

RESEARCH ARTICLE

Lifting and restricting t-structures

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Funding information

Deutsche Forschungsgemeinschaft;
Baden-Württemberg Stiftung

Abstract

We explore the interplay between t-structures in the bounded derived category of finitely presented modules and the unbounded derived category of all modules over a coherent ring A using homotopy colimits. More precisely, we show that every intermediate t-structure in $D^b(\text{mod}(A))$ can be lifted to a compactly generated t-structure in $D(\text{Mod}(A))$, by closing the aisle and the coaisle of the t-structure under directed homotopy colimits. Conversely, we provide necessary and sufficient conditions for a compactly generated t-structure in $D(\text{Mod}(A))$ to restrict to an intermediate t-structure in $D^b(\text{mod}(A))$, thus describing which t-structures can be obtained via lifting. We apply our results to the special case of HRS-t-structures. Finally, we discuss various applications to silting theory in the context of finite dimensional algebras.

MSC 2020

16E35 (primary), 18E10, 18E40, 18G80 (secondary)

1 | INTRODUCTION

A modern perspective on representation theory suggests to view the category of modules over a ring as an abelian subcategory of its derived category and to study properties of the ring via complexes of modules. Such a triangulated point of view allows to compare the representation theories of two given rings even if their module categories are rather different. Foundational work in this direction goes back to Happel [17], Rickard [32], and Keller [21], who extended classical Morita theory to the setup of derived categories, explaining when two derived module categories are triangle equivalent. The key notion in this context are tilting complexes. In recent years, tilting

theory was generalized to silting theory, which allows to study more generally the embeddings of module categories into derived categories as hearts of associated t-structures (see, e.g., [1, 23, 28, 31]).

In general, t-structures play a fundamental role in understanding the structure of triangulated categories such as derived module categories. Following [9], every t-structure gives rise to an abelian category—called the heart of the t-structure—connected to the ambient triangulated category via a cohomological functor. In other words, every t-structure induces a cohomology theory on the ambient triangulated category. Nowadays, t-structures are prominently used in different branches of mathematics such as algebraic geometry, representation theory, category theory, and topology.

In particular, understanding bounded t-structures on the bounded derived category of finitely presented modules $D^b(\text{mod}(A))$ over a finite dimensional algebra A or on the bounded derived category of coherent sheaves on a smooth projective variety \times is crucial for computing the space of Bridgeland stability conditions. In representation theory, bounded t-structures appear in the fundamental correspondence with silting complexes, co-t-structures and simple minded collections (see [22, 23, 33]). Nevertheless, even for bounded derived categories of finite dimensional algebras, many questions are still open, in particular, when it comes to constructing t-structures explicitly. On the contrary, in the unbounded derived category of all modules $D(\text{Mod}(A))$ over an arbitrary ring A , one can always generate a t-structure from any given set of objects [3, 35].

In this paper, we explore for a coherent ring A the interplay between t-structures in $D^b(\text{mod}(A))$ and in $D(\text{Mod}(A))$. We restrict ourselves to intermediate t-structures, sitting in an interval between shifts of the standard t-structure. Note that if A is a finite dimensional algebra, a t-structure in $D^b(\text{mod}(A))$ is intermediate if and only if it is bounded.

Building on [37], where it was proved that t-structures in $D(\text{Mod}(A))$ lift to coherent diagrams, we show that every intermediate t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(\text{mod}(A))$ with heart \mathcal{H} can be lifted to a compactly generated t-structure $(\text{hocolim}(\mathcal{X}), \text{hocolim}(\mathcal{Y}))$ in $D(\text{Mod}(A))$ by closing the classes \mathcal{X} and \mathcal{Y} under directed homotopy colimits. Moreover, the heart of this new t-structure is a locally coherent Grothendieck category with the set of finitely presented objects given by \mathcal{H} (see Section 3). This process of lifting t-structures from $D^b(\text{mod}(A))$ to $D(\text{Mod}(A))$ can be considered as a generalization of lifting torsion pairs $(\mathcal{T}, \mathcal{F})$ from $\text{mod}(A)$ to torsion pairs $(\varinjlim(\mathcal{T}), \varinjlim(\mathcal{F}))$ in $\text{Mod}(A)$ (see [13, Section 4.4]).

In Section 4, we discuss the problem of restricting t-structures from $D(\text{Mod}(A))$ to $D^b(\text{mod}(A))$. Note that except for the commutative case discussed in [2] not much seems to be known. We prove that a compactly generated and intermediate t-structure in $D(\text{Mod}(A))$ restricts to $D^b(\text{mod}(A))$ if and only if every object in the heart \mathbf{H} is a directed homotopy colimit of objects in $\mathbf{H} \cap D^b(\text{mod}(A))$ (see Theorem 4.1). Altogether, via lifting and restricting t-structures, we obtain the following result.

Theorem (see Corollary 4.2). *Lifting and restricting t-structures yields a bijection between*

- (1) *intermediate t-structures in $D^b(\text{mod}(A))$;*
- (2) *intermediate, compactly generated t-structures in $D(\text{Mod}(A))$ with a locally coherent Grothendieck heart \mathbf{H} , whose finitely presented objects are given by $\mathbf{H} \cap D^b(\text{mod}(A))$.*

As a consequence, we get that a derived equivalence $D(\text{Mod}(B)) \xrightarrow{F} D(\text{Mod}(A))$ preserves coherent rings if and only if the image under F of the standard t-structure in $D(\text{Mod}(B))$ restricts to $D^b(\text{mod}(A))$ (see Corollary 4.3 and Remark 4.4). Moreover, when restricting the bijection above

to the special case of noetherian rings and HRS-t-structures, that is intermediate t-structures with the smallest possible interval, the bijection takes a simpler form and recovers some results on lifting torsion pairs (see Section 5).

In Section 6, we discuss applications to silting theory. First of all, it turns out that every lifted t-structure in $D(\text{Mod}(A))$ is controlled by a not necessarily small cosilting complex. In case A is a finite dimensional algebra we can establish a dual way of lifting by cogenerated a t-structure in $D(\text{Mod}(A))$ from a t-structure in $D^b(\text{mod}(A))$. In this way, we obtain silting t-structures with respect to not necessarily compact silting complexes (see Theorem 6.2). In general, we do not know how to classify the silting t-structures of this form. Only in the case of HRS-t-structures, we get a complete picture (see Proposition 6.4). Finally, by combining both ways of lifting, we can show that for an intermediate t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(\text{mod}(A))$ there is a unique t-structure in $D(\text{Mod}(A))$ restricting to $(\mathcal{X}, \mathcal{Y})$ if and only if $(\mathcal{X}, \mathcal{Y})$ is algebraic, i.e. its heart is a finite length category (see Corollary 6.3).

2 | PRELIMINARIES

2.1 | Notation

Throughout, let A be a right coherent ring, that is a ring A such that every finitely generated submodule of a finitely presented right A -module is again finitely presented. The category of right A -modules is denoted by $\text{Mod}(A)$, its subcategory of finitely presented modules by $\text{mod}(A)$, its subcategory of projective modules by $\text{Proj}(A)$ and its subcategory of finitely generated projective modules by $\text{proj}(A)$. Note that A is right coherent if and only if $\text{mod}(A)$ is an abelian category. By $K^b(\text{proj}(A))$ (respectively, $K^-(\text{proj}(A))$), we denote the (right) bounded homotopy category. We write $D(A) = D(\text{Mod}(A))$ for the derived category of all A -modules and $D^b(A) = D^b(\text{mod}(A))$ for the bounded derived category of finitely presented A -modules.

All subcategories considered are strict and full. If \mathcal{C} is a subcategory of an additive category \mathcal{A} , we define \mathcal{C}^\perp to be the subcategory of \mathcal{A} formed by all objects Z with $\text{Hom}_{\mathcal{A}}(C, Z) = 0$ for all C in \mathcal{C} . Dually, we define ${}^\perp\mathcal{C}$. If \mathcal{T} is a triangulated category and C an object in \mathcal{T} , we define the following perpendicular classes for any subset I of \mathbb{Z} :

$${}^{\perp I}C := \{Z \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(Z, C[i]) = 0 \text{ for all } i \in I\}$$

$$C^{\perp I} := \{Z \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(C, Z[i]) = 0 \text{ for all } i \in I\}$$

2.2 | t-structures

For a triangulated category \mathcal{T} and a subcategory \mathcal{C} of \mathcal{T} , we call a morphism $f : Z \rightarrow C$ in \mathcal{T} a left \mathcal{C} -approximation of Z provided that C is in \mathcal{C} and the induced map $\text{Hom}_{\mathcal{T}}(C, C') \rightarrow \text{Hom}_{\mathcal{T}}(Z, C')$ is surjective for all C' in \mathcal{C} . We call f a minimal left \mathcal{C} -approximation, if f is a left-minimal morphism, i.e., every morphism $g : C \rightarrow C$ with $g \circ f = f$ is an isomorphism. Dually, we define (minimal) right approximations.

A pair of subcategories $(\mathcal{X}, \mathcal{Y})$ in \mathcal{T} is said to be a torsion pair, if both \mathcal{X} and \mathcal{Y} are closed under direct summands, $\text{Hom}_{\mathcal{T}}(X, Y) = 0$ for all X in \mathcal{X} and Y in \mathcal{Y} , and for all Z in \mathcal{T} there is a triangle

$$X \longrightarrow Z \longrightarrow Y \longrightarrow X[1] \tag{\star}$$

with X in \mathcal{X} and Y in \mathcal{Y} . Moreover, a torsion pair $(\mathcal{X}, \mathcal{Y})$ is called a t-structure (respectively, co-t-structure), if \mathcal{X} is closed under positive (respectively, negative) shifts. We call \mathcal{X} the aisle and \mathcal{Y} the coaisle of the (co-)t-structure.

For a t-structure $(\mathcal{X}, \mathcal{Y})$ the morphism $X \rightarrow Z$ in the triangle above is a minimal right \mathcal{X} -approximation, and the morphism $Z \rightarrow Y$ is a minimal left \mathcal{Y} -approximation. These approximations are induced by the truncation functors $\tau_{\mathcal{X}} : \mathcal{T} \rightarrow \mathcal{X}$, which is right adjoint to the inclusion of \mathcal{X} in \mathcal{T} , and $\tau^{\mathcal{Y}} : \mathcal{T} \rightarrow \mathcal{Y}$, which is left adjoint to the inclusion of \mathcal{Y} in \mathcal{T} . The triangle \star then takes the form

$$\tau_{\mathcal{X}}Z \longrightarrow Z \longrightarrow \tau^{\mathcal{Y}}Z \longrightarrow \tau_{\mathcal{X}}Z[1].$$

Moreover, for a t-structure $(\mathcal{X}, \mathcal{Y})$ we denote by $\mathcal{H} := \mathcal{X} \cap \mathcal{Y}[1]$ its heart, which is well-known to be an abelian category. Its exact structure is given by the triangles in \mathcal{T} with the first three terms in \mathcal{H} . Combining the truncation functors, we get a cohomological functor $H^0 : \mathcal{T} \rightarrow \mathcal{H}$ mapping an object $Z \in \mathcal{T}$ to $\tau^{\mathcal{Y}[1]}\tau_{\mathcal{X}}(Z)$, where $\tau^{\mathcal{Y}[1]}$ denotes the truncation functor with respect to the shifted t-structure $(\mathcal{X}[1], \mathcal{Y}[1])$. The functor H^0 sends triangles in \mathcal{T} to long exact sequences in \mathcal{H} .

If \mathcal{T} is a triangulated category with coproducts, we say that an object Z in \mathcal{T} is compact, if the functor $\text{Hom}_{\mathcal{T}}(Z, -)$ commutes with coproducts. Moreover, a t-structure $(\mathcal{X}, \mathcal{Y})$ in \mathcal{T} is called compactly generated, if $\mathcal{Y} = S^{\perp}$ for a set S of compact objects in \mathcal{T} .

For $\mathcal{T} = D(A)$, we denote by $(D(A)^{\leq 0}, D(A)^{\geq 1})$ the standard t-structure, where $D(A)^{\leq 0}$ (respectively, $D(A)^{\geq 1}$) denotes the subcategory of complexes with non-zero cohomologies only in non-positive (respectively, positive) degrees. For $n \in \mathbb{Z}$, we define $D(A)^{\leq n} := D(A)^{\leq 0}[-n]$ and $D(A)^{\geq n+1} := D(A)^{\geq 1}[-n]$. For the standard t-structure $(D(A)^{\leq 0}, D(A)^{\geq 1})$ we denote the truncation functors by $\tau^{\leq 0}$ and $\tau^{\geq 1}$, for the shift of the standard t-structure $(D(A)^{\leq n}, D(A)^{\geq n+1})$ by $\tau^{\leq n}$ and $\tau^{\geq n+1}$. A t-structure $(\mathcal{X}, \mathcal{Y})$ in $D(A)$ is called intermediate, if there are $m, n \in \mathbb{Z}$ such that $D(A)^{\leq m} \subseteq \mathcal{X} \subseteq D(A)^{\leq n}$, or equivalently, $D(A)^{\geq n+1} \subseteq \mathcal{Y} \subseteq D(A)^{\geq m+1}$. Note that $D^b(A)$ is equipped with a standard t-structure as well, and we can adapt the previous notation. In particular, we call a t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$ intermediate, if there are $m, n \in \mathbb{Z}$ such that $D^b(A)^{\leq m} \subseteq \mathcal{X} \subseteq D^b(A)^{\leq n}$. If A is a finite dimensional algebra, it can be checked that a t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$ is intermediate if and only if it is bounded, i.e.

$$\bigcup_{n \in \mathbb{Z}} \mathcal{X}[n] = D^b(A) = \bigcup_{n \in \mathbb{Z}} \mathcal{Y}[n].$$

2.3 | Homotopy colimits

We will be interested in taking homotopy colimits in $D(A)$. For that we will follow the approach based on the theory of derivators, without actually defining all the corresponding notions. Instead, we will give a short overview of facts and ideas used to define homotopy colimits in $D(A)$. For a more detailed overview on homotopy colimits in triangulated categories and the use of derivators, we refer to [16, 19, 37, 39].

Let Cat be the 2-category of all small categories. For $I \in Cat$ we consider the category $\text{Mod}(A)^I$, the category of I -shaped diagrams of A -modules. Objects of this category are functors $I \rightarrow \text{Mod}(A)$ and morphisms are natural transformations between these functors. The category $\text{Mod}(A)^I$ is again abelian, so we can consider the unbounded derived category $D(\text{Mod}(A)^I)$. This

category is the localization of the category of complexes $C(\text{Mod}(A)^I) \simeq C(\text{Mod}(A))^I$ with respect to quasi-isomorphisms. The objects of $D(\text{Mod}(A)^I)$ are called coherent diagrams and we think of them as I -shaped diagrams of complexes of A -modules.

A functor $u : I \rightarrow J$ in Cat yields a functor $u^* : \text{Mod}(A)^J \rightarrow \text{Mod}(A)^I$ via precomposition. As u^* is exact, it further induces a triangle functor $u^* : D(\text{Mod}(A)^J) \rightarrow D(\text{Mod}(A)^I)$. Now, let $*$ be the category with one object and one morphism. The category $D(\text{Mod}(A)^*)$ identifies with the category $D(A)$. For any $I \in Cat$ there exists a unique functor $\pi_I : I \rightarrow *$. The constant diagram functor $\pi_I^* : D(A) \rightarrow D(\text{Mod}(A)^I)$ maps an object $Z \in D(A)$ to the constant diagram of shape I with entries Z and identity maps everywhere. We can view π_I^* as a functor between the corresponding categories of complexes. In this case, its left (respectively, right) adjoint is the usual colimit (respectively, limit) functor in the category of complexes. The homotopy colimit is defined to be the left derived functor of this colimit functor, and thus the functor left adjoint to π_I^* on the level of derived categories.

$$\begin{array}{ccc}
 & \text{hocolim} & \\
 & \curvearrowright & \\
 D(A) & \xrightarrow{\pi_I^*} & D(\text{Mod}(A)^I)
 \end{array}$$

We will mostly be interested in the case, when I is a directed category. In this situation, we speak of a directed homotopy colimit and write $\text{hocolim}_{i \in I} Z_i$, where $(Z_i \mid i \in I)$ denotes a coherent diagram. Since direct limits in the category of complexes of A -modules are exact, directed homotopy colimits can be computed as usual direct limits of complexes.

For a small category I and an object $i \in I$ we can consider the functor $\iota_i : * \rightarrow I$ sending $*$ to i , and the corresponding functor $\iota_i^* : D(\text{Mod}(A)^I) \rightarrow D(A)$. Evaluating ι_i^* on a diagram of complexes of A -modules $(Z_i \mid i \in I)$ yields Z_i . Moreover, each morphism $f : i \rightarrow j$ in I gives rise to a natural transformation $f : \iota_i \Rightarrow \iota_j$ and thus to a natural transformation $f^* : \iota_i^* \Rightarrow \iota_j^*$ (via precomposition). Evaluating f^* on $(Z_i \mid i \in I)$ yields the corresponding morphism $Z_i \rightarrow Z_j$. We can extend this assignment to define the diagram functor $\text{dia}_I : D(\text{Mod}(A)^I) \rightarrow D(A)^I$. The objects of $D(A)^I$ are called incoherent diagrams.

Finally, for a subcategory C of $D(A)$, we denote by $\text{hocolim}(C)$ the smallest subcategory of $D(A)$ containing C and closed under directed homotopy colimits of objects from C , that is closed under directed homotopy colimits of diagrams $Z = (Z_i \mid i \in I)$ in $D(\text{Mod}(A)^I)$ with $\iota_i^*(Z) = Z_i \in C$ for all $i \in I$, for all directed categories $I \in Cat$.

3 | LIFTING T-STRUCTURES VIA HOMOTOPY COLIMITS

In the proof of the next lemma, we will use brutal truncations. For a complex $X = (X^i, d_i)$ with components X^i and a differential d_i of degree 1 we denote the brutal truncation at degree m by $\sigma^{\geq m}(X)$ (respectively $\sigma^{< m}(X)$), where $\sigma^{\geq m}(X) = (\sigma^{\geq m}(X)^i, d'_i)$ is the complex with $\sigma^{\geq m}(X)^i = X^i$ for $i \geq m$ and 0 otherwise, and $d'_i = d_i$ for $i \geq m$ and 0 otherwise. The truncation $\sigma^{< m}(X)$ is defined to be the complex $(\sigma^{< m}(X)^i, d'_i)$ with $\sigma^{< m}(X)^i = X^i$ for $i < m$ and 0 otherwise, and $d'_i = d_i$ for $i < m - 1$ and 0 otherwise.

Lemma 3.1. *Let $(\mathcal{X}, \mathcal{Y})$ be an intermediate t-structure in $D^b(A)$. Then*

$$(\mathbf{X}, \mathbf{Y}) := (\perp(\mathcal{X}^\perp), \mathcal{X}^\perp)$$

is a compactly generated intermediate t-structure in $D(A)$ and

$$(\mathbf{X} \cap D^b(A), \mathbf{Y} \cap D^b(A)) = (\mathcal{X}, \mathcal{Y}).$$

Proof. Since \mathcal{X} is skeletally small, the pair (\mathbf{X}, \mathbf{Y}) forms a t-structure in $D(A)$, where \mathbf{X} is given by the smallest subcategory of $D(A)$ containing \mathcal{X} and closed under extensions, coproducts and positive shifts (see [3]). By construction, the t-structure (\mathbf{X}, \mathbf{Y}) is again intermediate, i.e., there are $m, n \in \mathbb{Z}$ such that $D(A)^{\leq m} \subseteq \mathbf{X} \subseteq D(A)^{\leq n}$. This follows from the fact that taking the orthogonal in $D(A)$ yields $(D^b(A)^{\leq n})^\perp = D(A)^{\geq n+1}$. Now, take X in \mathcal{X} and view X as an object in $K^-(\text{proj}(A))$ by taking its projective resolution. Using brutal truncation, we obtain a triangle

$$\sigma^{< m}(X)[-1] \longrightarrow \sigma^{\geq m}(X) \longrightarrow X \longrightarrow \sigma^{< m}(X),$$

where $\sigma^{\geq m}(X)$, viewed as an object in $D(A)$, is compact. Define $S := S_1 \cup S_2$ to be the set of compact objects given by $S_1 := \{A[t] \mid t \geq -m\}$ and $S_2 := \{\sigma^{\geq m}(X) \mid X \in \mathcal{X}\}$. We claim that $S^\perp = \mathbf{Y}$, showing that (\mathbf{X}, \mathbf{Y}) is a compactly generated t-structure.

First, note that $\mathbf{Y} \subseteq D(A)^{\geq m+1} = S_1^\perp$. Hence, it suffices to observe that for an object Z in S_1^\perp , we have

$$Z \in S_2^\perp \Leftrightarrow Z \in \mathbf{Y} = \mathcal{X}^\perp.$$

Applying the functor $\text{Hom}_{D(A)}(-, Z)$ to the triangle above and using the fact that both $\text{Hom}_{D(A)}(\sigma^{< m}(X)[-1], Z)$ and $\text{Hom}_{D(A)}(\sigma^{< m}(X), Z)$ are zero, we get the claim. \square

We call the t-structure obtained in Lemma 3.1 the lifted t-structure associated to $(\mathcal{X}, \mathcal{Y})$. We want to better understand the t-structures in $D(A)$ that we obtain via lifting. Recall that an object X in a Grothendieck category \mathbf{H} is said to be finitely presented provided that $\text{Hom}_{\mathbf{H}}(X, -)$ commutes with direct limits. Moreover, \mathbf{H} is called locally coherent, if it has a set of finitely presented generators and if the finitely presented objects form an exact abelian subcategory.

Theorem 3.2. *Let $(\mathcal{X}, \mathcal{Y})$ be an intermediate t-structure in $D^b(A)$ with heart \mathcal{H} , and let (\mathbf{X}, \mathbf{Y}) be the associated lifted t-structure in $D(A)$ with heart \mathbf{H} . Then the following holds.*

- (1) $(\mathbf{X}, \mathbf{Y}) = (\text{hocolim}(\mathcal{X}), \text{hocolim}(\mathcal{Y}))$.
- (2) $\mathbf{H} = \text{hocolim}(\mathcal{H})$, and \mathbf{H} is a locally coherent Grothendieck category, whose finitely presented objects are given by $\mathcal{H} = \mathbf{H} \cap D^b(A)$.

Proof. (1): Every aisle of a t-structure in $D(A)$ is closed under homotopy colimits by [37, Proposition 4.2]. Moreover, since (\mathbf{X}, \mathbf{Y}) is compactly generated by Lemma 3.1, the coaisle \mathbf{Y} is closed under directed homotopy colimits (see [37, Proposition 5.6]). Indeed, if S is a compact object in $D(A)$ and $(Z_i \mid i \in I)$ is a directed coherent diagram, we have the canonical isomorphism

$$\text{Hom}_{D(A)}(S, \text{hocolim}_{i \in I} Z_i) \cong \varinjlim_{i \in I} \text{Hom}_{D(A)}(S, Z_i).$$

It follows that $\text{hocolim}(\mathcal{X}) \subseteq \mathbf{X}$ and $\text{hocolim}(\mathcal{Y}) \subseteq \mathbf{Y}$. Now, take an object Z in $D(A)$ and consider the approximation triangle

$$X \longrightarrow Z \longrightarrow Y \longrightarrow X[1]$$

with X in \mathbf{X} and Y in \mathbf{Y} . We will show that X is in $\text{hocolim}(\mathcal{X})$ and Y is in $\text{hocolim}(\mathcal{Y})$, proving that $(\mathbf{X}, \mathbf{Y}) = (\text{hocolim}(\mathcal{X}), \text{hocolim}(\mathcal{Y}))$. To this end, write $Z \cong \varinjlim_{i \in I} Z_i$ in $C(\text{Mod}(A))$, where $(Z_i \mid i \in I) \in C(\text{Mod}(A))^I$ is a direct system of bounded complexes of finitely presented modules (see [15, Lemmas 4.1.1(ii) and 5.1.1] for the existence of such a diagram, and [12, Section 4] for further context). We can view $(Z_i \mid i \in I)$ as an object in $D(\text{Mod}(A))^I$ and get $Z \cong \text{hocolim}_{i \in I} Z_i$. By [37, Theorem A], there is a t-structure $(\mathbf{X}_I, \mathbf{Y}_I)$ in $D(\text{Mod}(A))^I$, where \mathbf{X}_I , respectively \mathbf{Y}_I , is given by all I -shaped diagrams of complexes in \mathbf{X} , respectively \mathbf{Y} . Consider the associated approximation triangle

$$(X_i \mid i \in I) \longrightarrow (Z_i \mid i \in I) \longrightarrow (Y_i \mid i \in I) \longrightarrow (X_i \mid i \in I)[1]$$

in $D(\text{Mod}(A))^I$. Applying the functors t_i^* , we obtain triangles of the form

$$X_i \longrightarrow Z_i \longrightarrow Y_i \longrightarrow X_i[1]$$

in $D(A)$. Since all the Z_i belong to $D^b(A)$, so do all the X_i and Y_i . In fact, we have found the approximation triangles of Z_i with respect to the t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$. Taking the homotopy colimit of the triangle in $D(\text{Mod}(A))^I$ yields a triangle

$$\text{hocolim}_{i \in I} X_i \longrightarrow \text{hocolim}_{i \in I} Z_i \longrightarrow \text{hocolim}_{i \in I} Y_i \longrightarrow \text{hocolim}_{i \in I} X_i[1]$$

in $D(A)$ isomorphic to

$$X \longrightarrow Z \longrightarrow Y \longrightarrow X[1].$$

It follows that X belongs to $\text{hocolim}(\mathcal{X})$ and Y belongs to $\text{hocolim}(\mathcal{Y})$, as desired. Observe that the previous arguments applied to an arbitrary directed coherent diagram imply that truncation functors of compactly generated t-structures always commute with directed homotopy colimits.

(2): We know that $\text{hocolim}(\mathcal{H}) \subseteq \mathbf{H}$ by (1). Conversely, take an object H in \mathbf{H} and write again $H \cong \text{hocolim}_{i \in I} Z_i$ with Z_i in $D^b(A)$ and I directed. By $H^0 : D(A) \rightarrow \mathbf{H}$ we denote the cohomological functor associated to the t-structure (\mathbf{X}, \mathbf{Y}) . Note that on objects in $D^b(A)$ the functor H^0 coincides with the cohomological functor associated to $(\mathcal{X}, \mathcal{Y})$. Since H^0 is the composition of truncation functors, it commutes with directed homotopy colimits following (1). Hence, we get

$$H = H^0(H) \cong H^0(\text{hocolim}_{i \in I} Z_i) \cong \text{hocolim}_{i \in I} H^0(Z_i)$$

with $H^0(Z_i)$ in \mathcal{H} , as desired.

The heart \mathbf{H} of (\mathbf{X}, \mathbf{Y}) is a Grothendieck category by [7, Corollary 4.10] (see also [36, 37] for more general results in this direction). By [37, Theorem A and Corollary 5.8] we can lift incoherent diagrams in \mathbf{H} to coherent diagrams and compute direct limits in \mathbf{H} as directed homotopy colimits in $D(A)$. Thus, it only remains to show that the set of finitely presented objects in \mathbf{H} is given by $\mathcal{H} = \mathbf{H} \cap D^b(A)$. This follows from [34, Proposition 4.5]. In order to apply [34, Proposition 4.5] we only need to check that for each direct system $(H_i \mid i \in I)$ in \mathbf{H} and for each $k \in \mathbb{Z}$, the canonical map $\varinjlim_{i \in I} \text{Hom}_{D(A)}(A[k], H_i) \rightarrow \text{Hom}_{D(A)}(A[k], \varinjlim_{i \in I} H_i)$ is an isomorphism. After lifting $(H_i \mid i \in I)$ to a coherent diagram, this follows from the fact that $\text{Hom}_{D(A)}(A[k], -)$ is naturally isomorphic to the $-k$ th cohomology functor on complexes, which preserves direct limits. \square

We have seen so far that every intermediate t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$ lifts to an intermediate and compactly generated t-structure $(\underline{\text{hocolim}}(\mathcal{X}), \underline{\text{hocolim}}(\mathcal{Y}))$ in $D(A)$, and that this lifting process is compatible with the heart. Next, we would like to better understand which t-structures in $D(A)$ can be obtained in this way.

4 | RESTRICTING T-STRUCTURES

We say that a t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ restricts to $D^b(A)$, if $(\mathbf{X} \cap D^b(A), \mathbf{Y} \cap D^b(A))$ is a t-structure in $D^b(A)$. Clearly, every t-structure in $D(A)$ obtained via lifting restricts to the initial t-structure in $D^b(A)$.

Moreover, we call a t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ homotopically smashing, if the coaisle \mathbf{Y} is closed under directed homotopy colimits. As discussed in the proof of Theorem 3.2, every compactly generated t-structure is homotopically smashing.

Theorem 4.1. *Let (\mathbf{X}, \mathbf{Y}) be an intermediate, homotopically smashing t-structure in $D(A)$ with heart \mathbf{H} . Then the following are equivalent.*

- (1) (\mathbf{X}, \mathbf{Y}) restricts to $D^b(A)$;
- (2) $\mathbf{Y} = \underline{\text{hocolim}}(\mathbf{Y} \cap D^b(A))$;
- (3) $\mathbf{H} = \underline{\text{hocolim}}(\mathbf{H} \cap D^b(A))$.

In particular, if these conditions are fulfilled, we also have $\mathbf{X} = \underline{\text{hocolim}}(\mathbf{X} \cap D^b(A))$.

Proof. (1) \Rightarrow (2), (3): By lifting again, we obtain the t-structure

$$(\underline{\text{hocolim}}(\mathbf{X} \cap D^b(A)), \underline{\text{hocolim}}(\mathbf{Y} \cap D^b(A)))$$

in $D(A)$, which coincides with (\mathbf{X}, \mathbf{Y}) , as both \mathbf{X} and \mathbf{Y} are closed under directed homotopy colimits, and thus contain the respective classes $\underline{\text{hocolim}}(\mathbf{X} \cap D^b(A))$ and $\underline{\text{hocolim}}(\mathbf{Y} \cap D^b(A))$. Hence, we get (2), and also (3) by Theorem 3.2(2).

(2) \Rightarrow (1): Now, assume that $\mathbf{Y} = \underline{\text{hocolim}}(\mathbf{Y} \cap D^b(A))$. We show that for an object Z in $D^b(A)$ the approximation triangle

$$X \longrightarrow Z \xrightarrow{f} Y \longrightarrow X[1]$$

with respect to (\mathbf{X}, \mathbf{Y}) lies in $D^b(A)$. To this end, we first consider the case when Z is a compact object. By assumption, we can write $Y = \underline{\text{hocolim}}_{i \in I} Y_i$ with Y_i in $\mathbf{Y} \cap D^b(A)$ for all i . Since Z is compact, we have

$$f \in \text{Hom}_{D(A)}(Z, \underline{\text{hocolim}}_{i \in I} Y_i) \cong \varinjlim_{i \in I} \text{Hom}_{D(A)}(Z, Y_i).$$

Thus, the map f factors through some Y_i . Since f is a minimal left \mathbf{Y} -approximation, the object Y identifies with a direct summand of Y_i . In particular, the whole approximation triangle above lies in $D^b(A)$, as desired.

For the general case, assume $Z \in D^b(A)$. We view Z as an object in $K^-(\text{proj}(A))$ and use truncations. First, note that there exists $i \in \mathbb{Z}$ big enough such that $\tau^{>m} \sigma^{\geq -i}(Z) \simeq \tau^{>m} Z$, where m

is given by the intermediate condition $D(A)^{\leq m} \subseteq \mathbf{X} \subseteq D(A)^{\leq n}$. Since $\mathbf{Y} \subseteq D(A)^{> m}$, we get that $\tau^{\mathbf{Y}}\tau^{> m}W \simeq \tau^{\mathbf{Y}}W$ for any $W \in D(A)$. Thus

$$Y = \tau^{\mathbf{Y}}Z \simeq \tau^{\mathbf{Y}}\tau^{> m}Z \simeq \tau^{\mathbf{Y}}\tau^{> m}\sigma^{\geq -i}(Z) \simeq \tau^{\mathbf{Y}}\sigma^{\geq -i}(Z) \in D^b(A)$$

as $\sigma^{\geq -i}(Z)$ is compact.

(3) \Rightarrow (1): Finally, assume that $\mathbf{H} = \text{hocolim}(\mathbf{H} \cap D^b(A))$. Take Z in $D^b(A)$ and consider the approximation triangle

$$X \longrightarrow Z \longrightarrow Y \longrightarrow X[1] \tag{\dagger}$$

with respect to (\mathbf{X}, \mathbf{Y}) . Since (\mathbf{X}, \mathbf{Y}) is intermediate, there is some $k \geq 0$ with $Z[k]$ in \mathbf{X} . Instead of the triangle \dagger we consider the shifted triangle

$$X[k] \longrightarrow Z[k] \longrightarrow Y[k] \longrightarrow X[k + 1],$$

which is the approximation triangle of $Z[k]$ with respect to the t-structure $(\mathbf{X}[k], \mathbf{Y}[k])$. Hence, it is enough to show that for any object Z in $\mathbf{X} \cap D^b(A)$ and $k \geq 0$, the approximation triangle

$$X[k] \xrightarrow{f} Z \longrightarrow Y[k] \longrightarrow X[k + 1]$$

with respect to $(\mathbf{X}[k], \mathbf{Y}[k])$ belongs to $D^b(A)$. We proceed by induction on k . The case $k = 0$ is trivial. For the case $k = 1$, we argue in a similar way as in the proof of (2) \Rightarrow (1). First, consider a compact object Z in $\mathbf{X} \cap D^b(A)$ together with the approximation triangle

$$X[1] \longrightarrow Z \xrightarrow{\phi} Y[1] \longrightarrow X[2].$$

We have $Y[1] = H^0(Z)$, where H^0 denotes the cohomological functor associated to (\mathbf{X}, \mathbf{Y}) . By assumption, it follows that $H^0(Z) = \text{hocolim}_{i \in I} H_i$ with H_i in $\mathbf{H} \cap D^b(A)$ for all i . Since Z is compact, the map ϕ factors through some H_i and, thus, the previous triangle lies in $D^b(A)$. Now suppose that Z is an arbitrary object in $\mathbf{X} \cap D^b(A)$. We view Z as an object in $K^-(\text{proj}(A))$ so that the brutal truncations $Z_i := \sigma^{\geq -i}(Z)$ are compact for all i . For every i we get a triangle

$$\sigma^{< -i}(Z)[-1] \longrightarrow Z_i \longrightarrow Z \longrightarrow \sigma^{< -i}(Z).$$

Since (\mathbf{X}, \mathbf{Y}) is intermediate, the object $\sigma^{< -i}(Z)[-1]$ will belong to \mathbf{X} for sufficiently large i . In that case, Z_i will also be in \mathbf{X} , as an extension of objects in \mathbf{X} . Moreover, we can choose i big enough so that $\tau^{> m-1}Z \simeq \tau^{> m-1}\sigma^{\geq -i}(Z) = \tau^{> m-1}Z_i$, where m is given by the intermediate condition $D(A)^{\leq m} \subseteq \mathbf{X} \subseteq D(A)^{\leq n}$. Hence, as in the proof of (2) \Rightarrow (1) we get

$$Y[1] = \tau^{\mathbf{Y}[1]}Z \simeq \tau^{\mathbf{Y}[1]}\tau^{> m-1}Z \simeq \tau^{\mathbf{Y}[1]}\tau^{> m-1}Z_i \simeq \tau^{\mathbf{Y}[1]}Z_i \in D^b(A)$$

as Z_i is compact.

For the induction step, let

$$X'[k - 1] \xrightarrow{g} Z \longrightarrow Y'[k - 1] \longrightarrow X'[k]$$

be the approximation triangle of Z with respect to the t-structure $(\mathbf{X}[k - 1], \mathbf{Y}[k - 1])$, which belongs to $D^b(A)$ by the induction hypothesis. Moreover, let

$$X^\varepsilon[k] \xrightarrow{h} X'[k - 1] \longrightarrow Y^\varepsilon[k] \longrightarrow X^\varepsilon[k + 1]$$

be the approximation triangle of $X'[k - 1]$ with respect to $(\mathbf{X}[k], \mathbf{Y}[k])$. Note that, as $X'[k - 1]$ belongs to $\mathbf{X}[k - 1] \cap D^b(A)$, this triangle belongs to $D^b(A)$ by the base of induction ($k = 1$). We obtain the following commutative diagram of morphisms

$$\begin{array}{ccccccc} X^\varepsilon[k] & \xrightarrow{h} & X'[k - 1] & & & & \\ \vdots \downarrow c & & \downarrow g & & & & \\ X[k] & \xrightarrow{f} & Z & \longrightarrow & Y[k] & \longrightarrow & X[k + 1] \end{array}$$

Note that the composition gh factors through f , as f is a right $\mathbf{X}[k]$ -approximation. On the other hand, since $X[k]$ is in $\mathbf{X}[k] \subseteq \mathbf{X}[k - 1]$, the map f factors through g , as g is a right $\mathbf{X}[k - 1]$ -approximation, and thus through the composition gh , as h is a right $\mathbf{X}[k]$ -approximation. Hence, we get a map $d : X[k] \rightarrow X''[k]$ such that $f = gh d = f c d$. The minimality of f guarantees that cd is an isomorphism, showing that $X[k]$ lies in $D^b(A)$, as a direct summand of $X''[k]$. This finishes the proof. \square

As an immediate consequence, it follows that every intermediate, homotopically smashing t-structure in $D(A)$ that restricts to $D^b(A)$ is necessarily compactly generated. We can now determine precisely, which t-structures we obtain from lifting. In fact, it turns out that the necessary conditions discussed in Lemma 3.1 and Theorem 3.2 are also sufficient.

Corollary 4.2. *Lifting and restricting t-structures yields a bijection between*

- (1) *intermediate t-structures $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$;*
- (2) *intermediate, compactly generated t-structures (\mathbf{X}, \mathbf{Y}) in $D(A)$ with a locally coherent Grothendieck heart \mathbf{H} , whose finitely presented objects are given by $\mathbf{H} \cap D^b(A)$.*

Moreover, the t-structures in (2) are precisely the intermediate homotopically smashing t-structures in $D(A)$ that restrict to $D^b(A)$.

Proof. By Lemma 3.1 and Theorem 3.2, lifting intermediate t-structures in $D^b(A)$ yields a well-defined, injective map from (1) to (2). Now take a t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ as in (2). Using [37, Theorem A and Corollary 5.8], we can compute direct limits in \mathbf{H} as directed homotopy colimits in $D(A)$. Hence, it follows that $\mathbf{H} = \text{hocolim}(\mathbf{H} \cap D^b(A))$. By Theorem 4.1, the t-structure (\mathbf{X}, \mathbf{Y}) restricts to $D^b(A)$, and we can recover it via lifting, as (\mathbf{X}, \mathbf{Y}) is compactly generated. The final statement follows again from Theorem 4.1. \square

We finish this section with an application of Theorems 3.2 and 4.1 providing a criterion for a derived equivalence to preserve coherent rings. For that we need some further definitions.

A complex $T \in K^b(\text{proj}(A))$ is called tilting, if $\text{Hom}_{D(A)}(T, T[i]) = 0$ for any $i \neq 0$ and if the smallest triangulated subcategory of $K^b(\text{proj}(A))$ containing T and closed under direct summands is $K^b(\text{proj}(A))$ itself. We say that two rings A and B are derived equivalent if the derived categories

$D(A)$ and $D(B)$ are triangle equivalent. By [21, 32], this happens if and only if there exists a tilting complex T in $K^b(\text{proj}(A))$ such that $\text{End}_{D(A)}(T) \simeq B$. Under the triangle equivalence defined by the tilting complex T , the standard t-structure on $D(B)$ is mapped to the t-structure on $D(A)$ generated by T , that is the t-structure $(\mathbf{X}, \mathbf{Y}) = (T^{\perp > 0}, T^{\perp \leq 0})$. Its heart \mathbf{H} is equivalent to $\text{Mod}(B)$.

Corollary 4.3. *Let A be a right coherent ring and T be a tilting complex in $K^b(\text{proj}(A))$. Then the ring $B = \text{End}_{D(A)}(T)$ is right coherent if and only if the t-structure $(T^{\perp > 0}, T^{\perp \leq 0})$ restricts to $D^b(A)$.*

Proof. If the t-structure $(T^{\perp > 0}, T^{\perp \leq 0})$ restricts to $D^b(A)$, we can recover it via lifting. Thus, the finitely presented objects in its heart \mathbf{H} are of the form $\mathbf{H} \cap D^b(A)$ and they form an abelian category. Since \mathbf{H} is equivalent to $\text{Mod}(B)$, this implies that the finitely presented objects in $\text{Mod}(B)$ form an abelian category. So B is right coherent.

If B is right coherent the standard t-structure on $D(B)$ restricts to the standard t-structure on $D^b(B)$, and the triangle equivalence $D(B) \cong D(A)$ induced by T restricts to the level of bounded derived categories (see [32, Proposition 8.1]). Since the standard t-structure on $D(B)$ identifies with $(T^{\perp > 0}, T^{\perp \leq 0})$ under this equivalence, we get that $(T^{\perp > 0}, T^{\perp \leq 0})$ restricts to $D^b(A)$, as desired. \square

Note that Corollary 4.3 also appears as [29, Proposition 6.6].

Remark 4.4. In case we have a triangle equivalence $F : D(B) \longrightarrow D(A)$ that is hypothetically not given by a tilting complex T , we can still argue as above by considering the t-structure $(F(D(B)^{\leq 0}), F(D(B)^{\geq 1}))$ instead of $(T^{\perp > 0}, T^{\perp \leq 0})$ in $D(A)$. In particular, the ring B is right coherent if and only if this t-structure restricts to $D^b(A)$.

5 | CASE STUDY: HRS-T-STRUCTURES

In this section, we assume that A is a right noetherian ring and we specialize Corollary 4.2 to the setting of HRS-t-structures arising from torsion pairs in module categories. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}(A)$ induces a t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$ as follows:

$$\mathcal{X} = \{X \in D^b(A) \mid H^i(X) = 0 \text{ for } i > 0 \text{ and } H^0(X) \in \mathcal{T}\}$$

$$\mathcal{Y} = \{Y \in D^b(A) \mid H^i(Y) = 0 \text{ for } i < 0 \text{ and } H^0(Y) \in \mathcal{F}\},$$

where H^i denotes the cohomological functor with respect to the shifted standard t-structure $(D^b(A)^{\leq i}, D^b(A)^{\geq i+1})$. Analogously, a torsion pair (\mathbf{T}, \mathbf{F}) in $\text{Mod}(A)$ induces a t-structure in $D(A)$. We call these t-structures HRS-t-structures; see [18] for details. By [30, Lemma 1.1.2], HRS-t-structures are precisely the t-structures $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$ (respectively, $D(A)$) with $D^b(A)^{\leq -1} \subseteq \mathcal{X} \subseteq D^b(A)^{\leq 0}$ (respectively, $D(A)^{\leq -1} \subseteq \mathcal{X} \subseteq D(A)^{\leq 0}$), i.e. intermediate t-structures with the smallest non-trivial interval $[m, n]$ (see also [10]).

Starting with a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}(A)$, considering the associated HRS-t-structure in $D^b(A)$ and lifting this t-structure, we obtain an HRS-t-structure in $D(A)$ corresponding to some torsion pair (\mathbf{T}, \mathbf{F}) in $\text{Mod}(A)$. Since H^0 sends directed homotopy colimits to direct limits in $\text{Mod}(A)$, we get that

$$(\mathbf{T}, \mathbf{F}) = (\varinjlim(\mathcal{T}), \varinjlim(\mathcal{F})).$$

This recovers a well-known construction of lifting torsion pairs from $\text{mod}(A)$ to $\text{Mod}(A)$ by closing \mathcal{T} and \mathcal{F} under direct limits of objects in \mathcal{T} and \mathcal{F} , respectively (see [13, Section 4.4]). Recall that a torsion pair (\mathbf{T}, \mathbf{F}) in $\text{Mod}(A)$ is said to be of finite type, if the class \mathcal{F} is closed under direct limits. Torsion pairs of the form $(\varinjlim(\mathcal{T}), \varinjlim(\mathcal{F}))$ provide examples of torsion pairs of finite type. The following proposition makes precise the connection between lifting HRS-t-structures on the level of torsion pairs and on the level of t-structures.

Proposition 5.1. *Let A be a right noetherian ring. Then there is a commutative square of bijections as follows.*

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{t-structures } (\mathcal{X}, \mathcal{Y}) \text{ in } D^b(A) \text{ with} \\ D^b(A)^{\leq -1} \subseteq \mathcal{X} \subseteq D^b(A)^{\leq 0} \end{array} \right\} & \xleftarrow{1:1} & \{ \text{torsion pairs in } \text{mod}(A) \} \\
 \begin{array}{c} \downarrow \text{hocolim} \\ \uparrow \text{restriction} \end{array} & & \begin{array}{c} \downarrow \lim \\ \uparrow \text{restriction} \end{array} \\
 \left\{ \begin{array}{l} \text{compactly generated t-structures} \\ (\mathbf{X}, \mathbf{Y}) \text{ in } D(A) \text{ with} \\ D(A)^{\leq -1} \subseteq \mathbf{X} \subseteq D(A)^{\leq 0} \end{array} \right\} & \xleftarrow{1:1} & \left\{ \begin{array}{l} \text{torsion pairs in } \text{Mod}(A) \\ \text{of finite type} \end{array} \right\}
 \end{array}$$

Moreover, a t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ with $D(A)^{\leq -1} \subseteq \mathbf{X} \subseteq D(A)^{\leq 0}$ is compactly generated if and only if its heart \mathbf{H} is a Grothendieck category. In this case, \mathbf{H} is locally coherent and the finitely presented objects of \mathbf{H} are given by $\mathbf{H} \cap D^b(A)$.

Proof. The upper horizontal bijection was already mentioned above. Since A is right noetherian, every torsion pair in $\text{Mod}(A)$ restricts to a torsion pair in $\text{mod}(A)$. Thus, it is not hard to check that the vertical maps on the right hand side of the diagram induce mutually inverse bijections. By [11, Theorem 3.10], a torsion pair in $\text{Mod}(A)$ is of finite type if and only if the associated HRS-t-structure is compactly generated if and only if the heart of this t-structure is a Grothendieck category. In particular, we get the lower horizontal bijection. Now, it remains to observe that lifting t-structures on the left hand side yields a commutative diagram, so lifting induces a bijection with restriction being its inverse. Note that this bijection is a special case of the bijection appearing in Corollary 4.2. □

We have just seen that every compactly generated t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ fulfilling $D(A)^{\leq -1} \subseteq \mathbf{X} \subseteq D(A)^{\leq 0}$ restricts to $D^b(A)$. The following example shows that a similar statement fails, if we allow the interval to be any larger.

Example 5.2. Let A be the Kronecker algebra over an algebraically closed field k . Let (\mathbf{X}, \mathbf{Y}) be the t-structure in $D(A)$ generated by the stalk complexes $A[2]$ and S_λ , where S_λ denotes a simple regular A -module for λ in $k \cup \{\infty\}$. Thus, we have $\mathbf{Y} = S^\perp$ for $S = \{A[n + 2] \oplus S_\lambda[n] \mid n \geq 0\}$. It follows that (\mathbf{X}, \mathbf{Y}) is compactly generated with $D(A)^{\leq -2} \subseteq \mathbf{X} \subseteq D(A)^{\leq 0}$. But (\mathbf{X}, \mathbf{Y}) does not restrict to $D^b(A)$ by [38, Corollary 4.5(1)].

6 | CONNECTIONS WITH SILTING THEORY

An object T in $D(A)$ is called silting, if the pair $(T^{\perp > 0}, T^{\perp \leq 0})$ forms a t-structure in $D(A)$. Dually, an object C in $D(A)$ is called cosilting, if the pair $({}^{\perp \leq 0}C, {}^{\perp > 0}C)$ forms a t-structure in $D(A)$. We refer to these t-structures as silting and cosilting t-structures; further details can be found in [28, 31].

Note that it follows from [26, Theorem 4.6] that every intermediate and homotopically smashing t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ is cosilting with respect to a pure-injective cosilting object C in $D(A)$ with bounded cohomology. In fact, C has non-zero cohomology only in the interval determined by the intermediate condition, and we can choose C to be a bounded complex of injective A -modules. In particular, every t-structure in $D(A)$ obtained from lifting via homotopy colimits is controlled by such a cosilting complex. In general, however, we do not know how to deduce from the properties of the cosilting complex, if the associated t-structure restricts to $D^b(A)$; only partial results are available. For example, cosilting t-structures with respect to 2-term cosilting complexes of injective A -modules are (up to shift) precisely HRS-t-structures with respect to torsion pairs of finite type in $\text{Mod}(A)$ (see [4, Corollary 3.9]). Thus, if A is right noetherian, they restrict to $D^b(A)$ by Proposition 5.1. On the other hand, it follows from Example 5.2 that not every cosilting t-structure with respect to a 3-term cosilting complex restricts to $D^b(A)$.

Next, we discuss various conditions for the lifted t-structures to be silting as well. For this purpose, we restrict ourselves to the case when A is a finite dimensional algebra, which allows us to use the fundamental correspondences from [23].

Proposition 6.1. *Let A be a finite dimensional algebra over a field k . Let $(\mathcal{X}, \mathcal{Y})$ be an intermediate t-structure in $D^b(A)$ with heart \mathcal{H} , and let (\mathbf{X}, \mathbf{Y}) be the associated lifted t-structure in $D(A)$, whose heart is denoted by \mathbf{H} . Then the following conditions are equivalent.*

- (1) (\mathbf{X}, \mathbf{Y}) is a silting t-structure with respect to a silting object T ;
- (2) $\mathcal{X} \cap K^b(\text{proj}(A))$ is the coaisle of a co-t-structure in $K^b(\text{proj}(A))$;
- (3) $\mathcal{H} \cong \text{mod}(\Lambda)$ for a finite dimensional k -algebra Λ ;
- (4) $\mathbf{H} \cong \text{Mod}(\Lambda')$ for a finite dimensional k -algebra Λ' ;
- (5) (\mathbf{X}, \mathbf{Y}) is a silting t-structure with respect to a compact silting object T' .

Proof. (1) \Rightarrow (2): To show (2) it suffices to check that every object Z in $K^b(\text{proj}(A))$ admits a left \mathcal{X} -approximation $f : Z \rightarrow X$, where X lies in $K^b(\text{proj}(A))$. Indeed, since by assumption $K^b(\text{proj}(A))$ is a Krull-Schmidt category, it would follow from [25, Corollary 1.4] that we can choose f left-minimal, which by the dual of [20, Lemma 2.1] implies that there is a triangle

$$C \longrightarrow Z \xrightarrow{f} X \longrightarrow C[1]$$

in $K^b(\text{proj}(A))$ with C in ${}^{\perp}\mathcal{X}$, as desired. Now, take Z in $K^b(\text{proj}(A))$. Since (\mathbf{X}, \mathbf{Y}) is a silting t-structure, there is a co-t-structure $({}^{\perp}\mathbf{X}, \mathbf{X})$ in $D(A)$ and we get a left \mathbf{X} -approximation $f' : Z \rightarrow X'$ (see [28, Remark 4]). By assumption, we can write $X' = \text{hocolim}_{i \in I} X_i$ with X_i in \mathcal{X} . Since Z is compact, the map f' factors through some X_i via a map $f'_i : Z \rightarrow X_i$, which by construction is a left \mathcal{X} -approximation. Now, we can view X_i as an object in $K^-(\text{proj}(A))$ and use brutal truncations to write $X_i \cong \text{hocolim}_{j \in \mathbb{N}} \sigma^{\geq -j}(X_i)$ with $\sigma^{\geq -j}(X_i)$ in $K^b(\text{proj}(A))$. For sufficiently large j the objects $\sigma^{\geq -j}(X_i)$ belong to \mathcal{X} (see the proof of (3) \Rightarrow (1) in Theorem 4.1). Also, using again the fact that Z is compact, we get that f'_i factors through $\sigma^{\geq -j}(X_i)$ for j big enough. This way, we get the required approximation.

Remaining implications: Since $(\mathcal{X}, \mathcal{Y})$ is intermediate, the co-t-structure in (2) with coaisle $\mathcal{X} \cap K^b(\text{proj}(A))$ is intermediate with respect to the standard co-t-structure in $K^b(\text{proj}(A))$, so it is necessarily bounded. Therefore, (2) \Leftrightarrow (3) is part of the bijections in [23, Theorem 6.1], and it is further equivalent to stating that $(\mathcal{X}, \mathcal{Y}) = (T'^{\perp_{>0}}, T'^{\perp_{\leq 0}})$ for a silting object T' in $K^b(\text{proj}(A))$. The latter, however, is easily seen to be equivalent to (5). Moreover, (5) \Rightarrow (1) is trivial, and (5) \Rightarrow (4) holds by [28, Proposition 2]. Finally, (4) \Rightarrow (3) holds by Theorem 3.2, since \mathbf{H} is the subcategory of finitely presented objects of \mathbf{H} .

Note that if the equivalent conditions in the proposition are fulfilled, then Λ and Λ' are Morita equivalent, and T is equivalent to T' as a silting object, that is the smallest subcategory of $D(A)$ containing T and closed under coproducts and direct summands coincides with the corresponding subcategory containing T' . □

Note that Proposition 6.1 cannot be easily generalized. Using [8, Section 4], there are noetherian rings A admitting pure projective tilting modules that are not equivalent to finitely presented ones. These tilting modules induce compactly generated HRS-t-structures in $D(A)$, which necessarily restrict to $D^b(A)$ by Proposition 5.1, and which are both silting and cosilting. However, they are not silting with respect to a compact silting object and their hearts are Grothendieck categories with a projective generator, but not module categories.

By the discussion above, the process of lifting t-structures via homotopy colimits is closely related to cosilting theory. In the final part of this section, we will explore a dual way of lifting t-structures, which is closely connected to silting theory. For this purpose, we restrict ourselves again to the setting of finite dimensional algebras.

Theorem 6.2. *Let A be a finite dimensional algebra over a field k , and let $(\mathcal{X}, \mathcal{Y})$ be an intermediate t-structure in $D^b(A)$ with heart \mathbf{H} . Then the following holds.*

- (1) $(\mathbf{X}, \mathbf{Y}) = (\perp^1 \mathcal{Y}, (\perp^1 \mathcal{Y})^\perp)$ is a silting t-structure in $D(A)$ with respect to a bounded silting complex of projective A -modules.
- (2) An object $H \in \mathbf{H}$ satisfies $\text{Hom}_{\mathbf{H}}(H, \mathcal{H}) = 0$ if and only if $H = 0$.

Proof. (1): Since A is a finite dimensional algebra, dual arguments to those in the proof of Lemma 3.1 imply that $\perp^1 \mathcal{Y} = \perp^1 S$, where $S \subseteq \mathcal{Y}$ is a set of bounded complexes of finitely generated injective A -modules. Without loss of generality, we can assume that S is closed under negative shifts. Our aim is now to show that $(\perp^1 S, (\perp^1 S)^\perp)$ is a silting t-structure with respect to a bounded silting complex of projective A -modules. By ν and ν^{-1} we denote the total derived functors of the Nakayama functor $-\otimes_A \mathfrak{D}(A)$ and its right adjoint $\text{Hom}_A(\mathfrak{D}(A), -)$. Here, $\mathfrak{D} = \text{Hom}_k(-, k)$ denotes the standard duality. Recall that ν and ν^{-1} induce quasi-inverse triangle equivalences between $K^b(\text{proj}(A))$ and $K^b(\text{inj}(A))$, where the latter denotes the bounded homotopy category of finitely generated injective A -modules. Moreover, we have the following natural isomorphism

$$\mathfrak{D} \text{Hom}_{D(A)}(\nu^{-1}M, N) \cong \text{Hom}_{D(A)}(N, M)$$

for all M in $K^b(\text{inj}(A))$ and N in $D(A)$ (see [17, Ch.1, Section 4.6], [23, Section 2.3] and [24]). In our situation, it follows that the subcategory $\perp^1 S$ coincides with $(\nu^{-1}S)^\perp$, where all the objects in $\nu^{-1}S$ are compact in $D(A)$. Moreover, by assumption, we know that there are $m, n \in \mathbb{Z}$ with $D(A)^{\leq m} \subseteq \perp^1 S \subseteq D(A)^{\leq n}$. Hence, we can invoke [27, Theorem 3.6] to conclude that $(\perp^1 S, (\perp^1 S)^\perp)$ is a silting t-structure of the desired form.

(2): Take H in \mathbf{H} with $\text{Hom}_{\mathbf{H}}(H, \mathcal{H}) = 0$. First, observe that, by using approximation triangles with respect to $(\mathcal{X}, \mathcal{Y})$, every object $Y[1]$ in $\mathcal{Y}[1]$ appears in a triangle

$$H' \longrightarrow Y[1] \longrightarrow Y' \longrightarrow H'[1]$$

with H' in \mathcal{H} and Y' in \mathcal{Y} . Since $\text{Hom}_{D(A)}(H, \mathcal{Y}) = 0$ and $\text{Hom}_{\mathbf{H}}(H, \mathcal{H}) = 0$ by assumption, it follows that also $\text{Hom}_{D(A)}(H, \mathcal{Y}[1]) = 0$. Hence, H belongs to $\mathbf{X}[1]$. As H also belongs to $\mathbf{Y}[1]$, it must be zero, as desired. \square

We obtain the following general observation on restricting t-structures.

Corollary 6.3. *Let A be a finite dimensional algebra over a field k , and let $(\mathcal{X}, \mathcal{Y})$ be an intermediate t-structure in $D^b(A)$ with heart \mathcal{H} . Then there is a unique t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ that restricts to $(\mathcal{X}, \mathcal{Y})$ if and only if \mathcal{H} is equivalent to $\text{mod}(\Lambda)$ for a finite dimensional k -algebra Λ .*

Proof. First, let (\mathbf{X}, \mathbf{Y}) be the unique t-structure in $D(A)$ restricting to $(\mathcal{X}, \mathcal{Y})$. It follows that (\mathbf{X}, \mathbf{Y}) arises from $(\mathcal{X}, \mathcal{Y})$ via the two different ways of lifting discussed in Section 3 and Theorem 6.2. Thus, we can use Proposition 6.1 to conclude that \mathcal{H} is equivalent to $\text{mod}(\Lambda)$ for a finite dimensional k -algebra Λ . Conversely, suppose that $\mathcal{H} \cong \text{mod}(\Lambda)$ for a finite dimensional k -algebra Λ . We consider the t-structure (\mathbf{X}, \mathbf{Y}) in $D(A)$ obtained from the lifting procedure in Section 3. Again by Proposition 6.1, it follows that

$$(\mathbf{X}, \mathbf{Y}) = (T^{\perp_{>0}}, T^{\perp_{\leq 0}})$$

for a compact silting object T . As in the proof of Theorem 6.2, we will make use of the derived Nakayama functor ν and the natural isomorphism

$$\mathfrak{D} \text{Hom}_{D(A)}(M, N) \cong \text{Hom}_{D(A)}(N, \nu M)$$

for M in $K^b(\text{proj}(A))$ and N in $D(A)$. It follows that $T^{\perp_{>0}} = {}^{\perp_{\leq 0}}C$ for $C := \nu T[-1]$. Indeed, for an object X in $D(A)$ we have

$$\begin{aligned} X \in T^{\perp_{>0}} &\Leftrightarrow \text{Hom}_{D(A)}(T, X[n > 0]) = 0 \\ &\Leftrightarrow \text{Hom}_{D(A)}(X[n > 0], \nu T) = 0 \\ &\Leftrightarrow \text{Hom}_{D(A)}(X, \nu T[-1][n \leq 0]) = 0 \Leftrightarrow X \in {}^{\perp_{\leq 0}}C. \end{aligned}$$

Similarly, we have $T^{\perp_{\leq 0}} = {}^{\perp_{>0}}C$ and, thus, (\mathbf{X}, \mathbf{Y}) is the cosilting t-structure with respect to the cosilting complex C in $K^b(\text{inj}(A))$. Now, let $(\mathbf{X}', \mathbf{Y}')$ be any t-structure in $D(A)$ that restricts to $(\mathcal{X}, \mathcal{Y})$. Since T belongs to $\mathcal{X} \subset \mathbf{X}'$ and C lies in $\mathcal{Y} \subset \mathbf{Y}'$, we get $\mathbf{X} \subset \mathbf{X}'$ and $\mathbf{Y} \subset \mathbf{Y}'$ showing that $(\mathbf{X}, \mathbf{Y}) = (\mathbf{X}', \mathbf{Y}')$, as desired. \square

Corollary 6.3 and its proof show that the two presented ways of lifting t-structures in $D^b(A)$, in general, provide us with distinct classes of t-structures in $D(A)$. In fact, the heart of every t-structure in $D(A)$ that can be obtained via both lifting procedures is necessarily equivalent to a module category.

Moreover, note that part (2) of Theorem 6.2 is a weaker dual of the generation property of \mathcal{H} in Theorem 3.2(2), which was essential to describe the cosilting t-structures we obtain from lifting.

In general, we do not know how to classify the silting t-structures appearing in Theorem 6.2. One of the reasons is that cogenerating t-structures from a given set of objects seems to be a far more delicate task than generating t-structures. In particular, we cannot expect to simply replace homotopy colimits by homotopy limits to prove dual results to those before. In the case of HRS-t-structures, however, we can provide a complete classification. To make this precise, we need to introduce further terminology. Recall that a subcategory \mathbf{C} of $\text{Mod}(A)$ is called definable if it is closed under products, direct limits and pure submodules. Moreover, a submodule is called pure if the induced short exact sequence remains exact after tensoring with any other A -module. Note that for a torsion pair (\mathbf{T}, \mathbf{F}) in $\text{Mod}(A)$, the class \mathbf{T} is definable if and only if it is closed under products and pure submodules, and the class \mathbf{F} is definable if and only if it is closed under direct limits, i.e. the torsion pair is of finite type. We are now ready to state the result, which is dual to Proposition 5.1.

Proposition 6.4. *Let A be a finite dimensional algebra over a field k . By assigning to an intermediate t-structure $(\mathcal{X}, \mathcal{Y})$ in $D^b(A)$ (respectively, to a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}(A)$) the t-structure $({}^\perp\mathcal{Y}, ({}^\perp\mathcal{Y})^\perp)$ in $D(A)$ (respectively, the torsion pair $({}^\perp\mathcal{F}, ({}^\perp\mathcal{F})^\perp)$ in $\text{Mod}(A)$), we get a commutative square of bijections as follows.*

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{t-structures } (\mathcal{X}, \mathcal{Y}) \text{ in } D^b(A) \text{ with} \\ D^b(A)^{\leq -1} \subseteq \mathcal{X} \subseteq D^b(A)^{\leq 0} \end{array} \right\} & \xleftrightarrow{1:1} & \left\{ \text{torsion pairs } (\mathcal{T}, \mathcal{F}) \text{ in } \text{mod}(A) \right\} \\
 \begin{array}{c} \downarrow \text{cogeneration} \\ \uparrow \text{restriction} \end{array} & & \begin{array}{c} \downarrow \text{cogeneration} \\ \uparrow \text{restriction} \end{array} \\
 \left\{ \begin{array}{l} \text{silting t-structures } (\mathbf{X}, \mathbf{Y}) \text{ in } D(A) \\ \text{with } D(A)^{\leq -1} \subseteq \mathbf{X} \subseteq D(A)^{\leq 0} \end{array} \right\} & \xleftrightarrow{1:1} & \left\{ \begin{array}{l} \text{torsion pairs } (\mathbf{T}, \mathbf{F}) \text{ in } \text{Mod}(A) \\ \text{with } \mathbf{T} \text{ definable} \end{array} \right\}
 \end{array}$$

Proof. The upper horizontal bijection was already discussed in Section 5. Moreover, the lower horizontal bijection is a consequence of the fact that, in our setup, a torsion class \mathbf{T} in $\text{Mod}(A)$ is definable if and only if it is a silting class with respect to a silting A -module, which in turn is equivalent to stating that the associated HRS-t-structure is silting (see [5, Corollary 3.8] and [6, Section 4.2]). The right vertical map assigning $({}^\perp\mathcal{F}, ({}^\perp\mathcal{F})^\perp)$ to $(\mathcal{T}, \mathcal{F})$ is well-defined by [14, Section 2.3, Example 2]. Since in our setup all torsion pairs in $\text{Mod}(A)$ restrict to $\text{mod}(A)$, restriction yields a left inverse. Let us show that these two maps are bijections. It suffices to check that a torsion pair (\mathbf{T}, \mathbf{F}) in $\text{Mod}(A)$ with \mathbf{T} definable is of the form $({}^\perp\mathcal{F}, ({}^\perp\mathcal{F})^\perp)$, where $(\mathcal{T}, \mathcal{F}) := (\mathbf{T} \cap \text{mod}(A), \mathbf{F} \cap \text{mod}(A))$ denotes the restricted torsion pair in $\text{mod}(A)$. By construction, we know that $({}^\perp\mathcal{F})^\perp \subseteq \mathbf{F}$. It remains to check that ${}^\perp\mathcal{F} \subseteq \mathbf{T}$ as well. Let X be an object in ${}^\perp\mathcal{F}$. By [14, Section 2.2, Example 3], we can realize X as a pure subobject of $\prod X_i$, where X_i runs through all finite dimensional quotients of X . By assumption, all the X_i belong to \mathcal{T} . As $\mathcal{T} \subseteq \mathbf{T}$ and \mathbf{T} is definable, both $\prod X_i$ and X belong to \mathbf{T} as well. Finally, it is easy to see that the left vertical map in the diagram above, given by Theorem 6.2, makes the square commutative, hence it is a bijection, and its inverse is clearly given by restriction. This finishes the proof. \square

ACKNOWLEDGMENTS

The authors are grateful to Michal Hrbek, Simone Virili, and Jorge Vitória for helpful discussions on different parts of the presented material. The authors also thank the anonymous referee for

their careful reading and valuable comments. The first named author was supported by a post-doctoral fellowship of the Baden-Württemberg Stiftung. Both authors acknowledge support from DFG (Deutsche Forschungsgemeinschaft) through a scientific network on silting theory.

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The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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