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Sixteen-dimensional locally compact planes of Lenz-type V on which $SU_2\mathbb{H}$ acts as a group of collineations

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Abstract. We explicitly construct all the planes mentioned in the title.

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1. Introduction

We construct all 16-dimensional locally compact topological planes of Lenz-type V whose collineation groups contain a subgroup (locally) isomorphic to $SU_2\mathbb{H}$. If the classical plane over the algebra $\mathbb{O}$ of the octonions is put aside, the collineation group of such a plane will be shown to be a Lie group of dimension 37. In fact by results of Löwe these are the only 16-dimensional locally compact translation planes whose collineation groups have this dimension (see H. Löwe, Sixteen-dimensional locally compact translation planes with automorphism groups of dimension at least 36, Preprint, 2019). Note that the sixteen-dimensional locally compact translation planes with collineation groups of dimension at least 38 have been classified in [9], cf. [11, 82.26].

The planes which we determine here fall into three families, which are related among each other by dualization and transposition. The planes of one of these families are self-dual; this invites for further study of their polarities and their unitals, which however will not be pursued here.

1.1. Basic information

Basic facts about 16-dimensional locally compact translation planes in general and about the Lenz-type V planes among them are collected in [11, 81.0]. We make free use of them.

The translation line is unique and, hence, fixed by every collineation except if the plane is isomorphic to the classical plane over the algebra $\mathbb{O}$ of the octonions (in the latter plane, every line is a translation line). We represent such a translation plane as an affine plane, the translation line playing the rôle of the line L_∞ at infinity. Let $\mathbb{G}$ be the group of all affine collineations, that is, of the collineations fixing L_∞; it comprises all collineations except for the classical octonion plane, as we have already remarked.

The space of affine points is a 16-dimensional real vector space, the lines through the origin o are certain 8-dimensional $\mathbb{R}$-linear subspaces, and the other affine lines are the images of these under the (vector space) translations. The group $\mathbb{G}$ is the semidirect product of the normal subgroup T consisting of these translations by the stabilizer $\mathbb{G}_o$ of o:

$$\mathbb{G} = \mathbb{G}_o \cdot \mathrm{T}.$$

In a plane of Lenz type (at least) V, there is a point s at infinity such that for the line S joining o and s the group $\mathbb{G}_{[s,S]}$ of collineations with center s and axis S (shears) is sharply transitive on $L_\infty \backslash \{s\}$. If the plane is not the classical plane over the octonions, the point s is unique; it will be called the *shear center*. It is then fixed by every collineation. Hence, the shear group $\mathbb{G}_{[s,S]}$ is a normal subgroup of $\mathbb{G}_o$, and for a second line $W \neq S$ through o

$$\mathbb{G}_o = \mathbb{G}_{S,W} \cdot \mathbb{G}_{[s,S]}.$$

The action of $\mathbb{G}_{S,W}$ on the normal subgroup $\mathbb{G}_{[s,S]}$ by conjugation is equivalent to the action on $L_\infty \backslash \{s\}$, by sharp transitivity of $\mathbb{G}_{[s,S]}$ on $L_\infty \backslash \{s\}$.

With respect to W and S as first and second coordinate axis, a 16-dimensional plane of Lenz type V can be coordinatized by an 8-dimensional (non-associative) real division algebra $(D, +, \circ)$, see [11, 24.7, 25.8 and 64.14]. (In other terminology, a non-associative division algebra is also called a semifield). Its multiplication will be denoted by $\circ$ in order to distinguish it from the classical multiplication $\cdot$ of the octonions and the quaternions, which will also be used. In affine coordinates over D, the shear group $\mathbb{G}_{[s,S]}$ consists precisely of the transformations

$$(x, y) \mapsto (x, y + d \circ x) \quad \text{for} \quad d \in D,$$

see [11, 24.7 and 25.4]. Hence, $\mathbb{G}_{[s,S]}$ is isomorphic to the additive group $(\mathbb{R}^8, +)$ of D. In particular, it has no nontrivial compact subgroup, and the same is true for the translation group $\mathrm{T} \cong \mathbb{R}^{16}$. Therefore, in view of the semidirect products noted above, a maximal compact subgroup of $\mathbb{G}_{S,W}$ is a maximal compact subgroup of $\mathbb{G}$ (except in the classical octonion plane).

1.2. Actions of $\mathrm{SU}_2\mathbb{H}$ as a collineation group

Now assume that $\mathbb{G}_o$ has a closed connected subgroup Λ locally isomorphic to $\mathrm{SU}_2\mathbb{H}$. By what we have said just now, we may assume that Λ is contained in $\mathbb{G}_{S,W}$, since maximal compact subgroups are conjugate. The group Λ cannot act trivially on one of the lines W, S or L_∞, since then Λ could be described by a subgroup of the left nucleus, the middle nucleus, or the kernel of the division algebra D, see [11, 25.4], which is impossible by dimension reasons. Now $\mathrm{SU}_2\mathbb{H}$ is quasisimple, and the central involution ι generates the only nontrivial normal subgroup. Therefore, on any one of the lines W, S and L_∞, the group Λ acts effectively or as the factor group of $\mathrm{SU}_2\mathbb{H}$ by $\{\mathrm{id}, \iota\}$, which is isomorphic to $\mathrm{SO}_5\mathbb{R}$. These actions are $\mathbb{R}$-linear (in the case of the line L_∞, this has to be understood via the action on the shear group $\mathbb{G}_{[s,S]} \cong \mathbb{R}^8$). A nontrivial action of $\mathrm{SO}_5\mathbb{R}$ on $\mathbb{R}^8$ is the classical action on a 5-dimensional subspace and leaves a complementary 3-dimensional subspace pointwise fixed. Therefore, Λ cannot act as $\mathrm{SO}_5\mathbb{R}$ on two of the lines W, S and L_∞, since then Λ would fix a nondegenerate quadrangle and therefore would correspond to a subgroup of the automorphism group of a coordinatizing division algebra, see [11, 23.5]. But according to [5, 1.2] the automorphism group of such a division algebra does not have a subgroup isomorphic to $\mathrm{SO}_5\mathbb{R}$. Thus, Λ is isomorphic to $\mathrm{SU}_2\mathbb{H}$ and acts effectively on at least two of the lines W, S and L_∞. An effective linear action of $\mathrm{SU}_2\mathbb{H}$ on $\mathbb{R}^8$ is equivalent to the classical action on $\mathbb{H}^2$, and in this action, the central involution acts as $-\mathrm{id}$. Therefore, the central involution ι of Λ is not a Baer involution and hence is a reflection with one of the lines W, S or L_∞ as axis, see [11, 23.17]. On this line, then, Λ induces the group $\mathrm{SO}_5\mathbb{R}$. These considerations lead to the following case distinction.

Case 1 Λ acts on W and on S effectively as $\mathrm{SU}_2\mathbb{H}$ and on L_∞ as $\mathrm{SO}_5\mathbb{R}$.
Case 2 Λ acts on S and on L_∞ effectively as $\mathrm{SU}_2\mathbb{H}$ and on W as $\mathrm{SO}_5\mathbb{R}$.
Case 3 Λ acts on W and on L_∞ effectively as $\mathrm{SU}_2\mathbb{H}$ and on S as $\mathrm{SO}_5\mathbb{R}$.

We shall deal first with Case 1. The other cases will later be derived from Case 1 by dualization and spread transposition, see Sects. 3 and 4.

2. The planes of Case 1

2.1. The action of Λ on the shear group

We may identify $\mathbb{R}^{16}$ and $\mathbb{H}^2 \times \mathbb{H}^2$ as $\mathbb{R}$-vector spaces in such a way that S and W, considered as affine lines, are described as $S = \{0\} \times \mathbb{H}^2$ and $W = \mathbb{H}^2 \times \{0\}$, and that Λ acts on S and W effectively as the group $\mathrm{SU}_2\mathbb{H}$ in its classical action on $\mathbb{H}^2$. Thus,

$$\Lambda = \{(x, y) \mapsto (Ax, Ay); A \in \mathrm{SU}_2\mathbb{H}\}$$

for $(x, y) \in \mathbb{H}^2 \times \mathbb{H}^2$.

We now consider an 8-dimensional division algebra $(D, +, \circ)$ coordinatizing the plane with respect to W and S as first and second coordinate axis and the point $\left(\binom{1}{0}, \binom{1}{0}\right)$ as unit point. The fact that the shear group $\mathbb{G}_{[s,S]}$ is normal in $\mathbb{G}_o$ and its description in coordinates given in Sect. 1.1 imply that the following set of $\mathbb{R}$-linear transformations of D identified with $\mathbb{H}^2$ is invariant under conjugation by the elements of $\mathrm{SU}_2\mathbb{H}$:

$$\mathcal{D} = \{x \mapsto d \circ x; d \in D\}$$

for $x \in D = \mathbb{H}^2$, and that this action is equivalent to the action of Λ on the shear group $\mathbb{G}_{[s,S]}$. Moreover, since W and its point at infinity are fixed by Λ and since the shear group $\mathbb{G}_{[s,S]}$ acts sharply transitively on $L_\infty \backslash \{s\}$, this action is equivalent to the action of Λ on L_∞, and hence, by assumption of Case 1, to a nontrivial linear action as $\mathrm{SO}_5\mathbb{R}$ on $\mathbb{R}^8$. Hence, there is a 5-dimensional $\mathbb{R}$-linear subspace $\mathcal{C}$ of $\mathcal{D}$ on which Λ acts as $\mathrm{SO}_5\mathbb{R}$ in its classical action on $\mathbb{R}^5$, and a complementary 3-dimensional subspace $\mathcal{F}$ of $\mathcal{D}$ on which Λ acts trivially. First, we determine the possibilities for the subspace $\mathcal{C}$.

2.2 Lemma. *Let $\mathcal{C}$ be a 5-dimensional $\mathbb{R}$-linear subspace of the vector space $\mathrm{End}_\mathbb{R}\mathbb{H}^2$ of $\mathbb{R}$-linear endomorphisms of $\mathbb{H}^2$ (viewed as a real vector space) which is invariant under conjugation by $\mathrm{SU}_2\mathbb{H}$ and on which this group acts non-trivially. Then there is a quaternion $p \in \mathbb{H}\backslash\{0\}$ of norm 1, $\bar{p}p = 1$, such that*

$$\mathcal{C} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} s & \bar{h} \\ h & -s \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} p; \;\; s \in \mathbb{R}, h \in \mathbb{H} \right\},$$

where $x_1, x_2 \in \mathbb{H}$.

Proof. The only nontrivial representation of $\mathrm{SU}_2\mathbb{H}$ in dimension 5 is the classical action as $\mathrm{SO}_5\mathbb{R}$, the kernel of this representation being generated by the central involution ι. Consider the subgroup

$$\Psi = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix}; \; a, b \in \mathbb{H}, a\bar{a} = 1 = b\bar{b} \right\}$$

of $\mathrm{SU}_2\mathbb{H}$. Its quotient group by $\{\mathrm{id}, \iota\}$ is $\mathrm{SO}_4\mathbb{R}$. In its action on $\mathcal{C}$ by conjugation, its fixed elements form a 1-dimensional subspace $\mathcal{C}_1$. In other words, $\mathcal{C}_1$ consists precisely of the endomorphisms in $\mathcal{C}$ commuting with all elements of Ψ, in particular with the matrices $\begin{pmatrix} a & \\ & a \end{pmatrix}$, so that the elements of $\mathcal{C}_1$ are $\mathbb{H}$-linear endomorphisms of the left vector space $\mathbb{H}^2$ over $\mathbb{H}$. Furthermore, $\mathbb{H} \times \{0\}$ and $\{0\} \times \mathbb{H}$ are the eigenspaces of $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \Psi$ and hence are invariant under the elements of $\mathcal{C}_1$. Thus, there are $p, q \in \mathbb{H}$ such that

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 p \\ x_2 q \end{pmatrix} r; \;\; r \in \mathbb{R} \right\}.$$

The involution $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in \mathrm{SU}_2\mathbb{H}$ normalizes Ψ and hence leaves $\mathcal{C}_1$ invariant, but not elementwise fixed, because it does not belong to Ψ. Thus, $\mathcal{C}_1$ is contained in the eigenspace of this involution with eigenvalue -1, so that, in the given description of $\mathcal{C}$, we must have $q = -p$.

All the elements of $\mathcal{C}$ are obtained from the elements of $\mathcal{C}_1$ by conjugation with the elements of $\mathrm{SU}_2\mathbb{H}$. For an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}_2\mathbb{H}$ one has

$\overline{a}a + \overline{c}c = 1 = \overline{b}b + \overline{d}d$, $\overline{a}b + \overline{c}d = 0$, so that $A^{-1} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$, which in turn is equivalent to $a\overline{a} + b\overline{b} = 1 = c\overline{c} + d\overline{d}$ and $a\overline{c} + b\overline{d} = 0$.

Conjugation of an endomorphism from $\mathcal{C}_1$ by A gives the endomorphism

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} pr = \begin{pmatrix} \overline{a}a - \overline{c}c & \overline{a}b - \overline{c}d \\ \overline{b}a - \overline{d}c & \overline{b}b - \overline{d}d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} pr.$$

From the relations given above, we obtain that $\overline{b}b - \overline{d}d = -(\overline{a}a - \overline{c}c)$. Thus, the elements of $\mathcal{C}$ have the form stated in the lemma. It is clear that p may be chosen to be of norm 1. That for all $s \in \mathbb{R}, h \in \mathbb{H}$ the endomorphisms given in the lemma belong to $\mathcal{C}$ follows from the fact that they form a 5-dimensional vector space. □

2.3. Shears which are invariant under Λ

Now we determine the possibilities for a 3-dimensional subspace $\mathcal{F}$ of $\mathrm{End}_{\mathbb{R}}\mathbb{H}^2$ on which $\mathrm{SU}_2\mathbb{H}$ acts trivially by conjugation, which means that the endomorphisms in $\mathcal{F}$ commute with the elements of $\mathrm{SU}_2\mathbb{H}$. According to Schur's lemma, the elements of $\mathcal{F}$ are given by scalar multiplication from the right by quaternions; hence there is a 3-dimensional $\mathbb{R}$-linear subspace F of $\mathbb{H}$ such that

$$\mathcal{F} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} q;\ q \in F \right\}$$

(for $x_1, x_2 \in \mathbb{H}$). In the sequel, we think of F as the orthogonal space of a unit quaternion v, $\overline{v}v = 1$ with respect to the scalar product $\langle a, b\rangle = 1/2(\overline{a}b + \overline{b}a)$ on the $\mathbb{R}$-vector space $\mathbb{H}$:

$$F = v^{\perp}.$$

2.4. Lines through the origin

In coordinates over the division algebra $D = (\mathbb{H}^2, +, \circ)$, the affine lines through o are the subsets $\{(x, d \circ x); x \in \mathbb{H}^2\}$ for $d \in D$ together with the line $S = \{0\} \times \mathbb{H}^2$. Since the endomorphisms $x \mapsto d \circ x$ for $d \in D$ constitute the space $\mathcal{D} = \mathcal{C} + \mathcal{F}$ described in Sects. 2.2 and 2.3, the lines through the origin besides S are the following subsets of the point space $\mathbb{H}^2 \times \mathbb{H}^2$:

$$L_{s,q,h} = \left\{ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} p + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} q \right);\ x_1, x_2 \in \mathbb{H} \right\}$$

for $s \in \mathbb{R}, h \in \mathbb{H}, q \in F$.

The lines $L_{0,q,0}$ for $q \in F$ are obtained from the endomorphisms in $\mathcal{F}$, which are precisely the elements of $\mathcal{D}$ commuting with $\mathrm{SU}_2\mathbb{H}$. Thus, together with the line $S = \{0\} \times \mathbb{H}^2$, they are the fixed lines of the group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ of collineations. Necessarily, we must have that

$$p \notin F, \text{ that is } \langle p, v\rangle \neq 0;$$

otherwise, the point $\left(\binom{1}{0}, \binom{p}{0}\right)$ would belong to the two different lines $L_{1,0,0}$ and $L_{0,p,0}$ through the origin.

The question whether for every pair (v, p) of unit quaternions satisfying these conditions one obtains in this way the system of lines through the origin of an affine plane will be postponed until later, see Sect. 2.9.

2.5. Special isomorphisms

The plane for which we have developed this description will be called $\mathcal{P}$. We now consider a second plane $\mathcal{P}'$ with unit quaternions v' and p' instead of v and p as defining parameters. The lines through the origin of this plane described as in Sect. 2.4 will be called $L'_{s,q,h}$. We ask under which conditions on v, p, v', p' the map

$$\varphi : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \mapsto \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} c, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} d\right)$$

$(x_1, x_2, y_1, y_2 \in \mathbb{H})$ of the affine point space of $\mathcal{P}$ onto the affine point space of $\mathcal{P}'$ for unit quaternions c and d is an isomorphism between the two planes. The answer to this question serves two purposes: First, it will allow to restrict the parameters v and p up to isomorphism, and second, it will later help to determine all collineations of the planes of this type.

The map φ is deliberately chosen so as to commute with the action of the group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ on both planes. If φ is an isomorphism, it therefore must map fixed lines of Λ to fixed lines. Thus, the line $L_{0,q,0}$ for $q \in F$ must be mapped to the line $L'_{0,q',0}$ for some $q' \in F' := v'^{\perp}$. Now, φ maps $L_{0,q,0}$ to

$$\left\{\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} c, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} qd\right); x_1, x_2 \in \mathbb{H}\right\} = \left\{\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} c^{-1}qd\right); x_1, x_2 \in \mathbb{H}\right\}.$$

This is a line $L'_{0,q',0}$ if and only if $c^{-1}qd = q' \in F' = v'^{\perp}$. If φ is an isomorphism, this is true for all $q \in F = v^{\perp}$, which means that

$$c^{-1}vd = \pm v'.$$

In Sect. 2.4, the lines $L_{s,0,0}$ were constructed from the endomorphisms in $\mathcal{C}_1$, the endomorphisms in $\mathcal{C}$ which commute with the elements of Ψ, see 2.2. These lines therefore are fixed lines of the subgroup of $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ corresponding to Ψ. But this subgroup has more fixed lines through o than that. They are given by endomorphisms in all of $\mathcal{D}$ (not only $\mathcal{C}$) which commute with the elements of Ψ. Now $\mathcal{D} = \mathcal{C} + \mathcal{F}$, where $\mathcal{F}$ consists of the endomorphisms which commute with the whole group $\mathrm{SU}_2\mathbb{H}$, so that the endomorphisms in $\mathcal{D}$ commuting with Ψ are the elements of $\mathcal{C}_1 + \mathcal{F}$. The corresponding lines through o are the lines $L_{s,q,0}$ for $s \in \mathbb{R}, q \in F$. Now, if φ is an isomorphism, it must map $L_{1,0,0}$ to a line in $\mathcal{P}'$ which is also a fixed line of the subgroup of $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ corresponding to Ψ, that is to a line $L'_{s',q',0}$. The image of $L_{1,0,0}$ under φ is

$$\left\{\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} c, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} pd\right); x_1, x_2 \in \mathbb{H}\right\} =$$

$$\left\{\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}1&\\&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}c^{-1}pd\right);x_1,x_2\in\mathbb{H}\right\}.$$

This is a line $L'_{s',q',0}$ if and only if $c^{-1}pd = s'p' + q'$ and $-c^{-1}pd = -s'p' + q'$; but then $q' = 0$ and $c^{-1}pd = s'p'$. Since c and d are unit quaternions, this means that

$$c^{-1}pd = \pm p'.$$

Conversely, it is now easy to verify that if these conditions are fulfilled, φ maps all lines of $\mathcal{P}$ to lines of $\mathcal{P}'$ and thus is an isomorphism of $\mathcal{P}$ onto $\mathcal{P}'$.

2.6. Parameter restriction

The maps $\mathbb{H} \to \mathbb{H} : x \mapsto c^{-1}xd$ for unit quaternions c, d appearing in the conditions which have just been obtained describe all elements of $\mathrm{SO}_4\mathbb{R}$ in its classical action on $\mathbb{H} \cong \mathbb{R}^4$. They allow to transform any pair v, p of unit quaternions to any other such pair having the same scalar product. By the result of Sect. 2.5, we therefore may assume, up to isomorphism of planes, that

$$p = 1, \;\; v = \cos\alpha + \sin\alpha \cdot i$$

for $\alpha \notin (\mathbb{Z} + 1/2)\pi$ (where i, j, k is the usual Hamilton triple of $\mathbb{H}$). The last restriction comes from the condition that $1 = p \notin v^\perp$, see Sect. 2.4. Furthermore, one obtains an isomorphic plane if one replaces v by $\cos\alpha - \sin\alpha \cdot i$, since this quaternion has the same scalar product with 1 as v. Also, it is clear from the description of lines in Sect. 2.4 that $-v$ instead of v defines the same plane. So, finally, we may restrict v further by demanding that

$$0 \leq \alpha < \pi/2.$$

The plane obtained in this way shall be denoted by $\mathcal{P}_\alpha$.

In the sequel, we shall always use this choice of parameters. (The idea for this particular choice was given to us by H. Löwe in his preprint cited in the introduction, although one would have been tempted at first sight to use a choice with $1 \in v^\perp$ in order to ensure that the diagonal $\{(x, x); x \in \mathbb{H}^2\}$ is a line of the plane, which is not the case with $p = 1$. This slight disadvantage will be compensated, however, by far greater advantages.)

We shall see that different choices for α yield nonisomorphic planes; indeed we shall prove in Sect. 2.11 that no other isomorphisms than those already produced here can be found. With these parameters, the lines through the origin besides S are the following subsets of the point space $\mathbb{H}^2 \times \mathbb{H}^2$:

$$L_{s,q,h} = \left\{\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}s&\overline{h}\\h&-s\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}x_1\\x_2\end{pmatrix}q\right);\; x_1,x_2\in\mathbb{H}\right\}$$

for $s \in \mathbb{R}, q \in F = v^\perp, h \in \mathbb{H}$.

2.7. Special collineations

From Sect. 2.5 applied to $v' = v$ and $p = 1 = p'$ we know that in such a plane the map φ considered there is a collineation if the unit quaternions c, d defining φ satisfy $c^{-1}vd = \pm v$ and $c^{-1}d = \pm 1$, that is $d = \pm c$.

If $v = \pm 1$, this means that we obtain a collineation for all unit quaternions c if and only if $d = \pm c$. These collineations form a group isomorphic to the direct product of the multiplicative group of unit quaternions (which is $\mathrm{Spin}_3\mathbb{R}$) and the group generated by the involutory collineation given by $c = 1, d = -1$ which is a reflection with the line W as axis and, by the way, is contained in $\Lambda \cong \mathrm{SU}_2\mathbb{H}$. We shall see in Sect. 2.10 that the plane in this case ($v = \pm 1$) is isomorphic to the classical octonion plane.

If $v \neq \pm 1$, then under the restrictions of Sect. 2.6 one has $\mathbb{R}\cdot 1+\mathbb{R}v = \mathbb{R}\cdot 1+\mathbb{R}i$. By the conditions on c and d, this span and the one-dimensional subspaces $\mathbb{R}\cdot 1$ and $\mathbb{R}v$ are invariant under the orthogonal map $x \mapsto c^{-1}xd$. Since 1 and v are not orthogonal, this map either fixes both 1 and v or maps both to their antipodes, so that it induces either the identity or multiplication by -1 on the plane $\mathbb{R}\cdot 1 + \mathbb{R}i$. The second case may be reduced to the first case again by composition with the reflection given by $c = 1, d = -1$, which we have used just before and which is contained in $\Lambda \cong \mathrm{SU}_2\mathbb{H}$. In the first case $c = d$ and $c^{-1}ic = i$ which means $ic = ci$, so that $c \in \mathbb{R}\cdot 1 + \mathbb{R}i$. Thus we obtain the group of collineations

$$\{(x, y) \mapsto (xc, yc);\ c \in \mathbb{R}\cdot 1 + \mathbb{R}i, \bar{c}c = 1\},$$

a 1-dimensional torus whose intersection with Λ is generated by $(x, y) \mapsto (-x, -y)$.

2.8. A coordinatizing division algebra

In order to obtain a coordinatizing division algebra $(D, +, \circ)$, we change the identification of the affine point set with $\mathbb{H}^2 \times \mathbb{H}^2$ by the map $\left(\binom{x_1}{x_2}, \binom{y_1}{y_2}\right) \mapsto \left(\binom{x_1}{x_2}, \binom{y_1}{-y_2}\right)$. In new coordinates, the lines through the origin except S take the form

$$\left\{\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} s & \bar{h} \\ -h & s \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} q\right);\ x_1, x_2 \in \mathbb{H}\right\}.$$

The new coordinates present an advantage for coordinatization, namely that the diagonal $\{(x, x); x \in \mathbb{H}^2\}$ is one of the lines through the origin. There is a disadvantage however: The introduction of these coordinates deforms the action of the group Λ and of the torus group described in Sect. 2.7. Therefore we shall return to the old coordinates after this digression.

In coordinates over D, a line through the origin different from S is given by $\{(x, d \circ x); x \in D\}$. If this is the line described above, then d is obtained in terms of s, q, h by feeding the unit $\binom{1}{0}$ instead of $x = \binom{x_1}{x_2}$ into the second coordinate of this line:

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} s \\ -h \end{pmatrix} + \begin{pmatrix} q \\ 0 \end{pmatrix},$$

so that $d_1 = s + q$, $d_2 = -h$. By forming the scalar product of $d_1 = s + q$ with $v = \cos\alpha + \sin\alpha \cdot i$ and keeping in mind that $q \in v^\perp$ and that $\cos\alpha \neq 0$ we obtain conversely that

$$h = -d_2, \;\; s = \operatorname{Re} d_1 + \tan\alpha\langle d_1, i\rangle, \;\; q = d_1 - \operatorname{Re} d_1 - \tan\alpha\langle d_1, i\rangle.$$

Thus, the second coordinate of the point on the line described above with first coordinate $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{H}^2$ is

$$\begin{pmatrix} (\operatorname{Re} d_1 + \tan\alpha\langle d_1, i\rangle)x_1 - \overline{d_2}x_2 + x_1(d_1 - \operatorname{Re} d_1 - \tan\alpha\langle d_1, i\rangle) \\ d_2x_1 + (\operatorname{Re} d_1 + \tan\alpha\langle d_1, i\rangle)\, x_2 - x_2(d_1 - \operatorname{Re} d_1 - \tan\alpha\langle d_1, i\rangle) \end{pmatrix}.$$

This can be simplified:

$$d \circ x = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1d_1 - \overline{d_2}x_2 \\ x_2\overline{d_1} + d_2x_1 + \rho\langle d_1, i\rangle x_2 \end{pmatrix} = xd + \rho\langle d_1, i\rangle \begin{pmatrix} 0 \\ x_2 \end{pmatrix},$$

where xd is the classical octonion product and

$$\rho = 2\tan\alpha.$$

2.9. Verification of planarity

We still have to verify that for every parameter $\alpha \in [0, \pi/2)$ we obtain in this way a division algebra, so that the lines described in Sect. 2.6 are indeed the lines through the origin of a plane $\mathcal{P}_\alpha$ of Lenz type V. According to the criterion [11, 64.13], which is formulated for a more general situation, it suffices to prove that for $x \in D, x \neq 0$ the map $D \to D : d \mapsto d \circ x$ is bijective. This map is $\mathbb{R}$-linear; so by finite dimension it suffices to show that it is injective, or, in other words, that it has trivial kernel. This means that for $x \neq 0$, $d \circ x = 0$ implies $d = 0$. But this is equivalent to saying that $d \circ x = 0$ implies $d = 0$ or $x = 0$. This is what we are going to prove. Indeed, $d \circ x = 0$ means

$$\begin{aligned} x_1d_1 - \overline{d_2}x_2 &= 0 \\ x_2(\overline{d_1} + \rho\langle d_1, i\rangle) + d_2x_1 &= 0. \end{aligned}$$

If $\langle d_1, i\rangle = 0$ or $x_2 = 0$, then $d \circ x = xd$, and our assertion follows from the fact that the octonion algebra has no zero divisors.

So assume $\langle d_1, i\rangle \neq 0$ and $x_2 \neq 0$. From the first of the two equations above, we then obtain that $d_2 = \overline{x_2}^{-1}\overline{d_1\overline{x_1}}$. Inserting this into the second equation and multiplying by $\overline{x_2}$ gives $\overline{x_2}x_2\overline{d_1} + \overline{x_2}x_2\rho\langle d_1, i\rangle + \overline{d_1}\overline{x_1}x_1 = 0$. But then d_1 would be real, in contradiction to $\langle d_1, i\rangle \neq 0$.

2.10. The classical case

In order to decide isomorphism questions, we first have to determine for which admissible parameters α the plane $\mathcal{P}_\alpha$ described in Sect. 2.6 is isomorphic to the classical octonion plane, or equivalently, the division algebra in Sect. 2.8

is isomorphic to the classical octonion algebra. The latter is biassociative; so we test biassociativity of the multiplication $\circ$.

$\binom{i}{0} \circ \left(\binom{i}{0} \circ \binom{0}{1}\right) = \binom{0}{-1-\rho i}$ and $\left(\binom{i}{0} \circ \binom{i}{0}\right) \circ \binom{0}{1} = \binom{0}{-1}$.
If $(\mathbb{H}^2, +, \circ)$ is isomorphic to the classical octonion algebra, the result has to be the same, so that $\rho = 0$, which by definition of $\rho = 2\tan\alpha$ means $\alpha = 0$ and $v = 1$.

Conversely, it is clear that for $\rho = 0$ the division algebra $(\mathbb{H}^2, +, \circ)$ is isomorphic to the octonion algebra (by octonion conjugation). We formulate the result:

Proposition. *The division algebra constructed in Sect. 2.8 for* $0 \leq \alpha < \pi/2$ *is isomorphic to the classical octonion algebra if and only if* $\alpha = 0$*. This is equivalent to the plane* $\mathcal{P}_\alpha$ *described in Sect. 2.6 being isomorphic to the classical octonion plane.*

2.11. Isomorphisms and collineations

We now return to the question of isomorphisms between two such planes in the case they are nonclassical.

First, since the translation line L_∞ and the shear center s are unique, see Sect. 1.1, an isomorphism between two such planes $\mathcal{P} = \mathcal{P}_\alpha$ and $\mathcal{P}' = \mathcal{P}_{\alpha'}$ maps the translation line to the translation line and the shear center to the shear center. It can be modified by a translation in such a way that the origin o is mapped to the origin. Such an isomorphism is a semilinear transformation of the affine plane of $\mathcal{P}$ onto the affine plane of $\mathcal{P}'$ when the two affine planes are viewed as vector spaces over the respective kernels, see [1], [10, Theorem 1.18]. Now the kernel of such a plane can be obtained from the kernel of the coordinatizing division algebra D, which in the case of an 8-dimensional real division algebra consists of the real multiples of 1 only, see [2, Theorem 1]. Thus, an isomorphism φ mapping o to o is $\mathbb{R}$-linear.

As second coordinate axis in $\mathcal{P}$, we have used the line S through o having the shear center s as point at infinity; the isomorphism φ therefore maps S to the corresponding line of $\mathcal{P}'$. By the transitivity properties of the shear group, φ can be further modified by a shear so as to map the first coordinate axis W in $\mathcal{P}$ to the first coordinate axis in $\mathcal{P}'$, as well. We now represent both affine planes in the same real vector space $\mathbb{H}^2 \times \mathbb{H}^2$; then φ may be thought of as an $\mathbb{R}$-linear transformation leaving $W = \mathbb{H}^2 \times \{0\}$ and $S = \{0\} \times \mathbb{H}^2$ invariant.

In both planes, the group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ is a subgroup of the stabilizer $\mathbb{G}_{S,W}$ of W and S in the collineation group. We assert that in the nonclassical case it is a characteristic subgroup. Since D is a real division algebra, $\mathbb{G}_{S,W}$ contains the subgroup

$$\mathrm{Z} = \{(x, y) \mapsto (rx, ty);\ 0 < r, t \in \mathbb{R}\}.$$

By [4, 4.2], see also [11, 81.8], $\mathbb{G}_{S,W}$ has a largest compact subgroup M, and $\mathbb{G}_{S,W}$ is the product of M and Z. The group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ is contained in M. Since it acts irreducibly on W and on S, the same is true for the connected component M^1 of M. Thus, M^1 induces on $W \cong \mathbb{R}^8$ and $S \cong \mathbb{R}^8$ an irreducible compact connected subgroup of $\mathrm{GL}_8\mathbb{R}$ of dimension at least 10. Using information about representations of compact Lie groups in low dimensions these can easily be determined, see e.g. the list in [6, 2.8], from which the irreducible ones can be extracted. Up to conjugation, these are the groups $\mathrm{SO}_8\mathbb{R}$, $\mathrm{Spin}_7\mathbb{R}$, $\mathrm{U}_4\mathbb{C}$, $\mathrm{SU}_4\mathbb{C}$, $\mathrm{SU}_2\mathbb{H} \cdot \mathrm{Spin}_3\mathbb{R}$, $\mathrm{SU}_2\mathbb{H} \cdot \mathrm{SO}_2\mathbb{R}$, and $\mathrm{SU}_2\mathbb{H}$.

Now according to [7, 3.6 and 3.7] if M^1 induces on S and W the group $\mathrm{SO}_8\mathbb{R}$ or the group $\mathrm{Spin}_7\mathbb{R}$, the plane is the classical octonion plane, which we do not consider here. The same is true by [8, 3.1] if a subgroup of M induces the group $\mathrm{SU}_4\mathbb{C}$ on W and on S and if the group of shears with axis S has dimension at least 3. In the remaining groups, $\mathrm{SU}_2\mathbb{H}$ is a characteristic subgroup. This is what we have claimed.

As a consequence, conjugation by the isomorphism φ maps the group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ onto itself and induces an automorphism on Λ. Since $\mathrm{SU}_2\mathbb{H}$ has only inner automorphisms, φ can be modified by composition with an element of Λ such that conjugation by φ induces the identity on Λ, in other words, φ commutes with the elements of Λ. By Schur's lemma, φ is of the form $(x, y) \mapsto (xc, yd)$ for $x, y \in \mathbb{H}^2$ and some $c, d \in \mathbb{H}\backslash\{0\}$. By composition with one of the collineations in Z, we may assume that c and d are unit quaternions. But then, φ is just one of the special isomorphisms considered in Sect. 2.5. This means that we will not find any other isomorphisms and, in particular, collineations than those which we know already from Sects. 2.5 and 2.7.
We summarize our findings:

2.12 Theorem. *Up to isomorphism, the non-classical sixteen-dimensional locally compact planes of Lenz type V whose collineation groups contain a subgroup* $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ *acting according to Sect.* 1.2 *Case 1 are the planes* $\mathcal{P}_\alpha$ *whose lines through the origin are described in Sect.* 2.6 *for* $v = \cos\alpha + \sin\alpha \cdot i$, *where* $0 < \alpha < \pi/2$. *For different choices of* α, *one obtains non-isomorphic planes.*

The collineation group of such a plane is the product of the group of translations, the group of shears with axis S, *the group* Z *given in Sect.* 2.11, *the group* $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ *consisting of the collineations*

$$\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2 : (x, y) \mapsto (Ax, Ay) \quad \text{for} \quad A \in \mathrm{SU}_2\mathbb{H},$$

and the 1*-dimensional torus group consisting of the collineations*

$$\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2 : (x, y) \mapsto (xc, yc) \quad \text{for} \quad c \in \mathbb{R} \cdot 1 + \mathbb{R}i, \bar{c}c = 1.$$

The group of collineations is therefore connected and has dimension 37.

3. The planes of Case 2

These are obtained from the planes of Case 1 by passing to the dual planes. Indeed, the line at infinity L_∞ of a plane $\mathcal{P}_\alpha$ of Case 1 plays the rôle of the pencil of lines through the shear center in the dual plane $\mathcal{P}_\alpha^*$, so that the group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ acts on this pencil as $\mathrm{SO}_5\mathbb{R}$, and this action corresponds to the action on the first coordinate axis. It is well known and easy to see that the dual plane of a plane of Lenz type V is again of Lenz type V.

3.1. Dualization

The affine points of $\mathcal{P}_\alpha^*$ are the affine lines of $\mathcal{P}_\alpha$ which do not pass through the shear center s, that is which are not parallel to S. According to Sect. 2.6, these are the lines

$$[s,q,h;t] := \left\{ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} q + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right) ; \; x_1, x_2 \in \mathbb{H} \right\}$$

for $s \in \mathbb{R}, q \in v^\perp$ where $v = \cos\alpha + \sin\alpha \cdot i, h \in \mathbb{H}$, and

$$t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{H}^2.$$

Now $\mathbb{R} \cdot 1 + v^\perp = \mathbb{H}$, since $1 \notin v^\perp$. Therefore, we have an identification

$$[s,q,h;t] \mapsto \left(\begin{pmatrix} s+q \\ h \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right)$$

of the affine point set of the dual plane $\mathcal{P}_\alpha^*$ with $\mathbb{H}^2 \times \mathbb{H}^2$, which we shall use in the sequel.

3.2. The division algebra for the dual plane

The first coordinate axis of $\mathcal{P}_\alpha^*$ consists of the lines $[s,q,h;0]$ of $\mathcal{P}_\alpha$, the second coordinate axis carries the dual points $[0,0,0;t]$. We will use $[1,0,0;0]$ as unit element. The lines of the dual plane are the points of $\mathcal{P}_\alpha$. The line in $\mathcal{P}_\alpha^*$ incident with the origin $[0,0,0;0]$ and the point $[1,0,0;d]$ for $d = \binom{d_1}{d_2}$ is the point $\left(\binom{-d_1}{d_2}, 0\right)$ of $\mathcal{P}_\alpha$. We now determine the dual points $[s,q,h;t]$ which are incident with this dual line. They satisfy

$$\begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} \begin{pmatrix} -d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} -d_1 \\ d_2 \end{pmatrix} q + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = 0.$$

Thus, the dual points lying on the dual line through the origin $[0,0,0;0]$ and the dual point $[1,0,0;d]$ are the dual points $[s,q,h;t]$ satisfying

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} d_1(s+q) - \overline{h} d_2 \\ d_2(s-q) + h d_1 \end{pmatrix}.$$

In order to express $s-q$ in terms of $s+q$, recall that $q \in v^\perp$ and that $v = \cos\alpha + \sin\alpha \cdot i$. Thus, with the real part $\operatorname{Re} q$ of q, one has $0 = \langle q, v\rangle = \operatorname{Re} q \cdot \cos\alpha + \langle q, i\rangle \sin\alpha = \operatorname{Re} q \cdot \cos\alpha + \langle s+q, i\rangle \sin\alpha$, so that

$$\operatorname{Re} q = -\tan\alpha \langle s+q, i\rangle.$$

Furthermore, $s - q = \overline{s+q} - 2\operatorname{Re} q = \overline{s+q} + \rho\langle s+q, i\rangle$, where $\rho = 2\tan\alpha$. Passing to coordinates from $\mathbb{H}^2 \times \mathbb{H}^2$, one thus obtains that the dual point with coordinates $x = \binom{x_1}{x_2} = \binom{s+q}{h}$ and $y = \binom{y_1}{y_2} = \binom{t_1}{t_2}$ lies on the dual line through the dual points with coordinates $\binom{0}{0}$ and $\binom{1}{d}$ if and only if

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} d_1 x_1 - \overline{x_2} d_2 \\ d_2 \overline{x_1} + x_2 d_1 + \rho\langle x_1, i\rangle d_2 \end{pmatrix}.$$

The coordinatizing division algebra for the dual plane thus has multiplication

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1x_1 - \overline{x_2}d_2 \\ d_2\overline{x_1} + x_2d_1 + \rho\langle x_1, i\rangle d_2 \end{pmatrix} = dx + \rho\langle x_1, i\rangle \begin{pmatrix} 0 \\ d_2 \end{pmatrix}$$

where dx is the classical octonion product.

In preparation for Sect. 4, where we will use transposition to produce the planes of Case 3, we rewrite the multiplication $*$ by the real 8×8-matrix M_d of the endomorphism $x \mapsto d * x$ of the 8-dimensional real vector space $\mathbb{H}^2$. With respect to the $\mathbb{R}$-basis $1, i, j, k$ of $\mathbb{H}$, the matrices

$$K = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

when applied to a quaternion q, give $Kq = \overline{q}$ and $Tq = \langle q, i\rangle$. Furthermore, let L_a and R_a for $a \in \mathbb{H}$ be the real 4×4-matrices describing the endomorphisms $q \mapsto aq$ and $q \mapsto qa$ ($q \in \mathbb{H}$) of the real vector space $\mathbb{H}$ given by the classical multiplication of quaternions. With these matrices

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M_d \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{where} \quad M_d = \begin{pmatrix} L_{d_1} & -R_{d_2}K \\ L_{d_2}(K + \rho T) & R_{d_1} \end{pmatrix}.$$

3.3. Collineations

Next, we shall have to determine how the group $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ given in Sect. 2.12 acts on the affine points of $\mathcal{P}_\alpha^*$. The line $[s, q, h; t]$ (see Sect. 3.1) of $\mathcal{P}_\alpha$ is mapped by the collineation corresponding to $A \in \mathrm{SU}_2\mathbb{H}$ to

$$\begin{aligned} &\left\{\left(Ax, A\begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} x + Axq + At\right);\ x \in \mathbb{H}^2\right\} \\ &\quad = \left\{\left(x, A\begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} A^{-1}x + AA^{-1}xq + At\right);\ x \in \mathbb{H}^2\right\} \\ &\quad = \left\{\left(x, A\begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} A^{-1}x + xq + At\right);\ x \in \mathbb{H}^2\right\}. \end{aligned}$$

With

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \quad \text{one computes}$$

$$A\begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} A^{-1} = \begin{pmatrix} s' & \overline{h'} \\ h' & -s' \end{pmatrix} \quad \text{where}$$

$$s' = s(\overline{a}a - \overline{b}b) + 2\mathrm{Re}(bh\overline{a}), \qquad h' = 2sc\overline{a} + dh\overline{a} + c\overline{h}\,\overline{b}$$

(one uses that $c\overline{a} + d\overline{b} = 0$). Thus, in $\mathcal{P}$, the collineation corresponding to A maps the line

$$[s, q, h; t] \quad \text{to} \quad [s', q, h'; At].$$

In order to express this in quaternal coordinates, that is, in terms of $s+q$ and h only, we note that $s = \operatorname{Re}(s+q) - \operatorname{Re} q = \operatorname{Re}(s+q) + \tan\alpha\langle s+q, i\rangle$ and $q = s+q-s = s+q-\operatorname{Re}(s+q) - \tan\alpha\langle s+q,i\rangle$, so that $s'+q = (\operatorname{Re}(s+q)+\tan\alpha\langle s+q,i\rangle)(\overline{a}a-\overline{b}b) + s+q-\operatorname{Re}(s+q)-\tan\alpha\langle s+q,i\rangle + 2\operatorname{Re}(bh\overline{a}) = -2(\operatorname{Re}(s+q)+\tan\alpha\langle s+q,i\rangle)\overline{b}b + s+q+2\operatorname{Re}(bh\overline{a})$ (one uses that $\overline{a}a-\overline{b}b-1 = \overline{a}a-\overline{b}b-(\overline{a}a+\overline{b}b) = -2\overline{b}b$). Also, one obtains that $h' = 2(\operatorname{Re}(s+q)+\tan\alpha\langle s+q,i\rangle)c\overline{a} + dh\overline{a} + c\overline{h}\,\overline{b}$.

Again, for later use in Sect. 4, we rewrite the result by a real 8×8-matrix using the 4×4-blocks defined above and the matrices

$$S = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

(note that for a quaternion q one has $Sq = \operatorname{Re} q$). The collineation corresponding to A maps a dual point with quaternal coordinates

$$\left(\begin{pmatrix} s+q \\ h \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) \quad \text{to} \quad \left(\begin{pmatrix} s'+q \\ h' \end{pmatrix}, A\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \left(\widehat{A}\begin{pmatrix} s+q \\ h \end{pmatrix}, A\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right)$$

$$\text{where} \quad \widehat{A} = \begin{pmatrix} I-\overline{b}b(2S+\rho T) & 2SL_bR_{\overline{a}} \\ L_{c\overline{a}}(2S+\rho T) & L_dR_{\overline{a}} + L_cR_{\overline{b}}K \end{pmatrix};$$

(as above, we have put $\rho = 2\tan\alpha$).

Finally, we study the action of the torus subgroup of the collineation group of $\mathcal{P}_\alpha$ described in Sect. 2.12 on the dual plane $\mathcal{P}_\alpha^*$. The collineation $(x,y)\mapsto(xc,yc)$ for $c\in\mathbb{R}\cdot 1+\mathbb{R}i$, $\overline{c}c=1$ of $\mathcal{P}_\alpha$ belonging to this torus group maps the line $[s,q,h;t]$ to

$$\begin{aligned}
&\left\{\left(xc, \begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} xc + xqc + tc\right);\ x\in\mathbb{H}^2\right\} \\
&\quad = \left\{\left(x, \begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} xc^{-1}c + xc^{-1}qc + tc\right);\ x\in\mathbb{H}^2\right\} \\
&\quad = \left\{\left(x, \begin{pmatrix} s & \overline{h} \\ h & -s \end{pmatrix} x + xc^{-1}qc + tc\right);\ x\in\mathbb{H}^2\right\} = [s, c^{-1}qc, h; tc].
\end{aligned}$$

Thus, since $c^{-1}(s+q)c = s+c^{-1}qc$, the collineation in question maps a dual point having quaternal coordinates

$$\left(\begin{pmatrix} s+q \\ h \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) \quad \text{to} \quad \left(\begin{pmatrix} c^{-1}(s+q)c \\ h \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} c\right).$$

In quaternal coordinates, the result may be summarized as follows.

3.4 Theorem. *Up to isomorphism, the non-classical sixteen-dimensional locally compact planes of Lenz type V whose collineation groups contain a subgroup* $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ *acting according Sect. 1.2 Case 2 are the planes* $\mathcal{P}_\alpha^*$ *over the division* *algebras* $(\mathbb{H}^2, +, *)$ *described in Sect.* 3.2 *for* $0 < \alpha < \pi/2, \rho = 2\tan\alpha$*. For different choices of* α*, one obtains non-isomorphic planes.*

The collineation group of such a plane is the product of the group of translations, the group of shears with axis S*, the group* Z *given in Sect.* 2.11*, the group* $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ *consisting of the collineations*

$$\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2 : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto$$

$$\left(\begin{pmatrix} x_1 - \overline{b}b(2\mathrm{Re}\, x_1 + \rho\langle x_1, i\rangle) + 2\mathrm{Re}(bx_2\overline{a}) \\ (2\mathrm{Re}\, x_1 + \rho\langle x_1, i\rangle)c\overline{a} + dx_2\overline{a} + c\overline{x_2}\overline{b} \end{pmatrix}, \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} \right)$$

$$\textit{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}_2\mathbb{H},$$

and the 1*-dimensional torus group consisting of the collineations*

$$\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2 : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} c^{-1}x_1c \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} c \right)$$
$$\textit{for} \quad c \in \mathbb{R}\cdot 1 + \mathbb{R}i, \overline{c}c = 1.$$

The group of collineations is therefore connected and has dimension 37*.*

4. The planes of Case 3

These are obtained from the planes of Case 2 by spread transposition. It is proved in [3] (see the proof of Proposition 3 there) that transposition of a plane of Lenz-type V leads to a plane which is of Lenz type V again. The switch of the rôles of the first and second coordinate axis W and S as regards the action of Λ can be obtained from there, as well, but will also become clear in the sequel.

4.1. Transposition

Using the division algebra $(\mathbb{H}^2, +, *)$ of the dual plane $\mathcal{P}_\alpha^*$ obtained in Sect. 3.2 and the matrix M_d expressing its multiplication $*$, the lines through the origin of the dual plane $\mathcal{P}_\alpha^*$ (a plane of Case 2) can be described as

$$\left\{(x, M_d(x)); x \in \mathbb{H}^2\right\} \quad \text{for} \quad d \in \mathbb{H}^2$$

together with the line $\{(0, y); y \in \mathbb{H}^2\}$.

The transposed plane $\mathcal{P}_\alpha^{*t}$ can be obtained just by transposition of the matrices M_d, see [10, Proposition 1.33]. Thus, its affine point set is $\mathbb{H}^2 \times \mathbb{H}^2$, and its lines through the origin are

$$\{(x, M_d^t(x)); x \in \mathbb{H}^2\} \quad \text{for} \quad d \in \mathbb{H}^2$$

together with the line $\{(0, y); y \in \mathbb{H}^2\}$. To obtain the transpose of M_d as described in Sect. 3.2 by 4×4-block matrices, we need to know that $L_a^t =$

$L_{\overline{a}}, R_a^t = R_{\overline{a}}$; this is easily established by a little computation with quaternions. With these block matrices, one obtains

$$M_d^t = \begin{pmatrix} L_{\overline{d_1}} & (K+\rho T)^t L_{\overline{d_2}} \\ -KR_{\overline{d_2}} & R_{\overline{d_1}} \end{pmatrix},$$

where $\rho = 2\tan\alpha$. The corresponding line through the origin of the transposed plane $\mathcal{P}_\alpha^{*t}$ is

$$\left\{ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} \overline{d_1}x_1 + \overline{x_2}d_2 + \rho T^t(\overline{d_2}x_2) \\ -d_2\overline{x_1} + x_2\overline{d_1} \end{pmatrix} \right);\ x_1, x_2 \in \mathbb{H} \right\}.$$

For $d_1 = 1, d_2 = 0$ this is the diagonal of $\mathbb{H}^2 \times \mathbb{H}^2$. The line through the origin and the point $\left(\binom{1}{0}, \binom{u_1}{u_2}\right)$ is obtained for $d_1 = \overline{u_1}, d_2 = -u_2$. Therefore the multiplication of the coordinatizing division algebra is

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \diamond \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} u_1x_1 - \overline{x_2}u_2 - \rho T^t(\overline{u_2}x_2) \\ u_2\overline{x_1} + x_2u_1 \end{pmatrix} \\ &= \begin{pmatrix} u_1x_1 - \overline{x_2}u_2 - \rho\mathrm{Re}(\overline{u_2}x_2)i \\ u_2\overline{x_1} + x_2u_1 \end{pmatrix} \\ &= ux - \rho\mathrm{Re}(\overline{u_2}x_2)\begin{pmatrix} i \\ 0 \end{pmatrix}, \end{aligned}$$

where ux is the classical octonion product.

We note that the transposed plane is self-dual. Indeed, its dual plane is known to be coordinatized by the converse division algebra, with multiplication

$$u \bigtriangleup x = x \diamond u = xu - \rho\mathrm{Re}(\overline{x_2}u_2)\begin{pmatrix} i \\ 0 \end{pmatrix}.$$

There are involutory antiautomorphisms of the octonion algebra which fix i, leave $\{0\} \times \mathbb{H}$ invariant, and commute with conjugation so that they preserve real parts, and it is readily seen that such an antiautomorphism establishes an isomorphism beween $(\mathbb{H}^2, +, \diamond)$ and $(\mathbb{H}^2, +, \bigtriangleup)$. For instance,

$$\mathbb{H}^2 \to \mathbb{H}^2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -j\overline{x_1}j \\ -jx_2j \end{pmatrix}$$

is such an antiautomorphism.

4.2. Collineations of the transposed plane

We compare collineations fixing the two coordinate axes in the plane $\mathcal{P}_\alpha^*$ and in the transposed plane $\mathcal{P}_\alpha^{*t}$. For $\mathbb{R}$-linear transformations C, D of $\mathbb{H}^2$, the transformation

$$(x, y) \mapsto (Cx, Dy)$$

of the affine point set of $\mathcal{P}_\alpha^*$ is a collineation if and only if it maps every line through the origin onto such a line. Now the line $\{(x, M_d(x)); x \in \mathbb{H}^2\}$ is mapped to $\{(Cx, DM_d(x)); x \in \mathbb{H}^2\} = \{(x, DM_dC^{-1}(x)); x \in \mathbb{H}^2\}$, so that we have a collineation if for every $d \in \mathbb{H}^2$ there is $d' \in \mathbb{H}^2$ such that $DM_dC^{-1} =$

$M_{d'}$. But then, by transposition, $(C^{-1})^t M_d^t D^t = M_{d'}^t$, so that, by an analogous argument, the transformation

$$(x, y) \mapsto ((D^{-1})^t x, (C^{-1})^t y)$$

is a collineation of the transposed plane $\mathcal{P}_\alpha^{*t}$. Moreover, if C is orthogonal, then $(C^{-1})^t = C$. We now apply this to the collineations of $\mathcal{P}_\alpha^*$ studied in Sect. 3.3. The torus subgroup of the collineation group described in Sects. 3.3 and 3.4 acts orthogonally, so its actions on the first and second coordinate axis of $\mathcal{P}_\alpha^*$ and of the transposed plane $\mathcal{P}_\alpha^{*t}$ are interchanged.

The collineation in $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ corresponding to $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}_2\mathbb{H}$ induces the natural action of A on the second coordinate axis of $\mathcal{P}_\alpha^*$, and this action is orthogonal. The corresponding collineation of the transposed plane $\mathcal{P}_\alpha^{*t}$ therefore induces A in its natural action on the first coordinate axis.

On the second coordinate axis of $\mathcal{P}_\alpha^*$, this collineation induces the transformation $\widehat{A}$ described in Sect. 3.3. Now $\widehat{A}^{-1} = \widehat{A^{-1}}$ corresponds to $A^{-1} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$ $\in \mathrm{SU}_2\mathbb{H}$. Thus, we obtain a collineation of the transposed plane $\mathcal{P}_\alpha^{*t}$ inducing the natural action of A on the first coordinate axis and acting on the second coordinate axis by the matrix

$$(\widehat{A^{-1}})^t = \begin{pmatrix} I - \overline{c}c(2S + \rho T)^t & (2S + \rho T)^t L_{\overline{a}b} \\ 2R_{\overline{a}} L_c S & R_{\overline{a}} L_d + K R_{\overline{c}} L_b \end{pmatrix}.$$

In the following theorem, this will be translated into quaternal coordinates.

4.3 Theorem. *Up to isomorphism, the non-classical sixteen-dimensional locally compact planes of Lenz type V whose collineation groups contain a subgroup* $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ *acting according to Sect. 1.2 Case 3 are the planes* $\mathcal{P}_\alpha^{*t}$ *over the division algebra* $(\mathbb{H}^2, +, \diamond)$ *described in Sect. 4.1 for* $0 < \alpha < \pi/2, \rho = 2\tan\alpha$. *These planes are self-dual. For different choices of* α, *one obtains non-isomorphic planes.*

The collineation group of such a plane is the product of the group of translations, the group of shears with axis S, *the group* Z *given in Sect. 2.11, the group* $\Lambda \cong \mathrm{SU}_2\mathbb{H}$ *consisting of the collineations*

$$\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2 : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto$$

$$\left(\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \begin{pmatrix} y_1 - \overline{c}c\mathrm{Re}\, y_1(2 + \rho i) + \mathrm{Re}(\overline{a}by_2)(2 + \rho i) \\ 2\mathrm{Re}\, y_1 \cdot c\overline{a} + dy_2\overline{a} + c\overline{y_2}\overline{b} \end{pmatrix} \right)$$

$$\textit{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}_2\mathbb{H},$$

and the 1*-dimensional torus group consisting of the collineations*

$$\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2 : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} c, \begin{pmatrix} c^{-1} y_1 c \\ y_2 \end{pmatrix} \right)$$
$$\textit{for} \quad c \in \mathbb{R} \cdot 1 + \mathbb{R}i, \bar{c}c = 1.$$

The group of collineations is therefore connected and has dimension 37*.*

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