. cit.). As in [EY86], we take

$$u_0 := u_{\alpha_1+\alpha_2}(1) \cdot u_{2\alpha_1+\alpha_2}(1) \in \mathbf{U}_0^F \cap \mathsf{G}_2(a_1)^F$$

as a representative for this class, and let $\varphi_{x_1} \colon F^*A_1 \xrightarrow{\sim} A_1$ be the corresponding isomorphism. The characteristic function $\chi_{x_1} := \chi_{A_1,\varphi_{x_1}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ satisfies

$$\chi_{x_1}(u_0) = q^2.$$

On the other hand, we have

$$R_{x_1}(u_0) = (-1)^n q^2,$$

where $n \in \mathbb{N}$ is such that $q = p^n$. We conclude that $\xi_{x_1}(\varphi_{x_1}) = (-1)^n$.

(b) The character sheaf A_2 is supported by the unipotent variety $\mathbf{G}_{\mathrm{uni}} = \overline{\mathscr{O}}_{\mathrm{reg}} \subseteq \mathbf{G}$. We fix the representative

$$u_0 := u_{\alpha_1}(1) \cdot u_{\alpha_2}(1) \in \mathscr{O}_{\mathrm{reg}}^F.$$

(The $\mathbf{G}^F$-class of u_0 is named 'A_{51}' in [EY86].) Let $\varphi_{x_2} \colon F^*A_2 \xrightarrow{\sim} A_2$ be the corresponding isomorphism. Then the characteristic function $\chi_{x_2} := \chi_{A_2,\varphi_{x_2}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ is given by

$$\chi_{x_2}(g) = \begin{cases} q & \text{if } g \sim_{\mathbf{G}^F} u_0, \\ -q & \text{if } g \in \mathscr{O}_{\mathrm{reg}}^F \text{ is not } \mathbf{G}^F\text{-conjugate to } u_0, \\ 0 & \text{if } g \notin \mathscr{O}_{\mathrm{reg}}^F. \end{cases}$$

On the other hand, we have

$$R_{x_2}(u_0) = q,$$

so $\xi_{x_2}(\varphi_{x_2}) = +1$.

(c) The character sheaves A_3, A_4 are supported by the closure of the (F-stable) conjugacy class $\mathscr{C} \subseteq \mathbf{G}$ containing elements of the form $su = us$, where $s \in \mathbf{G}$ is semisimple such that $C_{\mathbf{G}}(s)$ has a root system of type A_2, and where u is a regular unipotent element of $C_{\mathbf{G}}(s)$. For $su = us \in \mathscr{C}^F$, we have

$$C_3 \cong A_{\mathbf{G}}(su) \supseteq A_{\mathbf{G}}(su)^F \cong C_{\mathbf{G}}(su)^F / C_{\mathbf{G}}^\circ(su)^F.$$

Here, $C_{\mathbf{G}}^\circ(su) = C_{C_{\mathbf{G}}(s)}^\circ(u)$ is a unipotent group of dimension 2, so

$$|C_{\mathbf{G}}(su)^F| = |A_{\mathbf{G}}(su)^F| \cdot |C_{\mathbf{G}}^\circ(su)^F| \leqslant |A_{\mathbf{G}}(su)| \cdot |C_{\mathbf{G}}^\circ(su)^F| = 3q^2.$$

There are three $\mathbf{G}^F$-conjugacy classes whose elements have centraliser order $3q^2$, so these must be exactly the three $\mathbf{G}^F$-classes into which $\mathscr{C}^F$ splits. Only one of these three classes is stable under taking inverses, and we fix a representative $g_0 = s_0u_0 = u_0s_0$ of this $\mathbf{G}^F$-class. (In [EY86], s_0 is denoted by '$h(\omega,\omega,\omega)$', u_0 by 'y_0', and the $\mathbf{G}^F$-class of g_0 is named '$B_2(0)$'.) For $i = 3, 4$, let $\varphi_{x_i} := \varphi_{A_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the choice of $g_0 \in \mathscr{C}^F$, and let $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}}$. From the tables in [EY86], we see that

$$R_{x_3}(g_0) = R_{x_4}(g_0) = q = \chi_{x_3}(g_0) = \chi_{x_4}(g_0).$$

We thus have $\xi_{x_3}(\varphi_{x_3}) = \xi_{x_4}(\varphi_{x_4}) = +1$.

Remark 4.3.5. For any prime p and any (F-stable, unipotent) cuspidal character sheaf

$A = \mathrm{IC}(\overline{\Sigma}, \mathscr{E})[\dim \Sigma]^{\#\mathbf{G}}$ on $\mathbf{G}$ (where $(\Sigma, \mathscr{E})$ is an F-stable cuspidal pair for $\mathbf{G}$), we have aimed to choose the representative $g_0 \in \Sigma^F$ in accordance with the guidelines in 3.2.22, 3.2.23: First of all, for any F-stable cuspidal pair $(\Sigma, \mathscr{E})$ for $\mathbf{G}$, it was possible to find a representative $g_0 \in \Sigma^F$ which is $\mathbf{G}^F$-conjugate to g_0^{-1}, and we have always chosen such a g_0. Together with the requirement for the centraliser order $|C_{\mathbf{G}^F}(g_0)|$ to be maximal among the elements of Σ^F, this uniquely specifies the $\mathbf{G}^F$-class of g_0 inside Σ^F in each of the cases 4.3.2(a),(c), 4.3.3(c) and 4.3.4(a),(c). (In particular, the representative u_3 in 4.3.2(a) is split unipotent.)

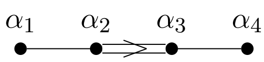
As for the remaining cases, we see that except for 4.3.3(a), Σ is always a conjugacy class consisting of regular elements of $\mathbf{G}$. So we know from 3.2.22(c) that the $\mathbf{G}^F$-class of g_0 inside Σ^F will be uniquely determined by the requirement (♡). The fact that our chosen representatives in 4.3.2(b), 4.3.3(b) and 4.3.4(b) satisfy this condition can (for instance) be deduced from evaluating (3.4.19.1) with $g = g_0$ and $w = w_{\mathrm{c}}$, as this is non-zero if and only if O_{g_0} has a non-empty intersection with $\mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F$, in view of (3.4.19.2).

Finally, let us consider the case 4.3.3(a). Here, the applicable conditions in 3.2.22 are not sufficient to single out a 'good' $\mathbf{G}^F$-class contained in $\mathscr{O}^F$ (where $\mathscr{O} = \mathsf{G}_2(a_1)$). However, using CHEVIE [MiChv, §6], we find the (unique) conjugacy class of $\mathbf{W}$ which is mapped to $\mathscr{O}$ under Lusztig's map (see 3.2.23). Choosing an element w of minimal length in this conjugacy class and evaluating (3.4.19.1) with this w and with elements of $\mathscr{O}^F$ then shows that the $\mathbf{G}^F$-class of our chosen u_0 is characterised by the property that O_{u_0} has a non-empty intersection with $\mathbf{B}_0^F w \mathbf{B}_0^F$ (see (♡′) in 3.2.23).

4.4. Groups of type F_4

In this section, let $\mathbf{G}$ be the simple group of type F_4 over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ (where q is a power of p), with corresponding Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$. We fix a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{G}$ and an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ which contains $\mathbf{T}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum attached to $\mathbf{G}$ and $\mathbf{T}_0$ (so $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), with bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$. Let $R^+ \subseteq R$ be the positive roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$, and let $\Pi = \{\alpha_1, \ldots, \alpha_4\} \subseteq R^+$ be the corresponding simple roots, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_4^\vee\}$ be the corresponding simple co-roots. We choose the order of $\alpha_1, \ldots, \alpha_4$ in such a way that the Dynkin diagram of $\mathbf{G}$ is as follows:

F_4



Let $\mathfrak{C} = (\langle \alpha_j, \alpha_i^\vee \rangle)_{1 \leqslant i,j \leqslant 4}$ be the associated Cartan matrix and $\mathbf{W} = {}^{N_{\mathbf{G}}(\mathbf{T}_0)}/_{\mathbf{T}_0}$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$. We identify $\mathbf{W}$ with a subgroup of $\mathrm{Aut}(X)$ (via

2.1.4) and write $s_i := w_{\alpha_i}$ (with w_{α_i} as defined in 2.1.2) for $1 \leqslant i \leqslant 4$; so $\mathbf{W}$ is a Coxeter group with Coxeter generators $S = \{s_1, s_2, s_3, s_4\}$. In order to be able to refer to Kondo's character table [Kon65] (cf. 4.1.5), we identify

$$s_1 \leftrightarrow \tau\sigma, \quad s_2 \leftrightarrow \tau, \quad s_3 \leftrightarrow a, \quad s_4 \leftrightarrow d,$$

that is, $\tau\sigma, \tau$ correspond to reflections in the long simple roots α_1, α_2, and a, d to reflections in the short simple roots α_3, α_4. We then denote by d_j the irreducible character of $\mathbf{W}$ corresponding to the jth irreducible character of degree d in Kondo's character table [Kon65] with the above identification. (This is the same notation and convention that Lusztig uses in [Lus84a, 4.10].) Furthermore, let $\mathbf{U}_0 = R_{\mathrm{u}}(\mathbf{B}_0)$ be the unipotent radical of $\mathbf{B}_0$, so that $\mathbf{B}_0 = \mathbf{U}_0 \rtimes \mathbf{T}_0$. As described in 2.1.19, F induces a p-isogeny of root data

$$\varphi\colon X \to X, \quad \lambda \mapsto \lambda \circ F|_{\mathbf{T}_0},$$

as well as a bijection $R \to R$, $\alpha \mapsto \alpha^\dagger$, so that $\varphi(\alpha^\dagger) = q\alpha$ for all $\alpha \in R$ (since $F\colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{G}$). The assignment $\alpha \mapsto \alpha^\dagger$ restricts to a graph automorphism of the Dynkin diagram, which is necessarily the identity since the Dynkin diagram does not have any non-trivial graph automorphisms (and again due to our assumption that F is a Frobenius map). Hence, $\mathbf{G}^F$ is the non-twisted group $\mathsf{F}_4(q)$ and $\sigma = \mathrm{id}_{\mathbf{W}}$. We are thus in the setting of Section 3.4 and adopt the further notation from there.

4.4.1. By [Lus78, 3.31] and [Lus80, 1.9], we have $|\mathfrak{X}(\mathbf{W})| = 37$, $|\operatorname{Irr}(\mathbf{W})| = 25$, and the set $\operatorname{Irr}(\mathbf{W})$ is partitioned into 11 families, as follows: There are 8 families consisting of a single character, 2 families consisting of 3 characters (the associated sets $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ have 4 elements each), and then there is one family consisting of 11 characters (the associated set $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ has 21 elements). We fix the bijections

$$\operatorname{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho, \tag{4.4.1.1}$$

and

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}, \quad x \mapsto A_x, \tag{4.4.1.2}$$

according to Corollary 3.4.8. For $x \in \mathfrak{X}(\mathbf{W})$, we will often write $\rho_x \in \operatorname{Uch}(\mathbf{G}^F)$ for the corresponding unipotent character under (4.4.1.1). There are exactly seven cuspidal character sheaves $A_1, A_2, \ldots, A_7$ in $\hat{\mathbf{G}}$, and all of them are in $\hat{\mathbf{G}}^{\mathrm{un}}$, see [Sho95a, §6, §7].

In particular, they are all F-stable (see 3.4.1). Let

$$\mathcal{F}_4 := \{12_1, 9_2, 9_3, 1_2, 1_3, 4_1, 4_3, 4_4, 6_1, 6_2, 16_1\} \subseteq \mathrm{Irr}(\mathbf{W})$$

be the unique family with 11 elements, so that $\mathcal{G}_{\mathcal{F}_4} = \mathfrak{S}_4$. By [Sho95a, §6], all the $x \in \mathfrak{X}(\mathbf{W})$ which parametrise cuspidal (unipotent) character sheaves on $\mathbf{G}$ under (4.4.1.2) are in $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_4})$, and they are given by

$$x_1 = (1, \lambda^3), \quad x_2 = (g_2, \varepsilon), \quad x_3 = (g_2', \varepsilon),$$
$$x_4 = (g_3, \omega), \quad x_5 = (g_3, \omega^2), \quad x_6 = (g_4, \mathrm{i}), \quad x_7 = (g_4, -\mathrm{i}).$$

Here and throughout this section, ω and i are assumed to be the same primitive roots of unity of order 3 and 4 as in 3.4.2, and the notation for the labels in $\mathfrak{M}(\mathfrak{S}_4)$ is as in [Lus84a, 4.3]: Thus, 1 denotes the trivial element of $\mathfrak{S}_4$; for $2 \leqslant j \leqslant 4$, g_j is a j-cycle in $\mathfrak{S}_4$; furthermore, $g_2' \in \mathfrak{S}_4$ is the product of two 2-cycles with disjoint support. We shall keep these elements fixed from now on. The irreducible characters of their centralisers are then named as follows: First, λ^3 denotes the sign character of $\mathfrak{S}_4 = C_{\mathfrak{S}_4}(1)$. If $g \in \{g_3, g_4\}$, we have $C_{\mathfrak{S}_4}(g) = \langle g \rangle$, and we identify the irreducible characters of this group with their values at g. Next, we have $C_{\mathfrak{S}_4}(g_2) = \langle g_2 \rangle \times \langle \tau \rangle$ where $\tau \in \mathfrak{S}_4$ is the transposition whose support is disjoint to the one of g_2; then ε is the restriction of λ^3 to $C_{\mathfrak{S}_4}(g_2)$. Finally, the group $C_{\mathfrak{S}_4}(g_2')$ is isomorphic to D_8 (the dihedral group of order 8), and ε is the restriction of λ^3 to $C_{\mathfrak{S}_4}(g_2')$. (This is sufficient to describe the seven pairs above; we refer to [Lus84a, 4.3] for the notation regarding the other elements in $\mathfrak{M}(\mathfrak{S}_4)$.)

As usual, it will be convenient to order the $A_1, A_2, \ldots, A_7 \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}$ in such a way that

$$A_i = A_{x_i} \quad \text{for} \quad 1 \leqslant i \leqslant 7, \tag{4.4.1.3}$$

even though this is different from the numberings chosen in [Sho95a, §6, §7], [Gec21, §7]. The (F-stable) cuspidal pairs $(\Sigma, \mathscr{E})$ corresponding to these seven A_i (see 3.1.17, 3.1.18) depend on whether $p = 2$, $p = 3$ or $p \geqslant 5$. They are provided by Shoji [Sho95a, §6, §7]. The root of unity λ_{A_i} attached to A_i (see 3.4.3(c)) is also determined in loc. cit., but note that the property $\lambda_{A_i} = \tilde{\lambda}_{x_i}$ (see Corollary 3.4.8) alone is not quite sufficient to distinguish the A_i. Indeed, we have

$$\tilde{\lambda}_{x_1} = 1,\ \tilde{\lambda}_{x_2} = -1,\ \tilde{\lambda}_{x_3} = 1,\ \tilde{\lambda}_{x_4} = \omega,\ \tilde{\lambda}_{x_5} = \omega^2,\ \tilde{\lambda}_{x_6} = \mathrm{i},\ \tilde{\lambda}_{x_7} = -\mathrm{i},$$

so $\tilde{\lambda}_{x_1} = \tilde{\lambda}_{x_3}$. It is argued in [Gec21, §7] to which cuspidal pairs the two character sheaves A_1, A_3 correspond, and our description of the results in loc. cit. below will of course take this into account.

On the other hand, using the notation of [Lus80, 1.9] (see also [Lus84a, p. 372]) for the cuspidal unipotent characters, we have

$$\rho_{x_1} = \mathsf{F}_4^{\mathrm{II}}[1], \quad \rho_{x_2} = \mathsf{F}_4[-1], \quad \rho_{x_3} = \mathsf{F}_4^{\mathrm{I}}[1],$$
$$\rho_{x_4} = \mathsf{F}_4[\omega], \quad \rho_{x_5} = \mathsf{F}_4[\omega^2], \quad \rho_{x_6} = \mathsf{F}_4[\mathrm{i}], \quad \rho_{x_7} = \mathsf{F}_4[-\mathrm{i}].$$

For any $x \in \mathfrak{X}(\mathbf{W})$, let $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ be an isomorphism as in 3.2.1(∗). (We will make an explicit choice for the cuspidal character sheaves below, depending on p.) Let

$$\chi_x := \chi_{A_x,\varphi_x} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$$

be the corresponding characteristic function. So by (3.4.12.2), we have

$$R_x = \xi_x \chi_x \text{ for } x \in \mathfrak{X}(\mathbf{W}), \text{ where } \xi_x = \xi_x(\varphi_x) \in \overline{\mathbb{Q}}_\ell^\times, \ |\xi_x| = 1. \tag{4.4.1.4}$$

For $1 \leqslant i \leqslant 7$, the scalars $\xi_{x_i} \in \overline{\mathbb{Q}}_\ell^\times$ in (4.4.1.4) (based on an explicit choice for the isomorphisms φ_{x_i}) have been determined in all characteristics, due to the work of Marcelo and Shinoda [MS95] (for those A_i whose support consists of unipotent elements) and Geck (see [Gec19, §5] for $p = 2$ and [Gec21, §7] for all the remaining cases). The purpose of this section is to describe these results (summarised in Proposition 4.4.7 below) and partly sketch the methods used in the proofs.

4.4.2. Let us describe the partition of the 37 elements of $\mathfrak{S}_\mathbf{W}$ into Harish-Chandra series, following [Lus78, 3.31] (and [Lus80, 1.9]). The set $\mathfrak{S}^\circ_{\mathbf{W}_J}$ is non-empty for the following subsets $J \subseteq S$: $J = \varnothing$, $J = \{s_2, s_3\}$ and $J = S$.

(a) The set $J = \varnothing$ gives rise to the 25 elements in the principal series, that is, the elements in the image of the embedding $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_\mathbf{W}$, $\phi \mapsto (\varnothing, \phi, (1,1))$.

(b) Let $J = \{s_2, s_3\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type B_2. We have $\mathfrak{S}^\circ_{\mathbf{W}_J} = \{(-1,2)\}$, and the relative Weyl group $W_\mathbf{G}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is isomorphic to $W(\mathsf{B}_2)$. So there are 5 elements in $\mathfrak{S}_\mathbf{W}$ of the form $(J, \epsilon, (-1,2))$, $\epsilon \in \mathrm{Irr}(W_\mathbf{G}(\mathbf{L}_J))$.

(c) For $J = S$, the set $\mathfrak{S}^\circ_{\mathbf{W}_J} = \mathfrak{S}^\circ_\mathbf{W}$ consists of the elements $(1,8)$, $(1,24)$, $(-1,4)$, $(\mathrm{i},4)$, $(-\mathrm{i},4)$, $(\omega,3)$, $(\omega^2,3)$ parametrising the seven cuspidal unipotent characters of $\mathbf{G}^F$ (and the seven cuspidal unipotent character sheaves on $\mathbf{G}$).

Type F_4 in characteristic $p \geqslant 5$

In this subsection (i.e., in 4.4.3–4.4.4 below), we assume that $p \geqslant 5$, that is, p is a good prime for $\mathbf{G}$.

4.4.3. The conjugacy classes of $\mathsf{F}_4(q)$ have been determined by Shoji [Sho74]. There is one cuspidal character sheaf whose support is contained in the unipotent variety $\mathbf{G}_{\text{uni}}$, and the corresponding scalar in (4.4.1.4) is determined by Marcelo and Shinoda [MS95]. The remaining six cuspidal character sheaves are dealt with by Geck [Gec21, §7]. (In fact, the unipotently supported character sheaf is considered there as well.)

Following [Gec21, 7.5], let us consider the F-stable unipotent conjugacy class $\mathscr{O}_0 \subseteq \mathbf{G}$ which is named $\mathsf{F}_4(a_3)$ in [Car85]. We have $\dim C_{\mathbf{G}}(u) = 12$ and $A_{\mathbf{G}}(u) \cong \mathfrak{S}_4$ for any $u \in \mathscr{O}_0$. A representative for the split unipotent $\mathbf{G}^F$-conjugacy class inside $\mathscr{O}_0^F$ (see 3.2.22(a)) is denoted by x_{14} in [Sho74, Table 5]. We shall write u_0 for this element, that is,

$$u_0 := u_{\alpha_1+\alpha_2}(1)u_{\alpha_2+2\alpha_3}(1)u_{\alpha_2+2\alpha_3+2\alpha_4}(-1)u_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}(-1) \in \mathbf{U}_0^F \cap \mathscr{O}_0^F$$

(with the structure constants $N_{\alpha,\beta}$ for $\alpha, \beta \in R$ as in [Sho74]). Thus, F induces the identity on $A_{\mathbf{G}}(u_0) \cong \mathfrak{S}_4$, and u_0 is conjugate to u_0^{-1} in $\mathbf{G}^F$. It follows that $\mathscr{O}_0^F$ is the union of 5 conjugacy classes of $\mathbf{G}^F$ (parametrised by the conjugacy classes of $\mathfrak{S}_4$, or by the partitions of 4). Representatives for these $\mathbf{G}^F$-classes are denoted by $x_{14}, \ldots, x_{18}$ in [Sho74]. We have

$$\begin{aligned}
&A_{\mathbf{G}}(x_{14})^F \cong \mathfrak{S}_4 && \text{(cycle type (1111))},\\
&A_{\mathbf{G}}(x_{15})^F \cong D_8 && \text{(cycle type (22))},\\
&A_{\mathbf{G}}(x_{16})^F \cong C_2 \times C_2 && \text{(cycle type (211))},\\
&A_{\mathbf{G}}(x_{17})^F \cong C_4 && \text{(cycle type (4))},\\
&A_{\mathbf{G}}(x_{18})^F \cong C_3 && \text{(cycle type (31))}.
\end{aligned}$$

The relevance of the conjugacy class $\mathscr{O}_0 \subseteq \mathbf{G}$ is that for any of the seven cuspidal pairs $(\Sigma, \mathscr{E})$ for $\mathbf{G}$, the unipotent parts in the Jordan decompositions of the elements of Σ are in $\mathscr{O}_0$. By [Sho95a, (6.2.4)] (and [Gec21, §7]), the correspondence between the cuspidal (unipotent) character sheaves on $\mathbf{G}$ and the cuspidal pairs for $\mathbf{G}$ is described by the following list, using the notation of Remark 3.1.18 as well as the convention that 's' is always a semisimple element of $\mathbf{G}$ and 'u' is always a unipotent element of $\mathbf{G}$. The names for the unipotent classes of $\mathbf{G}$ are as in [Car85].

(a) $A_1 \leftrightarrow (u, \varsigma)$ where $u \in \mathscr{O}_0$ and ς is the sign character of $A_{\mathbf{G}}(u) \cong \mathfrak{S}_4$.

(b) $A_2 \leftrightarrow (su, \varsigma)$ where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{C}_3 \times \mathsf{A}_1$ and $u \in C_{\mathbf{G}}(s) \cap \mathscr{O}_0$; the group $A_{\mathbf{G}}(su)$ is of the form $\langle \overline{su} \rangle \times \langle a \rangle = \langle \overline{s} \rangle \times \langle a \rangle \cong C_2 \times C_2$ (where $\overline{su} = \overline{s}$ is the image in $A_{\mathbf{G}}(su)$ of su, s, respectively, and a is another element of $A_{\mathbf{G}}(su)$ of order 2), and ς is the linear character which satisfies $\varsigma(\overline{su}) = -1$, $\varsigma(a) = 1$.

(c) $A_3 \leftrightarrow (su, \varsigma)$ where $C_{\mathbf{G}}(s)$ has a root system of type B_4 and $u \in C_{\mathbf{G}}(s) \cap \mathscr{O}_0$; we have $A_{\mathbf{G}}(su) \cong D_8$ (the dihedral group of order 8), and ς is the sign character of D_8.

(d) $A_i \leftrightarrow (su, \varsigma_i)$ for $i = 4, 5$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_2 \times \mathsf{A}_2$, and where $u \in C_{\mathbf{G}}(s) \cap \mathscr{O}_0$ is a regular unipotent element in $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_3$, and ς_4, ς_5 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_4(\overline{su}) = \omega$, $\varsigma_5(\overline{su}) = \omega^2$.

(e) $A_i \leftrightarrow (su, \varsigma_i)$ for $i = 6, 7$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_3 \times \mathsf{A}_1$, and where $u \in C_{\mathbf{G}}(s) \cap \mathscr{O}_0$ is a regular unipotent element in $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_4$, and ς_6, ς_7 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_6(\overline{su}) = \mathrm{i}$, $\varsigma_7(\overline{su}) = -\mathrm{i}$.

4.4.4. Following [Gec21, §7], for each of the cases (a)–(e) in 4.4.3, we now describe the choice of a 'good' $\mathbf{G}^F$-conjugacy class inside the given class of $\mathbf{G}$ and sketch how the corresponding scalars ξ_{x_i} ($1 \leqslant i \leqslant 7$) in (4.4.1.4) are determined in loc. cit. As is also noted there, the determination of the scalar ξ_{x_1} yields the values of all the unipotent characters at unipotent elements of $\mathbf{G}^F$.

(a) We have
$$A_1 = A_{(1,\lambda^3)} \cong \mathrm{IC}(\overline{\mathscr{O}}_0, \mathscr{E})[\dim \mathscr{O}_0]^{\#\mathbf{G}},$$
where $\mathscr{E}$ is the F-stable cuspidal local system on $\mathscr{O}_0$ described by the sign character λ^3 of $A_{\mathbf{G}}(u) \cong \mathfrak{S}_4$ (via 3.2.20(b)). Of course we choose the split unipotent element
$$u_0 = u_{\alpha_1+\alpha_2}(1)u_{\alpha_2+2\alpha_3}(1)u_{\alpha_2+2\alpha_3+2\alpha_4}(-1)u_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4}(-1) \in \mathbf{U}_0^F \cap \mathscr{O}_0^F$$
and its $\mathbf{G}^F$-conjugacy class inside $\mathscr{O}_0^F$ to define the isomorphism $\varphi_{x_1} \colon F^*A_1 \xrightarrow{\sim} A_1$ via 3.2.21 (as in [Gec21, 7.8], see also [MS95, p. 309]). The associated characteristic function $\chi_{x_1} := \chi_{A_1,\varphi_{x_1}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ is given by
$$\chi_{x_1}(g) = \begin{cases} q^6\lambda^3(a) & \text{if } g \sim_{\mathbf{G}^F} (u_0)_a \text{ for some } a \in \mathfrak{S}_4, \\ 0 & \text{if } g \notin \mathscr{O}_0^F. \end{cases}$$

By [Gec21, 7.8], the scalar $\xi_{x_1} = \xi_{x_1}(\varphi_{x_1})$ in (4.4.1.4) can be determined using the method described in 3.4.19(1). Indeed, since u_0 is $\mathbf{G}^F$-conjugate to u_0^{-1}, we know that $\xi_{x_1} \in \{\pm 1\}$ (see 3.4.18). In order to fix the sign, one can consider the unipotent character $\rho_{x_6} = \mathsf{F}_4[\mathrm{i}]$ and write it as a linear combination of almost characters: With the notation of 3.4.19, we have $\chi_x(u_0) = 0$ for any $x_1 \neq x \in \mathfrak{X}^\circ(\mathbf{W})$ and $\{x_6, x\} = 0$ for any $x \in \mathfrak{X}'(\mathbf{W})$, but $\{x_6, x_1\} \neq 0$. Since the values of the Green functions for groups of type F_4 are known, one can compute $R_\phi(u_0)$ for any $\phi \in \mathrm{Irr}(\mathbf{W})$ and obtains

$$\mathsf{F}_4[\mathrm{i}](u_0) = -\tfrac{1}{4}q^4(\xi_{x_1}q^2 - 1) \in \mathbb{Z}.$$

Since q is odd, we have

$$q^4(\xi_{x_1}q^2 - 1) \equiv \xi_{x_1} - 1 \pmod 4,$$

so $\xi_{x_1} = +1$. As noted in [Gec21], the scalar ξ_{x_1} could also be determined by making use of Kawanaka's results on generalised Gelfand–Graev representations [Kaw86, §4] and the fact that these are known to hold for any power q of a good prime p for $\mathbf{G}$, due to Taylor [Tay16].

(b) Consider the character sheaf $A_2 = A_{(g_2,\varepsilon)}$. Let $s \in \mathbf{G}^F$ be a semisimple element such that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{C}_3 \times \mathsf{A}_1$. (By the results of [Sho74], this condition uniquely determines the $\mathbf{G}^F$-conjugacy class of s.) There is a natural isogeny $\beta\colon \mathrm{Sp}_6(k) \times \mathrm{SL}_2(k) \to C_{\mathbf{G}}(s)$, and β is defined over $\mathbb{F}_q$. Let $u \in C_{\mathbf{G}}(s)^F_{\mathrm{uni}}$ be in the image under β of the unipotent class of $\mathrm{Sp}_6(k) \times \mathrm{SL}_2(k)$ whose elements have Jordan type $(4,2) \times (2)$, and denote by $\mathscr{C}$ the (F-stable) $\mathbf{G}$-conjugacy class of $g := su = us$. Then

$$\dim C_{\mathbf{G}}(g) = 6, \quad |C_{\mathbf{G}}(g)^F| = 4q^6 \quad \text{and} \quad \mathscr{C} = \mathscr{C}^{-1}.$$

Moreover, F acts trivially on $A_{\mathbf{G}}(g)$, and there exists some $1 \neq a \in A_{\mathbf{G}}(g)$ such that

$$A_{\mathbf{G}}(g) = \langle \overline{g} \rangle \times \langle a \rangle \cong C_2 \times C_2.$$

Hence, the set $\mathscr{C}^F$ splits into four $\mathbf{G}^F$-classes. We have $\{(g')^2 \mid g' \in \mathscr{C}\} = \mathscr{O}_0$. It is shown in [Gec21, 7.11] that one can choose an element $g_0 \in \mathscr{C}^F$ with the property that $g_0^2 = u_0 \in \mathscr{O}_0^F$, and this condition uniquely determines the $\mathbf{G}^F$-conjugacy class of g_0. In order to have a consistent notation, we also write a_0 instead of a from now on, so that $A_{\mathbf{G}}(g_0) = \langle \overline{g}_0 \rangle \times \langle a_0 \rangle$. Let $\mathscr{E}$ be the local system on $\mathscr{C}$ parametrised by the irreducible character of $A_{\mathbf{G}}(g_0)$ which takes the value -1 at $\overline{g}_0$ and $+1$ at a_0. Then

$$A_2 \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E})[\dim \mathscr{C}]^{\#\mathbf{G}}.$$

Let $\varphi_{x_2} \colon F^*A_2 \xrightarrow{\sim} A_2$ be the isomorphism corresponding to the $\mathbf{G}^F$-class of g_0, and let $\chi_{x_2} := \chi_{A_2,\varphi_{x_2}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic function. Recall from Theorem 3.1.13 that χ_{x_2} is identically zero outside of $\mathscr{C}^F$. Hence, χ_{x_2} is completely described by the following table, where a $\mathbf{G}^F$-conjugacy class inside $\mathscr{C}^F$ is named by the corresponding element of $A_{\mathbf{G}}(g_0)$ in the top line.

	1	a_0	$\overline{g}_0$	$\overline{g}_0 a_0$
χ_{x_2}	q^3	q^3	$-q^3$	$-q^3$

The determination of the scalar $\xi_{x_2} = \xi_{x_2}(\varphi_{x_2})$ is rather subtle: As a first step, applying the results in [Gec21, §3] to compute the R_ϕ ($\phi \in \mathrm{Irr}(\mathbf{W})$), Geck evaluates the character $\rho_{x_2} = \mathsf{F}_4[-1]$ at the elements of $\mathscr{C}^F$ and deduces $\xi_{x_2} \in \{\pm 1\}$ from the fact that $\mathsf{F}_4[-1]$ is rational-valued. Then, in order to fix the sign, he exploits certain congruence relations to compare suitable character values at elements of $\mathscr{C}^F$ and $\mathscr{O}_0^F$ (recall that we have $\mathscr{O}_0 = \{g^2 \mid g \in \mathscr{C}\}$), using the known values of unipotent characters at elements of $\mathscr{O}_0^F$. With the choice of g_0 as above, one obtains $\xi_{x_2} = +1$; we refer to [Gec21, 7.11] for the details. Let us just note that g_0 is $\mathbf{G}^F$-conjugate to g_0^{-1}: Indeed, since $g_0^2 = u_0$, we have $g_0^{-2} = u_0^{-1}$, which is $\mathbf{G}^F$-conjugate to u_0; so both $g_0, g_0^{-1} \in \mathscr{C}^F$ have the property that their squares are $\mathbf{G}^F$-conjugate to u_0, so they must be $\mathbf{G}^F$-conjugate as mentioned above.

(c) Consider the character sheaf $A_3 = A_{(g_2',\varepsilon)}$. Let $s \in \mathbf{G}^F$ be a semisimple element such that $C_{\mathbf{G}}(s)$ has a root system of type B_4. (The $\mathbf{G}^F$-class of s is uniquely determined by this property.) There is a natural isogeny $\beta \colon C_{\mathbf{G}}(s) \to \mathrm{SO}_9(k)$, defined over $\mathbb{F}_q$. Let $u \in C_{\mathbf{G}}(s)^F_{\mathrm{uni}}$ be such that the (unipotent) element $\beta(u) \in \mathrm{SO}_9(k)$ has Jordan type $(5,3,1)$, and let $\mathscr{C}$ be the (F-stable) conjugacy class of $\mathbf{G}$ which contains the element $su = us$. For $g \in \mathscr{C}^F$, the group $A_{\mathbf{G}}(g)$ is isomorphic to D_8, so $\mathscr{C}^F$ splits into five $\mathbf{G}^F$-conjugacy classes. There exists an element $g_0 \in \mathscr{C}^F$ for which F acts trivially on $A_{\mathbf{G}}(g_0)$ (which is equivalent to $C_{\mathbf{G}}(g_0)^F$ being of order $8q^8$), but there are in fact two $\mathbf{G}^F$-classes inside $\mathscr{C}^F$ whose elements have this property — as the value of the scalar $\xi_{x_3} = \xi_{x_3}(\varphi_{x_3})$ turns out to be the same regardless of which of these two classes one chooses to define $\varphi_{x_3} \colon F^*A_3 \xrightarrow{\sim} A_3$, we just pick out a representative g_0 from one of them. Denoting by $\mathscr{E}$ the local system on $\mathscr{C}$ described by the sign character sgn of $A_{\mathbf{G}}(g_0) \cong D_8$, we have

$$A_3 \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E})[\dim \mathscr{C}]^{\#\mathbf{G}}.$$

Let $\varphi_{x_3}\colon F^*A_3 \xrightarrow{\sim} A_3$ be the isomorphism corresponding to the choice of g_0, and let $\chi_{x_3} := \chi_{A_3,\varphi_{x_3}}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic function. Then

$$\chi_{x_3}(g) = \begin{cases} q^4\,\mathrm{sgn}(a) & \text{if } g \sim_{\mathbf{G}^F} (g_0)_a \text{ for some } a \in D_8, \\ \quad 0 & \text{if } g \notin \mathscr{C}^F. \end{cases}$$

It is easy to see that $\overline{R}_{x_3} = R_{x_3}$, so one can argue exactly as in 3.4.18 to get $\xi_{x_3} \in \{\pm 1\}$. Using the results of [Gec21, §3] to compute the uniform almost characters R_ϕ ($\phi \in \mathrm{Irr}(\mathbf{W})$) and, as in (a), considering the unipotent character $\rho_{x_6} = \mathsf{F}_4[\mathrm{i}]$ (but this time evaluating it at the element g_0), Geck concludes that

$$\mathsf{F}_4[\mathrm{i}](g_0) = \tfrac{1}{4}q^2(1 - \xi_{x_3}q^2) \in \mathbb{Z}.$$

Since q is odd, we have

$$q^2(1 - \xi_{x_3}q^2) \equiv 1 - \xi_{x_3} \pmod 4,$$

which only holds for $\xi_{x_3} = +1$. See [Gec21, 7.10] for the details.

(d) The character sheaves $A_4 = A_{(g_3,\omega)}$ and $A_5 = A_{(g_3,\omega^2)}$ are supported by the closure of the (F-stable) conjugacy class $\mathscr{C} \subseteq \mathbf{G}$ containing elements of the form $su = us$ where $s \in \mathbf{G}^F$ is semisimple such that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_2 \times \mathsf{A}_2$ and $u \in C_{\mathbf{G}}(s)^F \cap \mathscr{O}_0^F$ is a regular unipotent element of $C_{\mathbf{G}}(s)$. We have $\mathscr{C} = \mathscr{C}^{-1}$. Since $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle \cong C_3$, it is clear that F acts trivially on $A_{\mathbf{G}}(su)$, so $\mathscr{C}^F$ splits into three $\mathbf{G}^F$-conjugacy classes. There is a unique $\mathbf{G}^F$-class inside $\mathscr{C}^F$ which is stable under taking inverses, and we choose a representative $g_0 = s_0u_0 = u_0s_0 \in \mathscr{C}^F$ of this class (so that g_0, g_0^{-1} are conjugate by an element of $\mathbf{G}^F$). For $i = 1, 2$, let $\mathscr{E}_i$ be the local system on $\mathscr{C}$ whose isomorphism class corresponds to the irreducible character of $A_{\mathbf{G}}(g_0)$ which takes the value ω^i at $\overline{g}_0$. Then

$$A_4 \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E}_1)[\dim \mathscr{C}]^{\#\mathbf{G}} \quad \text{and} \quad A_5 \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E}_2)[\dim \mathscr{C}]^{\#\mathbf{G}}.$$

For $i = 4, 5$, let $\varphi_{x_i}\colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphisms corresponding to the choice of g_0, and let $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic functions. Then, for $g \in \mathscr{C}^F$, the values of χ_{x_4}, χ_{x_5} are given by the following table, where we describe the $\mathbf{G}^F$-classes inside $\mathscr{C}^F$ by giving the corresponding elements of $A_{\mathbf{G}}(g_0)$ in the top line.

	1	$\overline{g}_0$	$\overline{g}_0^2$
χ_{x_4}	q^2	ωq^2	$\omega^2 q^2$
χ_{x_5}	q^2	$\omega^2 q^2$	ωq^2

Since g_0 is $\mathbf{G}^F$-conjugate to g_0^{-1}, it follows from 3.4.18 that $\xi_{x_4}, \xi_{x_5} \in \{\pm 1\}$, and a similar argument shows that we have in fact $\xi_{x_4} = \xi_{x_5}$ (by comparing the almost characters R_{x_4} and R_{x_5}). Expressing the unipotent character $\rho_{x_4} = \mathsf{F}_4[\omega]$ as a linear combination of almost characters, one checks that the assumptions in 3.4.19(1) are met. By evaluating the uniform almost characters R_ϕ ($\phi \in \operatorname{Irr}(\mathbf{W})$) at g_0, Geck shows that

$$\mathsf{F}_4[\omega](g_0) = \tfrac{1}{3}(-1 + \xi_{x_4} q^2) \in \mathbb{Z}.$$

Since 3 does not divide q, we have

$$-1 + \xi_{x_4} q^2 \equiv -1 + \xi_{x_4} \pmod 3,$$

so ξ_{x_4} cannot be -1. One thus obtains $\xi_{x_4} = \xi_{x_5} = +1$. For the details, see [Gec21, 7.6].

(e) The character sheaves $A_6 = A_{(g_4,\mathrm{i})}$ and $A_7 = A_{(g_4,-\mathrm{i})}$ are supported by the closure of the (F-stable) conjugacy class $\mathscr{C} \subseteq \mathbf{G}$ containing elements of the form $su = us$ where $s \in \mathbf{G}^F$ is semisimple such that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_3 \times \mathsf{A}_1$ and $u \in C_{\mathbf{G}}(s)^F \cap \mathscr{O}_0^F$ is a regular unipotent element of $C_{\mathbf{G}}(s)$. We have $\mathscr{C} = \mathscr{C}^{-1}$ and $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle \cong C_4$, so F acts trivially on $A_{\mathbf{G}}(su)$. Thus, $\mathscr{C}^F$ splits into four $\mathbf{G}^F$-classes. Consider the Coxeter element

$$w_{\mathrm{c}} := s_1 s_2 s_3 s_4 \in \mathbf{W}.$$

By [Gec21, 4.9, 4.10], there exists a unique $\mathbf{G}^F$-conjugacy class $C_0 \subseteq \mathscr{C}^F$ such that $C_0 \cap \mathbf{B}_0^F w_{\mathrm{c}} \mathbf{B}_0^F \neq \varnothing$, which in particular implies that $C_0^{-1} = C_0$. So let us choose a representative $g_0 \in C_0$. Let $\mathscr{E}_1$ be the local system on $\mathscr{C}$ described by the irreducible character of $A_{\mathbf{G}}(g_0)$ which takes the value i at $\overline{g}_0$, and let $\mathscr{E}_2$ be the local system on $\mathscr{C}$ described by the irreducible character of $A_{\mathbf{G}}(g_0)$ which takes the value $-$i at $\overline{g}_0$. Then

$$A_6 \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E}_1)[\dim \mathscr{C}]^{\#\mathbf{G}} \quad \text{and} \quad A_7 \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E}_2)[\dim \mathscr{C}]^{\#\mathbf{G}}.$$

For $i = 6, 7$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphisms corresponding to the $\mathbf{G}^F$-class of g_0; the values of the associated characteristic functions $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ at elements of $\mathscr{C}^F$ are given by the following table where, as usual, we denote the $\mathbf{G}^F$-conjugacy classes inside $\mathscr{C}^F$ by the corresponding elements of $A_{\mathbf{G}}(g_0)$.

	1	$\overline{g}_0$	$\overline{g}_0^2$	$\overline{g}_0^3$
χ_{x_6}	q^2	$\mathrm{i}q^2$	$-q^2$	$-\mathrm{i}q^2$
χ_{x_7}	q^2	$-\mathrm{i}q^2$	$-q^2$	$\mathrm{i}q^2$

Since g_0 is $\mathbf{G}^F$-conjugate to g_0^{-1}, one deduces $\xi_{x_6} = \xi_{x_7} \in \{\pm 1\}$ as in (d), using the almost characters R_{x_6}, R_{x_7} instead of R_{x_4}, R_{x_5}. Fixing the sign is then achieved in a way similar to the case of E_7 with $p \neq 2$ (cf. 4.2.4), by considering the Hecke algebra of $\mathbf{G}^F$ with respect to the BN-pair $(\mathbf{B}_0^F, N_{\mathbf{G}}(\mathbf{T}_0)^F)$ and explicitly evaluating the right side of the formula (3.4.19.2). One finds that $\xi_{x_6} = \xi_{x_7} = +1$. We refer to [Gec21, 7.7] for the details.

Type F_4 in characteristic $p = 3$

4.4.5. Let us assume that $p = 3$. The conjugacy classes of $\mathsf{F}_4(q)$ have been determined by Shoji [Sho74], and the description is very similar to the case where $p \geqslant 5$. The only difference appears as far as regular unipotent elements of $\mathbf{G}$ are concerned, as the set $\mathscr{O}_{\mathrm{reg}}^F$ of F-stable regular unipotent elements splits into three $\mathbf{G}^F$-classes if $p = 3$ (while it forms a single $\mathbf{G}^F$-conjugacy class in case $p \geqslant 5$). There are three cuspidal character sheaves whose support is contained in $\mathbf{G}_{\mathrm{uni}}$ (namely, A_1, A_4 and A_5), and the corresponding scalars ξ_{x_1}, ξ_{x_4} and ξ_{x_5} in (4.4.1.4) have been determined by Marcelo and Shinoda [MS95]. As mentioned in [Gec21, 7.12], the remaining four cuspidal character sheaves A_2, A_3, A_6 and A_7 can be handled in a way completely analogous to the good characteristic case and in particular give rise to the same scalars $\xi_{x_2}, \xi_{x_3}, \xi_{x_6}$ and ξ_{x_7} in (4.4.1.4); in fact, this also holds for A_1 and ξ_{x_1}.

The support of the remaining two cuspidal character sheaves A_4 and A_5 is the unipotent variety $\mathbf{G}_{\mathrm{uni}}$. (Recall that this is not the case if $p \geqslant 5$.) By [Sho95a, 7.2], A_4 and A_5 are described as follows (with the same conventions as in 4.4.3):

(d) $A_i \leftrightarrow (u, \varsigma_i)$, $i = 4, 5$, where $u \in \mathscr{O}_{\mathrm{reg}}$; we have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_3$, and ς_4, ς_5 are the linear characters of $A_{\mathbf{G}}(u)$ which satisfy $\varsigma_4(\overline{u}) = \omega$, $\varsigma_5(\overline{u}) = \omega^2$.

Let us set

$$u_0 := u_{\alpha_1}(1)u_{\alpha_2}(1)u_{\alpha_3}(1)u_{\alpha_4}(1) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F.$$

(Note that the same choice is made in [MS95, p. 309].) Since $A_{\mathbf{G}}(u_0)$ is generated by $\overline{u}_0$ and $F(u_0) = u_0$, F acts trivially on $A_{\mathbf{G}}(u_0)$, so $\mathscr{O}_{\mathrm{reg}}^F$ splits into three $\mathbf{G}^F$-conjugacy classes. For $j = 1, 2$, let $\mathscr{E}_j$ be the $\mathbf{G}$-equivariant F-stable irreducible local system on

$\mathscr{O}_{\mathrm{reg}}$ described by the irreducible character of $A_{\mathbf{G}}(u_0)$ which takes the value ω^j at $\overline{u}_0$. Then

$$A_4 \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_1)[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}} \quad \text{and} \quad A_5 \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_2)[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}}.$$

For $i = 4, 5$, let $\varphi_{x_i}\colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of $u_0 \in \mathscr{O}_{\mathrm{reg}}^F$, and let $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}}\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic function. Then, denoting the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$ by the corresponding elements of $A_{\mathbf{G}}(u_0)$, the values of χ_{x_4}, χ_{x_5} are given by the following table.

	1	$\overline{u}_0$	$\overline{u}_0^2$
χ_{x_4}	q^2	ωq^2	$\omega^2 q^2$
χ_{x_5}	q^2	$\omega^2 q^2$	ωq^2

By [MS95, 4.2], we have $\xi_{x_4} = \xi_{x_5} = +1$. This can also be obtained with an argument similar to the one in the proof of Proposition 4.1.19.

Type F_4 in characteristic $p = 2$

4.4.6. Let us assume that $p = 2$. The conjugacy classes of $\mathsf{F}_4(q)$ have been determined by Shinoda [Shi74]. The support of five of the seven cuspidal character sheaves is contained in the unipotent variety $\mathbf{G}_{\mathrm{uni}}$, and the scalars ξ_{x_i} in (4.4.1.4) for those character sheaves are determined by Marcelo and Shinoda [MS95]. As far as the remaining two cuspidal character sheaves A_i are concerned, the scalars ξ_{x_i} have been determined by Geck, see [Gec19, §5]. This is achieved by first looking at the base case $p = q = 2$, using the known character table of $\mathsf{F}_4(2)$ (which is contained in the GAP library [Bre22]), and then extending the result to arbitrary powers q of $p = 2$ by applying Proposition 3.4.13. It should be noted that this involves some relatively deep computational arguments, as it is not always easy to identify a conjugacy class in [Shi74] with the corresponding one in the GAP table, see [Gec19, 5.2]. However, once this is accomplished, the values of the almost characters R_{x_i} with respect to $p = q = 2$ at the chosen elements in [Gec19, §5] can be computed, and one immediately obtains the scalars ξ_{x_i} in (4.4.1.4) by applying Proposition 3.4.13, for any $i \in \{1, 2, \dots, 7\}$. As in 4.4.3 (and with the same conventions), we first give a list describing the cuspidal (unipotent) character sheaves on $\mathbf{G}$ in terms of the corresponding cuspidal pairs for $\mathbf{G}$, following Shoji [Sho95a, 7.2].

(a) $A_1 \leftrightarrow (u, \varsigma)$ where $u \in \mathsf{F}_4(a_3)$; we have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_3$, and ς is the sign character of $\mathfrak{S}_3$.

(b) $A_2 \leftrightarrow (u, \varsigma)$ where $u \in \mathsf{F}_4(a_1)$; we have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_2$, and ς is the non-trivial linear character of $A_{\mathbf{G}}(u)$.

(c) $A_3 \leftrightarrow (u, \varsigma)$ where $u \in \mathsf{F}_4(a_2)$; we have $A_{\mathbf{G}}(u) \cong D_8$ (the dihedral group of order 8), and ς is the sign character of D_8.

(d) $A_i \leftrightarrow (su, \varsigma_i)$ for $i = 4, 5$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_2 \times \mathsf{A}_2$, and u is a regular unipotent element of $C_{\mathbf{G}}(s)$; we have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_3$, and ς_4, ς_5 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_4(\overline{su}) = \omega$, $\varsigma_5(\overline{su}) = \omega^2$.

(e) $A_i \leftrightarrow (u, \varsigma_i)$ for $i = 6, 7$, where $u \in \mathscr{O}_{\mathrm{reg}}$ is a regular unipotent element of $\mathbf{G}$; we have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_4$, and ς_6, ς_7 are the linear characters of $A_{\mathbf{G}}(u)$ which satisfy $\varsigma_6(\overline{u}) = \mathrm{i}$, $\varsigma_7(\overline{u}) = -\mathrm{i}$.

We now look at the cuspidal (unipotent) character sheaves described in (a)–(e) more closely, following [Gec19, §5].

(a) The (F-stable) unipotent conjugacy class $\mathsf{F}_4(a_3) \subseteq \mathbf{G}$ is uniquely determined by the property that $\dim C_{\mathbf{G}}(u) = 12$ for any $u \in \mathsf{F}_4(a_3)$. Let $u_0 \in \mathsf{F}_4(a_3)^F$ be the element denoted by x_{17} in [Shi74], that is,

$$u_0 := u_{\alpha_1+\alpha_2+\alpha_3}(1) u_{\alpha_1+2\alpha_2+2\alpha_3}(1) u_{\alpha_3+\alpha_4}(1) u_{\alpha_2+2\alpha_3+2\alpha_4}(1) \in \mathbf{U}_0^F \cap \mathsf{F}_4(a_3)^F.$$

(Note that the same choice is made in [MS95, p. 308].) Then $|C_{\mathbf{G}}(u_0)^F| = 6q^{12}$, and the $\mathbf{G}^F$-conjugacy class of u_0 is uniquely determined by this property. Thus, the elements $u \in \mathsf{F}_4(a_3)^F$ for which F acts trivially on $A_{\mathbf{G}}(u)$ are precisely the elements of the $\mathbf{G}^F$-class of u_0; in particular, u_0 must be $\mathbf{G}^F$-conjugate to u_0^{-1}. The $\mathbf{G}^F$-conjugacy classes inside $\mathsf{F}_4(a_3)^F$ are therefore parametrised by the elements of $A_{\mathbf{G}}(u_0)$, and $\mathsf{F}_4(a_3)^F$ splits into three $\mathbf{G}^F$-classes. Let us denote by $\mathscr{E}$ the $\mathbf{G}$-equivariant F-stable irreducible local system on $\mathsf{F}_4(a_3)$ whose isomorphism class corresponds to the sign character ε of $A_{\mathbf{G}}(u_0) \cong \mathfrak{S}_3$. We have

$$A_1 = A_{(1,\lambda^3)} \cong \mathrm{IC}(\overline{\mathsf{F}_4(a_3)}, \mathscr{E})[\dim \mathsf{F}_4(a_3)]^{\#\mathbf{G}}.$$

Let $\varphi_{x_1} \colon F^* A_1 \xrightarrow{\sim} A_1$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of u_0, and let $\chi_{x_1} := \chi_{A_1, \varphi_{x_1}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic function. For $g \in \mathbf{G}^F$, we have

$$\chi_{x_1}(g) = \begin{cases} q^6 \varepsilon(a) & \text{if } g \sim_{\mathbf{G}^F} (u_0)_a \text{ for some } a \in \mathfrak{S}_3, \\ 0 & \text{if } g \notin \mathsf{F}_4(a_3)^F. \end{cases}$$

It is shown in [MS95, 4.1] that $\xi_{x_1} = \xi_{x_1}(\varphi_{x_1}) = 1$.

(b) The (F-stable) unipotent conjugacy class $\mathsf{F}_4(a_1) \subseteq \mathbf{G}$ is uniquely determined by the property that $\dim C_{\mathbf{G}}(u) = 6$ for any $u \in \mathsf{F}_4(a_1)$. Let $u_0 \in \mathsf{F}_4(a_1)^F$ be the element denoted by x_{29} in [Shi74], that is,

$$u_0 := u_{\alpha_2+2\alpha_3+2\alpha_4}(1)u_{\alpha_1}(1)u_{\alpha_2}(1)u_{\alpha_3}(1) \in \mathbf{U}_0^F \cap \mathsf{F}_4(a_1)^F.$$

(The same choice is made in [MS95, p. 308].) Since F acts trivially on $A_{\mathbf{G}}(u_0) = \langle \overline{u}_0 \rangle \cong C_2$, we have $|C_{\mathbf{G}}(u_0)^F| = 2q^6$ and $\mathsf{F}_4(a_1)^F$ splits into two $\mathbf{G}^F$-conjugacy classes. Let $\mathscr{E}$ be the ($\mathbf{G}$-equivariant, F-stable) irreducible local system on $\mathsf{F}_4(a_1)$ described by the non-trivial linear character of $A_{\mathbf{G}}(u_0)$. We have

$$A_2 = A_{(g_2,\varepsilon)} \cong \mathrm{IC}(\overline{\mathsf{F}_4(a_1)}, \mathscr{E})[\dim \mathsf{F}_4(a_1)]^{\#\mathbf{G}}.$$

Let $\varphi_{x_2} \colon F^*A_2 \xrightarrow{\sim} A_2$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of $u_0 \in \mathsf{F}_4(a_1)^F$. Then the characteristic function $\chi_{x_2} := \chi_{A_2,\varphi_{x_2}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ is given by

$$\chi_{x_2}(g) = \begin{cases} q^3 & \text{if } g \in \mathsf{F}_4(a_1)^F \text{ and } g \text{ is } \mathbf{G}^F\text{-conjugate to } u_0, \\ -q^3 & \text{if } g \in \mathsf{F}_4(a_1)^F \text{ and } g \text{ is not } \mathbf{G}^F\text{-conjugate to } u_0, \\ 0 & \text{if } g \notin \mathsf{F}_4(a_1)^F. \end{cases}$$

By [MS95, 4.1], we have $\xi_{x_2} = \xi_{x_2}(\varphi_{x_2}) = 1$.

(c) The (F-stable) unipotent conjugacy class $\mathsf{F}_4(a_2) \subseteq \mathbf{G}$ is uniquely determined by the property that $\dim C_{\mathbf{G}}(u) = 8$ for any $u \in \mathsf{F}_4(a_2)$. Let $u_0 \in \mathsf{F}_4(a_2)^F$ be the element denoted by x_{24} in [Shi74], that is,

$$u_0 := u_{\alpha_1+\alpha_2}(1)u_{\alpha_2+2\alpha_3}(1)u_{\alpha_4}(1)u_{\alpha_3+\alpha_4}(1) \in \mathbf{U}_0^F \cap \mathsf{F}_4(a_2)^F.$$

(The same choice is made in [MS95, p. 308].) Thus, $|C_{\mathbf{G}}(u_0)^F| = 8q^8$, and F acts trivially on $A_{\mathbf{G}}(u_0) \cong D_8$, so $\mathsf{F}_4(a_2)^F$ splits into five $\mathbf{G}^F$-conjugacy classes. Let $\mathscr{E}$ be the ($\mathbf{G}$-equivariant, F-stable) irreducible local system on $\mathsf{F}_4(a_2)$ described by the sign character sgn of $A_{\mathbf{G}}(u_0) \cong D_8$. Then

$$A_3 = A_{(g_2',\varepsilon)} \cong \mathrm{IC}(\overline{\mathsf{F}_4(a_2)}, \mathscr{E})[\dim \mathsf{F}_4(a_2)]^{\#\mathbf{G}}.$$

Let $\varphi_{x_3} \colon F^*A_3 \xrightarrow{\sim} A_3$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of

$u_0 \in \mathsf{F}_4(a_2)^F$. The characteristic function $\chi_{x_3} := \chi_{A_3,\varphi_{x_3}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ is given by

$$\chi_{x_3}(g) = \begin{cases} q^4 \operatorname{sgn}(a) & \text{if } g \sim_{\mathbf{G}^F} (u_0)_a \text{ for some } a \in D_8, \\ 0 & \text{if } g \notin \mathsf{F}_4(a_2)^F. \end{cases}$$

By [MS95, 4.1], we have $\xi_{x_3}(\varphi_{x_3}) = 1$.

(d) Let $\mathscr{C} \subseteq \mathbf{G}$ be the (F-stable) conjugacy class containing elements of the form $su = us$ where $s \in \mathbf{G}^F$ is semisimple such that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_2 \times \mathsf{A}_2$ and $u \in C_{\mathbf{G}}(s)^F$ is a regular unipotent element of $C_{\mathbf{G}}(s)$. Arguing as in [Gec19, 5.2], one finds that there is an element $g_0 = s_0u_0 = u_0s_0 \in \mathscr{C}^F$ where s_0 has the same property as s above (which uniquely determines the $\mathbf{G}^F$-conjugacy class of s_0), and where

$$u_0 := u_{\alpha_1+\alpha_2+\alpha_3}(1)u_{\alpha_1+2\alpha_2+2\alpha_3}(1)u_{\alpha_3+\alpha_4}(1)u_{\alpha_2+2\alpha_3+2\alpha_4}(1) \in \mathbf{U}_0^F \cap C_{\mathbf{G}}(s)^F$$

is as in (a). With this choice, g_0 is $\mathbf{G}^F$-conjugate to g_0^{-1}, and the $\mathbf{G}^F$-class of g_0 is uniquely characterised by the above properties (when $q = p = 2$, it is the one named $\mathsf{12o}$ in the GAP character table). Since $A_{\mathbf{G}}(g_0) = \langle \overline{g}_0 \rangle \cong C_3$, F acts trivially on $A_{\mathbf{G}}(g_0)$ and $\mathscr{C}^F$ splits into three $\mathbf{G}^F$-conjugacy classes. For $i = 1, 2$, let $\mathscr{E}_i$ be the ($\mathbf{G}$-equivariant, F-stable) irreducible local system on $\mathscr{C}$ whose isomorphism class corresponds to the irreducible character of $A_{\mathbf{G}}(g_0)$ which takes the value ω^i at $\overline{g}_0$. Then

$$A_4 = A_{(g_3,\omega)} \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E}_1)[\dim \mathscr{C}]^{\#\mathbf{G}} \quad \text{and} \quad A_5 = A_{(g_3,\omega^2)} \cong \mathrm{IC}(\overline{\mathscr{C}}, \mathscr{E}_2)[\dim \mathscr{C}]^{\#\mathbf{G}}.$$

For $i = 4, 5$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphisms corresponding to the $\mathbf{G}^F$-conjugacy class of $g_0 \in \mathscr{C}^F$, and let $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic functions. The values of χ_{x_4}, χ_{x_5} at elements of $\mathscr{C}^F$ are given by the following table, where we describe the $\mathbf{G}^F$-conjugacy classes inside $\mathscr{C}^F$ by giving the corresponding element of $A_{\mathbf{G}}(g_0)$ at the top of each column.

	1	$\overline{g}_0$	$\overline{g}_0^2$
χ_{x_4}	q^4	ωq^4	$\omega^2 q^4$
χ_{x_5}	q^4	$\omega^2 q^4$	ωq^4

By [Gec19, 5.4], we have $\xi_{x_4}(\varphi_{x_4}) = \xi_{x_5}(\varphi_{x_5}) = 1$.

(e) Consider the regular unipotent class $\mathscr{O}_{\mathrm{reg}} \subseteq \mathbf{G}$, and let us fix the representative

$$u_0 := u_{\alpha_1}(1)u_{\alpha_2}(1)u_{\alpha_3}(1)u_{\alpha_4}(1) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F.$$

(In [Shi74], u_0 is denoted by x_{31}. The same choice is made in [MS95, p. 308].) Since $A_{\mathbf{G}}(u_0) = \langle \overline{u}_0 \rangle \cong C_4$, $\mathscr{O}_{\mathrm{reg}}^F$ splits into four $\mathbf{G}^F$-conjugacy classes. Let $\mathscr{E}_1$ be the $\mathbf{G}$-equivariant F-stable irreducible local system on $\mathscr{O}_{\mathrm{reg}}$ described by the irreducible character of $A_{\mathbf{G}}(u_0)$ which takes the value i at $\overline{u}_0$, and let $\mathscr{E}_2$ be the ($\mathbf{G}$-equivariant, F-stable) irreducible local system on $\mathscr{O}_{\mathrm{reg}}$ described by the irreducible character of $A_{\mathbf{G}}(u_0)$ which takes the value $-$i at $\overline{u}_0$. We have

$$A_6 = A_{(g_4,i)} \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_1)[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}}, \quad A_7 = A_{(g_4,-i)} \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \mathscr{E}_2)[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}}.$$

For $i = 6, 7$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphisms corresponding to the $\mathbf{G}^F$-conjugacy class of u_0, and let $\chi_{x_i} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic functions. The values of χ_{x_6}, χ_{x_7} at elements of $\mathscr{O}_{\mathrm{reg}}^F$ are given by the following table where, as usual, we name the $\mathbf{G}^F$-conjugacy classes inside $\mathscr{O}_{\mathrm{reg}}^F$ by the corresponding element of $A_{\mathbf{G}}(u_0)$.

	1	$\overline{u}_0$	$\overline{u}_0^2$	$\overline{u}_0^3$
χ_{x_6}	q^2	$\mathrm{i}q^2$	$-q^2$	$-\mathrm{i}q^2$
χ_{x_7}	q^2	$-\mathrm{i}q^2$	$-q^2$	$\mathrm{i}q^2$

By [MS95, 4.1], we have $\xi_{x_6}(\varphi_{x_6}) = \xi_{x_7}(\varphi_{x_7}) = 1$.

Assuming again that p is arbitrary, the results of Marcelo–Shinoda [MS95] and Geck [Gec19], [Gec21] on groups of type F_4 described in this section are thus summarised by the following proposition.

Proposition 4.4.7 (Marcelo–Shinoda [MS95, §4], Geck [Gec19, §5], [Gec21, §7])**.** *Let $\mathbf{G}$ be the simple group of type F_4, and let $F \colon \mathbf{G} \to \mathbf{G}$ be a Frobenius map which defines a (non-twisted) $\mathbb{F}_q$-rational structure on $\mathbf{G}$, where q is any power of any prime p. With the choices for the isomorphisms $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ in 4.4.4, 4.4.5, 4.4.6 for $p \geqslant 5$, $p = 3$, $p = 2$, respectively, let $\xi_{x_i} := \xi_{x_i}(\varphi_{x_i}) \in \overline{\mathbb{Q}}_\ell^\times$ be defined by (4.4.1.4) for $1 \leqslant i \leqslant 7$. Then we have*

$$\xi_{x_i} = 1 \quad \textit{for} \quad 1 \leqslant i \leqslant 7,$$

that is, the characteristic function $\chi_{x_i} := \chi_{A_i, \varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ coincides with the unipotent almost character R_{x_i} for $1 \leqslant i \leqslant 7$.

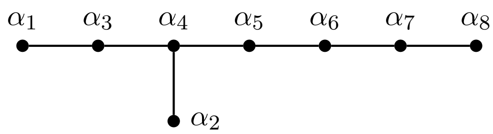
Remark 4.4.8. The problem of determining the values of unipotent characters at unipotent elements for the groups $\mathsf{F}_4(q)$ (where q is any power of any prime p) has been solved by Marcelo and Shinoda in [MS95]. Except when $p = 2$, this follows from the knowledge of the Green functions for groups of type F_4 (see 2.2.5) combined with the knowledge of the scalars ξ_{x_i} in (4.4.1.4) for those cuspidal character sheaves A_i whose support is contained in the unipotent variety. In the case where $p = 2$, one additionally needs to compute the five 'intermediate' unipotent almost characters R_x at unipotent elements, where $x \in \mathfrak{X}(\mathbf{W})$ corresponds to an element of $\mathfrak{S}_{\mathbf{W}}$ as in 4.4.2(b).

Hence, together with our results in Section 4.1 and Section 4.2 (and since the full character tables for groups of type G_2 are known anyway), the above problem is completely solved for the groups $\mathsf{G}_2(q)$, $\mathsf{F}_4(q)$, $\mathsf{E}_6(q)$, $\mathsf{E}_7(q)$.

4.5. Groups of type E_8

In this section, we denote by $\mathbf{G}$ the simple group of type E_8 over $k = \overline{\mathbb{F}}_p$. We assume that $\mathbf{G}$ is defined over $\mathbb{F}_q \subseteq k$ (where q is a power of p) and that $F\colon \mathbf{G} \to \mathbf{G}$ is the corresponding Frobenius map. Let us fix a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{G}$ and an F-stable Borel subgroup $\mathbf{B}_0 \subseteq \mathbf{G}$ which contains $\mathbf{T}_0$. Let $\mathscr{R} = (X, R, Y, R^\vee)$ be the root datum attached to $\mathbf{G}$ and $\mathbf{T}_0$ (so $X = X(\mathbf{T}_0)$ and $Y = Y(\mathbf{T}_0)$), with underlying bilinear pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{Z}$. Furthermore, let $R^+ \subseteq R$ be the positive roots determined by $\mathbf{B}_0 \supseteq \mathbf{T}_0$, and let $\Pi = \{\alpha_1, \ldots, \alpha_8\} \subseteq R^+$ be the corresponding simple roots, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_8^\vee\}$ be the corresponding simple co-roots. We choose the order of $\alpha_1, \ldots, \alpha_8$ in such a way that the Dynkin diagram of $\mathbf{G}$ is as follows:

E_8



Let $\mathfrak{C} = (\langle \alpha_j, \alpha_i^\vee \rangle)_{1 \leqslant i,j \leqslant 8}$ be the associated Cartan matrix. As usual, we denote by $\mathbf{U}_0 = R_{\mathrm{u}}(\mathbf{B}_0)$ the unipotent radical of $\mathbf{B}_0$. For $1 \leqslant i \leqslant 8$, let $u_i := u_{\alpha_i}$ be the homomorphism $\mathbf{G}_{\mathrm{a}} \to \mathbf{G}$ whose image is the root subgroup $\mathbf{U}_{\alpha_i} \subseteq \mathbf{U}_0$ (see 2.1.4). Let $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$. Identifying $\mathbf{W}$ with a subgroup of $\mathrm{Aut}(X)$ (via 2.1.4) and setting $s_i = w_{\alpha_i}$ for $1 \leqslant i \leqslant 8$ (see 2.1.2), $\mathbf{W}$ is thus a Coxeter group with Coxeter generators $S = \{s_1, \ldots, s_8\}$, arranged in the Coxeter diagram with the analogous numbering as in the Dynkin diagram of $\mathbf{G}$ printed above (see 2.1.5). We use the notation of Lusztig [Lus84a, 4.13] for the irreducible characters of $\mathbf{W}$ (which is based on Frame's in [Fra70]); see also [GP00, Table C.6 (pp. 415–416)]. By 2.1.19, F induces a p-isogeny of root data

$$\varphi\colon X \to X, \quad \lambda \mapsto \lambda \circ F|_{\mathbf{T}_0},$$

and a bijection $R \to R$, $\alpha \mapsto \alpha^\dagger$, so that $\varphi(\alpha^\dagger) = q\alpha$ for all $\alpha \in R$ (since $F\colon \mathbf{G} \to \mathbf{G}$ is a Frobenius map with respect to an $\mathbb{F}_q$-rational structure on $\mathbf{G}$). The assignment $\alpha \mapsto \alpha^\dagger$ gives rise (by restriction) to a graph automorphism of the Dynkin diagram, and since the only such automorphism in type E_8 is the identity, we must have $\alpha = \alpha^\dagger$ for all $\alpha \in R$, so $\mathbf{G}^F$ is necessarily the non-twisted group $\mathsf{E}_8(q)$ and $\sigma = \mathrm{id}_{\mathbf{W}}$. We are thus in the setting of Section 3.4 and adopt the further notation from there.

4.5.1. By the results of [Lus79] (see also the appendix of [Lus84a]), we have $|\mathfrak{X}(\mathbf{W})| = 166$, $|\operatorname{Irr}(\mathbf{W})| = 112$, and the irreducible characters $\operatorname{Irr}(\mathbf{W})$ fall into the following 46 families: There are 23 families consisting of a single character, 16 families consisting of 3 characters (the associated sets $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ have 4 elements each), 4 families consisting of 5 characters (the associated sets $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ have 8 elements each), the 2 exceptional families (see 3.4.2) consisting of 2 characters (the associated sets $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ have 4 elements each) and, finally, one family consisting of 17 characters (the associated set $\mathfrak{M}(\mathcal{G}_\mathcal{F})$ has 39 elements). As usual, let us fix the bijections

$$\operatorname{Uch}(\mathbf{G}^F) \xrightarrow{\sim} \mathfrak{X}(\mathbf{W}), \quad \rho \mapsto x_\rho, \tag{4.5.1.1}$$

and

$$\mathfrak{X}(\mathbf{W}) \xrightarrow{\sim} \hat{\mathbf{G}}^{\mathrm{un}}, \quad x \mapsto A_x, \tag{4.5.1.2}$$

through Corollary 3.4.8. We will then also write $\rho_x \in \operatorname{Uch}(\mathbf{G}^F)$ for the unipotent character parametrised by $x \in \mathfrak{X}(\mathbf{W})$ under (4.5.1.1). There are exactly 13 cuspidal character sheaves $A_1, A_2, \ldots, A_{13}$ in $\hat{\mathbf{G}}$, and all of them lie in $\hat{\mathbf{G}}^{\mathrm{un}}$, see [LuCS4, §21] and [Sho95b, 4.7, 5.3]. In particular, they are all F-stable (see 3.4.1). The elements of $\mathfrak{X}(\mathbf{W})$ which label cuspidal character sheaves under the parametrisation (4.5.1.2) are given as follows: They are all in the same $\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})$ where $\mathcal{F}_{16} \subseteq \operatorname{Irr}(\mathbf{W})$ is the unique family in $\mathbf{W} = W(\mathsf{E}_8)$ which contains 17 irreducible characters. We have $\mathcal{G}_{\mathcal{F}_{16}} = \mathfrak{S}_5$ and $|\mathfrak{M}(\mathcal{G}_{\mathcal{F}_{16}})| = 39$. Let us fix primitive roots of unity $\omega, \mathrm{i}, \zeta_5 \in \mathcal{R}$ of order $3, 4, 5$, respectively, which we always assume to be the same as the ones in 3.4.2. As in [Sho95b, 4.7], we set

$$\begin{gathered}
x_1 = (1, \lambda^4), \quad x_2 = (g_2, -\varepsilon), \quad x_3 = (g_3, -\omega), \quad x_4 = (g_3, -\omega^2), \quad x_5 = (g_4, \mathrm{i}), \\
x_6 = (g_4, -\mathrm{i}), \quad x_7 = (g_5, \zeta_5), \quad x_8 = (g_5, \zeta_5^2), \quad x_9 = (g_5, \zeta_5^3), \\
x_{10} = (g_5, \zeta_5^4), \quad x_{11} = (g_6, -\omega), \quad x_{12} = (g_6, -\omega^2), \quad x_{13} = (g_2', \varepsilon),
\end{gathered}$$

where the notation for the labels in $\mathfrak{M}(\mathfrak{S}_5)$ is essentially the same as the one in [Lus84a, 4.3]: So 1 denotes the trivial element of $\mathfrak{S}_5$; for $2 \leqslant j \leqslant 5$, g_j is a j-cycle in $\mathfrak{S}_5$, and we assume that g_2 and g_3 have disjoint supports, so that $g_6 := g_2 g_3 \in \mathfrak{S}_5$ has order 6; finally,

$g_2' \in \mathfrak{S}_5$ is the product of two 2-cycles with disjoint support. We keep these elements fixed from now on. Then the irreducible characters of their centralisers are named as follows: First, let λ^4 be the sign character of $\mathfrak{S}_5 = C_{\mathfrak{S}_5}(1)$. If $g \in \{g_4, g_5\}$, we have $C_{\mathfrak{S}_5}(g) = \langle g \rangle$, and we identify the irreducible characters of this group with their values at g. If $g \in \{g_3, g_6\}$, we have $C_{\mathfrak{S}_5}(g) = \langle g_6 \rangle = \langle g_2 \rangle \times \langle g_3 \rangle$, and we denote the irreducible characters of this group by their values at g_6 (i.e., by $\pm 1, \pm\omega, \pm\omega^2$). The centraliser of g_2 can be canonically identified with $\langle g_2 \rangle \times \mathfrak{S}_3$, and we write the characters of this group as

$$\pm 1 := (\pm 1) \boxtimes 1,\ \pm\varepsilon := (\pm 1) \boxtimes \varepsilon,\ \pm r := (\pm 1) \boxtimes r,$$

where the first factor gives the value at g_2, and where $1, \varepsilon, r \in \mathrm{Irr}(\mathfrak{S}_3)$ are the trivial, sign, reflection characters, respectively. Finally, we have $C_{\mathfrak{S}_5}(g_2') \cong D_8$ (the dihedral group of order 8). Since of the 13 pairs above, $x_{13} = (g_2', \varepsilon)$ is the only one whose first component is g_2', it will be sufficient for our purposes to say that ε is the unique non-trivial linear character of $C_{\mathfrak{S}_5}(g_2')$ which takes the value 1 at 4-cycles (and refer to [Lus84a, 4.3] or Appendix B for the notation regarding the other elements of $\mathfrak{M}(\mathfrak{S}_5)$). We number the cuspidal character sheaves $A_1, A_2, \ldots, A_{13}$ in such a way that

$$A_i = A_{x_i} \quad \text{for} \quad 1 \leqslant i \leqslant 13.$$

The $\tilde{\lambda}_{x_i} \in \overline{\mathbb{Q}}_\ell$ (see 3.4.2) are given by

$$\tilde{\lambda}_{x_1} = 1, \quad \tilde{\lambda}_{x_2} = -1, \quad \tilde{\lambda}_{x_3} = \omega, \quad \tilde{\lambda}_{x_4} = \omega^2, \quad \tilde{\lambda}_{x_5} = \mathrm{i}, \quad \tilde{\lambda}_{x_6} = -\mathrm{i},$$
$$\tilde{\lambda}_{x_7} = \zeta_5, \quad \tilde{\lambda}_{x_8} = \zeta_5^2, \quad \tilde{\lambda}_{x_9} = \zeta_5^3, \quad \tilde{\lambda}_{x_{10}} = \zeta_5^4, \quad \tilde{\lambda}_{x_{11}} = -\omega, \quad \tilde{\lambda}_{x_{12}} = -\omega^2, \quad \tilde{\lambda}_{x_{13}} = 1.$$

We note that our numbering of the A_i coincides with the one in [Sho95b, 4.7, 5.3], in any characteristic. (This is not quite implied by the property $\lambda_{A_i} = \tilde{\lambda}_{x_i}$ for $1 \leqslant i \leqslant 13$, as $\tilde{\lambda}_{x_1} = \tilde{\lambda}_{x_{13}} = 1$, but it can be deduced from the tables in [DLM14, Appendix C].) On the other hand, using the notation of [Lus84a, p. 370] for the cuspidal unipotent characters of $\mathbf{G}^F$, we have

$$\rho_{x_1} = \mathsf{E}_8^{\mathrm{II}}[1], \quad \rho_{x_2} = \mathsf{E}_8[-1], \quad \rho_{x_3} = \mathsf{E}_8[\omega], \quad \rho_{x_4} = \mathsf{E}_8[\omega^2], \quad \rho_{x_5} = \mathsf{E}_8[\mathrm{i}],$$
$$\rho_{x_6} = \mathsf{E}_8[-\mathrm{i}], \quad \rho_{x_7} = \mathsf{E}_8[\zeta_5], \quad \rho_{x_8} = \mathsf{E}_8[\zeta_5^2], \quad \rho_{x_9} = \mathsf{E}_8[\zeta_5^3],$$
$$\rho_{x_{10}} = \mathsf{E}_8[\zeta_5^4], \quad \rho_{x_{11}} = \mathsf{E}_8[-\omega], \quad \rho_{x_{12}} = \mathsf{E}_8[-\omega^2], \quad \rho_{x_{13}} = \mathsf{E}_8^{\mathrm{I}}[1].$$

For any $x \in \mathfrak{X}(\mathbf{W})$, let us for now fix an isomorphism $\varphi_x \colon F^*A_x \xrightarrow{\sim} A_x$ as in 3.2.1(∗) (for some of the cuspidal character sheaves, we will make an explicit choice later, but this depends on the characteristic p of k) and denote by

$$\chi_x := \chi_{A_x,\varphi_x} : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$$

the associated characteristic function. So by (3.4.12.2), we have

$$R_x = \xi_x\chi_x \text{ for } x \in \mathfrak{X}(\mathbf{W}), \text{ where } \xi_x = \xi_x(\varphi_x) \in \overline{\mathbb{Q}}_\ell^\times, \ |\xi_x| = 1. \tag{4.5.1.3}$$

As in the previous sections, we aim to get hold of the scalars ξ_x (after having chosen specific isomorphisms φ_x) for $x \in \{x_1, x_2, \ldots, x_{13}\}$. However, this endeavour appears to be considerably more difficult than for the other simple algebraic groups of exceptional type. Especially regarding those cuspidal character sheaves whose support is the closure of a non-unipotent conjugacy class of $\mathbf{G}$, the tools used in this work do not seem to be sufficient. One reason for this is the following: For example, recall that in the cases of E_6 with $p = 3$ and E_7 with $p = 2$, we exploited the formula (2.3.9.2) for a suitable element $u \in \mathbf{G}^F_{\text{uni}}$ in the support of a given cuspidal character sheaf A_i to obtain information on the unknown scalar ξ_{x_i}, but this heavily relied on the fact that we knew from the outset that many of the characteristic functions χ_x ($x \in \mathfrak{X}(\mathbf{W})$) vanish at u; see Corollary 3.2.8 or Remark 3.4.24. These resources are only available for elements in $\mathbf{G}^F_{\text{uni}}$ however. Hence, for the cuspidal character sheaves with a non-unipotent support, a different method seems to be necessary. One might try to find arguments along the lines of [Gec21], but at the very least the computations there will presumably be more elaborate. We hope to treat this elsewhere.

4.5.2. Following [Lus79], [Lus80, 1.12], let us explain how the 166 unipotent characters of $\mathbf{G}^F$ (respectively, the 166 unipotent character sheaves on $\mathbf{G}$) fall into Harish-Chandra series by describing the set $\mathfrak{S}_\mathbf{W}$ (see Corollary 3.4.8). The set $\mathfrak{S}^\circ_{\mathbf{W}_J}$ is non-empty for the following subsets $J \subseteq S$: $J = \varnothing$, $J = \{s_2, s_3, s_4, s_5\}$, $J = \{s_1, s_2, \ldots, s_6\}$, $J = \{s_1, s_2, \ldots, s_7\}$ and $J = S$.

(a) The set $J = \varnothing$ gives rise to the 112 elements in the principal series, that is, the elements in the image of the embedding $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{S}_\mathbf{W}$, $\phi \mapsto (\varnothing, \phi, (1,1))$.

(b) Let $J = \{s_2, s_3, s_4, s_5\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type D_4. We have $\mathfrak{S}^\circ_{\mathbf{W}_J} = \{(-1,2)\}$, and the relative Weyl group $W_\mathbf{G}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is isomorphic to $W(\mathsf{F}_4)$. So there are 25 elements in $\mathfrak{S}_\mathbf{W}$ of the form $(J, \epsilon, (-1,2))$, $\epsilon \in \mathrm{Irr}(W_\mathbf{G}(\mathbf{L}_J))$.

(c) Let $J = \{s_1, s_2, \ldots, s_6\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type E_6. We have $\mathfrak{S}^{\circ}_{\mathbf{W}_J} = \{(\omega, 3), (\omega^2, 3)\}$, and the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is isomorphic to $W(\mathsf{G}_2)$. This leads to the 12 elements $(J, \epsilon, (\omega, 3))$, $(J, \epsilon, (\omega^2, 3))$ of $\mathfrak{S}_{\mathbf{W}}$ where ϵ runs through the irreducible characters of $W_{\mathbf{G}}(\mathbf{L}_J)$.

(d) Let $J = \{s_1, s_2, \ldots, s_7\} \subseteq S$, so that the group $\mathbf{L}_J/\mathbf{Z}(\mathbf{L}_J)$ is simple of type E_7. In this case, we have $\mathfrak{S}^{\circ}_{\mathbf{W}_J} = \{(\mathrm{i}, 2), (-\mathrm{i}, 2)\}$, and the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is isomorphic to $W(\mathsf{A}_1) \cong C_2$. This leads to the 4 elements $(J, \pm 1, (\mathrm{i}, 2))$, $(J, \pm 1, (-\mathrm{i}, 2))$ of $\mathfrak{S}_{\mathbf{W}}$.

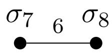
(e) For $J = S$, the set $\mathfrak{S}^{\circ}_{\mathbf{W}_J} = \mathfrak{S}^{\circ}_{\mathbf{W}}$ consists of the 13 elements $(1, 8)$, $(1, 120)$, $(-1, 12)$, $(\mathrm{i}, 4)$, $(-\mathrm{i}, 4)$, $(\omega, 6)$, $(-\omega, 6)$, $(\omega^2, 6)$, $(-\omega^2, 6)$, $(\zeta_5, 5)$, $(\zeta_5^2, 5)$, $(\zeta_5^3, 5)$, $(\zeta_5^4, 5)$ parametrising the 13 cuspidal unipotent characters of $\mathbf{G}^F$ (and the 13 cuspidal unipotent character sheaves on $\mathbf{G}$).

If no confusion may arise, it will sometimes be convenient to use the following notation below: We write $J = \mathsf{D}_4$, $J = \mathsf{E}_6$, $J = \mathsf{E}_7$, $J = \mathsf{E}_8$ if J is as in (b), (c), (d), (e), respectively. As observed in [GH22], there are two different conventions in the existing literature on finite groups of Lie type as far as the labelling of characters of Weyl groups of type F_4 are concerned. (We have already encountered this when dealing with the twisted groups of type E_6 in 4.1.5.) A similar thing happens for Weyl groups of type G_2. Since $W(\mathsf{F}_4)$ and $W(\mathsf{G}_2)$ occur as relative Weyl groups in (b) and (c) above, let us now specify our conventions for these groups.

4.5.3. Let $J = \{s_1, s_2, \ldots, s_6\} \subseteq S$. Thus, the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is of type G_2. Recall from 3.4.2 that $\mathbf{W}^{S/J}$ is a Coxeter group with Coxeter generators

$$\sigma_i := w_0^{J \cup \{s_i\}} w_0^J = w_0^J w_0^{J \cup \{s_i\}} \quad \text{for } i = 7, 8,$$

so that the associated Coxeter diagram is

G_2 

We note that this is compatible with [Lus84b, Thm. 9.2(a)] in the sense that we have a canonical isomorphism $\mathbf{W}^{S/J} \cong W_{\mathbf{G}}(\mathbf{L}_J)$ under which σ_i corresponds to the unique non-trivial element of $N_{\mathbf{L}_{J\cup\{s_i\}}}(\mathbf{L}_J)/\mathbf{L}_J$ for $i = 7, 8$. (The latter will be relevant when referring to the generalised Springer correspondence in characteristic 3 below; see [Lus19] and [Het22b] (Theorem 4.5.13), where the same convention as above is made.) With this notation, we use the following names for the irreducible characters of $\mathbf{W}^{S/J} \cong W_{\mathbf{G}}(\mathbf{L}_J)$, as in [Het22b, 4.6]: Let 1 be the trivial character and sgn the sign character; let ϵ, ϵ' be the two remaining irreducible characters of degree 1 such that

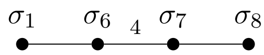
$$\epsilon(\sigma_7) = \epsilon'(\sigma_8) = 1, \quad \epsilon(\sigma_8) = \epsilon'(\sigma_7) = -1;$$

finally, let ρ be the character of the reflection representation and ρ' be the other irreducible character of degree 2.

4.5.4. Let $J = \{s_2, s_3, s_4, s_5\} \subseteq S$. Thus, the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ is of type F_4. Denoting by

$$\sigma_i := w_0^{J\cup\{s_i\}} w_0^J = w_0^J w_0^{J\cup\{s_i\}} \quad \text{for } i = 1, 6, 7, 8$$

the Coxeter generators of $\mathbf{W}^{S/J}$ (see 3.4.2), the associated Coxeter diagram is as follows (cf. [Lus76b, p. 154]):



Again, this is compatible with [Lus84b, Thm. 9.2(a)] in the sense as explained in 4.5.3. As in 4.1.5 and Section 4.4, we want to be able to refer to Kondo's character table [Kon65] of the Weyl group of type F_4. Recall that Kondo denotes the Coxeter generators by $\tau\sigma, \tau, a, d$ (with the corresponding nodes in the associated Coxeter diagram arranged accordingly), but there are two possibilities to match these Coxeter generators with the σ_i, $i \in \{1, 6, 7, 8\}$. In order to conform with the notation of Lusztig [Lus84a, p. 361, 4.10], we choose

$$\sigma_1 \leftrightarrow \tau\sigma, \quad \sigma_6 \leftrightarrow \tau, \quad \sigma_7 \leftrightarrow a, \quad \sigma_8 \leftrightarrow d.$$

Having fixed this convention, we will from now on denote by d_j the jth irreducible character of degree d in Kondo's table when referring to $\mathrm{Irr}(\mathbf{W}^{S/J})$. On the other hand, while Spaltenstein [Spa85] also refers to Kondo's table for $\mathrm{Irr}(\mathbf{W}^{S/J})$, he does not explicitly provide a correspondence between his Coxeter generators of $\mathbf{W}^{S/J}$ and Kondo's $\tau\sigma, \tau, a, d$; such a correspondence only implicitly follows from the references and results of [Spa85]. As observed in [GH22, §6, Summary B], it turns out that the tables in [Spa85] with respect to $\mathbf{L}_J \subseteq \mathbf{G}$ as above do in fact match with our conventions; see also Remark 4.5.33 below.

4.5.5. As indicated in 4.5.1 (see also the introduction to this chapter), we will only focus on those cuspidal character sheaves on $\mathbf{G}$ whose support is given by the closure of a *unipotent* conjugacy class of $\mathbf{G}$. In good characteristic (that is, $p \geqslant 7$), there is only one such character sheaf, namely, $A_1 = A_{(1,\lambda^4)}$. In fact, the cuspidal pair corresponding to this character sheaf (via Proposition 3.1.17) can be described in a uniform way for all characteristics (see [Sho95b, 4.7, 5.2]). So before splitting our consideration according to whether $p \geqslant 7$, $p = 5$, $p = 3$ or $p = 2$ and giving the lists containing the correspondence between the cuspidal (unipotent) character sheaves and the cuspidal pairs for $\mathbf{G}$, let us first look at the character sheaf A_1, without any restriction on p. The unipotent conjugacy classes for groups of type E_8 in bad characteristic have been classified by Mizuno [Miz80]; as noted in 2.1.11(d), the unipotent classes which appear in good characteristic permit an analogous parametrisation in bad characteristic, so we may still use the notation of [Car85, Chap. 5] for these classes, regardless of p.

Let us consider the (F-stable) unipotent conjugacy class $\mathsf{E}_8(a_7) \subseteq \mathbf{G}$ (in [Miz80], this class is denoted by $2\mathsf{A}_4$). For $u \in \mathsf{E}_8(a_7)$, we have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_5$. Following Mizuno [Miz80, Lm. 70] (see also [LS12, p. 361]), there exists an element $z_{117} \in \mathsf{E}_8(a_7)^F$ such that

$$|C_{\mathbf{G}}(z_{117})^F| = 120q^{40} \quad \text{(so } F \text{ acts trivially on } A_{\mathbf{G}}(z_{117}) \cong \mathfrak{S}_5\text{)},$$

and among the $\mathbf{G}^F$-conjugacy classes inside $\mathsf{E}_8(a_7)^F$, the class of z_{117} is uniquely determined by this property. In particular, since $C_{\mathbf{G}}(z_{117}) = C_{\mathbf{G}}(z_{117}^{-1})$, we conclude that z_{117} must be $\mathbf{G}^F$-conjugate to z_{117}^{-1}. So in good characteristic, z_{117} is a split unipotent element of $\mathsf{E}_8(a_7)^F$; but also if $p \leqslant 5$, the choice of z_{117} meets the requirements for a 'good' representative in 3.2.22.

Now let $\mathscr{E}$ be the F-stable cuspidal local system on $\mathsf{E}_8(a_7)$ whose isomorphism class is parametrised by the sign character of $A_{\mathbf{G}}(z_{117}) \cong \mathfrak{S}_5$. We have

$$A_1 = A_{(1,\lambda^4)} \cong \mathrm{IC}(\overline{\mathsf{E}_8(a_7)}, \mathscr{E})[\dim \mathsf{E}_8(a_7)]^{\#\mathbf{G}} \in \hat{\mathbf{G}}^{\circ,\mathrm{un}}.$$

Let $\varphi_{x_1} \colon F^*A_1 \xrightarrow{\sim} A_1$ be the isomorphism corresponding to the choice of z_{117} (see 3.2.21), and let $\chi_{x_1} := \chi_{A_1,\varphi_{x_1}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ be the associated characteristic function; we then have $\chi_{x_1}(z_{117}) = q^{20}$. From 3.4.18, we see that $\xi_{x_1} \in \{\pm 1\}$. As in 3.4.19 (and with the notation there), we can write any unipotent character $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ as a linear combination of unipotent almost characters (or of characteristic functions of character sheaves):

$$\Delta(x_\rho)\cdot\rho = \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_\rho, x_\phi\} R_\phi + \sum_{x\in\mathfrak{X}'(\mathbf{W})} \{x_\rho, x\}\xi_x\chi_x + \sum_{x\in\mathfrak{X}^\circ(\mathbf{W})} \{x_\rho, x\}\xi_x\chi_x.$$

By inspection of the Fourier matrix, we see that we can actually find several $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ so that $\{x_\rho, x_1\} \neq 0$ and $\{x_\rho, x\} = 0$ for all $x \in \mathfrak{X}'(\mathbf{W})$. For instance, we may take

$\rho := [420_y]$ and obtain

$$\begin{aligned}[420_y](z_{117}) &= \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_{420_y}, x_\phi\} R_\phi(z_{117}) + \{x_{420_y}, x_1\}\xi_{x_1}\chi_{x_1}(z_{117}) \\ &= \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_{420_y}, x_\phi\} R_\phi(z_{117}) + \tfrac{1}{5}\xi_{x_1} q^{20}.\end{aligned}$$

On the other hand, taking $\rho := \mathsf{E}_8[\mathrm{i}]$, we find that

$$\mathsf{E}_8[\mathrm{i}](z_{117}) = \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_{\mathsf{E}_8[\mathrm{i}]}, x_\phi\} R_\phi(z_{117}) - \tfrac{1}{4}\xi_{x_1} q^{20}.$$

We have

$$\xi_{x_1} q^{20} \in \mathbb{Z} \quad \text{and} \quad \{x_{420_y}, x_\phi\} \in \mathbb{Q},\ R_\phi(z_{117}) \in \mathbb{Q} \quad \text{for all } \phi \in \mathrm{Irr}(\mathbf{W}),$$

so $[420_y](z_{117}) \in \mathbb{Q}$. But at the same time, we know that character values are algebraic integers, so we deduce that $[420_y](z_{117}) \in \mathbb{Z}$. This shows, first of all, that

$$5 \cdot \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_{420_y}, x_\phi\} R_\phi(z_{117}) = 5 \cdot [420_y](z_{117}) - \xi_{x_1} q^{20} \in \mathbb{Z}.$$

Now, if $p \neq 5$, we have $q^{20} \equiv 1 \pmod 5$. We thus obtain

$$\xi_{x_1} \equiv \xi_{x_1} q^{20} \equiv -5 \cdot \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_{420_y}, x_\phi\} R_\phi(z_{117}) \pmod 5 \qquad \text{if } p \neq 5. \tag{4.5.5.1}$$

In the case where $p = 5$, we consider $\mathsf{E}_8[\mathrm{i}]$ instead. Since q is a power of 5, we certainly have $q^{20} \equiv 1 \pmod 4$, so an argument entirely analogous to the one for $[420_y]$ yields that

$$\xi_{x_1} \equiv \xi_{x_1} q^{20} \equiv 4 \cdot \sum_{\phi\in\mathrm{Irr}(\mathbf{W})} \{x_{\mathsf{E}_8[\mathrm{i}]}, x_\phi\} R_\phi(z_{117}) \pmod 4 \qquad \text{if } p = 5. \tag{4.5.5.2}$$

Hence, for a given p (and q), the knowledge of the values $R_\phi(z_{117})$ for $\phi \in \mathrm{Irr}(\mathbf{W})$ immediately gives rise to the sign $\xi_{x_1} \in \{\pm 1\}$ in (4.5.1.3), using either (4.5.5.1) or (4.5.5.2) above. (But recall from 2.2.5 that the Green functions are not yet completely known for groups of type E_8 in characteristic $p \leqslant 5$.)

Type E_8 in characteristic $p \geqslant 7$

In this subsection (i.e., in 4.5.6–4.5.7 below), we assume that $p \geqslant 7$, that is, p is a good prime for $\mathbf{G}$.

4.5.6. By [LuCS4, 21.2], [Sho95b, 4.7] and [DLM14, Appendix C], the following list describes the correspondence between the 13 cuspidal (unipotent) character sheaves on $\mathbf{G}$ and the cuspidal pairs for $\mathbf{G}$; we use the notation of Remark 3.1.18 and the convention that 's' is always a semisimple element of $\mathbf{G}$ and 'u' is always a unipotent element of $\mathbf{G}$. The names of the unipotent classes are as in [Car85, Chap. 5], but we also provide the ones of [Miz80] in case they are different.

(a) $A_1 \leftrightarrow (u, \varsigma)$ where u is an element of the class $\mathsf{E}_8(a_7)$ (named $2\mathsf{A}_4$ in [Miz80]). We have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_5$ and, under this identification, ς corresponds to the sign character of $A_{\mathbf{G}}(u)$.

(b) $A_2 \leftrightarrow (su, \varsigma)$ where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{E}_7$, and the class of u in $C_{\mathbf{G}}(s)$ is of type $\mathscr{O}_{\mathrm{reg}}^{\mathsf{A}_1} \times \mathsf{E}_7(a_5)$. (In [Miz80], the class $\mathsf{E}_7(a_5)$ is denoted by $\mathsf{D}_6(a_2)+\mathsf{A}_1$.) We have $A_{\mathbf{G}}(su) \cong \mathfrak{S}_3 \times C_2$ and, under this identification, ς corresponds to the character $\varepsilon \boxtimes (-1)$, with ε denoting the sign character of $\mathfrak{S}_3$ and -1 the non-trivial linear character of C_2.

(c) $A_j \leftrightarrow (su, \varsigma_j)$, $j = 3, 4$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_2 \times \mathsf{E}_6$, and the class of u in $C_{\mathbf{G}}(s)$ is of type $\mathscr{O}_{\mathrm{reg}}^{\mathsf{A}_2} \times \mathsf{E}_6(a_3)$. (In [Miz80], the class $\mathsf{E}_6(a_3)$ is denoted by $\mathsf{A}_5{+}\mathsf{A}_1$.) We have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_6$, and ς_3, ς_4 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_3(\overline{su}) = -\omega$, $\varsigma_4(\overline{su}) = -\omega^2$.

(d) $A_j \leftrightarrow (su, \varsigma_j)$, $j = 5, 6$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_3 \times \mathsf{D}_5$, and the class of u in $C_{\mathbf{G}}(s)$ is of type $\mathscr{O}_{\mathrm{reg}}^{\mathsf{A}_3} \times (3, 7)$ (where we write $(3, 7)$ for the unipotent class with this Jordan type in groups of type D_5). We have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_4$, and ς_5, ς_6 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_5(\overline{su}) = \mathrm{i}$, $\varsigma_6(\overline{su}) = -\mathrm{i}$.

(e) $A_j \leftrightarrow (su, \varsigma_j)$, $7 \leqslant j \leqslant 10$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_4 \times \mathsf{A}_4$, and u is a regular unipotent element of $C_{\mathbf{G}}(s)$. We have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_5$, and ς_j are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_j(\overline{su}) = \zeta_5^{j-6}$ for $7 \leqslant j \leqslant 10$.

(f) $A_j \leftrightarrow (su, \varsigma_j)$, $j = 11, 12$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{A}_2 \times \mathsf{A}_5$, and u is a regular unipotent element of $C_{\mathbf{G}}(s)$. We have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle = \langle \overline{s} \rangle \cong C_6$, and ς_{11}, ς_{12} are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_{11}(\overline{su}) = -\omega$, $\varsigma_{12}(\overline{su}) = -\omega^2$.

(g) $A_{13} \leftrightarrow (su, \varsigma)$ where $C_{\mathbf{G}}(s)$ has a root system of type D_8, and the conjugacy class of u in $C_{\mathbf{G}}(s)$ corresponds to the (unipotent) class with Jordan type $(1, 3, 5, 7)$ in groups of type D_8. We have $A_{\mathbf{G}}(su) \cong D_8$ (the dihedral group of order 8) and, under this identification, ς corresponds to the sign character of D_8.

In particular (as already mentioned in 4.5.5), A_1 is the only cuspidal character sheaf on $\mathbf{G}$ whose support contains unipotent elements.

Proposition 4.5.7. *Assume that $p \geqslant 7$. As in* 4.5.5*, let $\varphi_{x_1}: F^*A_1 \xrightarrow{\sim} A_1$ be defined with respect to the split unipotent element $z_{117} \in \mathsf{E}_8(a_7)^F$ (via* 3.2.21*). Then we have*

$$\xi_{x_1}(\varphi_{x_1}) = +1$$

in (4.5.1.3)*, that is, the characteristic function $\chi_{x_1} := \chi_{A_1,\varphi_{x_1}} : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ coincides with the unipotent almost character R_{x_1}.*

Proof. We apply (4.5.5.1). The values of the Green functions for groups of type E_8 in good characteristic are known and available via Lübeck's electronic library [Lüb]. We obtain

$$-5 \cdot \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \{x_{420_y}, x_\phi\} R_\phi(z_{117}) = q^{16}.$$

Hence, by (4.5.5.1), the sign $\xi_{x_1} \in \{\pm 1\}$ satisfies

$$\xi_{x_1} \equiv q^{16} \equiv 1 \pmod 5,$$

which proves that $\xi_{x_1} = +1$. □

Type E_8 in characteristic $p = 5$

In this subsection (i.e., in 4.5.8–4.5.11 below), we assume that $p = 5$.

4.5.8. By [Sho95b, 5.1], the description of the cuspidal character sheaves in (a), (b), (c), (d), (f) and (g) of 4.5.6 for $p \geqslant 7$ transfers to this case in exactly the same way; the remaining four cuspidal character sheaves are as follows (with the same conventions as in 4.5.6):

(e) $A_j \leftrightarrow (u, \varsigma_j)$, $j = 7, 8, 9, 10$, where $u \in \mathscr{O}_{\mathrm{reg}}$ is a regular unipotent element of $\mathbf{G}$. We have $A_{\mathbf{G}}(u) = \langle \overline{u} \rangle \cong C_5$, and ς_j is the linear character of $A_{\mathbf{G}}(u)$ which satisfies $\varsigma_j(\overline{u}) = \zeta_5^{j-6}$ (for $7 \leqslant j \leqslant 10$).

We consider the case (e) and set

$$z_1 := u_1(1) \cdot u_2(1) \cdot u_3(1) \cdot u_4(1) \cdot u_5(1) \cdot u_6(1) \cdot u_7(1) \cdot u_8(1) \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F. \qquad (4.5.8.1)$$

(Mizuno [Miz80, Lm. 37] defines z_1 in a slightly different way, but it is $\mathbf{G}^F$-conjugate to our chosen representative regardless of which Chevalley basis in the Lie algebra underlying $\mathbf{G}$ we choose, as one sees with an argument similar to that in Lemma 4.1.13 or Lemma 4.2.8.)

Lemma 4.5.9. *The element* $z_1 \in \mathbf{U}_0^F \cap \mathscr{O}_{\mathrm{reg}}^F$ *defined in* (4.5.8.1) *is* $\mathbf{G}^F$*-conjugate to* z_1^{-1}*.*

Proof. This is similar to the proof of Lemma 4.1.13 (cf. also Lemma 4.2.8). Indeed, taking

$$t := \alpha_3^\vee(-1)\alpha_4^\vee(-1)\alpha_7^\vee(-1)\alpha_8^\vee(-1) \in \mathbf{T}_0^F,$$

we have

$$tz_1t^{-1} = u_1(-1) \cdot u_2(-1) \cdot \ldots \cdot u_8(-1).$$

In order to reverse the order of the $u_i(-1)$ in this expression, we again mimic the proof of [Cas17, 1.4] (using elements of $\mathbf{U}_0^F$ of the form $u_i(1)$ and $u_i(-1) = u_i(1)^{-1}$). Specifically, setting $u_i := u_i(1)$ for $1 \leqslant i \leqslant 8$ and

$$u := u_8^{-1}u_7^{-1}u_8^{-1}u_6^{-1}u_7^{-1}u_8^{-1}u_5^{-1}u_6^{-1}u_7^{-1}u_8^{-1}u_4^{-1}u_2^{-1}u_5^{-1}u_6^{-1}u_7^{-1}u_8^{-1}u_2u_1,$$

we have $(ut)z_1(ut)^{-1} = z_1^{-1}$ and $u \in \mathbf{U}_0^F$, so $ut \in \mathbf{B}_0^F \subseteq \mathbf{G}^F$, as desired. □

Proposition 4.5.10. *Assume that* $p = 5$*. For* $7 \leqslant i \leqslant 10$*, let* $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ *be the isomorphism corresponding to the* $\mathbf{G}^F$*-conjugacy class of the element* $z_1 \in \mathscr{O}_{\mathrm{reg}}^F$ *defined in* (4.5.8.1) *(via* 3.2.21*). Then the scalars* $\xi_{x_i} = \xi_{x_i}(\varphi_{x_i})$ *in* (4.5.1.3) *are given by*

$$\xi_{x_7} = \xi_{x_8} = \xi_{x_9} = \xi_{x_{10}} = +1,$$

that is, the characteristic function $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ *coincides with the unipotent almost character* R_{x_i} *for* $i = 7, 8, 9, 10$*.*

Proof. By 3.2.21, the values of the characteristic functions χ_{x_i} ($7 \leqslant i \leqslant 10$) at elements of $\mathscr{O}_{\mathrm{reg}}^F$ are given by the following table where, as usual, we describe the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$ by giving the corresponding element of $A_{\mathbf{G}}(z_1) = \langle \overline{z}_1 \rangle \cong C_5$ at the top of each column.

	1	$\overline{z}_1$	$\overline{z}_1^2$	$\overline{z}_1^3$	$\overline{z}_1^4$
χ_{x_7}	q^4	$\zeta_5 q^4$	$\zeta_5^2 q^4$	$\zeta_5^3 q^4$	$\zeta_5^4 q^4$
χ_{x_8}	q^4	$\zeta_5^2 q^4$	$\zeta_5^4 q^4$	$\zeta_5 q^4$	$\zeta_5^3 q^4$
χ_{x_9}	q^4	$\zeta_5^3 q^4$	$\zeta_5 q^4$	$\zeta_5^4 q^4$	$\zeta_5^2 q^4$
$\chi_{x_{10}}$	q^4	$\zeta_5^4 q^4$	$\zeta_5^3 q^4$	$\zeta_5^2 q^4$	$\zeta_5 q^4$

Now let $\mathcal{U}' := \mathrm{Uch}(\mathbf{G}^F) \setminus \{\mathsf{E}_8[\zeta_5], \mathsf{E}_8[\zeta_5^2], \mathsf{E}_8[\zeta_5^3], \mathsf{E}_8[\zeta_5^4]\}$. The unipotent almost characters R_{x_i} are given by

$$R_{x_i} = \sum_{j=1}^{4} \{x_{j+6}, x_i\} \cdot \mathsf{E}_8[\zeta_5^j] + \sum_{\rho \in \mathcal{U}'} \{x_\rho, x_i\} \cdot \rho \quad \text{for } 7 \leqslant i \leqslant 10.$$

We have already seen in 3.4.11 that $\overline{\mathsf{E}_8[\zeta_5^j]} = \mathsf{E}_8[\zeta_5^{-j}]$ for $1 \leqslant j \leqslant 4$ and that $\rho \mapsto \overline{\rho}$ restricts to a bijection on $\mathcal{U}'$. Furthermore, for any $\rho \in \mathcal{U}'$ and any $i \in \{1,2,3,4\}$, we have $\{x_\rho, x_i\} = \{x_{\overline{\rho}}, x_i\}$. We deduce that

$$\overline{R}_{x_i} = \sum_{j=1}^{4} \{x_{j+6}, x_i\} \cdot \mathsf{E}_8[\zeta_5^{-j}] + \sum_{\rho \in \mathcal{U}'} \{x_\rho, x_i\} \cdot \rho \quad \text{for } 7 \leqslant i \leqslant 10.$$

By inspection of the values $\{x_{j+6}, x_i\}$, we conclude that

$$\overline{R}_{x_7} = R_{x_{10}} \quad \text{and} \quad \overline{R}_{x_8} = R_{x_9}.$$

Evaluating the $R_{x_i} = \xi_{x_i}\chi_{x_i}$ $(7 \leqslant i \leqslant 10)$ at z_1 thus shows that $\overline{\xi}_{x_7} = \xi_{x_{10}}$ and $\overline{\xi}_{x_8} = \xi_{x_9}$; since $\xi_{x_i} \in \{\pm 1\}$ (see 3.4.18), we get

$$\xi_{x_7} = \xi_{x_{10}} \in \{\pm 1\} \quad \text{and} \quad \xi_{x_8} = \xi_{x_9} \in \{\pm 1\}.$$

We evaluate (3.4.19.1) with $g \in \left\{ (z_1)_{\overline{z}_1^j} \;\middle|\; 0 \leqslant j \leqslant 4 \right\}$. For $0 \leqslant j \leqslant 4$ and any $w \in \mathbf{W}$, we obtain

$$m\Big((z_1)_{\overline{z}_1^j}, w\Big) = \sum_{x \in \mathfrak{X}(\mathbf{W})} c_x(w) R_x\Big((z_1)_{\overline{z}_1^j}\Big) = \sum_{x \in \mathfrak{X}(\mathbf{W})} c_x(w) \xi_x \chi_x\Big((z_1)_{\overline{z}_1^j}\Big), \tag{4.5.10.1}$$

where

$$c_x(w) = \sum_{\phi \in \mathrm{Irr}(\mathbf{W})} \Delta(x_\phi)\{x_\phi, x\}\,\mathrm{Trace}(T_w, V_\phi).$$

Now we see from Remark 3.4.24 that for $x \in \mathfrak{X}(\mathbf{W})$, the characteristic function χ_x of A_x

can only be non-zero at elements of $\mathscr{O}_{\mathrm{reg}}^F$ if x corresponds to some $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ of the form $\mathfrak{i} = (\mathscr{O}_{\mathrm{reg}}, \mathscr{E})$ under the generalised Springer correspondence. The isomorphism classes of ($\mathbf{G}$-equivariant, F-stable) irreducible local systems on $\mathscr{O}_{\mathrm{reg}}$ are parametrised by the irreducible characters of $A_{\mathbf{G}}(z_1) = \langle \overline{z}_1 \rangle \cong C_5$; for $0 \leqslant j \leqslant 4$, let us denote by ζ_5^j the local system on $\mathscr{O}_{\mathrm{reg}}$ described by the linear character of $A_{\mathbf{G}}(z_1)$ which takes the value ζ_5^j at $\overline{z}_1$. Then, in view of 4.5.8(e), we have

$$A_i \cong \mathrm{IC}(\mathbf{G}_{\mathrm{uni}}, \zeta_5^{i-6})[\dim \mathscr{O}_{\mathrm{reg}}]^{\#\mathbf{G}} \quad \text{for} \quad 7 \leqslant i \leqslant 10.$$

So by (3.2.13.2), A_i corresponds to $(\mathscr{O}_{\mathrm{reg}}, \zeta_5^{i-6})$ for $7 \leqslant i \leqslant 10$; regarding the trivial local system on $\mathscr{O}_{\mathrm{reg}}$, the pair $(\mathscr{O}_{\mathrm{reg}}, 1)$ corresponds to the triple $(\mathbf{T}_0, \{1\}, \overline{\mathbb{Q}}_\ell) \in \mathcal{M}_{\mathbf{G}}^F$ and to the trivial character of $\mathbf{W} = W_{\mathbf{G}}(\mathbf{T}_0)$. Hence, (4.5.10.1) reads

$$\begin{aligned} m\Big((z_1)_{\overline{z}_1^j}, w\Big) &= c_{1_\mathbf{W}}(w) R_{1_\mathbf{W}}\Big((z_1)_{\overline{z}_1^j}\Big) + \sum_{i=7}^{10} c_{x_i}(w)\xi_{x_i}\chi_{x_i}\Big((z_1)_{\overline{z}_1^j}\Big) \\ &= c_{1_\mathbf{W}}(w) + \sum_{i=7}^{10} c_{x_i}(w)\xi_{x_i}\zeta_5^{j(i-6)} q^4 \end{aligned}$$

(since $R_{1_\mathbf{W}}$ is the trivial character of $\mathbf{G}^F$). Taking for w the Coxeter element

$$w = w_{\mathrm{c}} := s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \in \mathbf{W},$$

we compute the $c_x(w_{\mathrm{c}})$ using CHEVIE [MiChv] and get

$$\begin{aligned} m\Big((z_1)_{\overline{z}_1^j}, w_{\mathrm{c}}\Big) &= q^8 + \sum_{i=7}^{10} \xi_{x_i}\zeta_5^{j(i-6)} q^8 = q^8\big(1 + \xi_{x_7}(\zeta_5^j + \zeta_5^{4j}) + \xi_{x_8}(\zeta_5^{2j} + \zeta_5^{3j})\big) \\ &= q^8\big(1 + 2\xi_{x_7}\mathrm{Re}(\zeta_5^j) + 2\xi_{x_8}\mathrm{Re}(\zeta_5^{2j})\big), \end{aligned}$$

where Re denotes the real part of a complex number. Now $m\Big((z_1)_{\overline{z}_1^j}, w_{\mathrm{c}}\Big)$ is certainly non-negative for any $j \in \{0,1,2,3,4\}$. Setting $j = 0$, we get

$$0 \leqslant 1 + 2\xi_{x_7} + 2\xi_{x_8},$$

which implies that at least one of ξ_{x_7}, ξ_{x_8} must be $+1$. Assume, if possible, that they are not both equal to $+1$, say $\xi_{x_7} = -1$, $\xi_{x_8} = +1$. Then, setting $j = 1$, we obtain

$$0 \leqslant 1 - 2\mathrm{Re}(\zeta_5) + 2\mathrm{Re}(\zeta_5^2) = 1 - \sqrt{5},$$

a contradiction. The case $\xi_{x_7} = +1$, $\xi_{x_8} = -1$ is excluded similarly, setting $j = 2$. So we must have $\xi_{x_7} = \xi_{x_8} = +1$. □

Remark 4.5.11. The proof of Proposition 4.5.10 shows that, for $u \in \mathscr{O}_{\mathrm{reg}}^F$, we have $R_x(u) = 0$ unless $x \in \{x_{1_\mathbf{W}}, x_7, x_8, x_9, x_{10}\}$. Since $R_{1_\mathbf{W}} = 1_{\mathbf{G}^F}$ and $R_{x_i} = \chi_{x_i}$, expressing a unipotent character $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ as a linear combination of unipotent almost characters allows us to explicitly write down the values of ρ at the elements of $\mathscr{O}_{\mathrm{reg}}^F$. For example, for $1 \leqslant j \leqslant 4$, we get

$$\begin{aligned}\mathsf{E}_8[\zeta_5^j](u) &= \{(g_5, \zeta_5^j), x_{1_\mathbf{W}}\} \cdot R_{1_\mathbf{W}}(u) + \sum_{i=7}^{10} \{(g_5, \zeta_5^j), x_i\} \cdot R_{x_i}(u) \\ &= \sum_{i=7}^{10} \{(g_5, \zeta_5^j), x_i\} \cdot \chi_{x_i}(u).\end{aligned}$$

The values of the $\mathsf{E}_8[\zeta_5^j]$ $(1 \leqslant j \leqslant 4)$ are given by Table 4.10 where, as in the proof of Proposition 4.5.10, the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$ are named by the corresponding element of $A_\mathbf{G}(z_1)$ at the top of each column. In particular, we see that the $\mathbf{G}^F$-conjugacy class of z_1 is the only one inside $\mathscr{O}_{\mathrm{reg}}^F$ on which the above four cuspidal characters take real values, so the $\mathbf{G}^F$-class O_{z_1} of z_1 is in fact the unique one inside $\mathscr{O}_{\mathrm{reg}}^F$ which is stable under taking inverses (similarly to Remark 4.1.22, but in contrast to Remark 4.2.14).

On the other hand (still in the setting of the proof of Proposition 4.5.10), knowing that $\xi_{x_i} = +1$ for $7 \leqslant i \leqslant 10$ yields that

$$m\Big((z_1)_{\bar{z}_1^j}, w_\mathrm{c}\Big) = \begin{cases} 5q^8 & \text{if } j = 0, \\ 0 & \text{if } j \in \{1,2,3,4\}. \end{cases}$$

Hence, in view of (3.4.19.2), we conclude that if $u \in \mathscr{O}_{\mathrm{reg}}^F$ is $\mathbf{G}^F$-conjugate to $(z_1)_{\bar{z}_1^j}$ for some $j \in \{1,2,3,4\}$, we must have $O_u \cap \mathbf{B}_0^F w_\mathrm{c} \mathbf{B}_0^F = \varnothing$, while $O_u \cap \mathbf{B}_0^F w_\mathrm{c} \mathbf{B}_0^F \neq \varnothing$ in case u is $\mathbf{G}^F$-conjugate to z_1. Thus, the $\mathbf{G}^F$-class of z_1 in $\mathscr{O}_{\mathrm{reg}}^F$ satisfies condition (♡) in 3.2.22(c) and is uniquely determined by this property among the $\mathbf{G}^F$-classes inside $\mathscr{O}_{\mathrm{reg}}^F$.

Type E_8 in characteristic $p = 3$

In this subsection (or, rather, in 4.5.14–4.5.23 below), we assume that $p = 3$.

Before starting with our investigation of the cuspidal character sheaves (with a unipotent support), let us recall that up until very recently, there was one last indeterminacy in the generalised Springer correspondence, which occurs for groups of type E_8 in characteristic

	1	$\overline{z}_1$	$\overline{z}_1^2$
$\mathsf{E}_8[\zeta_5]$	$\frac{1}{5}q^4$	$\frac{1}{5}q^4(5\zeta_5+1)$	$\frac{1}{5}q^4(5\zeta_5^3+1)$
$\mathsf{E}_8[\zeta_5^2]$	$\frac{1}{5}q^4$	$\frac{1}{5}q^4(5\zeta_5^2+1)$	$\frac{1}{5}q^4(5\zeta_5+1)$
$\mathsf{E}_8[\zeta_5^3]$	$\frac{1}{5}q^4$	$\frac{1}{5}q^4(5\zeta_5^3+1)$	$\frac{1}{5}q^4(5\zeta_5^4+1)$
$\mathsf{E}_8[\zeta_5^4]$	$\frac{1}{5}q^4$	$\frac{1}{5}q^4(5\zeta_5^4+1)$	$\frac{1}{5}q^4(5\zeta_5^2+1)$

	$\overline{z}_1^3$	$\overline{z}_1^4$
$\mathsf{E}_8[\zeta_5]$	$\frac{1}{5}q^4(5\zeta_5^2+1)$	$\frac{1}{5}q^4(5\zeta_5^4+1)$
$\mathsf{E}_8[\zeta_5^2]$	$\frac{1}{5}q^4(5\zeta_5^4+1)$	$\frac{1}{5}q^4(5\zeta_5^3+1)$
$\mathsf{E}_8[\zeta_5^3]$	$\frac{1}{5}q^4(5\zeta_5+1)$	$\frac{1}{5}q^4(5\zeta_5^2+1)$
$\mathsf{E}_8[\zeta_5^4]$	$\frac{1}{5}q^4(5\zeta_5^3+1)$	$\frac{1}{5}q^4(5\zeta_5+1)$

Table 4.10.: Values of $\mathsf{E}_8[\zeta_5^j]$ ($1 \leqslant j \leqslant 4$) on $\mathscr{O}_{\mathrm{reg}}^F$ for $p = 5$

3 (see 3.2.13 and also the global introduction). These last open cases have been settled by the author in [Het22b], so let us begin by stating this result.

4.5.12. Here (and only here), $\mathbf{G}$ may for now be assumed to be an arbitrary connected reductive group over $k = \overline{\mathbb{F}}_p$ where p is any prime. Recall from 3.2.13 that the generalised Springer correspondence is the bijection

$$\coprod_{[(\mathbf{L},\mathscr{O}_0,\mathscr{E}_0)]\in\mathcal{M}_\mathbf{G}} \mathrm{Irr}(W_\mathbf{G}(\mathbf{L})) \cong \biguplus_{\mathfrak{j}\in\mathcal{M}_\mathbf{G}} \tau^{-1}(\mathfrak{j}) = \mathcal{N}_\mathbf{G}. \tag{4.5.12.1}$$

More precisely, $\tau\colon \mathcal{N}_\mathbf{G} \to \mathcal{M}_\mathbf{G}$ is a surjective map, any $\mathfrak{j} \in \mathcal{M}_\mathbf{G}$ may be represented as $(\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0)$ for some $J \subseteq S$, and the elements in the associated fibre $\tau^{-1}(\mathfrak{j}) \subseteq \mathcal{N}_\mathbf{G}$ are naturally parametrised by the irreducible characters of the relative Weyl group $W_\mathbf{G}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$. Thus, (4.5.12.1) arises from the bijections

$$\mathrm{Irr}(W_\mathbf{G}(\mathbf{L}_J)) \xrightarrow{\sim} \tau^{-1}((\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0)) \quad \text{where} \quad (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}_\mathbf{G}. \tag{4.5.12.2}$$

These bijections have been determined *explicitly* by Lusztig [Lus84b], Lusztig–Spaltenstein [LS85], Spaltenstein [Spa85] and again Lusztig [Lus19], up to the following exception: In the case where $\mathbf{G}$ is of type E_8, $p = 3$, $J = \mathsf{E}_6$ and $(\mathscr{O}_0, \mathscr{E}_0)$ is any of the two cuspidal pairs for $\mathbf{L}_J$ (which in particular implies that $\mathscr{O}_0$ is the regular unipotent class of $\mathbf{L}_J$), the results of [Spa85] do not allow the necessary distinction between the two non-linear irreducible characters of $W_\mathbf{G}(\mathbf{L}_J)$ (see 4.5.3) in (4.5.12.2). Namely, we have $A_{\mathbf{L}_J}(u) = \langle \overline{u} \rangle \cong C_3$ for any $u \in \mathscr{O}_0$, so we denote the ($\mathbf{G}$-equivariant) irreducible local systems on $\mathscr{O}_0$ by the values of the corresponding linear characters of $A_{\mathbf{G}}(u)$ at $\overline{u}$. With the notation of 4.5.3, Spaltenstein's results imply that

$$\{\rho,\rho'\} \leftrightarrow \{(\mathsf{E}_8(a_3),\omega),(\mathsf{E}_7,\omega)\} \quad \text{with respect to} \quad (\mathbf{L}_J,\mathscr{O}_0,\omega)\in\mathcal{M}_{\mathbf{G}},$$
$$\{\rho,\rho'\} \leftrightarrow \{(\mathsf{E}_8(a_3),\omega^2),(\mathsf{E}_7,\omega^2)\} \quad \text{with respect to} \quad (\mathbf{L}_J,\mathscr{O}_0,\omega^2)\in\mathcal{M}_{\mathbf{G}}$$

under (4.5.12.2). (In [Spa85], the class $\mathsf{E}_8(a_3)$ is denoted by $\mathsf{E}_7{+}\mathsf{A}_1$.) Furthermore, we have $A_{\mathbf{G}}(u)\cong C_6$ for $u\in\mathsf{E}_8(a_3)$ and $A_{\mathbf{G}}(u)\cong C_3$ for $u\in\mathsf{E}_7$, so we can again denote the local systems on $\mathsf{E}_8(a_3)$ and E_7 by the values of the corresponding irreducible characters of $A_{\mathbf{G}}(u)$ at a (fixed) generator of $A_{\mathbf{G}}(u)$.

These last indeterminacies have been removed by the author in [Het22b]. The result is given by the following theorem, which thus completes the determination of the generalised Springer correspondence.

Theorem 4.5.13 (see [Het22b])**.** *Let* $\mathbf{G}$ *be the simple algebraic group of type* E_8 *over* $\overline{\mathbb{F}}_3$. *In the setting and with the notation of* 4.5.12, *we have*

$$\rho\leftrightarrow(\mathsf{E}_8(a_3),\omega) \quad \textit{and} \quad \rho'\leftrightarrow(\mathsf{E}_7,\omega) \quad \textit{with respect to} \quad (\mathbf{L}_{\mathsf{E}_6},\mathscr{O}_0,\omega)\in\mathcal{M}_{\mathbf{G}},$$
$$\rho\leftrightarrow(\mathsf{E}_8(a_3),\omega^2) \quad \textit{and} \quad \rho'\leftrightarrow(\mathsf{E}_7,\omega^2) \quad \textit{with respect to} \quad (\mathbf{L}_{\mathsf{E}_6},\mathscr{O}_0,\omega^2)\in\mathcal{M}_{\mathbf{G}}$$

under the bijection (4.5.12.2).

Proof. This originated from the observation that formula (3.4.19.2) provides strong constraints on the relation between the character values of finite groups of Lie type and the values of the characteristic functions associated to character sheaves (cf. also Remark 4.5.33 below). The proof is a combination of the main methods used in this thesis, based on computations which are very similar to the ones already executed numerous times before, so let us only give the essential idea here and refer to [Het22b] for the detailed argument. Let $F\colon\mathbf{G}\to\mathbf{G}$ be a Frobenius map and consider the Hecke algebra associated to $\mathbf{G}^F$ and its BN-pair as in Section 2.3. Assuming one of the four possibilities for the bijections

$$\{\rho,\rho'\}\leftrightarrow\{(\mathsf{E}_8(a_3),\omega),(\mathsf{E}_7,\omega)\}, \quad \{\rho,\rho'\}\leftrightarrow\{(\mathsf{E}_8(a_3),\omega^2),(\mathsf{E}_7,\omega^2)\} \tag{$*$}$$

and using 3.4.18–3.4.24, one can compute the numbers $m(u,w)$ (for any $w\in\mathbf{W}$ and $u\in\mathbf{G}^F_{\mathrm{uni}}$) up to certain roots of unity. Specifically, the element

$$w:=s_2s_3s_4s_3s_5s_4s_1s_2s_3s_4s_5s_6s_7s_8\in\mathbf{W}$$

is of minimal length in its conjugacy class $C \subseteq \mathbf{W}$, and C is sent to the unipotent class $\mathsf{E}_7{+}\mathsf{A}_1$ under Lusztig's map (see 3.2.23); for $u \in \mathsf{E}_7^F$, we thus have $m(u,w) = 0$. This leads to an equation in which the only unknowns are the aforementioned roots of unity, and one obtains a contradiction for three of the four possibilities for the bijections in $(*)$, which yields the desired result. □

Let us return to the assumption that $\mathbf{G}$ is the simple group of type E_8 over $\overline{\mathbb{F}}_3$, with the further notation as introduced in the beginning of this section.

4.5.14. By [Sho95b, 5.1] and [DLM14, Appendix C], the description of the cuspidal character sheaves in (a), (b), (d), (e) and (g) of 4.5.6 for $p \geqslant 7$ transfers to this case in exactly the same way; the remaining four cuspidal character sheaves are as follows (with the same conventions as in 4.5.6).

(c) $A_j \leftrightarrow (su, \varsigma_j)$, $j = 3, 4$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{E}_7$, and u is a regular unipotent element of $C_{\mathbf{G}}(s)$. We have $A_{\mathbf{G}}(su) \cong \langle \overline{su} \rangle \cong C_3$, and ς_3, ς_4 are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_3(\overline{su}) = \omega$, $\varsigma_4(\overline{su}) = \omega^2$.

(f) $A_j \leftrightarrow (u, \varsigma_j)$, $j = 11, 12$, where u is an element of the class $\mathsf{E}_8(a_3)$ (named $\mathsf{E}_7{+}\mathsf{A}_1$ in [Miz80]). We have $A_{\mathbf{G}}(u) = \langle a \rangle \cong C_6$ where $a \in A_{\mathbf{G}}(u)$ is such that $a^2 = \overline{u}^2$, and ς_{11}, ς_{12} are the linear characters of $A_{\mathbf{G}}(u)$ which satisfy $\varsigma_{11}(a) = -\omega$, $\varsigma_{12}(a) = -\omega^2$.

In (c), the fact that $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_1 \times \mathsf{E}_7$ is noted in [Lus22, §3]. As for the case (f), it is claimed in [Sho95b, 5.1] that $A_{\mathbf{G}}(u) \cong C_6$ is generated by $\overline{u}$, but this cannot be true since the order of $\overline{u}$ must be a power of $p = 3$. Instead, we can argue as follows: We have $A_{\mathbf{G}}(u) \cong C_6$ by [Miz80, Lm. 40] (see also [LS12, Table 22.1.1]); since $u \notin C_{\mathbf{G}}^\circ(u)$ (see [LS12, Table 17.4]), $\langle \overline{u} \rangle \subseteq A_{\mathbf{G}}(u)$ is a subgroup of index 2. Hence, for a fixed $u \in \mathsf{E}_8(a_3)$, there exists a generator $a = a(u)$ of $A_{\mathbf{G}}(u)$ which satisfies $a^2 = \overline{u}^2$, and we may denote the local systems on $\mathsf{E}_8(a_3)$ by the values of the corresponding irreducible characters of $A_{\mathbf{G}}(u)$ at a. (This is consistent with [Spa85, pp. 328, 337].)

Let us consider the case (f). Following Mizuno [Miz80, Lm. 40] (and using his structure constants $N_{\alpha,\beta}$ for $\alpha, \beta \in R$), we set

$$z_{21} := u_{\alpha_1+\alpha_3}(1)u_{\alpha_2+\alpha_4}(1)u_{\alpha_3+\alpha_4}(1)u_{\alpha_4+\alpha_5}(1)u_{\alpha_3+\alpha_4+\alpha_5}(1)u_{\alpha_5+\alpha_6}(1)u_7(1)u_8(1).$$

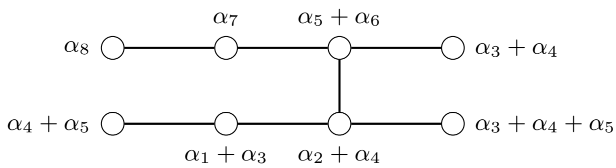
We have $z_{21} \in \mathbf{U}_0^F \cap \mathsf{E}_8(a_3)^F$.

Lemma 4.5.15. *The element z_{21} is $\mathbf{G}^F$-conjugate to z_{21}^{-1}.*

Proof. If $\alpha \in R^+$ is of the form $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_n}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \leqslant 8$, we write $u_{i_1 i_2 \ldots i_n} := u_\alpha(1) \in \mathbf{U}_\alpha \subseteq \mathbf{U}_0$, and also $u_{i_1 i_2 \ldots i_n}(-1) := u_\alpha(-1)$. So we have

$$z_{21} = u_{13} u_{24} u_{34} u_{45} u_{345} u_{56} u_7 u_8.$$

First, we want to conjugate with an element of $\mathbf{U}_0^F$ in such a way that the order of the u's in this product will be reversed. To this end, we define a graph consisting of 8 nodes labelled by the roots appearing as indices of the u's in the definition of z_{21}; an edge between two different nodes α, β is drawn if and only if $\alpha + \beta \in R$, which happens precisely when the elements of $\mathbf{U}_\alpha$ and $\mathbf{U}_\beta$ do not pairwise commute with each other. This graph is pictured as follows:



The idea consists in conjugating z_{21} by suitable u_α, using only such roots α which appear in the above graph, to reverse the order of the u_β in the product for z_{21}. This will be done step by step: We start by bringing the element in the first position of z_{21} to the last position, then the initially second element to the second to last position, then the initially third element to the third to last position, and so on. We thus begin by conjugating with u_{13}^{-1} to bring u_{13} to the last position. This gives

$$u_{24} u_{34} u_{45} u_{345} u_{56} u_7 u_8 u_{13}.$$

Next, we want to bring u_{24} to the second to last position. To achieve this, we note that we have

$$u_{24} u_{34} u_{45} u_{345} u_{56} u_7 u_8 u_{13} = u_{34} u_{45} u_{24} u_{13} u_{345} u_{56} u_7 u_8.$$

So we conjugate with $u_{345} u_{56} u_7 u_8$, getting

$$u_{345} u_{56} u_7 u_8 u_{34} u_{45} u_{24} u_{13} = u_{56} u_7 u_8 u_{345} u_{45} u_{34} u_{24} u_{13}.$$

To get u_{56} to its desired position, we move $u_7 u_8$ to the right and conjugate with $u_7 u_8$ to obtain

$$u_7 u_8 u_{56} u_{345} u_{45} u_{34} u_{24} u_{13}.$$

It remains to switch the positions of u_7, u_8. This is achieved by moving u_8 to the right and conjugating with u_8, thus arriving at

$$u_8 u_7 u_{56} u_{345} u_{45} u_{34} u_{24} u_{13}.$$

We have shown that

$$u z_{21} u^{-1} = u_8 u_7 u_{56} u_{345} u_{45} u_{34} u_{24} u_{13} \quad \text{where} \quad u := u_8 u_7 u_8 u_{345} u_{56} u_7 u_8 u_{13}^{-1} \in \mathbf{U}_0^F.$$

In order to obtain $z_{21}^{-1} = u_8(-1)u_7(-1)u_{56}(-1)u_{345}(-1)u_{45}(-1)u_{34}(-1)u_{24}(-1)u_{13}(-1)$, we conjugate with the torus element

$$t := \alpha_3^\vee(-1)\alpha_7^\vee(-1)\alpha_8^\vee(-1) \in \mathbf{T}_0^F.$$

Hence, we have

$$(tu)z_{21}(tu)^{-1} = z_{21}^{-1},$$

as desired. □

Lemma 4.5.16. *Let*

$$w_{21} := s_1 s_3 s_1 s_2 s_4 s_2 s_5 s_3 s_4 s_6 s_5 s_6 s_7 s_8 \in \mathbf{W}.$$

This is a reduced expression for w_{21}, w_{21} is of minimal length in its conjugacy class $C_{w_{21}} \subseteq \mathbf{W}$, and $C_{w_{21}}$ is sent to $\mathsf{E}_8(a_3)$ under Lusztig's map (see 3.2.23*). Moreover, the element z_{21} is $\mathbf{G}^F$-conjugate to an element of $\mathbf{B}_0^F w_{21} \mathbf{B}_0^F$.*

Proof. The statements concerning w_{21} and $C_{w_{21}}$ are easily verified using CHEVIE [MiChv]. Let us consider the element $z_{21} \in \mathsf{E}_8(a_3)^F$. As in the proof of Lemma 4.5.15, if $\alpha \in R^+$ is of the form $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_n}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \leqslant 8$, we write $u_{i_1 i_2 \ldots i_n} := u_\alpha(1)$ and $u_{i_1 i_2 \ldots i_n}(-1) := u_\alpha(-1)$. So we have

$$z_{21} = u_{13} u_{24} u_{34} u_{45} u_{345} u_{56} u_7 u_8 = u_{13} u_{24} u_{45} u_{34} u_{345} u_{56} u_7 u_8.$$

(Note that $u_{34}u_{45} = u_{45}u_{34}$, as $\alpha_3 + 2\alpha_4 + \alpha_5 \notin R$.) We first rewrite this by applying Chevalley's commutator relations; but we have to be careful with the structure constants $N_{\alpha,\beta}$ here since we are in odd characteristic. As mentioned above, we use Mizuno's [Miz80] choices for the $N_{\alpha,\beta}$. In particular, we have

$$N_{\alpha_3+\alpha_4,\alpha_5} = N_{\alpha_4,\alpha_5} = N_{\alpha_5,\alpha_6} = N_{\alpha_2,\alpha_4} = N_{\alpha_3,\alpha_4} = +1, \quad N_{\alpha_3,\alpha_1} = -1.$$

(These are the only ones which we will need below, and they coincide with those underlying the CHEVIE [MiChv] function `UnipotentGroup`, so this function can be used to verify our computation below.) We deduce that

$$u_{24} = u_2^{-1}u_4^{-1}u_2u_4, \quad u_{45} = u_4^{-1}u_5^{-1}u_4u_5, \quad u_{345} = u_{34}^{-1}u_5^{-1}u_{34}u_5, \quad u_{56} = u_5^{-1}u_6^{-1}u_5u_6,$$

so we obtain

$$z_{21} = u_{13}u_2^{-1}u_4^{-1}u_2u_5^{-1}u_4u_{34}u_6^{-1}u_5u_6u_7u_8.$$

Now $u_4u_{34} = u_{34}u_4 = u_3u_4u_3^{-1}$, so we get

$$z_{21} = u_{13}u_2^{-1}u_4^{-1}u_2u_5^{-1}u_3u_4u_3^{-1}u_6^{-1}u_5u_6u_7u_8 = u_{13}u_2^{-1}u_4^{-1}u_2u_5^{-1}u_3u_4u_6^{-1}u_5u_6u_7u_8u_3^{-1}.$$

Conjugating with u_3^{-1} thus gives

$$u_3^{-1}z_{21}u_3 = u_3^{-1}u_{13}u_2^{-1}u_4^{-1}u_2u_5^{-1}u_3u_4u_6^{-1}u_5u_6u_7u_8.$$

We would like to simplify the expression $u_3^{-1}u_{13}$, but to achieve this, we need to replace u_{13} by $u_{13}^{-1} = u_{13}(-1)$. Hence, we conjugate with the torus element $t := \alpha_3^\vee(-1) \in \mathbf{T}_0^F$ to get

$$t(u_3^{-1}z_{21}u_3)t^{-1} = u_3^{-1}u_{13}^{-1}u_2^{-1}u_4u_2u_5^{-1}u_3u_4^{-1}u_6^{-1}u_5u_6u_7u_8.$$

Since $N_{\alpha_3,\alpha_1} = -1$, we have $u_{13}^{-1} = u_{13}(-1) = u_3u_1u_3^{-1}u_1^{-1}$, so we obtain

$$\begin{aligned} t(u_3^{-1}z_{21}u_3)t^{-1} &= u_1u_3^{-1}u_1^{-1}u_2^{-1}u_4u_2u_5^{-1}u_3u_4^{-1}u_6^{-1}u_5u_6u_7u_8 \\ &= u_1u_3(-1)u_1(-1)u_2(-1)u_4u_2u_5(-1)u_3u_4(-1)u_6(-1)u_5u_6u_7u_8. \end{aligned}$$

It follows from Lemma 3.2.24 (combined with the sharp form of the Bruhat decomposition [Car85, 2.5.14]) that the latter is $\mathbf{G}^F$-conjugate to an element of $\mathbf{B}_0^F w_{21}\mathbf{B}_0^F$ (note that $-w_0(\alpha) = \alpha$ for any $\alpha \in R$). The lemma is proved. □

4.5.17. For $i = 11, 12$, let $\varphi_{x_i}\colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the choice of z_{21} (via 3.2.21), and let $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}}$. Taking $u := z_{21}$ and $a = a(z_{21}) \in A_{\mathbf{G}}(z_{21})$ such that $a^2 = \overline{z}_{21}^2$ in the description of (f) in 4.5.14, the discussion in 3.2.21 shows that the values of $\chi_{x_{11}}$ and $\chi_{x_{12}}$ at elements of $\mathsf{E}_8(a_3)^F$ are given by the following table, where we describe the $\mathbf{G}^F$-conjugacy classes contained in $\mathsf{E}_8(a_3)^F$ by giving the corresponding elements of $A_{\mathbf{G}}(z_{21}) = \langle a\rangle \cong C_6$ at the top of each column:

	1	a	a^2	a^3	a^4	a^5
$\chi_{x_{11}}$	q^7	$-q^7\omega$	$q^7\omega^2$	$-q^7$	$q^7\omega$	$-q^7\omega^2$
$\chi_{x_{12}}$	q^7	$-q^7\omega^2$	$q^7\omega$	$-q^7$	$q^7\omega^2$	$-q^7\omega$

The unipotent almost characters $R_{x_{11}}$, $R_{x_{12}}$ are given by

$$R_{x_{11}} = \tfrac{1}{3}\mathsf{E}_8[-\omega] - \tfrac{1}{6}\mathsf{E}_8[-\omega^2] + \sum_{x\in\mathfrak{X}(\mathbf{W})\setminus\{x_{11},x_{12}\}} \{x, x_{11}\}\rho_x$$

and

$$R_{x_{12}} = \tfrac{1}{3}\mathsf{E}_8[-\omega^2] - \tfrac{1}{6}\mathsf{E}_8[-\omega] + \sum_{x\in\mathfrak{X}(\mathbf{W})\setminus\{x_{11},x_{12}\}} \{x, x_{12}\}\rho_x.$$

Using (3.4.11.1) (in particular note that $\overline{\mathsf{E}_8[-\omega]} = \mathsf{E}_8[-\omega^2]$), we obtain

$$\overline{R}_{x_{11}} = R_{x_{12}}.$$

Evaluating $R_{x_i} = \xi_{x_i}\chi_{x_i}$ (for $i = 11, 12$) at z_{21} thus shows that $\overline{\xi}_{x_{11}} = \xi_{x_{12}}$ and, since $\xi_{x_i} \in \{\pm 1\}$ (see 3.4.18), we get

$$\xi := \xi_{x_{11}} = \xi_{x_{12}} \in \{\pm 1\}.$$

4.5.18. In order to determine the sign $\xi \in \{\pm 1\}$ in 4.5.17, we apply the method described in 3.4.19(2). So for any $w \in \mathbf{W}$ and any $g \in \mathbf{G}^F$, we have

$$m(g, w) = \sum_{x\in\mathfrak{X}(\mathbf{W})\setminus\{x_{11},x_{12}\}} c_x(w)R_x(g) + \xi \sum_{i=11}^{12} c_{x_i}(w)\chi_{x_i}(g). \tag{4.5.18.1}$$

We want to evaluate (4.5.18.1) with $w = w_{21}$ and $g = u \in \mathsf{E}_8(a_3)^F$, so we need a detailed information on $R_x|_{\mathsf{E}_8(a_3)^F}$ for $x \in \mathfrak{X}(\mathbf{W})$. Recall from Remark 3.4.24 that for this purpose, we only have to consider those $x \in \mathfrak{X}(\mathbf{W})$ which correspond to a pair of the form $\mathfrak{i} = (\mathscr{O}, \mathscr{E}) \in \mathscr{N}_{\mathbf{G}}^F$ (under the generalised Springer correspondence) where $\mathsf{E}_8(a_3) \subseteq \overline{\mathscr{O}}$. So we only have to consider the generalised Springer correspondence as far as the unipotent classes $\mathsf{E}_8(a_3)$, $\mathsf{E}_8(a_2)$, $\mathsf{E}_8(a_1)$ and $\mathscr{O}_{\mathrm{reg}}$ are concerned.

Combining the results of [Spa85] with Theorem 4.5.13, the part of the generalised Springer correspondence which we need below is thus given by Table 4.11, with the conventions for $\mathbf{W}^{S/\mathsf{E}_6} \cong W(\mathsf{G}_2)$ as in 4.5.3, and with the further notation as follows: Any u which appears is assumed to be an element of the unipotent class on the left, and then d_u is the dimension of the variety consisting of all Borel subgroups which contain u;

furthermore, when writing $\chi : \phi$, χ is the irreducible character of $A_{\mathbf{G}}(u)$ describing the local system on the class of u, and ϕ denotes the irreducible character of $W_{\mathbf{G}}(\mathbf{L}_J) \cong \mathbf{W}^{S/J}$ corresponding to (u, χ).

Class of u	d_u	$A_{\mathbf{G}}(u)$	$\mathbf{W}^{S/\varnothing} = \mathbf{W}$	$\mathbf{W}^{S/\mathsf{E}_6} \cong W(\mathsf{G}_2)$	$\mathbf{W}^{S/\mathsf{E}_8} = \{1\}$
$\mathscr{O}_{\mathrm{reg}}$	0	C_3	$1 : 1_x$	$\omega : 1$ $\omega^2 : 1$	—
$\mathsf{E}_8(a_1)$	1	C_3	$1 : 8_z$	$\omega : \epsilon$ $\omega^2 : \epsilon$	—
$\mathsf{E}_8(a_2)$	2	$\{1\}$	$1 : 35_x$	—	—
$\mathsf{E}_8(a_3)$	3	C_6	$1 : 112_z$ $-1 : 28_x$	$\omega : \rho$ $\omega^2 : \rho$	$-\omega : 1$ $-\omega^2 : 1$

Table 4.11.: Part of the generalised Springer correspondence for E_8, $p = 3$

4.5.19. In view of the discussion in 4.5.18 and Table 4.11, we need to consider $R_x|_{\mathsf{E}_8(a_3)^F}$ for 13 different $x \in \mathfrak{X}(\mathbf{W})$ in order to evaluate $m(u, w_{21})$ for $u \in \mathsf{E}_8(a_3)^F$. Let us first look at the $R_\phi|_{\mathsf{E}_8(a_3)^F}$ for $\phi \in \mathrm{Irr}(\mathbf{W})$. Using CHEVIE [MiChv], we see that

$$c_{112_z}(w_{21}) = c_{28_x}(w_{21}) = 0, \tag{4.5.19.1}$$

so it remains to consider $\phi \in \{1_x, 8_z, 35_x\}$. We know that $R_{1_x} = 1_{\mathbf{G}^F}$; for $R_{8_z}|_{\mathsf{E}_8(a_3)^F}$ and $R_{35_x}|_{\mathsf{E}_8(a_3)^F}$, we use the notation of 3.4.22 to get

$$R_{8_z}|_{\mathsf{E}_8(a_3)^F} = q \sum_{\phi' \in \mathrm{Irr}(\mathbf{W})} p_{\phi', 8_z} Y_{\phi'}|_{\mathsf{E}_8(a_3)^F}$$

and

$$R_{35_x}|_{\mathsf{E}_8(a_3)^F} = q^2 \sum_{\phi' \in \mathrm{Irr}(\mathbf{W})} p_{\phi', 35_x} Y_{\phi'}|_{\mathsf{E}_8(a_3)^F}.$$

Table 4.11 shows that the only $Y_{\phi'}$ which take non-zero values at elements of $\mathsf{E}_8(a_3)^F$ are the ones with $\phi' \in \{112_z, 28_x\}$. We have

$$Y_{112_z}\Big((z_{21})_{a^i}\Big) = \delta_{112_z} \quad \text{and} \quad Y_{28_x}\Big((z_{21})_{a^i}\Big) = \delta_{28_x}(-1)^i \quad \text{for } 0 \leqslant i \leqslant 5,$$

where $\delta_{112_z}, \delta_{28_x} \in \{\pm 1\}$. The coefficients $p_{\phi', \phi}$ above are $p_{28_x, 8_z} = q$, $p_{28_x, 35_x} = 0$ and

$p_{112_z,8_z} = p_{112_z,35_x} = 1$, so we get

$$R_{8_z}\Big((z_{21})_{a^i}\Big) = (-1)^i q^2 \delta_{28_x} + q\delta_{112_z} \quad \text{and} \quad R_{35_x}\Big((z_{21})_{a^i}\Big) = q^2\delta_{112_z} \quad \text{for } 0 \leqslant i \leqslant 5.$$

We have $c_{1_x}(w_{21}) = q^{14}$, $c_{8_z}(w_{21}) = q^{12}$ and $c_{35_x}(w_{21}) = -q^{11}$, so we obtain

$$\sum_{\phi \in \mathrm{Irr}(\mathbf{W})} c_\phi(w_{21}) R_\phi\Big((z_{21})_{a^i}\Big) = q^{14}\Big(1 + (-1)^i \delta_{28_x}\Big) \quad \text{for } 0 \leqslant i \leqslant 5. \tag{4.5.19.2}$$

4.5.20. Let $J := \{s_1, s_2, \ldots, s_6\}$. Thus, Table 4.11 contains six $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ for which $\tau(\mathfrak{i}) \in \mathcal{M}_{\mathbf{G}}^F$ is either $(\mathbf{L}_J, \mathscr{O}_0, \omega)$ or $(\mathbf{L}_J, \mathscr{O}_0, \omega^2)$. (Here, $\mathscr{O}_0 \subseteq \mathbf{L}_J$ is the regular unipotent class, and we have $A_{\mathbf{L}_J}(u) = \langle \overline{u} \rangle \cong C_3$ for $u \in \mathscr{O}_0$.) We identify $\mathfrak{X}(\mathbf{W})$ and $\mathfrak{S}_{\mathbf{W}}$ via Corollary 3.4.8; recall our conventions for $\mathbf{W}^{S/J} \cong W(\mathsf{G}_2)$ from 4.5.3. Let us denote the almost characters corresponding to the elements

$$(J, 1, (\omega, 3)),\ (J, \epsilon, (\omega, 3)),\ (J, \rho, (\omega, 3)),\ (J, 1, (\omega^2, 3)),\ (J, \epsilon, (\omega^2, 3)),\ (J, \rho, (\omega^2, 3)),$$

by

$$R_{\mathsf{E}_6[\omega,1]}, \quad R_{\mathsf{E}_6[\omega,\epsilon]}, \quad R_{\mathsf{E}_6[\omega,\rho]}, \quad R_{\mathsf{E}_6[\omega^2,1]}, \quad R_{\mathsf{E}_6[\omega^2,\epsilon]}, \quad R_{\mathsf{E}_6[\omega^2,\rho]},$$

respectively. We have $c_{(J,\rho,(\omega,3))}(w_{21}) = c_{(J,\rho,(\omega^2,3))}(w_{21}) = 0$, so we only need to consider the almost characters $R_{\mathsf{E}_6[\omega,1]}$, $R_{\mathsf{E}_6[\omega,\epsilon]}$, $R_{\mathsf{E}_6[\omega^2,1]}$ and $R_{\mathsf{E}_6[\omega^2,\epsilon]}$. Let us fix an isomorphism $\varphi_0 \colon F^*(\omega) \xrightarrow{\sim} \omega$ which induces a map of finite order at the stalk of ω at any element of $\mathscr{O}_0^F$. For any of the two $x \in \mathfrak{X}(\mathbf{W})$ corresponding to $(J, 1, (\omega, 3)), (J, \epsilon, (\omega, 3)) \in \mathfrak{S}_{\mathbf{W}}$ and for the associated $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ under the generalised Springer correspondence, we define the isomorphisms $\varphi_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$, $\overline{\varphi}_{A_\mathfrak{i}} \colon F^*A_\mathfrak{i} \xrightarrow{\sim} A_\mathfrak{i}$ as in 3.2.25 and put $\varphi_x := \overline{\varphi}_{A_\mathfrak{i}}$, $\chi_x := \chi_{A_x,\varphi_x}$. Setting $\zeta(\omega) := \zeta(J, (\omega, 3))$ in 3.4.23, we get

$$R_{\mathsf{E}_6[\omega,1]}|_{\mathbf{G}^F_{\mathrm{uni}}} = q^3\zeta(\omega) X_{(\mathscr{O}_{\mathrm{reg}},\omega)} \quad \text{and} \quad R_{\mathsf{E}_6[\omega,\epsilon]}|_{\mathbf{G}^F_{\mathrm{uni}}} = q^4\zeta(\omega) X_{(\mathsf{E}_8(a_1),\omega)}.$$

In order to evaluate these functions at elements of $\mathsf{E}_8(a_3)^F$, we only need the respective coefficients of $Y_{(\mathsf{E}_8(a_3),\omega)}$ in $X_{(\mathscr{O}_{\mathrm{reg}},\omega)}$ and $X_{(\mathsf{E}_8(a_1),\omega)}$. These coefficients are known in view of the results of [Het22b] (cf. Theorem 4.5.13), and they can be extracted from there. We obtain

$$X_{(\mathscr{O}_{\mathrm{reg}},\omega)}|_{\mathsf{E}_8(a_3)^F} = q^2 Y_{(\mathsf{E}_8(a_3),\omega)}|_{\mathsf{E}_8(a_3)^F} \quad \text{and} \quad X_{(\mathsf{E}_8(a_1),\omega)}|_{\mathsf{E}_8(a_3)^F} = 0,$$

so we have

$$R_{\mathsf{E}_6[\omega,1]}|_{\mathsf{E}_8(a_3)^F} = q^5\zeta(\omega)Y_{(\mathsf{E}_8(a_3),\omega)}|_{\mathsf{E}_8(a_3)^F} \quad \text{and} \quad R_{\mathsf{E}_6[\omega,\epsilon]}|_{\mathsf{E}_8(a_3)^F} = 0.$$

By 3.4.23, there exists a root of unity $\gamma = \gamma_{(\mathsf{E}_8(a_3),\omega)} \in \mathcal{R}$ such that

$$Y_{(\mathsf{E}_8(a_3),\omega)}\Big((z_{21})_{a^i}\Big) = \gamma\omega^i \quad \text{for} \quad 0 \leqslant i \leqslant 5,$$

so we get

$$R_{\mathsf{E}_6[\omega,1]}\Big((z_{21})_{a^i}\Big) = q^5\zeta(\omega)\gamma\omega^i \quad \text{for} \quad 0 \leqslant i \leqslant 5. \tag{4.5.20.1}$$

Taking $i = 0$ and using Lemma 4.5.15 (and the fact that any unipotent almost character is an $\mathbb{R}$-linear combination of unipotent characters), we deduce that the root of unity $\zeta(\omega)\gamma$ lies in $\mathbb{R}$, so we have $\zeta(\omega)\gamma \in \{\pm 1\}$. From 3.4.11 and with an argument analogous to the one in 4.2.10, we see that $R_{\mathsf{E}_6[\omega^2,1]} = \overline{R_{\mathsf{E}_6[\omega,1]}}$, so we get

$$R_{\mathsf{E}_6[\omega^2,1]}\Big((z_{21})_{a^i}\Big) = \overline{R_{\mathsf{E}_6[\omega,1]}}\Big((z_{21})_{a^i}\Big) = q^5\zeta(\omega)\gamma\omega^{-i} \quad \text{for } 0 \leqslant i \leqslant 5.$$

We have $c_{(J,1,(\omega,3))}(w_{21}) = c_{(J,1,(\omega^2,3))}(w_{21}) = q^9$ and $c_{x_{11}}(w_{21}) = c_{x_{12}}(w_{21}) = q^7$, so we obtain

$$\sum_{x \in \mathfrak{X}(\mathbf{W})\setminus \mathrm{Irr}(\mathbf{W})} c_x(w_{21})R_x(u) = q^9 \sum_{i=1}^{2} R_{\mathsf{E}_6[\omega^i,1]}(u) + \xi q^7 \sum_{i=11}^{12} \chi_{x_i}(u) \quad \text{for } u \in \mathsf{E}_8(a_3)^F.$$

Combining this with (4.5.19.2) and using the values of $\chi_{x_{11}}$, $\chi_{x_{12}}$ at elements of $\mathsf{E}_8(a_3)^F$ given in 4.5.17, we thus get, for $0 \leqslant i \leqslant 5$:

$$m\Big((z_{21})_{a^i}, w_{21}\Big) = q^{14}\Big(1 + (-1)^i\delta_{28_x} + \zeta(\omega)\gamma\cdot(\omega^i + \omega^{-i}) + \xi\cdot((-\omega)^i + (-\omega^2)^i)\Big).$$

In particular, we have

$$m(z_{21}, w_{21}) = q^{14}\Big(1 + \delta_{28_x} + 2\zeta(\omega)\gamma + 2\xi\Big) \tag{4.5.20.2}$$

and (since $\omega + \omega^2 = -1$)

$$m\Big((z_{21})_a, w_{21}\Big) = q^{14}\Big(1 - \delta_{28_x} - \zeta(\omega)\gamma + \xi\Big). \tag{4.5.20.3}$$

Proposition 4.5.21. *Assume that $p = 3$. For $i = 11, 12$, let $\varphi_{x_i}\colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of the element $z_{21} \in \mathsf{E}_8(a_3)^F$*

defined in 4.5.14. Then the scalars $\xi_{x_i} = \xi_{x_i}(\varphi_{x_i})$ *in* (4.5.1.3) *are given by*

$$\xi_{x_{11}} = \xi_{x_{12}} = +1,$$

that is, the characteristic function $\chi_{x_i} = \chi_{A_i,\varphi_{x_i}} \colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ *coincides with the unipotent almost character* R_{x_i} *for* $i = 11, 12$.

Proof. We already know that $\xi = \xi_{x_{11}} = \xi_{x_{12}} \in \{\pm 1\}$. Assume, if possible, that $\xi = -1$. By (4.5.20.2) and since $m(z_{21}, w_{21}) \geqslant 0$, we then have $\zeta(\omega)\gamma = +1$. In turn, (4.5.20.3) and the fact that $m((z_{21})_a, w_{21}) \geqslant 0$ imply that $\delta_{28_x} = -1$. Thus, (4.5.20.2) reads

$$0 = m(z_{21}, w_{21}) = \frac{|O_{z_{21}} \cap \mathbf{B}_0^F w_{21} \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(z_{21})|}{|\mathbf{B}_0^F|}.$$

But this contradicts the fact that $O_{z_{21}} \cap \mathbf{B}_0^F w_{21} \mathbf{B}_0^F \neq \varnothing$ (see Lemma 4.5.16). So we must have $\xi = +1$, as desired. □

Corollary 4.5.22. *The values of* $R_{\mathsf{E}_6[\omega,1]}$ *and* $R_{\mathsf{E}_6[\omega^2,1]}$ *at elements of* $\mathsf{E}_8(a_3)^F$ *are given by the following table, where we describe the* $\mathbf{G}^F$*-classes inside* $\mathsf{E}_8(a_3)^F$ *by giving the corresponding element of* $A_{\mathbf{G}}(z_{21}) = \langle a \rangle \cong C_6$ *(with* $a \in A_{\mathbf{G}}(z_{21})$ *such that* $a^2 = \overline{z}_{21}^2$*) at the top of each column.*

	1	a	a^2	a^3	a^4	a^5
$R_{\mathsf{E}_6[\omega,1]}$	q^5	$q^5\omega$	$q^5\omega^2$	q^5	$q^5\omega$	$q^5\omega^2$
$R_{\mathsf{E}_6[\omega^2,1]}$	q^5	$q^5\omega^2$	$q^5\omega$	q^5	$q^5\omega^2$	$q^5\omega$

Proof. By Proposition 4.5.21, we have $\xi = +1$, so the discussion in 4.5.20 shows that

$$m\Big((z_{21})_{a^i}, w_{21}\Big) = q^{14}\Big(1 + (-1)^i \delta_{28_x} + \zeta(\omega)\gamma \cdot (\omega^i + \omega^{-i}) + (-\omega)^i + (-\omega^2)^i\Big)$$

for $0 \leqslant i \leqslant 5$. Taking $i = 3$, we get

$$0 \leqslant m\Big((z_{21})_{a^3}, w_{21}\Big) = q^{14}\Big(-1 - \delta_{28_x} + 2\zeta(\omega)\gamma\Big),$$

with $\delta_{28_x} \in \{\pm 1\}$ and $\zeta(\omega)\gamma \in \{\pm 1\}$, so we must have $\zeta(\omega)\gamma = +1$. It remains to refer to (4.5.20.1). □

Remark 4.5.23. Having shown that $\xi = \zeta(\omega)\gamma = +1$ in the proofs of Proposition 4.5.21 and Corollary 4.5.22, we see that

$$0 \leqslant m\Big((z_{21})_{a^2}, w_{21}\Big) = q^{14}\Big(1 + \delta_{28_x} + 2(\omega^2 + \omega)\Big) = q^{14}(\delta_{28_x} - 1),$$

so we must have $\delta_{28_x} = +1$. We can thus evaluate $m\Big((z_{21})_{a^i}, w_{21}\Big)$ for any $0 \leqslant i \leqslant 5$; we get

$$m\Big((z_{21})_{a^i}, w_{21}\Big) = \begin{cases} 6q^{14} & \text{if } i = 0, \\ 0 & \text{if } i \in \{1,2,3,4,5\}. \end{cases}$$

Thus, the $\mathbf{G}^F$-class of z_{21} in $\mathsf{E}_8(a_3)^F$ satisfies the condition $(\heartsuit')$ in 3.2.23 and is uniquely determined by this property among the $\mathbf{G}^F$-classes inside $\mathsf{E}_8(a_3)^F$. So we declare the $\mathbf{G}^F$-conjugacy class of z_{21} as the *good* class among the $\mathbf{G}^F$-classes contained in $\mathsf{E}_8(a_3)^F$.

Type E_8 in characteristic $p = 2$

In this subsection (i.e., in 4.5.24–4.5.33 below), we assume that $p = 2$.

4.5.24. By [Sho95b, 5.1] and [DLM14, Appendix C], the description of the cuspidal character sheaves in (a), (c) and (e) of 4.5.6 for $p \geqslant 7$ transfers to this case in exactly the same way; the remaining six cuspidal character sheaves are as follows (with the same conventions as in 4.5.6); see [Lus22, §3] for the case (f).

(b) $A_2 \leftrightarrow (u, \varsigma)$ where u is an element of the class $\mathsf{E}_8(b_5)$ (named $\mathsf{E}_7(a_2)$+A_1 in [Miz80]). We have $A_{\mathbf{G}}(u) \cong \mathfrak{S}_3 \times C_2$ and, under this identification, ς corresponds to the character $\varepsilon \boxtimes (-1)$, with ε denoting the sign character of $\mathfrak{S}_3$ and -1 the non-trivial linear character of C_2.

(d) $A_j \leftrightarrow (u, \varsigma_j)$, $j = 5, 6$, where $u \in \mathsf{E}_8(a_1)$. We have $A_{\mathbf{G}}(u) \cong \langle \overline{u} \rangle \cong C_4$, and ς_5, ς_6 are the linear characters of $A_{\mathbf{G}}(u)$ which satisfy $\varsigma_5(\overline{u}) = \mathrm{i}$, $\varsigma_6(\overline{u}) = -\mathrm{i}$.

(f) $A_j \leftrightarrow (su, \varsigma_j)$, $j = 11, 12$, where $C_{\mathbf{G}}(s)$ has a root system of type $\mathsf{A}_2 \times \mathsf{E}_6$, and u is a regular unipotent element of $C_{\mathbf{G}}(s)$. We have $A_{\mathbf{G}}(su) = \langle \overline{su} \rangle \cong C_6$, and ς_{11}, ς_{12} are the linear characters of $A_{\mathbf{G}}(su)$ which satisfy $\varsigma_{11}(\overline{su}) = -\omega$, $\varsigma_{12}(\overline{su}) = -\omega^2$.

(g) $A_{13} \leftrightarrow (u, \varsigma)$ where u is an element of the class $\mathsf{E}_8(a_5)$ (named $\mathsf{D}_8(a_1)$ in [Miz80]). We have $A_{\mathbf{G}}(u) \cong D_8$ (the dihedral group of order 8) and, under this identification, ς corresponds to the sign character of D_8.

We consider the case (d). As usual, we first want to find a representative in $\mathsf{E}_8(a_1)^F$ which is $\mathbf{G}^F$-conjugate to its inverse. Following Mizuno [Miz80, Lm. 38], we set

$$z_{11} := u_1(1) \cdot u_2(1) \cdot u_{\alpha_2+\alpha_4}(1) \cdot u_{\alpha_3+\alpha_4}(1) \cdot u_5(1) \cdot u_6(1) \cdot u_7(1) \cdot u_8(1) \in \mathsf{E}_8(a_1)^F.$$

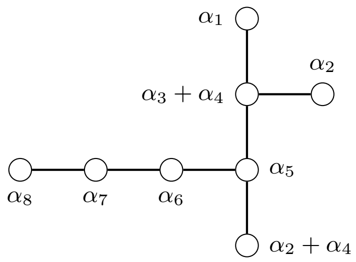
(Note that we do not have to refer to any convention for the choice of certain signs in a Chevalley basis in the Lie algebra underlying $\mathbf{G}$ since k is of characteristic 2.)

Lemma 4.5.25. *The element z_{11} is conjugate to z_{11}^{-1} in $\mathbf{U}_0^F \subseteq \mathbf{G}^F$.*

Proof. For $\alpha \in R$, let us write $u_\alpha := u_\alpha(1)$. We also set $u_i := u_i(1)$ for $1 \leqslant i \leqslant 8$. So

$$z_{11} = u_1 \cdot u_2 \cdot u_{\alpha_2+\alpha_4} \cdot u_{\alpha_3+\alpha_4} \cdot u_5 \cdot u_6 \cdot u_7 \cdot u_8.$$

Our strategy is the same as the one in the proof of Lemma 4.5.15, that is, we want to conjugate with an element of $\mathbf{U}_0^F$ in such a way that the order of the u's in the above product will be reversed. Thus, we define a graph consisting of 8 nodes labelled by the roots appearing as indices of the u's in the definition of z_{11}; an edge between two different nodes α, β is drawn if and only if $\alpha + \beta \in R$, which happens precisely when the elements of $\mathbf{U}_\alpha$ and $\mathbf{U}_\beta$ do not pairwise commute with each other. This graph is pictured as follows:



So we have to conjugate z_{11} by a product of suitable u_α, using only such roots α which appear in the above graph, to reverse the order of the u_β in the product for z_{11}. (Note that $u_\alpha = u_\alpha^{-1}$ for all $\alpha \in R$ since $1 = -1$ in $k = \overline{\mathbb{F}}_2$.) We begin by conjugating with u_1 to bring this element to the last position. This gives

$$u_2 \cdot u_{\alpha_2+\alpha_4} \cdot u_{\alpha_3+\alpha_4} \cdot u_5 \cdot u_6 \cdot u_7 \cdot u_8 \cdot u_1.$$

Now we conjugate with u_2 and use the fact that u_1 commutes with u_2, so that we can switch the two of them afterwards. We get

$$u_{\alpha_2+\alpha_4} \cdot u_{\alpha_3+\alpha_4} \cdot u_5 \cdot u_6 \cdot u_7 \cdot u_8 \cdot u_2 \cdot u_1.$$

Next we see that $u_{\alpha_2+\alpha_4}$ commutes with both u_1 and u_2, so conjugation with $u_{\alpha_2+\alpha_4}$ yields

$$u_{\alpha_3+\alpha_4} \cdot u_5 \cdot u_6 \cdot u_7 \cdot u_8 \cdot u_{\alpha_2+\alpha_4} \cdot u_2 \cdot u_1.$$

We conjugate with $u_{\alpha_3+\alpha_4}$ and get

$$u_5 \cdot u_6 \cdot u_7 \cdot u_8 \cdot u_{\alpha_2+\alpha_4} \cdot u_2 \cdot u_1 \cdot u_{\alpha_3+\alpha_4}.$$

Here, $u_{\alpha_3+\alpha_4}$ neither commutes with u_1 nor with u_2. But, since both u_1 and u_2 commute with any factor in the product except for $u_{\alpha_3+\alpha_4}$, we can shift them to the beginning and then conjugate with u_1u_2 to obtain

$$u_5 \cdot u_6 \cdot u_7 \cdot u_8 \cdot u_{\alpha_3+\alpha_4} \cdot u_{\alpha_2+\alpha_4} \cdot u_2 \cdot u_1.$$

(We have also used that $u_{\alpha_2+\alpha_4}$ and $u_{\alpha_3+\alpha_4}$ commute with each other.) To get u_5 to its desired position, we first shift $u_6u_7u_8$ to the very end and then conjugate with $u_6u_7u_8$. This gives

$$u_6 \cdot u_7 \cdot u_8 \cdot u_5 \cdot u_{\alpha_3+\alpha_4} \cdot u_{\alpha_2+\alpha_4} \cdot u_2 \cdot u_1.$$

Now we can shift u_7u_8 to the very end and conjugate with u_7u_8 to get

$$u_7 \cdot u_8 \cdot u_6 \cdot u_5 \cdot u_{\alpha_3+\alpha_4} \cdot u_{\alpha_2+\alpha_4} \cdot u_2 \cdot u_1.$$

It remains to shift u_8 to the last position and conjugate with u_8 to get

$$u_8 \cdot u_7 \cdot u_6 \cdot u_5 \cdot u_{\alpha_3+\alpha_4} \cdot u_{\alpha_2+\alpha_4} \cdot u_2 \cdot u_1,$$

which is equal to z_{11}^{-1}. We have shown that

$$uz_{11}u^{-1} = z_{11}^{-1} \quad \text{where} \quad u = u_8u_7u_8u_6u_7u_8u_1u_2u_{\alpha_3+\alpha_4}u_{\alpha_2+\alpha_4}u_2u_1 \in \mathbf{U}_0^F,$$

as desired. □

4.5.26. For $i = 5, 6$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the choice of z_{11} (via 3.2.21), and let $\chi_{x_i} := \chi_{A_i,\varphi_{x_i}}$. Taking $u := z_{11}$ in the description of (d) in 4.5.24, the non-zero values of χ_{x_5} and χ_{x_6} are given by the following table, where we denote a $\mathbf{G}^F$-class inside $\mathsf{E}_8(a_1)^F$ by the corresponding element of $A_{\mathbf{G}}(z_{11}) = \langle \overline{z}_{11} \rangle \cong C_4$:

	1	$\overline{z}_{11}$	$\overline{z}_{11}^2$	$\overline{z}_{11}^3$
χ_{x_5}	q^5	$\mathrm{i}q^5$	$-q^5$	$-\mathrm{i}q^5$
χ_{x_6}	q^5	$-\mathrm{i}q^5$	$-q^5$	$\mathrm{i}q^5$

The unipotent almost characters R_{x_5}, R_{x_6} are given by

$$R_{x_5} = \tfrac{1}{2}\mathsf{E}_8[\mathrm{i}] - \tfrac{1}{2}\mathsf{E}_8[-\mathrm{i}] + \sum_{x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_5, x_6\}} \{x, x_5\}\rho_x$$

and

$$R_{x_6} = \tfrac{1}{2}\mathsf{E}_8[-\mathrm{i}] - \tfrac{1}{2}\mathsf{E}_8[\mathrm{i}] + \sum_{x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_5, x_6\}} \{x, x_6\}\rho_x.$$

Using (3.4.11.1) (in particular note that $\overline{\mathsf{E}_8[\mathrm{i}]} = \mathsf{E}_8[-\mathrm{i}]$), we deduce that

$$\overline{R}_{x_5} = R_{x_6}.$$

Evaluating $R_{x_i} = \xi_{x_i}\chi_{x_i}$ (for $i = 5, 6$) at z_{11} thus gives $\overline{\xi}_{x_5} = \xi_{x_6}$; since $\xi_{x_5}, \xi_{x_6} \in \{\pm 1\}$ (see 3.4.18), we have

$$\xi := \xi_{x_5} = \xi_{x_6} \in \{\pm 1\}.$$

4.5.27. In order to determine the sign $\xi \in \{\pm 1\}$ in 4.5.26, we will argue similarly as in 4.5.18–4.5.21. With the notation in 3.4.19(2) we have, for any $w \in \mathbf{W}$ and any $g \in \mathbf{G}^F$:

$$m(g, w) = \sum_{x \in \mathfrak{X}(\mathbf{W}) \setminus \{x_5, x_6\}} c_x(w) R_x(g) + \xi \sum_{i=5}^{6} c_{x_i}(w)\chi_{x_i}(g). \tag{4.5.27.1}$$

We want to evaluate (4.5.27.1) with $g = z_{11}$. For $x \in \mathfrak{X}(\mathbf{W})$, the almost character R_x (or the characteristic function χ_x) vanishes identically on $\mathsf{E}_8(a_1)^F$ unless $x \in \mathfrak{X}(\mathbf{W})$ corresponds to some $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ of the form $(\mathscr{O}, \mathscr{E})$ with $\mathscr{O} \in \{\mathsf{E}_8(a_1), \mathscr{O}_{\mathrm{reg}}\}$ under the generalised Springer correspondence (see Remark 3.4.24). We have $A_{\mathbf{G}}(u) \cong C_4$ for any $u \in \mathsf{E}_8(a_1) \cup \mathscr{O}_{\mathrm{reg}}$, so there are 8 pairs in $\mathcal{N}_{\mathbf{G}}^F$ to consider. The generalised Springer correspondence with respect to these $\mathfrak{i} \in \mathcal{N}_{\mathbf{G}}^F$ is given in Table 4.12, with the conventions for $\mathbf{W}^{S/\mathsf{D}_4} \cong W(\mathsf{F}_4)$ as in 4.5.4, and with the further notation as in Table 4.11.

Class of u	d_u	$A_{\mathbf{G}}(u)$	$\mathbf{W}$	$\mathbf{W}^{S/\mathsf{D}_4} \cong W(\mathsf{F}_4)$	$\mathbf{W}^{S/\mathsf{E}_7}$	$\mathbf{W}^{S/\mathsf{E}_8} = \{1\}$
$\mathscr{O}_{\mathrm{reg}}$	0	C_4	$1 : 1_x$	$-1 : 1_1$	$\mathrm{i} : 1$ $-\mathrm{i} : 1$	—
$\mathsf{E}_8(a_1)$	1	C_4	$1 : 8_z$	$-1 : 2_1$	—	$\mathrm{i} : 1$ $-\mathrm{i} : 1$

Table 4.12.: Part of the generalised Springer correspondence for E_8, $p = 2$

4.5.28. We set

$$w_{11} := s_1 \cdot s_4 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_3 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \in \mathbf{W}, \tag{4.5.28.1}$$

a reduced expression for w_{11}. Using CHEVIE [MiChv, §6], we see that the conjugacy class of w_{11} in $\mathbf{W}$ is sent to $\mathsf{E}_8(a_1)$ under Lusztig's map (see 3.2.23), w_{11} is of minimal length in its conjugacy class, and we have $c_x(w_{11}) = 0$ for 4 of the 8 elements $x \in \mathfrak{X}(\mathbf{W})$ corresponding to the pairs $(\mathscr{O}, \mathrm{i}^r)$ $(0 \leqslant r \leqslant 3)$ described above. The 4 elements $x \in \mathfrak{X}(\mathbf{W})$ for which $c_x(w_{11})$ is non-zero are x_5, x_6, the image of the trivial character of $\mathbf{W}$ under the embedding $\mathrm{Irr}(\mathbf{W}) \hookrightarrow \mathfrak{X}(\mathbf{W})$ and one further element $x_0 \in \mathfrak{X}(\mathbf{W})$; the corresponding pairs in $\mathcal{N}_{\mathbf{G}}^F$ (in the same order) are $(\mathsf{E}_8(a_1), \mathrm{i})$, $(\mathsf{E}_8(a_1), -\mathrm{i})$, $(\mathscr{O}_{\mathrm{reg}}, 1)$ and $(\mathsf{E}_8(a_1), -1)$. We have

$$c_{x_5}(w_{11}) = c_{x_6}(w_{11}) = q^5, \quad c_{1_{\mathbf{W}}}(w_{11}) = q^{10} \quad \text{and} \quad c_{x_0}(w_{11}) = q^7.$$

Recall that $R_{1_{\mathbf{W}}}$ is the trivial character of $\mathbf{G}^F$. Hence, we get

$$\begin{aligned} m(z_{11}, w_{11}) &= c_{1_{\mathbf{W}}}(w_{11}) R_{1_{\mathbf{W}}}(z_{11}) + c_{x_0}(w_{11}) R_{x_0}(z_{11}) + \xi \sum_{i=5}^{6} c_{x_i}(w_{11}) \chi_{x_i}(z_{11}) \\ &= q^{10}(1 + 2\xi) + q^7 R_{x_0}(z_{11}), \end{aligned}$$

where $\xi = \xi_{x_5} = \xi_{x_6} \in \{\pm 1\}$ (see 4.5.26). As for the element $x_0 \in \mathfrak{X}(\mathbf{W})$ corresponding to $\mathfrak{i}_0 := (\mathsf{E}_8(a_1), -1) \in \mathcal{N}_{\mathbf{G}}^F$, we have $\tau(\mathfrak{i}_0) = (\mathbf{L}_J, \mathscr{O}_0, \mathscr{E}_0) \in \mathcal{M}_{\mathbf{G}}^F$ where $J = \mathsf{D}_4$ and $(\mathscr{O}_0, \mathscr{E}_0)$ is uniquely determined by J (with $\mathscr{O}_0 \subseteq \mathbf{L}_J$ the regular unipotent class). Let us fix an isomorphism $\varphi_0 \colon F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ which induces a map of finite order at the stalk of $\mathscr{E}_0$ at any element of $\mathscr{O}_0^F$, and let $\varphi_{A_{\mathfrak{i}_0}} : F^*A_{\mathfrak{i}_0} \xrightarrow{\sim} A_{\mathfrak{i}_0}$, $\overline{\varphi}_{A_{\mathfrak{i}_0}} : F^*A_{\mathfrak{i}_0} \xrightarrow{\sim} A_{\mathfrak{i}_0}$ be as in 3.2.25. We set $\varphi_{x_0} := \overline{\varphi}_{A_{\mathfrak{i}_0}}$ and $\chi_{x_0} := \chi_{A_{x_0}, \varphi_{x_0}}$. The discussion in 3.4.23 shows that we have

$$R_{x_0}|_{\mathsf{E}_8(a_1)^F} = q^3 \zeta_{\mathsf{D}_4} X_{\mathfrak{i}_0}|_{\mathsf{E}_8(a_1)^F} = q^3 \zeta_{\mathsf{D}_4} Y_{\mathfrak{i}_0}|_{\mathsf{E}_8(a_1)^F},$$

where $\zeta_{\mathsf{D}_4} \in \mathcal{R}$. We thus obtain

$$m(z_{11}, w_{11}) = q^{10}(1 + 2\xi + \zeta_{\mathsf{D}_4} Y_{\mathfrak{i}_0}(z_{11})). \tag{4.5.28.2}$$

Since $m(z_{11}, w_{11}) \in \mathbb{R}$, the root of unity $\zeta_{\mathsf{D}_4} Y_{\mathfrak{i}_0}(z_{11})$ must be ± 1.

Proposition 4.5.29. *Assume that $p = 2$. For $i = 5, 6$, let $\varphi_{x_i} \colon F^*A_i \xrightarrow{\sim} A_i$ be the isomorphism corresponding to the $\mathbf{G}^F$-conjugacy class of the element $z_{11} \in \mathsf{E}_8(a_1)^F$ defined in* 4.5.24*. Then the scalars $\xi_{x_i} = \xi_{x_i}(\varphi_{x_i})$ in* (4.5.1.3) *are given by*

$$\xi_{x_5} = \xi_{x_6} = +1,$$

that is, the characteristic function $\chi_{x_i} = \chi_{A_i, \varphi_{x_i}} : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ coincides with the unipotent almost character R_{x_i} for $i = 5, 6$.

Proof. We already know from 4.5.26 that $\xi = \xi_{x_5} = \xi_{x_6} \in \{\pm 1\}$. By (3.4.19.2) and the discussion in 4.5.28, we have

$$\frac{|O_{z_{11}} \cap \mathbf{B}_0^F w_{11} \mathbf{B}_0^F| \cdot |C_{\mathbf{G}^F}(z_{11})|}{|\mathbf{B}_0^F|} = q^{10}(1 + 2\xi + \zeta_{\mathsf{D}_4} Y_{\mathrm{i}_0}(z_{11})), \tag{4.5.29.1}$$

where $w_{11} \in \mathbf{W}$ is as defined in (4.5.28.1), and where $\zeta_{\mathsf{D}_4} Y_{\mathrm{i}_0}(z_{11}) \in \{\pm 1\}$. Hence, in order to show that $\xi = +1$, it suffices to find an element in $O_{z_{11}} \cap \mathbf{B}_0^F w_{11} \mathbf{B}_0^F$. Let $w_0 \in \mathbf{W}$ be the longest element with respect to the length function on $\mathbf{W}$ determined by $\Pi = \{\alpha_1, \ldots, \alpha_8\} \subseteq R^+$, and let $\dot{w}_0 \in N_{\mathbf{G}}(\mathbf{T}_0)^F$ be a representative of w_0. Recall that w_0 is characterised by the property $w_0(R^+) = -R^+$, so $-w_0(\Pi) = \Pi$, and in this way $-w_0$ defines a graph automorphism of the Dynkin diagram of $\mathbf{G}$. Since the only such automorphism is the identity, we have $-w_0(\alpha_i) = \alpha_i$ for $1 \leqslant i \leqslant 8$. Using Chevalley's commutator relations [Che55], we can write

$$z_{11} = u_1 \cdot u_4 \cdot u_2 \cdot u_3 \cdot u_4 \cdot u_3 \cdot u_5 \cdot u_6 \cdot u_7 \cdot u_8$$

(where $u_i = u_{\alpha_i}(1)$ for $1 \leqslant i \leqslant 8$). Comparing this expression with

$$w_{11} = s_1 \cdot s_4 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_3 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8,$$

we deduce from Lemma 3.2.24 that $\dot{w}_0 z_{11} \dot{w}_0^{-1} \in \mathbf{B}_0 w_{11} \mathbf{B}_0$. Since $F(\dot{w}_0) = \dot{w}_0$ and $F(z_{11}) = z_{11}$, the uniqueness of expressions in the sharp form of the Bruhat decomposition [Car85, 2.5.14] implies that $\dot{w}_0 z_{11} \dot{w}_0^{-1} \in \mathbf{B}_0^F w_{11} \mathbf{B}_0^F \cap O_{z_{11}}$, as desired. □

Corollary 4.5.30. *Let* $x_0 \in \mathfrak{X}(\mathbf{W})$ *be as in* 4.5.28 *and* z_{11} *as in* 4.5.24. *The values of* R_{x_0} *at elements of* $\mathsf{E}_8(a_1)^F$ *are given by the following table, where we denote a* $\mathbf{G}^F$*-class inside* $\mathsf{E}_8(a_1)^F$ *by the corresponding element of* $A_{\mathbf{G}}(z_{11})$ *at the top of each column.*

	1	$\overline{z}_{11}$	$\overline{z}_{11}^2$	$\overline{z}_{11}^3$
R_{x_0}	q^3	$-q^3$	q^3	$-q^3$

Proof. Using the values of $R_{x_5} = \chi_{x_5}$ and $R_{x_6} = \chi_{x_6}$ at $(z_{11})_{\overline{z}_{11}^j} \in \mathsf{E}_8(a_1)^F$ for $0 \leqslant j \leqslant 3$ given in 4.5.26, the discussion in 4.5.28 shows that we have

$$m\Big((z_{11})_{\overline{z}_{11}^j}, w_{11}\Big) = q^{10}\Big(1 + \mathrm{i}^j + (-\mathrm{i})^j + \zeta_{\mathsf{D}_4} Y_{\mathrm{i}_0}\Big((z_{11})_{\overline{z}_{11}^j}\Big)\Big) \quad \text{for } 0 \leqslant j \leqslant 3.$$

Taking $j = 2$, we get

$$m\Big((z_{11})_{\overline{z}_{11}^2}, w_{11}\Big) = q^{10}\Big(-1 + \zeta_{\mathsf{D}_4} Y_{\mathrm{i}_0}\Big((z_{11})_{\overline{z}_{11}^2}\Big)\Big).$$

This forces the root of unity $\zeta_{\mathsf{D}_4} Y_{\mathsf{i}_0}\Big((z_{11})_{\overline{z}_{11}^2}\Big)$ to be a real number, so it must be ± 1. Since $m\Big((z_{11})_{\overline{z}_{11}^2}, w_{11}\Big) \geqslant 0$, we deduce that $\zeta_{\mathsf{D}_4} Y_{\mathsf{i}_0}\Big((z_{11})_{\overline{z}_{11}^2}\Big) = +1$. Recall from 3.4.23 that this determines all the values of $\zeta_{\mathsf{D}_4} Y_{\mathsf{i}_0}$ at elements of $\mathsf{E}_8(a_1)^F$, and these values are given by the following table:

	1	$\overline{z}_{11}$	$\overline{z}_{11}^2$	$\overline{z}_{11}^3$
$\zeta_{\mathsf{D}_4} Y_{\mathsf{i}_0}$	1	-1	1	-1

The discussion in 4.5.28 thus yields the values of $R_{x_0}|_{\mathsf{E}_8(a_1)^F}$ as stated. □

Remark 4.5.31. The proof of Corollary 4.5.30 also shows that we have

$$m\Big((z_{11})_{\overline{z}_{11}^j}, w_{11}\Big) = \begin{cases} 4q^{10} & \text{if } j = 0, \\ 0 & \text{if } j \in \{1, 2, 3\}. \end{cases}$$

Thus, the $\mathbf{G}^F$-class of z_{11} in $\mathsf{E}_8(a_1)^F$ satisfies condition ($\heartsuit'$) in 3.2.23 and is uniquely determined by this property among the $\mathbf{G}^F$-classes contained in $\mathsf{E}_8(a_1)^F$. So we declare the $\mathbf{G}^F$-conjugacy class of z_{11} as the *good* class among the $\mathbf{G}^F$-classes inside $\mathsf{E}_8(a_1)^F$.

Example 4.5.32. Let us consider the two cuspidal unipotent characters $\rho_{x_5} = \mathsf{E}_8[\mathrm{i}]$, $\rho_{x_6} = \mathsf{E}_8[-\mathrm{i}]$. As noted in 4.5.27, if $u \in \mathsf{E}_8(a_1)^F$, there are only 8 elements $x \in \mathfrak{X}(\mathbf{W})$ for which $R_x(u)$ can possibly be non-zero; of these 8 elements, only 2 parametrise unipotent characters which are in the same family as $\mathsf{E}_8[\pm\mathrm{i}]$, namely, $x_5 = (g_4, \mathrm{i})$ and $x_6 = (g_4, -\mathrm{i})$. So for $u \in \mathsf{E}_8(a_1)^F$, we get

$$\mathsf{E}_8[\mathrm{i}](u) = \tfrac{1}{2}(R_{x_5}(u) - R_{x_6}(u)) = \tfrac{1}{2}(\chi_{x_5}(u) - \chi_{x_6}(u)) \qquad \text{and}$$
$$\mathsf{E}_8[-\mathrm{i}](u) = \overline{\mathsf{E}_8[\mathrm{i}]}(u) = \mathsf{E}_8[\mathrm{i}](u^{-1}).$$

Thus, the values of $\mathsf{E}_8[\pm\mathrm{i}]$ at elements of $\mathsf{E}_8(a_1)^F$ are given by Table 4.13, where the $\mathbf{G}^F$-conjugacy classes inside $\mathsf{E}_8(a_1)^F$ are described by the corresponding elements of $A_{\mathbf{G}}(z_{11})$.

	1	$\overline{z}_{11}$	$\overline{z}_{11}^2$	$\overline{z}_{11}^3$
$\mathsf{E}_8[\mathrm{i}]$	0	$\mathrm{i}q^5$	0	$-\mathrm{i}q^5$
$\mathsf{E}_8[-\mathrm{i}]$	0	$-\mathrm{i}q^5$	0	$\mathrm{i}q^5$

Table 4.13.: Values of $\mathsf{E}_8[\pm\mathrm{i}]$ at $\mathsf{E}_8(a_1)^F$ in characteristic $p = 2$

Remark 4.5.33. Coming back to the discussion in 4.5.4 on the labelling of the irreducible characters of the relative Weyl group $W_{\mathbf{G}}(\mathbf{L}_J) \cong W(\mathsf{F}_4)$ (where $J = \{s_2, s_3, s_4, s_5\} \subseteq S$), let $d, a, \tau, \tau\sigma$ be the Coxeter generators of this group as in [Kon65]. Following [GH22, 2.1], we consider the involution $\iota\colon W_{\mathbf{G}}(\mathbf{L}_J) \xrightarrow{\sim} W_{\mathbf{G}}(\mathbf{L}_J)$ defined by $\iota(d) = \tau\sigma$, $\iota(a) = \tau$, $\iota(\tau) = a$ and $\iota(\tau\sigma) = d$. Thus, ι induces a permutation of $\operatorname{Irr}(W_{\mathbf{G}}(\mathbf{L}_J))$, which we again denote by ι.

We have stated in 4.5.4 (by referring to [GH22]) that our conventions turn out to be compatible with Spaltenstein's for $\mathbf{L}_J \subseteq \mathbf{G}$. Let us check what would happen if we assumed otherwise: The only place where this affects our argument for the proof of Proposition 4.5.29 occurs as far as the parametrisation of the character sheaf $A_{(\mathsf{E}_8(a_1),-1)}$ is concerned, as the corresponding irreducible character of $W_{\mathbf{G}}(\mathbf{L}_J)$ would be $2_3 = \iota(2_1)$ instead of 2_1; let $x_0' \neq x_0$ be the element of $\mathfrak{X}(\mathbf{W})$ which parametrises $A_{(\mathsf{E}_8(a_1),-1)}$. Then we see that $c_{x_0'}(w_{11}) = 0$. Under the above hypothesis, the right side of (4.5.29.1) would thus be equal to $q^{10}(1 + 2\xi)$ with $\xi \in \{\pm 1\}$; since the left side is certainly non-negative, we must have $\xi = +1$. Evaluating $m(u, w_{11})$ with $u \in \mathsf{E}_8(a_1)^F$, we then obtain

$$0 \leqslant m(u, w_{11}) = c_{1_{\mathbf{W}}}(w_{11}) R_{1_{\mathbf{W}}}(u) + \sum_{i=5}^{6} c_{x_i}(w_{11}) \chi_{x_i}(u) = q^{10} + q^5 \sum_{i=5}^{6} \chi_{x_i}(u).$$

Taking $u := (z_{11})_{\overline{z}_{11}^2}$, we have $\chi_{x_5}\Big((z_{11})_{\overline{z}_{11}^2}\Big) = \chi_{x_6}\Big((z_{11})_{\overline{z}_{11}^2}\Big) = -q^5$ (see 4.5.26), so we get $0 \leqslant -q^{10}$, a contradiction. This provides a consistency check for our claim concerning Spaltenstein's conventions in 4.5.4, and thus for the statement in [GH22, §6, Summary B].

A. Finite Coxeter groups and generic Iwahori–Hecke algebras

The purpose of this chapter is to fix our notation as far as finite Coxeter groups and (generic) Iwahori–Hecke algebras are concerned, which are both omnipresent throughout this thesis. All of the notions and results presented here are of course well known, and most of them can be found in [GP00], which will be our main reference.

A.1. Coxeter systems, Cartan matrices and Weyl groups

In this section, we collect some properties of Coxeter groups and their relation with reflection groups associated to Cartan matrices, without giving any proofs. The content of this entire section is taken from [GP00, Chap. 1], where more details and proofs (or further references) are provided.

Definition A.1.1. Let S be a finite set, and let $M = (m_{st})_{s,t\in S}$ be a symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ss} = 1$ and $m_{st} > 1$ for all $s, t \in S$ with $s \neq t$. We define a group $W(M)$ by the presentation

$$W(M) := \langle S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ such that } m_{st} < \infty\rangle. \tag{A.1.1.1}$$

(In particular, we have $S \subseteq W(M)$, and any element of S has order 2 in $W(M)$. More generally, for any $s, t \in S$ such that $m_{st} < \infty$, it can be shown that m_{st} is in fact the order of st.) A group W with a distinguished finite subset $S \subseteq W$ of elements of order 2 is called a *Coxeter group* (and (W, S) is called a *Coxeter system*) if W is isomorphic to $W(M)$ as above (where for any $s, t \in S$, m_{st} is the order of $st \in W$, with the convention $m_{st} := \infty$ if $st \in W$ is an element of infinite order). We will then sometimes also refer to S as a set of *Coxeter generators* of W. If the group W is finite, we call (W, S) a *finite Coxeter system* and W a *finite Coxeter group.*

A.1.2. Let (W, S) be a Coxeter system. Thus, any $w \in W$ can be written as

$$w = s_1 \cdot s_2 \cdot \ldots \cdot s_e \quad \text{with } e \in \mathbb{N}_0 \text{ and } s_1, \ldots, s_e \in S.$$

The sequence $(s_1, \ldots, s_e) \in S^e$ is not uniquely determined by w. If $e \in \mathbb{N}_0$ is minimal with the property that there exist $s_1, \ldots, s_e \in S$ such that $w = s_1 \cdot \ldots \cdot s_e$, the latter is referred to as a *reduced expression for* w (*in terms of* S), and $e := l(w)$ is called the *length of* w (*with respect to* S). This defines a *length function*

$$l\colon W \to \mathbb{N}_0, \quad w \mapsto l(w).$$

In particular, we have $l(1_W) = 0$ and $l(s) = 1$ for any $s \in S$.

A.1.3. Let (W, S) be a finite Coxeter system.

(a) Since W is finite, it certainly contains an element of maximal length with respect to S. As it turns out, there is a unique element $w_0 \in W$ of maximal length, which is thus called *the longest element of* W (*with respect to* S). The longest element $w_0 \in W$ is characterised by the property that $l(sw_0) < l(w_0)$ for any $s \in S$; furthermore, we have $w_0^2 = 1_W$.

(b) Let $s_1, s_2, \ldots, s_r$ $(r = |S|)$ be any fixed order of the elements of S. Then

$$w_{\mathrm{c}} := s_1 \cdot s_2 \cdot \ldots \cdot s_r \in W$$

is called a *Coxeter element* of (W, S). The element w_{c} depends on the chosen order of $s_1, s_2, \ldots, s_r$, but any two Coxeter elements of (W, S) are conjugate in W. For a constructive proof of this well-known fact, we refer to [Cas17].

A.1.4. Let us assume that (W, S) is a finite Coxeter system. Associated to any such (W, S) is its *Coxeter diagram* (or *Coxeter graph*), a graph whose vertices are labelled by the elements of S and whose edges are given as follows: If $s, t \in S$ are such that the order m_{st} of st is at least 3, we draw an edge between s and t; if $m_{st} \geqslant 4$, we label this edge with the number m_{st}. Thus, the Coxeter diagram of (W, S) is an undirected graph which encodes precisely the information needed to describe the presentation of W in terms of generators and relations, as in (A.1.1.1).

In the case where $W \neq \{1\}$, the (finite) Coxeter system (W, S) (or the group W) is called *irreducible* if its Coxeter graph is connected; otherwise, (W, S) (or W) is called *reducible*. (The trivial group $W = \{1\}$ is neither irreducible nor reducible.) Clearly, any finite Coxeter group W is isomorphic to the direct product of the irreducible Coxeter groups corresponding to the connected components of the Coxeter graph of W. Hence, the classification of finite Coxeter groups is reduced to that of the finite irreducible Coxeter groups, that is, to the classification of connected Coxeter graphs. These are given by Figure A.1, where the numbers on the vertices describe a chosen labelling of the elements of S.

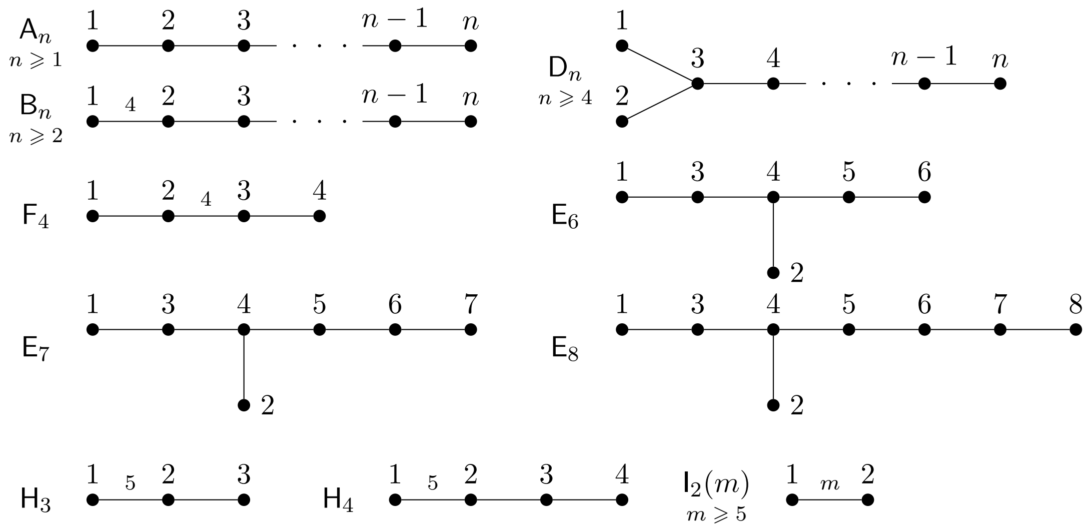


Figure A.1.: Connected Coxeter graphs

Definition A.1.5. Let S be a finite set, and let $\mathfrak{C} = (c_{st})_{s,t \in S}$ be a matrix with entries in $\mathbb{R}$. Then $\mathfrak{C}$ is called a *Cartan matrix* if the following two conditions are satisfied:

($\mathfrak{C}$1) For $s, t \in S$ with $s \neq t$, we have $c_{st} \leqslant 0$; moreover, $c_{st} \neq 0$ if and only if $c_{ts} \neq 0$.

($\mathfrak{C}$2) For all $s, t \in S$, we have $c_{ss} = 2$ and $c_{st}c_{ts} = 4\cos^2(\pi/m_{st})$ where $m_{st} \in \mathbb{N} \cup \infty$.

The Cartan matrix $\mathfrak{C}$ is said to be *of finite type* if (in addition to ($\mathfrak{C}$1) and ($\mathfrak{C}$2)) the following condition is satisfied:

($\mathfrak{C}$fin) The matrix $(-\cos(\pi/m_{st}))_{s,t \in S}$ is positive-definite.

A Cartan matrix $\mathfrak{C} = (c_{st})_{s,t \in S}$ is called *decomposable* if there exist non-empty subsets S_1, S_2 of S and a partition $S = S_1 \uplus S_2$ such that for any $s \in S_1, t \in S_2$, we have $c_{st} = 0$ (and therefore also $c_{ts} = 0$). So if we order the elements of S according to the decomposition $S = S_1 \uplus S_2$, $\mathfrak{C}$ becomes a block diagonal matrix, with diagonal blocks given by $\mathfrak{C}_1 = (c_{st})_{s,t \in S_1}$ and $\mathfrak{C}_2 = (c_{st})_{s,t \in S_2}$. A Cartan matrix $\mathfrak{C} = (c_{st})_{s,t \in S}$ is called *indecomposable* if $S \neq \varnothing$ and if no partition of S as above exists.

A.1.6. Let $\mathfrak{C} = (c_{st})_{s,t \in S}$ be a Cartan matrix, and let us consider a vector space V over $\mathbb{R}$ with basis $\{\alpha_s \mid s \in S\}$. Any $s \in S$ gives rise to a linear map

$$s \colon V \to V, \quad \alpha_t \mapsto \alpha_t - c_{st}\alpha_s \quad (\text{for } t \in S\text{; extended to } V \text{ by } \mathbb{R}\text{-linearity}),$$

called a *reflection* (with root α_s), which is justified by the fact that we have $s(\alpha_s) = -\alpha_s$, $\mathrm{Trace}(s) = |S| - 2$ and $s^2 = \mathrm{id}_V$. So we obtain a group

$$W(\mathfrak{C}) := \langle S \rangle \subseteq \mathrm{GL}(V),$$

called the *reflection group associated with* $\mathfrak{C}$. The set

$$R(\mathfrak{C}) := \{w(\alpha_s) \mid w \in W(\mathfrak{C}), s \in S\} \subseteq V \setminus \{0\}$$

is called the *root system associated with* $\mathfrak{C}$, and the elements of $R(\mathfrak{C})$ are called the *roots* (*associated with* $\mathfrak{C}$). We clearly have $\Pi(\mathfrak{C}) := \{\alpha_s \mid s \in S\} \subseteq R(\mathfrak{C})$. The elements of $\Pi(\mathfrak{C})$ are referred to as the *simple roots* in $R(\mathfrak{C})$, and the number $|S|$ is called the *rank* of $W(\mathfrak{C})$ or of $R(\mathfrak{C})$. We also note that $R(\mathfrak{C}) = -R(\mathfrak{C})$. Finally, the set $R(\mathfrak{C})$ is finite if and only if $W(\mathfrak{C})$ is a finite group. So if $W(\mathfrak{C})$ is finite, the set $R(\mathfrak{C})$ is indeed an abstract root system (not necessarily reduced or crystallographic) in the $\mathbb{R}$-vector space V, in the sense of [Bou68, Chap. VI, §1, Déf. 1].

Now, it can be shown that, given any Cartan matrix $\mathfrak{C}$, the group $W(\mathfrak{C})$ is a Coxeter group with presentation

$$W(\mathfrak{C}) = \langle S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ such that } m_{st} < \infty \rangle$$

(where $m_{st} \in \mathbb{N} \cup \{\infty\}$ are the numbers appearing in condition ($\mathfrak{C}$2) in Definition A.1.5). Conversely, if $W = W(M)$ is a Coxeter group as in (A.1.1.1), the matrix

$$\mathfrak{C} = (-2\cos(\pi/m_{st}))_{s,t \in S}$$

is a Cartan matrix, and the associated reflection group $W(\mathfrak{C})$ is isomorphic to W. The Cartan matrix $\mathfrak{C}$ is indecomposable if and only if $W(\mathfrak{C})$ is irreducible. Furthermore, the *finite* Coxeter groups are precisely the reflection groups associated to Cartan matrices *of finite type*.

A.1.7. Let us assume that $\mathfrak{C}$ is a Cartan matrix of finite type, so that the associated reflection group $W = W(\mathfrak{C})$ is finite. It should be noted that, in general, there will be several different Cartan matrices (of finite type) whose reflection group is isomorphic to W, so it is desirable to specify a 'standard' Cartan matrix attached to W.

In accordance with the notions for abstract root systems, $R(\mathfrak{C})$ is called *reduced* if, for any $c \in \mathbb{R}$ and any $\alpha \in R(\mathfrak{C})$ such that $c\alpha \in R(\mathfrak{C})$, we have $c \in \{\pm 1\}$. This happens precisely when $c_{st} = c_{ts}$ for all $s, t \in S$ with m_{st} odd. Let us consider the following two conditions for a Cartan matrix $\mathfrak{C}$ of finite type, where $s, t \in S$:

($\mathfrak{C}$red) Whenever m_{st} is odd, we have $c_{st} = c_{ts}$. (In this case, $\mathfrak{C}$ is called *reduced*.)

($\mathfrak{C}$nor) Whenever $m_{st} \geqslant 4$ is even, we have $c_{st} = -1$ or $c_{ts} = -1$.

Given any finite Coxeter group W, there exists a unique Cartan matrix $\mathfrak{C}$ of finite type which satisfies ($\mathfrak{C}$red) and ($\mathfrak{C}$nor), and whose associated reflection group $W(\mathfrak{C})$ is isomorphic to W. Associated to any such $\mathfrak{C}$ is its *Dynkin diagram* $\mathfrak{D}(\mathfrak{C})$, that is, a graph which encodes precisely the information needed to determine $\mathfrak{C}$ (under the assumption that $\mathfrak{C}$ satisfies ($\mathfrak{C}$1), ($\mathfrak{C}$2), ($\mathfrak{C}$fin), ($\mathfrak{C}$red) and ($\mathfrak{C}$nor)); we will provide the definition of Dynkin diagrams for an important subclass of Cartan matrices of finite type in A.1.8 below and refer to [GP00, 1.3.7] for the general case.

A.1.8. There is another natural condition on a Cartan matrix $\mathfrak{C}$, namely:

($\mathfrak{C}$crys) We have $c_{st} \in \mathbb{Z}$ for all $s, t \in S$.

If $\mathfrak{C}$ satisfies ($\mathfrak{C}$crys), we say that $\mathfrak{C}$ (or the Dynkin diagram $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{C}$) is *of crystallographic type*. (This is equivalent to $R(\mathfrak{C})$ being a *crystallographic* abstract root system.) In this case, the corresponding reflection group $W(\mathfrak{C})$ is called a *Weyl group*. As this notation suggests, not every finite Coxeter group is a Weyl group (and not every Cartan matrix of finite type is of crystallographic type): Indeed, any (indecomposable) Cartan matrix which gives rise to a reflection group whose Coxeter diagram is one of H_3, H_4, $\mathsf{I}_2(5)$ or $\mathsf{I}_2(m)$ with $m \geqslant 7$ does not satisfy ($\mathfrak{C}$crys); any such Cartan matrix is said to be of *non-crystallographic type*.

In this thesis, we only have to deal with finite Weyl groups, thus with Cartan matrices of both finite and crystallographic type, so it will be sufficient for our purposes to describe the Dynkin diagrams of such Cartan matrices. So let us make the following assumption on the Cartan matrices $\mathfrak{C}$ that we consider here:

$\mathfrak{C} = (c_{st})_{s,t \in S}$ satisfies ($\mathfrak{C}$1), ($\mathfrak{C}$2), ($\mathfrak{C}$fin) and ($\mathfrak{C}$crys).

Then, first of all, we must have $m_{st} \in \{2, 3, 4, 6\}$ for all $s, t \in S$ with $s \neq t$. Since $c_{st}c_{ts} = 4\cos^2(\pi/m_{st})$, we obtain, for any $s, t \in S$ with $s \neq t$:

$$\begin{aligned} m_{st} = 2 \quad &\Longleftrightarrow \quad c_{st}c_{ts} = 0, \\ m_{st} = 3 \quad &\Longleftrightarrow \quad c_{st}c_{ts} = 1, \\ m_{st} = 4 \quad &\Longleftrightarrow \quad c_{st}c_{ts} = 2, \\ m_{st} = 6 \quad &\Longleftrightarrow \quad c_{st}c_{ts} = 3. \end{aligned}$$

We deduce that $\mathfrak{C}$ automatically satisfies the conditions ($\mathfrak{C}$red) and ($\mathfrak{C}$nor).

The *Dynkin diagram* $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{C}$ is the directed graph whose vertices are in bijection with the elements of S and whose edges are defined as follows: If $s, t \in S$ with $s \neq t$, the vertices labelled by s, t are joined by $c_{st}c_{ts}$ lines; moreover, if $|c_{ts}| > 1$ (so $c_{st} = -1$ and $c_{st}c_{ts} > 1$), an arrow pointing towards the vertex labelled by t is added. Note that $\mathfrak{D}(\mathfrak{C})$ encodes all the information needed to recover $\mathfrak{C}$. The Dynkin diagram $\mathfrak{D}(\mathfrak{C})$ is closely related to the Coxeter diagram of the reflection group associated with $\mathfrak{C}$. If $\mathfrak{C}$ is indecomposable, $\mathfrak{D}(\mathfrak{C})$ will get the same name as the Coxeter diagram of $W(\mathfrak{C})$ in most cases; the main exception occurs as far as the Coxeter diagram of type B_n is concerned: As it turns out, for one of the leaves of this graph, labelled by $s_1 \in S$, say, and its neighbour $s_2 \in S$, we can have either $c_{s_1s_2} = -2$ or $c_{s_2s_1} = -2$; the two different Dynkin diagrams thus obtained are named B_n, C_n, respectively. Finally, it is common to write G_2 instead of $\mathsf{I}_2(6)$.

For a general Cartan matrix $\mathfrak{C} = (c_{st})_{s,t\in S}$ (of finite and crystallographic type, but not necessarily indecomposable), let $S = S_1 \uplus S_2 \uplus \ldots \uplus S_e$ ($e \in \mathbb{N}_0$) be a decomposition into non-empty subsets $S_i \subseteq S$ such that for any $1 \leqslant i, j \leqslant e$ with $i \neq j$, we have $c_{st} = c_{ts} = 0$ for all $s \in S_i$, $t \in S_j$, and such that each $\mathfrak{C}_i = (c_{st})_{s,t\in S_i}$ is an indecomposable Cartan matrix. Thus, if we order the elements of S according to the decomposition $S = S_1 \uplus S_2 \uplus \ldots \uplus S_e$, $\mathfrak{C}$ is a block diagonal matrix, with diagonal blocks given by $\mathfrak{C}_i = (c_{st})_{s,t\in S_i}$, $1 \leqslant i \leqslant e$. Then the Dynkin diagrams $\mathfrak{D}(\mathfrak{C}_1), \mathfrak{D}(\mathfrak{C}_2), \ldots, \mathfrak{D}(\mathfrak{C}_e)$ are the connected components of the Dynkin diagram $\mathfrak{D}(\mathfrak{C})$. (In particular, $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_e$ are uniquely determined by $\mathfrak{C}$ up to their order.) If $\mathfrak{D}_i$ is the name of the Dynkin diagram $\mathfrak{D}(\mathfrak{C}_i)$ as described above ($1 \leqslant i \leqslant e$), we will name $\mathfrak{D}(\mathfrak{C})$ by $\mathfrak{D}_1 \times \mathfrak{D}_2 \times \ldots \times \mathfrak{D}_e$ and say that $\mathfrak{C}$ *is of type* $\mathfrak{D}_1 \times \mathfrak{D}_2 \times \ldots \times \mathfrak{D}_e$. So the classification of Cartan matrices of finite and crystallographic type is reduced to indecomposable ones; the corresponding list of connected Dynkin diagrams is pictured in Figure A.2. The Dynkin diagrams named A_n, B_n, C_n and D_n are said to be *of classical type*, while E_6, E_7, E_8, F_4 and G_2 are said to be *of exceptional type*. The analogous notion is used for the corresponding Cartan matrices as well as for the corresponding Weyl groups. Accordingly, if W is an irreducible Weyl group with associated Cartan matrix of type

$$\mathsf{A}_n,\ \mathsf{B}_n,\ \mathsf{C}_n,\ \mathsf{D}_n,\ \mathsf{E}_6,\ \mathsf{E}_7,\ \mathsf{E}_8,\ \mathsf{F}_4,\ \mathsf{G}_2,$$

we will sometimes denote W by

$$W(\mathsf{A}_n),\ W(\mathsf{B}_n),\ W(\mathsf{C}_n),\ W(\mathsf{D}_n),\ W(\mathsf{E}_6),\ W(\mathsf{E}_7),\ W(\mathsf{E}_8),\ W(\mathsf{F}_4),\ W(\mathsf{G}_2),$$

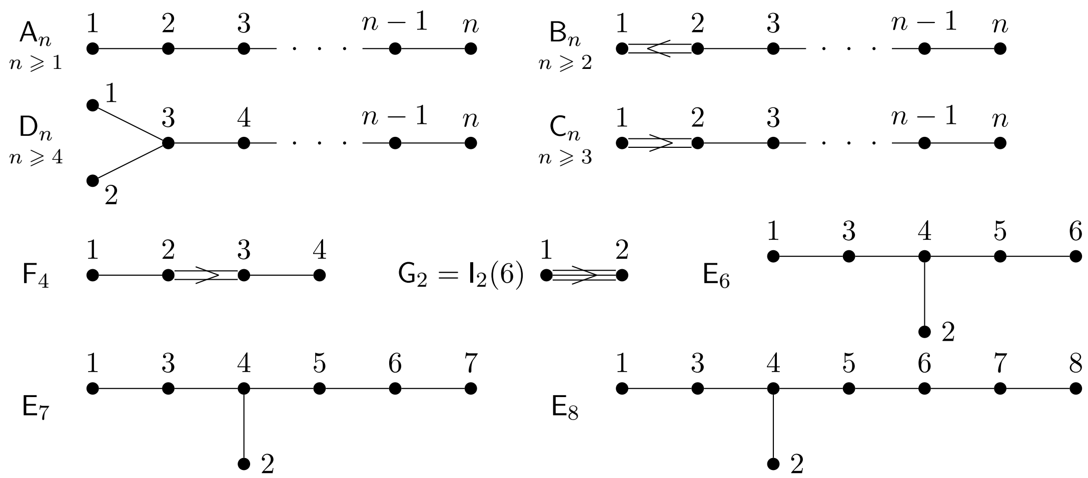


Figure A.2.: Connected Dynkin diagrams of crystallographic type

respectively.

A.2. Generic Iwahori–Hecke algebras

Definition A.2.1. Let (W, S) be a finite Coxeter system and $M = (m_{st})_{s,t\in S}$ be the associated matrix, as in Definition A.1.1. Following [GM20, 3.1.19] (see [GP00, §8.1]), let us consider a set $(\mathbf{v}_s \mid s \in S)$ of indeterminates such that $\mathbf{v}_s = \mathbf{v}_t$ whenever $s, t \in S$ are conjugate in W, and let

$$\mathbf{x}_s := \mathbf{v}_s^2 \quad \text{for } s \in S.$$

The *generic Iwahori–Hecke algebra* $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ over the ring $\mathbf{A} := \mathbb{Z}[\mathbf{v}_s^{\pm 1} \mid s \in S]$ is defined as the associative $\mathbf{A}$-algebra (with identity) generated by elements T_s, $s \in S$, subject to the relations

$$(T_s - \mathbf{x}_s)(T_s + 1) = 0 \qquad \text{for } s \in S,$$

$$T_sT_tT_s \cdots = T_tT_sT_t \cdots \quad (m_{st} \text{ terms on either side}) \qquad \text{for } s, t \in S.$$

A.2.2. Let (W, S) be a finite Coxeter system, and let $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ be the corresponding generic Iwahori–Hecke algebra over $\mathbf{A} = \mathbb{Z}[\mathbf{v}_s^{\pm 1} \mid s \in S]$ (where $\mathbf{x}_s = \mathbf{v}_s^2$ for $s \in S$), as in Definition A.2.1. Let $w \in W$, and let $w = s_1 \cdot s_2 \cdot \ldots \cdot s_n$ ($s_i \in S$, $n \in \mathbb{N}_0$) be a reduced expression for w in terms of S. By [GP00, 4.4.3, 4.4.6], the element

$$T_w := T_{s_1} \cdot T_{s_2} \cdot \ldots \cdot T_{s_n} \in \mathcal{H}(W, \mathbf{x}_s \mid s \in S)$$

does not depend upon the choice of the reduced expression for w, and the algebra $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ is free over $\mathbf{A}$, with $\mathbf{A}$-basis $\{T_w \mid w \in W\}$. (In particular, since the neutral element 1 of W is the empty product of elements in S, T_1 is the identity element of $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$.) Furthermore, the multiplication in $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ is uniquely determined by the following equations, where $w \in W$ and $s \in S$:

$$T_w \cdot T_s = \begin{cases} T_{ws} & \text{if} \quad l(ws) = l(w) + 1, \\ \mathbf{x}_s T_{ws} + (\mathbf{x}_s - 1)T_w & \text{if} \quad l(ws) = l(w) - 1. \end{cases}$$

A.2.3. In the setting of A.2.2, let us assume that we are given a commutative ring R with identity and a ring homomorphism (preserving the identity element) $\varphi\colon \mathbf{A} \to R$. Thus, φ naturally gives R the structure of a left $\mathbf{A}$-module, and we may define

$$\mathcal{H}_R(W, \varphi(\mathbf{x}_s) \mid s \in S) := \mathcal{H}(W, \mathbf{x}_s \mid s \in S) \otimes_{\mathbf{A}} R.$$

Following [GM20, 3.1.20], this is called the *specialisation of* $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ *along* φ. The ring homomorphism φ induces an algebra homomorphism

$$\mathcal{H}(W, \mathbf{x}_s \mid s \in S) \cong \mathcal{H}(W, \mathbf{x}_s \mid s \in S) \otimes_{\mathbf{A}} \mathbf{A} \xrightarrow{\mathrm{id} \otimes \varphi} \mathcal{H}_R(W, \varphi(\mathbf{x}_s) \mid s \in S),$$

which we will also denote by φ. It is uniquely determined by the values $\varphi(\mathbf{v}_s)$, $s \in S$. For any φ as above, $\mathcal{H}_R(W, \varphi(\mathbf{x}_s) \mid s \in S)$ is called an *Iwahori–Hecke algebra* associated to the Coxeter group W. Now assume that $R = L := \mathbb{Q}(\mathbf{v}_s \mid s \in S)$ is the field of fractions of $\mathbf{A}$ and that the following condition holds:

$$\text{The algebra } \mathcal{H}_L(W, \varphi(\mathbf{x}_s) \mid s \in S) \text{ is split and semisimple.} \tag{$\spadesuit$}$$

Then a fundamental result of Tits (Tits' Deformation Theorem, see [GP00, 7.4.6] or [CR87, (68.17)]) yields a natural bijection between the irreducible characters:

$$\mathrm{Irr}(\mathcal{H}_L(W, \mathbf{x}_s \mid s \in S)) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_L(W, \varphi(\mathbf{x}_s) \mid s \in S)).$$

(Here, on the left side, $\mathcal{H}_L(W, \mathbf{x}_s \mid s \in S) = \mathcal{H}(W, \mathbf{x}_s \mid s \in S) \otimes_{\mathbf{A}} L$ is the specialisation of $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ along the embedding $\mathbf{A} \hookrightarrow L$, that is, $\mathcal{H}_L(W, \mathbf{x}_s \mid s \in S)$ is obtained from $\mathcal{H}(W, \mathbf{x}_s \mid s \in S)$ by extension of scalars to L. We now use the fact that $\mathcal{H}_L(W, \mathbf{x}_s \mid s \in S)$ is a split algebra (see [GP00, 9.3.5 and 6.3.8]), so that [GP00, 7.4.6] is applicable.)

Example A.2.4. Let us consider the field extension $K := \mathbb{Q}(\cos(2\pi/m_{st}) \mid s, t \in S)$ of $\mathbb{Q}$. Then by [GP00, 6.3.8], every $\chi \in \mathrm{Irr}(W)$ can be realised over K, that is, there is an irreducible representation of $K[W]$ which affords the character χ. (In the special case where W is the Weyl group attached to a finite group of Lie type, see Chapter 2, we have in fact $K = \mathbb{Q}$.) Let $L := K(\mathbf{v}_s \mid s \in S)$, and consider the homomorphism $\varphi_1 \colon \mathbf{A} \to L$ defined by $\varphi_1(\mathbf{v}_s) := 1$ for all $s \in S$. Hence, we also have $\varphi_1(\mathbf{x}_s) = 1$ for all $s \in S$, and the multiplication rules for $T_w \cdot T_s$ in A.2.2 show that

$$\varphi_1(T_w) \cdot \varphi_1(T_s) = \varphi_1(T_{ws}) \quad \text{for all } w \in W,\, s \in S.$$

Thus, we have recovered the group algebra $\mathcal{H}_L(W, \varphi_1(\mathbf{x}_s) \mid s \in S) \cong L[W]$. This algebra is certainly split (since $K \subseteq L$) and also semisimple, due to Maschke's Theorem. Hence, the condition (♠) in A.2.3 is satisfied, and so we obtain a natural bijection

$$\mathrm{Irr}(\mathcal{H}_L(W, \mathbf{x}_s \mid s \in S)) \xrightarrow{\sim} \mathrm{Irr}(W), \quad \phi \mapsto \phi_1.$$

A.3. A note on different conventions for Weyl groups of type F_4 and G_2

The character table of the Weyl group of type F_4 has been determined by Kondo [Kon65]. However, as observed in [GH22] and as we have mentioned several times in the text (cf. 4.1.5, Section 4.4, 4.5.2, 4.5.4, Remark 4.5.33), there are different conventions for the labelling of characters of (relative) Weyl groups of type F_4 arising from reductive groups in the existing literature. In order to have a uniform reference, we decided to conform with Lusztig's book [Lus84a] by always specifying the same conversion to Kondo's notation [Kon65] as Lusztig. While Spaltenstein [Spa85] also uses the labels of [Kon65] for irreducible characters of (relative) Weyl groups of type F_4, he does not explicitly provide a correspondence between his Coxeter generators and Kondo's $\tau\sigma, \tau, a, d$. As noted in [GH22] (see also 4.5.4, Remark 4.5.33), it implicitly follows from the references and results of [Spa85] that Spaltenstein uses the same conversion scheme as Lusztig does in [Lus84a], which thus also coincides with the one chosen in this text. In order to be able to transfer this to the convention with respect to the root lengths that Carter fixes in [Car85, p. 414] and the arising identification of Carter's labels for the irreducible characters with Kondo's printed in [Car85, p. 413], the reader should remember that Kondo's a and d are always declared to be reflections in long roots in Carter's set-up — in particular, regarding the simple algebraic groups of type F_4 themselves, one would have to reverse the arrow in the Dynkin diagram printed in the beginning of Section 4.4. We refer to [GH22] for a much more detailed analysis of these different conventions.

A similar thing happens for (relative) Weyl groups of type G_2, but here this is less problematic since the description of the irreducible characters of these groups is explicitly provided in all of the above-mentioned sources [Lus84a], [Car85], [Spa85].

B. Tables

In this concluding chapter, we collect some of the bigger tables which we did not want to include in the main text to avoid unnecessary disturbance.

Table B.1 displays the full 39×39 block of the Fourier matrix in type E_8 (see 2.2.11), which we could not find in printed form in the existing literature. Here, our notation for the labels in $\mathfrak{M}(\mathfrak{S}_5)$ is similar to the one in [Lus84a, 4.3] and extends the one introduced in 4.5.1: So first of all, let us fix primitive roots of unity $\omega, \mathrm{i}, \zeta_5 \in \mathcal{R}$ of order $3, 4, 5$, respectively. As for the elements of $\mathfrak{S}_5$, $1 \in \mathfrak{S}_5$ is the neutral element; for $2 \leqslant j \leqslant 5$, g_j is a j-cycle in $\mathfrak{S}_5$, and we assume that g_2 and g_3 have disjoint supports, so that $g_6 := g_2 g_3 \in \mathfrak{S}_5$ has order 6; finally, $g_2' \in \mathfrak{S}_5$ is the product of two 2-cycles with disjoint support. Regarding the irreducible characters of the centralisers of $\mathfrak{S}_5 = C_{\mathfrak{S}_5}(1)$, let 1 be the trivial character of $\mathfrak{S}_5$; let λ^1 be the character of the reflection representation of $\mathfrak{S}_5$ and $\lambda^j \in \mathrm{Irr}(\mathfrak{S}_5)$ be the jth exterior power of λ^1 for $j = 2, 3, 4$ (so λ^4 is the sign character of $\mathfrak{S}_5$); furthermore, let $\nu \in \mathrm{Irr}(\mathfrak{S}_5)$ be the irreducible character of degree 5 which takes the value $+1$ on reflections, and let $\nu' := \nu \otimes \lambda^4$. If $g \in \{g_4, g_5\}$, we have $C_{\mathfrak{S}_5}(g) = \langle g \rangle$, and we denote the irreducible characters of this group by their values at g. If $g \in \{g_3, g_6\}$, we have $C_{\mathfrak{S}_5}(g) = \langle g_6 \rangle = \langle g_2 \rangle \times \langle g_3 \rangle$, and we denote the irreducible characters of this group by their values at g_6 (i.e., by $\pm 1, \pm\omega, \pm\omega^2$). The centraliser of g_2 can canonically be identified with $\langle g_2 \rangle \times \mathfrak{S}_3$, and we write the characters of this group as

$$\pm 1 := (\pm 1) \boxtimes 1, \ \pm\varepsilon := (\pm 1) \boxtimes \varepsilon, \ \pm r := (\pm 1) \boxtimes r,$$

where the first factor gives the value at g_2, and where $1, \varepsilon, r \in \mathrm{Irr}(\mathfrak{S}_3)$ are the trivial, sign, reflection characters, respectively. Finally, we have $C_{\mathfrak{S}_5}(g_2') \cong D_8$ (the dihedral group of order 8). Let 1 be the trivial character of $C_{\mathfrak{S}_5}(g_2')$ and $r \in \mathrm{Irr}(C_{\mathfrak{S}_5}(g_2'))$ be the unique irreducible character of degree 2; let ε' be the character of degree 1 which takes the value -1 on any transposition and on any 4-cycle in $C_{\mathfrak{S}_5}(g_2')$; let ε'' be the character of degree 1 which takes the value $+1$ on any transposition and -1 on any 4-cycle in $C_{\mathfrak{S}_5}(g_2')$; let $\varepsilon := \varepsilon' \otimes \varepsilon''$.

Table B.2 contains all the values of the unipotent characters at unipotent elements for the groups $\mathsf{E}_6(q)$ where q is a power of the prime $p = 2$, with Malle's notation in

[Mal93] for the representatives for the conjugacy classes of $\mathsf{E}_6(q)$. In addition to the Green functions computed by Malle [Mal93], this requires the determination of the unipotent almost characters $R_{\mathsf{D}_4[1]}$, $R_{\mathsf{D}_4[\varepsilon]}$ and $R_{\mathsf{D}_4[r]}$ at unipotent elements, which we have accomplished in 4.1.26, see Table 4.4.

Tables B.3 and B.4 are needed for the determination of the values of unipotent characters at unipotent elements for the groups $\mathsf{E}_7(q)$ where q is a power of the prime $p = 2$; see 4.2.21–4.2.28.

	$(1,1)$	$(g_3,1)$	$(g_2',1)$	$(1,\nu)$	$(1,\lambda^1)$	$(g_5,1)$	$(g_3,-1)$	$(1,\nu')$	(g_2',ε'')	$(1,\lambda^2)$	(g_2',ε')	$(1,\lambda^3)$	$(g_2,1)$
$(1,1)$	1/120	1/6	1/8	1/24	1/30	1/5	1/6	1/24	1/8	1/20	1/8	1/30	1/12
$(g_3,1)$	1/6	1/3	0	−1/6	1/6	0	1/3	−1/6	0	0	0	1/6	1/6
$(g_2',1)$	1/8	0	3/8	1/8	0	0	0	1/8	−1/8	−1/4	3/8	0	1/4
$(1,\nu)$	1/24	−1/6	1/8	5/24	1/6	0	−1/6	5/24	1/8	1/4	1/8	1/6	1/12
$(1,\lambda^1)$	1/30	1/6	0	1/6	2/15	−1/5	1/6	1/6	0	1/5	0	2/15	1/6
$(g_5,1)$	1/5	0	0	0	−1/5	4/5	0	0	0	1/5	0	−1/5	0
$(g_3,-1)$	1/6	1/3	0	−1/6	1/6	0	1/3	−1/6	0	0	0	1/6	−1/6
$(1,\nu')$	1/24	−1/6	1/8	5/24	1/6	0	−1/6	5/24	1/8	1/4	1/8	1/6	−1/12
(g_2',ε'')	1/8	0	−1/8	1/8	0	0	0	1/8	3/8	−1/4	−1/8	0	1/4
$(1,\lambda^2)$	1/20	0	−1/4	1/4	1/5	1/5	0	1/4	−1/4	3/10	−1/4	1/5	0
(g_2',ε')	1/8	0	3/8	1/8	0	0	0	1/8	−1/8	−1/4	3/8	0	−1/4
$(1,\lambda^3)$	1/30	1/6	0	1/6	2/15	−1/5	1/6	1/6	0	1/5	0	2/15	−1/6
$(g_2,1)$	1/12	1/6	1/4	1/12	1/6	0	−1/6	−1/12	1/4	0	−1/4	−1/6	1/3
$(g_4,1)$	1/4	0	1/4	−1/4	0	0	0	1/4	−1/4	0	−1/4	0	0
$(g_6,1)$	1/6	1/3	0	1/6	−1/6	0	−1/3	−1/6	0	0	0	1/6	1/6
(g_2,r)	1/6	−1/6	0	1/6	1/3	0	1/6	−1/6	0	0	0	−1/3	1/6
(g_2,ε)	1/12	1/6	−1/4	1/12	1/6	0	−1/6	−1/12	−1/4	0	1/4	−1/6	−1/6
$(g_4,-1)$	1/4	0	1/4	−1/4	0	0	0	1/4	−1/4	0	−1/4	0	0
$(g_6,-1)$	1/6	1/3	0	1/6	−1/6	0	−1/3	−1/6	0	0	0	1/6	−1/6
$(g_2,-r)$	1/6	−1/6	0	1/6	1/3	0	1/6	−1/6	0	0	0	−1/3	−1/6
$(g_2,-1)$	1/12	1/6	−1/4	1/12	1/6	0	−1/6	−1/12	−1/4	0	1/4	−1/6	1/6
(g_2',r)	1/4	0	−1/4	1/4	0	0	0	1/4	−1/4	−1/2	−1/4	0	0
(g_3,ω)	1/6	−1/6	0	−1/6	1/6	0	−1/6	−1/6	0	0	0	1/6	1/6
(g_3,ω^2)	1/6	−1/6	0	−1/6	1/6	0	−1/6	−1/6	0	0	0	1/6	1/6
(g_6,ω)	1/6	−1/6	0	1/6	−1/6	0	1/6	−1/6	0	0	0	1/6	1/6
(g_6,ω^2)	1/6	−1/6	0	1/6	−1/6	0	1/6	−1/6	0	0	0	1/6	1/6
(g_5,ζ_5)	1/5	0	0	0	−1/5	−1/5	0	0	0	1/5	0	−1/5	0
(g_5,ζ_5^2)	1/5	0	0	0	−1/5	−1/5	0	0	0	1/5	0	−1/5	0
(g_5,ζ_5^3)	1/5	0	0	0	−1/5	−1/5	0	0	0	1/5	0	−1/5	0
(g_5,ζ_5^4)	1/5	0	0	0	−1/5	−1/5	0	0	0	1/5	0	−1/5	0
$(g_6,-\omega)$	1/6	−1/6	0	1/6	−1/6	0	1/6	−1/6	0	0	0	1/6	−1/6
$(g_6,-\omega^2)$	1/6	−1/6	0	1/6	−1/6	0	1/6	−1/6	0	0	0	1/6	−1/6
$(g_3,-\omega)$	1/6	−1/6	0	−1/6	1/6	0	−1/6	−1/6	0	0	0	1/6	−1/6
$(g_3,-\omega^2)$	1/6	−1/6	0	−1/6	1/6	0	−1/6	−1/6	0	0	0	1/6	−1/6
(g_4,i)	1/4	0	−1/4	−1/4	0	0	0	1/4	1/4	0	1/4	0	0
$(g_4,-\mathrm{i})$	1/4	0	−1/4	−1/4	0	0	0	1/4	1/4	0	1/4	0	0
$(1,\lambda^4)$	1/120	1/6	1/8	1/24	1/30	1/5	1/6	1/24	1/8	1/20	1/8	1/30	−1/12
$(g_2,-\varepsilon)$	1/12	1/6	1/4	1/12	1/6	0	−1/6	−1/12	1/4	0	−1/4	−1/6	−1/3
(g_2',ε)	1/8	0	−1/8	1/8	0	0	0	1/8	3/8	−1/4	−1/8	0	−1/4

Table B.1.: The 39×39 block of the Fourier matrix in type E_8. Here, $A := -\frac{1}{5}\zeta_5 - \frac{2}{5}\zeta_5^2 - \frac{2}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^4$, $B := \frac{2}{5}\zeta_5 + \frac{2}{5}\zeta_5^4$, $C := \frac{2}{5}\zeta_5^2 + \frac{2}{5}\zeta_5^3$, $D := -\frac{2}{5}\zeta_5 - \frac{1}{5}\zeta_5^2 - \frac{1}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^4$.

	$(g_4,1)$	$(g_6,1)$	(g_2,r)	(g_2,ε)	$(g_4,-1)$	$(g_6,-1)$	$(g_2,-r)$	$(g_2,-1)$	(g_2',r)	(g_3,ω)	(g_3,ω^2)	(g_6,ω)	(g_6,ω^2)
$(1,1)$	$1/4$	$1/6$	$1/6$	$1/12$	$1/4$	$1/6$	$1/6$	$1/12$	$1/4$	$1/6$	$1/6$	$1/6$	$1/6$
$(g_3,1)$	0	$1/3$	$-1/6$	$1/6$	0	$1/3$	$-1/6$	$1/6$	0	$-1/6$	$-1/6$	$-1/6$	$-1/6$
$(g_2',1)$	$1/4$	0	0	$-1/4$	$1/4$	0	0	$-1/4$	$-1/4$	0	0	0	0
$(1,\nu)$	$-1/4$	$1/6$	$1/6$	$1/12$	$-1/4$	$1/6$	$1/6$	$1/12$	$1/4$	$-1/6$	$-1/6$	$1/6$	$1/6$
$(1,\lambda^1)$	0	$-1/6$	$1/3$	$1/6$	0	$-1/6$	$1/3$	$1/6$	0	$1/6$	$1/6$	$-1/6$	$-1/6$
$(g_5,1)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$(g_3,-1)$	0	$-1/3$	$1/6$	$-1/6$	0	$-1/3$	$1/6$	$-1/6$	0	$-1/6$	$-1/6$	$1/6$	$1/6$
$(1,\nu')$	$1/4$	$-1/6$	$-1/6$	$-1/12$	$1/4$	$-1/6$	$-1/6$	$-1/12$	$1/4$	$-1/6$	$-1/6$	$-1/6$	$-1/6$
(g_2',ε'')	$-1/4$	0	0	$-1/4$	$-1/4$	0	0	$-1/4$	$-1/4$	0	0	0	0
$(1,\lambda^2)$	0	0	0	0	0	0	0	0	$-1/2$	0	0	0	0
(g_2',ε')	$-1/4$	0	0	$1/4$	$-1/4$	0	0	$1/4$	$-1/4$	0	0	0	0
$(1,\lambda^3)$	0	$1/6$	$-1/3$	$-1/6$	0	$1/6$	$-1/3$	$-1/6$	0	$1/6$	$1/6$	$1/6$	$1/6$
$(g_2,1)$	0	$1/6$	$1/6$	$-1/6$	0	$-1/6$	$-1/6$	$1/6$	0	$1/6$	$1/6$	$1/6$	$1/6$
$(g_4,1)$	$1/2$	0	0	0	$-1/2$	0	0	0	0	0	0	0	0
$(g_6,1)$	0	$1/3$	$-1/6$	$1/6$	0	$-1/3$	$1/6$	$-1/6$	0	$-1/6$	$-1/6$	$-1/6$	$-1/6$
(g_2,r)	0	$-1/6$	$1/3$	$1/6$	0	$1/6$	$-1/3$	$-1/6$	0	$-1/6$	$-1/6$	$-1/6$	$-1/6$
(g_2,ε)	0	$1/6$	$1/6$	$1/3$	0	$-1/6$	$-1/6$	$-1/3$	0	$1/6$	$1/6$	$1/6$	$1/6$
$(g_4,-1)$	$-1/2$	0	0	0	$1/2$	0	0	0	0	0	0	0	0
$(g_6,-1)$	0	$-1/3$	$1/6$	$-1/6$	0	$1/3$	$-1/6$	$1/6$	0	$-1/6$	$-1/6$	$1/6$	$1/6$
$(g_2,-r)$	0	$1/6$	$-1/3$	$-1/6$	0	$-1/6$	$1/3$	$1/6$	0	$-1/6$	$-1/6$	$1/6$	$1/6$
$(g_2,-1)$	0	$-1/6$	$-1/6$	$-1/3$	0	$1/6$	$1/6$	$1/3$	0	$1/6$	$1/6$	$-1/6$	$-1/6$
(g_2',r)	0	0	0	0	0	0	0	0	$1/2$	0	0	0	0
(g_3,ω)	0	$-1/6$	$-1/6$	$1/6$	0	$-1/6$	$-1/6$	$1/6$	0	$1/3$	$-1/6$	$1/3$	$-1/6$
(g_3,ω^2)	0	$-1/6$	$-1/6$	$1/6$	0	$-1/6$	$-1/6$	$1/6$	0	$-1/6$	$1/3$	$-1/6$	$1/3$
(g_6,ω)	0	$-1/6$	$-1/6$	$1/6$	0	$1/6$	$1/6$	$-1/6$	0	$1/3$	$-1/6$	$1/3$	$-1/6$
(g_6,ω^2)	0	$-1/6$	$-1/6$	$1/6$	0	$1/6$	$1/6$	$-1/6$	0	$-1/6$	$1/3$	$-1/6$	$1/3$
(g_5,ζ_5)	0	0	0	0	0	0	0	0	0	0	0	0	0
(g_5,ζ_5^2)	0	0	0	0	0	0	0	0	0	0	0	0	0
(g_5,ζ_5^3)	0	0	0	0	0	0	0	0	0	0	0	0	0
(g_5,ζ_5^4)	0	0	0	0	0	0	0	0	0	0	0	0	0
$(g_6,-\omega)$	0	$1/6$	$1/6$	$-1/6$	0	$-1/6$	$-1/6$	$1/6$	0	$1/3$	$-1/6$	$-1/3$	$1/6$
$(g_6,-\omega^2)$	0	$1/6$	$1/6$	$-1/6$	0	$-1/6$	$-1/6$	$1/6$	0	$-1/6$	$1/3$	$1/6$	$-1/3$
$(g_3,-\omega)$	0	$1/6$	$1/6$	$-1/6$	0	$1/6$	$1/6$	$-1/6$	0	$1/3$	$-1/6$	$-1/3$	$1/6$
$(g_3,-\omega^2)$	0	$1/6$	$1/6$	$-1/6$	0	$1/6$	$1/6$	$-1/6$	0	$-1/6$	$1/3$	$1/6$	$-1/3$
(g_4,i)	0	0	0	0	0	0	0	0	0	0	0	0	0
$(g_4,-\mathrm{i})$	0	0	0	0	0	0	0	0	0	0	0	0	0
$(1,\lambda^4)$	$-1/4$	$-1/6$	$-1/6$	$-1/12$	$-1/4$	$-1/6$	$-1/6$	$-1/12$	$1/4$	$1/6$	$1/6$	$-1/6$	$-1/6$
$(g_2,-\varepsilon)$	0	$-1/6$	$-1/6$	$1/6$	0	$1/6$	$1/6$	$-1/6$	0	$1/6$	$1/6$	$-1/6$	$-1/6$
(g_2',ε)	$1/4$	0	0	$1/4$	$1/4$	0	0	$1/4$	$-1/4$	0	0	0	0

The 39×39 block of the Fourier matrix in type E_8. Here, $A := -\frac{1}{5}\zeta_5 - \frac{2}{5}\zeta_5^2 - \frac{2}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^4$, $B := \frac{2}{5}\zeta_5 + \frac{2}{5}\zeta_5^4$, $C := \frac{2}{5}\zeta_5^2 + \frac{2}{5}\zeta_5^3$, $D := -\frac{2}{5}\zeta_5 - \frac{1}{5}\zeta_5^2 - \frac{1}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^4$. (2/3)

	(g_5,ζ_5)	(g_5,ζ_5^2)	(g_5,ζ_5^3)	(g_5,ζ_5^4)	$(g_6,-\omega)$	$(g_6,-\omega^2)$	$(g_3,-\omega)$	$(g_3,-\omega^2)$	(g_4,i)	$(g_4,-\mathrm{i})$	$(1,\lambda^4)$	$(g_2,-\varepsilon)$	(g_2',ε)
$(1,1)$	1/5	1/5	1/5	1/5	1/6	1/6	1/6	1/6	1/4	1/4	1/120	1/12	1/8
$(g_3,1)$	0	0	0	0	−1/6	−1/6	−1/6	−1/6	0	0	1/6	1/6	0
$(g_2',1)$	0	0	0	0	0	0	0	0	−1/4	−1/4	1/8	1/4	−1/8
$(1,\nu)$	0	0	0	0	1/6	1/6	−1/6	−1/6	−1/4	−1/4	1/24	1/12	1/8
$(1,\lambda^1)$	−1/5	−1/5	−1/5	−1/5	−1/6	−1/6	1/6	1/6	0	0	1/30	1/6	0
$(g_5,1)$	−1/5	−1/5	−1/5	−1/5	0	0	0	0	0	0	1/5	0	0
$(g_3,-1)$	0	0	0	0	1/6	1/6	−1/6	−1/6	0	0	1/6	−1/6	0
$(1,\nu')$	0	0	0	0	−1/6	−1/6	−1/6	−1/6	1/4	1/4	1/24	−1/12	1/8
(g_2',ε'')	0	0	0	0	0	0	0	0	1/4	1/4	1/8	1/4	3/8
$(1,\lambda^2)$	1/5	1/5	1/5	1/5	0	0	0	0	0	0	1/20	0	−1/4
(g_2',ε')	0	0	0	0	0	0	0	0	1/4	1/4	1/8	−1/4	−1/8
$(1,\lambda^3)$	−1/5	−1/5	−1/5	−1/5	1/6	1/6	1/6	1/6	0	0	1/30	−1/6	0
$(g_2,1)$	0	0	0	0	−1/6	−1/6	−1/6	−1/6	0	0	−1/12	−1/3	−1/4
$(g_4,1)$	0	0	0	0	0	0	0	0	0	0	−1/4	0	1/4
$(g_6,1)$	0	0	0	0	1/6	1/6	1/6	1/6	0	0	−1/6	−1/6	0
(g_2,r)	0	0	0	0	1/6	1/6	1/6	1/6	0	0	−1/6	−1/6	0
(g_2,ε)	0	0	0	0	−1/6	−1/6	−1/6	−1/6	0	0	−1/12	1/6	1/4
$(g_4,-1)$	0	0	0	0	0	0	0	0	0	0	−1/4	0	1/4
$(g_6,-1)$	0	0	0	0	−1/6	−1/6	1/6	1/6	0	0	−1/6	1/6	0
$(g_2,-r)$	0	0	0	0	−1/6	−1/6	1/6	1/6	0	0	−1/6	1/6	0
$(g_2,-1)$	0	0	0	0	1/6	1/6	−1/6	−1/6	0	0	−1/12	−1/6	1/4
(g_2',r)	0	0	0	0	0	0	0	0	0	0	1/4	0	−1/4
(g_3,ω)	0	0	0	0	1/3	−1/6	1/3	−1/6	0	0	1/6	1/6	0
(g_3,ω^2)	0	0	0	0	−1/6	1/3	−1/6	1/3	0	0	1/6	1/6	0
(g_6,ω)	0	0	0	0	−1/3	1/6	−1/3	1/6	0	0	−1/6	−1/6	0
(g_6,ω^2)	0	0	0	0	1/6	−1/3	1/6	−1/3	0	0	−1/6	−1/6	0
(g_5,ζ_5)	A	B	C	D	0	0	0	0	0	0	1/5	0	0
(g_5,ζ_5^2)	B	D	A	C	0	0	0	0	0	0	1/5	0	0
(g_5,ζ_5^3)	C	A	D	B	0	0	0	0	0	0	1/5	0	0
(g_5,ζ_5^4)	D	C	B	A	0	0	0	0	0	0	1/5	0	0
$(g_6,-\omega)$	0	0	0	0	1/3	−1/6	−1/3	1/6	0	0	−1/6	1/6	0
$(g_6,-\omega^2)$	0	0	0	0	−1/6	1/3	1/6	−1/3	0	0	−1/6	1/6	0
$(g_3,-\omega)$	0	0	0	0	−1/3	1/6	1/3	−1/6	0	0	1/6	−1/6	0
$(g_3,-\omega^2)$	0	0	0	0	1/6	−1/3	−1/6	1/3	0	0	1/6	−1/6	0
(g_4,i)	0	0	0	0	0	0	0	0	1/2	−1/2	−1/4	0	−1/4
$(g_4,-\mathrm{i})$	0	0	0	0	0	0	0	0	−1/2	1/2	−1/4	0	−1/4
$(1,\lambda^4)$	1/5	1/5	1/5	1/5	−1/6	−1/6	1/6	1/6	−1/4	−1/4	1/120	−1/12	1/8
$(g_2,-\varepsilon)$	0	0	0	0	1/6	1/6	−1/6	−1/6	0	0	−1/12	1/3	−1/4
(g_2',ε)	0	0	0	0	0	0	0	0	−1/4	−1/4	1/8	−1/4	3/8

The 39×39 block of the Fourier matrix in type E_8. Here, $A := -\frac{1}{5}\zeta_5 - \frac{2}{5}\zeta_5^2 - \frac{2}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^4$, $B := \frac{2}{5}\zeta_5 + \frac{2}{5}\zeta_5^4$, $C := \frac{2}{5}\zeta_5^2 + \frac{2}{5}\zeta_5^3$, $D := -\frac{2}{5}\zeta_5 - \frac{1}{5}\zeta_5^2 - \frac{1}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^4$. (3/3)

	u_0	u_1	u_2
$[1_p]$	1	1	1
$[6_p]$	$q\Phi_8\Phi_9$	$q\Phi_5(q^3-q+1)$	$q(q^6+q^4+q^3+1)$
$[20_p]$	$q^2\Phi_4\Phi_5\Phi_8\Phi_{12}$	$q^2\Phi_5(q^6+q^4+1)$	$q^2(q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+q+1)$
$[64_p]$	$q^4\Phi_2^3\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$	$q^4\Phi_2^2\Phi_4\Phi_5\Phi_6(q^3-q+1)$	$q^4\Phi_2^2\Phi_4\Phi_6(q^4+q^3+1)$
$[60_p]$	$q^5\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{12}$	$q^5\Phi_5(q^9+q^7+2q^6+q^4+q^3+1)$	$q^5(q^{10}+q^9+2q^8+3q^7+4q^6+2q^5+3q^4+2q^3+q^2+q+1)$
$[81_p]$	$q^6\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}$	$q^6\Phi_3^2\Phi_5\Phi_6(q^3-q+1)$	$q^6\Phi_3\Phi_6(q^6+q^5+2q^4+q^3+q^2+q+1)$
$[24_p]$	$q^6\Phi_4^2\Phi_8\Phi_9\Phi_{12}$	$q^6\Phi_4(q^{10}+q^7+2q^6+q^4+q^3+1)$	$q^6\Phi_4(q^6+q^4+q^3+1)$
$[1_p']$	q^{36}	.	.
$[6_p']$	$q^{25}\Phi_8\Phi_9$	q^{25}	.
$[20_p']$	$q^{20}\Phi_4\Phi_5\Phi_8\Phi_{12}$	$q^{20}\Phi_5$	q^{20}
$[64_p']$	$q^{13}\Phi_2^3\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$	$q^{13}\Phi_2^2\Phi_4\Phi_6(q^4+q^3+1)$	$q^{13}\Phi_2^2\Phi_4\Phi_6$
$[60_p']$	$q^{11}\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{12}$	$q^{11}\Phi_5(2q^6+q^4+q^3+1)$	$q^{11}(q^7+2q^6+q^5+2q^4+2q^3+q^2+q+1)$
$[81_p']$	$q^{10}\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}$	$q^{10}\Phi_3^2\Phi_6(q^6+q^4+q^3+1)$	$q^{10}\Phi_3\Phi_5\Phi_6$
$[24_p']$	$q^{12}\Phi_4^2\Phi_8\Phi_9\Phi_{12}$	$q^{12}\Phi_4\Phi_5(q^3-q+1)$	$q^{12}\Phi_4\Phi_8$
$[30_p]$	$\frac{1}{2}q^3\Phi_4^2\Phi_5\Phi_9\Phi_{12}$	$\frac{1}{2}q^3\Phi_4\Phi_5(2q^6+q^3+1)$	$\frac{1}{2}q^3\Phi_4(q^8+2q^7+2q^6+q^5+2q^4+2q^3+q^2+q+1)$
$[15_p]$	$\frac{1}{2}q^3\Phi_5\Phi_6^2\Phi_8\Phi_9$	$\frac{1}{2}q^3\Phi_5\Phi_6(2q^6+q^3+q^2-q+1)$	$\frac{1}{2}q^3\Phi_6(q^8+q^7+3q^6+q^5+2q^4+2q^3+q^2+1)$
$[15_q]$	$\frac{1}{2}q^3\Phi_5\Phi_8\Phi_9\Phi_{12}$	$\frac{1}{2}q^3\Phi_2\Phi_5\Phi_6(2q^4-q^2+1)$	$\frac{1}{2}q^3(q^{10}+q^8+3q^7+2q^6+q^5+3q^4+q^3+q+1)$
$\mathsf{D}_4[1]$	$\frac{1}{2}q^3\Phi_1^4\Phi_3^2\Phi_5\Phi_9$	$-\frac{1}{2}q^3\Phi_1^3\Phi_3\Phi_5$	$\frac{1}{2}q^3\Phi_1^2\Phi_3(q^6-q^5+1)$
$[30_p']$	$\frac{1}{2}q^{15}\Phi_4^2\Phi_5\Phi_9\Phi_{12}$	$\frac{1}{2}q^{15}\Phi_2\Phi_4\Phi_5\Phi_6$	$\frac{1}{2}q^{15}\Phi_3\Phi_4$
$[15_p']$	$\frac{1}{2}q^{15}\Phi_5\Phi_6^2\Phi_8\Phi_9$	$\frac{1}{2}q^{15}\Phi_5\Phi_6(q^3+q^2-q+1)$	$\frac{1}{2}q^{15}\Phi_4\Phi_6$
$[15_q']$	$\frac{1}{2}q^{15}\Phi_5\Phi_8\Phi_9\Phi_{12}$	$\frac{1}{2}q^{15}\Phi_5(-q^5+2q^4+q^3-q^2+1)$	$\frac{1}{2}q^{15}\Phi_2^2\Phi_6$
$\mathsf{D}_4[\varepsilon]$	$\frac{1}{2}q^{15}\Phi_1^4\Phi_3^2\Phi_5\Phi_9$	$-\frac{1}{2}q^{15}\Phi_1^3\Phi_3\Phi_5$	$\frac{1}{2}q^{15}\Phi_1^2\Phi_3$
$[80_s]$	$\frac{1}{6}q^7\Phi_2^4\Phi_5\Phi_8\Phi_9\Phi_{12}$	$\frac{1}{6}q^7\Phi_2^3\Phi_5(2q^7-q^6+q^5+3q^4-q^2+q+1)$	$\frac{1}{6}q^7\Phi_2^3(3q^7+q^6+2q^5+4q^4+q^3+q^2+2q+1)$
$[20_s]$	$\frac{1}{6}q^7\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9$	$\frac{1}{6}q^7\Phi_4\Phi_5\Phi_6(2q^6+q^5+q^4+2q^2-q+1)$	$\frac{1}{6}q^7\Phi_4\Phi_6(3q^6+q^5+3q^4+2q^3+2q^2+1)$
$[60_s]$	$\frac{1}{2}q^7\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}$	$\frac{1}{2}q^7\Phi_4\Phi_5(q^7+2q^6+q^4+q^3+1)$	$\frac{1}{2}q^7\Phi_4(q^8+2q^7+3q^6+2q^5+3q^4+2q^3+q^2+q+1)$
$[10_s]$	$\frac{1}{3}q^7\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	$\frac{1}{3}q^7\Phi_5\Phi_6(-q^8+q^7+3q^6-2q^5+2q^3-q+1)$	$\frac{1}{3}q^7\Phi_6(q^7+3q^6+2q^4+2q^3+1)$
$[90_s]$	$\frac{1}{3}q^7\Phi_3^3\Phi_5\Phi_6^2\Phi_8\Phi_{12}$	$\frac{1}{3}q^7\Phi_3^2\Phi_5\Phi_6(2q^4+1)$	$\frac{1}{3}q^7\Phi_3^2\Phi_6(3q^4+q^3+q^2+q+1)$
$\mathsf{D}_4[r]$	$\frac{1}{2}q^7\Phi_1^4\Phi_3^2\Phi_5\Phi_8\Phi_9$	$-\frac{1}{2}q^7\Phi_1^3\Phi_3\Phi_5\Phi_8$	$\frac{1}{2}q^7\Phi_1^2\Phi_3(q^6-q^5+q^4+1)$
$\mathsf{E}_6[\omega]$	$\frac{1}{3}q^7\Phi_1^6\Phi_2^4\Phi_4^2\Phi_5\Phi_8$	$-\frac{1}{3}q^7\Phi_1^5\Phi_2^3\Phi_4\Phi_5$	$\frac{1}{3}q^7\Phi_1^4\Phi_2^3\Phi_4$
$\mathsf{E}_6[\omega^2]$	$\frac{1}{3}q^7\Phi_1^6\Phi_2^4\Phi_4^2\Phi_5\Phi_8$	$-\frac{1}{3}q^7\Phi_1^5\Phi_2^3\Phi_4\Phi_5$	$\frac{1}{3}q^7\Phi_1^4\Phi_2^3\Phi_4$

Table B.2.: Values of $\mathrm{Uch}(\mathsf{E}_6(q))$, $q=2^n$, at unipotent elements. The u_i $(0 \leqslant i \leqslant 27)$ are the representatives for the conjugacy classes as in [Mal93], Φ_m $(m \in \mathbb{N})$ is the mth cyclotomic polynomial evaluated at q.

	u_3	u_4	u_5
$[1_p]$	1	1	1
$[6_p]$	$q(q^4+q^3+1)$	$q(2q^4+q^3+1)$	$q\Phi_2\Phi_6$
$[20_p]$	$q^2(q^6+q^5+2q^4+q^3+q^2+q+1)$	$q^2(q^6+2q^5+3q^4+q^3+q^2+q+1)$	$q^2\Phi_4(q^4+q+1)$
$[64_p]$	$q^4\Phi_2(q^6+q^5+q^4+2q^3+q^2+1)$	$q^4\Phi_2(2q^6+q^5+2q^4+3q^3+q^2+1)$	$q^4\Phi_2^2\Phi_4\Phi_6$
$[60_p]$	$q^5(q^7+2q^6+2q^5+3q^4+2q^3+q^2+q+1)$	$q^5(q^6+3q^5+5q^4+2q^3+q^2+q+1)$	$q^5\Phi_2^2\Phi_4\Phi_6$
$[81_p]$	$q^6\Phi_3(q^4+q^3+q^2+1)$	$q^6\Phi_3^2(2q^2-q+1)$	$q^6\Phi_2\Phi_3\Phi_6$
$[24_p]$	$q^6\Phi_5\Phi_6$	$q^6(q^6+q^4+2q^3+q^2+1)$	$q^6\Phi_4\Phi_8$
$[1'_p]$	.	.	.
$[6'_p]$	.	.	.
$[20'_p]$	.	.	.
$[64'_p]$	$q^{13}\Phi_2$	$2q^{13}\Phi_2$	.
$[60'_p]$	$q^{11}\Phi_5$	$q^{11}\Phi_2^2$	$q^{11}\Phi_4$
$[81'_p]$	$q^{10}\Phi_3\Phi_4$	$q^{10}\Phi_3^2$	$q^{10}\Phi_3\Phi_6$
$[24'_p]$	q^{12}	$2q^{12}$	.
$[30_p]$	$\frac{1}{2}q^3(q^7+3q^6+2q^5+3q^4+3q^3+2q^2+q+1)$	$\frac{1}{2}q^3(q^7+5q^6+4q^5+4q^4+4q^3+2q^2+q+1)$	$\frac{1}{2}q^3\Phi_4(-q^5+q^4+q^3+q^2+q+1)$
$[15_p]$	$\frac{1}{2}q^3(-q^7+3q^6+q^4+q^3+2q^2-q+1)$	$\frac{1}{2}q^3(q^7+5q^6+2q^4+2q^3+2q^2-q+1)$	$\frac{1}{2}q^3\Phi_6(-q^5+q^3+q^2+1)$
$[15_q]$	$\frac{1}{2}q^3\Phi_2(q^6+2q^4+q^3+1)$	$\frac{1}{2}q^3(-q^7+q^6+4q^5+4q^4+q+1)$	$\frac{1}{2}q^3\Phi_2^3\Phi_6^2$
$\mathsf{D}_4[1]$	$-\frac{1}{2}q^3\Phi_1\Phi_{18}$	$-\frac{1}{2}q^3\Phi_1^3\Phi_3^2$	$\frac{1}{2}q^3(q^7+q^6-q+1)$
$[30'_p]$	$\frac{1}{2}q^{15}\Phi_2$	q^{15}	.
$[15'_p]$	$-\frac{1}{2}q^{15}\Phi_1$	q^{15}	.
$[15'_q]$	$\frac{1}{2}q^{15}\Phi_2$	.	q^{15}
$\mathsf{D}_4[\varepsilon]$	$-\frac{1}{2}q^{15}\Phi_1$	.	q^{15}
$[80_s]$	$\frac{1}{6}q^7\Phi_2^2(3q^5+3q^4+2q^3+3q^2+3q+1)$	$\frac{1}{6}q^7\Phi_2^3\Phi_3^2$	$\frac{1}{6}q^7\Phi_2^2(-q^5+3q^4-q^3+q^2+3q+1)$
$[20_s]$	$\frac{1}{6}q^7(-3q^7+3q^6-q^5+4q^4-q^3+4q^2-q+1)$	$\frac{1}{6}q^7(q^7+5q^6+6q^5+6q^4+6q^2-q+1)$	$-\frac{1}{6}q^7\Phi_1\Phi_3\Phi_4\Phi_6$
$[60_s]$	$\frac{1}{2}q^7\Phi_4(q^5+3q^4+2q^3+q^2+q+1)$	$\frac{1}{2}q^7(-q^7+q^6+2q^5+6q^4+4q^3+2q^2+q+1)$	$\frac{1}{2}q^7\Phi_2\Phi_3\Phi_4\Phi_6$
$[10_s]$	$\frac{1}{3}q^7\Phi_6(3q^4+2q^3+1)$	$\frac{1}{3}q^7(q^7-q^6-3q^5+3q^4+3q^3-q+1)$	$\frac{1}{3}q^7\Phi_6(-q^5+2q^3+q^2+1)$
$[90_s]$	$\frac{1}{3}q^7\Phi_3(3q^4+2q^3+2q^2+q+1)$	$\frac{1}{3}q^7\Phi_3^2(q^3+3q^2+1)$	$\frac{1}{3}q^7\Phi_3\Phi_6(-q^3+q^2+2q+1)$
$\mathsf{D}_4[r]$	$\frac{1}{2}q^7\Phi_1(-q^6-q^4+q^3-1)$	$-\frac{1}{2}q^7\Phi_1^3\Phi_3^2$	$\frac{1}{2}q^7(q^7+q^6+2q^4-q+1)$
$\mathsf{E}_6[\omega]$	$-\frac{1}{3}q^7\Phi_1^3\Phi_2^2$	$\frac{1}{3}q^7\Phi_1^4\Phi_2^3$	$-\frac{1}{3}q^7\Phi_1^3\Phi_2^2\Phi_4$
$\mathsf{E}_6[\omega^2]$	$-\frac{1}{3}q^7\Phi_1^3\Phi_2^2$	$\frac{1}{3}q^7\Phi_1^4\Phi_2^3$	$-\frac{1}{3}q^7\Phi_1^3\Phi_2^2\Phi_4$

Values of $\mathrm{Uch}(\mathsf{E}_6(q))$, $q=2^n$, at unipotent elements. The u_i $(0\leqslant i\leqslant 27)$ are the representatives for the conjugacy classes as in [Mal93], Φ_m $(m\in\mathbb{N})$ is the mth cyclotomic polynomial evaluated at q. (2/5)

	u_6	u_7	u_8	u_9	u_{10}
$[1_p]$	1	1	1	1	1
$[6_p]$	$q(q^4+q^3+1)$	$q\Phi_8$	$q\Phi_2\Phi_6$	$q(q^3+q^2+1)$	q
$[20_p]$	$q^2(q^5+2q^4+q^3+q^2+q+1)$	$q^2(q^4+q^2+q+1)$	$q^2\Phi_5$	$q^2\Phi_3\Phi_4$	$q^2\Phi_3$
$[64_p]$	$q^4(q^6+q^5+3q^4+3q^3+q^2+q+1)$	$q^4\Phi_2^2\Phi_6$	$q^4\Phi_2(q^3+q^2+1)$	$q^4\Phi_2^2\Phi_4$	$q^4\Phi_2$
$[60_p]$	$q^5(q^5+3q^4+2q^3+q^2+q+1)$	$q^5(q^4+q^2+q+1)$	$q^5\Phi_2(q^3+q^2+1)$	$q^5\Phi_2\Phi_4$	$q^5\Phi_3$
$[81_p]$	$q^6\Phi_3(q^3+q^2+1)$	$q^6\Phi_3\Phi_6$	$q^6\Phi_2\Phi_4$	$q^6(q^3+q^2+q+2)$	q^6
$[24_p]$	$q^6(q^3+q^2+1)$	q^6	$q^6\Phi_4$	$q^6\Phi_4$	q^6
$[1'_p]$	.	.	.	.	.
$[6'_p]$	.	.	.	.	.
$[20'_p]$	.	.	.	.	.
$[64'_p]$	q^{13}	.	.	.	.
$[60'_p]$	$q^{11}\Phi_2$	.	q^{11}	.	.
$[81'_p]$	$q^{10}\Phi_3$	.	q^{10}	q^{10}	.
$[24'_p]$	q^{12}	q^{12}	.	.	.
$[30_p]$	$\frac{1}{2}q^3\Phi_4(2q^4+2q^3+q^2+q+1)$	$\frac{1}{2}q^3\Phi_2^2\Phi_4\Phi_6$	$\frac{1}{2}q^3(2q^4+2q^3+2q^2+q+1)$	$\frac{1}{2}q^3\Phi_4(2q^2+2q+1)$	$\frac{1}{2}q^3\Phi_2\Phi_4$
$[15_p]$	$\frac{1}{2}q^3\Phi_6(2q^4+2q^3+q^2+1)$	$\frac{1}{2}q^3\Phi_6\Phi_8$	$\frac{1}{2}q^3(2q^2-q+1)$	$\frac{1}{2}q^3\Phi_4(2q^2+1)$	$-\frac{1}{2}q^3\Phi_1\Phi_4$
$[15_q]$	$\frac{1}{2}q^3\Phi_2(2q^4+q^3+1)$	$\frac{1}{2}q^3\Phi_3\Phi_8$	$\frac{1}{2}q^3\Phi_2(2q^3+1)$	$\frac{1}{2}q^3\Phi_2(2q^2-q+1)$	$\frac{1}{2}q^3\Phi_2\Phi_4$
$\mathsf{D}_4[1]$	$\frac{1}{2}q^3\Phi_1^2\Phi_3$	$\frac{1}{2}q^3\Phi_1^2\Phi_3\Phi_4$	$-\frac{1}{2}q^3\Phi_1$	$\frac{1}{2}q^3\Phi_1^2$	$-\frac{1}{2}q^3\Phi_1\Phi_4$
$[30'_p]$	.	.	.	.	.
$[15'_p]$	.	.	.	.	.
$[15'_q]$	.	.	.	.	.
$\mathsf{D}_4[\varepsilon]$	.	.	.	.	.
$[80_s]$	$\frac{1}{6}q^7\Phi_2^2\Phi_3(2q+1)$	$\frac{1}{6}q^7\Phi_2^2\Phi_3$	$\frac{1}{6}q^7\Phi_2(4q^2+4q+1)$	$\frac{1}{2}q^7\Phi_2^2$	$\frac{1}{6}q^7\Phi_2(2q+1)$
$[20_s]$	$\frac{1}{6}q^7\Phi_6(2q^3+3q^2+1)$	$\frac{1}{6}q^7\Phi_1^2\Phi_6$	$\frac{1}{6}q^7\Phi_1(-2q^2-1)$	$\frac{1}{2}q^7\Phi_4$	$\frac{1}{6}q^7\Phi_1(2q-1)$
$[60_s]$	$\frac{1}{2}q^7(3q^4+3q^3+2q^2+q+1)$	$\frac{1}{2}q^7\Phi_2^2\Phi_6$	$\frac{1}{2}q^7\Phi_2(2q^2+1)$	$\frac{1}{2}q^7\Phi_4$	$\frac{1}{2}q^7\Phi_2$
$[10_s]$	$\frac{1}{3}q^7(-q^5+q^4+2q^3+q^2-q+1)$	$\frac{1}{3}q^7\Phi_3\Phi_6$	$\frac{1}{3}q^7(q^3+2q^2-q+1)$	.	$\frac{1}{3}q^7(2q^2+1)$
$[90_s]$	$\frac{1}{3}q^7\Phi_2\Phi_3(2q^2+1)$	$\frac{1}{3}q^7\Phi_3\Phi_6$	$\frac{1}{3}q^7\Phi_2\Phi_3$	$q^7\Phi_3$	$-\frac{1}{3}q^7\Phi_1\Phi_2$
$\mathsf{D}_4[r]$	$\frac{1}{2}q^7\Phi_1^2\Phi_3$	$\frac{1}{2}q^7\Phi_1^2\Phi_3$	$-\frac{1}{2}q^7\Phi_1$	$\frac{1}{2}q^7\Phi_1^2$	$-\frac{1}{2}q^7\Phi_1$
$\mathsf{E}_6[\omega]$	$-\frac{1}{3}q^7\Phi_1^3\Phi_2^2$	$\frac{1}{3}q^7\Phi_1^2\Phi_2^2$	$\frac{1}{3}q^7\Phi_1^2\Phi_2$	.	$-\frac{1}{3}q^7\Phi_1\Phi_2$
$\mathsf{E}_6[\omega^2]$	$-\frac{1}{3}q^7\Phi_1^3\Phi_2^2$	$\frac{1}{3}q^7\Phi_1^2\Phi_2^2$	$\frac{1}{3}q^7\Phi_1^2\Phi_2$	.	$-\frac{1}{3}q^7\Phi_1\Phi_2$

Values of Uch($\mathsf{E}_6(q)$), $q = 2^n$, at unipotent elements. The u_i ($0 \leqslant i \leqslant 27$) are the representatives for the conjugacy classes as in [Mal93], Φ_m ($m \in \mathbb{N}$) is the mth cyclotomic polynomial evaluated at q. (3/5)

	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
$[1_p]$	1	1	1	1	1
$[6_p]$	$q\Phi_4$	$q(2q^2+1)$	$-q\Phi_1\Phi_2$	q	$q\Phi_4$
$[20_p]$	$q^2(2q^2+q+1)$	$q^2(3q^2+q+1)$	$q^2\Phi_2$	$q^2\Phi_3$	$q^2\Phi_3$
$[64_p]$	$q^4\Phi_2^2$	$q^4\Phi_2(2q+1)$	$-q^4\Phi_1\Phi_2$	$q^4\Phi_2$	$q^4\Phi_2$
$[60_p]$	$q^5\Phi_3$	$q^5\Phi_2$	$q^5\Phi_2$	$q^5\Phi_2$	q^5
$[81_p]$	$2q^6$	$3q^6$	.	q^6	q^6
$[24_p]$	q^6	q^6	q^6	q^6	.
$[1'_p]$	.	.	.	.	.
$[6'_p]$	.	.	.	.	.
$[20'_p]$	.	.	.	.	.
$[64'_p]$	.	.	.	.	.
$[60'_p]$	.	.	.	.	.
$[81'_p]$	.	.	.	.	.
$[24'_p]$	.	.	.	.	.
$[30_p]$	$\frac{1}{2}q^3\Phi_2\Phi_3$	$\frac{1}{2}q^3\Phi_2^3$	$\frac{1}{2}q^3\Phi_2\Phi_6$	$\frac{1}{2}q^3(-q^3+q^2+q+1)$	$\frac{1}{2}q^3\Phi_2^2$
$[15_p]$	$\frac{1}{2}q^3(-q^3+2q^2+1)$	$\frac{1}{2}q^3(q^3+3q^2+q+1)$	$\frac{1}{2}q^3\Phi_1(q^2+q-1)$	$-\frac{1}{2}q^3\Phi_1\Phi_4$	$\frac{1}{2}q^3\Phi_4$
$[15_q]$	$\frac{1}{2}q^3(q^3+2q^2+1)$	$\frac{1}{2}q^3(-q^3+3q^2-q+1)$	$\frac{1}{2}q^3\Phi_2(-q^2+q+1)$	$\frac{1}{2}q^3\Phi_2\Phi_4$	$\frac{1}{2}q^3\Phi_4$
$\mathsf{D}_4[1]$	$-\frac{1}{2}q^3\Phi_1\Phi_6$	$-\frac{1}{2}q^3\Phi_1^3$	$-\frac{1}{2}q^3\Phi_1\Phi_3$	$\frac{1}{2}q^3(q^3+q^2-q+1)$	$\frac{1}{2}q^3\Phi_1^2$
$[30'_p]$	.	.	.	.	.
$[15'_p]$	.	.	.	.	.
$[15'_q]$	.	.	.	.	.
$\mathsf{D}_4[\varepsilon]$	.	.	.	.	.
$[80_s]$	$\frac{1}{2}q^7\Phi_2$	q^7	.	.	.
$[20_s]$	$-\frac{1}{2}q^7\Phi_1$	q^7	.	.	.
$[60_s]$	$\frac{1}{2}q^7\Phi_2$	.	.	q^7	.
$[10_s]$	.	.	q^7	.	.
$[90_s]$	q^7	$2q^7$	.	.	.
$\mathsf{D}_4[r]$	$-\frac{1}{2}q^7\Phi_1$	.	.	q^7	.
$\mathsf{E}_6[\omega]$	.	.	q^7	.	.
$\mathsf{E}_6[\omega^2]$	.	.	q^7	.	.

Values of $\mathrm{Uch}(\mathsf{E}_6(q))$, $q=2^n$, at unipotent elements. The u_i $(0 \leqslant i \leqslant 27)$ are the representatives for the conjugacy classes as in [Mal93], Φ_m $(m \in \mathbb{N})$ is the mth cyclotomic polynomial evaluated at q. (4/5)

	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}	u_{26}	u_{27}
$[1_p]$	1	1	1	1	1	1	1	1	1	1	1	1
$[6_p]$	$q\Phi_2$	$q\Phi_2$	q	$q\Phi_2$	q	$q\Phi_2$	$-q\Phi_1$	q	q	q	.	.
$[20_p]$	$q^2\Phi_3$	$q^2\Phi_3$	$q^2\Phi_2$	$q^2\Phi_2$	q^2	q^2	q^2	q^2	q^2	.	.	.
$[64_p]$	$q^4\Phi_2$	$q^4\Phi_2$	q^4	q^4	.	.	.	.	.	.	.	.
$[60_p]$	.	.	q^5	.	.	.	.	.	.	.	.	.
$[81_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[24_p]$	q^6	q^6	.	.	.	.	.	.	.	.	.	.
$[1'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[6'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[20'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[64'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[60'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[81'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[24'_p]$	.	.	.	.	.	.	.	.	.	.	.	.
$[30_p]$	$\frac{1}{2}q^3(q^2+2)$	$\frac{1}{2}q^3(-q^2+2)$	$\frac{1}{2}q^3\Phi_2$	q^3	$\frac{1}{2}q^3\Phi_2$	q^3	.	$\frac{1}{2}q^3$	$-\frac{1}{2}q^3$	.	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$
$[15_p]$	$\frac{1}{2}q^3(-q^2+2)$	$\frac{1}{2}q^3(q^2+2)$	$-\frac{1}{2}q^3\Phi_1$	q^3	$-\frac{1}{2}q^3\Phi_1$	q^3	.	$-\frac{1}{2}q^3$	$\frac{1}{2}q^3$	.	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$
$[15_q]$	$-\frac{1}{2}q^5$	$\frac{1}{2}q^5$	$\frac{1}{2}q^3\Phi_2$	.	$\frac{1}{2}q^3\Phi_2$	.	q^3	$-\frac{1}{2}q^3$	$\frac{1}{2}q^3$	.	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$
$\mathsf{D}_4[1]$	$\frac{1}{2}q^5$	$-\frac{1}{2}q^5$	$-\frac{1}{2}q^3\Phi_1$	.	$-\frac{1}{2}q^3\Phi_1$	.	q^3	$\frac{1}{2}q^3$	$-\frac{1}{2}q^3$	.	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$
$[30'_p]$	$\frac{1}{2}q^8$	$-\frac{1}{2}q^8$	.	.	.	.	.	.	.	.	.	.
$[15'_p]$	$-\frac{1}{2}q^8$	$\frac{1}{2}q^8$	.	.	.	.	.	.	.	.	.	.
$[15'_q]$	$-\frac{1}{2}q^8$	$\frac{1}{2}q^8$	.	.	.	.	.	.	.	.	.	.
$\mathsf{D}_4[\varepsilon]$	$\frac{1}{2}q^8$	$-\frac{1}{2}q^8$	.	.	.	.	.	.	.	.	.	.
$[80_s]$	$\frac{1}{2}q^6\Phi_2$	$-\frac{1}{2}q^6\Phi_2$	.	.	.	.	.	$\frac{1}{2}q^4$	$-\frac{1}{2}q^4$	.	.	.
$[20_s]$	$-\frac{1}{2}q^6\Phi_2$	$\frac{1}{2}q^6\Phi_2$	.	.	.	.	.	$-\frac{1}{2}q^4$	$\frac{1}{2}q^4$	.	.	.
$[60_s]$	$-\frac{1}{2}q^6\Phi_2$	$\frac{1}{2}q^6\Phi_2$	.	.	.	.	.	$-\frac{1}{2}q^4$	$\frac{1}{2}q^4$	.	.	.
$[10_s]$	.	.	.	.	.	.	.	.	.	.	.	.
$[90_s]$	.	.	.	.	.	.	.	.	.	.	.	.
$\mathsf{D}_4[r]$	$\frac{1}{2}q^6\Phi_2$	$-\frac{1}{2}q^6\Phi_2$	.	.	.	.	.	$\frac{1}{2}q^4$	$-\frac{1}{2}q^4$	.	.	.
$\mathsf{E}_6[\omega]$	.	.	.	.	.	.	.	.	.	.	.	.
$\mathsf{E}_6[\omega^2]$	.	.	.	.	.	.	.	.	.	.	.	.

Values of $\mathrm{Uch}(\mathsf{E}_6(q))$, $q = 2^n$, at unipotent elements. The u_i $(0 \leqslant i \leqslant 27)$ are the representatives for the conjugacy classes as in [Mal93], Φ_m $(m \in \mathbb{N})$ is the mth cyclotomic polynomial evaluated at q. (5/5)

	w_{13}	w_{21}	w_{28}	w_{41}	w_{38}	w_{52}	w_{77}	w_{85}
1_a	q^{11}	q^6	q^6	q^8	q^6	q^5	q^5	q^4
$7'_a$	0	$q^6 - q^5$	$q^6 - q^5$	q^8	$q^6 - 2q^5$	$2q^5 - q^4$	$2q^5 - 2q^4$	$3q^4 - q^3$
27_a	0	$q^6 - q^5$	$q^6 - q^5$	$q^8 - 2q^6$	$2q^6 - 2q^5 + q^4$	$3q^5 - 2q^4$	$4q^5 - 4q^4 + q^3$	$6q^4 - 3q^3$
$21'_b$		q^6	$-q^5$	$-q^6$	$q^6 - 2q^5$	$2q^5 - q^4$	$2q^5 - 4q^4$	$4q^4 - 2q^3$
$56'_a$			$q^6 - q^5$	$q^8 - q^6 + q^5$	$q^6 - 3q^5 + q^4$	$2q^5 - 3q^4$	$4q^5 - 6q^4 + 2q^3$	$6q^4 - 6q^3$
21_a			$-q^5 + q^4$	0	$-2q^5 + 2q^4$	$q^5 - 2q^4 + q^3$	$q^5 - 4q^4 + 2q^3$	$3q^4 - 3q^3 + q^2$
120_a			$-q^5 + q^4$	$-2q^6 + q^5$	$q^6 - 3q^5 + 3q^4$	$2q^5 - 4q^4 + q^3$	$4q^5 - 10q^4 + 6q^3$	$8q^4 - 10q^3 + 2q^2$
$105'_a$			$-q^5 + q^4$	$-2q^6 + 2q^5$	$q^6 - 3q^5 + 3q^4 - q^3$	$2q^5 - 4q^4 + 2q^3$	$4q^5 - 9q^4 + 6q^3 - q^2$	$8q^4 - 9q^3 + 3q^2$
35_b			q^6	$q^8 - q^6$	$q^6 - q^5$	$q^5 - q^4$	$3q^5 - 2q^4 + q^3$	$3q^4 - 3q^3$
$189'_b$				$-q^6 + 2q^5$	$q^6 - 2q^5 + 2q^4$	$q^5 - 3q^4 + q^3$	$4q^5 - 10q^4 + 6q^3 - q^2$	$6q^4 - 12q^3 + 3q^2$
105_b				$-q^6$	$-q^5$	$-q^4$	$q^5 - 4q^4 + 3q^3$	$q^4 - 6q^3 + q^2$
210_a				$-q^6 + 2q^5 - q^4$	$-3q^5 + 4q^4 - q^3$	$q^5 - 4q^4 + 3q^3$	$3q^5 - 12q^4 + 10q^3 - 2q^2$	$7q^4 - 14q^3 + 6q^2$
168_a					$q^6 - q^5 + q^4 - q^3$	$q^5 - 2q^4 + q^3$	$4q^5 - 6q^4 + 6q^3 - 2q^2$	$6q^4 - 8q^3 + 4q^2$
$315'_a$						$-2q^4 + 2q^3$	$2q^5 - 8q^4 + 10q^3 - 5q^2$	$3q^4 - 13q^3 + 8q^2 - q$
$280'_a$						$-3q^4 + 4q^3 - q^2$	$2q^5 - 8q^4 + 14q^3 - 6q^2$	$4q^4 - 14q^3 + 10q^2 - 2q$
$35'_a$						$-q^4 + 2q^3 - q^2$	$-2q^4 + 4q^3 - 2q^2$	$q^4 - 3q^3 + 3q^2 - q$
$189'_c$						$q^5 - 2q^4 + 2q^3$	$2q^5 - 9q^4 + 8q^3 - 2q^2$	$6q^4 - 9q^3 + 6q^2$
280_b							$2q^5 - 6q^4 + 8q^3 - 4q^2$	$2q^4 - 10q^3 + 8q^2$
405_a							$q^5 - 10q^4 + 15q^3 - 6q^2 + q$	$3q^4 - 15q^3 + 12q^2 - 3q$
189_a							$q^5 - 6q^4 + 10q^3 - 6q^2 + q$	$3q^4 - 9q^3 + 9q^2 - 3q$
$378'_a$							$2q^5 - 6q^4 + 10q^3 - 7q^2 + 2q$	$3q^4 - 9q^3 + 12q^2 - 3q$
420_a							$q^5 - 6q^4 + 13q^3 - 12q^2 + 2q$	$3q^4 - 10q^3 + 15q^2 - 6q$
$336'_a$							$-6q^4 + 14q^3 - 10q^2 + 2q$	$2q^4 - 10q^3 + 14q^2 - 6q$
84_a							$q^5 + q^3 - 2q^2$	$q^4 + 3q^2$
$105'_c$								$q^4 + 4q^2 - 3q$
$(3,0)$	q^8	q^4	q^4	$-q^5$	q^4	q^3	q^3	q^2
$(0,3)$	$-q^7$	q^4	0	q^5	$-q^3$	q^3	$-q^2$	q^2
$(2,1)$	0	$q^4 - q^3$	$q^4 - q^3$	$-q^5$	$q^4 - 2q^3$	$2q^3 - q^2$	$2q^3 - q^2$	$3q^2$
$(1,2)$		$q^4 - q^3$	$-q^3$	q^5	$q^4 - q^3 + q^2$	$2q^3 - q^2$	$q^3 - 2q^2$	$3q^2$
$(21,0)$			q^4	$-q^5 + q^3$	$q^4 - q^3$	$q^3 - q^2$	$2q^3$	$2q^2$
$(0,21)$				$q^5 - q^3$	$-q^3 + q^2$	$q^3 - q^2$	$-2q^2$	$2q^2$
$(1^2,1)$					$q^4 - q^3 + q^2$	$q^3 - 2q^2$	$2q^3 - q^2$	$3q^2$
$(1,1^2)$						$q^3 - 2q^2$	$q^3 - 2q^2$	$3q^2$
$(1^3,0)$							q^3	q^2
$(0,1^3)$								q^2

Table B.3.: The values $c_x(w)$ for certain $x \in \mathfrak{X}(\mathbf{W})$ and $w \in \mathbf{W}$, required for the proof of Proposition 4.2.26. An empty cell does not necessarily mean that this entry is zero, but only that we do not need it for the proof.

	$\mathsf{E}_7(a_1)$	$\mathsf{E}_7(a_2)$	E_6	D_6	$\mathsf{D}_6(a_1)$	$\mathsf{D}_5{+}\mathsf{A}_1$	D_5	$\mathsf{D}_4{+}\mathsf{A}_1$	D_4
1_a	1	1	1	1	1	1	1	1	1
$7'_a$	q	q	q	q	$q\Phi_4$	q	$q\Phi_4$	$q\Phi_4$	$q\Phi_3\Phi_6$
27_a	0	q^2	q^2	q^2	$q^2\Phi_4$	$q^2\Phi_4$	$q^2(2q^2+1)$	$q^2\Phi_4^2$	$q^2\Phi_3\Phi_4\Phi_6$
$21'_b$	0	0	q^3	0	q^3	q^3	$q^3\Phi_4$	$q^3\Phi_4$	$q^3\Phi_4\Phi_8$
$56'_a$	0	0	0	q^3	$q^3\Phi_4$	q^3	$q^3\Phi_4$	$q^3\Phi_4^2$	$q^3\Phi_3\Phi_4\Phi_6$
21_a	0	0	0	0	q^4	0	q^4	q^4	$q^4\Phi_3\Phi_6$
120_a	0	0	0	0	q^4	q^4	$q^4\Phi_4$	$q^4\Phi_4^2$	$q^4(2q^4+q^2+1)\Phi_4$
$105'_a$	0	0	0	0	q^5	q^5	$2q^5$	$2q^5\Phi_4$	$q^5(q^4+q^2+2)\Phi_4$
35_b	0	0	0	q^4	q^4	q^4	q^4	$q^4\Phi_3\Phi_6$	$q^4\Phi_3\Phi_6$
$189'_b$	0	0	0	0	q^5	q^5	q^5	$q^5\Phi_4^2$	$q^5\Phi_3\Phi_4\Phi_6$
105_b	0	0	0	0	0	0	0	q^6	q^6
210_a	0	0	0	0	q^6	0	q^6	$q^6(q^2+2)$	$q^6(q^6+2q^4+2q^2+2)$
168_a	0	0	0	0	0	q^6	q^6	$q^6\Phi_4^2$	$q^6\Phi_3\Phi_4\Phi_6$
$315'_a$	0	0	0	0	0	0	0	$q^7\Phi_4$	$q^7\Phi_3\Phi_6$
$280'_a$	0	0	0	0	0	0	0	$q^7\Phi_4$	$q^7\Phi_4^2$
$35'_a$	0	0	0	0	0	0	0	0	q^9
$189'_c$	0	0	0	0	0	0	q^7	$q^7\Phi_4$	$q^7\Phi_3\Phi_4\Phi_6$
280_b	0	0	0	0	0	0	0	$q^8\Phi_4$	$q^8\Phi_4$
405_a	0	0	0	0	0	0	0	q^8	$q^8\Phi_3\Phi_6$
189_a	0	0	0	0	0	0	0	q^8	$q^8\Phi_3\Phi_6$
$378'_a$	0	0	0	0	0	0	0	$q^9\Phi_4$	$q^9\Phi_3\Phi_6$
420_a	0	0	0	0	0	0	0	q^{10}	$q^{10}\Phi_3\Phi_6$
$336'_a$	0	0	0	0	0	0	0	0	$q^{11}\Phi_4$
84_a	0	0	0	0	0	0	0	q^{12}	q^{12}
$105'_c$	0	0	0	0	0	0	0	0	q^{15}

Table B.4.: Values of certain R_ϕ, $\phi \in \mathrm{Irr}(\mathbf{W})$, at the 9 unipotent classes considered in 4.2.25, for the groups $\mathbf{G}^F = \mathsf{E}_7(q)$ where $q = 2^n$. These functions are constant on a given $\mathscr{O}^F$, so it suffices to write $\mathscr{O}$ at the top of a column; Φ_m $(m \in \mathbb{N})$ is the mth cyclotomic polynomial evaluated at q.

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