On two Problems for the Stark Laplacian on Domains

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines Doktors der Naturwissenschaften
(Dr. rer. nat.) genehmigte Abhandlung

vorgelegt von

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Prüfungsdatum:	26.10.2022

Institut für Analysis, Dynamik und Modellierung
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2023

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Acknowledgements

I would like to express my special thanks and gratitude to my supervisor Timo Weidl for teaching me mathematics, in particular analysis, since the beginning of my university studies. He drove my interest into the field of spectral theory and mathematical physics and constantly supported me during my whole time as a doctoral student. I had the fortune to profit from his help in mathematical as well as personal matters and his devotion to teaching mathematics.

Special thanks go to my whole group of the IADM including but not limited to Peter Lesky and Jens Wirth for their encouragement, helpful advice and fruitful discussions.

Last but not least I would like to express my thanks to Irene Wecker for her love and constant support each day and I am looking forward to our new adventure in autumn.

Abstract

The present paper is divided into two parts, each of them dedicated to a problem for the Stark operator $-\Delta + E \cdot x$ on some domain $\Omega \subset \mathbb{R}^d$, equipped with either Dirichlet or Neumann boundary conditions.

In the first part we deal with spectral estimates, i.e. estimates for the Riesz means

$$\mathrm{Tr}_\gamma \left(H^i_{\varepsilon_0}(\Omega) - \Lambda \right) := \sum_{j \in \mathbb{N}} (\Lambda - \lambda_j(\Omega; \varepsilon_0))^\gamma_+,$$

$i \in \{D, N\}$ and $\gamma \geq 0$. Here $(\lambda^i_j(\Omega; \varepsilon_0))_{j \in \mathbb{N}}$ denotes the sequence of Dirichlet or Neumann eigenvalues of $H^i_{\varepsilon_0}(\Omega) = -\Delta + \varepsilon_0 x_1$ on $\Omega \subset \mathbb{R} \times \mathbb{R}^{d-1}$ where we have chosen the x_1-axis to be the direction of the electric field. Our starting point will be Berezin's approach to an estimate of $\mathrm{Tr}_\gamma \left(H^D_0(\Omega) - \Lambda \right)$ where $H^D_0(\Omega) = -\Delta$ is the classical Dirichlet Laplacian on Ω. We generalize Berezin's inequality to the case $\varepsilon_0 > 0$ and additionally improve it by subtracting positive terms of lower order in Λ. This result can be accompanied by corresponding Kröger-type estimates on $\mathrm{Tr}_\gamma \left(H^N_{\varepsilon_0}(\Omega) - \Lambda \right)$ from below. In the case $\gamma < 1$, in particular $\gamma = 0$ where $\mathrm{Tr}_\gamma \left(H^i_{\varepsilon_0}(\Omega) - \Lambda \right)$ coincides with the counting function of all eigenvalues below Λ, we present estimates which are obtained by explicitly using the Airy functions, i.e. the fundamental solutions of the differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) + x\, u(x) = 0.$$

The second part of this paper outlines steps towards a Faber-Krahn

inequality for the Stark Laplacian. Due to the lack of symmetry of the Stark potential $\varepsilon_0 x_1$, the situation becomes less clear compared to the case of the classical Laplacian operator. Since the eigenvalues of $H^D_{\varepsilon_0}(\Omega)$ do not only depend on the area or the volume of Ω, but additionally on the position of Ω along the x_1-axis, minimizing the first eigenvalue $\lambda^D_1(\Omega;\varepsilon_0)$ only makes sence among Ω with fixed volume and center of mass. Yet we will show that these assumptions do not guarantee the existence of minimizers and additional assumptions are required. Finally we show the existence of a minimizing domain for $\lambda^D_1(\Omega;\varepsilon_0)$ in the set $\{\Omega \subset \mathbb{R}^d : \Omega \text{ convex}, |\Omega| = V, \int_\Omega x \,\mathrm{d}x = 0\}$ for fixed $V > 0$ in $d = 2$ and $d = 3$. However our proof does not provide any information on how these minimizers might look like. In order to get an idea of their shapes we perform numerical shape optimizations and observe how our candidates change in various regimes for $\varepsilon_0 > 0$. Since our optimization is based on a gradient descent for $\lambda^D_1(\Omega;\varepsilon_0)$ we have to quantify it's change when perturbing the shape of Ω and therefore prove a Hadamard-type formula for the Stark Laplacian.

Zusammenfassung

Die Vorliegende Arbeit besteht aus zwei Teilen, jedes einem Themenkreis für den Stark-Operator $-\Delta + E \cdot x$ auf einem Gebiet $\Omega \subset \mathbb{R}^d$, versehen mit Dirichlet- oder Neumann-Randbedingungen, gewidmet.

Im ersten Teil behandeln wir Spektralabschätzungen, d.h. Abschätzungen an die Riesz-Summen

$$\operatorname{Tr}_\gamma \left(H^i_{\varepsilon_0}(\Omega) - \Lambda \right) := \sum_{j \in \mathbb{N}} (\Lambda - \lambda_j(\Omega; \varepsilon_0))^\gamma_+,$$

$i \in \{D, N\}$ für $\gamma \geq 0$. Dabei bezeichnet $(\lambda^i_j(\Omega; \varepsilon_0))_{j \in \mathbb{N}}$ die Folge der Dirichlet- oder Neumann-Eigenwerte von $H^i_{\varepsilon_0}(\Omega) = -\Delta + \varepsilon_0 x_1$ auf $\Omega \subset \mathbb{R} \times \mathbb{R}^{d-1}$ wobei wir die x_1-Achse als Richtung für das elektrische Feld gewählt haben. Unser Ausgangspunkt ist Berezins Ansatz für eine Abschätzung an $\operatorname{Tr}_\gamma \left(H^D_0(\Omega) - \Lambda \right)$ für den klassischen Laplace-Operator $H^D_0(\Omega) = -\Delta$ auf Ω. Wir werden die Berezin-Ungleichung auf den Fall $\varepsilon_0 > 0$ verallgemeinern und zusätzlich durch einen zweiten Term mit kleinerer Ordnung in Λ verbessern. Ergänzend dazu zeigen wir untere Abschätzungen an $\operatorname{Tr}_\gamma \left(H^N_{\varepsilon_0}(\Omega) - \Lambda \right)$ basierend auf Arbeiten von Kröger. Im Fall $\gamma < 1$, insbesondere $\gamma = 0$ wo $\operatorname{Tr}_\gamma \left(H^i_{\varepsilon_0}(\Omega) - \Lambda \right)$ gerade die Zählfunktion aller Eigenwerte unterhalb von Λ ist, beweisen wir Ungleichungen durch explizites Rechnen mit Airy-Funktionen, also den Fundamentallösungen von

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) + x\, u(x) = 0.$$

Der zweite Teil der Arbeit beinhaltet Überlegungen zur Faber-Krahn-

Ungleichung für den Stark-Operator. Da das elektrische Potential $\varepsilon_0 x_1$ die Symmetrie des Operators bricht, ist die Situation hier, verglichen mit der beim klassischen Laplace-Operator, komplizierter. Eigenwerte von $H^D_{\varepsilon_0}(\Omega)$ skalieren nicht nur von mit der Fläche oder dem Volumen von Ω sondern verschieben sich zusätzlich mit der Lage von Ω auf der x_1-Achse. Das Minimierungsproblem ist daher nur unter Gebieten mit gleichem Volumen und fixiertem Schwerpunkt wohlgestellt. Diese Bedingung ist jedoch noch nicht hinreichend für die Existenz eines minimierenden Gebietes für $\lambda^D_1(\Omega; \varepsilon_0)$ und wir zeigen dessen Existenz innerhalb der Klasse $\{\Omega \subset \mathbb{R}^d \,:\, \Omega \text{ convex}, |\Omega| = V, \int_\Omega x \, \mathrm{d}x = 0\}$ mit fest gewähltem $V > 0$ in $d = 2$ und $d = 3$. Unser Beweis liefert jedoch keinerlei Informationen über die Gestalt dieser Minimierer. Um einen Überblick zu erhalten, führen wir eine Gebietsoptimierung durch und beobachten wie sich die Minimierer in verschiedenen Regimen für $\varepsilon_0 > 0$ verändern. Da unsere Optimierung auf einem Gradientenabstieg für $\lambda^D_1(\Omega; \varepsilon_0)$ beruht, müssen wir die Änderung von $\lambda^D_1(\Omega; \varepsilon_0)$ unter Störung von Ω kontrollieren und zeigen dafür eine Hadamard-Formel für den Stark-Operator.

Contents

Chapter 1

Introduction

In 1913 the experimental physicists J. Stark studied the influence of electric fields on the spectral lines of hydrogen and helium atoms. He observed that the presence of an electric field causes splitting of several spectral lines and described his results in a series of papers beginning with [85]. Later, in 1919, he was rewarded with the Nobel Prize in Physics. Since then the effect of splitting and shifting of spectral lines of atoms and molecules when exposing them to an external electric field is known as the *Stark effect.* The theoretical description of this effect occurs in the context of quantum mechanics with the help of atomic Schrödinger operators as e.g.

$$H = -\Delta + E \cdot x - \frac{Z}{|x|} \tag{1.1}$$

for the hydrogen atom. Here the linear part $E \cdot x$ describes the electric field. Throughout the physics literature, e.g. [20, 61], this operator is treated as a perturbation of the atomic Schrödinger operator without the term $E \cdot x$, however, this approach cannot be justified mathematically by any means. Hence, in the mathematical literature it is more common to introduce

$$H_0 = -\Delta + E \cdot x$$

as the unperturbed operator and then add some multiplication operator $V(x)$, a *potential* in physical terminology.

From a mathematical point of view (1.1) was first studied by E. C. Titchmarsh [87] where it is shown that the operator has no discrete eigenvalues if $|E| > 0$, but the spectrum covers the whole real line $]-\infty, \infty[$. However, the proof uses methods that are highly tailored to the symmetry of the hydrogen atom operator, such as adapted coordinates, and cannot be carried over to any other cases. Nevertheless, the work motivated J.E. Avron and I.W. Herbst to study operators of the form $H_0 + V$ in a more general setting [10]. They mainly focus on the operator in $\mathbb{R}^3$ but many of their results were later carried over to the one-dimensional case [51] or to arbitrary dimensions. To simplify things we choose the direction of the electric field along the x_1-axis, i.e. $E = \varepsilon_0 e_1$ for some coupling constant (or field strength) $\varepsilon_0 > 0$ and decompose $x \in \mathbb{R}^d$ into $x = (x_1, x_\perp)$ where $x_1 \in \mathbb{R}$ and $x_\perp \in \mathbb{R}^{d-1}$ if $d \geq 2$. The starting point in [10] is the introduction of an unitary mapping on $L^2(\mathbb{R}^3)$ which transforms

$$H_0 = -\Delta + \varepsilon_0 x_1$$

into the operator of multiplication by $\varepsilon_0 x_1 + |x_\perp|^2$. This transformation will be discussed in Section 1.4.2 in more detail. As a consequence the spectrum of H_0 is absolutely continuous with $\sigma(H_0) =]-\infty, \infty[$. Furthermore H_0 with $D(H_0) = C_0^\infty(\mathbb{R}^3)$ is essentially self-adjoint. For the sake of simplicity we will also denote its closure also by H_0. Although $D(H_0)$ remains uncharacterized, it is still possible to formulate sufficient conditions on the potential V to be a relatively compact perturbation of H_0. Therefore let

$$L^p_{0,c}(\mathbb{R}^3) := \{f \,:\, \forall_{\varepsilon>0} \exists_{f=f_{1,\varepsilon}+f_{2,\varepsilon}}\ f_{1,\varepsilon} \in L^p_0(\mathbb{R}^3) \text{ and } \|f_{2,\varepsilon}\| \leq \varepsilon\},$$

where $L^p_0(\mathbb{R}^3)$ is the set of all functions in $L^p(\mathbb{R}^3)$ with compact support.

Theorem 1.0.1 ([10, Theorem 4.2])**.** *Let G be a multiplication operator with $G\mathrm{e}^{-\alpha|x|} \in L^p(\mathbb{R}^3)$ for some $\alpha > 0$ and let H_0 be a self-adjoint extension of $-\Delta + G$ on $C_0^\infty(\mathbb{R}^3)$. If V is an operator of multiplication by $V \in L^q_{0,c}(\mathbb{R}^3)$ and either*

- *$1/p + 1/q = 2/3$, $p, q > 2$ or*
- *$p = 2$ and $q > 6$ or*
- *$q = 2$ and $p > 6$,*

then $V(H_0 + \mathrm{i})^{-1}$ is compact.

Since $x_1 \mathrm{e}^{-\alpha|x|} \in L^p(\mathbb{R}^3)$ for all $p \in \mathbb{N}$, it follows that $H = H_0 + V$ is essentially self-adjoint with $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$ for all $V \in L^q_{0,c}(\mathbb{R}^3)$, $q \in \mathbb{N}$, cf. [48].

From that J.E. Avron and I.W. Herbst established the first results on the scattering theory in the presence of an electric field. This work then was continued by D.A.W. White in [93], K. Yajima [96, 97] or more recently by T. Adachi, K. Itakura, K. Ito and E. Skibsted [3, 46], just to mention a few results. On the other hand there are accompanying works on trace formulas [52] or the spectral shift function [80] by E. Korotyaev, A. Pushnitski and V. Sloushch. The influence of an electric field in waveguides are studied by P. Briet and M. Gharsalli in [15, 16]. In the following two sections we want to highlight two other subjects of study, namely those of resonances which occur as limits of eigenvalues of dilated operators as a model for physical Stark effect and the study of eigenvalue bounds for complex perturbated Stark operators. At the end of this chapter, in 1.3 and 1.4, we proceed to the foundations of our work on the Stark operator on bounded domains.

1.1 Analytic potentials and resonances

A type of potentials for which the analysis of the spectrum is well advanced is the class of analytic potentials introduced in [10]. Consider a multiplication operator V such that for almost every $x_\perp$ the mapping $z \mapsto V(z, x_\perp)$ is analytic in a strip $\{z \in \mathbb{C} : |\mathrm{Im} z| < \beta\}$ where $\beta > 0$ is independent of $x_\perp$. If $V_z(H_0 + \mathrm{i})^{-1}$ for $V_z(x) = V(x + z, x_\perp)$ is a compact operator-valued function, then V is called H_0*-translation analytic*. This is for instance the case when $V_z(H_0 + \mathrm{i})^{-1}$ is uniformly bounded on each compact subset of the strip $\{z \in \mathbb{C} : |\mathrm{Im} z| < \beta\}$, see [10, Lemma 4.5].

Theorem 1.1.1 ([10, Theorem 4.7]). *1. Let $H = -\Delta + \varepsilon_0 x_1 + V$ for $\varepsilon_0 > 0$. If V is H_0-translation analytic in the strip $\{z \in \mathbb{C} : |\,\mathrm{Im} z| < \beta\}$, then $\sigma_p(H)$ does not have finite accumulation points. Furthermore $\sigma_{sing}(H) = \emptyset$ and $\sigma_{ac}(H) =]-\infty, \infty[$.*

2. Consider the operators $H_z = -\Delta + \varepsilon_0(x + z) + V_z$, then these form an analytic family of type A in the sense of Kato [48, VII, 2] *with the spectral properties*

$$\sigma_{ess}(H_z) = \mathbb{R} + \mathrm{i}\varepsilon_0 \,\mathrm{Im} z$$
$$\sigma_{disc}(H_z) \subset \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \in [0, \varepsilon_0 \,\mathrm{im} z[\}.$$

Thereby the singularities of $\lambda \mapsto (H_z - \lambda)^{-1}$ are poles with finite rank residues, their location in $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \in [0, \varepsilon_0 \,\mathrm{im} z[\}$ is independent of z.

The eigenvalues of H_z are called the resonances associated with the system described by H. If resonances exist, by physical intuition, they should converge to the eigenvalues of $-\Delta+V$ as $\varepsilon_0 \to 0$. The main technical difficulty at this point is the presence of the essential spectrum of H_z which converges to the real line for $\varepsilon_0 \to 0$. In order to overcome these difficulties I.W. Herbst [43] used the technique of complex scaling. We refer to [22, Chapter 8] for a comprehensive introduction to complex scaling with various applications also to the Stark effect. The idea is to somehow "rotate" the potential away from the real axis. Consider the free Stark operator $h(\alpha) = -\Delta + \alpha x_1$ for $\alpha \in \mathbb{C}$ on $C_0^\infty(\mathbb{R}^d)$. Then $h(\alpha)$ is closeable, but in contrast to the case $\alpha > 0$ the spectrum of $\overline{h(\alpha)}$ is empty if $\mathrm{Im}\alpha \neq 0$ [43, Theorem II.1]. Furthermore, if $\mathrm{Im}\alpha \neq 0$, the numerical range of $h(\alpha)$ is given by

$$W(h(\alpha)) = \{z \in \mathbb{C} : \mathrm{Re} z > \tfrac{\mathrm{Re}\alpha}{\mathrm{Im}\alpha} \,\mathrm{Im} z\}.$$

This allows the introduction of an operator

$$H(\varepsilon_0, \theta) = -\Delta \mathrm{e}^{-2\theta} + \varepsilon_0 x_1 \mathrm{e}^{\theta} + V(x\mathrm{e}^{\theta}) \tag{1.2}$$

for θ which needs to be specified further. Denote by $H_0(\varepsilon_0, \theta) := \mathrm{e}^{-2\theta}\overline{h(\varepsilon_0 \mathrm{e}^{3\theta})}$

the free Stark operator and define the dilation group on $L^2(\mathbb{R}^d)$ by $\{U(\theta) : \theta \in \mathbb{R}\}$ with

$$U(\theta)f(x) = \mathrm{e}^{d\theta/2} f(\mathrm{e}^{\theta} x). \tag{1.3}$$

Let $V(\theta) = U(\theta)VU(-\theta)$ for $\theta \in \mathbb{R}$ and some multiplication operator V and assume that V is self-adjoint with $D(V) \subset D(-\Delta)$ and $V(\theta)$ is compact and extends to a compact analytic operator-valued function on the strip $\{\theta \in \mathbb{C} : |\operatorname{Im}\theta| < \theta_0\}$ for some $\theta_0 \in]0, \pi/3]$ (in this case V is called *dilation-analytic*). Then the operator

$$V(\theta)(H_0(\varepsilon_0, \theta) - z)^{-1}$$

is compact and the mapping $(z, \theta) \mapsto V(\theta)(H_0(\varepsilon_0, \theta) - z)^{-1}$ is analytic on $\{\theta \in \mathbb{C} : 0 < \operatorname{Im}\theta < \theta_0\}$. Moreover

$$V(\theta)(H_0(\varepsilon_0, \theta) - z)^{-1} \stackrel{\|\cdot\|}{\longrightarrow} V(\theta)(H_0(0, \theta) - z)^{-1}$$

as $\varepsilon_0 \to 0$ uniformly for (z, θ) on each compact subset of $\{(z, \theta) \in \mathbb{C}^2 : d(z, \theta) > 0,\ 0 < \operatorname{Im}\theta < \theta_0\}$ where $d(z, \theta)$ is the distance from z to the numerical range of $H_0(1, \theta)$, see [43, Proposition III.1]. Finally it follows that

Theorem 1.1.2 ([43, Theorem III.2])**.** *For each $0 < \operatorname{Im}\theta < \theta_0$ for some $\theta_0 \in]0, \pi/3]$ and $\varepsilon_0 > 0$ the operator $H(\varepsilon_0, \theta)$ from 1.2 is closed on $D(H(\varepsilon_0, \theta)) = D(H_0(\varepsilon_0, \theta))$ and the family of operators*

$$\{H(\varepsilon_0, \theta) : 0 < \operatorname{Im}\theta < \theta_0\}$$

is an analytic family of operators of type A in the sense of Kato [48]*. Moreover, the spectrum of $H(\varepsilon_0, \theta)$ consists of eigenvalues of finite multiplicities where each of the eigenvalues multiplicity does not depend on θ.*

The eigenvalues of $H(\varepsilon_0, \theta)$ are interpreted as the resonances which occur when a system, which is described by the Hamiltonian $H = -\Delta + V$, is exposed to an electric field with constant field strength:

Theorem 1.1.3 ([43, Theorem III.3])**.** *Let λ be an eigenvalue of $-\Delta + V$ with multiplicity m. Then for each $\varepsilon_0 > 0$ (small enough) there are m eigenvalues* *of* $H(\varepsilon_0, \theta)$*, counting according to their multiplicity, that converge to* λ *as* $\varepsilon_0 \to 0$*.*

Finally the eigenvalues of $H(\varepsilon_0, \theta)$ can be reconnected with the operator $H(\varepsilon_0, 0) = -\Delta + \varepsilon_0 x_1 + V$ via dilation-analytic vectors. In this context *Dilation-analytic vectors* are functions $\phi \in L^2(\mathbb{R}^d)$, such that there is an analytic function ϕ_θ on $\{\theta \in \mathbb{C} : 0 < \mathrm{Im}\theta < \theta_0\}$ for which $\phi_\theta = U(\theta)\phi$ on $]0, \infty[$.

Theorem 1.1.4 ([43, Theorem III.4])**.** *Let* $\varepsilon_0 > 0$ *and* V *such that* $V(-\Delta + \varepsilon_0 x_1 + \mathrm{i})^{-1}$ *is compact. If* $\mathrm{Im}\theta > 0$*, all eigenvalues of* $H(\varepsilon_0, \theta)$ *are contained in the lower half plane* $\{z \in \mathbb{C} : \mathrm{Im}z \leq 0\}$*. For a pair of dilation vectors* ϕ, ψ *consider the mapping*

$$\{z \in \mathbb{C} : \mathrm{Im}z > 0\} \to \mathbb{C}, \qquad z \mapsto f_{\phi,\psi}(z) := (\phi, (z - H(\varepsilon_0, 0))^{-1}\psi)$$

has a meromorphic continuation on $\mathbb{C}$*. There, the only possible poles of this meromorphic continuation are the eigenvalues of* $H(\varepsilon_0, \theta)$*. More precisely: Given* $z \in \mathbb{C}$*, then there are dilation-analytic vectors* ϕ, ψ *such that* $f_{\phi,\psi}$ *has a pole in* z *if and only if* $z \in \sigma(H(\varepsilon_0, \theta))$*.*

Up to this point the results do not hold for the Coulomb potential $Z/|x|$, the case of the hydrogen atom, since it is not translation analytic. But when replacing the point charge Z by a Gaussian charge density, the new potential

$$V(x) = \int \frac{\rho(y)}{|x - y|}\,\mathrm{d}y$$

for $\rho(y) = -Z/(2\pi\tau^2)^{d/2}\mathrm{e}^{-y^2/(2\tau^2)}$, $\tau > 0$, is again translation analytic, cf. [10]. The following theorem shows that in this case the resonance eigenvalues of the smoothened system converge as $\tau \to 0$:

Theorem 1.1.5 ([43, Theorem III.5])**.** *1. Suppose* $V(H_0(\varepsilon, 0) + \mathrm{i})^{-1}$ *is compact and* $\theta \mapsto V(\theta)(-\Delta + 1)^{-1}$ *is an analytic compact operator-valued function on* $\{\theta \in \mathbb{C} : |\mathrm{Im}\theta| < \theta_0\}$ *for some* $\theta_0 \in]0, \pi/4]$*. Then these conditions also apply to* $\rho * V$ *and*

$$z \mapsto (\rho * V)^z(H_0(\varepsilon_0, 0) + \mathrm{i})^{-1}$$

is an entire compact operator-valued function. Let $a < 0$ *and* $Q_a := \{z \in \mathbb{C} : \mathrm{Im} z > \varepsilon_0 a\}$*, then*

$$\sigma(H_0(\varepsilon_0, 0) + \mathrm{i}\varepsilon_0 a + (\rho * V)^{\mathrm{i}a}) \cap Q_a = \sigma(H_0(\varepsilon_0, \theta) + (\rho * V)(\theta)) \cap Q_a$$

for all $\theta \in \mathbb{R}$ *with* $0 < \mathrm{Im}\theta < \theta_0$ *and the multiplicities of the eigenvalues coincide.*

2. *Let* λ *be an eigenvalue of* $H(\varepsilon_0, \theta)$ *with multiplicity* m*, then for small enough* τ *and large enough* $-a$ *there are exact* m *eigenvalues of* $H(\varepsilon_0, 0) + \mathrm{i}\varepsilon_0 a + (\rho * V)^{\mathrm{i}a}$*, counting according to their multiplicity, that converge to* λ *as* $\tau \to 0$.

We remark that the Stark operator with Coulomb potential in $\mathbb{R}^3$ was independently treatened in [36] by S. Graffi and V. Grecchi. While their results on the occurence and convergence of resonances coincide with [43], their methodes are very different. As in [87] the proof in [36] is adapted to the symmetry of the Coulomb potential and uses squared parabolic coordinates. Using a dilation based on squared parabolic coordinates, S. Graffi and V. Grecchi [37] proved the Stark effect as in Theorem 1.1.3 for N-body Schrödinger operators

$$H_N(\varepsilon_0) := \sum_{j=1}^{N} \Delta_j + \sum_{j=1}^{N} V_j(x^{(j)}) + \sum_{\substack{i,j=1 \\ i<j}}^{N} V_{ij}(x^{(i)} - x^{(j)}) + \varepsilon_0 \sum_{j=1}^{N} x_1^{(j)}.$$

Here $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$ denote the position of the j-th particle in $\mathbb{R}^3$ and Δ_j the Laplace operator with respect to $(x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$. Thereby the particle interacts with a fixed center via the potential V_j, the potentials V_{ij} describe the interaction between two particles. Restrictions on V_j and V_{ij} are given in [37, Assumptions 3.5] and include Coulomb and Yukawa potentials. As in Theorem 1.1.3 above, [37, Theorem 4.1] states that, for any eigenvalue λ_0 of the Schrödinger operator $H_N(0)$ without electric field, the dilated operator $H_N(\varepsilon_0, \theta, \phi)$ (see [37, (3.7)]) for its definition) has exactly as many eigenvalues according to the multiplicity of λ_0 which converge to λ_0 as $\varepsilon_0 \to 0$. A similar result for N-body Schrödinger operators, but within the framework of [43], was proven by I. W. Herbst and B. Simon in [44, Theorem 4.1].

1.2 Bounds on eigenvalues for complex potentials

Since the essential spectrum of $H_0 = -\Delta + x_1$ covers the complete real axis, isolated eigenvalues of $H = H_0 + V$ can only occur when H_0 is perturbated by a complex potential with nonzero imaginary part. The resulting operator will not be normal, in particular not self-adjoint, such that many methods which arise from the spectral theorem are no longer applicable. Beginning with the work of A. A. Abramov, A. Aslanyan and E. B. Davies [1], there was growing interest in bounds of eigenvalues of Schrödinger operators $-\Delta + V$ where V is a complex valued potential, satisfying various other conditions, see [1, 26, 27, 34, 63] to mention only a few examples. Among many useful tools for the calculation of eigenvalue bounds is the *Birman-Schwinger principle*: Given a potential V and multiplication operators W_1, W_2 satisfying $V = W_1 W_2$ and $|W_1| = |W_2|$, then λ is an eigenvalue of $H_0 + V$ if and only if -1 is an eigenvalue of

$$Y_0(\lambda) = W_1 R_0(\lambda) W_2,$$

where $R_0(\lambda) = (H_0 - \lambda)^{-1}$ is the resolvent. Note that $Y_0(\lambda)$ is bounded, if -1 is an eigenvalue of $Y_0(\lambda)$, then $\|Y_0(\lambda)\| \geq 1$. Thus, eigenvalues of $H_0 + V$ only occur in regions $\{\lambda \in \mathbb{C} : \|Y_0(\lambda)\| \geq 1\}$. In order to proof bounds on eigenvalues or characterize regions in $\mathbb{C}$ where no eigenvalues are located it remains to calculate the norm of $Y_0(\lambda)$. In the case when H_0 is the one-dimensional Laplace operator on $L^2(\mathbb{R})$ the Birman-Schwinger operator $Y_0(\lambda)$ is an integral operator with kernel

$$\operatorname{sgn}(V(x))|V(x)|^{1/2} \frac{\mathrm{e}^{-\sqrt{\lambda}|x-y|}}{2\sqrt{\lambda}} |V(y)|^{1/2}.$$

Thus

$$\frac{\|V\|^2_{L^1(\mathbb{R})}}{4|\lambda|} \geq \int_{\mathbb{R}^2} |V(x)| \frac{\mathrm{e}^{-\operatorname{Re}(\sqrt{\lambda})|x-y|}}{4|\lambda|} |V(y)| \,\mathrm{d}x\,\mathrm{d}y \geq 1.$$

Skipping some technical details around the definition of $\sqrt{\lambda}$ on $\mathbb{C} \setminus [0, \infty[$ this proves:

Theorem 1.2.1 ([1, Theorem 4]). *If $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then any eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty[$ of $-\Delta + V$ on $L^2(\mathbb{R})$ satisfies*

$$|\lambda|^{1/2} \leq \|V\|_{L^1(\mathbb{R})}/2.$$

When $H_0 = -\Delta + x_1$, this approach becomes more subtle since the kernel of $R_0(\lambda) = (H_0 - \lambda)^{-1}$ has a more complicated structure. So far bounds on the eigenvalues of the complex perturbated Stark operator have only been studied by E. Korotyaev and O. Safronov in [53]. There, V is a complex valued function in $L^\infty(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} |V(x)|^r \,\mathrm{d}x < \infty. \tag{1.4}$$

for some $r > 0$. Using the representation of $\mathrm{e}^{-\mathrm{i}tH_0}$ as the product

$$\mathrm{e}^{-\mathrm{i}tH_0} = \mathrm{e}^{-\mathrm{i}t^3/12}\,\mathrm{e}^{-\mathrm{i}tx_1/2}\,\mathrm{e}^{\mathrm{i}t\Delta}\,\mathrm{e}^{-\mathrm{i}tx_1/2}$$

for all $t \in \mathbb{R}$, see [53, Proposition 3.1], one can write the resolvent $R_0(\lambda) = (H_0 - \lambda)^{-1}$ as an integral operator with kernel

$$\frac{\mathrm{e}^{-3\mathrm{i}\pi/4}}{(4\pi)^{3/2}} \int_0^\infty \mathrm{e}^{\mathrm{i}|x-y|^2/(4t)}\,\mathrm{e}^{-\mathrm{i}t^3/12}\,\mathrm{e}^{\mathrm{i}t\Lambda}\,t^{\zeta-1}\,\frac{\mathrm{d}t}{t^{3/2}}$$

for all $x, y \in \mathbb{R}^3$, $\mathrm{Re}\zeta > 3/2$ and $\lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$, where $\Lambda = \lambda - (x_1 + y_1)/2$, see [53, Proposition 3.1]. That way the norm of the Birman-Schwinger operator $Y_0(\lambda)$ can be estimated by

$$\|Y_0(\lambda)\| \leq \frac{C}{(1 + |\lambda|)^{1/4}} \left(\int_{\mathbb{R}^3} (1 + |x|)^4 |V(x)| \,\mathrm{d}x + \|V\|_{L^2(\mathbb{R}^3)} \right) \tag{1.5}$$

for some constant $C > 0$ which shows

Theorem 1.2.2 ([53, Theorem 1.6]). *Let $V \in L^\infty(\mathbb{R}^3)$, then there exists a constant $C > 0$ such that all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator $-\Delta + x_1 + V$*

on $L^2(\mathbb{R}^3)$ satisfy

$$|\lambda| \le C\left(\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\,\mathrm{d}x + \|V\|_{L^2(\mathbb{R}^3)}\right)^4. \tag{1.6}$$

Note that by Hölder's inequality

$$\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\,\mathrm{d}x + \|V\|_{L^2(\mathbb{R}^3)} \le C_p \left(\int_{\mathbb{R}^3} (1+|x|)^p |V(x)|^2\,\mathrm{d}x\right)^{1/2}$$

for some constant C_p, $p > 11$, such that the right hand side of (1.6) is actually bounded under the assumption (1.4). Moreover, it can be shown that $Y_0(\lambda)$ is in the $\mathcal{S}_{2r}$-Schatten class for $r > 3/2$, see [53, Theorem 6.4], i.e.

$$\|Y_0(\lambda)\|_{\mathcal{S}_{2r}}^{2r} = \mathrm{Tr}\,(Y_0(\lambda)^* Y_0(\lambda))^r < \infty.$$

Thus, the perturbation determinant

$$D_n(\lambda) = \det{}_n\,(I + Y_0(\lambda))$$

is defined for any $n \ge 2r$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$ which gives a representation of the eigenvalues λ as zeros of an analytic function. This analytic function then can be treated with methods from function theory. From (1.5) it follows that $D_5(\lambda) = 1 + O(|\lambda|^{-5/4})$ as $|\lambda| \to \infty$ in $\mathrm{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$ and since

$$|D_5(\lambda)| = |\det{}_5\,(I + X)| \le \mathrm{e}^{C_p \|X\|_{\mathcal{S}_p}^p}$$

for any $X \in \mathcal{S}_p$, where $4 < p \le 5$ with some constant $C_p > 0$, see [53, Proposition 2.1], $D_5(\lambda + \mathrm{i}\epsilon)$ is bounded by a family of functions $\mathrm{e}^{C_p \|Y_0(\lambda+\mathrm{i}\epsilon)\|_{\mathcal{S}_p}^p}$.

Lemma 1.2.1 ([53, Proposition 3.11])**.** *Let a be an analytic function on the upper half plane $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$ such that*

$$a(\lambda) = 1 + o(|\lambda|^{-1})$$

as $|\lambda| \to \infty$ in $\mathbb{C}_+$. Suppose that there is a family of functions $f_\epsilon \in L^1(\mathbb{R})$,

$\epsilon \in]0, \epsilon_0[$ *for some* $\epsilon_0 > 0$ *satisfying*

$$|a(\lambda + \mathrm{i}\epsilon)| \leq \mathrm{e}^{\,f_\epsilon(\lambda)}$$

for all $\lambda \in \mathbb{R}$*, then the zeros* (λ_j) *of* a*, appearing in this sequence according to their multiplicity, satisfy*

$$\sum_j |\operatorname{Im}\lambda_j| \leq \frac{1}{2\pi} \sup_{0<\epsilon<\epsilon_0} \int_{\mathbb{R}} f_\epsilon(\lambda)\,\mathrm{d}\lambda.$$

Choose $f_\epsilon(\lambda) = C_p\|Y_0(\lambda + \mathrm{i}\epsilon)\|^p_{\mathcal{S}_p}$, then by estimating the $L^1(\mathbb{R})$-norm of f_ϵ as in [53, Theorem 3.10] one obtains:

Theorem 1.2.3 ([53, Theorem 1.1])**.** *Let* V *be a complex valued, bounded function such that* (1.4) *holds with* $r = p/(p-2)$ *for some* $4 < p \leq 5$*, then the eigenvalues* (λ_j) *of the operator* $-\Delta + x_1 + V$ *on* $L^2(\mathbb{R}^3)$ *satisfy*

$$\sum_j |\operatorname{Im}\lambda_j| \leq C_p \left(\left(\int_{\mathbb{R}^3} |V(x)|^{p/2}\,\mathrm{d}x \right)^2 + \left(\int_{\mathbb{R}^3} |V(x)|^{p/(p-2)}\,\mathrm{d}x \right)^{p-2} \right)$$

for some constant $C_p > 0$*. Thereby the eigenvalues appear in the sum as often according to their multipicity.*

With a few modifications it is possible to prove bounds for slower decaying potentials. If $q > 1$, then

$$\sum_j |\operatorname{Im}\lambda_j|^q \leq C_{p,q} \left(\int_{\mathbb{R}^3} |V(x)|^{p/2}\,\mathrm{d}x \right)^{2q/(p-3)} \tag{1.7}$$

for each p with $4 < p < q + 3$, $q > 1$, and a constant $C_{p,q} > 0$ as long as V is a bounded complex valued potential such that the right hand side in (1.7) is bounded.

When restricting to potentials that are exponential decreasing in the direction of the negative x_1-axis, i.e.

$$\int_{\mathbb{R}^3} |V(x)|^{p/2}\,\tau(x)\,\mathrm{d}x < \infty \tag{1.8}$$

where $p > 5$ and $\tau(x) := (1+\mathrm{e}^{-px_1/2})(1+|x_1|)^2$, the perturbation determinant $D_n(\lambda)$ behaves as $1 + o(|\lambda|^{-2})$ as $|\lambda| \to \infty$ in the corner $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \alpha + \epsilon, \mathrm{Im}\lambda \geq -\epsilon\}$ for $\alpha, \epsilon > 0$, see [53, Theorem 8.13]. A refined estimate on $\|Y_0(\lambda)\|_{\mathcal{S}_p}$ then allows to prove a bound on the number of zeros of $D_n(\lambda)$ within $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \alpha, \mathrm{Im}\lambda \geq 0\}$ and thus gives a bound on the number of eigenvalues of $-\Delta + x_1 + V$ in that half plane.

Theorem 1.2.4 ([53, Theorem 1.4])**.** *Let V be a complex valued, bounded function such that* (1.8) *holds for a fixed $p > 5$ and $\delta > 0$. For $\alpha > 0$ denote by $N(\alpha)$ the number of eigenvalues of $-\Delta + x_1 + V$ on $L^2(\mathbb{R}^3)$ located in the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < \alpha\}$. Then there is a constant $C_{\alpha,p,\delta} > 0$ such that*

$$N(\alpha) \leq C_{\alpha,p,\delta} \left(\int_{\mathbb{R}^3} |V(x)|^{p/2}\, \tau(x)\, \mathrm{d}x \right)^{2(1+\delta)}. \tag{1.9}$$

A combination of theorem 1.2.2 and theorem 1.2.4 thus yields that the total number of eigenvalues of $-\Delta+x_1++V$ on $L^2(\mathbb{R}^3)$ is finite for potentials satisfying both regularity conditions (1.4) and (1.8). The total count of the eigenvalues is then bounded by (1.9) where α is equal to the expression on the right hand side in (1.6).

1.3 Definition of the Stark Laplacian and general spectral properties

So far the Stark-Laplacian $-\Delta + E \cdot x$ for $E, x \in \mathbb{R}^d$ has not been studied on some domains $\Omega \subset \mathbb{R}^d$ equipped with either Dirichlet or Neumann boundary conditions. Since the spectral properties of this operator are independet of the direction of E, we choose $E = \varepsilon_0(1, 0, \ldots, 0)^T$ with some coupling constant $\varepsilon_0 > 0$ and consider

$$H_{\varepsilon_0} = -\Delta + \varepsilon_0 x_1.$$

Throughout this chapter we will use the notation $x = (x_1, x_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$. We define H_{ε_0} via its quadratic form

$$h_{\varepsilon_0}[u] = \int_\Omega |\nabla u|^2 \,\mathrm{d}(x_1, x_\perp) + \varepsilon_0 \int_\Omega x_1 |u|^2 \,\mathrm{d}(x_1, x_\perp)$$

either on $d[h_{\varepsilon_0}] = W_2^1(\Omega)$, where

$$W_2^1(\Omega) = \{u \in L^2(\Omega) \,:\, \partial_j u \in L^2(\Omega),\, j = 1, \ldots, d\}$$

for Neumann boundary conditions, or on $d[h_{\varepsilon_0}] = \mathring{W}_2^1(\Omega)$, that is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{W_2^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \sum_{j=1}^d \|\partial_j u\|^2_{L^2(\Omega)},$$

for Dirichlet boundary conditions. Both forms are densely defined and closed. If, in addition, Ω is bounded from below along the direction of the Stark-potential, that is

$$\lfloor\Omega := \inf\left\{x_1 \in \mathbb{R} \,:\, \exists_{x_\perp \in \mathbb{R}^{d-1}}\, (x_1, x_\perp) \in \Omega\right\} \tag{1.10}$$

is bounded from below, then

$$h_{\varepsilon_0}[u] \geq \varepsilon_0 \lfloor\Omega \int_\Omega |u|^2 \,\mathrm{d}(x_1, x_\perp)$$

and h_{ε_0} is semibounded from below. In that case, both quadratic forms give rise to the self adjoint operators $H^N_{\varepsilon_0}(\Omega)$ and $H^D_{\varepsilon_0}(\Omega)$ where $H^N_{\varepsilon_0}(\Omega)$ corresponds to h_{ε_0} with $d[h_{\varepsilon_0}] = W_2^1(\Omega)$ and $H^D_{\varepsilon_0}(\Omega)$ corresponds to h_{ε_0} with $d[h_{\varepsilon_0}] = \mathring{W}_2^1(\Omega)$.

Since there is a compact embedding $\mathring{W}_2^1(\Omega) \hookrightarrow L^2(\Omega)$ for any domain Ω with finite Lebesgue measure $|\Omega|$ (cf. Rellichs Theorem), the spectrum of $H^D_{\varepsilon_0}(\Omega)$ is purely discrete and accumulates to infinity only. More precisely, it holds that $\sigma_{\mathrm{ess}}(H^D_{\varepsilon_0}(\Omega)) = \emptyset$.

In the case of Neumann boundary conditions one needs additional assumptions on Ω. Let $\Omega \subset \mathbb{R}^d$ be open, then Ω has the W_2^1-*extension property* if there is a linear bounded operator $E : W_2^1(\Omega) \to W_2^1(\mathbb{R}^d)$ such that $(Eu)(x) = u(x)$ for all $u \in W_2^1(\Omega)$ and almost any $x \in \Omega$. This condition corresponds to geometric conditions on the boundary $\partial\Omega$ of Ω. For instance, it is known that all Ω with Lipschitz boundary satisfy the W_2^1-extension property. If Ω has the W_2^1-extension property, then there is a compact embedding $W_2^1(\Omega) \hookrightarrow L^2(\Omega)$ and again $\sigma_{\mathrm{ess}}(H_{\varepsilon_0}^N(\Omega)) = \emptyset$.

In what follows we arrange the eigenvalues of $H_{\varepsilon_0}^N(\Omega)$ and $H_{\varepsilon_0}^D(\Omega)$ to be monotonic increasing and denote the increasing sequence of eigenvalues of $H_{\varepsilon_0}^N(\Omega)$ by $(\mu_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$, respectively by $(\lambda_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$ for $H_{\varepsilon_0}^D(\Omega)$. Here any eigenvalue appears in the sequences according to its multiplicity.

If $\varepsilon_0 = 0$, the case of the classical Laplace operator, the eigenvalues do not depend on the position or orientation of Ω in $\mathbb{R}^d$. This symmetry is broken by the Stark potential and the eigenvalues $\lambda_j(\Omega,\varepsilon_0)$ or $\mu_j(\Omega,\varepsilon_0)$ are shifted when Ω is moved along the x_1-direction.

Lemma 1.3.1. *Let Ω be open such that $H_{\varepsilon_0}^D(\Omega)$, respectively $H_{\varepsilon_0}^N(\Omega)$ has pure discrete spectrum. For $h \in \mathbb{R}^d$ denote by $\Omega + h := \{x + h : x \in \Omega\}$ the image of Ω under the translation by h, then*

$$\lambda_j(\Omega + h, \varepsilon_0) = \lambda_j(\Omega, \varepsilon_0) + h_1\varepsilon_0,$$

respectively

$$\mu_j(\Omega + h, \varepsilon_0) = \mu_j(\Omega, \varepsilon_0) + h_1\varepsilon_0$$

where $h_1 = h \cdot (1, 0, \ldots, 0)^T$ is the component of h along the x_1-direction.

Proof. Let $u \in \mathring{W}_2^1(\Omega)$ be a solution of

$$(-\Delta + \varepsilon_0 x_1)u = \lambda_j(\Omega, \varepsilon_0)u,$$

then $\tilde{u}(x) := u(x - h)$ satisfies the Dirichlet boundary condition on $\partial(\Omega + h)$ and

$$\begin{aligned}(-\Delta + \varepsilon_0 x_1)\tilde{u} &= (-\Delta + \varepsilon_0(x_1 - h_1))u + h_1\varepsilon_0 u \\ &= (\lambda_j(\Omega, \varepsilon_0) + h_1\varepsilon_0)u \\ &= (\lambda_j(\Omega, \varepsilon_0) + h_1\varepsilon_0)\tilde{u}.\end{aligned}$$

Thus, $\tilde{u} \in \mathring{W}_2^1(\Omega + h)$ is a solution of the eigenvalue equation on $\Omega + h$ with eigenvalue

$$\lambda_j(\Omega + h, \varepsilon_0) = \lambda_j(\Omega, \varepsilon_0) + h_1 \varepsilon_0.$$

In the case of Neumann boundary conditions we follow the same arguments for $u \in W_2^1(\Omega)$ and $\mu_j(\Omega, \varepsilon_0)$. □

As in the case of the classical Laplace operator the eigenvalues scale when scaling the domain. While doing so, we additionally have to adjust the coupling constant by the right factor.

Lemma 1.3.2. *Let Ω be open such that $H_{\varepsilon_0}^D(\Omega)$, respectively $H_{\varepsilon_0}^N(\Omega)$ has pure discrete spectrum. For $\alpha > 0$ let $\alpha\Omega := \{\alpha \cdot x : x \in \Omega\}$ be the scaled domain, then*

$$\lambda_j(\alpha\Omega, \alpha^{-3}\varepsilon_0) = \frac{1}{\alpha^2}\lambda_j(\Omega, \varepsilon_0),$$

respectively

$$\mu_j(\alpha\Omega, \alpha^{-3}\varepsilon_0) = \frac{1}{\alpha^2}\mu_j(\Omega, \varepsilon_0).$$

Proof. Let $u \in \mathring{W}_2^1(\Omega)$ be a solution of

$$(-\Delta + \alpha^{-3}\varepsilon_0 x_1)u = \lambda_j(\Omega, \varepsilon_0)u,$$

then $\tilde{u}(x) := u(x/\alpha)$ satisfies the Dirichlet boundary condition on $\partial(\alpha\Omega)$ and

$$(-\Delta + \alpha^{-3}\varepsilon_0 x_1)\tilde{u} = \frac{1}{\alpha^2}(-\Delta + \varepsilon_0 x_1/\alpha)u = \frac{1}{\alpha^2}\lambda_j(\Omega, \varepsilon_0)u = \frac{1}{\alpha^2}\lambda_j(\Omega, \varepsilon_0)\tilde{u}.$$

Thus, $\tilde{u} \in \mathring{W}_2^1(\alpha\Omega)$ is a solution of the eigenvalue equation on $\alpha\Omega$ with eigenvalue

$$\lambda_j(\alpha\Omega, \alpha^{-3}\varepsilon_0) = \frac{1}{\alpha^2}\lambda_j(\Omega, \varepsilon_0).$$

In the case of Neumann boundary conditions the arguments do not change. □

In the later chapters we will make use of a slightly different formulation of these statements. A consequence of Lemma 1.3.2 is that

$$|\Omega|^{2/d}\,\lambda_j(\Omega, |\Omega|^{-3/d}\varepsilon_0)$$

does not depend on the volume $|\Omega|$ of $\Omega \subset \mathbb{R}^d$. For a translational invariant term we introduce the x_1-component of the center of mass

$$m_{x_1}(\Omega) := \int_\Omega x_1 \, \mathrm{d}x.$$

From Lemma 1.3.1 we then obtain that

$$|\Omega|^{2/d} \lambda_j(\Omega, |\Omega|^{-3/d}\varepsilon_0) + |\Omega|^{-1/d}\varepsilon_0 \, m_{x_1}(\Omega) \tag{1.11}$$

additionally does not depend on the position of Ω in $\mathbb{R}^d$.

1.4 Airy functions and Airy transform

The Airy transform was first introduced by J. E. Avron and I. W. Herbst in [10]. Using this transformation the operator $-\Delta + \varepsilon_0 x_1$ can be transformed into a multiplication operator in the same sense as $-\Delta$ can be transformed into the multiplication operator $u \mapsto |x|^2 u$ via the Fourier transform. Before introducing this transformation we want to summarize some bounds on the Airy functions which will be useful later.

1.4.1 Bounds on the Airy functions

The Airy function may be defined as the decreasing solution of

$$-u'' + \varepsilon_0 x \, u = 0.$$

More precisely, the differential equation has two linearly independent solutions. The solution satisfying $u \to 0$ as $x \to +\infty$ and

$$u(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$$

is called *Airy function of the first kind* and denoted by Ai. The remaining solution, denoted by Bi, is the *Airy function of the second kind.* Both of these functions have multiple representations as series or improper integrals. We refer to [2] for a complete survey or [83] for more details. For the sake of

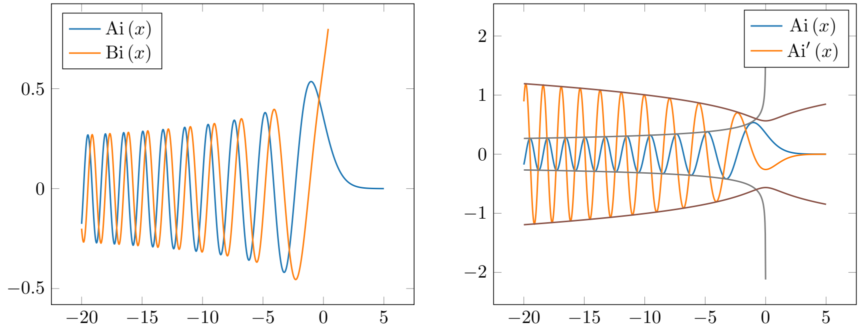


Figure 1.1: Plots of the Airy functions Ai and Bi (left figure), respectively Ai and Ai′ with their envelopes from (1.12) and (1.13) (right figure).

completeness, we want to summarize the most important properties for our survey: On the positive half of the real line Ai is exponentially decreasing, on the negative half Ai remains oscillating and behaves as

$$\operatorname{Ai}(-x) \sim \frac{\cos(\zeta - \pi/4)}{\sqrt{\pi}\, x^{1/4}} \tag{1.12}$$

for $\zeta = 2x^{3/2}/3$ as $x \to +\infty$ (see [2, 10.4.60] or [83, (14.5.54)]). For Ai' one obtains that

$$\operatorname{Ai}'(-x) \sim \frac{x^{1/4} \sin(\zeta - \pi/4)}{\sqrt{\pi}} \tag{1.13}$$

(see [2, 10.4.62] or [83, (14.5.55)]). These asymptotic formulas can be used to prove (sharp) bounds on Ai or Ai'. Some of these bounds, as e.g.

$$|\operatorname{Ai}(-x)| \leq \frac{1}{\sqrt{\pi}} x^{-1/4} \tag{1.14}$$

if $x \geq 0$, can be found in [58] which might not be the first reference for (1.14), but contents a simple proof of (1.14) using *Sonin functions*: If f is a solution of some ordinary differential equation

$$f'' + a(x)\, f' + b(x)\, f = 0$$

with positive b, then the so-called Sonin function $S(x)$ is given by

$$S(x) = f^2(x) + \frac{(f'(x))^2}{b(x)}.$$

It is easy to see that S is an envelope of f^2 coinciding with it in all local maxima. The derivative of S is given by

$$S'(x) = -(2a(x)b(x) + b'(x))\frac{(f'(x))^2}{b^2(x)},$$

thus, the sign of S' depends only on a and b. Moreover, if $S' > 0$, then

$$f^2(x) \leq S(x) \leq \lim_{x\to\infty} S(x).$$

That way (1.14) follows from (1.12) when $f(x) = x^{1/4}\,\mathrm{Ai}\,(-x)$ such that $S(x) = f^2(x) + 16x^2/(16x^3+5)$ is increasing. For the sake of completeness we also want to sketch the proof of

$$|\,\mathrm{Ai}'\,(-x)| \leq \frac{1}{\sqrt{\pi}}(1+x^2)^{1/8} \tag{1.15}$$

which somehow is missing in the literature, at least with the explicite constant $1/\sqrt{\pi}$. Note that the inequality $|\,\mathrm{Ai}'\,(-x)| \leq x^{1/4}/\sqrt{\pi}$ is violated for $x = 0$ as well as in a small neighbourhood of $x = 0$. Consider $f(x) = (1+x^2)^{-1/8}\,\mathrm{Ai}'\,(-x)$ which is a solution of

$$f'' - \frac{2+x^2}{2x+2x^3}\,f' + \frac{x(16-7x+32x^2+16x^4)}{16(1+x^2)^2}\,f = 0$$

where $16-7x+32x^2+64x^4 > 0$ for $x \geq 0$. The derivative of the corresponding Sonin function then satisfies

$$S'(x) = \frac{16+32x^2-21x^3+16x^4}{16(1+x^2)^3}\,\frac{(f')^2}{b^2}$$

and, again, $16+32x^2-21x^3+16x^4 > 0$ for $x \geq 0$. Finally from (1.13) follows that

$$f^2(x) \leq S(x) \leq \frac{1}{\pi}.$$

Although (1.14) and (1.15) capture the right asymptotical behaviour of Ai and Ai$'$, their oscillatory nature is not depicted. Using again (1.12), one can improve (1.14) with a more careful analysis of the Airy function and its corresponding differential equation as done by I. Krasikov:

Lemma 1.4.1 ([58, Lemma 13])**.** *Suppose f satisfies the differential equation*

$$f'' + b^2(x)\, f = 0$$

where b is twice differentiable and $b > 0$ on some interval I. If $g(x) = f(x)\sqrt{b(x)}$, then for each $x \in \mathbb{R}$ there is $\theta_x \in [-1, 1]$ such that

$$g(x) = c_1 \sin\left(B(x)\right) + c_2 \cos\left(B(x)\right) + \theta_x \int_a^x \left| \frac{3(b'(t))^2 - 2b(t)b''(t)}{4(b(t))^3}\, g(t) \right| \mathrm{d}t$$

for all $a \in I$, provided the integral exists. Here $B(x) = \int_a^x b(t)\,\mathrm{d}t$ is a primitive function of b and $c_1, c_2 \in \mathbb{R}$ are some constants.

From that one obtains

$$\mathrm{Ai}\,(-x) = \frac{\cos\left(\zeta - \pi/4\right)}{\sqrt{\pi}x^{1/4}} + \theta_x \frac{5}{24\pi^{1/2}x^{7/4}}, \qquad \zeta = 2x^{3/2}/3, \tag{1.16}$$

for all $x \geq 0$ with some constant $|\theta_x| \leq 1$ which depends on x. The key to Lemma 1.4.1 is the substitution $g = f\sqrt{b}$ which already captures the asymptotic of Ai in (1.14). The resulting equation for g is then solved by the trigonometric functions sin and cos where one chooses the solution according to (1.12). In view of (1.15), a corresponding bound for the derivative Ai$'$ should follow from a substitution $g = f/\sqrt{b}$, thus, we modify Lemma 1.4.1 as follows:

Lemma 1.4.2. *Suppose f satisfies the differential equation*

$$f'' - 2\frac{b'(x)}{b(x)}\, f' + b^2(x)\, f = 0 \tag{1.17}$$

where b is twice differentiable and $b > 0$ on some interval I. If $g(x) =$

$f(x)/\sqrt{b(x)}$, then for each $x \in \mathbb{R}$ there is $\theta_x \in [-1,1]$ such that

$$g(x) = c_1 \sin(B(x)) + c_2 \cos(B(x)) + \theta_x \int_a^x \left| \frac{5(b'(t))^2 - 2b(t)b''(t)}{4(b(t))^3} g(t) \right|^2 \mathrm{d}t$$

for all $a \in I$, provided the integral exists. Here $B(x) = \int_a^x b(t)\,\mathrm{d}t$ is a primitive function of b and $c_1, c_2 \in \mathbb{R}$ are some constants.

Proof. If f is a solution of (1.17), then $g(x) = f(x)/\sqrt{b(x)}$ satisfies

$$g'' - \frac{b'(x)}{b(x)} g' + b^2(x)\, g(1 - \epsilon(x)) = 0$$

where

$$\epsilon(x) = \frac{5(b'(x))^2 - 2b(x)b''(x)}{4b^4(x)}.$$

The solution of the homogeneous problem $g_0'' - (b'/b)g_0' + b^2 g_0 = 0$ is then given by

$$g_0(x) = c_1 \sin(B(x)) + c_2 \cos(B(x))$$

where $B(x) = \int_a^x b(t)\,\mathrm{d}t$. Formally, we treat $\epsilon b^2 g$ as an inhomogenity and search for a particular solution with variation of constants which yields

$$g(x) = c_1 \sin(B(x)) + c_2 \cos(B(x)) + \int_a^x \frac{1}{b(t)} \sin(B(t))\, \epsilon(t) b^2(t) g(t)\, \mathrm{d}t$$

with

$$\begin{aligned} \int_a^x \frac{1}{b(t)} \sin(B(t))\, \epsilon(t) b^2(t) g(t)\, \mathrm{d}t &= \int_a^x \epsilon(t) b(t) g(t)\, \mathrm{d}t \\ &= \theta_x \int_a^x \left| \frac{5(b')^2 - 2bb''}{4b^3} g(t) \right| \mathrm{d}t. \end{aligned}$$

□

Let $f(x) = \mathrm{Ai}'(-x)$, then f satisfies (1.4.2) with $b(x) = \sqrt{x}$, thus

$$g(x) = \frac{\mathrm{Ai}'(-x)}{x^{1/4}} = c_1 \sin(B(x)) + c_2 \cos(B(x)) + \theta_x \int_x^\infty \left| \frac{7}{16t^{5/2}} g(t) \right| \mathrm{d}t.$$

From (1.15) we obtain

$$|g(t)| \leq \frac{1}{t^{1/4}} \frac{(1+t^2)^{1/8}}{\sqrt{\pi}} \leq \frac{1}{\sqrt{\pi}} \left(\frac{1}{t^{1/4}} + 1 \right),$$

thus

$$\mathrm{Ai}'(-x) = \frac{x^{1/4} \sin(\zeta - \pi/4)}{\sqrt{\pi}} + \theta_x \left(\frac{7}{24\pi^{1/2} x^{5/4}} + \frac{7}{28\pi^{1/2} x^{3/2}} \right) \tag{1.18}$$

holds for all $x \geq 0$.

1.4.2 Airy transform and properties

Let $\mathcal{S}(\mathbb{R})$ be the space of all Schwartz functions on $\mathbb{R}$, then we can introduce the *Airy transform*

$$\mathcal{A}[u](z) = \varepsilon_0^{1/3} \int_{\mathbb{R}} \mathrm{Ai}\left(\varepsilon_0^{1/3}(x-z)\right) u(x) \, \mathrm{d}x$$

for any $u \in \mathcal{S}(\mathbb{R})$. Since

$$\int_{\mathbb{R}} \mathrm{Ai}(x-z) \, \mathrm{Ai}(z-\tilde{x}) \, \mathrm{dz} = \delta(x-\tilde{x}),$$

we can extend $\mathcal{A}$ to an unitary mapping on $L^2(\mathbb{R})$, see [10, Theorem 1.1]. Furthermore, integration by parts and using the differential equation for Ai yields that

$$\mathcal{A}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) + \varepsilon_0 x \, u(x) \right](z) = \varepsilon_0 z \, \mathcal{A}[u](z)$$

for all $u \in \mathbb{S}(\mathbb{R})$. In that sense the one-dimensional operator can be rewritten as

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \varepsilon_0 x = \mathcal{A}^{-1} \varepsilon_0 z \mathcal{A}.$$

If $u \in \mathcal{S}(\mathbb{R}^d)$, $d > 1$, we apply the Airy transform along the direction of the Stark potential. For the remaining components we use the Fourier transform and set

$$\mathcal{A}[u](z_1, z_\perp) = \frac{\varepsilon_0^{1/3}}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \mathrm{e}^{-\mathrm{i}\, x_\perp z_\perp} \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) u(x_1, x_\perp) \,\mathrm{d}(x_1, x_\perp). \tag{1.19}$$

As above $\mathcal{A}$ can be extended to an unitary mapping on $L^2(\mathbb{R}^d)$ where

$$-\Delta + \varepsilon_0 x_1 = \mathcal{A}^{-1}(\varepsilon_0 z_1 + |z_\perp|^2)\mathcal{A}.$$

1.5 Structure of this work and Main Theorems

As for the classical Laplacian operator the structure of the Stark operators spectrum on a domain with either Dirichlet or Neumann boundary conditions is purely discrete. This gives rise to several problems related to the eigenvalue sequences. But unlike in the case of the classical Laplacian, these problems were not addressed so far and are studied in what follows:

The structure of this work is two-fold. In the first part we want to deal with the so-called Riesz means

$$\operatorname{Tr}_\gamma \left(H^i_{\varepsilon_0}(\Omega) - \Lambda\right) := \sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega; \varepsilon_0))_+^\gamma,$$

$i \in \{D, N\}$. Using the Airy transform from above, we follow Berezin's approach of decomposing the free wave to the orthonormal basis of eigenfunctions in Chapter 2. This gives a bound in the case $\gamma = 1$ in terms of integrals over the Airy function. Estimating the envelope of the Airy function from above, our result can be compared to the Lieb-Thirring inequality obtained by interpreting the electric field term $\varepsilon_0 x_1$ as a potential (cf. Corollary 2.1.1 and Corollary 2.1.2). Additionally, our bound can be improved by subtracting terms of lower order in Λ which is shown in Theorem 2.2.2 or Theorem 2.2.3.

In Chapter 3 we follow P. Kröger's test function arguments, respectively the averaging principle from E. M. Harrell and J. Stubbe in order to prove corresponding bounds on the Neumann eigenvalues, see Corollary 3.0.1.

Chapter 4 is dedicated to the case $\gamma = 0$ which is the counting function of all eigenvalues below Λ. Our starting point will be an inequality which we reproduce from A. Pushnitski and V. Sloushch in Theorem 4.1.1. From that, we apply techniques known from the study of the classical Laplacian operator on domains, such as decomposition on product domains (Theorem 4.2.1), the Dirichlet-Neumann bracketing (Theorem 4.3.2) or Glazman's Lemma in order to prove inequalities between the Dirichlet and Neumann eigenvales (Theorem 4.4.1).

In the second part of this work we want to approach the Faber-Krahn inequality for the Stark Laplacian. Unlike the classical Laplacian operator, the Stark Laplacian lacks of symmetry, thus, symmetrisation techniques can only be applied perpendicular to the direction of the electric field. Additionally, the spectrum depends on the position of the domain. This already restricts the class of domains for which minimizers for the first eigenvalue exists. However, in Theorem 5.0.1 we prove that minimizers exist among convex domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ with fixed area and center of mass. In order to gain some idea of how this minimizing domains might look like, we close our work with numerical experiments based on a gradient descent. The necessary Hadamard-type formula for the change of eigenvalues is shown in Theorem 6.2.4, whereas our candidates for optimal domains are plotted in Figure 6.5.

Part I

Spectral estimates for the Stark Laplacian

Chapter 2

Berezin-Li-Yau-Type inequalities for Dirichlet Eigenvalues

Let $\Omega \subset \mathbb{R}^d$ be an open domain. We consider the monotonic sequence $(\lambda_j(\Omega, \varepsilon_0))_{j\in\mathbb{N}}$ of eigenvalues of the Stark Laplacian $H^D_{\varepsilon_0}(\Omega) = -\Delta + \varepsilon_0 x_1$ with Dirichlet boundary conditions as defined in Section 1.3 and want to study the so-called Riesz means given by

$$\operatorname{Tr}_\gamma \left(H^D_{\varepsilon_0}(\Omega) - \Lambda \right) := \sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega; \varepsilon_0))^\gamma_+.$$

Here and throughout the rest of this work, $x_\pm = (|x| \pm x)/2$ denotes the positive or negative part of real numbers or functions. Even in the simplest case $\varepsilon_0 = 0$, where our operator coincides with the Dirichlet Laplacian, the eigenvalues on the right hand side can only be computed explicitely for a very special shapes of Ω such as balls, rectangles or certain triangles. For more arbitrary shapes one focuses on estimating the Riesz means in terms of the phase space volume

$$\frac{1}{(2\pi)^d}\int\int_{\Omega\times\mathbb{R}^d}(|\xi|^2-\Lambda)_-^\gamma\,\mathrm{d}\xi\mathrm{d}x = L_{\gamma,d}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+d/2}$$

where

$$L_{\gamma,d}^{\mathrm{cl}} = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\,\Gamma(\gamma+1+d/2)} \tag{2.1}$$

is known as the classical Lieb-Thirring constant. Indeed, the phase space volume is the asymptotical limit as Λ approches to $+\infty$, i.e.

$$\sum_{j\in\mathbb{N}}(\Lambda-\lambda_j(\Omega;0))_+^\gamma = L_{\gamma,d}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+d/2}+o(\Lambda^{\gamma+d/2}) \tag{2.2}$$

as $\Lambda\to+\infty$. This is the well-known Weyl asymptotics and was already shown in 1912 by H. Weyl in [92]. Strictly speaking H. Weyl proved (2.2) in the case $\gamma=0$ where the Riesz means are just the counting function $N_0(\Omega;\Lambda)=\#\{\lambda_j(\Omega;0)\leq\Lambda\}$ of all eigenvalues below Λ. From there (2.2) follows if one notes

$$\sum_{j\in\mathbb{N}}(\Lambda-\lambda_j(\Omega;0))_+^\gamma = \gamma\int_0^\Lambda(\Lambda-t)^{\gamma-1}t^{d/2}\,\mathrm{d}t,$$

whereas integrating the right hand side in (2.2) gives

$$L_{0,d}^{\mathrm{cl}}|\Omega|\gamma\int_0^\Lambda(\Lambda-t)^{\gamma-1}t^{d/2}\,\mathrm{d}t = L_{0,d}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+d/2}\gamma\frac{\Gamma(\gamma)\Gamma(d/2+1)}{\Gamma(\gamma+d/2+1)} = L_{\gamma,d}^{\mathrm{cl}}|\Omega|.$$

In the literature this argument is known as the Lieb-Aizenman trick [4; 33, Section 5.1.1]. Besides the asymptotical result (2.2) one is interested in estimates of the form

$$\sum_{j\in\mathbb{N}}(\Lambda-\lambda_j(\Omega;0))_+^\gamma \leq L_{\gamma,d}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+d/2}. \tag{2.3}$$

This was first shown in 1972 by F. A. Berezin [11; 33, Section 3.5.1] in the case $\gamma\geq1$. For $\gamma\geq0$ (2.3) was shown by G. Pólya in 1961 [33, Theorem 3.23; 77] if Ω is a tiling domain. This result can be extended to cylinders over tiling domains, see [62, Theorem 2.8], but the famous Pólya conjecture, suggesting that (2.3) holds for any bounded set $\Omega \subset \mathbb{R}^d$ and all $\gamma \geq 0$, remains open to this day. Using the Legendre transform in Λ in (2.3) for $\gamma = 1$, one obtains that

$$\sum_{j=1}^{N} \lambda_j(\Omega; 0) \geq \frac{d}{d+2} (L_{0,d}^{\mathrm{cl}} |\Omega|)^{-2/d} N^{1+2/d}. \tag{2.4}$$

The latter was shown by P. Li and S. T. Yau in 1983 by other means (see [33, Section 3.5.2; 68]) and is therefore known as the Li-Yau inequality. From there, by applying Hölders inequality

$$\sum_{j=1}^{N} \lambda_j(\Omega; 0) \leq \left(\sum_{j=1}^{N} \lambda_j(\Omega; 0)^\gamma \right)^{1/\gamma} \cdot N^{1/\tilde{\gamma}}$$

where $\gamma^{-1} + \tilde{\gamma}^{-1} = 1$ and passing to the limit $\gamma \to \infty$ respectively $\tilde{\gamma} \to 1$, it follows that

$$\sum_{j \in \mathbb{N}} (\Lambda - \lambda_j(\Omega; 0))_+^\gamma \leq \left(1 + \frac{2}{d}\right)^{d/2} L_{\gamma,d}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma + d/2} \tag{2.5}$$

for $\gamma = 0$ and, again by the Lieb-Aizenman trick, for all $\gamma \geq 0$ (see [90] for details).

Bounds of the form (2.3) and (2.5) can be seen as special cases of a much larger class of inequalities known in the literature as *Lieb-Thirring inequalities*: We consider an Schrödinger-type operator of the form $H(V; \alpha) = -\Delta - \alpha V(x)$ on $L^2(\mathbb{R}^d$ for $\alpha \geq 0$ and suitable potential V. Usually such operators are definded via their quadratic forms

$$h[u, u] = \int_{\mathbb{R}^d} |\nabla u(x)|^2 + \alpha V(x) |u(x)|^2 \, \mathrm{d}x$$

for $u \in W_2^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If $\alpha = 0$, then $H(V; \alpha)$ has pure essential spectrum $\sigma_{\mathrm{ess}}(H(V; \alpha)) = [0, \infty[$ which is stable as α increases if V is relatively compact in a form sense, i.e.

$$\left| \int_{\mathbb{R}^d} V(x) \, |u(x)|^2 \, \mathrm{d}x \right| \leq \varepsilon \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, \mathrm{d}x + C(\varepsilon) \, \|u\|^2$$

holds true for all $\varepsilon > 0$ and $u \in W_2^1(\mathbb{R}^d)$ with some positive constant $C(\varepsilon)$. If $h[u,u] < 0$ for some coupling constant $\alpha > 0$ and a test function $u \in W_2^1(\mathbb{R}^d)$, the operator $H(V;\alpha)$ will have negative spectrum consisting of a sequence of eigenvalues $(\lambda_j(V;\alpha))_{j\in\mathbb{N}}$ with zero as the only possible accumulation point. A more detailed introduction of this subject can be found in [13, Chapter 9 and Chapter 10] or [81, Chapter XIII]. As above, one is interested in bounds on the moments of these eigenvalues

$$\mathrm{Tr}_\gamma\left(H_-^\gamma(V;\alpha)\right) := \sum_{j\in\mathbb{N}} (-\lambda_j(V;\alpha))^\gamma$$

in terms of the phase-space average

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (|\xi|^2 - \alpha V(x))_-^\gamma \,\mathrm{d}x\,\mathrm{d}\xi = L_{\gamma,d}^{\mathrm{cl}}\,\alpha^{\gamma+d/2}\int_{\mathbb{R}^d} (V(x))_+^{\gamma+d/2}\,\mathrm{d}x$$

where $L_{\gamma,d}^{\mathrm{cl}}$ is the classical constant from (2.1). The Weyl asymptotic for this class of operators reads as

$$\sum_{j\in\mathbb{N}} (-\lambda_j(V;\alpha))^\gamma = L_{\gamma,d}^{\mathrm{cl}}\,\alpha^{\gamma+d/2}\int_{\mathbb{R}^d} (V(x))_+^{\gamma+d/2}\,\mathrm{d}x\,(1+o(1)), \qquad \alpha\to\infty, \tag{2.6}$$

which can be shown for $V \in C_c^\infty(\mathbb{R}^d)$ and extended other to classes of potentials by approximation arguments [70]. Thus, one is interested in proving bounds of the form

$$\sum_{j\in\mathbb{N}} (-\lambda_j(V;\alpha))^\gamma \le R(\gamma,d)\,L_{\gamma,d}^{\mathrm{cl}}\,\alpha^{\gamma+d/2}\int_{\mathbb{R}^d} (V(x))_+^{\gamma+d/2}\,\mathrm{d}x \tag{2.7}$$

with some positive constants $R(\gamma,d)$ which we will comment below in more detail. These inequalities have awoken the interest of many authors. An almost complete survey over this topic can be found in [33] or [64] such that we only focus on a brief summary. In 1976 E. H. Lieb and W. Thirring established not only the type of notion, but also showed (2.7) for $\gamma > 1/2$ if $d = 1$ and $\gamma > 0$ if $d \ge 2$. On the other hand, results from M. S. Birman [12] and B. Simon [84] disproved the validity of (2.7) in the cases $0 \le \gamma < 1/2$, if $d = 1$, and $\gamma = 0$, if $d = 2$. The remaining case in dimension $d = 1$ is due to T. Weidl, in [91] it is shown that (2.7) holds for $d = 1$ and $\gamma = 1/2$. The case $\gamma = 0$ for $d \geq 3$ was independently proven by M. Cwikel [21], E. H. Lieb [69] and G. V. Rozenblum [82] and therefore is also known as Cwickel-Lieb-Rozenblum-bound.

Besides the validity of (2.7) one is interested in the sharp value of $R(\gamma, d)$, i.e. the smallest possible value if $R(\gamma, d)$ such that (2.7) still holds. Due to a result of M. Aizenman and E. H. Lieb [4] the sharp value of $R(\gamma, d)$ is monotonic decreasing in γ. More precisely, if (2.7) holds for a pair (γ, d), then (2.7) also holds for any other pair $(\tilde{\gamma}, d)$ if $\tilde{\gamma} \geq \gamma$ and

$$R(\tilde{\gamma}, d) \leq R(\gamma, d).$$

Apart from that, the problem appears to be challenging and many of the sharp values of $R(\gamma, d)$ remain still open. In the interest of simplification we denote the sharp value of $R(\gamma, d)$ also by $R(\gamma, d)$ and summarize some of the known facts: In $d = 1$ then $R(1/2, 1) = 2$ [45] and $R(3/2, 1) = 1$ [70], respectively $R(\gamma, 1)$ for $\gamma \geq 3/2$ or

$$1 \leq R(\gamma, 1) \leq 2 \tag{2.8}$$

for $1/2\gamma < 3/2$. Besides (2.8) nothing is known about $R(\gamma, 1)$ in that case. For arbitrary dimension $d \in \mathbb{N}$ A. Laptev and T. Weidl [65] proved $R(\gamma, 1) = 1$ if $\gamma \geq 3/2$. Alongside we know $R(1, d) \leq \pi/\sqrt{3}$ (see [29] for the case $d = 1$ and [28] if $d \geq 1$).

2.1 Approach to the leading order and comparison with Lieb-Thirring inequalities

As a first step we want to follow the proof of Berezin's inequality for the eigenvalues of the classical Laplacian operator. In this proof, the general idea is to decompose the free wave solution $\mathrm{e}^{-\mathrm{i}xz}$ to the orthonormal basis $(\phi_j)_{j \in \mathbb{N}}$ consisting of eigenfunctions ϕ_j of the Laplacian operator. Thereby, the coefficients $\langle \mathrm{e}^{-\mathrm{i}z\bullet}, \phi_j \rangle_{L^2(\Omega)}$ coincide with the Fourier transforms of the ϕ_j. For the Stark operator the free wave is given by the Airy functions $\mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)$

in the direction of the electric field and by $\mathrm{e}^{-\mathrm{i}x_\perp z_\perp}$ in the perpendicular directions. The projection of this solutions onto an eigenfunction ϕ_j gives the Airy transform of ϕ_j. Thus, by replacing the Fourier transform with the Airy transform introduced in Section 1.4.2, we obtain a bound for the eigenvalues of $H_{\varepsilon_0} = -\Delta + \varepsilon_0 x_1$ on $L^2(\Omega)$, $\Omega \in \mathbb{R}^d$ with dirichlet boundary conditions on $\partial\Omega$. In the final part of this section we want to compare our result with the Lieb-Thirring inequality. Throughout this chapter we use our usual decomposition $x = (x_1, x_\perp) \in \mathbb{R} \times \mathbb{R}^\perp$ in a component parallel to the direction of the Stark potential, i.e. the x_1-direction and the remaining perpendicular components $x_\perp$.

Let $(\phi_j)_{j=1,\dots,k}$ be a sequence of orthonormal functions in $L^2(\Omega)$, then by Parseval's inequality

$$\begin{aligned}
\sum_{j=1}^{k} |\mathcal{A}[\phi_j](z)|^2 &= \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \sum_{j=1}^{k} \left| \int_\Omega \mathrm{e}^{-\mathrm{i}x_\perp z_\perp} \ \mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \phi_j(x)\,\mathrm{d}x \right|^2 \\
&= \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \sum_{j=1}^{k} \left| \left\langle \mathrm{e}^{-\mathrm{i}x_\perp z_\perp} \ \mathrm{Ai}\left(\varepsilon_0^{1/3}(\bullet_1 - z_1)\right), \phi_j \right\rangle_{L^2(\Omega)} \right|^2 \\
&\le \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \left\| \mathrm{e}^{-\mathrm{i}\bullet_\perp z_\perp} \ \mathrm{Ai}\left(\varepsilon_0(\bullet_1 - z_1)\right) \right\|_{L^2(\Omega)}^2 \\
&= \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_\Omega \left| \mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \right|^2 \mathrm{d}x. \qquad (2.9)
\end{aligned}$$

If $\lambda_j = \lambda_j(\Omega; \varepsilon_0)$ is an eigenvalue of H_{ε_0} with corresponding eigenfunction ϕ_j, it follows that

$$\begin{aligned}
\lambda_j = \lambda_j \|\phi_j\|^2 &= \langle H_{\varepsilon_0}\phi_j, \phi_j \rangle \\
&= \langle \mathcal{A}[H_{\varepsilon_0}\phi_j], \mathcal{A}[\phi_j] \rangle = \langle (\varepsilon_0 z_1 + |z_\perp|^2)\, \mathcal{A}[\phi_j], \mathcal{A}[\phi_j] \rangle,
\end{aligned}$$

respectively using (2.9)

$$\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j)_+ = \sum_{j\in\mathbb{N}} \left(\Lambda \|\mathcal{A}[\phi_j]\|^2 - \langle (\varepsilon_0 z_1 + |z_\perp|^2)\, \mathcal{A}[\phi_j], \mathcal{A}[\phi_j] \rangle \right)_+$$

$$= \sum_{j\in\mathbb{N}} \left(\int_{\mathbb{R}^d} (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2) \, |\mathcal{A}[\phi_j](z)|^2 \, \mathrm{d}z \right)_+ \tag{2.10}$$

$$\leq \int_{\mathbb{R}^d} (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+ \sum_{j\in\mathbb{N}} |\mathcal{A}[\phi_j](z)|^2 \, \mathrm{d}z$$

$$\leq \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+ \int_\Omega | \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)|^2 \, \mathrm{d}x \, \mathrm{d}z. \tag{2.11}$$

The expression on the right hand side can be simplified by computing the integral over the $d-1$ components of $z_\perp$ which yields

$$\int_{\mathbb{R}^{d-1}} (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+ \, \mathrm{d}z_\perp = (\Lambda - \varepsilon_0 z_1)_+^{1+(d-1)/2} \int_{\mathbb{R}^{d-1}} (1 - |z_\perp|^2)_+ \, \mathrm{d}z_\perp$$

$$= (\Lambda - \varepsilon_0 z_1)_+^{(d+1)/2} \, \frac{\pi^{(d-1)/2}}{\Gamma((d+3)/2)}.$$

In summary we have shown

Theorem 2.1.1. *Denote by $(\lambda_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$ the sequence of eigenvalues of $H_{\varepsilon_0} = -\Delta + \varepsilon_0 x_1$ on $L^2(\Omega)$ for $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions on $\partial\Omega$, then*

$$\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;\varepsilon_0))_+$$

$$\leq L(d,\varepsilon_0) \int_{\mathbb{R}} (\Lambda - \varepsilon_0 z_1)_+^{(d+1)/2} \int_\Omega \left| \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\right|^2 \mathrm{d}(x_1, x_\perp) \, \mathrm{d}z_1 \tag{2.12}$$

with $L(d,\varepsilon_0) = \varepsilon_0^{2/3}/((4\pi)^{(d-1)/2}\,\Gamma((d+3)/2))$ holds for all $\Lambda \in \mathbb{R}$.

Remark 2.1.1. If the domain Ω is shifted along the x_1-direction, i.e. Ω is replaced by $\Omega + he_1$, according to Lemma 1.3.1 the eigenvalues change by $\varepsilon_0 h$ and the left hand side of (2.12) behaves as

$$\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega + h;\varepsilon))_+ = \sum_{j\in\mathbb{N}} ((\Lambda - \varepsilon_0 h) - \lambda_j(\Omega;\varepsilon))_+.$$

That way a shift of the domain corresponds to a shift of the bound Λ in

(2.12). At the same time the right hand side of (2.12) behaves as

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda - \varepsilon_0 z_1)_+^{(d+1)/2} \int_{\Omega+h} \left| \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right|^2 \mathrm{d}x \, \mathrm{d}z_1 \\
&= \int_{\mathbb{R}} (\Lambda - \varepsilon_0 z_1)_+^{(d+1)/2} \int_{\Omega} \left| \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - (z_1 - h))\right|^2 \mathrm{d}x \, \mathrm{d}z_1 \\
&= \int_{\mathbb{R}} (\Lambda - \varepsilon_0 (z_1 + h))_+^{(d+1)/2} \int_{\Omega} \left| \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right|^2 \mathrm{d}x \, \mathrm{d}z_1 \\
&= \int_{\mathbb{R}} ((\Lambda - \varepsilon_0 h) - \varepsilon_0 z_1)_+^{(d+1)/2} \int_{\Omega} \left| \operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right|^2 \mathrm{d}x \, \mathrm{d}z_1 .
\end{aligned}$$

Thus, (2.12) is translational invariant in the sense that neither the validity nor the strength of (2.12) depends on the location of Ω.

An alternative approach to estimate $\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;\varepsilon_0))_+$ is to simply use the Lieb-Thirring inequality (2.7) for $\gamma = 1$ and

$$V_\Omega(x) = \begin{cases} \Lambda - \varepsilon_0 x_1 & \text{if } (x_1, x_\perp) \in \Omega, \\ -\infty & \text{else.} \end{cases}$$

Inserting this potential then yields

$$\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;\varepsilon_0))_+ \leq R(1,d)\, L^{\mathrm{cl}}_{1,d} \int_\Omega (\Lambda - \varepsilon_0 x_1)_+^{1+d/2} \, \mathrm{d}x \qquad (2.13)$$

with some constant $1 \leq R(1,d) \leq 2$ whose optimal value is still unknown. Note that introducing new Dirichlet boundary conditions on $\partial\Omega$ increases individual eigenvalues and thus lowers the sum on the right hand side in (2.13). That way (2.13) holds for our Stark Laplacian $H^D_{\varepsilon_0}(\Omega)$.

For the rest of this section we aim at a comparison between (2.13) and (2.12), thus, we have to make the order of $\Lambda - \varepsilon_0 x_1$ in the integral on the right hand side of (2.12) more visible. Note that

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda - \varepsilon_0 z_1)_+^{(d+1)/2} \int_{\Omega} \left| \operatorname{Ai}\left(\varepsilon_0(x_1 - z_1)\right)\right|^2 \mathrm{d}x\, \mathrm{d}z_1 \\
=&\varepsilon_0^{-1/3} \int_{\Omega} \mathrm{d}x\, (\Lambda - \varepsilon_0 x_1)_+^{(d+1)/2} \odot \\
&\qquad \odot \int_{-\varepsilon_0^{-2/3}(\Lambda - \varepsilon_0 x_1)_+}^{\infty} \left(1 + \frac{\varepsilon_0^{2/3}}{\Lambda - \varepsilon_0 x_1}\, z_1\right)^{(d+1)/2} \operatorname{Ai}^2(z_1)\, \mathrm{d}z_1.
\end{aligned}$$

In this context capturing the order of $\Lambda - \varepsilon_0 x_1$ in (2.12) requires an analysis of

$$h_d(a) = a^{(d+1)/2} \int_{-a}^{\infty} \left(1 + \frac{z}{a}\right)^{(d+1)/2} \operatorname{Ai}^2(z)\, \mathrm{d}z.$$

Lemma 2.1.1. *For all $a > 0$ and $d \in \mathbb{N}$ is $h_d'(a) = \frac{d+1}{2}\, h_{d-2}(a)$.*

Proof. The statement is a simple consequence from Leibnitz' rule. Direct computation yields

$$\begin{aligned}
h_d'(a) &= \frac{d+1}{2}\, a^{(d-1)/2} \int_{-a}^{\infty} \left(1 + \frac{z}{a}\right)^{(d+1)/2} \operatorname{Ai}^2(z)\, \mathrm{d}z \\
&\quad - \frac{d+1}{2}\, a^{(d-1)/2} \int_{-a}^{\infty} \frac{z}{a} \left(1 + \frac{z}{a}\right)^{(d-1)/2} \\
&= \frac{d+1}{2}\, a^{(d-1)/2} \int_{-a}^{\infty} \left(1 + \frac{z}{a}\right)^{(d-1)/2} \operatorname{Ai}^2(z)\, \mathrm{d}z.
\end{aligned}$$

□

That way, bounds on $h_d(a)$ follow inductively by integrating the previous bound. Starting with $d = 1$ we obtain

$$h_1'(a) = h_{-1}(a) = \int_{-a}^{\infty} \operatorname{Ai}^2(z)\, \mathrm{d}z = a \operatorname{Ai}^2(-a) + \operatorname{Ai}'^2(-a),$$

and proceeding with (1.14) and (1.15), respectively (1.16) and (1.18), yields $h_1'(a) \leq \sqrt{a}/\pi + r(a)$ where

$$r(a) = \frac{1}{\pi} \cdot \min\left\{ (1 + a^2)^{1/2}, \frac{1}{a} + \frac{\pi^{1/2}}{2a^{1/2}} + \frac{37}{288 a^{5/2}} + \frac{1}{16 a^3} + \frac{7}{48 a^{11/4}} \right\}.$$

Thus, h_1' is of order $a^{1/2}$, whereas the remainder is of class $O(1)$, if $a \to 0$, and $o(a^{1/2})$, if $a \to \infty$. Integrating this bound and collecting all the constants from (2.12) results in

Corollary 2.1.1. *If $d = 1$, then*

$$\sum_{j \in \mathbb{N}} (\Lambda - \lambda_j(\Omega; \varepsilon_0))_+ \leq \frac{2}{3\pi} \int_\Omega (\Lambda - \varepsilon_0 x_1)_+^{3/2} + R(\Lambda - \varepsilon_0 x_1) \, \mathrm{d}x \tag{2.14}$$

where R is a positive function satisfying $R \in O(1)$ as $\Lambda - \varepsilon_0 x_1 \to 0$ and $R \in o((\Lambda - \varepsilon_0 x_1)^{3/2})$ as $\Lambda - \varepsilon_0 x_1 \to \infty$.

Compared with (2.13) the bound (2.14) captures the right order in Λ, but with the sharp constant of $L_{1,1}^{\mathrm{cl}} = 2/(3\pi)$. Taking this approach one step furher gives a bound in $d = 3$ with the correct asymptotics and again the sharp constant of $L_{1,2}^{\mathrm{cl}} = 1/(15\pi^2)$:

Corollary 2.1.2. *If $d = 3$, then*

$$\sum_{j \in \mathbb{N}} (\Lambda - \lambda_j(\Omega; \varepsilon_0))_+ \leq \frac{1}{15\pi^2} \int_\Omega (\Lambda - \varepsilon_0 x_1)_+^{5/2} + R(\Lambda - \varepsilon_0 x_1) \, \mathrm{d}x$$

where R is a positive function satisfying $R \in O(1)$ as $\Lambda - \varepsilon_0 x_1 \to 0$ and $R \in o((\Lambda - \varepsilon_0 x_1)^{5/2})$ as $\Lambda - \varepsilon_0 x_1 \to \infty$.

Remark 2.1.2. Since

$$h_d(0) = \int_0^\infty z^{(d+1)/2} \operatorname{Ai}^2(z) \, \mathrm{d}z > 0,$$

the bound $h_d(a) \leq a^{(d+1)/2}$ cannot be true and will be violated at least in a small neighbourhood of $a = 0$. Thus, any uniform bound we derive from (2.12) will always contain lower order terms. Nevertheless, (2.12) is valuable in dimension $d = 1$ since the sharp value of $R(1,1)$ in (2.13) is unknown.

2.2 Improvement of Berezin type inequalities

Despite the fact that the validity of (2.3) for $0 \leq \gamma < 1$ and arbitrary domains remains open, many works focus on improving the inequality for

$\gamma \geq 1$. Due to the Weyl law (2.2), the right hand side in (2.3) cannot be improved in terms of an uniform constant in the leading order without further assumptions on the geometry of Ω or restricting the range of Λ. But (2.3) may be improved by subtracting positive terms of lower order in Λ on the right hand side. A first step in this direction was done by A. D. Melas [71], improving (2.4) by

$$\sum_{j=1}^{N} \lambda_j(\Omega;0) \geq \frac{d}{d+2}(L_{0,d}^{\mathrm{cl}}|\Omega|)^{-2/d} N^{1+2/d} + M_d \frac{|\Omega|}{I(\Omega)} N \tag{2.15}$$

where M_d is some constant that only depends on the dimension d and

$$I(\Omega) = \min_{y \in \mathbb{R}} \int_\Omega |x-y|^2 \, \mathrm{d}x.$$

One objection of this result might be that the second term does not capture the right order in N. In view of the second term of the Weyl asymptotics,

$$\begin{aligned} &\sum_{j \in \mathbb{N}} (\Lambda - \lambda_j(\Omega;0))_+^\gamma \\ &= L_{\gamma,d}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+d/2} - \frac{1}{4} L_{\gamma,d-1}^{\mathrm{cl}}|\partial\Omega|\Lambda^{\gamma+(d-1)/2} + o(\Lambda^{\gamma+(d-1)/2}), \end{aligned} \tag{2.16}$$

the correct order, when formulated in the Li-Yau type way, should be $N^{1+1/d}$. This two term asymptotic formula was shown 1980 independently by V. Ja. Ivriĭ [47] and R. B. Melrose [72] under additional geometric constraints of Ω and its boundary. Note that some restrictions on $\partial\Omega$ are necessary since otherwise the right hand side of (2.16) might be negative for fixed $\Lambda > 0$ and domains with bounded area $|\Omega|$ but unbounded surface boundary measure $|\partial\Omega|$ as for instance the Koch snowflake in $d = 2$. Also, the quotient $|\Omega|/I(\Omega)$ in (2.15) does not make this circumstances clearly visible. The first result capturing the correct order of $\Lambda^{\gamma+(d-1)/2}$ was shown by T. Weidl

Theorem 2.2.1 ([90, Theorem 2.1]). *For each $d \geq 2$ and $\gamma \geq 3/2$ exists a* *constant $\nu(\gamma, d)$ such that*

$$\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;0))_+^\gamma \leq L_{\gamma,d}^{cl}|\Omega_\Lambda|\Lambda^{\gamma+d/2} - \frac{\nu(\gamma,d)}{4} L_{\gamma,d-1}^{cl} d_\Lambda(\Omega)\Lambda^{\gamma+(d-1)/2}$$

holds for any $\Omega \subset \mathbb{R}^d$ and $\Lambda > 0$.

Here $\Omega_\Lambda \subset \Omega$ denotes the subset containing only intervalls (along some fixed axis) of length less or equal than $\pi\Lambda^{-1/2}$. I.e. for the x_1-direction that is

$$\Omega_\Lambda := \bigcup_{x'\in\mathbb{R}^{d-1}} \Omega_\Lambda(x') \times \{x'\}$$

if

$$\Omega_\Lambda(x') := \bigcup_{k\in\kappa(x',\Lambda)} J_k(x')$$

where $J_k(x')$ are the connected components, respectively intervals, of

$$\Omega(x') := \{x_1 \in \mathbb{R} \,:\, (x_1, x') \in \Omega\}$$

and $\kappa(x', \Lambda)$ is the set of indices for which the length of $J_k(x')$ is bounded from above by $\pi\Lambda^{-1/2}$. Note that $\kappa(x', \Lambda)$ is finite for almost every $x' \in \mathbb{R}^{d-1}$ as long as $|\Omega_\Lambda|$ is finite and therefore

$$d_\Lambda(\Omega) := \int_{\mathbb{R}^{d-1}} \#\kappa(x', \Lambda)\, \mathrm{d}x'$$

exists. In order to prove Theorem 2.2.1 one can make use of an estimate for the eigenvalues of the one-dimensional Laplacian on the intervals $J_k(x')$ and therefore reduce the problem to a spectral estimate for a Schrödinger type operator with operator valued potential. This remaining part of the problem can be treated with the help of Lieb-Thirring bounds as in Theorem 3.1 in [65], which restricts the result to the case $\gamma \geq 3/2$. When searching for inequalities for $\gamma < 3/2$, in particular $\gamma = 1$, the situation becomes even less straightforward. In dimension two H. Kovařík, S. Vugalter and T. Weidl

[54] proposed

$$\sum_{j=1}^{N} \lambda_j(\Omega;0) \geq \frac{2\pi}{|\Omega|}N^2 + \alpha C_\Omega |\Omega|^{-3/2} N^{3/2-\varepsilon(N)} + (1-\alpha)\frac{|\Omega|}{32I}N$$

for any $\alpha \in]0,1[$ where C_Ω is a constant that reflects some of the geometry of the boundary $\partial\Omega$ and

$$\varepsilon(N) = \frac{2}{\sqrt{\log_2(2\pi N/c)}}, \qquad c = \sqrt{\frac{3\pi}{14}}\,10^{-11}$$

reaching the order of the second term arbitrary close as $N \to \infty$. Their proof is based on the proof of (2.4) by P. Li and S. T. Yau and uses a more careful application of the bath tube principle. An alternative idea is to start with the Berezin type variant (2.3) and to give more attention to the remainders when applying Parseval's inequality. This was carried out by H. Kovařík and T. Weidl in [55]. The result is

$$\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;0))_+$$

$$\leq L^{\mathrm{cl}}_{1,d}|\Omega|\Lambda^{1+d/2} - L^{\mathrm{cl}}_{1,d}K(\Omega)\sigma(\Omega)\left(\frac{\sigma(\Omega)}{|\Omega|}\right)^{\mu/(\mu+2)}\Lambda^{d/2+1/(\mu+2)}$$

where

$$K(\Omega) = \frac{2+\mu}{\mu}(4+4\mu)^{-(2+2\mu)/(2+\mu)}$$

for $\mu = \mu(\Omega) = \sqrt{c_h(\Omega)}$ and $c_h(\Omega)$ is the constant in the Hardy inequality commented below in more detail. Thereby the quantity

$$\sigma(\Omega) = \inf_{0<\beta<\mathrm{Ri}\,(\Omega)} \frac{|\Omega_\beta|}{\beta},$$

with $\mathrm{Ri}\,(\Omega)$ for the inner radius of Ω, is related to the geometry of Ω.

For the rest of this section we follow their approach from [55] in order to improve our bound (2.12). Beforehand, we also have to introduce additional parameters related to the domain which somehow reflect the curvature of the boundary and make further assumptions on Ω. As before we make use of the notation $x = (x_1, x_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$ for $x \in \mathbb{R}^d$, $d > 1$. Along the x_1-direction consider the slices

$$\Omega(x_1) := \{y \in \mathbb{R}^{d-1} : (x_1, y) \in \Omega\}$$

for any $x_1 \in \mathbb{R}$. Note that by this definition $\Omega(x_1)$ might be empty. On every slice let $\Omega_\beta(x_1)$ be the set of all points which are close to the boundary $\partial\Omega(x_1)$. More precisely

$$\Omega_\beta(x_1) := \{y \in \Omega(x_1) : \delta_\perp(y) < \beta\},$$

where $\delta_\perp(y) = \operatorname{dist}(y, \partial\Omega(x_1))$ is the euclidean distance of any $y \in \Omega(x_1)$ to the boundary $\partial\Omega(x_1)$, measured in the $d-1$-dimensional submanifold $\Omega(x_1)$. Finally let

$$\Omega_\beta := \bigcup_{x_1 \in \mathbb{R}} \Omega_\beta(x_1).$$

In our proof we shall make use of

$$\sigma(\Omega, z_1) := \inf_{\beta > 0} \frac{1}{\beta} \int_{\Omega_\beta} \mathrm{Ai}^2 \left(\varepsilon_0^{1/3} (x_1 - z_1) \right) \mathrm{d}x, \tag{2.17}$$

as an additional restriction on Ω we will assume that

$$\sigma(\Omega, z_1) > 0$$

holds pointwise for any $z_1 \in \mathbb{R}$. Since

$$\int_{\Omega_\beta + h} \mathrm{Ai}^2 \left(\varepsilon_0^{1/3} (x_1 - z_1) \right) \mathrm{d}x = \int_{\Omega_\beta} \mathrm{Ai}^2 \left(\varepsilon_0^{1/3} (x_1 - (z_1 - h)) \right) \mathrm{d}x$$

for any $\beta > 0$, it follows that

$$\sigma(\Omega + h, z_1) = \sigma(\Omega, z_1 - h), \tag{2.18}$$

which will be important for the fact that our inequality reflects the behaviour of the eigenvalues while shifting the domain. If the slices $\Omega(x_1)$ are convex, (2.17) can be expressed in terms of the inner radii

$$\mathrm{Ri}\,(\Omega(x_1)) := \sup_{y\in\Omega(x_1)} \mathrm{dist}(y, \partial\Omega(x_1))$$

since by [55, Lemma 4.2] $\inf_{\beta>0} |\Omega_\beta(x_1)|/\beta = |\Omega(x_1)|/\,\mathrm{Ri}\,(\Omega(x_1))$ holds for convex domains and, thus,

$$\sigma(\Omega, z_1) = \int_{\mathbb{R}} \frac{|\Omega(x_1)|}{\mathrm{Ri}\,(\Omega(x_1))} \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - (z_1 - h))\right) \mathrm{d}x_1.$$

Another ingredient of our proof will be the Hardy inequality. Let $\Omega \subset \mathbb{R}^d$ be open and $u \in \mathring{W}_2^1(\Omega)$, then

$$\int_{\Omega(x_1)} \frac{|u(x_1, x_\perp)|^2}{\delta_\perp^2(x_\perp)} \mathrm{d}x_\perp \le c_h(\Omega(x_1)) \int_{\Omega(x_1)} |\nabla_{x_\perp} u(x_1, x_\perp)|^2 \,\mathrm{d}x_\perp \tag{2.19}$$

holds for all $x_1 \in \mathbb{R}$. We will assume that the optimal constants $c_h(\Omega(x_1))$ are uniformly bounded, i.e. $c_h(\Omega(x_1)) \le c_h(\Omega) < \infty$ for all $x_1 \in \mathbb{R}$, respectively

$$\inf_{u\in\mathring{W}_2^1(\Omega),\, u\neq 0} \frac{\int_{\mathbb{R}} \int_{\Omega(x_1)} |\nabla_{x_\perp} u|^2 \,\mathrm{d}x_\perp \mathrm{d}x_1}{\int_{\mathbb{R}} \int_{\Omega(x_1)} |u|^2/\delta_\perp^2 \,\mathrm{d}x_\perp \mathrm{d}x_1} \ge \frac{1}{c_h(\Omega)} > 0. \tag{2.20}$$

This holds true for instance if any slice $\Omega(x_1)$ is convex (then $c_h(\Omega) = 4$) or simply connected (then $c_h(\Omega) \le 16$), see [24] for a complete survey.

Theorem 2.2.2. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain such that*

$$\sigma(\Omega, z_1) := \inf_{\beta>0} \frac{1}{\beta} \int_{\Omega_\beta} \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x > 0$$

for all $z_1 \in \mathbb{R}$ *and* (2.20) *holds true, then*

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;\varepsilon_0))_+ \\
&\leq \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+ \int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x\, \mathrm{d}z \qquad (2.21) \\
&\quad - \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \frac{1}{16\, c_h(\Omega)} \odot \\
&\quad \odot \int_{\mathbb{R}^d} \sigma^2(\Omega, z_1) \frac{(\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+}{(\Lambda - \varepsilon_0 \lfloor\Omega)_+} \left[\int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\right]^{-1} \mathrm{d}z
\end{aligned}$$

for all $\Lambda > \varepsilon_0 \lfloor\Omega$ *where*

$$\lfloor\Omega = \inf\{x_1 \in \mathbb{R} : \Omega(x_1) \neq \emptyset\}$$

is the left border of Ω *as defined in* (1.10).

Note that with (2.18) in mind, one can easily see that (2.21) reflects the behaviour of the eigenvalues when shifting the domain along the direction of the Stark potential.

Proof. Fix $\Lambda > 0$ and let

$$n(\Lambda) := \#\{\lambda_j(\Omega;\varepsilon_0) : \lambda_j(\Omega;\varepsilon_0) < \Lambda\}$$

be the counting function of all eigenvalues below Λ. For $\lambda_j = \lambda_j(\Omega;\varepsilon_0)$ we choose a sequence of orthonormal eigenfunctions ϕ_j. Then

$$\begin{aligned}\sum_{j\le n(\Lambda)}(\Lambda-\lambda_j) &= \sum_{j\le n(\Lambda)}\int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)\,|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z \\ &= \int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_+\sum_{j\le n(\Lambda)}|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z \\ &\quad -\int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_-\sum_{j\le n(\Lambda)}|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z \\ &= \int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_+\sum_{j\in\mathbb{N}}|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z \\ &\quad -\int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_+\sum_{j> n(\Lambda)}|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z \\ &\quad -\int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_-\sum_{j\le n(\Lambda)}|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z\end{aligned} \tag{2.22}$$

where we have already seen that

$$\begin{aligned}&\int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_+\sum_{j\in\mathbb{N}}|\mathcal{A}[\phi_j](z)|^2\,\mathrm{d}z \\ &\qquad\le \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}}\int_{\mathbb{R}^d}(\Lambda-\varepsilon_0 z_1-|z_\perp|^2)_+\int_\Omega|\operatorname{Ai}\left(\varepsilon_0^{1/3}(x_1-z_1)\right)|^2\,\mathrm{d}x\,\mathrm{d}z,\end{aligned}$$

cf. (2.11). For the lower order terms we want to estimate

$$R(\Lambda,z)=\sum_{j>n(\Lambda)}|\mathcal{A}[\phi_j](z)|^2$$

from below. The last term on the right hand side will be omitted. Since

$(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$, it follows that

$$\sum_{j>n(\Lambda)} |\mathcal{A}[\phi_j](z)|^2 + \sum_{j\le n(\Lambda)} |\mathcal{A}[\phi_j](z)|^2 = \int_\Omega \left| \frac{\varepsilon_0^{1/3}}{(2\pi)^{(d-1)/2}} \, \mathrm{e}^{-\mathrm{i}x_\perp z_\perp} \, \mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \right|^2 \mathrm{d}x,$$

and from that by the Pythagorean theorem

$$\begin{aligned} &R(\Lambda, z) \\ &= \int_\Omega \left| \frac{\varepsilon_0^{1/3}}{(2\pi)^{(d-1)/2}} \, \mathrm{e}^{-\mathrm{i}x_\perp z_\perp} \, \mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) - \sum_{j\le n(\Lambda)} \mathcal{A}[\phi_j](z)\, \phi_j(x) \right|^2 \mathrm{d}x \\ &\ge \frac{1}{2} \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_{\Omega_\beta} \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x - \int_{\Omega_\beta} \left| \sum_{j\le n(\Lambda)} \mathcal{A}[\phi_j](z)\, \phi_j(x) \right|^2 \mathrm{d}x \end{aligned} \tag{2.23}$$

where in the last step we have used that $\Omega_\beta \subset \Omega$ and $|a-b|^2 \ge |a|^2/2 - |b|^2$ for each $a, b \in \mathbb{C}$. We introduce the abbreviation

$$F_\Lambda(z, x) = \sum_{j\le n(\Lambda)} \mathcal{A}[\phi_j](z)\, \phi_j(x)$$

and note that $F_\Lambda(z, (x_1, \cdot)) \in \mathring{W}_2^1(\Omega(x_1))$ for each $\Lambda > 0$, $z \in \mathbb{R}^d$ and $x_1 \in \mathbb{R}$. Hence we can make use of the Hardy inequality (2.19) in combination with (2.20) and obtain

$$\begin{aligned} \int_{\Omega_\beta} |F_\Lambda(z,x)|^2 \, \mathrm{d}x &= \int_\mathbb{R} \int_{\Omega_\beta(x_1)} |F_\Lambda(z,x)|^2 \mathrm{d}x_\perp \mathrm{d}x_1 \\ &\le \beta^2 \int_\mathbb{R} \int_{\Omega(x_1)} \frac{|F_\Lambda(z,x)|^2}{\delta_\perp^2(x_\perp)} \, \mathrm{d}x_\perp \mathrm{d}x_1 \qquad (2.24) \\ &\le \beta^2 \, c_h(\Omega) \int_\mathbb{R} \int_{\Omega(x_1)} |\nabla_{x_\perp} F_\Lambda(z,x)|^2 \, \mathrm{d}x_\perp \mathrm{d}x_1 \end{aligned}$$

where $c_h(\Omega) = \sup_{x_1 \in \mathbb{R}} c_h(\Omega(x_1))$ (if $\Omega(x_1) = \emptyset$ we formally set $c_h(\Omega(x_1)) = 0$). Recall that $F_\Lambda(z,x)$ is a linear combination of eigenfunctions, therefore

$$\begin{aligned}\int_\Omega |\nabla_{x_\perp} F_\Lambda(z,x)|^2 \,\mathrm{d}x &= \int_\Omega |\nabla_x F_\Lambda(z,x)|^2 + \varepsilon_0 x_1 |F_\Lambda(z,x)|^2 \,\mathrm{d}x \\ &\quad - \int_\Omega \left| \frac{\partial}{\partial x_1} F_\Lambda(z,x) \right|^2 \mathrm{d}x - \int_\Omega \varepsilon_0 x_1 |F_\Lambda(z,x)|^2 \,\mathrm{d}x \\ &\leq \sum_{j \leq n(\Lambda)} |\mathcal{A}[\phi_j](z)|^2 \,\lambda_j - \varepsilon_0 \lfloor \Omega \int_\Omega |F_\Lambda(z,x)|^2 \,\mathrm{d}x.\end{aligned}$$

With

$$\int_\Omega |F_\Lambda(z,x)|^2 \,\mathrm{d}x = \sum_{j \leq n(\Lambda)} |\mathcal{A}[\phi_j](z)|^2$$

it follows that

$$\int_{\Omega_\beta} |F_\Lambda(z,x)|^2 \,\mathrm{d}x \leq \beta^2 \, c_h(\Omega) \sum_{j \leq n(\Lambda)} (\lambda_j - \varepsilon_0 \lfloor \Omega) \, |\mathcal{A}[\phi_j](z)|^2.$$

Since $\lambda_j < \Lambda$ for $j \leq n(\Lambda)$ and there are no eigenvalues below $\varepsilon_0 \lfloor \Omega$, we conclude from that

$$\begin{aligned}\int_{\Omega_\beta} |F_\Lambda(z,x)|^2 \,\mathrm{d}x &\leq \beta^2 \, c_h(\Omega) \, (\Lambda - \varepsilon_0 \lfloor \Omega)_+ \sum_{j \leq n(\Lambda)} |\mathcal{A}[\phi_j](z)|^2 \\ &= \beta^2 \, c_h(\Omega) \, (\Lambda - \varepsilon_0 \lfloor \Omega)_+ \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_\Omega \mathrm{Ai}^2 \left(\varepsilon_0^{1/3} (x_1 - z_1) \right) \mathrm{d}x.\end{aligned}$$

Inserting this into (2.23) gives

$$
\begin{aligned}
&R(\Lambda, z)\\
&\geq \frac{\varepsilon_0^{2/3}\beta}{(2\pi)^{d-1}} \odot\\
&\quad \odot \left[\frac{1}{2\beta}\int_{\Omega_\beta} \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x\right.\\
&\qquad\qquad \left. -\beta\, c_h(\Omega)\,(\Lambda - \varepsilon_0 \lfloor\Omega)_+ \int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x\right]\\
&\geq \frac{\varepsilon_0^{2/3}\beta}{(2\pi)^{d-1}}\left[\frac{1}{2}\sigma(\Omega, z_1) - \beta\, c_h(\Omega)\,(\Lambda - \varepsilon_0\lfloor\Omega)_+ \int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x\right]
\end{aligned}
$$

where in the last step we have used that

$$\sigma(\Omega, z_1) \leq \frac{1}{\beta}\int_{\Omega_\beta} \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x$$

for each $\beta > 0$ and $z_1 \in \mathbb{R}$. So far we have not made any assumptions on $\beta > 0$. If we choose

$$\beta = \beta(z_1) = \frac{\sigma(\Omega, z_1)}{4\, c_h(\Omega)}\,(\Lambda - \varepsilon_0\lfloor\Omega)_+^{-1}\left[\int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x\right]^{-1}, \quad (2.25)$$

which will be commented below, this yields

$$R(\Lambda, z) \geq \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}}\,\frac{\sigma^2(\Omega, z_1)}{16\, c_h(\Omega)}\,(\Lambda - \varepsilon_0\lfloor\Omega)_+^{-1}\left[\int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x\right]^{-1}.$$

By inserting this into (2.22) we finally obtain our result. □

Remark 2.2.1. Let

$$\mathrm{Ri}\,(\Omega(x_1)) := \sup_{y\in\Omega(x_1)} \mathrm{dist}(y, \partial\Omega(x_1))$$

be the inner radius of each slice $\Omega(x_1)$ and

$$\mathrm{Ri}_\perp(\Omega) := \sup_{x_1\in\mathbb{R}} \mathrm{Ri}\,(\Omega(x_1))$$

where we set $\mathrm{Ri}\,(\Omega(x_1)) = 0$ if $\Omega(x_1)$ is empty. We want to show that our choice of $\beta(z_1)$ in (2.25) satisfies

$$\beta(z_1) \le \frac{\mathrm{Ri}_\perp(\Omega)}{4} \tag{2.26}$$

which means that Ω_β indeed consists of points close to the boundary of Ω.

Since $\beta = \mathrm{Ri}_\perp(\Omega)$ is also a valid choice in the definition of Ω_β or $\sigma(\Omega, z)$, it follows that

$$\frac{1}{\mathrm{Ri}_\perp(\Omega)} \int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x \ge \sigma(\Omega, z_1),$$

thus, our choice of $\beta(z_1)$ in (2.25) satisfies

$$\beta(z_1) \le \frac{1}{4\, c_h(\Omega)} \left(\lambda_1 - \varepsilon_0 \lfloor\Omega\right)_+^{-1} \frac{1}{\mathrm{Ri}_\perp(\Omega)}. \tag{2.27}$$

From our Hardy inequality in (2.20) we obtain

$$\frac{1}{\mathrm{Ri}_\perp^2} \int_\Omega |u(x)|^2\,\mathrm{d}x \le c_h(\Omega) \int_\Omega \left(|\nabla u(x)|^2 + (\varepsilon_0 x_1 - \varepsilon_0 \lfloor\Omega)_+ |u(x)|^2\right) \mathrm{d}x$$

for each $u \in \mathring{W}_2^1(\Omega)$ and choosing u to be the eigenfunction for $\lambda_1(\Omega; \varepsilon_0)$ leads to

$$1 \le \mathrm{Ri}_\perp^2(\Omega)\, c_h(\Omega)\, (\lambda_1 - \varepsilon_0 \lfloor\Omega)_+$$

which, together with (2.27), finishes the proof of (2.26). Note that from the last inequality it also follows that

$$\lambda_1 \ge \varepsilon_0 \lfloor\Omega + (\mathrm{Ri}_\perp^2(\Omega)\, c_h(\Omega))^{-1}.$$

Although the order of Λ in the second term on the right hand side in (2.21) is not that clearly visible, we assume that it is far from being optimal. Indeed, replacing the Hardy inequality in the step (2.24) by the improved bound

$$\int_{\{x\in\Omega\,:\,\delta(x)<\beta\}} |u(x)|^2 \,\mathrm{d}x \le (\mu\beta)^{2+2/\mu}\, \|Hu\|_2 \cdot \|H^{1/\mu}u\|_2, \tag{2.28}$$

which was shown by E. B. Davies in [25, Theorem 4], reduces the order of Λ. Here $u \in \operatorname{dom} H$ where $H = -\Delta + V$ is a non-negative operator on a domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions such that the Hardy inequality

$$\int_\Omega \frac{|u(x)|^2}{\delta^2(x)} \,\mathrm{d}x \le \mu^2 h[u],$$

h being the quadratic form of H, holds for any $u \in C_0^\infty(\Omega)$. Actually (2.28) is proven in [25] under much milder assumptions on H and can also be extended to magnetic operators, see [55, Proposition 5.1]. Also δ does not need to be a proper distance function, it is sufficient that δ is a continous function on Ω satisfying $|\delta(x) - \delta(y)| \le |x - y|$ for all $x, y \in \Omega$. Applying (2.28) with $H = -\Delta + \varepsilon_0(x_1 - \lfloor\Omega)$ and δ being the euclidean distance of any $x \in \Omega$ to the domains boundary, we obtain:

Theorem 2.2.3. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain such that*

$$\sigma(\Omega) := \inf_{\beta>0} \frac{1}{\beta} \int_{\{x\in\Omega\,:\,\operatorname{dist}(\partial\Omega,x)<\beta\}} \operatorname{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x > 0,$$

then

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} (\Lambda - \lambda_j(\Omega;\varepsilon_0))_+ \\
&\le \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+ \int_\Omega \operatorname{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x\,\mathrm{d}z \qquad (2.29)\\
&\quad - C(\Omega) \int_{\mathbb{R}^d} \mathrm{d}z\, (\Lambda - \varepsilon_0 z_1 - |z_\perp|^2)_+ (\Lambda - \varepsilon_0 \lfloor\Omega)_+^{-\frac{1+c}{2+c}} \odot \\
&\quad \odot \left[\int_\Omega \operatorname{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \mathrm{d}x\right]^{-\frac{c}{2+c}}
\end{aligned}$$

for all $\Lambda > \varepsilon_0 \lfloor \Omega$ *where*

$$C(\Omega) := \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \frac{1}{c} (4+4c)^{-\frac{2+2c}{2+c}} \sigma(\Omega)^{\frac{2+2c}{2+c}}$$

and $c^2 = c_h(\Omega)$ *is the constant in the classical Hardy inequality of* $-\Delta$ *on* Ω.

Proof. Denote by $\mathrm{dist}(\partial\Omega, x)$ the euclidean distance of any $x \in \Omega$ to the boundary $\partial\Omega$. From the classical Hardy inequality it follows that

$$\int_\Omega \frac{|u(x)|^2}{(\mathrm{dist}(\partial\Omega, x))^2} \,\mathrm{d}x \leq c^2 \left[\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x + \varepsilon_0 \int_\Omega (x_1 - \lfloor \Omega) \, |u(x)|^2 \,\mathrm{d}x \right]$$

if $u \in \mathring{W}_2^1(\Omega)$ where $c^2 = c_h(\Omega)$. On the right hand side we have used the quadratic form of $-\Delta + \varepsilon_0(x_1 - \lfloor \Omega)$. Since $x_1 \geq \lfloor \Omega$ for each $(x_1, x_\perp) \in \Omega$, this operator is non-negative and its eigenvalues are given by $\lambda_j(\Omega; \varepsilon_0) - \varepsilon_0 \lfloor \Omega$ where $\lambda_j(\Omega; \varepsilon_0)$ are the eigenvalues of $-\Delta + \varepsilon_0 x_1$ on Ω with Dirichlet boundary conditions. Thus, the assumptions of [25, Theorem 4] are satisfied and applying (2.28) to $F_\Lambda(z, \cdot) \in \mathring{W}_2^1(\Omega)$ from the proof of Theorem 2.2.2 yields

$$\begin{aligned} &\int_{\{x \in \Omega \,:\, \mathrm{dist}(\partial\Omega, x) < \beta\}} |F_\Lambda(z, x)|^2 \,\mathrm{d}x \\ &\qquad \leq (c\beta)^{2+2/c} (\Lambda - \varepsilon_0 \lfloor \Omega)_+^{1+1/c} \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_\Omega \mathrm{Ai}^2 \left(\varepsilon_0^{1/3} (x_1 - z_1 \right) \mathrm{d}x. \end{aligned}$$

Proceeding as in the proof of Theorem 2.2.2, let

$$R(\Lambda, z) = \sum_{j > n(\Lambda)} |\mathcal{A}[\phi_j](z)|^2$$

then

$$\begin{aligned} &R(\Lambda, z) \\ &\geq \frac{\varepsilon_0^{2/3} \beta}{(2\pi)^{d-1}} \left[\frac{\sigma(\Omega)}{2} - \beta^{1+2/c} c^{2+2/c} (\Lambda - \varepsilon_0 \lfloor \Omega)_+^{1+1/c} \int_\Omega \mathrm{Ai}^2 \left(\varepsilon_0^{1/3} (x_1 - z_1 \right) \mathrm{d}x \right]. \end{aligned} \tag{2.30}$$

Minimizing the right hand side for β yields

$$\beta = \frac{1}{c}(4+4c)^{-\frac{c}{2+c}}\sigma(\Omega)^{\frac{c}{2+c}}(\Lambda - \varepsilon_0\lfloor\Omega)_+^{-\frac{1+c}{2+c}}\left(\int_\Omega \mathrm{Ai}^2\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\mathrm{d}x\right)^{-\frac{c}{2+c}}$$

Since $c \geq 2$, this choice of β again satisfies the condition

$$\beta \leq \left(\frac{c}{4+c}\right)^{\frac{c}{2+c}} \mathrm{Ri}\,(\Omega) \leq \mathrm{Ri}\,(\Omega)$$

where $\mathrm{Ri}\,(\Omega) := \sup_{y\in\Omega} \mathrm{dist}(\partial\Omega, y)$. Inserting (2.30) into (2.22) finishes the proof of (2.29). □

Chapter 3

Kröger type estimates

This chapter aimes at a series of inequalities for the sums of eigenvalues. For a moment, denote

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \ldots$$

the sequence of eigenvalues of the classical Laplacian on $\Omega \subset \mathbb{R}^d$ equipped with Dirichlet boundary conditions and

$$0 = \mu_1 < \mu_2 \leq \mu \leq \cdots \leq \mu_k \leq \ldots$$

the eigenvales of $-\Delta$ on Ω with Neumann boundary conditions where the necessary restrictions on Ω apply in this case. As usual, the eigenvalues appear according to their multiplicity. For the Dirichlet Laplacian the sum of eigenvalues can be estimated from below by

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2}(2\pi)^2 |B^d|^{-2/d} |\Omega|^{-2/d} k^{1+2/d}$$

where $|B^d| = \pi^{d/2}/\Gamma(d/2+1)$ is the volume of the unit ball in $\mathbb{R}^d$. The latter is known as *Li-Yau inequality*, cf. [68]. In fact it is a consequence from (2.3) if $\gamma = 1$, and can be proved by applying the Legendre transformation on both sides of (2.3).

In 1992 P. Kröger introduced a test function argument proving

$$\sum_{j=1}^{k} \mu_j \leq \frac{d}{d+2}(2\pi)^2|B^d|^{-2/d}|\Omega|^{-2/d}k^{1+2/d} \tag{3.1}$$

for any $\Omega \subset \mathbb{R}^d$ with piece-wise smooth boundary, see [59, Corollary 1]. Most interesting about this result is that the right hand side matches the right hand side of the Li-Yau inequality. The proof of (3.1) is based on the estimate

Theorem 3.0.1 ([59, Theorem 1])**.** *Let $\Omega \subset \mathbb{R}^d$ with piece-wise smooth boundary, then*

$$\mu_{k+1} \leq \inf_{r>2\pi(k/|\Omega|)^{1/d}} \frac{(d/(d+2))r^{d+2}|B^d|\,|\Omega| - (2\pi)^d \sum_{j=1}^{k} \mu_j}{r^d\,|B^d|\,|\Omega| - (2\pi)^d k}.$$

The proof of this theorem relies on the Rayleigh-Ritz formula

$$\mu_k = \min_{\substack{\phi\in W_2^1(\Omega)\\ \phi\perp\phi_1,\dots,\phi_{k-1}}} \frac{\int_\Omega |\nabla\phi(y)|^2\,\mathrm{d}y}{\int_\Omega |\phi(y)|^2\,\mathrm{d}y} \leq \frac{\int_\Omega |\nabla\phi(y)|^2\,\mathrm{d}y}{\int_\Omega |\phi(y)|^2\,\mathrm{d}y} \tag{3.2}$$

where we choose $\phi(y)$ to be

$$H_z(y) = h_z(y) - (2\pi)^{d/2}\,\mathcal{F}_x[\Phi_k](z,y) \in W_2^1(\Omega)$$

for $h_z(y) = \mathrm{e}^{\mathrm{i}zy}$. Here $\mathcal{F}_x[\Phi_k]$ is the Fourier transform of

$$\Phi_k(x,y) = \sum_{j=1}^{k} \phi_j(x)\phi_i(y)$$

with respect to the x variable, and $(\phi_j)_{j=1,\dots,k}$ denotes the corresponding sequence of eigenfunctions of the eigenvalues $(\mu_j)_{j=1,\dots,k}$. Thus, $\mathcal{F}_x[\Phi_k]$ is the projection of h_z onto the subspace which is spanned by $(\phi_j)_{j=1,\dots,k}$, a fact which is heavily used during the calculations. For any details we refer to [59]. Using this approach with a truncated variant of $h_z(y)$ satisfying $h_z(y) = 0$ for $y \in \partial\Omega$ as a test function in the variational quotient for the Dirichlet eigenvalues leads to

Theorem 3.0.2 ([60, Theorem 1]). *For $\Omega \subset \mathbb{R}^d$ let $\Omega_r := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < 1/r\}$ and suppose that*

$$|\Omega_r| \leq \frac{C_0(\Omega)}{r} |\Omega|^{1-1/d}$$

holds for some constant $C_0(\Omega)$ and any $r > |\Omega|^{-1/d}$, then there is a constant $c_d > 0$ such that

$$\sum_{j=1}^{k} \lambda_j \leq \frac{d}{d+2} (2\pi)^2 |B^d|^{-2/d} |\Omega|^{-2/d} (k^{1+2/d} + c_d\, C_0(\Omega) k^{1+1/d}).$$

In [39] E. M. Harrell and J. Stubbe observed that the essence of Kröger's test function arguments is an averaging of different parts of a variational estimate simplifying some of the coefficients and formulated a generalized version. This generalized version can be used to prove inequalities for the sums of eigenvalues of the Stark Laplacian which is why we want to present it in more detail. Let $H_V = -\Delta + V$ be a self adjoint operator on $\Omega \subset \mathbb{R}^d$ and h_V its corresponding quadratic form. Suppose H_V has discrete spectrum $-\infty < \mu_1 \leq \mu_2 \leq \ldots$ with a corresponding sequence $(\phi_j)_{j=1,2,\ldots}$ of orthonormal eigenfunctions. For any $f \in d[h_V]$ denote by

$$P_k f := \sum_{j=1}^{k} \langle \phi_j, f \rangle\, \phi_j$$

the projection of f onto the subspace spanned by $(\phi_j)_{j=1,\ldots,k}$.

Lemma 3.0.1 ([39]). *If either*

- *H_V is equipped with Neumann boundary conditions on Ω or*

- *H_V is equipped with Dirichlet boundary conditions and $f \equiv 0$ on $\partial\Omega$,*

then

$$h_V[f - P_k f, f - P_k f] = h_V[f, f] - h_V[P_k f, P_k f].$$

Proof. Observe that

$$\begin{aligned} h_V[f - P_k f, f - P_k f] &= h_V[f, f] - 2\,\mathrm{Re}\, h_V[f, P_k f] + h_V[P_k f, P_k f] \\ &= h_V[f, f] - 2\,\mathrm{Re}\, h_V[f - P_k f, P_k f] - h_V[P_k f, P_k f]. \end{aligned}$$

From that the statement follows as long as $h_V[f - P_k f, P_k f] = 0$. Thus, we proceed with

$$\begin{aligned} & h[f - P_k f, \phi_j] \\ &= \int_\Omega \nabla(f - P_k f)\,\nabla\phi_j \,\mathrm{d}x + \langle f - P_k f, V\phi_j\rangle_{L^2(\Omega)} \\ &= \int_{\partial\Omega} (f - P_k f)\,\frac{\partial}{\partial n}\phi_j \,\mathrm{d}\sigma - \int_\Omega (f - P_k f)\,\Delta\phi_j \,\mathrm{d}x + \langle f - P_k f, V\phi_j\rangle_{L^2(\Omega)}. \end{aligned}$$

If $\frac{\partial}{\partial n}\phi_j = 0$ or $f \equiv 0$ on $\partial\Omega$, the integral over $\partial\Omega$ on the right hand side vanishes and it follows that

$$h_V[f - P_k f, \phi_j] = \langle f - P_k f, (-\Delta + V)\phi_j\rangle_{L^2(\Omega)} = \lambda_j \langle f - P_k f, \phi_j\rangle_{L^2(\Omega)} = 0$$

for each $j = 1, \dots, k$ and $k \in \mathbb{N}$. □

This intermediate result can now be used in various test function arguments.

Theorem 3.0.3 ([30, Theorem 2.1])**.** *Consider $H_V = -\Delta + V$ on a bounded domain $\Omega \subset \mathbb{R}^d$ with Neumann boundary conditions on $\partial\Omega$ such that the spectrum of H_V (or at least its lower part) is discrete. Let $(\phi_j)_{j=1,2,\dots}$ be a orthonormal sequence of eigenfunctions for the eigenvalues $(\mu_j)_{j=1,2,\dots}$ and for $z \in \mathbb{R}^d$ be f_z a family of functions in the form domain $d[h_V]$, then*

$$\sum_{j\in\mathbb{N}} (\Lambda - \mu_j)_+ \int_{\mathbb{R}^d} |\langle f_z, \phi_j\rangle_{L^2(\Omega)}|^2 \,\mathrm{d}z \geq \int_{M_0} \left(\Lambda \,\|f_z\|^2_{L^2(\Omega)} - h_V[f_z, f_z]\right) \mathrm{d}z \tag{3.3}$$

holds for all $M_0 \subset \mathbb{R}^d$ and $\Lambda \in \mathbb{R}$.

Proof. If $\Lambda \leq \mu_1$, the left hand side in (3.3) vanishes, whereas the right hand side is negative and the result becomes trivial. For $\Lambda > \mu_1$ let k be the smallest integer such that $\Lambda \leq \mu_k$ or $\Lambda \in]\mu_{k-1}, \mu_k]$. Since $\langle f_z - P_{k-1} f_z, \phi_j \rangle_{L^2(\Omega)} = 0$ for all $j = 1, \dots, k-1$, the function $f_z - P_{k-1} f_z$ is a valid test function for μ_k in (3.2), and together with Lemma 3.0.1 it follows that

$$\begin{aligned} \Lambda \langle f_z - P_{k-1} f_z, f_z - P_{k-1} f_z \rangle &\leq \mu_k \langle f_z - P_{k-1} f_z, f_z - P_{k-1} f_z \rangle \\ &\leq h_V [f_z - P_{k-1} f_z, f_z - P_{k-1} f_z] \qquad (3.4) \\ &= h_V [f_z, f_z] - h_V [P_{k-1} f_z, P_{k-1} f_z]. \end{aligned}$$

We have chosen $(\phi_j)_{j=1,2,\dots}$ to be orthonormal such that $\langle P_{k-1} f_z, P_{k-1} f_z \rangle = \sum_{j=1}^{k-1} |\langle f_z, \phi_j \rangle|^2$ and

$$h_V [P_{k-1} f_z, P_{k-1} f_z] = \sum_{j=1}^{k-1} \mu_j \, |\langle f_z, \phi_j \rangle|^2.$$

If we note $\langle f_z - P_{k-1} f_z, f_z - P_{k-1} f_z \rangle = \|f_z\|^2 - \langle P_{k-1} f_z, P_{k-1} f_z \rangle$, we obtain

$$\Lambda \, \|f_z\|^2 - h_V [f_z, f_z] \leq \sum_{j=1}^{k-1} (\Lambda - \mu_j) \, |\langle f_z, \phi_j \rangle|^2$$

from (3.4), and integration over M_0 yields

$$\begin{aligned} \int_{M_0} \left(\Lambda \, \|f_z\|^2 - h_V [f_z, f_z] \right) \mathrm{d}z &\leq \sum_{j=1}^{k-1} (\Lambda - \mu_j) \int_{M_0} |\langle f_z, \phi_j \rangle|^2 \, \mathrm{d}z \\ &\leq \sum_{j=1}^{k-1} (\Lambda - \mu_j) \int_{\mathbb{R}^d} |\langle f_z, \phi_j \rangle|^2 \, \mathrm{d}z. \end{aligned}$$

□

A remarkable consequence of Theorem 3.0.3 is

$$\mu_k \leq \frac{\displaystyle\int_{M_0} h_V [f_z, f_z] \, \mathrm{d}z - \sum_{j=1}^{k-1} \mu_j \int_{\mathbb{R}^d} |\langle f_z, \phi_j \rangle_{L^2(\Omega)}|^2 \, \mathrm{d}z}{\displaystyle\int_{M_0} \|f_z\|^2_{L^2(\Omega)} \, \mathrm{d}z - \sum_{j=1}^{k-1} \int_{\mathbb{R}^d} |\langle f_z, \phi_j \rangle_{L^2(\Omega)}|^2 \, \mathrm{d}z} \qquad (3.5)$$

which was already proved in [39] but can also be obtained by setting $\Lambda = \mu_k$ in (3.3). Following Krögers induction argument in the proof of Theorem 1 in [59] from (3.5) on we show:

Theorem 3.0.4. *Let $(\mu_j)_{j=1,2,\ldots}$ be the sequence of eigenvalues of $H_V = -\Delta + V$ on $\Omega \subset \mathbb{R}^d$ with Neumann boundary conditions where the same restrictions as in Theorem 3.0.3 apply. If f_z is in the form domain $d[h_V]$, then*

$$\mu_k \leq \frac{\displaystyle\int_{M_0} h_V[f_z, f_z]\,\mathrm{d}z - \sum_{j=1}^{k-1} \mu_j}{\displaystyle\int_{M_0} \|f_z\|^2_{L^2(\Omega)}\,\mathrm{d}z - (k-1)}. \tag{3.6}$$

Proof. If $k = 1$, then (3.6) follows from the variational principle (3.2) since $f_z \in d[h_V]$ is a valid test function. Assume (3.6) holds for some $k \in \mathbb{N}$, then equivalently

$$\mu_k \leq \frac{\int_{M_0} h_V[f_z, f_z]\,\mathrm{d}z - \sum_{j=1}^{k} \mu_j}{\int_{M_0} \|f_z\|^2\,\mathrm{d}z - k}.$$

From the monotonicity of the eigenvalue sequence we then obtain

$$\frac{\sum_{j=1}^{k} \mu_j \left(1 - \int_{M_0} |\langle f_z, \phi_j\rangle|^2\,\mathrm{d}z\right)}{\sum_{j=1}^{k} \left(1 - \int_{M_0} |\langle f_z, \phi_j\rangle|^2\,\mathrm{d}z\right)} \leq \mu_k \leq \frac{\int_{M_0} h_V[f_z, f_z]\,\mathrm{d}z - \sum_{j=1}^{k} \mu_j}{\int_{M_0} \|f_z\|^2\,\mathrm{d}z - k},$$

and

$$\begin{aligned}
\mu_{k+1} &\leq \frac{\int_{M_0} h_V[f_z, f_z]\,\mathrm{d}z - \sum_{j=1}^{k} \mu_j \int_{M_0} |\langle f_z, \phi_j\rangle|^2\,\mathrm{d}z}{\int_{M_0} \|f_z\|^2\,\mathrm{d}z - \sum_{j=1}^{k} \int_{M_0} |\langle f_z, \phi_j\rangle|^2\,\mathrm{d}z} \\
&= \frac{\int_{M_0} h_V[f_z, f_z]\,\mathrm{d}z - \sum_{j=1}^{k} \mu_j + \sum_{j=1}^{k} \mu_j \left(1 - \int_{M_0} |\langle f_z, \phi_j\rangle|^2\,\mathrm{d}z\right)}{\int_{M_0} \|f_z\|^2\,\mathrm{d}z - k + \sum_{j=1}^{k} \left(1 - \int_{M_0} |\langle f_z, \phi_j\rangle|^2\,\mathrm{d}z\right)} \\
&\leq \frac{\int_{M_0} h_V[f_z, f_z]\,\mathrm{d}z - \sum_{j=1}^{k} \mu_j}{\int_{M_0} \|f_z\|^2\,\mathrm{d}z - k}
\end{aligned}$$

follows from (3) and the basic statement $c/d \leq a/b \Rightarrow (a+c)/(b+d)$ if $b, d > 0$. □

Our initial goal was to provide estimates for the Stark Laplacian which falls into the class of operators in Theorem 3.0.3 and it remains to choose an appropriate expression for f_z. In the case of the classical Laplacian, Krögers result (3.0.1) follows from (3.6) if $f_z(x) = (2\pi)^{-d/2}\,\mathrm{e}^{\mathrm{i}xz}$. In that case $\langle f_z, \phi_j\rangle_{L^2(\Omega)}$ is the Fourier transform of ϕ_j. Since the corresponding transform for Stark Laplacian is the Airy transform introduced in Section 1.4.2, we choose f_z to be the integral kernel of $\mathcal{A}$, that is

$$f_z(x) = \frac{\varepsilon_0^{1/3}}{(2\pi)^{(d-1)/2}}\,\mathrm{e}^{-\mathrm{i}x_\perp z_\perp}\,\mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)$$

in our usual notation $x = (x_1, x_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Hence

$$\langle f_z, \phi_j\rangle_{L^2(\Omega)} = \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_\Omega \mathrm{e}^{-\mathrm{i}x_\perp z_\perp}\,\mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) \phi_j(x)\,\mathrm{d}x = \mathcal{A}[\phi_j](z)$$

(cf. (1.19)), respectively

$$\int_{\mathbb{R}^d} |\langle f_z, \phi_j\rangle_{L^2(\Omega)}|^2\,\mathrm{d}z = \|\mathcal{A}[\phi_j]\|^2_{L^2(\mathbb{R}^d)} = \|\phi_j\|^2_{L^2(\Omega)} = 1$$

if the corresponding sequence of eigenfunctions $(\phi_j)_{j=1,2,\dots}$ is orthonormal. Since

$$\left(-\frac{\partial}{\partial x_1} + \varepsilon_0 x_1\right) \mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right) = \varepsilon_0 z_1\;\mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right),$$

it follows that

$$\begin{aligned} h[f_z, f_z] &= \langle H f_z, f_z\rangle_{L^2(\Omega)} \\ &= (|z_\perp|^2 + \varepsilon_0 z_1)\,\|f_z\|^2_{L^2(\Omega)} \\ &= (|z_\perp|^2 + \varepsilon_0 z_1) \int_\Omega \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \left|\mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\right|^2 \mathrm{d}x, \end{aligned}$$

and (3.3) reads as follows:

Corollary 3.0.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain such that the spectrum of $-\Delta + \varepsilon_0 x_1$ on Ω with Neumann boundary conditions is a discrete sequence* *of eigenvalues* $(\mu_j(\Omega;\varepsilon_0))_{j=1,2,\ldots}$, *then*

$$\sum_{j\in\mathbb{N}} (\Lambda - \mu_j(\Omega;\varepsilon_0))_+ \\ \geq \frac{\varepsilon_0^{2/3}}{(2\pi)^{d-1}} \int_{M_0} (\Lambda - |z_\perp|^2 - \varepsilon_0 z_1) \int_\Omega \left|\mathrm{Ai}\left(\varepsilon_0^{1/3}(x_1 - z_1)\right)\right|^2 \mathrm{d}x\, \mathrm{d}z$$

holds for all $M_0 \subset \mathbb{R}^d$ *and* $\Lambda \in \mathbb{R}$.

As in the case of the classical Laplacian, the constants on the right hand side of this inequality match their counterparts on the right hand side of the Berezin type inequality for the Dirichlet eigenvalues, cf. (2.12). But unlike (2.12) the right hand side is not translational invariant and can be maximized in respect to M_0.

Chapter 4

Estimates on the Counting Function

Let A be a self-adjoint operator on some separable Hilbertspace $\mathcal{H}$ and E_A the spectral measure on $\mathcal{H}$ associated to A via the spectral theorem, see [13, Chapter 6, Theorem 1]. The dimension of $E_A(]-\infty,\Lambda[)\mathcal{H}$ corresponds to the number of eigenvalues below a bound $\Lambda \in \mathbb{R}$ and thus gives rise to the counting function

$$N(A,\Lambda) := \dim E_A(]-\infty,\Lambda[)\mathcal{H} = \#\{\lambda(A) \leq \Lambda\}.$$

Note that this function counts the eigenvalues according to their multiplicities. In the case of the classical Dirichlet (or Neumann) Laplacian on a domain $\Omega \subset \mathbb{R}^d$ this function was studied for a long period. As mentioned before, calculating its asymptotic was the starting point for the whole field of spectral estimates addressing inequalities which compare various Riesz means to their corresponding phase space volume. Thereby Weyl's asymptotical result [92] from 1912 shows that the phase space volume is also the limit of the counting function $N_0^D(\Omega;\Lambda)$ as $\Lambda \to \infty$, more precisely

$$N_0^D(\Omega;\Lambda) = L_{0,d}^{\mathrm{cl}}\,|\Omega|\Lambda^{d/2} + O(\Lambda^{(d-1)/2}), \qquad \Lambda \to \infty.$$

Here we used our shorthand notation

$$N^i_{\varepsilon_0}(\Omega;\Lambda) := N(H^i_{\varepsilon_0}(\Omega),\Lambda) \tag{4.1}$$

for the counting function of all eigenvalues of $H^i_{\varepsilon_0}(\Omega)$ as introduced in Section 1.3, on $\Omega \subset \mathbb{R}^d$, equipped with Dirichlet boundary conditions ($i = D$) or Neumann boundary conditions ($i = N$). Regarding sharp bounds on $N^i_0(\Omega;\Lambda)$, there is not much known. In 1961 G. Pólya proved that the leading order term in (4.1) is an upper bound on $N^D_0(\Omega;\Lambda)$, i.e.

$$N^D_0(\Omega;\Lambda) \leq L^{\mathrm{cl}}_{0,d}\,|\Omega|\,\Lambda^{d/2} \tag{4.2}$$

for all $\Lambda \geq 0$, but under the additional assumption that Ω is a tiling domain. Tiling domains are those sets $\Omega \subset \mathbb{R}^d$ whose inifinite copies completely fill the whole $\mathbb{R}^d$ up to a set of Lebesgue measure zero, only by translating and rotating the set. Since balls do not have this property, (4.2) for balls remained open for a long period, despite the fact that the eigenvalues can be explicitly computed via the zeros of the Bessel function. Progress has only been made recently by M. Levitin, I. Polterovich and D. A. Sher in [67] by a computer-assisted proof. In the general case the problem remains still open and it is not known if (4.2) holds for surprisingly simple domains as for example certain polygons. So far the only attempt in proving (4.2) for any non-tiling domains has been published by A. Laptev [62, Theorem 2.8] where it has been shown that if (4.2) holds on any domain $\Omega_1 \subset \mathbb{R}^{d_1}$, then (4.2) also holds on any cylindrical domain $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^d$, $d = d_1 + d_2$. Here $\Omega_2 \subset \mathbb{R}^{d_2}$ is assumed to have finite d_2-Lebesque measure. For arbitrary open bounded sets holds

$$N^D_0(\Omega;\Lambda) \leq L^{\mathrm{cl}}_{0,d}\left(1+\frac{2}{d}\right)^{d/2}|\Omega|\,\Lambda^{d/2}$$

which is obtained from (2.5) and captures the currently best constants for a bound without any additionally assumptions on the shape of Ω.

One purpose of this chapter is to give an estimate on the counting functions $N^i_{\varepsilon_0}(\Omega;\Lambda)$ of $H^i_{\varepsilon_0}(\Omega)$ for $\varepsilon > 0$. As a first step we want to count the eigenvalues of the one dimensional operator on some bounded interval $[a, b]$. In the case $\varepsilon_0 = 0$ this can be done easily since the eigenfunctions of $H_0^i([a, b])$ are known to be

$$\varphi_k(x) = \begin{cases} \sin\left(\frac{k\pi}{b-a}(x-a)\right) & \text{if } i = D, \\ \cos\left(\frac{(k-1)\pi}{b-a}(x-a)\right) & \text{if } i = N, \end{cases}$$

$k \in \mathbb{N}$. That way the eigenvalues of $H_0^D([a, b])$ can be computed explicitely. By separation of variables one can also compute the eigenvalues on cubes $[a, b]^d \subset \mathbb{R}^d$ which then are given by

$$\lambda_k^i([a, b]^d, 0) = \frac{\pi^2}{(b-a)^2} \sum_{j=1}^{d} n_j^2$$

for $n_j \in \mathbb{N}$ if $i = D$, respectively $n_j \in \mathbb{N}_0$ if $i = N$. Thus, $N_0^i([a, b]^d, ; \Lambda)$ is related to the number of grid points within a d-dimensional sphere of radius $\Lambda^{1/2}(b-a)/\pi$ and can be estimated by the volume of the sphere.

If $\varepsilon_0 > 0$, any solution of

$$-\varphi''(x) + \varepsilon_0\, x\varphi(x) = \nu\, \varphi(x)$$

is given by

$$\varphi(x) = c_1 \operatorname{Ai}\left(\varepsilon_0^{-2/3}(\varepsilon_0 x - \nu)\right) + c_2 \operatorname{Bi}\left(\varepsilon_0^{-2/3}(\varepsilon_0 x - \nu)\right)$$

with constants $c_1, c_2 \in \mathbb{R}$. Inserting the Dirichlet conditions in a and b then yields

$$c_1 \operatorname{Ai}\left(\varepsilon_0^{-2/3}(\varepsilon_0 a - \nu)\right) + c_2 \operatorname{Bi}\left(\varepsilon_0^{-2/3}(\varepsilon_0 a - \nu)\right) = 0$$
$$c_1 \operatorname{Ai}\left(\varepsilon_0^{-2/3}(\varepsilon_0 b - \nu)\right) + c_2 \operatorname{Bi}\left(\varepsilon_0^{-2/3}(\varepsilon_0 b - \nu)\right) = 0$$

which, unfortunately, cannot be reduced any further, even in the simplest cases. Therefore other methods are needed.

In the next section we will follow the ideas of A. Pushnitski and V. Sloushch from [80] and reproduce an inequality from their proof of [80, Proposition 7.2].

The remaining parts are dedicated to an extension of the result to higher dimensions and to explore various other techniques which are associated to the counting function $N_0^D(\Omega;\Lambda)$.

4.1 The one dimensional case

Consider the operator $-\mathrm{d}^2/\mathrm{d}x^2+\varepsilon_0 x$ on $[\gamma,\infty)$, $\gamma\in\mathbb{R}$, with a Dirichlet condition in $x=\gamma$. If ν is an eigenvalue of this operator, the corresponding eigenfunction is given by

$$\varphi(x)=\mathrm{Ai}\left(\varepsilon_0^{-2/3}(\varepsilon_0 x-\nu)\right).$$

Thus, the eigenvalues of our operator can be computed by solving $\varphi(\gamma)=0$, respectively if $\varepsilon_0^{-2/3}(\varepsilon_0\gamma-\nu)$ is a zero of the Airy function. Let $(a_n)_{n\in\mathbb{N}}$ be the monotonic decreasing sequence of zeros of Ai, then $a_n=-f(v_n)$ where $v_n=3\pi(4n-1)/8$ and

$$f(z)=z^{2/3}+O(z^{-4/3})$$

as $z\to\infty$ (see [2, 10.4.94, 10.4.105]). From that we obtain

$$\nu_n=\varepsilon_0\gamma-\varepsilon_0^{2/3}a_n=\varepsilon_0\gamma+\varepsilon_0^{2/3}v_n^{2/3}+O(n^{-4/3})$$

for the corresponding sequence of eigenvalues $(\nu_n)_{n\in\mathbb{N}}$, respectively

$$|\nu_n-\varepsilon_0\gamma-\varepsilon_0^{2/3}v_n^{2/3}|\le C$$

for some constant $C>0$. Our next goal is to estimate the counting function $n_\gamma(0)$ for the eigenvalues ν_n below zero, i.e.

$$n_\gamma(0):=\#\{\nu_n\le 0\}.$$

If we replace ν_n by $\varepsilon_0\gamma + \varepsilon_0^{2/3} v_n^{2/3}$, the corresponding counting function differs only by a constant. With

$$\varepsilon_0\gamma + \varepsilon_0^{2/3} v_n^{2/3} \leq 0 \qquad \Leftrightarrow \qquad n \leq \frac{2}{3\pi}\varepsilon_0^{1/2}\gamma_-^{3/2} + \frac{1}{4}$$

in mind, we obtain that

$$\left| n_\gamma(0) - \frac{2}{3\pi}\varepsilon_0^{1/2}\gamma_-^{3/2} \right| \leq C \tag{4.3}$$

for $\gamma \in \mathbb{R}$ and some constant $C > 0$.

Theorem 4.1.1. *Let $N^i(\Lambda, [a,b])$ for $i = D$ be the counting function of all eigenvalues λ_n of $H^D_{\varepsilon_0}([a,b])$ below $\Lambda \in \mathbb{R}$, respectively the counting function for the eigenvalues μ_n of $H^N_{\varepsilon_0}([a,b])$ below Λ if $i = N$, then*

$$\left| N^i(\Lambda, [a,b]) - \frac{2}{3\pi}\frac{1}{\varepsilon_0}\left[(\Lambda - a\varepsilon_0)_+^{3/2} - (\Lambda - b\varepsilon_0)_+^{3/2}\right] \right| \leq C \tag{4.4}$$

for some constant $C > 0$.

Proof. If our interval is shifted by Λ/ε_0, the eigenvalues of $H^D_{\varepsilon_0}([a,b])$ or $H^N_{\varepsilon_0}([a,b])$ are shifted by Λ. That way our problem of counting the eigenvalues below Λ is equivalent to counting the eigenvalues below zero of the shifted operator $H^i_{\varepsilon_0}([a - \Lambda/\varepsilon_0, b - \Lambda/\varepsilon_0])$, $i = D, N$, more precisely it holds that

$$N^i(\Lambda, [a,b]) = N^i(0, [\tilde{a}, \tilde{b}]) \tag{4.5}$$

with $\tilde{a} = a - \Lambda\varepsilon_0$ and $\tilde{b} = b - \Lambda/\varepsilon_0$.

For $\gamma = \tilde{a}, \tilde{b}$ we denote by $H(\gamma)$ the operator $-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon_0 x$ on $[\gamma, \infty)$ with a Dirichlet boundary condition in γ from above and by $n_\gamma(0)$ its corresponding counting function of all eigenvalues below zero. Just the same let $\tilde{H}(\tilde{a})$ be the operator on $[\tilde{a}, \infty[$ with Dirichlet conditions in both points $\tilde{a} < \tilde{b}$ and $\tilde{n}_{\tilde{a}}(0)$ is corresponding counting function. Any eigenfunction of $H^D_{\varepsilon_0}([\tilde{a}, \tilde{b}])$ can be extended by zero onto $[\tilde{a}, \infty[$ which yields an eigenfunction of $\tilde{H}(\tilde{a})$. Conversely the restriction of any eigenfunction φ of $\tilde{H}(\tilde{a})$ onto $[\tilde{a}, \tilde{b}]$ is an eigenfunction of $H^D_{\varepsilon_0}([\tilde{a}, \tilde{b}])$ as long as $\varphi \not\equiv 0$ on $[\tilde{a}, \tilde{b}]$. In addition, the extension of any eigenfunction of $H(\tilde{b})$ onto $[\tilde{a}, \tilde{b}]$ by zero also yields an eigenfunction of $\tilde{H}(\tilde{a})$ but not of $H^D_{\varepsilon_0}([\tilde{a}, \tilde{b}])$. In summary it follows that

$$N^D(0, [\tilde{a}, \tilde{b}]) = \tilde{n}_{\tilde{a}}(0) - n_{\tilde{b}}(0).$$

If we compare the operators $\tilde{H}(\tilde{a})$ and $H(\tilde{a})$, they differ by an additional Dirichlet condition in $\tilde{b}$. Since applying new conditions for one dimensional Schrödinger operators is a perturbation of finite rank, their corresponding counting functions differ only by a constant. Therefore, it follows that

$$\left| N^D(0, [\tilde{a}, \tilde{b}]) - (n_{\tilde{a}}(0) - n_{\tilde{b}}(0)) \right| \leq C$$

for some constant $C > 0$. Since changing of boundary conditions for one dimensional Schrödinger operators is also a finite rank perturbation, we can replace $N^D(0, [\tilde{a}, \tilde{b}])$ by $N^N(0, [\tilde{a}, \tilde{b}])$ in the last inequality, and our result follows after inserting (4.3) and (4.5). □

Remark 4.1.1. Consider

$$\frac{2}{3\pi} \lim_{\varepsilon_0 \to 0} \frac{1}{\varepsilon_0} \left[(\Lambda - a\varepsilon_0)_+^{3/2} - (\Lambda - b\varepsilon_0)_+^{3/2} \right] = \frac{1}{\pi} \Lambda^{1/2} (b - a)$$

which matches the counting function for the Laplace operator on $[a, b]$ with Dirichlet or Neumann boundary conditions. Furthermore, from (4.4) follows that

$$\lim_{\Lambda \to \infty} \Lambda^{-1/2} N(\Lambda, [a, b]) = \frac{b - a}{\pi}$$

which corresponds to the one dimensional Weyl asymptotics.

4.2 Product domains

So far, the only non tiling domains for which (4.2) is known are products $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1 + d_2}$ where $\Omega_1 \subset \mathbb{R}^{d_1}$ is a tiling domain and $d_1 \geq 2$. This result is due to A. Laptev [62, Theorem 2.8]. The argument is based on separation of variables; due to the product structure of Ω the eigenvalues λ of $H^D_0(\Omega)$ can be written as $\lambda = \lambda_1 + \lambda_2$ where λ_1 is an eigenvalue of $H^D_0(\Omega_1)$

and λ_2 is an eigenvalue of $H_0^D(\Omega_2)$. Applying (4.2) for Ω_1, that way obtains

$$N_0^D(\Omega;\Lambda) = \sum_{\lambda_2} N_0^D(\Omega_1, \Lambda - \lambda_2) \leq L^{\mathrm{cl}}_{0,d_1} |\Omega_1| \sum_{\lambda_2} (\Lambda - \lambda_2)_+^{d_1/2}.$$

Since $d_1/2 > 1$, the Riesz means $\sum_{\lambda_2} (\Lambda - \lambda_2)_+^{d_1/2}$ on the right hand side are bounded by $L^{\mathrm{cl}}_{d_1/2,d_2} |\Omega_2| \, \Lambda^{(d_1+d_2)/2}$. Simplifying the constants yields (4.2) for $\Omega = \Omega_1 \times \Omega_2$.

In what follows we will use this idea in order to extend our result from Theorem 4.1.1 onto domains of the form $\Omega = [a,b] \times \omega$ with $\omega \subset \mathbb{R}^{d-1}$.

Theorem 4.2.1. *Let $\Omega = [a,b] \times \omega$ where ω is a bounded domain in $\mathbb{R}^{d-1}$ and $N(H^D_{\varepsilon_0}(\Omega), \Lambda)$ be the counting function for the eigenvalues of $H^D_{\varepsilon_0}(\Omega)$ below $\Lambda \in \mathbb{R}$, then*

$$N(H^D_{\varepsilon_0}(\Omega), \Lambda) \leq L^{cl}_{0,d} \left(1 + \frac{2}{d-1}\right)^{(d-1)/2} |\Omega| \, (\Lambda - b\varepsilon_0)_+^{d/2} + R(\Lambda, \Omega) \quad (4.6)$$

where

$$R(\Lambda,\Omega) = 1 + L^{cl}_{0,d-1} \left(1 + \frac{2}{d-1}\right)^{(d-1)/2} |\omega| \odot$$
$$\odot \left[(C + \varepsilon_0^{3/2}(b-a)^{3/2})(\Lambda - a\varepsilon_0)_+^{(d-1)/2} + \frac{1}{2}(\Lambda - b\varepsilon_0)_+^{(d-1)/2} \right]$$

and $C > 0$ is the constant from (4.4).

Proof. If $\Omega = [a,b] \times \omega$, then, by separation of variables, any eigenvalue λ of $H^D_{\varepsilon_0}(\Omega)$ can be written as the sum $\lambda = \nu_j + \tilde{\lambda}_k$. Thereby ν_j is an eigenvalue of the one dimensional Stark Laplacian $H^D_{\varepsilon_0}([a,b])$ and $\tilde{\lambda}_k$ is an eigenvalue of the classical Laplacian operator with Dirichlet boundary conditions an ω.

Thus, by Theorem 4.1.1 we obtain

$$
\begin{aligned}
N(H^D_{\varepsilon_0}(\Omega),\Lambda) &= \#\{(j,k)\in\mathbb{N}^2 \,:\, \nu_j+\tilde\lambda_k\le\Lambda\}\\
&= \sum_{k\in\mathbb{N}} N(H^D_{\varepsilon_0}([a,b]),\Lambda-\tilde\lambda_k)\\
&= \frac{2}{3\pi}\frac{1}{\varepsilon_0}\sum_{k\in\mathbb{N}}\Big((\Lambda-a\varepsilon_0-\tilde\lambda_k)_+^{3/2}-(\Lambda-b\varepsilon_0-\tilde\lambda_k)_+^{3/2}\Big)+D
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
|D| &\le C\cdot\#\{k\in\mathbb{N}\,:\,\tilde\lambda_k\le\Lambda-a\varepsilon_0\}\\
&\le 1+C\,L^{\mathrm{cl}}_{0,d-1}\left(1+\frac{2}{d-1}\right)^{(d-1)/2}|\omega|\,(\Lambda-a\varepsilon_0)_+^{(d-1)/2}
\end{aligned}
$$

and $C>0$ is the constant from (4.4). We split the sum in (4.7) into

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}}\Big((\Lambda-a\varepsilon_0-\tilde\lambda_k)_+^{3/2}-(\Lambda-b\varepsilon_0-\tilde\lambda_k)_+^{3/2}\Big)\\
&\quad=\sum_{\tilde\lambda_k<\Lambda-b\varepsilon_0}\Big((\Lambda-a\varepsilon_0-\tilde\lambda_k)^{3/2}-(\Lambda-b\varepsilon_0-\tilde\lambda_k)^{3/2}\Big)\\
&\qquad+\sum_{\Lambda-b\varepsilon_0<\tilde\lambda_k<\Lambda-a\varepsilon_0}(\Lambda-a\varepsilon_0-\tilde\lambda_k)^{3/2}.
\end{aligned}
$$

While the first sum on the right hand side will provide the leading order term, the second sum is of order $O((\Lambda-a\varepsilon_0)^{(d-1)/2})$ as $\Lambda\to\infty$. The latter can be seen from

$$
\begin{aligned}
&\sum_{\Lambda-b\varepsilon_0<\tilde\lambda_k<\Lambda-a\varepsilon_0}(\Lambda-a\varepsilon_0-\tilde\lambda_k)^{3/2}\\
&\quad\le\varepsilon_0^{3/2}(b-a)^{3/2}\left(\#\{\tilde\lambda_k\le\Lambda-a\varepsilon_0\}-\#\{\tilde\lambda_k\le\Lambda-b\varepsilon_0\}\right)\\
&\quad\le\varepsilon_0^{3/2}(b-a)^{3/2}L^{\mathrm{cl}}_{0,d-1}\left(1+\frac{2}{d-1}\right)^{(d-1)/2}|\omega|\,(\Lambda-a\varepsilon_0)_+^{(d-1)/2},
\end{aligned}
$$

whereas for the leading order we note that

$$
\begin{aligned}
&(\Lambda - a\varepsilon_0 - \tilde{\lambda}_k)^{3/2} - (\Lambda - b\varepsilon_0 - \tilde{\lambda}_k)^{3/2} \\
&\quad = (\Lambda - b\varepsilon_0 - \tilde{\lambda}_k + (b-a)\varepsilon_0)^{3/2} - (\Lambda - b\varepsilon_0 - \tilde{\lambda}_k)^{3/2} \\
&\quad \le \frac{3}{2}\,\varepsilon_0(b-a)\,(\Lambda - b\varepsilon_0 - \tilde{\lambda}_k)^{1/2} + \frac{3}{2}\,\varepsilon_0^{3/2}(b-a)^{3/2}
\end{aligned}
$$

if $\Lambda - b\varepsilon_0 - \tilde{\lambda}_k \ge 0$ and therefore

$$
\begin{aligned}
&\sum_{\Lambda - b\varepsilon_0 < \tilde{\lambda}_k < \Lambda - a\varepsilon_0} \Big((\Lambda - a\varepsilon_0 - \tilde{\lambda}_k)^{3/2} - (\Lambda - b\varepsilon_0 - \tilde{\lambda}_k)^{3/2} \Big) \\
&\quad \le \frac{3}{2}(b-a)\varepsilon_0 \sum_{k\in\mathbb{N}} (\Lambda - b\varepsilon_0 - \tilde{\lambda}_k)_+^{1/2} + O((\Lambda - b\varepsilon_0)^{(d-2)/2}) \qquad (4.8) \\
&\qquad + \frac{3}{2}\varepsilon_0^{3/2}(b-a)^{3/2} \cdot \#\{\tilde{\lambda}_k \le \Lambda - b\varepsilon_0\}.
\end{aligned}
$$

Collecting all of the lower order terms gives

$$
\begin{aligned}
R(\Lambda,\Omega) = 1 + (C + \varepsilon_0^{3/2}(b-a)^{3/2}) \cdot \#\{\tilde{\lambda}_k \le \Lambda - a\varepsilon_0\} \\
+ \frac{1}{2}\varepsilon_0^{3/2}(b-a)^{3/2} \cdot \#\{\tilde{\lambda}_k \le \Lambda - b\varepsilon_0\}.
\end{aligned}
$$

Thus, inserting (2.5) with $\gamma = 0$ for the counting functions in $R(\Lambda,\Omega)$ and with $\gamma = 1/2$ for the sum in (4.8) finally yields the desired result

$$
N(H_{\varepsilon_0}^D(\Omega),\Lambda) = \frac{1}{\pi} L_{1/2,d-1}^{\mathrm{cl}} \left(1 + \frac{2}{d-1}\right)^{(d-1)/2} |\Omega|\,(\Lambda - b\varepsilon_0)_+^{d/2} + R(\Lambda,\Omega)
$$

where

$$
\begin{aligned}
\frac{1}{\pi} L_{1/2,d-1}^{\mathrm{cl}} &= \frac{1}{\pi}\frac{\Gamma(3/2)}{(4\pi)^{(d-1)/2}\Gamma(3/2 + (d-1)/2)} \\
&= \frac{1}{2\sqrt{\pi}}\frac{\Gamma(1)}{(4\pi)^{(d-1)/2}\Gamma(1 + d/2)} \\
&= L_{0,d}^{\mathrm{cl}}.
\end{aligned}
$$

□

The limit of the right hand side in (4.6) as $\varepsilon_0 \to 0$ coincides, up to terms of lower order, with (2.5). Thus, the order of $\Lambda - b\varepsilon_0$ is the expected one from the asymptotics of the counting function for the classical Dirichlet Laplacian. In terms of a sharp constant, (4.6) contains the same additional factor $(1+2(d-1))^{(d-1)/2}$ as in (2.5). This is due to the usage of (2.5) in (4.8). If we additionally suppose that

$$\sum_{k\in\mathbb{N}} (\Lambda - \lambda_k(\omega;0))_+^{1/2} \leq L^{\mathrm{cl}}_{1/2,d-1}|\omega|\,\Lambda^{d/2}$$

holds for $\omega \subset \mathbb{R}^{d-1}$ from the decomposition $\Omega = [a,b]\times\omega$, which is for instance the case if ω is a tiling domain, then

$$N(H^D_{\varepsilon_0}(\Omega),\Lambda) \leq L^{\mathrm{cl}}_{0,d}|\Omega|\,(\Lambda - b\varepsilon_0)_+^{d/2} + O((\Lambda - b\varepsilon_0)^{(d-1)/2}).$$

In particular this is the case for any box $[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_d,b_d]\subset\mathbb{R}^d$. Instead of inserting a bound for the Riesz means $\sum_{k\in\mathbb{N}}(\Lambda-\lambda(\Omega;0))_+^{1/2}$ in (4.8), one can directly take the limit as $\Lambda\to\infty$ and use the Weyl asymptotics (2.2) for the eigenvalues of the classical Laplacian operator. Since both asymptotic formulas for the Stark Laplacian and for the classical Laplacian coincide in the case of Dirichlet and Neumann boundary conditions, one obtains

Corollary 4.2.1. *If $\Omega = [a,b]\times\omega$ for some domain $\omega\in\mathbb{R}^{d-1}$, then*

$$\lim_{\Lambda\to\infty}\Lambda^{-d/2}N(H^i_{\varepsilon_0}(\Omega),\Lambda) = L^{cl}_{0,d}\,|\Omega| \tag{4.9}$$

in both cases, $i = D$ and $i = N$.

In the following section we will extend this result to general $\Omega\in\mathbb{R}^d$ using the Dirichlet Neumann bracketing technique based on (4.9) for boxes in $\mathbb{R}^d$.

4.3 Applications of the variational principle

In this section we want to discuss several consequences and techniques linked with variational principles such as Glazman's lemma:

Theorem 4.3.1 ([13, 10.2, Theorem 3]). *Let A be a selfadjoint operator which is bounded from below and*

$$N(A, \Lambda) := \#\{\lambda_n(A) \leq \Lambda\}$$

be the counting function of the eigenvalues of A below Λ. Then

$$N(A, \Lambda) = \sup_F \dim F,$$

where $F \subset d[a]$ is a linear set that is contained in the domain of the corresponding quadratic form a of A such that

$$a[\phi, \phi] < \Lambda \|\phi\|^2$$

holds for all $\phi \in F \setminus \{0\}$.

Recall that the classical Lapalcian operator on a bounded domain $\Omega \subset \mathbb{R}^d$ is defined via the quadratic form

$$h_0[\phi] = \int_\Omega |\nabla\phi|^2 \, \mathrm{d}x \tag{4.10}$$

with $d[h_0] = \mathring{W}_2^1(\Omega)$ in the dirichlet case and with $d[h_0] = W_2^1(\Omega)$ in the case of Neumann boundary conditions. This gives rise to the operators $H_0^D(\Omega)$ and $H_0^N(\Omega)$. Since $\mathring{W}_2^1(\Omega) \subset W_2^1(\Omega)$ the corresponding operators satisfy

$$H_0^D(\Omega) \succ H_0^N(\Omega)$$

in the form sense, and by Glazman's lemma it follows that

$$N(H_0^D(\Omega), \Lambda) \leq N(H_0^N(\Omega), \lambda)$$

for all $\Lambda \in \mathbb{R}$. Roughly speaking, changing Dirichlet to Neumann boundary conditions will increase the counting function or equivalently reduce individual eigenvalues. The same holds if one introduces new Neumann boundary conditions in the inner part of a domain. Note that this reasoning is only based upon the inclusion of the form domains $d[h_0^D] \subset d[h_0^N]$ and not on the algebraic expression in (4.10). When replacing h_0 by

$$h_{\varepsilon_0}[\phi] = \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + \varepsilon_0 \int_\Omega x_1 |\phi|^2 \,\mathrm{d}x, \tag{4.11}$$

the reasoning can be repeated word by word and then yields

$$N(H^D_{\varepsilon_0}(\Omega), \Lambda) \leq N(H^N_{\varepsilon_0}(\Omega), \Lambda) \tag{4.12}$$

for each $\varepsilon_0 \geq 0$ and $\Lambda \in \mathbb{R}$. In the same way the principles for introduction of new Diriclet boundary conditions or enlarging a domain with Dirichlet boundary conditions can be extended from the case $\varepsilon_0 = 0$ to $\varepsilon_0 \geq 0$:

Let $\Omega \subset \mathbb{R}^d$ be open and $\Gamma \subset \Omega$ of Lebesgue measure zero such that $\tilde{\Omega} = \Omega \setminus \Gamma$ is open. Denote by $H^D_{\varepsilon_0}(\Omega)$ or $H^D_{\varepsilon_0}(\tilde{\Omega})$ the corresponding operators to the form (4.11) on $d[h^D_{\varepsilon_0}(\Omega)] = \mathring{W}^1_2(\Omega)$ and $d[h^D_{\varepsilon_0}(\tilde{\Omega})] = \mathring{W}^1_2(\tilde{\Omega})$ then $d[h^D_{\varepsilon_0}(\Omega)] \subset d[h^D_{\varepsilon_0}(\tilde{\Omega})]$ and therefore

$$N(H^D_{\varepsilon_0}(\tilde{\Omega}), \Lambda) \leq N(H^D_{\varepsilon_0}(\Omega), \Lambda) \tag{4.13}$$

for each $\varepsilon_0 \geq 0$ and $\Lambda \in \mathbb{R}$ if $\tilde{\Omega} = \Omega \setminus \Gamma$.

If $\Omega, \hat{\Omega} \subset \mathbb{R}^d$ are open, $\Omega \subset \hat{\Omega}$ and $H^D_{\varepsilon_0}(\Omega)$, $H^D_{\varepsilon_0}(\hat{\Omega})$ are the operators defined via the quadratic form (4.11) on $d[h^D_{\varepsilon_0}(\Omega)] = \mathring{W}^1_2(\Omega)$ and $d[h^D_{\varepsilon_0}(\hat{\Omega})] = \mathring{W}^1_2(\hat{\Omega})$, any test function from $d[h^D_{\varepsilon_0}(\Omega)]$ is, by extension with zero, also contained in $d[h^D_{\varepsilon_0}(\hat{\Omega})]$. Again by Glazman's lemma, it follows that

$$N(H^D_{\varepsilon_0}(\Omega), \Lambda) \leq N(H^D_{\varepsilon_0}(\hat{\Omega}), \Lambda) \tag{4.14}$$

for all ε_0 and $\Lambda \in \mathbb{R}$ if $\Omega \subset \hat{\Omega}$.

In the case of the classical Laplacian these three principles, (4.12), (4.13) and (4.14), give rise to the Weyl asymptotics (4.1) and are used in Pólya's proof of (4.2) for tiling domains. For the rest of this section we will take a closer look at these applications and extend the results to the case $\varepsilon_0 \geq 0$.

Theorem 4.3.2. *Let $\Omega \subset \mathbb{R}^d$ be an open Jordan measurable set, then for*

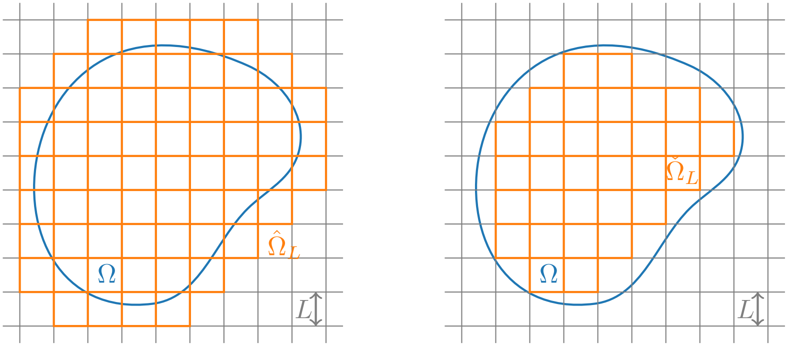


Figure 4.1: Covering and partial filling of Ω by $\hat{\Omega}_L$ and $\check{\Omega}_L$.

each $\varepsilon_0 \geq 0$ *holds*

$$\lim_{\Lambda\to\infty} \Lambda^{-d/2} N^D_{\varepsilon_0}(\Omega; \Lambda) = L^{cl}_{0,d}\, |\Omega|. \tag{4.15}$$

Proof. We have already seen in Corollary 4.2.1 that (4.15) holds for any box

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \subset \mathbb{R}^d.$$

Thus, it remains to extend (4.15) to arbitrary bounded domains $\Omega \subset \mathbb{R}^d$. This is done by a Dirichlet Neumann bracketing technique using the principles (4.12), (4.13), (4.14). We fix $L > 0$. With each vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$ we associate the box

$$B_\alpha(L) := \{x \in \mathbb{R}^d \, : \, L\alpha_j < x_j < L(\alpha_j + 1),\ j = 1, \ldots, d\}.$$

Let $A = \{\alpha \in \mathbb{Z}^d \, : \, B_\alpha(L) \cap \Omega \neq \emptyset\}$, thus, A is the set of vectors whose boxes cover Ω, and

$$\hat{\Omega}_L := \operatorname{int}\left(\bigcup_{\alpha \in A} B_\alpha(L) \right)$$

is a superset of Ω (cf. Figure 4.1). Conversely, let $B = \{\alpha \in \mathbb{Z}^d \, : \, B_\alpha(L) \subset$

$\Omega\}$, then

$$\check{\Omega}_L := \operatorname{int}\left(\bigcup_{\alpha\in B} B_\alpha(L)\right)$$

is contained in Ω. By extending Ω to $\hat{\Omega}_L$ with Dirichlet boundary conditions, changing at $\partial\hat{\Omega}_L$ Dirichlet to Neumann boundary conditions and introducing new Neumann boundary conditions at all $\partial B_\alpha(L)$, $\alpha \in A$, we obtain

$$\begin{aligned} N(H^D_{\varepsilon_0}(\Omega), \Lambda) &\overset{(4.14)}{\leq} N(H^D_{\varepsilon_0}(\hat{\Omega}_L), \Lambda) \overset{(4.12)}{\leq} N(H^N_{\varepsilon_0}(\hat{\Omega}_L), \Lambda) \\ &\overset{(4.12)}{\leq} N(\textstyle\bigoplus_{\alpha\in A} H^N_{\varepsilon_0}(B_\alpha(L)), \Lambda) = \sum_{\alpha\in A} N(H^N_{\varepsilon_0}(B_\alpha(L)), \Lambda). \end{aligned}$$

Although $N(H^N_{\varepsilon_0}(B_\alpha(L)), \Lambda)$ depends on the location of $B_\alpha(L)$ along the x_1-axis, according to Corollary 4.2.1 the limit of $\Lambda^{-d/2}N(H^N_{\varepsilon_0}(B_\alpha(L)), \Lambda)$ as $\Lambda \to \infty$ is for all boxes $B_\alpha(L)$ the same. Thus

$$\limsup_{\Lambda\to\infty} \Lambda^{-d/2} N(H^D_{\varepsilon_0}(\Omega), \Lambda) \leq L^{\mathrm{cl}}_{0,d} \sum_{\alpha\in A} |B_\alpha(L)| = L^{\mathrm{cl}}_{0,d}\, |\hat{\Omega}_L|.$$

Conversely, restricting Ω to $\check{\Omega}_L$ and introducing new Dirichlet boundary conditions on $\partial B_\alpha(L)$ for $\alpha \in B$ yields

$$\begin{aligned} N(H^D_{\varepsilon_0}(\Omega), \Lambda) &\overset{(4.14)}{\geq} N(H^D_{\varepsilon_0}(\check{\Omega}_L), \Lambda) \\ &\overset{(4.13)}{\geq} N(\textstyle\bigoplus_{\alpha\in B} H^D_{\varepsilon_0}(B_\alpha(L)), \Lambda) = \sum_{\alpha\in B} N(H^D_{\varepsilon_0}(B_\alpha(L)), \Lambda). \end{aligned}$$

As above, we proceed to the limit as $\Lambda \to \infty$ and obtain from Corollary 4.2.1 that

$$\liminf_{\Lambda\to\infty} \Lambda^{-d/2} N(H^D_{\varepsilon_0}(\Omega), \Lambda) \geq L^{\mathrm{cl}}_{0,d-1} \sum_{\alpha\in B} |B_\alpha(L)| = L^{\mathrm{cl}}_{0,d}\, |\check{\Omega}_L|.$$

Since Ω is Jordan measurable, $|\hat{\Omega}_L|$ and $|\check{\Omega}_L|$ converge to $|\Omega|$ as $L \to 0$ which closes the proof. $\square$

4.4 An inequality between individual eigenvalues

Let $(\lambda_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$ be the increasing sequence of Dirichlet eigenvalues of the Stark Laplacian $H^D_{\varepsilon_0}(\Omega)$ and $(\mu_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$ the increasing sequence of the Neumann eigenvalues of $H^N_{\varepsilon_0}(\Omega)$ with their multiplicities taken into account. By $N(H^D_{\varepsilon_0}(\Omega),\Lambda)$, respectively $N(H^N_{\varepsilon_0}(\Omega),\Lambda)$, we denote their counting functions of all eigenvalues which do not exceed a bound $\Lambda\in\mathbb{R}$. In the previous section we have seen that $N(H^D_{\varepsilon_0}(\Omega),\Lambda)$ is bounded from above by $N(H^N_{\varepsilon_0}(\Omega),\Lambda)$ for each Λ, see (4.12). For the individual eigenvalues this means

$$\mu_j(\Omega;\varepsilon_0)\le\lambda_j(\Omega;\varepsilon_0).$$

For the classical Laplacian operator for $\varepsilon_0=0$ this inequality was improved in various ways. Note that $\mu_1(\Omega;0)=0$ and $\lambda_1(\Omega;0)>0$ hold for each $\Omega\subset\mathbb{R}^d$, thus, the question arises whether the first non trivial Neumann eigenvalue is still below $\lambda_1(\Omega;0)$. The minimal value of $\lambda_1(\Omega;0)$ is attained if Ω is a disk, respectively

$$\Lambda_1(\Omega;0)\ge\lambda_1(B_r;0)=\frac{1}{R^2}j^2_{d/2-1,1}$$

where $B_R=\{x\in\mathbb{R}^d\,:\,|x|<1\}$ such that $|\Omega|=|B_R|$ and $j_{n,k}$ denotes the k-th zero of the Bessel function J_n. This is known as the *Rayleigh-Faber-Krahn inequality* and will be treated in the second part of this work in more detail. In [50] E. T. Kornhauser and I. Stakgold conjectured that $\mu_2(\Omega;0)$ is maximized by the disk if $d=2$ and

$$\mu_2(\Omega;0)\le\frac{1}{R^2}{j'}^2_{0,1}$$

where $j'_{n,k}$ denotes the k-th zero of the derivative J'_n of the Bessel function. From these bounds it follows that

$$\mu_2(\Omega;0)\le\left(\frac{j'_{0,1}}{j_{0,1}}\right)^2\lambda_1(\Omega;0). \tag{4.16}$$

Remarking on [50], G. Pólya showed this assertion with a slightly weaker constant in [78]. The first rigorous proof of (4.16) was presented by G. Szegö in [86] under the assumption that $\Omega \subset \mathbb{R}^2$ is bounded by an analytic curve. Involving higher eigenvalues L. E. Payne, [75], proved that

$$\mu_{j+2}(\Omega; 0) \leq \mu_j(\Omega; 0) \tag{4.17}$$

for $d = 2$ if $\Omega \subset \mathbb{R}^2$ is convex and its boundary is twice continously differentiable. For such domains the curvature of the boundary is non-negative. In order to generalize these results to higher dimensions $d > 2$, one has to deal with $d-1$ principal curvatures at each point of $\partial\Omega$. In [66] H. A. Levine and H. F. Weinberger present multiple extensions of (4.17), the most important being

$$\mu_{j+d}(\Omega; 0) \leq \lambda_j(\Omega; 0)$$

which holds for bounded convex domains $\Omega \subset \mathbb{R}^d$ or

$$\mu_{j+1}(\Omega; 0) \leq \lambda_j(\Omega; 0)$$

if $\partial\Omega$ is of class $C^{2,\alpha}$ for some $\alpha \in]0,1[$ and has non-negative mean curvature. The latter was already contained in [9]. For a more general class of domains, the first result is due to L. Friedlander; in [35] it is shown that

$$\mu_{j+1}(\Omega; 0) \leq \lambda_j(\Omega; 0) \tag{4.18}$$

holds for all bounded domains $\Omega \subset \mathbb{R}^d$ with C^1-boundary. Much later (4.18) was shown by N. Filonov in [32] without any assumptions on the smoothness of the domains boundary except of the compactness of the embedding $W_2^1(\Omega) \subset L^2(\Omega)$ which ensures that the spectra of $H_{\varepsilon_0}^D(\Omega)$ and $H_{\varepsilon_0}^D(\Omega)$ are purely discrete. Filonov's proof is fairly simple and relies only on Glazman's lemma and upon the fact that $\phi(x) = \mathrm{e}^{\mathrm{i}\kappa x}$ solves $-\Delta\phi = \kappa^2\phi$ on $L^2(\mathbb{R}^d)$. For the remaining part of this section we want to extend the result of [32] to the case $\varepsilon > 0$. Beforehand, we need to proof a technical lemma showing that no eigenfunction can satisfy the Neumann and Dirichlet boundary conditions simultaneously:

Lemma 4.4.1. *For each* $\lambda \in \mathbb{R}$ *holds*

$$\mathring{W}_2^1(\Omega) \cap \ker\left(H_{\varepsilon_0}^N(\Omega) - \Lambda\right) = \{0\}.$$

Proof. Let $u \in \mathring{W}_2^1(\Omega) \cap \ker\left(H_{\varepsilon_0}^N(\Omega) - \Lambda\right)$ and $v \in W_2^1(\mathbb{R}^d)$ be the continuation of u by zero onto $\mathbb{R}^d$, i.e.

$$v(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Since

$$\begin{aligned} \int_{\mathbb{R}^d} (\nabla v \cdot \overline{\nabla \phi} + \varepsilon_0 x_1\, v\overline{\phi})\,\mathrm{d}x &= \int_\Omega (\nabla u \cdot \overline{\nabla \phi} + \varepsilon_0 x_1 u\overline{\phi})\,\mathrm{d}x \\ &= \int_\Omega ((-\Delta u)\overline{\phi} + \varepsilon_0 x_1\, u\overline{\phi})\,\mathrm{d}x \\ &= \Lambda \int_\Omega u\overline{\phi}\,\mathrm{d}x = \Lambda \int_{\mathbb{R}^d} v\overline{\phi}\,\mathrm{d}x \end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$, it follows that $(-\Delta + \varepsilon_0 x_1)v = \Lambda v$ on $\mathbb{R}^d$. Thus,

$$v(x) = c \cdot \mathrm{Ai}\,(\varepsilon_0^{-2/3}(\varepsilon_0 x_1 + \omega_1))\,\mathrm{e}^{\mathrm{i}\omega_\perp x_\perp}$$

for some constant $c \in \mathbb{R}$ and $\omega = (\omega_1, \omega_\perp)$ such that $\omega_1 + |\omega_\perp|^2 = \Lambda$, respectively, $v(x) = 0$, since this is the only solution in $L^2(\mathbb{R}^d)$. □

Theorem 4.4.1. *Let* $d \geq 2$ *and* $\Omega \subset \mathbb{R}^d$*, such that the spectra of* $H_{\varepsilon_0}^D(\Omega)$ *and* $H_{\varepsilon_0}^N(\Omega)$ *are dicrete. Denote by* $(\lambda_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$*, respectively by* $(\mu_j(\Omega;\varepsilon_0))_{j\in\mathbb{N}}$ *their increasing sequences of eigenvalues, then*

$$\mu_{j+1}(\Omega;\varepsilon_0) \leq \lambda_j(\Omega;\varepsilon_0) \tag{4.19}$$

for all $j \in \mathbb{N}$ *where* $\lambda_j(\Omega;\varepsilon_0)$ *are the Dirichlet eigenvalues and* $\mu_j(\Omega;\varepsilon_0)$ *are the Neumann eigenvalues.*

Proof. By Glazman's lemma (Theorem 4.3.1) it holds that

$$N(H^D_{\varepsilon_0}(\Omega), \Lambda) = \sup_F \dim F$$

where the supremum is taken over all $F \subset \mathring{W}^1_2(\Omega)$ such that

$$\int_\Omega (|\nabla\phi|^2 + \varepsilon_0 x_1 \, |\phi|^2) \, \mathrm{d}x \leq \Lambda \int_\Omega |\phi|^2 \, \mathrm{d}x$$

holds for all $\phi \in F$. Fix $\Lambda \in \mathbb{R}$ and choose F such that $N(H^D_{\varepsilon_0}(\Omega), \Lambda) = \dim F$. According to lemma 4.4.1 above, $F \dot{+} \ker(H^N_{\varepsilon_0}(\Omega) - \Lambda)$ is a direct sum. For each $\omega = (\omega_1, \omega_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$ such that $\omega_1 + |\omega_\perp|^2 = \Lambda$, consider the function

$$\varphi_\omega(x) := \mathrm{Ai}\,(\varepsilon_0^{-2/3}(\varepsilon_0 x_1 + \omega_1))\, \mathrm{e}^{\mathrm{i}\omega_\perp x_\perp}.$$

From the orthogonality of Ai and the exponential function it follows that the family of all φ_ω, where $\omega_1 + |\omega_\perp|^2 = \Lambda$, is linearly independet. Thus, there exists some ω such that φ_ω is not contained in $F \dot{+} \ker(H^N_{\varepsilon_0}(\Omega) - \Lambda)$ (note that $F \dot{+} \ker(H^N_{\varepsilon_0}(\Omega) - \Lambda)$ has finite dimension). Let

$$G := F \dot{+} \ker(H^N_{\varepsilon_0}(\Omega) - \Lambda) \dot{+} \{c \cdot \varphi_\omega \; : \; c \in \mathbb{R}\}.$$

If $u + v + c \cdot \varphi_\omega \in F \dot{+} \ker(H^N_{\varepsilon_0}(\Omega) - \Lambda) \dot{+} \{c \cdot \varphi_\omega \; : \; c \in \mathbb{R}\}$, then

$$\begin{aligned}
&\int_\Omega (|\nabla(u + v + c \cdot \varphi_\omega)|^2 + \varepsilon_0 x_1 \, |u + v + c \cdot \varphi_\omega|^2) \, \mathrm{d}x \\
&= \int_\Omega (|\nabla u|^2 + |\nabla v|^2 + |\nabla(c \cdot \varphi_\omega)|^2 + \varepsilon_0 x_1 \, |u|^2 + \varepsilon_0 x_1 \, |v|^2 + \varepsilon_0 x_1 \, |c \cdot \varphi_\omega|^2) \, \mathrm{d}x \\
&\quad + 2 \,\mathrm{Re} \int_\Omega (\nabla v \cdot \overline{\nabla(u + c \cdot \varphi_\omega)} + \nabla(c \cdot \varphi_\omega) \cdot \overline{\nabla u} \, \mathrm{d}x \\
&\quad + 2 \,\mathrm{Re} \int_\Omega + \varepsilon_0 x_1 \, v \, \overline{(u + c \cdot \varphi_\omega)} + \varepsilon_0 x_1 \, c \cdot \varphi_\omega \overline{u}) \, \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
\leq & \Lambda \int_\Omega (|u|^2 + |v|^2 + |c \cdot \varphi_\omega|^2)\,\mathrm{d}x \\
& + 2\,\mathrm{Re}\int_\Omega ((-\Delta v)\overline{(u + c \cdot \varphi_\omega)} - \Delta(c \cdot \varphi_\omega)\,\overline{u} + \varepsilon_0 x_1\, v\,\overline{(u + c \cdot \varphi_\omega)} + \varepsilon_0 x_1\, c \cdot \varphi_\omega \overline{u})\,\mathrm{d}x \\
= & \Lambda \int_\Omega (|u|^2 + |v|^2 + |c \cdot \varphi_\omega|^2)\,\mathrm{d}x + 2\Lambda\,\mathrm{Re}\int_\Omega (v\,\overline{(u + c \cdot \varphi_\omega)} + c \cdot \varphi_\omega\,\overline{u})\,\mathrm{d}x \\
= & \Lambda \int_\Omega |u + v + c \cdot \varphi_\omega|^2\,\mathrm{d}x.
\end{aligned}
$$

Again by using Glazman's lemma, we obtain

$$
N(H^N_{\varepsilon_0}(\Omega), \Lambda) \geq \dim G = N(H^D_{\varepsilon_0}(\Omega), \Lambda) + \dim\ker(H^N_{\varepsilon_0}(\Omega) - \Lambda) + 1,
$$

and choosing $\Lambda = \lambda_j(\Omega; \varepsilon_0)$ yields

$$
\begin{aligned}
N(H^N_{\varepsilon_0}(\Omega), \lambda_j(\Omega; \varepsilon_0)) - \dim\ker(H^N_{\varepsilon_0}(\Omega) - \lambda_j(\Omega; \varepsilon_0)) \\
\geq N(H^D_{\varepsilon_0}(\Omega), \lambda_j(\Omega; \varepsilon_0)) + 1 = j + 1.
\end{aligned}
$$

From that follows the assertion (4.19). □

Part II

Steps towards the Faber-Krahn-Inequality

Chapter 5

On the existence of a minimizing Domain

We consider the operator

$$H_{\varepsilon_0}^D(\Omega) = -\Delta + \varepsilon_0 x_1, \tag{5.1}$$

$\varepsilon_0 \geq 0$ on $L^2(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions as defined in Section 1.3. It is well known that the lowest eigenvalue $\lambda_1(\Omega; \varepsilon_0)$ of this operator is simple and the corresponding eigenfunction has constant sign on Ω, such that it can be choosen to be positive. A popular question is the problem of finding the domain Ω which minimalizes the lowest eigenvalue. In the case $\varepsilon_0 = 0$ Lord Rayleigh conjectured that the minimalizing domain in $d = 2$ should be the disc $B^2 = \{x \in \mathbb{R}^d : |x| \leq 1\}$, i.e.

$$\lambda_1(B^2; 0) = \min\{\lambda_1(\Omega; 0) : \Omega \subset \mathbb{R}^2 \text{ open}, |\Omega| = \omega_2\}$$

where $\omega_2 = \pi^2$ is the volume of B^2, due to some explicite calculations close to the boundary of Ω and physical evidence. The physical or rather musical interpretation of this would be that among all drums with the same volume the circular drum has the lowest voice. Almost 30 years later Rayleigh's conjecture was independetly proven by G. Faber [31] and E. Krahn [56]. The corresponding result for $d \geq 2$ was proven by E. Krahn in [57]. Both results are often written in the form of an inequality

$$\lambda_1(\Omega;\varepsilon_0) \geq C_d^{2/d} j_{d/2-1,1} |\Omega|^{-2/d},$$

where $C_d = \pi^{d/2}/\Gamma(d/2+1)$ and $j_{k,1}$ is the first zero of the Bessel function J_k. In the literature this inequality is known as the *Rayleigh-Faber-Krahn inequality.* Its proof is based upon a variational characterization of the first eigenvalue, namely

$$\lambda_1(\Omega;0) = \inf_{u \in \mathring{W}_2^1(\Omega), u \not\equiv 0} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega |u|^2 \, \mathrm{d}x},$$

and the fact that the symmetrical decreasing rearrangement of a function decreases the quotient on the right hand side. Let Ω^* be the Schwarz symmetrization of Ω and u^* the corresponding rearrangement of $u \in \mathring{W}_2^1(\Omega)$, see [49, (2.6); 49, (2.1)], then

$$\int_{\Omega^*} |u^*|^2 \, \mathrm{d}x = \int_\Omega |u|^2 \, \mathrm{d}x$$

and

$$\int_{\Omega^*} |\nabla u^*|^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

see [49, (C)]. The latter is due to G. Pólya and G. Szegő and was first proven in [79].

In the case for $\varepsilon_0 > 0$ we cannot rely on Schwarz symmetrization since we cannot expect the minimizing domain to be symmetrical along the direction of the Stark potential, in our case the x_1-direction. Nevertheless to make use of some rearrangement techniques we introduce *Steiner symmetrization*: Consider the decomposition $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ with respect to the last component. For a bounded domain $\Omega \subset \mathbb{R}^d$ and $(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}$ we introduce

$$D(x) = \{y \in \mathbb{R} \,:\, (x, y) \in \Omega\}.$$

Note that $D(x)$ might be empty if the line (x, y) for $y \in \mathbb{R}$ and fixed $x \in \mathbb{R}^{d-1}$

does not intersect Ω at all. The length of the part of the line which lies inside of Ω can be computed by

$$l(D(x)) = \int_{\mathbb{R}} \chi_{D(x)}(x)\,\mathrm{d}x$$

where $\chi_{D(x)}$ is the characteristic function of $D(x)$. Let

$$D'(x) := \begin{cases} \{(x,y) \,:\, 0 \le |y| \le l(D(x))/2\} & \text{if } D(x) \ne \emptyset \\ \emptyset & \text{if } D(x) = \emptyset \end{cases}$$

and

$$\Omega' := \bigcup_{D'(x) \ne \emptyset} D'(x),$$

then Ω' is clearly symmetric along the y-direction. For a Lipschitz continuous function $u : \mathbb{R}^d \to \mathbb{R}_0^+$ with compact support $\operatorname{supp} u \subset \overline{\Omega}$ we define the corresponding rearrangement by

$$u'(x) := \sup\{c \in \mathbb{R} \,:\, x \in \Omega'_c\} \tag{5.2}$$

where Ω'_c are the symmetrizations of the level sets

$$\Omega_c := \{x \in \overline{\Omega} \,:\, u(x) \ge c\}.$$

This process can be repeated to any other component of $\mathbb{R}^d$. Let $\Omega^* \subset \mathbb{R} \times \mathbb{R}^{d-1}$ be the symmetrized domain with respect to any component perpendicular to the Stark field, respectively the x_1-direction and u^* the corresponding rearrangement of a compactly supported Lipschitz function $u : \Omega \to \mathbb{R}_0^+$. As in the case of Schwarz symmetrization it holds that

$$\int_{\Omega^*} F(u^*)\,\mathrm{d}x = \int_{\Omega} F(u)\,\mathrm{d}x$$

for any continous function $F : \mathbb{R}_0^+ \to \mathbb{R}$ and

$$\int_{\Omega^*} |\nabla u^*|^2\,\mathrm{d}x \le \int_{\Omega} |\nabla u|^2\,\mathrm{d}x$$

since both relations hold in any step of the symmetrization process, see [49, (C)]. Taking into account that

$$\lambda_1(\Omega,\varepsilon_0) = \inf_{u\in \mathring{W}_2^1(\Omega), u\not\equiv 0} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \varepsilon_0 \int_\Omega x_1 |u|^2 \,\mathrm{d}x}{\int_\Omega |u|^2 \,\mathrm{d}x}$$

we have shown that

Lemma 5.0.1. *If there exists a minimizing domain Ω for the first Dirichlet eigenvalue $\lambda_1(\Omega;\varepsilon_0)$ of the Stark Laplacian* (5.1) *it is symmetric along any direction perpendicular to the x_1-axis.*

As in the case of the classical Dirichlet Laplacian, the problem of minimizing the domain for the lowest eigenvalue makes only sense in the class of all domains with the same volume. Since the spectrum of the Stark Laplacian depends upon the position of the domain along the direction of the Stark potential, respectively $\lambda_1(\Omega + h;\varepsilon_0) \to -\infty$ as $h \to -\infty$ where

$$\Omega + h = \{(x+h, y) \in \mathbb{R}\times\mathbb{R}^{d-1} \,:\, (x,y) \in \Omega\}$$

(cf. Lemma 1.3.1), we additionally have to fix the center of mass for example in the origin such that

$$\int_\Omega x_1 \,\mathrm{d}x = 0.$$

Example 5.0.1. Let $B_r(a) \subset \mathbb{R}^d$ be the ball with radius $r > 0$ and center in $(a, 0, \ldots, 0) \in \mathbb{R}^d$. Consider $\Omega_a := B_r(-a) \cup B_r(a)$ whose center of mass is fixed in the origin for any $a \in \mathbb{R}$. If u is an eigenfunction for the lowest eigenvalue $\lambda_1(B_r(0);\varepsilon_0)$ of the Stark Laplacian on $B_r(0)$, then $u(\cdot + a\varepsilon_0)$ is an eigenfunction for the lowest eigenvalue $\lambda_1(B_r(-a);\varepsilon_0) = \lambda_1(B_r(0);\varepsilon_0) - a\varepsilon_0$ of the operator on $B_r(-a)$. By continuation with zero we obtain an eigenfunction of the Stark operator on Ω_a and it follows that

$$\lambda_1(\Omega_a;\varepsilon_0) \le \lambda_1(B_r(0);\varepsilon_0) - a\varepsilon_0 \to -\infty$$

as $a \to \infty$.

One may argue that the domain from Example 5.0.1 is not connected.

But, as shown in the following example, considering connected domains does not change the situation substantially.

Example 5.0.2. We connect both components of Ω_a from Example 5.0.1 by a thin tube $[-a, a] \times [-\delta, \delta]^{d-1}$ and consider

$$\Omega_{a,\delta} := \Omega_a \cup [-a, a] \times [-\delta, \delta]^{d-1}.$$

Since $\Omega_{a,\delta} \to \Omega_a$ as $\delta \to 0$ in respect to the Hausdorff-distance, it follows that

$$\lambda_1(\Omega_{a,\delta}; \varepsilon_0) \to \lambda_1(\Omega_a; \varepsilon_0)$$

as $\delta \to 0$ (see details below in Section 5.1). As $\lambda_1(\Omega_a; \varepsilon_0)$ for $a \to -\infty$ is not bounded from below, the same holds for $\lambda_1(\Omega_{a,\delta}; \varepsilon_0)$ if $\delta > 0$ is chosen sufficiently small.

If we additionally assume our domain $\Omega \subset \mathbb{R}^d$ to be convex, we might prove the existence of a minimizing domain. Therefore we first restrict ourselves to boxes in $\mathbb{R}^d$. Since

$$\lambda_1([-a, a]; \varepsilon_0) \leq -a\varepsilon_0 + \varepsilon_0^{2/3}\zeta + \frac{C}{a}$$

for some constant $C > 0$, see Lemma 5.2.1, there exists a box in $d = 2$ with optimal aspect ratio which minimizes the first eigenvalue, see Lemma 5.2.2. In $d = 3$ an optimal box exists only for small $\varepsilon_0 > 0$. If ε_0 exceeds a certain bound, we will disprove the existence of an optimal box in Lemma 5.2.2. Finally the existence of a box with optimal aspect ratio for the first eigenvalue of the Stark Laplacian leads to an optimal domain among the class of convex sets with fixed volume and center of mass.

Theorem 5.0.1. *Let* $V > 0$ *and* $\mathcal{C}_V := \{\Omega \subset \mathbb{R}^d : \Omega \text{ convex}, |\Omega| = V, \int_\Omega x \,\mathrm{d}x = 0\}$. *If either*

- $d = 2$ *or*

- $d = 3$ *and* $0 < \varepsilon_0 < \pi^2/2$, *then there exists some domain* $\Omega^*_{\varepsilon_0} \in \mathcal{C}_V$ *such that*

$$\lambda_1(\Omega^*_{\varepsilon_0};\varepsilon_0) = \inf_{\Omega \in \mathcal{C}_V} \lambda_1(\Omega;\varepsilon_0).$$

5.1 Hausdorff convergence and Mosco convergence for the Stark Laplacian

Before proceeding with our proof of Theorem 5.0.1 we want to summarize the results which allow us to obtain convergence for the sequence of the first eigenvalues for varying domains which converge in respect to the Hausdorff distance.

Let $(\Omega_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence of convex sets that are contained in a ball B_r of radius $r > 0$. Denote by $\lambda_n = \lambda_1(\Omega_n;\varepsilon_0)$ the first Dirichlet eigenvalue of $H^D_{\varepsilon_0}(\Omega)$ on $L^2(\Omega_n)$. In order to prove convergence of $(\lambda_n)_{n\in\mathbb{N}}$ to $\lambda_1(\Omega;\varepsilon_0)$ as $\Omega_n \to \Omega$ in some sense, it is sufficient to prove strong convergence of the sequence of resolvent operators

$$R_n = i_{\Omega_n}(\lambda I + H^D_{\varepsilon_0}(\Omega))^{-1} r_{\Omega_n}$$

to $R(\lambda) = i_\Omega(\lambda I + H^D_{\varepsilon_0}(\Omega))^{-1} r_\Omega$ for some $\lambda \in \mathbb{C}$, see [23, Theorem 4.3.1 and Corrolar 4.3.2], where $i_\Omega : \mathring{W}^1_2(\Omega) \to W^1_2(\mathbb{R}^d)$ denotes the trivial extension and $r_\Omega : L^2(\mathbb{R}^d) \to L^2(\Omega)$ the restriction operator. The latter is less dependent on the operator than on the sequence $(\Omega_n)_{n\in\mathbb{N}}$ and can be connected to geometric properties of the domains for a wide range of operators, see [23, 41] for a detailed survey.

If we restrict ourselves to convex sets in $\mathbb{R}^d$ the natural topology on those sets is induced by the *Hausdorff metric*

Definition 5.1.1. • Let $K_1, K_2 \subset \mathbb{R}^d$ be compact sets, then

$$d^H(K_1,K_2) := \max\left\{\sup_{x\in K_1} d(x,K_2), \sup_{x\in K_2} d(x,K_1)\right\},$$

where

$$d(x,K) := \inf_{y\in K} |x-y|.$$

- Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets that are contained in a bounded domain B. We say that $(K_n)_{n\in\mathbb{N}}$ *converges in the sense of Hausdorff to* K if
$$\lim_{n\to\infty} d^H(K_n, K) = 0.$$

- Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets that are contained in some compact domain B, then
$$d_H(\Omega_1, \Omega_2) := d^H(B \setminus \Omega_1, B \setminus \Omega_2),$$
respectively a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of open sets that is uniformly bounded by a compact domain B, i.e. $\Omega_n \subset B$ holds for each $n \in \mathbb{N}$, *converges to* Ω *in the sense of Hausdorff* if,
$$\lim_{n\to\infty} d_H(\Omega_n, \Omega) = 0.$$

An almost complete survey of this notion of Hausdorff convergence can be found in [41]. For our purposes it is sufficient that Hausdorff convergence preserves convexity. If $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of open sets, uniformly bounded by a compact set $B \subset \mathbb{R}^d$ and converging to Ω in the sense of Hausdorff, then Ω is also convex, see [41, (2.19)].

A common phenomenon when dealing with Hausdorff convergent sequences of open sets is collapsing at the limit, which is to say, that Hausdorff convergence does not preserve the volume. For arbitrary sequences of open sets the volume is only a lower semicontinous function, see [41, Proposition 2.2.23]. Preservation of the volume is related to L^1-convergence of the characteristic functions $\chi_{\Omega_n} \to \chi_\Omega$ which requires assumptions on the boundaries of Ω_n. Therefore we want to introduce the ϵ-cone property.

Definition 5.1.2. For an open set $\Omega \subset \mathbb{R}^d$ and $\epsilon > 0$ we consider the cones
$$C(y, \xi, \epsilon) := \{z \in \mathbb{R}^d \, : \, \langle z - y, \xi \rangle \geq |z - y| \cos(\epsilon) \, \wedge \, 0 < |z - y| \leq \epsilon\}$$
with vertex $y \in \mathbb{R}^d$ along the direction $\xi \in \mathbb{R}^d$, $|\xi| = 1$. An open set has the ϵ*-cone property* if for all $x \in \partial\Omega$ there is $\xi_x \in \mathbb{R}^d$, $|\xi_x| = 1$ such that $C(y, \xi_x, \epsilon) \subset \bar{\Omega} \cap B_\epsilon(x)$.

The ϵ-cone property is also related to Lipschitz-continuity of the boundary, see [41, Theorem 2.4.7]. If Ω is convex, then Ω has the ϵ-cone property for any

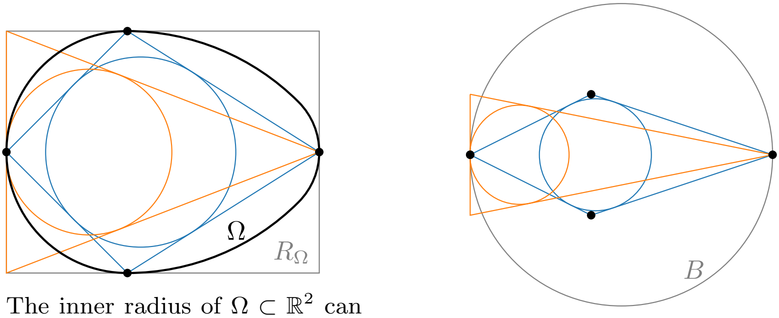
$$\epsilon \leq \min\left\{\frac{\mathrm{Ri}\,(\Omega)}{2}, \arcsin\left(\frac{\mathrm{Ri}\,(\Omega)}{2\sup_{x\in\partial\Omega}|x_0 - x|}\right)\right\} \tag{5.3}$$

where $\mathrm{Ri}\,(\Omega)$ is the inner radius of Ω, i.e. the radius of the largest open ball which is fully contained in Ω, and $x_0 \in \Omega$ its center, see [41, Proposition 2.4.4]. When considering Hausdorff convergent sequences $(\Omega_n)_{n\in\mathbb{N}}$ of open, convex sets we need an uniform bound of $\epsilon > 0$ such that any Ω_n shares the ϵ-cone property uniform for any $n \in \mathbb{N}$. Since convex domains may have arbitrary sharp corners, we have to take into account that in our case all Ω_n are contained in a compact domain B and have fixed volume $|\Omega_n| = V$. In that case the inner radii of the Ω_n are bounded from below and the Ω_n satify the ϵ-cone property simultanously for any ϵ below some bound. In view of Theorem 5.0.1 and Lemma 5.2.2 we will give a proof only in the case of symmetric domains in dimension $d \in \{2, 3\}$.

Lemma 5.1.1. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open convex sets in $\mathbb{R}^d$ that are contained in some compact domain $B \subset \mathbb{R}^d$, $d \in \{2, 3\}$ and that are symmetric along any direction perpendicular to the x_1-axis. Suppose that $|\Omega_n| = V$ for each $n \in \mathbb{N}$. Then the inner radii of the Ω_n are uniformly bounded from below and therefore have the ϵ-cone property for any ϵ below some bound independent of $n \in \mathbb{N}$.*

Proof. In view of (5.3), it suffices to show that $\mathrm{Ri}\,(\Omega_n)$ is uniformly bounded from below. We begin our proof in the case $d = 2$. Fix some domain $\Omega \subset \mathbb{R}^2$ and let $R_\Omega \subset \mathbb{R}^2$ be the minimal bounding box, touching Ω in at least four points on the boundary of Ω (cf. Figure 5.1a). Since Ω is symmetric along the x_2-axis, these touching points are also aligned symmetrically and, by connecting them, we arrive at a kite whose inner radius does not exceed the inner radius of Ω. Since $|\Omega_n| = V$ for all $n \in \mathbb{N}$, the smallest possible inner radius of the kite is attained when the kite degenerates to a triangle whose base and vertex lies on the boundary of B (cf. Figure 5.1b).

In the case $d = 3$ the domains $\Omega \subset \mathbb{R}^3$ are rotational symmetric with respect to the x_1-axis, and we can cut Ω along the x_1-x_2-plane. Let R_Ω be



(a) The inner radius of $\Omega \subset \mathbb{R}^2$ can be estimated from below by the inner radius of the blue kite, respectively, the inner radius of the red triangle.

(b) The thinnest possible kite, that fits into $B \subset \mathbb{R}^2$ with volume of $V/2$ touches B at the boundary.

Figure 5.1: Visualization of the construction in the proof of Lemma 5.1.1. For a domain $\Omega \subset \mathbb{R}^2$ we consider the minimal bounding box R_Ω touching Ω in at least four points. Connecting these points leads to the blue kite. The kite with minimal inner radius is the degenerated one, respectively, the red triangle.

the minimal bounding box of the cut as in the case $d = 2$. The inner radius of the cut is bounded from below by the inner radius of the corresponding kite, respectively the degenerated kite. Rotating around the x_1-axis does not change the inner radii at all such that the inner radius of the cone is a lower bound for the inner radius of Ω. □

Consider a bounded sequence $(\Omega_n)_{n\in\mathbb{N}}$ of domains. In our proof of Theorem 5.0.1 we will rely on the existence of a Hausdorf-convergent subsequence of $(\Omega_n)_{n\in\mathbb{N}}$. This is guaranteed by the *Blaschke selection theorem*, see [38, Theorem 6.3] for instance. The following variant additionally ensures the preservation of volume when proceeding to the limit:

Theorem 5.1.1 ([41, Theorem 2.4.10]). *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open domains in $\mathbb{R}^d$ which are uniformly bounded by some compact set $B \subset \mathbb{R}^d$ and uniformly satisfy the ϵ-cone condition. Then there is an open set $\Omega \subset B$ which satisfies the ϵ-cone condition and a subsequence $(\Omega_{n_k})_{k\in\mathbb{N}}$ which converges to Ω in the sense of Hausdorff. Furthermore the closures $(\bar{\Omega}_{n_k})_{k\in\mathbb{N}}$ converge to $\bar{\Omega}$ in the sense of Hausdorff and $\chi_{\Omega_{n_k}} \to \chi_\Omega$ in $L^p(B)$ for each $p < \infty$.*

Remark 5.1.1. Convergence of $\chi_{\Omega_n} \to \chi_\Omega$ in $L^2(B)$ also ensures that the center of masses of the Ω_n converge to the center of mass of Ω since

$$\left| \int_{\Omega_n} x \, \mathrm{d}x - \int_\Omega x \, \mathrm{d}x \right| = \left| \int_B x \cdot [\chi_{\Omega_n}(x) - \chi_\Omega(x)] \right| \\ \leq \|\chi_{\Omega_n} - \chi_\Omega\|_{L^2(B)} \cdot \left[\int_B x^2 \, \mathrm{d}x \right]^{1/2} .$$

The Hausdorff-topology again can be related to various other notions of convergence between open sets. In fact, the relation between the Hausdorff topology on convex sets and convergence of solutions to variational equations on those sets was studied by U. Mosco in [73] in a much more general setting. Thereby two conditions appear naturally which are today known under the term *Mosco convergence*:

Definition 5.1.3. The sequence of spaces $\mathring{W}_2^1(\Omega_n)$ converges to $\mathring{W}_2^1(\Omega)$ as $n \to \infty$ *in the sense of Mosco* if the following two properties hold:

- For all $u \in \mathring{W}_2^1(\Omega)$ exists a sequence $(u_n)_{n\in\mathbb{N}}$ satisfying $u_n \in \mathring{W}_2^1(\Omega_n)$ for each $n \in \mathbb{N}$ such that $u_n \to u$ in $W_2^1(\mathbb{R}^d)$.
- If $(u_n)_{n\in\mathbb{N}}$ is a sequence of functions satisfying $u_n \in \mathring{W}_2^1(\Omega_n)$ for each $n \in \mathbb{N}$ that converges weakly to a function $u \in W_2^1(\mathbb{R}^d)$, then $u \in \mathring{W}_2^1(\Omega)$.

An equivalent statement can be formulated in terms of the unique solutions $u = u_\Omega^f$ of the Dirichlet problem $-\Delta u = f$ on some open set Ω for given $f \in (\mathring{W}_2^1(\Omega))'$:

Lemma 5.1.2 ([41, Proposition 3.5.5])**.** *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^d$ that are contained in some bounded domain $B \subset \mathbb{R}^d$ and $\Omega \subset B$. The sequence of spaces $\mathring{W}_2^1(\Omega_n)$ converges to $\mathring{W}_2^1(\Omega)$ as $n \to \infty$ in the sense of Mosco if and only if $u_{\Omega_n}^f \to u_\Omega^f$ in $\mathring{W}_2^1(B)$ for all $f \in (\mathring{W}_2^1(B))'$.*

In the literature convergence of $u_{\Omega_n}^f \to u_\Omega^f$ is usually reffered to as γ-*convergence*. Under the constraints of Theorem 5.1.1 it can be related to Hausdorff-convergence:

Lemma 5.1.3 ([41, Theorem 3.4.12])**.** *Let* $(\Omega_n)_{n\in\mathbb{N}}$ *be a sequence of open domains in* $\mathbb{R}^d$ *that are uniformly bounded by some compact set* $B \subset \mathbb{R}^d$ *and uniformly satisfy the* ϵ*-cone condition. If* $\Omega_n \to \Omega$ *in the sense of Hausdorff, then* $u^f_{\Omega_n} \to u^f_\Omega$ *in* $\mathring{W}^1_2(B)$ *for all* $f \in (\mathring{W}^1_2(B))'$*.*

Convergence of the spaces $\mathring{W}^1_2(\Omega_n)$ in the sense of Mosco is again an equivalent condition to strong convergence of the corresponding resolvent operators as long as the sequence $(\Omega_n)_{n\in\mathbb{N}}$ is uniformly bounded by some open bounded set $B_r \subset \mathbb{R}^d$. If $(\Omega_n)_{n\in\mathbb{N}}$ is unbounded, Mosco convergence of $\mathring{W}^1_2(\Omega_n)$ still ensures convergence of the operators $R_n(\lambda)$ in a weaker sense, see [23, Theorem 5.2.4].

Lemma 5.1.4 ([23])**.** *Let* $(\Omega_n)_{n\in\mathbb{N}}$ *be a sequence of open sets in* $\mathbb{R}^d$ *that are contained in some bounded domain* B_r *and* λ *in the resolvent set of* (5.1) *on* $L^2(\Omega)$*. If* $\mathring{W}^1_2(\Omega_n)$ *converges to* $\mathring{W}^1_2(\Omega)$ *as* $n \to \infty$ *in the sense of Mosco, then* λ *is contained in the resolvent set of* (5.1) *on* $L^2(\Omega_n)$ *for large enough* n *and* $R_n(\lambda) \to R(\lambda)$ *in* $\mathcal{L}(H^{-1}(B_r), L^2(B_r))$*.*

Remark 5.1.2. The Lemma is a slight modification of [23, Corollary 5.2.5] where it is shown that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(B_r), L^q(B_r))$ for all $q \in [1, 2d/(d-2))$ if $\mathring{W}^1_2(B_r)$ converges to $\mathring{W}^1_2(B_r)$ in the sense of Mosco. The proof there relies on the compactness of the embedding $\mathring{W}^1_2(B_r) \hookrightarrow L^q(B_r)$ for all $q \in [1, 2d/(d-2))$ (Rellich-Kondrachov theorem). If we make use of the compactness of the embedding $\mathring{W}^1_2(B_r) \hookrightarrow L^2(B_r)$ (Rellich's theorem) the claim follows as statet in Lemma 5.1.4 above.

Although strong convergence of the resolvent operators does not ensure convergence of the complete spectrum of the corresponding operators, convergence holds for any compact part of the spectrum which can be separated by a simple, rectifiable curve from the remaining parts of the spectra. The basic framework for this idea is already set up in [48] and statet more explicitly in [23].

Lemma 5.1.5 ([23, Corollary 4.3.2])**.** *Suppose that* $R_n(\lambda) \to R(\lambda)$ *for some* $\lambda \in \mathbb{C}$ *and* Σ *is a compact subset of the spectrum of* (5.1) *on* $L^2(\Omega)$ *which could be enclosed by a simple, rectifiable curve* Γ *seperating it from the* *remaining spectrum. Then for large enough n this curve Γ is seperating a subset Σ_n of the spectrum of (5.1) on $L^2(\Omega_n)$. Let*

$$P_n = -\frac{1}{2\pi \mathrm{i}} \int_\Gamma R_n(\lambda)\,\mathrm{d}\lambda, \qquad P = -\frac{1}{2\pi \mathrm{i}} \int_\Gamma R(\lambda)\,\mathrm{d}\lambda$$

be the corresponding spectral projections, then for large enough n their images coincide and $P_n \to P$.

5.2 Proof of Theorem 5.0.1

As a first step we want to observe if there is a box $[-a, a] \times [-a_1, a_1] \times \cdots \times [-a_{d-1}, a_{d-1}] \subset \mathbb{R}^d$ with optimal aspect ratio which minimizes the first Dirichlet eigenvalue of the Stark operator when the volumes of the boxes are fixed. Therefore we need an estimate on the first eigenvalue on an interval $[-a, a]$. Since the Stark potential $\varepsilon_0 x$ is bounded on $[-a, a]$ a bound on $\lambda_1([-a, a]; \varepsilon_0)$ is given in terms of the lowest eigenvalue for the Dirichlet Laplacian, i.e.

$$-\varepsilon_0 a + \frac{\pi^2}{4a^2} \leq \lambda_1([-a, a]; \varepsilon_0) \leq \varepsilon_0 a + \frac{\pi^2}{4a^2}$$

holds for each $a > 0$ and $\varepsilon_0 \geq 0$. While this inequality reflects the behaviour of $\lambda_1([-a, a]; \varepsilon_0)$ on small intervals it fails to give the correct asymptotics as $a \to \infty$.

Lemma 5.2.1. *Let $-\zeta$ be the first zero of the Airy function, then*

$$\lambda_1([-a, a]; \varepsilon_0) \leq -a\varepsilon_0 + \varepsilon_0^{2/3}\zeta + \frac{C}{a}$$

holds for some constant $C > 0$ as long as $-a\varepsilon_0 + \varepsilon_0^{2/3}\zeta$ is negative, in particular if $a > \varepsilon_0^{-1/3}\zeta$ for fixed $\varepsilon_0 > 0$.

Proof. Consider the test function

$$\phi(x) = \mathrm{Ai}\left(\varepsilon_0^{1/3}(x + a) - \zeta\right) \chi(x/a)$$

where χ is a smooth, monotonic decreasing, cutt-off function satisfying $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Then $\phi \in \mathring{W}_2^1([-a, a])$ is non-negative, satisfies the Dirichlet boundary conditions in $\pm a$ and

$$\begin{aligned}
&\left\langle (-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon_0 x)\, \phi(x), \phi(x) \right\rangle \\
&\leq \left\langle (-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon_0 x)\ \mathrm{Ai}\left(\varepsilon_0^{1/3}(x+a) - \zeta\right), \mathrm{Ai}\left(\varepsilon_0^{1/3}(x+a) - \zeta\right) \right\rangle \\
&\qquad + \frac{1}{a} \max_{x \in [0,1]} \chi'(x) \int_0^\infty \mathrm{Ai}^2(x)\, \mathrm{d}x \\
&= \left(-a\varepsilon_0 + \varepsilon_0^{2/3}\zeta\right) \left\| \mathrm{Ai}\left(\varepsilon_0^{1/3}(x+a) - \zeta\right) \right\|^2 + \frac{1}{a} \max_{x \in [0,1]} \chi'(x) \int_0^\infty \mathrm{Ai}^2(x)\, \mathrm{d}x \\
&\leq \left(-\varepsilon_0 a + \varepsilon_0^{2/3}\zeta\right) \|\phi\|^2 + \frac{C}{a}.
\end{aligned}$$

The estimate now follows from the variational principle and the fact

$$\|\phi\|^2 \geq \int_{-\zeta}^0 \mathrm{Ai}^2(x)\, \mathrm{d}x.$$

□

By separation of variables the lowest eigenvalue $\lambda_1(R_d; \varepsilon_0)$ of $-\Delta + \varepsilon_0 x_1$ on a box $R_d = [-a, a] \times [-a_1, a_1] \times \cdots \times [-a_{d-1}, a_{d-1}]$ is given by

$$\lambda_1(R_d; \varepsilon_0) = \lambda_1([-a, a]; \varepsilon_0) + \sum_{j=1}^{d-1} \frac{\pi^2}{4a_j^2}.$$

The sum on the right hand side becomes minimal if $a_1 = \cdots = a_{d-1}$ such that we can restrict ourselves without loss of generality to the case $R_d = [-a, a] \times [-b, b]^{d-1}$ where $V = 2^d a b^{d-1}$ is fixed.

Lemma 5.2.2. *Let $V > 0$ be fixed and $\mathcal{R}_{V,d}$ be the set of all boxes $R_d = [-a, a] \times [-b, b]^{d-1}$ satisfying $V = 2^d a b^{d-1}$. If either*

- *$d = 2$ with $\varepsilon_0 > 0$ or*
- *$d = 3$ with $0 < \varepsilon_0 < \pi^2/2$*

there exists some $R^* \in \mathcal{R}_{V,d}$ *such that*

$$\lambda_1(R^*;\varepsilon_0) = \inf_{R\in\mathcal{R}_{V,d}} \lambda_1(R;\varepsilon_0).$$

If either

- $d = 3$ *with* $\varepsilon_0 > \pi^2/2$ *or*
- $d > 3$ *with* $\varepsilon_0 > 0$

then $\lambda_1(R;\varepsilon_0)$ *is not bounded from below for* $R \in \mathcal{R}_{V,d}$.

Proof. Let $R_a = [-a,a] \times [-a^{1/(1-d)}, a^{1/(1-d)}]$, then the mapping $a \mapsto \lambda_1(R_a;\varepsilon_0)$ is continuous. Since

$$\begin{aligned}\lambda_1(R_a;\varepsilon_0) &= \lambda_1([-a,a];\varepsilon_0) + \frac{\pi^2(d-1)}{4}\, a^{2/(d-1)}\\ &\geq -\varepsilon_0 a + \frac{\pi^2}{4a^2} + \frac{\pi^2(d-1)}{4}\, a^{2/(d-1)}\end{aligned}$$

for each $\varepsilon_0 > 0$ and $a > 0$, it follows that $\lambda_1(R_a;\varepsilon_0) \to +\infty$ as $a \to 0$ for all $d \geq 1$. If $a \to 0$, then $\lambda_1(R_a;\varepsilon_0) \to +\infty$ if $d = 2$ or $d = 3$ and $\varepsilon_0 < \pi^2/2$. In this case $a \mapsto \lambda_1(R_a;\varepsilon_0)$ attains its minimum for some $a \in (0,\infty)$.

Let $d > 3$ or $d = 3$ and $\varepsilon_0 > \pi^2/2$, then $a \mapsto \lambda_1(R;\varepsilon_0)$ cannot be bounded from below since Lemma 5.2.1 states that

$$\lambda_1(R_a;\varepsilon_0) \leq -\varepsilon_0 a + \varepsilon_0^{2/3}\zeta + \frac{C}{a} + \frac{\pi^2(d-1)}{4}\, a^{2/(d-1)}$$

for some uniform constant $C > 0$ and $-\zeta$ being the first zero of the Airy function. □

Remark 5.2.1. Lemma 5.2.2 also suggests that the minimalizing domain for the first Dirichlet eigenvalue of the Stark Laplacian cannot be to thin. For a domain $\Omega \subset \mathbb{R}^d$ consider the one-parametric sequence of domains

$$T_a\Omega := \{(a^{1-d}x_1, ax_2, \ldots, ax_d) \,:\, (x_1,\ldots,x_d) \in \Omega\},$$

where all $T_a\Omega$, $a > 0$ share the same volume and center of mass. If R is a minimizing bounding box of Ω, that is the smallest axis-parallel box containing Ω, each $T_a R$ is the minimal bounding box of $T_a\Omega$, and since $T_a\Omega \subset T_a R$ for each $a > 0$, it follows that

$$\lambda_1(T_\alpha\Omega; \varepsilon_0) \geq \lambda_1(T_\alpha R; \varepsilon_0)$$

where $\lambda_1(T_\alpha R; \varepsilon_0) \to \infty$ as $\alpha \to 0$ or $\alpha \to \infty$ if $d = 2$ or $d = 3$ and $0 < \varepsilon_0 < \pi^2/2$. In that sense a domain which is to thin might be replaced by a thicker domain with smaller first eigenvalue.

Proof of Theorem 5.0.1. In order to apply Theorem 5.1.1, we first want to show that there is an uniform bound on $\lambda_1(\Omega; \varepsilon_0)$ for $\Omega \in \mathcal{C}_V$. Let therefore $R_\Omega \subset \mathbb{R}^d$ be a minimal bounding box for $\Omega \in \mathcal{C}_V$ touching Ω in at least $2d$ points of the boundary of R_Ω. Since Ω can be considered to be symmetrical along any component perpendicular to the x_1-direction, these touching points are also arranged in a symmetric way such that their convex hull consists of two hyperpyramides, each of volume $|R_\Omega|/(2\,d!)$. The convex hull of the touching points is again a subset of Ω, thus

$$|R_\Omega| \leq d! \cdot |\Omega| = d! \cdot V$$

and it follows that $\lambda_1(\Omega; \varepsilon_0) \geq \lambda_1(R_\Omega; \varepsilon_0)$ for the first Dirichlet eigenvalues of our operator on Ω, respectively R_Ω. If we replace R_Ω by a box R^*_Ω with the same volume as R_Ω but optimal aspect ratio (see Lemma 5.2.2), we obtain $\lambda_1(\Omega; \varepsilon_0) \geq \lambda_1(R^*_\Omega; \varepsilon_0)$. These various boxes for $\Omega \in \mathcal{C}_V$ might be shifted to each other but are otherwise independent of Ω. By shifting them to the left such that their right border is alining with the origin, we obtain an uniform bound on $\lambda_1(\Omega; \varepsilon_0)$, i.e.

$$\lambda_1(\Omega; \varepsilon_0) \geq \lambda_1(R^*; \varepsilon_0),$$

where R^* is a box of the type $[a_1, 0] \times [a_2, b_2] \times \ldots [a_d, b_d]$ with $|R^*| = d! \cdot V$ and optimal aspect ratio.

Denote by $\mathcal{U}_r$ the set of all convex domains that contain a ball of radius $r > 0$ and let $(\Omega^r_j)_{j\in\mathbb{N}}$ be a minimizing sequence for $\inf_{\Omega\in\mathcal{C}_V\cap\mathcal{U}_r} \lambda_1(\Omega; \varepsilon_0)$. Since $\sup_{\Omega\in\mathcal{U}_r} \operatorname{diam}\Omega < \infty$, all $\Omega^r_j \in \mathcal{C}_V \cap \mathcal{U}_r$ are contained in some bounded region. From Theorem 5.1.1 it follows that there is a subsequence of $(\Omega_j^r)_{j\in\mathbb{N}}$ converging to a convex set $\Omega_r^* \in \mathcal{C}_V$ with respect to the Hausdorff distance. Thus, from the results in Section 5.1 we obtain

$$\lambda_1(\Omega_r^*;\varepsilon_0) = \inf_{\Omega\in\mathcal{C}_V\cap\mathcal{U}_r} \lambda_1(\Omega;\varepsilon_0).$$

If $r_1 \leq r_2$, then $\mathcal{U}_{r_1} \subset \mathcal{U}_{r_2}$ and $\lambda_1(\Omega_{r_1}^*;\varepsilon_0) \geq \lambda_1(\Omega_{r_2}^*;\varepsilon_0)$ and $\lambda_1(\Omega_r^*;\varepsilon_0)$ is decreasing in r. According to Remark 5.2.1, the minimizing domain of $\lambda_1(\Omega_r^*;\varepsilon_0)$ cannot be too thin such that $\lambda_1(\Omega_r^*;\varepsilon_0)$ becomes constant for sufficiently small $r > 0$, i.e. there is a $\Omega^* \in \mathcal{C}_V$ satisfying

$$\lambda_1(\Omega^*;\varepsilon_0) = \inf_{r>0} \lambda_1(\Omega_r^*;\varepsilon_0) = \inf_{\Omega\in\mathcal{C}_V} \lambda_1(\Omega;\varepsilon_0).$$

$\square$

Chapter 6

Numerical experiments

As discussed in the previous chapter, the problem of finding the domain $\Omega \subset \mathbb{R}^2$ with given area $|\Omega| > 0$ which minimizes the lowest eigenvalue $\lambda_1(\Omega)$ of the classical Laplacian operator with Dirichlet boundary conditions is solved by the circle [31, 56]. The solution to the same question concerning the second eigenvalue $\lambda_2(\Omega)$ was attributed to P. Szegö in [76] but appeared in a more or less explicite form in Krahn's paper [57], see [40] for more references. Here the solution is the union of two disjoint discs in $\mathbb{R}^2$ with identical radii since the restriction of any eigenfunction ϕ for $\lambda_2(\Omega)$ to its nodal domains Ω_+, Ω_-, i.e. the domains where the sign of ϕ does not change, is an eigenfunction to $\lambda_1(\Omega_+)$ and $\lambda_1(\Omega_-)$. The latter is minimized if Ω_+, Ω_- are discs, see [40, 4.1.1] for a more detailed presentation of the proof. Thus, the minimzing domain of $\lambda_2(\Omega)$ is not connected. Moreover, a consequence of this fact is that the problem of finding

$$\inf \{\lambda_2(\Omega) \,:\, \Omega \subset \mathbb{R}^2 \text{ open and connected with } |\Omega| = c\}$$

for given $c > 0$ has no solution. Indeed, connecting two identical disjoint discs B_1, B_2 by a thin tube of width $\varepsilon > 0$ as in example 5.0.2 above yields a sequence of domains Ω_ε for which $|\Omega_\varepsilon|\lambda_2(\Omega_\varepsilon) \to |B_1 \cup B_2|\lambda_2(B_1 \cup B_2)$ as $\varepsilon \to 0$. Thus, the Ω_ε form a minimizing sequence with a limitting domain that violates the connectivity condition. Nevertheless, under the additional assumption of convexity a solution to the problem of finding

$$\inf \{\lambda_2(\Omega) \,:\, \Omega \subset \mathbb{R}^2 \text{ open and convex with } |\Omega| = c\} \tag{6.1}$$

does exist [40, Theorem 2.4.1], but the challenge remains to find it. A natural conjecture would be that its solution $\check{\Omega}$ is the convex hull of two disjoint discs, usually referred to as a stadium. This conjecture was stated by B. A. Troesch in 1973 who supported it with numerical experiments, see [88]. It remained for quiet some time until it was disproved by A. Henrot and E. Oudet in [42] showing that the boundary of the minimizer $\check{\Omega}$ of $\lambda_2(\Omega)$ cannot contain arcs of a circle. Regarding of the geometry of $\check{\Omega}$, they give the following properties:

Theorem 6.0.1 ([42, Theorem 2])**.** *The solution $\check{\Omega}$ to the problem of finding* (6.1) *for a given $c > 0$ is*

- *at least of class C^1 and at most of class C^2*
- *the boundary $\partial\check{\Omega}$ contains exactly two straight lines if $\check{\Omega}$ is of class $C^{1,1}$.*

Appart from that, nothing is known about $\check{\Omega}$ and numerical methods and shape optimization algorithms come in handy in order to gain some principal ideas of how $\check{\Omega}$ might look like. In the example of minimizing $\lambda_2(\Omega)$ numerical experiments in [74] revealed that $\check{\Omega}$ still looks very much like a stadium with two axes of symmetry. But the question if $\check{\Omega}$ is actually symmetric remains open.
Other examples where numerical experiments lead to new conjectures on the solutions of related shape optimization problems are given in [7], where the problem of minimizing the first Dirichlet eigenvalue $\lambda_1(\Omega)$ among all polygons in $\mathbb{R}^2$ is treatened, or in [6], concerning the gap conjecture, i.e. bounding the difference $\lambda_1(\Omega) - \lambda_2(\Omega)$ (spectral gap) from below in terms of various quantities related to the domain as for instance the perimeter.
Besides minimizing $\lambda_2(\Omega)$ one is interested in finding the optimal domains for $\lambda_j(\Omega)$ if $j \geq 3$. Here, the existence of a minimizer $\check{\Omega}_j$ is guaranteed by the Buttazzo-Dal Maso-Theorem among the quasi open sets that are contained in some open region $D \subset \mathbb{R}^d$, see [19], which also includes the definition of

a quasi open set. We note that the restriction $\Omega \subset D$ can be dropped as shown by D. Bucur in [17, Theorem 3]. Just like in the case of $\lambda_2(\Omega)$, there are numerical studys investigating possible candidates for the minimizing domains, beginning with E. Oudet in [74] to the works of P. Autunes and P. Freitas [8] who confirmed most of Oudet's candidates but also improved a few of them.
Many of the optimizations we mentioned above are based on gradient descent methods which try to find local minima of the desired functions usually referred as cost functions. Let $\Omega \mapsto J(\Omega)$ be this cost function for $\Omega \subset \mathbb{R}^d$. In order to minimize J, we disturb Ω by a vector field V on $\mathbb{R}^d$ and consider

$$\Omega_t := \{x + t \cdot V \; : \; x \in \Omega\}.$$

The derivative of J in Ω along the direction V is then given by

$$D_V J(\Omega) = \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

If $\check{\Omega} \subset \mathbb{R}^d$ is a minimizer of J, then $D_V J(\check{\Omega}) = 0$ for all vector fields V. In case of $J(\Omega) = |\Omega|$ or $J(\Omega) = \lambda_j(\Omega)$ there are more useful representations for this derivative, the latter known as *Hadamard's formula*, see Theorem 6.2.2. The various approaches mostly differ in the methods used to evaluate J or the representation of Ω and the corresponding vector field. Whereas in [74] the eigenvalue function $\lambda_2(\Omega)$ is evaluated with the help of a finite element algorithm, C. Alves and P. Autunes presented in [5] a more efficient method for the evaluation of Dirichlet Laplacian eigenvalues based on a formulation as a boundary value problem. Therefore $\Omega \subset \mathbb{R}^d$ is considered as a domain enclosed by a simple boundary curve $\partial\Omega$ which is represented by truncated Fourier series. Unfortunately, this method is not available for the Stark Laplacian. Thus, in what follows we will approximate Ω by polygons with large counts of vertices and stick to the finite element approach which will be described in the following section. In Section 6.2 we will derive a Hadamard-type formula for the Stark Laplacian and use it in the remaining part of this chapter in order to find candidates for the minimizing domains of $\lambda_1(\Omega; \varepsilon_0)$ for various regimes of the coupling strength $\varepsilon_0 > 0$.

6.1 Evaluation of eigenvalues

Throughout our experiments we will use the finite element method (FEM) to approximate the eigenvalues and eigenfunctions of our operator on certain domains $\Omega \subset \mathbb{R}^2$. A standard finite element approximation is obtained by formulating the eigenvalue problem

$$(-\Delta + \varepsilon_0 x_1)u = \lambda(\Omega, \varepsilon_0)u, \qquad u \in \mathring{W}_2^1(\Omega)$$

in a variational way. Therefore let

$$\begin{aligned} a_{\varepsilon_0} &: \mathring{W}_2^1(\Omega) \times \mathring{W}_2^1(\Omega) \to \mathbb{C}, \\ &(u, v) \mapsto a_{\varepsilon_0}[u, v] = \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)} + \varepsilon_0 x_1\, u(x)\overline{v(x)}\, \mathrm{d}x \end{aligned}$$

be the corresponding sesquilinear form. If $u \in \mathring{W}_2^1(\Omega)$ is an eigenfunction with eigenvalue λ it follows that

$$a_{\varepsilon_0}[u, v] = \lambda\, (u, v)_{L^2(\Omega)}$$

for all $v \in \mathring{W}_2^1(\Omega)$. Instead of searching for u and λ we want to find approximate solutions by solving an appropriate finite dimensional problem: By choosing a finite dimensional subspace $V_h \subset \mathring{W}_2^1(\Omega)$ we search for u_h, λ_h such that

$$a_{\varepsilon_0}[u_h, v_h] = \lambda_h\, (u_h, v_h)_{L^2(\Omega)} \tag{6.2}$$

for all $v_h \in V_h$, where $u_h \in V_h$ is the finite element approximation of the eigenfunction u. A more detailed introduction into the subject and results concerning the convergence $\lambda_h \to \lambda$ can be found in [14]. Since $\dim(V_h) = N$ is finite, the latter can be expressed as a problem from linear algebra. To emphasize this, let $(\phi_j)_{j=1,\dots,N}$ a basis of V_h and

$$u_h = \sum_{i=1}^{N} \alpha_i \phi_i.$$

From (6.2) we then obtain that

$$\sum_{i=1}^{N} \alpha_j \, a_{\varepsilon_0}[\phi_i, \phi_j] = \lambda \sum_{i=1}^{N} \alpha_j \, (\phi_i, \phi_j)_{L^2(\Omega)}$$

for all ϕ_j, respectively in a vectorized form

$$A\alpha = \lambda M\alpha \tag{6.3}$$

if $\alpha = (\alpha_i)_{i=1,\ldots,N}$, $A = (a_{\varepsilon_0}[\phi_i, \phi_j])_{i,j=1,\ldots,N}$ and $M = ((\phi_i, \phi_j)_{L^2(\Omega)})_{i,j=1,\ldots,N}$. For the remaining problem a wide range of methods in various software packages is available. Apart from solving this equation, the assembling of the matrizes A and M is the most expensive part of the calculation. In the later sections we will use methods in which the eigenvalues have to be evaluated multiple times on slightly changing domains. So, choice of a good FEM approximation space is crucial where we have to compromise between achieving descent accuracy while letting computation times not to grow to large. On the one hand the basis (ϕ_i) of V_h has to be simple enough to allow assembling of A and M in a small time, on the other hand the FEM space needs to exhaust the space $\mathring{W}_2^1(\Omega)$ best possible for increasing dimension of V_h to gain accurency.

Many choices of V_h depend on a meshing of Ω, that is a decomposition of Ω in a union of non overlapping triangular cells. The space V_h can then be choosen as the space of continuous functions that vanish on the boundary $\partial\Omega$ and are of degree one in each triangle cell. That way any element of V_h is uniquely described by the set of values at inner nodes (the vertices of the triangle cells). Therefore $\dim(V_h)$ is equal to the number of inner nodes and a basis of V_h is given by the hat functions that are 1 on a single node and 0 in any other nodes, respectively vanish on any triangle which shares no vertex with that node. When assembling A and M one has only take into account adjacent or equal hat functions, any other entries of the matrizes will be zero which makes A and M sparse matrizes and simplifies the problem of solving (6.3).

Knowing the structure of V_h, also makes it possible to precompute the integrals for the entries of A and M and thus further reduce the computation times. Let $\triangle(p_0, p_1, p_2)$ be one of the triangle cells of the mesh. Then $\triangle(p_0, p_1, p_2)$ can be transformed to the unit triangle $\triangle(O, e_1, e_2)$ with vertices $O = (0,0)^T$, $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$ by the transform

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad x \mapsto D^{-1}(x - p_0)$$

where the matrix D consists of the column vectors $p_1 - p_0$ and $p_2 - p_0$. That way

$$\int_{\triangle(p_0,p_1,p_2)} \phi_i(x)\phi_j(x)\,\mathrm{d}x = \det(D)^{-1} \int_{\triangle(O,e_1,e_2)} \phi_i(\Psi(x))\phi_j(\Psi(x))\,\mathrm{d}x$$

The integral on the right hand side now depends only upon the question if ϕ_i and ϕ_j coincide or not. Hence, we obtain that

$$\int_{\triangle(p_0,p_1,p_2)} \phi_i(x)\phi_j(x)\,\mathrm{d}x = \frac{1}{24}\begin{cases} 2\det(D)^{-1} & \phi_i \equiv \phi_j, \\ \det(D)^{-1} & \phi_i \not\equiv \phi_j. \end{cases}$$

For the Laplacian part it follows that

$$\begin{aligned} &\int_{\triangle(p_0,p_1,p_2)} \nabla\phi_i(x) \cdot \nabla\phi_j(x)\,\mathrm{d}x \\ &\quad= \det(D)^{-1} \int_{\triangle(O,e_1,e_2)} (\nabla\phi_i)(\Psi(x)) \cdot (\nabla\phi_j)(\Psi(x))\,\mathrm{d}x \\ &\quad= \det(D)^{-1} \int_{\triangle(O,e_1,e_2)} \big(D^{-1}\,\nabla(\phi_i(\Psi(x)))\big) \cdot \big(D^{-1}\,\nabla(\phi_i(\Psi(x)))\big)\,\mathrm{d}x. \end{aligned}$$

The gradients in the integral on the right hand side are all gradients of hat functions on $\triangle(O, e_1, e_2)$. Depending on the node there are only three possible values for these gradients. Let

$$\beta_i, \beta_j \in \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

then

$$\int_{\triangle(p_0,p_1,p_2)} \nabla\phi_i(x)\cdot\nabla\phi_j(x)\,\mathrm{d}x = \frac{1}{2}\det(D)^{-1}(D^{-1}\beta_i)\cdot(D^{-1}\beta_j).$$

Unfortunately, the integral for the part of the Stark potential cannot be written in a compact form. After transformation to the unit triangle

$$\int_{\triangle(p_0,p_1,p_2)} x_1\,\phi_i(x)\phi_j(x)\,\mathrm{d}x \\ = \det(D)^{-1}\int_{\triangle(O,e_1,e_2)} (\Psi(x))_1\,\phi_i(\Psi(x))\phi_j(\Psi(x))\,\mathrm{d}x$$

the integral takes the form

$$\det(D)^{-1}\int_{\triangle(O,e_1,e_2)} (\alpha x_1+\beta x_2+\gamma)\,\phi_i(\Psi(x))\phi_j(\Psi(x))\,\mathrm{d}x$$

and can also be easily precomputed using one of the integrals

$$\begin{aligned}
\int_\triangle x_1^3\,\mathrm{d}x = \int_\triangle x_2^3\,\mathrm{d}x &= \frac{1}{20},\\
\int_\triangle x_1^2x_2\,\mathrm{d}x = \int_\triangle x_1x_2^2\,\mathrm{d}x &= \frac{1}{60},\\
\int_\triangle x_1x_2(1-x_1-x_2)\,\mathrm{d}x &= \frac{1}{120},\\
\int_\triangle x_1(1-x_1-x_2)^2\,\mathrm{d}x = \int_\triangle x_2(1-x_1-x_2)^2\,\mathrm{d}x &= \frac{1}{60},\\
\int_\triangle x_1^2(1-x_1-x_2)\,\mathrm{d}x = \int_\triangle x_2^2(1-x_1-x_2)\,\mathrm{d}x &= \frac{1}{60}.
\end{aligned}$$

We do not want to go into any further detail and refer to questions concerning efficient mesh generation and convergence to the literature, e.g. [14] and references therein.

For our numerical experiments below we will use Mathematica's functions `NDEigenvalues` and `NDEigensystem`, see [94, 95]. To gain an orientation on their numerical accuracy we first want to compare their outputs to the exact eigenvalues which are known for a particular family of rectangles in $\mathbb{R}^2$. Let

$(a_n)_{n\in\mathbb{N}}$ be the monotonic decreasing sequence of the Airy functions' zeros, then

$$\varphi_1(x_1) = \mathrm{Ai}\left(\varepsilon_0^{-2/3}(\varepsilon_0 x_1 - \nu)\right)$$

is an eigenfunction of the one-dimensional operator $-\mathrm{d}^2/\mathrm{d}x_1^2 + \varepsilon_0 x_1$ for the eigenvalue ν satisfying the Dirichlet condition on the interval $I_1 = [\varepsilon_0^{-1/3} a_{n+1} + \nu/\varepsilon, \varepsilon_0^{-1/3} a_n + \nu/\varepsilon_0]$. Since φ_1 has no zeros in the inner part of this interval, ν is the lowest eigenvalue $\lambda_1(I_1, \varepsilon_0)$. By seperation of variables $\varphi(x_1, x_2) = \varphi_1(x_1)\cdot\varphi_2(x_2)$ where $\varphi_2(x_2) = \cos(\mu x_2)$, $\mu = \pi(a_{n+1} - a_n)\varepsilon^{-1/3}$ is the Dirichlet eigenfunction for $-\Delta + \varepsilon_0 x_1$ on

$$R_{\varepsilon_0,n} := [\varepsilon_0^{-1/3} a_{n+1} + \nu/\varepsilon, \varepsilon_0^{-1/3} a_n + \nu/\varepsilon_0] \times [-\pi/(2\mu), \pi/(2\mu)]$$

for the eigenvalue

$$\lambda_1(R_{\varepsilon_0,n}, \varepsilon_0) = \nu + \pi^2(a_{n+1} - a_n)^2 \varepsilon_0^{-2/3}.$$

Denote by $\lambda_1^{\approx}(R_{\varepsilon_0,n}, \varepsilon_0)$ Mathematica's approximative value of $\lambda_1(R_{\varepsilon_0,n}, \varepsilon_0)$, then their difference

$$\mathrm{Err}\,(\nu, \varepsilon_0, n) := \lambda_1^{\approx}(R_{\varepsilon_0,n}, \varepsilon_0) - \lambda_1(R_{\varepsilon_0,n}, \varepsilon_0)$$

does not change with the position of $\mathcal{R}_{\nu,\varepsilon_0}$ along the x_1 direction, i.e. the value of ν. Regarding the influence of n and ε_0, Figure 6.1 shows plots of $\mathrm{Err}\,(0, \varepsilon_0, n)$ over a wide range of ε_0 and various values of n. As one would expect from the min-max principle, $\lambda_1^{\approx}(R_{\varepsilon_0,n}, \varepsilon_0)$ is an upper bound on $\lambda_1(R_{\varepsilon_0,n}, \varepsilon_0)$, since the FEM approximation of $\varphi(x_1, x_2)$ on $R_{\varepsilon_0,n}$ could be seen as a test function. If $R_{\varepsilon_0,n}$ has extreme aspect ratios, $\mathrm{Err}\,(0, \varepsilon_0, n)$ is larger than for rectangles which do look more like a square. Accordingly, $\mathrm{Err}\,(0, \varepsilon_0, n)$ seems to grow with larger values of ε_0, but in which follows below we do not want to deal with the strong coupling limit such that the numerical errors of Mathematica are reasonable small for our purposes.

As a first application we want to search for minimizing domains for the lowest eigenvalue of the Dirichlet Stark Laplacian among triangles, rectangles

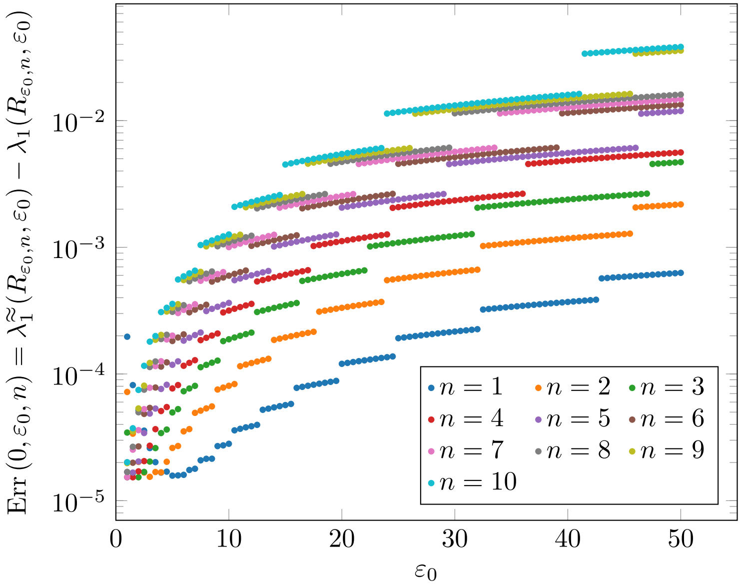
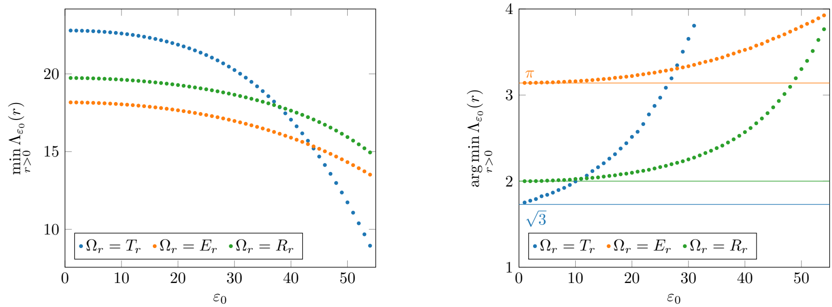


Figure 6.1: The difference between Mathematica's approximate value for the lowest eigenvalue $\lambda_1(R_{\varepsilon_0,n},\varepsilon_0)$ using `NDEigenvalues` and $\lambda_1(R_{\varepsilon_0,n},\varepsilon_0)$ on various rectangles characterized by the n-th zero of the Airy function.



(a) Minimal values of $\Lambda_{\varepsilon_0}(r)$ among triangles, ellipses and rectangles as a function of ε_0.

(b) The corresponding minimal values of r for triangles, ellipses and rectangles.

Figure 6.2: Minimizing $\Lambda_{\varepsilon_0}(r) = \Lambda_{\varepsilon_0}(\Omega_r)$ where Ω_r belongs to one of the families from (6.4).

and ellipses. Introducing the families of domains

$$\begin{aligned} E_r &:= \{(x_1, x_2) \in \mathbb{R}^2 \,:\, (\pi^2/r^2)x_1^2 + (r^2/\pi^2)x_2^2 \le 1\} \\ R_r &:= \text{poly}\,\{(r,1),(0,1),(0,-1),(r,-1)\} \\ T_r &:= \text{poly}\,\{(r,0),(0,1),(0,-1)\} \end{aligned} \tag{6.4}$$

for $r > 0$, these one dimensional problems can be solved by a simple golden section search. Since we are rather interested in the shape or aspect ratios of the minimizers, we would have to normalize the areas and center of masses of the domains E_r, R_r or T_r. As an alternative we use the invariant form

$$\Lambda_{\varepsilon_0}(r) = \Lambda_{\varepsilon_0}(\Omega_r) = |\Omega_r| \cdot \lambda_1(\Omega_r, |\Omega_r|^{-3/2}\varepsilon_0) + |\Omega_r|^{-1/2}\,\varepsilon_0\, m_{x_1}(\Omega_r),$$

$\Omega_r \in \{E_r, R_r, T_r\}$ from (1.11) which does not change when scalling or shifting a domain. Figure 6.2a shows the minimal values of $\Lambda_{\varepsilon_0}(r)$ we found among the three families using the golden section search with an accuracy smaller than 10^{-5}. The minimizing domain for $\varepsilon_0 = 0$ is known to be the circle. With this in mind, it is not suprising that the lowest eigenvalues are found among ellipses when $\varepsilon_0 > 0$ is small. For stronger coupling of the Stark potential optimal triangles have lower first eigenvalues than optimal ellipses. Figure

6.2b is a plot of the corresponding values of r for the minimizing domains for various values of ε_0. Again, as one would expect from the case $\varepsilon_0 = 0$, the minimizers are close to the circle E_π, the equilateral triangle $T_{\sqrt{3}}$ and the square R_2. For larger values of ε_0 the minimizers are stretched along the direction of the Stark potential and the optimal values of r grow.

6.2 Change of eigenvalues with respect to the domain

When searching for minimizing domains, one needs to know how eigenvalues behave under certain changes of the domains boundary. A helpful tool at this point is *Hadamard's Formula* which was already used by P. Autunes and P. Freitas when exploring the optimal domains for minimizing various means of Dirichlet- and Neumann eigenvalues of the classical Laplacian [8]. In this section we want to give a version of Hadamard's Formula for our operator and apply it in a gradient descent step in order to find minimizing domains for the first eigenvalue in various regimes of the coupling strength of the Stark potential.

Let $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ be the space of bounded Lipschitz maps from $\mathbb{R}^d$ into $\mathbb{R}^d$ equipped with the norm

$$\|\theta\|_{W^{1,\infty}} := \|\theta\|_\infty + \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|\theta(x) - \theta(y)|}{|x-y|}.$$

Functions from this space are differentiable a.e., but $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ can also be replaced by $C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) := C^1(\mathbb{R}^d, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ if one likes to work with stronger assumptions on the differentiability. For $t \in [0, T[$ we consider the family of functions $\Phi(t) : \mathbb{R}^d \to \mathbb{R}^d$ where $\Phi(0) = I$ is the identity on $\mathbb{R}^d$ and $t \mapsto \Phi(t) - I \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is differentiable at $t = 0$ satisfying

$$\Phi'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)\bigg|_{t=0} = V$$

for some vector field V on $\mathbb{R}^d$. A common choice might be $\Phi(t) = I + t\theta$ for some $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ or even $\Phi(t) = I + t \cdot V$. By applying $\Phi(t)$ on sets

$\Omega \subset \mathbb{R}^d$, we generate new sets

$$\Omega_t := \Phi(t)(\Omega) = \{\Phi(t)(x) \, : \, x \in \Omega\}.$$

Note that, depending on V, Ω_t does not have to be convex even if Ω is a convex set. Nevertheless, this approach allows us to quantify the change of volume of Ω in terms of V:

Theorem 6.2.1 ([40, Theorem 2.5.3]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, then $t \mapsto |\Omega_t|$ is differentiable at $t = 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t|\bigg|_{t=0} = \int_{\partial\Omega} V \cdot n \, \mathrm{d}\sigma$$

where n is the outward pointing unit normal at each point on the boundary curve $\partial\Omega$.

Hadamard's Formula gives a characterization of the change for the k-th eigenvalue of the Dirichlet (or Neumann) Laplacian on Ω_t. Let k be fixed and u_t be the solution of $-\Delta u_t = \lambda_k(\Omega_t) u_t$ in $\mathring{W}_2^1(\Omega_t)$ with the usual normalization

$$\int_{\Omega_t} u_t^2 \, \mathrm{d}x = 1,$$

then it follows that

Theorem 6.2.2 ([40, Theorem 2.5.1; 41, Theorem 5.7.1]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set that is convex or has C^2-boundary and $u_t \in \mathring{W}_2^1(\Omega_t)$ a normalized solution of $-\Delta u_t = \lambda_k(t) u_t$ for each fixed $t \in [0, T[$. Then $t \mapsto \lambda_k(\Omega_t)$ is differentiable at $t = 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(\Omega_t)\bigg|_{t=0} = -\int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n}\right)^2 V \cdot n \, \mathrm{d}\sigma \tag{6.5}$$

as long as $\lambda_k(t)$ is a simple eigenvalue.

A detailed proof of this theorem can be found in [41]. It makes use of the variational characterization of the eigenvalue equation on $\Omega_\theta = (I + \theta)(\Omega)$

for some $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, i.e.

$$\int_{\Omega_\theta} \nabla u_\theta \cdot \nabla \varphi_\theta \, \mathrm{d}x = \lambda_k(\Omega_\theta) \int_{\Omega_\theta} u_\theta \varphi_\theta \, \mathrm{d}x$$

for all $\varphi_\theta \in \mathring{W}^1_2(\Omega_\theta)$. This variational problem is then transformed onto Ω which gives the equivalent formulation

$$-\operatorname{div}\left(A(\theta)\nabla \nu_\theta\right) = \lambda_k(\Omega_\theta)\nu_\theta J_\theta$$

where $A(\theta) = J_\theta(I + D\theta)^{-1}(I + (D\theta)^T)^{-1}$ and $\nu_\theta = u_\theta \circ (I + \theta) \in \mathring{W}^1_2(\Omega)$ with the normalization

$$\int_\Omega \nu_\theta^2 \, J_\theta \mathrm{d}y = 1.$$

Again, we refer to [41] for any details and notation. The operator $\mathcal{F}$: $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \times \mathring{W}^1_2(\Omega) \times \mathbb{R} \to (\mathring{W}^1_2(\Omega))' \times \mathbb{R}$ defined by

$$\mathcal{F}(\theta, \nu, \lambda) = \left(-\operatorname{div}\left(A(\theta)\nabla \nu\right) - \lambda \nu J_\theta, \int_\Omega \nu^2 \, J_\theta \mathrm{d}y - 1\right)$$

is of class C^∞, moreover,

$$D_{\nu,\lambda}\mathcal{F}(0, u_0, \lambda_k(\Omega))(\hat{\nu}, \hat{\lambda}) = \left(-\Delta\hat{\nu} - \hat{\lambda}u_0 - \lambda_k(\Omega)\hat{\nu}, 2\int_\Omega u_0 \hat{\nu} \, \mathrm{d}y\right)$$

is an isomorphism from $\mathring{W}^1_2(\Omega) \times \mathbb{R}$ onto $(\mathring{W}^1_2(\Omega))' \times \mathbb{R}$ and it follows from the inverse function theorem that

$$W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \to \mathring{W}^1_2(\Omega) \times \mathbb{R}, \qquad \theta \mapsto (\nu_\theta, \lambda_k(\Omega_\theta))$$

is differentiable in a neighbourhood of $0 \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. By choosing $\theta = \Phi(t) - I$ this gives the differentiability of $t \mapsto \lambda_k(\Omega_t)$ and $t \mapsto u_t$. The representation (6.5) for the derivative then follows from differentiating either the eigenvalue equation or the integral representation

$$\lambda_k(\Omega_t) = \int_{\Omega_t} |\nabla u_t|^2 \, \mathrm{d}x.$$

In what follows we want to adapt this procedure in order to prove the corresponding result for the Dirichlet eigenvalues of the Stark Laplacian.

Theorem 6.2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $\lambda_k(\Omega, \varepsilon_0)$ be a simple eigenvalue of the Stark Laplacian with Dirichlet boundary conditions and u its corresponding eigenfunction. For $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ let u_θ be a Dirichlet eigenfunction of the Stark Laplacian on $\Omega_\theta = (I + \theta)(\Omega)$ for the k-th eigenvalue $\lambda_k(\Omega_\theta, \varepsilon_0)$, then the mappings $\theta \mapsto \lambda_k(\Omega_\theta, \varepsilon_0)$ and $\theta \mapsto u_\theta$ are differentiable in a small neighbourhood of* $0 \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

Proof. Our proof follows the ideas from the proof of [41, Theorem 5.7.1] which we will present in a more elaborated way. Recall that $\Omega_\theta = (I + \theta)(\Omega)$ and $u_\theta \in \mathring{W}_2^1(\Omega_\theta)$, satisfying $\|u_\theta\|_{L^2(\Omega_\theta)} = 1$, is a solution of

$$\int_{\Omega_\theta} \left[|\nabla u_\theta|^2 + \varepsilon_0 x_1 |u_\theta|^2 \right] \mathrm{d}x = \lambda_k(\Omega_\theta, \varepsilon_0) \int_{\Omega_\theta} |u_\theta|^2 \,\mathrm{d}x. \tag{6.6}$$

Thereby we may choose the sign of u_θ such that

$$W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \to \mathring{W}_2^1(\Omega_\theta) \times \mathbb{R}, \qquad \theta \mapsto (u_\theta, \lambda_k(\Omega_\theta, \varepsilon_0))$$

is continuous, cf. [41, Chapter 4]. Problem (6.6) can be written in a variational formulation as

$$\int_{\Omega_\theta} \left[\nabla u_\theta \cdot \overline{\nabla \varphi_\theta} + \varepsilon_0 x_1 u_\theta \overline{\varphi_\theta} \right] \mathrm{d}x = \lambda_k(\Omega_\theta, \varepsilon_0) \int_{\Omega_\theta} u_\theta \overline{\varphi_\theta} \,\mathrm{d}x$$

for all $\varphi_\theta \in \mathring{W}_2^1(\Omega_\theta)$. The basic idea is to substitute $y = (I + \theta)^{-1} x$ and thus transfer this problem onto Ω. That way we set $\varphi_\theta = \varphi \circ (I + \theta)^{-1}$ with $\varphi \in \mathring{W}_2^1(\Omega)$ where

$$\nabla \varphi_\theta = [(I + (D\theta)^T)^{-1} \cdot \nabla \varphi] \circ (I + \theta)^{-1}$$

and $D\theta$ is the derivative of θ. Similary, we obtain for $\nu_\theta = u_\theta \circ (I + \theta)$ that

$$\nabla u_\theta = [(I + (D\theta)^T)^{-1} \cdot \nabla u_\theta] \circ (I + \theta)^{-1}$$

and thus

$$\int_{\Omega_\theta} \nabla u_\theta \cdot \nabla \varphi_\theta \,\mathrm{d}x = \int_\Omega [(I + (D\theta)^T)^{-1} \nabla \nu_\theta] \cdot [(I + (D\theta)^T)^{-1} \nabla \varphi] J_\theta \mathrm{d}y$$

with the Jacobian $J_\theta = \det(I + D\theta)$. The same way follows that

$$\int_{\Omega_\theta} x_1 u_\theta \varphi_\theta \,\mathrm{d}x = \int_\Omega [(I + \theta)y]_1 \nu_\theta \varphi J_\theta \mathrm{d}y.$$

In summary we have shown that $\nu_\theta \in \mathring{W}_2^1(\Omega)$ is a solution of

$$-\operatorname{div}(A(\theta)\nabla \nu_\theta) + \varepsilon_0[(I + \theta)y]_1 J_\theta \nu_\theta = J_\theta \nu_\theta \lambda_k(\Omega_\theta, \varepsilon_0)$$

satisfying the normalization condition $\int_\Omega \nu_\theta^2 J_\theta \mathrm{d}y = 1$. In this shorthand notation we used the abbrevation $A(\theta) = J_\theta (I + D\theta)^{-1}(I + (D\theta)^T)^{-1}$. Let

$$\mathcal{F}(\theta, \nu, \lambda) := \left(-\operatorname{div}(A(\theta)\nabla \nu) + \varepsilon_0[(I + \theta)y]_1 J_\theta \nu - \lambda J_\theta \nu, \int_\Omega \nu^2 J_\theta \mathrm{d}y \right)$$

which is indeed a mapping from $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \times \mathring{W}_2^1(\Omega) \times \mathbb{R}$ to $(\mathring{W}_2^1(\Omega))' \times \mathbb{R}$ (see below). Moreover, this mapping is of class C^1 and even C^∞ as a composition of differentiable mappings.
Note that if $\theta = 0$ and $\lambda = \lambda_k(\Omega, \varepsilon_0)$ then $F(\theta, \nu_\theta, \lambda) = 0$. Our next goal is to apply the inverse function theorem in a neighbourhood of $\theta = 0$ and $\lambda = \lambda_k(\Omega, \varepsilon_0)$ which yields a differential mapping $\theta \mapsto (\nu(\theta), \lambda(\theta))$ that necessarily coincides with the continuous function $\theta \mapsto (\nu_\theta, \lambda_k(\Omega_\theta, \varepsilon_0))$. Therefore we have to show that

$$\begin{aligned} D_{\nu,\lambda}\mathcal{F}(0, u, \lambda_k(\Omega, \varepsilon_0)) &: \mathring{W}_2^1(\Omega) \times \mathbb{R} \to (\mathring{W}_2^1(\Omega))' \times \mathbb{R} \\ &(\hat{\nu}, \hat{\lambda}) \mapsto \left(-\Delta \hat{\nu} + \varepsilon_0 y_1 \hat{\nu} - \hat{\lambda} u - \lambda_k(\Omega, \varepsilon_0)\hat{\nu}, 2\int_\Omega u \hat{\nu} \,\mathrm{d}y \right) \end{aligned}$$

is an isomorphism. Since $D_{\nu,\lambda}\mathcal{F}(0, u, \lambda_k(\Omega, \varepsilon_0))$ is continouos, it remains to show that it is one-to-one. In order to do so we will construct a unique solution $(\hat{\nu}, \hat{\lambda}) \in \mathring{W}_2^1(\Omega) \times \mathbb{R}$ to the system

$$\begin{aligned} -\Delta\hat{\nu} + \varepsilon_0 y_1 \hat{\nu} - \hat{\lambda} u - \lambda_k(\Omega, \varepsilon_0)\hat{\nu} &= Z \\ 2\int_\Omega u\hat{\nu}\,\mathrm{d}y &= \Lambda \end{aligned}$$

for any $(Z, \Lambda) \in (\mathring{W}_2^1(\Omega))' \times \mathbb{R}$: Recall that

$$\begin{aligned} \|(-\Delta + \varepsilon_0 y_1)u\|_{(\mathring{W}_2^1(\Omega))'} &= \sup_{v\in\mathring{W}_2^1(\Omega), \|v\|=1} |(v, (-\Delta + \varepsilon_0 y_1)u)| \\ &= \sup_{v\in\mathring{W}_2^1(\Omega), \|v\|=1} \left| \int_\Omega \nabla v \cdot \overline{\nabla u}\,\mathrm{d}y + \varepsilon_0 \int_\Omega y_1 v\overline{u}\,\mathrm{d}y \right| \\ &\leq \sup_{v\in\mathring{W}_2^1(\Omega), \|v\|=1} \|u\|_{\mathring{W}_2^1(\Omega)} \|v\|_{\mathring{W}_2^1(\Omega)} \\ &\qquad + \varepsilon_0 \Omega\rfloor \cdot \|u\|_{\mathring{W}_2^1(\Omega)} \|v\|_{\mathring{W}_2^1(\Omega)} \\ &= (1 + \varepsilon_0\Omega\rfloor) \cdot \|u\|_{\mathring{W}_2^1(\Omega)} \end{aligned}$$

where $\Omega\rfloor := \sup\{x_1 \in \mathbb{R} : \exists_{x_\perp \in \mathbb{R}^{d-1}} (x_1, x_\perp) \in \Omega\}$ denotes the right bound of Ω along the x_1-direction. Thus $(-\Delta + \varepsilon_0 y_1)$ mapps $\mathring{W}_2^1(\Omega)$ indeed onto $(\mathring{W}_2^1(\Omega))'$.

From now on we assume that the spectrum of $-\Delta + \varepsilon_0 y_1$ is positive, more precisely, we demand

$$\lambda_k(\Omega, \varepsilon_0) \geq \lambda_1(\Omega, \epsilon_0) \geq \lambda_1(\Omega, 0) - \varepsilon\Omega\rfloor > 0.$$

This can be achieved by shifting the domain which does not affect the differentiability of the eigenvalue function $\theta \mapsto \lambda_k(\Omega_\theta, \varepsilon_0)$ or the mapping to the corresponding eigenfunction. In this case we deduce

$$\begin{aligned}\|(-\Delta + \varepsilon_0 y_1)u\|_{(\mathring{W}_2^1(\Omega))'} &= \sup_{v\in\mathring{W}_2^1(\Omega), v\neq 0} \frac{|(v, (-\Delta + \varepsilon_0 y_1)u)|}{\|v\|_{\mathring{W}_2^1(\Omega)}} \\ &\geq \frac{1}{\|u\|}\left|\int_\Omega |\nabla u|^2 \,\mathrm{d}y + \varepsilon_0 \int_\Omega y_1|u|^2 \,\mathrm{d}y\right| \\ &\geq \frac{1}{\|u\|}\int_\Omega |\nabla u|^2 \,\mathrm{d}y - \frac{\epsilon_0}{\|u\|}\int_\Omega y_1|u|^2 \,\mathrm{d}y \\ &\geq (\lambda_1(\Omega, 0) - \varepsilon_0 \Omega\rfloor)\|u\|_{\mathring{W}_2^1(\Omega)},\end{aligned}$$

thus $(-\Delta + \varepsilon_0 y_1)^{-1} : (\mathring{W}_2^1(\Omega))' \to \mathring{W}_2^1(\Omega) \subset (\mathring{W}_2^1(\Omega))'$ is well defined and by Rellichs embedding theorem compact and we can apply the Fredholm alternative to the operator $(-\Delta+\varepsilon_0 y_1 - \lambda_k(\Omega, \varepsilon_0))$. By assumption the kernel of this operator is one-dimensional such that $\varphi \in (\mathring{W}_2^1(\Omega))'$ belongs to its range if and only if $(\varphi, u) = 0$. In particular $\varphi = Z + \hat{\lambda}u \in \operatorname{ran}(-\Delta + \varepsilon_0 y_1 - \lambda_k(\Omega, \varepsilon_0))$, thus,

$$0 = (Z + \hat{\lambda}u, u) = (Z, u) + \hat{\lambda}$$

which defines $\hat{\lambda}$ in a unique way. Let

$$\nu_0 = (-\Delta + \varepsilon_0 y_1 - \lambda_k(\Omega, \varepsilon_0))^{-1}(Z + \hat{\lambda}u),$$

then any other element of the preimage of $Z + \hat{\lambda}u$ by $(-\Delta + \varepsilon_0 y_1 - \lambda_k(\Omega, \varepsilon_0))$ is given by $\hat{\nu} = \nu_0 + s \cdot u$, $s \in \mathbb{R}$. But since the relation

$$\Lambda = 2\int_\Omega u(\nu_0 + s\cdot u)\,\mathrm{d}y = 2\int_\Omega u\nu_0 \,\mathrm{d}y + 2s$$

defines s in a unique way, $\hat{\nu}$ is unique. □

When searching for optimal domains that minimize the eigenvalues of the Dirichlet Laplacian, one is more interested in the domains shape rather than the areas, respectively, the minimization problems as (6.1) are only well-posed in the class of domains with fixed area. In order to control the influence of rescaling during the optimization procedure, the cost function is usually chosen to be $t \mapsto |\Omega_t|^{2/d}\lambda_k(\Omega_t)$. In the case of the Stark Laplacian one has additionally to take into account that eigenvalues depend on the position of the domains along the x_1-direction and the value of the coupling constant $\varepsilon_0 > 0$. With (1.11) in mind our cost function for $d = 2$ will be

$$\Lambda(t) = |\Omega_t| \cdot \lambda_1(\Omega_t, |\Omega_t|^{-3/2}\varepsilon_0) + |\Omega_t|^{-1/2}\varepsilon_0\, m_{x_1}(\Omega_t). \tag{6.7}$$

Thus, instead of differentiating the Stark eigenvalues $\lambda_k(\Omega_t, \varepsilon_0)$, we rather want to give the derivation of $t \mapsto \lambda_k(\Omega_t, |\Omega_t|^{-3/2}\varepsilon_0)$. As above let u_t be a normalized solution of $-\Delta u_t + |\Omega_t|^{-3/2}\varepsilon_0 x_1 u_t = \lambda(t) u_t$ in $\mathring{W}_2^1(\Omega_t)$, then

Theorem 6.2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set that is convex or has C^2-boundary and $u_t \in \mathring{W}_2^1(\Omega_t)$ a normalized solution of $-\Delta u_t + |\Omega_t|^{-3/2}\varepsilon_0 u_t = \lambda_k(t) u_t$ for each fixed $t \in [0, T[$. Then $t \mapsto \lambda_k(t)$ is differentiable at $t = 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(t)\bigg|_{t=0} = -\int_{\partial\Omega} \left[\left(\frac{\partial u_0}{\partial n} \right)^2 + \frac{3}{2} \frac{\varepsilon_0}{|\Omega|^{5/2}} \int_\Omega x_1\, u_0^2 \,\mathrm{d}x \right] V \cdot n \,\mathrm{d}\sigma$$

as long as $\lambda_k(t)$ is simple.

Proof. Since $u_t \equiv 0$ on $\partial\Omega_t$, it follows that

$$u_t' = -\frac{\partial u_t}{\partial n} V \cdot\, n.$$

From the normalizing condition $\int_{\Omega_t} u_t^2 \,\mathrm{d}x$ we obtain that

$$\int_{\Omega_t} u_t u_t' \,\mathrm{d}x = 0.$$

Differentiating both sides of the eigenvalue equation leads to

$$-\Delta u_t' - \frac{3}{2} \frac{\varepsilon_0}{|\Omega_t|^{5/2}} |\Omega_t|' x_1 u_t + \frac{\varepsilon_0}{|\Omega_t|^{3/2}} x_1 u_t' = \lambda_k' u_t + \lambda_k u_t'.$$

From there we multiply both sides by u_t and integrate over Ω_t, hence

$$\lambda_k'(t) = -\int_{\Omega_t} u_t\, \Delta u_t' \,\mathrm{d}x - \frac{3}{2} \frac{\varepsilon_0}{|\Omega_t|^{5/2}} |\Omega_t|' \int_{\Omega_t} x_1\, u_t^2 \,\mathrm{d}x + \frac{\varepsilon_0}{|\Omega_t|^{3/2}} \int_{\Omega_t} x_1\, u_t' u_t \,\mathrm{d}x.$$

The first term on the right hand side can be replaced by Green's identity

$$\begin{aligned}\int_{\Omega_t} u_t' \,\Delta u_t \,\mathrm{d}x - \int_{\Omega_t} u_t \,\Delta u_t' \,\mathrm{d}x &= \int_{\partial\Omega_t} u_t' \left(\frac{\partial u_t}{\partial n}\right) \mathrm{d}\sigma - \int_{\partial\Omega_t} u_t \left(\frac{\partial u_t'}{\partial n}\right) \mathrm{d}\sigma \\ &= \int_{\partial\Omega_t} u_t' \left(\frac{\partial u_t}{\partial n}\right) \mathrm{d}\sigma\end{aligned}$$

where one of the integrals vanishes due to the Dirichlet condition. Since

$$-\int_{\Omega_t} u_t' \,\Delta u_t \,\mathrm{d}x + \frac{\varepsilon_0}{|\Omega|^{3/2}} \int_{\Omega_t} x_1 \, u_t' u_t \,\mathrm{d}x = \lambda_k \int_{\Omega_t} u_t' u_t = 0,$$

it follows that

$$\lambda_k'(t) = -\int_{\partial\Omega_t} \left(\frac{\partial u_0}{\partial n}\right)^2 V \cdot n \,\mathrm{d}\sigma - \frac{3}{2} \frac{\varepsilon_0}{|\Omega|^{5/2}} \int_{\Omega} x_1 u_0^2 \,\mathrm{d}x \,|\Omega_t|'$$

and inserting the formula from Theorem 6.2.1 finishes the proof. □

In order to find a formula for the derivation of $\Lambda(t)$ in (6.7), it remains to differentiate the integral

$$m_{x_1}(\Omega_t) = \int_{\Omega_t} x_1 \,\mathrm{d}x$$

for the center of mass of Ω_t. This can be done the same way as in Theorem 6.2.1 and it follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} m_{x_1}(\Omega_t)\right|_{t=0} = \int_{\partial\Omega} x_1 \, V \cdot n \,\mathrm{d}\sigma.$$

In summary we obtain

$$\begin{aligned}&\left.\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t)\right|_{t=0} \\ &\quad= \int_{\partial\Omega} \left[\lambda_1 - |\Omega| \left(\frac{\partial u}{\partial n}\right)^2 + \frac{\varepsilon_0}{|\Omega|^{1/2}} x_1\right] V \cdot n \,\mathrm{d}\sigma \\ &\qquad - \int_{\partial\Omega} \left[\frac{1}{2} \frac{\varepsilon_0}{|\Omega|^{3/2}} m_{x_1}(\Omega) + \frac{3}{2} \frac{\varepsilon_0}{|\Omega|^{3/2}} \int_{\Omega} x_1 \, u^2 \,\mathrm{d}x\right] V \cdot n \,\mathrm{d}\sigma\end{aligned} \tag{6.8}$$

where $u \in \mathring{W}_2^1(\Omega)$ satisfies $(-\Delta + |\Omega|^{-3/2}\varepsilon_0\, x_1)u = \lambda u$.

6.3 Gradient descent method

Since our algorithm for evaluating the eigenvalues is based on meshing our domain, we do not restrict ourselves too much when searching for minimizing domains among polygons with large numbers of vertices (in our case 120 up to 200). Let

$$\Omega = \Omega((x_1, y_1), (x_2, y_2), \dots, (x_N, y_N))$$

be the polygon with vertices $(x_j, y_j) \in \mathbb{R}^2$, $j = 1, \dots, N$. With each polygon we associate the vector of vertices $\boldsymbol{x} = (x_1, y_1, x_2, y_2, \dots, x_N, y_N)$. To make the dependency of Ω from $\boldsymbol{x}$ more explicite, we use the notation $\Omega(\boldsymbol{x})$. Consider the mapping

$$\begin{aligned} \Lambda : \mathbb{R}^{2N} &\to \mathbb{R}, \\ \boldsymbol{x} &\mapsto \Lambda(\boldsymbol{x}) := |\Omega(\boldsymbol{x})|\lambda_1(\Omega(\boldsymbol{x}), |\Omega(\boldsymbol{x})|^{-3/2}\varepsilon_0) + |\Omega(\boldsymbol{x})|^{-1/2}\varepsilon_0\, m_{x_1}(\Omega(\boldsymbol{x})). \end{aligned}$$

Our goal is to optimize Λ in a gradient descent procedure by replacing $\Omega(\boldsymbol{x})$ by the polygon

$$\Omega_\beta = \Omega(\boldsymbol{x} + \beta \cdot \nabla\Lambda)$$

where β is chosen such that $\Lambda(\Omega_\beta)$ is minimized. This one-dimensional optimization problem for β is solved by a simple golden section search in each step.

Evaluation of the directional derivatives is done by choosing the appropiate vector fields for V in (6.8). If $P_0 = (x_0, y_0)$ is one of the vertices of $\Omega(\boldsymbol{x})$, then let $P_L = (x_L, y_L)$ and $P_R = (x_R, y_R)$ be its neighbours as shown in Figure 6.3. If P_0 is moved along the direction $(1, 0)^T$ or $(0, 1)^T$, the triangles $\triangle(P_R, O, P_0)$ and $\triangle(O, P_L, P_0)$ change while the rest of the polygon remains untouched. Here $O = (0, 0)$ is the origin of our coordinate system and without loss of generality we can shift our domain $\Omega(\boldsymbol{x})$ such that O is contained in $\Omega(\boldsymbol{x})$. In order to obtain a sufficiently smooth transition we shift any other point in $\triangle(P_R, O, P_0)$ or $\triangle(O, P_L, P_0)$ proportional to its distance from the

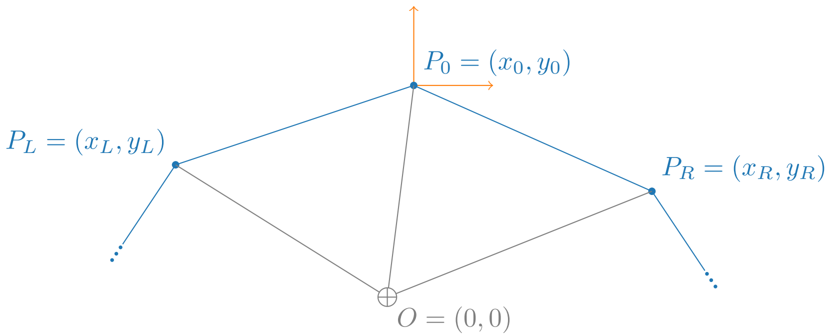


Figure 6.3: Relative position of the vertices P_0, P_L, P_R of $\Omega(\boldsymbol{x})$.

edge $[P_R, O]$ or $[O, P_L]$. Thus, we choose V to be

$$V(x,y) = \left|\frac{x_R y - x y_R}{x_R y_0 - x_0 y_R}\right| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{resp.} \quad V(x,y) = \left|\frac{x_R y - x y_R}{x_R y_0 - x_0 y_R}\right| \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

on $\triangle(P_R, O, P_0)$,

$$V(x,y) = \left|\frac{x_L y - x y_L}{x_L y_0 - x_0 y_L}\right| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{resp.} \quad V(x,y) = \left|\frac{x_L y - x y_L}{x_L y_0 - x_0 y_L}\right| \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

on $\triangle(O, P_L, P_0)$ and $V(x,y) = 0$ in any other point of $\Omega(\boldsymbol{x})$. This way

$$\begin{aligned}
&\int_{\partial\Omega} V \cdot n \,\mathrm{d}\sigma \\
&= \int_{[P_R,P_0]} V \cdot n \,\mathrm{d}\sigma + \int_{[P_0,P_L]} V \cdot n \,\mathrm{d}\sigma \\
&= \int_0^1 t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y_0 - y_R \\ -x_0 + x_R \end{pmatrix} \mathrm{d}t + \int_0^1 (1-t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y_L - y_0 \\ -x_L + x_0 \end{pmatrix} \mathrm{d}t \\
&= \frac{1}{2}(y_L - y_R)
\end{aligned}$$

when P_0 is moved along $(1,0)^T$, respectively

$$\int_{\partial\Omega} V \cdot n \,\mathrm{d}\sigma = \frac{1}{2}(x_R - x_L)$$

when P_0 is shifted along $(0,1)^T$. The remaining integrals in (6.8) can be computed in a similar manner.

6.4 Results

In what follows we want to summarize our results when searching for candidates for optimal domains which minimize the lowest eigenvalue of $-\Delta+\varepsilon_0 x_1$ among the set of all convex domains in $\mathbb{R}^2$ with fixed area and center of mass. Instead of minimizing $\lambda_1(\Omega,\varepsilon_0)$ directly, we consider the invariant form

$$\Lambda(\Omega) = |\Omega| \cdot \lambda_1(\Omega, |\Omega|^{-3/2}\varepsilon_0) + |\Omega|^{-1/2}\varepsilon_0\, m_{x_1}(\Omega)$$

which does not change when scaling or shifting the domain. However, any domains shown below are scaled to share $|\Omega| = 1$ and shifted such that $m_{x_1}(\Omega) = 0$ and Λ coincides with the first Stark eigenvalue $\lambda_1(\Omega,\varepsilon_0)$. As mentioned previously we want to use a gradient descent procedure while representing Ω as polygons with high counts of vertices that approximate a convex shape. Let

$$\boldsymbol{x} = (x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$$

be the vector of vertices $(x_j, y_j) \in \mathbb{R}^2$, $j = 1,\ldots,N$ of $\Omega = \Omega(\boldsymbol{x})$ and $\nabla\Lambda$ the gradient of $\Lambda(\boldsymbol{x}) = \Lambda(\Omega(\boldsymbol{x}))$ as given in (6.8) with vector field from Section 6.3, then Ω is replaced by the new polygon

$$\Omega' = \min_{\beta>0} \Lambda(\boldsymbol{x} + \beta\,\nabla\Lambda)$$

in each step. In doing so the corresponding one dimensional optimization is done with the help of a simple golden section search.

To begin with we run a quick test for our algorithm in the case of triangular domains, where we have evaluated the optimal aspect ratios above. Table 6.1 shows the drop of the value for Λ after performing one single gradient descent step for a starting triangle with vertices $(0,\pm1)$, $(1,0)$. The posterior values of Λ are fitting the optimal values quiet close and the corresponding

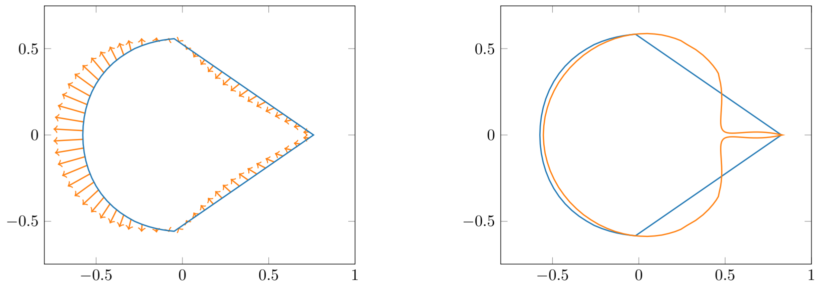
ε_0	start domain	after one gradient step	optimal triangle
0	23.2056	22.7935	22.7935
10	23.4208	22.6003	22.6003
20	23.4976	21.8804	21.8803
30	23.4451	20.2488	20.2485
40	23.2722	17.0550	17.0549
50	22.9878	11.7334	11.7314

Table 6.1: Values of Λ before and after a single gradient step performed with starting triangle with vertices $(0, \pm 1)$, $(1, 0)$ compared to those of the optimal triangles with area 1 for various values of ε_0.

triangles cannot be distinguished by eye from their optimal counterparts.

When performing a gradient descent step it might occur that we fall into domains lying at the boundary of the convex shapes in the sense that any distortion of the domain leads to a non convex shape. An example of such a domain is shown in Figure 6.4a. In fact, this domain occured when searching for an optimal domain if $\varepsilon_0 = 20$ while starting with random convex domains, performing a genetic optimization algorithm first (as suggested in [7]) and then proceeding with the gradient descent algorithm. The prior value of Λ is approximately 18.2435, after the gradient step it will fall to 18.0045 which is close to the optimal value of 17.65816 we found for $\varepsilon_0 = 20$. However, a plot of the new domain in Figure 6.4b shows that it is clearly not convex and shows the first signs that continuing this procedure will turn the domain in a handle bar like shape similar to the one we used above to disprove the existence of a minimizing domain among the non convex shapes. To avoid this effect we will replace the new domains by their convex hull and filling large line segments of the boundary with interpolation points to keep up the vertex count during our procedure.

In order to find candidates for the optimal domains we experimented with various random generated (convex) starting domains (the proof of [89, Theorem 1] gives a neat algorithm to create them), but the best results were found when starting with the ellipses or triangles with optimal aspect ratios in the appropriate regimes for ε_0. Figure 6.5 shows the fittest domains



(a) Start domain sketched in blue. The orange arrows indicate the components of the normalized Gradient $\nabla\Lambda$ attached to the corresponding vertices.

(b) Prior (blue) and posterior domain (orange) of the gradient step. Both domains are scaled to the same area and aligned to share the center of mass.

Figure 6.4: Result of a gradient step if $\varepsilon_0 = 20$ when starting with the domain sketched in blue ($\Lambda = 18.2435$). The orange domain in the right picture is the result for $\beta = 6.6$ with $\Lambda = 18.0145$.

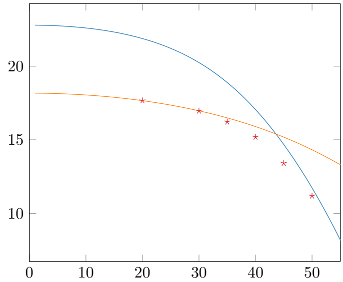
we found for $\varepsilon_0 \in \{20, 30, 35, 40, 45, 50\}$. Since the minimizing domain for $\varepsilon_0 = 0$ is the circle, we expect the fittest domain for small ε_0 also to look circular or at least elliptical. In fact up to $\varepsilon_0 = 20$, the domains we found do exactly look like an ellipse and can only be distinguished when comparing the stretched absolute values of the boundary points. Therefore, let

$$\partial \check{E}_{\varepsilon_0} := \{(x, y) \in \mathbb{R}^2 \,:\, (\pi^3/r_{\varepsilon_0}^2)x^2 + (r_{\varepsilon_0}^2/\pi)y^2 = 1\}$$

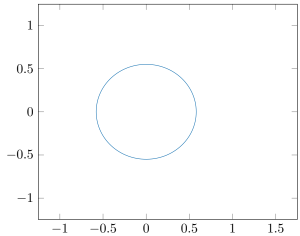
be the boundary of the optimal ellipses $\check{E}_{\varepsilon_0}$ with aspect ratios r_{ε_0} we found in Section 6.1 and $(x, y) \in \partial\check{\Omega}_{\varepsilon_0}$ a boundary point of the fittest domain $\check{\Omega}_{\varepsilon_0}$ we found for some ε_0. Figure 6.6b then shows a plot of

$$\mathrm{abs}_{r_{\varepsilon_0}}(x, y) := \frac{\pi^3}{r_{\varepsilon_0}^2}x^2 + \frac{r_{\varepsilon_0}^2}{\pi}y^2$$

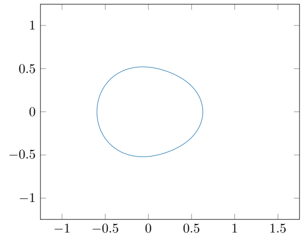
versus the polar angle $\varphi = \arg(x, y)$ when (x, y) is represented in polar form. It can be seen that $\check{\Omega}_{20}$ with $\Lambda(\check{\Omega}_{20}) = 17.65816$ clearly differs from $\check{E}_{20}$ with $\Lambda(\check{E}_{20}) = 17.65820$, but the difference $|\Lambda(\check{E}_{20}) - \Lambda(\check{\Omega}_{20})| < 10^{-4}$ is expected to lie in the range of the discretisation error that occurs from the polygonal



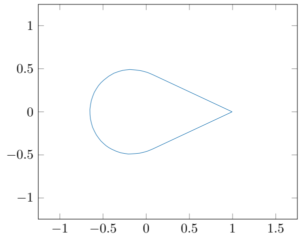
(a) Comparison of the lowest possible eigenvalues among right facing triangles (blue), ellipses (orange) and various other domains marked with a red star which are shown below.



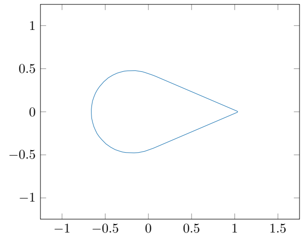
(b) $\check{\Omega}_{20}$ with $\Lambda(\check{\Omega}_{20}) = 17.65816$



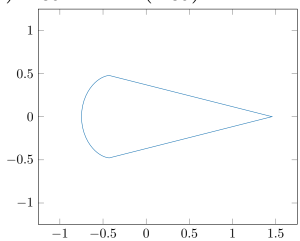
(c) $\check{\Omega}_{30}$ with $\Lambda(\check{\Omega}_{30}) = 16.9602$



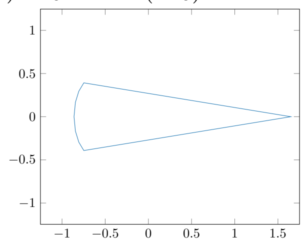
(d) $\check{\Omega}_{35}$ with $\Lambda(\check{\Omega}_{35}) = 16.2184$



(e) $\check{\Omega}_{40}$ with $\Lambda(\check{\Omega}_{40}) = 15.1909$

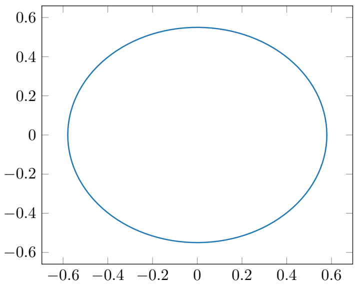


(f) $\check{\Omega}_{45}$ with $\Lambda(\check{\Omega}_{45}) = 13.4101$

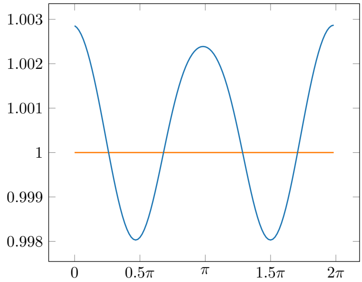


(g) $\check{\Omega}_{50}$ with $\Lambda(\check{\Omega}_{50}) = 11.1844$

Figure 6.5: Plots of the fittest domains $\check{\Omega}_{\varepsilon_0}$ for various values of ε_0 and $\Lambda(\check{\Omega}_{\varepsilon_0})$ in comparison with $\Lambda(\check{E}_{\varepsilon_0})$ and $\Lambda(\check{T}_{\varepsilon_0})$.



(a) The plot does exactly look like the plot of the ellipse with optimal aspect ratio.

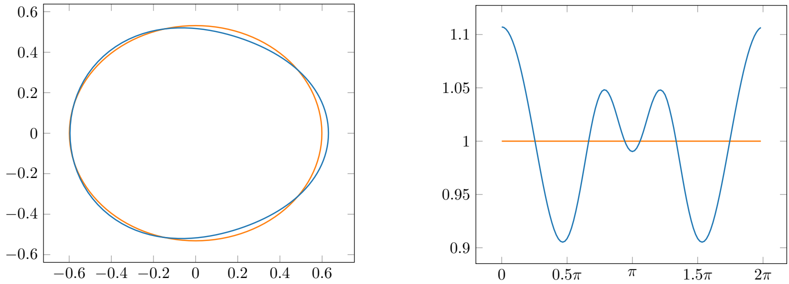


(b) Plot of $\text{abs}_{r_{\varepsilon_0}}(x, y)$ versus the polar angle $\varphi = \arg(x, y)$.

Figure 6.6: Fittest domain $\check{\Omega}_{20}$ for $\varepsilon_0 = 20$ with $|\check{\Omega}_{20}| = 1$ and $m_{x_1}(\check{\Omega}_{20}) = 0$ (blue). The domain differs only slightly from the ellipse with optimal aspect ratio.

approximations of the domains. Proceeding in the same way for $\varepsilon_0 = 30$ reveals that $\check{\Omega}_{30}$ differs even more from the optimal ellipse $\check{E}_{30}$ and is rather egg-shaped.

It is noteworthy that any egg-shaped domain in Figure 6.5 arises from an elliptical start domain and slices of a pie appear when starting with optimal triangles $\check{T}_{\varepsilon_0}$. In order to inspect this circumstances closer we compare the values of Λ for our limiting domains from our gradient descent when starting with optimal triangles, respectively optimal ellipses. Figure 6.8 shows that up to $\varepsilon_0 = 31.4$ egg-shaped domains have a lower value for Λ than slices of a pie. If $\varepsilon_0 > 31.5$ slices of a pie result in lower values for Λ than egg-shaped domains. The transition between both domains is located somewhere in $]31.4, 31.5[$. However, egg-shaped domains resulting from the gradient descent when starting with optimal ellipses still occur as local minima up to some point $\varepsilon_0 \in]33.70, 33.75[$ where the value of Λ drops significantly and coincides to those of the limiting domains when starting the gradient descent with optimal triangles, cf. Figure 6.9. In fact, examining the domains in the gradient descent for $\varepsilon_0 = 35$ and $\check{E}_{35}$ as starting domain after each step reveales egg-shaped domains as intermediate stages which then transform into a slice of a pie as the evaluation continues, see Figure 6.10.



(a) Both domains are scaled to the same area and aligned to share the center of mass.

(b) Plot of $\mathrm{abs}_{r_{\varepsilon_0}}(x, y)$ versus the polar angle $\varphi = \arg(x, y)$.

Figure 6.7: Fittest domain $\check{\Omega}_{30}$ for $\varepsilon_0 = 30$ with $|\check{\Omega}_{30}| = 1$ and $m_{x_1}(\check{\Omega}_{30}) = 0$ (blue). The domain clearly differs from the ellipse $\check{E}_{30}$ with optimal ratio (orange).

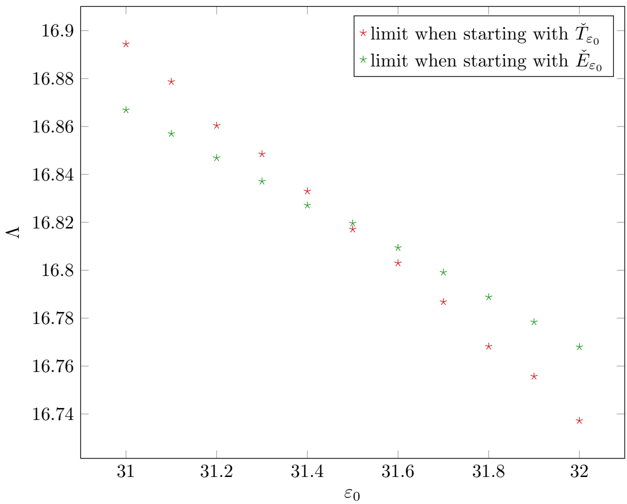


Figure 6.8: Minimal values for Λ achieved in the gradient descent when starting with the optimal triangle (red stars) or with the optimal ellipse (green stars).

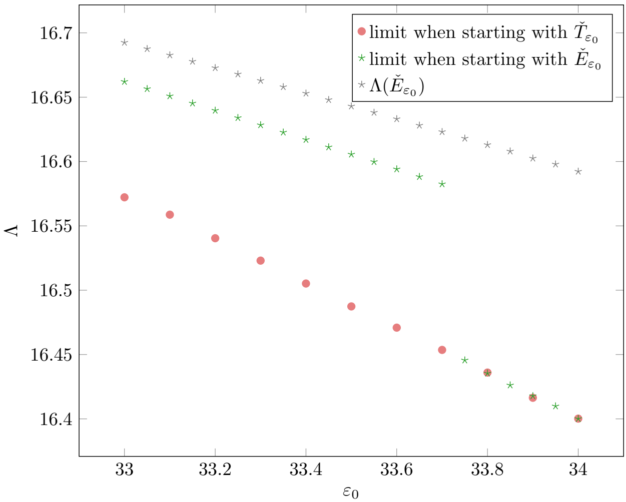
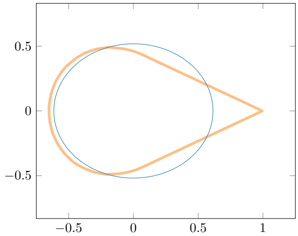
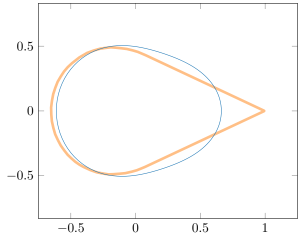


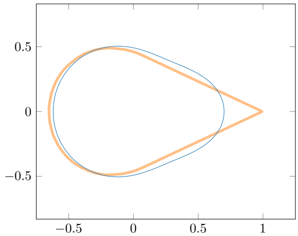
Figure 6.9: Minimal values for Λ achieved in the gradient descent when starting with the optimal triangle (red dots) or with the optimal ellipse (green stars). Between $\varepsilon_0 = 33.7$ and $\varepsilon_0 = 33.75$ the limiting values when starting with the optimal ellipse drop significantly and coincide with those when starting with the optimal triangle if $\varepsilon_0 \geq 33.75$.



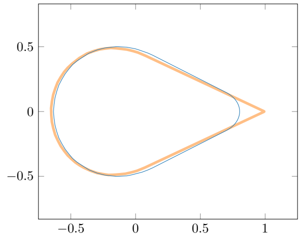
(a) Initial domain.



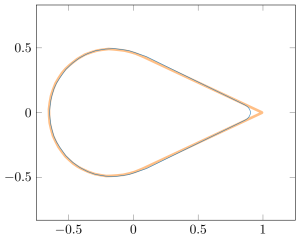
(b) Domain after iteration step 1.



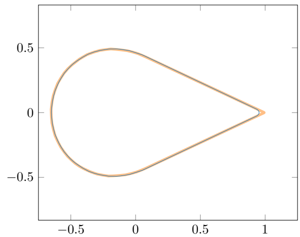
(c) Domain after iteration step 2.



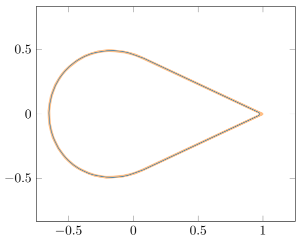
(d) Domain after iteration step 5.



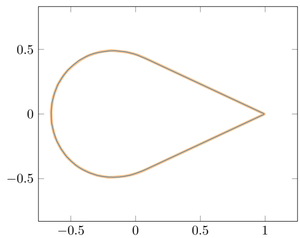
(e) Domain after iteration step 10.



(f) Domain after iteration step 15.



(g) Domain after iteration step 20.



(h) Domain after iteration step 30.

Figure 6.10: Various intermediate domains (blue) when performing the gradient descent for $\varepsilon_0 = 35$ with $\check{E}_{35}$ as starting domain. During the first few steps we arrive at egg-shaped domains which then turn into a slice of a pie. The limiting domain coincides with $\check{\Omega}_{35}$ (orange), shown in each step for comparison. All domains Ω are scaled such that $|\Omega| = 1$ and shifted to match $m_{x_1}(\Omega) = 0$.

Bibliography

[1] A. A. Abramov, A. Aslanyan, and E. B. Davies, *Bounds on complex eigenvalues and resonances*, J. Phys. A **34** (2001), no. 1, 57–72, DOI 10.1088/0305-4470/34/1/304. MR1819914

[2] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. MR0167642

[3] T. Adachi, K. Itakura, K. Ito, and E. Skibsted, *Spectral theory for 1-body Stark operators*, J. Differential Equations **268** (2020), no. 9, 5179–5206, DOI 10.1016/j.jde.2019.11.006. MR4066044

[4] Michael Aizenman and Elliott H. Lieb, *On semiclassical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. A **66** (1978), no. 6, 427–429, DOI 10.1016/0375-9601(78)90385-7. MR598768

[5] Carlos J. S. Alves and Pedro R. S. Antunes, *The method of fundamental solutions applied to boundary value problems on the surface of a sphere*, Comput. Math. Appl. **75** (2018), no. 7, 2365–2373, DOI 10.1016/j.camwa.2017.12.015. MR3777106

[6] Pedro Antunes and Pedro Freitas, *A numerical study of the spectral gap*, J. Phys. A **41** (2008), no. 5, 055201, 19, DOI 10.1088/1751-8113/41/5/055201. MR2433425

[7] ———, *New bounds for the principal Dirichlet eigenvalue of planar regions*, Experiment. Math. **15** (2006), no. 3, 333–342. MR2264470

[8] Pedro R. S. Antunes and Pedro Freitas, *Numerical optimization of low eigenvalues of the Dirichlet and Neumann Laplacians*, J. Optim. Theory Appl. **154** (2012), no. 1, 235–257, DOI 10.1007/s10957-011-9983-3. MR2931377

[9] Patricio Aviles, *Symmetry theorems related to Pompeiu's problem*, Amer. J. Math. **108** (1986), no. 5, 1023–1036, DOI 10.2307/2374594. MR859768

[10] J. E. Avron and I. W. Herbst, *Spectral and scattering theory of Schrödinger operators related to the Stark effect*, Comm. Math. Phys. **52** (1977), no. 3, 239–254. MR0468862

[11] F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1134–1167 (Russian). MR0350504

[12] M. Š. Birman, *On the spectrum of singular boundary-value problems*, Mat. Sb. (N.S.) **55 (97)** (1961), 125–174 (Russian). MR0142896

[13] M. Š. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, 1987. Translated from the 1980 Russian original by S. Khrushchëv and V. Peller. MR1192782

[14] Daniele Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer. **19** (2010), 1–120, DOI 10.1017/S0962492910000012. MR2652780

[15] Philippe Briet and Mounira Gharsalli, *Stark resonances in 2-dimensional curved quantum waveguides*, Rep. Math. Phys. **76** (2015), no. 3, 317–338, DOI 10.1016/S0034-4877(15)30036-7. MR3441149

[16] ———, *Stark resonances in a quantum waveguide with analytic curvature*, J. Phys. A **49** (2016), no. 49, 495202, 13, DOI 10.1088/1751-8113/49/49/495202. MR3584387

[17] Dorin Bucur, *Minimization of the k-th eigenvalue of the Dirichlet Laplacian*, Arch. Ration. Mech. Anal. **206** (2012), no. 3, 1073–1083, DOI 10.1007/s00205-012-0561-0. MR2989451

[18] Dorin Bucur and Giuseppe Buttazzo, *Variational methods in shape optimization problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 65, Birkhäuser Boston, Inc., Boston, MA, 2005. MR2150214

[19] Giuseppe Buttazzo and Gianni Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Rational Mech. Anal. **122** (1993), no. 2, 183–195, DOI 10.1007/BF00378167. MR1217590

[20] Claude Cohen-Tannoudji, Bernard Diu, and Franck Laloë, *Quantenmechanik. - Band 2.*, 5. Auflage, De Gruyter, Berlin, Boston, 2019.

[21] Michael Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. of Math. (2) **106** (1977), no. 1, 93–100, DOI 10.2307/1971160. MR473576

[22] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer Study Edition, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987. MR883643

[23] Daniel Daners, *Domain perturbation for linear and semi-linear boundary value problems*, Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 1–81, DOI 10.1016/S1874-5733(08)80018-6. MR2569323

[24] E. B. Davies, *A review of Hardy inequalities*, The Maz′ya anniversary collection, Vol. 2 (Rostock, 1998), Oper. Theory Adv. Appl., vol. 110, Birkhäuser, Basel, 1999, pp. 55–67. MR1747888

[25] ______, *Sharp boundary estimates for elliptic operators*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 1, 165–178, DOI 10.1017/S0305004100004400. MR1757786

[26] E. B. Davies and Jiban Nath, *Schrödinger operators with slowly decaying potentials*, J. Comput. Appl. Math. **148** (2002), no. 1, 1–28, DOI 10.1016/S0377-0427(02)00570-8. On the occasion of the 65th birthday of Professor Michael Eastham. MR1946184

[27] Michael Demuth, Marcel Hansmann, and Guy Katriel, *Eigenvalues of non-selfadjoint operators: a comparison of two approaches*, Mathematical physics, spectral theory and stochastic analysis, Oper. Theory Adv. Appl., vol. 232, Birkhäuser/Springer Basel AG, Basel, 2013, pp. 107–163, DOI 10.1007/978-3-0348-0591-9_2. MR3077277

[28] Jean Dolbeault, Ari Laptev, and Michael Loss, *Lieb-Thirring inequalities with improved constants*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 4, 1121–1126, DOI 10.4171/JEMS/142. MR2443931

[29] A. Eden and C. Foias, *A simple proof of the generalized Lieb-Thirring inequalities in one-space dimension*, J. Math. Anal. Appl. **162** (1991), no. 1, 250–254, DOI 10.1016/0022-247X(91)90191-2. MR1135275

[30] Ahmad El Soufi, Evans M. Harrell II, Saïd Ilias, and Joachim Stubbe, *On sums of eigenvalues of elliptic operators on manifolds*, J. Spectr. Theory **7** (2017), no. 4, 985–1022, DOI 10.4171/JST/183. MR3737886

[31] G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungsber. Bayer. Akad. Wiss. München, Math.-Phys. Kl. (1923), 169–172 (German).

[32] N. Filonov, *On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator*, Algebra i Analiz **16** (2004), no. 2, 172–176, DOI 10.1090/S1061-0022-05-00857-5 (Russian); English transl., St. Petersburg Math. J. **16** (2005), no. 2, 413–416. MR2068346

[33] Rupert L. Frank, Ari Laptev, and Timo Weidl, *Schrödinger Operators: Eigenvalues and LiebThirring Inequalities*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2022.

[34] Rupert L. Frank, Ari Laptev, Elliott H. Lieb, and Robert Seiringer, *Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials*, Lett. Math. Phys. **77** (2006), no. 3, 309–316, DOI 10.1007/s11005-006-0095-1. MR2260376

[35] Leonid Friedlander, *Some inequalities between Dirichlet and Neumann eigenvalues*, Arch. Rational Mech. Anal. **116** (1991), no. 2, 153–160, DOI 10.1007/BF00375590. MR1143438

[36] Sandro Graffi and Vincenzo Grecchi, *Resonances in Stark effect and perturbation theory*, Comm. Math. Phys. **62** (1978), no. 1, 83–96. MR506369

[37] S. Graffi and V. Grecchi, *Resonances in the Stark effect of atomic systems*, Comm. Math. Phys. **79** (1981), no. 1, 91–109. MR609230

[38] Peter M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007. MR2335496

[39] Evans M. Harrell and Joachim Stubbe, *On sums of graph eigenvalues*, Linear Algebra Appl. **455** (2014), 168–186. MR3217405

[40] Antoine Henrot, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006. MR2251558

[41] Antoine Henrot and Michel Pierre, *Shape variation and optimization*, EMS Tracts in Mathematics, vol. 28, European Mathematical Society (EMS), Zürich, 2018. A geometrical analysis; English version of the French publication [MR2512810] with additions and updates. MR3791463

[42] Antoine Henrot and Edouard Oudet, *Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions*, Arch. Ration. Mech. Anal. **169** (2003), no. 1, 73–87, DOI 10.1007/s00205-003-0259-4. MR1996269

[43] Ira W. Herbst, *Dilation analyticity in constant electric field. I. The two body problem*, Comm. Math. Phys. **64** (1979), no. 3, 279–298. MR520094

[44] Ira W. Herbst and B. Simon, *Dilation analyticity in constant electric field. II. N-body problem, Borel summability*, Comm. Math. Phys. **80** (1981), no. 2, 181–216. MR623157

[45] Dirk Hundertmark, Elliott H. Lieb, and Lawrence E. Thomas, *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator*, Adv. Theor. Math. Phys. **2** (1998), no. 4, 719–731, DOI 10.4310/ATMP.1998.v2.n4.a2. MR1663336

[46] K. Ito and E. Skibsted, *Stationary scattering theory for one-body Stark operators, II*, Ann. Henri Poincaré **23** (2022), no. 2, 513–548, DOI 10.1007/s00023-021-01101-9. MR4386442

[47] V. Ja. Ivriĭ, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34 (Russian). MR575202

[48] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition. MR1335452

[49] Bernhard Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985. MR810619

[50] E. T. Kornhauser and I. Stakgold, *A variational theorem for* $\nabla^2 u + \lambda u = 0$ *and its applications*, J. Math. Physics **31** (1952), 45–54. MR0047236

[51] Evgeny L. Korotyaev, *Resonances for 1d Stark operators*, J. Spectr. Theory **7** (2017), no. 3, 699–732, DOI 10.4171/JST/175. MR3713023

[52] E. Korotyaev and A. Pushnitski, *Trace formulae and high energy asymptotics for the Stark operator*, Comm. Partial Differential Equations **28** (2003), no. 3-4, 817–842, DOI 10.1081/PDE-120020498. MR1978316

[53] Evgeny Korotyaev and Oleg Safronov, *Eigenvalue bounds for Stark operators with complex potentials*, Trans. Amer. Math. Soc. **373** (2020), no. 2, 971–1008, DOI 10.1090/tran/7873. MR4068256

[54] Hynek Kovařík, Semjon Vugalter, and Timo Weidl, *Two-dimensional Berezin-Li-Yau inequalities with a correction term*, Comm. Math. Phys. **287** (2009), no. 3, 959–981, DOI 10.1007/s00220-008-0692-1. MR2486669

[55] Hynek Kovařík and Timo Weidl, *Improved Berezin-Li-Yau inequalities with magnetic field*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 1, 145–160. MR3304579

[56] E. Krahn, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94** (1925), no. 1, 97–100 (German). MR1512244

[57] ———, *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat) **A9** (1926), 1–44 (German).

[58] Ilia Krasikov, *Approximations for the Bessel and Airy functions with an explicit error term*, LMS J. Comput. Math. **17** (2014), no. 1, 209–225, DOI 10.1112/S1461157013000351. MR3230865

[59] Pawel Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space*, J. Funct. Anal. **106** (1992), no. 2, 353–357. MR1165859

[60] ———, *Estimates for sums of eigenvalues of the Laplacian*, J. Funct. Anal. **126** (1994), no. 1, 217–227. MR1305068

[61] Lev Davidovič Landau and Evgenij M. Lifšic, *Lehrbuch der theoretischen Physik 3, Quantenmechanik, Unvernd. Nachdr. d. 9. Aufl.*, Edition Harri Deutsch, Frankfurt am Main, 2007.

[62] A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. **151** (1997), no. 2, 531–545, DOI 10.1006/jfan.1997.3155. MR1491551

[63] Ari Laptev and Oleg Safronov, *Eigenvalue estimates for Schrödinger operators with complex potentials*, Comm. Math. Phys. **292** (2009), no. 1, 29–54, DOI 10.1007/s00220-009-0883-4. MR2540070

[64] Ari Laptev and Timo Weidl, *Recent results on Lieb-Thirring inequalities*, Journées "Équations aux Dérivées Partielles" (La Chapelle sur Erdre, 2000), Univ. Nantes, Nantes, 2000, pp. Exp. No. XX, 14. MR1775696

[65] ———, *Sharp Lieb-Thirring inequalities in high dimensions*, Acta Math. **184** (2000), no. 1, 87–111, DOI 10.1007/BF02392782. MR1756570

[66] Howard A. Levine and Hans F. Weinberger, *Inequalities between Dirichlet and Neumann eigenvalues*, Arch. Rational Mech. Anal. **94** (1986), no. 3, 193–208, DOI 10.1007/BF00279862. MR846060

[67] Michael Levitin, Iosif Polterovich, and David A. Sher, *Pólya's conjecture for the disk: a computer-assisted proof*, arXiv, posted on 15 Mar 2022, DOI 10.48550/ARXIV.2203.07696.

[68] Peter Li and Shing Tung Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), no. 3, 309–318. MR701919

[69] Elliott H. Lieb, *The number of bound states of one-body Schroedinger operators and the Weyl problem*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 241–252. MR573436

[70] Elliott H. Lieb and Walter E. Thirring, *Inequalities for the Moments of the Eigenvalues of the Schrödinger Hamiltonian and Their Relation to Sobolev Inequalities*, The Stability of Matter: From Atoms to Stars: Selecta of Elliott H. Lieb, Springer Berlin Heidelberg, 1991, pp. 135–169, DOI 10.1007/978-3-662-02725-7_13.

[71] Antonios D. Melas, *A lower bound for sums of eigenvalues of the Laplacian*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 631–636, DOI 10.1090/S0002-9939-02-06834-X. MR1933356

[72] R. B. Melrose, *Weyl's conjecture for manifolds with concave boundary*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 257–274. MR573438

[73] Umberto Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math. **3** (1969), 510–585, DOI 10.1016/0001-8708(69)90009-7. MR298508

[74] Édouard Oudet, *Numerical minimization of eigenmodes of a membrane with respect to the domain*, ESAIM Control Optim. Calc. Var. **10** (2004), no. 3, 315–330, DOI 10.1051/cocv:2004011. MR2084326

[75] L. E. Payne, *Inequalities for eigenvalues of membranes and plates*, J. Rational Mech. Anal. **4** (1955), 517–529, DOI 10.1512/iumj.1955.4.54016. MR70834

[76] G. Pólya, *On the characteristic frequencies of a symmetric membrane*, Math. Z. **63** (1955), 331–337, DOI 10.1007/BF01187944. MR73047

[77] ———, *On the eigenvalues of vibrating membranes*, Proc. London Math. Soc. (3) **11** (1961), 419–433, DOI 10.1112/plms/s3-11.1.419. MR129219

[78] ———, *Remarks on the foregoing paper*, J. Math. Physics **31** (1952), 55–57. MR0047237

[79] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, No. 27, Princeton University Press, Princeton, N. J., 1951. MR0043486

[80] A. Pushnitski and V. Sloushch, *Spectral shift function for the Stark operator in the large coupling constant limit*, Asymptot. Anal. **51** (2007), no. 1, 63–89. MR2294105

[81] Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR0493421

[82] G. V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Dokl. Akad. Nauk SSSR **202** (1972), 1012–1015 (Russian). MR0295148

[83] Barry Simon, *Advanced complex analysis*, A Comprehensive Course in Analysis, Part 2B, American Mathematical Society, Providence, RI, 2015. MR3364090

[84] ———, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Physics **97** (1976), no. 2, 279–288, DOI 10.1016/0003-4916(76)90038-5. MR404846

[85] Joachim Stark, *Beobachtungen über den Effekt des elektrischen Feldes auf Spektrallinien. I. Quereffekt*, Annalen der Physik **348** (1914), no. 7, 965-982, DOI https://doi.org/10.1002/andp.19143480702.

[86] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. **3** (1954), 343–356, DOI 10.1512/iumj.1954.3.53017. MR61749

[87] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations. Vol. 2*, Oxford, at the Clarendon Press, 1958. MR0094551

[88] B. Andreas Troesch, *Elliptical membranes with smallest second eigenvalue*, Math. Comp. **27** (1973), 767–772, DOI 10.2307/2005510. MR421277

[89] P. Valtr, *Probability that n random points are in convex position*, Discrete Comput. Geom. **13** (1995), no. 3-4, 637–643, DOI 10.1007/BF02574070. MR1318803

[90] Timo Weidl, *Improved Berezin-Li-Yau inequalities with a remainder term*, Spectral theory of differential operators, Amer. Math. Soc. Transl. Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 253–263, DOI 10.1090/trans2/225/17. MR2509788

[91] ———, *On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$*, Comm. Math. Phys. **178** (1996), no. 1, 135–146. MR1387945

[92] Hermann Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479 (German). MR1511670

[93] D. A. W. White, *The Stark effect and long range scattering in two Hilbert spaces*, Indiana Univ. Math. J. **39** (1990), no. 2, 517–546, DOI 10.1512/iumj.1990.39.39029. MR1089052

[94] Wolfram Research, *NDEigensystem* (2015), `https://reference.wolfram.com/language/ref/NDEigensystem.html`. Accessed May 2022, 16.

[95] ———, *NDEigenvalues* (2015), `https://reference.wolfram.com/language/ref/NDEigenvalues.html`. Accessed May 2022, 16.

[96] Kenji Yajima, *Spectral and scattering theory for Schrödinger operators with Stark effect*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **26** (1979), no. 3, 377–390. MR560003

[97] ———, *Spectral and scattering theory for Schrödinger operators with Stark effect. II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 1, 1–15. MR617860