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Polar representation of material elasticity applied to wood and engineered wood products

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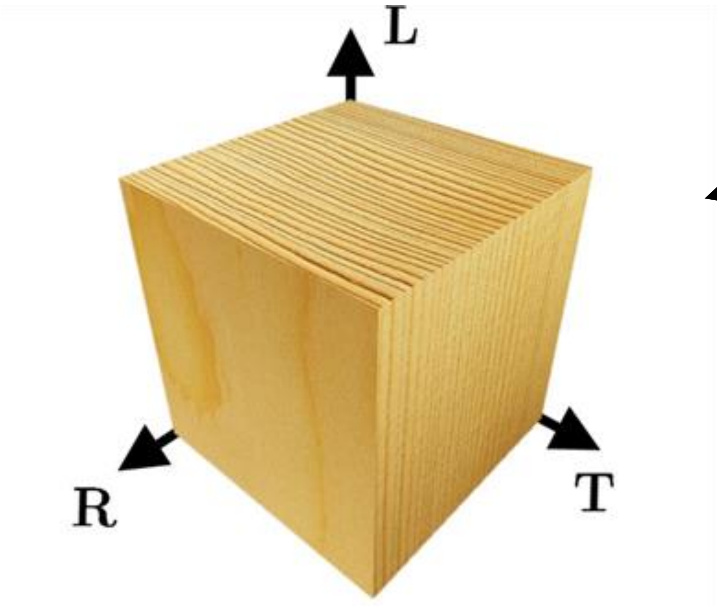
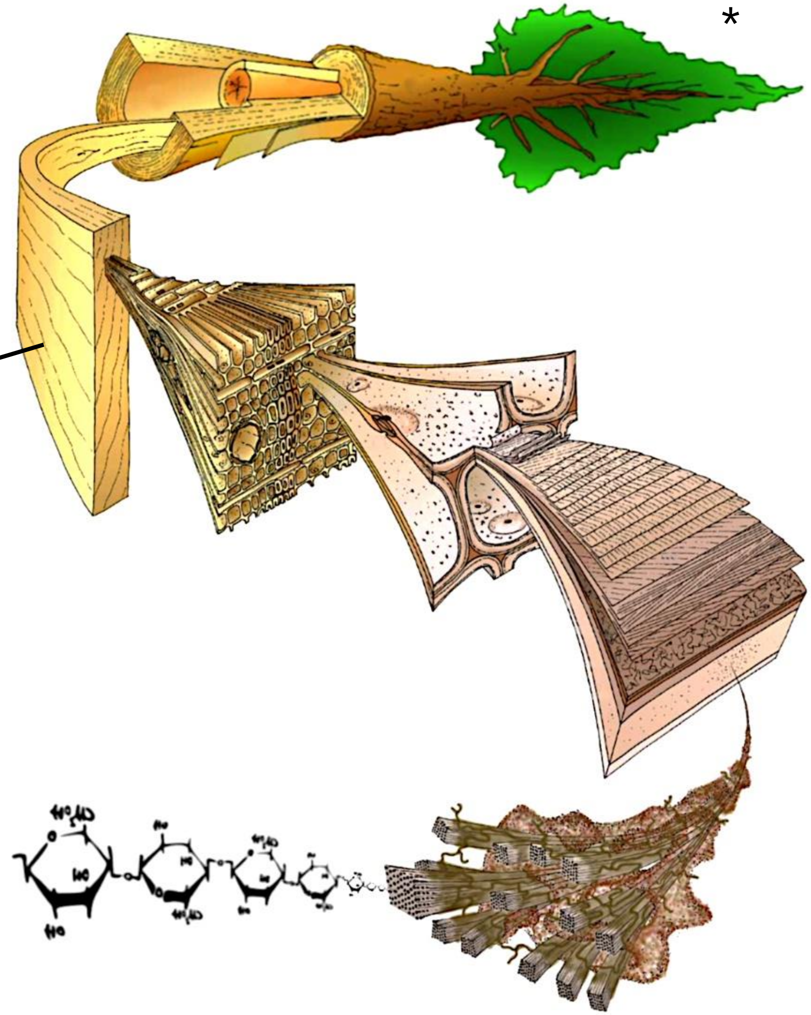
Wood





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- 3 anatomical growth directions (L, R, T)
- Anisotropic / Orthotropic: Assuming orthogonal system in clearwood

* Image modified from M. Harrington and J.J. Harrington

Background / History

- W. Voigt (1910): „*Lehrbuch der Kristallphysik*“
- J. Bodig, B. A. Jayne (1982): “*Mechanics of Wood and Wood Composites*”
- G. Schickhofer (1994): ”*Starrer und nachgiebiger Verbund bei geschichteten, flächenhaften Holzstrukturen*“
- M. Grimsel (1999): „*Mechanisches Verhalten von Holz : Struktur- und Parameteridentifikation eines anisotropen Werkstoffes*“
- D. Keunecke, S. Hering, P. Niemz (2008): “*Three-dimensional elastic behavior of common yew and norway spruce*”

(List of selected works)

What is missing?

- Comprehensive derivations, “hard to understand” for inexperienced readers
- No substantial use of the potential of polar representation for parametric analysis

Method

Generalized Hooke's law

- 2nd order stress tensor: $\boldsymbol{\sigma} \equiv \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$
- 2nd order strain tensor: $\boldsymbol{\varepsilon} \equiv \varepsilon_{kl}\mathbf{e}_k \otimes \mathbf{e}_l$

$\rightarrow$ $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$

- 4th order stiffness tensor:
  $\mathbf{C} \equiv C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$

- Compliance tensor: $\mathbf{S} \equiv \mathbf{C}^{-1}$

In matrix-notation & orthotropic material

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{pmatrix}$$

Orthotropic compliance matrix

$$\mathbf{S} = \begin{pmatrix} E_1^{-1} & -\nu_{21}E_2^{-1} & -\nu_{31}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & -\nu_{32}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{13}E_1^{-1} & -\nu_{23}E_2^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{31}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12}^{-1} \end{pmatrix}$$

- 9 (independent) engineering constants for orthotropic materials
- Determined by experimental tests

3D rotation of a compliance matrix?

2 Options:

- Rotation as a 4th-order tensor, using 4th-order tensor rotation rules and subscript notation, followed by a backwards transformation of the rotated tensor into Voigt matrix notation → ?

- Direct rotation of the compliance matrix in Voigt notation (Bond 1943*) → Easier

*W. L. Bond, The mathematics of the physical properties of crystals, Bell System Technical Journal 22 (1) (1943) 1–72. doi:10.1002/j.1538-7305.1943.tb01304.x

Simplified method

Method by Bond 1943* :

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

Any arbitrary rotation matrix

↓ Rotation of a 6x6 Voigt-notation matrix becomes:

$$\mathbf{R}^V = \begin{pmatrix} R_{11}^2 & R_{12}^2 & R_{13}^2 & R_{12}R_{13} & R_{13}R_{11} & R_{11}R_{12} \\ R_{21}^2 & R_{22}^2 & R_{23}^2 & R_{22}R_{23} & R_{23}R_{21} & R_{21}R_{22} \\ R_{31}^2 & R_{32}^2 & R_{33}^2 & R_{32}R_{33} & R_{33}R_{31} & R_{31}R_{32} \\ 2R_{21}R_{31} & 2R_{22}R_{32} & 2R_{23}R_{33} & R_{22}R_{33} + R_{23}R_{32} & R_{21}R_{33} + R_{23}R_{31} & R_{22}R_{31} + R_{21}R_{32} \\ 2R_{31}R_{11} & 2R_{32}R_{12} & 2R_{33}R_{13} & R_{12}R_{33} + R_{13}R_{32} & R_{13}R_{31} + R_{11}R_{33} & R_{11}R_{32} + R_{12}R_{31} \\ 2R_{11}R_{21} & 2R_{12}R_{22} & 2R_{13}R_{23} & R_{12}R_{23} + R_{13}R_{22} & R_{13}R_{21} + R_{11}R_{23} & R_{11}R_{22} + R_{12}R_{12} \end{pmatrix}$$

*W. L. Bond, The mathematics of the physical properties of crystals,
Bell System Technical Journal 22 (1) (1943) 1–72. doi:10.1002/j.1538-7305.1943.tb01304.x

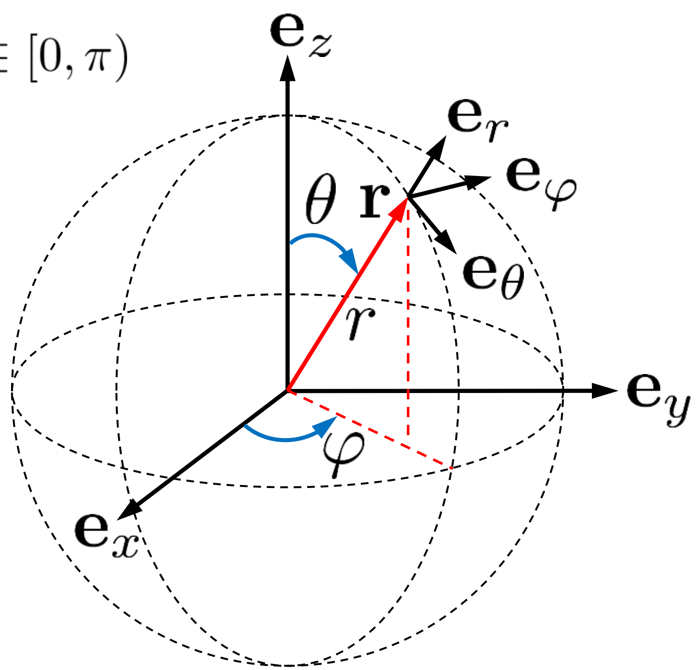
Polar representation

Material compliance is a vector **r** that can be rotated around in space depending on direction of acting load parametrized by φ and θ

$r \in [0, \infty)$

$\varphi \in [0, 2\pi)$

$\theta \in [0, \pi)$



$$\mathbf{r} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$$

$$\mathbf{r} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

$$\mathbf{r} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\mathbf{R}_{\varphi\theta} \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \equiv (\mathbf{e}_\theta, \mathbf{e}_\varphi, \mathbf{e}_r) \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

Rotation matrix

Rotation of compliance matrix

In conjunction with polar representation:

$$\mathbf{R} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

→ Build $\mathbf{R}^V$ using Bond rule

The 3D rotation of the compliance matrix reads:

$$\mathbf{S}' = \mathbf{R}^V \mathbf{S} (\mathbf{R}^V)^\intercal$$

Rotation of uniaxial load

Material compliance is a vector **r** that can be rotated around in space depending on direction of acting load parametrized by φ and θ

If uniaxial load (tensile or compression), then element S_{11} of **S** is rotated

$$\mathbf{S}' = \mathbf{R}^V \mathbf{S} (\mathbf{R}^V)^\intercal$$

Yields

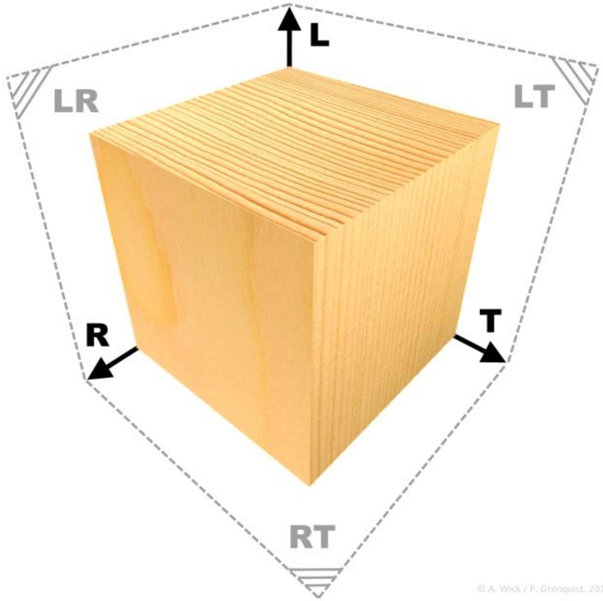
$$\begin{aligned} S'_{11} = S_{11} \cos^4 \varphi \sin^4 \theta + S_{22} \sin^4 \varphi \sin^4 \theta + S_{33} \cos^4 \theta \\ + 2S_{12} \sin^2 \varphi \cos^4 \varphi \sin^4 \theta + 2S_{13} \cos^2 \varphi \sin^2 \theta \cos^2 \theta \\ + 2S_{23} \sin^2 \varphi \sin^2 \theta \cos^2 \theta + S_{44} \sin^2 \varphi \sin^2 \theta \cos^2 \theta \\ + S_{55} \cos^2 \varphi \sin^2 \theta \cos^2 \theta + S_{66} \cos^2 \varphi \sin^2 \varphi \sin^4 \theta \end{aligned}$$

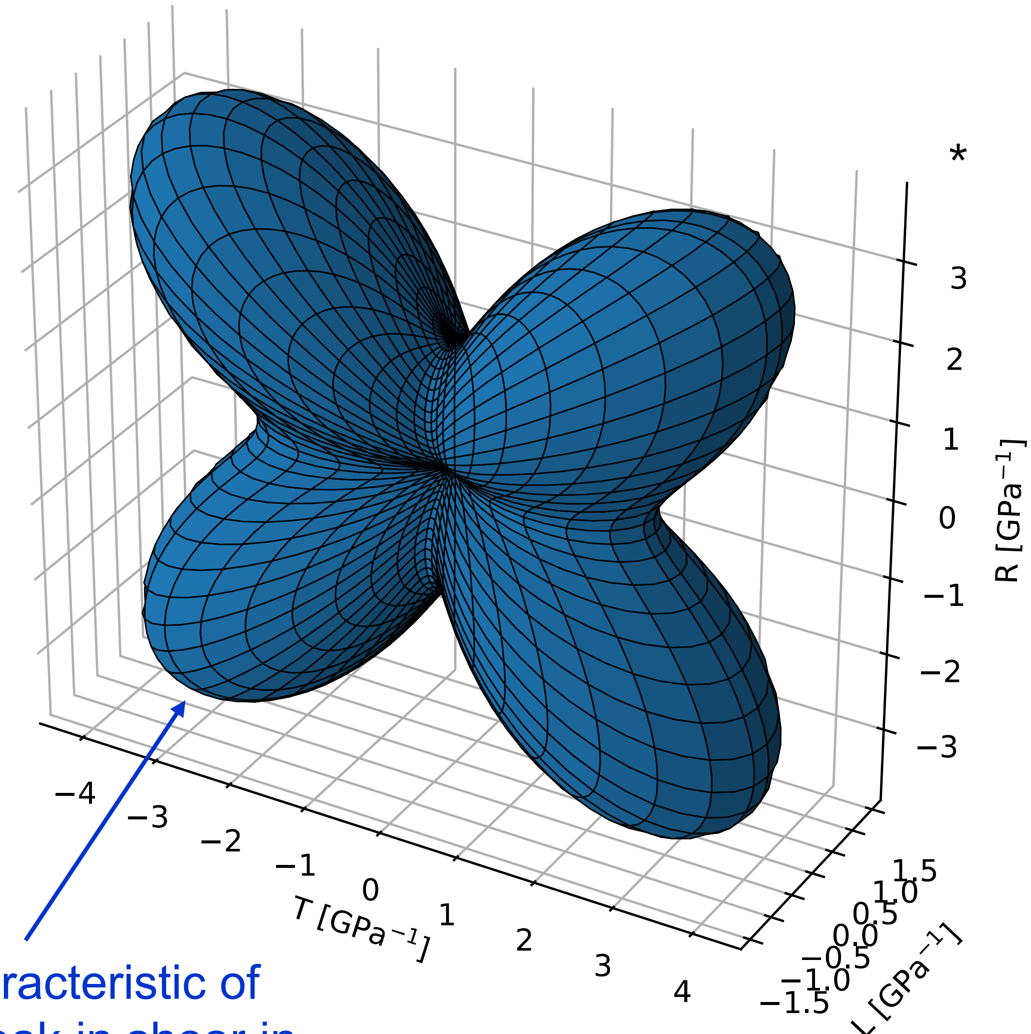
Educational example

“Clearwood”

Norway spruce

Typical European softwood species

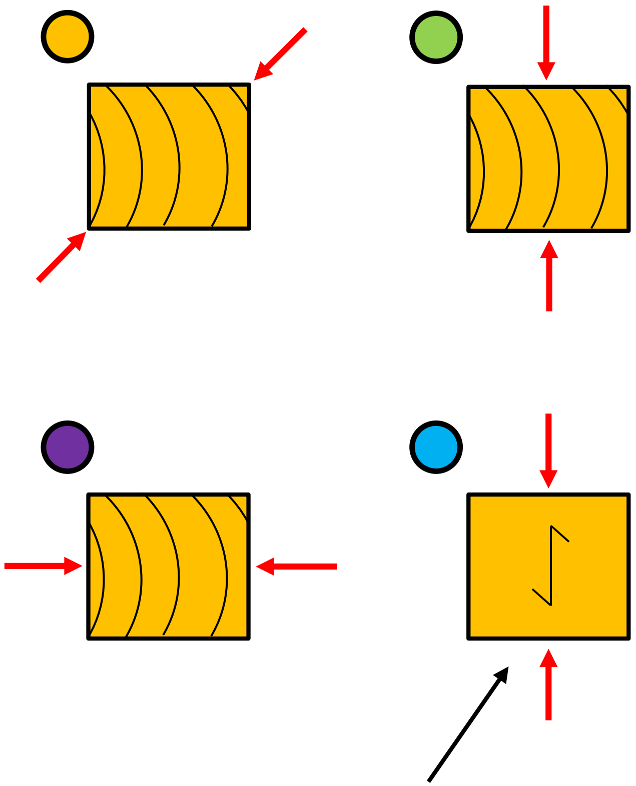




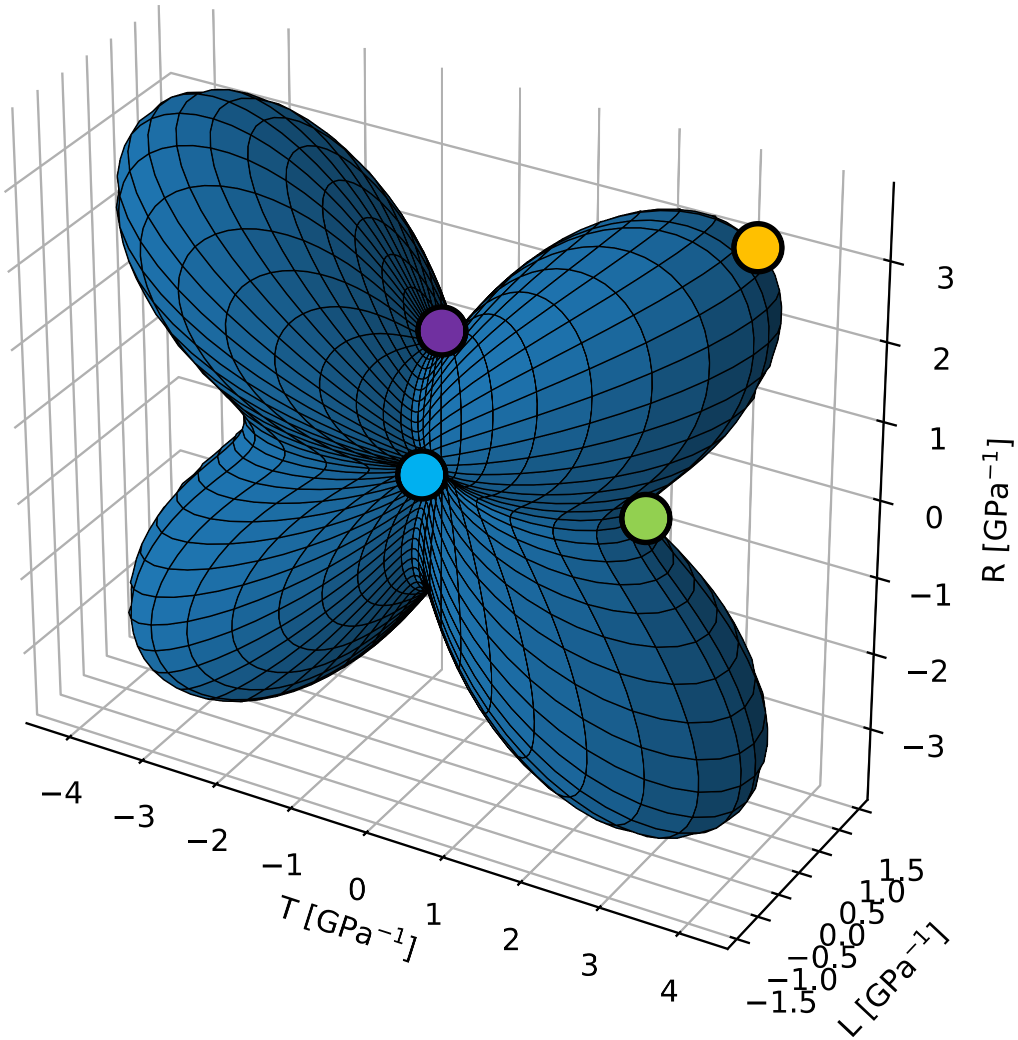
"Rolling shear" characteristic of softwoods: Very weak in shear in RT plane (45° load in RT plane)

*Using literature data parameters for clearwood at 12% MC

Interpretation

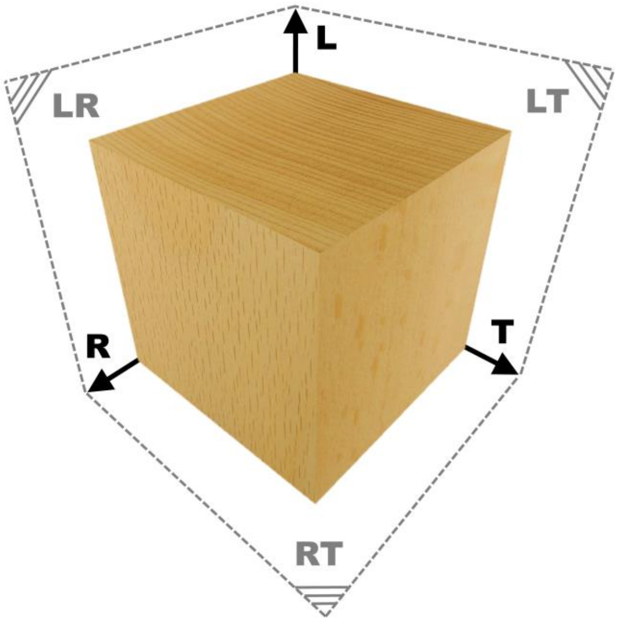


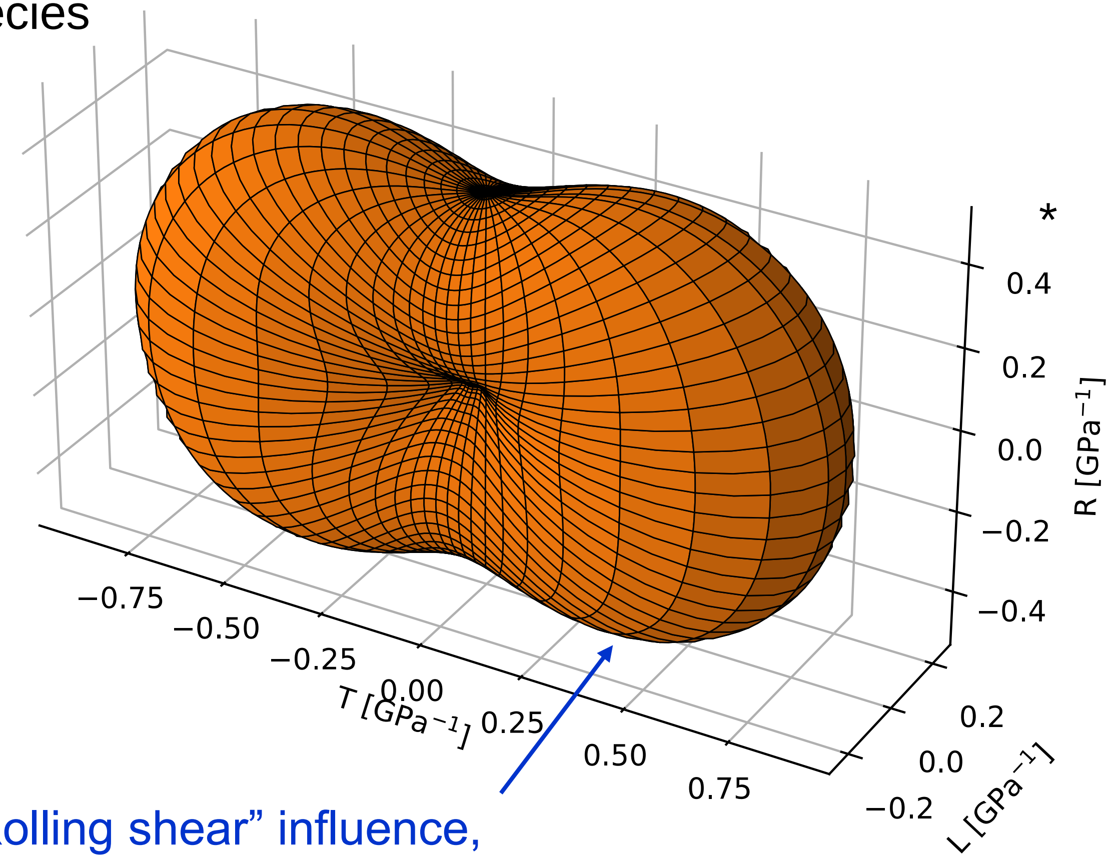
Load in wood fiber direction, least compliant



European beech

Typical European hardwood species

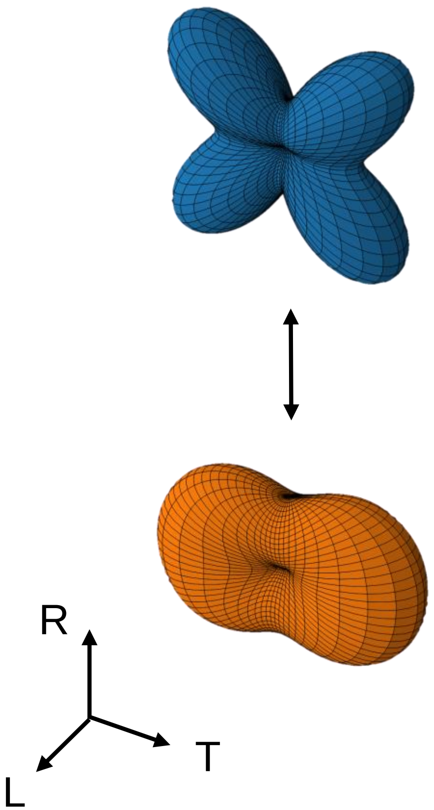


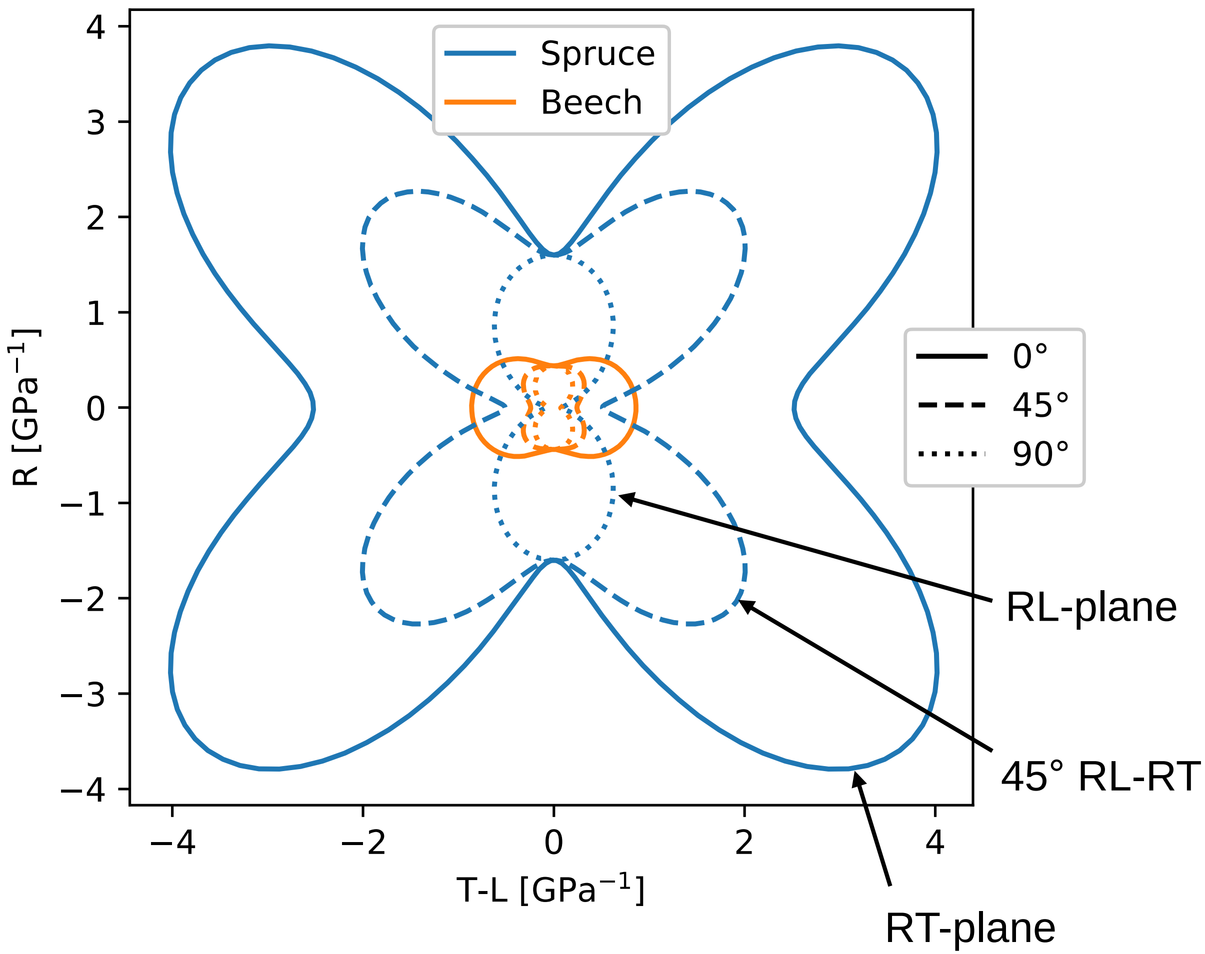


Much less "Rolling shear" influence,
weakest direction is T

*Using literature data parameters for clearwood at 12% MC

Comparison



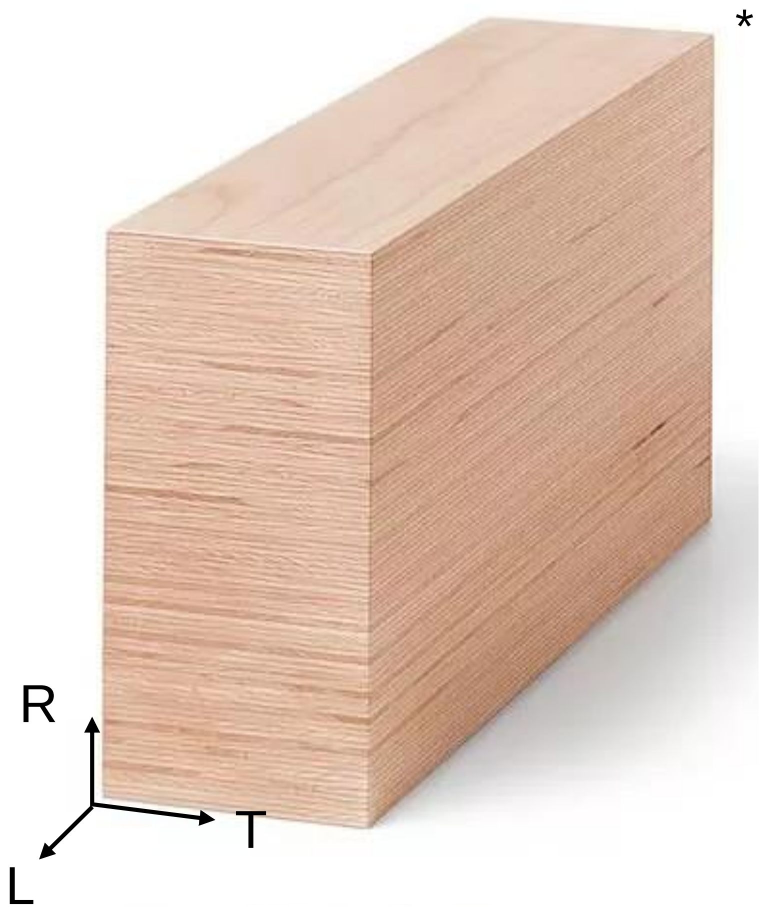


- Hardwoods typically show lower compliance than softwoods due to higher density
- Wood structure differences strongly affect rolling shear

Educational example

Beech laminated veneer lumber (LVL)

Beech LVL



High-performance engineered wood product, increasing use

- Main Applications: Columns, beams, reinforcements
- Wood anisotropy is somewhat retained due to veneer lamination, but very high "**homogenization / lamination effect**" (strength & stiffness)

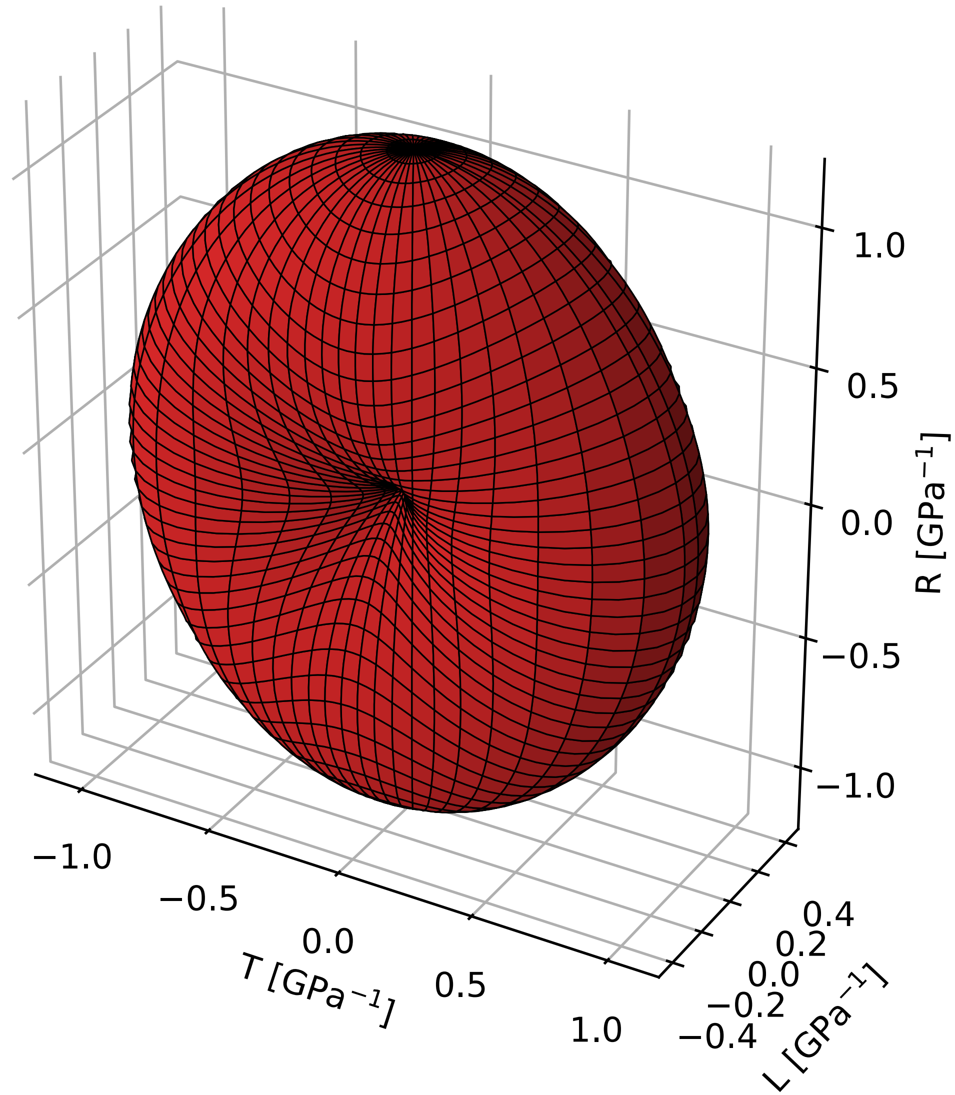
* Image source: https://www.pollmeier.com

Beech LVL

BauBuche (S) GL75
(ETA-14/0354)

WMC = 5.5%, n = 6-12*

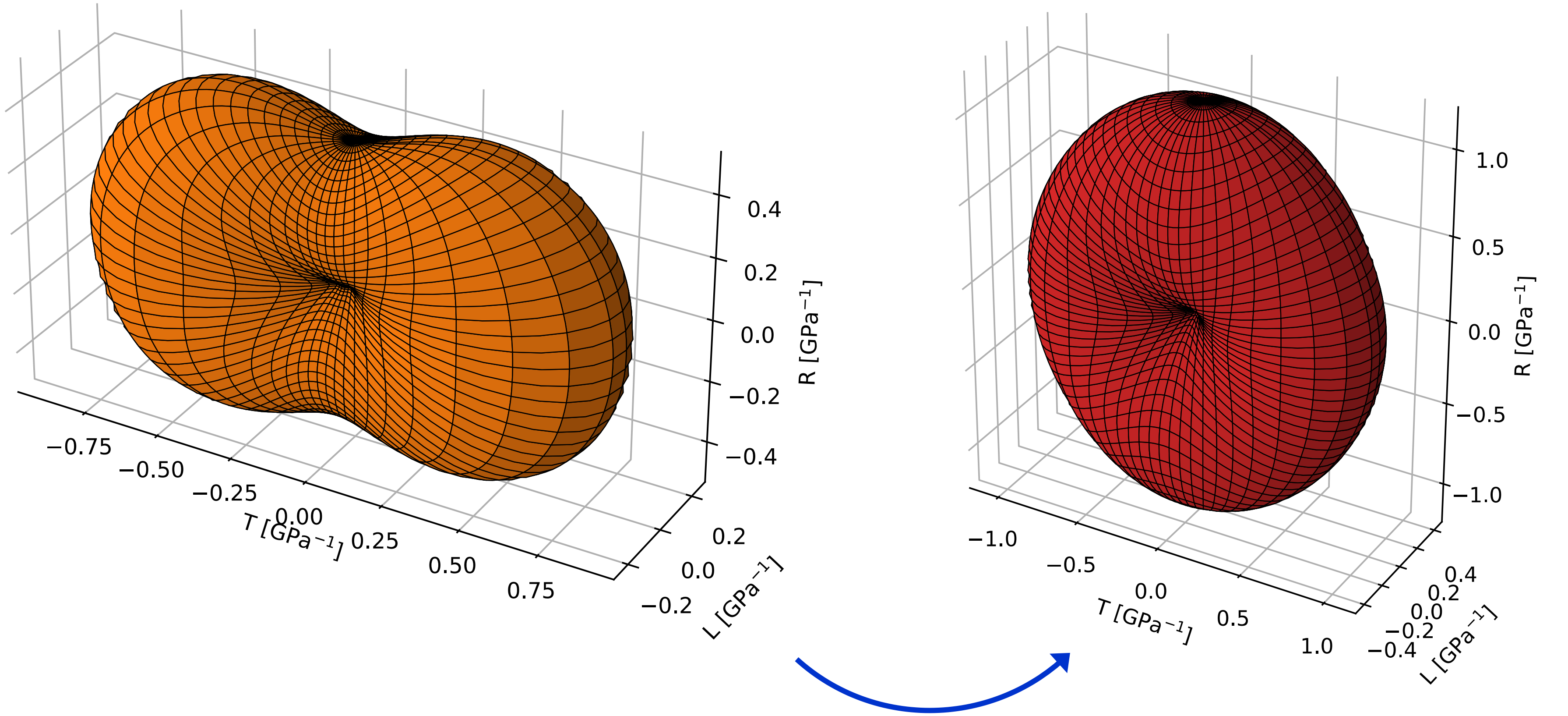
$$E_L = 17'259 \text{ MPa}$$
$$E_R = 840 \text{ MPa}$$
$$E_T = 966 \text{ MPa}$$
$$G_{LR} = 909 \text{ MPa}$$
$$G_{LT} = 1'006 \text{ MPa}$$
$$G_{RT} = 365^1 \text{ MPa}$$
$$\nu_{LR} = 0.305 \text{ MPa}$$
$$\nu_{LT} = 0.500 \text{ MPa}$$
$$\nu_{RT} = 0.208 \text{ MPa}$$



*U. Kuhlmann, J. Töpler, Experimentelle und numerische Untersuchungen von Brettschichtholz aus Buchen-Furnierschichtholz (BauBuche), Doktorandenkolloquium Holzbau "Forschung und Praxis“ 2022

[1]: Assumption from beech clearwood

Beech clearwood vs. LVL

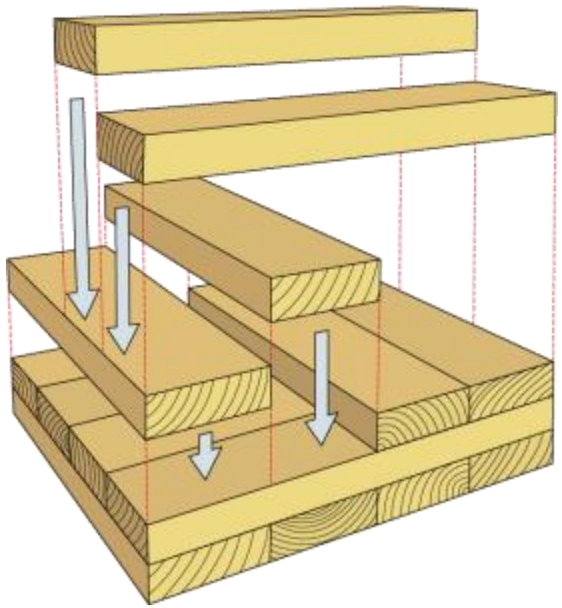


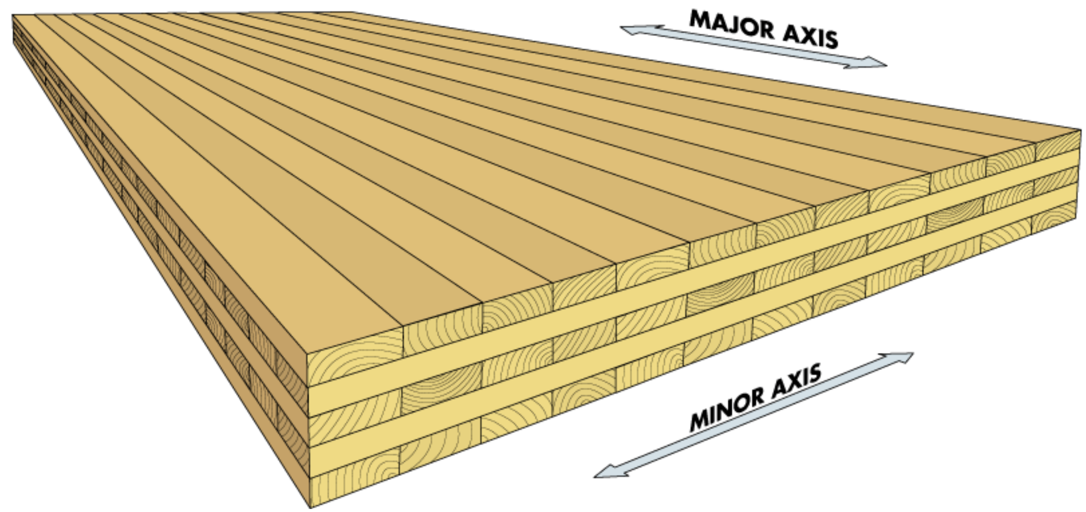
- Homogenization in transverse directions, beech LVL more compliant
- Beech LVL less compliant in L direction (“homogenization/lamination effect”)

Practical example

Cross-laminated timber (CLT)

Cross-laminated timber (CLT)



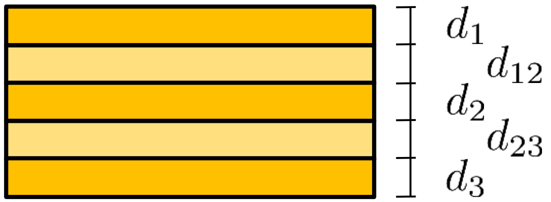


One of the most relevant engineered wood products, increasing use worldwide

- Main Applications: Slabs and walls
- Wood anisotropy is somewhat retained due to major vs minor axis

Image source: https://www.structuremag.org/

CLT calculation: Bending



"Gamma method" according to Eurocode 5 (EN 1995-1-1) for 5-layer CLT:

$$\gamma_1 = \frac{1}{1 + \frac{\pi^2 E_1 d_1 d_{12}}{l_{ref}^2 G_{R,12}}} \qquad a_1 = (\frac{d_1}{2} + d_{12} + \frac{d_2}{2}) - a_2$$

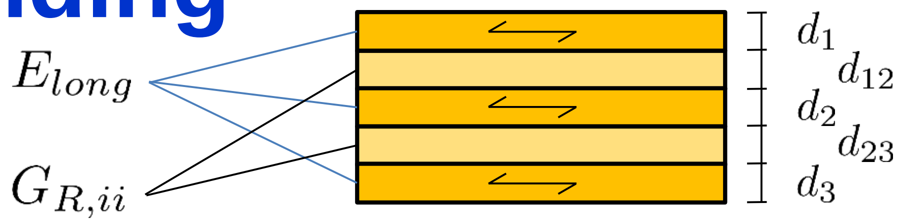
$$\gamma_2 = 1.0 \qquad a_2 = \frac{\gamma_1 E_1 d_1 (\frac{d_1}{2} + d_{12} + \frac{d_2}{2}) - \gamma_3 E_3 d_3 (\frac{d_2}{2} + d_{23} + \frac{d_3}{2})}{\sum_{i=1}^{3} \gamma_i E_i d_i}$$

$$\gamma_3 = \frac{1}{1 + \frac{\pi^2 E_3 d_3 d_{23}}{l_{ref}^2 G_{R,23}}} \qquad a_3 = (\frac{d_2}{2} + d_{23} + \frac{d_3}{2}) + a_2$$

Effective bending E-modulus

$$EI_{ef} = \sum_{i=1}^{3} E_i \frac{d_i^3}{12} + \sum_{i=1}^{3} \gamma_i d_i a_i^2 \longrightarrow E_{ef} = \frac{EI_{ef}}{I_0} = \frac{EI_{ef}}{\frac{(d_1+d_{12}+d_2+d_{23}+d_3)^3}{12}}$$

CLT calculation: Bending



Example parameters for CLT made of softwood C24 graded boards:

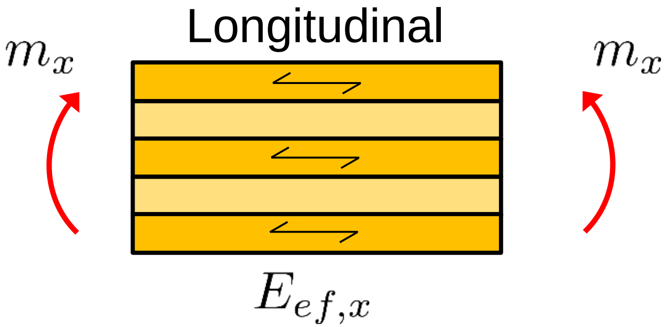
- E-Modulus in fiber direction: $E_{long} = 11'000 \text{ N/mm}^2$
- Rolling shear modulus: $G_{R,ii} = 50 \text{ N/mm}^2$

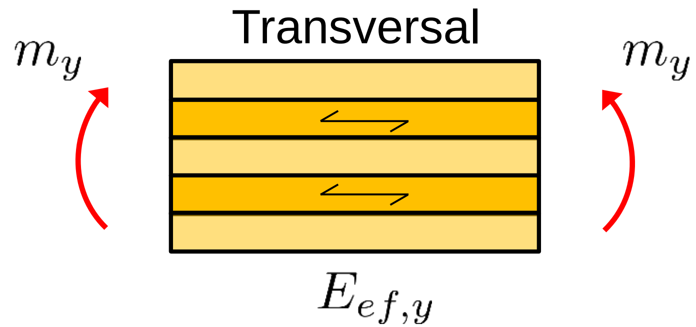
$\Big\} \; EI_{ef}$

Effective bending E-modulus

$$E_{ef} = \frac{EI_{ef}}{I_0} = \frac{EI_{ef}}{\frac{(d_1+d_{12}+d_2+d_{23}+d_3)^3}{12}}$$

CLT: Rotation of uniaxial bending moment





$$\begin{pmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{pmatrix} = \frac{1}{I_0} \begin{pmatrix} E_{ef,x}^{-1} & 0 & 0 \\ 0 & E_{ef,y}^{-1} & 0 \\ 0 & 0 & (k_t G)^{-1} \end{pmatrix} \begin{pmatrix} m_x \\ m_y \\ m_{xy} \end{pmatrix}$$

: Simplified (uniaxial) moment-curvature relation*

2D Polar rotation

→

$$S'_{ef} = E_{ef,x}^{-1} \cos^4 \varphi + E_{ef,y}^{-1} \sin^4 \varphi + (k_t G)^{-1} \cos^2 \varphi \sin^2 \varphi$$

* No coupling terms with axial forces (no biaxial moments), no lateral strain (v=0)

CLT: Edge-gluing

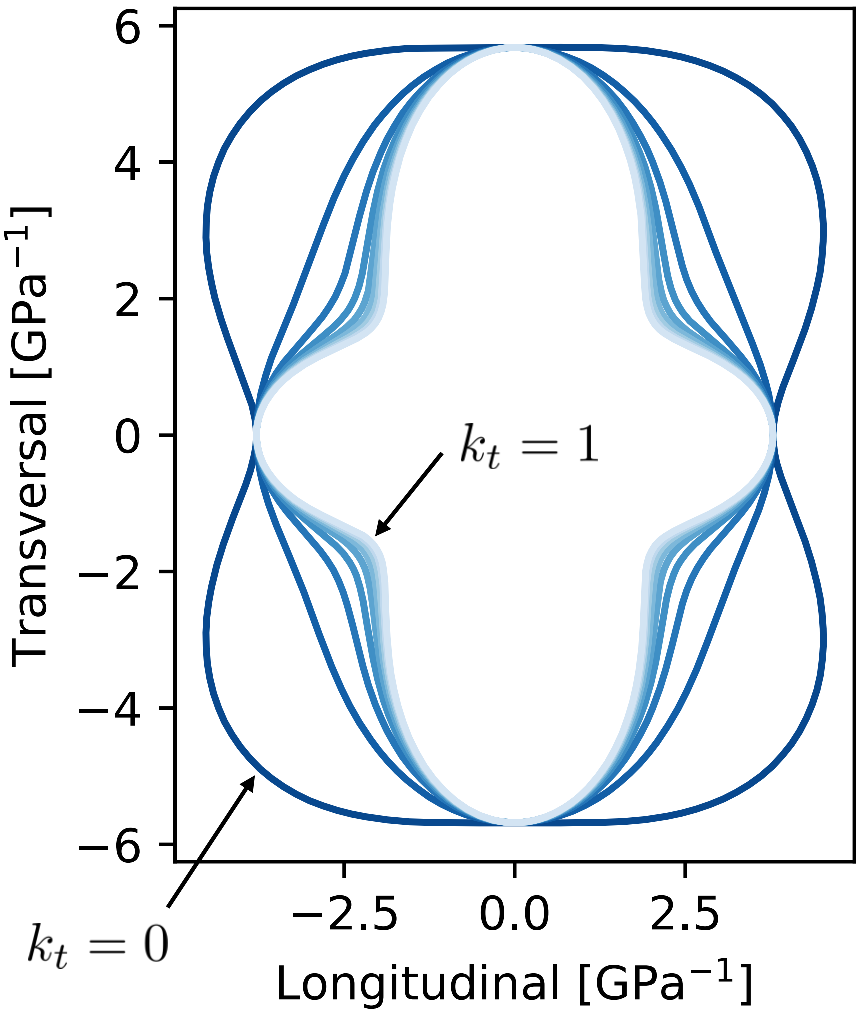
Example analysis: Effect of edge-gluing of layers

- Variation of torsional moment factor:

—	$k_t = 0$
—	$k_t = 1$

(Shades in between: Steps of 0.1)

Typical assumptions in practice:
- Edge-glued CLT: $k_t = 0.65$
- Non-edge-glued: $k_t = 0$



➢ "Strong axis" changes whether edge-glued or not!

CLT: Layer size

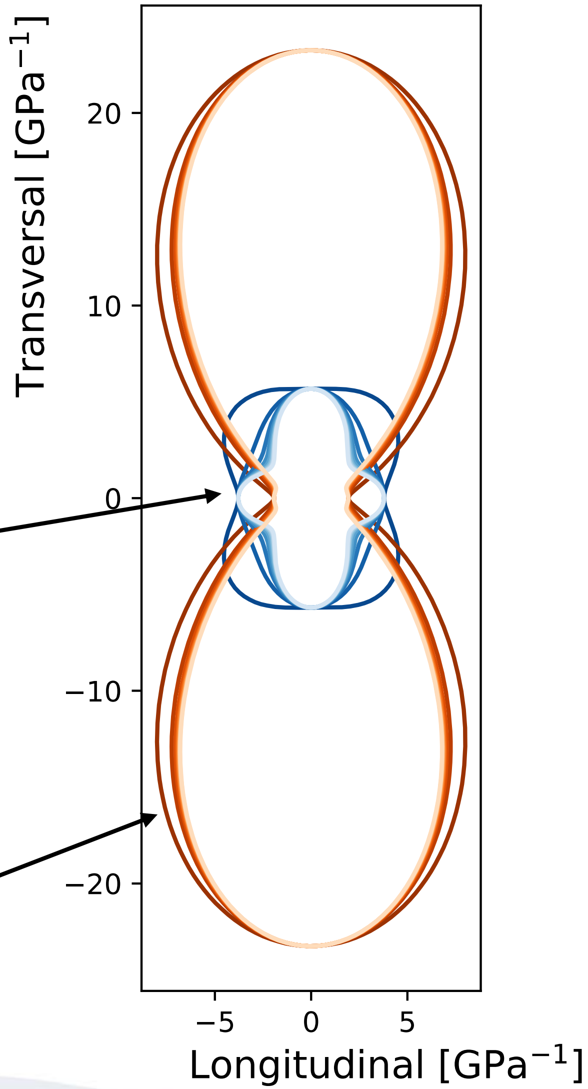
Example analysis: Effect of layer sizes / configuration in 5-layer CLT:

$[40_0, 40_{90}, 40_0, 40_{90}, 40_0]$ mm

Less compliant in transverse direction, less anisotropic, observable effect of edge-gluing

$[40_0, 20_{90}, 40_0, 20_{90}, 40_0]$ mm

Less compliant in longitudinal direction, more anisotropic, no effect of edge-gluing



Conclusions

Conclusions

- Polar representation is a powerful tool for visualization of anisotropic material behavior (e.g. wood, engineered wood products)

- Polar representation is not new, but its potential still remains somewhat unexploited

  - Science education for students: Building intuition and comprehension for material behavior

  - Materials selection and design for engineers: Compare and optimize design solutions, assess different effects

Thanks for your attention!

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