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A Characterisation of Morita Algebras in Terms of Covers

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Abstract

A pair (A, P) is called a cover of $\mathrm{End}_A(P)^{op}$ if the Schur functor $\mathrm{Hom}_A(P, -)$ is fully faithful on the full subcategory of projective A-modules, for a given projective A-module P. By definition, Morita algebras are the covers of self-injective algebras and then P is a faithful projective-injective module. Conversely, we show that A is a Morita algebra and $\mathrm{End}_A(P)^{op}$ is self-injective whenever (A, P) is a cover of $\mathrm{End}_A(P)^{op}$ for a faithful projective-injective module P.

Keywords Morita algebras · Covers · Self-injective algebras

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1 Introduction

Morita algebras were introduced in [8] to better understand and generalize self-injective algebras. The definition is based on a theorem by Morita (see [9, section 16], [8, p. 185]) and it says that a Morita algebra is the endomorphism algebra of a generator over a self-injective algebra. Moreover, Morita showed that this generator can be chosen to be projective-injective of the form $Ae \simeq D(eA)$ when regarded as a left module over the Morita algebra A, for some idempotent e of A. Modules containing the regular module as a direct summand are examples of generators.

Morita algebras occur in several contexts, including cover theory and the Morita-Tachikawa correspondence.

A cover, in Rouquier's sense [11], of an algebra B is a pair (A, P) consisting of the endomorphism algebra A of a generator over B and a certain projective A-module P. Covers

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are useful to transfer properties from the cover to B through a Schur functor $\mathrm{Hom}_A(P, -)$. This construction allows us to view the module category of B as a kind of quotient of the module category of its cover A. It follows from their definition that Morita algebras are exactly the covers of self-injective algebras.

On the other hand, generators over self-injective algebras are also cogenerators. The endomorphism algebras of generators-cogenerators are described by the Morita-Tachikawa correspondence, which classifies the finite-dimensional algebras with dominant dimension at least two as the endomorphism algebras of a generator-cogenerator. The famous Nakayama conjecture claims that finite-dimensional algebras with infinite dominant dimension are self-injective.

Many interesting covers arise as endomorphism algebras of generators-cogenerators. In this situation, the following questions arise. Given a faithful projective-injective A-module P:

- When is (A, P) a cover of $\mathrm{End}_A(P)^{op}$?
- When is A a Morita algebra?
- When is $\mathrm{End}_A(P)^{op}$ a self-injective algebra?

Our main result provides answers to these questions and it provides several characterisations of Morita algebras with fewer assumptions than the theorem by Morita that motivated the definition of Morita algebras in [8, pages 185-186]:

Theorem 1 *Let A be a finite-dimensional algebra. Assume that P is a faithful projective-injective left A-module. Then the following assertions are equivalent:*

(i) (A, P) *is a cover of* $\mathrm{End}_A(P)^{op}$*;*
(ii) A *is a Morita algebra;*
(iii) *The endomorphism algebra* $\mathrm{End}_A(P)^{op}$ *is a self-injective algebra and* $\mathrm{domdim} A \geq 2$*;*
(iv) $\mathrm{domdim} A \geq 2$ *and* $\mathrm{add}_A DA \otimes_A P = \mathrm{add}_A P$.
(v) $\mathrm{domdim} A \geq 2$ *and the Nakayama functor restricts to* $DA \otimes_A -\colon \mathrm{add}_A P \to \mathrm{add}_A P$.

The implications $(ii) \Leftrightarrow (iii) \implies (i)$ are already known by [9, section 16] and Morita-Tachikawa correspondence. The equivalence $(v) \Leftrightarrow (iv) \Leftrightarrow (ii)$ is related to the study of strongly projective modules in [4]. Here, we present a shorter proof. The proof of Theorem 1 involves the study of double centralizer properties and a reformulation of the definition of Morita algebras using the Nakayama functor. Prominent examples of double centralizer properties are Soergel's double centralizer theorem [12], classical Schur–Weyl duality [6] and its many generalizations (see for example [2]).

As a byproduct of Theorem 1, we clarify in Remark 13 some situations where a double centralizer property on a module Ae is equivalent to a double centralizer property on eA, for some idempotent e of a given finite-dimensional algebra A. Further, although it does not come as a surprise, we see in Example 17 that if P is only projective the assertion (i) together with $(A, \mathrm{Hom}_A(P, A))$ being a cover of $\mathrm{End}_A(P)^{op}$ is not sufficient for A to be a Morita algebra.

As application of Theorem 1, we give in Corollary 14 a criterion for a *QF-1* algebra to be a self-injective algebra.

2 Notation

We will assume throughout this paper that k is a field and A and B are finite-dimensional k-algebras. By A-mod (resp. mod-A) we mean the category of finitely generated left (resp. right) A-modules and by A-proj the full subcategory of A-mod whose modules are the finitely generated projective A-modules. We denote by (resp. addM_A) (or just add$M$ when A is fixed) the full subcategory of A-mod (resp. mod-A) whose modules are direct summands of finite direct sums of $M \in A$-mod (resp. $M \in$ mod-A). We write A-proj to denote add$A$. For any $M \in A$-mod and $f, g \in \mathrm{End}_A(M)$ the multiplication fg is the composite $f \circ g$ of g and f. The opposite algebra of A will be denoted by A^{op}.

Given a finitely generated (A, B)-bimodule M, there is a *double centralizer property* on M between A and B provided that the multiplication maps on M induce isomorphisms $A \simeq \mathrm{End}_B(M)$ and $B \simeq \mathrm{End}_A(M)^{op}$. By the standard duality D we mean the functor $\mathrm{Hom}_k(-, k)\colon A\text{-mod} \to A^{op}\text{-mod}$.

An algebra B is called *self-injective* if the regular module B is an injective left B-module, or, equivalently, if the regular module B is a right B-module. If there exists a (B, B)-bimodule isomorphism between DB and B then B is called a *symmetric algebra*.

3 Dominant Dimension

Let

$$0 \to {}_AA \to I_0 \to I_1 \to \cdots \to I_n \to \cdots \tag{1}$$

be a minimal injective resolution of the regular module ${}_AA$. We say that the *dominant dimension* of the algebra A, denoted by domdim$A$, is $n \in \mathbb{N} \cup \{\infty\}$ if I^t is projective for $t < n$ and I_n is not. In particular, domdim$A$ is infinite if all injective modules I_t are projective. Analogously, we can define the dominant dimension using the right regular module A_A. This (right) dominant dimension is equal to domdim$A$. A detailed account on dominant dimension can be found in [10, 13]. A is called *QF-3 algebra* if domdim$A \geq 1$. In such a case, I_0 is a faithful projective-injective module. Moreover, given another faithful projective-injective module $X \in A$-mod, add$X$ = addI_0 [7, Lemma 2.3] and domdim$A \geq n$ if there exists an exact sequence

$$0 \to A \to X_0 \to X_1 \to \cdots \to X_{n-1}, \tag{2}$$

where $X_i \in$ add$X$, $i = 0, \ldots, n-1$. This last claim follows from [13, 7.7]. In particular, there exists an idempotent e such that Ae is a projective-injective faithful module which is a direct sum of pairwise non-isomorphic indecomposable modules. Under these conditions, Ae is called a *projective-injective minimal faithful* module. Furthermore, a minimal faithful projective-injective module Ae (if it exists) has a double centralizer property if and only if domdim$A \geq 2$ (see for example [10, Theorem 2]). A module $M \in A$-mod is called a *generator* if ${}_AA \in \mathrm{add}_AM$. Analogously, a module $M \in A$-mod is called a *cogenerator* if $DA \in \mathrm{add}_AM$. For self-injective algebras the notions of generator and cogenerator coincide.

Theorem 2 (Morita-Tachikawa correspondence) [10, Theorem 2] *There is a bijection:*

$$\left\{(B, M)\colon \begin{array}{c} B \textit{ finite dimensional} \\ k\textit{-algebra} \\ M \textit{ a } B\textit{-generator-cogenerator} \end{array}\right\} \Big/ \sim_1 \longleftrightarrow \left\{A\colon \begin{array}{c} A \textit{ finite dimensional} \\ k\textit{-algebra} \\ \textit{domdim} A \geq 2 \end{array}\right\} \Big/ \sim_2$$

Here, $A \sim_2 A'$ *if and only if* A *and* A' *are isomorphic, whereas,* $(B, M) \sim_1 (B', M')$ *if and only if there is an equivalence of categories* $F: B\text{-mod} \to B'\text{-mod}$ *such that* $M' = FM$.

$$\begin{aligned} (B, M) &\mapsto A = End_B(M)^{op} \\ (End_A(N), N) &\mapsfrom A \end{aligned}$$

where N *is a minimal projective-injective faithful right* A*-module.*

Usually, the Morita-Tachikawa correspondence is formulated for basic algebras. However, the above formulation is also equivalent due to a double centralizer property being a Morita invariant property.

Theorem 3 [13, 10.1] *Let* A *and* B *be finite-dimensional* k*-algebras. Suppose that there is an equivalence* $H: A\text{-mod} \to B\text{-mod}$*. If there is a double centralizer property on* $M \in A\text{-mod}$ *then there is a double centralizer property on* $HM \in B\text{-mod}$*.*

4 Covers

The theory of covers was introduced by Rouquier [11].

Lemma 4 [11, Proposition 4.33] *Let* A *and* B *be finite-dimensional* k*-algebras such that* $B = \mathrm{End}_A(P)^{op}$*, for some* $P \in A\text{-proj}$*. Denote by* F *the Schur functor* $\mathrm{Hom}_A(P, -): A\text{-mod} \to B\text{-mod}$ *and denote by* G *its right adjoint* $\mathrm{Hom}_B(FA, -)$*. The following assertions are equivalent.*

(i) *The canonical map of algebras* $A \to \mathrm{End}_B(FA)^{op}$*, given by* $a \mapsto (f \mapsto f(-)a)$*,* $a \in A$*,* $f \in FA$*, is an isomorphism of* k*-algebras.*
(ii) *For all* $M \in A\text{-proj}$*, the unit* $\eta_M: M \to GFM$ *is an isomorphism of* A*-modules.*
(iii) *The restriction of* F *to* $A\text{-proj}$ *is full and faithful.*

Definition 5 We say that (A, P) is a **cover** of B if the restriction of $F = \mathrm{Hom}_A(P, -): A\text{-mod} \to B\text{-mod}$ to $A\text{-proj}$ is full and faithful.

Remark 6 In the situation of Definition 5, a double centralizer property holds on FA, but not necessarily on P.

Before we proceed with some basic results about covers, recall the following result.

Proposition 7 *Let* $M, N \in A\text{-mod}$ *with* $\mathrm{add}_A M = \mathrm{add}_A N$*. Then the algebras* $B := \mathrm{End}_A(M)^{op}$ *and* $C := \mathrm{End}_A(N)^{op}$ *are Morita equivalent and the algebras* $\mathrm{End}_B(M)$ *and* $\mathrm{End}_C(N)$ *are isomorphic.*

Proof See for example [8, Proposition 1.3]. □

Proposition 8 *Let* A *be a QF-3 algebra with a projective-injective faithful right* A*-module* V*. If* $\mathrm{domdim} A \geq 2$ *then* $(A, \mathrm{Hom}_A(V, A))$ *is a cover of* $B := End_A(V)$*.*

Proof Let eA be the minimal right projective-injective faithful A-module. Since domdim$A \geq 2$ there is a double centralizer property $\mathrm{End}_{eAe}(eA)^{op} \simeq A$. Because of V being faithful projective-injective, addV_A = addeA_A. By Proposition 7, $\mathrm{End}_{eAe}(eA)^{op} \simeq \mathrm{End}_B(V)^{op}$. Thus,

$$A \simeq \mathrm{End}_{eAe}(eA)^{op} \simeq \mathrm{End}_B(V)^{op} \simeq \mathrm{End}_B(\mathrm{Hom}_A(\mathrm{Hom}_A(V, A), A))^{op}. \quad (3)$$

The last isomorphism follows from V being right A-projective and therefore V being reflexive, that is, $V \simeq \mathrm{Hom}_A(\mathrm{Hom}_A(V, A), A)$. Further, this isomorphism is also an isomorphism of B-modules. So, the claim follows. □

The definition of cover can be formulated in general for finitely generated projective algebras over commutative Noetherian rings. Unlike the general case, covers of finite-dimensional algebras can always be reduced to covers arising from idempotents.

Proposition 9 *If (A, P) is a cover of B then there exists an idempotent $e \in A$ such that (A, Ae) is a cover of eAe and eAe is Morita equivalent to B.*

Proof We can decompose P into a direct sum of projective indecomposables $P_1 \oplus \cdots \oplus P_n$. By the Krull-Remak-Schmidt Theorem, there is a subset I of $\{1, \ldots, n\}$ so that $Q := \bigoplus_{i\in I} P_i$ is an A-summand of A, where the modules P_i, $i \in I$, are pairwise non-isomorphic and add$Q$ = add$P$. Moreover, there exists an idempotent $e \in A$ such that $Ae \simeq Q$. Hence, the algebras B and eAe are Morita equivalent. The functor $\mathrm{Hom}_B(\mathrm{Hom}_A(P, Ae), -)\colon B\text{-mod} \to eAe\text{-mod}$ is an equivalence of categories. On the other hand, the canonical map $\mathrm{Hom}_A(Ae, A) \to \mathrm{Hom}_B(F(Ae), FA)$ is bijective. Moreover, it is an eAe-isomorphism. Therefore,

$$\begin{aligned} A &\simeq \mathrm{End}_B(\mathrm{Hom}_A(P, A))^{op} \simeq \mathrm{End}_{eAe}(\mathrm{Hom}_B(\mathrm{Hom}_A(P, Ae), \mathrm{Hom}_A(P, A)))^{op} \quad (4)\\ &= \mathrm{End}_{eAe}(\mathrm{Hom}_B(F(Ae), FA))^{op} \simeq \mathrm{End}_{eAe}(\mathrm{Hom}_A(Ae, A))^{op}. \end{aligned}$$

□

As mentioned, covers can be used to obtain properties of the module category of an algebra using one of its covers, for example, the number of blocks, or classification of simple modules, among many others. Although we do not pursue this direction here, cover theory really shines when the cover has finite global dimension and the algebra B has not. For self-injective algebras B, covers of B with finite global dimension are the non-commutative resolutions of [3]. As in their particular case, covers are non-commutative unless the cover of B is isomorphic to B itself.

Proposition 10 *Suppose that A is a finite-dimensional commutative k-algebra. If (A, Ae) is a cover of eAe, for some idempotent e in A, then A is isomorphic to eAe.*

Proof The commutativity of A implies that e is a central idempotent and eAe is commutative. If (A, Ae) is a cover of eAe then

$$A \simeq \mathrm{End}_{eAe}(eA) = \mathrm{End}_{eAe}(e^2A) = \mathrm{End}_{eAe}(eAe) \simeq eAe. \quad □$$

5 Morita Algebras and Nakayama Functor

Morita algebras were introduced by Kerner and Yamagata in [8]. A finite-dimensional k-algebra A is called a *Morita algebra* if it can be written as the endomorphism ring of a generator-cogenerator over some self-injective algebra. A detailed account on Morita algebras and double centralizer properties can also be found in [15]. A characterization of dominant dimension over Morita algebras in terms of cohomology over self-injective algebras was given in [5].

For the proof of the main result, we require the following characterisation of Morita algebras. Theorem 11 is an extension of Proposition 2.9 of [4] (formulated in a different terminology).

Theorem 11 *Let A be a QF-3 k-algebra. Let P be a faithful projective-injective left A-module. The following assertions are equivalent.*

(a) $\operatorname{domdim} A \geq 2$ *and the Nakayama functor restricts to* $DA \otimes_A -\colon \operatorname{add}_A P \to \operatorname{add}_A P$.
(b) $\operatorname{domdim} A \geq 2$ *and* $\operatorname{add}_A DA \otimes_A P = \operatorname{add}_A P$.
(c) *The endomorphism algebra $B = \operatorname{End}_A(P)^{op}$ is self-injective with generator $P \in \operatorname{mod}(B)$ and $A \simeq \operatorname{End}_B(P)$, that is, A is a Morita algebra.*

(a') *The Nakayama functor restricts to* $-\otimes_A DA\colon \operatorname{add} DP_A \to \operatorname{add} DP_A$ *and* $\operatorname{dom\,dim} A \geq 2$.
(b') $\operatorname{domdim} A \geq 2$ *and* $\operatorname{add} DP \otimes_A DA_A = \operatorname{add} DP_A$.

Proof We will show $(b) \implies (a) \implies (c) \implies (b)$. The implications $(b') \implies (a') \implies (c) \implies (b')$ are analogous since $\operatorname{End}_B(P) \simeq \operatorname{End}_B(DP)^{op}$ and cogenerators are exactly the generators for self-injective finite dimensional algebras.

The implication $(b) \implies (a)$ is clear since $DA \otimes_A X \in \operatorname{add} DA \otimes_A P = \operatorname{add} P$, for all $X \in \operatorname{add}_A P$.

Assume that (a) holds. Write $B = \operatorname{End}_A(P)^{op}$. Let Ae be a minimal faithful projective-injective module. Then $\operatorname{add} Ae = \operatorname{add} P$. By Proposition 7, $\operatorname{End}_B(P) \simeq \operatorname{End}_{eAe}(Ae)$. By Morita-Tachikawa correspondence,

$$\operatorname{End}_B(P) \simeq \operatorname{End}_{eAe}(Ae) \simeq A, \tag{5}$$

and Ae is a generator of eAe. Since equivalence of categories preserves generators, P is a generator of B. It remains to show that B is self-injective. This follows by observing that, as right B-modules,

$$B = \operatorname{Hom}_A(P, P) \simeq \operatorname{Hom}_A(P, A) \otimes_A P \simeq D(DA \otimes_A P) \otimes_A P \in \operatorname{add}(DP \otimes_A P)_B, \tag{6}$$

where the third isomorphism is obtained by applying the functor $-\otimes_A P\colon \text{mod-}A \to \text{mod-}B$ and using Tensor-Hom adjunction. Moreover, by Tensor-Hom adjunction there exists a (B, B)-bimodule isomorphism $D(DP \otimes_A P) = \operatorname{Hom}_k(DP \otimes_A P, k) \simeq \operatorname{Hom}_A(P, DDP) = B$. In particular, as right B-modules, $DB \simeq DD(DP \otimes_A P) \simeq DP \otimes_A P$. So, $B \in \operatorname{add} DB_B$. Hence B is B-injective.

Finally, assume that (c) holds. Let Ae be a minimal faithful projective-injective module. Again, since $\operatorname{add}_A Ae = \operatorname{add}_A P$, eAe is Morita equivalent to B. So Ae is a generator of eAe and $A \simeq \operatorname{End}_B(P) \simeq \operatorname{End}_{eAe}(Ae)$. By Morita-Tachikawa correspondence, $\operatorname{domdim} A \geq 2$. Again, since $A \simeq \operatorname{End}_B(P)$ there exists an (A, A)-bimodule isomorphism $DA \simeq P \otimes_B DP$.

Moreover, as left A-modules,

$$DA \otimes_A P \simeq P \otimes_B DP \otimes_A P \simeq P \otimes_B DB. \tag{7}$$

Since DB is B-projective and $B \in \mathrm{add}DB$, DB is a B-progenerator. Hence, $\mathrm{add}_A DA \otimes_A P = \mathrm{add}_A P$. This completes the proof. $\square$

Remark 12 By Tensor-Hom adjunction, for each $M, N \in A$-mod, the $(\mathrm{End}_A(N)^{op}, \mathrm{End}_A(M)^{op})$-bimodules $\mathrm{Hom}_k(DM \otimes_A N, k)$ and $\mathrm{Hom}_A(N, \mathrm{Hom}_k(DM, k))$ are isomorphic.

Using the terminology of [4], Theorem 11 says that all faithful projective-injective modules over a Morita algebra are strongly projective-injective. In particular, this provides a new and shorter proof for Proposition 2.9 of [4].

6 Proof of the Main Theorem

Proof of Theorem 1 The equivalence $(ii) \Leftrightarrow (iii)$ follows from the definition of Morita algebras and the Morita-Tachikawa correspondence. The equivalence $(ii) \Leftrightarrow (iv) \Leftrightarrow (v)$ is the content of Theorem 11.

Assume that A is a Morita algebra. By Theorem 11, $\mathrm{add}DA \otimes_A P = \mathrm{add}P$. Let Ae be a minimal projective-injective faithful module. Then $\mathrm{add}\mathrm{Hom}_A(P, A)_A = \mathrm{add}DP_A = \mathrm{add}D(Ae)_A$. Since $\mathrm{domdim}A \geq 2$, we can write

$$A \simeq \mathrm{End}_{eAe}(Ae) \simeq \mathrm{End}_{eAe}(D(Ae))^{op} \simeq \mathrm{End}_B(\mathrm{Hom}_A(P, A))^{op}. \tag{8}$$

This shows that (A, P) is a cover of B.

Conversely, suppose that (A, P) is a cover of $B := \mathrm{End}_A(P)^{op}$. By Lemma 4, there is a double centralizer property on $\mathrm{Hom}_A(P, A)$. More precisely,

$$\mathrm{End}_A(\mathrm{Hom}_A(P, A)) \simeq B \quad \mathrm{End}_B(\mathrm{Hom}_A(P, A))^{op} \simeq A. \tag{9}$$

In particular, $\mathrm{Hom}_A(P, A)$ is faithful-projective as right A-module. Hence, there exists an injective A-homomorphism $A \to \mathrm{Hom}_A(P, A)^s$, for some $s > 0$. Since DP is projective as right A-module, there is a monomorphism $DP \to A^t \to \mathrm{Hom}_A(P, A)^{st}$. DP is injective as right A-module. Hence, $DP \in \mathrm{add}\mathrm{Hom}_A(P, A)_A$.

We claim now that $DA \otimes_A P$ is a left A-projective module. To see this, define P' to be the direct sum of all non-isomorphic indecomposable A-modules that belong to the additive closure of P. So, $\mathrm{add}P = \mathrm{add}P'$ and $P' \in \mathrm{add}_A DA \otimes_A P = \mathrm{add}_A DA \otimes_A P'$. By Krull-Remak-Schmidt theorem, we can write $DA \otimes_A P' \simeq P' \oplus X$, for some A-module X. On the other hand,

$$\mathrm{End}_A(P' \oplus X)^{op} \simeq \mathrm{End}_A(DA \otimes_A P')^{op} \simeq \mathrm{End}_A(\mathrm{Hom}_A(P', A)) \simeq \mathrm{End}_A(P')^{op}. \tag{10}$$

So, by comparing k-dimensions, X must be the zero module. Hence, $DA \otimes_A P'$ is a faithful projective-injective module. Consequently, $DA \otimes_A P$ is also a faithful projective-injective module. Now, the double centralizer property (9) implies that $\mathrm{domdim}A \geq 2$. Since both P and $DA \otimes_A P$ are faithful projective-injective modules, $\mathrm{add}_A P = \mathrm{add}_A DA \otimes_A P$. So, A is a Morita algebra by Theorem 11. $\square$

Remark 13 For a idempotent e of A, $\operatorname{Hom}_A(Ae, A) \simeq eA$ as (eAe, A)-bimodules. In addition, assume that e is an idempotent so that Ae or eA is projective-injective. By Theorem 1, a double centralizer property on Ae is not equivalent to a double centralizer property on eA unless A is a Morita algebra. In particular, if A is a Morita algebra then $A = \operatorname{End}_{eAe}(eA)^{op} = \operatorname{End}_{eAe}(Ae)$.

7 An Application and Two Examples

A finite-dimensional k-algebra is called *QF-1 algebra* if all faithful A-modules have the double centralizer property (see [14]).

Corollary 14 *Let A be a QF-1 k-algebra. Assume that P is a faithful projective-injective left A-module. Then, A is self-injective if and only if* $\operatorname{Hom}_A(P, A)$ *is a faithful right A-module.*

Proof One direction is clear. Assume that $\operatorname{Hom}_A(P, A)$ is faithful. Since A is a QF-1 algebra, (A, P) is a cover of $\operatorname{End}_A(\operatorname{Hom}_A(P, A)) \simeq \operatorname{End}_A(P)^{op}$. By Theorem 1, A is a Morita algebra. By [1, Proposition 2.2], a Morita algebra is QF-1 if and only if it is self-injective. Therefore, A is self-injective. □

Example 15 For a QF-3 algebra A with dominant dimension two and with a projective-injective faithful module P the pair (A, P) is not, in general, a cover of $\operatorname{End}_A(P)^{op}$.

Let k be an algebraically closed field. Let A be the following bound quiver k-algebra

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3,\ \alpha_2\alpha_1 = 0$$

Note that we read the arrows in a path like morphisms, that is, from right to left.

Denote by $P(i)$ the projective indecomposable module associated with the vertex i and denote by $I(i)$ the indecomposable injective module associated with the vertex i.

The indecomposable projective (left) modules are given by

$$P(1) = I(2) = \begin{matrix}1\\2\end{matrix}, \quad P(2) = I(3) = \begin{matrix}2\\3\end{matrix}, \quad P(3) = 3\,. \tag{11}$$

$$0 \to A \to P(1) \oplus P(2) \oplus P(2) \to P(1) \to I(1) \to 0 \tag{12}$$

is a minimal injective resolution of A. Denote by P the projective module $P(1) \oplus P(2)$. Hence, A is a QF-3 algebra with minimal faithful projective-injective left A-module P and with $\operatorname{domdim} A \geq 2$. Here $B = \operatorname{End}_A(P)^{op}$ is the path algebra with quiver

$$1 \xrightarrow{\alpha_1} 2.$$

But B is not self-injective. By Theorem 1, (A, P) is not a cover of B. But, $(A, P(2) \oplus P(3))$ is a cover of $\operatorname{End}_A(P)^{op}$ by Proposition 8. In fact, $P(2) \oplus P(3) \simeq \operatorname{Hom}_A(DA, P) = \operatorname{Hom}_A(DP, A)$ as left A-modules.

This example also shows that $\operatorname{End}_B(\operatorname{Hom}_A(P, A))^{op}$ is not isomorphic to $\operatorname{End}_B(P)$, in general.

Remark 16 If we drop the injectivity of Ae and of eA in Remark 13, the statement is false. This can be seen in the next example.

Example 17 There are idempotents e and non-Morita algebras A so that there are double centralizer properties on Ae and on eA.

Let k be an algebraically closed field. Let A be the following bound quiver k-algebra

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\theta}{\overset{\gamma}{\rightleftarrows}} 3 \,, \quad \gamma\alpha = \beta\theta = \alpha\beta = \gamma\theta = 0. \tag{13}$$

We are using the same notation as in the previous example. So, the indecomposable projective (left) modules are given by

$$P(1) = Ae_1 = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix}, \quad P(2) = Ae_2 = \begin{matrix} & 2 & \\ 1 & & 3 \\ & & 2 \end{matrix}, \quad P(3) = Ae_3 = \begin{matrix} 3 \\ 2 \end{matrix}. \tag{14}$$

The projective $P(3)$ has dominant dimension zero so A cannot be a Morita algebra. We can see that A has an involution fixing the primitive idempotents and interchanging α with β and γ with θ. Fix $e = e_1 + e_2$. By a direct computation, we can see that $(A, P(1) \oplus P(2))$ is a cover of $B = eAe$. Here, B is the bound quiver k-algebra

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \circlearrowleft t \,, \quad \alpha\beta = \beta t = t\alpha = 0. \tag{15}$$

Again by a direct computation or by observing that the duality of A restricts to one of B fixing e it follows that $\mathrm{End}_{eAe}(Ae) \simeq \mathrm{End}_{eAe}(eA)^{op} \simeq A$.

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