Existence and non-existence of breather solutions on necklace graphs

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Abstract

In this thesis we are interested in dispersive systems posed on periodic graphs. For instance, periodic graphs are used as phenomenological models for more complex physical structures such as photonic crystals, nano-tubes or graphene. We focus on the existence or non-existence of spatially localized and time-periodic solutions, so called breathers, of nonlinear Klein-Gordon equations posed on necklace graphs.

This thesis is divided into three parts. In the first part we consider a discrete nonlinear Klein-Gordon system on a discrete necklace graph with additional localized potential. We improve existing dispersive estimates up to a temporal decay rate of $(1+t)^{-\frac{3}{2}}$ w.r.t. the ℓ^∞-norm for small symmetric initial conditions. The proof requires suitable integral representations of the linear semigroup and van der Corput's Lemma. Although anti-symmetric initial conditions correspond to eigenstates we are now able to prove asymptotic stability of the zero state for small localized initial data. The energy loss only occurs due to a nonlinear coupling into the absolutely continuous spectrum. This leads to a weaker temporal decay rate for small localized initial data in the nonlinear problem as compared to the temporal decay rate in the linear problem.

In contrast to the non-existence result of spatially localized and time-periodic solutions in the first part, we show two existence results for breather solutions in nonlinear Klein-Gordon systems on a large class of discrete periodic graphs in the second part of this thesis. The proofs are based on the Theorem of Crandall and Rabinowitz. In order to prove the existence results we request a non-resonance condition and invariance conditions depending on the nonlinearity and on the topological structure of the discrete graph.

Finally, in the last part we prove the convergence of generalized breather solutions on discrete necklace graphs towards breather solutions on the metric necklace graph as the discretization parameter goes to zero. This result is relevant for the numerical computation of breather solutions since discrete necklace graphs can be seen as discretizations of the metric necklace graph. For the proof we use spatial dynamics, bifurcation theory and a center manifold reduction.

Zusammenfassung

In dieser Arbeit interessieren wir uns für dispersive Systeme, die auf periodischen Graphen gestellt sind. Periodische Graphen werden beispielsweise als phänomenologische Modelle für komplexere physikalische Strukturen wie etwa Photonische Kristalle, Nanoröhren oder Graphen benutzt. Wir fokussieren uns auf die Existenz oder Nicht-Existenz von räumlich lokalisierten und zeitperiodischen Lösungen, sogenannten Breathern, von nichtlinearen Klein-Gordon Gleichungen, die auf Perlenschnur-Graphen gestellt sind.

Diese Arbeit ist in drei Teile gegliedert. Im ersten Teil betrachten wir ein diskretes nichtlineares Klein-Gordon System auf einem diskreten Perlenschnur-Graphen mit einem zusätzlichen lokalisierten Potential. Wir verbessern vorhandene dispersive Abschätzungen bis zu einer zeitlichen Abfallrate von $(1+t)^{-\frac{3}{2}}$ bezüglich der ℓ^∞-Norm für kleine symmetrische Anfangsbedingungen. Der Beweis benötigt geeignete Integraldarstellungen der linearen Halbgruppe und van der Corput's Lemma. Obwohl antisymmetrische Anfangsbedingungen zu Eigenfunktionen gehören, sind wir nun in der Lage asymptotische Stabilität der Nulllösung für kleine lokalisierte Anfangsdaten zu beweisen. Der Energieverlust tritt nur aufgrund einer nichtlinearen Kopplung in das absolutstetige Spektrum auf. Dies führt zu einer schwächeren zeitlichen Abfallrate für kleine lokalisierte Anfangsdaten des nichtlinearen Problems im Vergleich zu der zeitlichen Abfallrate des linearen Problems.

Im Kontrast zum Nicht-Existenzresultat von räumlich lokalisierten und zeitperiodischen Lösungen im ersten Teil zeigen wir im zweiten Teil dieser Arbeit zwei Existenzresultate für Breather Lösungen in nichtlinearen Klein-Gordon Systemen für eine große Klasse diskreter periodischer Graphen. Die Beweise basieren auf dem Theorem von Crandall und Rabinowitz. Um die Existenzresultate zu beweisen, fordern wir eine Nicht-Resonanzbedingung und Invarianzbedingungen, die von der Nichtlinearität und der topologischen Struktur des diskreten Graphen abhängen.

Schließlich beweisen wir im letzten Teil die Konvergenz von verallgemeinerten Breather Lösungen auf diskreten Perlenschnur-Graphen gegen Breather Lösungen auf metrischen Perlenschnur-Graphen, während der Diskretisierungsparameter gegen Null geht. Dieses Resultat ist relevant für die numerische Berechnung von Breather Lösungen, da diskrete Perlenschnur-Graphen als Diskretisierungen des metrischen Perlenschnur-Graphen gesehen werden können. Für den Beweis benutzen wir räumliche Dynamik, Bifurkationstheorie und eine Zentrumsmannigfaltigkeitenreduktion.

Acknowledgments and declaration

First and foremost I would like to thank my advisor Guido Schneider for his ongoing advice, support and encouragement over the last years. I am very grateful for many fruitful mathematical discussions and thank him very much for introducing me to exciting areas of research.

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My colleagues at the Institut für Analysis, Dynamik und Modellierung in Stuttgart deserve my special thanks for creating an enjoyable and inspiring atmosphere. Finally, I would like to thank my family and friends. Without their ongoing support this endeavor would not have been possible.

Ich erkäre hiermit, dass ich diese Dissertation selbständig angefertigt und keine anderen als die angegebenen Hilfsmittel benutzt habe. Alle Stellen, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen sind, sind von mir durch Angabe der Quelle als Entlehnung kenntlich gemacht.

I hereby certify that this thesis has been composed by myself and describes my own work unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted and all sources of information have been specifically acknowledged.

Stuttgart, 2023 Tobias Kielwein

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Chapter 1

Introduction

The knowledge about the existence or non-existence of spatially localized solutions in dispersive systems in periodic media is of great importance for numerous fields in nature and technology. Applications for dispersive systems in periodic media arise in many fields of science, for instance as phenological models for more complex physical structures such as photonic crystals, nano-tubes or graphene, cf. [BVFL$^+$07]. A prototypical dispersive system is the cubic Klein-Gordon equation

$$\partial_t^2 u(t,x) = \partial_x^2 u(t,x) - u(t,x) + u^3(t,x), \quad t \in \mathbb{R},\, x \in \mathbb{R},$$

posed on the real axis. Periodic media is often accounted by addition of periodic potentials to the differential equation. A different approach is to capture the properties of the periodic media through the underlying topological structure of a discrete or metric graph. Discrete or metric graphs are networks of edges connected by vertices. In the case of a metric graph we identify the edges by intervals on the real axis that are joined together on the vertices. Against this background, we consider the cubic Klein-Gordon equation

$$\partial_t^2 u(t,x) = \partial_x^2 u(t,x) - u(t,x) + u^3(t,x), \quad t \in \mathbb{R},\, x \in \Gamma,$$

on a one-dimensional periodic metric graph such as, e.g., the periodic necklace graph Γ, cf. Figure 1.1. We impose Kirchhoff boundary conditions at the vertex points, which consist of continuity and conservation of the fluxes.

In this thesis we focus on the existence or non-existence of spatially localized and time-periodic solutions, so called breathers, of nonlinear Klein-Gordon equations posed on discrete and metric necklace graphs. In the case that these structures do not exist for all times, we are interested in the decay rates their energy is radiated to the environment with.

This thesis is divided into three parts. First, we show the non-persistence of small solutions, cf. Chapter 2. Afterwards, we deal with the existence of breather solutions on discrete periodic graphs, cf. Chapter 3. In the last part we investigate the continuum limit of breather solutions on discrete necklace graphs, cf. Chapter 4.

We refrain from giving references to the literature in the introduction. We refer to each respective chapter.

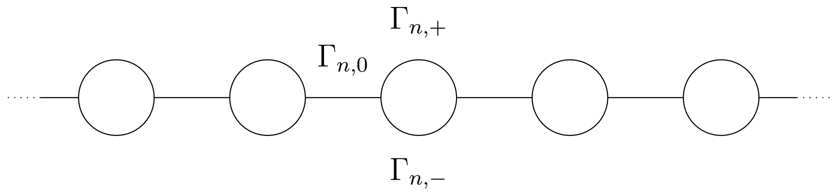

Figure 1.1: The periodic metric necklace graph Γ.

In Chapter 2 we consider the Klein-Gordon system

$$\begin{aligned}
\partial_t^2 u_j(t) &= f(v_j^+(t) - u_j(t)) + f(v_j^-(t) - u_j(t)) - h(u_j(t) - w_{j-1}(t)) + r_u(u_j(t)),\\
\partial_t^2 v_j^+(t) &= g(w_j(t) - v_j^+(t)) - f(v_j^+(t) - u_j(t)) + r_v(v_j^+(t)),\\
\partial_t^2 v_j^-(t) &= g(w_j(t) - v_j^-(t)) - f(v_j^-(t) - u_j(t)) + r_v(v_j^-(t)),\\
\partial_t^2 w_j(t) &= h(u_{j+1}(t) - w_j(t)) - g(v_j^+(t) - w_j(t)) - g(v_j^-(t) - w_j(t)) + r_w(w_j(t)),
\end{aligned}$$

with interaction potentials $f, g, h : \mathbb{R} \to \mathbb{R}$, local potentials $r_u, r_v, r_w : \mathbb{R} \to \mathbb{R}$ and coordinates $u_j, v_j^{\pm}, w_j \in \mathbb{R}$ for all $j \in \mathbb{Z}$ which correspond to the horizontal displacement of the mass particles of its equilibrium positions on the discrete graph Γ with periodic junctions, cf. Figure 1.2.

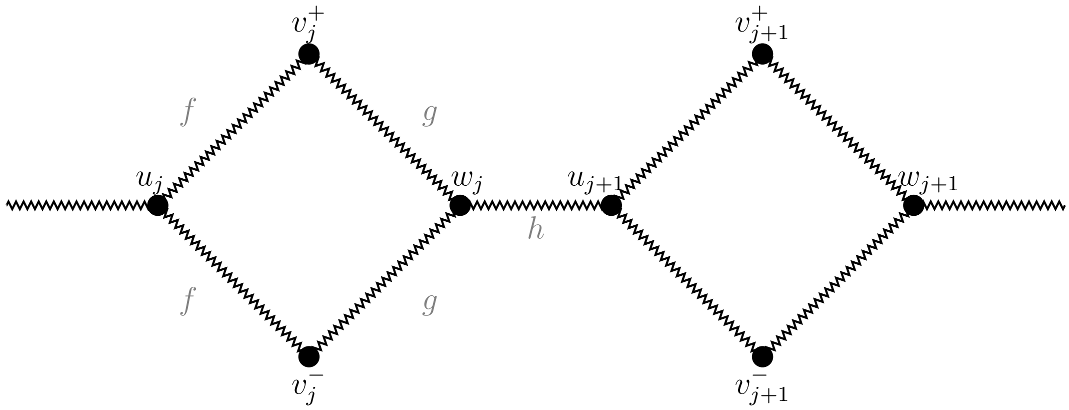

Figure 1.2: Discrete necklace graph Γ with nodes u_j, v_j^+, v_j^- and w_j and interaction forces f, g and h.

We show that in the presence of an additional localized potential solutions of the linear problem to localized initial data that are symmetric w.r.t. the semicircles decay with a rate of $(1+t)^{-\frac{3}{2}}$ in the ℓ^∞-norm.

In addition to that, we prove asymptotic stability of the trivial state, i.e., the zero solution, for localized initial data without restrictions on the symmetry w.r.t. the semicircles. Since anti-symmetric initial data correspond to eigenstates of the linear problem we can not expect the same decay rate for the nonlinear problem. In particular, we show that solutions to localized initial data decay with a rate of $(1+t)^{-\frac{1}{2p-2}}$ in the $\ell^2_{-\sigma}$-norm, cf. equation (2.3), where p is the power of the nonlinear terms and $\sigma > \frac{7}{2}$.

In contrast to the non-existence of time-periodic and spatially localized solutions in Chapter 2 we prove two abstract existence results for breather solutions on periodic discrete graphs in Chapter 3. Since we handle a much larger class of graphs in Chapter 3 it is advantageous to use a different notation in comparison to the notation in Chapter 2. We consider a nonlinear Klein-Gordon type differential equation on a discrete periodic graph Γ. The nonlinear part of the differential equation is proportional to a polynomial of order p.

First, we deduce appropriate non-resonance conditions and invariance conditions depending on the nonlinear terms and the structure of the discrete graph. These conditions are used to apply the Theorem of Crandall and Rabinowitz in order to show the existence of breather solutions.

Furthermore, we illustrate a second existence result for breather solutions with weakened invariance conditions if we introduce a suitable localized potential to our differential equation.

We illustrate the use of these two existence results by providing application examples.

In Chapter 4, we present a result which is relevant for numerical computations of breather solutions of the cubic Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - \alpha u - u^3,$$

with $\alpha > 0$, $x \in \Gamma$, $t \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$, posed on an infinite necklace graph Γ, cf. Figure 1.1. In order to do numerical computations we need to discretize the underlying metric necklace graph Γ. Hence, we introduce discrete necklace graphs which can be seen as discretizations of the metric necklace graph, cf. Figure 1.3.

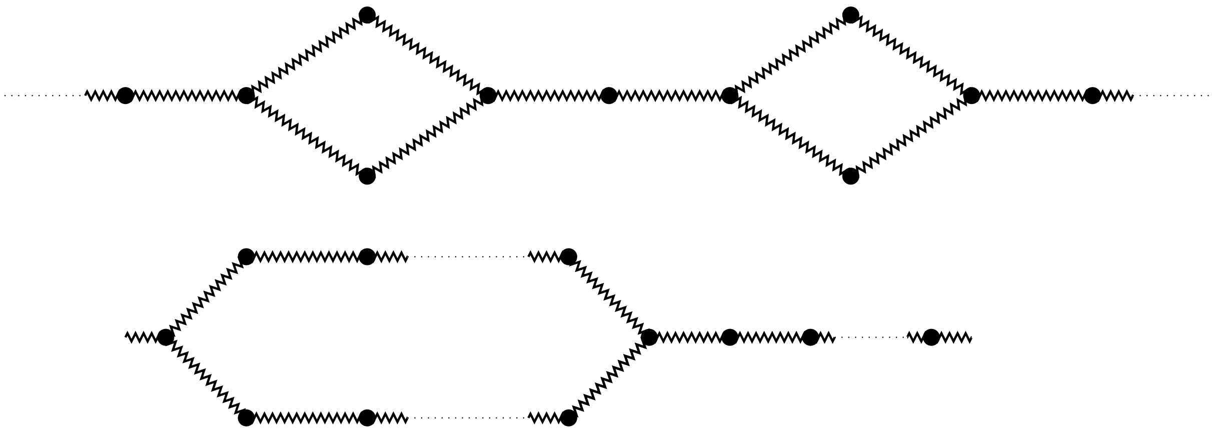

Figure 1.3: Two discretized versions of the metric necklace graph Γ.

On these discrete necklace graphs we consider discrete cubic Klein-Gordon systems. With the help of spatial dynamics and center manifold reduction we show the existence of generalized breather solutions on discrete necklace graphs. Furthermore, we prove the convergence of the generalized breather solutions on discrete necklace graphs towards breather solutions on the metric necklace graph as the discretization parameter goes to zero.

For the reader's convenience we keep the chapters self-contained and introduce the setting at the beginning of each chapter.

Chapter 2

Asymptotic stability on discrete necklace graphs

In this chapter we consider a discrete Klein-Gordon system on a discrete necklace graph with additional localized potential V_{loc}. We show that localized symmetric solutions decay in the linear case with a rate of $(1+t)^{-\frac{3}{2}}$ w.r.t. the ℓ^∞-norm. Based on this estimate, we prove asymptotic stability of the trivial state for small non-symmetric initial data. In particular, small localized initial data decay with the rate $(1+t)^{-\frac{1}{2p-2}}$ w.r.t. the $\ell^2_{-\sigma}$-norm, cf. equation (2.3), where the power p of the nonlinearity is of degree higher than four and even and $\sigma > \frac{7}{2}$.

2.1 Introduction

The question as to whether spatially localized structures can exist for all times is fundamental for many fields in nature. In the case that these structures do not exist for all times, one is interested in the decay rates their energy is radiated to the environment with.

Soffer and Weinstein [SW99] and Prill [Pri15] showed that spatially localized and time-periodic solutions of the linear problem are destroyed by generic nonlinear Hamiltonian perturbations via slow radiation of energy to infinity. This is referred to as *meta-stability*. The dispersive decay is strongly reduced in comparison to the associated linear problem.

The same effects are expected in discrete systems. Cuccagna [CT09] derived asymptotic stability of the trivial state on the lattice $\mathbb{Z}$. In the case of a discrete necklace graph, cf. Figure 2.1, Maier [Mai19] showed that solutions to symmetric initial conditions show a dispersive decay with a rate of $(1+t)^{-1/3}$ in the ℓ^∞-norm. We are interested whether solutions with non-symmetric initial conditions also show dispersive behavior or if such initial conditions lead to time-periodic solutions. Note that anti-symmetric initial conditions correspond to eigenvalues. Thus, temporal decay of anti-symmetric initial conditions can not be expected. However, if we add a localized potential V_{loc} and a suitable nonlinearity we show that the trivial state is asymptotically stable. Hence, the spatially localized time-periodic solutions of the linear system get destroyed. For small localized initial data we show a decay with the rate $(1+t)^{-\frac{1}{2p-2}}$ w.r.t. the $\ell^2_{-\sigma}$-norm where the power p of the nonlinearity is of degree higher than four and even and $\sigma > \frac{7}{2}$.

We consider the Klein-Gordon system

$$\begin{aligned}
\partial_t^2 u_j(t) &= f(v_j^+(t) - u_j(t)) + f(v_j^-(t) - u_j(t)) - h(u_j(t) - w_{j-1}(t)) + r_u(u_j(t)),\\
\partial_t^2 v_j^+(t) &= g(w_j(t) - v_j^+(t)) - f(v_j^+(t) - u_j(t)) + r_v(v_j^+(t)),\\
\partial_t^2 v_j^-(t) &= g(w_j(t) - v_j^-(t)) - f(v_j^-(t) - u_j(t)) + r_v(v_j^-(t)),\\
\partial_t^2 w_j(t) &= h(u_{j+1}(t) - w_j(t)) - g(v_j^+(t) - w_j(t)) - g(v_j^-(t) - w_j(t)) + r_w(w_j(t)),
\end{aligned} \tag{2.1}$$

with interaction potentials $f, g, h : \mathbb{R} \to \mathbb{R}$, local potentials $r_u, r_v, r_w : \mathbb{R} \to \mathbb{R}$ and coordinates $u_j, v_j^{\pm}, w_j \in \mathbb{R}$ for all $j \in \mathbb{Z}$ which correspond to the horizontal displacement of the mass particles of its equilibrium positions on the subsequent discrete graph Γ with periodic junctions, cf. Figure 2.1.

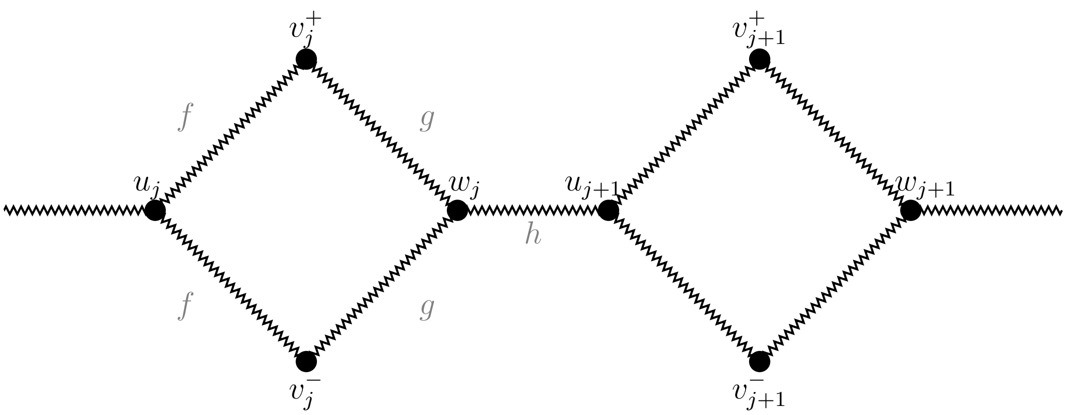

Figure 2.1: Discrete necklace graph Γ with nodes u_j, v_j^+, v_j^- and w_j and interaction forces f, g and h.

We use the Taylor expansion of the forces $f(x) = f_1 x + f_2 x^2 + \ldots$ to split the right-hand side of (2.1) into a linear part L and a nonlinear part N. Thus, by collecting all nodes of the j-th periodicity cell in a vector $(u_j(t), v_j^+(t), v_j^-(t), w_j(t))^\intercal = X(t,j) \in \mathbb{R}^4$ we rewrite the Klein-Gordon system (2.1) as

$$\partial_t^2 X(t,j) + (LX)(t,j,j+1,j-1) = N(X)(t,j,j+1,j-1).$$

As in [Mai19] the linear problem shows dispersive behavior for symmetric initial conditions, i.e., the nodes v_j^+ and v_j^- coincide. However, the linear part L possesses anti-symmetric eigenstates $(0, v_j, -v_j, 0)^\intercal$ and any time-decay can not be expected for the linear problem with anti-symmetric initial conditions. In the nonlinear problem (2.1) there occurs a nonlinear interaction between the anti-symmetric discrete mode (bound state) and the symmetric continuous modes (dispersive radiation). The dispersive decay of the continuous mode is not strong enough to remove all energy from the discrete mode. This mechanism is responsible for the eventual time-decay and non-persistence of trapped states. It turns out that by adding a localized linear potential V_{loc} and a suitable space-weight function γ in front of the nonlinear part N we can show that solutions to small localized initial data show a better time-decay. The localized potential V_{loc} should be symmetric w.r.t. the semicircles and only be supported in one periodicity cell. Hence, we consider the following Cauchy problem

$$\begin{aligned} \partial_t^2 X(t,j) - (HX)(t,j,j+1,j-1) &= \gamma(j)N(X)(t,j,j+1,j-1), \quad t \geq 0, j \in \mathbb{Z}, \\ X(0,j) &= X_0(j), \quad j \in \mathbb{Z}, \\ \partial_t X(0,j) &= X_1(j), \quad j \in \mathbb{Z}, \end{aligned} \tag{2.2}$$

on the discrete graph Γ where $H = -L + V_{\text{loc}}$. We assume $N(X) = \mathcal{O}(|X|^p)$.

Notation. We use $\langle t \rangle := \left(1 + |t|^2\right)^{\frac{1}{2}}$ and equip the vector-valued spaces with the norm

$$\|X\|_{\ell^p_\sigma}^p := \sum_{j \in \mathbb{Z}} |X(j)|^p \langle j \rangle^{p\sigma}. \tag{2.3}$$

Here, $\left\lceil \frac{p-3}{2} \right\rceil$ denotes the smallest integer which is greater than or equal $\frac{p-3}{2}$.

The result concerning the asymptotic stability is formulated in the following theorem.

Theorem 2.1.1. *We consider the Cauchy problem* (2.2)*. Let ω_0^2 be an eigenvalue of the linear operator $-L$. We make the subsequent assumptions:*

(A0) Let $p \in \mathbb{N}_{\text{even}}$, $p \geq 4$ be the exponent of the nonlinear part N and $\gamma(j) = \langle j \rangle^{-\zeta}$ with $\zeta > \frac{1}{2} + (p+1)\sigma$ and $\sigma > \frac{7}{2}$.

(A1) For some $\rho \in \left\{0, \dots, \left\lceil \frac{p-3}{2} \right\rceil \right\}$ the term $(p-2\rho)\omega_0$ falls into the continuous spectrum of the operator $\sqrt{H}$.

(A2) For ρ from assumption (A1) we have

$$\operatorname{Im}\left(\Lambda^-_{p-2\rho} + \Lambda^+_{p-2\rho}\right) > 0,$$

where $\Lambda^\pm_{p-2\rho}$ will be defined in (2.40)*.*

(A3) For every $m \in \mathbb{N}_0$ let $m^2\omega_0^2 \notin \partial\sigma_c(-L)$.

(A4) The localized potential V_{loc} is symmetric w.r.t the semicircles. The eigenvalue Ω^2 generated by V_{loc} does not possess any resonances at the thresholds of $-L$, i.e.,

$$m^2\Omega^2 \notin \partial\sigma_c(-L) \cup \sigma_p(-L), \quad \textit{for every } m \in \mathbb{Z}.$$

Then there is a $\delta > 0$ such that for initial data X_0 and X_1 with $\|X_0\|_{\ell^2_\sigma} < \delta$ and $\|X_1\|_{\ell^2_\sigma} < \delta$ the following holds

$$\|X(t,\cdot)\|_{\ell^2_{-\sigma}} \leq C\langle t \rangle^{-\frac{1}{2p-2}}.$$

Remark 2.1.2. The system (2.2) possesses localized eigenstates and its energy is conserved over time. Theorem 2.1.1 implies that the energy of small initial conditions gets dispersed to infinity through time despite the presence of the localized eigenstates.

Remark 2.1.3. The arguments used in the proof heavily rely on the structure of the discrete necklace graph. Although, discrete necklace graphs with more nodes per periodicity cell can possess more than one eigenvalue the structure remains the same. Thus, the proof can be adapted for discrete necklace graphs with more nodes per periodicity cell if we guarantee a certain symmetry for the semicircles, cf. Figure 2.2.

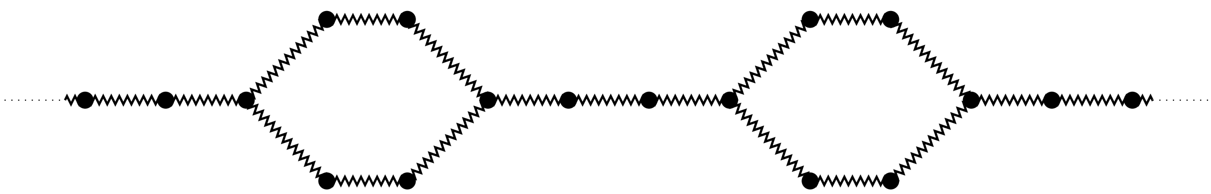

Figure 2.2: A discrete necklace graph with eight nodes per periodicity cell.

Remark 2.1.4. We do not expect that this proof can be adapted for other discrete periodic graphs without certain assumptions concerning their structure.

Remark 2.1.5. The nonlinear interaction between the discrete mode and the continuous modes is important and guaranteed by assumptions (A0) and (A1). Assumption (A2) assures that energy can be transferred from the discrete mode to the continuous modes. In principle, we have a nonlinear analogon of Fermi's Golden rule.

Remark 2.1.6. The assumptions (A3) and (A4) ensure that we improve the linear dispersive estimates from [Mai19].

Although the strategy used to establish Theorem 2.1.1 is similar to the one used in [SW99,Pri15], the following new challenges occur. First, the underlying periodic structure of the necklace graph already possesses an eigenvalue $-\omega_0^2$. Second, we have to request an even power for the nonlinear term since the eigenspace of the eigenvalue $-\omega_0^2$ is invariant under odd power nonlinearities.

An extensive insight in the long-time behavior of the solutions corresponding to the linear equation forms the basis for the subsequent investigation of the nonlinear problem. Therefore, we establish initially a subsidiary result, cf. Lemma 2.4.1, improving the dispersive decay rate from [Mai19] for symmetric initial conditions. We obtain a time decay of $(1+t)^{-\frac{3}{2}}$ w.r.t. to a $\ell^2_{-\sigma}$-norm. Thus, the energy of the initial conditions gets transported to infinity.

Starting with the discussion of the spectral picture of the underlying operators L and H in Section 2.2 and Section 2.3, we introduce appropriate integral representations for the operators $-L$ and $H = -L + V_{\text{loc}}$. With the help of von Neumann's spectral theorem and the functional calculus for self-adjoint operators we write the solution of the linear problem as

$$P_c X(t,j) = \cos(\sqrt{H}t) P_c X_0(j) + \frac{\sin(\sqrt{H}t)}{\sqrt{H}} P_c X_1(j),$$

where P_c denotes the projection onto the absolutely continuous spectral subspace of ℓ^2 w.r.t. the operator H since we can only expect decay for initial conditions from the absolutely continuous spectral subspace of ℓ^2 w.r.t. the operator H. Thus, this identity and the representation of $\cos(x)$ and $\sin(x)$ via the exponential function e^x allow us to deal with operators of the form

$$\sqrt{H}^{-\beta} \mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c, \quad \beta \in \{0,1\}.$$

Their corresponding integral representations can be derived with the help of suitable transformations introduced in Lemma 2.3.1 respectively Lemma 2.3.4. The integral kernels consist of integrands which display a stronger decay behavior. However, we have to restrict ourselves such that the initial data have to be chosen from a space with faster decay at infinity. This amounts to showing that

$$\left\| \sqrt{H}^{-\beta} \mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c \right\|_{\ell^2_\sigma \to \ell^2_{-\sigma}} \le c \langle t \rangle^{-\frac{3}{2}}.$$

For this purpose, we use the generalized eigenfunctions of H, given by

$$\psi_n(j,l) := \frac{1}{2\pi} \begin{cases} T_n(l) F_{n,+}(j,l) & \text{for } l \ge 0, \\ T_n(-l) F_{n,-}(j,-l) & \text{for } l < 0, \end{cases}$$

where $T_n(l)$ is the transmission coefficient introduced in Lemma 2.3.3. This leads to the subsequent integral representation of the operator $\sqrt{H}^{-\beta} \mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c$ when it acts on a function $X \in \ell^2$:

$$\frac{\mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c}{\sqrt{H}^{\beta}} X(j) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} X(m) \sum_{n=1}^{3} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}}{\omega_n^\beta(l)} \mathrm{e}^{-4\mathrm{i}l(m-j)} \overline{\psi_n(l)}\ \psi_n(l) \mathrm{d}l.$$

The decay estimate from Lemma 2.4.1 is proved if we show for the integral kernel that

$$\left| \frac{\mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c}{\sqrt{H}^{\beta}} X(j) \right| \le c \, \langle t \rangle^{-\frac{3}{2}} \, \langle j \rangle^2 \, \|X\|_{\ell^1_2}.$$

In fact, we have an oscillatory integral which can be estimated using the method of stationary phase. It demonstrates that the existence of a zero of some derivative of the dispersion relation provides the temporal decay behavior of the integral. The generalized eigenfunctions of the operator H possess useful properties which can be drawn from scattering theory results from [DT79, CV11, PS08, KKK06]. The assumptions on the localized potential V_{loc} ensure that the transmission coefficients T_n vanish at the border of the spectral bands. This guarantees that the energy decays sufficiently fast at the band edges. The remaining part can be treated by repeated integration by parts and displays an even stronger temporal decay.

In a similar way we prove dispersive estimates for operators which include singular resolvents of the form $\left(\sqrt{H}-\lambda\pm \mathrm{i}0\right)^{-1}$ for $\lambda\in\sigma_c(\sqrt{H})$ in Lemma 2.4.5.

Before we start with the decay estimate of the nonlinear problem, we derive a priori estimates for the solution of the Cauchy problem (2.2) in Lemma 2.5.2 in Section 2.5.

Next, we outline the strategy for the decay estimate of the nonlinear problem in Section 2.6. Since the operator H possesses an eigenvalue the linear dispersive estimates only hold for the part of the solution which gets projected onto the absolutely continuous spectral subspace. Hence, we only can expect a much weaker decay in the nonlinear problem. Due to the structure of the discrete necklace graph and the even exponent of the nonlinear part N, cf. Assumption (A1) and Remark 2.6.1 and Remark 2.6.2, we can neglect without loss of generality the presence of eigenstates to the eigenvalue Ω^2 in the initial data. Hence, starting with the ansatz

$$X(t,j)=a(t)\Phi(j)+\eta(t,j),$$

where $a(t)$ is the one-dimensional component of the solution corresponding to the eigenvalue ω_0^2 and Φ is the corresponding eigenfunction and $\eta(t,j)$ is the infinite-dimensional component of the solution corresponding to the absolutely continuous spectral subspace, we obtain two coupled evolutionary systems

$$\partial_t^2 a(t)+\omega_0^2 a(t)=\langle\gamma N(a(t)\Phi+\eta(t)),\Phi\rangle, \tag{1DS}$$
$$\partial_t^2\eta(t,j)+H\eta(t,j)=P_c\gamma(j)N(a(t)\Phi(j)+\eta(t,j)). \tag{uDS}$$

We have a one-dimensional oscillator a with frequency ω_0 which interacts with the continuous medium, where η corresponds to the continuous spectrum. The oscillator and medium are in resonance since some of the harmonics $(p-\rho)\omega_0$ fall into the continuous spectrum of the operator H, cf. assumption (A1). Thus, there takes place a significant energy exchange between the two evolutionary systems. Due to assumption (A2), energy is taken from the discrete mode to the continuous modes from where it is dispersed to infinity. We reduce the one-dimensional evolutionary system to a first order differential equation

$$\partial_t A(t)=\frac{1}{2\mathrm{i}\omega_0}\mathrm{e}^{\mathrm{i}\omega_0 t}F(A,\overline{A},\eta t)=F(a,\eta),$$

where we have $a(t)=A(t)\mathrm{e}^{\mathrm{i}\omega_0 t}+\overline{A}(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}$ and $F(a,\eta)$ is the right-hand side of (1DS). Carefully separating the resonant terms from the non-resonant terms in $F(a,\eta)$ by using integration by parts yields

$$\partial_t A(t)=\sum_{k+l=2p-1}\alpha_{kl}A(t)^k\overline{A}(t)^l\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 t}+E,$$

where we collect the remaining higher order terms in E. We want to simplify this equation further. This is achieved by a normal form transformation K where we eliminate all oscillatory terms of order $\mathcal{O}(|A|^{2p-1})_{A\to 0}$, cf. Lemma 2.6.4. The transformed equation, a dispersive Hamiltonian normal form, then reads as

$$\partial_t \tilde{A}(t) = \alpha_{pp-1}\left|\tilde{A}(t)\right|^{2p-2}\tilde{A}(t) + \mathcal{O}\left(\left|\tilde{A}\right|^{3p-2}\right)_{\tilde{A}\to 0} + \tilde{E}_K, \tag{2.4}$$

where the complex coefficient α_{pp-1} has negative real part due to assumption (A2), cf. Remark 2.6.5. If we disregard the error terms and terms of higher order we obtain the equation

$$\partial_t \tilde{A}(t) = \alpha_{pp-1}\left|\tilde{A}\right|^{2p-2}\tilde{A}(t), \tag{2.5}$$

which leads to a time decay rate of $(1+t)^{-\frac{1}{2p-2}}$ for $\tilde{A}$. We prove with the help of Lemma 2.6.6 that (2.4) shows the same time decay as (2.5) if the higher order terms $Q(t) := \mathcal{O}\left(\left|\tilde{A}\right|^{3p-2}\right)_{\tilde{A}\to 0} + \tilde{E}_K$ satisfy

$$|Q(t)| \leq Q_0 \langle t\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}.$$

The proof for this estimate is part of Lemma 2.6.7. Hence, we obtain the decay rate of $(1+t)^{-\frac{1}{2p-2}}$ for the discrete mode $A(t)$.

The anticipated decay rate of $(1+t)^{-1-\frac{1}{2p-2}}$ for the continuous mode η can be made rigorous with the help of Duhamel's principle. Finally, combining these decay estimates of the solution components yields that the solution vanishes with a decay rate of $(1+t)^{-\frac{1}{2p-2}}$ w.r.t. the $\ell^2_{-\sigma}$-norm.

2.2 Spectrum and Floquet-Bloch transform

The spectrum of the linear operator $-L$ on the discrete necklace graph can be computed with the help of the Floquet-Bloch transform which is given by

$$\mathcal{B}(X)(k,l) = \check{X}(k,l) = \sum_{j\in\mathbb{Z}} X_k(j)\mathrm{e}^{\mathrm{i}l\cdot j}, \quad l \in [-\pi,\pi),\ 1 \leq k \leq 4,$$

where k denotes the k-th component of the vector X_j. The Floquet-Bloch transform turns the linear operator $-L$ into a family of multiplication operators $M_{-L}(l)$, $l \in [-\pi,\pi)$. If we fix the parameter l in $M_{-L}(l)$ we obtain a positive definite self-adjoint square matrix

$$M_{-L}(l) = \begin{pmatrix} 2f_1 + h_1 + r_{u,1} & -f_1 & -f_1 & -h_1\mathrm{e}^{-\mathrm{i}l} \\ -f_1 & f_1 + g_1 + r_{v,1} & 0 & -g_1 \\ -f_1 & 0 & f_1 + g_1 + r_{v,1} & -g_1 \\ -h_1\mathrm{e}^{\mathrm{i}l} & -g_1 & -g_1 & 2g_1 + h_1 + r_{w,1} \end{pmatrix}.$$

There exists an orthonormal basis in $\mathbb{R}^4$ of eigenvectors of $M_{-L}(l)$. We denote these eigenvectors by $\phi_n(l) \in \mathbb{R}^4$, $n = 0,1,2,3$ with corresponding eigenvalues $\omega_n^2(l)$. The eigenvectors ϕ_n satisfy

$$\langle \phi_{n_1}, \phi_{n_2}\rangle_{L^2_{\mathrm{per}}} = \delta(n_1 - n_2)$$

for $n_1, n_2 \in \{0,1,2,3\}$ and

$$\sum_{n=0}^{3} (\phi_n(l))_k \left(\overline{\phi_n(l)}\right)_m = \delta(k-m)$$

for $1 \le k, m \le 4$ where k, m denote the k-th resp. m-th entry of a vector.

The well-known Floquet-Bloch theory implies that the spectrum of $-L$ and M_{-L} coincide and possesses band-gap structure

$$\sigma(-L) = \sigma(M_{-L}) = \bigcup_{l\in[-\pi,\pi)} \sigma(M_{-L}(l)).$$

We call

$$\omega_j^2 : [-\pi,\pi) \to \mathbb{R}, \quad l \mapsto \omega_j^2(l)$$

the j-th spectral band of $-L$. We distinguish between two forms of spectral bands. On the one hand we have a flat spectral band if

$$\inf_{l\in[-\pi,\pi)} \omega_j^2(l) = \sup_{l\in[-\pi,\pi)} \omega_j^2(l),$$

and on the other hand we call a spectral band non-flat if

$$\inf_{l\in[-\pi,\pi)} \omega_j^2(l) < \sup_{l\in[-\pi,\pi)} \omega_j^2(l).$$

Flat spectral bands correspond to eigenvalues and therefore are part of the point spectrum of $-L$. The remaining part of the spectrum coincides with the absolutely continuous spectrum of $-L$. We denote the eigenspace of the point spectrum $\sigma_p(-L)$ by E_p and the eigenspace corresponding to the absolutely continuous spectrum $\sigma_{ac}(-L)$ by E_{ac}.

2.3 Explicit integral representations

We deduce explicit integral representations of the operators $-L$ and H. The explicit integral representation of the operator H will be used in Section 2.4 to prove better dispersive decay estimates.

2.3.1 Explicit integral representation of the operator L

We introduce the transformation $\mathfrak{F}_{-L}$ to calculate the integral kernel of $-L$ through the following lemma.

Lemma 2.3.1. *We introduce the transformation*

$$\begin{aligned} \mathfrak{F}_{-L} :& \ell^2(\mathbb{Z},\mathbb{C}^4) \to L^2_{\text{per}}([-\pi,\pi),\mathbb{C}^4), \\ & X \mapsto \check{X}, \end{aligned}$$

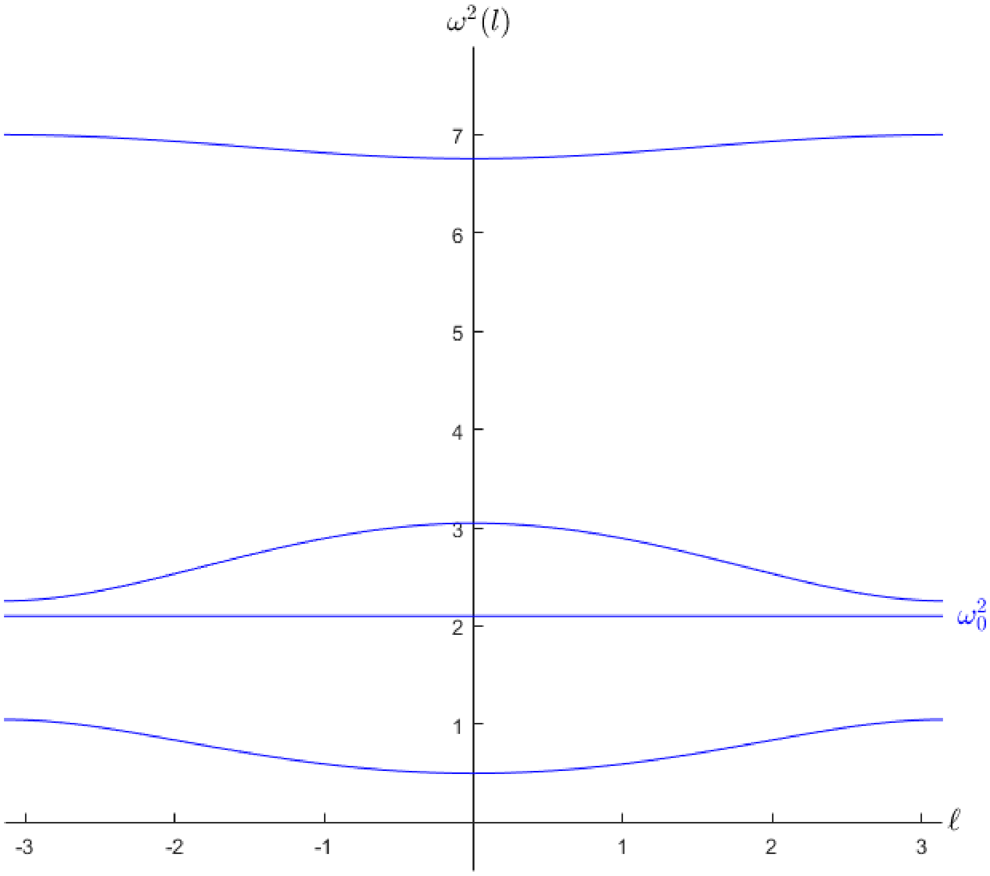

Figure 2.3: The spectral picture of $-L$ consists of four Floquet bands. Here, we have chosen $f_1 = 1.3$, $g_1 = 0.3$, $h_1 = 2$ and $r_1 = 0.5$.

by

$$(\mathfrak{F}_{-L}X)_k(l) := \check{X}_k(l) := \sum_{n=0}^{3}\sum_{j\in\mathbb{Z}} X_k(j)\mathrm{e}^{-\mathrm{i}l(4j+k)}\left(\overline{\phi_n(l)}\right)_k,$$

for $X \in \ell^2(\mathbb{Z},\mathbb{C}^4)$ *. This mapping possesses the properties*

- $\|X\|_{\ell^2}^2 = \|\mathfrak{F}_{-L}X\|_{L^2_{\mathrm{per}}}^2$,
- $X_k(j) = \frac{1}{2\pi}\sum_{n=0}^{3}\int_{-\pi}^{\pi}\left(\mathfrak{F}_{-L}X\right)_k(l)\mathrm{e}^{\mathrm{i}l(4j+k)}\left(\phi_n(l)\right)_k \mathrm{d}l$,
- $\left(\mathfrak{F}_{-L}(-LX)\right)_k(l) = \sum_{n=0}^{3}\omega_n^2(l)\sum_{j\in\mathbb{Z}} X_k(j)\mathrm{e}^{-\mathrm{i}l(4j+k)}\left(\overline{\phi_n(l)}\right)_k$.

Proof. See [Mai19]. □

We are able to formulate an explicit integral representation of the linear operator

$$(-LX)(j) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} X(m)\left(\sum_{n=0}^{3}\int_{-\pi}^{\pi}\omega_n^2(l)\mathrm{e}^{-4\mathrm{i}lm}\overline{\phi_n(l)}\mathrm{e}^{4\mathrm{i}lj}\phi_n(l)\mathrm{d}l\right).$$

It is possible to insert any Borel-measurable function G_t depending on $-L$ in this integral representation instead of $-L$. For a Borel-measurable time-dependent function G_t we obtain

$$(G_t(-L)X)(j) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} X(m)\left(\sum_{n=0}^{3}\int_{-\pi}^{\pi}G_t(\omega_n^2(l))\mathrm{e}^{-4\mathrm{i}lm}\overline{\phi_n(l)}\mathrm{e}^{4\mathrm{i}lj}\phi_n(l)\mathrm{d}l\right),$$

for $X \in \ell^2$.

We introduce the spectral projections P_n onto the n-th spectral band of the operator $-L$. The spectral theorem implies

$$-L = \sum_{n=0}^{3} P_n := P_c + P_p,$$

where P_c is the spectral projection onto the absolutely continuous part and P_p is the projection onto the flat spectral band corresponding to the eigenvalue $\omega_0^2(l)$. The integral representation for the absolutely continuous part is

$$P_c(-LX)(j) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} X(m)\left(\sum_{n=1}^{3}\int_{-\pi}^{\pi}\omega_n^2(l)\mathrm{e}^{-4\mathrm{i}lm}\overline{\phi_n(l)}\mathrm{e}^{4\mathrm{i}lj}\phi_n(l)\mathrm{d}l\right).$$

With the help of Neumann's spectral theorem and the functional calculus the subsequent dispersive estimates were proven in [Mai19].

Remark 2.3.2. For symmetric initial data the solutions to the linear initial value problem obey

$$\|P_cX(t)\|_{\ell^\infty_{-1}} \lesssim \langle t\rangle^{-\frac{1}{2}}\left(\|X_0\|_{\ell^1_1} + \|X_1\|_{\ell^1_1}\right),$$
$$\|P_cX(t)\|_{\ell^\infty} \lesssim \langle t\rangle^{-\frac{1}{3}}\left(\|X_0\|_{\ell^1} + \|X_1\|_{\ell^1}\right),$$

for all $t \geq 0$ and initial data X_0 and X_1. The proof uses van der Corput's Lemma. For the second inequality we use the fact that the spectral bands either satisfy $\partial_l^2 \omega(l) > 0$ or $\partial_t^3 \omega(l) > 0$. This leads to a uniform estimate. The first inequality is a pointwise estimate due to the fact that we use the identity $\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t} = \frac{1}{\pm \mathrm{i}\partial_l \omega_n(l)t} \partial_l \left(\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}\right)$, which leads to a factor $|jm|$ if we use integration by parts.

These dispersive estimates are not strong enough if we want to show that the trivial solution is asymptotically stable. Therefore, we need to improve those estimates. In order to do this we introduce a localized potential V_{loc} in the next section.

2.3.2 Properties of the operator H

In this section we consider the linear initial value problem

$$\begin{aligned} \partial_t^2 X(t,j) &= (HX)(t,j,j+1,j-1), \quad t \geq 0,\ j \in \mathbb{Z}, \\ X(0,j) &= X_0(j), \quad j \in \mathbb{Z}, \\ \partial_t X(0,j) &= X_1(j), \quad j \in \mathbb{Z}. \end{aligned} \tag{2.6}$$

We assume that $V_{\mathrm{loc}} \in \ell_1^1(\mathbb{Z}, \mathbb{R}^4)$ is a generic localized potential, i.e., that V_{loc} generates at most one eigenvalue $\Omega^2 \in \mathbb{R}$. Examples for V_{loc} are sufficiently small positive potentials which are syymetric with respect to the semi-circles, cf. [FK98] Accordingly to assumption (A4) the eigenvalue Ω^2 possesses no resonances at the thresholds of $-L$, i.e.,

$$m^2\Omega^2 \notin \partial\sigma_c(-L), \quad \text{for every } m \in \mathbb{Z}.$$

The new operator H is self-adjoint due to the theorem of Kato-Rellich since $-L$ is a self-adjoint operator and V_{loc} is a bounded symmetric operator. It is essential that the absolutely continuous spectra of H and $-L$ coincide. Since V_{loc} is a compact operator and the resolvent $(-L-z)^{-1}$ is bounded for $z \in \rho(-L)$ the concatenation of these operators is a compact operator. Thus, the following equality for the essential spectrum of H and $-L$ holds

$$\sigma_{ac}(H) = \sigma_{ess}(H) = \sigma_{ess}(-L) = \sigma_{ac}(-L).$$

Since V_{loc} is a localized potential there still exist eigenfunctions corresponding to the eigenvalue ω_0^2 with

$$H\phi_0(j,l) = \omega_0^2 \phi_0(j,l).$$

There exist generalized eigenfunctions $F_{n,\pm}$, $n = 1,2,3$, for the operator H with

$$HF_{n,\pm}(j,l) = \omega_n^2(l) F_{n,\pm}(j,l).$$

It is possible to construct these functions with the help of an analogue of Volterra's integral equation. In our case we obtain

$$\begin{aligned} F_{n,+}(j,l) &= \mathrm{e}^{\mathrm{i}l}\phi_n(l) \\ &\quad - \sum_{m=j}^{\infty} \left(V_{\mathrm{loc}}(m) \cdot F_{n,+}(m,l)\right) \cdot \frac{\mathrm{e}^{\mathrm{i}l(j-m)}\left(\phi_n(l)\overline{\phi_n(l)}\right) - \mathrm{e}^{-\mathrm{i}l(j-m)}\left(\overline{\phi_n(l)}\phi_n(l)\right)}{[\mathrm{e}^{\mathrm{i}l\cdot}\phi_n(l), \mathrm{e}^{-\mathrm{i}l\cdot}\phi_n(l)]}, \end{aligned}$$

and

$$F_{n,-}(j,l) = \mathrm{e}^{\mathrm{i}l}\phi_n(l) \\ - \sum_{-\infty}^{m=j} \left(V_{\mathrm{loc}}(m) \cdot F_{n,-}(m,l)\right) \cdot \frac{\mathrm{e}^{\mathrm{i}l(j-m)}\left(\phi_n(l)\overline{\phi_n(l)}\right) - \mathrm{e}^{-\mathrm{i}l(j-m)}\left(\overline{\phi_n(l)}\phi_n(l)\right)}{[\mathrm{e}^{\mathrm{i}l\cdot}\phi_n(l), \mathrm{e}^{-\mathrm{i}l\cdot}\phi_n(l)]}.$$

We emphasize that the functions $F_{n,\pm}(\cdot,l)$ solve the eigenvalue problem for the operator H with corresponding eigenvalues $\omega_n^2(l)$, $n=1,2,3$. Furthermore, these functions obey the following asymptotic property

$$\lim_{j\to\pm\infty} \mathrm{e}^{-\mathrm{i}lj} F_{n,\pm}(j,l) = \phi_n(l).$$

We want to construct a suitable transformation to obtain an explicit integral representation for which we show improved dispersive estimates. A key element are the so-called transmission coefficients $T_n(l)$ and reflection coefficients $R_{n,\pm}$ from scattering theory which satisfy the equation

$$F_{n,\mp}(j,l) = \frac{1}{T_n(l)}\overline{F_{n,\pm}(j,l)} + \frac{R_{n,\pm}(l)}{T_n(l)}F_{n,\pm}(j,l), \tag{2.7}$$

for $l \notin \{0, \pm\pi\}$. We collect some properties for the transmission coefficients and reflection coefficients in the following lemma.

Lemma 2.3.3. *The transmission coefficients and reflection coefficients can be continuously extended onto the interval $[-\pi,\pi]$. It holds*

- $T_n(-\pi) = T_n(0) = T_n(\pi) = 0$, *if V_{loc} is generic,*
- $\overline{T_n(l)} = T_n(-l), \quad \overline{R_{n,\pm}(l)} = R_{n,\pm}(l),$
- $|T_n(l)|^2 + |R_{n,\pm}(l)|^2 = 1. \quad T_n(l)\overline{R_{n,\pm}(l)} + R_{n,\mp}(l)\overline{T_n(l)} = 0.$

Proof. The general case on the real line was handled in [DT79] and improved for the addition of periodic potentials in [Pri14, CV11]. A similar result for the discrete case can be found in [KKK06, PS08]. Our problem on the discrete necklace graph can be interpreted as a Klein-Gordon equation with additional periodic potential on $\mathbb{Z}$. $\square$

We introduce the notion of the Wronskian for functions $u(\cdot), v(\cdot) : \mathbb{Z} \to \mathbb{C}$ through

$$[u(j), v(j)] = \langle u(j), v(j+1)\rangle - \langle u(j+1), v(j)\rangle .$$

We call $F_{n,\pm}$ Jost functions if they satisfy

$$[F_{n,+}(j,l_s), F_{n,-}(j,l_s)] \neq 0,$$

for $l_s \in \{0, \pm\pi\}$ and $1 \leq n \leq 3$ and $j \in \mathbb{Z}$. The Wronskian of u and v is constant if u and v are solutions to the equation $Hu = \omega_n^2(l)u$. Thus, we have

$$[F_{n,+}(\cdot,l), F_{n,-}(\cdot,l)] = const. \neq 0,$$

for $1 \leq n \leq 3$ and all $l \in [-\pi, \pi]$ since $F_{n,+}(l)$ and $F_{n,-}(l)$ are linearly independent. Further we obtain

$$\left[\mathrm{e}^{\mathrm{i}l\cdot}\phi_n(l), \mathrm{e}^{-\mathrm{i}l\cdot}\phi_n(l)\right] = \left(\mathrm{e}^{-\mathrm{i}l} - \mathrm{e}^{\mathrm{i}l}\right)\left\langle \phi_n(l), \overline{\phi_n(l)}\right\rangle_{\mathbb{C}^4}.$$

These two identities are used to define the transmission coefficients

$$T_n(l) = \frac{\left[\mathrm{e}^{\mathrm{i}l\cdot}\phi_n(l), \mathrm{e}^{-\mathrm{i}l\cdot}\phi_n(l)\right]}{[F_{n,+}(\cdot, l), F_{n,-}(\cdot, l)]},$$

such that (2.7) is satisfied and the properties from Lemma 2.3.3 are fulfilled.

2.3.3 Explicit integral representation of the operator H

In this section we compute the integral kernel for the absolutely continuous part of $H = -L + V_{\mathrm{loc}}$. The time dependency in this section is neglected.

We establish new functions ψ_n, $1 \leq n \leq 3$, which we call Bloch waves or generalized eigenfunctions to the operator H. They are given by

$$\psi_n(j, l) := \frac{1}{2\pi}\begin{cases} T_n(l)F_{n,+}(j, l) & \text{for } l \geq 0, \\ T_n(-l)F_{n,-}(j, -l) & \text{for } l < 0. \end{cases}$$

In particular, the functions ψ_n solve the eigenvalue problem

$$H\psi_n(j, l) = \omega_n^2(l)\psi_n(j, l),$$

for $1 \leq n \leq 3$. Let P_c be the spectral projection onto the absolutely continuous part of H. We introduce a transformation in the following lemma.

Lemma 2.3.4. *We introduce the transformation*

$$\begin{aligned} \mathfrak{F}_H : & P_c\ell^2(\mathbb{Z}, \mathbb{C}^4) \to L^2_{\mathrm{per}}([-\pi, \pi), \mathbb{C}^4), \\ & X \mapsto \check{X}, \end{aligned}$$

by

$$\left(\mathfrak{F}_H X\right)_k(l) := \check{X}_k(l) := \sum_{n=1}^{3}\sum_{j\in\mathbb{Z}} X_k(j)\mathrm{e}^{-\mathrm{i}l(4j+k)}\left(\overline{\psi_n(j, l)}\right)_k,$$

for $X \in P_c\ell^2(\mathbb{Z}, \mathbb{C}^4)$. This mapping possesses the following properties

- $\|X\|^2_{\ell^2} = \|\mathfrak{F}_H X\|^2_{L^2_{\mathrm{per}}}$,
- $X_j(k) = \sum_{n=1}^3 \int_{-\pi}^{\pi} \left(\mathfrak{F}_H X\right)_k(l)\mathrm{e}^{\mathrm{i}l(4j+k)}\left(\psi_n(j, l)\right)_k \mathrm{d}l$,
- $\left(\mathfrak{F}_H H X\right)_k(l) = \sum_{n=1}^3 \omega_n^2(l)\sum_{j\in\mathbb{Z}} X_k(j)\mathrm{e}^{-\mathrm{i}l(4j+k)}\left(\overline{\psi_n(j, l)}\right)_k.$

Proof. We have on the one hand

$$\|X\|_{\ell^2}^2 = \sum_{j\in\mathbb{Z}} \sum_{k=1}^{4} |X_k(j)|^2,$$

and on the other hand

$$\begin{aligned}
\|\mathfrak{F}_H X\|_{L^2\mathrm{per}}^2 &= \int_{-\pi}^{\pi} \overline{\mathfrak{F}_H X(l)} \mathfrak{F}_H X(l) \mathrm{d}l \\
&= \int_{-\pi}^{\pi} \sum_{k=1}^{4} \overline{\mathfrak{F}_H X(l,k)} \mathfrak{F}_H X(l,k) \mathrm{d}l \\
&= \int_{-\pi}^{\pi} \sum_{k=1}^{4} \left(\sum_{n_1=1}^{3} \sum_{j\in\mathbb{Z}} \overline{X_k(j) \mathrm{e}^{-\mathrm{i}l(4j+k)} \left(\overline{\psi_{n_1}(j,l)}\right)_k} \right) \\
&\quad \times \left(\sum_{n_2=1}^{3} \sum_{m\in\mathbb{Z}} X_k(m) \mathrm{e}^{-\mathrm{i}(4m+k)} \left(\overline{\psi_{n_2}(m,l)}\right)_k \right) \mathrm{d}l.
\end{aligned}$$

The integral vanishes for all terms of the sum with $j \neq m$ due to the factor $\mathrm{e}^{-\mathrm{i}l(4(m-j)+k-k)}$. Thus, we obtain

$$\begin{aligned}
\|\mathfrak{F}_H X\|_{L^2_{\mathrm{per}}}^2 &= \int_{-\pi}^{\pi} \sum_{j\in\mathbb{Z}} \sum_{k=1}^{4} |X_k(j)|^2 \sum_{n_1,n_2=1}^{3} (\psi_{n_1}(j,l))_k \left(\overline{\psi_{n_2}(j,l)}\right)_k \mathrm{d}l \\
&= \sum_{j\in\mathbb{Z}} \sum_{k=1}^{4} |X_k(j)|^2 \sum_{n_1,n_2=1}^{3} \langle \psi_{n_1}(j), \psi_{n_2}(j) \rangle_{L^2_{\mathrm{per}}} \\
&= \sum_{j\in\mathbb{Z}} \sum_{k=1}^{4} |X_k(j)|^2 = \|X\|_{\ell^2}.
\end{aligned}$$

We show that

$$\left(\mathfrak{F}_H^{-1} \check{X}\right)_k (l) = \sum_{n=1}^{3} \int_{-\pi}^{\pi} \check{X}_k(l) \mathrm{e}^{\mathrm{i}l(4j+k)} (\psi_n(j,l))_k \mathrm{d}l$$

is the inverse of $\mathfrak{F}_H$ by computing

$$\begin{aligned}
\left(\mathfrak{F}_H^{-1} \check{X}\right)_k (j) &= \sum_{n_1=1}^{3} \int_{-\pi}^{\pi} \sum_{n_2=1}^{3} \sum_{m\in\mathbb{Z}} X_k(m) \mathrm{e}^{-\mathrm{i}l(4m+k)} \left(\overline{\psi_{n_1}(m,l)}\right)_k \mathrm{e}^{\mathrm{i}l(4j+k)} (\psi_{n_2}(j,l))_k \mathrm{d}l \\
&= \int_{-\pi}^{\pi} X_k(j) \sum_{n_1=1}^{3} \sum_{n_2=1}^{3} \left(\overline{\psi_{n_1}(j,l)}\right)_k (\psi_{n_2}(j,l))_k \mathrm{d}l \\
&= X_k(j) \sum_{n_1,n_2=1}^{3} \langle \psi_{n_1}(j), \psi_{n_2}(j) \rangle_{L^2_{\mathrm{per}}} \\
&= X_k(j),
\end{aligned}$$

where we have used that all addends in the integral with $j \neq m$ vanish due to the factor $\mathrm{e}^{-\mathrm{i}l(4(m-j)+k-k)}$.

In the last step we obtain an integral representation of the action of the operator H. In order to proof this representation we use the self-adjointness of H in

$$\begin{aligned}
\left(\mathfrak{F}_H HX\right)_k(l) &= \sum_{n=1}^{3}\sum_{j\in\mathbb{Z}} HX_k(j)\mathrm{e}^{-\mathrm{i}l(4j+k)}\left(\overline{\psi_n(j,l)}\right)_k \\
&= \sum_{n=1}^{3}\left\langle HX_k, \mathrm{e}^{\mathrm{i}l(4\cdot+k)}\left(\overline{\psi_n(\cdot,l)}\right)_k\right\rangle_{\ell^2} \\
&= \sum_{n=1}^{3}\left\langle X_k, H\mathrm{e}^{\mathrm{i}l(4\cdot+k)}\left(\overline{\psi_n(\cdot,l)}\right)_k\right\rangle_{\ell^2} \\
&= \sum_{n=1}^{3}\left\langle X_k, \omega_n^2(l)\mathrm{e}^{\mathrm{i}l(4\cdot+k)}\left(\overline{\psi_n(\cdot,l)}\right)_k\right\rangle_{\ell^2} \\
&= \sum_{n=1}^{3}\omega_n^2(l)\left\langle X_k, \mathrm{e}^{\mathrm{i}l(4\cdot+k)}\left(\overline{\psi_n(\cdot,l)}\right)_k\right\rangle_{\ell^2} \\
&= \sum_{n=1}^{3}\omega_n^2(l)\sum_{j\in\mathbb{Z}} X_k(j)\mathrm{e}^{-\mathrm{i}l(4j+k)}\left(\overline{\psi_n(j,l)}\right)_k .
\end{aligned}$$

□

With the help of this lemma we deduce an integral representation for the operator H. However, this is only valid for $X \in P_c\ell^2$.

$$\begin{aligned}
(HX)(j) &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} X(m)\left(\sum_{n=1}^{3}\int_{-\pi}^{\pi}\omega_n^2(l)\mathrm{e}^{-4\mathrm{i}lm}\overline{\psi_n(l)}\mathrm{e}^{4\mathrm{i}lj}\psi_n(l)\mathrm{d}l\right) \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} X(m)\left(\sum_{n=1}^{3}\int_{-\pi}^{\pi}\omega_n^2(l)\Psi_n(l,j,m)\mathrm{d}l\right).
\end{aligned}$$

It is possible to insert any Borel-measurable function G_t depending on H in the integral representation of the operator H which is then given by

$$(G_t(H)X)(j) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} X(m)\left(\sum_{n=1}^{3}\int_{-\pi}^{\pi}G_t(\omega_n^2(l))\Psi_n(l,j,m)\mathrm{d}l\right), \tag{2.8}$$

for $X \in P_c\ell^2$.

2.4 Linear dispersive estimates

In the first part of this section we improve the existing dispersive estimates from [Mai19]. Furthermore, we show singular resolvent estimates in the second part.

2.4.1 Dispersive estimates for the operator e^{Ht}

That the absolutely continuous part of a solution to (2.2) decays with rate of $(1+t)^{-\frac{3}{2}}$, is part of the following lemma.

Lemma 2.4.1. *Assume that the initial data X_0 and X_1 coincide in the upper and lower semicircles. Then the solutions to the linear problem (2.6) obey the subsequent decay estimate*

$$\|P_c X(t)\|_{\ell^\infty_{-2}} \lesssim \langle t\rangle^{-\frac{3}{2}} \left(\|X_0\|_{\ell^1_2} + \|X_1\|_{\ell^1_2}\right),$$

for $t \geq 0$.

With the help of Neumann's spectral theorem and the functional calculus for self-adjoint operators we write the absolutely continuous part of the solutions of (2.6) as

$$P_c X(t,j) = \cos(\sqrt{H}t) P_c X_0(j) + \frac{\sin(\sqrt{H}t)}{\sqrt{H}} P_c X_1(j). \tag{2.9}$$

The constituents of (2.9) can be rewritten via Euler's formula by

$$G_t^\beta(\sqrt{H}) := \sqrt{H}^{-\beta} \mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c, \quad \beta \in \{0,1\},$$

and are explicitly given

$$\begin{aligned}\left(G_t^\beta(\sqrt{H})X\right)(j) &= \sqrt{H}^{-\beta} \mathrm{e}^{\pm \mathrm{i}\sqrt{H}t} P_c X(j) \\ &= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} X(m) \left(\sum_{n=1}^{3} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\pm\omega_n(l)t}}{\omega_n(l)^\beta} \Psi_n(l,j,m)\mathrm{d}l\right) \\ &= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} X(m) G_{j,m}^\beta(t),\end{aligned} \tag{2.10}$$

for $X \in P_c\ell^2$ by the integral representation (2.8). Thus, it suffices to estimate the equation (2.10). We emphasize that $\beta = 1$ causes no problems since $\omega_n(l) > 0$ for $l \in [-\pi, \pi]$.

A natural approach to such a problem is to use van der Corput's Lemma which is stated in the subsequent lemma.

Lemma 2.4.2. *Let $\varphi(l)$ be a smooth function on the interval $[a,b]$ with $|\partial_l^\nu \varphi(l)| \geq c_\nu > 0$ in (a,b) for $\nu \in \mathbb{N}_{\geq 2}$. Then follows for some constant $C_\nu := 5 \cdot 2^{\nu-1} - 2$ that*

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}\varphi(l)t} \alpha(l)\mathrm{d}l\right| \leq C_\nu (c_\nu t)^{-\frac{1}{\nu}} \left(\min\{|\alpha(a)|, |\alpha(b)|\} + \int_a^b |\partial_l \alpha(l)|\mathrm{d}l\right). \tag{2.11}$$

Proof. For the proof we refer to [SM93]. ◻

For fixed $n \in \{1,2,3\}$, we observe that $\partial_l \omega_n(l)$ possesses zeros at the border of the spectrum, i.e., for $l \in \{0, \pm\pi\}$ we have $\partial_l \omega_n(l) = 0$. Since $\omega_n(l)$ is a smooth function it follows, that $\partial_l^2 \omega_n(l) > 0$ in a small neighborhood of the zeros of $\partial_l \omega_n(l)$. Thus, it is

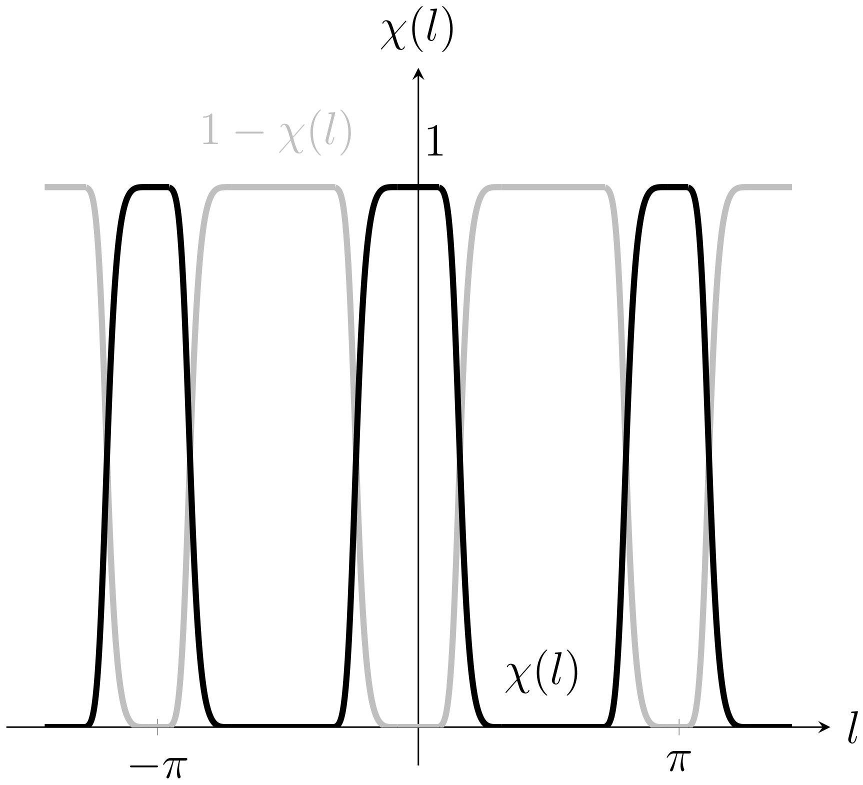

Figure 2.4: The cut-off functions χ and $1-\chi$.

possible to choose a smooth cut-off function $\chi(l)$ such that the second derivative of $\omega_n(l)$ does not vanish in the support of $\chi(l)$, cf. Figure 2.4. As direct consequence the first derivative of $\omega_n(l)$ does not vanish in the support of $1-\chi(l)$. We split $G^{\beta}_{j,m}(t)$ into the parts $A^{\beta}_1(t,j,m)$ and $A^{\beta}_2(t,j,m)$ by

$$\begin{aligned}
G^{\beta}_{j,m}(t) &= \left(\int_{-\pi}^{\pi}\sum_{n=1}^{3}\frac{\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}}{\omega_n(l)^{\beta}}\Psi_n(l,j,m)\mathrm{d}l\right)\\
&= \left(\int_{-\pi}^{\pi}\sum_{n=1}^{3}\frac{\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}}{\omega_n(l)^{\beta}}\chi(l)\Psi_n(l,j,m)\mathrm{d}l\right)\\
&\quad+\left(\int_{-\pi}^{\pi}\sum_{n=1}^{3}\frac{\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}}{\omega_n(l)^{\beta}}(1-\chi(l))\Psi_n(l,j,m)\mathrm{d}l\right)\\
&=: A^{\beta}_1(t,j,m)+A^{\beta}_2(t,j,m).
\end{aligned}$$

First, we analyze the term including $1-\chi$. Since the first derivative of $\omega_n(l)$ does not vanish in the support of $1-\chi$ we use the identity

$$\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}=\frac{1}{\pm\mathrm{i}\partial_l\omega_n(l)t}\partial_l\left(\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\right), \tag{2.12}$$

to estimate the term A_2^β as follows

$$\begin{aligned}\left|A_2^\beta(t,j,m)\right| &= \left|\int_{-\pi}^{\pi}\sum_{n=1}^{3}\frac{1}{\mathrm{i}\partial_l\omega_n(l)t}\partial_l\left(\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\right)\frac{1}{\omega_n(l)^\beta}(1-\chi(l))\Psi_n(l,j,m)\mathrm{d}l\right|\\ &\leq \left|\left[\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta t}\right]_{l=-\pi}^{l=\pi}\right.\\ &\quad\left.-\int_{-\pi}^{\pi}\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta t}\right)\mathrm{d}l\right|\\ &\leq \langle t\rangle^{-1}\left|\int_{-\pi}^{\pi}\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\mathrm{d}l\right|.\end{aligned}$$

We integrated by parts and used the fact that the cut-off function $1-\chi$ vanishes in a small neighborhood of $l\in\{0,\pm\pi\}$. Thus, the term in square brackets vanishes and $\partial_l\omega_n(l)$ in the denominator causes no problems.

Next we repeat these steps to obtain

$$\begin{aligned}\left|A_2^\beta(t,j,m)\right| &\leq \langle t\rangle^{-1}\left|\int_{-\pi}^{\pi}\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\mathrm{d}l\right|\\ &= \langle t\rangle^{-1}\left|\int_{-\pi}^{\pi}\sum_{n=1}^{3}\frac{1}{\mathrm{i}\partial_l\omega_n(l)t}\partial_l\left(\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\right)\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\mathrm{d}l\right|\\ &\leq \langle t\rangle^{-2}\left|\left[\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\frac{1}{\mathrm{i}\partial_l\omega_n(l)}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\right]_{l=-\pi}^{l=\pi}\right.\\ &\quad\left.-\int_{-\pi}^{\pi}\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\partial_l\left(\frac{1}{\mathrm{i}\partial_l\omega_n(l)}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\right)\mathrm{d}l\right|\\ &\leq \langle t\rangle^{-2}\left|\int_{-\pi}^{\pi}\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\partial_l\left(\frac{1}{\mathrm{i}\partial_l\omega_n(l)}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\right)\mathrm{d}l\right|.\end{aligned}$$

The functions $1-\chi$, ω_n and Ψ_n are smooth in l and can be bounded by a constant C which does not depend on l. The same holds for the derivatives of $1-\chi$ and ω_n. Due to the factor $\mathrm{e}^{-4il(m-j)}$ the derivatives of Ψ_n contain powers of $(m-j)$. We bound these terms by $\langle j\rangle^2\langle m\rangle^2$. Therefore, we estimate the integral term via

$$\left|\int_{-\pi}^{\pi}\sum_{n=1}^{3}\mathrm{e}^{\pm\mathrm{i}\omega_n(l)t}\partial_l\left(\frac{1}{\mathrm{i}\partial_l\omega_n(l)}\partial_l\left(\frac{(1-\chi(l))\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta}\right)\right)\mathrm{d}l\right| \leq C\langle j\rangle^2\langle m\rangle^2,$$

from where we deduce the inequality

$$\left|A_2^\beta(t,j,m)\right| \leq C\langle t\rangle^{-2}\langle j\rangle^2\langle m\rangle^2.$$

We write the function $\chi(l)=\chi_1(l)+\chi_2(l)+\chi_3(l)$ as sum of three cut-off functions which are localized around $-\pi$, 0 and π respectively, cf. Figure 2.4. We split the term

$A_1^\beta(t,j,m)$ accordingly to the partition of χ into χ_1, χ_2 and χ_3 into

$$A_{1,\mu}^\beta(t,j,m) := \int_{-\pi}^{\pi} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}}{\omega_n(l)^\beta} \chi_\mu(l)\Psi_n(l,j,m)\mathrm{d}l, \quad \mu = 1,2,3.$$

We start by investigating $A_{1,2}^\beta$ by using the identity (2.12) and integration by parts. This leads to

$$\begin{aligned}
\left|A_{1,2}^\beta(t,j,m)\right| &= \left| \int_{-\pi}^{\pi} \sum_{n=1}^{3} \frac{1}{\mathrm{i}\partial_l\omega_n(l)t} \partial_l \left(\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}\right) \frac{1}{\omega_n(l)^\beta} \chi_2(l)\Psi_n(l,j,m)\mathrm{d}l \right| \\
&\leq \left| \left[\sum_{n=1}^{3} \mathrm{e}^{\pm \mathrm{i}\omega_n(l)t} \frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta t} \right]_{l=-\pi}^{l=\pi} \right. \\
&\quad \left. - \int_{-\pi}^{\pi} \sum_{n=1}^{3} \mathrm{e}^{\pm \mathrm{i}\omega_n(l)t} \partial_l \left(\frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta t} \right) \mathrm{d}l \right| \\
&\leq \langle t\rangle^{-1} \left| \int_{-\pi}^{\pi} \sum_{n=1}^{3} \mathrm{e}^{\pm \mathrm{i}\omega_n(l)t} \partial_l \left(\frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta} \right) \mathrm{d}l \right|.
\end{aligned}$$

The boundary terms from the integration by parts vanish since the support of χ_2 is contained in an interval $[-\delta,\delta]$ for some small $\delta > 0$. The zero of $\partial_l\omega_n(l)$ at $l = 0$ in the denominator causes no problems since the terms

$$\partial_l^r \left(\frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta} \right) \leq C, \quad r = 0,1,$$

are bounded for small l due to the fact that $\Psi_n(0) = 0$ thanks to the transmission coefficients $T_n(l) = \overline{T_n(-l)}$ which are zero for $l = 0$.

It is not possible to repeat the integration by parts. However, we use van der Corput's Lemma to further estimate $A_{1,2}^\beta$ since $\partial_l^2\omega_n(l) \neq 0$ for $l \in \operatorname{supp}\chi_2$. We use (2.11) with $\nu = 2$ and $\varphi = \omega_n$ for $n = 1,2,3$ and obtain

$$\begin{aligned}
\left|A_{1,2}^\beta(t,j,m)\right| &\leq \langle t\rangle^{-1} \left| \int_{-\delta}^{\delta} \sum_{n=1}^{3} \mathrm{e}^{\pm \mathrm{i}\omega_n(l)t} \partial_l \left(\frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta} \right) \mathrm{d}l \right| \\
&\leq C\langle t\rangle^{-\frac{3}{2}} \left(\int_{-\delta}^{\delta} \sum_{n=1}^{3} \left| \partial_l^2 \left(\frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta} \right) \right| \mathrm{d}l \right. \\
&\quad \left. + \sum_{n=1}^{3} \min_{l=\pm\delta} \left\{ \left| \partial_l \left(\frac{\chi_2(l)\Psi_n(l,j,m)}{i\left(\partial_l\omega_n(l)\right)\omega_n(l)^\beta} \right) \right| \right\} \right) \\
&\leq C\langle t\rangle^{-\frac{3}{2}} \langle j\rangle^2 \langle m\rangle^2.
\end{aligned}$$

The cut-off functions χ_1 and χ_3 can be considered simultaneously. We start with a change of coordinates

$$\tilde{l} := \begin{cases} l+\pi & , l \leq 0, \\ l-\pi & , l > 0. \end{cases}$$

We recall that the functions ω_n and Ψ_n are 2π-periodic in l. In addition we have $\operatorname{supp}\chi_1 \subset [-\pi-\delta, -\pi+\delta]$ and $\operatorname{supp}\chi_3 \subset [\pi-\delta, \pi+\delta]$. Thus, we compute with the coordinate change

$$\begin{aligned}
A^\beta_{1,1}(t,l,m) + A^\beta_{1,3}(t,j,m) &= \int_{-\pi}^{\pi} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}}{\omega_n(l)^\beta} \chi_1(l)\Psi_n(l,j,m)\mathrm{d}l \\
&\quad + \int_{-\pi}^{\pi} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}}{\omega_n(l)^\beta} \chi_3(l)\Psi_n(l,j,m)\mathrm{d}l \\
&= \int_{-\pi}^{-\pi+\delta} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}}{\omega_n(l)^\beta} \chi_1(l)\Psi_n(l,j,m)\mathrm{d}l \\
&\quad + \int_{\pi-\delta}^{\pi} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(l)t}}{\omega_n(l)^\beta} \chi_3(l)\Psi_n(l,j,m)\mathrm{d}l \\
&= \int_{0}^{\delta} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(\tilde{l})t}}{\omega_n(\tilde{l})^\beta} \chi_1(\tilde{l})\mathrm{e}^{-4\mathrm{i}\tilde{l}m}\overline{\psi_n(\tilde{l})}\mathrm{e}^{4\mathrm{i}\tilde{l}j}\psi_n(\tilde{l})\mathrm{d}\tilde{l} \\
&\quad + \int_{-\delta}^{0} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\omega_n(\tilde{l})t}}{\omega_n(\tilde{l})^\beta} \chi_3(\tilde{l})\mathrm{e}^{-4\mathrm{i}\tilde{l}m}\overline{\psi_n(\tilde{l})}\mathrm{e}^{4\mathrm{i}\tilde{l}j}\psi_n(\tilde{l})\mathrm{d}\tilde{l} \\
&= \int_{-\delta}^{\delta} \sum_{n=1}^{3} \frac{\mathrm{e}^{\pm \mathrm{i}\tilde{\omega}_n(\tilde{l})t}}{\tilde{\omega}_n(\tilde{l})^\beta} \tilde{\chi}(\tilde{l})\mathrm{e}^{-4\mathrm{i}\tilde{l}m}\overline{\tilde{\psi}_n(\tilde{l})}\mathrm{e}^{4\mathrm{i}\tilde{l}j}\tilde{\psi}_n(\tilde{l})\mathrm{d}\tilde{l},
\end{aligned}$$

where we have taken together the cut-off functions χ_1 and χ_3 as a new function $\tilde{\chi}$ as well as the amplitude functions. The last integral is of the same form as $A^\beta_{1,2}$ and can be treated analogously. Therefore we obtain

$$\left| A^\beta_{1,\mu}(t,j,m) \right| \le C \langle t \rangle^{-\frac{3}{2}} \langle j \rangle^2 \langle m \rangle^2, \quad \mu = 1,3.$$

Combining the estimates for A^β_1 and A^β_2 we get the desired result for $G^\beta_{j,m}(t)$ in

$$\left| G^\beta_{j,m}(t) \right| \le C \langle t \rangle^{-\frac{3}{2}} \langle j \rangle^2 \langle m \rangle^2.$$

As a direct consequence we obtain that

$$\left\| G^\beta_t(\sqrt{H})X \right\|_{\ell^\infty_{-2}} \lesssim \langle t \rangle^{-\frac{3}{2}} \|X\|_{\ell^1_2}.$$

Finally, it follows with (2.9) that

$$\|P_c X(t)\|_{\ell^\infty_{-2}} \lesssim \langle t \rangle^{-\frac{3}{2}} \left(\|X_0\|_{\ell^1_2} + \|X_1\|_{\ell^1_2} \right), \quad t \ge 0,$$

which proves Lemma 2.4.1.

Corollary 2.4.3. *For $\sigma > \frac{5}{2}$ and symmetric initial data X_0 and X_1 the decay estimate from Lemma 2.4.1 can be transferred into a decay estimate for ℓ^2_σ-spaces*

$$\|P_c X(t)\|_{\ell^2_{-\sigma}} \lesssim \langle t \rangle^{-\frac{3}{2}} \left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma} \right), \tag{2.13}$$

for $t \ge 0$.

Proof. For any $\sigma \in \mathbb{R}$ the norm corresponding to the space ℓ^2_σ is given by

$$\|X\|_{\ell^2_\sigma} = \left(\sum_{j\in\mathbb{Z}} \left|\langle j\rangle^\sigma X(j)^2\right|^2\right)^{\frac{1}{2}}.$$

We compute for $\sigma > \frac{5}{2}$ that

$$\begin{aligned}
\left\|G_t^\beta(\sqrt{H})X\right\|^2_{\ell^2_{-\sigma}} &= \sum_{j\in\mathbb{Z}} \left|\langle j\rangle^{-\sigma} \sum_{m\in\mathbb{Z}} X(m) G^\beta_{j,m}(t)\right|^2 \\
&\le \sum_{j\in\mathbb{Z}} \langle j\rangle^{-2\sigma} \left|\sum_{m\in\mathbb{Z}} \langle m\rangle^\sigma X(m) \langle m\rangle^{-\sigma} G^\beta_{j,m}(t)\right|^2 \\
&\le c\,\langle t\rangle^{-3} \left(\sum_{j\in\mathbb{Z}} \langle j\rangle^{-2\sigma+4}\right) \left|\sum_{m\in\mathbb{Z}} \langle m\rangle^\sigma X(m) \langle m\rangle^{-\sigma+2}\right|^2 \\
&\le c\,\langle t\rangle^{-3} \left(\sum_{j\in\mathbb{Z}} \langle j\rangle^{-2\sigma+4}\right) \left(\sum_{m\in\mathbb{Z}} \langle m\rangle^{2\sigma} X(m)^2\right) \left(\sum_{m\in\mathbb{Z}} \langle m\rangle^{-2\sigma+4}\right) \\
&\le c\,\langle t\rangle^{-3} \left(\sum_{j\in\mathbb{Z}} \langle j\rangle^{-2\sigma+4}\right) \left(\sum_{m\in\mathbb{Z}} \langle m\rangle^{-2\sigma+4}\right) \|X\|^2_{\ell^2_\sigma} \\
&\le C\,\langle t\rangle^{-3} \|X\|^2_{\ell^2_\sigma}
\end{aligned}$$

By taking the root and using the identity (2.9) we get the desired estimate (2.13). □

Remark 2.4.4. The decay estimate (2.13) still holds if we replace P_c by $G_t(H)P_c$ where $G_t(H)$ is some time-dependent Borel-measurable function.

2.4.2 Singular resolvent estimates

In order to prove the asymptotic stability result from Theorem 2.1.1 in Section 2.6 we need estimates of terms which include singular resolvents, i.e., operators of the form $\mathrm{e}^{\pm \mathrm{i}t\sqrt{H}}(\sqrt{H} - \lambda \pm \mathrm{i}0)^{-1}P_c$ with $\lambda \in \sigma_c(\sqrt{H})$. The case $\lambda \notin \sigma(\sqrt{H})$ can be handled by (2.13) and Remark 2.4.4. We state the desired decay estimate for the singular resolvents in

Lemma 2.4.5. *All assumptions from Theorem 2.1.1 are met and let λ be an arbitrary point inside the continuous spectrum $\sigma_c(\sqrt{H})$. Then we obtain the decay estimate*

$$\left\|\langle\cdot\rangle^{-\sigma}\mathrm{e}^{\pm\mathrm{i}t\sqrt{H}}(\sqrt{H} - \lambda \pm \mathrm{i}0)^{-1}P_c\langle\cdot\rangle^{-\sigma}g\right\|_{\ell^2} \le c\langle t\rangle^{-\frac{3}{2}}\|g\|_{\ell^2}, \tag{2.14}$$

for $t \in [0,\infty)$ and for all $g \in \ell^2$.

Remark 2.4.6. A similar resolvent estimate is proved in [Pri15] on the real line with a periodic potential and a localized potential. We adapt this proof such that it suits the discrete case on the periodic necklace graph.

Remark 2.4.7. In order to keep the notation simple we only treat the operator with positive sign since the case with a negative sign can be treated analogously. However, if the operator possesses a negative sign it is necessary to approach the real axis from below.

Proof of Lemma 2.4.5. We start by introducing a smooth cut-off function χ_Δ of an open interval Δ which contains λ and which is compactly contained in $\sigma_c(\sqrt{H})$. The counter part is denoted by $\chi_\Delta^c := 1 - \chi_\Delta$. With the help of these two functions we decompose the operator $\langle\cdot\rangle^{-\sigma}\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}0)^{-1}P_c\langle\cdot\rangle^{-\sigma}$ into

$$\langle\cdot\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}0)^{-1}P_c\langle\cdot\rangle^{-\sigma}, \tag{2.15}$$

and

$$\langle\cdot\rangle^{-\sigma}\chi_\Delta^c(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}0)^{-1}P_c\langle\cdot\rangle^{-\sigma}. \tag{2.16}$$

We will estimate these two parts separately. The reason for this decomposition is that the energy in (2.15) is localized around λ whereas the energy in (2.16) is localized away from λ. Thus, we estimate the part in (2.16) with the help of (2.13) since the only difference is the additional factor

$$\frac{\chi_\Delta^c(\sqrt{H})}{\sqrt{H}-\lambda+\mathrm{i}\varepsilon}.$$

There arise no problems from this factor because it is smooth and possesses no singularities in the support of the cut-off function $\chi_\Delta^c(\sqrt{H})$.

Therefore, we focus on the part (2.15). We introduce the formal identity

$$\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)^{-1} = -\mathrm{i}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\varepsilon t}\int_t^\infty \mathrm{e}^{\mathrm{i}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)\tau}\mathrm{d}\tau,$$

which can be verified by integrating the right-hand side. We use this formal identity on the regularization of (2.15) which is then given by

$$\begin{aligned}&\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\,\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)^{-1}P_c\langle j\rangle^{-\sigma}g(j)\\ &\qquad= -\mathrm{i}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\varepsilon t}\int_t^\infty \langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)\tau}P_c\langle j\rangle^{-\sigma}g(j)\mathrm{d}\tau.\end{aligned}$$

Next, we rewrite $\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)\tau}P_c\langle j\rangle^{-\sigma}$ as an integral operator with the help of the integral representation for the operator H as in (2.8). This yields

$$\begin{aligned}&\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\,\mathrm{e}^{\mathrm{i}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)\tau}P_c\langle j\rangle^{-\sigma}g(j)\\ &\qquad= \sum_{m\in\mathbb{Z}}\langle j\rangle^{-\sigma}\left(\int_{-\pi}^{\pi}\chi_\Delta(\omega(l))\mathrm{e}^{\mathrm{i}\omega(l)\tau}\mathrm{e}^{-\mathrm{i}\lambda\tau}\mathrm{e}^{-\varepsilon\tau}\Psi(l,j,m)\mathrm{d}l\langle m\rangle^{-\sigma}g(m)\right),\end{aligned}$$

where λ lies in the spectral band $\omega(l)$ and $\Psi(l,j,m) := \frac{1}{2\pi}\mathrm{e}^{-\mathrm{i}l(4m+k)}\overline{\psi(l)_k}\mathrm{e}^{\mathrm{i}l(4j+k)}\psi(l)_k$. The integral representations for the remaining non-flat spectral bands do vanish since χ_Δ is compactly supported in only one spectral band.

Our goal is to bound the integral kernel of $\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)\tau}P_c\langle j\rangle^{-\sigma}$ by

$$\left|\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}\chi_\Delta(\sqrt{H})\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m)\mathrm{d}l\mathrm{e}^{-\mathrm{i}\lambda\tau}\mathrm{e}^{-\varepsilon\tau}\right|\leq c\mathrm{e}^{-\varepsilon\tau}(1+\tau)^{-r}\langle j\rangle^r\langle m\rangle^r. \tag{2.17}$$

The prime cause for this estimate is the fact that χ_Δ concentrates the energy on an interval $\Delta\subset\subset\sigma_c(\sqrt{H})$ around $\lambda\in\Delta$ in exactly one spectral band. Thus, the thresholds $l_s\in\{0,\pm\pi\}$ become irrelevant. Since those are the only points for which $\partial_l\omega(l_s)=0$ holds, we only integrate over non-critical points of $\omega(l)$. The support of χ_Δ can be chosen such that it is contained in $[\lambda-\delta,\lambda+\delta]\subset\subset\{\omega(l):\, l\in[-\pi,\pi)\}$, $\delta>0$. The spectral band function $\omega(l)$ is bijective, continuous in both directions and strictly monotonic on $(-\pi,0)$ respectively $(0,\pi)$. Hence, the compact interval $[\lambda-\delta,\lambda+\delta]$ is mapped by ω^{-1} onto a compact interval $\tilde{K}:=[\tilde{k}-\tilde{\delta}_1,\tilde{k}+\tilde{\delta}_2]$, which is compactly contained in $(-\pi,0)$ and $(0,\pi)$, respectively. By introducing $B:=\frac{1}{\mathrm{i}\partial_l\omega(l)}\partial_l$ and $B^\dagger:=-\partial_l\frac{1}{\mathrm{i}\partial_l\omega(l)}$ we are led to

$$\begin{aligned}\int_{\tilde{K}}&\mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}\chi_\Delta(\omega(l))\,\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m)\mathrm{d}l\\&=\left(\frac{1}{1+\tau}\right)^r\int_{\tilde{K}}B^r\left(\mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}\right)\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m)\mathrm{d}l\\&=\left(\frac{1}{1+\tau}\right)^r\int_{\tilde{K}}\mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}(B^\dagger)^r\left(\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m)\right)\mathrm{d}l.\end{aligned}$$

Accordingly, the integral kernel of (2.15) obeys the estimate

$$\begin{aligned}\left|\int_{\tilde{K}}\mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m)\mathrm{d}l\mathrm{e}^{-\mathrm{i}\lambda\tau}\mathrm{e}^{-\varepsilon\tau}\right|&\leq(1+\tau)^{-r}\mathrm{e}^{-\varepsilon\tau}\int_{\tilde{K}}1\cdot C\langle j\rangle^r\langle m\rangle^r\mathrm{d}l\\&\leq c\mathrm{e}^{-\varepsilon\tau}(1+\tau)^{-r}\langle j\rangle^r\langle m\rangle^r,\end{aligned}$$

provided that we prove that the r-fold application of the formal adjoint of B can be controlled by

$$\left|(B^\dagger)^r\left(\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m)\right)\right|\leq C\langle j\rangle^r\langle m\rangle^r. \tag{2.18}$$

In order to verify this estimate we consider the action of the operator $(B^\dagger)^r$ on a function $f(l)$ which is given by

$$(B^\dagger)^r(f(l))=(-1)^r\partial_l\left(\frac{1}{\mathrm{i}\partial_l\omega(l)}\partial_l\left(\frac{1}{\mathrm{i}\partial_l\omega(l)}\dots\partial_l\left(\frac{1}{\mathrm{i}\partial_l\omega(l)}f(l)\right)\dots\right)\right).$$

This expression has the structure of a fraction with denominator $(\partial_l\omega(l))^{2r}$. The numerator is a sum of products which are made up of a factor $\partial^\varrho f$ with $\varrho\in\{1,\dots,r\}$ and some factors that are derivatives of $\omega(l)$ up to order r. In our case the l-dependent function f is given by

$$f(l):=\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l,j,m),$$

which contains j and m as parameters. We only need to study the behavior of the function $f(l)$ and its derivatives with respect to l to complete the verification of (2.18). Due to the factor $\chi_\Delta(\omega(l))$ the domain of integration $\tilde{K}$ is restricted such that $\omega(l)$ has no critical points. Thus, we have $\partial_l\omega(l) \geq \tilde{c} > 0$, whence

$$\frac{1}{\partial_l\omega(l)} \leq \frac{1}{\tilde{c}} \text{ for } l \in \tilde{K}. \tag{2.19}$$

Furthermore, $\omega(l)$ has continuous derivatives of order $n \in \{0, 1, 2, \ldots\}$ on $\tilde{K}$ which yields

$$|\partial_l^n \omega(l)| \leq c \text{ for } l \in \tilde{K} \text{ and } n \in \{0, 1, 2, \ldots\}. \tag{2.20}$$

We observe that the derivatives of order n of $\mathrm{e}^{-\mathrm{i}\omega(l)}$ and $\chi_\Delta(\omega(l))$ are composed of sums and products of themselves and derivatives of $\omega(l)$ up to order n. The derivative of order n of $\Psi(l, j, m)$ obeys the estimate

$$|\partial_l^n \Psi(l, j, m)| \leq c\langle j\rangle^n \langle m\rangle^n. \tag{2.21}$$

With the help of (2.19), (2.20), (2.21) and the observations above we are able to provide the validity of

$$|\partial^\varrho f(l)| \leq c\langle j\rangle^\varrho \langle m\rangle^\varrho \leq c\langle j\rangle^r \langle m\rangle^r \text{ and } |\partial_l^\varrho \omega(l)| \leq c.$$

As a result we obtain the desired bound for $(B^\dagger)^r(f(l))$ as in (2.18). If we apply this estimate to the regularized integral kernel representation of the operator (2.15) we get

$$\begin{aligned}
&\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H} - \lambda + \mathrm{i}\varepsilon)^{-1} P_c \langle j\rangle^{-\sigma} g(j) \\
&\quad = -\mathrm{i}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\varepsilon t} \int_t^\infty \langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}(\sqrt{H} - \lambda + \mathrm{i}\varepsilon)\tau} P_c \langle j\rangle^{-\sigma} g(j) \mathrm{d}\tau \\
&\quad = -\mathrm{i}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\varepsilon t} \int_t^\infty \sum_{m\in\mathbb{Z}} \langle j\rangle^{-\sigma} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega(l)\tau}\mathrm{e}^{\mathrm{i}\omega(l)}\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\mathrm{e}^{-\mathrm{i}\lambda\tau}\mathrm{e}^{-\varepsilon\tau}\Psi(l, j, m)\mathrm{d}l \langle m\rangle^{-\sigma} g(m)\mathrm{d}\tau \\
&\quad = -\mathrm{i}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\varepsilon t} \sum_{m\in\mathbb{Z}} \int_t^\infty \langle j\rangle^{-\sigma} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l, j, m)\mathrm{d}l \mathrm{e}^{-\mathrm{i}\lambda\tau}\mathrm{e}^{-\varepsilon\tau}\langle m\rangle^{-\sigma} g(m)\mathrm{d}\tau,
\end{aligned}$$

which is absolutely convergent. With (2.17) we estimate the regularization of (2.15) according to

$$\begin{aligned}
&\left|\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H} - \lambda + \mathrm{i}\varepsilon)^{-1} P_c \langle j\rangle^{-\sigma} g(j)\right| \\
&\qquad \leq \mathrm{e}^{\varepsilon t} \sum_{m\in\mathbb{Z}} \langle j\rangle^{-\sigma} \int_t^\infty \left| \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega(l)(1+\tau)}\chi_\Delta(\omega(l))\mathrm{e}^{-\mathrm{i}\omega(l)}\Psi(l, j, m)\mathrm{d}l \mathrm{e}^{-\mathrm{i}\lambda\tau}\mathrm{e}^{-\varepsilon\tau}\right| \langle m\rangle^{-\sigma}|g(m)|\mathrm{d}\tau \\
&\qquad \leq \mathrm{e}^{\varepsilon t} \sum_{m\in\mathbb{Z}} \langle j\rangle^{-\sigma} \int_t^\infty c\mathrm{e}^{-\varepsilon\tau}(1+\tau)^{-r}\langle j\rangle^r\langle m\rangle^r\langle m\rangle^{-\sigma}|g(m)|\mathrm{d}\tau \\
&\qquad = c\mathrm{e}^{\varepsilon t} \cdot \langle j\rangle^{r-\sigma} \cdot \left(\sum_{m\in\mathbb{Z}} \langle m\rangle^{r-\sigma}|g(m)|\right) \cdot \int_t^\infty \mathrm{e}^{-\varepsilon\tau}(1+\tau)^{-r}\mathrm{d}\tau.
\end{aligned}$$

We handle the integral with respect to τ by integration by parts and obtain

$$\begin{aligned}\int_t^\infty \mathrm{e}^{-\varepsilon\tau}(1+\tau)^{-r}\mathrm{d}\tau &= \left(\mathrm{e}^{-\varepsilon\tau}\frac{(1+\tau)^{-r+1}}{-r+1}\Big|_{\tau=t}^{\infty} - \int_t^\infty -\varepsilon\mathrm{e}^{-\varepsilon\tau}\frac{(1+\tau)^{-r+1}}{-r+1}\mathrm{d}\tau\right)\\ &= \mathrm{e}^{-\varepsilon t}\frac{(1+t)^{-r+1}}{-r+1} - \int_t^\infty \varepsilon\mathrm{e}^{-\varepsilon\tau}\frac{(1+\tau)^{-r+1}}{r-1}\mathrm{d}\tau\\ &\leq \mathrm{e}^{-\varepsilon t}\frac{(1+t)^{-r+1}}{-r+1},\end{aligned}$$

since for $r > 1$, the last integral has a positive integrand so that it can be omitted. We continue with the original estimate

$$\begin{aligned}&\left|\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)^{-1}P_c\langle j\rangle^{-\sigma}g(j)\right|\\ &\quad\leq c\mathrm{e}^{\varepsilon t}\cdot\langle j\rangle^{r-\sigma}\cdot\left(\sum_{m\in\mathbb{Z}}\langle m\rangle^{r-\sigma}|g(m)|\right)\cdot\mathrm{e}^{-\varepsilon t}\frac{(1+t)^{-r+1}}{-r+1}\\ &\quad\leq c_r\langle t\rangle^{-r+1}\cdot\langle j\rangle^{r-\sigma}\cdot\left(\sum_{m\in\mathbb{Z}}\langle m\rangle^{r-\sigma}|g(m)|\right)\\ &\quad\leq c_r\langle t\rangle^{-r+1}\|g\|_{\ell^2}\langle j\rangle^{r-\sigma},\end{aligned}$$

where we applied the Cauchy-Schwarz inequality in the last step with $\sigma > r+\frac{1}{2}$. By extension we arrive at

$$\begin{aligned}&\left\|\langle\cdot\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)^{-1}P_c\langle\cdot\rangle^{-\sigma}g\right\|_{\ell^2}\\ &\quad\leq\left(\sum_{j\in\mathbb{Z}}\left|\langle j\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}\varepsilon)^{-1}P_c\langle j\rangle^{-\sigma}g(j)\right|^2\right)^{\frac{1}{2}}\\ &\quad\leq c_r\langle t\rangle^{-r+1}\|g\|_{\ell^2}\left(\sum_{j\in\mathbb{Z}}\langle j\rangle^{2(r-\sigma)}\right)^{\frac{1}{2}}\\ &\quad\leq c_r\langle t\rangle^{-r+1}\|g\|_{\ell^2}.\end{aligned}\tag{2.22}$$

The sum in the last line is finite for $\sigma > r+\frac{1}{2}$. Since we are interested in a time-decay rate of at least $\langle t\rangle^{-\frac{3}{2}}$ we have to choose $r \geq 3$. For example taking $r = 3$ leads to a time-decay of $\langle t\rangle^{-2}$. This results in the requirement $\sigma > 3+\frac{1}{2} = \frac{7}{2}$. In order to get the estimate for the operator (2.15) we remark that the right-hand side in (2.22) no longer depends on ε. Thus, we take the limit $\varepsilon \to 0$ and obtain the desired result

$$\left\|\langle\cdot\rangle^{-\sigma}\chi_\Delta(\sqrt{H})\mathrm{e}^{\mathrm{i}t\sqrt{H}}(\sqrt{H}-\lambda+\mathrm{i}0)^{-1}P_c\langle\cdot\rangle^{-\sigma}g\right\|_{\ell^2} \leq c\langle t\rangle^{-2}\|g\|_{\ell^2}.$$

□

2.5 A priori estimates

We need a priori estimates for the solution of the Cauchy problem (2.2) in order to ensure that our solution exists for all $t \geq 0$.

Remark 2.5.1. The Cauchy problem (2.2) consists of infinitely many coupled second order ordinary differential equations. Thus, the Picard-Lindelöf theorem can be applied. Hence, for this Cauchy problem exists a unique solution on the interval $[0, T)$ for some $T > 0$.

We consider in $\ell^2 \oplus \ell^2$ the following quantity

$$\begin{aligned}
\mathcal{E}(t) &= \mathcal{E}(X(t), \partial_t X(t)) \\
&= \sum_{j \in \mathbb{Z}} \frac{1}{2}(\partial_t X(t,j))^2 + \frac{1}{2}\left(\sqrt{H} X(t,j)\right)^2 - \frac{1}{p+1}\gamma(j) L_1^{-1} N_{p+1}(X)(t,j) \\
&= \frac{1}{2}\langle \partial_t X(t), \partial_t X(t)\rangle_{\ell^2} + \frac{1}{2}\langle \sqrt{H} X(t), \sqrt{H} X(t)\rangle_{\ell^2} \\
&\quad - \sum_{j \in \mathbb{Z}} \frac{1}{p+1}\gamma(j) L_1^{-1} N_{p+1}(X)(t,j),
\end{aligned}$$

where N_{p+1} is the nonlinear part of (2.2) with exponent $p+1$ instead of p. The operator L_1^{-1} is the inverse of the following self-adjoint and bounded operator

$$L_1 Y(j) = L_1 \begin{pmatrix} Y_1(j) \\ Y_2(j) \\ Y_3(j) \\ Y_4(j) \end{pmatrix} = \begin{pmatrix} Y_2(j) + Y_3(j) + Y_4(j-1) - 4Y_1(j) \\ Y_4(j) + Y_1(j) - 3Y_2(j) \\ Y_4(j) + Y_1(j) - 3Y_3(j) \\ Y_1(j+1) + Y_2(j) + Y_3(j) - 4Y_4(j) \end{pmatrix}.$$

In our case $\mathcal{E}(t)$ is a conserved quantity, i.e., $\frac{d}{dt}\mathcal{E}(t) = 0$. We state the a priori estimate for solutions of the Cauchy problem (2.2) in

Lemma 2.5.2. *For sufficiently small initial conditions X_0, X_1 with*

$$\|X_0\|_{\ell^2} + \|X_1\|_{\ell^2} \leq \delta,$$

where $\delta > 0$ is sufficiently small, there exists a constant $\tilde{c}$ such that the solution X of the Cauchy problem (2.2) *on the interval* $[0, T)$ *satisfies the a priori bound*

$$\sup_{t \in [0,T)} \left(\|X(t)\|_{\ell^2} + \|\partial_t X(t)\|_{\ell^2}\right) \leq \tilde{c}\left(\|X_0\|_{\ell^2} + \|X_1\|_{\ell^2}\right). \tag{2.23}$$

Proof. We set

$$G(X) = \frac{2}{p+1}\gamma L_1^{-1} N_{p+1}(X),$$

with $G(0) = 0$ such that $G'(x) = g(X) = 2\gamma N(X)$. Since $\mathcal{E}(t)$ is a conserved quantity we obtain

$$\|(X(t), \partial_t X(t))\|^2_{\ell^2 \oplus \ell^2} - \sum_{j \in \mathbb{Z}} G(X(t,j)) = \|X_0, X_1\|^2_{\ell^2 \oplus \ell^2} - \sum_{j \in \mathbb{Z}} G(X_0(j)). \tag{2.24}$$

We start by estimating

$$G(X(t,j)) \le |G(X(t,j))| = \left|\frac{2}{p+1}\gamma(j)L_1^{-1}N_{p+1}(X)(t,j)\right| \le c|N_{p+1}(X)(t,j,j+1,j-1)|.$$

By adding up over $j \in \mathbb{Z}$ we receive

$$\sum_{j\in\mathbb{Z}} G(X(t,j)) \le c\sum_{j\in\mathbb{Z}} |N_{p+1}(X)(t,j)| \le c\|N_{p+1}(X)(t)\|_{\ell^1}.$$

The bound of the nonlinear term N_{p+1}

$$|N_{p+1}(X)(j)| \le c_{f,g,h}|X(j)|^{p+1},$$

yields

$$c\|N_{p+1}(X)(t)\|_{\ell^1} \le c\|X(t)\|_{\ell^{p+1}}^{p+1} \le c\|X(t)\|_{\ell^2}^{p+1}.$$

The last inequality holds due to the embedding $\ell^{p+1} \hookrightarrow \ell^2$. We estimate further

$$\sum_{j\in\mathbb{Z}} G(X(t,j)) \le c\|X(t)\|_{\ell^2}^{p+1} \le c\|(X(t),\partial_t X(t))\|_{\ell^2\oplus\ell^2}^{p+1}, \tag{2.25}$$

and

$$-\sum_{j\in\mathbb{Z}} G(X_0(j)) \le \left|\sum_{j\in\mathbb{Z}} G(X_0(j))\right| \le c\|X_0\|_{\ell^2}^{p+1} \le c\|(X_0,X_1)\|_{\ell^2\oplus\ell^2}^{p+1}. \tag{2.26}$$

With the help of the inequalities (2.25) and (2.26) we rearrange (2.24) to

$$\begin{aligned}\|(X(t),\partial_t X(t))\|_{\ell^2\oplus\ell^2}^2 &= \|(X_0,X_1)\|_{\ell^2\oplus\ell^2}^2 - \sum_{j\in\mathbb{Z}} G(X_0(j)) + \sum_{j\in\mathbb{Z}} G(X(t,j)) \\ &\le \|(X_0,X_1)\|_{\ell^2\oplus\ell^2}^2 + c\|(X_0,X_1)\|_{\ell^2\oplus\ell^2}^{p+1} + c\|(X(t),\partial_t X(t))\|_{\ell^2\oplus\ell^2}^{p+1}.\end{aligned} \tag{2.27}$$

In order to bound $\|(X(t),\partial_t X(t))\|_{\ell^2\oplus\ell^2}$ independently of T for all $t \in [0,T)$ we follow the arguments from [CH98]. We set $M := M(t) := \|(X(t),\partial_t X(t))\|_{\ell^2\oplus\ell^2}^2$ and introduce for $M \ge 0$ the function

$$\vartheta(M) := M - cM^{\frac{p+1}{2}} = M(1 - cM^{\frac{p-1}{2}}),$$

which is motivated by the inequality (2.27). For all $t \in [0,T)$ we find

$$\vartheta(M) \le \vartheta_0 := \|(X_0,X_1)\|_{\ell^2\oplus\ell^2}^2 + c\|(X_0,X_1)\|_{\ell^2\oplus\ell^2}^{p+1}.$$

Let ϑ_m be the unique positive maximum of the concave function ϑ with corresponding maximizing element M_m. For every $\vartheta_0 \in (0,\vartheta_m)$ there exist two elements M_1 and M_2 with $0 < M_1 < M_m < M_2$ such that $\vartheta(M_1) = \vartheta(M_2) = \vartheta_0$ holds. Since the exponent p is even and $p \ge 4$ we have especially $\frac{p+1}{2} \ge 2$. Hence, we write $\frac{p+1}{2} = 1 + \varsigma$ for some $\varsigma > 0$.

Since our initial condition $\vartheta_0 < \vartheta_m$ is small we have $M(t) \in [0, M_1) \cup (M_2, \infty)$. We claim that the following two inequalities are satisfied

$$\vartheta_0 \overset{(I)}{<} M_1 \overset{(II)}{<} \vartheta_0 \frac{1+\varsigma}{\varsigma}. \tag{2.28}$$

For the inequality (I) we consider $\partial_M \vartheta(M) = 1 - (1+\varsigma)cM^\varsigma$ and $\partial_M^2 \vartheta(M) = -\varsigma(1+\varsigma)cM^{\varsigma-1}$. The first derivative evaluated at zero has the value $\partial_M \vartheta(0) = 1$ and for positive values of M we obtain for the second derivative $\partial_M^2 \vartheta(M) < 0$. The identity map $\mathrm{Id}_M : M \mapsto M$ as well as the map ϑ possess at $M = 0$ the function value zero and their derivatives possess the function value one. Thus, the map Id_M is a tangent to the function ϑ at value zero. Since ϑ is concave the graph of Id_M lies above the graph of the function ϑ. This yields $\vartheta_0 = \vartheta(M_1) < M_1$.

In order to show the inequality (II) we observe that the maximizing element M_m can be computed with the help of the equation $1-(1+\varsigma)cM^\alpha = 0$. We obtain $M_m = (c(1+\varsigma))^{-\frac{1}{\varsigma}}$ and the corresponding maximum $\vartheta(M_m) = M_m - cM_m^{1+\varsigma}$. We compute

$$\frac{\vartheta(M_m)}{M_m} = \frac{M_m - cM_m^{1+\varsigma}}{M_m} = 1 - cM_m^\varsigma = 1 - c\frac{1}{c(1+\varsigma)} = \frac{\varsigma}{1+\varsigma}.$$

Thus, $M \mapsto \frac{\varsigma}{1+\varsigma} M$ is the secant of ϑ through the origin and the maximum of ϑ. The graph of $M \mapsto \frac{\varsigma}{1+\varsigma} M$ lies as a consequence of the concavity of ϑ below the graph of the function ϑ on the interval $[0, M_m]$. Hence, (2.28) can be shown by considering these two functions at the value M_1.

Since $M = M(t)$ depends continuously on t and we have for sufficiently small initial data $M(0) = \|(X_0, X_1)\|^2_{\ell^2 \oplus \ell^2} < M_1$ that

$$M(t) \leq M_1 \leq \frac{1+\varsigma}{\varsigma} \vartheta_0,$$

for all $t \in [0, T)$. Due to the fact that the bound $\frac{1+\varsigma}{\varsigma} \vartheta_0$ does not depend on T we finally obtain

$$\sup_{t \in [0,\infty)} \|(X(t), \partial_t X(t))\|^2_{\ell^2 \oplus \ell^2} \leq c\|(X_0, X_1)\|^2_{\ell^2 \oplus \ell^2}.$$

□

Remark 2.5.3. With the help of this a priori estimate the unique solution $X(t)$ of the Cauchy problem (2.2) can be extended globally in time.

2.6 Asymptotic stability

We recall the Cauchy problem (2.2)

$$\begin{aligned} \partial_t^2 X(t,j) - (HX)(t,j,j+1,j-1) &= \gamma(j) N(X)(t,j,j+1j,-1), \quad t \geq 0, j \in \mathbb{Z}, \\ X(0,j) &= X_0(j), \quad j \in \mathbb{Z}, \\ \partial_t X(0,j) &= X_1(j), \quad j \in \mathbb{Z}, \end{aligned}$$

on our discrete periodic necklace graph Γ with linear part $H = -L + V_{\text{loc}}$ and the function $\gamma(j) = \langle j \rangle^{-\zeta} := (1+|j|^2)^{-\frac{\zeta}{2}}$ for some $\zeta > 1/2 + (p+1)\sigma$ with $\sigma > 5/2$ and nonlinear part

$$N(X)_j = \begin{pmatrix} f_p(v_j^+(t) - u_j(t))^p + f_p(v_j^-(t) - u_j(t))^p - h_p(u_j(t) - w_{j-1}(t))^p, \\ g_p(w_j(t) - v_j^+(t))^p - f_p(v_j^+(t) - u_j(t))^p, \\ g_p(w_j(t) - v_j^-(t))^p - f_p(v_j^-(t) - u_j(t))^p, \\ h_p(u_{j+1}(t) - w_j(t))^p - g_p(v_j^+(t) - w_j(t))^p - g_p(v_j^-(t) - w_j(t))^p, \end{pmatrix}.$$

Due to assumption (A4) the operator H possesses two eigenvalues ω_0^2 and Ω^2, where ω_0^2 is also an eigenvalue of the operator $-L$ and Ω^2 is generated by the localized potential V_{loc}. The eigenspaces associated to these two eigenvalues consist of anti-symmetric sequences with respect to the semicircles, i.e., the nodes v_j^+ and v_j^- possess the same value with opposing sign. We denote with $e_k \in \ell^2(\mathbb{Z}, \mathbb{R}^4)$ the normalized eigenstate of $-L$ which is localized in the k-th periodicity cell

$$e_k(k) = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)^\intercal, \quad e_k(j) = (0,0,0,0)^\intercal \text{ for } j \neq k.$$

Then, $E_{\omega_0^2} = \text{span}\{e_k\}_{k \notin \text{supp}\, V_{\text{loc}}}$ is the eigenspace associated to the eigenvalue ω_0^2. Due to the symmetry of V_{loc}, we characterize the eigenspace associated to the eigenvalue Ω^2 by $E_{\Omega^2} = \text{span}\{e_k\}_{k \in \text{supp}\, V_{\text{loc}}}$.

The eigenspace associated to the absolutely continuous spectrum consists of all functions which are symmetric with respect to the semicircles, i.e., the values at the nodes v_j^+ and v_j^- coincide for all $j \in \mathbb{Z}$. We will denote this eigenspace by E_{ac}. Thus, it is possible to separate the sequence space into a symmetric and an anti-symmetric part

$$E_{ac} \oplus (\oplus_{k \in \mathbb{Z}} \text{span}\{e_k\}) .$$

We obtain the following properties for the eigenspaces under the action of H

$$\begin{aligned} X \in E_{\omega_0^2} &\Rightarrow HX = \omega_0^2 X, \\ X \in E_{\Omega^2} &\Rightarrow HX = \Omega^2 X, \\ X \in E_{ac} &\Rightarrow HX \in E_{ac}. \end{aligned}$$

Since the exponent p of the nonlinearity N is even we verify that

$$\begin{aligned} X \in E_{\omega_0^2} &\Rightarrow N(X) \in E_{ac}, \\ X \in E_{\Omega^2} &\Rightarrow N(X) \in E_{ac}, \\ X \in E_{ac} &\Rightarrow N(X) \in E_{ac}. \end{aligned}$$

In particular, E_{ac} is an invariant subspace under N.

Remark 2.6.1. Hence, only the eigenstates to ω_0^2 and Ω^2 which are present in the initial data will be excited. Any eigenstate which is not excited in the initial data will never be excited.

Therefore, we assume without loss of generality that only one fixed

$$\Phi \in \operatorname{span}\{e_k\}_{k \notin \operatorname{supp} V_{\mathrm{loc}}}$$

will be present in the initial data. We choose $\Phi = e_{k_0}$ for a fixed $k_0 \notin \operatorname{supp} V_{\mathrm{loc}}$.

Remark 2.6.2. The same approach is also valid if eigenstates to the eigenvalue Ω^2 are excited in the initial data provided that Ω^2 satisfies the same assumptions as ω_0^2 in Theorem 2.1.1.

The linearized system of (2.2) can be solved by $a_{\mathrm{lin}}(t)\Phi(j)$ with $a_{\mathrm{lin}}(t)$ a time dependent function of the form

$$a_{\mathrm{lin}}(t) = R\sin(\omega_0 t + \theta),$$

with constants $R, \theta \in \mathbb{R}$. For solving the whole system we choose the ansatz

$$X(t,j) = a(t)\Phi(j) + \eta(t,j), \tag{2.29}$$

where $a(t) = R(t)\sin(\omega_0 t + \theta(t))$ is the one-dimensional component of the solution and $\eta(t,j)$ is the infinite-dimensional component of the solution. The function $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^\intercal$ satisfies

$$\langle \eta(t), \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} = 0, \quad \text{for all } t \in [0,\infty), \tag{2.30}$$

and thus we have $\eta_2(t,k_0) = \eta_3(t,k_0)$. The initial conditions for the functions a and η can be expressed via the initial conditions X_0 and X_1

$$\begin{aligned} a(0) &= \langle X_0, \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},\\ \partial_t a(0) &= \langle X_1, \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},\\ \eta(0,j) &= P_c X_0(j),\\ \partial_t \eta(0,j) &= P_c X_1(j), \end{aligned}$$

where P_c is the projection onto the eigenspace E_{ac}. With the help of ansatz (2.29) we rewrite the system (2.2) into a one-dimensional evolutionary system (1DS) and an infinite dimensional evolutionary system (uDS) which can be treated separately

$$\partial_t^2 a(t) + \omega_0^2 a(t) = \langle \gamma N(a(t)\Phi + \eta(t)), \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}, \tag{1DS}$$

$$\partial_t^2 \eta(t,j) + H\eta(t,j) = P_c \gamma(j) N(a(t)\Phi(j) + \eta(t,j)). \tag{uDS}$$

We start with the investigation of (1DS) and remark that the system

$$\begin{aligned} \partial_t^2 a(t) + \omega_0^2 a(t) &= \langle \gamma N(a(t)\Phi + \eta(t)), \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} =: F(a,\eta),\\ a(0) &= \langle X_0, \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},\\ \partial_t a(0) &= \langle X_1, \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}, \end{aligned} \tag{2.31}$$

is just a second order ordinary differential equation where the right-hand side $F(a,\eta)$ can be explicitly computed. Since $\Phi(j) = (0,0,0,0)^\intercal$ for $j \neq k_0$ we obtain

$$\langle \gamma N(a(t)\Phi + \eta(t)), \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} = \langle \gamma(k_0) N(a(t)\Phi(k_0) + \eta(t,k_0)), \Phi(k_0)\rangle_{\mathbb{R}^4},$$

where $\Phi(k_0) = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)^\intercal$. We start by evaluating the nonlinear term at k_0

$$\begin{aligned}
&N(a(t)\Phi(k_0) + \eta(t,k_0)) \\
&= \begin{pmatrix}
f_p \cdot \left(\frac{1}{\sqrt{2}}a(t) + \eta_2(t,k_0) - \eta_1(t,k_0)\right)^p - h_p \cdot (\eta_4(t,k_0-1) - \eta_1(t,k_0))^p \\
\quad + f_p \cdot \left(-\frac{1}{\sqrt{2}}a(t) + \eta_2(t,k_0) - \eta_1(t,k_0)\right)^p \\
g_p \cdot \left(\eta_4(t,k_0) - \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p - f_p \cdot \left(\eta_1(t,k_0) - \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p \\
g_p \cdot \left(\eta_4(t,k_0) + \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p - f_p \cdot \left(\eta_1(t,k_0) + \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p \\
h_p \cdot (\eta_1(t,k_0+1) - \eta_4(t,k_0))^p - g_p \cdot \left(\frac{1}{\sqrt{2}}a(t) + \eta_2(t,k_0) - \eta_4(t,k_0)\right)^p \\
\quad - g_p \cdot \left(-\frac{1}{\sqrt{2}}a(t) + \eta_2(t,k_0) - \eta_4(t,k_0)\right)^p
\end{pmatrix},
\end{aligned}$$

where we have used $\eta_2(t,k_0) = \eta_3(t,k_0)$. We obtain for $F(a,\eta)$ from (2.31) that

$$\begin{aligned}
F(a,\eta) = \frac{\gamma(k_0)}{\sqrt{2}} &\left[g_p \cdot \left(\eta_4(t,k_0) - \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p \right. \\
&\left. - f_p \cdot \left(\eta_1(t,k_0) - \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p \right] \\
- \frac{\gamma(k_0)}{\sqrt{2}} &\left[g_p \cdot \left(\eta_4(t,k_0) + \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p \right. \\
&\left. - f_p \cdot \left(\eta_1(t,k_0) + \frac{1}{\sqrt{2}}a(t) - \eta_2(t,k_0)\right)^p \right].
\end{aligned}$$

We further expand $F(a,\eta)$ by using the binomial identity to the power p

$$\begin{aligned}
F(a,\eta) = \frac{\gamma(k_0)}{\sqrt{2}} &\left[g_p \sum_{\rho=0}^{p} \sum_{l=0}^{\rho} \binom{p}{\rho}\binom{\rho}{l} \eta_4(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l} \left(-\frac{1}{\sqrt{2}}a(t)\right)^l \right. \\
&- g_p \sum_{\rho=0}^{p} \sum_{l=0}^{\rho} \binom{p}{\rho}\binom{\rho}{l} \eta_4(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l} \left(\frac{1}{\sqrt{2}}a(t)\right)^l \\
&- f_p \sum_{\rho=0}^{p} \sum_{l=0}^{\rho} \binom{p}{\rho}\binom{\rho}{l} \eta_1(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l} \left(-\frac{1}{\sqrt{2}}a(t)\right)^l \\
&\left. + f_p \sum_{\rho=0}^{p} \sum_{l=0}^{\rho} \binom{p}{\rho}\binom{\rho}{l} \eta_1(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l} \left(\frac{1}{\sqrt{2}}a(t)\right)^l \right].
\end{aligned}$$

We condense the terms with respect to the factor f_p respectively g_p into

$$F(a,\eta) = \frac{\gamma(k_0)}{\sqrt{2}} \left(f_p \sum_{\rho=0}^{p} \sum_{l=0}^{\rho} \binom{p}{\rho} \binom{\rho}{l} \eta_1(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l} \left(\frac{a(t)}{\sqrt{2}}\right)^l \left[(-1)^{l+1}+1\right] \right.$$
$$\left. - g_p \sum_{\rho=0}^{p} \sum_{l=0}^{\rho} \binom{p}{\rho} \binom{\rho}{l} \eta_4(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l} \left(\frac{a(t)}{\sqrt{2}}\right)^l \left[(-1)^{l+1}+1\right] \right).$$

Our goal is to investigate the long term behavior of $a(t)$. We expect that $a(t)$ consists on the one-hand side of fast oscillations which come from the eigenfrequency ω_0 and the nonlinear harmonics, and on the other-hand side the amplitude which is exposed to small variations. We use the same approach as in [Pri15]. Thus, we start with extracting the dominant frequencies with the help of the ansatz

$$a(t) = A(t)\mathrm{e}^{\mathrm{i}\omega_0 t} + \overline{A}(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}, \tag{2.32}$$

which turns the second order differential equation (2.31) into a first order differential equation

$$\begin{aligned} \partial_t A(t) &= \frac{1}{2\mathrm{i}\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0 t} F(A,\overline{A},\eta,t), \\ A(0) = A_0 &= \frac{1}{2}\langle X_0, \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} - \frac{\mathrm{i}}{2\omega_0}\langle X_1, \Phi\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}, \end{aligned} \tag{2.33}$$

with $F(A,\overline{A},\eta,t) = F(a,\eta)$. We want to identify the resonant and non-resonant terms of the right-hand side of (2.33). In order to do this, it is suitable to decompose $F(a,\eta)$ into four parts

$$F(a,\eta) = F_1(a) + F_2(a,\eta) + F_3(a,\eta) + F_4(\eta). \tag{2.34}$$

The term $F_1(a)$ contains all terms without a component of $\eta(t,j)$. $F_2(a,\eta)$ is composed of all terms which are linear in η. $F_4(\eta)$ is made up of all terms which do not contain $a(t)$. The remaining terms are all part of $F_3(a,\eta)$.

Remark 2.6.3. If the power p of the nonlinearity N is odd we get $F_2(a,\eta) = 0$. Hence, the constant α_{pp-1} in the dispersive Hamiltonian normal form would be zero which would contradict assumption (A2), cf. Remark 2.6.5. Therefore we need an even power p in N.

The most crucial part in the decomposition of $F(a,\eta)$ is

$$F_2(a,\eta) = 2^{-\frac{p}{2}} p a(t)^{p-1} \langle \beta(\cdot), \gamma(\cdot)\eta(t,\cdot)\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},$$

where we have $\beta(k_0) = (-2f_p, -f_p+g_p, -f_p+g_p, -2g_p)^\intercal$ and $\beta(j) = (0,0,0,0)^\intercal$ for $j \neq k_0$. The remaining terms are all contained in $F_3(a,\eta)$ since the structure of $F(a,\eta)$ in combination with an even power p of N yields

$$F_1(a) = 0 \quad \text{and} \quad F_4(\eta) = 0.$$

The resonant contribution of η has a dominant effect on the oscillation of $a(t)$ through the term $F_2(a,\eta) = 2^{-\frac{p}{2}} p a(t)^{p-1} \langle \beta(\cdot), \gamma(\cdot)\eta(t,\cdot)\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}$, which is linear in η. We want to investigate the energy transfer between the two evolutionary systems (1DS) and (uDS). It is convenient to also decompose η into three parts

$$\eta(t,j) = \eta_{1,\cdot}(t,j) + \eta_{2,\cdot}(t,j) + \eta_{3,\cdot}(t,j),$$

where the $\cdot$ in the index indicates that $\eta_{j,\cdot}$ is not a component of the vector η and is instead a vector in $\mathbb{R}^4$. Each part of η satisfies its own differential equation. For $\eta_{1,\cdot}$ we obtain the linear problem

$$\begin{aligned} \partial_t^2 \eta_{1,\cdot}(t,j) + H\eta_{1,\cdot}(t,j) &= 0, && \text{for } t \geq 0,\ j \in \mathbb{Z}, \\ \eta_{1,\cdot}(0,j) &= P_c X_0(j), && \text{for } j \in \mathbb{Z}, \\ \partial_t \eta_{1,\cdot}(0,j) &= P_c X_1(j), && \text{for } j \in \mathbb{Z}. \end{aligned}$$

The second part $\eta_{2,\cdot}$ satisfies

$$\begin{aligned} \partial_t^2 \eta_{2,\cdot}(t,j) + H\eta_{2,\cdot}(t,j) &= 2^{-p/2}\gamma(j) a(t)^p \xi(j), && \text{for } t \geq 0,\ j \in \mathbb{Z}, \\ \eta_{2,\cdot}(0,j) &= 0, && \text{for } j \in \mathbb{Z}, \\ \partial_t \eta_{2,\cdot}(0,j) &= 0, && \text{for } j \in \mathbb{Z}. \end{aligned}$$

where we introduce $\xi(k_0) = (2f_p, g_p - f_p, g_p - f_p, -2g_p)^\intercal$ and $\xi(j) = (0,0,0,0)^\intercal$ for $j \neq k_0$. In $\eta_{3,\cdot}$ we collect the remaining dynamics of η which leads to the system

$$\begin{aligned} \partial_t^2 \eta_{3,\cdot}(t,j) + H\eta_{3,\cdot}(t,j) &= P_c \gamma(j) N(a\Phi + \eta)(t,j) - 2^{-p/2}\gamma(j) a(t)^p \xi(j), && \text{for } t \geq 0,\ j \in \mathbb{Z}, \\ \eta_{3,\cdot}(0,j) &= 0, && \text{for } j \in \mathbb{Z}, \\ \partial_t \eta_{3,\cdot}(0,j) &= 0, && \text{for } j \in \mathbb{Z}. \end{aligned} \tag{2.35}$$

We are most interested in $\eta_{2,\cdot}$ since its contribution due to the linearity in $\eta_{2,\cdot}$ is the biggest for small amplitude solutions. With the help of Duhamel's formula we write $\eta_{2,\cdot}$ as

$$\eta_{2,\cdot}(t,j) = \int_0^t \frac{\sin\left(\sqrt{H}(t-s)\right)}{\sqrt{H}} a(s)^p 2^{-p/2} \gamma(j)\xi(j) \mathrm{d}s.$$

In the next step we substitute $a(s)^p$ with the help of (2.32) as

$$a(s)^p = \sum_{\rho=0}^{p} \binom{p}{\rho} \left(A(s)\mathrm{e}^{\mathrm{i}\omega_0 s}\right)^{p-\rho} \left(\overline{A}(s)\mathrm{e}^{-\mathrm{i}\omega_0 s}\right)^{\rho},$$

and use Euler's formula for

$$\sin(\sqrt{H}(t-s)) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}(t-s)} - \mathrm{e}^{-\mathrm{i}\sqrt{H}(t-s)}}{2\mathrm{i}}.$$

This leads to

$$\begin{aligned}
\eta_{2,\cdot}(t,j) &= \int_0^t \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}(t-s)} - \mathrm{e}^{-\mathrm{i}\sqrt{H}(t-s)}}{2\mathrm{i}\sqrt{H}} \sum_{\rho=0}^{p} \binom{p}{\rho} \left(A(s)\mathrm{e}^{\mathrm{i}\omega_0 s}\right)^{p-\rho} \left(\overline{A}(s)\mathrm{e}^{-\mathrm{i}\omega_0 s}\right)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&= \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{\rho=0}^{p} \binom{p}{\rho} \mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&\quad - \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{\rho=0}^{p} \binom{p}{\rho} \mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s.
\end{aligned}$$

We decompose the terms by separating the resonant terms with $\pm(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})$ from the non-resonant terms with $\pm(p-2\rho)\omega_0 \notin \sigma_c(\sqrt{H})$. This yields the following decomposition

$$\begin{aligned}
\eta_{2,\cdot}(t,j) &= \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&\quad + \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{(p-2\rho)\omega_0 \notin \sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&\quad - \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{-(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&\quad - \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{-(p-2\rho)\omega_0 \notin \sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&=: \eta_{2,\cdot}^{r}(t,j) + \eta_{2,\cdot}^{nr}(t,j),
\end{aligned} \tag{2.36}$$

where we collect the resonant terms in $\eta_{2,\cdot}^{r}(t,j)$ and the non-resonant terms in $\eta_{2,\cdot}^{nr}(t,j)$. For the purpose of the investigation of $\eta_{2,\cdot}^{r}(t,j)$ around the resonant points $\pm(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})$ we introduce its regularization $\eta_{2\varepsilon,\cdot}^{r}(t,j)$. For $\varepsilon > 0$ let

$$\begin{aligned}
\eta_{2\varepsilon,\cdot}^{r}(t,j) &= \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\
&\quad - \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \sum_{-(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)s} A(s)^{p-\rho}\overline{A}(s)^{\rho} 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s.
\end{aligned} \tag{2.37}$$

With the help of the spectral representation of the self-adjoint operators and Lebesgue's dominated convergence theorem and the unitary operator $\mathrm{e}^{\pm\mathrm{i}(\sqrt{H}\mp(p-2\rho)\omega_0\pm\mathrm{i}\varepsilon)s}$, which is bounded independently of s and ε, we show that for $\varepsilon \to 0$ the regularized term $\eta_{2\varepsilon,\cdot}^{r}(t)$ converges in ℓ^2 towards $\eta_{2,\cdot}^{r}(t)$. The choice of $\pm\mathrm{i}\varepsilon$ as regularization is motivated by the fact that the resulting operators in the limit

$$\mathrm{e}^{\pm\mathrm{i}\sqrt{H}t} \left(\sqrt{H} \mp (p-2\rho)\omega_0 \pm \mathrm{i}0\right)^{-1},$$

satisfy certain time-decay estimates for $t \to \infty$ due to the singular resolvent estimates in (2.14). We use integration by parts in (2.37) and obtain

$$
\begin{aligned}
\eta^r_{2\varepsilon,\cdot}(t,j) = & \sum_{(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \Bigg(\frac{\mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)t}}{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)} \\
& \quad \times A(t)^{p-\rho}\overline{A}(t)^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \frac{1}{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)} A_0^{p-\rho}\overline{A}_0^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \int_0^t \frac{\mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)s}}{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)} \partial_t \left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right) 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \Bigg) \\
& - \sum_{-(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \Bigg(\frac{\mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)t}}{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)} \\
& \quad \times A(t)^{p-\rho}\overline{A}(t)^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \frac{1}{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)} A_0^{p-\rho}\overline{A}_0^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \int_0^t \frac{\mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)s}}{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)} \partial_t \left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right) 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \Bigg) \\
= & \sum_{(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{1}{2\sqrt{H}} \Bigg[(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)^{-1} \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 t}\mathrm{e}^{\varepsilon t} \\
& \quad \times A(t)^{p-\rho}\overline{A}(t)^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \mathrm{e}^{\mathrm{i}\sqrt{H}t}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)^{-1} A_0^{p-\rho}\overline{A}_0^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \int_0^t \mathrm{e}^{\mathrm{i}\sqrt{H}(t-s)}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}\varepsilon)^{-1} \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s}\mathrm{e}^{\varepsilon s} \\
& \quad \times \partial_t \left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right) 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \Bigg] \\
& + \sum_{-(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{1}{2\sqrt{H}} \Bigg[(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)^{-1} \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 t}\mathrm{e}^{\varepsilon t} \\
& \quad \times A(t)^{p-\rho}\overline{A}(t)^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \mathrm{e}^{-\mathrm{i}\sqrt{H}t}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)^{-1} A_0^{p-\rho}\overline{A}_0^{\rho} 2^{-p/2}\gamma(j)\xi(j) \\
& - \int_0^t \mathrm{e}^{-\mathrm{i}\sqrt{H}(t-s)}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}\varepsilon)^{-1} \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s}\mathrm{e}^{\varepsilon s} \\
& \quad \times \partial_t \left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right) 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \Bigg] \\
=: & \ \eta^r_{*\varepsilon,\cdot}(t,j) + \eta^{nr1}_{*\varepsilon,\cdot}(t,j) + \eta^{nr2}_{*\varepsilon,\cdot}(t,j),
\end{aligned}
$$

where $\eta^r_{*\varepsilon,\cdot}(t,j)$ contains the first and fourth line, $\eta^{nr1}_{*\varepsilon,\cdot}(t,j)$ contains the second and fifth line and $\eta^{nr2}_{*\varepsilon,\cdot}(t,j)$ contains the third and sixth line. Accordingly we split $\eta^r_{2,\cdot}$ up and obtain

$$\eta^r_{2,\cdot}(t,j) = \eta^r_{*,\cdot}(t,j) + \eta^{nr1}_{*,\cdot}(t,j) + \eta^{nr2}_{*,\cdot}(t,j). \tag{2.38}$$

Next, we consider the term

$$\begin{aligned}\eta^{nr}_{2,\cdot}(t,j) = &\sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \mathrm{e}^{-\mathrm{i}(\sqrt{H}-(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^\rho 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \\ &- \sum_{-(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{2\mathrm{i}\sqrt{H}} \int_0^t \mathrm{e}^{\mathrm{i}(\sqrt{H}+(p-2\rho)\omega_0)s} A(s)^{p-\rho}\overline{A}(s)^\rho 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s.\end{aligned}$$

We decompose the term $\eta^{nr}_{2,\cdot}(t,j)$ into three parts by using integration by parts

$$\begin{aligned}\eta^{nr}_{2,\cdot}(t,j) = &\sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{1}{2\sqrt{H}} \Bigg[(\sqrt{H}-(p-2\rho)\omega_0)^{-1}\mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 t} \\ &\quad \times A(t)^{p-\rho}\overline{A}(t)^\rho 2^{-p/2}\gamma(j)\xi(j) \\ &- \mathrm{e}^{\mathrm{i}\sqrt{H}t}(\sqrt{H}-(p-2\rho)\omega_0)^{-1} A_o^{p-\rho}\overline{A}_0^\rho 2^{-p/2}\gamma(j)\xi(j) \\ &- \int_0^t \mathrm{e}^{\mathrm{i}\sqrt{H}(t-s)}(\sqrt{H}-(p-2\rho)\omega_0)^{-1}\mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s} \\ &\quad \times \partial_t \left(A(s)^{p-\rho}\overline{A}(s)^\rho\right) 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \Bigg] \\ &+ \sum_{-(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})} \binom{p}{\rho} \frac{1}{2\sqrt{H}} \Bigg[(\sqrt{H}+(p-2\rho)\omega_0)^{-1}\mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 t} \\ &\quad \times A(t)^{p-\rho}\overline{A}(t)^\rho 2^{-p/2}\gamma(j)\xi(j) \\ &- \mathrm{e}^{-\mathrm{i}\sqrt{H}t}(\sqrt{H}+(p-2\rho)\omega_0)^{-1} A_0^{p-\rho}\overline{A}_0^\rho 2^{-p/2}\gamma(j)\xi(j) \\ &- \int_0^t \mathrm{e}^{-\mathrm{i}\sqrt{H}(t-s)}(\sqrt{H}+(p-2\rho)\omega_0)^{-1}\mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s} \\ &\quad \times \partial_t \left(A(s)^{p-\rho}\overline{A}(s)^\rho\right) 2^{-p/2}\gamma(j)\xi(j)\mathrm{d}s \Bigg] \\ =:\ &\eta^{nr}_{\star,\cdot}(t,j) + \eta^{nr1}_{\star,\cdot}(t,j) + \eta^{nr2}_{\star,\cdot}(t,j),\end{aligned} \tag{2.39}$$

where $\eta^{nr}_{\star,\cdot}(t,j)$ includes the first and fourth line, $\eta^{nr1}_{\star,\cdot}(t,j)$ includes the second and fifth line and $\eta^{nr2}_{\star,\cdot}(t,j)$ includes the third and sixth line.

We now turn to the analysis of $F_2(a,\eta)$ which we split according to the decomposition of η since $F_2(a,\eta)$ is linear in η. We obtain

$$F_2(a,\eta) = F_2(a,\eta_{1,\cdot}+\eta_{2,\cdot}+\eta_{3,\cdot}) = F_2(a,\eta_{1,\cdot}) + F_2(a,\eta_{2,\cdot}) + F_2(a,\eta_{3,\cdot}).$$

The term $\eta_{2,\cdot}$ has again its own decomposition according to (2.36), (2.38) and (2.39) which can be transferred to $F_2(a,\eta_{2,\cdot})$. This leads to

$$\begin{aligned}F_2(a,\eta_{2,\cdot}) = F_2(a,\eta^r_{*,\cdot}) + F_2(a,\eta^{nr}_{\star,\cdot})+ \\ + F_2(a,\eta^{nr1}_{*,\cdot}) + F_2(a,\eta^{nr2}_{*,\cdot}) + F_2(a,\eta^{nr1}_{\star,\cdot}) + F_2(a,\eta^{nr2}_{\star,\cdot}).\end{aligned}$$

The next step is to calculate the exact expressions for the components of $F_2(a,\eta_{2,\cdot})$. We will need these exact formulas to convert the amplitude equation for $A(t)$ into a dispersive Hamiltonian normal form. The regularized version of $\eta^r_{*,\cdot}$ yields

$$\begin{aligned}F_2(a,\eta^r_{*\varepsilon,\cdot}) = & \frac{p}{2^{p+1}}\left(A(t)\mathrm{e}^{\mathrm{i}\omega_0 t} + \overline{A}(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}\right)^{p-1} \sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 t}\mathrm{e}^{\varepsilon t} \\ & \times A(t)^{p-\rho}\overline{A}(t)^{\rho}\left\langle \gamma(\cdot)\beta(\cdot), \sqrt{H}^{-1}(\sqrt{H}-(p-2\rho)\omega_0 + \mathrm{i}\varepsilon)^{-1}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} \\ & + \frac{p}{2^{p+1}}\left(A(t)\mathrm{e}^{\mathrm{i}\omega_0 t} + \overline{A}(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}\right)^{p-1} \sum_{-(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})} \binom{p}{\rho} \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 t}\mathrm{e}^{\varepsilon t} \\ & \times A(t)^{p-\rho}\overline{A}(t)^{\rho}\left\langle \gamma(\cdot)\beta(\cdot), \sqrt{H}^{-1}(\sqrt{H}+(p-2\rho)\omega_0 - \mathrm{i}\varepsilon)^{-1}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}.\end{aligned}$$

We introduce the limit

$$\Lambda^{\mp}_{p-2\rho} := \lim_{\varepsilon\downarrow 0}\frac{p}{2^{p+1}}\left\langle \gamma(\cdot)\beta(\cdot), \sqrt{H}^{-1}(\sqrt{H}\mp(p-2\rho)\omega_0 \pm \mathrm{i}\varepsilon)^{-1}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}, \tag{2.40}$$

to obtain

$$\begin{aligned}F_2(a,\eta^r_{*,\cdot}) = & \sum_{\sigma=0}^{p-1}\sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})}\binom{p-1}{\sigma}\binom{p}{\rho}\Lambda^-_{p-2\rho}\mathrm{e}^{\mathrm{i}(2p-1-2\rho-2\sigma)\omega_0 t}A(t)^{2p-1-\rho-\sigma}\overline{A}(t)^{\rho+\sigma} \\ & + \sum_{\sigma=0}^{p-1}\sum_{-(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})}\binom{p-1}{\sigma}\binom{p}{\rho}\Lambda^+_{p-2\rho}\mathrm{e}^{\mathrm{i}(2p-1-2\rho-2\sigma)\omega_0 t}A(t)^{2p-1-\rho-\sigma}\overline{A}(t)^{\rho+\sigma}.\end{aligned} \tag{2.41}$$

The term $F_2(a,\eta^{nr}_{\star,\cdot})$ can be treated similarly as $F_2(a,\eta^r_{*,\cdot})$. We introduce

$$\Upsilon^{\mp}_{p-2\rho} := \frac{p}{2^{p+1}}\left\langle \gamma(\cdot)\beta(\cdot), \sqrt{H}^{-1}(\sqrt{H}\mp(p-2\rho)\omega_0)^{-1}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},$$

to get

$$\begin{aligned}F_2(a,\eta^{nr}_{\star,\cdot}) = & \sum_{\sigma=0}^{p-1}\sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})}\binom{p-1}{\sigma}\binom{p}{\rho}\Upsilon^-_{p-2\rho}\mathrm{e}^{\mathrm{i}(2p-1-2\rho-2\sigma)\omega_0 t}A(t)^{2p-1-\rho-\sigma}\overline{A}(t)^{\rho+\sigma} \\ & + \sum_{\sigma=0}^{p-1}\sum_{-(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})}\binom{p-1}{\sigma}\binom{p}{\rho}\Upsilon^+_{p-2\rho}\mathrm{e}^{\mathrm{i}(2p-1-2\rho-2\sigma)\omega_0 t}A(t)^{2p-1-\rho-\sigma}\overline{A}(t)^{\rho+\sigma}.\end{aligned} \tag{2.42}$$

The terms $F_2(a,\eta^{nr1}_{*,\cdot})$ and $F_2(a,\eta^{nr2}_{*,\cdot})$ can be handled similarly

$$
\begin{aligned}
F_2(a,\eta^{nr1}_{*,\cdot}) = &- \sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})} \binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}A_0^{p-\rho}\overline{A}_0^{\rho} \\
&\times \left\langle \gamma(\cdot)\beta(\cdot), \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{\sqrt{H}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} \\
&- \sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})} \binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}A_0^{p-\rho}\overline{A}_0^{\rho} \\
&\times \left\langle \gamma(\cdot)\beta(\cdot), \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{\sqrt{H}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},
\end{aligned}
\tag{2.43}
$$

and

$$
\begin{aligned}
F_2(a,\eta^{nr2}_{*,\cdot}) = &- \sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})} \binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}\int_0^t \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s}\partial_t\left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right) \\
&\times \left\langle \gamma(\cdot)\beta(\cdot), \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}(t-s)}}{\sqrt{H}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} \mathrm{d}s \\
&- \sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})} \binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}\int_0^t \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s}\partial_t\left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right) \\
&\times \left\langle \gamma(\cdot)\beta(\cdot), \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}(t-s)}}{\sqrt{H}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} \mathrm{d}s.
\end{aligned}
\tag{2.44}
$$

The last two remaining terms can also be considered in similar fashion

$$
\begin{aligned}
F_2(a,\eta^{nr1}_{\star,\cdot}) = &- \sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})} \binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}A_0^{p-\rho}\overline{A}_0^{\rho} \\
&\times \left\langle \gamma(\cdot)\beta(\cdot), \frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{\sqrt{H}(\sqrt{H}-(p-2\rho)\omega_0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)} \\
&- \sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})} \binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}A_0^{p-\rho}\overline{A}_0^{\rho} \\
&\times \left\langle \gamma(\cdot)\beta(\cdot), \frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{\sqrt{H}(\sqrt{H}+(p-2\rho)\omega_0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)},
\end{aligned}
\tag{2.45}
$$

and

$$
\begin{aligned}
F_2(a,\eta^{nr2}_{\star,\cdot}) = &-\sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})}\binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}\int_0^t \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s}\partial_t\left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right)\\
&\times\left\langle\gamma(\cdot)\beta(\cdot),\frac{\mathrm{e}^{\mathrm{i}\sqrt{H}(t-s)}}{\sqrt{H}(\sqrt{H}-(p-2\rho)\omega_0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}\mathrm{d}s\\
&-\sum_{(p-2\rho)\omega_0\notin\sigma_c(\sqrt{H})}\binom{p}{\rho}\frac{p}{2^{p+1}}a(t)^{p-1}\int_0^t \mathrm{e}^{\mathrm{i}(p-2\rho)\omega_0 s}\partial_t\left(A(s)^{p-\rho}\overline{A}(s)^{\rho}\right)\\
&\times\left\langle\gamma(\cdot)\beta(\cdot),\frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}(t-s)}}{\sqrt{H}(\sqrt{H}+(p-2\rho)\omega_0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2(\mathbb{Z},\mathbb{R}^4)}\mathrm{d}s.
\end{aligned}
\tag{2.46}
$$

With the help of (2.34), (2.41), (2.42), (2.43), (2.44), (2.45) and (2.46) we rewrite (2.33) as

$$
\begin{aligned}
\partial_t A(t) &= \frac{1}{2\mathrm{i}\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0 t}\left(F_2(a,\eta^{r}_{*,\cdot})+F_2(a,\eta^{nr}_{\star,\cdot})\right)+E\\
&=\sum_{k+l=2p-1}\alpha_{kl}A(t)^k\overline{A}(t)^l\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 t}+E,
\end{aligned}
\tag{2.47}
$$

with

$$
\begin{aligned}
E := \frac{1}{2\mathrm{i}\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0 t}\Big(&F_3(a,\eta)+F_2(a,\eta^{nr1}_{*,\cdot}+\eta^{nr2}_{*,\cdot})+F_2(a,\eta^{nr1}_{\star,\cdot}+\eta^{nr2}_{\star,\cdot})\\
&+F_2(a,\eta_{1,\cdot})+F_2(a,\eta_{3,\cdot})\Big).
\end{aligned}
$$

In (2.47) we have $k=2p-1-\rho-\sigma$ and $l=p+\sigma$ for $p-2\rho\in\sigma_c(\sqrt{H})$ and $0\le\sigma\le p-1$.

2.6.1 Dispersive Hamiltonian normal form

Our aim is to transform the amplitude equation (2.47) into a dispersive Hamiltonian normal form.

Lemma 2.6.4. *If $|A|$ is sufficiently small then there exists a smooth normal form transformation K which maps $A\to\tilde{A}=A-h(A,t)$ with $h(A,t)=h(A,t+\frac{2\pi}{\omega_0})$ and $h(A,t)=\mathcal{O}\left(|A|^{2p-1}\right)$ such that*

$$
\partial_t A(t)=\sum_{k+l=2p-1}\alpha_{kl}A(t)^k\overline{A}(t)^l\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 t}+E,
$$

turns into

$$
\partial_t\tilde{A}(t)=\alpha_{pp-1}|\tilde{A}(t)|^{2p-2}\tilde{A}(t)+\mathcal{O}\left(|\tilde{A}|^{3p-2}\right)_{\tilde{A}\to 0}+\tilde{E}_K.
\tag{2.48}
$$

The coefficient α_{pp-1} is given by

$$
\alpha_{pp-1}=-\mathrm{i}\sum_{\substack{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})\\ \rho+\sigma=p-1}}\binom{p-1}{\sigma}\binom{p}{\rho}\frac{1}{2\omega_0}\left(\Lambda^-_{p-2\rho}+\Lambda^+_{p-2\rho}\right).
$$

Proof. We start by rewriting (2.47) as

$$\partial_t A(t) = \alpha_{pp-1}|A(t)|^{2p-2}A(t) + \underbrace{\sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}A(t)^k\overline{A}(t)^l\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 t}}_{=:O_{2p-1}(A)} + E. \tag{2.49}$$

Hence, we want to transform the terms contained in $O_{2p-1}(A)$ such that they are of order $\mathcal{O}(|A|^{3p-2})_{A\to 0}$. This can be achieved with the help of a normal form transformation. By integrating the equation (2.49) we obtain

$$A(t) = A_0 + \int_0^t \alpha_{pp-1}|A(s)|^{2p-2}A(s) + O_{2p-1}(A) + E\,\mathrm{d}s.$$

We investigate the integral term $\int_0^t O_{2p-1}(A)\mathrm{d}s$ by using integration by parts

$$\begin{aligned}
\int_0^t O_{2p-1}(A)\mathrm{d}s &= \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\int_0^t A(s)^k\overline{A}(s)^l\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}\mathrm{d}s\\
&= \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\left[\frac{1}{\mathrm{i}(k-l-1)\omega_0}\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}A(s)^k\overline{A}(s)^l\Big|_{s=0}^t\right.\\
&\qquad\left.-\int_0^t \frac{\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}}{\mathrm{i}(k-l-1)\omega_0}\partial_t\left(A(s)^k\overline{A}(s)^l\right)\mathrm{d}s\right]\\
&= \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\frac{\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}}{\mathrm{i}(k-l-1)\omega_0}A(t)^k\overline{A}(t)^l - \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\frac{A_0^k\overline{A}_0^l}{\mathrm{i}(k-l-1)\omega_0}\\
&\qquad - \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\int_0^t \frac{\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}}{\mathrm{i}(k-l-1)\omega_0}\partial_t\left(A(s)^k\overline{A}(s)^l\right)\mathrm{d}s\\
&= h(A,t) - h(A_0,0) - \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\int_0^t \frac{\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}}{\mathrm{i}(k-l-1)\omega_0}\\
&\qquad \times A(s)^{k-1}\overline{A}(s)^{l-1}\left(k\overline{A}(s)\partial_t A(s) + lA(s)\partial_t\overline{A}(s)\right)\mathrm{d}s,
\end{aligned}$$

where we introduce the function $h(A,t)$ to abbreviate the terms which are at most of order $\mathcal{O}(|A|^{2p-1})$ in

$$h(A,t) = \sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl}\frac{\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}}{\mathrm{i}(k-l-1)\omega_0}A(t)^k\overline{A}(t)^l$$

It is sufficient to estimate the higher order terms roughly which are not in $h(A,t)$ by

$$\begin{aligned}\sum_{\substack{k+l=2p-1\\k-l\neq 1}} \alpha_{kl} \int_0^t \frac{\mathrm{e}^{\mathrm{i}(k-l-1)\omega_0 s}}{\mathrm{i}(k-l-1)\omega_0} A(s)^{k-1}\overline{A}(s)^{l-1} \\ \times \Big[(k\overline{A}(s)\left(\alpha_{pp-1}|A(s)|^{2p-2}A(s)+O_{2p-1}(A)+E\right) \\ +lA(s)\left(\overline{\alpha}_{pp-1}|A(s)|^{2p-2}\overline{A}(s)+\overline{O}_{2p-1}(A)+\overline{E}\right)\Big]\,\mathrm{d}s \\ = \int_0^t \mathcal{O}\left(|A|^{2p-2}\left(|A|^p+|E|\right)\right)_{A\to 0}\mathrm{d}s,\end{aligned}$$

where we have used the following relations

$$\begin{aligned}A^{k-1}\overline{A}^{l-1}k\overline{A} &= \mathcal{O}\left(|A|^{2p-2}\right)_{A\to 0},\\ A^{k-1}\overline{A}^{l-1}lA &= \mathcal{O}\left(|A|^{2p-2}\right)_{A\to 0},\\ \alpha_{pp-1}|A|^{2p-2}A+O_{2p-1}(A) &= \mathcal{O}\left(|A|^{p}\right)_{A\to 0},\\ \overline{\alpha}_{pp-1}|A|^{2p-2}\overline{A}+\overline{O}_{2p-1}(A) &= \mathcal{O}\left(|A|^{p}\right)_{A\to 0}.\end{aligned}$$

We obtain

$$\int_0^t O_{2p-1}(A)\mathrm{d}s = h(A,t)-h(A_0,0)+\int_0^t \mathcal{O}\left(|A|^{2p-2}(|A|^p+|E|)\right)_{A\to 0}\mathrm{d}s.$$

Hence, we write $A(t)$ as

$$\begin{aligned}A(t)-h(A,t) &= A_0+\int_0^t \alpha_{pp-1}|A(s)|^{2p-2}A(s)\mathrm{d}s - h(A_0,0)\\ &\quad+\int_0^t \mathcal{O}\left(|A|^{2p-2}(|A|^p+|E|)\right)_{A\to 0}\mathrm{d}s+\int_0^t E\mathrm{d}s.\end{aligned}$$

We introduce the normal form transformation K through

$$K : A \to \tilde{A} = A - h(A,t),$$

under which the amplitude equation (2.49) takes the form

$$\tilde{A}(t) = \tilde{A}_0+\int_0^t \alpha_{pp-1}|\tilde{A}(s)|^{2p-2}\tilde{A}(s)\mathrm{d}s+\int_0^t \mathcal{O}\left(|\tilde{A}|^{3p-2}\right)_{\tilde{A}\to 0}+\mathcal{O}\left(|\tilde{E}|\right)_{\tilde{A}\to 0}+\tilde{E}\mathrm{d}s,$$

since $h(A,t)$ is of order $\mathcal{O}(|A|^{2p-1})$ the remaining terms are of order $\mathcal{O}(|\tilde{A}|^{3p-2})_{\tilde{A}\to 0}$. If we set $\tilde{E}_K := \mathcal{O}\left(|\tilde{E}|\right)_{\tilde{A}\to 0}+\tilde{E}$ then we obtain the dispersive Hamiltonian normal form from (2.48)

$$\partial_t\tilde{A}(t) = \alpha_{pp-1}|\tilde{A}(t)|^{2p-2}\tilde{A}(t)+\mathcal{O}\left(|\tilde{A}|^{3p-2}\right)_{\tilde{A}\to 0}+\tilde{E}_K.$$

□

Remark 2.6.5. According to assumption (A2) the constant α_{pp-1} possesses negative real part. This can be verified by considering the terms of the sum of $\frac{1}{2\mathrm{i}\omega_0}\mathrm{e}^{-\mathrm{i}\omega_0 t}F_2(a,\eta^r_{*,\cdot})$ for which $2p-1-\rho-\sigma = p$ and $\rho+\sigma = p-1$ holds. This leads to

$$\alpha_{pp-1} = -\mathrm{i} \sum_{\substack{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})\\ \rho+\sigma=p-1}} \binom{p-1}{\sigma}\binom{p}{\rho}\frac{1}{2\omega_0}\left(\Lambda^-_{p-2\rho}+\Lambda^+_{p-2\rho}\right).$$

The real part of the coefficient α_{pp-1} is determined through

$$\operatorname{Re}\alpha_{pp-1} = -\operatorname{Im} \sum_{\substack{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})\\ \rho+\sigma=p-1}} \binom{p-1}{\sigma}\binom{p}{\rho}\frac{1}{2\omega_0}\left(\Lambda^-_{p-2\rho}+\Lambda^+_{p-2\rho}\right).$$

Thus, the fact that α_{pp-1} has a negative real part is equivalent to

$$\operatorname{Im}\left(\Lambda^-_{p-2\rho}+\Lambda^+_{p-2\rho}\right) > 0$$

for at least one ρ with $(p-2\rho)\omega_0 \in \sigma_c(\sqrt{H})$.

2.6.2 Asymptotics of the one-dimensional solution component

We cite Proposition 4.12 from [Pri15] which estimates the asymptotic behavior of the solution from (2.48) for $t\to\infty$ if the higher-order terms $Q(t) = \mathcal{O}(|\tilde{A}|^{3p-2}) + \tilde{E}_K$ in (2.48) satisfy certain decay estimates. These estimates will be proved in Lemma 2.6.7. In particular, the higher-order terms do not affect the long time behavior of the solution of (2.48).

Lemma 2.6.6. *Consider the ordinary differential equation*

$$\partial_t\tilde{A}(t) = \alpha_{pp-1}|\tilde{A}(t)|^{2p-2}\tilde{A}(t) + Q(t),$$

to the initial datum $\tilde{A}(0) = \tilde{A}_0$. Suppose that $\tilde{m}_* := \max\{1, 2(p-1)c_\Gamma\tilde{A}_0^{2p-2}\}$ and

$$|Q(t)| \le Q_0\langle t\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}. \tag{2.50}$$

Then its solution satisfies the estimate

$$|\tilde{A}(t)| \le (1+2(p-1)c_\Gamma|\tilde{A}_0|^{2p-2}\cdot t)^{-\frac{1}{2p-2}}\cdot\left(2^{\frac{2}{p-1}}|\tilde{A}_0|^4 + C\frac{Q_0^{\frac{2p+2}{2p-1}}\tilde{m}_*^{P+1}}{c_\Gamma^{\frac{3}{2p-1}}\langle t\rangle^{\frac{p+1}{2p-1}}}\right)^{\frac{1}{4}}, \tag{2.51}$$

where $P := (\frac{1}{2p-2}+\frac{3}{2p+2})\frac{2p+2}{2p-1}$. In particular, the solution displays the same large-time behavior as in the case $Q(t)\equiv 0$.

In order to apply Lemma 2.6.6 we need to show that the inequality (2.50) holds, which is in our case equivalent to the inequality

$$|Q(t)| := \left|\mathcal{O}(|\tilde{A}^{3p-2}|)_{\tilde{A}\to 0} + \tilde{E}_K\right| \leq Q_0\langle t\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}. \tag{2.52}$$

We know that $h(A,t) = \mathcal{O}(|A|^{2p-1})$, and thus $\tilde{A} = A(t) + \mathcal{O}(|A|^{2p-1})$. The normal form transformation K is invertible in a sufficiently small neighborhood of the origin. It follows that $A(t) = \tilde{A}(t) + \mathcal{O}(|\tilde{A}|^{2p-1})$ and there exists a constant $c_A > 0$ such that

$$\frac{1}{c_A}|\tilde{A}(t)| \leq |A(t)| \leq c_A|\tilde{A}(t)|.$$

Since the error term E depends on $A(t)$ we obtain for small $A(t)$ respectively small $\tilde{A}(t)$ that

$$\begin{aligned}\tilde{E}(\tilde{A}) &= E((\mathrm{Id}+h)(\tilde{A})) = E(\tilde{A} + \mathcal{O}(|\tilde{A}|^{2p-1}))\\ &= E(\tilde{A}) + \mathcal{O}(|\tilde{A}|^{2p-1}) = E(A) + \mathcal{O}(|A|^{2p-1}).\end{aligned}$$

Consequently we have $E(A) = \tilde{E}(\tilde{A}) + \mathcal{O}(|\tilde{A}|^{2p-1})$ and there exists a constant $c_E > 0$ such that

$$\frac{1}{c_E}|\tilde{E}(\tilde{A})| \leq |E(A)| \leq c_E|\tilde{E}(\tilde{A})|.$$

Since $\tilde{E}_K = \tilde{E} + \mathcal{O}(|\tilde{E}|)$ we can also verify the inequality (2.52) with E instead of $\tilde{E}_K$. Furthermore, the estimate (2.51) then turns into

$$|A(t)| \leq c\langle t\rangle^{-\frac{1}{2p-2}}\left(|A_0|^4 + Q_0^{\frac{2p+2}{2p-1}}\right)^{\frac{1}{4}}.$$

Before we start estimating the term E we introduce the subsequent notation

$$[A](T) := \sup_{0\leq t\leq T}\left(\langle t\rangle^{\frac{1}{2p-2}}|A(t)|\right) \quad\text{and}\quad [\eta](T) := \sup_{0\leq t\leq T}\left(\langle t\rangle^{1+\frac{1}{2p-2}}\|\eta(t)\|_{\ell^2_{-\sigma}}\right). \tag{2.53}$$

This new notation serves to identify the anticipated decay rates of the solution components $a(t)$ and $\eta(t,\cdot)$ as the actual ones. In particular, we obtain

$$|a(t)| \leq 2|A(t)| \leq 2[A](T)\langle t\rangle^{-\frac{1}{2p-2}} \quad\text{and}\quad \|\eta(t)\|_{\ell^2_{-\sigma}} \leq \langle t\rangle^{-1-\frac{1}{2p-2}}[\eta](T).$$

We need to estimate the term $2\mathrm{i}\omega_0\mathrm{e}^{\mathrm{i}\omega_0 t}E$ which can be decomposed into

$$\begin{aligned}2\mathrm{i}\omega_0\mathrm{e}^{\mathrm{i}\omega_0 t}E = \Big(&F_3(a,\eta) + F_2(a,\eta^{nr1}_{*,\cdot}) + F_2(a,\eta^{nr2}_{*,\cdot}) + F_2(a,\eta^{nr1}_{\star,\cdot}) + F_2(a,\eta^{nr2}_{\star,\cdot})\\ &+ F_2(a,\eta_{1,\cdot}) + F_2(a,\eta_{3,\cdot})\Big).\end{aligned}$$

The estimates for the constituents of E are collected in the subsequent lemma.

Lemma 2.6.7. *The constituents of* $2\mathrm{i}\omega_0 \mathrm{e}^{\mathrm{i}\omega_0 t} E$ *fulfill the estimates*

- $|F_3(a,\eta)| \leq c\langle t\rangle^{-2-\frac{3}{2p-2}}[A](T)[\eta](T)^2 \left(\sum\limits_{\rho=0}^{p} \sum\limits_{\substack{l=1 \\ l\,\text{odd} \\ l<p-1}}^{\rho} (c_\alpha^{2(p-\rho)-2} + c_\alpha^{2(\rho-l)-2})[A](T)^{\rho-1} \right)$,
- $|F_2(a,\eta_{1,\cdot})| \leq c\langle t\rangle^{-2}[A](T)^{p-1}\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right)$,
- $|F_2(a,\eta_{3,\cdot})| \leq c\langle t\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}[A](T)^{p-1}\left(\sum\limits_{\rho=1}^{p}[\eta](T)^\rho [A](T)^{p-\rho}\right)$,
- $\left|F_2(a,\eta^{nr1}_{*,\cdot})\right| \leq c\langle t\rangle^{-\frac{5}{2}-\frac{1}{2p-2}}[A](T)^{2p-1}$,
- $\left|F_2(a,\eta^{nr2}_{*,\cdot})\right| \leq c\langle t\rangle^{-2-\frac{1}{2p-2}}[A](T)^{3p-2}$,
- $\left|F_2(a,\eta^{nr1}_{\star,\cdot})\right| \leq c\langle t\rangle^{-\frac{5}{2}-\frac{1}{2p-2}}[A](T)^{2p-1}$,
- $\left|F_2(a,\eta^{nr2}_{\star,\cdot})\right| \leq c\langle t\rangle^{-2-\frac{1}{2p-2}}[A](T)^{3p-2}$.

We need to prove the subsequent auxiliary result before we prove Lemma 2.6.7.

Lemma 2.6.8. *Let* $\delta > 0$*. For sufficiently small initial data* $X_0, X_1 \in \ell^2_\sigma$ *with*

$$\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma} \leq \delta,$$

there exists a constant $c_\alpha \geq \delta > 0$ *such that*

$$\sup_{t\in[0,\infty)} |a(t)| \leq c_\alpha, \qquad \sup_{t\in[0,\infty)} \|\eta(t)\|_{\ell^2} \leq c_\alpha, \tag{2.54}$$

holds.

Proof. We have

$$\begin{aligned}
\|X(t)\|^2_{\ell^2} &= \langle X(t), X(t)\rangle_{\ell^2} = \langle a(t)\Phi + \eta(t), a(t)\Phi + \eta(t)\rangle_{\ell^2} \\
&= \langle a(t)\Phi, a(t)\Phi\rangle_{\ell^2} + \langle a(t)\Phi, \eta(t)\rangle_{\ell^2} + \langle \eta(t), a(t)\Phi\rangle_{\ell^2} + \langle \eta(t), \eta(t)\rangle_{\ell^2} \\
&= a(t)^2\langle \Phi, \Phi\rangle_{\ell^2} + a(t)\langle \Phi, \eta(t)\rangle_{\ell^2} + a(t)\langle \eta(t), \Phi\rangle_{\ell^2} + \langle \eta(t), \eta(t)\rangle_{\ell^2} \\
&= a(t)^2\|\Phi\|^2_{\ell^2} + \|\eta(t)\|^2_{\ell^2} \geq \|\eta(t)\|^2_{\ell^2},
\end{aligned}$$

where we used that $\langle \Phi, \eta(t)\rangle_{\ell^2} = 0$ due to (2.30). Thus, it follows from (2.23) that

$$\sup_{t\in[0,\infty)} \|\eta(t)\|_{\ell^2} \leq \sup_{t\in[0,\infty)} \|X(t)\|_{\ell^2} \leq \tilde{c}_\alpha \left(\|X_0\|_{\ell^2} + \|X_1\|_{\ell^2}\right).$$

It also holds

$$|a(t)|^2 = \frac{1}{\|\Phi\|^2_{\ell^2}}\left(\|X(t)\|^2_{\ell^2} - \|\eta(t)\|^2_{\ell^2}\right) \leq c\|X(t)\|^2_{\ell^2},$$

whereas it follows

$$\sup_{t\in[0,\infty)} |a(t)| \le c \sup_{t\in[0,\infty)} \|X(t)\|_{\ell^2} \le c\tilde{c}_\alpha \left(\|X_0\|_{\ell^2} + \|X_1\|_{\ell^2}\right).$$

We obtain the desired estimates from (2.54) if we choose

$$c_\alpha := \delta\left(1 + \tilde{c}_\alpha + c\tilde{c}_\alpha\right). \tag{2.55}$$

□

Proof of Lemma 2.6.7. We start by estimating

$$\begin{aligned}
|F_3(a,\eta)| = \frac{2\gamma(k_0)}{\sqrt{2}}\Bigg| & g_p \sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} \binom{p}{\rho}\binom{\rho}{l} \eta_4(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l}\left(\frac{a(t)}{\sqrt{2}}\right)^l \\
& - f_p \sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} \binom{p}{\rho}\binom{\rho}{l} \eta_1(t,k_0)^{p-\rho}(-\eta_2(t,k_0))^{\rho-l}\left(\frac{a(t)}{\sqrt{2}}\right)^l \Bigg| \\
\le c\Bigg(& \sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} \gamma(k_0)|\eta_4(t,k_0)|^{p-\rho}|\eta_2(t,k_0)|^{\rho-l}|a(t)|^l \\
& + \sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} \gamma(k_0)|\eta_1(t,k_0)|^{p-\rho}|\eta_2(t,k_0)|^{\rho-l}|a(t)|^l \Bigg) \\
\le c\Bigg(& \sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} \gamma(k_0)|a(t)|^l \left(|\eta_4(t,k_0)|^{2(p-\rho)} + |\eta_2(t,k_0)|^{2(\rho-l)}\right) \\
& + \sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} \gamma(k_0)|a(t)|^l \left(|\eta_1(t,k_0)|^{2(p-\rho)} + |\eta_2(t,k_0)|^{2(\rho-l)}\right) \Bigg) \\
\le c & \langle t\rangle^{-\frac{1}{2p-2}}[A](T)\|\eta(t)\|^2_{\ell^2_{-\sigma}} \\
& \times \Bigg(\sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} (c_\alpha^{2(p-\rho)-2} + c_\alpha^{2(\rho-l)-2})[A](T)^{l-1} \Bigg) \\
\le c & \langle t\rangle^{-2-\frac{3}{2p-2}}[A](T)[\eta](T)^2 \Bigg(\sum_{\rho=0}^{p} \sum_{\substack{l=1\\ l\,\text{odd}\\ l<p-1}}^{\rho} (c_\alpha^{2(p-\rho)-2} + c_\alpha^{2(\rho-l)-2})[A](T)^{\rho-1} \Bigg).
\end{aligned}$$

With the help of (2.54) and the embedding $\ell^2 \subset \ell^\infty$ we estimate

$$\begin{aligned}\gamma(k_0)|\eta_1(t,k_0)|^{2(p-\rho)} &= \gamma(k_0)(1+k_0^2)^\sigma|\eta_1(t,k_0)|^{2(p-\rho)-2}(1+k_0^2)^{-\sigma}|\eta_1(t,k_0)|^2\\ &\le \gamma(k_0)(1+k_0^2)^\sigma|\eta_1(t,k_0)|^{2(p-\rho)-2}\|\eta(t)\|^2_{\ell^2_{-\sigma}}\\ &\le c\|\eta(t)\|^{2(p-\rho)-2}_{\ell^\infty}\|\eta(t)\|^2_{\ell^2_{-\sigma}}\\ &\le cc_\alpha^{2(p-\rho)-2}\|\eta(t)\|^2_{\ell^2_{-\sigma}}\\ &\le cc_\alpha^{2(p-\rho)-2}\langle t\rangle^{-2-\frac{2}{2p-2}}[\eta](T)^2.\end{aligned}$$

Furthermore, we have

$$|a(t)|^l \le 2^l\langle t\rangle^{-\frac{l}{2p-2}}[A](T)^l \le c\langle t\rangle^{-\frac{1}{2p-2}}[A](T)^l$$

for $1 \le l \le p-2$.

Since $P_c\eta_{1,\cdot} = \eta_{1,\cdot}$, we use the improved decay estimate (2.13) for the estimate of $F_2(a,\eta_{1,\cdot})$ and obtain

$$\begin{aligned}|F_2(a,\eta_{1,\cdot})| &\le c|a(t)|^{p-1}\|\beta\|_{\ell^2}\left\|\gamma(\cdot)\langle\cdot\rangle^\sigma\eta_{1,\cdot}(t,\cdot)\langle\cdot\rangle^{-\sigma}\right\|_{\ell^2}\\ &\le c\langle t\rangle^{-\frac12}[A](T)^{p-1}\|\gamma(\cdot)\langle\cdot\rangle^\sigma\|_{\ell^\infty}\left\|\eta_{1,\cdot}(t,\cdot)\langle\cdot\rangle^{-\sigma}\right\|_{\ell^2}\\ &\le c\langle t\rangle^{-2}[A](T)^{p-1}\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}\right).\end{aligned}$$

The third term $F_2(a,\eta_{3,\cdot})$ can be estimated by

$$\begin{aligned}|F_2(a,\eta_{3,\cdot})| &\le \left|2^{-\frac{p}{2}}pa(t)^{p-1}\langle\beta(\cdot),\gamma(\cdot)\eta_{3,\cdot}(t,\cdot)\rangle_{\ell^2}\right|\\ &\le c|a(t)|^{p-1}\|\beta\|_{\ell^2}\|\gamma(\cdot)\eta_{3,\cdot}(t,\cdot)\|_{\ell^2}\\ &\le c\langle t\rangle^{-\frac12}[A](T)^{p-1}\|\gamma(\cdot)\langle\cdot\rangle^\sigma\|_{\ell^2}\left\|\langle\cdot\rangle^{-\sigma}\eta_{3,\cdot}(t,\cdot)\right\|_{\ell^2}\\ &\le c\langle t\rangle^{-\frac12}[A](T)^{p-1}\left\|\langle\cdot\rangle^{-\sigma}\eta_{3,\cdot}(t,\cdot)\right\|_{\ell^2}.\end{aligned}$$

Since $\eta_{3,\cdot}$ solves the differential equation (2.35) we express $\eta_{3,\cdot}$ with the help of Duhamel's formula

$$\eta_{3,\cdot}(t,j) = \int_0^t \frac{\sin(\sqrt{H}(t-s))}{\sqrt{H}}\left[P_c\gamma(j)N(a\Phi+\eta)(s,j) - 2^{-\frac{p}{2}}\gamma(j)a(s)^p\xi(j)\right]\mathrm{d}s.$$

We use the improved decay estimate (2.13) to get

$$\left\|\langle\cdot\rangle^{-\sigma}\eta_{3,\cdot}(t,\cdot)\right\|_{\ell^2} \le c\int_0^t\langle t-s\rangle^{-\frac32}\left\|P_c\gamma(\cdot)N(a\Phi+\eta)(s,\cdot) - 2^{-\frac{p}{2}}\gamma(\cdot)a(s)^p\xi(\cdot)\right\|_{\ell^2_\sigma}\mathrm{d}s.$$

We estimate the norm in the integral further via

$$\begin{aligned}
&\left\|P_c\gamma(\cdot)N(a\Phi+\eta)(s,\cdot)-2^{-\frac{p}{2}}\gamma(\cdot)a(s)^p\xi(\cdot)\right\|_{\ell^2_\sigma}\\
&\quad\le c\left\|\gamma(\cdot)\langle\cdot\rangle^{\sigma}\sum_{\rho=1}^{p}\binom{p}{\rho}\left(\frac{1}{\sqrt{2}}a(s)\right)^{p-\rho}\langle\cdot\rangle^{\sigma p}\langle\cdot\rangle^{-\sigma p}\eta(s,\cdot)^{\rho}\right\|_{\ell^2}\\
&\quad\le c\sum_{\rho=1}^{p}\left\|\langle\cdot\rangle^{\sigma(p+1)}\gamma(\cdot)\right\|_{\ell^\infty}\left\|\langle\cdot\rangle^{-\sigma p}\eta(s,\cdot)^{\rho}\right\|_{\ell^2}|a(s)|^{p-\rho}\\
&\quad\le c\sum_{\rho=1}^{p}\|\eta(s)\|^{\rho}_{\ell^2_{-\sigma}}|a(s)|^{p-\rho}\\
&\quad\le c\sum_{\rho=1}^{p}\langle s\rangle^{\rho(-1-\frac{1}{2p-2})}[\eta](T)^{\rho}[A](T)^{p-\rho}\langle s\rangle^{-\frac{p-\rho}{2p-2}}\\
&\quad\le c\sum_{\rho=1}^{p}[\eta](T)^{\rho}[A](T)^{p-\rho}\langle s\rangle^{-1-\frac{p}{2p-2}}.
\end{aligned}$$

By integrating this estimate we obtain

$$|F_2(a,\eta_{3,\cdot})|\le c\langle t\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}[A](T)^{p-1}\sum_{\rho=1}^{p}[\eta](T)^{\rho}[A](T)^{p-\rho}.$$

Using the singular resolvent estimate (2.14) we estimate the term $F_2(a,\eta^{nr1}_{*,\cdot})$ through

$$\begin{aligned}
\left|F_2(a,\eta^{nr1}_{*,\cdot})\right|\le&\sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})}\binom{p}{\rho}\frac{p}{2^{p+1}}\left|a(t)^{p-1}A_0^{p-\rho}\overline{A}_0^{\rho}\right|\\
&\times\left|\left\langle\gamma(\cdot)\beta(\cdot),\frac{\mathrm{e}^{\mathrm{i}\sqrt{H}t}}{\sqrt{H}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2}\right|\\
&+\sum_{-(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})}\binom{p}{\rho}\frac{p}{2^{p+1}}\left|a(t)^{p-1}A_0^{p-\rho}\overline{A}_0^{\rho}\right|\\
&\times\left|\left\langle\gamma(\cdot)\beta(\cdot),\frac{\mathrm{e}^{-\mathrm{i}\sqrt{H}t}}{\sqrt{H}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}0)}\gamma(\cdot)\xi(\cdot)\right\rangle_{\ell^2}\right|\\
\le c&\sum_{(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})}\langle t\rangle^{-1-\frac{1}{2p-2}}[A](T)^{2p-1}\|\gamma(\cdot)\langle\cdot\rangle^{\sigma}\|_{\ell^\infty}\|\beta\|_{\ell^2}\\
&\times\left\|\langle\cdot\rangle^{-\sigma}\mathrm{e}^{\mathrm{i}\sqrt{H}t}\sqrt{H}^{-1}(\sqrt{H}-(p-2\rho)\omega_0+\mathrm{i}0)^{-1}\gamma(\cdot)\xi(\cdot)\right\|_{\ell^2}\\
&+c\sum_{-(p-2\rho)\omega_0\in\sigma_c(\sqrt{H})}\langle t\rangle^{-1-\frac{1}{2p-2}}[A](T)^{2p-1}\|\gamma(\cdot)\langle\cdot\rangle^{\sigma}\|_{\ell^\infty}\|\beta\|_{\ell^2}\\
&\times\left\|\langle\cdot\rangle^{-\sigma}\mathrm{e}^{-\mathrm{i}\sqrt{H}t}\sqrt{H}^{-1}(\sqrt{H}+(p-2\rho)\omega_0-\mathrm{i}0)^{-1}\gamma(\cdot)\xi(\cdot)\right\|_{\ell^2}\\
\le c&\langle t\rangle^{-1-\frac{1}{2p-2}}[A](T)^{2p-1}c_\sigma\langle t\rangle^{-\frac{3}{2}}\|\gamma(\cdot)\xi(\cdot)\|_{\ell^2}\le c\langle t\rangle^{-\frac{5}{2}-\frac{1}{2p-2}}[A](T)^{2p-1}.
\end{aligned}$$

The remaining three terms $F_2(a,\eta^{nr2}_{*,\cdot})$, $F_2(a,\eta^{nr1}_{\star,\cdot})$ and $F_2(a,\eta^{nr2}_{\star,\cdot})$ possess a similar structure and can be treated analogously. $\square$

By combining all these estimates we obtain the desired estimate

$$|E| \le c\langle t\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}|E_0|.$$

If we set $Q(t)$ as in (2.52) with $Q_0 := c([A](T)^{3p-2}+E_0)$ the inequality (2.50) is satisfied. Hence, we obtain the decay rate of $t^{-\frac{1}{2p-2}}$ for the one-dimensional component of the solution as in (2.51) which immediately implies

$$|a(t)| \le c\langle t\rangle^{-\frac{1}{2p-2}}\left(|A_0|^4 + Q_0^{\frac{2p+2}{2p-1}}\right)^{\frac{1}{4}}. \tag{2.56}$$

2.6.3 Asymptotics of the infinite-dimensional solution component

In this section we prove the decay estimate for the infinite-dimensional solution component. We anticipate a decay rate of $t^{-1-\frac{1}{2p-2}}$. In order to establish this decay rate we estimate η as follows

$$\begin{aligned}
\|\eta(t)\|_{\ell^2_{-\sigma}} &\le \|\eta_{1,\cdot}(t)\|_{\ell^2_{-\sigma}} + \|\eta_{2,\cdot}(t)\|_{\ell^2_{-\sigma}} + \|\eta_{3,\cdot}(t)\|_{\ell^2_{-\sigma}} \\
&\le \left\|\cos(\sqrt{H}t)P_cX_0 + \sqrt{H}^{-1}\sin(\sqrt{H}t)P_cX_1\right\|_{\ell^2_{-\sigma}} \\
&\quad + \left\|\int_0^t \sqrt{H}^{-1}\sin(\sqrt{H}(t-s))2^{-\frac{p}{2}}\gamma(\cdot)a(s)^p\xi(\cdot)\mathrm{d}s\right\|_{\ell^2_{-\sigma}} \\
&\quad + \left\|\int_0^t \frac{\sin(\sqrt{H}(t-s))}{\sqrt{H}}\left[P_c\gamma(\cdot)N(a\Phi+\eta)(s,\cdot) - 2^{-\frac{p}{2}}\gamma(\cdot)a(s)^p\xi(\cdot)\right]\mathrm{d}s\right\|_{\ell^2_{-\sigma}} \\
&\le c\langle t\rangle^{-\frac{3}{2}}\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) \\
&\quad + c\int_0^t \left\|\sqrt{H}^{-1}\sin(\sqrt{H}(t-s))\gamma(\cdot)\xi(\cdot)\right\|_{\ell^2_{-\sigma}}|a(s)|^p\mathrm{d}s \\
&\quad + c\int_0^t \left\|\frac{\sin(\sqrt{H}(t-s))}{\sqrt{H}}\left[P_c\gamma(\cdot)N(a\Phi+\eta)(s,\cdot) - 2^{-\frac{p}{2}}\gamma(\cdot)a(s)^p\xi(\cdot)\right]\right\|_{\ell^2_{-\sigma}}\mathrm{d}s \\
&\le c\langle t\rangle^{-\frac{3}{2}}\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) \\
&\quad + c\int_0^t \langle t-s\rangle^{-\frac{3}{2}}\langle s\rangle^{-\frac{1}{2}-\frac{1}{2p-2}}[A](T)^p\mathrm{d}s \\
&\quad + c\int_0^t \langle t-s\rangle^{-\frac{3}{2}}\left\|P_c\gamma(\cdot)N(a\Phi+\eta)(s,\cdot) - 2^{-\frac{p}{2}}\gamma(\cdot)a(s)^p\xi(\cdot)\right\|_{\ell^2_\sigma}\mathrm{d}s.
\end{aligned}$$

The integrals on the right-hand side of the inequality exist. Thus, we obtain with the help of (2.53) that

$$\begin{aligned}
\|\eta(t)\|_{\ell^2_{-\sigma}} &\le c\langle t\rangle^{-\frac{3}{2}}\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}\right)\\
&\quad + c\langle t\rangle^{-1-\frac{1}{2p-2}}[A](T)^p\\
&\quad + c\sum_{\rho=1}^{p}\int_0^t\langle t-s\rangle^{-\frac{3}{2}}[\eta](T)^\rho[A](T)^{p-\rho}\langle s\rangle^{-\frac{3}{2}-\frac{1}{2p-2}}\mathrm{d}s\\
&\le c\langle t\rangle^{-\frac{3}{2}}\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}\right)+c\langle t\rangle^{-1-\frac{1}{2p-2}}[A](T)^p\\
&\quad + c\langle t\rangle^{-1-\frac{1}{2p-2}}\left(\sum_{\rho=1}^{p}[\eta](T)^\rho[A](T)^{p-\rho}\right).
\end{aligned}$$

This yields

$$\|\eta(t)\|_{\ell^2_{-\sigma}} \le c\langle t\rangle^{-1-\frac{1}{2p-2}}\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}+[A](T)^p+\sum_{\rho=1}^{p}[\eta](T)^\rho[A](T)^{p-\rho}\right). \tag{2.57}$$

2.6.4 Asymptotic stability of the solution with respect to the ℓ^2_σ-norm

To conclude the proof of Theorem 2.1.1 we have to combine the decay estimates for the one-dimensional solution component (2.56) and the infinite-dimensional solution component (2.57). The first step is to find a constant C_* such that

$$|a(t)| \le C_*\langle t\rangle^{-\frac{1}{2p-2}}, \quad \|\eta(t,\cdot)\|_{\ell^2_{-\sigma}} \le C_*\langle t\rangle^{-1-\frac{1}{2p-2}}$$

holds. We set $C(T) := [\eta](T)+[A](T)+c_\alpha$. Our aim is to show that there exists a constant C_* independent of $T>0$ such that

$$C(T) \le C_* \tag{2.58}$$

is satisfied. At first, we estimate

$$\begin{aligned}
Q_0 &\le c[A](T)^{3p-2}+c|E_0|\\
&\le c\left(C(T)^{3p-2}+C(T)^{2p-1}+C(T)^{p-1}\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}\right)+C(T)^3C(T)^{p-3}\right)\\
&\le c\left(C(T)^{3p-2}+C(T)^{2p-1}+C(T)^{2p-2}+\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}\right)^2+C(T)^3C(T)^{p-3}\right).
\end{aligned}$$

However, the constant Q_0 possesses the exponent $\frac{2p+2}{2p-1}\frac{1}{4}$ in the inequality (2.56). Hence, we compute

$$\begin{aligned}
Q_0^{\frac{2p+2}{2p-1}\frac{1}{4}} \le c\Bigg(&C(T)^{\frac{2p+2}{2p-1}\frac{3p-2}{4}}+C(T)^{\frac{2p+2}{2p-1}\frac{2p-1}{4}}+C(T)^{\frac{2p+2}{2p-1}\frac{2p-2}{4}}\\
&+\left(\|X_0\|_{\ell^2_\sigma}+\|X_1\|_{\ell^2_\sigma}\right)^{\frac{2p+2}{2p-1}\frac{2}{4}}+C(T)^{\frac{2p+2}{2p-1}\frac{p}{4}}\Bigg).
\end{aligned}$$

This leads to an analogous estimate as in [SW99] for the one-dimensional solution component

$$\begin{aligned}
|a(t)| &\leq c\langle t\rangle^{-\frac{1}{2p-2}} \left(|A_0|^4 + Q_0^{\frac{2p+2}{2p-1}}\right)^{\frac{1}{4}} \\
&\leq c\langle t\rangle^{-\frac{1}{2p-2}} \left(|A_0| + Q_0^{\frac{2p+2}{2p-1}\frac{1}{4}}\right) \\
&\leq c\langle t\rangle^{-\frac{1}{2p-2}} \left(\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) + \left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right)^{b_0} + \sum_{k=1}^{K} C(T)^{b_k}\right),
\end{aligned} \tag{2.59}$$

with $b_0 > \frac{1}{2}$ and $b_k > 1$ since we assumed $p \geq 4$. We use $|A_0| \leq \left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right)$. Eventually, we obtain from (2.55), (2.57) and (2.59) that

$$\begin{aligned}
[\eta](T) &\leq c\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) + cC(T)^p, \\
[A](T) &\leq c\left(\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) + \left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right)^{b}\right) + c\sum_{k=1}^{K} C(T)^{b_k}, \\
c_\alpha &\leq c\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right).
\end{aligned}$$

If we add up these three inequalities this yields

$$C(T) \leq c\left(\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) + \left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right)^{b}\right) + cC(T)\sum_{k=1}^{K} C(T)^{c_k},$$

for certain $c_k > 0$. We rearrange the terms to

$$C(T)\left(1 - c\sum_{k=1}^{K} C(T)^{c_k}\right) \leq c\left(\left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right) + \left(\|X_0\|_{\ell^2_\sigma} + \|X_1\|_{\ell^2_\sigma}\right)^{b}\right).$$

We can repeat the arguments from the proof for the a priori bounds in Lemma 2.5.2 to conclude that there exists a constant C_* which is independent of $T > 0$ which satisfies the inequality (2.58). Finally, this implies

$$|a(t)| \leq C_*\langle t\rangle^{-\frac{1}{2p-2}}, \quad \|\eta(t,\cdot)\|_{\ell^2_{-\sigma}} \leq C_*\langle t\rangle^{-1-\frac{1}{2p-2}}.$$

The claim from Theorem 2.1.1 follows from these two inequalities and the representation of the solution $X(t,j) = a(t)\Phi(j) + \eta(t,j)$.

Chapter 3

Existence of breather solutions on discrete periodic graphs

We show two abstract existence results of breather solutions in nonlinear Klein-Gordon systems on discrete periodic graphs. The proofs are based on the Theorem of Crandall-Rabinowitz. We give examples of non-trivial periodic discrete graphs for which these results are applicable.

3.1 Introduction

Starting with [AKNS73] the existence of localized time-periodic solutions of finite energy, so called breathers, for dispersive systems has received a lot of attention in the past five decades. Breathers are a very rare phenomenon in nonlinear PDEs. For instance Denzler [Den93] and Birnir, McKean and Weinstein [BMW94] showed that the breather solutions of the Sine-Gordon equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \sin(u(x,t)) = 0, \quad x, t \in \mathbb{R},$$

do not persist if the nonlinearity gets perturbed. However, this changes if we consider lattices and discrete graphs. MacKay and Aubry [MA94] constructed breathers in Hamiltonian lattices with anharmonic on-site potentials and weak coupling

$$\partial_t^2 x_n + V'(x_n) = \alpha(x_{n+1} - 2x_n + x_{n+1}), \quad n \in \mathbb{Z},\ t \in \mathbb{R},$$

with $V'(0) = 0$, $V''(0) = \omega_0^2 > 0$. In their proof breathers are obtained by the anti-continuum limit, i.e., by continuation from the uncoupled case in which trivial breathers exist. This means that only one oscillator is excited and the others are at rest. With the same ideas the existence of breathers was established for diatomic Fermi-Pasta-Ulam (FPU) chains, cf. [LSM97]. Aubry et al. [AK01] showed the existence of breathers in FPU lattices with frequencies above the phonon spectrum if the potential is a strictly convex polynomial of degree 4.

In the special case of the discrete necklace graph, cf. Figure 3.1, it was proved in [Mai20a] that breathers exist under certain non-resonance conditions. Maier considered a Klein-Gordon system

$$\begin{aligned}
\partial_t^2 u_k(t) &= f(v_k^+(t) - u_k(t)) + f(v_k^-(t) - u_k(t)) - h(u_k(t) - w_k(t)) + r(u_k(t)),\\
\partial_t^2 v_k^+(t) &= g(x_k(t) - v_k^+(t)) - f(v_k^+(t) - u_k(t)) + r(v_k^+(t)),\\
\partial_t^2 v_k^-(t) &= g(x_k(t) - v_k^-(t)) - f(v_k^-(t) - u_k(t)) + r(v_k^-(t)),\\
\partial_t^2 w_k(t) &= h(u_{k+1}(t) - w_k(t)) - g(w_k(t) - v_k^+(t)) - g(w_k(t) - v_k^-(t)) + r(w_k(t)),
\end{aligned}$$

on the discrete necklace graph, cf. Figure 3.1, with interaction potentials f, g, h, r which possess Taylor expansions of the form $f(x) = f_1 x + \ldots + f_p x^p$. The proof relies on the use of the theorem of Crandall-Rabinowitz.

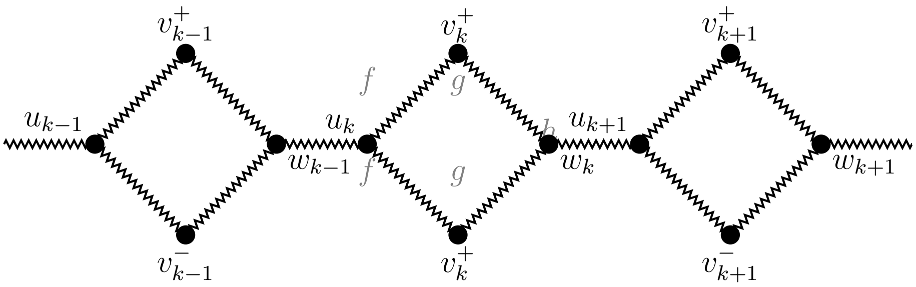

Figure 3.1: The discrete necklace graph from [Mai20a].

Our goal is to show that breathers also exist on more complex discrete graphs than the aforementioned necklace graph or diatomic FPU chains. We prove with the help of the theorem of Crandall-Rabinowitz that under certain non-resonance conditions strongly localized breathers exist on periodic discrete graphs. We will formulate the theorems and proofs in an abstract way in order to underline the great range of discrete graphs for which these statements hold. The abstract setup will be displayed in Section 3.2. Due to the abstractness we will use notation that differs from the notation in Chapter 2.

The main result, cf. Theorem 3.4.1, can be stated as follows: Let $-\omega_0^2$ be an eigenvalue of the linear part for which the non-resonance condition $-m^2\omega_0^2 \notin \sigma_{ac}(L)$, $m \in \mathbb{Z}$ holds. Suppose that the corresponding eigenspace $E_{-\omega_0^2}$ and the absolutely continuous eigenspace E_{ac} satisfy some invariance conditions. Then there exists a one-parameter family of real-valued solutions that are periodic in time and spatially localized. These breather solutions are strongly localized since they bifurcate from eigenstates that are localized in a single periodicity cell of the periodic graph, cf. Figure 3.2.

It is possible to weaken the invariance conditions in Theorem 3.4.1 by introducing a localized potential V_{loc} in the Klein-Gordon system. The corresponding result is formulated in Theorem 3.6.1.

This chapter is organized as follows. We start by introducing the notation for discrete periodic graphs and the corresponding differential equations. The spectral situation of

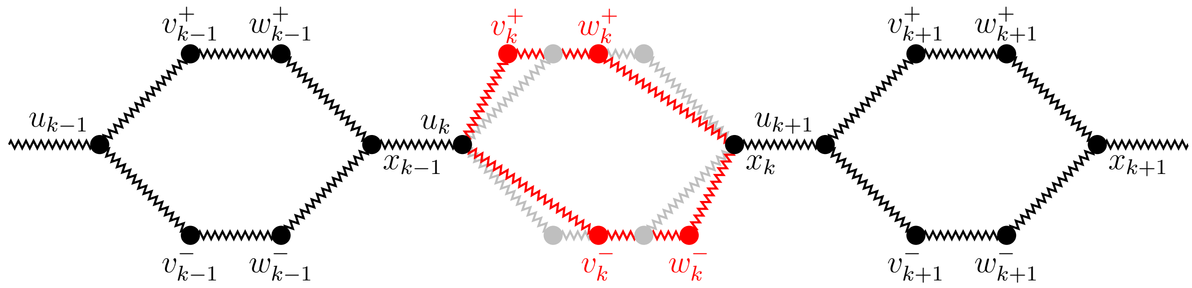

Figure 3.2: An anti-symmetric eigenstate in a discrete necklace graph with six nodes per periodicity cell. Only the masses $v_k^\pm$ and $w_k^\pm$ are displaced from their equilibrium positions.

the linearized system is discussed in Section 3.3. In Section 3.4 we formulate and prove the main result, cf. Theorem 3.4.1. In order to emphasize the convenience of our result we give an example of a periodic discrete graph for which breathers exist in Section 3.5. In Section 3.6 we formulate and prove a similar result with an additional localized potential, cf. Theorem 3.6.1. In the last part we illustrate the use of this theorem by giving an example of a periodic discrete graph for which this result is applicable.

3.2 Abstract setup

This section lays the foundation for the theorems and proofs by introducing an abstract notion of discrete periodic graphs and their associated Klein-Gordon systems. A weighted discrete graph $\Gamma = (V, E, \alpha)$ is a triple consisting of a set of nodes V and a set of edges $E \subseteq \{(u, v) \in V^2 \text{ and } u \neq v\}$ and a weight function $\alpha : E \to \mathbb{R}$. For $u, v \in V$ we write $u \sim v$ if $(u, v) \in E$.

We introduce the notion of a discrete $\mathbb{Z}^n$-periodic graph $\Gamma = (V, E, \alpha)$ and its associated translation over the periodicity cells T.

Definition 3.2.1. A discrete graph $\Gamma = (V, E, \alpha)$ is periodic or $\mathbb{Z}^n$-periodic if and only if the set of nodes V can be equipped with the action of a free abelian group $G = \mathbb{Z}^n$. To be precise, there exists a map

$$\begin{aligned} T : (G, V) &\to V, \\ (g, v) &\mapsto T_g v, \end{aligned}$$

with the following properties:

(G1) **Group action**. For every $g \in G$ the map $v \mapsto T_g v$ is a bijection from V onto itself. It holds $T_0 v = v$ for every $v \in V$ if $0 \in G$ is the neutral element. Furthermore, we have $T_{g_1 g_2} v = T_{g_1}(T_{g_2} v)$ for every $g_1, g_2 \in G$ and $v \in V$.

(G2) **Free action**. If $T_g v = v$ for some $v \in V$ then g is the neutral element, i.e., $g = 0$.

(G3) **Discreteness**. For every $v \in V$ there exists a neighborhood U of v such that $T_g v \notin U$ for $g \neq 0$.

(G4) **Co-compactness**. The set of orbits V/G is finite. The whole set of nodes V can be obtained by G-shifts of a finite subset of V.

(G5) **Structure preserving**. We have $T_g u \sim T_g v$, i.e., $(T_g u, T_g v) \in E$, if and only if $(u, v) \in E$. The weight function α is preserved under the action of G, i.e.,

$$\alpha(u, v) = \alpha(T_g u, T_g v),$$

for any $u, v \in V$ and $g \in G$.

We call the finite set $W \subsetneq V$ a *periodicity cell* if the union of all G-shifts of W covers the whole set of nodes V and if there is no subset $X \subsetneq W$ for which the union of all G-shifts of X covers the whole set of nodes V.

Remark 3.2.2. The map T_g is a translation over the periodicity cells of Γ.

Remark 3.2.3. There exist infinitely many distinct periodicity cells. However, all periodicity cells contain the same number of nodes.

Remark 3.2.4. We illustrate the action of T using the example of the $\mathbb{Z}$-periodic necklace graph $\Gamma = (V, E, \alpha)$ with six nodes per periodicity cell, cf. Figure 3.3. The set of nodes V is given by

$$V = \bigcup_{j \in \mathbb{Z}} \left\{ u_j, v_j^+, v_j^-, w_j^+, w_j^-, x_j \right\}.$$

The action of T for some $g \in G = \mathbb{Z}$ is given by $T_g z_j = z_{j+g}$ for $z \in \{u, v^+, v^-, w^+, w^-, x\}$.

A possible choice for a periodicity cell is $W = \{u_0, v_0^+, v_0^-, w_0^+, w_0^-, x_0\}$. However, this choice is not unique since for example the set $\tilde{W} = \{w_0^+, w_0^-, x_0, u_1, v_1^+, v_1^-\}$ is also a periodicity cell of the graph Γ.

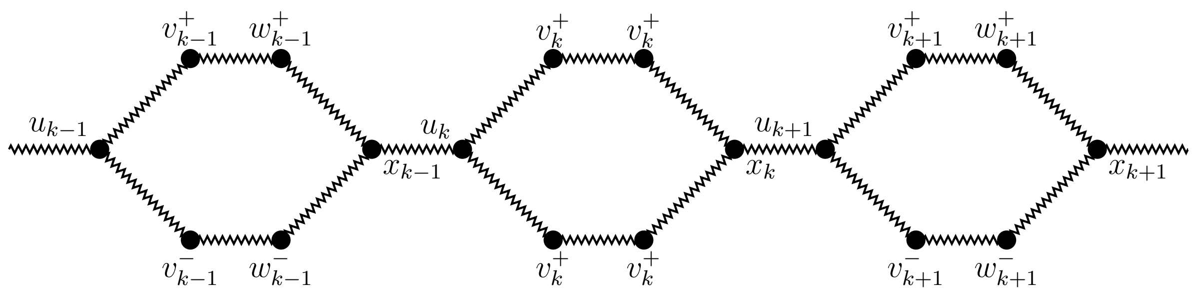

Figure 3.3: Three periodicity cells of the discrete necklace graph Γ with six nodes per periodicity cell.

We consider a nonlinear Klein-Gordon type differential equation

$$\partial_t^2 f(t,v) = Lf(t,v) + N(f(t,v)), \quad v \in V,\, t \geq 0, \tag{3.1}$$

on a discrete periodic graph $\Gamma = (V, E, \alpha)$. We use the symbol $L = \Delta_\alpha - r$ for the linear part, where $r \in \mathbb{R}$ and Δ_α denotes the weighted discrete Laplacian. The weighted discrete Laplacian consists of linear nearest-neighbor interactions between the nodes of the discrete graph. Hence, Δ_α is given by

$$\Delta_\alpha f(v) = \sum_{v \sim w} \alpha(v,w)(f(w) - f(v)),$$

for some function $f : V \to \mathbb{R}$, where the value $f(v)$ corresponds to the horizontal displacement of the mass particles $v \in V$ from its equilibrium positions. The nonlinear part N of (3.1) is of power type structure with power $p \in \mathbb{N}$ as in

$$N(f(v)) = \sum_{v \sim w} \alpha(v,w)(f(w) - f(v))^p, \tag{3.2}$$

or as in

$$N(f(v)) = r_p(v) f(v)^p, \tag{3.3}$$

where $r_p : V \to \mathbb{R}$ is some periodic weight function on the nodes of the discrete graph Γ.

3.3 Spectral situation

In order to gain a better understanding for the non-resonance conditions we collect information about the spectral situation of the linear problem. We consider the associated linearized problem of (3.1) on the discrete $\mathbb{Z}^n$-periodic graph $\Gamma = (V, E, \alpha)$ where we collect the nodes of the g-th periodicity cell in the vector $T_g W = W_g$, $g \in \mathbb{Z}^n$, which is given by

$$\partial_t^2 W_g(t) = L W_g(t), \quad g \in \mathbb{Z}^n,\, t \geq 0, \tag{3.4}$$

with $L = \Delta_\alpha - r$. The system (3.4) is solved by so-called Bloch waves

$$W_g(t) = \mathrm{e}^{\mathrm{i}(l \cdot g - \omega t)} \check{W}(l), \quad l \in \mathbb{R}^n,\, \omega \in \mathbb{R},$$

where $\check{W}(l)$, l and ω solve the eigenvalue problem

$$M_L(l) \check{W}(l) = -\omega^2 \check{W}(l).$$

For fixed l the operator $M_L(l)$ is a self-adjoint matrix. Its size is determined by the size of the periodicity cell. The coefficients of $M_L(l)$ can be computed explicitly by

$$(M_L(l))\,(j,j) = -\left(\sum_{(w_j, v) \in E} \alpha(w_j, v) \right),$$

$$(M_L(l))\,(j,k) = \left(\sum_{\substack{g \in \mathbb{Z}^n \\ (w_j, T_g w_k) \in E}} \alpha(w_j, T_g w_k) \mathrm{e}^{-\mathrm{i} l \cdot g} \right),$$

where w_j and w_k are the j-th respectively k-th node of the periodicity cell W_g. In particular, the matrix $M_L(l)$ is periodic for $l \in [-\pi, \pi)^n$.

The Floquet-Bloch theory, cf. [Eas75, RS79], implies that the spectrum of L has band gap structure and coincides with the spectrum of M_L

$$\sigma(L) = \sigma(M_L) = \bigcup_{l \in [-\pi,\pi)^n} \sigma(M_L(l)).$$

Furthermore, we have $\sigma(L) \subset \mathbb{R}$ since $M_L(l)$ is self-adjoint for every $l \in [-\pi, \pi)^n$. We denote the spectral bands by $\omega_j(l)$, for $1 \leq j \leq \dim(W)$. We distinguish between two types of spectral bands. We call a spectral band flat if

$$\inf_{l \in [-\pi,\pi)^n} \omega_j(l) = \sup_{l \in [-\pi,\pi)^n} \omega_j(l),$$

and non-flat if

$$\inf_{l \in [-\pi,\pi)^n} \omega_j(l) < \sup_{l \in [-\pi,\pi)^n} \omega_j(l).$$

The flat spectral bands correspond to the eigenvalues of L whereas the non-flat spectral bands represent the absolutely continuous spectrum $\sigma_{ac}(L)$. We denote the associated eigenspaces by E_p for the point spectrum respectively E_{ac} for the absolutely continuous spectrum. For the existence of breathers we need at least one flat spectral band, i.e., the point spectrum $\sigma_p(L) \neq \emptyset$.

3.4 Existence of breather solutions on discrete periodic graphs

In this section we will prove the existence of non-trivial discrete breathers on discrete periodic graphs by means of bifurcation theory. Breathers in discrete settings arise from the combined effects of the nonlinearity and the discreteness. Under a number of non-resonance conditions there exists a one-parameter family of breather solutions mainly supported in one periodicity cell of the discrete graph. The existence result is captured in the subsequent theorem.

Theorem 3.4.1. *We consider the following assumptions:*

(B1) The operator L possesses at least one flat spectral band corresponding to the eigenvalue $-\omega_0^2$.

(B2) For any element Z of the eigenspace $E_{-\omega_0^2}$ it holds

$$N(Z) \in E_{ac} \oplus \operatorname{span}\{Z\}.$$

(B3) The absolutely continuous eigenspace E_{ac} is invariant under the nonlinear part N.

(B4) The non-resonance condition

$$-m^2\omega_0^2 \notin \sigma_{ac}(L), \quad \textit{for all } m \in \mathbb{N}_0,$$

is fulfilled.

If the assumptions are met, there exists a one-parameter family of real-valued solutions of (3.1) *which are periodic in time and spatially localized.*

Remark 3.4.2. If the exponent p of the nonlinear part is even then the assumption (B4) contains the condition $0 \notin \sigma_{ac}(L)$ due to $m = 0 \in \mathbb{N}_0$. Thus, the factor r in (3.1) must be non-zero. But if the exponent p is odd the theorem still holds if $0 \in \sigma_{ac}(L)$, respectively $r = 0$.

Remark 3.4.3. The assumptions (B1) and (B4) guarantee that there exists an eigenvalue to which we can construct a breather. The non-resonance condition ensures that there is no energy loss through a coupling into the absolutely continuous spectrum.

Remark 3.4.4. The assumptions (B2) and (B3) guarantee that only the originally excited eigenstate resonates. If (B2) is not met it can lead to the excitation of other eigenstates in the same periodicity cell. Disregarding (B3) can lead to the excitation of eigenstates in other periodicity cells which would result in spatially non-localized solutions.

Remark 3.4.5. The existence result of breathers in [Mai20a] on the discrete necklace graph with four nodes per periodicity cell, cf. Figure 3.1, is a special case of Theorem 3.4.1. Due to the structure of the discrete necklace graph with four nodes per periodicity cell and a symmetry assumption on the interaction potentials, i.e., in our case the weight function α, there exists exactly one flat spectral band of the operator L. In particular, Theorem 3.4.1 is also applicable if there exist multiple flat spectral bands, i.e., the linear part possesses multiple eigenvalues.

The idea of the proof is the application of the Theorem of Crandall-Rabinowitz within an appropriately chosen invariant subspace of solutions. We recall the theorem of Crandall-Rabinowitz from [Kie11] which is stated as follows.

Theorem 3.4.6. *We consider a map $F : U \times V \to Y$. Let $U \times V \subset X \times \mathbb{R}$ be an open subset around $(0, 0)$ and X, Y be Banach spaces. We assume that*

(H1) $F(0, \mu) = 0$ for all $\mu \in \mathbb{R}$,

(H2) $F \in C^2(U \times V; Y)$,

(H3) $F(\cdot, 0)$ is a Fredholm operator with index 0 with

$$\dim(\mathrm{Ker}(D_Z F(0, 0))) = \mathrm{codim}(\mathrm{Ran}(D_Z F(0, 0))) = 1,$$

(H4) Let $E \in X$ with $\|E\|_X = 1$ such that $\mathrm{span}\{E\} = \mathrm{Ker}(D_Z F(0, 0))$. Then it holds

$$\left[D^2_{\mu Z} F(0, 0)\right](E) \notin \mathrm{Ran}(D_Z F(0, 0)).$$

Then there exists a non-trivial branch of solutions described by a C^1-curve

$$\{(Z_s, \mu_s) \,:\, s \in (-s_0, s_0),\, (Z_0, \mu_0) = (0,0)\},$$

which satisfies $F(Z_s, \mu_s) = 0$ locally. All solutions in a neighborhood of $(0,0)$ are either trivial solutions or lie on the non-trivial curve.

Proof of Theorem 3.4.1. Let $I = \left[-\frac{\pi}{\omega_0}, \frac{\pi}{\omega_0}\right]$ be an interval with $-\omega_0^2$ the eigenvalue of L. We fix $k_0 \in \mathbb{Z}^n$ and denote by $W \subset V$ a periodicity cell of our graph Γ and identify the k_0-th periodicity cell by $W_{k_0} := T_{k_0} W$. We know there exists exactly one normalized eigenfunction f_{k_0} to the eigenvalue $-\omega_0^2$ which satisfies (B2). In the next step we introduce the time dependent spaces

$$X(k_0) := C^2_{\mathrm{per}}(I, E_{ac} \oplus \mathrm{span}\{f_{k_0}\}),$$

with associated norm

$$\|Z\|_{X(k_0)} := \max_{t \in I} \|Z(t)\|_{\ell^2} + \max_{t \in I} \left\|\dot{Z}(t)\right\|_{\ell^2} + \max_{t \in I} \left\|\ddot{Z}(t)\right\|_{\ell^2},$$

and

$$Y(k_0) := C^0_{\mathrm{per}}(I, E_{ac} \oplus \mathrm{span}\{f_{k_0}\}),$$

with associated norm

$$\|Z\|_{Y(k_0)} := \max_{t \in I} \|Z(t)\|_{\ell^2},$$

of periodically extendable functions with values in $S(k_0) := E_{ac} \oplus \mathrm{span}\{f_{k_0}\}$. If the exponent p of the nonlinear term N is even we denote the space of even functions in time by

$$X_{\mathrm{even}}(k_0) := C^2_{\mathrm{per,even}}(I, S(k_0)),$$

and respectively for odd exponents we denote the space of odd functions in time by

$$X_{\mathrm{odd}}(k_0) := C^2_{\mathrm{per,odd}}(I, S(k_0)).$$

Then the map F from the theorem of Crandall and Rabinowitz is given by

$$\begin{aligned} F :& X(k_0) \times \mathbb{R} \to Y(k_0), \\ & F(Z, \mu)(t) = (1+\mu)\ddot{Z}(t) - LZ(t) - N(Z)(t). \end{aligned}$$

The map F is well-defined since we have the estimate

$$\|N(Z)(t)\|^2_{\ell^2} \le C(\alpha) \|Z(t)\|^{2p}_{\ell^{2p}} \le C(\alpha) \|Z(t)\|^{2p}_{\ell^2},$$

which is valid since the embedding $\ell^2 \subset \ell^{2p}$ and the assumption (B3) hold. To finish the proof we have to verify the assumptions (H1) - (H4) on our map F from the theorem of Crandall and Rabinowitz.

It is obvious that $F(0,\mu) = 0$ for all $\mu \in \mathbb{R}$. Thus, a trivial solution branch exists. The assumption (H2) is satisfied due to the polynomial structure of F, respectively N. In the next step we compute the Fréchet derivatives which are necessary for checking the remaining assumptions

$$\begin{aligned}[D_Z F(0,\mu)]\,(H) &= (1+\mu)\partial_t^2 H(t) - LH(t),\\ \left[D^2_{\mu Z} F(0,\mu)\right](H) &= \partial_t^2 H(t).\end{aligned}$$

We introduce the spaces

$$X_{ac} := C^2_{\mathrm{per}}(I, E_{ac}), \quad X_{ac,\mathrm{even}} := C^2_{\mathrm{per,even}}(I, E_{ac}), \quad X_{ac,\mathrm{odd}} := C^2_{\mathrm{per,odd}}(I, E_{ac}),$$

and

$$Y_{ac} := C^0_{\mathrm{per}}(I, E_{ac}), \quad Y_{ac,\mathrm{even}} := C^0_{\mathrm{per,even}}(I, E_{ac}), \quad Y_{ac,\mathrm{odd}} := C^0_{\mathrm{per,odd}}(I, E_{ac}).$$

It follows directly that

$$D_Z F(0,0) X_{ac} \subset Y_{ac}. \tag{3.5}$$

The time dependent function

$$f_{k_0}(t) = f_{k_0}\sin(\omega_0 t) \in X(k_0)$$

solves the linear problem

$$\partial_t^2 f_{k_0} = L f_{k_0}.$$

As a direct consequence from (B2) we obtain

$$\mathrm{Ker}(D_Z F(0,0)) = \mathrm{span}\{f_{k_0}(t)\}. \tag{3.6}$$

Next, we check the assumption (H4) which can be stated with the help of (3.6) as

$$D^2_{\mu Z} F(0,0) f_{k_0}(t) = \partial_t^2 f_{k_0}(t) \notin \mathrm{Ran}(D_z F(0,0)).$$

Hence, the assumption (H4) is satisfied if the equation

$$-\omega_0^2 f_{k_0}(t) = \partial_t^2 H(t) - LH(t),$$

for $H \in X(k_0)$ does not possess a solution. Thanks to the observation (3.5) the function H must be of the form

$$H(t) = \kappa f_{k_0}(t),$$

for some $\kappa \in \mathbb{R}$. The resulting equation

$$-\omega_0^2 f_{k_0}(t) = \kappa\left(\partial_t^2 f_{k_0}(t) - L f_{k_0}(t)\right) = \kappa \cdot 0$$

does not possess a solution for any $\kappa \in \mathbb{R}$. Thus, assumption (H4) is satisfied.

The last remaining point is to check if

$$\operatorname{codim}(\operatorname{Ran}(D_Z F(0,0))) = 1$$

holds. From (3.5) we conclude that

$$\operatorname{span}\{f_{k_0}(t)\} \nsubseteq [D_Z F(0,0)]\, X_{ac}.$$

Thus, the following equivalence is valid

$$\begin{aligned} &\operatorname{codim}(\operatorname{Ran}(D_Z F(0,0))) = 1 \\ &\qquad \Leftrightarrow D_Z F(0,0) \text{ is invertible on } X_{ac} \to Y_{ac}. \end{aligned} \tag{3.7}$$

The right-hand side of (3.7) is satisfied if the equation

$$\xi = [D_Z F(0,0)]\, \eta = \left(\partial_t^2 - L\right) \eta \tag{3.8}$$

is solvable on the subspace X_{ac}. At this point it is important to distinguish between even and odd functions in time since we can represent the $2\pi/\omega_0$ periodic and even functions through

$$\xi(t) = \sum_{m\in\mathbb{N}_0} \xi_m \cos(m\omega_0 t), \quad \eta(t) = \sum_{m\in\mathbb{N}_0} \eta_m \cos(m\omega_0 t),$$

with $\xi_m, \eta_m \in E_{ac}$ for $m \in \mathbb{N}_0$, and respectively the $2\pi/\omega_0$ periodic and odd functions through

$$\xi(t) = \sum_{m\in\mathbb{N}} \xi_m \sin(m\omega_0 t), \quad \eta(t) = \sum_{m\in\mathbb{N}} \eta_m \sin(j\omega_0 t),$$

with $\xi_m, \eta_m \in E_{ac}$ for $m \in \mathbb{N}$.

By inserting these representations in (3.8) we obtain the time independent equations in the even case

$$\xi_m = \left(-m^2\omega_0^2 - L\right) \eta_m, \quad m \in \mathbb{N}_0,$$

or respectively in the odd case

$$\xi_m = \left(-m^2\omega_0^2 - L\right) \eta_m, \quad m \in \mathbb{N}.$$

These equations are solvable if and only if the non-resonance condition from (B4) is satisfied. Thus, there exists a non-trivial branch of solutions to the equation

$$(1+\mu^2)\ddot{Z}_s(t) - LZ_s(t) - N(Z_s)(t) = 0,$$

with $Z_s \in X(k_0)$ for $s \in (-s_0, s_0)$ for sufficiently small $s_0 > 0$. With the help of a suitable rescaling in time the proof of this theorem is done. □

3.5 Breathers for discrete necklace graphs

We want to emphasize the use of this theorem by giving an application example and checking the assumptions.

We apply Theorem 3.4.1 to discrete periodic necklace graphs with more than four nodes per periodicity cell. The case with four nodes was handled in [Mai20a]. We consider discrete periodic necklace graph as in Figure 3.4 with $2+2n+m$ nodes for $n \geq 1$ and $m \geq 0$ and its associated Klein-Gordon system

$$\partial_t^2 f(t,v) = \underbrace{\Delta_\alpha f(t,v) + r \cdot f(t,v)}_{=:Lf(t,v)} + N(f(t,v)), \quad v \in V, t \geq 0.$$

We compute the spectrum via the multiplication operator M_L which is obtained by using

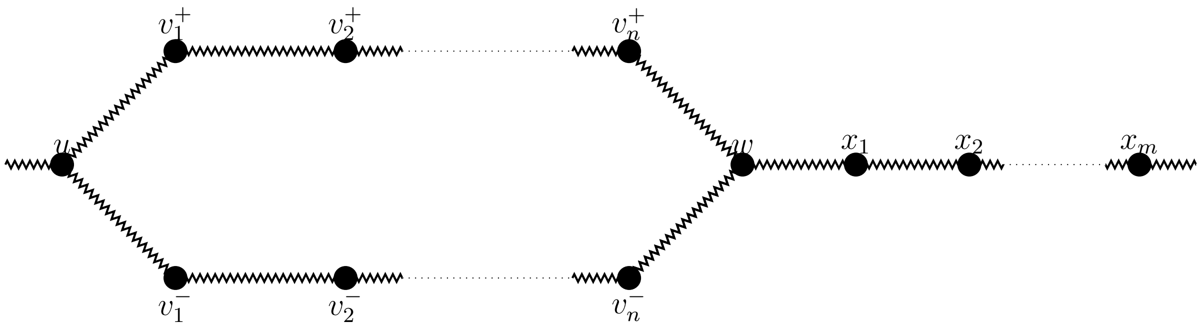

Figure 3.4: A periodicity of a necklace graph with $2+2n+m$ nodes

the Floquet-Bloch transform. In the case with $2+2n+m$ nodes the spectrum of L consists of $2+2n+m$ spectral bands. Due to the structure of the necklace graph we characterize the absolutely continuous eigenspace E_{ac} by

$$E_{ac} := \left\{ f : \ f(v_j^+) = f(v_j^-), 1 \leq j \leq n \right\}.$$

Thus, E_{ac} is symmetric with respect to the semicircles. In contrast to that the eigenspaces corresponding to the eigenvalues are anti-symmetric with respect to the semicircles. If there is more than one flat spectral band, i.e., more than one eigenvalue, then the functions from two distinct eigenspaces are orthogonal to each other.

In order to apply Theorem 3.4.1 we need to check the assumptions. The first assumption (B1) and the last assumption (B4) can be checked by studying the spectral picture of L. The two remaining conditions rely heavily on the nonlinear term N. We distinguish between two types for N. On the one hand we have

$$N(f(v)) = \sum_{w \sim v} \alpha_p(v,w)(f(w) - f(v))^p,$$

where $\alpha_p : E \to \mathbb{R}_+$ is a periodic weight function on the edges of the graph Γ as in (3.2). On the other hand the nonlinearity is given by

$$N(f(v)) = r_p(v) f(v)^p,$$

where $r_p : V \to \mathbb{R}_+$ is a periodic weight function on the nodes of the graph Γ as in (3.3). For both types of nonlinearity we need to assume that α_p respectively r_p are symmetric with respect to the semicircles.

We start by verifying (B3) for (3.2). We choose an arbitrary $Z \in E_{ac}$ and show that $N(Z) \in E_{ac}$. The absolutely continuous eigenspace is characterized through the fact that the nodes opposite each other v_j^+ and v_j^-, for $1 \le j \le n$ possess the same value. Thus, it is sufficient to compute the nonlinearity N at the nodes $v_j^\pm$ for some $1 \le j \le n$. At the nodes $v_1^\pm$ and $v_n^\pm$ we obtain

$$\begin{aligned} N(v_1^+) &= \alpha_p(v_1^+, u) \cdot (u - v_1^+)^p + \alpha_p(v_1^+, v_2^+) \cdot (v_2^+ - v_1^+)^p \\ &= \alpha_p(v_1^-, u) \cdot (u - v_1^-)^p + \alpha_p(v_1^-, v_2^-) \cdot (v_2^- - v_1^-)^p = N(v_1^-), \end{aligned}$$

respectively

$$\begin{aligned} N(v_n^+) &= \alpha_p(v_n^+, w) \cdot (w - v_n^+)^p + \alpha_p(v_n^+, v_{n-1}^+) \cdot (v_{n-1}^+ - v_n^+)^p \\ &= \alpha_p(v_n^-, w) \cdot (w - v_n^-)^p + \alpha_p(v_n^-, v_{n-1}^-) \cdot (v_{n-1}^- - v_n^-)^p = N(v_n^-). \end{aligned}$$

The computation at $v_j^\pm$ for $1 < j < n$ reads as

$$\begin{aligned} N(v_j^+) &= \alpha_p(v_j^+, v_{j-1}^+) \cdot (v_{j-1}^+ - v_j^+)^p + \alpha_p(v_j^+, v_{j+1}^+) \cdot (v_{j+1}^+ - v_j^+)^p \\ &= \alpha_p(v_j^-, v_{j-1}^-) \cdot (v_{j-1}^- - v_j^-)^p + \alpha_p(v_j^-, v_{j+1}^-) \cdot (v_{j+1}^- - v_j^-)^p = N(v_j^-). \end{aligned}$$

Thus, we have $N(Z) \in E_{ac}$ for every $Z \in E_{ac}$, and the absolutely continuous eigenspace is invariant under the nonlinearity N which concludes the check for (B3). The last assumption (B2) can be checked in a similar way. We recall that for an arbitrary $Z \in E_{-\omega_0^2}$ we must show that

$$N(Z) \in E_{ac} \oplus \operatorname{span}\{Z\}.$$

An element of the space $E_{-\omega_0^2}$ can be characterized by the fact that it has value zero outside the semicircles and is anti-symmetric with respect to the semicircles. Therefore it is again sufficient to compute the values of N at the nodes $v_j^\pm$ for some $1 \le j \le n$. We start with the nodes $v_1^\pm$ and $v_n^\pm$ and recall that $v_j^+ = -v_j^-$ for all $1 \le j \le n$. It follows

$$\begin{aligned} N(v_1^-) &= \alpha_p(v_1^-, u) \cdot (0 - v_1^-)^p + \alpha_p(v_1^-, v_2^-) \cdot (v_2^- - v_1^-)^p \\ &= (-1)^p \cdot \alpha_p(v_1^+, u) \cdot (v_1^+)^p + \alpha_p(v_1^+, v_2^+) \cdot (-1)^p \cdot (v_2^+ - v_1^+)^p = (-1)^p N(v_1^+), \end{aligned}$$

respectively

$$\begin{aligned} N(v_n^-) &= \alpha_p(v_n^-, w) \cdot (0 - v_n^-)^p + \alpha_p(v_n^-, v_{n-1}^-) \cdot (v_{n-1}^- - v_n^-)^p \\ &= (-1)^p \cdot \alpha_p(v_n^+, w) \cdot (v_n^+)^p + \alpha_p(v_n^+, v_{n-1}^+) \cdot (-1)^p \cdot (v_{n-1}^+ - v_n^+)^p = (-1)^p N(v_n^+). \end{aligned}$$

For the nodes $v_j^\pm$ with $1 < j < n$ we obtain

$$\begin{aligned} N(v_j^-) &= \alpha_p(v_j^-, v_{j-1}^-) \cdot (v_{j-1}^- - v_j^-)^p + \alpha_p(v_j^-, v_{j+1}^-) \cdot (v_{j+1}^- - v_j^-)^p \\ &= \alpha_p(v_j^+, v_{j-1}^+) \cdot (-1)^p \cdot (v_{j-1}^+ - v_j^+)^p + \alpha_p(v_j^+, v_{j+1}^+) \cdot (-1)^p \cdot (v_{j+1}^+ - v_j^+)^p \\ &= (-1)^p N(v_j^+). \end{aligned}$$

Thus, if the exponent p is even we have for $Z \in E_{-\omega_0^2}$ that $N(Z) \in E_{ac}$. If the exponent p is odd we obtain for $Z \in E_{-\omega_0^2}$ that $N(Z) \in \text{span}\{Z\}$. Hence, the assumption (B2) holds and the Theorem 3.4.1 can be applied. Therefore, breather solutions exist on such discrete periodic necklace graphs with nonlinearity N of type (3.2).

We also want to check the assumptions for nonlinear terms N of type (3.3). In this case it suffices to compute the nonlinearity N at $v_j^\pm$ for some $1 \leq j \leq n$ since there is no nearest neighbor interaction in the nonlinear terms. Hence, if we choose an arbitrary $Z \in E_{ac}$ we obtain

$$N(v_j^+) = r_p(v_j^+) \cdot (v_j^+)^p = r_p(v_j^-) \cdot (v_j^-)^p = N(v_j^-).$$

This verifies assumption (B3). In order to check assumption (B2) we compute for an arbitrary $Z \in E_{-\omega_0^2}$ that

$$N(v_j^+) = r_p(v_j^+) \cdot (v_j^+)^p = r_p(v_j^-) \cdot (-v_j^-)^p = (-1)^p N(v_j^-).$$

Hence, we have again that

$$Z \in E_{-\omega_0^2} \Rightarrow \begin{cases} N(Z) \in E_{ac}, & p \text{ even}, \\ N(Z) \in E_{-\omega_0^2}, & p \text{ odd}. \end{cases}$$

We conclude that assumption (B2) is also satisfied. Thus, we apply Theorem 3.4.1 to show the existence of breather solutions on such discrete necklace graphs with nonlinearity of type (3.3).

3.6 Existence of breather solutions on discrete periodic graphs with a localized potential

We consider the following Klein-Gordon system with additional localized potential V_{loc}

$$\partial_t^2 f(t,v) = \Delta_\alpha f(t,v) - r \cdot f(t,v) + V_{\text{loc}} f(t,v) + N(f(t,v)), \quad v \in V, t \geq 0, \tag{3.9}$$

on the $\mathbb{Z}^n$-periodic discrete graph $\Gamma = (V, E, \alpha)$. The associated linearized problem of (3.9) is given by

$$\partial_t^2 W_g(t) = L W_g(t) + V_{\text{loc}} W_g(t), \quad g \in \mathbb{Z}^n,\ t \geq 0,$$

with $L = \Delta_\alpha - r$ and $T_g W = W_g$ a periodicity cell of Γ. The spectrum of the operator L can be computed as in Section 3.3. The operator $L + V_{\text{loc}}$ is self-adjoint due to the theorem of Kato-Rellich since L is a self-adjoint operator and V_{loc} is a bounded symmetric operator. It is essential that the absolutely continuous spectra of $L + V_{\text{loc}}$ and L coincide. Since V_{loc} is a compact operator and the resolvent $(L - z)^{-1}$ is bounded for $z \in \rho(L)$ the concatenation of these operators is a compact operator. Thus, the following equality for the essential spectrum of $L + V_{\text{loc}}$ and L holds

$$\sigma_{ac}(L + V_{\text{loc}}) = \sigma_{ess}(L + V_{\text{loc}}) = \sigma_{ess}(L) = \sigma_{ac}(L).$$

Theorem 3.6.1. *We consider the following assumptions:*

(C1) The operator L possesses one flat spectral band corresponding to the eigenvalue $-\omega_0^2$.

(C2) The localized potential has support in exactly one periodicity cell, i.e., for one $k_0 \in \mathbb{Z}^n$ we have

$$\operatorname{supp} V_{\mathrm{loc}} \subseteq W_{k_0}.$$

(C3) The localized potential V_{loc} generates the eigenvalue $-\omega_1^2 = -\omega_0^2 + \delta$, for some $\delta \neq m\omega_0^2$, $m \in \mathbb{Z}$. The spectrum of $L + V_{\mathrm{loc}}$ is given by

$$\sigma(L + V_{\mathrm{loc}}) = \sigma(L) \cup \{-\omega_1^2\}.$$

(C4) The non-resonance condition

$$-m^2\omega_1^2 \notin \sigma(L), \quad \text{for all } m \in \mathbb{N}_0,$$

is fulfilled.

If the assumptions are met, there exists a one-parameter family of real-valued solutions of (3.9) *which are periodic in time and spatially localized.*

Remark 3.6.2. If the exponent p of the nonlinear part is even then the assumption (C4) contains the condition $0 \notin \sigma_{ac}(L)$ due to $m = 0 \in \mathbb{N}_0$. Thus, the factor r in (3.9) must be non-zero. But if the exponent p is odd the theorem still holds if $0 \in \sigma_{ac}(L)$, respectively $r = 0$.

Remark 3.6.3. The assumptions guarantee that there exists an eigenvalue to which we can construct a breather. The non-resonance condition (C4) ensures that there is no energy loss through a coupling into the absolutely continuous spectrum.

Remark 3.6.4. The assumptions (C2) and (C3) replace the invariance assumptions (B2) and (B3) from Theorem 3.4.1. The invariance assumptions are no longer needed since the eigenvalue $-\omega_1^2$ only possesses one eigenstate in one periodicity cell. Furthermore, other eigenvalues do not generate solutions of the linear problem with frequency $\frac{2\pi}{\omega_1}$.

The idea of the proof is the application of the Theorem of Crandall-Rabinowitz within an appropriately chosen space of solutions. We recall the theorem of Crandall-Rabinowitz from [Kie11] which is stated in Theorem 3.4.6.

Proof of Theorem 3.6.1. Let $I = \left[-\frac{\pi}{\omega_1}, \frac{\pi}{\omega_1}\right]$ be an interval with $-\omega_1^2$ the eigenvalue generated by the localized potential V_{loc}. Let $d = |W|$ be the size of the periodicity cell W. We fix $k_0 \in \mathbb{Z}^n$ accordingly to assumption (C2). Thus, there exists exactly one normalized eigenfunction f_{k_0} to the eigenvalue $-\omega_1^2$. We introduce the time dependent spaces

$$X(k_0) := C^2_{\mathrm{per}}(I, \Upsilon \oplus \operatorname{span}\{f_{k_0}\}),$$

with associated norm

$$\|Z\|_{X(k_0)} := \max_{t\in I}\|Z(t)\|_{\ell^2} + \max_{t\in I}\left\|\dot{Z}(t)\right\|_{\ell^2} + \max_{t\in I}\left\|\ddot{Z}(t)\right\|_{\ell^2},$$

and

$$Y(k_0) := C^0_{\text{per}}(I, \Upsilon \oplus \operatorname{span}\{f_{k_0}\}),$$

with associated norm

$$\|Z\|_{Y(k_0)} := \max_{t\in I}\|Z(t)\|_{\ell^2},$$

of periodically extendable functions with values in $S(k_0) := \Upsilon \oplus \operatorname{span}\{f_{k_0}\}$. The space Υ is given by

$$\Upsilon = \ell^2(\mathbb{Z}^n, \mathbb{R}^d) \setminus \operatorname{span}\{f_{k_0}\}.$$

If the power p of the nonlinear term N is even we denote the space of even functions in time by

$$X_{\text{even}}(k_0) := C^2_{\text{per,even}}(I, S(k_0)),$$

and respectively for odd powers p we denote the space of odd functions in time by

$$X_{\text{odd}}(k_0) := C^2_{\text{per,odd}}(I, S(k_0)).$$

Then the map F from the theorem of Crandall and Rabinowitz is given by

$$\begin{aligned} &F : X(k_0) \times \mathbb{R} \to Y(k_0), \\ &\quad F(Z,\mu)(t) = (1+\mu)\ddot{Z}(t) - LZ(t) - V_{\text{loc}}Z(t) - N(Z)(t). \end{aligned}$$

The map F is well-defined since we have the estimate

$$\|N(Z)(t)\|^2_{\ell^2} \leq C(\alpha)\|Z(t)\|^{2p}_{\ell^{2p}} \leq C(\alpha)\|Z(t)\|^{2p}_{\ell^2},$$

which is valid since the embedding $\ell^2 \subset \ell^{2p}$ holds. To finish the proof we have to verify the assumptions (H1) - (H4) on our map F from the theorem of Crandall and Rabinowitz.

It is obvious that $F(0,\mu) = 0$ for all $\mu \in \mathbb{R}$. Thus, a trivial solution branch exists. The assumption (H2) is satisfied due to the polynomial structure of F, respectively N. In the next step we compute the Fréchet derivatives which are necessary for checking the remaining assumptions

$$\begin{aligned} \left[D_Z F(0,\mu)\right](H) &= (1+\mu)\partial_t^2 H(t) - LH(t) - V_{\text{loc}}H(t), \\ \left[D^2_{\mu Z} F(0,\mu)\right](H) &= \partial_t^2 H(t). \end{aligned}$$

We introduce the spaces

$$X_\Upsilon := C^2_{\text{per}}(I,\Upsilon), \quad X_{\Upsilon,\text{even}} := C^2_{\text{per,even}}(I,\Upsilon), \quad X_{\Upsilon,\text{odd}} := C^2_{\text{per,odd}}(I,\Upsilon),$$

and

$$Y_\Upsilon := C^0_{\text{per}}(I,\Upsilon), \quad Y_{\Upsilon,\text{even}} := C^0_{\text{per,even}}(I,\Upsilon), \quad Y_{\Upsilon,\text{odd}} := C^0_{\text{per,odd}}(I,\Upsilon).$$

It follows directly that

$$D_Z F(0,0) X_\Upsilon \subset Y_\Upsilon. \tag{3.10}$$

The time dependent function

$$f_{k_0}(t) = f_{k_0} \sin(\omega_1 t) \in X(k_0)$$

solves the linear problem

$$\partial_t^2 f_{k_0} = (L + V_{\mathrm{loc}}) f_{k_0}.$$

As a direct consequence from (C2) we obtain

$$\mathrm{Ker}(D_Z F(0,0)) = \mathrm{span}\{f_{k_0}(t)\}. \tag{3.11}$$

Next, we check the assumption (H4) which can be stated with the help of (3.11) as

$$D^2_{\mu Z} F(0,0) f_{k_0}(t) = \partial_t^2 f_{k_0}(t) \notin \mathrm{Ran}(D_Z F(0,0)).$$

Hence, the assumption (H4) is satisfied if the equation

$$-\omega_1^2 f_{k_0}(t) = \partial_t^2 H(t) - (L + V_{\mathrm{loc}}) H(t),$$

for $H \in X(k_0)$ does not possess a solution. Due to the observation (3.10) the function H must be of the form

$$H(t) = \kappa f_{k_0}(t),$$

for some $\kappa \in \mathbb{R}$. The resulting equation

$$-\omega_1^2 f_{k_0}(t) = \kappa \left(\partial_t^2 f_{k_0}(t) - (L + V_{\mathrm{loc}}) f_{k_0}(t) \right) = \kappa \cdot 0$$

does not possess a solution for any $\kappa \in \mathbb{R}$. Thus, assumption (H4) is satisfied.

The last remaining point is to check if

$$\mathrm{codim}(\mathrm{Ran}(D_Z F(0,0))) = 1$$

holds. From (3.10) we conclude that

$$\mathrm{span}\{f_{k_0}(t)\} \not\subseteq [D_Z F(0,0)]\, X_\Upsilon.$$

Hence, the following equivalence is valid

$$\begin{aligned} &\mathrm{codim}(\mathrm{Ran}(D_Z F(0,0))) = 1 \\ &\qquad \Leftrightarrow D_Z F(0,0) \text{ is invertible on } X_\Upsilon \to Y_\Upsilon. \end{aligned} \tag{3.12}$$

The right-hand side of (3.12) is satisfied if the equation

$$\xi = [D_Z F(0,0)]\, \eta = \left(\partial_t^2 - L \right) \eta \tag{3.13}$$

is solvable on the subspace X_Υ. At this point it is important to distinguish between even and odd functions in time since we can represent the $2\pi/\omega_1$ periodic and even function through

$$\xi(t) = \sum_{m\in\mathbb{N}_0} \xi_m \cos(m\omega_1 t), \quad \eta(t) = \sum_{m\in\mathbb{N}_0} \eta_m \cos(m\omega_1 t),$$

with $\xi_m, \eta_m \in \Upsilon$ for $m \in \mathbb{N}_0$, and respectively the $2\pi/\omega_1$ periodic and odd functions through

$$\xi(t) = \sum_{m\in\mathbb{N}} \xi_m \cos(m\omega_1 t), \quad \eta(t) = \sum_{m\in\mathbb{N}} \eta_m \cos(m\omega_1 t),$$

with $\xi_m, \eta_m \in \Upsilon$ for $m \in \mathbb{N}$.

By inserting these representations in (3.13) we obtain the time independent equations in the even case

$$\xi_m = \left(-m^2\omega_1^2 - (L + V_{\mathrm{loc}})\right) \eta_m, \quad m \in \mathbb{N}_0,$$

or respectively in the odd case

$$\xi_m = \left(-m^2\omega_1^2 - (L + V_{\mathrm{loc}})\right) \eta_m, \quad m \in \mathbb{N}.$$

These equations are solvable if and only if the non-resonance conditions from (C3) and (C4) are satisfied. Thus, there exists a non-trivial branch of solutions to the equation

$$(1 + \mu^2)\ddot{Z}_s(t) - (L + V_{\mathrm{loc}})Z_s(t) - N(Z_s)(t) = 0,$$

with $Z_s \in X(k_0)$ for $s \in (-s_0, s_0)$ for sufficiently small s_0. With the help of a suitable rescaling in time the proof of this theorem is done. $\square$

3.7 Breathers for a discrete $\mathbb{Z}^2$-periodic graph

We give an application example for Theorem 3.6.1. We consider the discrete $\mathbb{Z}^2$-periodic graph in Figure 3.5 and its associated Klein-Gordon system with additional localized potential V_{loc}

$$\partial_t^2 u(t,v) = \underbrace{\Delta_\alpha u(t,v) + r \cdot u(t,v)}_{=:Lu(t,v)} + V_{\mathrm{loc}} u(t,v) + N(u(t,v)), \quad v \in V,\ t \geq 0.$$

Our goal is to verify the assumptions (C1)-(C4) from Theorem 3.6.1 for this discrete periodic graph. In order to check the assumption (C1) we compute the spectrum of L via the multiplication operator M_L and the Floquet-Bloch transform. The spectrum of L consists of eight spectral bands. There exists exactly one flat spectral band if the following

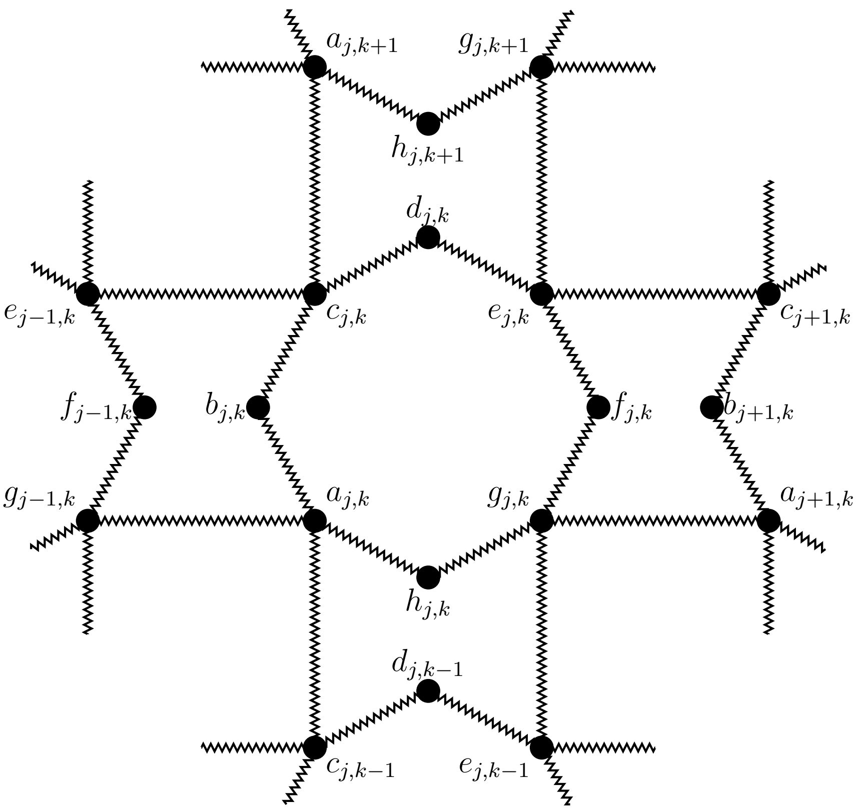

Figure 3.5: A discrete $\mathbb{Z}^2$-periodic graph.

relations for the weight function $\alpha : E \to \mathbb{R}$ are satisfied:

$$\alpha(a_{j,k}, b_{j,k}) = \alpha(g_{j,k}, f_{j,k}), \tag{3.14}$$
$$\alpha(b_{j,k}, c_{j,k}) = \alpha(f_{j,k}, e_{j,k}), \tag{3.15}$$
$$\alpha(a_{j,k}, h_{j,k}) = \alpha(c_{j,k}, d_{j,k}), \tag{3.16}$$
$$\alpha(h_{j,k}, g_{j,k}) = \alpha(d_{j,k}, e_{j,k}), \tag{3.17}$$
$$\alpha(a_{j,k}, b_{j,k}) + \alpha(b_{j,k}, c_{j,k}) = \alpha(a_{j,k}, h_{j,k}) + \alpha(h_{j,k}, g_{j,k}), \tag{3.18}$$
$$\alpha(a_{j,k}, c_{j,k-1}) = \alpha(g_{j,k}, e_{j,k-1}) = \alpha(a_{j,k}, g_{j-1,k}) = \alpha(c_{j,k}, e_{j-1,k}), \tag{3.19}$$
$$\alpha(a_{j,k}, b_{j,k}) + \alpha(b_{j,k}, c_{j,k}) < 2\alpha(a_{j,k}, c_{j,k-1}), \tag{3.20}$$

for all $j, k \in \mathbb{Z}$. Then, the operator L possesses an eigenvalue with value $-\omega_0^2 = -\alpha(a_{j,k}, b_{j,k}) - \alpha(b_{j,k}, c_{j,k}) - r$ and thus, the first assumption is satisfied. The associated eigenstates $u_{j,k}$ to the eigenvalue $-\omega_0^2$ affect only the nodes $b_{j,k}, d_{j,k}, f_{j,k}$ and $h_{j,k}$ in one periodicity cell of the graph.

Remark 3.7.1. The relations (3.14), (3.15), (3.16) and (3.17) mean that opposing edges in a periodicity cell possess the same weight. Furthermore, the sum of the weights of the two horizontal links must coincide with the sum of the two vertical links, cf. (3.18). The edges between periodicity cells all possess the same value due to (3.19). The last relation (3.20) ensures that we have an isolated flat spectral band.

We choose V_{loc} in a way such that for fixed $k_0 \in \mathbb{Z}^2$ we have

$$V_{\text{loc}} f(v) = \begin{cases} -\delta, & v \in \{b_{k_0}, d_{k_0}, f_{k_0}, h_{k_0}\}, \\ 0, & \text{else}, \end{cases}$$

for some $\delta > 0$ with $\delta \neq m\omega_0^2$, $m \in \mathbb{N}$. In this case the localized potential V_{loc} generates the eigenvalue $-\omega_1^2 = -\omega_0^2 - \delta$ with associated eigenstate u_{k_0}. Hence, the assumptions (C2) and (C3) are satisfied.

Since the spectrum of L only consists of finitely many spectral bands the last remaining assumption (C4) can be achieved through an appropriate choice of δ and a careful study of the spectral pictures of L respectively $L + V_{\text{loc}}$.

Thus, we apply Theorem 3.6.1 to show the existence of breather solutions on this discrete necklace graph with an additional localized potential.

Chapter 4

Breather solutions on discrete necklace graphs - The continuum limit

We are interested in spatially localized time-periodic solutions of the cubic Klein-Gordon equation posed on an infinite necklace graph. In [Mai20b] such solutions have been constructed via bifurcation theory, spatial dynamics and center manifold reduction. We consider the discretized version of this problem and prove the existence of generalized breather solutions on discrete necklace graphs and the convergence of generalized breather solutions towards the breather solution of the continuum problem in the continuum limit. The results are relevant for numerical computations of breather solutions.

4.1 Introduction

Breathers are spatially localized time-periodic solutions of finite energy. We are interested in the existence of such solutions for the cubic Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - \alpha u - u^3, \tag{4.1}$$

with $\alpha > 0$, $x \in \Gamma$, $t \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$, posed on an infinite necklace graph Γ and their connection to the (generalized) breather solutions for the discretized cubic Klein-Gordon equation posed on an infinite discrete necklace graph. The infinite necklace graph Γ, see Figure 4.1, consists of bonds (or edges) connected at the vertices. At the vertices the solutions have to satisfy so called Kirchhoff boundary conditions. The breather solutions which we are interested in are the same in the upper and lower circles shown in Figure 4.1 and so an equivalent formulation for such solutions is to solve (4.1) on the real line with the continuity and jump conditions

$$\begin{aligned} u(n\pi-,t) &= u(n\pi+,t), && n \in \mathbb{Z}, && (4.2)\\ \partial_x u(n\pi-,t) &= 2\partial_x u(n\pi+,t), && n \in \mathbb{Z}_{odd}, && (4.3)\\ 2\partial_x u(n\pi-,t) &= \partial_x u(n\pi+,t), && n \in \mathbb{Z}_{even}, && (4.4) \end{aligned}$$

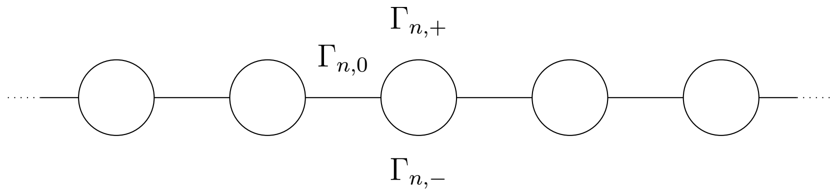

Figure 4.1: The periodic metric necklace graph Γ.

for $t \in \mathbb{R}$, where

$$u(x-) = \lim_{h\to 0, h>0} u(x-h), \qquad u(x+) = \lim_{h\to 0, h>0} u(x+h).$$

Such solutions have been constructed by bifurcation theory, spatial dynamics and invariant manifold theory in [Mai20b]. Using Fourier series the time-periodic solutions can be written as

$$u(x,t) = \sum_{m\in\mathbb{Z}_{odd}} u_m(x) e^{im\Omega t},$$

where the continuous u_m satisfy

$$-m^2\Omega^2 u_m = u_m'' - u_m - \langle u^3, e^{im\Omega t}\rangle, \qquad m \in \mathbb{Z}_{odd}, \tag{4.5}$$

with the jump conditions

$$\begin{aligned} u_m'(n\pi-) &= 2u_m'(n\pi+) \quad \text{for} \quad n \in \mathbb{Z}_{odd}, \\ 2u_m'(n\pi-) &= u_m'(n\pi+) \quad \text{for} \quad n \in \mathbb{Z}_{even}, \end{aligned} \tag{4.6}$$

where $\langle u, v\rangle = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} u(t)\overline{v(t)}dt$. Writing the infinitely many ODEs for the u_m as evolutionary system w.r.t. x is called the spatial dynamics formulation. The breather solutions correspond to bifurcating non-trivial homoclinic solutions of the spatial dynamics formulation. Due to the jump-conditions the spatial dynamics formulation is periodic w.r.t the evolutionary variable x.

The linearization of this formulation around the trivial solution possesses two central Floquet exponents, whereas all other Floquet exponents are bounded away from the imaginary axis. It turns out that a reduction to a two-dimensional center manifold is possible. On this two-dimensional center manifold non-trivial bifurcating homoclinic solutions can be found.

Remark 4.1.1. There is a one-to-one correspondence between the spectrum of the spatial dynamics formulation (4.5) and the spectrum of the time-evolutionary problem (4.1). On the one hand, the linearized cubic Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - u, \tag{4.7}$$

with the continuity and jump conditions (4.2) is solved by

$$u(x,t) = e^{i\omega t} e^{i\ell x} f_n(\ell, x),$$

with $\omega = \omega_n(\ell)$ plotted in Figure 4.2. The Bloch functions satisfy

$$-\omega^2 f_n = (\partial_x + i\ell)^2 f_n - f_n, \qquad f_n(\ell, x) = f_n(\ell, x + 2\pi), \tag{4.8}$$

and the counterparts to the continuity and jump conditions (4.2). On the other hand, the linearized spatial dynamics formulation

$$-m^2\Omega^2 u_m = u_m'' - u_m$$

with the jump conditions (4.6) is solved by $u_m(x) = e^{\mu x} B_m(x)$ where the B_m satisfy

$$-m^2\Omega^2 B_m = (\partial_x + \mu)^2 B_m - B_m, \qquad B_m(x) = B_m(x + 2\pi). \tag{4.9}$$

Therefore, comparing (4.8) and (4.9) central eigenvalues $\mu = i\ell$ of the spatial dynamics formulation 4.5 can be obtained if

$$\omega_n(\ell)^2 = m^2\Omega^2.$$

Remark 4.1.2. In lowest order the breather solutions can be approximated through the associated NLS approximation

$$u(x,t) = \varepsilon A(\varepsilon(x - ct), \varepsilon^2 t) e^{i\ell_0 x - i\omega_{n_0}(\ell_0)t} f_{n_0}(\ell_0, x) + c.c. + \mathcal{O}(\varepsilon^2),$$

with $c = \omega_{n_0}'(\ell_0)$ the group velocity and where in lowest order $A = A(X,T)$ satisfies an NLS equation

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2,$$

with coefficients

$$\nu_1 = \omega_{n_0}''(\ell_0)/2 \qquad \text{and} \qquad \nu_2 = -\langle f_{n_0}(\ell_0, x), f_{n_0}(\ell_0, x)^3 \rangle.$$

Hence, for obtaining standing time periodic solutions of small amplitude we have to choose ℓ_0 such that $\omega_{n_0}'(\ell_0) = 0$ which implies $\ell_0 \in \{0, 1/2\}$ and yields $\Omega = \omega_{n_0}(\ell_0)$ at the bifurcation point.

It was observed in [Mai20b], following [BCBLS11], that for the necklace graph it is possible to choose α in such a way that for $n_0 = 1$ and $\ell_0 = 1/2$ we have

$$m^2\Omega^2 \neq \omega_n(\ell)^2$$

for all $|m| \geq 2$, all $n, m \in \mathbb{Z}_{odd}$, and all $\ell \in [-1/2, 1/2)$. Therefore, all odd integer multiples of Ω fall in spectral gaps and so we only have four central Floquet exponents which can be reduced further to two by using the reflection symmetry of the problem. As a consequence the complete problem can be reduced to a two-dimensional center manifold where bifurcating homoclinic solutions can be found. The result of [Mai20b] can be summarized as follows.

Theorem 4.1.3. *For $n_0 = 1$ and $\ell_0 = 1/2$ and $\alpha = 1/4 + \varepsilon^2$ and $\Omega = \omega_{n_0}(l_0) + \mathcal{O}(\varepsilon^2)$ system* (4.1) *possesses time periodic solutions $u(x,t) = u_{per}(x,t)$ of order $\mathcal{O}(\varepsilon)$, with $u_{per}(x,t) = u_{per}(x, t + 2\pi/\Omega)$ and $\lim_{|x|\to\infty} u_{per}(x,t) = 0$.*

In this paper we consider uniform discretizations of the above spatial dynamics formulation and we are interested in the existence of breather solutions for the discretized version and the convergence of these (generalized) 'discrete' breathers towards the 'continuous' breather solutions from Theorem 4.1.3 if the discretization parameter gets smaller and smaller. The solutions of the discretized system are denoted with u_h and are evaluated at $x_h = jh$ where $h = 2\pi/N$ with $j \in \mathbb{Z}$ and $N \in \mathbb{N}$. With the help of these considerations we obtain the discrete Klein-Gordon systems

$$\partial_t^2 u_h(x_h,t) = \Delta_h u_h(x_h,t) - \alpha u_h(x_h,t) - u_h^3(x_h,t), \tag{4.10}$$

where we denote the discrete Laplacian Δ_h by

$$\Delta_h u(v) := \sum_{v\sim w} h^2(u(w) - u(v)).$$

Remark 4.1.4. Again there is a one-to-one correspondence between the spectrum of the discretized time-evolutionary system and the spectrum of the discretized spatial dynamics formulation. The linearized discretized cubic Klein-Gordon equation is solved by

$$u_h(x_h,t) = e^{i\omega_h t} e^{i\ell x_h} f_{h,n}(\ell, x_h),$$

with $\omega = \omega_{h,n}(\ell)$ plotted subsequently in Figure 4.6 and where the Bloch functions satisfy

$$-\omega^2 f_{h,n} = (\partial_{h,x} + i\ell)^2 f_{h,n} - \alpha f_{h,n}, \qquad f_{h,n}(\ell, x_h) = f_{h,n}(\ell, x_h + 2\pi) \tag{4.11}$$

and the counterparts to the continuity and jump conditions (4.2). On the other hand, the discretized linearized spatial dynamics formulation

$$-m^2\Omega^2 u_{h,m} = \partial_{h,x}^2 u_{h,m} - \alpha u_{h,m}$$

with the jump conditions (4.6) is solved by $u_{h,m}(x_h) = e^{\mu x_h} B_{h,m}(x_h)$ where the $B_{h,m}$ satisfy

$$-m^2\Omega^2 B_{h,m} = (\partial_{h,x} + \mu)^2 B_{h,m} - \alpha B_{h,m}, \qquad B_{h,m}(x) = B_{h,m}(x + 2\pi). \tag{4.12}$$

Therefore, comparing again (4.11) and (4.12) central eigenvalues $\mu = i\ell$ of the discretized spatial dynamics formulation can be obtained if

$$\omega_{h,n}(\ell)^2 = m^2\Omega^2.$$

Plotting the curves of eigenvalues, cf. Figure 4.2, shows that not all odd integer multiples of Ω fall in spectral gaps. However, for small m this remains true, i.e., there exists an $N_0 = N_0(N)$ such that $m\Omega$ falls in spectral gaps for $3 \le m \le N_0(N)$. Hence, ignoring terms of order $\varepsilon^{N_0(N)+1}$ and higher still gives a two-dimensional center manifold. Due to the convolution structure the ignored higher order central modes lead at most to some growth proportional to $\varepsilon^{N_0(N)+1}|x|$ in the discretized spatial dynamics formulation. Hence, the solutions possess for large $|x_h|$ tails of order $\mathcal{O}(\varepsilon^{2/N_0(N)})$. Thus, for the discretized systems we can prove the existence of generalized breather solutions.

Theorem 4.1.5. *For $n_0 = 1$ and $\ell_0 = 1/2$ and $\alpha = 1/4 + \varepsilon^2$ the discretized systems* (4.10) *possesses time-periodic solutions $u_h(x_h, t) = u_{per,h}(x_h, t)$ of order $\mathcal{O}(\varepsilon)$ for $|x_h| \leq \varepsilon^{-N_0(N)/2}$, with $u_{per,h}(x_h, t) = u_{per,h}(x_h, t + 2\pi/\Omega)$ and tails of order $\mathcal{O}(\varepsilon^{N_0(N)/2})$ for large $|x_h|$. In lowest order the solutions are given by the associated NLS approximation of the discretized systems* (4.10).

Since the spectrum and the eigenfunctions of the discretized problem converge in some weak sense against the spectrum of the continuous problem, since the NLS approximation of the discretized System (4.10) converges towards the NLS approximation of continuous system (4.1) finally we have

Theorem 4.1.6. *For $N \to \infty$, respectively $h \to 0$, the breather solutions $u_{per,h}(x_h, t)$ of the discretized systems* (4.10) *converge towards the breather solutions $u_{per,h}(x_h, t)$ of the continuous system* (4.1).

Precise formulations of Theorem 4.1.3 to Theorem 4.1.6 will be given in subsequent sections.

4.2 Breathers on the continuous necklace graph

We recall existing results about the validity of the NLS approximation for a periodic metric necklace graph, cf. [Gil17], as well as the existence result of small-amplitude breathers for a Klein-Gordon system posed on a periodic metric necklace graph, cf. [Mai20b].

i) The periodic metric necklace graph Γ can be split into $\Gamma = \oplus_{n\in\mathbb{Z}}\Gamma_n$ with $\Gamma_n = \Gamma_n^0 \oplus \Gamma_n^+ \oplus \Gamma_n^-$ where Γ_n^0 is the horizontal link whereas $\Gamma_n^\pm$ are the upper and lower semicircles, cf. Figure 4.1. We identify the horizontal links Γ_n^0 isometrically with the interval $I_n^0 = [2n\pi, 2n\pi + \pi]$ and the semicircles with the interval $I_n^\pm = [2n\pi + \pi, 2(n+1)\pi]$. The periodicity of the necklace graph is 2π, since the length of a horizontal link is π and the semicircles have length π. We denote the part of a function $U : \Gamma \to \mathbb{C}$ on the interval I_n^0 with u_n^0 and on the intervals $I_n^\pm$ with $u_n^\pm$.

We pose the following cubic Klein-Gordon equation on the necklace graph

$$\partial_t^2 U(x,t) = \partial_x^2 U(x,t) - (\alpha + \varepsilon^2)U(x,t) + U(x,t)^3, \quad t \geq 0, x \in \operatorname{int}\Gamma, \tag{4.13}$$

with some real-valued constant α and small enough $\varepsilon > 0$. We need to impose Kirchhoff boundary conditions at the vertex points $\{2n\pi\}_{n\in\mathbb{Z}}$ and $\{2n\pi + \pi\}_{n\in\mathbb{Z}}$ which consist of the continuity condition at the vertex points

$$\begin{aligned} u_n^0(2n\pi + \pi, t) &= u_n^\pm(2n\pi + \pi, t), \quad n \in \mathbb{Z}, \\ u_{n+1}^0(2(n+1)\pi, t) &= u_n^\pm(2(n+1)\pi, t), \quad n \in \mathbb{Z}, \end{aligned}$$

and of the conservation of the fluxes at the vertex points

$$\begin{aligned} \partial_x u_n^0(2n\pi + \pi, t) &= \partial_x u_n^+(2n\pi + \pi, t) + \partial_x u_n^-(2n\pi + \pi, t), \quad n \in \mathbb{Z}, \\ \partial_x u_{n+1}^0(2(n+1)\pi, t) &= \partial_x u_n^+(2(n+1)\pi, t) + \partial_x u_n^-(2(n+1)\pi, t), \quad n \in \mathbb{Z}. \end{aligned}$$

It is possible to restrict this problem to the case where functions coincide on the upper and lower semicircles of the necklace graph. Thus, conservation conditions of the Kirchhoff boundary conditions turn into

$$\begin{aligned}\partial_x u_n^0(2n\pi+\pi,t) &= 2\partial_x u_n^{\pm}(2n\pi+\pi,t), \quad n\in\mathbb{Z},\\ \partial_x u_{n+1}^0(2(n+1)\pi,t) &= 2\partial_x u_n^{\pm}(2(n+1)\pi,t), \quad n\in\mathbb{Z}.\end{aligned}$$

ii) Next we summarize shortly the spectral properties of the linearized problem

$$\partial_t^2 U(x,t) = \partial_x^2 U(x,t) - (\alpha+\varepsilon^2)U(x,t).$$

This equation can be solved by so called Bloch modes

$$U(x,t) = \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}lx} f(l,x), \quad l,\, x\in\mathbb{R},$$

where $f(l,\cdot) = (f^0, f^+, f^-)(l,\cdot)$ is a 2π-periodic function for every $l\in\mathbb{R}$. The Bloch functions f solve the eigenvalue problem

$$-(\partial_x+\mathrm{i}l)^2 f(l,x) + (\alpha+\varepsilon^2)f(l,x) = \omega^2(l)f(l,x), \tag{4.14}$$

with corresponding boundary conditions

$$\begin{aligned}f^0(l,2n\pi+\pi) &= f^{\pm}(l,2n\pi+\pi), \quad n\in\mathbb{Z},\\ f^0(l,2(n+1)\pi) &= f^{\pm}(l,2(n+1)\pi), \quad n\in\mathbb{Z},\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\partial_x f^0(l,2n\pi+\pi) &= 2\partial_x f^{\pm}(l,2n\pi+\pi), \quad n\in\mathbb{Z},\\ \partial_x f^0(l,2(n+1)\pi) &= 2\partial_x f^{\pm}(l,2(n+1)\pi), \quad n\in\mathbb{Z}.\end{aligned} \tag{4.16}$$

Thus, the spectrum of the linear problem is given by the spectral bands $\omega(l)$ which is illustrated in Figure 4.2. These spectral bands $\omega_m(l)$ can be explicitly computed and are here given by

$$\omega_m(l) = \pm\sqrt{\left(\frac{1}{2\pi}\arccos\left(\frac{8}{9}\cos(2\pi l)+\frac{1}{9}\right)+m\right)^2 + (\alpha+\varepsilon^2)},$$

for every $m\in\mathbb{Z}$. If we do not restrict ourselves to the case where the functions coincide in the semicircles then we additionally obtain flat spectral bands given by

$$\omega_m(l) = \pm\sqrt{m^2+(\alpha+\varepsilon^2)}, \quad \text{for every } m\in\mathbb{Z}.$$

We introduce L^2-based spaces for the periodic eigenvalue problem (4.14) by

$$\begin{aligned}L^2_\Gamma := \Big\{\tilde U = (\tilde u^0,\tilde u^+,\tilde u^-) \in & L^2([0,\pi])\oplus L^2([\pi,2\pi])\oplus L^2([\pi,2\pi]),\\ & \operatorname{supp}\tilde u^j = I_0^j, \quad j\in\{0,+,-\}\Big\},\end{aligned}$$

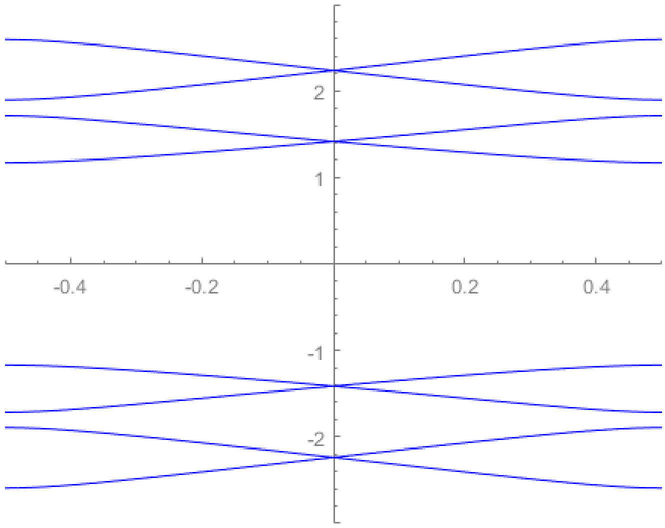

Figure 4.2: The spectral curves $\omega_m(l)$ for the symmetric linear problem on the necklace graph Γ with $\alpha + \varepsilon^2 = 1$.

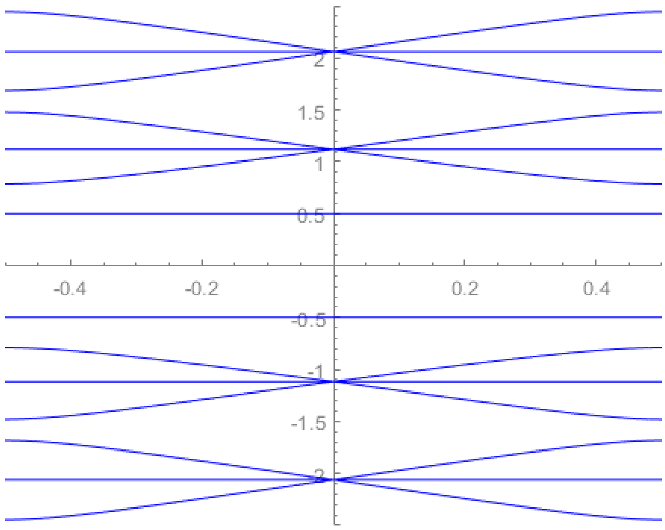

Figure 4.3: The spectral curves $\omega_m(l)$ for the non-symmetric linear problem on the necklace graph Γ with $\alpha + \varepsilon^2 = 1/4$.

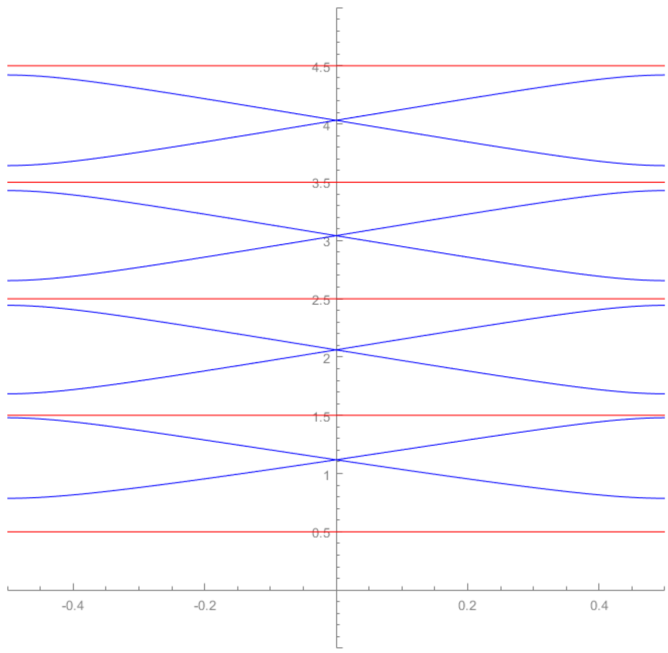

Figure 4.4: The spectral curves $\omega_m(l)$ for the symmetric linear problem in blue. The odd multiples of the frequency ω in red.

and

$$H^2_\Gamma := \left\{\tilde{U} \in L^2_\Gamma,\ \tilde{u}^j \in H^2(I^j_0), \quad j \in \{0,+,-\},\ (4.15) \text{ and } (4.16) \text{ are satisfed}\right\}.$$

iii) In [Gil17] it was shown that in lowest order the cubic Klein-Gordon equation (4.13) on the periodic necklace graph Γ possesses a NLS approximation of the form

$$U(x,t) = \varepsilon A(\varepsilon(x-ct), \varepsilon^2 t) f_{m_0}(l_0,x) \mathrm{e}^{\mathrm{i}l_0 x} \mathrm{e}^{\mathrm{i}\omega_{m_0}(l_0)t} + c.c. + \mathcal{O}(\varepsilon^2),$$

with $c = \omega'_{m_0}(l_0)$ the group velocity and where $A = A(T,X)$ satisfies an NLS equation

$$\mathrm{i}\partial_T A = \nu_1 \partial_X^2 A + \nu_2 |A|^2 A, \tag{4.17}$$

with coefficients

$$\nu_1 = -\frac{1}{2}\partial_l^2 \omega_{m_0}(l_0), \quad \nu_2 = \frac{3\left\langle f_{m_0}(l_0,x), f_{m_0}(l_0,x)^3\right\rangle_{L^2_\Gamma}}{2\omega_{m_0}(l_0)}. \tag{4.18}$$

Hence, for obtaining standing time periodic solutions of small amplitude we have to choose l_0 such that $\omega'_{m_0}(l_0) = 0$ which implies $l_0 \in \{0, \pm 1/2\}$ and yields $\Omega = \omega_{m_0}(l_0)$ at the bifurcation point.

Remark 4.2.1. To prove the validity of the NLS approximation we consider the problem (4.13) in Bloch space

$$\partial_t^2 \tilde{U}(l,x,t) = \tilde{L}(l)\tilde{U}(l,x,t) + \left(\tilde{U} * \tilde{U} * \tilde{U}\right)(l,x,t),$$

with $\tilde{L}(l) = -(\partial_x + \mathrm{i}l)^2 - \alpha$. We decompose the solution of this equation into two parts

$$\tilde{U}(l,x,t) = \tilde{V}(l,t) f_{m_0}(l,x) + \tilde{U}^\perp(l,x,t),$$

where the second part satisfies the orthogonality condition

$$\left\langle \tilde{U}^\perp(l,\cdot,t), f_{m_0}(l,\cdot)\right\rangle_{L^2_\Gamma} = 0,$$

to ensure the uniqueness of the decomposition. This ansatz leads to

$$\begin{aligned}
\partial_t^2 \tilde{V}(l,t) &= -\left(\omega_{m_0}(l)\right)^2 \tilde{V}(l,t) - \left\langle \left(\tilde{U} * \tilde{U} * \tilde{U}\right)(l,\cdot,t), f_{m_0}(l,\cdot)\right\rangle_{L^2_\Gamma}, \\
\partial_t^2 \tilde{U}^\perp(l,x,t) &= -\tilde{L}(l)(l,x,t) - \left(\tilde{U} * \tilde{U} * \tilde{U}\right)(l,x,t) \\
&\quad + \left\langle \left(\tilde{U} * \tilde{U} * \tilde{U}\right)(l,\cdot,t), f_{m_0}(l,\cdot)\right\rangle_{L^2_\Gamma}.
\end{aligned}$$

It is sufficient to evaluate this equation at $\tilde{U}^\perp = 0$ since there are no quadratic terms in the original system (4.13). If we insert the ansatz

$$\tilde{V}(l,t) = \tilde{A}_1\left(\frac{l-l_0}{\varepsilon}, \varepsilon^2 t\right) \mathrm{e}^{\mathrm{i}\omega_{m_0}(l_0)t} \mathrm{e}^{\mathrm{i}\partial_l \omega_{m_0}(l_0)(l-l_0)t} + c.c.,$$

we obtain in leading order $\varepsilon^2 \mathrm{e}^{\mathrm{i}\omega_{m_0}(l_0)t}\mathrm{e}^{\mathrm{i}\partial_l\omega_{m_0}(l_0)(l-l_0)t}$ the equation

$$\begin{aligned}\mathrm{i}\partial_T\tilde{A}_1(\xi,T) = &-\frac{1}{2}\partial_l^2\omega_{m_0}(l_0)\xi^2\tilde{A}_1(\xi,T)\\ &- \nu_2\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\tilde{A}_1(\xi_1,T)\tilde{A}_1(\xi_2,T)\tilde{A}_{-1}(\xi-\xi_1-\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned}$$

where $l = l_0 + \varepsilon\xi$, $T = \varepsilon^2 t$ and ν_1, ν_2 according to (4.18). By taking the limit $\varepsilon \to 0$ this yields the equation

$$\mathrm{i}\partial_T\hat{A}_1(\xi,T) = -\nu_1\xi^2\hat{A}_1(\xi,T) - \nu_2\left(\hat{A}_1 * \hat{A}_1 * \hat{A}_{-1}\right)(\xi,T)$$

in Fourier space which corresponds to the amplitude equation (4.17) in physical space.

With the help of the NLS approximation it is possible to construct breather solutions to the frequency $\omega = \sqrt{\alpha}$ which correspond to the minimum of the smallest positive spectral band. This result from [Mai20b] is captured in the subsequent theorem.

Theorem 4.2.2. *Let $L = \pi$ be the length of the horizontal links. For an odd integer k and a sufficiently small $\varepsilon > 0$ the nonlinear, cubic Klein-Gordon equation*

$$\partial_t^2 u(t,x) = \partial_x^2 u(t,x) - \left(\frac{k^2}{4} + \varepsilon^2\right)u(t,x) + u(t,x)^3, \quad t \geq 0,\ x \in \mathrm{int}\Gamma,$$

with Kirchhoff boundary conditions at the vertices possesses breather solutions of amplitude $\mathcal{O}(\varepsilon)$ and frequency $\omega = k/2$. These solutions are coincide in the upper and lower semicircles. Precisely, there exist functions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

- $u(x,t) = u(x, t + \frac{2\pi}{\omega})$ *for all $t \geq 0$, $x \in \mathbb{R}$.*
- $\lim_{|x|\to\infty} u(x,t)\mathrm{e}^{\beta|x|} = 0$ *for all $t \geq 0$ and a constant $\beta > 0$.*

Remark 4.2.3. We give a short summary of the proof. The key idea is to perform a center manifold reduction in order to reduce (4.5) to a finite dimensional system. The construction on the center manifold heavily relies on the spectral properties of the linear system, which allows us to choose a frequency in the spectrum, whose harmonics fall into spectral gaps, cf. Figure 4.4. However, because of the Kirchhoff boundary conditions, the system is non-autonomous and the first derivatives of the solutions have jumps and the flow on the center manifold is no longer continuous.

4.3 The discrete necklace graph

We introduce a discretization scheme for the periodic metric necklace graph from Figure 4.1 by discretizing the intervals I_n^0, $I_n^\pm$ with discretization parameter h. Thus, the periodic metric necklace graph Γ will be converted into discrete periodic necklace graphs Γ_h. The first two steps are illustrated in Figure 4.5.

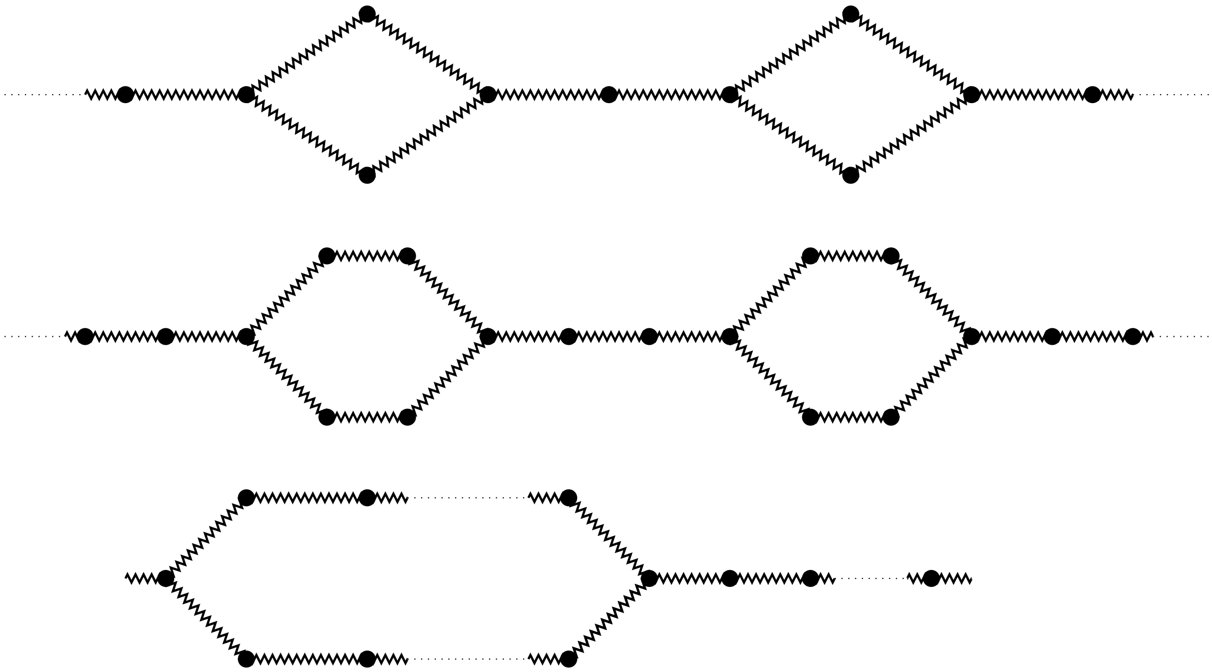

Figure 4.5: Three discretized versions of the metric necklace graph Γ.

We observe that the discrete graphs possess nodes at each vertex point of the quantum graph and every interval is replaced by a number of additional nodes which coincides with the number of steps. Thus, for each discretization step we obtain three additional nodes which results in $2 + 3N$ nodes in the N-th step. We denote a whole periodicity cell of the discrete graph as a vector $U_h(n) \in \mathbb{R}^{2+3N}$ for $h = \pi/(N+1)$. A single node from the periodicity cell $U_h(j)$ will be called $u_h(j,k)$ with $1 \leq k \leq 2+3N$.

The discretized version of (4.13) is given by

$$\partial_t^2 u_h(j,t,k) = \Delta_h u_h(j,t,k) - (\alpha + \varepsilon^2) u_h(j,t,k) + u_h(j,t,k)^3, \quad t \geq 0,\ j \in \mathbb{Z}, \tag{4.19}$$

with Δ_h the discrete Laplacian defined via

$$\Delta_h f(v) := \sum_{w \sim v} h^2 (f(w) - f(v)),$$

and the constants α, ε from Theorem 4.2.2. The system (4.19) in terms of U_h, for the k-th component of the vector U_h with $1 \leq k \leq 2+3N$, reads as

$$\partial_t^2 \left(U_h(j,t)\right)_k = \Delta_h \left(U_h(j,t)\right)_k - (\alpha + \varepsilon^2) \left(U_h(j,t)\right)_k + \left(U_h(j,t)\right)_k^3, \tag{4.20}$$

for $t \geq 0$, $j \in \mathbb{Z}$.

We want to establish a relation between functions on the quantum graph and functions on the discrete graphs. In order to do this we introduce two projections P_h and p_h. P_h maps a function $f \in L^2(\Gamma, \mathbb{C})$ onto a function $f_h : \Gamma_h \to \mathbb{C}$ by

$$P_h f(x_h) = \frac{1}{h} \int_{x_h}^{x_h + h} f(x) \mathrm{d}x,$$

with $x_h = nh \in \Gamma_h$. The linear interpolation operator p_h which maps a function $f : \Gamma_h \to \mathbb{C}$ onto a function on the metric graph is defined by

$$(p_h f)(x) := f(x_h) + \frac{f(x_h + h) - f(x_h)}{h}(x - x_h), \quad \text{for all } x \in x_h + h, \tag{4.21}$$

where $x_h = jh \in \Gamma_h$ accordingly to the notation in (4.10).

4.4 The spectral situation and its continuum limit

We investigate the spectral situation of the discrete necklace graph and its continuum limit for a fixed discretization parameter h and consider the associated linear problem

$$\partial_t^2 U_h(j,t) = \Delta_h U_h(j,t) - (\alpha + \varepsilon^2) U_h(j,t), \quad t \geq 0,\, j \in \mathbb{Z},$$

which is solved by Bloch waves of the form

$$U_h(j,t) = \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{e}^{\mathrm{i}jl} f(l), \quad l, \omega \in \mathbb{R}.$$

The function f solves the associated eigenvalue problem

$$M_h(l) \check{U}(l) = -\omega^2(l) f(l),$$

where $M_h(l)$ is a self adjoint matrix of size $2 + 3N$. Thus, for fixed l we obtain $2 + 3N$ eigenvalues. The well-known Floquet-Bloch theory then implies that the spectrum of the linear problem possesses band gap structure

$$\sigma(\Delta_h - (\alpha + \varepsilon^2)) = \bigcup_{l \in [-1/2, 1/2)} \sigma(M_h(l)).$$

We observe that for $h \to 0$ the number of spectral bands increases $2 + 3N \to \infty$. Furthermore, the spectral bands *converge* against the corresponding spectral bands of the quantum graph, cf. Figure 4.3, 4.6, 4.7 and 4.8 and [NT21].

Remark 4.4.1. We are interested in breathers with frequency $\Omega = k/2$. In the continuous case we know that all odd integer multiples $m\Omega$ fall into spectral gaps, cf. Remark 4.2.3. In the discrete case we can not expect that all odd integer multiples of the frequency $\Omega = k/2$ will fall into spectral gaps. However, there exists $N_0(h)$ such that for $3 \leq m \leq N_0(h)$ the multiples $m\Omega$ fall into spectral gaps. This is possible since the spectral gaps in the continuous case open linearly and the spectral gaps of the discrete steps *converge* against the continuous case.

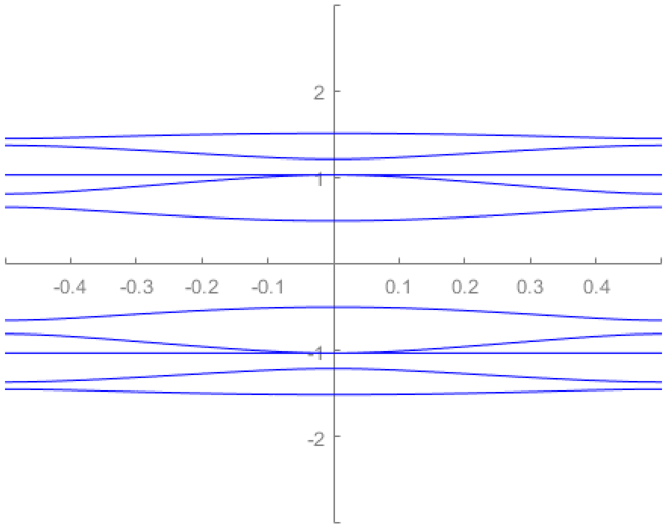

Figure 4.6: The spectral curves $\omega(l)$ of the discrete necklace graph Γ_h with $h = \pi/2$.

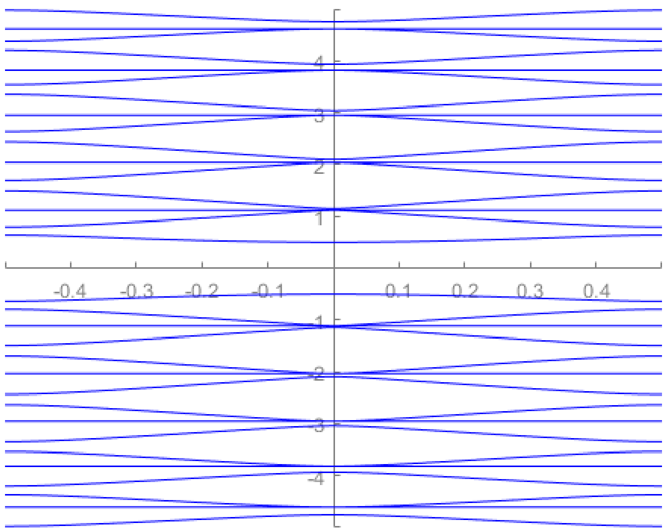

Figure 4.7: The spectral curves $\omega(l)$ of the discrete necklace graph Γ_h with $h = \pi/11$.

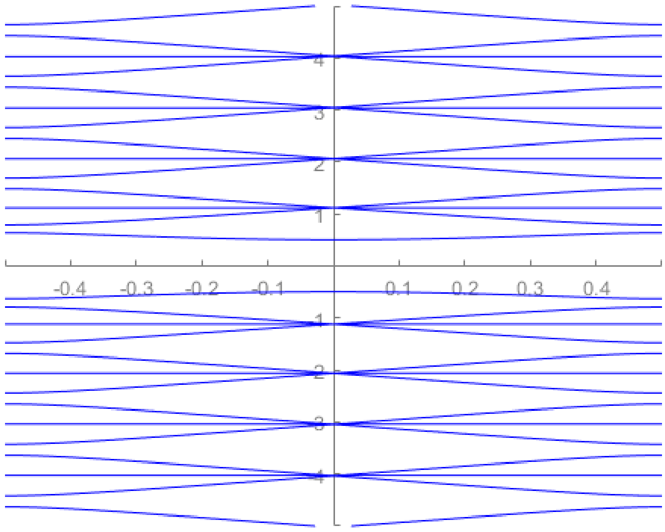

Figure 4.8: The spectral curves $\omega(l)$ of the discrete necklace graph Γ_h with $h = \pi/101$.

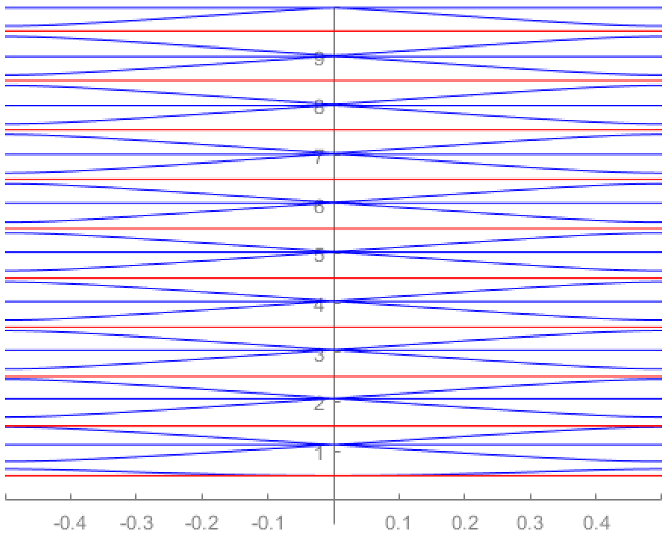

Figure 4.9: The spectral curves $\omega(l)$ of the discrete necklace graph Γ_h with $h = \pi/101$ in blue and the odd multiples of the frequency Ω in red.

4.5 Derivation of the NLS equation and its continuum limit

The validity of the NLS approximation for the discrete system (4.19) can be justified in a similar manner as the validity of the NLS approximation of the continuous system (4.1). Without loss of generality we only consider the case with five nodes per periodicity cell. We start by looking for solutions of the linear problem

$$\partial_t^2 U_h(j,t) = \Delta_h U_h(j,t) - (\alpha+\varepsilon^2)U_h(j,t), \quad t \geq 0,\, j \in \mathbb{Z},$$

which are so-called Bloch waves

$$U_h(j,t) = \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}lj} f(l), \quad \text{with } f(l) \in \mathbb{C}^5,$$

for all $l \in \mathbb{R}$. We choose f, ω and l such that the dispersion relation is satisfied. In this case it is the eigenvalue problem

$$M_L(l)f(l) = M_{\Delta_h-(\alpha+\varepsilon^2)}(l)f(l) = \lambda f(l), \tag{4.22}$$

with $\lambda = -\omega^2$ and $M_L(l)$ is the operator corresponding to the Bloch transform of the linear operator $L := \Delta_h - (\alpha+\varepsilon^2)$. For fixed l the operator $M_L(l)$ is in the case of five nodes per periodicity cell just a self-adjoint matrix in $\mathbb{C}^{5\times 5}$. Thus, we find five eigenvalue curves $l \mapsto \lambda_k(l)$, $1 \leq k \leq 5$ with corresponding eigenfunctions $f_k(l)$.

Next we introduce the concept of the discrete Fourier transform $\mathcal{F}$ on the discrete necklace graph

$$\mathcal{F}(U)(l) = \check{U}(l) = \sum_{j\in\mathbb{Z}} U(j)\mathrm{e}^{2\pi\mathrm{i}lj},$$

and its inverse

$$\mathcal{F}(\check{U})(j) = U(j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \check{U}(l)\mathrm{e}^{-2\pi\mathrm{i}lj}\mathrm{d}l.$$

The Fourier transform connects the two spaces

$$\ell^2(\mathbb{Z},\mathbb{R}^5) = \left\{ U : \mathbb{Z} \to \mathbb{R}^5 \;:\; \|U(\cdot)\|_{\ell^2}^2 := \sum_{n\in\mathbb{Z}} |U(n)|^2 < \infty \right\}$$

and

$$L^2([-\frac{1}{2},\frac{1}{2}),\mathbb{C}^5) = \left\{ \check{U} : [-\frac{1}{2},\frac{1}{2}) \to \mathbb{C}^5 \;:\; \left\|\check{U}\right\|_{L^2}^2 := \int_{-\frac{1}{2}}^{\frac{1}{2}} |U(l)|^2 \mathrm{d}l < \infty \right\}$$

isometrically. We apply the discrete Fourier transform to our discrete Klein-Gordon system (4.20) and obtain

$$\partial_t^2 \left(\check{U}_h(l,t)\right)_k = M_L(l)\left(\check{U}_h(l,t)\right)_k + \mathcal{F}\left((U_h(\cdot,t)^3)\right)_k(l), \tag{4.23}$$

whereas the nonlinear part is given by

$$\mathcal{F}\left((U_h(\cdot,t)^3)\right)_k(l) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\check{U}_h(l-l_1-l_2,t)\right)_k \\ \times \left(\check{U}_h(l_1,t)\right)_k \left(\check{U}_h(l_2,t)\right)_k \mathrm{d}l_1 \mathrm{d}l_2.$$

In order to proceed the matrix $M_L(l)$ needs to be diagonalized. Since $M_L(l)$ is self-adjoint and negative semi-definite there exists a unitary matrix $S(l) \in \mathbb{C}^{5\times 5}$ such that the transformed matrix $\Lambda^2(l)$ is diagonalized

$$\Lambda^2(l) := \begin{pmatrix} -\omega_1^2(l) & 0 & 0 & 0 & 0 \\ 0 & -\omega_2^2(l) & 0 & 0 & 0 \\ 0 & 0 & -\omega_3^2(l) & 0 & 0 \\ 0 & 0 & 0 & -\omega_4^2(l) & 0 \\ 0 & 0 & 0 & 0 & -\omega_5^2(l) \end{pmatrix} = S^{-1}(l)M_L(l)S(l).$$

The columns of the matrix $S(l)$ are exactly the eigenfunctions $f_k(l)$ which solve the eigenvalue problem (4.22). Since $S^{-1}(l) = S^{\intercal}(l)$ the rows of $S^{-1}(l)$ also coincide with these eigenfunctions.

We insert the ansatz $\check{U}_h(l,t) = S(l)\check{V}(l,t)$ into (4.23) and apply $S^{-1}(l)$ from the left side in order to obtain

$$\begin{aligned} \partial_t^2 \left(\check{V}(l,t)\right)_k &= \left(\Lambda^2(l)\check{V}(l,t)\right)_k \\ &+ \left(S^{-1}(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \check{U}_h(l-l_1-l_2,t)\check{U}_h(l_1,t)\check{U}_h(l_2,t)\mathrm{d}l_1\mathrm{d}l_2\right)_k. \end{aligned} \tag{4.24}$$

The system (4.24) consists of five coupled scalar ordinary differential equations of second order. We rephrase this system with respect to the components $\check{v}_k(t,l)$ of $\check{V}(t,l)$ for $1 \leq k \leq 5$. This leads to

$$\partial_t^2 \check{v}_k(l,t) = -\omega_k^2(l)\check{v}_k(l,t) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\langle f_k(l), \check{U}_h(l-l_1-l_2,t)\check{U}_h(l_1,t)\check{U}_h(l_2,t)\right\rangle \mathrm{d}l_1\mathrm{d}l_2.$$

If we take a closer look at the nonlinear terms we observe that we can rewrite this differential equation as

$$\begin{aligned} \partial_t^2 \check{v}_k(l,t) = -\omega_k^2(l)\check{v}_k(l,t) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k_1=1}^{5} \sum_{k_2=1}^{5} \sum_{k_3=1}^{5} \check{v}_{k_1}(l-l_1-l_2,t) \\ \times \check{v}_{k_2}(l_1,t)\check{v}_{k_3}(l_2,t)\beta(l,l-l_1-l_2,l_1,l_2)\mathrm{d}l_1\mathrm{d}l_2, \end{aligned} \tag{4.25}$$

with β given by

$$\beta(l,l-l_1-l_2,l_1,l_2) = \left\langle f_k(l), f_{k_1}(l-l_1-l_2)f_{k_2}(l_1)f_{k_3}(l_2)\right\rangle.$$

Now fix $k_0 \in \{1,2,3,4,5\}$. We want to insert the ansatz

$$\hat{v}_{k_0}(l,t) = \hat{A}_1(\frac{l-l_0}{\varepsilon}, \varepsilon^2 t)\mathrm{e}^{\mathrm{i}\omega_{k_0}(l_0)t}\mathrm{e}^{\mathrm{i}\omega'_{k_0}(l_0)(l-l_0)t} + c.c. + \mathcal{O}(\varepsilon^2), \tag{4.26}$$

$$\hat{v}_k(l,t) = \mathcal{O}(\varepsilon^2), \quad k \neq k_0, \tag{4.27}$$

into (4.25) to deduce a NLS equation in lowest ε order. We introduce new coordinates $T = \varepsilon^2 t$ and $\xi = \frac{l-l_0}{\varepsilon}$. The left-hand side of (4.25) turns under (4.26) into

$$\begin{aligned}
\partial_t^2 \hat{v}_{k_0}(l,t) = &-\left(\omega_{k_0}(l_0) + \omega'_{k_0}(l-l_0)\right)^2 \hat{A}_1(\xi,T)\cdot E_1(t,l) \\
&+ 2\mathrm{i}\varepsilon^2\left(\omega_{k_0}(l_0) + \omega_{k_0}(l_0)(l-l_0)\right)\partial_T \hat{A}_1(\xi,T)\cdot E_1(l,t) \\
&+ \varepsilon^4 \partial_T^2 \hat{A}_1(\xi,T)\cdot E_1(l,t) \\
= &-\left(\omega_{k_0}(l_0)\right)^2 \hat{A}_1(\xi,T)\cdot E_1(l,t) \\
&- 2\varepsilon\omega_{k_0}(l_0)\omega'_{k_0}(l_0)\xi\hat{A}_1(\xi,T)\cdot E_1(l,t) \\
&+ 2\mathrm{i}\varepsilon^2\omega_{k_0}(l_0)\partial_T\hat{A}_1(\xi,T)\cdot E_1(l,t) \\
&- \varepsilon^2\left(\omega'_{k_0}(l_0)\right)^2\xi^2\hat{A}_1(\xi,T)\cdot E_1(l,t) \\
&+ 2\mathrm{i}\varepsilon^3\omega'_{k_0}(l_0)\xi\partial_T\hat{A}_1(\xi,T)\cdot E_1(l,t) \\
&+ \varepsilon^4\partial_T^2\hat{A}_1(\xi,T)\cdot E_1(l,t),
\end{aligned}$$

with

$$E_{\pm 1}(l,t) = \mathrm{e}^{\pm\mathrm{i}\omega_{k_0}(l_0)t}\mathrm{e}^{\pm\mathrm{i}\omega'_{k_0}(l_0)(l-l_0)t}.$$

We expand $\omega_{k_0}(l)$ around l_0

$$\begin{aligned}
\omega_{k_0}(l) &= \omega_{k_0}(l_0) + \omega'_{k_0}(l_0)(l-l_0) + \frac{1}{2}\omega''_{k_0}(l_0)(l-l_0)^2 \\
&= \omega_{k_0}(l_0) + \varepsilon\omega'_{k_0}(l_0)\xi + \frac{1}{2}\varepsilon^2\omega''_{k_0}(l_0)\xi^2.
\end{aligned} \tag{4.28}$$

With the help of ansatz (4.26) and (4.28) we rewrite the term $\omega_{k_0}^2(l)\hat{v}_{k_0}(l,t)$ as

$$\begin{aligned}
\omega_{k_0}^2(l)\hat{v}_{k_0}(l,t) = -&\Big[\left(\omega_{k_0}(l_0)\right)^2 + 2\varepsilon\omega_{k_0}(l_0)\omega'_{k_0}(l_0)\xi \\
&+\varepsilon^2\left(\omega_{k_0}(l_0)\omega''_{k_0}(l_0) + \left(\omega'_{k_0}(l_0)\right)^2\right)\xi^2 \\
&+\varepsilon^3\left(\omega'_{k_0}(l_0)\omega''_{k_0}(l_0)\right)\xi^3 \\
&+\frac{1}{4}\varepsilon^4\left(\omega''_{k_0}(l_0)\right)^2\xi^4\Big]\hat{A}_1(\xi,T)\cdot E_1(l,t).
\end{aligned}$$

If we use the ansatz (4.27) in the nonlinear terms we observe that

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{k_1=1}^{5}\sum_{k_2=1}^{5}\sum_{k_3=1}^{5}\check{v}_{k_1}(l-l_1-l_2,t)\check{v}_{k_2}(l_1,t)\check{v}_{k_3}(l_2,t)\\
&\qquad\times\beta(l,l-l_1-l_2,l_1,l_2)\mathrm{d}l_1\mathrm{d}l_2\\
&\quad=\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\check{v}_{k_0}(l-l_1-l_2,t)\check{v}_{k_0}(l_1,t)\check{v}_{k_0}(l_2,t)\\
&\qquad\times\beta(l,l-l_1-l_2,l_1,l_2)\mathrm{d}l_1\mathrm{d}l_2+\mathcal{O}(\varepsilon^2).
\end{aligned} \tag{4.29}$$

By inserting (4.26) into the first addend of (4.29) we obtain

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\check{v}_{k_0}(l-l_1-l_2,t)\check{v}_{k_0}(l_1,t)\check{v}_{k_0}(l_2,t)\beta(l,l-l_1-l_2,l_1,l_2)\mathrm{d}l_1\mathrm{d}l_2\\
&\quad=3\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\beta(l,l-l_1-l_2,l_1,l_2)\hat{A}_1\left(\frac{l-l_1-l_2-l_0}{\varepsilon},T\right)\\
&\qquad\times\hat{A}_1\left(\frac{l_1-l_0}{\varepsilon},T\right)\hat{A}_{-1}\left(\frac{l_2+l_0}{\varepsilon},T\right)\mathrm{d}l_1\mathrm{d}l_2\cdot E_1(l,t)\\
&\qquad+3\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\beta(l,l-l_1-l_2,l_1,l_2)\hat{A}_1\left(\frac{l-l_1-l_2-l_0}{\varepsilon},T\right)\\
&\qquad\times\hat{A}_{-1}\left(\frac{l_1+l_0}{\varepsilon},T\right)\hat{A}_{-1}\left(\frac{l_2+l_0}{\varepsilon},T\right)\mathrm{d}l_1\mathrm{d}l_2\cdot E_{-1}(l,t)\\
&\qquad+\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\beta(l,l-l_1-l_2,l_1,l_2)\hat{A}_1\left(\frac{l-l_1-l_2-l_0}{\varepsilon},T\right)\\
&\qquad\times\hat{A}_1\left(\frac{l_1-l_0}{\varepsilon},T\right)\hat{A}_1\left(\frac{l_2-l_0}{\varepsilon},T\right)\mathrm{d}l_1\mathrm{d}l_2\cdot E_3(l,t)\\
&\qquad+\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\beta(l,l-l_1-l_2,l_1,l_2)\hat{A}_{-1}\left(\frac{l-l_1-l_2+l_0}{\varepsilon},T\right)\\
&\qquad\times\hat{A}_{-1}\left(\frac{l_1+l_0}{\varepsilon},T\right)\hat{A}_{-1}\left(\frac{l_2+l_0}{\varepsilon},T\right)\mathrm{d}l_1\mathrm{d}l_2\cdot E_{-3}(l,t).
\end{aligned} \tag{4.30}$$

The next step is to substitute $\xi=\frac{l-l_0}{\varepsilon}$, $\xi_1=\frac{l_1-l_0}{\varepsilon}$ and $\xi_2=\frac{l_2+l_0}{\varepsilon}$. The first addend of (4.30) is then given by

$$\begin{aligned}
&3\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\beta(l_0+\varepsilon\xi,l_0+\varepsilon(\xi-\xi_1-\xi_2),l_0+\varepsilon\xi_1,-l_0+\varepsilon\xi_2)\\
&\qquad\qquad\times\hat{A}_1(\xi-\xi_1-\xi_2,T)\hat{A}_1(\xi_1,T)\hat{A}_{-1}(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\varepsilon^2E_1(l,t).
\end{aligned}$$

If we expand the term $\beta(l_0+\varepsilon\xi,l_0+\varepsilon(\xi-\xi_1-\xi_2),l_0+\varepsilon\xi_1,-l_0+\varepsilon\xi_2)$ in every component around l_0 respectively $-l_0$ we obtain in the lowest ε order

$$3\beta(l_0,l_0,l_0,-l_0)\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\hat{A}_1(\xi-\xi_1-\xi_2,T) \\ \times\hat{A}_1(\xi_1,T)\hat{A}_{-1}(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\varepsilon^2E_1(l,t)+\mathcal{O}(\varepsilon^3).$$

We repeat these computations for the other summands of (4.30) and obtain the expression for the nonlinear terms

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\check{v}_{k_0}(l-l_1-l_2,t)\check{v}_{k_0}(l_1,t)\check{v}_{k_0}(l_2,t)\beta(l,l-l_1-l_2,l_1,l_2)\mathrm{d}l_1\mathrm{d}l_2\\
&\quad=3\varepsilon^2\beta(l_0,l_0,l_0,-l_0)\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\hat{A}_1(\xi-\xi_1-\xi_2,T)\\
&\qquad\times\hat{A}_1(\xi_1,T)\hat{A}_{-1}(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot E_1(l,t)\\
&\quad+3\varepsilon^2\beta(-l_0,-l_0,-l_0,l_0)\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\hat{A}_{-1}(\xi-\xi_1-\xi_2,T)\\
&\qquad\times\hat{A}_{-1}(\xi_1,T)\hat{A}_1(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot E_{-1}(l,t)\\
&\quad+\varepsilon^2\beta(3l_0,l_0,l_0,l_0)\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\hat{A}_1(\xi-\xi_1-\xi_2,T)\\
&\qquad\times\hat{A}_1(\xi_1,T)\hat{A}_1(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot E_3(l,t)\\
&\quad+\varepsilon^2\beta(-3l_0,-l_0,-l_0,-l_0)\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\hat{A}_{-1}(\xi-\xi_1-\xi_2,T)\\
&\qquad\times\hat{A}_{-1}(\xi_1,T)\hat{A}_{-1}(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot E_{-3}(l,t).
\end{aligned}$$

Since we are interested in standing solutions we choose l_0 such that $\omega_{k_0}'(l_0)=0$. Therefore the k_0-th spectral band has to possess an extremum at l_0. For the discrete necklace graph this is satisfied for $l_0\in\{0,\pm\frac{1}{2}\}$. With this consideration in mind we get at $\varepsilon^2E_1(t,l)$ the NLS equation

$$\begin{aligned}
\mathrm{i}\partial_T\hat{A}_1(\xi,T)={}&-\frac{1}{2}\omega_{k_0}''(l_0)\xi^2\hat{A}_1(\xi,T)\\
&+\frac{\nu}{2\omega_{k_0}(l_0)}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}}\hat{A}_1(\xi-\xi_1-\xi_2,T)\hat{A}_1(\xi_1,T)\hat{A}_{-1}(\xi_2,T)\mathrm{d}\xi_1\mathrm{d}\xi_2,
\end{aligned}\tag{4.31}$$

with $\nu=3\beta(l_0,l_0,l_0,-l_0)$. The terms of order $\varepsilon^2E_{-1}(l,t)$ satisfy the complex conjugate NLS equation.

If we consider the sequence of NLS equations for every step in the discretization scheme for the quantum necklace graph, then the coefficients of (4.31) will converge against the coefficients of the NLS approximation for the quantum graph. For the linear coefficient $-\frac{1}{2}\omega_{k_0}''(l_0)$ it is obvious due to the observations made at the end of Section 4.4. In order to reach the same conclusion for the nonlinear coefficient $\nu/(2\omega_{k_0}(l_0))$ we take a closer look at the structure of β. It turns out that β converges against the inner product in the space L^2_Γ. Thus, it seems natural that it is possible that solutions for the system on the quantum graph can be represented as the limit of solutions of the discrete systems. An analogous result was shown for NLS equations on the real line in [HY19].

4.6 Breathers on discrete necklace graphs

For the construction of the generalized breather for the discrete necklace graph we again use a spatial dynamics formulation. We look for time-periodic solutions in (4.10) which is a system of the form

$$\partial_t^2 U_n = \kappa_{n,+} U_{n+1} + f(n, U_n) + \kappa_{n,-} U_{n-1},$$

with $f(n, U_n) = f(n+N, U_n)$ and $\kappa_{n,\pm} = \kappa_{n+N,\pm}$. We search for $2\pi/\omega$-time periodic solutions of this system and so we make the Fourier ansatz

$$U_n(t) = \sum_{m \in \mathbb{Z}_{odd}} U_{n,m} e^{im\omega t}.$$

We find

$$-m^2\omega^2 U_{n,m} = \kappa_{n,+} U_{n+1,m} + f_m(n, U_n) + \kappa_{n,-} U_{n-1,m}, \tag{4.32}$$

for all $m \in \mathbb{Z}_{odd}$ which is an infinite-dimensional discrete dynamical system with discrete time $n \in \mathbb{Z}$. With $V_{n,m} = U_{n-1,m}$ we write (4.32) as a first order system, namely as

$$\begin{aligned} V_{n+1,m} &= U_{n,m}, \\ U_{n+1,m} &= \kappa_{n,+}^{-1}(m^2\omega^2 U_{n,m} - f_m(n, U_n) - \kappa_{n,-} V_{n,m}), \end{aligned}$$

which we abbreviate with $W_{n,m} = (U_{n,m}, V_{n,m})$ and $W_n = (W_{n,m})_{m \in \mathbb{Z}_{odd}}$ in the following as

$$W_{n+1,m} = F_m(n, W_n) = A_{n,m} W_{n,m} + G_m(n, W_n),$$

where the linear operator $A_{n,m}$ satisfies $A_{n,m} = A_{n+N,m}$ and the nonlinear function $G_m(n, W_n) = \mathcal{O}(\|W_n\|^2)$ satisfies $G_m(n, W_n) = G_m(n+N, W_n)$. By Floquet's theorem the linear system

$$W_{n+1,m} = A_{n,m} W_{n,m}$$

is solved by

$$W_{n,m} = P_{n,m} M_m W_{0,m},$$

with $P_{n,m} = P_{n+N,m}$ and constant coefficients monodromy matrix M_m. We make the linear part autonomous by introducing $Z_{n,m}$ by

$$W_{n,m} = P_{n,m} Z_{n,m}.$$

We find

$$Z_{n+1,m} = M_m Z_{n,m} + P_{n,m}^{-1} G_m(n, P_n Z_n).$$

By construction M_1 and M_{-1} have two Floquet multipliers zero. In the following this part is denoted with $Z_{n,c,\leq}$. By the analysis of the previous sections there is a $N_* \leq N$ where for all $m \in \mathbb{Z}_{odd}$ with $|m| \leq N_*$ the numbers $m\omega$ fall into spectral gaps, i.e., the associated Floquet exponents are off the imaginary axis. They are denoted with $Z_{n,u,\leq}$and $Z_{n,s,\leq}$ for the unstable and stable part. The modes with $|m| > N_*$ are denoted with $Z_{n,u,>}$ and $Z_{n,s,>}$ for the unstable and stable part and with $Z_{n,c,>}$ for the central part. The associated projections are denoted with $P_{c,\leq}$, $P_{u,\leq}$, $P_{s,\leq}$, $P_{u,>}$, $P_{s,>}$, and $P_{c,>}$. Then we obtain

$$\begin{aligned}
Z_{n+1,c,\leq} &= M_{c,\leq} Z_{n,c,\leq} + P_{c,\leq} P_n^{-1} G(n, P_n Z_n), \\
Z_{n+1,u,\leq} &= M_{u,\leq} Z_{n,u,\leq} + P_{u,\leq} P_n^{-1} G(n, P_n Z_n), \\
Z_{n+1,s,\leq} &= M_{s,\leq} Z_{n,s,\leq} + P_{s,\leq} P_n^{-1} G(n, P_n Z_n), \\
Z_{n+1,u,>} &= M_{u,>} Z_{n,u,>} + P_{u,>} P_n^{-1} G(n, P_n Z_n), \\
Z_{n+1,s,>} &= M_{s,>} Z_{n,s,>} + P_{s,>} P_n^{-1} G(n, P_n Z_n), \\
Z_{n+1,c,>} &= M_{c,>} Z_{n,c,>} + P_{c,>} P_n^{-1} G(n, P_n Z_n).
\end{aligned}$$

We use the reversibility of the original equations as shown in [Mai20b] and write the equations for the stable part as

$$\begin{aligned}
Z_{n-1,s,\leq} &= M_{s,\leq} Z_{n,s,\leq} + P_{s,\leq} P_n^{-1} G(n, P_n Z_n), \\
Z_{n-1,s,>} &= M_{s,>} Z_{n,s,>} + P_{s,>} P_n^{-1} G(n, P_n Z_n).
\end{aligned}$$

The time-periodic solutions, we are interested in, are a part of the center manifold and so we use the contraction mapping for the construction of the center manifold, namely

$$\begin{aligned}
Z_{n,c,\leq} &= M_{c,\leq}^n Z_{0,c,\leq} + \sum_{k=0}^{n} M_{c,\leq}^{n-k} P_{c,\leq} P_k^{-1} G(k, P_k Z_k), \\
Z_{n,u,\leq} &= -\sum_{k=0}^{\infty} M_{u,\leq}^{-k} P_{u,\leq} P_k^{-1} G(k, P_k Z_k), \\
Z_{n,s,\leq} &= \sum_{k=0}^{\infty} M_{s,\leq}^{k} P_{s,\leq} P_k^{-1} G(k, P_k Z_k), \\
Z_{n,c,>} &= M_{c,>}^n Z_{0,c,>} + \sum_{k=0}^{n} M_{c,>}^{n-k} P_{c,>} P_k^{-1} G(k, P_k Z_k), \\
Z_{n,u,>} &= -\sum_{k=0}^{\infty} M_{u,>}^{-k} P_{u,>} P_k^{-1} G(k, P_k Z_k), \\
Z_{n,s,>} &= \sum_{k=0}^{\infty} M_{s,>}^{k} P_{s,>} P_k^{-1} G(k, P_k Z_k).
\end{aligned} \tag{4.33}$$

Lemma 4.6.1. *The truncated system* (4.33) *with* $Z_{n,c,>} = 0$, $Z_{n,u,>} = 0$, *and* $Z_{n,s,>} = 0$ *possesses an exact homoclinic solution to the origin of order* $\mathcal{O}(\varepsilon)$. *This solution corresponds to a time-periodic solutions* $u_h(x_h, t) = u_{approx,h}(h_h, t)$ *of order* $\mathcal{O}(\varepsilon)$ *with* $u_{approx,h}(x_h, t) = u_{approx,h}(x_h, t + 2\pi/\Omega)$ *in the discretized systems* (4.10). *This solution is an approximate breather for the discrete systems.*

If we set $Z_{n,c,>} = 0$, $Z_{n,u,>} = 0$, and $Z_{n,s,>} = 0$ and ignore the last three equations we have a finite-dimensional dynamical system with a four-dimensional center manifold. On this four-dimensional center manifold the vector-field is given in lowest order by the stationary NLS equation. Thus, we find an exact homoclinic solution to the origin of order $\mathcal{O}(\varepsilon)$ which corresponds to a time-periodic solution in the discretized systems (4.10).

Remark 4.6.2. The truncated system with $Z_{n,c,>} = 0$, $Z_{n,u,>} = 0$, and $Z_{n,s,>} = 0$ coincides with the full system (4.33) if for all $m \in \mathbb{Z}_{\text{odd}}$ the numbers $m\omega$ fall into spectral gaps. This condition is satisfied for the system (4.1).

Lemma 4.6.3. *For N big enough there exists $N_* = N_*(N)$ such that $m\Omega$ falls into spectral gaps for $3 \le m \le N_*$ such that for $n_0 = 1$ and $l_0 = 1/2$ and $\alpha = 1/4 + \varepsilon^2$ the discretized systems* (4.10) *possess time-periodic solutions $u_h(x_h, t) = u_{\text{per},h}(h_h, t)$ of order $\mathcal{O}(\varepsilon)$ for $|x_h| \le \varepsilon^{-N_*/2}$, with $u_{\text{per},h}(x_h, t) = u_{\text{per},h}(x_h, t + 2\pi/\Omega)$ and tails of order $\mathcal{O}(\varepsilon^{N_*/2})$ for large $|x_h|$. In lowest order the solutions are given by the associated NLS approximation of the discretized systems. These solutions are called generalized breather solutions.*

Exactly with the same arguments as in [GS01, GS05, GS08] in the full system we can achieve a transversal intersection of the center-stable manifold with the fixed space of reversibility. However due to the fact that we have the additional central modes $Z_{n,c,>}$ we have a possible growth for $|n| \to \infty$. Since the bifurcating homoclinic solutions are of order $\mathcal{O}(\varepsilon)$ and the lowest harmonic which may fall into the spectrum is $N_*\Omega$ and since we have a convolution structure the term $P_{c,>}P_k^{-1}G(k, P_k Z_k)$ is at most of order $\mathcal{O}(\varepsilon^{N_*})$. Hence, the term $\sum_{k=0}^{n} M_{c,>}^{n-k} P_{c,>} P_k^{-1} G(k, P_k Z_k)$ is at most of order $\mathcal{O}(\varepsilon^{N_*} n)$. Hence the tails are less than $\mathcal{O}(\varepsilon^{N_*/2})$ for $|n| \le \mathcal{O}(\varepsilon^{-N_*/2})$.

4.7 Convergence result

We formulate the convergence of the generalized discrete breathers against the breather solution on the continuous necklace graph in the following theorem.

Theorem 4.7.1. *For $\varepsilon > 0$ small enough we have on every compact interval K that for $h \to 0$ respectively $N \to \infty$ the generalized discrete breather solutions $u_{\text{per},h}(x_h, t)$ of the discretized system* (4.10) *converge against the breather solutions $u_{\text{per}}(x_h, t)$ of the system* (4.1).

Proof. Since the flow on the center manifold in the first step is described by the stationary NLS equation and since the coefficients of the stationary NLS equations in the discrete case converge towards the coefficients of the stationary NLS equation in the continuous case for $h \to 0$ the approximate breather of the discrete system converges towards the breather of the continuous system. Moreover, since the generalized breathers of the discrete systems are $\mathcal{O}(\varepsilon^{N_*/2})$ close to the approximate breather of the discrete system for $|n| \le \mathcal{O}(\varepsilon^{-N_*/2})$ and since $N^* \to \infty$ for $h \to 0$ we have the convergence of the generalized discrete breather towards the breather of the continuous system through

$$\begin{aligned}|u_{\text{per}}(x_h,t)-u_{\text{per},h}(x_h,t)| \leq{}& |u_{\text{per}}(x_h,t)-u_{approx,h}(x_h,t)| \\ &+|u_{approx,h}(x_h,t)-u_{\text{per},h}(x_h,t)| \to 0,\end{aligned}$$

at the vertex points x_h inside some compact interval K. The equation above still holds true for any $x \in K$ if we use the linear interpolation p_h defined in (4.21). We obtain

$$\begin{aligned}|u_{\text{per}}(x,t)-(p_h u_{\text{per},h})(x,t)| \leq{}& |u_{\text{per}}(x_h,t)-(p_h u_{approx,h})(x,t)| \\ &+|(p_h u_{approx,h})(x,t)-(p_h u_{\text{per},h})(x,t)| \to 0,\end{aligned}$$

for any $x \in K$ for $h \to 0$. $\square$

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