STABILITY OF EINSTEIN METRICS ON HOMOGENEOUS SPACES

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Vorgelegt von:

Paul Valentin Schwahn

Geb. in Tübingen

Hauptberichter:
Prof. Dr. Uwe Semmelmann (Universität Stuttgart)

Mitberichter:
Prof. Dr. Christoph Böhm (WWU Münster)
Prof. Dr. Emilio Lauret (Universidad Nacional del Sur, Argentinien)

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Institut für Geometrie und Topologie
Universität Stuttgart

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"Habe nun, ach! Mathematik –
Geometrie, Analysis –
Und leider auch etwas Physik
Durchaus studiert, mit Mut und Biss.
Da steh ich nun, ich armer Tor!
Und bin so klug als wie zuvor;
Heiße Master, heiße Doktor gar
Und ziehe schon an die neun Jahr
Herauf, herab und quer und krumm
Die stuvus[1] *an der Nas herum."*
Frei nach Johann Wolfgang von Goethe (Faust I).

This is the place and time to say *thank you*. But where to start? Let us sort this out chronologically.

First, I would like to thank my parents, Hartmut and Susanne, for establishing my existence in 1999 and relentlessly supporting me throughout my mathematical career.

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[1] Studierendenvertretung Universität Stuttgart

Spezi and Markgrafenbräu (depending on the time of day).

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Now, let us dive in.

Abstract

Einstein metrics originally arose in the context of General Relativity when considering the field equations of gravitation in a vacuum, but have since been noted by Riemannian geometers to be excellent objects of study. This is because the Einstein condition – i.e. the constancy of Ricci curvature – is an ideal intermediate condition between the very restrictive setting of constant sectional curvature on the one hand, only allowing for spherical, Euclidean and hyperbolic geometry, and the very lax condition of constant scalar curvature on the other hand, being achievable by conformal transformations from any given Riemannian metric.

It is possible to characterize Einstein metrics as the critical points of the Einstein–Hilbert functional S restricted to metrics of a fixed volume. This raises the question of what the critical point type for a given Einstein metric g is. As it stands, this question is swiftly answered: it turns out that g is always a saddle point. Only after restricting to the subset of constant scalar curvature metrics, g may (or may not) turn out to be *stable*, i.e. a local maximum of S. The second variation of S at g allows in any case for a finite coindex and nullity. The behavior of S is, at least to second order, governed by the spectral properties on an elliptic Laplace-type differential operator on symmetric 2-tensors, called the Lichnerowicz Laplacian.

Closely related to stability is the rigidity problem – that is, whether a given Einstein metric g is isolated in the moduli space $\mathscr{E}$ of Einstein metrics. The null directions of S correspond to infinitesimal perturbations of the metric, and it is a priori not clear whether a given such perturbation is in fact integrable, that is, tangent to a proper curve in $\mathscr{E}$.

Both problems are particularly interesting in the presence of symmetry, for example on a homogeneous space. The Einstein equation, being an elliptic partial differential equation, reduces for invariant metrics on homogeneous spaces to a polynomial equation. Likewise, the homogeneous structure allows reducing many analytic problems to problems in representation theory, owing to the applicability of harmonic analysis. The Lichnerowicz Laplacian, an invariant differential operator, may be analyzed in the Fourier image, opening the door to an algebraic treatment of the stability problem on homogeneous spaces.

The rigidity problem for homogeneous Einstein metrics is tied to the Finiteness Conjecture which states that there are essentially only finitely many solutions of the invariant

Einstein equations on any compact homogeneous space. Given an infinitesimal perturbation of the metric, there are polynomial obstructions to integrability which may, in the homogeneous setting, also be tackled using representation-theoretic methods.

The idea of employing harmonic analysis to study the stability and deformability of homogeneous Einstein manifolds was first brought forward by Koiso in the 1980s, who settled the stability question for most of the irreducible symmetric spaces of compact type. He left open a few gaps which were only recently closed, partially by the results of the current thesis and partially by Semmelmann–Weingart.

The symmetric setting is particularly nice since the Licherowicz Laplacian coincides with a natural representation-theoretic object called the Casimir operator. This fails to hold if the space under consideration is not symmetric. However, for the much broader class of normal homogeneous spaces, Casimir operators are still readily available. As a first expedition into the non-symmetric world, the current thesis studies homogeneous Gray manifolds, that is, strict nearly Kähler manifolds in dimension six. Here, the Lichnerowicz Laplacian is still more or less explicitly related to a Casimir operator. The coindex of S as well as the rigidity are completely determined for the known list of homogeneous Gray manifolds.

For other normal homogeneous Einstein manifolds such an explicit relation has not been available. At the heart of the current work lies a new explicit formula for the Lichnerowicz Laplacian in terms of various Casimir operators, which is of course considerably more complicated than in the symmetric setting. This is subsequently used to systematically analyze the stability of normal homogeneous spaces – in particular, using an estimation-based approach and with computer aid we obtain explicit results for many normal homogeneous spaces with simple transitive group, these having previously been classified by Wang, Wolf and Ziller.

An interesting member of the class of normal homogeneous spaces is the generalized Wallach space $\mathrm{E}_7/\,\mathrm{PSO}(8)$. As one of the few generalized Wallach spaces that have been observed by E. Lauret, J. Lauret and Will to be invariantly stable in a more restrictive sense, the current thesis shows its stability by analyzing a certain curvature operator. This is notable not only because it provides the first known example of a non-symmetric stable Einstein metric of positive scalar curvature, but also because the curvature term and the Lichnerowicz Laplacian are much more elusive than in the symmetric case: calculating their eigenvalues is in general very difficult, forcing us to resort to estimates that are not always sharp.

Zusammenfassung

Der Begriff der Einstein-Metrik stammt ursprünglich aus der Allgemeinen Relativitätstheorie und bezeichnet dort eine Lösung der Feldgleichungen der Gravitation im Vakuum. Seitdem erwiesen sich Einstein-Metriken auch im Kontext der Riemannschen Geometrie als exzellente Studienobjekte. Dies rührt daher, dass die Einstein-Bedingung – also die Eigenschaft konstanter Ricci-Krümmung – einen idealen Mittelweg bildet zwischen der sehr starren Bedingung konstanter Schnittkrümmung, welche im Wesentlichen zur sphärischen, euklidischen oder hyperbolischen Geometrie führt, und der eher lockeren Bedingung konstanter Skalarkrümmung, von welcher jede beliebige Riemannsche Metrik nur eine konforme Transformation entfernt ist.

Einstein-Metriken können als kritische Punkte des Einstein–Hilbert-Funktionals S, eingeschränkt auf Metriken festen Volumens, charakterisiert werden. Dies wirft die Frage auf, was für eine gegebene Einstein-Metrik g der Typ des kritischen Punktes ist. Die so gestellte Frage lässt sich leicht beantworten: wie sich herausstellt, ist g stets ein Sattelpunkt. Schränken wir S jedoch auf die Teilmenge der Metriken konstanter Skalarkrümmung ein, dann kann es passieren, dass g ein lokales Maximum ist. In jedem Falle lässt die zweite Variation von S an der Stelle g nur endlichen Koindex und Nullität zu. Das Verhalten von S ist zumindest zur zweiten Ordnung durch die spektralen Eigenschaften eines gewissen elliptischen Differentialoperators vom Laplace-Typ auf symmetrischen 2-Tensoren bestimmt, dem sogenannten Lichnerowicz–Laplace-Operator.

Eng verwoben mit der Stabilität ist das Problem der Starrheit – das heißt, ob eine gegebene Einstein-Metrik g im Modulraum $\mathscr{E}$ der Einstein-Metriken isoliert ist. Die Nullrichtungen von S entsprechen infinitesimalen Störungen der Metrik und es ist a priori nicht klar, ob eine solche Störung integrabel ist, also ob man sie als tangential zur einer echten Kurve in $\mathscr{E}$ verstehen kann.

Beide Probleme sind von besonderem Interesse, wenn Symmetrien ins Spiel kommen, wie zum Beispiel auf homogenen Räumen. Die Einstein-Gleichung, welche ja eine elliptische partielle Differentialgleichung ist, reduziert sich für invariante Metriken auf homogenen Räumen auf eine polynomielle Gleichung. Ähnlich erlaubt die homogene Struktur, viele analytische Probleme auf darstellungstheoretische Probleme herunterzubrechen, was letztlich der Anwendbarkeit harmonischer Analysis zu verdanken ist. Der Lichnerowicz–Laplace-Operator kann als invarianter Differentialoperator im Fourier-Bild analysiert wer-

den, was Tür und Tor zu einer algebraischen Betrachtung des Stabilitätsproblems für homogene Räume öffnet.

Das Starrheitsproblem für homogene Einstein-Mannigfaltigkeiten hängt überdies mit der Endlichkeitsvermutung zusammen, welche besagt, dass es auf jedem kompakten homogenen Raum im Wesentlichen höchstens endlich viele Lösungen der invarianten Einstein-Gleichungen gibt. Zu jeder infinitesimalen Deformation der Metrik gibt es polynomielle Obstruktionen gegen die Integrabilität, welche im homogenen Fall auch mit darstellungstheoretischen Methoden untersucht werden können.

Die Idee, harmonische Analysis zur Untersuchung der Stabilität und Starrheit homogener Räume einzusetzen, wurde erstmalig von Koiso in den 1980er-Jahren vorgebracht, welcher die Stabilitätsfrage für die meisten irreduziblen symmetrischen Räume kompakten Typs klärte. Er ließ allerdings ein paar Lücken, welche erst vor Kurzem geschlossen werden konnten – teils durch diese Dissertation, teils durch die Ergebnisse von Semmelmann–Weingart.

Der symmetrische Fall ist besonders angenehm, da der Lichnerowicz–Laplace-Operator dort mit einem natürlichen darstellungstheoretischen Objekt zusammenfällt – dem Casimir-Operator. Dies gilt nicht mehr, wenn der Raum nicht symmetrisch ist. Für die größere Klasse von normalen homogenen Räumen bieten sich Casimir-Operatoren jedoch immer noch an. Als erste Expedition in die nicht-symmetrische Welt untersucht die vorliegende Dissertation homogene Gray-Mannigfaltigkeiten, das heißt, strikte *nearly-Kähler*-Mannigfaltigkeiten in Dimension sechs. Der Lichnerowicz–Laplace-Operator steht hier immer noch mehr oder weniger explizit in Relation mit einem Casimir-Operator. Der Koindex von S sowie die Starrheit werden für die bekannte Liste homogener Gray-Mannigfaltigkeiten vollständig bestimmt.

Eine entsprechend explizite Relation gab es für allgemeine normale homogene Einstein-Mannigfaltigkeiten bisher nicht. Das Herzstück dieser Arbeit bildet eine neue Formel, welche den Lichnerowicz–Laplace-Operator durch verschiedene Casimir-Operatoren ausdrückt. Diese ist natürlich deutlich komplizierter als im symmetrischen Fall. Im Weiteren wird diese genutzt, um systematisch die Stabilität normaler homogener Räume zu analysieren – insbesondere erhalten wir durch einen auf Abschätzungen basierenden Ansatz und mit Computerunterstützung explizite Ergebnisse für viele normal homogene Räume mit einfacher transitiver Gruppe. Letztere wurden zuvor von Wang, Wolf und Ziller klassifiziert.

Ein interessanter normaler homogener Raum ist der verallgemeinerte Wallach-Raum $\mathrm{E}_7/\mathrm{PSO}(8)$. Er ist einer der wenigen verallgemeinerten Wallach-Räume, die von E. Lauret, J. Lauret und Will als stabil in einem restriktiveren, invarianten Sinne identifiziert wurden. In der vorliegenden Dissertation zeigen wir seine Stabilität, indem ein gewisser Krümmungsoperator analysiert wird. Dies ist aus zwei Gründen bemerkenswert: erstens, weil damit das erste bekannte Beispiel einer nicht-symmetrischen stabilen Einstein-Metrik positiver Skalarkrümmung gefunden ist, und zweitens, weil der Krümmungsterm und der Lichnerowicz–Laplace-Operator sich einer einfachen Behandlung wie im symmetrischen Fall entziehen: die Berechnung ihrer Eigenwerte ist im Allgemeinen schwierig, was uns dazu zwingt, Abschätzungen einzusetzen, die nicht immer scharf sein müssen.

Preface

By the nature of a publication-based dissertation, it is bound to contain some amount of redundancy. Each of the contributed articles [Sch22a; Sch22b; SSW22; Sch23] contains its own introduction and a section where the necessary preliminaries are discussed in appropriate technical detail, often overlapping contentwise. Moreover, the contributed articles were in their original form aimed at an audience of readers proficient in the field. Of course a doctoral thesis should give a more comprehensive account of the background than a mere collection of research articles usually does.

In Chapter 1 my aim is to bridge that gap, to go beyond a boiled-down version of the introductory parts of the contributed articles and instead tell a consistent story, beginning with the historical motivation for Einstein manifolds and then steadily building up the background knowledge and context required to appreciate the new results. For the sake of story-telling I will for the most part refrain from regurgitating known proofs and getting tangled up in technicalities, supplying comprehensive references instead. This shall, however, not prevent me from spelling out the necessary terms and notions as precisely as possible, so as not to lapse into handwaving.

Chapter 2 will give an overview over the methods and strategies used in the contributed articles, the hurdles that arose and how they were overcome. It will also put the articles in the contemporary research context, explaining the motivation for their precise objects of study and give an exposition of the new results – all that not without paying respect to the earlier work they have built on. Moreover, an outlook on potential further developments, open questions and interesting cases to study next is provided.

Owing to the requirements on the dissertation, the introductory part is closely tied to the contributed articles and necessarily limited in scope. It does thus not aspire to present a complete survey of the by now quite broad topics of stability of Einstein metrics and of homogeneous Einstein manifolds. It may, however, serve as a loose *leitfaden* to those that are unacquainted with the subject, presupposing only that the reader has acquired basic knowledge and interest in Riemannian geometry and perhaps also representation theory – although the latter may be *black-boxed* in large parts. Ideally, Chapters 1 and 2 explain everything (or at least something!) that the contributed articles do not.

Having prepared the stage, the remaining chapters constitute the cumulative part of the dissertation, that is, a reproduction of the contributed articles. We note that the last of the four contributed articles [Sch23] has, as of today, not undergone peer review – we thus expect it to be subject to a certain amount of change during its publication process.

Contents

1 Introduction

"Dem Zauber dieser Theorie wird sich kaum jemand entziehen können, der sie wirklich erfaßt hat: sie bedeutet einen wahren Triumph der durch Gauss, Riemann, Christoffel, Ricci und Levi-Civiter [sic] *begründeten Methode des allgemeinen Differentialkalküls."*
Albert Einstein (Zur allgemeinen Relativitätstheorie [Ein15b]).

1.1 Physical motivation

Our story opens in 1915 with Einstein first proposing the field equations of gravitation

$$\operatorname{Ric}_g - \frac{1}{2} \operatorname{scal}_g g = T \tag{1.1}$$

in a curved spacetime, thus putting the theory of general relativity on a solid foundation [Ein15a]. Here g stands for the "gravitational potential" – in other words a pseudo-Riemannian metric with Lorentzian signature – which determines the curvature quantities Ric_g (Ricci curvature) and scal_g (scalar curvature), and T denotes the stress-energy tensor of matter. Putting $T = 0$ in (1.1) we obtain the *vacuum field equations*. These are equivalent to g being Ricci-flat, that is $\operatorname{Ric}_g = 0$.

In the same year Hilbert showed [Hil15] that Einstein's field equations emerge via the *principle of stationary action* as the Euler–Lagrange equation of the total scalar curvature functional

$$S(g) = \int \operatorname{scal}_g \operatorname{vol}_g,$$

today appropriately called the *Einstein–Hilbert action*. More concretely: Equation (1.1) with $T = 0$ describes precisely the solutions of the variational problem $\delta S = 0$.

In order to meet the assumption that the universe does not expand over time, Einstein added a "cosmological term" to (1.1) [Ein17], leading to the modified field equations

$$\operatorname{Ric}_g - \frac{1}{2} \operatorname{scal}_g g + Eg = T \tag{1.2}$$

with *cosmological constant* $E \in \mathbb{R}$. Folklore has it that Einstein later referred to the introduction of E as the "greatest blunder" of his life, for at least since the observations of Hubble in 1929 the postulate of a static universe was no longer tenable. Only many decades later did the cosmological term prove to be of great importance for modern cosmology.

If we restrict the Einstein–Hilbert functional S to metrics with a fixed volume, the variational problem $\delta S = 0$ leads precisely to the field equations with cosmological term (1.2) in vacuum ($T = 0$), which can be further simplified to

$$\mathrm{Ric}_g = Eg. \tag{1.3}$$

A pseudo-Riemannian metric g satisfying condition (1.3) is called an *Einstein metric*, and E is also called the *Einstein constant* of g. Applying the trace tr_g it follows immediately that $E = \frac{\mathrm{scal}_g}{n}$, where n denotes the dimension of the underlying manifold. In particular Einstein metrics have constant scalar curvature. In fact (1.3) is in dimension $n \geq 3$ equivalent to the vanishing of the trace-free part of Ric_g, that is

$$\mathrm{Ric}_g - \frac{\mathrm{scal}_g}{n} g = 0. \tag{1.4}$$

In the case of Lorentzian signature, (1.4) is a *hyperbolic* partial differential equation. A usual approch to this type of problem is to specify Cauchy initial data on a suitable spacelike hypersurface and view (1.4) as an evolution equation.

In contrast, we will exclusively deal with *Riemannian* (i.e. positive definite) metrics. Furthermore we will assume that the underlying space is compact, so that integration does not cause any difficulties. Equation (1.4) is then an *elliptic* partial differential equation. In order to study Einstein metrics we can and will exploit the full power of elliptic theory.

It should be mentioned that the Einstein condition (1.3) presents a reasonable notion of "constant Ricci curvature". This can be illustrated as follows: Let $SM \subset TM$ denote the sphere bundle of unit tangent vectors of some Riemannian manifold (M, g). The Ricci tensor Ric_g now defines a quadratic form $r : TM \to \mathbb{R}$ by $r(X) := \mathrm{Ric}_g(X, X)$. Then r is constant on SM if and only if g is an Einstein metric.

1.2 The Einstein–Hilbert action

In this and the next two sections, before we come to the special case of homogeneous manifolds, we will briefly lay out the groundwork for what is to come. A comprehensive and detailed account of the following topics is available in [Bes87].

Let us first agree on the setting. Let M^n be a closed and oriented manifold which we will assume to be of dimension $n \geq 3$. (On surfaces there is no traceless part of Ric_g and the Einstein condition is equivalent to the constancy of the Gaussian curvature.)

The set $\mathscr{M}$ of Riemannian metrics on M is a convex open cone (in the compact-open topology) inside the space $\mathscr{S}^2(M) = \Gamma(\operatorname{Sym}^2 T^*M)$ of covariant symmetric 2-tensor fields. The total scalar curvature functional (or Einstein–Hilbert action) does what its name says: it maps

$$S: \ \mathscr{M} \to \mathbb{R}, \qquad g \mapsto \int_M \operatorname{scal}_g \operatorname{vol}_g .$$

Denote with

$$\mathscr{M}_1 = \{g \in \mathscr{M} \mid \textstyle\int_M \operatorname{vol}_g = 1\}$$

the subset of volume one metrics. By a theorem of Hilbert [Bes87, Thm. 4.21], Einstein metrics in $\mathscr{M}_1$ are precisely the critical points of the restricted functional $S\big|_{\mathscr{M}_1}$. In terms of the first variation (or gradient)

$$S'_g: \ T_g\mathscr{M}_1 \to \mathbb{R}$$

at g, this means that $S'_g(h) = 0$ for all $h \in T_g\mathscr{M}_1$. Here

$$T_g\mathscr{M}_1 = \{h \in \mathscr{S}^2(M) \mid \textstyle\int_M (\operatorname{tr}_g h) \operatorname{vol}_g = 0\}$$

is the formal tangent space of $\mathscr{M}_1$ at g.

At this point we should talk about what “critical point”, or more precisely, “taking a derivative” means inside $\mathscr{M}_1$. Certainly one can take directional derivatives since $\mathscr{M}_1$ is a subset of the topological (even Fréchet!) vector space $\mathscr{S}^2(M)$, which is sufficient for defining formal tangent spaces and critical points. However, we have even more: $\mathscr{M}_1$ is an ILH-manifold (that is, modeled on an inverse limit of Hilbert spaces, a notion introduced by Omori [Omo68]), which yields a version of the implicit function theorem.

Suppose now that $g \in \mathscr{M}_1$ is Einstein. One may ask under which conditions it is possible for g to be a local maximum or minimum of S – that is, whether g is *stable* for the Einstein–Hilbert action. But alas, it turns out that this is too much to ask for at this stage. We will need to pass to a more refined setting.

First, let us remark that S is invariant under diffeomorphisms – that is

$$S(g) = S(\varphi^* g)$$

for any $g \in \mathscr{M}$ and $\varphi \in \operatorname{Diff}(M)$. Hence S'_g will annihilate all directions tangent to the orbit $\operatorname{Diff}(M)g \subset \mathscr{M}_1$, i.e. the space $L_{\mathfrak{X}(M)}g \subset T_g\mathscr{M}_1$ of Lie derivatives of g. We can think of $\operatorname{Diff}(M)$ as a gauge group that should be “quotiented out”. This is in fact justified by Ebin’s Slice Theorem [Ebi68] which essentially states that the quotient $\mathscr{M}/\operatorname{Diff}(M)$ can locally be treated as a manifold.

Instead of $\mathscr{M}_1$, consider the ILH-submanifold $\mathfrak{S}$ of unit volume metrics with constant scalar curvature. Tangent to it is the space

$$\mathscr{S}^2_{\mathrm{tt}}(M) = \{h \in \mathscr{S}^2(M) \mid \operatorname{tr}_g h = 0,\ \delta_g h = 0\}$$

of *tt-tensors* (short for *transverse* and *traceless*), the term tracing back to General Relativity where it describes a certain canonical gauge for small perturbations of a background metric. This is no coincidence – by the Slice Theorem and the decomposition

$$\mathscr{S}^2(M) = L_{\mathfrak{X}(M)} g \oplus \ker \delta_g,$$

the requirement $\delta_g h = 0$ does precisely the job of fixing a gauge with respect to $\mathrm{Diff}(M)$.

Let $C^\infty_g(M)$ denote the set of smooth functions on M with average zero. A theorem due to Berger and Koiso states that, under the generic assumption that (M, g) is not isometric to a round sphere, there is a decomposition

$$T_g \mathscr{M}_1 = C^\infty_g(M) g \oplus L_{\mathfrak{X}(M)} g \oplus \mathscr{S}^2_{\mathrm{tt}}(M), \tag{1.5}$$

which is orthogonal with respect to the second variation (or Hessian) S''_g of the total scalar curvature functional [Bes87, Thm. 4.60]. This allows us to discuss S''_g separately on each component. Moreover the second and third summand in (1.5) constitute together the formal tangent space $T_g \mathfrak{S}$.

We will not look under the hood, i.e. at the proof, of this result. However it shall be said that in order to prove decomposition (1.5) one needs to apply elliptic theory to various differential operators as well as employ the ILH implicit function theorem for $\mathfrak{S}$.

It turns out that S''_g is positive definite when restricted to $C^\infty_g(M) g$, implying that Einstein metrics are local minima of S under (volume-preserving) conformal change. As per a previous remark the summand $L_{\mathfrak{X}(M)} g$ is contained in the null space of S''_g. Finally, on the infinitesimal space of tt-tensors, S''_g is *almost* negative definite in the sense that it has only finite coindex and nullity. This is a consequence of the formula

$$S''_g(h, h) = -\frac{1}{2} \left(\Delta_L h - 2Eh, h\right)_{L^2}, \qquad h \in \mathscr{S}^2_{\mathrm{tt}}(M), \tag{1.6}$$

involving the peculiar *Lichnerowicz Laplacian* Δ_{L}, a self-adjoint elliptic operator with spectrum accumulating at positive infinity.

This answers our original question about the nature of critical points g of S on $\mathscr{M}_1$ – all of them are saddle points! The problem gets more interesting once we restrict S to the space $\mathfrak{S}$ of constant scalar curvature metrics, where we can have both saddle points and local maxima. The latter is what is usually meant when people talk about *stable* Einstein metrics.

The *linearized* stability problem is thus decided by the behaviour of S''_g on tt-tensors. An Einstein metric g is called *linearly (strictly) stable* if S''_g is indeed negative definite on $\mathscr{S}^2_{\mathrm{tt}}(M)$, and *unstable* if $S''_g(h,h) > 0$ for some $h \in \mathscr{S}^2_{\mathrm{tt}}(M)$ (such a tt-tensor h is called a *destabilizing direction*). Some authors only require semidefiniteness for linear stability, while others call Einstein metrics that satisfy this weaker condition *semistable*.

What about null directions for S''_g in $\mathscr{S}^2_{\mathrm{tt}}(M)$? These are called the *essential infinitesimal deformations* (EID) of an Einstein metric g and can be thought of as potential first-order jets to a genuine curve of Einstein metrics. We will address this issue of deformability of Einstein metrics in §1.4.

We emphasize that semistability is not sufficient for g to be a local maximum of $S\big|_{\mathscr{M}_1}$. Indeed, two counterexamples are the symmetric spaces $\mathrm{SU}(n)$, $n \geq 3$, and $\mathrm{SU}(2n)/\operatorname{Sp}(n)$, $n \geq 2$ [LW22a], see also [BWZ04, Ex. 6.8] for the latter.

It shall not go unmentioned that the investigation of stability in the Einstein–Hilbert sense is also motivated by the close relation to stability under the Ricci flow. The Ricci flow, a much-praised and important tool in Riemannian geometry and geometric topology, is a dynamical system on $\mathscr{M}$ defined by the evolution equation

$$\frac{d}{dt} g_t = -2 \operatorname{Ric}_{g_t} \tag{1.7}$$

for a curve $g_t \in \mathscr{M}$. The critical points of this flow are Ricci-flat metrics, while Einstein metrics with $E \neq 0$ correspond to shrinking or expanding solutions (depending on the sign of E). An even larger class of self-similar solutions to (1.7) are the *gradient Ricci solitons* defined by the equation

$$\operatorname{Ric}_g + \nabla^2 f = \frac{1}{2\tau} g$$

where $f \in C^\infty(M)$ is some potential function (whose gradient flow consists of diffeomorphisms that realize the aforementioned self-similarity up to scaling). In order to study the shrinking case ($\tau > 0$), Perelman introduced a functional $\nu : \mathscr{M} \to \mathbb{R}$ (called the *ν-entropy*) that is monotonically increasing under the Ricci flow and whose critical points are exactly the (shrinking) gradient Ricci solitons [Per02]. Remarkably, this functional turns out to have quite similar properties to the Einstein–Hilbert action. As shown by Cao–Hamilton–Ilmanen [CHI04], if (M,g) is an Einstein manifold not isometric to a round sphere, then decomposition (1.5) is still orthogonal for the second variation ν''_g – moreover ν''_g and S''_g coincide on $L_{\mathfrak{X}(M)} g \oplus \mathscr{S}^2_{\mathrm{tt}}(M)$, and ν''_g is negative definite on $C^\infty_g(M) g$ if and only if the first nonzero eigenvalue of the Laplace–Beltrami operator Δ on $C^\infty(M)$ is greater than $2E$. This *ν-linear stability* of an Einstein metric is thus a slightly stronger condition than linear stability in the Einstein–Hilbert sense.

It should be emphasized that the ν-functional is not just some arbitrary entropy quantity, but captures the dynamical behaviour of the Ricci flow quite well. This is illustrated in a result of Kröncke which states that any Einstein metric g of positive scalar curvature is *dynamically stable* under the Ricci flow if and only if g locally maximizes the ν-entropy [Krö15]. This dynamical stability (in the sense of Sesum [Ses06]) roughly means that any arbitrary metric close to g will, under (1.7), eventually flow towards some Einstein metric near g. As a consequence of Kröncke's result, ν-linear strict stability implies dynamical stability, and ν-linear instability implies dynamical instability.

We are thus forced to admit that the study of the Ricci flow may benefit from a better understanding of the S- and thus the ν-functional.

1.3 The Lichnerowicz Laplacian

For now, let us take a closer look at the operator Δ_{L} lying at the heart of the (linearized) stability problem and being the central object of study in the present work. Fix a Riemannian metric g on M, let ∇ denote its Levi–Civita connection and R its Riemannian curvature tensor. On differential forms there is the well-known Weitzenböck formula

$$d^*d + dd^* = \nabla^*\nabla + q(R) \tag{1.8}$$

where $\nabla^*\nabla$ is the *rough* or *Bochner Laplacian*, and $q(R)$ is a fibrewise term called the *standard curvature endomorphism*. As Lichnerowicz realized [Lic61], the right hand side of (1.8) makes sense when applied to tensor fields of arbitrary type, which led to the definition

$$\Delta_{\mathrm{L}} = \nabla^*\nabla + q(R) \tag{1.9}$$

of the Lichnerowicz Laplacian. Even more generally, this formula defines the *standard Laplace operator* on any geometric vector bundle over a Riemannian manifold with metric connection, as introduced by Semmelmann–Weingart in [SW18], where they also discussed its remarkable functorial and commutative properties. We will encounter another such Laplacian in §1.8.

The uncommon term "geometric vector bundle" refers simply to a vector bundle associated to the holonomy-reduced frame bundle via some representation of the holonomy group. This includes all tensor (and spinor!) bundles, but also all holonomy-invariant subbbundles. Homomorphisms of geometric vector bundles are then just vector bundle homomorphisms that are equivariant under the action of the holonomy group, i.e. parallel bundle maps.

Similar to the Hodge–de Rham Laplacian, the Lichnerowicz Laplacian possesses excellent commutation properties in that it commutes not only with all homomorphisms of geometric vector bundles, but also with many important differential operators. In particular Δ_L commutes with both tr_g and δ_g and thus preserves the space of tt-tensors.

By (1.6) the linear stability of a given Einstein metric g lies entirely in the hands of the spectrum of Δ_L – namely, g is stable if and only if $\Delta_\mathrm{L} > 2E$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$, and unstable if and only if $\Delta_\mathrm{L}\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)}$ has an eigenvalue smaller than $2E$. We note that if $(M = M_1 \times M_2, g = g_1 \oplus g_2)$ is a Riemannian product, a tt-tensor in the kernel of Δ_L may be constructed from the *canonical variation* in which the metrics g_1 and g_2 are scaled by different constants. If g is of positive scalar curvature (i.e. $E > 0$), this produces a destabilizing direction.

As a short digression from the Riemannian world, we remark that the Lichnerowicz Laplacian also turns up in the investigation of the gravitational stability of various higher-dimensional spacetimes [GH02]. The condition for this so-called *physical stability* is almost the same as for linear stability in the Einstein–Hilbert sense, albeit with the critical eigenvalue $\frac{9-n}{4}E$ instead of $2E$.

In general the spectrum of Δ_L is difficult to get ahold of. Boucetta managed to compute the spectrum on $\mathscr{S}^2(M)$ for the standard metrics on S^n and $\mathbb{RP}^n$ [Bou99]. For the more general case of compact symmetric spaces there is an astonishing representation-theoretic interpretation of the Lichnerowicz Laplacian which will be discussed in §1.5. However there are some a priori estimates involving only the curvature of (M, g). Let us thus take another look at the curvature term $q(R)$.

On the tangent bundle TM, the endomorphism $q(R)$ simply coincides with the Ricci endomorphism Ric_g. Passing to the bundle of covariant symmetric 2-tensors $\mathrm{Sym}^2 T^*M$, we have instead

$$q(R) = -2\mathring{R} - \mathrm{Der}_{\mathrm{Ric}_g}.$$

Here $\mathring{R} \in \mathrm{End}\,\mathrm{Sym}^2 T^*M$ is the *curvature operator of second kind* introduced by Bourguignon–Karcher [BK78] (along with the one of the first kind, $\widehat{R} \in \mathrm{End}\,\Lambda^2 T^*M$), and $\mathrm{Der}_{\mathrm{Ric}_g}$ denotes the extension of the Ricci endomorphism to $\mathrm{Sym}^2 T^*M$ as a derivation. Using the two beautiful Weitzenböck formulae

$$\delta\delta^* - \delta^*\delta = \Delta_\mathrm{L} - 2q(R), \tag{1.10}$$

$$\delta^\nabla d^\nabla + d^\nabla \delta^\nabla = \Delta_\mathrm{L} - \frac{1}{2}q(R), \tag{1.11}$$

Koiso proved that a sufficient criterion for the linear stability of an Einstein metric g is $\mathring{R} < \max\{-E, E/2\}$ (or equivalently $q(R) > \min\{E, 4E\}$) on traceless 2-tensors [Koi78, Thm. 3.3]. As Fujitani showed [Fuj79], the eigenvalues of $\mathring{R}$ are controlled by the sectional curvature of g. In particular, a suitably pinched or a negative sectional curvature is sufficient for stability (see [Bes87, §12.H] for details).

1.4 Einstein deformations

Let us ask a different question: given some Einstein metric g on a compact orientable smooth manifold M, are there any other Einstein metrics nearby? Obviously one can produce continous deformations by pulling back through a curve of diffeomorphisms or by homothetic scaling. In order to discuss only the interesting directions, we thus quotient out diffeomorphisms and scaling to obtain the moduli space $\mathscr{M}_1/\operatorname{Diff}(M)$ of (unit volume) Riemannian structures on M. The subset of $\mathscr{M}_1/\operatorname{Diff}(M)$ consisting of equivalence classes of Einstein metrics is called the *moduli space of Einstein structures* on M and shall be denoted by $\mathscr{E}$.

Instead of $\mathscr{E}$, it is often easier to work with the *premoduli space* of Einstein structures around g, obtained by applying Ebin's Slice Theorem [Ebi68] in order to take a slice around $g \in \mathscr{M}_1$ to the action of $\operatorname{Diff}(M)$ and considering the Einstein metrics contained therein. However this technicality shall not concern us here, as we may take the freedom to skip all the difficult arguments and showcase only the results.

What can we say about the geometric structure of the moduli space $\mathscr{E}$? Most prominently, this question has been investigated by Koiso and Berger–Ebin [Koi80; BE69]. The space $\mathscr{E}$ may in general not be a manifold, but is at least Hausdorff. Moreover, it is locally arcwise connected (arcs meaning real analytic curves). In particular, since Einstein metrics are critical points of S, it follows that S descends to a locally constant function on $\mathscr{E}$.

Another consequence of the arcwise connectedness is that an Einstein metric is *rigid*, i.e. isolated in the moduli space $\mathscr{E}$, if and only if there exists no real analytic curve through g in $\mathscr{E}$. Suppose, on the other hand, that we are given such a curve (g_t) of Einstein metrics with $g_0 = g$. An equivalent reformulation of the Einstein condition (1.4) is $E(g_t) = 0$, where

$$E:\ \mathscr{M}_1 \to \mathscr{S}^2(M):\ g \mapsto \operatorname{Ric}_g - \frac{S(g)}{n} g$$

denotes the *Einstein operator*. Then its first order jet

$$h = \frac{d}{dt}\Big|_{t=0} g_t$$

satisfies the *linearized Einstein equation*

$$E'_g(h) = \frac{d}{dt}\Big|_{t=0} E(g_t) = 0. \tag{1.12}$$

With the additional requirement that h is infinitesimally volume-preserving (i.e. $h \in T_g\mathscr{M}_1$) and transverse to the orbit of $\operatorname{Diff}(M)$ (i.e. $\delta_g h = 0$), (1.12) is precisely equivalent to the definition of EID in §1.2 – that is, h is a tt-tensor satisfying $\Delta_{\mathrm{L}} h = 2Eh$ [BE69].

We denote the space of EID of g with $\varepsilon(g)$. By the previous discussion we see that a sufficient condition for the rigidity of an Einstein metric g is the vanishing of $\varepsilon(g)$. In particular if g is strictly linearly stable, it must also be rigid.

Pondering the case $\varepsilon(g) \neq 0$, we are led to ask whether any EID $h \in \varepsilon(g)$ actually integrates into a corresponding curve g_t with h as first order jet. As we shall see shortly, there are obstructions to integrability. Consider again a curve (g_t) with $E(g_t) = 0$ and all of its jets

$$h_j = \frac{d^j}{dt^j}\Big|_{t=0} g_t, \qquad j \in \mathbb{N}.$$

It then follows that

$$E_g^{(j)}(h_1, \dots, h_j) = \frac{d^j}{dt^j}\Big|_{t=0} E(g_t) = 0 \tag{1.13}$$

for all $j \in \mathbb{N}$. Note that the $E_g^{(k)}$ are polynomials in $h_1, \dots, h_k$, of degree one in h_k.

This gives countably many obstructions against integrability: an EID $h_1 \in \varepsilon(g)$ is now appropriately called *formally integrable to order k* if there exist $h_2, \dots, h_k$ such that (1.13) is fulfilled for all $j = 2, \dots, k$. Moreover, these obstructions are all independent – there is no single "obstruction space" that we can require to vanish in order for (1.13) to always admit a solution h_j (see [Bes87, §12.E] for a detailed explanation). This puts a damper on our ambitions to easily decide the integrability of a given EID – for if it was in fact integrable, we would have to check countably many polynomial conditions! This is in stark contrast to, for example, the deformation theory of complex structures, where we have only one integrability obstruction (manifesting as the second sheaf cohomology of the sheaf of holomorphic vector fields). At the very least we are granted the mercy that the difficulties are only of this algebraic nature and there are no convergence issues: an EID integrates into a genuine curve of Einstein metrics if and only if it is formally integrable to all orders [Bes87, Cor. 12.50].

In light of this difficulty occurring at the infinitesimal level it is perhaps not surprising that the general structure of $\mathscr{E}$ remains nebulous to this day. For example it is unknown whether $\mathscr{E}$ may actually be singular or whether it is always locally a manifold.

One moral of the deformation theory above is that there is in general no reason to expect EID to be integrable. Indeed, it often suffices to look at the second order obstruction in order to produce examples of Einstein metrics which are infinitesimally deformable but rigid. By a result of Koiso this obstruction can be restated as follows: any $h \in \varepsilon(g)$ is formally integrable to order two if and only if $E_g''(h, h)$ is orthogonal to $\varepsilon(g)$ [Koi82, Lem. 4.7]. It thus seems prudent to investigate the trilinear form

$$\Psi : \ \varepsilon(g) \times \varepsilon(g) \times \varepsilon(g) \to \mathbb{R} : \ (h_1, h_2, h_3) \mapsto \left(E_g''(h_1, h_2), h_3\right)_{L^2} \tag{1.14}$$

which is actually symmetric (shown in [NS23]) and thus can be thought of as a homogeneous cubic polynomial on $\varepsilon(g)$. EID that are integrable to second order would correspond to nonzero critical points of this polynomial.

The earliest example of an Einstein metric which is rigid but infinitesimally deformable is the symmetric space $\mathbb{CP}^1 \times \mathbb{CP}^{2n}$, established by Koiso using the second order obstruction combined with a representation-theoretic argument [Koi82, Thm. 6.12]. Another more recent such example is the bi-invariant metric on $\mathrm{SU}(2n+1)$ due to Batat et al. [BH+21].

A similar discussion applies to the question of *solitonic rigidity*, i.e. whether a given Einstein metric might be deformable through a curve of gradient Ricci solitons, if not Einstein metrics. In this case the infinitesimal deformation space is enlarged by the $2E$-eigenspace of the Laplace–Beltrami operator – the deformation theory, developed by Podestà–Spiro [PS15], plays out similarly. As Kröncke showed, the second order obstruction for these infinitesimal solitonic deformations is again given in terms of a cubic polynomial [Krö16, Thm. 1.2]. This has been used, for example, to prove the solitonic rigidity of the symmetric metric on $\mathbb{CP}^{2n}$ [Krö16, Thm. 6.1]. Very recently, Li–Zhang proved solitonic rigidity of $\mathbb{CP}^{2n+1}$ by taking it one order higher and working out the integrability obstruction to third order [LZ22] – tedious but worth it!

This obstructed deformation theory of Einstein metrics may be a curiosity in the Riemannian world, but it should be mentioned that there is a similar phenomenon in the Lorentzian setting, even though the theory behind it is completely different. The keyword is *linearization stability* of a Lorentzian Einstein metric g (not to be confused with the linear stability from §1.2), which is the property that every infinitesimal perturbation of g can be integrated into a smooth curve of Einstein metrics. At least for the Ricci-flat case, this has first been investigated by Fischer–Marsden [FM73]. As shown by Moncrief, the existence of so-called *spurious solutions* (i.e. nonintegrable infinitesimal deformations) is closely linked to the presence of Killing vector fields on the underlying spacetime [Mon75; Mon76].

1.5 Homogeneous Einstein manifolds

A class of great importance, and the source of many examples in Riemannian geometry, are the *homogeneous manifolds*. To clarify our intent, we call a smooth manifolds M homogeneous if there exists a Lie group G acting transitively on M. For any point $p \in M$ we call the stabilizer G_p the *isotropy group* at p. The homogeneous manifold M can thus be presented as the coset space G/G_p.

We reserve the name *homogeneous space* for the *pair* (G, H) of a Lie group G and a closed subgroup H giving rise to the quotient $M = G/H$. Here H is the isotropy group of the *base point* $o = eH$. In general a homogeneous manifold M may have many different presentations G/H as a homogeneous space. Usually they can be narrowed down by the requirement that G is simply connected and acts almost effectively, and the relaxation that H needs only be determined up to an automorphism of G – both steps preserve the notion of a G-invariant structure on M. But even then there may be several homogeneous spaces with the same underlying manifold. The most prominent examples of this are the spheres, admitting transitive effective actions of various Lie groups, as classified by Montgomery–Samelson [MS43]. This example is of particular significance for Riemannian geometry: it is a crucial ingredient for Berger's classification of holonomy groups of irreducible, simply connected, non-symmetric Riemannian manifolds [Ber55].

The transitive action on a homogeneous space $M = G/H$ simplifies many matters of Riemannian geometry, or just renders them possible to tackle in the first place. First, suppose we restrict our attention to objects which are *invariant* under the G-action. Because the action is transitive, invariant objects are determined by their behavior at the base point. For example G-invariant Riemannian metrics on M correspond in a one-to-one manner to H-invariant inner products on the *isotropy representation* T_oM. Under this correspondence, the Einstein equation (1.4) reduces to a polynomial equation in the structure constants of the Riemannian homogeneous space in question, eliminating the need for hard analysis and leaving only algebraic problems.

Second, even for the description of non-invariant objects we gain a new useful tool: harmonic analysis. This relates objects on (more precisely: sections of homogeneous vector bundles over) M to the representation theory of the transitive group G by the generalized *Fourier transform*. This is especially useful in the compact case, where the set of Fourier modes is countable and any G-module carries an invariant inner product, guaranteeing a decomposition into finite-dimensional irreducible modules. All this is condensed into a classic theorem by Peter–Weyl [Wal73, Thm. 2.8.2]: if G is a compact Lie group, acting on $L^2(G)$ from the left and right by translations, then

$$L^2(G) \cong \overline{\bigoplus_{\gamma\in\hat{G}}} V_\gamma^* \otimes V_\gamma, \tag{1.15}$$

the overline indicating L^2-closure. Here $(V_\gamma)_{\gamma\in\hat{G}}$ is the sequence of equivalence classes of finite-dimensional representations of G. This directly generalizes the concept of a Fourier series. Any invariant operator will respect this decomposition, so "invariant problems" can, in principle, be broken down into their irreducible parts. We shall see in §1.8 how this applies to homogeneous vector bundles and how it helps us close in on the spectrum of the Lichnerowicz Laplacian.

Let us return to the homogeneous Einstein equation. As easy as the promised reduction to algebra in the invariant case may sound, the resulting equation can in actuality still be quite hard to solve explicitly. In low dimensions, some classifications of compact homogeneous Einstein manifolds are known (see [BK06] for an overview). There are also homogeneous spaces which do not admit any invariant Einstein metric [WZ86]. It is, however, still unknown whether any compact simply connected homogeneous space carries only finitely many invariant Einstein metrics up to isometry (which is false in the noncompact setting). This is known today as the *Finiteness Conjecture*.

There are, however, some general structure results on homogeneous Einstein manifolds, in terms of the sign of the Einstein constant E, which we shall recall. If $E = 0$, i.e. if (M, g) is Ricci-flat, then it is in fact flat and thus the product of a flat torus with Euclidean space, as shown by Alekseevskii–Kimelfeld [AK75]. If $E < 0$, then M (and thus G) are noncompact. Indeed, a theorem of Bochner shows that compact Riemannian manifolds with negative Ricci curvature does not admit Killing vector fields and thus there cannot be a transitive group of isometries [Boc46]. Conversely, if $E > 0$, then it follows from Myers' Theorem that M is compact and has finite fundamental group [Mye41]. In this case it is known that the moduli space of G-invariant Einstein metrics has only finitely many components which are all compact [BWZ04].

In the compact setting, the group G can itself be assumed to be compact and semisimple. This is an excellent starting position for doing harmonic analysis thanks to (1.15) and the fact that the representation theory of compact semisimple Lie groups is especially nice: the set $\hat{G}$ of Fourier modes admits a powerful description in terms of the *dominant integral weights* of G.

Ultimately we shall only consider a special type of homogeneous metrics called *normal*, where the Einstein condition takes a particularly simple form and where representation theory provides us with friendly and helpful entities known as *Casimir operators* which will ultimately provide us with a way of gaining computational information about the spectrum of the Lichnerowicz Laplacian. But let us take one thing at a time.

1.6 Invariant connections

We mentioned in §1.5 the isotropy representation $H \to \mathfrak{gl}(T_oM)$ of a homogeneous space $M = G/H$. Let us denote with $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H, respectively. Recall that G acts on $\mathfrak{g}$ through the adjoint representation $\mathrm{Ad}: \ G \to \mathrm{GL}(\mathfrak{g})$. If we restrict this to the subgroup H, the subalgebra $\mathfrak{h}$ is of course left invariant – thus the action of H descends to the quotient $\mathfrak{g}/\mathfrak{h}$. This quotient space is now naturally and H-equivariantly identified with the isotropy representation on T_oM.

In general, we cannot reconstruct $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ as a H-module, for there is not always an $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. If such a complement exists, we call the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ and thus the homogeneous space G/H *reductive*. For our purposes this requirement poses no problem: it is already sufficient that H is compact, which makes it possible to endow any H-module with an H-invariant inner product which we can use to take the orthogonal complement.

Since a compact Riemannian manifold has compact isometry group $\mathrm{Iso}(M, g)$, if (M, g) is a compact Riemanian homogeneous manifold, the transitive group $G \subset \mathrm{Iso}(M, g)$ may be chosen to be compact as well. Hence H is automatically compact as a closed subgroup. However if (M, g) is a non-compact homogeneous Einstein manifold, then a conjecture of Alekseevskii states that M is diffeomorphic to Euclidean space – equivalently, H is a maximal compact subgroup of G [Ale75]. Conveniently, this conjecture was very recently positively resolved by Böhm–Lafuente [BL23].

So suppose that we are given a reductive Riemannian homogeneous space (M, g). Choose once and for all a reductive (i.e. $\mathrm{Ad}(H)$-invariant) complement $\mathfrak{m}$ of $\mathfrak{h} \subset \mathfrak{g}$. Henceforth we tacitly identify $\mathfrak{m} \cong T_oM$ and in the same vein refer to it as *isotropy representation*.

If we want to understand what it means for the invariant metric g to be Einstein, we first need to know how to determine the Riemannian curvature. Thus it becomes important to understand the Levi-Civita connection ∇ in the homogeneous setting, which is an instance of an *invariant connection*. We shall also see later that there is another invariant connection that is much more convenient for doing harmonic analysis on reductive homogeneous spaces. So let us embark on a short digression about invariant connections.

There is a one-to-one correspondence between the set of invariant metric connections on M and the set $\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{so}(\mathfrak{m}))$ of H-equivariant linear maps $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$, i.e. those satisfying

$$\forall h \in H,\ X \in \mathfrak{m} : \ \Lambda_\mathfrak{m}(\mathrm{Ad}(h)X) = \mathrm{Ad}(h) \circ \Lambda_\mathfrak{m}(X) \circ \mathrm{Ad}(h)^{-1},$$

the so-called *Nomizu maps* (see [KN69, §X.1/2] for a detailed description of the correspondence). We remark that this formalism can be generalized to handle invariant connections subordinate to any invariant K-structure on M, in which case the $\mathfrak{so}(\mathfrak{m})$ in the definition of the Nomizu map is replaced by the structure algebra $\mathfrak{k} \subset \mathfrak{gl}(\mathfrak{m})$.

The simplest choice of such an invariant connection, in this context, is of course the one corresponding to the trivial Nomizu map $\Lambda_\mathfrak{m} = 0$. This connection, henceforth denoted by $\bar{\nabla}$, is called the *canonical (reductive) connection*, also called the *Ambrose–Singer connection*, as it is precisely the connection appearing in their famous theorem: any connected, simply connected, complete Riemannian manifold is Riemannian homogeneous if and only if it admits a metric connection $\bar{\nabla}$ with parallel torsion and curvature [AS58].

Indeed, the torsion T and curvature R of an invariant metric connection with Nomizu map $\Lambda_{\mathfrak{m}}$ are themselves invariant tensors, given at the base point by

$$\begin{aligned} T_o(X,Y) &= \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}}, \\ R_o(X,Y) &= [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{ad}([X,Y]_{\mathfrak{h}}) \end{aligned} \tag{1.16}$$

for $X, Y \in \mathfrak{m}$ [KN69, Prop. X.2.3], where the subscripts $X_{\mathfrak{h}}, X_{\mathfrak{m}}$ denote projection onto the $\mathfrak{h}$- or $\mathfrak{m}$-part, respectively. Setting $\Lambda_{\mathfrak{m}} = 0$ in (1.16) we obtain a simple expression for the torsion $\bar{T}$ and curvature $\bar{R}$ of $\bar{\nabla}$:

$$\begin{aligned} \bar{T}_o(X,Y) &= -[X,Y]_{\mathfrak{m}}, \\ \bar{R}_o(X,Y)Z &= -[[X,Y]_{\mathfrak{h}}, Z], \end{aligned} \qquad X, Y, Z \in \mathfrak{m}. \tag{1.17}$$

It should be emphasized that $\bar{\nabla}$ depends crucially on the initial choice of the reductive complement $\mathfrak{m} \subset \mathfrak{g}$. Moreover, among the invariant connections on M, the canonical connection $\bar{\nabla}$ has the unique property that it leaves *every* invariant tensor (and in particular $\bar{T}$ and $\bar{R}$) parallel.

Another way of characterizing the connection $\bar{\nabla}$ is to regard it as coming from a principal connection on the H-principal bundle $G \to G/H$. Any principal connection is (in the sense of an Ehresmann connection) determined by a choice of a horizontal distribution $\mathcal{H} \subset TG$ [KN63, §II.1], and the principal connection underlying $\bar{\nabla}$ corresponds to choosing as horizontal spaces the left-translates of the reductive complement $\mathfrak{m} \subset \mathfrak{g}$.

Taking a closer look at the expression for $\bar{T}$ in (1.17), we see that its vanishing is precisely equivalent to the third Cartan relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Also, since $\bar{\nabla}$ is a metric connection, the vanishing of its torsion would mean that it coincides with the Levi-Civita connection ∇. In the context of the Ambrose–Singer theorem stated above we thus recover the classical theorem of Cartan which states that a Riemannian manifold is locally symmetric if and only if its curvature tensor is ∇-parallel. In this sense the torsion tensor $\bar{T}$ can be viewed as measuring how (M, g) fails to be symmetric.

But what if the space under consideration is not symmetric? The only missing ingredient is now the Nomizu map of the Levi-Civita connection ∇, which is given by

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y), \qquad X, Y \in \mathfrak{m}$$

[KN69, Thm. X.3.3], where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the symmetric bilinear mapping defined via

$$2g_o(U(X,Y), Z) = g_o([Z,X]_{\mathfrak{m}}, Y) + g_o(X, [Z,Y]_{\mathfrak{m}}).$$

We shall restrict our attention further to an important subclass of reductive homogeneous spaces for which the Nomizu map takes on a simpler form. A reductive Riemannian homogeneous space (M, g) is said to be *naturally reductive* if $U = 0$. This seemingly technical condition has a neat geometric interpretation: it is equivalent to $\bar{\nabla}$ and ∇ having the same geodesics. By virtue of (1.17), it is furthermore equivalent to the torsion $\bar{T}$ of the canonical connection being *totally skew-symmetric* (or *skew* for short). Metric connections with skew torsion play a vital role in the study of *non-integrable geometries* such as nearly Kähler or nearly parallel G_2-manifolds, in that they are often more adapted to the structure at hand as the Levi-Civita connection (see [Agr06] for a survey of the topic).

In order to properly do harmonic analysis later on, we need to impose one further restriction on the metric. Given a naturally reductive Riemannian homogeneous space (M, g), let $\mathfrak{g}' = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$. We assume without loss of generality that $\mathfrak{g} = \mathfrak{g}'$, else we simply restrict the presentation $M = G/H$ as a homogeneous space to the (still transitive) subgroup $G' \subset G$ with Lie algebra $\mathfrak{g}'$. A theorem of Kostant tells us that there now exists a unique nondegenerate and, most importantly, $\mathrm{Ad}(G)$-invariant bilinear form Q on $\mathfrak{g}$ such that $Q(\mathfrak{h}, \mathfrak{m}) = 0$ and the invariant metric g on M is induced by the restriction $Q\Big|_{\mathfrak{m}}$ [Kos56].

If this bilinear form Q is positive definite, then we call (M, g) a *normal* homogeneous space. A necessary consequence of this requirement is that $\mathfrak{g}$ is the Lie algebra of a compact group, and that (M, g) has nonnegative sectional curvature [Bes87, Prop. 7.87].

An important $\mathrm{Ad}(G)$-invariant bilinear form on $\mathfrak{g}$ is the *Killing form* $B_{\mathfrak{g}}$ defined by

$$B_{\mathfrak{g}}(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)), \qquad X, Y \in \mathfrak{g}.$$

The Killing form is negative-semidefinite if and only if G is compact, and nondegenerate if and only if G is semisimple. Thus, in the compact semisimple case, a possible choice of invariant metric on M is the one induced by the inner product $Q = -B_{\mathfrak{g}}$. This metric is called the *standard metric*. Later, we will mainly be interested in the case where G is simple. Then, by Schur's Lemma, any $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$ will be a scalar multiple of the Killing form. Thus it is natural to normalize the metric on a normal homogeneous space with simple transitive group G to be the standard metric.

1.7 Representation theory and Casimir operators

The $\mathrm{Ad}(G)$-invariance of the inner product Q on $\mathfrak{g}$ is crucial because it allows us to define the *Casimir operators* advertised earlier. Given any representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, its associated Casimir operator is a $\mathfrak{g}$-equivariant endomorphism of V, defined with respect to Q as

$$\mathrm{Cas}_V^{\mathfrak{g},Q} = -\sum_i \rho_*(e_i)^2,$$

where (e_i) denotes some Q-orthonormal basis of $\mathfrak{g}$.

Casimir operators arise ubiquitously in geometry once a Lie algebra with invariant inner product is available. Expressing geometric objects in terms of Casimir operators carries the great advantage that the spectrum of a Casimir operator is very easy to compute. In order to do that, we need to recapitulate a tad of representation theory.

As a quick side note before we get started: we shall use the terms *G-representation* and *G-module* somewhat interchangeably, although strictly speaking "module" refers to the underlying space V and "representation" to the Lie group homomorphism $G \to \mathrm{GL}(V)$. The same goes, of course, for representations/modules of some Lie algebra $\mathfrak{g}$.

Now, suppose that $\mathfrak{g}$ is a (real or complex) semisimple Lie algebra with a Cartan subalgebra (i.e. a maximal toral subalgebra) $\mathfrak{t} \subset \mathfrak{g}$. The dimension of $\mathfrak{t}$ is called the *rank* $\operatorname{rk}\mathfrak{g}$ of $\mathfrak{g}$. Since $\mathfrak{t}$ is abelian, every finite-dimensional complex representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ can be simultaneously diagonalized when restricted to the subalgebra $\mathfrak{t} \subset \mathfrak{g}$. The elements of the dual space $\mathfrak{t}^*$ that occur on the diagonal are called the *weights* of the representation ρ_*. The *roots* of $\mathfrak{g}$ are then nothing but the nonzero weights of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, and the set of roots is called the *root system* of $\mathfrak{g}$.

A striking discovery made by Killing in 1888 is that the structure of a semisimple Lie algebra $\mathfrak{g}$ is completely determined by specifying its root system, facilitating the famous Killing–Cartan classification of complex simple Lie algebras through the classification of irreducible root systems [Kil90; Car94], a remarkable feat for the time. But roots and weights can do even more: thanks to the theorem of the highest weight, they provide us with complete insight into the representation theory of such a Lie algebra.

Let Q^* denote the inner product on $\mathfrak{t}^*$ dually induced by the restriction $Q\big|_{\mathfrak{t}}$. An element $\lambda \in \mathfrak{t}^*$ is called *integral* if

$$2\frac{Q^*(\lambda,\alpha)}{Q^*(\alpha,\alpha)} \in \mathbb{Z}$$

for any root α. The set of integral elements of $\mathfrak{t}^*$ is suggestively called the *weight lattice* since it contains all possible weights of representations of $\mathfrak{g}$.

One more choice is necessary. Consider to each root α its orthogonal hyperplane $P_\alpha = \alpha^\perp \subset \mathfrak{t}^*$. The union of all the P_α partitions $\mathfrak{t}^*$ into a set of disjoint open cones called *Weyl chambers*. We choose one of these chambers and call it the *dominant chamber*. This equally amounts to declaring a set of *positive roots*, which is preferred by some authors, and gives us also a partial ordering on the weights of any representation: we say that $\lambda \geq \mu$ if and only if $\lambda - \mu$ is contained in the dominant chamber. Finally, we define the set of *dominant integral weights* as the intersection between the dominant chamber and the weight lattice.

The theorem of the highest weight now states that every irreducible complex representation of $\mathfrak{g}$ has a unique weight which is dominant and maximal under $\geq$, called the *highest weight*. Conversely, for each dominant integral weight λ of $\mathfrak{g}$ there exists an, up to equivalence, unique irreducible $\mathfrak{g}$-module V_λ having λ as its highest weight, the so-called *highest weight module* to λ.

A slight complication in passing from the Lie algebra level back to that of a Lie group is that not every representation of $\mathfrak{g}$ necessarily corresponds to a representation of G. In fact, the set $\hat{G}$ of equivalence classes is again parametrized by the dominant elements in the weight lattice of G, which is generally a sublattice of the weight lattice of $\mathfrak{g}$. The two lattices coincide if and only if G is simply connected. As a classical example, the spinor representation of a spin group does not descend to a representation of the corresponding orthogonal group.

The usefulness of the theorem of the highest weight is evident because the set of dominant integral weights is very simple to characterize: it is an (additive) semigroup generated by the so-called *fundamental weights*, usually denoted $\omega_1, \ldots, \omega_r$, where $r = \operatorname{rk} \mathfrak{g}$. That is, the dominant integral weights are precisely the linear combinations of the fundamental weights with nonnegative integer coefficients. A standard convention for enumerating the fundamental weights of the *simple* Lie algebras is the one of Bourbaki, specified in [Bou81, Planches I–IX]. We shall follow it as well.

We can now finally state Freudenthal's formula for the Casimir operator [Fre54]. By the Lemma of Schur any equivariant endomorphism of an irreducible complex $\mathfrak{g}$-module is a constant multiple of the identity. In particular the Casimir operator on a highest weight module V_λ acts as multiplication by the *Casimir constant* $\operatorname{Cas}_\lambda^{\mathfrak{g},Q}$. Denote by $\delta_{\mathfrak{g}} = \omega_1 + \ldots + \omega_r$ the sum of fundamental weights of $\mathfrak{g}$ – this is the same as the half-sum of positive roots and is sometimes called the *Weyl element* of $\mathfrak{g}$. Then the Casimir constant is given by the very simple formula

$$\operatorname{Cas}_\lambda^{\mathfrak{g},Q} = Q^*(\lambda + 2\delta_{\mathfrak{g}}, \lambda). \tag{1.18}$$

This immediately yields the spectrum of the Casimir operator on any representation of a semisimple Lie algebra. Sometimes we may also be interested in the case where $\mathfrak{g}$ is abelian, but this is even simpler: any finite-dimensional complex representation of an abelian Lie algebra splits into one-dimensional weight spaces, on each of which formula (1.18) holds true, with the only difference that there are no roots, so one has to set $\delta_{\mathfrak{g}} = 0$.

1.8 Homogeneous vector bundles

To see the particular relevance of the Casimir operator in the context of our endeavor, we first need to define the representation on which it acts. Let again $M = G/H$ be a homogeneous space with G compact and recall viewing G as a H-principal bundle over M.

To each finite-dimensional representation $\rho:\ H \to \mathrm{GL}(V)$ one associates a homogeneous vector bundle

$$VM = G \times_\rho V = (G \times V)/_{\forall h\in H:\ (x,v)\sim(xh^{-1},\rho(h)v)}$$

over M with fiber V. Taking V to be the isotropy representation $\mathfrak{m}$, we recover the tangent bundle TM. The associated bundle construction hence already covers all tensor bundles over M by taking V to be a tensor power of $\mathfrak{m}$.

In order to do harmonic analysis on VM, we need to pass to a more analytical viewpoint. Sections of the bundle VM are in a canonical way identified with V-valued functions on G that are equivariant under the H-action – that is $\Gamma(VM) \cong C^\infty(G,V)^H$. Left-translation by G gives rise to the *left-regular representation* of G on $C^\infty(G,V)^H$, defined by

$$\ell:\ G \to \mathrm{GL}(C^\infty(G,V)^H):\ (\ell(x)f)(y) = f(x^{-1}y).$$

We are finally set to introduce the generalized Fourier transform alluded to earlier. By combining the classical Peter–Weyl theorem (1.15) with the Frobenius reciprocity theorem

$$\mathrm{Hom}_G(V_\gamma, L^2(G,V)^H) \cong \mathrm{Hom}_H(V_\gamma, V)$$

we obtain the Peter–Weyl theorem for homogeneous vector bundles:

$$L^2(G,V)^H \cong \overline{\bigoplus_{\gamma\in\hat{G}}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V). \tag{1.19}$$

The factor $\mathrm{Hom}_H(V_\gamma, V)$ in each term (called the space of *matrix coefficients*) does the job of counting the multiplicity of the module V_γ inside $L^2(G,V)^H$. We note that (1.19) works equally well with complex or real representations in place of V_γ and V. By virtue of the theorem of the highest weight, however, the complex finite-dimensional irreducible representations of G are easier to enumerate.

Consider now some G-equivariant differential operator $D:\ \Gamma(VM) \to \Gamma(WM)$ between homogeneous vector bundles. As a consequence of (1.19) together with Schur's Lemma, D can be presented as a sequence of linear maps $(D\big|_\gamma)_{\gamma\in\hat{G}}$ with

$$D\big|_\gamma:\ \mathrm{Hom}_H(V_\gamma, V) \longrightarrow \mathrm{Hom}_H(V_\gamma, W).$$

Recall that we are ultimately interested in the Lichnerowicz Laplacian Δ_{L} on $\mathscr{S}^2_{\mathrm{tt}}(M)$. It thus suffices to work out the maps

$$\Delta_{\mathrm{L}}\big|_\gamma:\ \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}) \longrightarrow \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m})$$

in order to access the spectrum of Δ_{L}. The issue raised by the first-order condition of divergence-freeness adds another subtlety that we will get back to in §2.1.2.

The immediate question is, for a given Fourier mode $\gamma \in \hat{G}$, how to deduce the linear map $D\big|_{\gamma}$ from the differential operator D. Now comes the time to shine for the canonical reductive connection. By its properties of being left-invariant and coming from a principal connection on the H-principal bundle $G \to M$, it behaves very neatly under the identification $\Gamma(VM) \cong C^{\infty}(G, V)^H$: a covariant derivative on $\Gamma(VM)$ becomes a directional derivative on $C^{\infty}(G, V)^H$. Expressed in terms of the left-regular representation, the correspondence is

$$\bar{\nabla}_X \rightsquigarrow -\ell_*(X) \tag{1.20}$$

for any left-invariant vector field $X \in \mathfrak{m}$. This key fact is ultimately what enables us to describe invariant differential operators on homogeneous spaces.

The next step is to notice that the Casimir operator $\mathrm{Cas}_{\ell}^{\mathfrak{g},Q}$ of the left-regular representation coincides with the second order differential operator defined by

$$\bar{\Delta} = \bar{\nabla}^* \bar{\nabla} + q(\bar{R})$$

with respect to the normal metric on M induced by $Q\big|_{\mathfrak{m}}$ [MS10]. Note the similarity of this expression with the Lichnerowicz Laplacian (1.9). Both operators are instances of the standard Laplace operator mentioned in §1.3. In this sense, $\bar{\Delta}$ and Δ_{L} are the "most natural" Laplace-type operators to study on a Riemannian homogeneous space.

Because $\bar{\Delta} = \mathrm{Cas}_{\ell}^{\mathfrak{g},Q}$, its spectrum is now easy to compute via Freudenthal's formula (1.18) – on each term of (1.19), $\bar{\Delta}$ simply acts as multiplication by a Casimir constant!

The astute reader may remember from §1.6 that the connections $\bar{\nabla}$ and ∇ coincide if (M, g) is Riemannian symmetric. Consequentially, the two standard Laplacians $\bar{\Delta}$ and Δ_{L} coincide as well, and by the preceding discussion the spectrum of Δ_{L} is served to us on a plate. This relation between Δ_{L} and the Casimir operator on symmetric spaces had been noticed by Koiso and subsequently utilized in his investigation of the stability of symmetric spaces of compact type [Koi80], thereby pioneering the study of stability of homogeneous Einstein manifolds.

The assumption that the metric on M is normal is crucial in order to employ the Casimir operator $\mathrm{Cas}_{\ell}^{\mathfrak{g},Q}$. Of course, symmetric spaces are not the only normal homogeneous Einstein manifolds. Consider a homogeneous space $M = G/H$ which is *isotropy irreducible*, that is, the isotropy representation $\mathfrak{m}$ is irreducible as a H-module. By the Lemma of Schur, the space $\mathrm{Sym}^2\,\mathfrak{m}^H$ of H-invariant symmetric bilinear forms on $\mathfrak{m}$ is one-dimensional – thus any G-invariant Riemannian metric on M is automatically normal and Einstein.

In 1968, all compact, simply connected, non-symmetric isotropy irreducible homogeneous spaces were classified by Wolf [Wol68] (and independently by Manturov in 1961 [Man61a; Man61b; Man66] and Krämer in 1975 [Krä75]). Going further, Wang–Ziller classified in 1985 all the compact, simply connected, non-symmetric *isotropy reducible* homogeneous spaces with simple G and carrying a normal Einstein metric [WZ85]. We see that there are many more examples of normal homogeneous Einstein manifolds besides the symmetric spaces, and it is these classes of spaces that we will ultimately turn our attention to. Of course, the two Laplacians $\bar{\Delta}$ and Δ_{L} will now differ by some first order operator, which makes matters more difficult. This issue will be addressed later (see §§2.2, 2.4). The spirit stays the same: through the exploitation of symmetry, an analytic problem is transformed into one of representation theory.

1.9 G-stability

Another important notion of stability arising in the homogeneous case shall not go unmentioned. Instead of $\mathscr{M}_1$, consider only the finite-dimensional manifold $\mathscr{M}_1^G$ of G-invariant unit volume Riemannian metrics on some compact, connected homogeneous space $M = G/H$. The variational characterization of Einstein metrics is retained in this invariant setting: a metric $g \in \mathscr{M}_1^G$ is Einstein if and only if it is a critical point of the restricted Einstein–Hilbert functional $S\big|_{\mathscr{M}_1^G}$. This has been used in many cases to find new invariant Einstein metrics or rule out their existence [Jen71; WZ86; BWZ04; Böh04].

An Einstein metric $g \in \mathscr{M}_1^G$ is now called *G-stable* if it is a local maximum of $S\big|_{\mathscr{M}_1^G}$. This is a strictly weaker condition than the “classical” stability with respect to the functional $S\big|_{\mathfrak{S}}$, introduced in §1.2 – in particular, G-instability implies classical instability. On the other hand, there are G-stable homogeneous Einstein manifolds that are unstable in the wider sense, such as the Berger space $\mathrm{SO}(5)/\,\mathrm{SO}(3)_{\mathrm{irr}}$ [SWW22, §5].

For the Einstein–Hilbert action restricted to invariant variations the situation is distinctly simpler than in §1.2 since the only invariant conformal variation is constant scaling. In fact, the formal tangent space to $\mathscr{M}_1^G$ is given by

$$T_g\mathscr{M}_1^G = \mathscr{S}_0^2(M)^G = T_g(\mathrm{Aut}(G/H)\cdot g) \oplus \mathscr{S}_{\mathrm{tt}}^2(M)^G,$$

where $\mathrm{Aut}(G/H) \subset \mathrm{Diff}(M)$ is the group of automorphisms of G mapping H to itself. As before, this decomposition is orthogonal with respect to the second variation S_g''. We note that $T_g(\mathrm{Aut}(G/H)\cdot g)$ vanishes if the isotropy representation $\mathfrak{m}$ is multiplicity-free or if it contains no trivial part, i.e. $\mathfrak{m}^H = 0$.

The G-stability of compact homogeneous Einstein manifolds has been investigated by Gutiérrez, E. Lauret, J. Lauret and Will [Lau22; LW22b; LW22a; LL23; GL23]. In the context of (1.19) everything plays out on the Fourier mode “0” corresponding to the trivial representation. The Lichnerowicz Laplacian thus reduces to a linear endomorphism

$$\Delta_{\mathrm{L}}\Big|_0 : \ (\mathrm{Sym}^2_0\,\mathfrak{m})^H \longrightarrow (\mathrm{Sym}^2_0\,\mathfrak{m})^H$$

whose spectrum may be computed with the help of the formulas developed in [LW22c, Cor. 6.7]. This enabled the aforementioned authors to determine the G-stability type of many homogeneous Einstein manifolds, even non-normal ones.

It is also sensible to study the G-invariant analogue of rigidity, especially in the light of the Finiteness Conjecture mentioned in §1.5 which is equivalent to all compact homogeneous Einstein manifolds being G-rigid. To clarify, an invariant Einstein metric is called G-rigid if it is isolated in the moduli space $\mathscr{E}^G \subset \mathscr{M}_1^G/\operatorname{Aut}(G/H)$. In the same way as before, the nonexistence of G-invariant EID implies G-rigidity. In several cases it is possible to examine the Einstein–Hilbert functional $S\Big|_{\mathscr{M}_1^G}$ directly and show that its critical points are isolated [BWZ04, Ex. 6.7–6.11]. Another notable method of Derdzinski–Gal involves the algebraic variety of possible Levi-Civita connections of G-invariant metrics and yields, among others, the G-rigidity of the standard metric on $\mathrm{SU}(n)$ among left-invariant metrics [DG14]. We note that the latter approach even encompasses pseudo-Riemannian metrics.

1.10 What else?

There is, of course, a great deal more to say about the by now numerous stability and instability results for Einstein metrics, in particular in the homogeneous case. Our ambition is not to give a complete picture – nevertheless we summarize a few more themes that are especially relevant for the cumulative part of the current thesis.

The canonical variation, mentioned in §1.3 in the case of Riemannian products, applies in fact to the broader case of (M, g) being the total space of a Riemannian submersion with totally geodesic fibers. Denote with (B, g_B) and (F, g_F) the base and fiber, respectively. Then $g = g_B \oplus g_F$ with respect to a suitable decomposition of TM. The canonical variation is the family of metrics on M given by

$$g_t = g_B \oplus t g_F, \qquad t \in \mathbb{R}. \tag{1.21}$$

The curvature of Riemannian submersion metrics is well understood largely thanks to the theory developed by O'Neill [ONe66]. See [Bes87, §9] for an in-depth discussion.

The trace-free part of the first order jet to the curve of metrics defined by (1.21) yields a tt-tensor $h \in \mathscr{S}^2_{\mathrm{tt}}$ which is also a *Killing tensor* – that is, it satisfies $\delta^* h = 0$, where $\delta^* : \mathscr{S}^p(M) \to \mathscr{S}^{p+1}(M)$ is the symmetrized covariant derivative, also called *Killing* *operator*. The estimate

$$\Delta_{\mathrm{L}} \geq 2q(R)$$

derived from (1.10) on $\ker \delta$ turns into an equality on Killing tensors [HMS16, Prop. 6.2]. This is one reason why Killing tensors are natural candidates to test against the Lichnerowicz Laplacian. Simultaneously it raises the question whether the lowest tt-modes of Δ_{L} always consist of Killing tensors. Indeed, Killing tensors have frequently served as a source for instabilities. See [WW21] for many examples from the canonical variation, but also [SWW22, §5] for a direct utilization of the Killing property. Moreover all G-invariant tensors on a naturally reductive homogeneous space are automatically Killing.

Going back to the canonical variation (1.21), it is worth noting that under certain conditions it can contain another Einstein metric. One of the two is then automatically unstable by a-priori considerations [Bes87, Prop. 9.72, Lem. 9.74]. The most prominent examples of such new Einstein metrics have been constructed on $S^{4n+3} \twoheadrightarrow \mathbb{HP}^n$ by Jensen [Jen73] (sometimes called *squashed spheres*), $S^{15} \twoheadrightarrow \mathbb{OP}^1$ by Bourguignon–Karcher [BK78] and $\mathbb{CP}^{2n+1} \twoheadrightarrow \mathbb{HP}^n$ by Ziller [Zil82].

If the homogeneous world becomes too boring, one may dare to abandon a bit of symmetry and consider spaces of *cohomogeneity one*, that is Riemannian manifolds (M, g) with an isometric action of a Lie group G that has orbits of codimension one. This setting retains some of the advantages of homogeneous spaces. The Einstein equation (1.4) essentially reduces to an ordinary differential equation on the (one-dimensional) orbit space M/G, and the greatest difficuly is usually posed by finding examples that are compact, i.e. solutions that "close up" properly.

It took until 1978 to find the first example of a compact Einstein manifold of cohomogeneity one, when Page constructed such a metric on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$ [Pag78]. Since then, this approach has been generalized in various ways and many new such spaces have been found. A notable (infinite) family of examples are the metrics found by Böhm on spheres in dimension $5 \leq n \leq 9$, some products of spheres and other spaces of low dimension [Böh98]. Their stability has been partially studied by Gibbons–Hartnoll–Pope [GHP03], who showed instability for certain products of spheres and performed numerical experiments that suggest instability of most of the Böhm metrics.

One last very interesting class of Einstein manifolds where stability has been studied shall be mentioned here: these are the spin manifolds (M, g) carrying a *Killing spinor*, that is a section of the complex spinor bundle $\psi \in \Gamma(\Sigma M)$ such that

$$\nabla_X \psi = cX \cdot \psi, \qquad X \in \mathfrak{X}(M),$$

where $\cdot$ denotes Clifford multiplication, for some $c \in \mathbb{C}$ (called the *Killing constant*). Strikingly, the existence of a nonzero Killing spinor forces (M, g) to be Einstein and moreover determines the Einstein constant to be $E = 4(n-1)c^2$. In particular c can only be real or purely imaginary.

Killing spinors usually bring some special structure with them. Compact simply connected manifolds with Killing spinors (which automatically have $c \in \mathbb{R}$, thus are called *real* Killing spinors) have been classified by Bär in 1993 [Bär93]: they are either standard spheres (which are stable) or Einstein–Sasaki manifolds, except in dimensions 6 and 7 which allow for nearly Kähler manifolds and nearly parallel G_2-manifolds, respectively. All of these are instances of the non-integrable geometries briefly mentioned in §1.6 and carry with them a canonical metric connection with parallel skew torsion. Riemannian manifolds admitting such a connection have been studied by Cleyton–Moroianu–Semmelmann in [CMS21].

All this extra structure has been put to use to produce destabilizing directions from harmonic forms: sufficient for instability is $b_2(M) > 0$ in the Einstein–Sasaki case, $b_3(M) > 0$ in the nearly parallel G_2 case [SWW22], and $b_2(M) + b_3(M) > 0$ in the strict nearly Kähler case [SWW20], the latter providing an analogue for the Kähler case with $b_2(M) > 1$ [CHI04, p. 6]. Various other instability results in the presence of real Killing spinors have been produced by C. Wang and M. Y. Wang [Wan17; WW18].

Making more explicit use of the spinor bundle ΣM, a lower estimate for Δ_L on tt-tensors can be given in terms of a twisted Dirac operator on $\Sigma M \otimes T^*M$ [Wan91], see also [GHP03, §IV.C]. A similar method has been used by Dai–Wang–Wei to prove the semistability of all compact manifolds with parallel spinors (i.e. Killing spinors with $c = 0$) [DWW05]. These are necessarily Ricci-flat – in fact, all known compact, simply connected Ricci-flat manifolds belong to this class.

This shall conclude our excursion into the jungle of stability and instability results. Let us next turn to a discussion of the new original results of this work and the way to achieve them.

2 Methods and main results

"Tu as voulu de l'algèbre, et tu en auras jusqu'au menton!"
Jules Verne (Autour de la lune).

2.1 Symmetric spaces

The stability analysis of compact Riemannian locally symmetric spaces was initiated by Koiso [Koi80]. Recall that these spaces can, up to covering, be decomposed into a product of irreducible symmetric spaces, which are isotropy irreducible and thus Einstein. They are in turn classified, according to their curvature, into those of *compact type*, of *noncompact type*, and Euclidean space. The Euclidean case is not particularly interesting: flat tori are semistable, the kernel of $\Delta_{\mathrm{L}}\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)}$ consisting simply of constant traceless tensors. For spaces of noncompact type, Koiso noticed [Koi78, Cor. 3.5] that the estimate $\mathring{R} < -E$ holds provided there is no local two-dimensional factor, and this is sufficient for stability (see §1.3). Such curvature arguments do not work as easily for symmetric spaces of compact type – which is where representation theory steps in. Koiso's results, together with a result of Gasqui–Goldschmidt on the complex quadric $\mathrm{SO}(5)/(\mathrm{SO}(3)\times\mathrm{SO}(2))$, are summarized in Theorem 4.2.1. In particular Koiso determined which of the symmetric spaces of compact type are infinitesimally deformable.

A few cases, however, remained open. The stability of the spaces

$$\mathrm{SU}(n) \quad (n \geq 3), \qquad \mathrm{E}_6/\mathrm{F}_4, \qquad \frac{\mathrm{Sp}(p+q)}{\mathrm{Sp}(p)\times\mathrm{Sp}(q)} \quad (p \geq q \geq 2), \qquad \mathbb{HP}^2, \qquad \mathbb{OP}^2$$

was not completely settled. The issue was only recently resolved with our own article [Sch22b], addressing the infinitesimally deformable spaces $\mathrm{SU}(n)$ and $\mathrm{E}_6/\mathrm{F}_4$, and the related work by Semmelmann–Weingart [SW22], who considered the quaternionic Grassmannians and the Cayley-projective plane. These newer results are summarized in Theorem 4.2.2 and 4.2.3.

2.1.1 Determining stability

Let us now give an account of the basic procedure underlying all of this work. As explained in §§1.7, 1.8, the Lichnerowicz Laplacian reduces in this setting to a Casimir operator whose spectrum can be computed by means of the Peter–Weyl theorem (1.19) and Freudenthal's formula (1.18). The procedure goes as follows: fix a symmetric space G/H of compact type with simply connected and simple G, and endow it with the Killing form metric induced by $-B_{\mathfrak{g}}\big|_{\mathfrak{m}}$. This has the convenient side effect that the critical eigenvalue is normalized to $2E = 1$.

The set $\hat{G}$ may be identified with the semigroup generated by the fundamental weights. Using (1.18) and the Cartan matrix of G to find the inner product between the fundamental weights, one first lists all $\gamma \in \hat{G}$ whose Casimir constant lies below the critical eigenvalue, i.e. $\mathrm{Cas}^{\mathfrak{g}}_{\gamma} \leq 1$.

Next, write $\mathscr{S}^2_0(M)$ as a homogeneous vector bundle with fiber $\mathrm{Sym}^2_0\,\mathfrak{m}$. To find the multiplicity of the G-module V_γ inside $\mathscr{S}^2_0(M)$, one uses the known branching laws to decompose each V_γ on the list into irreducible H-modules and thus computes the dimension of $\mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m})$. If it vanishes, the Fourier mode γ does not appear inside $\mathscr{S}^2_0(M)$ and is thus ruled out as a potential instability.

2.1.2 The divergence operator

A substantial hurdle, which was also the main reason for the incompleteness of Koiso's analysis, is understanding and incorporating the divergence operator δ in this picture (after all, we are looking only for tt-tensors). As an invariant differential operator, δ induces a linear map

$$\delta\big|_{\gamma} : \ \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}) \longrightarrow \mathrm{Hom}_H(V_\gamma, \mathfrak{m})$$

on each Fourier mode $\gamma \in \hat{G}$. In some cases, the divergence can be shown to vanish by a-priori considerations, for example if $\mathrm{Hom}_H(V_\gamma, \mathfrak{m})$ is trivial. In the general case, however, more needs to be done.

A hands-on approach is to try and compute the divergence explicitly on the candidates for instabilities. Exploiting the properties of the canonical reductive connection, a formula could be developed, see Lemma 4.4.3. This approach ultimately works just fine (see §4.8, also carried out in [Sch22b]). However it is bound to be quite cumbersome since it first requires working out the space $\mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m})$ explicitly.

As it turns out, there is a more elegant method which is thoroughly explained in §4.5 and picked up again in §7.6. In essence, one can derive an exact dimension formula for the tt-eigenspaces of Δ_{L} in terms of the dimensions of the corresponding eigenspaces on $\mathscr{S}^2_0(M)$, $\Omega^1(M)$ and the space of conformal Killing vector fields (see Lemma 4.5.1). This dimension formula arises from a short exact sequence

$$0 \longrightarrow \{\text{conf. Killing v.f.}\} \hookrightarrow \Omega^1(M) \longrightarrow \mathscr{S}^2_0(M) \twoheadrightarrow \mathscr{S}^2_{\mathrm{tt}}(M) \longrightarrow 0$$

of invariant operators, all commuting with Δ_{L}.

We further note that on compact Einstein manifolds, apart from round spheres, all conformal Killing vector fields are in fact Killing. The space of Killing vector fields, however, is just given by $\mathfrak{iso}(M,g) = \mathfrak{g}$ in our case. Combining all this, one can finally calculate the dimensions of the tt-eigenspaces of Δ_{L} to the potential subcritical Casimir constants and thus arrive at the results.

2.2 A careful step into non-symmetric waters

The success of representation theory in determining the stability of symmetric spaces motivates us to extend this approach as far as possible to other homogeneous Einstein manifolds. The two connections ∇ and $\bar{\nabla}$ no longer coincide, and consequently the Lichnerowicz Laplacian is no longer simply a Casimir operator. If we assume the metric to be normal, however, the standard Laplacian $\bar{\Delta}$ retains its interpretation as a Casimir operator, and also commutes with Δ_{L}. The overarching hope is now to find a way to "simultaneously diagonalize" these operators – this shall be achieved by finding a relation between the eigenvalue problems for the two Laplacians, or by expressing the first-order difference term $\Delta_{\mathrm{L}} - \bar{\Delta}$ in a suitable manner.

Normal metrics are naturally reductive, and thus by the discussion of §1.6 the two connections at hand are related by

$$\nabla = \bar{\nabla} + \frac{1}{2}\mathcal{A}, \tag{2.1}$$

where $\mathcal{A} = -\bar{T}$ is the totally skew-symmetric G-invariant $(2,1)$-tensor given at the base point by

$$\mathcal{A}_o(X,Y) = [X,Y]_{\mathfrak{m}}, \qquad X,Y \in \mathfrak{m}.$$

Equation (2.1) immediately yields an expression for the difference $\Delta_{\mathrm{L}} - \bar{\Delta}$ – however, in order to utilize this (as in §2.4), we first need to gather some information about $\mathcal{A}$.

A good testing ground would be a class of homogeneous Einstein manifolds where the tensor $\mathcal{A}$ is explicitly known. One such class are the homogeneous Gray manifolds.

2.2.1 Homogeneous Gray manifolds

An almost Hermitian manifold (M,g,J) is called *nearly Kähler* if the covariant derivative ∇J is totally skew-symmetric. Strict (i.e. non-Kähler) nearly Kähler manifolds are the non-integrable analogues of Kähler manifolds in the sense that they carry a distinguished metric connection ∇^{h} with torsion called the *canonical Hermitian connection* which has holonomy $\mathrm{U}(m)$. This is closely related to the concept of *weak holonomy* originally introduced by Gray [Gra71]. He also initiated the study of nearly Kähler manifolds [Gra70], meriting the name *Gray manifolds* for compact, strict nearly Kähler manifolds of dimension six. This dimension is particularly interesting since it is a notable case in the Nagy classification of compact, simply connected nearly Kähler manifolds [Nag02]. Additionally, these manifolds are Einstein because they carry a Killing spinor (see §1.10), and their structure group can be further reduced to SU(3).

Not all Gray manifolds are homogeneous, but the homogeneous ones form a manageable class, consisting only of the four spaces

$$S^6, \qquad S^3\times S^3=\frac{\mathrm{SU}(2)^3}{\Delta\,\mathrm{SU}(2)}, \qquad \mathbb{CP}^3=\frac{\mathrm{SO}(5)}{\mathrm{U}(2)}, \qquad F_{1,2}=\frac{\mathrm{SU}(3)}{T^2},$$

all endowed with the Killing form metric. Moreover all of them are so-called 3-*symmetric spaces* (see §5.3.4), for which, marvellously, the two canonical connections $\bar\nabla$ and ∇^{h} coincide! In turn, the auxiliary tensor $\mathcal{A}$ defined in (2.1) can be expressed in terms of the almost complex structure J as

$$\mathcal{A}(X,Y)=J(\nabla_X J)Y, \qquad X,Y\in\mathfrak{X}(M).$$

Alternatively, $\mathcal{A}$ can be thought of as the imaginary part of the complex volume form manifesting the SU(3)-structure on M. In all the above cases, J and $\mathcal{A}$ can be written down explicitly.

In the strict nearly Kähler setting the eigenvalue problem

$$\Delta_{\mathrm{L}}h=\lambda h, \qquad h\in\mathscr{S}^2_{\mathrm{tt}}(M),\ \lambda\in\mathbb{R}$$

had been approached by Moroianu–Semmelmann in order to characterize infinitesimal Einstein deformations [MS11]. Their method is generalized in the contributed article [Sch22a], see §5.4. The strategy is as follows: first, comparison formulas developed in [MS10; MS11] are used to translate the eigenvalue problem for Δ_{L} into a problem involving Δ^{h}. This leads to a system of coupled equations for the Sym^+- and Sym^--parts of h (that is, the parts commuting or anticommuting with J, respectively). Second, we make use of the ∇^{h}-parallel bundle isomorphisms coming from the nearly Kähler structure, which all commute with Δ^{h} since it is a standard Laplacian for ∇^{h}, and transport the entire problem for tt-tensors to coclosed primitive forms of J-type $(1,1)$ and $(1,2)+(2,1)$. This has the advantage that the equations "decouple" in a sense (at least for sufficiently small λ), which finally makes it possible to describe the solution space in terms of eigenspaces of Δ^{h} (see Lemma 5.4.2). Remarkably, on the aforementioned spaces of forms, Δ^{h} is actually the same as the ordinary Hodge–de Rham Laplacian, so the three Laplace operators $\Delta,\Delta^{\mathrm{h}},\bar\Delta$ coincide here!

For the third step we use that $\bar{\Delta}$ is a Casimir operator and proceed as in §2.1.1 for each of the homogeneous Gray manifolds (except the round S^6 which is known to be stable). In a very similar fashion to the divergence operator, the codifferential poses an issue in determining whether a certain subcritical Casimir eigenvalue contributes to instability. However, an elegant dimension formula for the eigenspaces on coclosed forms is not readily available. This leaves us no choice but to compute the codifferential exactly in some cases. Once this is done we obtain a complete description of the destabilizing directions on each of the homogeneous Gray manifolds (see Propositions 5.5.3, 5.5.6, 5.5.8, summarized in Theorem 5.2.1). In particular, all of them except S^6 are unstable.

The instability results themselves are not new, as the stability of homogeneous Gray manifolds had previously been analyzed. Wang–Wang showed that $\mathbb{CP}^3$ and $F_{1,2}$ are destabilized by the canonical variation of the twistor fibrations $\mathbb{CP}^3 \twoheadrightarrow \mathbb{HP}^1$ and $F_{1,2} \twoheadrightarrow \mathbb{CP}^2$ [WW18]. Moreover, Semmelmann–Wang [SWW20] already established the lower bound $b_2(M) + b_3(M)$ on the coindex of S_g''. The first main achievement of [Sch22a] is to show that this bound is sharp for homogeneous Gray manifolds – that is, all destabilizing directions indeed come from harmonic 2- or 3-forms.

2.2.2 Other favorable settings

3-symmetric spaces are of course not the only homogeneous spaces where the tensor $\mathcal{A}$ is well understood. Another notable nonintegrable geometry is the class of *nearly parallel* G_2*-manifolds*. First, a G_2*-structure* on a 7-manifold is a nondegenerate 3-form $\sigma \in \Omega^3(M)$ of a certain algebraic type (namely such that its stabilizer under $\mathrm{GL}(7, \mathbb{R})$ is isomorphic to G_2). Such a G_2-structure always induces an orientation and a Riemannian metric by virtue of the inclusion $\mathrm{G}_2 \subset \mathrm{SO}(7)$. A G_2-structure σ is called *nearly parallel* if $*d\sigma = \tau_0 \sigma$ for some constant $\tau_0 \in \mathbb{R}$. Under this condition, we again have a canonical connection for which σ is parallel and which realizes the weak holonomy G_2.

In the normal homogeneous setting, this connection can again be brought to coincide with the Ambrose–Singer connection $\bar{\nabla}$ [AS12, Lem. 7.1], and the difference to the Levi-Civita connection is simply given by

$$\mathcal{A} = \frac{\tau_0}{6}\sigma.$$

Homogeneous nearly parallel G_2-manifolds have been classified by Friedrich et al. [FK+97], and their stability has been discussed by Semmelmann–Wang–Wang [WW18; SWW22].

An entirely different setting are the *generalized Wallach spaces*, that is, reductive homogeneous spaces G/H admitting a $B_{\mathfrak{g}}$-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ of the isotropy representation into three H-irreducible summands such that $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$ for $i = 1, 2, 3$. They have been classified by Nikonorov [Nik16] and generally admit several invariant Einstein metrics [LNF04]. A particularly interesting case for us is when $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are of the same dimension. Then the standard metric is Einstein, and there is always a permutational symmetry of some sort between the isotropy summands. This symmetry is heavily exploited in the contributed article [SSW22] in order to analyze the example $\mathrm{E}_7/\,\mathrm{PSO}(8)$. We will shortly discuss this particular space in §2.4.1.

For now, let us come to the second substantial part of [Sch22a]. The only of the homogeneous Gray manifolds that admits EID is the flag manifold $F_{1,2}$, and it shall be tested whether these EID are integrable in the sense of §1.4. For that, we need to talk about the integrability obstruction in the homogeneous setting.

2.3 Rigidity results

Recall Koiso's second order obstruction form $\Psi \in \operatorname{Sym}^2 \varepsilon(g)^* \otimes \varepsilon(g)^*$ defined in (1.14). The exact condition for integrability to second order can be stated as this: any $h \in \varepsilon(g)$ is formally integrable to second order if and only if

$$\Psi(h, h, k) = 0 \qquad \forall k \in \varepsilon(g). \tag{2.2}$$

Nagy–Semmelmann utilize in [NS23] the *Frölicher-Nijenhuis bracket* to show that Ψ is in fact totally symmetric, that is $\Psi \in \operatorname{Sym}^3 \varepsilon(g)^*$. It is thus via polarization determined by the cubic polynomial $\Psi(h, h, h)$, for which an expression was found by Koiso [Koi82] – see (5.2). The expression involves a contraction of h with its second covariant derivative $\nabla^2 h$ and integrating this over the manifold M. We note that the condition (2.2) above is equivalent to h being a *critical point* of the polynomial $\Psi(h, h, h)$.

2.3.1 Rigidity of homogeneous Einstein manifolds

Turning to the setting of a homogeneous space $M = G/H$ with an invariant Einstein metric g, we note that the space $\varepsilon(g)$ of EID is a G-submodule of $\mathscr{S}^2_{\mathrm{tt}}(M)$. By construction, the obstruction form Ψ itself becomes G-invariant as well, so we are now dealing with an element of $(\operatorname{Sym}^3 \varepsilon(g))^G$, the space of G-invariant cubic homogeneous polynomials over $\varepsilon(g)$. This observation opens the door to various representation-theoretic considerations. It was first utilized by Koiso himself to show that there are no nontrivial EID integrable to second order on the symmetric space $\mathbb{CP}^1 \times \mathbb{CP}^{2n}$ [Koi82, Thm. 6.12].

The next venture in this direction was made by Batat et al. who examined Ψ on the symmetric space $\mathrm{SU}(n)$ [BH+21]. They refrained from computing $\Psi(h, h, h)$ explicitly for a general EID $h \in \varepsilon(g)$, but instead showed that it must be a nonzero multiple of a certain easier to handle invariant symmetric trilinear form. Moreoever they exploit the fact that it is possible to construct the EID of $\mathrm{SU}(n)$ from Killing vector fields in an equivariant way, thus directly reducing the problem to an explicit computation on $\mathfrak{g}$. This strategy is also outlined in [BH+21, §3]. They found that for odd n there are no nontrivial critical points of the obstruction polynomial, thus showing that the biinvariant metric on $\mathrm{SU}(n)$ is rigid if n is odd.

We note that in all cases seen so far, including our investigation of the flag manifold $F_{1,2}$ [Sch22a], the total symmetry of $\Psi \in (\mathrm{Sym}^2 \varepsilon(g) \otimes \varepsilon(g))^G$ is established *a posteriori* by comparing the spaces

$$(\mathrm{Sym}^2 \varepsilon(g) \otimes \varepsilon(g))^G \subset (\mathrm{Sym}^3 \varepsilon(g))^G$$

and observing that they coincide, thus not relying on the eventual *a priori* result by Nagy–Semmelmann [NS23]. These authors apply their reformulation of the integrability obstruction to the complex 2-plane Grassmannian $\mathrm{SU}(n+2)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(n))$, again showing rigidity if n is odd.

2.3.2 Rigidity of the flag manifold $F_{1,2}$

The strategy pursued in [Sch22a] to show rigidity of the normal metric on $F_{1,2} = \mathrm{SU}(3)/T^2$ differs from the one of Batat et al. to the extent that the obstruction polynomial is in fact computed directly. A prerequisite for that is to have an explicit description of the space $\varepsilon(g)$ at hand. In our case, $\varepsilon(g)$ is as a G-module equivalent to the adjoint representation $\mathfrak{g} = \mathfrak{su}(3)$. This algebraic identification has already been described by Moroianu–Semmelmann [MS10, §6] and is picked up again in §5.6.1. In essence one chooses a suitable basis of $\mathfrak{su}(3)$ to write down the Fourier coefficient in $\mathrm{Hom}_H(\mathfrak{su}(3), \mathrm{Sym}^2_0 \mathfrak{m})$, under which every element $\xi \in \mathfrak{su}(3)$ corresponds to a function in $\hat{h}_\xi \in C^\infty(G, \mathrm{Sym}^2_0 \mathfrak{m})^H$ representing an EID $h_\xi \in \varepsilon(g)$.

The obstruction polynomial on h_ξ is worked out in §5.6.2. The first step is to rewrite the obstruction polynomial Ψ in a suitable form. In general, integration by parts yields a way to write $\Psi(h, h, h)$ in terms of only h and its *first* covariant derivative ∇h.

Now comes the part where the algebraic groundwork and the knowledge of the auxiliary tensor $\mathcal{A}$ is applied. Using (2.1) we obtain ∇h_ξ by computing $\bar{\nabla} h_\xi$ and $\mathcal{A} h_\xi$ separately. First, recall that computing $\bar{\nabla} h_\xi$ corresponds to taking directional derivatives of $\hat{h}_\xi$ by relation (1.20) – thus $\bar{\nabla} h_\xi$ can be worked out entirely in terms of the infinitesimal action of $\mathfrak{g} = \mathfrak{su}(3)$ on ξ. Second, the tensor $\mathcal{A}$ is also explicit thanks to [MS10, §6], so it remains to contract it with h_ξ to find $\mathcal{A} h_\xi$. Combining this, we can plug both h_ξ and ∇h_ξ into the expression (5.2) for $\Psi(h_\xi, h_\xi, h_\xi)$ and obtain an integral over a certain polynomial I on $\mathfrak{g}$.

Integrating, i.e. averaging this over G, means in algebraic terms nothing more than orthogonally projecting $I \in \mathrm{Sym}^3 \mathfrak{g}$ to its G-invariant part $\Psi \in (\mathrm{Sym}^3 \mathfrak{g})^G$. For $\mathfrak{g} = \mathfrak{su}(3)$,

$(\mathrm{Sym}^3\,\mathfrak{g})^G$ is in turn a one-dimensional space spanned by $\mathrm{i}\det$, where $\det : \mathfrak{su}(3) \to \mathrm{i}\mathbb{R}$ is the ordinary determinant viewed as a cubic homogeneous polynomial. We conclude that the polynomial Ψ has no nontrivial critical points, hence all EID of the normal metric on $F_{1,2}$ are obstructed to second order.

There is another notion of rigidity that has previously been analyzed, namely that of nearly Kähler SU(3)-structures on Gray manifolds. Any nearly Kähler structure induces an Einstein metric, and in fact the space of *infinitesimal nearly Kähler deformations* (INKD) of a given nearly Kähler structure has been characterized by Moroianu–Nagy–Semmelmann in [MNS08] as a certain subspace of the EID of the corresponding Einstein metric – for the latter, see [MS11]. The space of INKD on homogeneous Gray manifolds is determined in [MS10]. Foscolo developed a general deformation theory for nearly Kähler structures and formulated a second order integrability obstruction which again manifests as a cubic homogeneous polynomial [Fos17]. Moreover, he showed the rigidity of the homogeneous nearly Kähler structure on $F_{1,2}$. In this context, the rigidity result in [Sch22a] can be viewed as a stronger version of Foscolo's result.

2.4 Normal homogeneous spaces: a systematic approach

Successfully testing the waters of non-symmetric normal homogeneous spaces in the nearly Kähler setting motivates a systematic comparison of the two Laplacians Δ_{L} and $\bar{\Delta}$, which is the aim of the other two contributed articles [SSW22; Sch23]. In terms of the auxiliary tensor $\mathcal{A}$ defined in (2.1), we can a priori write the difference as

$$\Delta_{\mathrm{L}} - \bar{\Delta} = \mathcal{A}^*\bar{\nabla} + \frac{1}{4}\mathcal{A}^*\mathcal{A} + q(R) - q(\bar{R}).$$

A key observation is that the difference between the two curvature operators is itself just

$$q(R) - q(\bar{R}) = \frac{1}{4}\mathcal{A}^*\mathcal{A},$$

provided we apply this only to symmetric tensors (see Corollary 6.4.2), thus simplifying the difference formula above considerably (see Lemma 7.4.1).

As a first step in understanding the difference between the Laplacians, we focus on the zeroth order operator $\mathcal{A}^*\mathcal{A} \in \mathrm{End}_H(\mathfrak{m}^{\otimes p})$ (the discussion applies to tensors of arbitrary valence p, not just symmetric 2-tensors). Strikingly, we find that it can be expressed completely in terms of Casimir operators (see Lemma 6.4.3). By far the most headacheinducing of the occurring terms is the peculiar projection

$$\mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} \Big|_{\mathfrak{m}^{\otimes p}}$$

of the $\mathfrak{g}$-Casimir operator of $\mathfrak{g}^{\otimes p}$ back to $\mathfrak{m}^{\otimes p} \subset \mathfrak{g}^{\otimes p}$. Computing its spectrum on $\mathfrak{m}^{\otimes p}$ is unfortunately not straightforward, even if the decomposition of $\mathfrak{g}^{\otimes p}$ into $\mathfrak{g}$- and $\mathfrak{h}$-modules is completely known. This is roughly because the same $\mathfrak{h}$-isotype V might occur as a submodule in two distinct $\mathfrak{g}$-modules $W_1, W_2 \subset \mathfrak{g}^{\otimes p}$ with differing Casimir constants, and thanks to the projection $\mathrm{pr}_{\mathfrak{m}^{\otimes p}}$ the result depends crucially on the embedding of the submodule $V \subset \mathfrak{m}^{\otimes p}$ into $W_1 \oplus W_2 \subset \mathfrak{g}^{\otimes p}$. However, the spectrum is still bounded above and below by the respective Casimir constants of $\mathfrak{g}^{\otimes p}$, which ultimately leads to practical estimates of the Lichnerowicz Laplacian in [Sch23].

We will get back to this in §2.4.2. For now, let us come to a specific example – the normal homogeneous space $\mathrm{E}_7/\mathrm{PSO}(8)$ – which is the main object of study in the article [SSW22] and on which the additional symmetry alluded to in §2.2.2 is sufficient to find the spectrum of $\mathcal{A}^*\mathcal{A}$ on symmetric 2-tensors.

2.4.1 A first stable example

The G-stability of invariant Einstein metrics on generalized Wallach spaces, of which $\mathrm{E}_7/\mathrm{PSO}(8)$ is an example, had recently been studied by Lauret–Will [LW22b], showing that most of them are unstable. In fact, the only G-stable cases are the normal metrics on $\mathrm{SU}(2)$, $\mathrm{E}_7/\mathrm{PSO}(8)$ and $\mathrm{E}_8/(\mathrm{Spin}(8)^2/\mathbb{Z}_2)$, raising the question whether they are also stable in the sense of §1.2.

The structure of the generalized Wallach space $\mathrm{E}_7/\mathrm{PSO}(8)$ is described in detail in §6.5. To begin, there is a splitting

$$\mathfrak{e}_7 = \mathfrak{so}(8) \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

into $\mathfrak{so}(8)$-modules, and the permutational symmetry of the components $\mathfrak{m}_i$ is given by the *triality automorphism* $\Theta \in \mathrm{Aut}(\mathfrak{so}(8))$ of the root system D_4. This outer automorphism corresponds to the rotational symmetry in the Dynkin diagram of D_4, so in the context of highest weight modules V_γ it does the job of permuting the fundamental weights $\omega_1, \omega_3, \omega_4$ and fixing ω_2 (again adapting the enumerative convention of Bourbaki [Bou81]).

Setting $\mathfrak{su}(8)_a := \mathfrak{so}(8) \oplus \mathfrak{m}_a$ for $a = 0, 1, 2$ defines three non-conjugate subalgebras of $\mathfrak{e}_7$, all isomorphic to $\mathfrak{su}(8)$, that get mapped to one another by a suitable extension of Θ to $\mathfrak{e}_7$. The key idea for computing the spectrum of $\mathcal{A}^*\mathcal{A}$ lies in obtaining a second formula in terms of Casimir operators (Lemma 6.6.2), this time involving the Lie algebras $\mathfrak{su}(8)_a$. It turns out that together with the formula from Lemma 6.4.3, this is enough to piece together all the eigenvalues of $\mathcal{A}^*\mathcal{A}$ on $\operatorname{Sym}^2_0\mathfrak{m}$ (see §6.6).

Let us briefly sketch how this new formula arises. First, we note that $\mathcal{A}^*\mathcal{A}$ may, by the skew-symmetry of $\mathcal{A}$, be viewed as a contraction of $\mathcal{A}^2$. That is, it can be written as the composition

$$VM \xrightarrow{\mathcal{A}} T^*M \otimes VM \xrightarrow{\mathcal{A}} T^*M \otimes T^*M \otimes VM \xrightarrow{-\operatorname{tr}_g} VM$$

for any tensor bundle VM. Replacing the total trace tr_g by a partial trace over the subbundle of T^*M associated to $\mathfrak{m}_a \subset \mathfrak{m}$, $a = 0, 1, 2$, we obtain a new operator denoted by $(\mathcal{A}^*\mathcal{A})_a$. On the one hand, summing up these three operators recovers the original $\mathcal{A}^*\mathcal{A}$. On the other hand, $(\mathcal{A}^*\mathcal{A})_a$ can itself easily be seen to be

$$(\mathcal{A}^*\mathcal{A})_a = \operatorname{Cas}^{\mathfrak{su}(8)_a} - \operatorname{Cas}^{\mathfrak{so}(8)_a},$$

where $\mathfrak{so}(8)_a \subset \mathfrak{su}(8)_a$ denotes $\mathfrak{so}(8)$ together with a modified action on $\mathfrak{m}$ that has to be taken into account.

Now that we know the spectrum of $\mathcal{A}^*\mathcal{A}$ on $\operatorname{Sym}^2_0\mathfrak{m}$, we can derive that of the curvature endomorphism $q(R)$. It remains to observe that the estimate $q(R) > E$ is satisfied, which is already a sufficient criterion for stability (see §1.3). This yields the to our knowledge first non-symmetric example of a stable Einstein metric of positive scalar curvature!

One might now be tempted to try and apply a similar procedure to the other G-stable generalized Wallach space $\mathrm{E}_8/(\mathrm{Spin}(8)^2/\mathbb{Z}_2)$, but this is not really necessary. In fact, the stability of both spaces $\mathrm{E}_7/\mathrm{PSO}(8)$ and $\mathrm{E}_8/(\mathrm{Spin}(8)^2/\mathbb{Z}_2)$, among many more, pops out as a result of the systematic analysis of normal homogeneous spaces carried out in [Sch23].

2.4.2 An algorithm for estimating Δ_{L}

The theoretical centerpiece of the fourth contributed article [Sch23] is the exact representation-theoretic description of the Lichnerowicz Laplacian (see Corollary 7.4.5). As the zeroth order term $\mathcal{A}^*\mathcal{A}$ has already been taken care of, it remains to discuss the first order term $\mathcal{A}^*\bar{\nabla}$, which is done in Lemma 7.4.4. The operator $\mathcal{A}^*\bar{\nabla}$ may be viewed as an equivariant endomorphism of the left-regular representation $C^\infty(G, \mathfrak{m}^{\otimes p})^H$. In the same spirit as in the beginning of §2.4, a description in terms of Casimir operators can be achieved by enlarging the representation space. Consider the inclusion

$$C^\infty(G, \mathfrak{m}^{\otimes p})^H \subset C^\infty(G, \mathfrak{g}^{\otimes p}) \cong C^\infty(G) \otimes \mathfrak{g}^{\otimes p}.$$

By now it is no surprise that the Casimir operator $\operatorname{Cas}^{\mathfrak{g}}_{\ell\otimes\mathrm{Ad}^{\otimes p}}$ of the right-hand side occurs in the formula for $\mathcal{A}^*\bar{\nabla}$, again composed with the projection $\operatorname{pr}_{\mathfrak{m}^{\otimes p}}$. In total we have to cope with exactly two of these troublesome projected Casimir operators, namely

$$\mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} \qquad \text{and} \qquad \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\ell\otimes\mathrm{Ad}^{\otimes p}}\,.$$

As explained above, computing their spectrum (say, on a fixed Fourier mode $\gamma \in \hat{G}$) is not a trivial task. For this reason we resort to an estimation-based approach. First, a fibrewise estimate of $\mathcal{A}^*\mathcal{A}$ may be obtained by noting down the possible $\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}}$-eigenvalues on the $\mathfrak{g}$-orbit of $\mathfrak{m}^{\otimes p}$. This leads to a first very crude lower bound for Δ_{L} on symmetric tensors (see Theorem 7.4.6) which has the advantage of being very computationally efficient for varying $\gamma \in \hat{G}$, depending only on the Casimir constant $\mathrm{Cas}^{\mathfrak{g}}_{\gamma}$ itself. Second, similar considerations are applied to the other Casimir operator $\mathrm{Cas}^{\mathfrak{g}}_{\ell\otimes\mathrm{Ad}^{\otimes p}}$, resulting in a more refined estimate on Δ_{L} in Theorem 7.4.7.

The second estimate is generally better but has the downside that serious tensor product and branching calculations have to be carried out for each Fourier mode separately. All this is condensed into the algorithm described in §7.7 which operates roughly according to the following plan: at first, it is checked whether the fibrewise estimate on $\mathcal{A}^*\mathcal{A}$ is sufficient for the curvature estimate $q(R) > E$ which would imply stability, as is the case for the example discussed in §2.4.1. If this does not work, the crude estimate is applied to find a constant $C > 0$ with the property that for every $\gamma \in \hat{G}$, the inequality $\mathrm{Cas}^{\mathfrak{g}}_{\gamma} > C$ implies the inequality $\Delta_{\mathrm{L}}\big|_{\gamma} > 2E$. This narrows the potentially destabilizing Fourier modes down to the (finite!) set

$$\hat{G}_C = \{\gamma \in \hat{G} \mid \mathrm{Cas}^{\mathfrak{g}}_{\gamma} \leq C\}.$$

We restrict further to just the Fourier modes in $\hat{G}_C$ that actually occur within $\mathscr{S}^2_{\mathrm{tt}}(M)$, utilizing the same arguments as in Sections 2.1.1 and 2.1.2. Finally, we apply the refined estimate to each of the remaining modes. For the reasons described above, this step is responsible for the bulk of the computation time. It leaves us either with a computational proof that the space in question is stable or with a small remaining list of potentially destabilizing Fourier modes. On that note, our algorithm is only suited to show stability, not instability.

The algorithm thus described is implemented in the open-source computer algebra system SageMath, utilizing its interface to the software package LiE which takes care of the Lie-theoretic computations [Sage; LiE]. Issues that we will not touch upon here are how to find the right branching laws from $\mathfrak{g}$ to $\mathfrak{h}$ and how to calculate the relevant Casimir constants with the correct normalization (for the latter see §7.5).

Ultimately, we apply this procedure to the known list of normal, non-symmetric homogeneous Einstein manifolds G/H with simple G mentioned at the end of §1.8. These spaces were in their totality classified by Wolf and Wang–Ziller [Wol68; WZ85], consisting of 10 infinite families and 13 exceptions in the isotropy irreducible case, and additionally 9 infinite families and 22 exceptions in the isotropy reducible case. Of course, due to computational constraints, only a finite number of members of each family could be checked.

The results are manifold and surprising. Remarkably, many of the spaces turn out to be stable. Thus a plethora of new examples of non-symmetric stable Einstein metrics of positive scalar curvature is provided. In some cases the stability follows only after combining with the G-stability results of Lauret–Lauret–Will who investigated the same class of spaces [LL23]. The inclined reader is encouraged to skip directly to §7.8 where the results are discussed in due detail.

2.5 Outlook

There is a lot left to do. In the following we shall present a small selection of topics that might be worthwhile to study further.

An exact general procedure to compute eigenvalues of Δ_{L}, or even just the curvature term $q(R)$, on normal homogeneous spaces is still lacking. To blame are primarily the projected Casimir operators mentioned in §2.4.2. This might be remedied by keeping track of and linearly relating certain vectors inside $\mathfrak{m}^{\otimes p}$ and $\mathfrak{g}^{\otimes p}$ in a way that makes it possible to evaluate expressions of the form

$$c_{ij} = \langle \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} v_i, v_j \rangle, \qquad i, j = 1, \ldots, k$$

where $v_i \in V_i$ are suitable representatives (say, highest weight vectors) of some isotypical $\mathfrak{h}$-submodule $V = V_1 \oplus \ldots \oplus V_k \subset \mathfrak{m}^{\otimes p}$. The symmetric matrix with entries c_{ij} could then be diagonalized to find the spectrum of $\mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}}$ on V, and a similar procedure might be employed for the other projected Casimir operator.

Such a method would be called for in light of the several potentially destabilizing Fourier modes left open in our analysis of normal homogeneous Einstein manifolds [Sch23]. Another undertaking worthy of consideration is to apply the developed estimation algorithm to the infinite families of the Wolf–Wang–Ziller classification in their totality, as opposed to the case-by-case check on low-rank examples carried out in [Sch23].

There are other settings that might be fruitful to investigate. The groundwork by Lauret–Will showed G-stability not only for the particular generalized Wallach spaces described above, but also for the unique Kähler–Einstein metric on the generalized flag manifolds (also called *Kähler C-spaces*) with $b_2(M) = 1$. The Kähler–Einstein metric is not standard but has a rather simple and peculiar form. Concretely, there is always an irreducible decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$ such that the metric is, up to a factor, given by

$$g_o = -B_{\mathfrak{g}}\Big|_{\mathfrak{m}_1} \oplus -2B_{\mathfrak{g}}\Big|_{\mathfrak{m}_2} \oplus \ldots \oplus -kB_{\mathfrak{g}}\Big|_{\mathfrak{m}_k}.$$

Additionally, in the case $k = 2$, there exist additional commutator relations that might simplify the necessary calculations considerably. Kähler C-spaces with $b_2(M) = 1$ are of particular interest since any compact connected homogeneous Kähler–Einstein manifold is either a flat complex torus or a Kähler C-space [Bes87, Cor. 8.98]. Moreover we note that Kähler–Einstein metrics with $b_2(M) > 1$ are necessarily unstable [CHI04, p. 6]. In the same vein, one could carefully try to extend our methods to other homogeneous Einstein manifolds whose metric is very close to being normal.

In the spirit of §2.2.2 one could try to apply similar techniques as in [Sch22a] to other spaces where the tensor $\mathcal{A}$ is explicitly known – for example homogeneous nearly parallel G_2 manifolds – in order to compute their coindex, find possible EID and, if these exist, study their integrability.

Following Batat et al. and Nagy–Semmelmann [BH+21; NS23], it still remains to settle the rigidity of the infinesimally deformable symmetric spaces of compact type other than $\mathrm{SU}(n)$ and $\mathrm{SU}(n+2)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(n))$ with n odd. Batat et al. give a promising description of EID on spaces other the complex Grassmannians, namely via constructing them (fibrewise!) from Killing vector fields. For the Grassmannians, we expect a possible characterization of the EID as the image of Killing vector fields under a certain first-order differential operator. Viewing divergence-freeness as an algebraic condition after restricting to a suitable Fourier mode, as in §2.1.2, might prove useful here.

It remains, of course, to work out the obstruction integral in any case. In some cases such as $\mathrm{SU}(n)$ and $\mathrm{SU}(n+2)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(n))$ with n even, it also remains to work out the third order obstruction to integrability. Setting up a systematic framework for these integrability obstructions in the homogeneous case, possibly with the aid of symbolic computer calculations, might be a step towards a resolution of the Finiteness Conjecture.

A broader notion is the solitonic rigidity briefly mentioned in §1.4, which is, for instance, also still open for symmetric spaces. The spaces of infinitesimal solitonic deformations are known, and the obstruction to second order manifests again as a certain cubic integral.

In any of the above cases where integrability is investigated, we eventually obtain a system of homogeneous polynomial equations on some arbitrarily high-dimensional vector spaces. The set of solutions thus forms a projective variety, and it seems in general difficult to describe it or even to determine whether it is empty. Certainly, the discussion may benefit from the tools of real algebraic geometry.

3 Declaration to the cumulative part

3.1 List of contributed publications

The cumulative part of this thesis consists of four research articles. Their title and current publication status is as follows.

- Paul Schwahn. *Stability of Einstein metrics on symmetric spaces of compact type.* Published in: *Annals of Global Analysis and Geometry*, 2022, [Sch22b].

  Chapter 4 is a reproduction of this article.

  DOI: 10.1007/s10455-021-09810-4

- Paul Schwahn. *Coindex and Rigidity of Einstein Metrics on Homogeneous Gray Manifolds.* Published in: *The Journal of Geometric Analysis*, 2022, [Sch22a].

  Chapter 5 is a reproduction of this article.

  DOI: 10.1007/s12220-022-01061-4

- Paul Schwahn, Uwe Semmelmann, and Gregor Weingart. *Stability of the Non-Symmetric Space* $\mathrm{E}_7/\mathrm{PSO}(8)$. Accepted for publication in: *Advances in Mathematics.* arXiv preprint 2022, [SSW22].

  Chapter 6 is a reproduction of this article.

  DOI: 10.48550/ARXIV.2203.10138

- Paul Schwahn, *The Lichnerowicz Laplacian on normal homogeneous spaces.* Manuscript submitted to: *Annales de l'Institut Fourier.* arXiv preprint 2023, [Sch23].

  Chapter 7 is a reproduction of this article.

  DOI: 10.48550/ARXIV.2304.10607

Chapters 4 to 7 represent the original articles except for the following changes:

- slight unifying adjustments of notation,
- reformatting in the style of this thesis,
- correction of typographical errors,
- absorption of the bibliographies into the one of the current thesis.

Apart from these changes, Chapters 4 to 7 are reproductions of the corresponding published/submitted articles listed in §3.1.

3.2 Statement about contributed publications

I, Paul Schwahn, hereby declare that the coauthor lists are complete and that I have not reproduced, without acknowledgment, the work of another. Concerning the article *Stability of the Non-Symmetric Space* $\mathrm{E}_7/\,\mathrm{PSO}(8)$ [SSW22], I declare that I majorly contributed, including in particular the phase of problem selection, literature research, the derivation of theoretical and numerical results as well as the writing process. I declare that the contributed articles [Sch22b; Sch22a; SSW22] (see §3.1) have been accepted for publication and published online in a common international publication organ.

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4 Stability of Einstein metrics on symmetric spaces of compact type

4.1 Abstract

We prove the linear stability with respect to the Einstein-Hilbert action of the symmetric spaces $\mathrm{SU}(n)$, $n \geq 3$, and $\mathrm{E}_6/\mathrm{F}_4$. Combined with earlier results, this resolves the stability problem for irreducible symmetric spaces of compact type.

4.2 Introduction

Let M be a closed manifold of dimension $n > 2$. It is a well-known fact (see [Bes87]) that Einstein metrics are critical points of the total scalar curvature functional

$$g \mapsto S(g) = \int_M \mathrm{scal}_g \mathrm{vol}_g,$$

also called the Einstein–Hilbert action, restricted to the space of Riemannian metrics of a fixed volume. In general, these critical points are neither maximal nor minimal. If we, however, restrict S to the set $\mathfrak{S}$ of all Riemannian metrics on M of the same fixed volume that have constant scalar curvature, then some Einstein metrics are maximal, while others form saddle points. To examine this, one considers the second variation S''_g of S at a fixed Einstein metric g on M. If we exclude the case where (M, g) is a standard sphere, the tangent space of $\mathfrak{S}$ at g consists precisely of tt-tensors, i.e. symmetric 2-tensors that are **t**ransverse (divergence-free) and **t**raceless. In these directions, the coindex and nullity of S''_g are always finite. The *stability problem* is to decide whether they vanish for a given Einstein manifold (M, g).

The stability of an Einstein metric g is determined by the spectrum of a Laplace-type operator Δ_{L}, called the Lichnerowicz Laplacian, on tt-tensors. There is a critical eigenvalue, corresponding to null directions for S''_g, which is equal to $2E$, where E is the Einstein constant of g. The metric g is called *linearly (strictly) stable* if $\Delta_{\mathrm{L}} \geq 2E$ (resp. $\Delta_{\mathrm{L}} > 2E$) on tt-tensors, and *infinitesimally deformable* if there is a tt-eigentensor of Δ_{L} for the critical eigenvalue.

Suppose that (M, g) is a locally symmetric Einstein manifold of compact type. The Cartan–Ambrose–Hicks theorem implies that its universal cover $(\tilde{M}, \tilde{g})$ is a simply connected symmetric space. As such, $(\tilde{M}, \tilde{g})$ can be written as a Riemannian product of irreducible symmetric spaces of compact type. For many of these spaces, the stability problem has been decided by N. Koiso. The following theorem collects the results of Koiso in [Koi80] together with a result of J. Gasqui and H. Goldschmidt in [GG96] about the complex quadric $\mathrm{SO}(5)/(\mathrm{SO}(3) \times \mathrm{SO}(2))$.

4.2.1 Theorem. *1. The only irreducible symmetric spaces of compact type that are infinitesimally deformable are*

$$\mathrm{SU}(n),\ \mathrm{SU}(n)/\operatorname{SO}(n),\ \mathrm{SU}(2n)/\operatorname{Sp}(n) \quad (n \geq 3),$$

$$\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)) \quad (p \geq q \geq 2),$$

as well as $\mathrm{E}_6/\mathrm{F}_4$.

2. The irreducible symmetric spaces

$$\mathrm{Sp}(n) \quad (n \geq 2), \quad \mathrm{Sp}(n)/\operatorname{U}(n) \quad (n \geq 3),$$

as well as the complex quadric $\mathrm{SO}(5)/(\mathrm{SO}(3) \times \mathrm{SO}(2))$ *are unstable.*

3. Let (M, g) be an irreducible symmetric space of compact type. If (M, g) is none of the spaces from 1. and 2., nor one of

$$\mathrm{Sp}(p+q)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q)) \quad (p \geq q \geq 2 \text{ or } p = 2, q = 1)$$

nor $\mathrm{F}_4/\operatorname{Spin}(9)$*, then g is strictly stable.*

Moreover, the smallest eigenvalue of Δ_{L} on trace-free symmetric 2-tensors has been computed in each case (see [CH15]). Among the spaces that possess infinitesimal deformations, we have $\Delta_{\mathrm{L}} \geq 2E$ on $\mathscr{S}^2_0(M)$ on the spaces

$$\mathrm{SU}(n)/\operatorname{SO}(n),\ \mathrm{SU}(2n)/\operatorname{Sp}(n)\ (n \geq 3),\ \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))\ (p \geq q \geq 2),$$

which shows that they are linearly stable.

However, this did not fully settle the stability problem on irreducible symmetric spaces of compact type. In particular, it had not been decided whether unstable directions exist on the spaces

$$\mathrm{SU}(n) \quad (\text{where } n \geq 3), \qquad \mathrm{E}_6/\mathrm{F}_4, \qquad \mathrm{F}_4/\operatorname{Spin}(9),$$

$$\mathrm{Sp}(p+q)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q)) \quad (\text{where } p \geq q \geq 2 \text{ or } p = 2, q = 1).$$

In these cases, we know that Δ_{L} has eigenvalues smaller than $2E$ on the space of trace-free symmetric 2-tensors, but it had not been checked whether the corresponding eigentensors are also divergence-free. In a recent paper [SW22], U. Semmelmann and G. Weingart show the following results.

4.2.2 Theorem. *1. The quaternionic Grassmannians* $\mathrm{Sp}(p+q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$ *are linearly stable for* $p=2$ *and* $q=1$*, but unstable for* $p\geq q\geq 2$*.*

2. The Cayley plane $\mathbb{OP}^2=\mathrm{F}_4/\operatorname{Spin}(9)$ *is linearly stable.*

The current article finally resolves the question of stability for the last remaining cases by proving the following.

4.2.3 Theorem. *The symmetric spaces* $\mathrm{SU}(n)$*, where* $n\geq 3$*, as well as* $\mathrm{E}_6/\mathrm{F}_4$ *are linearly stable.*

Consider a manifold (M,g) that is a Riemannian product of Einstein manifolds. Then (M,g) is Einstein if and only if the factors have the same Einstein constant E. It turns out that if $E>0$, then (M,g) is always unstable (see [Krö13, Prop. 3.3.7]). For example, if (M,g) is the Riemannian product of two Einstein manifolds $(M_i^{n_i},g_i)$ $(i=1,2)$ with the same Einstein constant, then an unstable direction is given by

$$h:=n_2\pi_1^*g_1-n_1\pi_2^*g_2,$$

where $\pi_i:M\to M_i$ are the projections onto each factor, respectively. In particular, a product of symmetric spaces of compact type is always unstable since the factors have positive curvature.

If we take (M,g) to be locally symmetric of compact type, we cannot in general conclude its instability from the instability of its universal cover $(\tilde{M},\tilde{g})$. The same holds for the existence of infinitesimal Einstein deformations. On the other hand, if $(\tilde{M},\tilde{g})$ is infinitesimally non-deformable (resp. stable), then the same follows for (M,g). In [Koi82], N. Koiso has proved the infinitesimal non-deformability of a large class of such manifolds:

4.2.4 Theorem. *Let* (M,g) *be a locally symmetric Einstein manifold of compact type. Let* $(\tilde{M},\tilde{g})$ *be its universal cover and* $(\tilde{M},\tilde{g})=\prod_{i=1}^N(M_i,g_i)$ *its decomposition into irreducible symmetric spaces.*

1. *For* $N=1$*, see Theorem 4.2.1, 1.*

2. *If* $N=2$ *and* M_i *are neither of the spaces listed in Theorem 4.2.1, 1., nor* G_2 *or any Hermitian space except* S^2*, then* (M,g) *is infinitesimally non-deformable.*

3. *If $N \geq 3$ and M_i are neither of the above nor S^2, then (M, g) is infinitesimally non-deformable.*

A closely related notion of stability arises in the study of the Ricci flow. The fixed points (modulo diffeomorphisms and scaling) of the Ricci flow are called Ricci solitons. The *ν-entropy* defined by G. Perelman is a quantity that increases monotonically under the Ricci flow. Its critical points are the *shrinking gradient Ricci solitons*, which include Einstein manifolds. An Einstein metric is called *ν-linearly stable* if the second variation of the ν-entropy is negative-semidefinite. H.-D. Cao, R. Hamilton and T. Ilmanen first studied the ν-linear stability of Einstein metrics (see [CHI04]). It turns out that an Einstein metric is ν-linearly stable if and only if $\Delta_{\mathrm{L}} \geq 2E$ on tt-tensors and if the first nonzero eigenvalue of the ordinary Laplacian on functions is bounded below by $2E$ as well. In particular, ν-linear stability implies linear stability with respect to the Einstein–Hilbert action. In [CH15], the ν-linear stability of irreducible symmetric spaces of compact type is completely decided.

There is yet another notion of stability worth mentioning. It is motivated, for example, by the investigation of Anti-de Sitter product spacetimes and generalized Schwarzschild–Tangherlini spacetimes (see [Die13] or [GHP03]). An Einstein manifold (M^n, g) with Einstein constant E is called *physically stable* if

$$\Delta_{\mathrm{L}} \geq \frac{E}{n-1}\left(4 - \frac{1}{4}(n-5)^2\right) = \frac{9-n}{4}E$$

on tt-tensors. This critical eigenvalue is significantly smaller than the one from stability with respect to the Einstein–Hilbert action, and even negative for $n > 9$. As it turns out, all irreducible symmetric spaces of compact type are physically stable (see [Die13]). If (M, g) is a product of at least two symmetric spaces of compact type, then the smallest eigenvalue of Δ_{L} on tt-tensors is actually equal to 0; hence (M, g) is physically stable if and only if $n \geq 9$.

In §4.3 we fix the notation and definitions used throughout this work. In particular, we elaborate on the notion of stability of an Einstein metric. In §4.4 we recall some tools from the harmonic analysis of homogeneous spaces that are routinely employed. Furthermore we prove a technical lemma that allows us to make explicit computations involving the divergence operator. A helpful formula for the dimension of tt-eigenspaces of the Lichnerowicz Laplacian is worked out in §4.5, generalizing a proposition of Koiso and utilizing properties of Killing vector fields on Einstein manifolds. §4.6 uses representation theory to determine the stability of $\mathrm{SU}(n)$, making use of the formula from §4.5; in §4.7, the same is done for $\mathrm{E}_6/\mathrm{F}_4$. A different approach for proving the stability of both spaces that involves explicit computations of the divergence operator can be found in the Appendix (§4.8).

4.3 Preliminaries

Throughout what follows, let (M, g) be a compact, orientable Riemannian manifold. Let ∇ denote the Levi-Civita connection of g. The Riemannian curvature tensor, Ricci tensor and scalar curvature are in our convention given as

$$\begin{aligned} R(X,Y)Z &:= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ \mathrm{Ric}(X,Y) &:= \mathrm{tr}(Z \mapsto R(Z,X)Y), \\ \mathrm{scal} &:= \mathrm{tr}_g \mathrm{Ric}, \end{aligned}$$

respectively.[1] The action of the Riemannian curvature extends to an endomorphism on tensor bundles as

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where ∇ also denotes the induced connection on the respective tensor bundle. Furthermore, let $\mathscr{S}^p(M) = \Gamma(\mathrm{Sym}^p T^*M)$ for $p \geq 0$. We denote by

$$\delta : \ \mathscr{S}^{p+1}(M) \to \mathscr{S}^p(M)$$

the *divergence* operator on symmetric tensors, given by

$$\delta = -\sum_i e_i \lrcorner \nabla_{e_i}.$$

The space of tt-tensors, i.e. trace- and divergence-free symmetric 2-tensors on M, is denoted by $\mathscr{S}^2_{\mathrm{tt}}(M)$.

Let $\delta^* : \mathscr{S}^p(M) \to \mathscr{S}^{p+1}(M)$ be the formal adjoint[2] of the divergence operator. It can be written as

$$\delta^* = \sum_i e_i^\flat \odot \nabla_{e_i},$$

where (e_i) is a local orthonormal basis of TM. Here, $\odot$ denotes the (associative) symmetric product, defined by

$$\alpha \odot \beta := \frac{(k+l)!}{k!l!} \mathrm{sym}(\alpha \otimes \beta)$$

for $\alpha \in \mathrm{Sym}^k T$, $\beta \in \mathrm{Sym}^l T$, where T is any vector space and the symmetrization map $\mathrm{sym} : T^{\otimes k} \to \mathrm{Sym}^k T$ is given by

$$\mathrm{sym}(X_1 \otimes \ldots \otimes X_k) := \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(k)}$$

[1] We use the index g only when the metric-dependence of an object is to be emphasized.

[2] That is, with respect to the inner product $\langle \cdot, \cdot \rangle_g$ on $\mathrm{Sym}^p T^*M$ with orthonormal basis $(e_{i_1}^\flat \odot \ldots \odot e_{i_p}^\flat)$.

for $X_1, \ldots, X_k \in T$. This is analogous to the definition of the wedge product via the alternation map. For tensors α, β of rank 1, we have

$$\alpha \odot \beta = \alpha \otimes \beta + \beta \otimes \alpha.$$

It should be noted that $\delta^* X^\flat = L_X g$ for any vector field $X \in \mathfrak{X}(M)$. Consequently, the kernel of δ^* on $\Omega^1(M)$ is (via the metric) isomorphic to the space of Killing vector fields on (M, g). More generally, symmetric tensors $\alpha \in \mathscr{S}^k(M)$ with $\delta^*\alpha = 0$ are called *Killing tensors* of rank k, and δ^* is sometimes called the *Killing operator*.

4.3.1 Definition. On tensors of any rank, the following operators are defined:

1. The *curvature endomorphism* $q(R)$ is defined by

$$q(R) := \sum_{i<j} (e_i \wedge e_j)_* R(e_i, e_j),$$

   where (e_i) is a local orthonormal basis of TM and the asterisk indicates the natural action of $\Lambda^2 T \cong \mathfrak{so}(T)$.

2. The *Lichnerowicz Laplacian* Δ_{L} is defined by

$$\Delta_{\mathrm{L}} := \nabla^*\nabla + q(R).$$

   Recall that on $\Omega^p(M)$, $p \geq 0$, this coincides with the Hodge Laplacian Δ.

On the space of Riemannian metrics on M, which is an open cone in $\mathscr{S}^2(M)$, the *total scalar curvature functional* or *Einstein-Hilbert action* is given by

$$S(g) = \int_M \mathrm{scal}_g \, \mathrm{vol}_g$$

for any Riemannian metric g on M. As mentioned earlier, if we restrict this functional to the space of metrics of a fixed total volume, then Einstein metrics are precisely the critical points of the restriction of S.

Let (M, g) be an Einstein manifold with Einstein constant $E \in \mathbb{R}$, that is

$$\mathrm{Ric} = Eg,$$

and suppose that (M, g) is not isometric to a standard round sphere. Denote

$$C_g^\infty(M) = \left\{ f \in C^\infty(M) \,\middle|\, \int_M f \, \mathrm{vol}_g = 0 \right\}.$$

It is well known (see [Bes87]) that there is a decomposition of $\mathscr{S}^2(M)$, which is orthogonal with respect to the second variation S''_g of the total scalar curvature functional, into the four summands

$$\mathscr{S}^2(M) = \mathbb{R}g \oplus C^\infty_g(M)g \oplus \operatorname{im}\delta^* \oplus \mathscr{S}^2_{\mathrm{tt}}(M).$$

These correspond to infinitesimal changes in the metric by homothety, volume-preserving conformal scaling, the action of diffeomorphisms, and moving within $\mathfrak{S}$, respectively. The second variation S''_g is positive on $C^\infty_g(M)g$, zero on $\operatorname{im}\delta^*$ and is given by

$$S''_g(h,h) = -\frac{1}{2}\left(\Delta_{\mathrm{L}}h - 2Eh, h\right)_g$$

on $\mathscr{S}^2_{\mathrm{tt}}(M)$, where it has finite coindex and nullity; that is, the maximal subspace of $\mathscr{S}^2_{\mathrm{tt}}(M)$ where S''_g is nonnegative is finite-dimensional. In fact, the null directions in $\mathscr{S}^2_{\mathrm{tt}}(M)$ are precisely the *infinitesimal Einstein deformations* of g, i.e. infinitesimal deformations of g that preserve the Einstein property, the total volume and are orthogonal to the orbit of g under diffeomorphisms.

4.3.2 Definition. An Einstein metric g on M is called

1. *(linearly) stable* (with respect to the Einstein–Hilbert action) if $S''_g \leq 0$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$ or, equivalently, if $\Delta_{\mathrm{L}} \geq 2E$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$. Otherwise it is called *(linearly) unstable.*

2. *strictly (linearly) stable* (with respect to the Einstein–Hilbert action) if $S''_g < 0$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$ or, equivalently, if $\Delta_{\mathrm{L}} > 2E$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$.

3. *infinitesimally deformable* if $\Delta_{\mathrm{L}}h = 2Eh$ for some nonzero $h \in \mathscr{S}^2_{\mathrm{tt}}(M)$.

4.4 Invariant differential operators

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and K a closed subgroup such that $(M = G/K, g)$ is a reductive Riemannian homogeneous space with K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ is the reductive complement which is canonically identified with the tangent space T_oM at the base point $o := eK \in M$. Recall that for some representation $\rho : K \to \operatorname{Aut} V$, the *left-regular representation* on the space of K-equivariant smooth functions $C^\infty(G,V)^K$ is defined as

$$\ell : G \to \operatorname{Aut} C^\infty(G,V)^K : \ (\ell(x)f)(y) := f(x^{-1}y)$$

for $x, y \in G$. Furthermore, the space $C^\infty(G,V)^K$ is identified with the space of sections of the associated bundle $G \times_\rho V$ over M. The identification is given by

$$\Gamma(G \times_\rho V) \to C^\infty(G,V)^K : \ s \mapsto \hat{s},$$

where $\hat{s}$ is defined by $s([x]) = [x, \hat{s}(x)]$ for any $x \in G$. If V can be expressed in terms of the isotropy representation $\mathfrak{m}$, then $G \times_\rho V$ is a tensor bundle; for example, we have

$$\begin{aligned}
\mathfrak{X}(M) = \Gamma(TM) &\cong \Gamma(G \times_\rho \mathfrak{m}) \cong C^\infty(G, \mathfrak{m})^K, \\
\Omega^1(M) = \Gamma(T^*M) &\cong \Gamma(G \times_\rho \mathfrak{m}^*) \cong C^\infty(G, \mathfrak{m})^K, \\
\mathscr{S}^2(M) = \Gamma(\operatorname{Sym}^2 T^*M) &\cong \Gamma(G \times_\rho \operatorname{Sym}^2 \mathfrak{m}^*) \cong C^\infty(G, \operatorname{Sym}^2 \mathfrak{m})^K, \\
\mathscr{S}^2_0(M) = \Gamma(\operatorname{Sym}^2_0 T^*M) &\cong \Gamma(G \times_\rho \operatorname{Sym}^2_0 \mathfrak{m}^*) \cong C^\infty(G, \operatorname{Sym}^2_0 \mathfrak{m})^K,
\end{aligned}$$

where Sym^2_0, $\mathscr{S}^2_0$ denotes the space of trace-free elements with respect to the metric. Note that the invariant Riemannian metric yields an equivalence between $\mathfrak{m}$ and $\mathfrak{m}^*$.

Suppose that V is a complex representation. Choose a maximal torus T inside G with Lie algebra $\mathfrak{t}$. Recall that up to equivalence, every irreducible finite-dimensional complex representation of G is characterized by its highest weight $\gamma \in \mathfrak{t}^*$. By the Peter-Weyl theorem and Frobenius reciprocity (cf. [Wal73]), the left-regular representation $C^\infty(G, V)^K$ can be decomposed into irreducible summands as[3]

$$C^\infty(G, V)^K \cong \overline{\bigoplus_\gamma} V_\gamma \otimes \operatorname{Hom}_K(V_\gamma, V), \tag{4.1}$$

where γ runs over all highest weights of G-representations and (V_γ, ρ_γ) is the (up to equivalence) unique irreducible representation of G with highest weight γ. For any

$$\alpha \otimes A \in V_\gamma \otimes \operatorname{Hom}_K(V_\gamma, V),$$

the corresponding element of $C^\infty(G, V)^K$ is defined by

$$f^A_\alpha : \ G \to V : \ x \mapsto A(\rho_\gamma(x^{-1})\alpha).$$

Since the Lichnerowicz Laplacian Δ_{L} on $\Gamma(G \times_\rho V)$ is a G-invariant differential operator, Schur's Lemma implies that on each of the isotypical subspaces

$$V_\gamma \otimes \operatorname{Hom}_K(V_\gamma, V),$$

Δ_{L} acts as an endomorphism of the finite-dimensional vector space $\operatorname{Hom}_K(V_\gamma, V)$, that is,

$$\Delta_{\mathrm{L}} f^A_\alpha = f^{L_\gamma(A)}_\alpha$$

for some $L_\gamma \in \operatorname{End} \operatorname{Hom}_K(V_\gamma, V)$.

[3]Here, the bar over the direct sum denotes the closure in $C^\infty(G, V)^K$ (with the L^2 inner product). In other words, $\bigoplus_\gamma V_\gamma \otimes \operatorname{Hom}_K(V_\gamma, V)$ is dense in $C^\infty(G, V)^K$. In fact, it is dense in $L^2(G, V)^K$, but for our purposes, it suffices to consider smooth sections.

In order to obtain the spectrum of Δ_{L}, one would have to find the eigenvalues of each L_γ – a potentially very cumbersome task. We will shortly see that this matter is considerably simpler in the symmetric case.

Fix an Ad-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$. If we assume that G is semisimple, one such inner product is given by $-B_{\mathfrak{g}}$, where $B_{\mathfrak{g}}$ is the Killing form on $\mathfrak{g}$, defined by

$$B_{\mathfrak{g}}(X, Y) := \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$$

for $X, Y \in \mathfrak{g}$. Recall that for any representation $\pi : G \to \operatorname{Aut} W$, the Casimir operator Cas_π^G with respect to the chosen inner product is an equivariant endomorphism of W, defined as

$$\mathrm{Cas}_\pi^G := -\sum_i \pi_*(e_i) \circ \pi_*(e_i)$$

for any orthonormal basis (e_i) of $\mathfrak{g}$.

The following proposition combines two well-known results that allow us to compute the eigenvalues of Δ_{L} on compact symmetric spaces, the latter being a formula due to H. Freudenthal (cf. [FH91]).

4.4.1 Proposition. *Let $(M = G/K, g)$ be a compact Riemannian symmetric space where the Riemannian metric is induced by an* Ad*-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, and let $\rho : K \to \operatorname{Aut} V$ be a representation.*

1. *On the left-regular representation $\Gamma(G \times_\rho V)$, the Lichnerowicz Laplacian Δ_{L} coincides with the Casimir operator Cas_ℓ^G of the representation $\ell : G \to \operatorname{Aut} \Gamma(G \times_\rho V)$.*
2. *On each irreducible representation V_γ, the Casimir eigenvalue is given by*

$$\mathrm{Cas}_\gamma^G = \langle \gamma, \gamma + 2\delta_{\mathfrak{g}} \rangle_{\mathfrak{t}^*},$$

where $\delta_{\mathfrak{g}}$ is the half-sum of positive roots and $\langle \cdot, \cdot \rangle_{\mathfrak{t}^*}$ is the inner product on $\mathfrak{t}^*$ induced by the inner product on $\mathfrak{t} \subset \mathfrak{g}$.

4.4.2 Remark. The first statement is a consequence of a more general result. Let G be a compact Lie group and $(M = G/K, g)$ be a reductive Riemannian homogeneous space. To the reductive decomposition corresponds a canonical G-invariant connection on M (also called the *Ambrose–Singer connection*), which we denote by $\bar{\nabla}$. This connection in turn defines a curvature tensor $\bar{R}$ and an analogue to the Lichnerowicz Laplacian via

$$\bar{\Delta} := \bar{\nabla}^* \bar{\nabla} + q(\bar{R}),$$

called the *standard Laplacian* of this connection (introduced in [SW18]). Then, in fact, $\bar{\Delta} = \mathrm{Cas}_\ell^G$ on $\Gamma(G \times_\rho V)$. The above statement follows when we note that on Riemannian symmetric spaces, the Ambrose–Singer connection coincides with the Levi–Civita connection.

According to (4.1) we can write the complexified left-regular representation on trace-free symmetric 2-tensors as

$$\mathscr{S}^2_0(M)^{\mathbb{C}} \cong \overline{\bigoplus_{\gamma}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}).$$

Recall that irreducible symmetric spaces of compact type can be endowed with a Riemannian metric induced by the Killing form (the so-called standard metric). In this case, the critical eigenvalue of Δ_{L} is $2E = 1$. Supposing we have a representation V_γ with subcritical Casimir eigenvalue $\mathrm{Cas}^G_\gamma < 1$ occurring in this decomposition, it remains to check whether the tensors in the corresponding subspace are divergence-free. By Schur's Lemma, the G-invariant operator

$$\delta : \ \mathscr{S}^2_0(M)^{\mathbb{C}} \to \Omega^1(M)^{\mathbb{C}}$$

is constant on each irreducible subspace. This means that we can regard δ as a linear mapping

$$\delta\Big|_\gamma : \ \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_K(V_\gamma, \mathfrak{m}^{\mathbb{C}}),$$

the so-called *prototypical differential operator* associated to δ and V_γ. For a further discussion of invariant differential operators on homogeneous spaces, we refer the reader to [SW22, §2].

The following lemma is of use when we need to calculate δ explicitly. A derivation of essentially the same formula can also be found in [SW22, §2].

4.4.3 Lemma. *Suppose (M, g) is a Riemannian symmetric space. Let $h \in \mathscr{S}^2(M)^{\mathbb{C}}$ correspond to an element*

$$\alpha \otimes A \in V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2 \mathfrak{m}^{\mathbb{C}})$$

in the decomposition (4.1) *of $\mathscr{S}^2(M)^{\mathbb{C}}$. Let further (e_i) be an orthonormal basis of $\mathfrak{m}$. Then we have*

$$(\delta h)_o(X) = \sum_i \langle A((\rho_\gamma)_*(e_i)\alpha), e_i \odot X\rangle$$

for any $X \in \mathfrak{m} \cong T_oM$.

Proof. The element of $C^\infty(G, \mathrm{Sym}^2 \mathfrak{m}^{\mathbb{C}})^K$ corresponding to $h \in \mathscr{S}^2(M)$ is given by

$$\hat{h} = f^A_\alpha : \ G \to \mathrm{Sym}^2 \mathfrak{m}^{\mathbb{C}} : \ x \mapsto A(\rho_\gamma(x^{-1})\alpha),$$

where ρ_γ is the representation of G on V_γ. The covariant derivative of h at the base point may be expressed by

$$(\nabla h)_o(X,Y) = \langle d\hat{h}_e, X \odot Y\rangle$$

for $X, Y \in \mathfrak{m} \cong T_oM$, since ∇ coincides with the Ambrose-Singer connection on M as a reductive homogeneous space. This implies that

$$\begin{aligned}(\delta h)_o(X) &= -\sum_i e_i \lrcorner \nabla_{e_i} h(X) = -\sum_i \nabla_{e_i} h(e_i, X) = -\sum_i \langle d\hat{h}(e_i), e_i \odot X\rangle \\ &= -\sum_i \langle df_\alpha^A(e_i), e_i \odot X\rangle = \sum_i \langle A((\rho_\gamma)_*(e_i)\alpha), e_i \odot X\rangle.\end{aligned}$$

□

4.5 tt-Eigenspaces of the Lichnerowicz Laplacian

We return to the general setting of a compact Einstein manifold (M,g). Define

$$\theta : \Omega^1(M) \to \mathscr{S}^2_0(M) : \ \alpha \mapsto \delta^*\alpha + \frac{2}{n}\delta\alpha \cdot g,$$

so that $\theta\alpha$ is precisely the trace-free part of $\delta^*\alpha \in \mathscr{S}^2(M)$. The kernel of this operator is (via the metric) isomorphic to the space of *conformal Killing fields* on (M,g), that is, the space of vector fields $X \in \mathfrak{X}(M)$ such that $L_X g = fg$ for some $f \in C^\infty(M)$. We thus call θ the *conformal Killing operator.*

The following lemma is a generalization of a proposition by Koiso [Koi82, Prop. 3.3]. For the proof, we refer the reader to the Appendix.

4.5.1 Lemma. *Let (M,g) be a compact Einstein manifold of dimension $n \geq 3$. For any $\lambda \in \mathbb{R}$, the dimension of the eigenspace of Δ_{L} to the eigenvalue λ on tt-tensors is given by*

$$\begin{aligned}\dim\ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} =& \dim\ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)} - \dim\ker(\Delta - \lambda)\big|_{\Omega^1(M)} \\ &+ \dim\left(\ker(\Delta - \lambda)\big|_{\Omega^1(M)} \cap \ker\theta\right).\end{aligned}$$

At first glance the third term on the right hand side of the above formula does not look very amenable to computation. However, matters are made easier if we observe the following properties of (conformal) Killing vector fields on Einstein manifolds, both of which are proven in the Appendix.

4.5.2 Lemma. *On any compact Einstein manifold (M,g) not isometric to a standard round sphere, conformal Killing fields are actually Killing, that is, $L_X g = fg$ for some*

$f \in C^\infty(M)$ *implies* $f = 0$*. Equivalently,* $\ker \theta = \ker \delta^*$ *on* $\Omega^1(M)$*.*

4.5.3 Lemma. *Any Killing field* $X \in \mathfrak{X}(M)$ *on an Einstein manifold with Einstein constant* E *satisfies*

$$\Delta X^\flat = 2EX^\flat.$$

Equivalently, $\ker \delta^* \subset \ker(\Delta - 2E)$ *on* $\Omega^1(M)$*.*

If we assume that (M, g) is not isometric to a standard sphere, we can immediately conclude that the intersection $\ker(\Delta - \lambda)\big|_{\Omega^1(M)} \cap \ker \theta$ is trivial if $\lambda \neq 2E$. By virtue of Lemma 4.5.1, we obtain the following.

4.5.4 Corollary. *Let* (M, g) *be a compact Einstein manifold that is not isometric to a standard round sphere, and let* E *be its Einstein constant. For any* $\lambda \neq 2E$*, the dimension of the eigenspace of* Δ_{L} *to the eigenvalue* λ *on tt-tensors is given by*

$$\dim \ker(\Delta_{\mathrm{L}} - \lambda)\Big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} = \dim \ker(\Delta_{\mathrm{L}} - \lambda)\Big|_{\mathscr{S}^2_0(M)} - \dim \ker(\Delta - \lambda)\Big|_{\Omega^1(M)}.$$

4.5.5 Remark. If we set $\lambda = 2E$ in Lemma 4.5.1 and note that

$$\ker(\Delta - 2E)\Big|_{\Omega^1(M)} \cap \ker \theta = \ker \delta^*\Big|_{\Omega^1(M)}$$

(as Koiso did in his proof of [Koi82, Prop. 3.3]), we recover the original formula for the critical eigenvalue

$$\begin{aligned} \dim \ker(\Delta_{\mathrm{L}} - 2E)\Big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} = \dim \ker(\Delta_{\mathrm{L}} - 2E)\Big|_{\mathscr{S}^2_0(M)} &- \dim \ker(\Delta - 2E)\Big|_{\Omega^1(M)} \\ &+ \dim \ker \delta^*\Big|_{\Omega^1(M)}. \end{aligned}$$

4.5.6 Remark. Although the dimension formula of Lemma 4.5.1 works on any compact Einstein manifold (M, g), it is worth mentioning that if additionally, (M, g) carries the structure of a Riemannian homogeneous space $M = G/K$, the result can be refined in terms of irreducible representations of G. Namely, if V_γ is an irreducible representation of G, then the multiplicity of V_γ in the (complexified) left-regular representation on tt-tensors is given by

$$\begin{aligned} \dim \mathrm{Hom}_G(V_\gamma, \mathscr{S}^2_{\mathrm{tt}}(M)^{\mathbb{C}}) = \dim \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) &- \dim \mathrm{Hom}_K(V_\gamma, \mathfrak{m}^{\mathbb{C}}) \\ &+ \dim \mathrm{Hom}_G(V_\gamma, (\ker \theta)^{\mathbb{C}}). \end{aligned}$$

As in the proof of Lemma 4.5.1, the dimension formula essentially arises from the short exact sequence

$$0 \longrightarrow \ker \theta \xrightarrow{\subset} \Omega^1(M) \xrightarrow{\theta} \mathscr{S}^2_0(M) \xrightarrow{P} \mathscr{S}^2_{\mathrm{tt}}(M) \longrightarrow 0$$

and the fact that the Laplacian commutes with every arrow. In the homogeneous case, we note that we have a short exact sequence of G-representations and use Frobenius reciprocity to arrive at the statement.

4.6 The symmetric space $\mathrm{SU}(n)$

Throughout what follows, let $n \geq 3$. As a symmetric space, $\mathrm{SU}(n) = G/K$ where $G = \mathrm{SU}(n) \times \mathrm{SU}(n)$ and $K = \mathrm{SU}(n)$ is diagonally embedded, i.e. via

$$\mathrm{SU}(n) \hookrightarrow \mathrm{SU}(n) \times \mathrm{SU}(n): \; k \mapsto (k, k).$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the corresponding Lie algebras of G and K, respectively. We endow M with the standard metric g induced by the Killing form on $\mathfrak{g}$. Hence, M is Einstein with critical eigenvalue $2E = 1$. The reductive decomposition of $\mathfrak{g}$ with respect to g is given by

$$\mathfrak{g} = \tilde{\mathfrak{k}} \oplus \mathfrak{m},$$

where

$$\begin{aligned} \tilde{\mathfrak{k}} &= \{(X, X) \,|\, X \in \mathfrak{k}\}, \\ \mathfrak{m} &= \{(X, -X) \,|\, X \in \mathfrak{k}\}. \end{aligned}$$

The K-representations $\mathfrak{k}$, $\tilde{\mathfrak{k}}$ and $\mathfrak{m}$ are all equivalent. We denote by $E = \mathbb{C}^n$ the standard representation of K.

4.6.1 Lemma. *Let V_γ be an irreducible complex representation of G with $\mathrm{Cas}^G_\gamma < 1$ and*

$$\mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{k}^{\mathbb{C}}) \neq 0.$$

Then V_γ is equivalent to one of the G-representations $E \otimes E^*$ and $E^* \otimes E$. In fact,

$$\dim \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{k}^{\mathbb{C}}) = 1$$

and the Casimir eigenvalue is $\mathrm{Cas}^G_\gamma = \frac{(n-1)(n+1)}{n^2}$.

Proof. Let $\mathfrak{t}$ be the torus of diagonal matrices in $\mathfrak{k}$. The dual $\mathfrak{t}^*$ is generated by the weights $\varepsilon_1, \ldots, \varepsilon_n$ of the defining representation E. Explicitly,

$$\varepsilon_j(X) = X_j, \quad 1 \leq j \leq n$$

for $X = \mathrm{diag}(\mathrm{i}X_1, \ldots, \mathrm{i}X_n) \in \mathfrak{t}$. Note that $\varepsilon_1 + \ldots + \varepsilon_n = 0$.

Fix the ordering on roots and weights such that the simple roots of $\mathfrak{k}$ are given by

$$\varepsilon_j - \varepsilon_{j+1}, \quad 1 \leq j \leq n-1.$$

The semigroup of dominant integral weights is then generated by the fundamental weights

$$\omega_j = \sum_{k=1}^{j} \varepsilon_j, \quad 1 \leq j \leq n-1,$$

cf. [FH91, §15.1]. The highest weights of representations of K, i.e. all the dominant integral weights, are precisely the linear combinations

$$\gamma = \sum_{r=1}^{n-1} a_r \omega_r$$

with coefficients $a_r \in \mathbb{N}_0$. The fundamental weights themselves correspond to the representations

$$V_{\omega_r} = \Lambda^r E \cong \Lambda^{n-r} E^*.$$

Let $\gamma, \gamma' \in \mathfrak{t}^*$ be two dominant integral weights. In particular, they satisfy

$$\langle \gamma, \gamma' \rangle_{\mathfrak{t}^*} \geq 0.$$

Using Freudenthal's formula for the Casimir operator Cas_γ^K of a K-representation V_γ, this implies the estimate

$$\begin{aligned} \mathrm{Cas}_{\gamma+\gamma'}^K &= \langle \gamma + \gamma' + 2\delta_{\mathfrak{k}}, \gamma + \gamma' \rangle_{\mathfrak{t}^*} = \langle \gamma + 2\delta_{\mathfrak{k}}, \gamma \rangle_{\mathfrak{t}^*} + 2\langle \gamma, \gamma' \rangle_{\mathfrak{t}^*} + \langle \gamma' + 2\delta_{\mathfrak{k}}, \gamma' \rangle_{\mathfrak{t}^*} \\ &\geq \langle \gamma + 2\delta_{\mathfrak{k}}, \gamma \rangle_{\mathfrak{t}^*} + \langle \gamma' + 2\delta_{\mathfrak{k}}, \gamma' \rangle_{\mathfrak{t}^*} = \mathrm{Cas}_\gamma^K + \mathrm{Cas}_{\gamma'}^K . \end{aligned}$$

In particular we obtain

$$\mathrm{Cas}_\gamma^K \geq \sum_r a_r \,\mathrm{Cas}_{\omega_r}^K \tag{$*$}$$

for $\gamma = \sum_{r=1}^{n-1} a_r \omega_r$.

The Casimir eigenvalues of the fundamental representations are given as

$$\mathrm{Cas}_{\omega_r}^K = \frac{(n+1)r(n-r)}{2n^2}$$

for $r = 1, \ldots, n-1$. Note that this expression is symmetric around $r = \frac{n}{2}$ and strictly increasing for $r \leq \frac{n}{2}$. Furthermore, we can compute that

$$\begin{aligned}
\mathrm{Cas}^K_{\omega_1} &= \frac{(n+1)(n-1)}{2n^2} < 1, \\
\mathrm{Cas}^K_{\omega_2} &= \frac{(n+1)(n-2)}{n^2} < 1, \\
\mathrm{Cas}^K_{\omega_3} &= \begin{cases} \frac{7}{8} < 1, & n = 6, \\ \frac{48}{49} < 1, & n = 7, \\ \frac{3(n+1)(n-3)}{2n^2} > 1, & n \geq 8, \end{cases} \\
\mathrm{Cas}^K_{\omega_1} + \mathrm{Cas}^K_{\omega_2} &> 1, \ n \geq 4, \\
\mathrm{Cas}^K_{2\omega_1} &> 1, \\
\mathrm{Cas}^K_{\omega_1+\omega_{n-1}} &= 1,
\end{aligned}$$

cf. table on p. 15 of [SW22]. Combining the above with inequality $(*)$, we can deduce that if γ is a highest weight with $\mathrm{Cas}^K_\gamma < 1$, then necessarily

$$\gamma \in \{0, \omega_1, \omega_{n-1}, \omega_2, \omega_{n-2}, \underbrace{\omega_3, \omega_{n-3}}_{\text{if } n=6,7}\}.$$

These dominant integral weights are, respectively, highest weights of the representations $\mathbb{C}$, E, E^*, $\Lambda^2 E$, $\Lambda^2 E^*$, $\Lambda^3 E$, $\Lambda^3 E^*$ of K.

The irreducible representations of $G = K \times K$ are precisely the tensor products of irreducible representations of K. Let γ, γ' be highest weights of K-representations such that

$$\mathrm{Cas}^G_{(\gamma,\gamma')} = \mathrm{Cas}^K_\gamma + \mathrm{Cas}^K_{\gamma'} < 1$$

holds. Assuming that $\gamma, \gamma' \neq 0$, we conclude that $\gamma, \gamma' \in \{\omega_1, \omega_{n-1}\}$. This yields the four pairwise inequivalent G-representations $E \otimes E$, $E \otimes E^*$, $E^* \otimes E$ and $E^* \otimes E^*$. Furthermore, in the case of $\gamma = 0$ or $\gamma' = 0$ we obtain the representations of K that were listed above, composed with the projection onto one factor,

$$G \to K : \ (k_1, k_2) \mapsto k_1 \quad \text{or} \quad (k_1, k_2) \mapsto k_2,$$

respectively. By restricting the mentioned G-representations to K via the embedding

$$K \to G : \ k \mapsto (k, k),$$

we again obtain the irreducible K-representations $\mathbb{C}$, E, E^*, $\Lambda^2 E$, $\Lambda^2 E^*$, $\Lambda^3 E$, $\Lambda^3 E^*$ as well as the tensor product representations $E \otimes E$, $E \otimes E^*$ and $E^* \otimes E^*$. The latter are not irreducible, but decompose into irreducible summands as follows:

$$\begin{aligned} E \otimes E &= \operatorname{Sym}^2 E \oplus \Lambda^2 E, \\ E \otimes E^* &= E \otimes_0 E^* \oplus \mathbb{C}, \\ E^* \otimes E^* &= \operatorname{Sym}^2 E^* \oplus \Lambda^2 E^*. \end{aligned}$$

Here $E \otimes_0 E^*$ is the set of trace-free elements of $E \otimes E^*$ when regarded as $n \times n$-matrices over $\mathbb{C}$. As a representation of K, we have

$$E \otimes_0 E^* \cong V_{\omega_1+\omega_{n-1}} \cong \mathfrak{k}^{\mathbb{C}}.$$

The K-representation $\operatorname{Sym}^2 \mathfrak{k}^{\mathbb{C}} \cong \operatorname{Sym}^2(E \otimes_0 E^*)$ appears on one hand as a summand of

$$\operatorname{Sym}^2(E \otimes E^*) \cong \operatorname{Sym}^2(E \otimes_0 E^* \oplus \mathbb{C}) \cong \operatorname{Sym}^2(E \otimes_0 E^*) \oplus E \otimes_0 E^* \oplus \mathbb{C}.$$

On the other hand, the symmetric power of the tensor product is given by[4]

$$\operatorname{Sym}^2(E \otimes E^*) \cong \operatorname{Sym}^2 E \otimes \operatorname{Sym}^2 E^* \oplus \Lambda^2 E \otimes \Lambda^2 E^*.$$

The tensor products $\operatorname{Sym}^2 E \otimes \operatorname{Sym}^2 E^*$ and $\Lambda^2 E \otimes \Lambda^2 E^*$ can in turn be decomposed into

$$\begin{aligned} \operatorname{Sym}^2 E \otimes \operatorname{Sym}^2 E^* &\cong V_{2\omega_1+2\omega_{n-1}} \oplus V_{\omega_1+\omega_{n-1}} \oplus \mathbb{C}, \\ \Lambda^2 E \otimes \Lambda^2 E^* &\cong \begin{cases} E^* \otimes E \cong V_{\omega_1+\omega_{n-1}} \oplus \mathbb{C}, & n = 3, \\ V_{\omega_2+\omega_{n-2}} \oplus V_{\omega_1+\omega_{n-1}} \oplus \mathbb{C}, & n \geq 4. \end{cases} \end{aligned}$$

By comparing summands we see that

$$\operatorname{Sym}^2(E \otimes_0 E^*) \cong V_{2\omega_1+2\omega_{n-1}} \oplus \underbrace{V_{\omega_2+\omega_{n-2}}}_{\text{if } n \geq 4} \oplus E \otimes_0 E^* \oplus \mathbb{C}.$$

Hence the trace-free part is given by

$$\operatorname{Sym}^2_0(E \otimes_0 E^*) \cong V_{2\omega_1+2\omega_{n-1}} \oplus \underbrace{V_{\omega_2+\omega_{n-2}}}_{\text{if } n \geq 4} \oplus E \otimes_0 E^*.$$

Now that we have decomposed the relevant representations into irreducible summands, we recognize that $E \otimes E^*$ and $E^* \otimes E$ are the only two of the specified subcritical representations of G that, after restriction to K, have a common summand with $\operatorname{Sym}^2_0 \mathfrak{k}^{\mathbb{C}}$. In

[4]This is a consequence of, for example, the formula $\operatorname{Sym}^d(V \otimes W) = \bigoplus \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\lambda(W)$ in [FH91, Ex. 6.11].

each case, the summand in question $E \otimes_0 E^* \cong \mathfrak{k}^{\mathbb{C}}$ appears with multiplicity 1; hence we have

$$\dim \mathrm{Hom}_K(E \otimes E^*, \mathrm{Sym}^2_0 \mathfrak{k}^{\mathbb{C}}) = \dim \mathrm{Hom}_K(E^* \otimes E, \mathrm{Sym}^2_0 \mathfrak{k}^{\mathbb{C}}) = 1.$$

Moreover, both G-representations exhibit the same Casimir eigenvalue

$$\mathrm{Cas}^G_{(\omega_1,\omega_{n-1})} = \mathrm{Cas}^G_{(\omega_{n-1},\omega_1)} = \mathrm{Cas}^K_{\omega_1} + \mathrm{Cas}^K_{\omega_{n-1}} = \frac{(n-1)(n+1)}{n^2}.$$

□

According to Lemma 4.6.1, the only representations of G (up to equivalence) with subcritical Casimir eigenvalue that occur in decomposition (4.1) of $\mathscr{S}^2_0(M)^{\mathbb{C}}$ are $E \otimes E^*$ and $E^* \otimes E$, and we have

$$\dim \mathrm{Hom}_K(E \otimes E^*, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) = \dim \mathrm{Hom}_K(E^* \otimes E, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) = 1$$

(recall that $\mathfrak{m} \cong \mathfrak{k}$), i.e. the summand occurs with multiplicity 1. It remains to check whether the tensors in the corresponding subspaces are divergence-free. Since

$$E \otimes E^* \cong \mathfrak{k}^{\mathbb{C}} \oplus \mathbb{C}$$

as a representation of K, we have

$$\dim \mathrm{Hom}_K(E \otimes E^*, \mathfrak{m}^{\mathbb{C}}) = \dim \mathrm{Hom}_K(E^* \otimes E, \mathfrak{m}^{\mathbb{C}}) = 1,$$

meaning that both summands also occur in the left-regular representation $\Omega^1(M)$ with the same multiplicity. It now follows from Corollary 4.5.4 that

$$\dim \ker(\Delta_{\mathrm{L}} - \lambda)\Big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} = 0$$

for $\lambda = \frac{(n-1)(n+1)}{n^2}$. Since this is the only subcritical eigenvalue on $\mathscr{S}^2_0(M)$, we have shown the following.

4.6.2 Proposition. *The symmetric space* $\mathrm{SU}(n)$ *is linearly stable.*

4.7 The symmetric space $\mathrm{E}_6/\mathrm{F}_4$

Let $(\mathfrak{H}, \circ)$ be the Albert algebra, where $\mathfrak{H}$ is the set of Hermitian 3×3-matrices over the octonions, i.e.

$$\mathfrak{H} := \left\{ \begin{pmatrix} a & x & \bar{y} \\ \bar{x} & b & z \\ y & \bar{z} & c \end{pmatrix} \middle| a, b, c \in \mathbb{R},\ x, y, z \in \mathbb{O} \right\},$$

and with Jordan multiplication defined by

$$X \circ Y := \frac{1}{2}(XY + YX).$$

The exceptional Lie group E_6 can be realized as

$$\mathrm{E}_6 := \left\{ \alpha \in \mathrm{Aut}_{\mathbb{C}} \mathfrak{H}^{\mathbb{C}} \,\middle|\, \alpha \text{ preserves determinant and inner product} \right\},$$

while F_4 is defined as the set of algebra automorphisms

$$\mathrm{F}_4 := \mathrm{Aut}(\mathfrak{H}, \circ).$$

By complex-linearly extending linear automorphisms of $\mathfrak{H}$, one obtains the inclusion $\mathrm{Aut}_{\mathbb{R}} \mathfrak{H} \subset \mathrm{Aut}_{\mathbb{C}} \mathfrak{H}^{\mathbb{C}}$. In this sense, we have $\mathrm{F}_4 \subset \mathrm{E}_6$. In fact,

$$\mathrm{F}_4 = \mathrm{E}_6 \cap \mathrm{Aut}_{\mathbb{R}} \mathfrak{H}.$$

As a representation of E_6, $\mathfrak{H}^{\mathbb{C}}$ is irreducible. As an F_4-representation, $\mathfrak{H}$ decomposes into the irreducible summands

$$\mathfrak{H} \cong \mathfrak{H}_0 \oplus \mathbb{R},$$

where $\mathfrak{H}_0$ is the set of trace-free elements of $\mathfrak{H}$. An invariant inner product on $\mathfrak{H}$ is defined by

$$\langle A, B \rangle := \mathrm{tr}(A \circ B)$$

for $A, B \in \mathfrak{H}$. An orthogonal basis of $\mathfrak{H}$ (cf. [Yok09, §2.1]) is given by the matrices

$$E_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_1(x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \quad F_2(x) := \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) := \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where x runs through the standard basis of $\mathbb{O}$ as a real vector space.

In this section we consider the Riemannian symmetric space $M = \mathrm{E}_6/\mathrm{F}_4$ equipped with the standard metric (hence with critical eigenvalue $2E = 1$). The reductive decomposition of $\mathfrak{e}_6$ with respect to the standard metric is given by

$$\mathfrak{e}_6 = \mathfrak{f}_4 \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong \mathfrak{H}_0$ as a representation of F_4.

4.7.1 Lemma. *Let V_γ be an irreducible complex representation of E_6 with $\mathrm{Cas}_\gamma^{\mathrm{E}_6} < 1$ and*

$$\mathrm{Hom}_{\mathrm{F}_4}(V_\gamma, \mathrm{Sym}_0^2 \mathfrak{H}_0^{\mathbb{C}}) \neq 0.$$

Then V_γ is equivalent to one of the E_6-representations $\mathfrak{H}^{\mathbb{C}}$ and $\overline{\mathfrak{H}^{\mathbb{C}}}$. In fact,

$$\dim \mathrm{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \mathrm{Sym}_0^2 \mathfrak{H}_0^{\mathbb{C}}) = \dim \mathrm{Hom}_{\mathrm{F}_4}(\overline{\mathfrak{H}^{\mathbb{C}}}, \mathrm{Sym}_0^2 \mathfrak{H}_0^{\mathbb{C}}) = 1,$$

and the Casimir eigenvalue is $\mathrm{Cas}_\gamma^{\mathrm{E}_6} = \frac{13}{18}$.

Proof. We abstain from specifying a particular choice of simple root system and fundamental weights for E_6 and F_4, since we are merely interested in the corresponding fundamental representations of the respective Lie group. Following the enumerative convention of Bourbaki (as used by the software package LiE), if we denote the fundamental weights of E_6 by $\omega_1, \ldots, \omega_6$ and of F_4 by $\eta_1, \ldots, \eta_4$, then the associated representations are identified as

$$\begin{array}{llll}
V_{\omega_1} = \mathbf{27} \cong \mathfrak{H}^{\mathbb{C}}, & V_{\omega_2} = \mathbf{78} \cong \mathfrak{e}_6^{\mathbb{C}}, & V_{\omega_3} = \mathbf{351} \cong \Lambda^2\mathfrak{H}^{\mathbb{C}}, & \\
V_{\omega_4} = \mathbf{2925} \cong \Lambda^3\mathfrak{H}^{\mathbb{C}}, & V_{\omega_5} = \overline{\mathbf{351}} \cong \Lambda^2\overline{\mathfrak{H}^{\mathbb{C}}}, & V_{\omega_6} = \overline{\mathbf{27}} \cong \overline{\mathfrak{H}^{\mathbb{C}}}, & \\
V_{\eta_1} = \mathbf{52} \cong \mathfrak{f}_4^{\mathbb{C}}, & V_{\eta_2} = \mathbf{1274}, & V_{\eta_3} = \mathbf{273}, & V_{\eta_4} = \mathbf{26} \cong \mathfrak{H}_0^{\mathbb{C}},
\end{array}$$

where the number indicates the dimension.

As in the proof of Lemma 4.6.1, we have the estimate

$$\mathrm{Cas}_\gamma^{\mathrm{E}_6} \geq \sum_{r=1}^{6} a_r \,\mathrm{Cas}_{\omega_r}^{\mathrm{E}_6}$$

for any representation V_γ of E_6 with highest weight

$$\gamma = \sum_{r=1}^{6} a_r \omega_r.$$

Among the fundamental representations, only the Casimir eigenvalues

$$\mathrm{Cas}^{\mathrm{E}_6}_{\omega_1} = \mathrm{Cas}^{\mathrm{E}_6}_{\omega_6} = \frac{13}{18}$$

are smaller than 1 (see table on p. 16 of [SW22]). Since $\frac{13}{18} + \frac{13}{18} > 1$, it follows that only the representations to the highest weights $\mathbb{C}, \mathfrak{H}^{\mathbb{C}}, \overline{\mathfrak{H}^{\mathbb{C}}}$ come into question.

Consider now the F_4-representation $\mathfrak{H}_0^{\mathbb{C}} \cong V_{\eta_4}$. We obtain[5] the decomposition

$$\mathrm{Sym}^2 V_{\eta_4} \cong V_{2\eta_4} \oplus V_{\eta_4} \oplus \mathbb{C}$$

into irreducible summands, hence

$$\mathrm{Sym}^2_0 \mathfrak{H}_0^{\mathbb{C}} \cong V_{2\eta_4} \oplus \mathfrak{H}_0^{\mathbb{C}}.$$

Furthermore, we have

$$\mathfrak{H}^{\mathbb{C}} \cong \overline{\mathfrak{H}^{\mathbb{C}}} \cong \mathfrak{H}_0^{\mathbb{C}} \oplus \mathbb{C}$$

as a representation of F_4. The assertion follows by comparison of summands. □

Lemma 4.7.1 now tells us that the representations of E_6 with subcritical Casimir eigenvalue that occur in decomposition (4.1) of $\mathscr{S}^2_0(M)^{\mathbb{C}}$ are precisely $\mathfrak{H}^{\mathbb{C}}$ and $\overline{\mathfrak{H}^{\mathbb{C}}}$, both with multiplicity 1, i.e.

$$\dim \mathrm{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) = \dim \mathrm{Hom}_{\mathrm{F}_4}(\overline{\mathfrak{H}^{\mathbb{C}}}, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) = 1,$$

since $\mathfrak{m} \cong \mathfrak{H}_0$. Again, we have to check whether the tensors in the corresponding subspace are divergence-free. It follows from the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathbb{R}$ as a representation of F_4 that

$$\dim \mathrm{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}) = \dim \mathrm{Hom}_{\mathrm{F}_4}(\overline{\mathfrak{H}^{\mathbb{C}}}, \mathfrak{m}^{\mathbb{C}}) = 1,$$

so as in the previous section, the summand has the same multiplicity in the left-regular representation $\Omega^1(M)$. Again, it follows from Corollary 4.5.4 that

$$\dim \ker(\Delta_{\mathrm{L}} - \lambda)\Big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} = 0$$

for $\lambda = \frac{13}{18}$, and since this is the only subcritical eigenvalue on $\mathscr{S}^2_0(M)$, we have shown the following, which, together with Prop. 4.6.2, finishes the proof of the main theorem.

4.7.2 Proposition. *The symmetric space* $\mathrm{E}_6/\mathrm{F}_4$ *is linearly stable.*

[5]This has been verified through use of the software package LiE [LiE]. Simply enter the command `sym_tensor(2,[0,0,0,1],F4)` into the LiE shell.

4.8 Appendix

4.8.1 Proofs of general statements

Proof of Lemma 4.5.1. The following is a slightly generalized version of the proof of a result by N. Koiso [Koi82, Prop. 3.3]. We first note that

$$(\delta\alpha \cdot g, h)_g = \int_M \delta\alpha \langle g, h\rangle_g \operatorname{vol}_g = \frac{1}{2}(\delta\alpha, \operatorname{tr}_g h)_g = \frac{1}{2}(\alpha, d\operatorname{tr}_g h)_g$$

for $\alpha \in \Omega^1(M)$, $h \in \mathscr{S}^2(M)$, so the formal adjoint of θ is given by

$$\theta^* : \mathscr{S}^2_0(M) \to \Omega^1(M) : \; h \mapsto \delta h + \frac{1}{n} d\operatorname{tr}_g h.$$

We show that θ is overdetermined elliptic. The principal symbol of θ is

$$\sigma_\xi(\theta)\alpha = \sigma_\xi(\delta^*)\alpha + \frac{2}{n}\sigma_\xi(\delta)\alpha \cdot g = \xi \odot \alpha - \frac{2}{n}\langle \xi, \alpha\rangle_g g$$

for $\xi, \alpha \in T_p^*M$. If $\xi \neq 0$, then $\sigma_\xi(\theta)$ is injective: Suppose $\sigma_\xi(\theta)\alpha = 0$. Then

$$\xi \odot \alpha = \frac{2}{n}\langle \xi, \alpha\rangle_g g.$$

Take an orthonormal basis (e_i) with respect to g of T_pM and write

$$\xi = \sum_i \xi_i e_i^\flat, \;\; \alpha = \sum_i \alpha_i e_i^\flat.$$

For $i, j = 1, \dots, n$ it follows that

$$\xi_i \alpha_j + \xi_j \alpha_i = \frac{2}{n}\langle \xi, \alpha\rangle_g \delta_{ij}$$

and so $\xi_i\alpha_j = -\xi_j\alpha_i$ if $i \neq j$, as well as $\xi_i\alpha_i = \xi_j\alpha_j$ for any i, j. Then

$$\xi_i^2 \alpha_j = -\xi_i\alpha_i\xi_j = -\xi_j^2\alpha_j.$$

If $\alpha_j \neq 0$, this would imply that $\xi_i^2 + \xi_j^2 = 0$ and so $\xi_i = \xi_j = 0$ which contradicts the assumption that $\xi \neq 0$. Overall we conclude that $\alpha = 0$ and thus the injectivity is proven.

From ellipticity we obtain the orthogonal decomposition

$$\mathscr{S}^2_0(M) = \operatorname{im}\theta \oplus \ker\theta^*.$$

Let $h \in \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)}$. According to the above decomposition we can write h as

$$h = \theta\alpha + \psi$$

where $\theta^*\psi = 0$. Then also

$$\delta\psi = \theta^*\psi - \frac{1}{n} d \operatorname{tr}_g \psi = 0.$$

Since (M, g) is Einstein, Δ_{L} commutes with δ on $\mathscr{S}^2(M)$ and with δ^* on $\Omega^1(M)$ [Lic61, pp. 10.7/10.8]. Furthermore $\Delta_{\mathrm{L}}(fg) = (\Delta f)g$ for any $f \in C^\infty(M)$. We conclude that Δ_{L} commutes with θ and θ^* as well. This implies that

$$\begin{aligned} \theta(\Delta - \lambda)\alpha &= (\Delta_{\mathrm{L}} - \lambda)\theta\alpha = (\Delta_{\mathrm{L}} - \lambda)(h - \psi) = -(\Delta_{\mathrm{L}} - \lambda)\psi, \\ \theta^*(\Delta_{\mathrm{L}} - \lambda)\psi &= (\Delta - \lambda)\theta^*\psi = 0, \end{aligned}$$

and so

$$\theta^*\theta(\Delta - \lambda)\alpha = -\theta^*(\Delta_{\mathrm{L}} - \lambda)\psi = 0.$$

It follows that

$$\|\theta(\Delta - \lambda)\alpha\|_g^2 = \left(\theta^*\theta(\Delta - \lambda)\alpha, (\Delta - \lambda)\alpha\right)_g = 0$$

and so $\theta(\Delta - \lambda)\alpha = 0 = (\Delta_{\mathrm{L}} - \lambda)\psi$. In total $\psi \in \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)}$.

Also, if h is an element of $\ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)}$, then

$$\theta^* h = \delta h + \frac{1}{n} d \operatorname{tr}_g h = 0$$

and so $\psi = h$. This means that the mapping

$$P : \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)} \to \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} : \ h \mapsto \psi$$

defines a projection, and the dimension formula

$$\dim \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_{\mathrm{tt}}(M)} = \dim\left(\ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)}\right) - \dim \ker P$$

holds.

By definition, the kernel of P consists of those $h \in \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)}$ with $h = \theta\alpha$ for some $\alpha \in \Omega^1(M)$, i.e. $h \in \operatorname{im}\theta$. Hence we know that

$$\ker P = \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)} \cap \operatorname{im}\theta.$$

Let $\alpha \in \ker(\Delta - \lambda)\big|_{\Omega^1(M)}$. We have seen that Δ_{L} commutes with θ, so it follows that

$\theta\alpha \in \ker(\Delta_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)}$ and therefore

$$\theta\left(\ker(\Delta - \lambda)\big|_{\Omega^1(M)}\right) \subset \ker P.$$

Conversely, let $h \in \ker P$. Then there exists some $\alpha \in \Omega^1(M)$ such that $h = \theta\alpha$, and also $h \in \ker(\Delta^g_{\mathrm{L}} - \lambda)\big|_{\mathscr{S}^2_0(M)}$. By the ellipticity of the operator $\Delta - \lambda$ we can decompose α into

$$\alpha = \beta + (\Delta - \lambda)\gamma$$

with $\beta \in \ker(\Delta - \lambda)\big|_{\Omega^1(M)}$, $\gamma \in \Omega^1(M)$. Then

$$\begin{aligned}
0 &= (\Delta_{\mathrm{L}} - \lambda)\theta\alpha \\
&= (\Delta_{\mathrm{L}} - \lambda)\theta\beta + (\Delta_{\mathrm{L}} - \lambda)\theta(\Delta - \lambda)\gamma \\
&= \theta(\Delta - \lambda)\beta + (\Delta_{\mathrm{L}} - \lambda)^2\theta\gamma \\
&= (\Delta_{\mathrm{L}} - \lambda)^2\theta\gamma.
\end{aligned}$$

Since Δ_{L} is self-adjoint, we have

$$\|(\Delta_{\mathrm{L}} - \lambda)\theta\gamma\|_g^2 = \left((\Delta_{\mathrm{L}} - \lambda)^2\theta\gamma, \theta\gamma\right)_g = 0$$

and thus

$$\theta(\Delta - \lambda)\gamma = (\Delta_{\mathrm{L}} - \lambda)\theta\gamma = 0,$$

i.e. $(\Delta - \lambda)\gamma \in \ker\theta$. This implies that $h = \theta\alpha = \theta\beta$, so

$$\theta : \ker(\Delta - \lambda)\big|_{\Omega^1(M)} \to \ker P$$

is surjective and we obtain the dimension formula

$$\dim\ker P = \dim\ker(\Delta - \lambda)\big|_{\Omega^1(M)} - \dim\left(\ker(\Delta - \lambda)\big|_{\Omega^1(M)} \cap \ker\theta\right).$$

□

Proof of Lemma 4.5.2. Let E be the Einstein constant of (M, g). Let $\alpha \in \Omega^1(M)$ such that

$$\theta\alpha = \delta^*\alpha + \frac{2}{n}\delta\alpha \cdot g = 0.$$

Taking the divergence yields

$$\delta\theta\alpha = \delta\delta^*\alpha - \frac{2}{n}d\delta\alpha = 0,$$

since $\delta(fg) = -df$ for $f \in C^\infty(M)$. We make use of the well-known Weitzenböck identities

$$\delta\delta^* - \delta^*\delta = \nabla^*\nabla - q(R) \quad \text{on } \mathscr{S}^k(M),$$
$$\Delta = d^*d + dd^* = \nabla^*\nabla + q(R) \quad \text{on } \Omega^k(M).$$

For $k = 1$ and since $\delta^* = d = \nabla$ on functions and (M, g) is Einstein, these amount to

$$\delta\delta^*\alpha - d\delta\alpha = \nabla^*\nabla\alpha - E\alpha,$$
$$d^*d\alpha + d\delta\alpha = \nabla^*\nabla\alpha + E\alpha.$$

Putting these together we obtain

$$\delta\theta\alpha = \left(1 - \frac{2}{n}\right) d\delta\alpha + \nabla^*\nabla\alpha - E\alpha = \left(2 - \frac{2}{n}\right) d\delta\alpha + d^*d\alpha - 2E\alpha = 0.$$

Taking the L^2 inner product with α then yields

$$\left(2 - \frac{2}{n}\right) \|\delta\alpha\|_g^2 + \|d\alpha\|_g^2 - 2E\|\alpha\|_g = 0.$$

If $E < 0$, this directly implies that $\alpha = 0$. If $E = 0$, it implies $\delta\alpha = 0$ and $d\alpha = 0$, and since $\theta\alpha = 0$, it follows that $\delta^*\alpha = 0$. If $E > 0$, then applying the codifferential to $\delta\theta\alpha$ yields

$$\left(2 - \frac{2}{n}\right) d^*d\delta\alpha + (d^*)^2 d\alpha - 2Ed^*\alpha = \left(2 - \frac{2}{n}\right) \Delta\delta\alpha - 2E\delta\alpha = 0,$$

so $\delta\alpha$ would be an eigenfunction of the Laplacian to the eigenvalue $\frac{En}{n-1} = \frac{\mathrm{scal}_g}{n-1}$. By a theorem of Obata [BGM71, Thm. D.I.6], this eigenvalue can only be attained on the standard sphere, so necessarily $\delta\alpha = 0$. It follows again from $\theta\alpha = 0$ that $\delta^*\alpha = 0$. □

Proof of Lemma 4.5.3. Let $\alpha \in \Omega^1(M)$ such that $\delta^*\alpha = 0$. Then also $\delta\alpha = 0$, since $\delta\alpha = -\operatorname{tr}_g \delta^*\alpha = 0$. By virtue of the Weitzenböck formulae that were already employed in the proof of Lemma 4.5.2 we conclude that

$$\Delta\alpha = \nabla^*\nabla\alpha + E\alpha = \delta\delta^*\alpha - d\delta\alpha + 2E\alpha = 2E\alpha.$$

□

4.8.2 Alternative proof of the stability of $\mathrm{SU}(n)$

An alternative method of checking that the prototypical differential operators

$$\begin{aligned}\delta\Big|_{E\otimes E^*} &: \ \mathrm{Hom}_K(E\otimes E^*, \mathrm{Sym}^2_0\,\mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_K(E\otimes E^*, \mathfrak{m}^{\mathbb{C}}),\\ \delta\Big|_{E^*\otimes E} &: \ \mathrm{Hom}_K(E^*\otimes E, \mathrm{Sym}^2_0\,\mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_K(E^*\otimes E, \mathfrak{m}^{\mathbb{C}})\end{aligned}$$

are injective is an explicit computation by means of Lemma 4.4.3. To do so, we first pick out an explicit element

$$A \in \mathrm{Hom}_K(E\otimes E^*, \mathrm{Sym}^2_0\,\mathfrak{m}^{\mathbb{C}})$$

and then proceed to compute the divergence on the corresponding subspace of $\mathscr{S}^2_0(M)$.

4.8.1 Lemma. *Let* $\pi : \mathrm{Sym}^2(E\otimes E^*) \to E\otimes E^*$ *denote the mapping defined by*

$$\pi(A\odot B) := AB^* + BA^*,$$

where $A, B \in E\otimes E^*$ *are regarded as complex* $n\times n$*-matrices. Then*

$$\pi \in \mathrm{Hom}_K(\mathrm{Sym}^2(E\otimes E^*), E\otimes E^*).$$

Moreover, the restriction

$$\pi : \mathrm{Sym}^2_0(E\otimes_0 E^*) \to E\otimes_0 E^*$$

is surjective, and $W := \left(\ker \pi\Big|_{\mathrm{Sym}^2_0(E\otimes_0 E^*)}\right)^{\perp} \cong E\otimes_0 E^*$.

Proof. The equivariance of π under the action of K follows from

$$\pi(kAk^{-1}\odot kBk^{-1}) = kAk^{-1}(k^{-1})^*B^*k^* + k^{-1}Bk(k^{-1})^*A^*k^* = k(AB^*+BA^*)k^{-1}$$

for any $k \in K = \mathrm{SU}(n)$ and $A, B \in E\otimes E^*$. Furthermore, we have

$$\mathrm{tr}(\pi(A\odot B)) = \mathrm{tr}(AB^*+BA^*) = \langle A, B\rangle + \langle B, A\rangle = \mathrm{tr}(A\odot B),$$

where the last trace is taken with respect to the inner product on $E\otimes E^*$. This means that

$$\pi(\mathrm{Sym}^2_0(E\otimes E^*)) \subset E\otimes_0 E^*.$$

Next we want to show that π does not vanish when restricted to $\mathrm{Sym}^2_0(E\otimes_0 E^*)$. If we denote by E_{ij} the $n\times n$-matrix that has entry 1 at position (i,j) and 0 elsewhere, then we have for example $E_{21}, E_{31} \in E\otimes_0 E^*$ and $\langle E_{21}, E_{31}\rangle = 0$, so $E_{21}\odot E_{31} \in \mathrm{Sym}^2_0(E\otimes_0 E^*)$

and

$$\pi(E_{21} \odot E_{31}) = E_{21}E_{13} + E_{31}E_{12} = E_{23} + E_{32} \neq 0.$$

Now, since $E \otimes_0 E^*$ is irreducible, the mapping

$$\pi : \mathrm{Sym}^2_0(E \otimes_0 E^*) \to E \otimes_0 E^*$$

must be surjective. We have seen in the proof of Lemma 4.6.1 that $E \otimes_0 E^*$ appears in the decomposition of $\mathrm{Sym}^2_0(E \otimes_0 E^*)$ with multiplicity 1; hence $W := \left(\ker \pi\big|_{\mathrm{Sym}^2_0(E\otimes_0 E^*)}\right)^{\perp}$ must be the irreducible summand of $\mathrm{Sym}^2_0(E \otimes_0 E^*)$ that is equivalent to $E \otimes_0 E^*$. $\square$

Alternative proof of Prop. 4.6.2. The properties of π from Lemma 4.8.1 allow us to define

$$\tilde{A} := \pi\big|_W^{-1} \in \mathrm{Hom}_K(E \otimes_0 E^*, \mathrm{Sym}^2_0(E \otimes_0 E^*))$$

and extend it with zero to a mapping $\tilde{A} \in \mathrm{Hom}_K(E \otimes E^*, \mathrm{Sym}^2_0(E \otimes_0 E^*))$. Via the identification $\mathfrak{m}^{\mathbb{C}} \cong E \otimes_0 E^*$, this gives rise to a mapping

$$A \in \mathrm{Hom}_K(E \otimes E^*, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}).$$

From the equivariance of $\pi\big|_W$, the irreducibility of $W \cong E \otimes_0 E^*$ and Schur's Lemma it follows that $\pi\big|_W$ is unitary up to a positive constant, that is

$$\langle \pi(v), \pi(w) \rangle_{E\otimes_0 E^*} = c \cdot \langle v, w \rangle_{\mathrm{Sym}^2_0(E\otimes_0 E^*)}$$

for all $v, w \in W$ and some $c > 0$. Denote the tensor product representation of G on $E \otimes E^*$ by

$$\rho : G \to \mathrm{Aut}(E \otimes E^*) : \ \rho(k_1, k_2)F = k_1 F k_2^{-1}$$

for $F \in E \otimes E^*$. Its differential is given by

$$\rho_* : \mathfrak{g} \to \mathrm{End}(E \otimes E^*) : \ d\rho(X_1, X_2)F = X_1 F - F X_2$$

for $X_1, X_2 \in \mathfrak{k}$. In particular

$$\rho_*(X, -X)F = XF + FX.$$

Let (e_i) be an orthonormal basis of $\mathfrak{m}$, $e_i = (f_i, -f_i)$ with $f_i \in \mathfrak{k}$. Under the identification $\mathfrak{m}^{\mathbb{C}} \cong E \otimes_0 E^*$, the invariant inner product changes by some positive constant factor, and e_i is mapped to f_i. Hence, (f_i) is an orthonormal basis of $\mathfrak{k} \subset E \otimes_0 E^*$ up to a positive factor.

Now, let $X \in \mathfrak{k}$ and $F \in E \otimes E^*$. Using the formula from Lemma 4.4.3 it follows that

$$\begin{aligned}
(\delta h)_o(X, -X) &= \sum_i \langle A(d\rho(e_i)F), e_i \odot (X, -X)\rangle_{\mathrm{Sym}_0^2 \mathfrak{m}^{\mathbb{C}}} \\
&= c \cdot \sum_i \langle \tilde{A}(f_i F + F f_i), f_i \odot X\rangle_{\mathrm{Sym}_0^2(E \otimes_0 E^*)} \\
&= c \cdot \sum_i \langle \tilde{A}(f_i F + F f_i), \mathrm{pr}_W(f_i \odot X)\rangle_{\mathrm{Sym}_0^2(E \otimes_0 E^*)} \\
&= c' \cdot \sum_i \langle f_i F + F f_i, \pi(\mathrm{pr}_{\mathrm{Sym}_0^2(E \otimes_0 E^*)}(f_i \odot X))\rangle_{E \otimes_0 E^*}
\end{aligned}$$

for some $c, c' > 0$. Since the trivial summand of $\mathrm{Sym}^2(E \otimes_0 E^*)$ can only be mapped to the trivial summand of $E \otimes E^*$ under the equivariant map π, we have

$$\pi \circ \mathrm{pr}_{\mathrm{Sym}_0^2(E \otimes_0 E^*)} = \mathrm{pr}_{E \otimes_0 E^*} \circ \pi$$

on $\mathrm{Sym}^2(E \otimes_0 E^*)$, implying that

$$\begin{aligned}
(\delta h)_o(X, -X) &= c' \cdot \sum_i \langle f_i F + F f_i, \mathrm{pr}_{E \otimes_0 E^*}(f_i X^* + X f_i^*)\rangle \\
&= -c' \cdot \sum_i \langle f_i F + F f_i, \mathrm{pr}_{E \otimes_0 E^*}(f_i X + X f_i)\rangle.
\end{aligned}$$

Choose the (up to a positive factor) orthonormal basis (f_i) of $\mathfrak{k}$ in such a way that $f_1 = E_{12} - E_{21}$. Furthermore, let $X = F = E_{13} - E_{31}$. Then,

$$f_1 F + F f_1 = (E_{12} - E_{21})(E_{13} - E_{31}) + (E_{13} - E_{31})(E_{12} - E_{21}) = -E_{23} - E_{32} \in E \otimes_0 E^*$$

and we obtain

$$\begin{aligned}
&\sum_i \langle f_i F + F f_i, \mathrm{pr}_{E \otimes_0 E^*}(f_i X + X f_i)\rangle = \sum_i \langle f_i F + F f_i, \mathrm{pr}_{E \otimes_0 E^*}(f_i F + F f_i)\rangle \\
\geq &\langle f_1 F + F f_1, \mathrm{pr}_{E \otimes_0 E^*}(f_1 F + F f_1)\rangle = \langle E_{23} + E_{32}, E_{23} + E_{32}\rangle = 2 > 0.
\end{aligned}$$

In particular, we have found $Y \in \mathfrak{m}$ such that $(\delta h)_o(Y) \neq 0$, where $h \in \mathscr{S}_0^2(M)$ is associated to

$$F \otimes A \in (E \otimes E^*) \otimes \mathrm{Hom}_K(E \otimes E^*, \mathrm{Sym}_0^2 \mathfrak{m}^{\mathbb{C}}).$$

This means that the linear mapping

$$\delta\Big|_{E \otimes E^*} : \ \mathrm{Hom}_K(E \otimes E^*, \mathrm{Sym}_0^2 \mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_K(E \otimes E^*, \mathfrak{m}^{\mathbb{C}})$$

is nonzero. Hence, there are no tt-eigentensors for the subcritical Casimir eigenvalue. This proves the assertion. □

4.8.3 Alternative proof of the stability of $\mathrm{E}_6/\mathrm{F}_4$

As we did before in the situation of $\mathrm{SU}(n)$, we want to apply Lemma 4.4.3 to verify that the mappings

$$\begin{aligned} \delta\big|_{\mathfrak{H}^{\mathbb{C}}} &: \ \mathrm{Hom}_{F_4}(\mathfrak{H}^{\mathbb{C}}, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_{F_4}(\mathfrak{H}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}), \\ \delta\big|_{\overline{\mathfrak{H}^{\mathbb{C}}}} &: \ \mathrm{Hom}_{F_4}(\overline{\mathfrak{H}^{\mathbb{C}}}, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_{F_4}(\overline{\mathfrak{H}^{\mathbb{C}}}, \mathfrak{m}^{\mathbb{C}}) \end{aligned}$$

are injective. Surprisingly, the computation works very similar to the $\mathrm{SU}(n)$ case.

4.8.2 Lemma. *Let* $\pi : \mathrm{Sym}^2 \mathfrak{H} \to \mathfrak{H}$ *denote the mapping defined by*

$$\pi(A \odot B) := AB + BA = 2A \circ B.$$

Then we have

$$\pi \in \mathrm{Hom}_{\mathrm{F}_4}(\mathrm{Sym}^2 \mathfrak{H}_0, \mathfrak{H}).$$

The restriction

$$\pi : \mathrm{Sym}^2_0 \mathfrak{H}_0 \to \mathfrak{H}_0$$

is surjective, and $W := \left(\ker \pi\big|_{\mathrm{Sym}^2_0 \mathfrak{H}_0}\right)^{\perp} \cong \mathfrak{H}_0$.

Proof. The proof is completely analogous to the proof of Lemma 4.8.1. First, we note that π is well-defined since $(\mathfrak{H}, \circ)$ is a commutative algebra. The equivariance of π under the action of F_4 follows from

$$\pi(f(A) \odot f(B)) = 2f(A) \circ f(B) = f(2A \circ B) = f(\pi(A \odot B))$$

for any $f \in \mathrm{F}_4 = \mathrm{Aut}(\mathfrak{H}, \circ)$ and $A, B \in \mathfrak{H}$. Furthermore, we have

$$\mathrm{tr}(\pi(A \odot B)) = 2\,\mathrm{tr}(A \circ B) = 2\langle A, B\rangle = \mathrm{tr}(A \odot B),$$

where the last trace is taken with respect to the inner product on $\mathfrak{H}$. This means that

$$\pi(\mathrm{Sym}^2_0 \mathfrak{H}) \subset \mathfrak{H}_0.$$

Now we want to show that π does not vanish when restricted to $\mathrm{Sym}^2_0 \mathfrak{H}_0$. For example, take $F_1(1), F_2(1) \in \mathfrak{H}_0$. We have $\langle F_1(1), F_2(1)\rangle = 0$ and thus $F_1(1) \odot F_2(1) \in \mathrm{Sym}^2_0 \mathfrak{H}_0$. Also,

$$\pi(F_1(1) \odot F_2(1)) = 2F_1(1) \circ F_2(1) = F_3(1) \neq 0.$$

Since $\mathfrak{H}_0$ is irreducible over F_4, the mapping

$$\pi : \operatorname{Sym}^2_0 \mathfrak{H}_0 \to \mathfrak{H}_0$$

must be surjective. From the proof of Lemma 4.7.1, we know that $\mathfrak{H}_0$ appears in the decomposition of $\operatorname{Sym}^2_0 \mathfrak{H}_0$ with multiplicity 1; hence $W := \left(\ker \pi\big|_{\operatorname{Sym}^2_0 \mathfrak{H}_0}\right)^{\perp}$ must be the irreducible summand of $\operatorname{Sym}^2_0 \mathfrak{H}_0$ that is equivalent to $\mathfrak{H}_0$. □

Alternative proof of Prop. 4.7.2. By Lemma 4.8.2 we can define

$$A := \pi\big|_W^{-1} \in \operatorname{Hom}_{\mathrm{F}_4}(\mathfrak{H}_0, \operatorname{Sym}^2_0 \mathfrak{H}_0),$$

extend it with zero to $\mathfrak{H}$ and then complex-linearly to a mapping $A \in \operatorname{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \operatorname{Sym}^2_0 \mathfrak{H}_0^{\mathbb{C}})$. Again, we need that $\pi\big|_W$ is unitary up to a positive constant, which follows by Schur's Lemma from the equivariance of $\pi\big|_W$ and the irreducibility of $W \cong \mathfrak{H}_0$. By Theorem 3.2.4 in [Yok09], every element $\alpha \in \mathfrak{e}_6 \subset \operatorname{End}_{\mathbb{C}}(\mathfrak{H}^{\mathbb{C}})$ can be written as

$$\alpha = \beta + \mathrm{i}T\circ$$

with unique elements $\beta \in \mathfrak{f}_4 \subset \mathfrak{e}_6$ and $T \in \mathfrak{H}_0$. This corresponds to the F_4-invariant decomposition

$$\mathfrak{e}_6 \cong \mathfrak{f}_4 \oplus \mathfrak{H}_0.$$

Throughout what follows we identify $\mathfrak{m} \cong \mathfrak{H}_0$. If we denote the defining representation by

$$\rho : \ \mathrm{E}_6 \to \operatorname{Aut} \mathfrak{H}^{\mathbb{C}},$$

then in particular

$$\rho_*(X) = \mathrm{i}X\circ$$

for $X \in \mathfrak{m}$. Let (e_i) be an orthonormal basis of $\mathfrak{H}_0$ (again, under the identification $\mathfrak{m} \cong \mathfrak{H}_0$, the invariant inner product changes at most by some positive constant factor), $X \in \mathfrak{m}$ and $F \in \mathfrak{H}^{\mathbb{C}}$. Using Lemma 4.4.3, we thus obtain

$$\begin{aligned}(\delta h)_o(X) &= c \cdot \sum_i \langle A(d\rho(e_i)F), e_i \odot X\rangle_{\operatorname{Sym}^2_0 \mathfrak{H}_0^{\mathbb{C}}} = c \cdot \sum_i \langle A(\mathrm{i}e_i \circ F), e_i \odot X\rangle_{\operatorname{Sym}^2_0 \mathfrak{H}_0^{\mathbb{C}}} \\ &= c \cdot \sum_i \langle A(\mathrm{i}e_i \circ F), \mathrm{pr}_W(e_i \odot X)\rangle_{\operatorname{Sym}^2_0 \mathfrak{H}_0^{\mathbb{C}}} = c' \cdot \sum_i \langle \mathrm{i}e_i \circ F, \pi(\mathrm{pr}_{\operatorname{Sym}^2_0 \mathfrak{H}_0}(e_i \odot X))\rangle_{\mathfrak{H}_0^{\mathbb{C}}}\end{aligned}$$

for some $c, c' > 0$. The trivial summand of $\operatorname{Sym}^2 \mathfrak{H}_0$ can only be mapped to the trivial summand of $\mathfrak{H}$ under the equivariant map π, implying that

$$\pi \circ \mathrm{pr}_{\mathrm{Sym}^2_0 \mathfrak{H}_0} = \mathrm{pr}_{\mathfrak{H}_0} \circ \pi$$

on $\mathrm{Sym}^2 \mathfrak{H}_0$. Thus, we have

$$(\delta h)_o(X) = \mathrm{i}c' \cdot \sum_i \langle e_i \circ F, \mathrm{pr}_{\mathfrak{H}_0}(\pi(e_i \odot X)) \rangle = 2\mathrm{i}c' \sum_i \langle e_i \circ F, \mathrm{pr}_{\mathfrak{H}_0}(e_i \circ X) \rangle.$$

Now let $X = F = F_1(1)$. Choose the (up to a positive factor) orthonormal basis (e_i) of $\mathfrak{H}_0$ in such a way that $e_1 = F_2(1)$. Then we have

$$e_1 \circ F = F_2(1) \circ F_1(1) = \frac{1}{2} F_3(1) \in \mathfrak{H}_0$$

and it follows that

$$\begin{aligned} \sum_i \langle e_i \circ F, \mathrm{pr}_{\mathfrak{H}_0}(e_i \circ X) \rangle = \sum_i \langle e_i \circ F, \mathrm{pr}_{\mathfrak{H}_0}(e_i \circ F) \rangle &\geq \langle e_1 \circ F, \mathrm{pr}_{\mathfrak{H}_0}(e_1 \circ F) \rangle \\ &= \frac{1}{4} \langle F_3(1), F_3(1) \rangle = \frac{1}{2} > 0. \end{aligned}$$

In particular, we have found $Y \in \mathfrak{m}$ such that $(\delta h)_o(Y) \neq 0$, where $h \in \mathscr{S}^2_0(M)$ is associated to

$$F \otimes A \in \mathfrak{H}^{\mathbb{C}} \otimes \mathrm{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}).$$

This means that the linear mapping

$$\delta\Big|_{\mathfrak{H}^{\mathbb{C}}} : \ \mathrm{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) \to \mathrm{Hom}_{\mathrm{F}_4}(\mathfrak{H}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}})$$

is nonzero. The same argument works for the E_6-representation $\overline{\mathfrak{H}^{\mathbb{C}}}$, since we exclusively used real elements and automorphisms in the computation. In total, there are no tt-eigentensors for the subcritical Casimir eigenvalue, which proves the assertion. $\square$

5 Coindex and rigidity of Einstein metrics on homogeneous Gray manifolds

5.1 Abstract

Any 6-dimensional strict nearly Kähler manifold is Einstein with positive scalar curvature. We compute the coindex of the metric with respect to the Einstein-Hilbert functional on each of the compact homogeneous examples. Moreover, we show that the infinitesimal Einstein deformations on $F_{1,2} = \mathrm{SU}(3)/T^2$ are not integrable into a curve of Einstein metrics.

5.2 Introduction

The special case of dimension 6 has been a primary focus of nearly Kähler geometry since P.-A. Nagy showed that every nearly Kähler manifold is locally isometric to a Riemannian product of 6-dimensional nearly Kähler manifolds, nearly Kähler homogeneous spaces and twistor spaces over positive scalar curvature quaternionic-Kähler manifolds [Nag02]. Moreover, nearly Kähler manifolds that are non-Kähler (so-called *strict* nearly Kähler manifolds) of dimension 6 exhibit other notable properties, such as carrying a real Killing spinor and thus being Einstein with positive scalar curvature.

On a compact manifold M, Einstein metrics can be variationally characterized as critical points of the total scalar curvature functional S (also called *Einstein–Hilbert action*), defined on the set of all Riemannian metrics on M of a fixed volume. Given a compact Einstein manifold (M, g), one can ask whether g locally maximizes S (after restricting to a suitable subclass of Riemannian metrics). Such an Einstein metric g is called *stable* with respect to S. The linearized problem considers the Hessian S''_g of the Einstein-Hilbert action at g. Accordingly, an Einstein metric g is called *linearly stable* if $S''_g \leq 0$ on the space of tt-tensors (i.e. trace- and divergence-free symmetric 2-tensors on M). A closely related notion is that of *infinitesimal deformability* of the Einstein metric g – it is called infinitesimally deformable if S''_g is degenerate on tt-tensors.

A compact, 6-dimensional, strict nearly Kähler manifold (M,g,J) with scalar curvature normalized to $\mathrm{scal}_g = 30$ (hence Einstein constant $E=5$) is called a *Gray manifold*, after A. Gray who studied them in the 70s. The stability and infinitesimal deformability of Einstein metrics on Gray manifolds have already been investigated. In [SWW20], U. Semmelmann, C. Wang and M. Y.-K. Wang show linear instability if the second or third Betti number does not vanish – in fact, the coindex of g (see §5.3.2 for a definition) is bounded below by b_2+b_3. A. Moroianu and U. Semmelmann [MS11] give a description of the space of infinitesimal Einstein deformations in terms of eigenspaces of the Hodge Laplacian on coclosed primitive $(1,1)$-forms. The present article generalizes this result to a similar description of eigenspaces of the Lichnerowicz Laplacian on tt-tensors to arbitrary eigenvalues not exceeding a certain threshold (see Lemma 5.4.2).

Homogeneous Gray manifolds have been classified by J.-B. Buitruille [But05]. There are only four cases: $S^6 = \frac{\mathrm{G}_2}{\mathrm{SU}(3)}$, $S^3\times S^3 = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta\,\mathrm{SU}(2)}$, $\mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\,\mathrm{U}(1)} = \frac{\mathrm{SO}(5)}{\mathrm{U}(2)}$ and the flag manifold $F_{1,2} = \frac{\mathrm{SU}(3)}{T^2}$, all of them equipped with the Killing form metric (up to scaling). In [WW18], C. Wang and M. Y.-K. Wang show instability of the latter three spaces. S^6 carries the round metric and is thus strictly stable.

One aim of this article is to improve the coindex estimates from [SWW20] to equalities for the homogeneous examples. Our first main result can be stated as follows.

5.2.1 Theorem. *Let (M,g) be a homogeneous Gray manifold with standard metric g. The coindex of the Einstein metric g is*

- *equal to* 2 *if* $M = S^3\times S^3 = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta\,\mathrm{SU}(2)}$,
- *equal to* 1 *if* $M = \mathbb{CP}^3 = \frac{\mathrm{SO}(5)}{\mathrm{U}(2)}$,
- *equal to* 2 *if* $M = F_{1,2} = \frac{\mathrm{SU}(3)}{T^2}$.

The destabilizing directions, i.e. contributions to the coindex, can be viewed as arising from harmonic 3-forms in the first and from harmonic 2-forms in the second and third case via the construction in [SWW20]. For the last two cases there is an additional geometric explanation: consider the Riemannian submersions given by the twistor fibrations

$$\begin{aligned} \mathbb{CP}^3 = \frac{\mathrm{SO}(5)}{\mathrm{U}(2)} &\longrightarrow \frac{\mathrm{SO}(5)}{\mathrm{SO}(4)} = S^4,\\ F_{1,2} = \frac{\mathrm{SU}(3)}{T^2} &\longrightarrow \frac{\mathrm{SU}(3)}{\mathrm{S}(\mathrm{SU}(2)\,\mathrm{U}(1))} = \mathbb{CP}^2. \end{aligned}$$

In both cases the canonical variation (scaling the base against the fiber) yields a destabilizing direction by [WW18, Prop. 4.4]. For the flag manifold there are actually three such fibrations whose canonical variations give rise to a two-dimensional space of tt-tensors, explaining the coindex of 2 (see Remark 5.5.9).

Also worth noting is the G-invariant stability problem in which the Einstein–Hilbert functional S is restricted to the class of G-invariant metrics on a fixed homogeneous space $M = G/H$. Since the destabilizing directions on $\mathbb{CP}^3$ and on $F_{1,2}$ are G-invariant, it follows that these spaces are G-unstable. On the other hand, $S^3 \times S^3$ is G-stable. In fact, $\mathbb{CP}^3$ and $F_{1,2}$ are G-strongly unstable, i.e. *all* G-invariant variations of the metric are destabilizing and hence these metrics are local minima of S among G-invariant metrics. For $F_{1,2}$ this has already been shown by J. Lauret [Lau22, Table 2].

Let us return to the general setting of a compact manifold M. An Einstein metric g on M is called *rigid* if it is isolated in the moduli space of Einstein structures (disregarding variation by homothetic scaling and action of diffeomorphisms). If an Einstein manifold (M, g) admits infinitesimal Einstein deformations, one naturally asks whether they are integrable into a curve of Einstein metrics on M. In fact, not every infinitesimally deformable Einstein must lie within a nontrivial curve of Einstein metrics. The first example of such a metric is the canonical symmetric metric on $\mathbb{CP}^1 \times \mathbb{CP}^{2k}$ found by N. Koiso [Koi82], who started the investigation of stability and infinitesimal deformatibility of symmetric spaces [Koi80]. Another recent example due to Batat et. al is the bi-invariant metric on $\mathrm{SU}(2n+1)$ [BH+21]. We add one more example to this list by proving the following result.

5.2.2 Theorem. *The Einstein metric on the Gray manifold $F_{1,2}$ is rigid, that is, its infinitesimal Einstein deformations are not integrable.*

In all of the above examples integrability fails at an obstruction to second order (see the end of §5.3.2). We suspect that this phenomenon occurs generically. Given some infinitesimal Einstein deformation, i.e. an element of the null space of S''_g, the obstruction polynomial (5.2) has no immediate compulsion to vanish and should do so only coincidentally – see for example the case $\mathrm{SU}(2n)$ in [BH+21].

Since Gray manifolds are Einstein, every infinitesimal deformation of the nearly Kähler structure corresponds to an infinitesimal Einstein deformation, but not necessarily vice versa [MS11]. Infinitesimal deformability of the nearly Kähler structure has been investigated by A. Moroianu, P.-A. Nagy and U. Semmelmann [MNS08]. The question whether a given infinitesimal nearly Kähler deformation can be integrated into a curve of nearly Kähler structures has been studied by L. Foscolo in [Fos17], where a similar polynomial occurs as integrability obstruction to second order. In particular, he showed that the infinitesimal nearly Kähler deformations on $F_{1,2}$ are all obstructed. One can view Theorem 5.2.2 as a generalization of this result to the Einstein picture.

This article is organized as follows. In §5.3, notation is fixed and the necessary preliminaries are recapitulated. §5.4 concerns itself with a description of eigenspaces of the Lichnerowicz Laplacian on tt-tensors on general Gray manifolds as well as a discussion of the homogeneous case, in which explicit calculations are possible by means of harmonic analysis. These results are applied in §5.5 to each of the unstable Gray manifolds $S^3 \times S^3$, $\mathbb{CP}^3$ and $F_{1,2}$ to obtain the results collected in Theorem 5.2.1. Finally, §5.6 recalls the description of the infinitesimal Einstein deformations on $F_{1,2}$ given in [MS10] and proceeds to show the nonintegrability to second order, proving Theorem 5.2.2.

The author owes gratitude to Prof. U. Semmelmann for helpful exchanges about a gap in the argument given in the proof of [MS11, Thm. 5.1] (the corrected argument is the proof of Lemma 5.4.1, which includes the aforementioned as the special case $\lambda = 10$). Furthermore, the author would like to thank Prof. G. Weingart for his useful suggestions regarding the rigidity argument.

5.3 Preliminaries

5.3.1 Nearly Kähler manifolds

An almost Hermitian manifold (M, g, J) is an even-dimensional Riemannian manifold (M, g) with an almost complex structure J that is compatible with the metric, i.e.

$$g(JX, JY) = g(X, Y)$$

for any $X, Y \in T_pM$. The Kähler form ω is then defined by

$$\omega(X, Y) := g(JX, Y).$$

Any almost Hermitian structure has an associated *canonical Hermitian connection* ∇^{h} (see for example [But08, §2] for a general definition). In particular it satisfies $\nabla^{\mathrm{h}} g = 0$ and $\nabla^{\mathrm{h}} J = 0$.

Let ∇ denote the Levi-Civita connection of the Riemannian manifold (M, g). An almost Hermitian manifold (M, g, J) is called *nearly Kähler* if ∇J is skew-symmetric, or equivalently, if

$$(\nabla_X J)X = 0$$

for all $X \in T_pM$. In this case, the canonical Hermitian connection can be described by

$$\nabla^{\mathrm{h}}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y$$

for any two vector fields $X, Y \in \mathfrak{X}(M)$. A nearly Kähler manifold is called *strictly nearly*

Kähler if it is not Kähler. *Gray manifolds* are compact strict nearly Kähler manifolds of dimension 6.

As usual, the almost complex structure J defines a splitting of the complexified cotangent bundle $T^*M^{\mathbb{C}} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$ and hence of the bundle of k-forms into (p,q)-forms with $p+q=k$. The complex bundle of (p,q)-forms will be denoted with the prefix $\Lambda^{p,q}$, and the space of its smooth sections by $\Omega^{p,q}$. The Kähler form ω is of type $(1,1)$. A (p,q)-form α is called *primitive* if it vanishes under contraction with the Kähler form, i.e. if $\omega \lrcorner \alpha = 0$. We will denote the bundle of primitive (p,q)-forms by $\Lambda_0^{p,q}$. Furthermore, let $\Lambda_{\mathbb{R}}^{p,q}$ denote the projection of the complex bundle $\Lambda^{p,q}$ to the real bundle Λ^{p+q}.

Likewise, the bundle $\operatorname{Sym} TM$ of g-symmetric endomorphisms of the tangent bundle splits into a direct sum $\operatorname{Sym}^+ TM \oplus \operatorname{Sym}^- TM$, where the elements of $\operatorname{Sym}^\pm TM$ commute (resp. anticommute) with J. We further denote by $\operatorname{Sym}_0^+ TM$ the subbundle of trace-free endomorphisms in $\operatorname{Sym}^+ TM$, and with $\mathscr{S}^\pm$, $\mathscr{S}_0^+$ the spaces of smooth sections in the respective bundles.

Let $\mathscr{S}^k = \Gamma(\operatorname{Sym}^k T^*M)$ denote the space of symmetric k-tensor fields. Note that the metric yields a natural identification $\operatorname{Sym}^2 T^*M \cong \operatorname{Sym} TM$. The subspace of tt-tensors in $\mathscr{S}^2$ (i.e. $h \in \mathscr{S}^2$ satisfying $\operatorname{tr}_g h = 0$ and $\delta h = 0$) will be denoted by $\mathscr{S}^2_{\mathrm{tt}}$.

If (M,g,J) is nearly Kähler, then the tensor $\Psi^+ := \nabla\omega$ is totally skew-symmetric and in fact the real part of a ∇^{h}-parallel complex volume form $\Psi^+ + \mathrm{i}\Psi^-$. The imaginary part Ψ^- can be described by $X \lrcorner \Psi^- = J \circ (\nabla_X J)$ for all $X \in TM$. The strict nearly Kähler case is characterized by the non-vanishing of Ψ^+.

Let (M,g,J) be a strict nearly Kähler manifold of dimension 6. There are ∇^{h}-parallel isomorphisms

$$\begin{array}{ccc} TM \cong \Lambda_{\mathbb{R}}^{2,0}M, & \operatorname{Sym}_0^+ TM \cong \Lambda_{0,\mathbb{R}}^{1,1}M, & \operatorname{Sym}^- TM \cong \Lambda_{\mathbb{R}}^{2,1}M \\ X \mapsto X \lrcorner \Psi^+ & h \mapsto J \circ h & h \mapsto h_* \Psi^+ \end{array} \tag{5.1}$$

of vector bundles with structure group $\mathrm{SU}(3)$, each arising from an equivalence of $\mathrm{SU}(3)$-representations. Here, h_* denotes the extension of the endomorphism $h \in \operatorname{End} TM$ to tensor bundles as a derivation.

5.3.2 Stability and rigidity

The *Lichnerowicz Laplacian* Δ_{L} of a Riemannian manifold (M,g) is an operator that generalizes the Hodge Laplacian Δ on differential forms to tensor fields of any rank. It is defined by

$$\Delta_{\mathrm{L}} := \nabla^*\nabla + q(R),$$

where $q(R)$ is the curvature endomorphism acting on tensors by

$$q(R) := \sum_{i<j} (e_i \wedge e_j)_* R(e_i, e_j)$$

for some local orthonormal frame (e_i) of TM. The asterisk denotes the natural action of $\Lambda^2 T \cong \mathfrak{so}(T)$. In particular, $q(R) = \mathrm{Ric}$ on 1-forms.

On an almost Hermitian manifold we analogously define the *Hermitian Laplace operator* Δ^{h} by replacing the Levi-Civita connection ∇ in the above definition by the canonical Hermitian connection ∇^{h}, i.e.

$$\Delta^{\mathrm{h}} := \nabla^{\mathrm{h}*} \nabla^{\mathrm{h}} + q(R^{\mathrm{h}}).$$

Here, R and R^{h} denote the curvature tensors of the connections ∇ and ∇^{h}, respectively. Both Δ_{L} and Δ^{h} are instances of the *standard Laplacian* of a given connection (see [SW18]), an operator with several neat properties – for example, it commutes with parallel bundle maps. Comparison formulas for the two Laplace operators in the setting of 6-dimensional nearly Kähler manifolds can be found in [MS10] and [MS11]. For our purposes, it is important to note that Δ and Δ^{h} coincide on coclosed primitive $(1,1)$-forms, as well as on coclosed $(2,1)$- and $(1,2)$-forms, which follows from combining [MS11, Prop. 3.4, Cor. 4.4].

Consider a fixed compact orientable smooth manifold M of dimension $n > 2$. On the set of all Riemannian metrics on M, the *total scalar curvature functional* (or *Einstein–Hilbert action*) is defined by

$$g \mapsto S(g) = \int_M \mathrm{scal}_g \, \mathrm{vol}_g \, .$$

Einstein metrics on M are then precisely the critical points of the restriction of S to metrics of a fixed total volume. Let (M, g) be an Einstein manifold with $\mathrm{Ric} = Eg$. If (M, g) not isometric to the standard sphere, there is a well-known decomposition

$$\mathscr{S}^2 = \mathbb{R}g \oplus C^\infty_g g \oplus L_{\mathfrak{X}} g \oplus \mathscr{S}^2_{\mathrm{tt}}$$

that is orthogonal with respect to the second variation S''_g (see [Bes87]). Furthermore,

$$\begin{aligned}
S''_g &> 0 \text{ on } C^\infty_g g, \text{ where} && C^\infty_g = \{f \in C^\infty(M) \,|\, (f, \mathbf{1})_{L^2} = 0\},\\
S''_g &= 0 \text{ on} && L_{\mathfrak{X}} g = \{L_X g \,|\, X \in \mathfrak{X}(M)\},\\
S''_g(h,h) &= -\frac{1}{2} \left(\Delta_{\mathrm{L}} h - 2Eh, h\right)_{L^2} \text{ on} && \mathscr{S}^2_{\mathrm{tt}} = \{h \in \mathscr{S}^2 \mid \mathrm{tr}_g h = 0, \ \delta h = 0\}.
\end{aligned}$$

On the latter space S''_g has finite coindex and nullity, i.e. the maximal subspace of $\mathscr{S}^2_{\mathrm{tt}}$ on which S''_g is nonnegative is finite-dimensional. The sum $L_{\mathfrak{X}} g \oplus \mathscr{S}^2_{\mathrm{tt}} = T_g \mathfrak{S}$ can also be regarded as formal tangent space to the set $\mathfrak{S}$ of metrics with constant scalar curvature and fixed total volume.

The stability problem is to decide whether an Einstein metric g is a local maximum or a saddle point of $S\big|_{\mathfrak{S}}$. We are primarily concerned with the linearized version, considering only the second variation of S at g. An Einstein metric g is called *(linearly) stable* if $S''_g\big|_{\mathscr{S}^2_{\mathrm{tt}}} \leq 0$, or, equivalently, if $\Delta_{\mathrm{L}} \geq 2E$ on $\mathscr{S}^2_{\mathrm{tt}}$. If strict inequality holds, we call g *strictly stable*. On the other hand, g is called *(linearly) unstable* if there exists $h \in \mathscr{S}^2_{\mathrm{tt}}$ such that $S''_g(h,h) > 0$, or, equivalently, if $(\Delta_{\mathrm{L}} h, h)_{L^2} < 2E\|h\|^2_{L^2}$. The dimension of the maximal subspace of $\mathscr{S}^2_{\mathrm{tt}}$ on which $S''_g > 0$ is called the *coindex* of g.

A closely related notion is that of rigidity. An Einstein metric g is called *rigid* if it is isolated in the moduli space, i.e. the space of Einstein metrics modulo diffeomorphisms and homotheties. This is equivalent to the nonexistence of a smooth curve (g_t) of Einstein metrics through $g = g_0$ with nonvanishing first-order jet $\dot{g}_0 \in \mathscr{S}^2_{\mathrm{tt}}$.

Denote by $\varepsilon(g) = \{h \in \mathscr{S}^2_{\mathrm{tt}} \,|\, \Delta_{\mathrm{L}} h = 2Eh\}$ the null space of S''_g, also called the space of *(essential) infinitesimal Einstein deformations (EID)*. If $\varepsilon(g) \neq 0$, we call g *infinitesimally deformable*. A metric with $\varepsilon(g) = 0$ is automatically rigid – in particular, strict stability implies rigidity.

In general EID need not be integrable into a curve of Einstein metrics. On the set of unit volume Riemannian metrics, define the Einstein operator E by

$$E(g) := \mathrm{Ric}_g - \frac{S(g)}{n} g.$$

Then a metric g is Einstein if and only if $E(g) = 0$. An EID $h \in \varepsilon(g)$ is called *formally integrable to order k* if there exist $h_2, \dots, h_k \in \mathscr{S}^2$ such that

$$E\left(g + th + \sum_{j=2}^{k} \frac{t^k}{k!} h_k\right) = 0.$$

A classical result [Bes87] is that an EID $h \in \varepsilon(g)$ can be integrated into a curve (g_t) of Einstein metrics with $\dot{g}_0 = h$ if and only if it is formally integrable to all orders $k \geq 2$.

The integrability criterion to each order can be expressed in terms of derivatives of E. By a result of N. Koiso [Koi82, Lem. 4.7], $h \in \varepsilon(g)$ is integrable to order 2 if and only if $E''_g(h,h) \perp \varepsilon(g)$ in the L^2 sense. Also due to N. Koiso [Koi82, Lem. 4.3] is the formula

$$\begin{aligned} 2\left(E''_g(h,h), h\right)_{L^2} = \int_M \Big(&2E h_{ij} h_{ik} h_{jk} + 3(\nabla_{e_i}\nabla_{e_j} h)_{kl} h_{ij} h_{kl} \\ &- 6(\nabla_{e_i}\nabla_{e_j} h)_{kl} h_{ik} h_{jl}\Big) \mathrm{vol}_g \end{aligned} \tag{5.2}$$

for the second order obstruction, where we implicitly sum over a local orthonormal frame (e_i) of TM. The vanishing of the quantity in (5.2) is a necessary condition for the integrability of h.

5.3.3 Harmonic analysis

Let $(M = G/H, g)$ be a Riemannian homogeneous space, where G is some Lie group and H is a closed subgroup. We will always denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. The homogeneous space M is called *reductive* if there exists an $\mathrm{Ad}\big|_H$-invariant complement $\mathfrak{m}$ of $\mathfrak{h} \subset \mathfrak{g}$. This is always the case if H is compact (in particular if G is compact). Through the canonical projection $\pi: \; G \to M$, the reductive complement $\mathfrak{m} \subset \mathfrak{g} \cong T_eG$ is canonically identified with the tangent space T_oM at the base point $o = eH$.

The G-invariant metric on a reductive Riemannian homogeneous space (M, g) is determined by an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$. Suppose that Q is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Then $\mathfrak{m} := \mathfrak{h}^\perp$ is an $\mathrm{Ad}(H)$-invariant subspace. We call (M, g) a *normal* homogeneous space if the metric is induced by the restriction Q to $\mathfrak{m}$, i.e. $g_o = Q\big|_{\mathfrak{m}\times\mathfrak{m}}$. If G is compact and semisimple, then the Killing form $B_\mathfrak{g}$ is negative-definite. In this case, the *standard* metric is defined by $g_o = -B_\mathfrak{g}\big|_{\mathfrak{m}\times\mathfrak{m}}$.

A normal homogeneous space is in particular *naturally reductive*, i.e.

$$g_o([X, Y]_\mathfrak{m}, Z) + g_o(Y, [X, Z]_\mathfrak{m}) = 0$$

(where $X_\mathfrak{m}$ denotes the projection of X to $\mathfrak{m}$) holds for all $X, Y, Z \in \mathfrak{m}$.

Let $\rho : H \to \mathrm{Aut}\, V$ be a finite-dimensional (real or complex) representation. Denote by $VM = G \times_\rho V$ the associated homogeneous vector bundle over M. Its sections can be viewed as H-equivariant smooth V-valued functions on G – the isomorphism is explicitly given by

$$\Gamma(VM) \xrightarrow{\cong} C^\infty(G, V)^H : \; s \mapsto \hat{s},$$

where $s(xH) = [x, \hat{s}(x)]$ for any $x \in G$. Left-translation on sections of VM gives rise to the *left-regular representation* on $C^\infty(G, V)^H$, explicitly given by

$$\ell : \; G \to \mathrm{Aut}\, C^\infty(G, V)^H : \; (\ell(x)f)(y) = f(x^{-1}y)$$

for $x, y \in G$.

If M is reductive, we can write every tensor bundle as an associated bundle of some tensor power of the reductive complement. For example

$$\mathscr{S}^2 = \Gamma(\mathrm{Sym}^2 T^*M) \cong \Gamma(G \times_\rho \mathrm{Sym}^2 \mathfrak{m}) \cong C^\infty(G, \mathrm{Sym}^2 \mathfrak{m})^H$$

(note that the H-representations $\mathfrak{m}$ and $\mathfrak{m}^*$ are equivalent via the Riemannian metric).

For a compact Lie group G we denote by $\hat{G}$ the set of dominant integral weights of G (after choosing a suitable maximal torus $T \subset G$). Recall that the elements of $\hat{G}$ are in one-to-one correspondence with equivalence classes of irreducible complex representations of G. Any representative of such a class with highest weight $\gamma \in \hat{G}$ will be denoted by (V_γ, ρ_γ). Let V be a unitary representation of H. The homogeneous version of the Peter–Weyl theorem [Wal73, Thm. 5.3.6] states that the left-regular representation decomposes into

$$L^2(G,V)^H \cong \overline{\bigoplus_{\gamma\in\hat{G}}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V). \tag{5.3}$$

Here $\mathrm{Hom}_H(V_\gamma, V)$ simply counts the multiplicity of V_γ inside $L^2(G,V)^H$ and is called the space of *Fourier (matrix) coefficients*. The equivalence in (5.3) is made explicit by

$$V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V) \hookrightarrow C^\infty(G,V)^H : \; v \otimes F \mapsto \left(x \mapsto F(\rho_\gamma^{-1}(x)v)\right). \tag{5.4}$$

Let V, W be unitary representations of H and $D : \Gamma(VM) \to \Gamma(WM)$ be a G-invariant differential operator. Combining (5.3) with Schur's Lemma, the operator D acts as a linear mapping

$$D\Big|_\gamma : \; \mathrm{Hom}_H(V_\gamma, V) \longrightarrow \mathrm{Hom}_H(V_\gamma, W)$$

for each fixed $\gamma \in \hat{G}$. We call this mapping the *prototypical differential operator* associated to D and γ (as introduced by U. Semmelmann and G. Weingart in [SW22]).

On a reductive homogeneous space, a choice of reductive complement $\mathfrak{m}$ determines a G-invariant connection $\bar{\nabla}$ on VM, called the *canonical reductive* (or *Ambrose–Singer*) connection, by stipulating that

$$\widehat{\bar{\nabla}_X s} = \tilde{X}(\hat{s}) \tag{5.5}$$

for all $X \in TM$, $s \in \Gamma(VM)$, where the *horizontal lift* $\tilde{X} \in TG$ is the unique vector in the *canonical horizontal distribution* $\mathcal{H} = \bigcup_{x\in G} dl_x(\mathfrak{m})$ such that $d\pi(\tilde{X}) = X$. This connection has the important property that all G-invariant sections of VM are parallel. If (M,g) is naturally reductive, $\bar{\nabla}$ is a metric connection with parallel totally skew torsion tensor $\bar{T}$, given (at the base point) by

$$\bar{T}_o(X,Y) = -[X,Y]_{\mathfrak{m}}.$$

On any representation $\rho : G \to \mathrm{Aut}\, V$ of a compact Lie group G, the *Casimir operator* with respect to a fixed $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ is the equivariant endomorphism of V defined by

$$\mathrm{Cas}_\rho^{\mathfrak{g},Q} = -\sum_i \rho_*(e_i)^2,$$

where (e_i) is an orthonormal basis of $\mathfrak{g}$ with respect to Q. We omit the superscript Q if the choice of inner product is clear from context. For $\gamma \in \hat{G}$ the Casimir operator on V_γ acts as multiplication with the *Casimir constant*

$$\mathrm{Cas}_\gamma^{\mathfrak{g},Q} = \langle \gamma, \gamma + 2\delta_{\mathfrak{g}} \rangle_{\mathfrak{t}^*,Q} \tag{5.6}$$

by Freudenthal's formula, cf. [FH91]. Here $\langle \cdot, \cdot \rangle_{\mathfrak{t}^*,Q}$ is the inner product induced by Q on the dual $\mathfrak{t}^*$ of the Lie algebra $\mathfrak{t}$ of the torus $T \subset G$, while $\delta_{\mathfrak{g}}$ denotes the half-sum of positive roots of $\mathfrak{g}$.

A crucial fact [MS10, Lem. 5.2] is that on a normal homogeneous space with Riemannian metric induced by an $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$, the standard Laplacian of $\bar{\nabla}$ is precisely the Casimir operator of G acting on the left-regular representation, i.e.

$$\bar{\Delta} := (\bar{\nabla})^* \bar{\nabla} + q(\bar{R}) = \mathrm{Cas}_\ell^{\mathfrak{g},Q}. \tag{5.7}$$

In particular the prototypical differential operator associated to $\bar{\Delta}$ and γ is simply multiplication by the Casimir constant. In other words, the eigenspaces of $\bar{\Delta}$ are the isotypical components

$$V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, V)$$

in the Peter-Weyl decomposition (5.3). The eigenvalues are readily computable by means of Freudenthal's formula (5.6).

It should be noted that in the symmetric case, the torsion of $\bar{\nabla}$ vanishes. Hence $\bar{\nabla}$ coincides with the Levi-Civita connection ∇. It follows that $\Delta_{\mathrm{L}} = \bar{\Delta}$, so the spectrum of the Lichnerowicz Laplacian on any tensor bundle is easily computable, facilitating the foundational work by N. Koiso on the stability of symmetric spaces [Koi80].

5.3.4 3-symmetric spaces

A homogeneous space $M = G/H$ is called *3-symmetric* if there exists an automorphism $\sigma \in \mathrm{Aut}\, G$ of order 3 such that $G_0^\sigma \subset H \subset G^\sigma$, where G^σ is the fixed point set of σ and G_0^σ is the connected component of the identity in G^σ.

The complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ decomposes into eigenspaces of the differential at the base point $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-.$$

The eigenvalues of σ_* are 1 on $\mathfrak{h}^{\mathbb{C}}$, $\mathrm{j} := e^{\frac{2\pi \mathrm{i}}{3}}$ on $\mathfrak{m}^+$ and $\mathrm{j}^2 = \bar{\mathrm{j}} = e^{\frac{4\pi \mathrm{i}}{3}}$ on $\mathfrak{m}^-$, respectively. M then carries a natural G-invariant almost complex structure J with $\pm\mathrm{i}$-eigenspaces $\mathfrak{m}^\pm$, given by

$$\sigma_*\Big|_{\mathfrak{m}} = \frac{1}{2}\,\mathrm{Id}_{\mathfrak{m}} + \frac{\sqrt{3}}{2} J_o$$

at the base point, where $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$. Furthermore, M is a reductive homogeneous space, since $\mathfrak{m}$ is invariant under the adjoint action of $H \subset G^\sigma$.

When endowed with a G-invariant Riemannian metric g compatible with J, (M, g) is called a *Riemannian 3-symmetric space*. In particular, (M, g, J) is almost Hermitian. Furthermore, the almost Hermitian structure (M, g, J) is nearly Kähler if and only if (M, g) is naturally reductive [But08, Prop. 3.8].

For an extensive treatment of 3-symmetric spaces see [But08]. The final thing we need for our purposes is the key observation [But08, Prop. 3.5] that on a Riemannian 3-symmetric space, the canonical reductive connection $\bar{\nabla}$ associated to $\mathfrak{m}$ coincides with the canonical Hermitian connection ∇^{h} defined in §5.3.1.

5.4 Small Lichnerowicz eigenvalues on Gray manifolds

Throughout what follows, let (M, g, J) be a Gray manifold. In order to find destabilizing directions for the Einstein–Hilbert functional, we need to solve the system

$$\begin{cases} \Delta_{\mathrm{L}} h = \lambda h, \\ \delta h = 0 \end{cases} \tag{L1}$$

in $h \in \mathscr{S}^2_0$ for some $\lambda < 2E = 10$. We follow the discussion in [MS11, §5] to transform (L1) into an eigenvalue problem for the more familiar Hodge–de Rham Laplacian. Viewed as a section of $\operatorname{Sym} TM$, the tensor h splits into $h = h^+ + h^-$ with $h^+ \in \mathscr{S}^+_0$ and $h^- \in \mathscr{S}^-$. By applying the bundle isomorphisms given in (5.1), we obtain tensors $\varphi := h^+ \circ J \in \Omega^{1,1}_{0,\mathbb{R}}$ and $\sigma := h^-_* \Psi^+ \in \Omega^{2,1}_{0,\mathbb{R}}$ carrying the information of h.

5.4.1 Lemma. *Under the isomorphisms $h^+ \mapsto \varphi$ and $h^- \mapsto \sigma$ above, if $\lambda < 16$, the system of equations* (L1) *is equivalent to*

$$\begin{cases} \Delta \varphi = (\lambda - 6)\varphi - \delta\sigma, \\ \Delta\sigma = (\lambda - 4)\sigma - 4d\varphi, \\ \delta\varphi = 0, \\ \delta\sigma \in \Omega^{1,1}_{0,\mathbb{R}}. \end{cases} \tag{L2}$$

Proof. Using the formulae from [MS11, Prop. 3.4, Cor. 4.4], the first equation of (L1) can be rewritten as

$$(\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}}))(h^+ + h^-) = \lambda(h^+ + h^-) - (3h^+ + s) - (2h^- - (\delta h^- \lrcorner \Psi^+ + \delta\sigma) \circ J) \\ - 3h^+ - 2h^-,$$

with $s \in \mathscr{S}^-$ defined by $s_*\Psi^+ = 2\delta h^+ \wedge \omega + 4d\varphi$. We note that $(\delta h^- \lrcorner \Psi^+ + \delta\sigma) \circ J$ is necessarily a traceless symmetric 2-tensor, hence automatically $\delta h^- \lrcorner \Psi^+ + \delta\sigma \in \Omega^{1,1}_{0,\mathbb{R}}$, or, equivalently,

$$\delta h^- \lrcorner \Psi^+ + (\delta\sigma)_{2,0} = 0.$$

Using that $\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}})$ preserves the spaces $\mathscr{S}^\pm$ and $\Omega^{2,0}_{\mathbb{R}}$, we can write (L1) equivalently as

$$\begin{cases} (\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}}))h^+ = (\lambda - 6)h^+ + (\delta\sigma)_{1,1} \circ J, \\ (\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}}))h^- = (\lambda - 4)h^- - s, \\ \delta h^+ + \delta h^- = 0. \end{cases}$$

Let now $\eta \in \Omega^1$ such that $(\delta\sigma)_{2,0} = \eta \lrcorner \Psi^+$. Since $\delta h^+ = -J\delta\varphi$ and $\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}})$ commutes with the bundle isomorphisms from (5.1), we can apply them to obtain

$$\begin{cases} (\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}}))\varphi = (\lambda - 6)\varphi - (\delta\sigma)_{1,1}, \\ (\nabla^{\mathrm{h}*}\nabla^{\mathrm{h}} + q(R^{\mathrm{h}}))\sigma = (\lambda - 4)\sigma - 2\eta \wedge \omega - 4d\varphi, \\ \delta\varphi = J\eta, \\ (\delta\sigma)_{2,0} = \eta \lrcorner \Psi^+. \end{cases}$$

Using the remaining formulae in [MS11, Prop. 3.4, Cor. 4.4], we see that this is equivalent to

$$\begin{cases} \Delta\varphi = (\lambda - 6)\varphi - \delta\sigma, \\ \Delta\sigma = (\lambda - 4)\sigma - 4d\varphi - 4\eta \wedge \omega, \\ \delta\varphi = J\eta, \\ (\delta\sigma)_{2,0} = \eta \lrcorner \Psi^+. \end{cases}$$

Suppose now that $\lambda < 16$. By applying δ to the first line of the above, it follows that

$$\Delta\delta\varphi = \delta\Delta\varphi = (\lambda - 6)\delta\varphi.$$

The Lichnerowicz estimate $\Delta \geq 2q(R) = 2E = 10$ on coclosed 1-forms now implies that $\delta\varphi = 0$ and hence $\eta = 0$, simplifying the above to (L2).[1] □

[1]This bridges the gap in the proof of [MS11, Thm. 5.1], where $\eta = 0$ was assumed without justification.

The space of solutions is described by the following lemma, which is a generalization of [MS11, Lem. 5.2].

5.4.2 Lemma. *Suppose that $\lambda = 10 - \varepsilon$ in system* (L2) *for some $\varepsilon > 0$. Denote with $E(\mu) := \ker(\Delta - \mu)\big|_{\Omega^{1,1}_{0,\mathbb{R}}} \cap \ker \delta$ the μ-eigenspace of Δ on coclosed primitive* $(1,1)$*-forms.*

(i) *Suppose that $\varepsilon < \frac{25}{4}$ and $\varepsilon \neq 6$. Then the space of solutions to system* (L2) *is isomorphic to the direct sum $E(\mu_1) \oplus E(\mu_2) \oplus E(\mu_3)$ where $\mu_{1,2} = 7 - \varepsilon \pm \sqrt{25 - 4\varepsilon}$ and $\mu_3 = 6 - \varepsilon$. The isomorphism is given by*

$$\Psi : \ (\varphi, \sigma) \mapsto ((3 - \sqrt{25 - 4\varepsilon})\varphi + \delta\sigma, (3 + \sqrt{25 - 4\varepsilon})\varphi + \delta\sigma, *d\sigma).$$

The inverse is given by

$$\Phi : \ (\alpha, \beta, \gamma) \mapsto \left(\frac{\beta - \alpha}{2\sqrt{25 - 4\varepsilon}}, \frac{d\beta - d\alpha}{2(6 - \varepsilon)\sqrt{25 - 4\varepsilon}} + \frac{d\alpha + d\beta}{2(6 - \varepsilon)} - \frac{*d\gamma}{6 - \varepsilon} \right).$$

If $6 < \varepsilon < \frac{25}{4}$, then $E(\mu_3)$ becomes trivial and thus $\gamma = *d\sigma = 0$.

(ii) *If $\varepsilon = 6$, then the space of solutions to* (L2) *is isomorphic to $E(2) \oplus \ker \Delta\big|_{\Omega^3}$, with isomorphism given by*

$$\Psi_6 : \ (\varphi, \sigma + \tau) \mapsto (\varphi, \tau) = \left(-\frac{1}{4}\delta\sigma, \tau \right)$$

for any $\sigma \in \operatorname{im} \Delta\big|_{\Omega^3}$ and $\tau \in \ker \Delta\big|_{\Omega^3}$, and inverse

$$\Phi_6 : \ (\varphi, \tau) \mapsto (\varphi, -2d\varphi + \tau)\,.$$

(iii) *If $\varepsilon = \frac{25}{4}$, then the space of solutions to* (L2) *is isomorphic to $E(\frac{3}{4})$, with isomorphism given by*

$$\Psi_{\frac{25}{4}} : \ (\varphi, \sigma) \mapsto \varphi = -\frac{1}{3}\delta\sigma$$

and inverse

$$\Phi_{\frac{25}{4}} : \ \varphi \mapsto (\varphi, -4d\varphi)\,.$$

(iv) *If $\varepsilon > \frac{25}{4}$, then the space of solutions to* (L2) *is trivial.*

Proof. The proof of the first part works completely analogously to the one of [MS11, Lem. 5.2]. We observe that if (φ, σ) is a solution to (L2), then

$$\begin{pmatrix} \Delta\varphi \\ \Delta\delta\sigma \end{pmatrix} = A \begin{pmatrix} \varphi \\ \delta\sigma \end{pmatrix}, \quad A := \begin{pmatrix} 4 - \varepsilon & -1 \\ -4(4 - \varepsilon) & 10 - \varepsilon \end{pmatrix}.$$

The eigenvalues of the matrix A are $\mu_{1,2} = 7 - \varepsilon \pm \sqrt{25-4\varepsilon}$ with corresponding eigenvectors $v_{1,2} = \left(\begin{smallmatrix} 3\mp\sqrt{25-4\varepsilon} \\ 1 \end{smallmatrix}\right)$, if $\varepsilon \neq \frac{25}{4}$.

Let $(\alpha, \beta, \gamma) := \Psi(\varphi, \sigma)$. Then

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3-\sqrt{25-4\varepsilon} & 1 \\ 3+\sqrt{25-4\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \delta\sigma \end{pmatrix},$$

$$\begin{pmatrix} \varphi \\ \delta\sigma \end{pmatrix} = \frac{1}{2\sqrt{25-4\varepsilon}} \begin{pmatrix} -1 & 1 \\ 3+\sqrt{25-4\varepsilon} & -3+\sqrt{25-4\varepsilon} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

If $\varepsilon < \frac{25}{4}$, then $25 - 4\varepsilon > 0$ and we can always recover the data $(\varphi, \delta\sigma)$ from (α, β, γ). We also have

$$*d\gamma = *d * d\sigma = -\delta d\sigma$$

and thus

$$d\delta\sigma - *d\gamma + 4d\varphi = \Delta\sigma + 4d\varphi = (6-\varepsilon)\sigma.$$

Hence if $\varepsilon \neq 6$, we can also recover σ. In total, Ψ is invertible, and one can check that its inverse is given by Φ.

If $6 < \varepsilon < \frac{25}{4}$, then $\mu_3 < 0$ and since Δ is nonnegative, $E(\mu_3)$=0.

Let $\varepsilon = 6$. If (φ, σ) is a solution to (L2), then

$$\Delta d\sigma = d\Delta\sigma = -4d\Delta d\varphi = -4\Delta d^2\varphi = 0,$$

i.e. $d\sigma$ is harmonic. But this implies that $\delta d\sigma = 0$ and hence $d\sigma = 0$. Also, $2\varphi + \delta\sigma \in E(2)$ and $4\varphi + \delta\sigma \in E(0)$. At the same time,

$$4\varphi + \delta\sigma = 2(2\varphi + \delta\sigma) - \delta\sigma.$$

Since both $E(2)$ and $\operatorname{im}\delta$ are orthogonal to $E(0)$, it follows that $4\varphi + \delta\sigma = 0$. Now

$$2\varphi + \delta\sigma = -2\varphi = \frac{1}{2}\delta\sigma \in E(2).$$

Furthermore $d\sigma = 0$ and $\sigma \perp \ker \Delta\big|_{\Omega^3}$ imply that $\sigma \in \operatorname{im} d\big|_{\Omega^2}$. Since

$$(\sigma, d\eta)_{L^2} = (\delta\sigma, \eta)_{L^2} = (-4\varphi, \eta)_{L^2} = (-2\Delta\varphi, \eta)_{L^2} = (-2\delta d\varphi, \eta)_{L^2} = (-2d\varphi, d\eta)_{L^2}$$

for any $\eta \in \Omega^2$, it follows that $\sigma = -2d\varphi$. One can check that $(\varphi, -2d\varphi + \tau)$ solves (L2) for any $\varphi \in E(2)$ and $\tau \in \ker \Delta\big|_{\Omega^3}$. Note that $\ker \Delta\big|_{\Omega^3} \subset \Omega^{2,1}_{0,\mathbb{R}}$ by a theorem of Verbitsky [Ver05, Thm. 6.2].

Let $\varepsilon = \frac{25}{4}$. In this case, $\mu_1 = \mu_2$ and the matrix A is not diagonalizable. If we set

$$\alpha' := \frac{1}{3}\delta\sigma, \quad \beta' := 3\varphi + \delta\sigma,$$

then we obtain the system

$$\begin{pmatrix} \Delta\alpha' \\ \Delta\beta' \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 1 \\ 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}.$$

For any $\beta \in E(\frac{3}{4})$, we therefore need to solve $(\Delta - \frac{3}{4})\alpha' = \beta'$. But $\mathrm{im}(\Delta - \frac{3}{4}) \perp \ker(\Delta - \frac{3}{4})$, hence there can only exist a solution if $\beta' = 0$. In this case, we obtain $\Delta\alpha' = \frac{3}{4}\alpha'$. Furthermore, $*d\sigma \in E(-\frac{1}{4})$. Since Δ is nonnegative, this implies that $d\sigma = 0$. We can recover σ from α' by

$$\frac{1}{4}\sigma = -\Delta\sigma - 4d\varphi = -d\delta\sigma + \frac{4}{3}d\delta\sigma = d\alpha'$$

and φ by $3\varphi + \delta\sigma = \beta' = 0$. In total, the space of solutions is isomorphic to $E(\frac{3}{4})$.

Let $\varepsilon > \frac{25}{4}$. Then μ_1, μ_2 are imaginary and $\mu_3 < 0$. Since Δ is nonnegative, $E(\mu_i) = 0$ for $i = 1, 2, 3$. The mapping Ψ is still an isomorphism, hence the space of solutions is trivial. $\square$

5.4.3 Remark. Note that in case (i) of the above lemma, the eigenvalues μ_i are subject to the bounds $\mu_1 < 12$, $\mu_2 < 2$ and $\mu_3 < 6$ if we assume that $\varepsilon > 0$. In the critical case where we set $\varepsilon = 0$, we recover the description

$$\varepsilon(g) \cong E(2) \oplus E(6) \oplus E(12)$$

by A. Moroianu and U. Semmelmann [MS11, Thm. 5.1].

Recall from §5.3.4 that a naturally reductive Riemannian 3-symmetric space $(G/H, g)$ carries a nearly Kähler structure whose Hermitian connection ∇^{h} coincides with the canonical reductive connection $\bar{\nabla}$ of the homogeneous structure. In fact, these assumptions hold for all of the homogeneous Gray manifolds, namely S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold $\mathrm{SU}(3)/T_2$.

The Hermitian Laplace operator Δ^{h} therefore coincides with the standard Laplacian $\bar{\Delta}$. In light of §5.3.3 this enables us to describe the eigenspaces of Δ^{h} in terms of irreducible complex representations of G; in particular on $\Omega^{1,1}_0$, where $\Delta^{\mathrm{h}} = \Delta$.

However, the statement of Lemma 5.4.2 involves the eigenspaces of Δ restricted to the subspace of *coclosed* forms in $\Omega^{1,1}_{0,\mathbb{R}}$. To single out the coclosed elements in an eigenspace, we are going to perform explicit computations utilizing the following lemma. A similar formula for the divergence on symmetric tensors has already been employed to decide the stability of certain symmetric spaces, cf. [Sch22b, Lem. 3.3] and [SW22, §2].

5.4.4 Lemma. *Let $(M = G/H, g, J)$ be a homogeneous Gray manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as in §5.3.4. Let $\delta : \Omega_0^{1,1} \to \Omega_{\mathbb{C}}^1$ denote the codifferential. Its prototypical differential operator is given by*

$$\delta\Big|_\gamma : \ \mathrm{Hom}_H(V_\gamma, \Lambda_0^{1,1}\mathfrak{m}) \to \mathrm{Hom}_H(V_\gamma, \mathfrak{m}^{\mathbb{C}}) : \ F \mapsto \sum_i e_i \lrcorner F \circ (\rho_\gamma)_*(e_i)$$

for any orthonormal basis (e_i) of $\mathfrak{m}$.

Proof. By (5.5), the Ambrose–Singer connection $\bar{\nabla} = \nabla^{\mathrm{h}}$ translates into a directional derivative on $C^\infty(G, \Lambda_0^{1,1}\mathfrak{m})$. Fix some $\gamma \in \hat{G}$, a vector $v \in V_\gamma$ and a homomorphism $F \in \mathrm{Hom}_H(V_\gamma, \Lambda_0^{1,1}\mathfrak{m})$, and let $\alpha \in \Omega_0^{1,1}$ be associated to $v \otimes F$ via the equivalence (5.3). Differentiating the smooth function

$$\hat{\alpha} : \ G \to \Lambda_0^{1,1}\mathfrak{m} : \ x \mapsto F(\rho_\gamma^{-1}(x)v)$$

defined in (5.4), we find that, for any $x \in G$ and $X \in \mathfrak{m} \cong T_oM$,

$$\widehat{\nabla_X^{\mathrm{h}}\alpha} = X(\hat{\alpha}) = -F((\rho_\gamma)_*(X)v)$$

Let (e_i) denote some local orthonormal frame of TM. It follows from [MS11, Lem. 4.2] that

$$\delta\alpha = -\sum_i e_i \lrcorner \nabla_{e_i}\alpha = -\sum_i e_i \lrcorner \nabla_{e_i}^{\mathrm{h}}\alpha$$

(essentially using that ∇^{h} has skew torsion). Combining the above, we obtain

$$\widehat{\delta\alpha} = -\sum_i e_i \lrcorner \widehat{\nabla_{e_i}^{\mathrm{h}}\alpha} = \sum_i e_i \lrcorner F((\rho_\gamma)_*(e_i)v).$$

at the base point. The assertion now follows from the G-invariance of δ. $\square$

5.5 Case-by-case stability analysis

5.5.1 Nearly Kähler $S^3 \times S^3$

Let $K = \mathrm{SU}(2)$ with Lie algebra $\mathfrak{k} = \mathfrak{su}(2)$, let $G = K \times K \times K$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$ and let $H = \Delta K \subset G$ be the diagonal, with Lie algebra $\mathfrak{h} \cong \mathfrak{k}$. We consider the homogeneous space $M = G/H$. Let $B_{\mathfrak{k}}$ denote the Killing form of $\mathfrak{k}$. The inner product on $\mathfrak{g}$ that is given by $-\frac{1}{12}(B_{\mathfrak{k}} \oplus B_{\mathfrak{k}} \oplus B_{\mathfrak{k}})$ defines a normal Riemannian metric g on M, which has scalar curvature $\mathrm{scal}_g = 30$. The automorphism

$$\sigma : G \to G : \ (k_1, k_2, k_3) \mapsto (k_2, k_3, k_1)$$

that cyclically permutes the factors is of order three, fixes H and hence gives M the structure of a Riemannian 3-symmetric space.

We denote by $E = \mathbb{C}^2$ the standard representation of $K = \mathrm{SU}(2)$. Furthermore, we label the irreducible complex representations of K by $k \in \mathbb{N}_0$, where $V_k = \mathrm{Sym}^k E$ is the unique $(k+1)$-dimensional irreducible complex representation of K.

5.5.1 Lemma. *Let V_γ be an irreducible complex representation of G with $\mathrm{Cas}^G_\gamma < 12$ and*

$$\mathrm{Hom}_H(V_\gamma, \Lambda^{1,1}_0 \mathfrak{m}) \neq 0.$$

Then V_γ is equivalent to one of the representations $E \otimes E \otimes \mathbb{C}$, $E \otimes \mathbb{C} \otimes E$ and $\mathbb{C} \otimes E \otimes E$ of G. In any of those cases

$$\dim \mathrm{Hom}_H(V_\gamma, \Lambda^{1,1}_0 \mathfrak{m}) = 1$$

and the Casimir eigenvalue is $\mathrm{Cas}^G_\gamma = 9$.

Proof. Since irreducible representations of $G = K \times K \times K$ are precisely the threefold tensor products of irreducible representations of K, we can label them by

$$V_{(a,b,c)} := V_a \otimes V_b \otimes V_c,$$

where $a, b, c \in \mathbb{N}_0$. Restricting the representation $V_{(a,b,c)}$ to the diagonal $H \subset G$ simply yields the tensor product $V_a \otimes V_b \otimes V_c$ as a representation of K. The Clebsch–Gordan rules allow us to decompose these into irreducible summands. From [MS10, (31)] we know that the Casimir eigenvalues of G with respect to the inner product $-\frac{1}{12}(B_\mathfrak{k} \oplus B_\mathfrak{k} \oplus B_\mathfrak{k})$ are given by

$$\mathrm{Cas}^G_{(a,b,c)} = \mathrm{Cas}^{\mathrm{SU}(2)}_a + \mathrm{Cas}^{\mathrm{SU}(2)}_b + \mathrm{Cas}^{\mathrm{SU}(2)}_c = \frac{3}{2}(a(a+2) + b(b+2) + c(c+2))$$

for $a, b, c \in \mathbb{N}_0$. The results for the first few Casimir eigenvalues are listed in Table 5.1.

$\gamma = (a,b,c)$	Branching of V_γ to K	Cas^G_γ
$(0,0,0)$	$\mathbb{C} = V_0$	0
$(1,0,0), (0,1,0), (0,0,1)$	$V_1 \otimes \mathbb{C} \otimes \mathbb{C} = V_1$	$\frac{9}{2}$
$(1,1,0), (1,0,1), (0,1,1)$	$V_1 \otimes V_1 \otimes \mathbb{C} \cong V_2 \oplus V_0$	9
$(1,1,1)$	$V_1 \otimes V_1 \otimes V_1 \cong V_3 \oplus V_1 \oplus V_1$	$\frac{27}{2}$
$(2,0,0), (0,2,0), (0,0,2)$	$V_2 \otimes \mathbb{C} \otimes \mathbb{C} = V_2$	12

Table 5.1: The first few Casimir eigenvalues of $G = K \times K \times K$

By [MS10, Lem. 5.5] we know that $\Lambda^{1,1}_0 \mathfrak{m} \cong V_4 \oplus V_2$ as a representation of K. Comparing summands now yields that the only irreducible complex representations V_γ of G with

Casimir eigenvalue smaller than 12 and nontrivial $\mathrm{Hom}_H(V_\gamma, \Lambda_0^{1,1}\mathfrak{m})$ are $V_{(1,1,0)}$, $V_{(1,0,1)}$ and $V_{(0,1,1)}$. $\square$

Since Δ^{h} acts as the Casimir operator, this means that 9 is the only eigenvalue smaller than 12 in the spectrum of Δ^{h} on $\Omega_0^{1,1}$. The eigenvalue 12 itself does also occur on $\Omega_0^{1,1}$, but in [MS10] it is shown that the corresponding eigenforms are not coclosed, hence proving that (M, g) has no infinitesimal Einstein deformations. It now remains to check whether this is the case for the eigenforms to the eigenvalue 9.

5.5.2 Lemma. *The eigenspace of Δ^{h} on $\Omega_{0,\mathbb{R}}^{1,1}$ to the eigenvalue 9 contains no nontrivial coclosed forms.*

Proof. We explicitly calculate the codifferential on the summands in question using the formula from Lemma 5.4.4.

Lemma 5.5.1 tells us that the relevant summands of the left-regular representation on $\Omega_0^{1,1}$ are $V_{(1,1,0)}$, $V_{(1,0,1)}$ and $V_{(0,1,1)}$. First, the representation $V_{(1,1,0)} = E \otimes E \otimes \mathbb{C}$ is given by

$$\rho : G \to \mathrm{Aut}(E \otimes E) : \ \rho(k_1, k_2, k_3)(v_1 \otimes v_2) = k_1 v_1 \otimes k_2 v_2$$

for any $k_1, k_2, k_3 \in K$ and $v_1, v_2 \in E$. Recall that $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, where $\mathfrak{m}^+$ is the eigenspace of σ_* to the eigenvalue $\mathrm{j} = -\frac{1}{2} + \frac{3}{2}\mathrm{i}$, and $\mathfrak{m}^-$ is the eigenspace to $\mathrm{j}^2 = -\frac{1}{2} - \frac{3}{2}\mathrm{i}$. Explicitly,

$$\mathfrak{m}^+ = \{(Y, \mathrm{j}Y, \mathrm{j}^2 Y) \,|\, Y \in \mathfrak{k}\}, \quad \mathfrak{m}^- = \{(Y, \mathrm{j}^2 Y, \mathrm{j}Y) \,|\, Y \in \mathfrak{k}\}.$$

Let (Y_1, Y_2, Y_3) be an orthonormal basis of $\mathfrak{k}$ with respect to the inner product $-B_{\mathfrak{k}}$. Then

$$X_i := 2(Y_i, \mathrm{j}Y_i, \mathrm{j}^2 Y_i) \in \mathfrak{m}^+, \quad \overline{X_i} := 2(Y_i, \mathrm{j}^2 Y_i, \mathrm{j}Y_i) \in \mathfrak{m}^-, \quad i = 1, 2, 3$$

constitute an orthonormal basis of $\mathfrak{m}^{\mathbb{C}}$ with respect to $-\frac{1}{12}(B_{\mathfrak{k}} \oplus B_{\mathfrak{k}} \oplus B_{\mathfrak{k}})$. With respect to the basis

$$\mathcal{B} = (z_1 \otimes z_1, z_1 \otimes z_2, z_2 \otimes z_1, z_2 \otimes z_2) \quad \text{of} \quad E \otimes E,$$

we can represent $\rho_*(X_i)$ by the 4×4-matrices

$$\rho_*(X_1) = \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & \mathrm{j} & 1 & 0 \\ \mathrm{j} & 0 & 0 & 1 \\ 1 & 0 & 0 & \mathrm{j} \\ 0 & 1 & \mathrm{j} & 0 \end{pmatrix}, \qquad \rho_*(X_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{j} & -1 & 0 \\ \mathrm{j} & 0 & 0 & -1 \\ 1 & 0 & 0 & -\mathrm{j} \\ 0 & 1 & \mathrm{j} & 0 \end{pmatrix},$$

$$\rho_*(X_3) = \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 1+\mathrm{j} & 0 & 0 & 0 \\ 0 & 1-\mathrm{j} & 0 & 0 \\ 0 & 0 & -1+\mathrm{j} & 0 \\ 0 & 0 & 0 & -1-\mathrm{j} \end{pmatrix}. \tag{5.8}$$

This works similarly for $\overline{X_i}$ by means of simply replacing the symbol j with j^2.

Turning to the decomposition of $\Lambda^{1,1}\mathfrak{m}$ and $E \otimes E$ into K-irreducible summands, we have

$$\begin{aligned} \Lambda^{1,1}\mathfrak{m} &= \mathfrak{m}^+ \otimes \mathfrak{m}^- \cong \mathfrak{k}^{\mathbb{C}} \otimes \mathfrak{k}^{\mathbb{C}} \cong \operatorname{Sym}^2_0 \mathfrak{k}^{\mathbb{C}} \oplus \Lambda^2\mathfrak{k}^{\mathbb{C}} \oplus \mathbb{C}, \\ E \otimes E &= \operatorname{Sym}^2 E \oplus \Lambda^2 E \end{aligned}$$

with common summand $\Lambda^2\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{k}^{\mathbb{C}} \cong V_2 = \operatorname{Sym}^2 E$. If we choose the basis of the image of $\Lambda^2\mathfrak{k}^{\mathbb{C}}$ in $\Lambda^{1,1}\mathfrak{m}$ as

$$\mathcal{B}' = \left(X_1 \wedge \overline{X_2} - X_2 \wedge \overline{X_1},\ X_2 \wedge \overline{X_3} - X_3 \wedge \overline{X_2},\ X_3 \wedge \overline{X_1} - X_1 \wedge \overline{X_3}\right),$$

then a generator of $\operatorname{Hom}_K(E \otimes E, \Lambda^{1,1}_0\mathfrak{m})$ is represented by the matrix

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -\mathrm{i} & 0 & 0 & \mathrm{i} \end{pmatrix}$$

with respect to $\mathcal{B}$ and $\mathcal{B}'$. Taking

$$\mathcal{B}'' := \left(X_1, X_2, X_3, \overline{X_1}, \overline{X_2}, \overline{X_3}\right)$$

as a basis of $\mathfrak{m}^{\mathbb{C}}$, we compute $\delta\big|_{(1,1,0)} F$ according to Lemma 5.4.4:

$$\delta\Big|_{(1,1,0)} F = \sum_i X_i \lrcorner (F \circ \rho_*(X_i)) + \sum_i \overline{X_i} \lrcorner (F \circ \rho_*(\overline{X_i})) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1-\mathrm{j}^2 & -1+\mathrm{j}^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1-\mathrm{j} & -1+\mathrm{j} & 0 \end{pmatrix}$$

with respect to the bases $\mathcal{B}$ and $\mathcal{B}''$. We have thus shown that

$$\delta\Big|_{(1,1,0)} :\ \operatorname{Hom}_H(V_{(1,1,0)}, \Lambda^{1,1}_0\mathfrak{m}) \to \operatorname{Hom}_H(V_{(1,1,0)}, \mathfrak{m}^{\mathbb{C}})$$

does not vanish. For the other two representations $V_{(1,0,1)}$ and $V_{(0,1,1)}$ modeled on the vector space $E \otimes E$ with the same Casimir eigenvalue, the only thing that changes is the action of G, amounting to cyclic permutations of the factors $1, \mathrm{j}, \mathrm{j}^2$ in (5.8). The computations work out analogously. We conclude that the eigenspace of Δ^{h} on $\Omega_0^{1,1}$ to the eigenvalue 9 contains no nonzero coclosed forms. □

It remains to apply Lemma 5.4.2 in order to finally obtain the desired result.

5.5.3 Proposition. *On the nearly Kähler manifold $S^3 \times S^3$, the space of unstable directions for its Einstein metric g consists solely of the 2-dimensional Δ_{L}-eigenspace to the eigenvalue 4, the latter arising from harmonic 3-forms. In total, the coindex of g is 2.*

Proof. We recall the bounds $\mu_1 < 12$, $\mu_2 < 2$ and $\mu_3 < 6$ from Lemma 5.4.2. Lemmas 5.5.1 and 5.5.2 imply that $E(\mu)$ is trivial for all $\mu < 12$. However, 5.4.2, (ii) yields a space of solutions isomorphic to the space of harmonic 3-forms in the case $\varepsilon = 6$. Since $b_3 = 2$, this gives us a 2-dimensional subspace of $\mathscr{S}^2_{\mathrm{tt}}$ such that

$$\Delta_{\mathrm{L}} h = (10 - 6)h = 4h$$

for any element h. □

5.5.2 Nearly Kähler $\mathbb{CP}^3$

Let $G = \mathrm{SO}(5)$ and $H = \mathrm{U}(2)$. We consider H embedded into G via the natural inclusions $\mathrm{U}(2) \subset \mathrm{SO}(4) \subset \mathrm{SO}(5)$. The normal Riemannian metric g induced by $-\frac{1}{12}B_{\mathfrak{g}}$ on the homogeneous space $M = G/H$ is the nearly Kähler metric on $\mathbb{CP}^3$, normalized to $\mathrm{scal}_g = 30$. It should be noted that (M, g) is, again, naturally reductive and 3-symmetric, with reductive complement $\mathfrak{m} = \mathfrak{h}^{\perp}$.

Let $\mathfrak{t} = \{\mathrm{diag}(\mathrm{i}\theta_1, \mathrm{i}\theta_2) \,|\, \theta_1, \theta_2 \in \mathbb{R}\} \subset \mathfrak{h} \subset \mathfrak{g}$ be the maximal torus Lie algebra. The positive roots $\alpha_i \in \mathfrak{t}^*$ of G can then be expressed as

$$\alpha_1 = \theta_1, \;\; \alpha_2 = \theta_2, \;\; \alpha_3 = \theta_1 + \theta_2, \;\; \alpha_4 = \theta_1 - \theta_2.$$

In terms of root spaces of G we have

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{-\alpha_1} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_2} \oplus \mathfrak{g}^{\alpha_3} \oplus \mathfrak{g}^{-\alpha_3}. \tag{5.9}$$

The almost complex structure J can be defined by specifying its $\pm\mathrm{i}$-eigenspaces

$$\mathfrak{m}^{+} = \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_3}, \;\; \mathfrak{m}^{-} = \mathfrak{g}^{-\alpha_1} \oplus \mathfrak{g}^{-\alpha_2} \oplus \mathfrak{g}^{\alpha_3}.$$

In passing we note that the standard complex structure on $\mathbb{CP}^3$ has $\mathfrak{m}^{+} = \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{\alpha_3}$.

We label the irreducible complex representations V_γ of G by their highest weights $\gamma = (a, b)$, where $a, b \in \mathbb{N}_0$, $a \geq b$. For example $V_{(1,0)} = \mathbb{C}^5$ is the complexified standard representation of G, while $V_{(1,1)} = \mathfrak{so}(5)^{\mathbb{C}}$ is the complexified adjoint representation.

Again, let $E = \mathbb{C}^2$ be the standard representation of $\mathrm{SU}(2)$. Let furthermore $\mathbb{C}_k$ denote the representation of $\mathrm{U}(1)$ on $\mathbb{C}$ defined by

$$\mathrm{U}(1) \times \mathbb{C} \to \mathbb{C} : \ (z, w) \mapsto z^k w$$

for any $k \in \mathbb{Z}$. The irreducible complex representations of $H \cong (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ are then given by $E^a_b := \mathrm{Sym}^a E \otimes \mathbb{C}_b$ for $a \in \mathbb{N}_0$ and $b \in \mathbb{Z}$, $a \equiv b \mod 2$.

5.5.4 Lemma. *Let V_γ be an irreducible complex representation of G with $\mathrm{Cas}^G_\gamma < 12$ and*

$$\mathrm{Hom}_H(V_\gamma, \Lambda^{1,1}_0 \mathfrak{m}) \neq 0.$$

Then V_γ is equivalent to either the trivial representation $V_{(0,0)}$ or the standard representation $V_{(1,0)}$. In both cases,

$$\dim \mathrm{Hom}_H(V_\gamma, \Lambda^{1,1}_0 \mathfrak{m}) = 1$$

and the Casimir eigenvalues are $\mathrm{Cas}^G_{(0,0)} = 0$ and $\mathrm{Cas}^G_{(1,0)} = 8$, respectively.

Proof. We first work out how to decompose the restriction of $V_{(1,0)}$ to H into irreducible summands. We know that $V_{(1,0)} = \mathbb{C}^5$ is the complexified standard representation of G. The inclusion $\mathrm{U}(2) \subset \mathrm{SO}(4)$ can be understood as realification $(E^1_1)^{\mathbb{R}}$ of the defining representation E^1_1 of $\mathrm{U}(2)$. Furthermore, the inclusion $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ defines a five-dimensional real representation $\mathbb{R}^4 \oplus \mathbb{R}$ of $\mathrm{SO}(4)$, where the group acts as on its defining representation on the first summand and trivially on the second. In total, the restriction of the real standard representation of $G = \mathrm{SO}(5)$ to $H = \mathrm{U}(2)$ is given by $(E^1_1)^{\mathbb{R}} \oplus \mathbb{R}$. Complexifying then yields the decomposition

$$V_{(1,0)} = (E^1_1)^{\mathbb{R}\mathbb{C}} \oplus \mathbb{C} \cong E^1_1 \oplus E^1_{-1} \oplus E^0_0.$$

For the branching of $V_{(1,1)}$ under the restriction to H, we refer to [MS10, Lem. 5.9]. Using the decomposition

$$V_{(1,0)} \otimes V_{(1,0)} \cong V_{(2,0)} \oplus V_{(1,1)} \oplus V_{(0,0)}$$

of G-representations, the known branchings of $V_{(1,1)}$ and $V_{(1,0)}$ as well as the Clebsch–Gordan rules, we can also work out the branching of $V_{(2,0)}$ to H, although it will not be needed hereafter.

By [MS10, (33)], we know that on the irreducible representation $V_{(a,b)}$ of G, the Casimir eigenvalue of G with respect to the inner product $-\frac{1}{12}B_{\mathfrak{g}}$ is given by

$$\mathrm{Cas}^G_{(a,b)} = 2(a(a+3)+b(b+1)).$$

Table 5.2 lists the results for the smallest few Casimir eigenvalues of G.

$\gamma = (a,b)$	Branching of V_γ to H	Cas^G_γ
$(0,0)$	$\mathbb{C} = E^0_0$	0
$(1,0)$	$E^1_1 \oplus E^1_{-1} \oplus E^0_0$	8
$(1,1)$	$E^2_0 \oplus E^1_1 \oplus E^1_{-1} \oplus E^0_2 \oplus E^0_0 \oplus E^0_{-2}$	12
$(2,0)$	$E^2_2 \oplus E^2_0 \oplus E^2_{-2} \oplus E^1_1 \oplus E^1_{-1} \oplus E^0_0$	20

Table 5.2: The first few Casimir eigenvalues of $G = \mathrm{SO}(5)$

[MS10, Lem. 5.8] tells us that

$$\Lambda^{1,1}_0\mathfrak{m} \cong E^2_0 \oplus E^1_3 \oplus E^1_{-3} \oplus E^0_0$$

as a representation of $H = \mathrm{U}(2)$. By comparing summands, we conclude that $V_{(0,0)}$ and $V_{(1,0)}$ are the only irreducible complex representations V_γ of G with Casimir eigenvalue smaller than 12 and nontrivial $\mathrm{Hom}_H(V_\gamma, \Lambda^{1,1}_0\mathfrak{m})$. $\square$

Again, Δ^{h} acts as the Casimir operator. We therefore know that 0 and 8 are the only eigenvalues smaller than 12 in the spectrum of Δ^{h} on $\Omega^{1,1}_0$. As in the case $S^3 \times S^3$, the eigenvalue 12 does occur on $\Omega^{1,1}_0$, but the corresponding eigenforms are not coclosed (see [MS10]). Hence (M,g) has no infinitesimal Einstein deformations. It remains to check whether the eigenforms to the eigenvalues 0 and 8 are coclosed.

5.5.5 Lemma. *The eigenspace of Δ^{h} on $\Omega^{1,1}_{0,\mathbb{R}}$ to the eigenvalue 0 consists of coclosed forms, while the eigenspace to the eigenvalue 8 contains no nontrivial coclosed forms.*

Proof. The eigenspace of Δ to the eigenvalue 0 corresponds to the trivial summand in the left-regular representation, i.e. to G-invariant elements of $\Omega^{1,1}_0$. But these are parallel with respect to the Ambrose–Singer connection (which equals the canonical Hermitian connection ∇^{h}). Recall that

$$\delta = -\sum_i e_i \lrcorner \nabla_{e_i} = -\sum_i e_i \lrcorner \nabla^{\mathrm{h}}_{e_i} \qquad \text{on } \Omega^{1,1}_0$$

by [MS11, Lem. 4.2]. It follows any element in the 0-eigenspace of Δ^{h} on $\Omega^{1,1}_0$ is coclosed.

For the eigenspace to the eigenvalue 8 we again make use of the formula from Lemma 5.4.4 in an explicit calculation. Lemma 5.5.4 tells us that the relevant summand of the left-regular representation on $\Omega^{1,1}_0$ is $V_{(1,0)}$. A generator F of the one-dimensional space

$\mathrm{Hom}_H(V_{(1,0)}, \Lambda_0^{1,1}\mathfrak{m})$ must map the trivial summand E_0^0 in the H-representation

$$V_{(1,0)} \cong E_1^1 \oplus E_{-1}^1 \oplus E_0^0$$

to the trivial summand in

$$\Lambda_0^{1,1}\mathfrak{m} \cong E_0^2 \oplus E_3^1 \oplus E_{-3}^1 \oplus E_0^0.$$

The former is spanned by v_5, where $(v_1, \ldots, v_5)$ is the standard basis of $V_{(1,0)} = \mathbb{C}^5$. To describe the latter we remark that the root spaces in decomposition (5.9) can be written as

$$\begin{aligned}
\mathfrak{g}_{\alpha_1} &= \mathrm{span}\{e_1 - \mathrm{i}e_2\}, & \mathfrak{g}_{-\alpha_1} &= \mathrm{span}\{e_1 + \mathrm{i}e_2\},\\
\mathfrak{g}_{\alpha_2} &= \mathrm{span}\{e_3 - \mathrm{i}e_4\}, & \mathfrak{g}_{-\alpha_2} &= \mathrm{span}\{e_3 + \mathrm{i}e_4\},\\
\mathfrak{g}_{\alpha_3} &= \mathrm{span}\{f_1 - \mathrm{i}f_2\}, & \mathfrak{g}_{-\alpha_3} &= \mathrm{span}\{f_1 + \mathrm{i}f_2\}
\end{aligned}$$

in terms of the basis

$$\begin{aligned}
&e_1 := E_{15} - E_{51}, && e_2 := E_{25} - E_{52}, && e_3 := E_{35} - E_{53},\\
&e_4 := E_{45} - E_{54}, && f_1 := E_{13} - E_{24} - E_{31} + E_{42}, && f_2 := E_{14} + E_{23} - E_{32} - E_{41}
\end{aligned}$$

of $\mathfrak{m} \subset \mathfrak{so}(5)$, where E_{ij} denotes the 5×5-matrix with 1 at position (i, j) and zero at all other entries. Note that under the inner product $-\frac{1}{12}B_{\mathfrak{g}}$, the basis $(\sqrt{2}e_i, f_j)$ is orthonormal. It follows from the definition of J in terms of $\mathfrak{m}^{\pm}$ that the Kähler form can be written as

$$\omega = 2e_{12} + 2e_{34} - f_{12}.$$

By [MS10, (32)] the root space $\mathfrak{g}^{-\alpha_3} \subset \mathfrak{g}^{\mathbb{C}}$ is H-invariant and equivalent to E_{-2}^0. By conjugation, $\mathfrak{g}^{\alpha_3} \cong E_2^0$. Hence H acts trivially on $\mathfrak{g}^{-\alpha_3} \otimes \mathfrak{g}^{\alpha_3} \cong E_{-2}^0 \otimes E_2^0 = E_0^0$. Recall that $\mathfrak{g}^{-\alpha_3} \subset \mathfrak{m}^+$ and $\mathfrak{g}^{\alpha_3} \subset \mathfrak{m}^-$. It follows that $f_{12} = \frac{\mathrm{i}}{2}(f_1 + \mathrm{i}f_2) \wedge (f_1 - \mathrm{i}f_2)$ spans a trivial subspace of $\Lambda^{1,1}\mathfrak{m} = \mathfrak{m}^+ \wedge \mathfrak{m}^-$. Since $\Lambda_0^{1,1}\mathfrak{m}$ is the orthogonal complement of ω in $\Lambda^{1,1}\mathfrak{m}$, the remaining trivial summand in $\Lambda_0^{1,1}\mathfrak{m}$ must be spanned by

$$\eta = e_{12} + e_{34} + f_{12}.$$

Having found the H-trivial subspaces of $V_{(1,0)}$ and $\Lambda_0^{1,1}\mathfrak{m}$, we see that the space of equivariant homomorphisms $\mathrm{Hom}_H(V_{(1,0)}, \Lambda_0^{1,1}\mathfrak{m})$ is spanned by

$$F: \ \mathbb{C}^5 \to \Lambda_0^{1,1}\mathfrak{m}: \ (z_1, \ldots, z_5) \mapsto z_5\eta.$$

Observe that $f_j v_k \perp v_5$ for all j,k and $e_i v_k \perp v_5$ for all i,k, except for

$$e_1 v_1 = e_2 v_2 = e_3 v_3 = e_4 v_4 = -v_5.$$

Hence, by Lemma 5.4.4,

$$\langle (\delta\big|_{(1,0)} F)(v_i), X\rangle = 2\langle F(e_i v_i), e_i \wedge X\rangle = -2\langle \eta, e_i \wedge X\rangle = -2\langle e_i \lrcorner \eta, X\rangle$$

for any $X \in \mathfrak{m}^{\mathbb{C}}$. We have thus shown that

$$\delta\big|_{(1,0)} : \ \mathrm{Hom}_H(V_{1,0}, \Lambda_0^{1,1}\mathfrak{m}) \to \mathrm{Hom}_H(V_{1,0}, \mathfrak{m}^{\mathbb{C}})$$

is not the zero map – in fact, it maps $\delta\big|_{(1,0)} F = F'$, where

$$F'(v_i) = -2e_i \lrcorner \eta \text{ for } i = 1, \dots, 4, \quad F'(v_5) = 0.$$

Thus the eigenspace of Δ^{h} on $\Omega_0^{1,1}$ to the eigenvalue 8 contains no nonzero coclosed forms. $\square$

As before, the desired result follows from an application of Lemma 5.4.2.

5.5.6 Proposition. *On the nearly Kähler manifold $\mathbb{CP}^3$, the space of unstable directions for its Einstein metric g consists solely of the 1-dimensional Δ_{L}-eigenspace to the eigenvalue 6, arising from harmonic 2-forms. Consequently the coindex of g is 1.*

Proof. Lemmas 5.5.4 and 5.5.5 imply that $E(\mu)$ from Lemma 5.4.2 is trivial for all $\mu < 12$ except if $\mu = 0$. We have already seen that $E(0)$ consists of harmonic forms. In fact, [Ver05, Thm. 6.2] implies that all harmonic 2-forms on M lie in $E(0)$. If we solve

$$\mu_{1,2} = 7 - \varepsilon \pm \sqrt{25 - \varepsilon} = 0$$

for ε, we obtain $\varepsilon = 5 \pm 1$; from $\mu_3 = 6 - \varepsilon = 0$ we obtain $\varepsilon = 6$. If $\varepsilon = 6$, i.e. in case (ii) of Lemma 5.4.2, the space of solutions to (L2) is isomorphic to $E(2) \oplus \ker \Delta\big|_{\Omega^3}$. Since 2 does not appear in the spectrum on $\Omega_0^{1,1}$ and $b_3(\mathbb{CP}^3) = 0$, this space is trivial. The only possibility in which $E(0)$ contributes to the space of solutions of (L2) is in case (i) of Lemma 5.4.2 with $\varepsilon = 4$. Since $b_2 = 1$, we obtain a 1-dimensional subspace of $\mathscr{S}^2_{\mathrm{tt}}$ on which

$$\Delta_{\mathrm{L}} h = (10 - 4)h = 6h$$

for any of its elements h. $\square$

5.5.3 The flag manifold $F_{1,2}$

Let $G = \mathrm{SU}(3)$ and $H = T^2$, the latter embedded into G via

$$H = \mathrm{U}(1) \times \mathrm{U}(1) \hookrightarrow G: \ (z_1, z_2) \mapsto \mathrm{diag}(z_1, z_2, (z_1 z_2)^{-1}).$$

The homogeneous space $M = G/H$ is a description of the manifold $F_{1,2}$ of flags in $\mathbb{C}^3$. As in the previous examples we endow M with the normal Riemannian metric g induced by $-\frac{1}{12}B_{\mathfrak{g}}$, which has scalar curvature $\mathrm{scal}_g = 30$. The Riemannian homogeneous space (M, g) is naturally reductive, 3-symmetric and hence nearly Kähler. For more information on the nearly Kähler structure see [MS10, §5.6].

Denote by $E = \mathbb{C}^3$ the standard representation of G. Any irreducible complex representation of G can then be described as the Cartan summand $V_{(k,l)}$ of the tensor product $\mathrm{Sym}^k E \otimes \mathrm{Sym}^l \bar{E}$ for some $k, l \in \mathbb{N}_0$. For example $V_{(1,1)}$ is equivalent to the complexified adjoint representation $\mathfrak{su}(3)^{\mathbb{C}}$ of G.

5.5.7 Lemma. *Let V_γ be an irreducible complex representation of G with $\mathrm{Cas}^G_\gamma < 12$ and*

$$\mathrm{Hom}_H(V_\gamma, \Lambda^{1,1}_0 \mathfrak{m}) \neq 0.$$

Then V_γ is the trivial representation and $\dim \mathrm{Hom}_H(\mathbb{C}, \Lambda^{1,1}_0 \mathfrak{m}) = 2$.

Proof. It follows from [MS10, (35)] that the Casimir eigenvalue on the irreducible representation $V_{(k,l)}$ of G with respect to the inner product $-\frac{1}{12}B_{\mathfrak{g}}$ is given by

$$\mathrm{Cas}^G_{(k,l)} = 2(k(k+2) + l(l+2)).$$

(k,l)	$\dim \mathrm{Hom}_H(V_{(k,l)}, \Lambda^{1,1}_0 \mathfrak{m})$	$\mathrm{Cas}^G_{(k,l)}$
$(0,0)$	2	0
$(1,0)$	0	6
$(0,1)$	0	6
$(1,1)$	4	12

Table 5.3: The first few Casimir eigenvalues of $G = \mathrm{SU}(3)$

By analyzing the weights of the respective representations it has already been checked in [MS10, §5.6] that

$$\mathrm{Hom}_H(V_{(1,0)}, \Lambda^{1,1}_0 \mathfrak{m}) = \mathrm{Hom}_H(V_{(0,1)}, \Lambda^{1,1}_0 \mathfrak{m}) = 0$$

and $\dim \mathrm{Hom}_H(V_{(1,1)}, \Lambda^{1,1}_0 \mathfrak{m}) = 4$.

The maximal subspace of $\Omega_0^{1,1}$ on which G acts trivially is precisely the kernel of the Casimir operator. We can argue exactly as in the proof of Lemma 5.5.5 that the elements of this space are coclosed. Furthermore, we recall that $\Delta = \Delta^{\mathrm{h}}$ on coclosed primitive $(1,1)$-forms. Since Δ^{h} acts as the Casimir operator, we are simply talking about the subspace of harmonic forms in $\Omega_0^{1,1}$. By Verbitsky's theorem [Ver05, Thm. 6.2] all harmonic 2-forms lie in $\Omega_0^{1,1}$. Thus, the multiplicity of the trivial representation $\mathbb{C}$ in $\Omega_0^{1,1}$ is $b_2(F_{1,2}) = 2$. By Frobenius reciprocity this is also the dimension of $\mathrm{Hom}_H(\mathbb{C}, \Lambda_0^{1,1}\mathfrak{m})$.

The results of the above discussion are listed in Table 5.3. □

We have thus shown that 0 is the only eigenvalue smaller than 12 in the spectrum of Δ^{h} on $\Omega_0^{1,1}$. Besides, it has been proven in [MS10, §6] that an 8-dimensional subspace of the 32-dimensional eigenspace to the eigenvalue 12 consists of coclosed forms and hence yields infinitesimal Einstein deformations of (M, g). We will describe these more explicitly in §5.6.

5.5.8 Proposition. *On the nearly Kähler manifold $F_{1,2}$, the space of unstable directions for its Einstein metric g consists solely of a 2-dimensional Δ_{L}-eigenspace to the eigenvalue 6, arising from harmonic 2-forms. In total, the coindex of g is 2.*

Proof. One last time, we want to apply Lemma 5.4.2. By Lemma 5.5.7 and the fact that harmonic forms are coclosed, we see that for $\mu < 12$, the eigenspace $E(\mu)$ is only nontrivial if $\mu = 0$. With the same reasoning as in the proof of Proposition 5.5.6, we conclude that (L2) can be solved for $\varepsilon = 4$, yielding a 2-dimensional subspace of $\mathscr{S}^2_{\mathrm{tt}}$ such that

$$\Delta_{\mathrm{L}} h = (10 - 4)h = 6h$$

for all its elements h. □

5.5.9 Remark. The space of destabilizing directions (or of harmonic 2-forms) can be described rather explicitly. The 3-dimensional space

$$\Lambda^{1,1}\mathfrak{m}^H \cong \mathrm{Hom}_H(V_{(0,0)}, \Lambda^{1,1}\mathfrak{m})$$

of H-invariant elements of $\Lambda^{1,1}\mathfrak{m}$ corresponds to the space $(\Omega^{1,1})^G$ of G-invariant $(1,1)$-forms on $F_{1,2}$ and is spanned by

$$e_{12}, \quad e_{34}, \quad e_{56},$$

using the notation introduced in §5.6. The Kähler form ω corresponding to the strict nearly Kähler structure on $F_{1,2}$ is given by

$$\hat{\omega} = e_{12} - e_{34} + e_{56}.$$

The 2-dimensional space $\Lambda^{1,1}_0\mathfrak{m}^H \cong (\Omega^{1,1}_0)^G = \ker\Delta\big|_{\Omega^2}$ responsible for the destabilizing directions is now the orthogonal complement of $\hat{\omega}$ in $\Lambda^{1,1}\mathfrak{m}^H$.

The twistor space over $\mathbb{CP}^2$ can be identified with $F_{1,2}$ in three distinct ways, giving rise to three fibrations $F_{1,2} \to \mathbb{CP}^2$, which in turn induce six almost complex structures on $F_{1,2}$. Three of them are actually integrable with respective Kähler forms $\omega_1, \omega_2, \omega_3$, described by

$$\hat{\omega}_1 = -e_{12} - e_{34} + e_{56}, \quad \hat{\omega}_2 = e_{12} + e_{34} + e_{56}, \quad \hat{\omega}_3 = e_{12} - e_{34} - e_{56},$$

while the other three coincide with the almost complex structure with Kähler form ω. See [Mor14, §3.2.3] for a detailed description.

Thus, in light of the construction in §5.4 using the isomorphism

$$\mathscr{S}^+_0 \to \Omega^{1,1}_{0,\mathbb{R}} : \; h \mapsto h \circ J,$$

the destabilizing directions of the nearly Kähler metric g on $F_{1,2}$ can be viewed as coming from variations of $\omega = g(J\cdot,\cdot)$ in the directions of $\omega_1, \omega_2, \omega_3$ while fixing J.

Alternatively, consider the canonical variation, i.e. change of scale of fiber against base, on each of the three aforementioned fibrations

$$\pi_j : \; F_{1,2} = \frac{\mathrm{SU}(3)}{T^2} \longrightarrow \frac{\mathrm{SU}(3)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))} = \mathbb{CP}^2, \qquad j = 1,2,3.$$

In [WW18, Prop. 4.4] it is shown that these variations yield destabilizing tt-tensors. General destabilizing directions $h \in \mathscr{S}^2_{\mathrm{tt}}$ are thus (at the base point) of the form

$$h = t_1 g\big|_{\mathfrak{m}_1} + t_2 g\big|_{\mathfrak{m}_2} + t_3 g\big|_{\mathfrak{m}_3}, \qquad \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

with $t_1 + t_2 + t_3 = 0$, where each of the pairwise orthogonal subspaces $\mathfrak{m}_j$ is the vertical tangent space with respect to π_j. Since π_j are Riemannian submersions with totally geodesic fibers, the destabilizing directions are *Killing tensors* by [HMS16, Ex. 7.3], that is, they satisfy the Killing equation[2]

$$\nabla_X h(X,X) = 0 \qquad \forall X \in TM.$$

[2]This can be verified directly, using the description of $\nabla_X - \nabla^{\mathrm{h}}_X$ in §5.6.2 and the fact that h is G-invariant and hence ∇^{h}-parallel.

5.6 Rigidity of $F_{1,2}$

5.6.1 The infinitesimal Einstein deformations of $F_{1,2}$

We will utilize the explicit description of the infinitesimal Einstein deformations of $F_{1,2}$ given in [MS10, §6]. For this, it is helpful to represent $M = F_{1,2}$ as a quotient of $G = \mathrm{U}(3)$ by the diagonally embedded torus $H = T^3$.

Denote by E_{ij} the 3×3-matrix with a 1 at position (i,j) and zero entries elsewhere. Let $\{h_1, h_2, h_3, e_1, \dots, e_6\}$ be the basis of $\mathfrak{g} = \mathfrak{u}(3)$ given by

$$
\begin{aligned}
h_1 &= \mathrm{i}E_{11}, & h_2 &= \mathrm{i}E_{22}, & h_3 &= \mathrm{i}E_{33},\\
e_1 &= E_{12} - E_{21}, & e_2 &= \mathrm{i}(E_{12} + E_{21}), & e_3 &= E_{13} - E_{31},\\
e_4 &= \mathrm{i}(E_{13} + E_{31}), & e_5 &= E_{23} - E_{32}, & e_6 &= \mathrm{i}(E_{23} + E_{32}).
\end{aligned}
$$

Note that $\{h_1, h_2, h_3\}$ span the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, while the reductive complement $\mathfrak{m} \subset \mathfrak{g}$ is spanned by $\{e_1, \dots, e_6\}$. We now define the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ (and the induced bi-invariant metric on G) in such a way that $(e_i, \sqrt{2}h_j)$ is an orthonormal system. One easily checks that this coincides with $-\frac{1}{12}B_{\mathfrak{su}(3)}$ when restricted to $\mathfrak{su}(3) \subset \mathfrak{g}$, so we recover the same metric g on $F_{1,2}$.

The space $\varepsilon(g)$ of infinitesimal Einstein deformations of g is equivalent to $\mathfrak{su}(3)$ via the following prodecure. For a fixed element $\xi \in \mathfrak{su}(3) \subset \mathfrak{g}$, let $\xi^* \in C^\infty(G, \mathfrak{g})$ be given by $\xi^*(x) = \mathrm{Ad}(x)\xi$. This defines smooth, real-valued functions $x_1, x_2, x_3, v_1, \dots, v_6$ on G via

$$
\xi^* = \begin{pmatrix} 2\mathrm{i}v_1 & x_1 + \mathrm{i}x_2 & x_3 + \mathrm{i}x_4 \\ -x_1 + \mathrm{i}x_2 & 2\mathrm{i}v_2 & x_5 + \mathrm{i}x_6 \\ -x_3 + \mathrm{i}x_4 & -x_5 + \mathrm{i}x_6 & 2\mathrm{i}v_3 \end{pmatrix}.
$$

As before, we identify sections in a tensor bundle $E = G \times_H V$ over M with H-equivariant functions on G with values in V and denote this by

$$
\Gamma(E) \ni \varphi \mapsto \hat{\varphi} \in C^\infty(G, V)^H.
$$

As seen in Remark 5.5.9, the Kähler form $\omega \in \Omega^2$ corresponds to the (constant) function

$$
\hat{\omega} = e_{12} - e_{34} + e_{56} \in C^\infty(G, \Lambda^2\mathfrak{m})^H
$$

(we write $e_{ij} = e_i \wedge e_j$ to shorten notation). Define a real-valued function $\hat{\varphi} \in C^\infty(G, \Lambda^2\mathfrak{m})$ by

$$
\hat{\varphi}_\xi = v_1 e_{56} - v_2 e_{34} + v_3 e_{12}.
$$

Using the description of the Kähler form via $\hat{\omega}$ and the fact that $v_1 + v_2 + v_3 = 0$, it is easy to check that $\hat{\varphi}_\xi$ is in fact $\Lambda_0^{1,1}$-valued.

The functions $v_i \in C^\infty(G)$ are H-invariant since

$$v_i(x) = \langle \xi^*(x), h_i \rangle = \langle \mathrm{Ad}(x^{-1})\xi, h_i \rangle = \langle \xi, \mathrm{Ad}(x)h_i \rangle$$

and $\mathrm{ad}(h_j)h_i = [h_j, h_i] = 0$, hence $dv_i(h_j) = 0$. Using the commutator relations of $\mathfrak{u}(3)$, one can check that the 2-forms $e_{12}, e_{34}, e_{56} \in \Lambda^2\mathfrak{m}$ are H-invariant as well. This implies that the function $\hat{\varphi}_\xi$ is H-equivariant. In total, $\hat{\varphi}_\xi \in C^\infty(G, \Lambda_{0,\mathbb{R}}^{1,1}\mathfrak{m})^H$ and thus $\hat{\varphi}_\xi$ projects to a primitive $(1,1)$-form φ_ξ on M. The coclosedness of φ_ξ has also been checked in [MS10, §6].

It is worth noting that in the language of harmonic analysis on homogeneous spaces, φ_ξ is associated to the element

$$\xi \otimes F \in \mathfrak{su}(3)^{\mathbb{C}} \otimes \mathrm{Hom}_{T^2}(\mathfrak{su}(3)^{\mathbb{C}}, \Lambda_0^{1,1}\mathfrak{m}),$$

where the Fourier coefficient F is given by

$$F(X) = \langle X, h_1 \rangle e_{56} - \langle X, h_2 \rangle e_{34} + \langle X, h_3 \rangle e_{12}.$$

It is therefore no surprise that $\Delta^{\mathrm{h}}\varphi_\xi = 12\varphi_\xi$, since 12 is the eigenvalue of the Casimir operator on $V_{(1,1)} = \mathfrak{su}(3)^{\mathbb{C}}$ (see Table 5.3). The fact that each tensor φ_ξ thus obtained is coclosed amounts to $\delta\big|_{(1,1)} F = 0$, where $\delta\big|_{(1,1)}$ also denotes the prototypical differential operator

$$\delta\big|_{(1,1)} : \mathrm{Hom}_{T^2}(\mathfrak{su}(3), \Lambda_{0,\mathbb{R}}^{1,1}\mathfrak{m}) \longrightarrow \mathrm{Hom}_{T^2}(\mathfrak{su}(3), \mathfrak{m})$$

associated to the invariant differential operator $\delta : \Omega_{0,\mathbb{R}}^{1,1} \to \Omega^1$.

The corresponding symmetric 2-tensor, which is the actual infinitesimal Einstein deformation, is now given as $h_\xi = -J \circ \varphi_\xi$. By composing $\hat{\varphi}_\xi$ with $\hat{\omega}_\xi$, we obtain

$$\hat{h}_\xi = v_3 \cdot (e_1 \otimes e_1 + e_2 \otimes e_2) + v_2 \cdot (e_3 \otimes e_3 + e_4 \otimes e_4) + v_1 \cdot (e_5 \otimes e_5 + e_6 \otimes e_6).$$

In this way, each $\xi \in \mathfrak{su}(3)$ determines a unique element $\varphi_\xi \in \Omega_{0,\mathbb{R}}^{1,1}$ and hence a unique $h_\xi \in \varepsilon(g)$.

In passing, we note that the infinitesimal Einstein deformations of $F_{1,2}$ in fact coincide with the infinitesimal deformations of the nearly Kähler structure [MS10, Cor. 5.12]. Their nonintegrability in the nearly Kähler sense was already established [Fos17]. We now turn to the question whether the integrability in the Einstein sense is also obstructed.

5.6.2 The obstruction against integrability

We first note that via the equivalence $\varepsilon(g) \cong \mathfrak{su}(3)$ constructed in §5.6.1, the integrability obstruction to second order

$$\Psi : \varepsilon(g) \times \varepsilon(g) \times \varepsilon(g) \to \mathbb{R} : \Psi(h_1, h_2, h_3) := \left(E''_g(h_1, h_2), h_3\right)_{L^2}$$

can be viewed as a G-equivariant multilinear map that is symmetric in the first two entries, i.e. $\Psi \in (\operatorname{Sym}^2 \mathfrak{su}(3)^* \otimes \mathfrak{su}(3)^*)^G$. Both of the spaces

$$(\operatorname{Sym}^3 \mathfrak{su}(3))^G \subset (\operatorname{Sym}^2 \mathfrak{su}(3) \otimes \mathfrak{su}(3))^G$$

turn out to be one-dimensional and hence equal – in particular Ψ must be totally symmetric. Hence $\left(E''_g(h,h),k\right)_{L^2}$ can be recovered from expressions of the type $\Psi(h,h,h)$ via polarization. Concretely,

$$\left(E''_g(h,h),k\right)_{L^2} = \frac{1}{3}\frac{d}{dt}\Big|_{t=0} \Psi(h+tk, h+tk, h+tk) \tag{5.10}$$

for $h, k \in \varepsilon(g)$.

The space $(\operatorname{Sym}^3 \mathfrak{su}(3))^G$ is generated by the G-invariant cubic homogeneous polynomial $\mathrm{i}\det$. We therefore know that $\Psi(h_\xi, h_\xi, h_\xi) = c \cdot \mathrm{i}\det(\xi)$ for some $c \in \mathbb{R}$. Next, we proceed to show that $c \neq 0$.

Introducing the notation $\alpha^\sigma(X_1, \ldots, X_r) := \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(r)})$ for any permutation $\sigma \in S_r$ and any tensor α of rank r, we can rewrite formula (5.2) as

$$2\Psi(h,h,h) = \int_M 2E \operatorname{tr}_g(h^3) \operatorname{vol}_g + 3\left(\nabla^2 h, h \otimes h\right)_{L^2} - 6\left(\nabla^2 h, (h \otimes h)^{(23)}\right)_{L^2}.$$

Integrating by parts and computing

$$\begin{aligned}
\nabla^*(h \otimes h) &= -\sum_i f_i \lrcorner \nabla_{f_i}(h \otimes h) = -\sum_i f_i \lrcorner (\nabla_{f_i} h \otimes h + h \otimes \nabla_{f_i} h) \\
&= \delta h \otimes h - \sum_j f_j \otimes \nabla_{h(f_j)} h = -\nabla_{h(\cdot)} h, \\
\nabla^*(h \otimes h)^{(23)} &= -\sum_i f_i \lrcorner (\nabla_{f_i} h \otimes h + h \otimes \nabla_{f_i} h)^{(23)} \\
&= (\delta h \otimes h)^{(12)} - \sum_j (f_j \otimes \nabla_{h(f_j)} h)^{(12)} = -(\nabla_{h(\cdot)} h)^{(12)}
\end{aligned}$$

with some local orthonormal frame (f_i) of TM, we obtain

$$\Psi(h,h,h) = \frac{1}{2}\int_M (2EI_0 - 3I_1 + 6I_2)\,\mathrm{vol}_g$$

for $h \in \varepsilon(g)$, where $I_0, I_1, I_2 \in C^\infty(M)$ are defined by

$$I_0 := \mathrm{tr}_g(h^3), \quad I_1 := \langle \nabla h, \nabla_{h(\cdot)} h\rangle_g, \quad I_2 := \langle \nabla h, (\nabla_{h(\cdot)} h)^{(12)}\rangle_g.$$

The functions I_0, I_1, I_2 on M give rise to H-invariant functions $\hat{I}_0, \hat{I}_1, \hat{I}_2 \in C^\infty(G)^H$, the first of which can already be easily computed:

$$\hat{I}_0 = \mathrm{tr}(\hat{h}^3) = 2v_1^3 + 2v_2^3 + 2v_3^3 = 6v_1v_2v_3,$$

using that $v_1 + v_2 + v_3 = 0$. In order to obtain the other two terms we have to compute derivatives of h. Recall that the canonical Hermitian connection ∇^{h} and the Levi-Civita connection ∇ are related by

$$\nabla_X = \nabla^{\mathrm{h}}_X + \frac{1}{2}\mathcal{A}_X, \qquad X \in TM,$$

where $\mathcal{A}_X = J \circ (\nabla_X J)$ on TM and then extended as a derivation to tensors of arbitrary rank. Identifying 2-forms with skew-symmetric endomorphisms of TM, we can also write $\mathcal{A}_X = X \lrcorner \Psi^-$, where $\Psi^- \in \Omega^3$ is the imaginary part of the complex volume form of M, which is G-invariant and at the base point given by

$$\Psi^- = e_{236} - e_{146} - e_{135} - e_{245}$$

(see also [MS10, §6]).

The canonical horizontal distribution $\mathcal{H} \subset TG$ is spanned by the left-invariant vector fields $e_1, \ldots, e_6$. For any vector $X \in TM$, let $\tilde{X} \in \mathcal{H}$ denote its horizontal lift. Since ∇^{h} is the Ambrose–Singer connection of the homogeneous space $M = G/H$, it follows from (5.5) that

$$\begin{aligned}\widehat{\nabla^{\mathrm{h}}_X h} = \tilde{X}(\hat{h}) &= \tilde{X}(v_3)\cdot(e_1 \otimes e_1 + e_2 \otimes e_2) + \tilde{X}(v_2)\cdot(e_3 \otimes e_3 + e_4 \otimes e_4)\\ &\quad + \tilde{X}(v_1)\cdot(e_5 \otimes e_5 + e_6 \otimes e_6).\end{aligned}$$

We compute

$$\begin{aligned}
e_1(\hat{h}) &= x_2 \cdot (e_3 \otimes e_3 + e_4 \otimes e_4 - e_5 \otimes e_5 - e_6 \otimes e_6),\\
e_2(\hat{h}) &= x_1 \cdot (-e_3 \otimes e_3 - e_4 \otimes e_4 + e_5 \otimes e_5 + e_6 \otimes e_6),\\
e_3(\hat{h}) &= x_4 \cdot (e_1 \otimes e_1 + e_2 \otimes e_2 - e_5 \otimes e_5 - e_6 \otimes e_6),\\
e_4(\hat{h}) &= x_3 \cdot (-e_1 \otimes e_1 - e_2 \otimes e_2 + e_5 \otimes e_5 + e_6 \otimes e_6),\\
e_5(\hat{h}) &= x_6 \cdot (e_1 \otimes e_1 + e_2 \otimes e_2 - e_3 \otimes e_3 - e_4 \otimes e_4),\\
e_6(\hat{h}) &= x_5 \cdot (-e_1 \otimes e_1 - e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4).
\end{aligned}$$

Secondly it follows from the G-invariance of $\mathcal{A}$ that[3]

$$\begin{aligned}
\widehat{\mathcal{A}_X h} = \mathcal{A}_{\hat{X}} \hat{h} = {}& v_3 \cdot (\mathcal{A}_{\hat{X}} e_1 \odot e_1 + \mathcal{A}_{\hat{X}} e_2 \odot e_2) + v_2 \cdot (\mathcal{A}_{\hat{X}} e_3 \odot e_3 + \mathcal{A}_{\hat{X}} e_4 \odot e_4)\\
&+ v_1 \cdot (\mathcal{A}_{\hat{X}} e_5 \odot e_5 + \mathcal{A}_{\hat{X}} e_6 \odot e_6).
\end{aligned}$$

Using the above expression for Ψ^- we compute

$$\begin{aligned}
\mathcal{A}_{e_1} \hat{h} &= (v_1 - v_2) \cdot (e_3 \odot e_5 + e_4 \odot e_6),\\
\mathcal{A}_{e_2} \hat{h} &= (v_1 - v_2) \cdot (e_4 \odot e_5 - e_3 \odot e_6),\\
\mathcal{A}_{e_3} \hat{h} &= (v_3 - v_1) \cdot (e_1 \odot e_5 - e_2 \odot e_6),\\
\mathcal{A}_{e_4} \hat{h} &= (v_3 - v_1) \cdot (e_1 \odot e_6 + e_2 \odot e_5),\\
\mathcal{A}_{e_5} \hat{h} &= (v_2 - v_3) \cdot (e_1 \odot e_3 + e_2 \odot e_4),\\
\mathcal{A}_{e_6} \hat{h} &= (v_2 - v_3) \cdot (e_1 \odot e_4 - e_2 \odot e_3).
\end{aligned}$$

To obtain ∇h we simply combine:

$$\hat{X} \lrcorner \widehat{\nabla h} = \widehat{\nabla_X h} = \widehat{\nabla^{\mathrm{h}}_X h} + \frac{1}{2} \widehat{\mathcal{A}_X h} = \tilde{X}(\hat{h}) + \frac{1}{2} \mathcal{A}_{\hat{X}} \hat{h}.$$

The coefficients of $\widehat{\nabla h} \in C^\infty(G, \mathfrak{m}^{\otimes 3})$ with respect to the basis (e_i) are listed in Table 5.4.

Now we can finally tackle the terms I_1 and I_2 in the integrability obstruction. Let (f_i) be a local orthonormal frame of TM. Then

$$\begin{aligned}
I_1 = \langle \nabla h, \nabla_{h(\cdot)} h \rangle_{T^*M^{\otimes 3}} &= \sum_i \langle \nabla_{f_i} h, \nabla_{h(f_i)} h \rangle_{T^*M^{\otimes 2}}\\
&= \sum_{i,j} h(f_i, f_j) \langle \nabla_{f_i} h, \nabla_{f_j} h \rangle_{T^*M^{\otimes 2}}.
\end{aligned}$$

[3] We write $\alpha \odot \beta = \alpha \otimes \beta + \beta \otimes \alpha$ for $\alpha, \beta \in \mathfrak{m}$.

i	$\widehat{\nabla h}(e_i, e_1, \cdot)$	$\widehat{\nabla h}(e_i, e_2, \cdot)$	$\widehat{\nabla h}(e_i, e_3, \cdot)$
1	0	0	$x_2 e_3 + \frac{v_1 - v_2}{2} e_5$
2	0	0	$-x_1 e_3 + \frac{v_2 - v_1}{2} e_6$
3	$x_4 e_1 + \frac{v_3 - v_1}{2} e_5$	$x_4 e_2 + \frac{v_1 - v_3}{2} e_6$	0
4	$-x_3 e_1 + \frac{v_3 - v_1}{2} e_6$	$-x_3 e_2 + \frac{v_3 - v_1}{2} e_5$	0
5	$x_6 e_1 + \frac{v_2 - v_3}{2} e_3$	$x_6 e_2 + \frac{v_2 - v_3}{2} e_4$	$-x_6 e_3 + \frac{v_2 - v_3}{2} e_1$
6	$-x_5 e_1 + \frac{v_2 - v_3}{2} e_4$	$-x_5 e_2 + \frac{v_3 - v_2}{2} e_3$	$x_5 e_3 + \frac{v_3 - v_2}{2} e_2$

i	$\widehat{\nabla h}(e_i, e_4, \cdot)$	$\widehat{\nabla h}(e_i, e_5, \cdot)$	$\widehat{\nabla h}(e_i, e_6, \cdot)$
1	$x_2 e_4 + \frac{v_1 - v_2}{2} e_6$	$-x_2 e_5 + \frac{v_1 - v_2}{2} e_3$	$-x_2 e_6 + \frac{v_1 - v_2}{2} e_4$
2	$-x_1 e_4 + \frac{v_1 - v_2}{2} e_5$	$x_1 e_5 + \frac{v_1 - v_2}{2} e_4$	$x_1 e_6 + \frac{v_2 - v_1}{2} e_3$
3	0	$-x_4 e_5 + \frac{v_3 - v_1}{2} e_1$	$-x_4 e_6 + \frac{v_1 - v_3}{2} e_2$
4	0	$x_3 e_5 + \frac{v_3 - v_1}{2} e_2$	$x_3 e_6 + \frac{v_3 - v_1}{2} e_1$
5	$-x_6 e_4 + \frac{v_2 - v_3}{2} e_2$	0	0
6	$x_5 e_4 + \frac{v_2 - v_3}{2} e_1$	0	0

Table 5.4: Coefficients of $\widehat{\nabla h}$.

By the G-invariance of the Riemannian metric on M it follows that

$$\hat{I}_1 = \sum_{i,j} \widehat{h(f_i, f_j)} \langle \widehat{\nabla_{f_i} h}, \widehat{\nabla_{f_j} h} \rangle_{\mathfrak{m}^{\otimes 2}} = \sum_{i,j} \hat{h}(\hat{f}_i, \hat{f}_j) \langle \hat{f}_i \lrcorner \widehat{\nabla h}, \hat{f}_j \lrcorner \widehat{\nabla h} \rangle_{\mathfrak{m}^{\otimes 2}}.$$

Note that $(\hat{f}_i(x))$ forms an orthonormal basis of $\mathfrak{m}$ at each point $x \in G$. Since the above expression is independent of the choice of orthonormal basis, we can substitute in the orthonormal basis (e_i) of $\mathfrak{m}$. Hence the above is equal to

$$\hat{I}_1 = \sum_{i,j} \hat{h}(e_i, e_j) \langle e_i \lrcorner \widehat{\nabla h}, e_j \lrcorner \widehat{\nabla h} \rangle_{\mathfrak{m}^{\otimes 2}}.$$

Similarly we have

$$\hat{I}_2 = \sum_{i,j} \hat{h}(e_i, e_j) \langle e_i \lrcorner (\widehat{\nabla h})^{(12)}, e_j \lrcorner \widehat{\nabla h} \rangle_{\mathfrak{m}^{\otimes 2}}.$$

Plugging in the coefficients from Table 5.4 we obtain

$$\begin{aligned} \hat{I}_1 &= -18 v_1 v_2 v_3 + 4(x_1^2 + x_2^2) v_3 + 4(x_3^2 + x_4^2) v_2 + 4(x_5^2 + x_6^2) v_1, \\ \hat{I}_2 &= 9 v_1 v_2 v_3. \end{aligned}$$

One can check that these functions are indeed H-invariant and thus project to functions I_0, I_1, I_2 on M. Recall that $\mathrm{scal}_g = 30$, whence $E = 5$. Subsuming the above results, the full integrability obstruction is given by

$$\Psi(h,h,h) = \frac{1}{2}\int_M (10I_0 - 3I_1 + 6I_2)\,\mathrm{vol} = \frac{1}{\mathrm{Vol}(K)}\int_G I\,\mathrm{vol},$$
$$I = 84v_1v_2v_3 - 6(x_1^2+x_2^2)v_3 - 6(x_3^2+x_4^2)v_2 - 6(x_5^2+x_6^2)v_1.$$

Integrating over G amounts to projecting the integrand to its G-invariant, i.e. constant, part. If we view $v_1,\dots,x_6$ as linear forms on $\mathfrak{su}(3)$, the integrand I is an H-invariant cubic homogeneous polynomial in $\mathfrak{su}(3)^*$. Recall that the inner product on $\mathfrak{su}(3)$ is induced by $-\frac{1}{12}B_{\mathfrak{su}(3)}$, and

$$v_i = \langle \xi^*, h_i\rangle, \quad x_i = \langle \xi^*, e_i\rangle.$$

We therefore have the relations $\langle x_i, x_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ as well as $\langle x_i, v_j\rangle = \langle e_i, h_j\rangle = 0$, while

$$\langle v_i, v_j\rangle = \langle \mathrm{pr}_{\mathfrak{su}(3)} h_i, \mathrm{pr}_{\mathfrak{su}(3)} h_j\rangle = \begin{cases} \frac{1}{3} & i=j, \\ -\frac{1}{6} & i\neq j. \end{cases}$$

The generator $\mathrm{i}\det$ of $(\mathrm{Sym}^3\,\mathfrak{su}(3))^G$ can be written as

$$\begin{aligned}\mathrm{i}\det =\,& 8v_1v_2v_3 + 2(x_1x_3x_5 - x_1x_4x_6 - x_2x_3x_6 - x_2x_4x_5) \\ & - 2(x_1^2+x_2^2)v_3 - 2(x_3^2+x_4^2)v_2 - 2(x_5^2+x_6^2)v_1.\end{aligned}$$

The inner product on $\mathrm{Sym}^k\,\mathfrak{su}(3)$ is induced by the inner product on $\mathfrak{su}(3)$ via

$$\langle a_1\cdots a_k, b_1\cdots b_k\rangle = \sum_{\sigma\in\mathfrak{S}_k}\prod_{i=1}^k \langle a_i, b_{\sigma(i)}\rangle \qquad \text{for } a_1,\dots,a_k,b_1,\dots,b_k\in\mathfrak{su}(3).$$

We therefore see that

$$\begin{aligned}\langle I, \mathrm{i}\det\rangle_{\mathrm{Sym}^3\,\mathfrak{su}(3)} =\,& 84\cdot 3\cdot |v_1v_2v_3|^2_{\mathrm{Sym}^3\,\mathfrak{su}(3)} \\ & + 6\cdot 2\cdot (|x_1^2v_3|^2_{\mathrm{Sym}^3\,\mathfrak{su}(3)} + \ldots + |x_6^2v_1|^2_{\mathrm{Sym}^3\,\mathfrak{su}(3)}) \\ =\,& 672\cdot\frac{1}{18} + 12\cdot 6\cdot\frac{2}{3} = \frac{256}{3} \neq 0\end{aligned}$$

and hence $\Psi(h,h,h) = c\cdot\mathrm{i}\det(h)$ for some $c\neq 0$.

Suppose now that $\det(\xi)=0$ for some nonzero $\xi\in\mathfrak{su}(3)$. By equation (5.10), $E_g''(h_\xi,h_\xi)$ is orthogonal to $\varepsilon(g)$ if and only if ξ is a critical point of det, i.e. if

$$\frac{d}{dt}\Big|_{t=0}\det(\xi+t\eta) = 0$$

for all $\eta \in \mathfrak{su}(3)$. Equivalently, the rank of the complex 3×3-matrix ξ is equal to 1. However no such element of $\mathfrak{su}(3)$ exists since nonzero skew-hermitian matrices have rank at least 2.

This concludes the proof of Theorem 5.2.2.

6 Stability of the Non–Symmetric Space $\mathrm{E}_7/\,\mathrm{PSO}(8)$

6.1 Abstract

We prove that the normal metric on the homogeneous space $\mathrm{E}_7/\,\mathrm{PSO}(8)$ is stable with respect to the Einstein-Hilbert action, thereby exhibiting the first known example of a non-symmetric metric of positive scalar curvature with this property.

6.2 Introduction

Einstein metrics are Riemannian or pseudo-Riemannian metrics whose Ricci tensor is proportional to the metric, i.e. $\mathrm{Ric}_g = Eg$ for some constant E called the Einstein constant of g. It is a well-known fact that Einstein metrics on closed manifolds are precisely the critical points of the Einstein–Hilbert functional $S(g) := \int_M \mathrm{scal}_g \mathrm{vol}_g$ restricted to the space of metrics of unit volume. Einstein metrics are always saddle points but they can be local maxima if the functional is further restricted to the set of unit volume metrics with constant scalar curvature. Tangent to this is the space of tt-tensors, i.e. trace- and divergence-free symmetric 2-tensors. The second variation of the Einstein-Hilbert functional S on tt-tensors can be expressed in terms of the Lichnerowicz Laplacian Δ_{L} on symmetric 2-tensors as

$$S''_g(h,h) = -\frac{1}{2}\Big(\Delta_{\mathrm{L}}h - 2Eh, h\Big)_{L^2}.$$

Following Koiso [Koi80] we will call an Einstein metric g *stable* if g is a local maximum of the Einstein-Hilbert functional S restricted to the space of tt-tensors. In particular this is the case if $S''_g < 0$ on tt-tensors, or equivalently if $\Delta_{\mathrm{L}} > 2E$. If g is a saddle point instead, the metric g is called *unstable*. The existence of a tt-tensor h such that $S''_g(h,h) > 0$, or equivalently, $\Delta_{\mathrm{L}}h = \mu h$ for some eigenvalue $\mu < 2E$, implies instability of the metric. These eigentensors for eigenvalues less than $2E$ are also called *destabilizing directions*. Unstable Einstein metrics are particularly interesting since they turn out to also be unstable with respect to Perelman's ν-entropy as well as dynamically unstable with respect to the Ricci flow (see [CH15; Krö15]). Finally, metrics g with $S''_g \leq 0$ on tt-tensors, or equivalently $\Delta_{\mathrm{L}} \geq 2E$, will be called *linearly stable*.

In [Koi80] Koiso studied the stability question for symmetric spaces. It turned out that most of the irreducible symmetric spaces of compact type are linearly stable and only very few are unstable (see also [SW22; Sch22b] for the proof in the cases not covered by Koiso). Further examples of stable Einstein metrics are provided by Einstein metrics of negative sectional curvature (see [Bes87, Cor. 12.73]), or by Kähler–Einstein metrics of negative scalar curvature (see [DWW07]). All known compact manifolds of vanishing Ricci curvature, in other words all manifolds admitting parallel spinors, are linearly stable (see [DWW05]). On the other side there are many examples of unstable Einstein metrics, e.g. metrics on the total space of a Riemannian submersion over an unstable base (see [Böh05; WW21]). Sometimes destabilizing directions are related to harmonic forms, as on Kähler–Einstein manifolds of positive scalar curvature with $b_2 > 0$ (see [CHI04]), nearly Kähler manifolds in dimension 6 with $b_2 > 0$ or $b_3 > 0$ (see [SWW20]), or on Einstein–Sasaki manifolds with $b_2 > 0$ (see [SWW22]). Recently, many more unstable examples on homogeneous spaces appeared in the work of J. Lauret et al. (see [Lau22],[LW22b],[LL23]). It is interesting to note that all these unstable examples have positive scalar curvature. Indeed it is rather surprising that so far, except on the symmetric spaces, no example of a stable Einstein metric of positive scalar curvature was found.

In this article we will consider the generalized Wallach space $\mathrm{E}_7/\operatorname{PSO}(8)$ and its universal cover. The standard metric on this non-symmetric homogeneous space induced by minus the Killing form is known to be Einstein of positive scalar curvature. Moreover it was shown in [LW22b] that the standard metric is G-stable in the sense that it is a local maximum of the Einstein–Hilbert functional S restricted to the space of G-invariant metrics. The main result of our article is the stability of the standard metric on $\mathrm{E}_7/\operatorname{PSO}(8)$ in the much larger class of all Riemannian metrics. This provides the first example of a stable non-symmetric Einstein metric of positive scalar curvature.

6.2.1 Theorem.
Let g be the standard Riemannian metric of positive Einstein constant $E = \frac{\mathrm{scal}_g}{105} = \frac{13}{36}$ on the connected homogeneous space $M = \mathrm{E}_7/\operatorname{PSO}(8)$ or its universal cover. Then the Lichnerowicz Laplacian Δ_{L} restricted to the space of tt-tensors satisfies

$$\Delta_{\mathrm{L}} \geq \frac{30}{13}E > 2E.$$

Equality is realized exactly on left invariant, trace free symmetric 2-tensors. In particular, the Riemannian metric g is a stable, non-symmetric Einstein metric of positive scalar curvature.

The proof of the main theorem rests on an estimate of Δ_{L} against a curvature term $q(R)$. The strategy of the article is as follows. §6.3 sets up the necessary preliminaries about the Lichnerowicz Laplacian, normal homogeneous spaces and Casimir operators and introduces along the way the auxiliary operator $\mathcal{A}^*\mathcal{A}$ that depends on the torsion of the reductive connection on a homogeneous space. In §6.4 the curvature endomorphism $q(R)$ is related to $\mathcal{A}^*\mathcal{A}$ and a formula for the latter is given in terms of Casimir operators. The Lie algebra $\mathfrak{e}_7$ as well as the homogeneous space $\mathrm{E}_7/\operatorname{PSO}(8)$ and its relevant structure are discussed in §6.5. Finally, in §6.6 we compute the eigenvalues of $\mathcal{A}^*\mathcal{A}$ and thus $q(R)$, yielding a sufficient lower bound on Δ_{L} to prove Theorem 6.2.1.

6.3 Preliminaries

6.3.1 The Lichnerowicz Laplacian

Let (M, g) be a Riemannian manifold with Levi-Civita connection denoted by ∇. The Riemannian curvature tensor and Ricci tensor are given by

$$\begin{aligned} R(X,Y)Z &:= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\ \operatorname{Ric}(X,Y) &:= \operatorname{tr}(Z \mapsto R(Z,X)Y). \end{aligned}$$

We use the term *tensor bundle* to refer to a vector bundle VM that is associated to the frame bundle of (M, g) by some representation of $\mathrm{SO}(n)$. Equivalently, a tensor bundle is a $\mathrm{SO}(TM)$-invariant subbundle of some tensor power of TM.

On any tensor bundle VM the *standard curvature endomorphism* is the symmetric endomorphism $q(R) \in \operatorname{End}(VM)$ defined by

$$q(R) := \sum_{i<j} (e_i \wedge e_j)_* R(e_i, e_j)_*,$$

where (e_i) is a local orthonormal frame of TM. The asterisk denotes the natural action of $\mathfrak{so}(T)$ on tensors, i.e. extension as a derivation. We also implicitly identify $\Lambda^2 T \cong \mathfrak{so}(T)$ via

$$X \wedge Y \longmapsto (Z \mapsto g(X,Z)Y - g(Y,Z)X).$$

On TM the endomorphism $q(R)$ coincides with the Ricci endomorphism, i.e.

$$g(q(R)X, Y) = \operatorname{Ric}(X,Y).$$

Applied to the bundle $\operatorname{Sym}^2 T^*M$ of symmetric 2-tensors, $q(R)$ can be written as

$$q(R) = -2\mathring{R} - \operatorname{Der}_{\operatorname{Ric}},$$

where $\mathring{R}$ is the so-called *curvature operator of second kind* given by

$$(\mathring{R}h)(X,Y) = \sum_i h(R(e_i,X)Y,e_i), \qquad h \in \operatorname{Sym}^2 T^*M,$$

while Der_A denotes the extension of some endomorphism $A \in \operatorname{End}(T)$ to higher-rank tensors as a derivation.

The *Lichnerowicz Laplacian* Δ_{L} is now defined on tensor fields, i.e. smooth sections of VM, by

$$\Delta_{\mathrm{L}} := \nabla^*\nabla + q(R).$$

This is a Laplace type operator with a discrete spectrum accumulating only at positive infinity. On differential forms Δ_{L} coincides with the Hodge Laplacian $\Delta = d^*d + dd^*$, thus generalizing the latter to tensors of arbitrary algebraic type. Even more generally, the Lichnerowicz Laplacian is an instance of the *standard Laplace operator* on geometric vector bundles introduced in [SW18].

Since we aim to investigate the spectrum of Δ_{L} on tt-tensors, the bundle we will primarily consider is $\operatorname{Sym}^2 T^*M$. We will denote by

$$\mathscr{S}^p(M) := \Gamma(\operatorname{Sym}^p T^*M), \quad p \in \mathbb{N},$$

the space of smooth sections of $\operatorname{Sym}^p T^*M$.

The *divergence operator* on symmetric tensors is defined as the metric contraction of the covariant derivative, i.e.

$$\delta : \ \mathscr{S}^{p+1}(M) \to \mathscr{S}^p(M) : \ h \mapsto \delta h := -\sum_i e_i \lrcorner \nabla_{e_i} h$$

for a local orthonormal frame (e_i) of TM.

As explained in the introduction, the stability of an Einstein metric g is decided by a spectral property of the Lichnerowicz operator Δ_{L} on the space $\mathscr{S}^2_{\mathrm{tt}}(M)$ of tt-tensors, i.e. on symmetric 2-tensors h with $\operatorname{tr}_g h = 0$ and $\delta h = 0$. On this space we have the lower bound

$$\Delta_{\mathrm{L}} \geq 2q(R) \tag{6.1}$$

(see [HMS16, Prop. 6.2]), which will be the main tool for our proof of the stability of the standard metric on $\mathrm{E}_7/\operatorname{PSO}(8)$. The estimate is consequence of the Weitzenböck formula

$$\Delta_{\mathrm{L}} - 2q(R) = \nabla^*\nabla - q(R) = \delta\delta^* - \delta^*\delta,$$

where the symmetrized covariant derivative (or *Killing operator*) $\delta^* : \ \mathscr{S}^2(M) \to \mathscr{S}^3(M)$ is formally adjoint to the divergence δ. Tensors in the kernel of δ^* are called *Killing* *tensors* (see [HMS16] for further details). We see that a divergence-free tensor h satisfies the equality $\Delta_{\mathrm{L}} h = 2q(R)h$ if and only if it is Killing. In many cases, e.g. for the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)_{\mathrm{irr}}$ (see [SWW22]), destabilizing directions for Einstein metrics are realized by Killing tensors.

6.3.2 Normal homogeneous spaces

Let $M = G/H$ be a homogeneous space and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Let further Q be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, and let $\mathfrak{m} := \mathfrak{h}^{\perp_Q}$ denote the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, which is canonically identified with the tangent space T_oM at the base point $o = eH$. In particular, we obtain an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We will use subscripts $X_{\mathfrak{h}}, X_{\mathfrak{m}}$ to denote the projection of $X \in \mathfrak{g}$ to the respective direct summand. The unique G-invariant Riemannian metric g which coincides with the restriction $Q\big|_{\mathfrak{m}}$ at the base point is called the *normal* metric induced by Q.

For compact and semisimple G, the Killing form $B_{\mathfrak{g}}$ is negative-definite – hence, $-B_{\mathfrak{g}}$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. The metric g on M induced by $-B_{\mathfrak{g}}$ will be called the *standard metric*. Naturally, if G is simple, every normal metric will be a scalar multiple of the standard metric.

A normal homogeneous space is in particular naturally reductive, that is, it satisfies

$$g([X,Y]_{\mathfrak{m}}, Z) + g(Y, [X,Z]_{\mathfrak{m}}) = 0 \quad \text{for all } X,Y,Z \in \mathfrak{m}.$$

In other words, the G-invariant $(2,1)$-tensor $\mathcal{A}$ defined by $\mathcal{A}_X Y := [X,Y]_{\mathfrak{m}}$ is totally skew-symmetric.

Since $\mathfrak{m} \subset \mathfrak{g}$ is $\mathrm{Ad}(H)$-invariant, the decomposition is *reductive* – therefore, it defines a G-invariant connection $\bar{\nabla}$ on M, called the *canonical, reductive* (or *Ambrose–Singer*) connection, which is induced by the left-invariant principal connection

$$\mathrm{pr}_{\mathfrak{h}} \circ \theta : \; TG \longrightarrow \mathfrak{h},$$

where $\theta : \; TG \to \mathfrak{g}$ denotes the Maurer–Cartan form and $\mathrm{pr}_{\mathfrak{h}}$ some H-equivariant projection from $\mathfrak{g}$ to $\mathfrak{h}$. It can also be viewed as the affine Ehresmann connection corresponding to the horizontal distribution $\mathcal{H} = \bigcup_{x \in G} dl_x(\mathfrak{m})$ in TG. A distinctive property of the reductive connection is that every G-invariant tensor is $\bar{\nabla}$-parallel. The G-invariant torsion and curvature tensors of $\bar{\nabla}$ are given by

$$\begin{aligned} \bar{T}(X,Y) &= -[X,Y]_{\mathfrak{m}} = -\mathcal{A}_X Y, \\ \bar{R}(X,Y)Z &= -[[X,Y]_{\mathfrak{h}}, Z] \end{aligned} \qquad \text{for } X,Y,Z \in \mathfrak{m}. \tag{6.2}$$

In particular $\bar{\nabla}$ is a metric connection with parallel and totally skew-symmetric torsion. If we extend the endomorphism $\mathcal{A}_X \in \mathfrak{so}(\mathfrak{m})$ to tensors as a derivation $(\mathcal{A}_X)_*$, it induces a $\bar{\nabla}$-parallel bundle map

$$\mathcal{A}: \ VM \to T^*M \otimes VM : \qquad v \mapsto \sum_i e_i^\flat \otimes (\mathcal{A}_{e_i})_* v$$

for any tensor bundle VM, with metric adjoint

$$\mathcal{A}^*: \ T^*M \otimes VM \to VM : \quad \alpha \otimes v \mapsto \sum_i \alpha(e_i)(\mathcal{A}_{e_i})_* v.$$

This allows us to express the relation between the reductive connection and the Levi-Civita connection of g by

$$\nabla = \bar{\nabla} + \frac{1}{2}\mathcal{A}.$$

Recall that if (M, g) is a Riemannian symmetric space, it satisfies the Cartan relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, implying $\bar{T} = 0$ and thus $\nabla = \bar{\nabla}$. In this sense the tensor $\mathcal{A}$ measures the failure of a normal homogeneous space (M, g) to be (locally) symmetric.

It is worth noting that the standard Laplacian $\bar{\nabla}^*\bar{\nabla} + q(\bar{R})$ coincides with the action of the Casimir operator (see §6.3.3) on the left-regular representation of G on sections of tensor bundles over M. This fact has been vital for Koiso's study of the stability of compact symmetric spaces [Koi80].

We further note that the composition $\mathcal{A}^*\mathcal{A}$ is a $\bar{\nabla}$-parallel self-adjoint bundle endomorphism of VM that can, by combining the above, be written as

$$\mathcal{A}^*\mathcal{A} = -\sum_i (\mathcal{A}_{e_i})_*^2.$$

This auxiliary operator will be employed in order to compute the spectrum of $q(R)$ on the symmetric 2-tensors of the normal homogeneous space $\mathrm{E}_7/\operatorname{PSO}(8)$, utilizing the formulae in §6.4.

6.3.3 Casimir operators

The leitmotif of analysis and geometry on normal homogeneous spaces is to reduce calculations as far as possible to the computation of eigenvalues of Casimir operators. Fix some invariant inner product Q on a compact Lie algebra $\mathfrak{g}$. Given a representation (V, ρ_*) of $\mathfrak{g}$, its *Casimir operator* is the endomorphism defined by

$$\mathrm{Cas}_V^{\mathfrak{g},Q} := -\sum_i \rho_*(e_i)^2 \in \mathrm{End}(V).$$

This operator is $\mathfrak{g}$-equivariant. By Schur's Lemma it hence acts as multiplication with a constant when applied to an finite-dimensional irreducible complex representation of $\mathfrak{g}$. This constant can be computed by means of Freudenthal's formula. Choose a maximal torus $\mathfrak{t} \subset \mathfrak{g}$ and let $\langle \cdot, \cdot \rangle$ be the inner product on the dual $\mathfrak{t}^*$ that is induced by $Q\big|_{\mathfrak{t}\times\mathfrak{t}}$. We label the (equivalence classes of) finite-dimensional irreducible representations V_γ of $\mathfrak{g}$ by their highest weights $\gamma \in \mathfrak{t}^*$. If the complex representation V_γ has a real structure, we will sometimes abuse notation and denote the real form by V_γ as well. The Casimir eigenvalue on V_γ is then given by

$$\mathrm{Cas}_\gamma^{\mathfrak{g},Q} := \langle \gamma, \gamma + 2\delta_{\mathfrak{g}} \rangle, \tag{6.3}$$

where $\delta_{\mathfrak{g}}$ is the half-sum of positive roots of $\mathfrak{g}$. We omit the superscript Q if the inner product is clear from context. When working on a normal homogeneous space $M = G/H$ with metric induced by Q, we will encounter Casimir operators of both Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Unless otherwise stated, the inner product on $\mathfrak{g}$ will be Q and the inner product on $\mathfrak{h}$ will be the restriction $Q\big|_{\mathfrak{h}\times\mathfrak{h}}$.

Suppose $\mathfrak{g}$ is a compact Lie algebra, i.e. $B_{\mathfrak{g}}$ is negative definite, and $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra. Fix the standard inner product $-B_{\mathfrak{g}}$ on both $\mathfrak{g}$ and $\mathfrak{h}$ and consider the adjoint representation $\mathfrak{g}$ as a representation of $\mathfrak{h}$. An easy calculation then shows that

$$\mathrm{tr}_{\mathfrak{g}} \, \mathrm{Cas}_{\mathfrak{g}}^{\mathfrak{h},-B_{\mathfrak{g}}} = \dim \mathfrak{h}. \tag{6.4}$$

In particular the Casimir operator of $\mathfrak{g}$ on its adjoint representation satisfies the normalization condition

$$\mathrm{Cas}_{\mathfrak{g}}^{\mathfrak{g},-B_{\mathfrak{g}}} = 1.$$

On a normal homogeneous space $M = G/H$ the standard curvature endomorphism $q(\bar{R})$ of the reductive connection $\bar{\nabla}$ acts as

$$q(\bar{R}) = \mathrm{Cas}_V^{\mathfrak{h}} \tag{6.5}$$

on any tensor bundle VM. In particular the Ricci endomorphism $\overline{\mathrm{Ric}}$ of the reductive connection coincides with $\mathrm{Cas}_{\mathfrak{m}}^{\mathfrak{h}}$. It is well-known that if g is the standard metric, (M, g) is Einstein if and only if $\mathrm{Cas}_{\mathfrak{m}}^{\mathfrak{h}}$ has only one eigenvalue. In this case the Einstein constant E can easily be computed by means of the relation

$$\mathrm{Cas}_{\mathfrak{m}}^{\mathfrak{h}} = 2E - \frac{1}{2}, \tag{6.6}$$

cf. [Bes87, Prop. 7.89, 7.92].

6.4 Curvature formulae

In order to compute the endomorphism $q(R)$ on a normal homogeneous space, we would like to relate it to the curvature endomorphism $q(\bar{R})$ of the reductive connection, which coincides with the Casimir operator $\mathrm{Cas}^{\mathfrak{h}}$ on the fiber. As mentioned in §6.3.2, the reductive connection is an instance of a metric connection $\bar{\nabla}$ with parallel skew torsion $\bar{T}$. Such a connection can always be recovered from its torsion by means of the formula $\bar{\nabla} = \nabla + \frac{1}{2}\bar{T}$. Moreover there is a well-known relation (cf. [CMS21])

$$(R - \bar{R})(X, Y) = \frac{1}{4}([\bar{T}_X, \bar{T}_Y] - 2\bar{T}_{\bar{T}_X Y}) \tag{6.7}$$

between its curvature tensor $\bar{R}$ and the Riemannian curvature R, where $\bar{T}_X := \bar{T}(X, \cdot)$.

Note that with $\mathcal{A}_X Y = [X, Y]_{\mathfrak{m}}$ the torsion of the reductive connection is given by $\bar{T} = -\mathcal{A}$. Despite only the case of the reductive connection being necessary for our purposes, we state the following lemma in its full generality.

6.4.1 Lemma. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and another metric connection $\bar{\nabla} = \nabla + \frac{1}{2}\bar{T}$ with parallel skew torsion. On symmetric tensors of any rank,*

$$q(R) - q(\bar{R}) = -\frac{1}{4}\sum_i (\bar{T}_{e_i})^2_*.$$

Proof. Let (e_i) be an orthonormal basis of T_xM and denote $a_{ijk} := g(\bar{T}(e_i, e_j), e_k)$. Note that a_{ijk} is antisymmetric in the indices i, j, k. It follows from the definition of the curvature endomorphism and equation (6.7) that

$$q(R) - q(\bar{R}) = \frac{1}{4}\sum_{j<k}(e_j \wedge e_k)_*([\bar{T}_{e_j}, \bar{T}_{e_k}] - 2\bar{T}_{\bar{T}_{e_j} e_k})_*.$$

Looking at the individual terms,

$$\begin{aligned}
[\bar{T}_{e_j}, \bar{T}_{e_k}] &= \sum_{\substack{i\\ l<m}} (a_{kli}a_{jim} - a_{jli}a_{kim}) e_l \wedge e_m, \\
\bar{T}_{\bar{T}_{e_j} e_k} &= \sum_{\substack{i\\ l<m}} a_{jki}a_{ilm} e_l \wedge e_m.
\end{aligned}$$

It follows that

$$\sum_i (\bar{T}_{e_i})^2_* = \sum_{\substack{i\\ j<k\\ l<m}} a_{ijk}a_{ilm}(e_j \wedge e_k)_*(e_l \wedge e_m)_* = \sum_{j<k}(e_j \wedge e_k)_*(\bar{T}_{\bar{T}_{e_j} e_k})_*.$$

Let $S(X, Y) := [\bar{T}_X, \bar{T}_Y] - \bar{T}_{\bar{T}_X Y}$. It remains to show that $q(S) = 0$ on symmetric tensors.

Indeed,

$$\begin{aligned}
q(S)e_k &= \sum_{i<j}(e_i \wedge e_j)_*([\bar{T}_{e_i}, \bar{T}_{e_j}]e_k - \bar{T}_{\bar{T}_{e_i}e_j}e_k) \\
&= \sum_{i,j} g([\bar{T}_{e_i}, \bar{T}_{e_j}]e_k - \bar{T}_{\bar{T}_{e_i}e_j}e_k, e_i)e_j \\
&= \sum_{i,j,l}(a_{ili}a_{jkl} - a_{jli}a_{ikl} - a_{ijl}a_{lki})e_j = 0
\end{aligned}$$

using the antisymmetry of a_{ijk}, so $q(S)$ vanishes on T_xM. Let now $p \in \mathbb{N}$ and denote by $\odot$ the associative symmetric product. For $X_1, \ldots, X_p \in T_xM$,

$$\begin{aligned}
q(S)(X_1 \odot \ldots \odot X_p) &= \sum_{i<j}(e_i \wedge e_j)_* S(e_i \wedge e_j)_*(X_1 \odot \ldots \odot X_p) \\
&= \sum_{\substack{i<j \\ k}} X_1 \odot \ldots \odot (e_i \wedge e_j)_* S(e_i \wedge e_j) X_k \odot \ldots \odot X_p \\
&\quad + \sum_{\substack{i<j \\ k\neq l}} X_1 \odot \ldots \odot (e_i \wedge e_j)_* X_k \odot \ldots \odot S(e_i \wedge e_j)_* X_l \odot \ldots \odot X_p.
\end{aligned}$$

Summing over i, j in the first sum reduces it to having a factor of the form $q(S)X$ in each summand, which was just shown to vanish. The second sum, on the other hand, can be grouped to contain factors of the type

$$\begin{aligned}
&\sum_{i<j}((e_i \wedge e_j)_* X \odot S(e_i, e_j)Y + S(e_i, e_j)X \odot (e_i \wedge e_j)_* Y) \\
&= \sum_{i,j}(g(e_i, X)e_j \odot S(e_i, e_j)Y + g(e_i, Y)S(e_i, e_j)X \odot e_j) \\
&= \sum_j e_j \odot (S(X, e_j)Y + S(Y, e_j)X),
\end{aligned}$$

which vanishes as well since

$$\begin{aligned}
\langle S(e_i, e_j)e_k + S(e_k, e_j)e_i, e_l\rangle = &\sum_m (a_{jkm}a_{iml} - a_{ikm}a_{jml} - a_{ijm}a_{mkl} \\
&+ a_{jim}a_{kml} - a_{kim}a_{jml} - a_{kjm}a_{mil}) = 0.
\end{aligned}$$

Combining the above, we obtain $q(S) = 0$ on $\mathrm{Sym}^p T_xM$. The same calculation works for $\mathrm{Sym}^p T_x^*M$ up to sign changes in the action of $\mathfrak{so}(T_xM)$, which however cancel out in the end. In total, this proves the assertion. □

6.4.2 Corollary. *If (M, g) is normal homogeneous with reductive connection $\bar{\nabla}$, then*

$$q(R) - q(\bar{R}) = \frac{1}{4}\mathcal{A}^*\mathcal{A}.$$

From now on we stay in the normal homogeneous setting as introduced in §6.3.2, where $\bar{\nabla}$ is the reductive connection and $\mathcal{A}_X Y = [X,Y]_{\mathfrak{m}}$. The H-equivariant endomorphism $\mathcal{A}^*\mathcal{A}$ can itself be written in terms of Casimir operators, yielding an approach to the computation of its spectrum. For $p \in \mathbb{N}$, consider the p-fold tensor power $\mathfrak{m}^{\otimes p}$ embedded into $\mathfrak{g}^{\otimes p}$, and let

$$\mathrm{pr}_{\mathfrak{m}^{\otimes p}} : \ \mathfrak{g}^{\otimes p} \longrightarrow \mathfrak{m}^{\otimes p}$$

be the orthogonal projection onto $\mathfrak{m}^{\otimes p}$ with respect to the inner product naturally induced by Q on the tensor power.

6.4.3 Lemma. *On tensors of rank p,*

$$\mathcal{A}^*\mathcal{A} = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}}\Big|_{\mathfrak{m}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} - \mathrm{Der}_{\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}}.$$

Proof. Let $X \in \mathfrak{m}$. Since $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}$, the operator $\mathrm{ad}(X) \in \mathfrak{so}(\mathfrak{g})$ can be written as a block matrix

$$\mathrm{ad}(X) = \begin{pmatrix} 0 & r'_X \\ r_X & \mathcal{A}_X \end{pmatrix}$$

with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where

$$r_X = \mathrm{ad}(X)\Big|_{\mathfrak{h}} \quad \text{and} \quad r'_X = -(r_X)^* = \mathrm{pr}_{\mathfrak{h}} \mathrm{ad}(X)\Big|_{\mathfrak{m}}.$$

Consider now the p-fold tensor power

$$\mathfrak{g}^{\otimes p} = (\mathfrak{h} \oplus \mathfrak{m})^{\otimes p} = \bigoplus_{r=0}^{p} \mathfrak{v}_r \tag{6.8}$$

where $\mathfrak{v}_r \cong \binom{p}{r} \mathfrak{h}^{\otimes p-r} \otimes \mathfrak{m}^{\otimes r}$. In particular $\mathfrak{v}_p = \mathfrak{m}^{\otimes p}$. Note that the induced endomorphism $\mathrm{ad}(X)_* \in \mathfrak{so}(\mathfrak{g}^{\otimes p})$ is a derivation, changing only one factor in the tensor product at once. Hence

$$\mathrm{ad}(X)_* : \ \mathfrak{v}_r \to \mathfrak{v}_{r-1} \oplus \mathfrak{v}_r \oplus \mathfrak{v}_{r+1}$$

(we set $\mathfrak{v}_{-1} = \mathfrak{v}_{p+1} = 0$). In other words, it takes the block form

$$\mathrm{ad}(X)_* = \begin{pmatrix} 0 & * & 0 & \dots & 0 \\ * & * & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & * & * \\ 0 & \dots & 0 & * & (\mathcal{A}_X)_* \end{pmatrix}$$

with respect to decomposition (6.8). The nonzero entries of the last row and column are given by

$$a_{p-1,p} = \mathrm{pr}_{\mathfrak{v}_{p-1}} \mathrm{ad}(X)_*\Big|_{\mathfrak{m}^{\otimes p}} = (r'_X)_*,$$
$$a_{p,p-1} = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{ad}(X)_*\Big|_{\mathfrak{v}_{p-1}} = r_X \otimes \mathrm{Id}_{\mathfrak{m}^{\otimes(p-1)}},$$
$$a_{p,p} = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{ad}(X)_*\Big|_{\mathfrak{m}^{\otimes p}} = (\mathcal{A}_X)_*.$$

Combining these, the lowest rightmost entry of $\mathrm{ad}(X)_*^2$ is

$$\mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{ad}(X)_*^2\Big|_{\mathfrak{m}^{\otimes p}} = (r_X \otimes \mathrm{Id}_{\mathfrak{m}^{\otimes(p-1)}}) \circ (r'_X)_* + (\mathcal{A}_X)_*^2.$$

For $X_1, \dots, X_p \in \mathfrak{m}^{\otimes p}$ we have

$$\begin{aligned}
&(r_X \otimes \mathrm{Id}_{\mathfrak{m}^{\otimes(p-1)}}) \circ (r'_X)_*(X_1 \otimes \ldots \otimes X_p) \\
&= (r_X \otimes \mathrm{Id}_{\mathfrak{m}^{\otimes(p-1)}})([X, X_1]_{\mathfrak{h}} \otimes X_2 \otimes \ldots \otimes X_p + \ldots + X_1 \otimes \ldots \otimes X_{p-1} \otimes [X, X_p]_{\mathfrak{h}}) \\
&= [X, [X, X_1]_{\mathfrak{h}}] \otimes X_2 \otimes \ldots \otimes X_p + \ldots + X_1 \otimes \ldots \otimes X_{p-1} \otimes [X, [X, X_p]_{\mathfrak{h}}] \\
&= \mathrm{Der}_{[X,[X,\cdot]_{\mathfrak{h}}]}(X_1 \otimes \ldots \otimes X_p).
\end{aligned}$$

Together with (6.2) and (6.5) this implies

$$\sum_i (r_{e_i} \otimes \mathrm{Id}_{\mathfrak{m}^{\otimes(p-1)}}) \circ (r'_{e_i})_* = \sum_i \mathrm{Der}_{[e_i,[e_i,\cdot]_{\mathfrak{h}}]} = -\,\mathrm{Der}_{\overline{\mathrm{Ric}}} = -\,\mathrm{Der}_{\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}},$$

where (e_i) is an orthonormal basis of $\mathfrak{m}$. Note that (e_i) extends any orthormal basis of $\mathfrak{h}$ to an orthonormal basis of $\mathfrak{g}$. Thus by definition

$$\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} = -\sum_i \mathrm{ad}(e_i)_*^2 - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{g}^{\otimes p}}.$$

By virtue of $\mathfrak{m}^{\otimes p} \subset \mathfrak{g}^{\otimes p}$ being an H-invariant subspace,

$$\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{g}^{\otimes p}}\Big|_{\mathfrak{m}^{\otimes p}} = \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}}.$$

Putting everything together, we obtain

$$\begin{aligned}
\mathcal{A}^*\mathcal{A} &= -\sum_i (\mathcal{A}_{e_i})_*^2 = -\sum_i \Big(\mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{ad}(e_i)_*^2\Big|_{\mathfrak{m}^{\otimes p}} - (r_{e_i} \otimes \mathrm{Id}_{\mathfrak{m}^{\otimes(p-1)}}) \circ (r'_{e_i})_*\Big) \\
&= \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}}\Big|_{\mathfrak{m}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} - \mathrm{Der}_{\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}}.
\end{aligned}$$

□

6.5 The normal homogeneous space $\mathrm{E}_7/\,\mathrm{PSO}(8)$

We begin with a construction of the exceptional Lie algebra $\mathfrak{e}_7$ that has the advantage of introducing the chain of subalgebras $\mathfrak{so}(8) \subset \mathfrak{su}(8) \subset \mathfrak{e}_7$ along the way, which will be important later on. If $\mathrm{Sym}^2_0\,\mathbb{R}^8$ denotes the space of trace-free symmetric 8×8-matrices over $\mathbb{R}$, then

$$\mathfrak{su}(8) \longrightarrow \mathfrak{so}(8) \oplus \mathrm{Sym}^2_0\,\mathbb{R}^8 : \; X \mapsto (\operatorname{Re} X, \operatorname{Im} X)$$

is a vector space isomorphism. According to the classification of symmetric spaces, there exists a symmetric pair $\mathfrak{su}(8) \subset \mathfrak{e}_7$ whose complex isotropy representation is equal to $\Lambda^4\mathbb{C}^8$. In other words, there exists an $\mathrm{SU}(8)$-invariant real structure on $\Lambda^4\mathbb{C}^8$, i.e. a real $\mathrm{SU}(8)$-module W such that $\Lambda^4\mathbb{C}^8 = W^{\mathbb{C}}$, and $\mathfrak{e}_7 = \mathfrak{su}(8) \oplus W$.

Upon restriction to $\mathfrak{so}(8) \subset \mathfrak{su}(8)$, the isotropy representation $W \cong \Lambda^4_+\mathbb{R}^8 \oplus \Lambda^4_-\mathbb{R}^8$ decomposes into the self-dual and anti-self-dual forms, which in turn are equivalent to the trace-free second symmetric powers $\mathrm{Sym}^2_0\,\Sigma^\pm$ of the two half-spin representations $\Sigma^\pm$ (both isomorphic, but not equivalent to the defining representation $\mathbb{R}^8$).

Summarizing this argument, we can construct the exceptional Lie algebra $\mathfrak{e}_7$ as a Lie algebra with underlying vector space

$$\mathfrak{e}_7 := \mathfrak{so}(8) \oplus \mathfrak{m} := \mathfrak{so}(8) \oplus (\mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2), \qquad \mathfrak{m}_a := \mathrm{Sym}^2_0\,\mathbb{R}^8_a, \quad a = 0, 1, 2,$$

where $\mathbb{R}^8_0, \mathbb{R}^8_1, \mathbb{R}^8_2$ denote the three inequivalent representations of $\mathfrak{so}(8)$ on $\mathbb{R}^8$. Due to triality in dimension eight, it is actually immaterial which of the three representations $\mathbb{R}^8_0$, $\mathbb{R}^8_1$ and $\mathbb{R}^8_2$ we identify with the defining representation. Indeed there exists an outer automorphism $\Theta \in \mathrm{Aut}(\mathfrak{so}(8))$ of order 3, which cyclically permutes the $\mathbb{R}^8_a$ and extends to an automorphism of $\mathfrak{e}_7$ by cyclically permuting the summands $\mathfrak{m}_a$, $a = 0, 1, 2$.

Throughout this and the next chapter we will encounter several different representations of $\mathfrak{so}(8)$, $\mathfrak{su}(8)$ and $\mathfrak{e}_7$ and decompose some of their tensor products. As in §6.3.3 we will label irreducible finite-dimensional complex representations V_γ of some Lie algebra by their highest weights γ. It is therefore appropriate to introduce a basis of fundamental weights for each of the three relevant Lie algebras. Here we follow the convention of Bourbaki [Bou81, Planches I, IV, VI], using the same sets of fundamental weights in the same order. The fundamental weights are denoted as follows:

$\omega_1, \ldots, \omega_7$	for $\mathfrak{e}_7$	(type E_7)	with adjoint representation $V_{\omega_1} = \mathfrak{e}_7$,
$\zeta_1, \ldots, \zeta_7$	for $\mathfrak{su}(8)$	(type A_7)	with standard representation $V_{\zeta_1} = \mathbb{C}^8$,
$\eta_1, \ldots, \eta_4$	for $\mathfrak{so}(8)$	(type D_4)	with standard representation $V_{\eta_1} = \mathbb{R}^8$.

Under this convention we can write the $\mathfrak{so}(8)$-modules $\mathfrak{m}_a$ as

$$\mathfrak{m}_0 = \mathrm{Sym}^2_0 \mathbb{R}^8 = V_{2\eta_1}, \quad \mathfrak{m}_1 = \mathrm{Sym}^2_0 \Sigma^+ = V_{2\eta_3}, \quad \mathfrak{m}_2 = \mathrm{Sym}^2_0 \Sigma^- = V_{2\eta_4}. \tag{6.9}$$

It will become important that precomposing a $\mathfrak{so}(8)$-representation with the triality automorphism Θ cyclically permutes the weights η_1, η_3, η_4. The tensor product decompositions in §6.6 are computed with the help of the software package LiE[LiE] which uses the same enumerative convention.

In passing we remark that similar to the construction of the exceptional Lie algebras $\mathfrak{g}_2$ and $\mathfrak{f}_4$, the real division algebra of octonions plays a crucial role in the construction of the Lie algebra $\mathfrak{e}_7$. It provides both the automorphism Θ and the remaining parts of the Lie brackets that are not covered by the action of $\mathfrak{so}(8)$.[1] For our purposes it suffices to note that the Lie bracket satisfies the commutator relations

$$[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_c \quad \text{for distinct } a, b, c = 0, 1, 2, \qquad [\mathfrak{m}_a, \mathfrak{m}_a] \subset \mathfrak{so}(8). \tag{6.10}$$

These properties of the Lie bracket, which is constituted by $\mathfrak{so}(8)$-equivariant homomorphisms $\mathfrak{m}_a \otimes \mathfrak{m}_b \to \mathfrak{e}_7$, can be deduced directly using the decompositions of $\mathfrak{m}_a \otimes \mathfrak{m}_b$ into irreducible $\mathfrak{so}(8)$-modules combined with Schur's Lemma (note that the $\mathfrak{m}_a$ are self-dual as they are modules of an orthogonal group). For example

$$\mathfrak{m}_0 \otimes \mathfrak{m}_1 \cong V_{2\eta_1+2\eta_3} \oplus V_{\eta_1+\eta_3+\eta_4} \oplus \mathfrak{m}_2$$

implies that all $\mathfrak{so}(8)$-equivariant homomorphisms $\mathfrak{m}_0 \otimes \mathfrak{m}_1 \to \mathfrak{e}_7$ must map into $\mathfrak{m}_2$, since the two other summands in the above decomposition do not occur as $\mathfrak{so}(8)$-submodules of $\mathfrak{e}_7$.

We endow $\mathfrak{e}_7$ with the standard inner product $-B_{\mathfrak{e}_7}$, where $B_{\mathfrak{e}_7}$ is the Killing form of $\mathfrak{e}_7$, and fix the inner products on $\mathfrak{so}(8), \mathfrak{su}(8) \subset \mathfrak{e}_7$ as the respective restrictions of $-B_{\mathfrak{e}_7}$. Given an irreducible representation of any of the three Lie algebras, its Casimir eigenvalue is calculated using Freudenthal's formula (6.3). The calculation may be implemented with LiE. The scale factors coming from the choice of inner product on the Lie algebra have to be treated with particular caution. However we can always normalize the result using the

[1] Choosing an isometry $\mathbb{O} \cong \mathbb{R}^8$ or, equivalently, an orthonormal basis $(e_1, \dots, e_8) \subset \mathbb{O}$ with respect to $\langle X, Y\rangle := \mathrm{Re}(\bar{X}Y)$, we may in fact define a bilinear convolution product $\bullet$ on the vector space $\mathbb{R}^{8\times 8}$ by setting

$$A \bullet B := \sum_{i,j=1}^{8} A_{ij} L_i B L_j^\top,$$

where $L_i \in \mathbb{R}^{8\times 8}$ are the matrices representing the endomorphisms $x \mapsto \overline{e_i x}$. In terms of this convolution product, the triality automorphism on $\mathfrak{so}(8)$ reads $\Theta(X) := \frac{1}{4}\,\mathrm{Id}_{8\times 8} \bullet X$, while $[X, Y] := \frac{1}{2} X \bullet Y$ defines the partial Lie bracket $\mathrm{Sym}^2_0 \mathbb{R}^8_0 \times \mathrm{Sym}^2_0 \mathbb{R}^8_1 \to \mathrm{Sym}^2_0 \mathbb{R}^8_2$.

Casimir eigenvalues of the adjoint representation, since the ratio $c^{\mathfrak{g}}(V_\gamma) := \mathrm{Cas}^{\mathfrak{g}}_{\gamma} / \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}}$ is independent of the chosen multiple of the Killing form.

The proper Casimir eigenvalues of the adjoint representations are accessible to us by means of identity (6.4). Writing the trace in terms of eigenvalues, we have

$$\dim \mathfrak{h} = \mathrm{tr}_{\mathfrak{g}} \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{g}} = \sum_i \dim \mathfrak{g}_i \cdot \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{g}_i},$$

where $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ is a decomposition into irreducible $\mathfrak{h}$-modules. Note that in the cases we are interested in, $\mathfrak{h}$ is simple, so $\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{h}}$ can be treated as a constant. This constant can now be expressed as

$$\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{h}} = \frac{\dim \mathfrak{h}}{\sum_i \dim \mathfrak{g}_i \cdot c^{\mathfrak{h}}(\mathfrak{g}_i)}.$$

The ratios $c^{\mathfrak{h}}(\mathfrak{g}_i)$ on the right hand side can now be computed with whatever inner product on $\mathfrak{h}$ is convenient. We ultimately arrive at the normalizations

$$\mathrm{Cas}^{\mathfrak{e}_7}_{\mathfrak{e}_7} = 1, \quad \mathrm{Cas}^{\mathfrak{su}(8)}_{\mathfrak{su}(8)} = \frac{4}{9}, \quad \mathrm{Cas}^{\mathfrak{so}(8)}_{\mathfrak{so}(8)} = \frac{1}{6}.$$

Furthermore we find that the modules $\mathfrak{m}_a$ have the same Casimir eigenvalue $\mathrm{Cas}^{\mathfrak{so}(8)}_{\mathfrak{m}_a} = \frac{2}{9}$, $a = 0, 1, 2$, which is expected as the Casimir operator is invariant under automorphisms of the Lie algebra.

Let $\mathrm{E}_7 := \mathrm{Aut}^0(\mathfrak{e}_7) \subset \mathrm{SO}(\mathfrak{e}_7)$ be the compact adjoint form of $\mathfrak{e}_7$. The unique simply connected compact Lie group $\widetilde{\mathrm{E}}_7$ with Lie algebra $\mathfrak{e}_7$, which is the 2-fold universal cover of E_7, can be constructed as the preimage $\widetilde{\mathrm{E}}_7 \subset \mathrm{Spin}(\mathfrak{e}_7)$ under the spin covering. Inside both E_7 and $\widetilde{\mathrm{E}}_7$ one finds the projective special orthogonal group

$$\mathrm{PSO}(8) := \mathrm{SO}(8)/_{\{\pm \mathrm{Id}\}}$$

as the unique connected subgroup with Lie algebra $\mathfrak{so}(8)$. In consequence there are actually two connected homogeneous spaces

$$M := \mathrm{E}_7/_{\mathrm{PSO}(8)} = \widetilde{\mathrm{E}}_7/_{\mathrm{PSO}(8) \times \mathbb{Z}_2}, \qquad \widetilde{M} := \widetilde{\mathrm{E}}_7/_{\mathrm{PSO}(8)}$$

representing the pair $\mathfrak{so}(8) \subset \mathfrak{e}_7$ of Lie algebras, the latter the universal cover of the former. Note that $\mathrm{SO}(8)$ is not contained in either E_7 or $\widetilde{\mathrm{E}}_7$.

Let g denote the standard metric induced by $-B_{\mathfrak{e}_7}$ on both M and $\widetilde{M}$. The Casimir operator $\mathrm{Cas}^{\mathfrak{so}(8)}_{\mathfrak{m}}$ of the isotropy representation is a multiple of the identity, namely $\frac{2}{9}$, so g is an Einstein metric with Einstein constant $E = \frac{13}{36}$ by virtue of (6.6). Since there is a decomposition of the isotropy representation $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ into three pairwise orthogonal $\mathrm{PSO}(8)$-modules satisfying the commutator relations (6.10), the normal homogeneous space (M, g) is a so-called *generalized Wallach space* (see [LNF04]).

Finally we note that the normal homogenous space $\mathrm{E}_7/\,\mathrm{PSO}(8)$ fibres over the symmetric space $\mathrm{E}_7/(\mathrm{SU}(8)/\mathbb{Z}_4)$ in a Riemannian submersion of totally geodesic fibres. Notably, the fiber $(\mathrm{SU}(8)/\,\mathrm{SO}(8))/\mathbb{Z}_2$ is itself locally symmetric. It is easy to check that the conditions of [Bes87, Thm. 9.73] for the existence of a second Einstein metric in the canonical variation of metrics are satisfied. This is again an invariant Einstein metric belonging to the 3-dimensional family of invariant metrics on $\mathrm{E}_7/\,\mathrm{PSO}(8)$. In fact there are three distinct such submersions with vertical tangent spaces $\mathfrak{m}_0$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, yielding three invariant Einstein metrics on $\mathrm{E}_7/\,\mathrm{PSO}(8)$ besides the normal one. The G-instability of those was shown in [LW22b], but also follows from results of [WW21].

6.6 The spectrum of the standard curvature endomorphism

In this section we will calculate the eigenvalues and eigenspaces of the auxiliary curvature term $\mathcal{A}^*\mathcal{A}$ and thus, via Corollary 6.4.2, the standard curvature endomorphism $q(R)$ on the fiber $\mathrm{Sym}^2\,\mathfrak{m}^*$ of the vector bundle $\mathrm{Sym}^2\,T^*M$ over the base point of the homogeneous space $M = \mathrm{E}_7/\,\mathrm{PSO}(8)$ or its universal cover. The minimal eigenvalue of $q(R)$ will then give a lower bound for the Lichnerowicz Laplacian Δ_{L}, concluding the proof of Theorem 6.2.1. All subsequent calculations use the standard Riemannian metric g with Einstein constant $E = \frac{13}{16}$ as defined in §6.5. For any other normal metric $\frac{1}{c}g$ on M, the eigenvalues have to be multiplied by $c > 0$.

In order to compute the spectrum of the $\mathrm{PSO}(8)$-equivariant endomorphism $\mathcal{A}^*\mathcal{A}$ we exploit the inclusions $\mathfrak{so}(8) \subset \mathfrak{su}(8) \subset \mathfrak{e}_7$. In fact there are several distinct intermediate subalgebras of type $\mathfrak{su}(8)$, exhibiting a certain symmetry under triality.

6.6.1 Definition. For $a = 0, 1, 2$, let $\mathfrak{su}(8)_a := \mathfrak{so}(8) \oplus \mathfrak{m}_a$. By (6.10), these are Lie subalgebras of $\mathfrak{e}_7$ which are isomorphic to one another via the triality automorphism Θ. Denote by $\mathfrak{m}_a^\perp$ the orthogonal complement of $\mathfrak{m}_a \subset \mathfrak{m}$. We define a representation of $\mathfrak{su}(8)_a$ on $\mathfrak{m} = \mathfrak{m}_a \oplus \mathfrak{m}_a^\perp$ as follows:

(i) On $\mathfrak{m}_a$ the Lie algebra $\mathfrak{su}(8)_a$ acts trivially.

(ii) On $\mathfrak{m}_a^\perp$ the Lie algebra $\mathfrak{su}(8)_a$ acts through the Lie bracket of $\mathfrak{e}_7$.[2]

(iii) Further, when $\mathfrak{so}(8) \subset \mathfrak{su}(8)_a$ acts on $\mathfrak{m}$ through restriction of the action defined above, we indicate this by the subscript $\mathfrak{so}(8)_a$.

[2]This is well-defined by (6.10) since both $\mathfrak{so}(8)$ and $\mathfrak{m}_a$ preserve $\mathfrak{m}_a^\perp = \mathfrak{m}_b \oplus \mathfrak{m}_c$ (a, b, c distinct) under the Lie bracket of $\mathfrak{e}_7$.

6.6.2 Lemma. *On any tensor bundle over* $\mathrm{E}_7/\,\mathrm{PSO}(8)$,

$$\mathcal{A}^*\mathcal{A} = (\mathcal{A}^*\mathcal{A})_0 + (\mathcal{A}^*\mathcal{A})_1 + (\mathcal{A}^*\mathcal{A})_2 \quad \textit{where} \quad (\mathcal{A}^*\mathcal{A})_a = \mathrm{Cas}^{\mathfrak{su}(8)_a} - \mathrm{Cas}^{\mathfrak{so}(8)_a}.$$

Moreover the endomorphism $(\mathcal{A}^*\mathcal{A})_0$ *determines the other parts by*

$$(\mathcal{A}^*\mathcal{A})_{a+1} = \Theta_*^{-1} \circ (\mathcal{A}^*\mathcal{A})_a \circ \Theta_*, \qquad a \in \mathbb{Z}_3.$$

Proof. Recall that $\mathcal{A}^*\mathcal{A}$ is defined as a sum over an orthonormal basis of the isotropy representation $\mathfrak{m}$, which has the invariant orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$. In turn we can write $\mathcal{A}^*\mathcal{A}$ as a sum

$$\mathcal{A}^*\mathcal{A} = (\mathcal{A}^*\mathcal{A})_0 + (\mathcal{A}^*\mathcal{A})_1 + (\mathcal{A}^*\mathcal{A})_2$$

of $\mathrm{PSO}(8)$-equivariant self-adjoint endomorphisms $(\mathcal{A}^*\mathcal{A})_a$ defined by summing over an orthonormal basis $(e_i^{(a)})$ of $\mathfrak{m}_a$, i.e.

$$(\mathcal{A}^*\mathcal{A})_a := -\sum_i \left(\mathcal{A}_{e_i^{(a)}}\right)_*^2.$$

The extended triality automorphism $\Theta \in \mathrm{Aut}(\mathfrak{e}_7)$ maps the subspace $\mathfrak{m} \subset \mathfrak{e}_7$ isometrically to itself and permutes $\mathfrak{m}_0$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$. In consequence Θ preserves $\mathcal{A}$, that is,

$$\Theta(\mathcal{A}_X Y) = \Theta([X,Y]_{\mathfrak{m}}) = [\Theta X, \Theta Y]_{\mathfrak{m}} = \mathcal{A}_{\Theta X}(\Theta Y),$$

and maps any orthonormal basis of $\mathfrak{m}_a$ to an orthonormal basis of $\mathfrak{m}_{a-1}$. It is then easy to see that

$$(\mathcal{A}^*\mathcal{A})_{a+1} = \Theta^{-1} \circ (\mathcal{A}^*\mathcal{A})_a \circ \Theta, \qquad a \in \mathbb{Z}_3,$$

holds on $\mathfrak{m}$. Provided we replace Θ with its induced action Θ_* on tensors, these relations continue to hold on tensor powers of $\mathfrak{m}$,

Let now $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_k$ for some $a, b = 0, 1, 2$. By the commutator relations (6.10),

$$\mathcal{A}_X Y = [X,Y]_{\mathfrak{m}} = \begin{cases} 0 & a = b, \\ [X,Y] & a \neq b. \end{cases}$$

This means X acts on $\mathfrak{m}$ through the $\mathfrak{su}(8)_a$-action defined in 6.6.1. Completing $(e_i^{(a)})$ to an orthonormal basis of $\mathfrak{su}(8)_a$, we immediately obtain

$$\mathrm{Cas}^{\mathfrak{su}(8)_a} = (\mathcal{A}^*\mathcal{A})_a + \mathrm{Cas}^{\mathfrak{so}(8)_a}.$$

□

6.6.3 Isotypical decomposition of $\operatorname{Sym}^2 \mathfrak{m}$**.** Since $\mathcal{A}^*\mathcal{A}$ is a symmetric PSO(8)-equivariant endomorphism of $\operatorname{Sym}^2 \mathfrak{m}^*$, all its eigenspaces are necessarily PSO(8)-invariant subspaces. Hence we will begin by decomposing the fiber $\operatorname{Sym}^2 \mathfrak{m}^*$ into isotypical subspaces. We note that $\mathfrak{m}^* \cong \mathfrak{m}$ is self-dual via the invariant inner product, thus also $\operatorname{Sym}^2 \mathfrak{m} \cong \operatorname{Sym}^2 \mathfrak{m}^*$. The second symmetric power of $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ initially decomposes as

$$\operatorname{Sym}^2 \mathfrak{m} = \operatorname{Sym}^2 \mathfrak{m}_0 \oplus \operatorname{Sym}^2 \mathfrak{m}_1 \oplus \operatorname{Sym}^2 \mathfrak{m}_2 \oplus (\mathfrak{m}_0 \otimes \mathfrak{m}_1) \oplus (\mathfrak{m}_0 \otimes \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \otimes \mathfrak{m}_2).$$

Recall the description (6.9) of the PSO(8)-modules $\mathfrak{m}_a$ in terms of highest weights. With help of LiE, the above decomposition can be refined as follows:

$$\begin{aligned}
\operatorname{Sym}^2 \mathfrak{m}_0 &= \operatorname{Sym}^2 V_{2\eta_1} = \mathbb{R} \oplus V_{4\eta_1} \oplus V_{2\eta_1} \oplus V_{2\eta_2},\\
\operatorname{Sym}^2 \mathfrak{m}_1 &= \operatorname{Sym}^2 V_{2\eta_3} = \mathbb{R} \oplus V_{4\eta_3} \oplus V_{2\eta_3} \oplus V_{2\eta_2},\\
\operatorname{Sym}^2 \mathfrak{m}_2 &= \operatorname{Sym}^2 V_{2\eta_4} = \mathbb{R} \oplus V_{4\eta_4} \oplus V_{2\eta_4} \oplus V_{2\eta_2},\\
\mathfrak{m}_0 \otimes \mathfrak{m}_1 &= V_{2\eta_1} \otimes V_{2\eta_3} = V_{2\eta_1+2\eta_3} \oplus V_{\eta_1+\eta_3+\eta_4} \oplus V_{2\eta_4},\\
\mathfrak{m}_0 \otimes \mathfrak{m}_2 &= V_{2\eta_1} \otimes V_{2\eta_4} = V_{2\eta_1+2\eta_4} \oplus V_{\eta_1+\eta_3+\eta_4} \oplus V_{2\eta_3},\\
\mathfrak{m}_1 \otimes \mathfrak{m}_2 &= V_{2\eta_3} \otimes V_{2\eta_4} = V_{2\eta_3+2\eta_4} \oplus V_{\eta_1+\eta_3+\eta_4} \oplus V_{2\eta_1}.
\end{aligned} \tag{6.11}$$

Note that the last three lines imply the relations $[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_c$ in (6.10) for the $\mathfrak{e}_7$ Lie bracket, as hinted at in §6.5. Note also the symmetry under triality, i.e. under permutation of the weights η_1, η_3 and η_4.

These highest weight modules can be further interpreted as

$$V_{4\eta_1} = \mathfrak{m}_0 \boxdot \mathfrak{m}_0 = \operatorname{Sym}^4_0 \mathbb{R}^8_0, \qquad V_{2\eta_2} = \Lambda^2\mathbb{R}^8_0 \boxdot \Lambda^2\mathbb{R}^8_0,$$
$$V_{\eta_1+\eta_3+\eta_4} = \mathbb{R}^8_0 \boxdot \Lambda^3\mathbb{R}^8_0, \qquad V_{2\eta_1+2\eta_3} = \mathfrak{m}_0 \boxdot \mathfrak{m}_1$$

and similarly for permutations of η_1, η_3, η_4 (resp. $\mathbb{R}^8_0, \mathbb{R}^8_1, \mathbb{R}^8_2$). Here, $\boxdot$ denotes the Cartan product of irreducible representations,

$$V_\gamma \boxdot V_{\gamma'} := V_{\gamma+\gamma'} \subset V_\gamma \otimes V_{\gamma'}.$$

Moreover $V_{2\eta_2}$ can be identified with the space of algebraic Weyl tensors over any of the 8-dimensional representations of $\mathfrak{so}(8)$.

6.6.4 Actions of $\mathfrak{e}_7$ and $\mathfrak{su}(8)$. In order to compute the spectrum of $\mathcal{A}^*\mathcal{A}$ on $\operatorname{Sym}^2 \mathfrak{m}$ by means of Lemmas 6.4.3 and 6.6.2, one needs to evaluate Casimir operators of $\mathfrak{e}_7$, $\mathfrak{su}(8)_a$

and $\mathfrak{so}(8)_a$. It is therefore essential to identify how these Lie algebras act on each isotypical summand of $\operatorname{Sym}^2 \mathfrak{m}$, or, to be more precise, how each summand embeds into a module of each $\mathfrak{e}_7$, $\mathfrak{su}(8)_a$ and $\mathfrak{so}(8)_a$. We thus turn to decompositions of suitable modules that are invariant under $\mathfrak{e}_7$ or $\mathfrak{su}(8)$, respectively. All subsequent decompositions and branchings to subalgebras are computed with help of LiE.

First, consider the embedding $\operatorname{Sym}^2 \mathfrak{m} \subset \operatorname{Sym}^2 \mathfrak{e}_7$. The right hand side decomposes into irreducible $\mathfrak{e}_7$-modules as

$$\operatorname{Sym}^2 \mathfrak{e}_7 = \mathbb{R} \oplus V_{\omega_6} \oplus V_{2\omega_1}.$$

Branching to $\mathfrak{so}(8)$ gives

$$\begin{aligned} V_{\omega_6} &\cong V_{2\eta_1} \oplus V_{2\eta_3} \oplus V_{2\eta_4} \oplus 3V_{\eta_1+\eta_3+\eta_4} \oplus V_{2\eta_2} \oplus 3V_{\eta_2}, \\ V_{2\omega_1} &\cong 3\mathbb{R} \oplus 3V_{2\eta_1} \oplus 3V_{2\eta_3} \oplus 3V_{2\eta_4} \oplus V_{4\eta_1} \oplus V_{4\eta_3} \oplus V_{4\eta_4} \oplus 3V_{\eta_1+\eta_3+\eta_4} \oplus 3V_{2\eta_2} \\ &\quad \oplus V_{2\eta_1+2\eta_3} \oplus V_{2\eta_1+2\eta_4} \oplus V_{2\eta_3+2\eta_4} \oplus V_{\eta_2+2\eta_1} \oplus V_{\eta_2+2\eta_3} \oplus V_{\eta_2+2\eta_4}. \end{aligned} \tag{6.12}$$

By comparison with (6.11), we find that the summands $\mathfrak{m}_a \boxdot \mathfrak{m}_b$ necessarily embed into $V_{2\omega_1}$ for $a, b = 0, 1, 2$. Moreover, by considering the tracefree part $\operatorname{Sym}^2_0 \mathfrak{e}_7 \cong V_{\omega_6} \oplus V_{2\omega_1}$, we see that the 2-dimensional trivial submodule $(\operatorname{Sym}^2_0 \mathfrak{m})^{\mathfrak{so}(8)} \cong 2\mathbb{R}$ of $\operatorname{Sym}^2_0 \mathfrak{m}$ also lies inside $V_{2\omega_1}$. Thus on these summands the $\operatorname{Cas}^{\mathfrak{e}_7}_{\mathfrak{e}_7 \otimes \mathfrak{e}_7}$-term from Lemma 6.4.3 is simply multiplication by the constant $\operatorname{Cas}^{\mathfrak{e}_7}_{2\omega_1}$.

Second, recall that as an $\mathfrak{su}(8)_a$-representation

$$\mathfrak{m} = \mathfrak{m}_a \oplus \mathfrak{m}_a^\perp, \qquad a = 0, 1, 2,$$

where $\mathfrak{m}_a$ is trivial and $\mathfrak{m}_a^\perp \cong W$ with $W^\mathbb{C} \cong \Lambda^4 \mathbb{C}^8 = V_{\zeta_4}$. Thus

$$\begin{aligned} \operatorname{Sym}^2 \mathfrak{m} &= \underbrace{\operatorname{Sym}^2 \mathfrak{m}_a}_{\text{trivial}} \oplus \underbrace{(\mathfrak{m}_a \otimes \mathfrak{m}_a^\perp)}_{\cong 35 V_{\zeta_4}} \oplus \operatorname{Sym}^2 \mathfrak{m}_a^\perp, \\ \operatorname{Sym}^2 \mathfrak{m}_a^\perp &= \mathbb{R} \oplus V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}. \end{aligned}$$

Since $\mathfrak{so}(8)_0$ is embedded into $\mathfrak{su}(8)_0$ in the standard way, the branchings of the $\mathfrak{su}(8)_0$-representations $V_{2\zeta_4}, V_{\zeta_2+\zeta_6}$ to $\mathfrak{so}(8)_0$ can easily be computed:

$$\begin{aligned} V_{2\zeta_4} &\cong V_{4\eta_3} \oplus V_{4\eta_4} \oplus V_{2\eta_3+2\eta_4} \oplus V_{2\eta_2} \oplus V_{2\eta_1} \oplus \mathbb{R}, \\ V_{\zeta_2+\zeta_6} &\cong V_{2\eta_3} \oplus V_{2\eta_4} \oplus V_{2\eta_2} \oplus V_{\eta_1+\eta_3+\eta_4}. \end{aligned} \tag{6.13}$$

The branchings of $\mathfrak{su}(8)_{1,2}$-representations to $\mathfrak{so}(8)_{1,2}$ work similarly, but with η_1, η_3, η_4 permuted by triality. Comparing with the isotypical decomposition of $\operatorname{Sym}^2 \mathfrak{m}$, we can again identify the actions of $\mathfrak{su}(8)_a$ as well as $\mathfrak{so}(8)_a$ on some summands of (6.11). The results are collected in Table 6.1. Whenever a summand of $\operatorname{Sym}^2 \mathfrak{m}$ embeds into a unique isotypical module V_γ of $\mathfrak{e}_7$ or $\mathfrak{su}(8)_a$, the corresponding Casimir operator acts as multiplication by the constant Cas_γ. In each of those cases this constant is computed using LiE and listed in Table 6.2.

6.6.5 Eigenvalues of $\mathcal{A}^*\mathcal{A}$ on remaining components. On any isotypical summand of $\mathrm{Sym}^2\,\mathfrak{m}$ where the preceding has shown that the Casimir operators of either $\mathfrak{e}_7$ or $\mathfrak{su}(8)_a$ are multiples of the identity, we find the eigenvalue of $\mathcal{A}^*\mathcal{A}$ by one of the formulas from Lemmas 6.4.3 and 6.6.2 (see Table 6.3). This works for most summands of $\mathrm{Sym}^2\,\mathfrak{m}$, except for

(i) the three copies of the representation $V_{2\eta_2}$ of Weyl tensors on $\mathbb{R}^8$,

(ii) the trace part in $\mathrm{Sym}^2\,\mathfrak{m}$, i.e. the trivial summand spanned by $B_{\mathfrak{e}_7}\big|_{\mathfrak{m}}$.

Issue (ii) is swiftly remedied by noting that

$$\mathcal{A}_X g(Y,Z) = -g(\mathcal{A}_X Y, Z) - g(Y, \mathcal{A}_X Z) = 0$$

since the $(2,1)$-tensor $\mathcal{A}$ is totally skew-symmetric, and thus $\mathcal{A}^*\mathcal{A}g = 0$. However (i) requires a more careful analysis.

Denote by $\mathfrak{W}_a$ the copy of $V_{2\eta_2}$ occurring inside $\mathrm{Sym}^2\,\mathfrak{m}_a$, cf. (6.11), and

$$\mathfrak{W} := \mathfrak{W}_0 \oplus \mathfrak{W}_1 \oplus \mathfrak{W}_2 \subset \mathrm{Sym}^2\,\mathfrak{m}, \qquad \mathfrak{W} \cong 3V_{2\eta_2}.$$

Consider the operator $(\mathcal{A}^*\mathcal{A})_0$ on $\mathfrak{W}$. By the construction of $(\mathcal{A}^*\mathcal{A})_0$ and commutator relations (6.10), $(\mathcal{A}^*\mathcal{A})_0$ must annihilate $\mathfrak{W}_0 \subset \mathrm{Sym}^2\,\mathfrak{m}_0$ and preserve $\mathfrak{W}_1 \oplus \mathfrak{W}_2$. Combined with the symmetries under triality, it follows that $(\mathcal{A}^*\mathcal{A})_0$ takes the block form

$$(\mathcal{A}^*\mathcal{A})_0\big|_{\mathfrak{W}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & t \\ 0 & t & s \end{pmatrix}, \qquad s,t \in \mathbb{R}, \tag{6.14}$$

with respect to the above decomposition of $\mathfrak{W}$. This matrix has eigenvalues 0 and $s \pm t$. Since $(\mathcal{A}^*\mathcal{A})_0$ determines $(\mathcal{A}^*\mathcal{A})_1$ and $(\mathcal{A}^*\mathcal{A})_2$ by triality, these have a similar block form. Summing up, we find that the matrix of $\mathcal{A}^*\mathcal{A}$ reads

$$\mathcal{A}^*\mathcal{A}\big|_{\mathfrak{W}} = (\mathcal{A}^*\mathcal{A})_0\big|_{\mathfrak{W}} + (\mathcal{A}^*\mathcal{A})_1\big|_{\mathfrak{W}} + (\mathcal{A}^*\mathcal{A})_2\big|_{\mathfrak{W}} = \begin{pmatrix} 2s & t & t \\ t & 2s & t \\ t & t & 2s \end{pmatrix}.$$

We look for clues to determine s and t. First, recall that $\mathfrak{su}(8)_0$ acts on $\mathrm{Sym}^2\,\mathfrak{m}_0^\perp$ through the Lie bracket, thus $\mathfrak{W}_1 \oplus \mathfrak{W}_2 \cong 2V_{2\eta_2}$ as an $\mathfrak{so}(8)_0$-submodule of $\mathrm{Sym}^2\,\mathfrak{m}_0^\perp$. But the module $V_{2\eta_2}$ occurs with multiplicity 1 in each of $V_{2\zeta_4}, V_{\zeta_2+\zeta_6} \subset \mathrm{Sym}^2 \mathfrak{m}_0^\perp$, cf. (6.13). It follows from Lemma 6.6.2 that $(\mathcal{A}^*\mathcal{A})_0\big|_{\mathfrak{W}_1\oplus\mathfrak{W}_2}$ has eigenvalues

$$\begin{aligned}
(\mathcal{A}^*\mathcal{A})_0\Big|_{(\mathfrak{W}_1\oplus\mathfrak{W}_2)\cap V_{2\zeta_4}} &= \mathrm{Cas}^{\mathfrak{su}(8)}_{2\zeta_4} - \mathrm{Cas}^{\mathfrak{so}(8)}_{2\eta_2} = \frac{4}{9}\cdot\frac{5}{2} - \frac{1}{6}\cdot\frac{7}{3} = \frac{13}{18},\\
(\mathcal{A}^*\mathcal{A})_0\Big|_{(\mathfrak{W}_1\oplus\mathfrak{W}_2)\cap V_{\zeta_2+\zeta_6}} &= \mathrm{Cas}^{\mathfrak{su}(8)}_{\zeta_2+\zeta_6} - \mathrm{Cas}^{\mathfrak{so}(8)}_{2\eta_2} = \frac{4}{9}\cdot\frac{7}{4} - \frac{1}{6}\cdot\frac{7}{3} = \frac{7}{18}.
\end{aligned}$$

In light of (6.14) this implies that $s = \frac{10}{18}$ and $t = \pm\frac{3}{18}$. In turn $\mathcal{A}^*\mathcal{A}$ is of block form

$$\mathcal{A}^*\mathcal{A}\Big|_{\mathfrak{W}} = \frac{1}{18}\begin{pmatrix} 20 & \pm 3 & \pm 3\\ \pm 3 & 20 & \pm 3\\ \pm 3 & \pm 3 & 20\end{pmatrix},$$

which has eigenvalues $\frac{13}{9}, \frac{17}{18}, \frac{17}{18}$ or $\frac{23}{18}, \frac{23}{18}, \frac{7}{9}$.

Second, looking at the decompositions (6.12), we find that $V_{2\eta_2}$ has multiplicity 4 in $\mathrm{Sym}^2 \mathfrak{e}_7$. Denote the $V_{2\eta_2}$-isotypical component of $\mathrm{Sym}^2 \mathfrak{e}_7$, viewed as an $\mathfrak{so}(8)$-module, by $\mathfrak{W}' \cong 4V_{2\eta_2}$. Then $\mathfrak{W}' \cap V_{2\omega_1} \cong 3V_{2\eta_2}$. Since

$$\mathfrak{W} \cap V_{2\omega_1} = \mathfrak{W} \cap (\mathfrak{W}' \cap V_{2\omega_1}) \subset \mathfrak{W}'$$

and the intersection of any two 3-dimensional subspaces in $\mathbb{R}^4$ is at least 2-dimensional, it follows with Schur's Lemma that $\mathfrak{W}\cap V_{2\omega_1} \cong cV_{2\eta_2}$ with $c \geq 2$. On this subspace, $\mathrm{Cas}^{\mathfrak{e}_7}$ is just multiplication by the constant $\mathrm{Cas}^{\mathfrak{e}_7}_{2\omega_1} = \frac{19}{9}$. Thus the eigenvalue of $\mathcal{A}^*\mathcal{A}$ is readily computed as

$$\mathcal{A}^*\mathcal{A}\Big|_{\mathfrak{W}\cap V_{2\omega_1}} = \mathrm{Cas}^{\mathfrak{e}_7}_{2\omega_1} - \mathrm{Cas}^{\mathfrak{so}(8)}_{2\eta_2} - 2\,\mathrm{Cas}^{\mathfrak{so}(8)}_{\mathfrak{m}} = \frac{19}{9} - \frac{1}{6}\cdot\frac{7}{3} - 2\cdot\frac{2}{9} = \frac{23}{18}.$$

Combined with the considerations above, we conclude that $t = -\frac{3}{18}$ and the spectrum of $\mathcal{A}^*\mathcal{A}\big|_{\mathfrak{W}}$ is given by

$$\frac{7}{9} \text{ on } \mathrm{diag}(V_{2\eta_2}) \subset \mathfrak{W}, \quad \frac{23}{18} \text{ on } \mathrm{diag}(V_{2\eta_2})^\perp \cong 2V_{2\eta_2} \subset \mathfrak{W}.$$

6.6.6 Proof of Theorem 6.2.1. Now that the spectrum of $\mathcal{A}^*\mathcal{A}$ is assembled, we turn to the operator $q(R)$ on $\mathrm{Sym}^2 \mathfrak{m}$. Recall from (6.5) that $q(\bar{R}) = \mathrm{Cas}^{\mathfrak{so}(8)}_{\mathrm{Sym}^2 \mathfrak{m}}$, which is a constant on each isotypical component of $\mathrm{Sym}^2 \mathfrak{m}$. By virtue of Corollary 6.4.2, we now obtain $q(R)$ from

$$q(R) = \frac{1}{4}\mathcal{A}^*\mathcal{A} + \mathrm{Cas}^{\mathfrak{so}(8)}_{\mathrm{Sym}^2 \mathfrak{m}}.$$

The respective eigenvalues are listed in Table 6.3. Notice that $q(R) \geq \frac{5}{12}$ on $\operatorname{Sym}^2_0 \mathfrak{m}$ (excluding the trace part spanned by $B_{\mathfrak{e}_7}\big|_{\mathfrak{m}}$), and recall that $E = \frac{13}{36}$. Together with inequality (6.1) this implies that

$$\Delta_{\mathrm{L}} \geq 2q(R) \geq \frac{5}{6} = \frac{30}{13}E > 2E$$

holds true on $\mathscr{S}^2_{\mathrm{tt}}(M)$. Thus the strict stability of the standard metric on $\mathrm{E}_7/\,\mathrm{PSO}(8)$ is shown. $\square$

6.6.7 Remark. This bound on Δ_{L} is sharp and realized by E_7-invariant tensors arising from the canonical variation in the three Riemannian submersions

$$\left(\mathrm{SU}(8)/\mathrm{SO}(8)\right)/\mathbb{Z}_2 \longrightarrow \mathrm{E}_7/\mathrm{PSO}(8) \longrightarrow \mathrm{E}_7/\mathrm{SU}(8)/\mathbb{Z}_4$$

with totally geodesic fibres, or, equivalently, from scaling the standard metric on one of the summands in the decomposition $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Indeed, by results of [HMS16], these are Killing tensors and thus satisfy $\Delta_{\mathrm{L}} h = 2q(R)h$. The Lichnerowicz eigenvalue of these invariant tensors had originally been found by J. Lauret and C. Will [LW22b, Table 2], who also showed that these tensors constitute (up to tracelessness) destabilizing directions for any of the three non-normal Einstein metrics on $\mathrm{E}_7/\,\mathrm{PSO}(8)$.

	$\mathfrak{su}(8)_0$	$\mathfrak{so}(8)_0$	$\mathfrak{su}(8)_1$	$\mathfrak{so}(8)_1$	$\mathfrak{su}(8)_2$	$\mathfrak{so}(8)_2$	$\mathfrak{e}_7$
$\mathfrak{m}_0 \boxdot \mathfrak{m}_0$	trivial	trivial	$V_{2\zeta_4}$	$V_{4\eta_1}$	$V_{2\zeta_4}$	$V_{4\eta_1}$	$V_{2\omega_1}$
$\mathfrak{m}_1 \boxdot \mathfrak{m}_1$	$V_{2\zeta_4}$	$V_{4\eta_3}$	trivial	trivial	$V_{2\zeta_4}$	$V_{4\eta_3}$	$V_{2\omega_1}$
$\mathfrak{m}_2 \boxdot \mathfrak{m}_2$	$V_{2\zeta_4}$	$V_{4\eta_4}$	$V_{2\zeta_4}$	$V_{4\eta_4}$	trivial	trivial	$V_{2\omega_1}$
$V_{2\eta_2} \subset \operatorname{Sym}^2 \mathfrak{m}_0$	trivial	trivial	$V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}$	$V_{2\eta_2}$	$V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}$	$V_{2\eta_2}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$V_{2\eta_2} \subset \operatorname{Sym}^2 \mathfrak{m}_1$	$V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}$	$V_{2\eta_2}$	trivial	trivial	$V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}$	$V_{2\eta_2}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$V_{2\eta_2} \subset \operatorname{Sym}^2 \mathfrak{m}_2$	$V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}$	$V_{2\eta_2}$	$V_{2\zeta_4} \oplus V_{\zeta_2+\zeta_6}$	$V_{2\eta_2}$	trivial	trivial	$V_{2\omega_1} \oplus V_{\omega_6}$
$\mathfrak{m}_0 \boxdot \mathfrak{m}_1$	V_{ζ_4}	$V_{2\eta_3}$	V_{ζ_4}	$V_{2\eta_1}$	$V_{2\zeta_4}$	$V_{2\eta_1+2\eta_3}$	$V_{2\omega_1}$
$\mathfrak{m}_0 \boxdot \mathfrak{m}_2$	V_{ζ_4}	$V_{2\eta_4}$	$V_{2\zeta_4}$	$V_{2\eta_1+2\eta_4}$	V_{ζ_4}	$V_{2\eta_1}$	$V_{2\omega_1}$
$\mathfrak{m}_1 \boxdot \mathfrak{m}_2$	$V_{2\zeta_4}$	$V_{2\eta_3+2\eta_4}$	V_{ζ_4}	$V_{2\eta_4}$	V_{ζ_4}	$V_{2\eta_3}$	$V_{2\omega_1}$
$\mathfrak{m}_0 \subset \operatorname{Sym}^2 \mathfrak{m}_0$	trivial	trivial	$V_{\zeta_2+\zeta_6}$	$V_{2\eta_1}$	$V_{\zeta_2+\zeta_6}$	$V_{2\eta_1}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$\mathfrak{m}_1 \subset \operatorname{Sym}^2 \mathfrak{m}_1$	$V_{\zeta_2+\zeta_6}$	$V_{2\eta_3}$	trivial	trivial	$V_{\zeta_2+\zeta_6}$	$V_{2\eta_3}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$\mathfrak{m}_2 \subset \operatorname{Sym}^2 \mathfrak{m}_2$	$V_{\zeta_2+\zeta_6}$	$V_{2\eta_4}$	$V_{\zeta_2+\zeta_6}$	$V_{2\eta_4}$	trivial	trivial	$V_{2\omega_1} \oplus V_{\omega_6}$
$\mathfrak{m}_0 \subset \mathfrak{m}_1 \otimes \mathfrak{m}_2$	$V_{2\zeta_4}$	$V_{2\eta_1}$	V_{ζ_4}	$V_{2\eta_4}$	V_{ζ_4}	$V_{2\eta_3}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$\mathfrak{m}_1 \subset \mathfrak{m}_0 \otimes \mathfrak{m}_2$	V_{ζ_4}	$V_{2\eta_4}$	$V_{2\zeta_4}$	$V_{2\eta_3}$	V_{ζ_4}	$V_{2\eta_1}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$\mathfrak{m}_2 \subset \mathfrak{m}_0 \otimes \mathfrak{m}_1$	V_{ζ_4}	$V_{2\eta_3}$	V_{ζ_4}	$V_{2\eta_1}$	$V_{2\zeta_4}$	$V_{2\eta_4}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$V_{\eta_1+\eta_3+\eta_4} \subset \operatorname{Sym}^2 \mathfrak{m}_0$	$V_{\zeta_2+\zeta_6}$	$V_{\eta_1+\eta_3+\eta_4}$	V_{ζ_4}	$V_{2\eta_4}$	V_{ζ_4}	$V_{2\eta_3}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$V_{\eta_1+\eta_3+\eta_4} \subset \operatorname{Sym}^2 \mathfrak{m}_1$	V_{ζ_4}	$V_{2\eta_4}$	$V_{\zeta_2+\zeta_6}$	$V_{\eta_1+\eta_3+\eta_4}$	V_{ζ_4}	$V_{2\eta_1}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$V_{\eta_1+\eta_3+\eta_4} \subset \operatorname{Sym}^2 \mathfrak{m}_2$	V_{ζ_4}	$V_{2\eta_3}$	V_{ζ_4}	$V_{2\eta_1}$	$V_{\zeta_2+\zeta_6}$	$V_{\eta_1+\eta_3+\eta_4}$	$V_{2\omega_1} \oplus V_{\omega_6}$
$(\operatorname{Sym}^2_0 \mathfrak{m})^{\mathfrak{so}(8)}$	$\mathbb{R} \oplus V_{2\zeta_4}$	trivial	$\mathbb{R} \oplus V_{2\zeta_4}$	trivial	$\mathbb{R} \oplus V_{2\zeta_4}$	trivial	$V_{2\omega_1}$
$\mathbb{R}B_{\mathfrak{e}_7}\big\vert_{\mathfrak{m}}$	$\mathbb{R} \oplus V_{2\zeta_4}$	trivial	$\mathbb{R} \oplus V_{2\zeta_4}$	trivial	$\mathbb{R} \oplus V_{2\zeta_4}$	trivial	$\mathbb{R} \oplus V_{2\omega_1}$

Table 6.1: All $\mathfrak{so}(8)$-irreducible summands of $\operatorname{Sym}^2 \mathfrak{m}$ and the highest weight modules they embed into.

	$\mathrm{Cas}^{\mathfrak{su}(8)_a}$	$\mathrm{Cas}^{\mathfrak{so}(8)_a}$	$\mathrm{Cas}^{\mathfrak{su}(8)_b}$	$\mathrm{Cas}^{\mathfrak{so}(8)_b}$	$\mathrm{Cas}^{\mathfrak{e}_7}$	$\mathrm{Cas}^{\mathfrak{so}(8)}$
$\mathfrak{m}_a \boxdot \mathfrak{m}_a$	0	0	$\frac{10}{9}$	$\frac{5}{9}$	$\frac{19}{9}$	$\frac{5}{9}$
$V_{2\eta_2} \subset \mathrm{Sym}^2 \mathfrak{m}_a$	0	0	–	$\frac{7}{18}$	–	$\frac{7}{18}$
$\mathfrak{m}_a \boxdot \mathfrak{m}_c$	$\frac{1}{2}$	$\frac{2}{9}$	$\frac{10}{9}$	$\frac{1}{2}$	$\frac{19}{9}$	$\frac{1}{2}$
$\mathfrak{m}_a \subset \mathrm{Sym}^2 \mathfrak{m}_a$	0	0	$\frac{7}{9}$	$\frac{2}{9}$	–	$\frac{2}{9}$
$\mathfrak{m}_a \subset \mathfrak{m}_b \otimes \mathfrak{m}_c$	$\frac{10}{9}$	$\frac{2}{9}$	$\frac{1}{2}$	$\frac{2}{9}$	–	$\frac{2}{9}$
$V_{\eta_1+\eta_3+\eta_4} \subset \mathrm{Sym}^2 \mathfrak{m}_a$	$\frac{7}{9}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{9}$	–	$\frac{1}{3}$
$(\mathrm{Sym}^2_0 \mathfrak{m})^{\mathfrak{so}(8)}$	–	0	–	0	$\frac{19}{9}$	0
$\mathbb{R}B_{\mathfrak{e}_7}\big\|_{\mathfrak{m}}$	–	0	–	0	–	0

Table 6.2: Casimir eigenvalues on the summands in Table 6.1. Here a, b, c are distinct. A dash indicates that the summand might not be contained in a single eigenspace of the Casimir operator.

	$(\mathcal{A}^*\mathcal{A})_a$	$(\mathcal{A}^*\mathcal{A})_b$	$\mathcal{A}^*\mathcal{A}$	$q(\bar{R})$	$q(R)$
$\mathfrak{m}_a \boxdot \mathfrak{m}_a$	0	$\frac{5}{9}$	$\frac{10}{9}$	$\frac{5}{9}$	$\frac{5}{6}$
$\mathrm{diag}(V_{2\eta_2}) \subset \mathfrak{W}$	–	–	$\frac{7}{9}$	$\frac{7}{18}$	$\frac{7}{12}$
$\mathrm{diag}(V_{2\eta_2})^\perp \subset \mathfrak{W}$	–	–	$\frac{23}{18}$	$\frac{7}{18}$	$\frac{17}{24}$
$\mathfrak{m}_a \boxdot \mathfrak{m}_c$	$\frac{5}{18}$	$\frac{11}{18}$	$\frac{7}{6}$	$\frac{1}{2}$	$\frac{19}{24}$
$\mathfrak{m}_a \subset \mathrm{Sym}^2 \mathfrak{m}_a$	0	$\frac{5}{9}$	$\frac{10}{9}$	$\frac{2}{9}$	$\frac{1}{2}$
$\mathfrak{m}_a \subset \mathfrak{m}_b \otimes \mathfrak{m}_c$	$\frac{8}{9}$	$\frac{5}{18}$	$\frac{13}{9}$	$\frac{2}{9}$	$\frac{7}{12}$
$V_{\eta_1+\eta_3+\eta_4} \subset \mathrm{Sym}^2 \mathfrak{m}_a$	$\frac{4}{9}$	$\frac{5}{18}$	1	$\frac{1}{3}$	$\frac{7}{12}$
$(\mathrm{Sym}^2_0 \mathfrak{m})^{\mathfrak{so}(8)}$	–	–	$\frac{5}{3}$	0	$\frac{5}{12}$
$\mathbb{R}B_{\mathfrak{e}_7}\big\|_{\mathfrak{m}}$	–	–	0	0	0

Table 6.3: The eigenvalues of $\mathcal{A}^*\mathcal{A}$, $q(\bar{R})$ and $q(R)$ on the summands of $\mathrm{Sym}^2 \mathfrak{m}$. Here a, b, c are distinct.

7 The Lichnerowicz Laplacian on normal homogeneous spaces

7.1 Abstract

We give a new formula for the Lichnerowicz Laplacian on normal homogeneous spaces in terms of Casimir operators. We derive some practical estimates and apply them to the known list of non-symmetric, compact, simply connected homogeneous spaces G/H with G simple whose standard metric is Einstein. This yields many new examples of Einstein metrics which are stable in the Einstein–Hilbert sense, which have long been lacking in the positive scalar curvature setting.

7.2 Introduction

In 1961 André Lichnerowicz introduced a second order differential operator, known today as the *Lichnerowicz Laplacian* Δ_{L}, acting on tensor fields on any Riemannian manifold (M, g) [Lic61]. It is a generalization of the Hodge–deRham Laplacian on differential forms for which there is a Weitzenböck formula

$$d^*d + dd^* = \nabla^*\nabla + q(R),$$

where $q(R)$ is a fibrewise operator depending linearly on the Riemannian curvature R. The right hand side makes sense not only for alternating tensors fields (i.e. differential forms) but for tensor fields of any type and is thus taken as a definition for Δ_{L}.

Most notably, the Lichnerowicz Laplacian occurs in the stability analysis of Einstein manifolds [Bes87]. An *Einstein metric* g on M is a Riemannian metric whose Ricci tensor satisfies $\mathrm{Ric} = Eg$ for some constant $E \in \mathbb{R}$, called the *Einstein constant* of (M, g).

Let M be a compact and oriented Riemannian manifold. The *Einstein–Hilbert functional*, defined as

$$S(g) = \int_M \mathrm{scal}_g \mathrm{vol}_g,$$

assigns to each Riemannian metric g on M its total scalar curvature. It is well-known that Einstein metrics on M can be characterized as the critical points of S restricted to the ILH manifold[1] of unit volume metrics.

These critical points turn out to always be saddle points. It gets more interesting once we restrict to the manifold $\mathfrak{S}$ of unit volume metrics with constant scalar curvature – then an Einstein metric can also be a local maximum, in which case it is called *stable*.

Fix some Einstein metric g on M and consider the linearized problem. Tangent to $\mathfrak{S}$ lies the space of *tt-tensors* (short for **t**raceless and **t**ransverse), denoted $\mathscr{S}^2_{\mathrm{tt}}(M)$. The transversality is merely a gauge condition in light of the diffeomorphism invariance of S. For $h \in \mathscr{S}^2_{\mathrm{tt}}(M)$, the second variation of S takes the form

$$S''_g(h,h) = -\frac{1}{2}\left(\Delta_{\mathrm{L}}h - 2Eh, h\right)_{L^2}.$$

This demonstrates a direct relation between the *linear* stability of g and the spectrum of Δ_{L} on $\mathscr{S}^2_{\mathrm{tt}}(M)$. It follows from the spectral properties of Δ_{L} that S''_g has finite coindex and nullity, i.e. the maximal subspace of $\mathscr{S}^2_{\mathrm{tt}}(M)$ on which $S''_g \geq 0$ is finite-dimensional. Null directions for S''_g are the *infinitesimal Einstein deformations* of g, that is, those tt-perturbations of g which preserve the Einstein condition to first order.

For the purpose of this article we drop the prefix "linearly" and call an Einstein metric *stable* if $\Delta_{\mathrm{L}} > 2E$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$, *semistable* if $\Delta_{\mathrm{L}} \geq 2E$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$, and *unstable* if Δ_{L} has an eigenvalue $\mu < 2E$ on $\mathscr{S}^2_{\mathrm{tt}}(M)$.

In 1980 Koiso published a seminal article which treats the case of Riemannian symmetric spaces [Koi80]. Irreducible symmetric spaces are isotropy-irreducible, thus they carry only one invariant Riemannian metric up to homothety which, in addition, is Einstein. If (M, g) is a locally symmetric space of noncompact type with no local two-dimensional factors, it is stable thanks to a curvature criterion [Koi80, Cor. 2.9]. The case where (M, g) is of compact type required a more extensive analysis which is facilitated by the key fact that Δ_{L} coincides with a *Casimir operator*, a representation-theoretic entity whose spectrum is straightforward to compute thanks to the theorem of Peter–Weyl, the Frobenius reciprocity theorem, and a formula of Freudenthal. This enabled Koiso to carry out the stability analysis of irreducible symmetric spaces of compact type, leaving open some gaps that were filled recently [Sch22b; SW22].

Although the symmetric case is a particularly pleasant one, utilizing a Casimir operator is already possible once we are dealing with an Ad-invariant inner product on some Lie algebra. Thus an appropriate class of spaces to extend this approach to is that of *normal homogeneous spaces*, that is, homogeneous manifolds $M = G/H$ carrying an invariant Riemannian metric which is induced by an $\mathrm{Ad}(G)$-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. All normal homogeneous Einstein manifolds with G simple are known:

[1] That is, an (infinite-dimensional) manifold modeled on an **i**nverse **l**imit of **H**ilbert spaces.

they consist of

1. irreducible symmetric spaces of compact type, classified by Cartan in 1927.

2. (non-symmetric) *strongly isotropy irreducible* spaces in the sense that the identity component of the isotropy group H acts irreducibly on the tangent space of M, classified by Wolf [Wol68] in 1968. Here G is necessarily simple. These spaces were indepently classified by Manturov [Man61a; Man61b; Man66] in 1961 and are also contained in a more extensive list of Krämer [Krä75] from 1975.

3. (non-symmetric) normal homogeneous Einstein manifolds with G simple which are *not strongly isotropy irreducible*, classified by Wang and Ziller [WZ85] in 1985.

The purpose of the present article is to find a suitable description for the Lichnerowicz Laplacian in terms of Casimir operators and initiate the stability analysis of the second and third case. We remark that if we choose G connected such that G/H is simply connected, then H is automatically also connected. We shall thus tacitly assume these properties and speak simply of *isotropy irreducible* spaces.

The third of the above classes has been investigated by E. Lauret, J. Lauret and C. Will in [Lau22; LW22b; LL23] with regard to a weaker notion of stability, the so-called *G-stability*. An invariant Einstein metric on a homogeneous space G/H is called G-stable (or G-semistable, G-unstable) if the respective spectral properties of the Lichnerowicz Laplacian hold on the subspace of G-invariant tt-tensors. In particular a G-unstable metric is also unstable in the classical sense. Restricted to the G-invariant setting, the Lichnerowicz Laplacian reduces to a term of order zero ($\frac{1}{2}\mathcal{A}^*\mathcal{A}$ in our notation) for which, in the naturally reductive case, a formula in terms of structural constants was developed [Lau22, Thm. 5.3].

For a long time there were no known non-symmetric examples of stable Einstein metrics of *positive scalar curvature* (p.s.c.). This contrasts the fact that negative sectional curvature is sufficient for stability [Bes87, Cor. 12.73], or that all Einstein metrics coming from parallel spinors (which are Ricci-flat) are semistable [DWW05]. On the other hand all known examples of unstable Einstein metrics so far have p.s.c. In [SSW22] the stability of the p.s.c. standard Einstein metric on the generalized Wallach space $\mathrm{E}_7/\mathrm{PSO}(8)$ is proved, after its G-stability was already shown in [LW22b], yielding the first known example a stable p.s.c. Einstein metric. The result follows from the discussion of the zeroth order curvature term $q(R)$ and already utilizes Casimir operators in a crucial way.

Our aim is to treat the full second order operator Δ_{L} instead. We lay out the necessary preliminaries in Sec. 7.3 and develop an exact formula for Δ_{L} in terms of Casimir operators in §7.4, from which two useful estimates follow. After a short digression on how to compute the relevant Casimir eigenvalues in §7.5, we give in §7.7 an explicit algorithm employing the new estimates in order to find lower bounds on Δ_{L} on individual Fourier modes (see §7.3.4 for a clarification of this term) and single out potential sources of instability. This algorithm is then applied, case-by-case, to the lists of Wolf and Wang–Ziller of compact, simply connected standard homogeneous Einstein manifolds, all of which have nonnegative sectional curvature.

In order to carry out the necessary calculations, computer assistance has been indispensable. We implemented our algorithm in the software system SageMath [Sage] and heavily relied on its interface to the computer algebra package LiE [LiE]. Both systems are open source. The Sage code used to implement to the algorithm in §7.7 and with which the data in §7.8 was generated is available on GitHub[2].

By the nature of our approach we were only able to reap the rewards of Alg. 7.7.1 on a finite number of spaces. It also remains unclear in many cases whether the found potentially destabilizing Fourier modes actually contain destabilizing tt-tensors. Although our results are only partial, they produce a lot of stable examples, such as

- some members of the isotropy irreducible families I, II, III, VII and IX (see Tables 7.1 and 7.5),
- some members of the isotropy reducible families XV, XVI and XVIIa (see Tables 7.2 and 7.8), the latter being the full flag manifolds $\mathrm{SO}(2n)/T^n$,
- and many of the exceptional spaces in Tables 7.3, 7.4 and 7.6.

The results are listed and discussed in detail in §7.8. Overall we are led to the conclusion that stable p.s.c. Einstein metrics are not as scarce as previously believed.

7.3 Preliminaries

7.3.1 The Lichnerowicz Laplacian

We begin with a compact, oriented Riemannian manifold (M,g). A *tensor bundle* over M is a $\mathrm{SO}(TM)$-invariant subbundle of some tensor power of TM, or more abstractly, any vector bundle VM associated to the frame bundle of (M,g) via some representation of $\mathrm{SO}(n)$. On any such bundle, the *standard curvature endomorphism* $q(R)$ of the Riemannian curvature R is defined by

$$q(R) = \sum_{i<j} (e_i \wedge e_j)_* R(e_i, e_j)_* \in \mathfrak{gl}(VM),$$

[2] `https://github.com/PSchwahn/LLBounds`

where (e_i) is a local orthonormal frame of TM and A_* denotes the natural action of some $A \in \mathfrak{gl}(T)$ on tensors as a derivation. For the sake of notational clarity we will also write Der_A instead whenever appropriate.

Note that on TM itself $q(R)$ coincides with the Ricci endomorphism, i.e.

$$g(q(R)X, Y) = \mathrm{Ric}(X, Y).$$

Let ∇ denote the Levi-Civita connection of g, as well as the connection induced on the tensor bundle VM. The *Lichnerowicz Laplacian* is the self-adjoint elliptic operator defined by

$$\Delta_{\mathrm{L}} = \nabla^*\nabla + q(R)$$

on sections of VM. It is an instance of the *standard Laplace operator* on geometric vector bundles [SW18]. As for any Laplace-type operator, Δ_{L} has discrete spectrum accumulating only at positive infinity. The Lichnerowicz Laplacian generalizes the Hodge-deRham Laplacian in the sense that $\Delta_{\mathrm{L}} = d^*d + dd^*$ on $\Omega^p(M)$.

A tensor bundle of particular importance is $\mathrm{Sym}^p T^*M$, the bundle of covariant symmetric p-tensors. Its space of smooth sections will be denoted by $\mathscr{S}^p(M)$. Let δ denote the (metric) *divergence* operator defined by

$$\delta : \ \mathscr{S}^{p+1}(M) \to \mathscr{S}^p(M) : \quad \delta h = -\sum_i e_i \lrcorner \nabla_{e_i} h.$$

Symmetric 2-tensors h that are divergence-free ($\delta h = 0$, also *transverse*) and trace-free ($\mathrm{tr}_g h = 0$) are called *tt-tensors*. As explained in the introduction, the space $\mathscr{S}^2_{\mathrm{tt}}(M)$ of tt-tensors is the central stage for the stability analysis of an Einstein metric. There is the estimate

$$\Delta_{\mathrm{L}} \geq 2q(R) \qquad \text{on } \mathscr{S}^2_{\mathrm{tt}}(M), \tag{7.1}$$

cf. [HMS16, Prop. 6.2]. A sufficient criterion for stability is thus the condition $q(R) > E$ on trace-free symmetric 2-tensors, which will serve as an important shortcut in some cases. It provides the striking advantage of only having to analyze a fibrewise term instead of a second order differential operator.

7.3.2 Normal homogeneous spaces

Let $M = G/H$ be a reductive homogeneous space and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be its reductive (i.e. $\mathrm{Ad}(H)$-invariant) decomposition, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. As usual, $\mathfrak{m}$ is identified with the tangent space of M at the base point and called the *isotropy representation* of H. There is then a one-to-one correspondence between H-invariant inner products on $\mathfrak{m}$ and G-invariant Riemannian metrics on M.

Such an invariant metric is called *normal* if it is induced by the restriction $Q\big|_{\mathfrak{m}}$, where Q is some $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. If G is compact and semisimple, there is the canonical choice $Q = -B_{\mathfrak{g}}$, where $B_{\mathfrak{g}}$ is the (negative-definite) Killing form of $\mathfrak{g}$. This particular metric is called the *standard metric* on M. If G is simple, then clearly every normal metric is homothetic to the standard metric.

Let $\mathcal{A}$ be the G-invariant $(2,1)$-tensor field on M defined by $\mathcal{A}_X Y = \mathrm{ad}_{\mathfrak{m}}(X)Y = \mathrm{pr}_{\mathfrak{m}}[X,Y]$ for $X, Y \in \mathfrak{m}$. If (M,g) is normal homogeneous, then it is also naturally reductive – equivalently, $\mathcal{A}$ is totally skew-symmetric. The tensor field $\mathcal{A}$ can be thought of as measuring the failure of (M,g) to be locally symmetric since the vanishing of $\mathcal{A}$ is equivalent to the third Cartan relation $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$.

Let further $\bar{\nabla}$ denote the *canonical reductive* (or *Ambrose–Singer*) connection on M. This G-invariant connection has the distinctive property that it leaves every G-invariant tensor field parallel. In particular $\bar{\nabla}$ is a metric connection. It is, however, not torsion-free; notably, its torsion tensor is given by $-\mathcal{A}$.

For any $X \in \mathfrak{m}$, consider the endomorphism $\mathcal{A}_X = \mathrm{ad}_{\mathfrak{m}}(X) \in \mathfrak{so}(\mathfrak{m})$ and extend it to tensors of valence p as a derivation $\mathrm{Der}_{\mathcal{A}_X} = (\mathcal{A}_X)_* = \mathrm{ad}_{\mathfrak{m}}^{\otimes p}(X)$. Given some tensor bundle VM, this defines a $\bar{\nabla}$-parallel bundle map

$$\mathcal{A}\colon\ VM \to T^*M \otimes VM: \quad v \mapsto \sum_i e^i \otimes (\mathcal{A}_{e_i})_* v$$

with metric adjoint

$$\mathcal{A}^*\colon\ T^*M \otimes VM \to VM: \quad \alpha \otimes v \mapsto -\sum_i \alpha(e_i)(\mathcal{A}_{e_i})_* v,$$

where (e_i) again denotes a local orthonormal frame of TM. The Levi-Civita connection ∇ of a normal metric g can then be expressed in terms of $\bar{\nabla}$ and $\mathcal{A}$ as

$$\nabla = \bar{\nabla} + \frac{1}{2}\mathcal{A}. \tag{7.2}$$

7.3.3 Casimir operators

Consider a real Lie algebra $\mathfrak{g}$ equipped with an invariant inner product Q. Given a representation $\rho_* : \mathfrak{g} \to \mathrm{End}\, V$, the *Casimir operator* is a $\mathfrak{g}$-equivariant endomorphism of V defined by

$$\mathrm{Cas}_V^{\mathfrak{g},Q} := -\sum_i \rho_*(e_i)^2.$$

On an irreducible module, the Casimir operator acts as multiplication with a constant as a consequence of Schur's Lemma, henceforth called the *Casimir constant*. For compact semisimple $\mathfrak{g}$, the Casimir constant of an irreducible $\mathfrak{g}$-module V with highest weight $\lambda \in$

$\mathfrak{t}^*$, where $\mathfrak{t} \subset \mathfrak{g}$ is a suitably chosen maximal abelian subalgebra, is given by Freudenthal's formula:

$$\mathrm{Cas}_\lambda^{\mathfrak{g},Q} = Q^*(\lambda, \lambda + 2\delta_{\mathfrak{g}}), \tag{7.3}$$

where $\delta_{\mathfrak{g}}$ is the half-sum of positive roots and Q^* is the inner product on $\mathfrak{t}^*$ dual to $Q\big|_{\mathfrak{t}}$.

On the other hand, if $\mathfrak{g}$ is abelian, the Casimir constant on the weight space defined by the weight $\lambda \in \mathfrak{g}^*$ is simply given by the squared length of the weight, i.e.

$$\mathrm{Cas}_\lambda^{\mathfrak{g},Q} = Q^*(\lambda, \lambda). \tag{7.4}$$

Two issues arise in practice when the Casimir constants are to be computed: first, how to find and represent the highest weights; second, how to find the appropriate inner product on the weight lattice, especially when $\mathfrak{g}$ is not simple. Section 7.5 is devoted to handling these problems.

For our purposes, it suffices to express the weights of a semisimple Lie algebra of rank r in the basis of *fundamental weights* $(\omega_i)_{i=1}^r$, such that each dominant integral weight λ can be written as

$$\lambda = \sum_{i=1}^r a_i \omega_i, \qquad a_i \in \mathbb{Z}_{\geq 0}$$

(also called *coroot style notation*). We use Bourbaki's convention for the ordering of fundamental weights of a simple Lie algebra, as do LiE and Sage.

Throughout what follows we will omit the superscript Q in $\mathrm{Cas}_V^{\mathfrak{g},Q}$ if the inner product is clear from context. If Casimir operators of both $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ are present, the implied inner product on $\mathfrak{h}$ shall be the restriction $Q\big|_{\mathfrak{h}}$ unless otherwise stated. If $\mathfrak{g}$ is compact and $Q = -B_{\mathfrak{g}}$ is the standard inner product, the Casimir operator on the adjoint representation is the identity, that is

$$\mathrm{Cas}_{\mathfrak{g}}^{\mathfrak{g},-B_{\mathfrak{g}}} = 1, \tag{7.5}$$

which may serve as a normalization condition to find the "right" inner product on the weight lattice.

We remark that the Einstein condition for a standard homogeneous space is itself encoded in a Casimir operator – namely, the standard metric on a compact homogeneous space G/H is Einstein if and only if the Casimir operator of the isotropy representation $\mathrm{Cas}_{\mathfrak{m}}^{\mathfrak{h}}$ has only one eigenvalue. If this is the case, the eigenvalue is $2E - \frac{1}{2}$ where E is the Einstein constant [Bes87, Prop. 7.89, 7.92].

7.3.4 Harmonic analysis on homogeneous spaces

Let $M = G/H$ be a homogeneous space and $\rho : H \to \operatorname{Aut} V$ a finite-dimensional representation. We denote with $VM = G \times_\rho V$ the associated vector bundle over M with fiber V. Its sections are identified with H-equivariant V-valued functions on G, i.e.

$$\Gamma(VM) \xrightarrow{\cong} C^\infty(G, V)^H : \; s \mapsto \hat{s}, \qquad \text{where} \quad s(xH) = [x, \hat{s}(x)] \in G \times_\rho V.$$

This space is an infinite-dimensional G-module via the *left-regular representation*

$$\ell : \; G \to \operatorname{Aut} C^\infty(G, V)^H : \; (\ell(x)f)(y) = f(x^{-1}y), \qquad x, y \in G.$$

Every tensor bundle on M can be understood as associated to a suitable tensor power of the isotropy represention $\mathfrak{m}$ of M, for example $\operatorname{Sym}^p T^*M \cong G \times_\rho \operatorname{Sym}^p \mathfrak{m}^*$.

The canonical reductive connection $\bar{\nabla}$, acting as covariant derivative on sections of a tensor bundle $\Gamma(VM)$, translates simply into the directional derivative on $C^\infty(G, V)^H$, i.e.

$$\widehat{\bar{\nabla}_X s} = X(\hat{s}) = -\ell_*(X)\hat{s}, \qquad X \in \mathfrak{m}.$$

Suppose G is compact and denote with $\hat{G}$ the set of equivalence classes of finite-dimensional irreducible complex G-modules. Each such module V_γ is (up to equivalence) uniquely determined by its highest weight γ. The set $\hat{G}$ is thus parametrized by the dominant integral weights of G, after the necessary choices have been made.

If V is a unitary H-module, then an irreducible decomposition of the left-regular representation on sections of VM is given by a consequence of the classical Peter–Weyl theorem and Frobenius reciprocity, also known as the Peter–Weyl theorem for homogeneous vector bundles [Wal73, Thm. 5.3.6]. It states that

$$L^2(G, V)^H \cong \overline{\bigoplus_{\gamma \in \hat{G}}} V_\gamma \otimes \operatorname{Hom}_H(V_\gamma, V). \tag{7.6}$$

For each *Fourier mode* $\gamma \in \hat{G}$ we call $\operatorname{Hom}_H(V_\gamma, V) = (V_\gamma^* \otimes V)^H$ the space of *matrix coefficients*. Given $v \in V_\gamma$ and $F \in \operatorname{Hom}_H(V_\gamma, V)$, the equivariant (smooth) function corresponding to $v \otimes F$ is given by $x \mapsto F(x^{-1}v)$.

Any G-invariant differential operator $D : \; \Gamma(VM) \to \Gamma(WM)$ between such vector bundles can be analyzed in the Fourier image where it consists of a sequence $(D\big|_\gamma)_{\gamma \in \hat{G}}$ of linear operators

$$D\big|_\gamma : \; \operatorname{Hom}_H(V_\gamma, V) \longrightarrow \operatorname{Hom}_H(V_\gamma, W).$$

One important invariant differential operator is the standard Laplacian of the connection

$\bar{\nabla}$, defined by

$$\bar{\Delta} = \bar{\nabla}^*\bar{\nabla} + q(\bar{R}).$$

A key observation is that on normal homogeneous spaces this operator coincides with the Casimir operator of the left-regular representation [MS10, Lem. 5.2], that is

$$\bar{\Delta} = \mathrm{Cas}^{\mathfrak{g}}_{\ell} \tag{7.7}$$

(so that $\bar{\Delta}\big|_{\gamma}$ is just multiplication by the constant $\mathrm{Cas}^{\mathfrak{g}}_{\gamma}$). We remark that if the underlying space is symmetric, i.e. $\mathcal{A} = 0$, then $\bar{\nabla}$ coincides with the Levi-Civita connection ∇ and thus $\bar{\Delta}$ with the Lichnerowicz Laplacian Δ_{L}, a fact that has been of vital importance for the foundational work of Koiso [Koi80] on the stability of symmetric spaces. Our aim is to give a similarly satisfying formula for Δ_{L} also in the case $\mathcal{A} \neq 0$.

7.4 Formulas and estimates for the Lichnerowicz Laplacian

Let $M = G/H$ be a homogeneous space, where G is a compact Lie group, equipped with a normal Riemannian metric g. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. We begin with a description of the Lichnerowicz Laplacian on symmetric tensor fields in terms of the reductive connection $\bar{\nabla}$ and the tensor field $\mathcal{A}$.

7.4.1 Lemma. *On $\mathscr{S}^p(M)$, $\Delta_{\mathrm{L}} = \bar{\Delta} + \mathcal{A}^*\bar{\nabla} + \frac{1}{2}\mathcal{A}^*\mathcal{A}$.*

Proof. By definition, $\Delta_{\mathrm{L}} = \nabla^*\nabla + q(R)$ and $\bar{\Delta} = \bar{\nabla}^*\bar{\nabla} + q(\bar{R})$. We first compare the two rough Laplacians. Noting that $\bar{\nabla}^*\mathcal{A} = \mathcal{A}^*\bar{\nabla}$ since $\mathcal{A}$ is $\bar{\nabla}$-parallel, it follows from (7.2) that

$$\nabla^*\nabla = \bar{\nabla}^*\bar{\nabla} + \mathcal{A}^*\bar{\nabla} + \frac{1}{4}\mathcal{A}^*\mathcal{A}.$$

Combining this with [SSW22, Cor. 3.2], which states that $q(R) = q(\bar{R}) + \frac{1}{4}\mathcal{A}^*\mathcal{A}$ on symmetric tensors of any valence, we obtain the desired relation. $\square$

We recognize the standard Laplace operator $\bar{\Delta}$ of the reductive connection, which is nothing but the Casimir operator on the left-regular representation by (7.7). Our goal is to obtain an expression of Δ_{L} purely in terms of Casimir operators so that the calculation of its spectrum reduces to a representation-theoretic problem as in the symmetric case. Fortunately, this turns out to be possible. First, we need to recall an earlier result about the zeroth order term in the formula of Lemma 7.4.1.

7.4.2 Lemma ([SSW22], Lem. 3.3)**.** *On $\mathfrak{m}^{\otimes p}$,*

$$\mathcal{A}^*\mathcal{A} = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} - \mathrm{Der}_{\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}}.$$

Recall that $\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}$ simply acts as multiplication with the constant $c = 2E - \frac{1}{2}$ if (M, g) is Einstein with Einstein constant E. Extending this as a derivation to the p-fold tensor power results in multiplication with pc. This simplifies the formula in Lemma 7.4.2.

7.4.3 Corollary. *If (M, g) is Einstein, then on $\mathfrak{m}^{\otimes p}$,*

$$\mathcal{A}^*\mathcal{A} = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} - 2pE + \frac{p}{2}.$$

We turn now to a description of the first order differential operator $\mathcal{A}^*\bar{\nabla}$. By means of the inclusion $\mathfrak{m}^{\otimes p} \subset \mathfrak{g}^{\otimes p}$ and forgetting the H-invariance we can consider $C^\infty(G, \mathfrak{m}^{\otimes p})^H$ as a subspace of the G-module $C^\infty(G, \mathfrak{g}^{\otimes p}) \cong C^\infty(G) \otimes \mathfrak{g}^{\otimes p}$. On the level of matrix coefficients this corresponds to the inclusion $(V_\gamma^* \otimes \mathfrak{m}^{\otimes p})^H \subset V_\gamma^* \otimes \mathfrak{g}^{\otimes p}$. Suggestively denoting the representation of G on $C^\infty(G, \mathfrak{g}^{\otimes p})$ by $\ell \otimes \mathrm{Ad}^{\otimes p}$, it becomes possible to write the first order term $\mathcal{A}^*\bar{\nabla}$ in terms of Casimir operators.

7.4.4 Lemma. *On $C^\infty(G, \mathfrak{m}^{\otimes p})^H$,*

$$\mathcal{A}^*\bar{\nabla} = \frac{1}{2} \mathrm{Cas}^{\mathfrak{g}}_{\ell} + \frac{1}{2} \mathrm{pr}_{\mathfrak{m}^{\otimes p}}(\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{g}}_{\ell \otimes \mathrm{Ad}^{\otimes p}}) - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}}.$$

Proof. Let $F \in C^\infty(G, \mathfrak{m}^{\otimes p})^H$. The H-invariance means precisely that $(\ell \otimes \mathrm{Ad}^{\otimes p})\big|_H$ acts trivially on F. In particular $\mathrm{Cas}^{\mathfrak{h}}_{\ell \otimes \mathrm{Ad}^{\otimes p}} F = 0$ and thus, if (e_i) denotes an orthonormal basis of $\mathfrak{m}$,

$$\begin{aligned}
\mathrm{Cas}^{\mathfrak{g}}_{\ell \otimes \mathrm{Ad}^{\otimes p}} F &= -\sum_i (\ell \otimes \mathrm{Ad}^{\otimes p})_*(e_i)^2 F \\
&= -\sum_i \Big(\ell_*(e_i)^2 F + 2\,\mathrm{ad}^{\otimes p}(e_i)\ell_*(e_i) F + \mathrm{ad}^{\otimes p}(e_i)^2 F\Big).
\end{aligned}$$

Let us analyze the occurring terms separately. First,

$$-\sum_i \ell_*(e_i)^2 F = \mathrm{Cas}^{\mathfrak{g}}_{\ell} F - \mathrm{Cas}^{\mathfrak{h}}_{\ell} F$$

and $\mathrm{Cas}^{\mathfrak{h}}_{\ell} F = \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} F$ by the H-invariance of F. Second, recall that for any $X \in \mathfrak{m}$, $\bar{\nabla}_X$ on sections of $TM^{\otimes p}$ translates into $-\ell_*(X)$ on $C^\infty(G, \mathfrak{m}^{\otimes p})^H$ and thus

$$\mathcal{A}^*\bar{\nabla} F = \sum_i \mathrm{ad}^{\otimes p}_{\mathfrak{m}}(e_i)\ell_*(e_i) F = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \sum_i \mathrm{ad}^{\otimes p}(e_i)\ell_*(e_i) F.$$

Third,

$$-\sum_i \mathrm{ad}^{\otimes p}(e_i)^2 F = \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} F - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} F.$$

After orthogonally projecting to $\mathfrak{m}^{\otimes p}$ in the fiber, we thus obtain

$$\operatorname{pr}_{\mathfrak{m}^{\otimes p}} \operatorname{Cas}^{\mathfrak{g}}_{\ell\otimes\operatorname{Ad}^{\otimes p}} F = \operatorname{Cas}^{\mathfrak{g}}_{\ell} F - 2\mathcal{A}^*\bar{\nabla}F + \operatorname{pr}_{\mathfrak{m}^{\otimes p}} \operatorname{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} - 2\operatorname{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}}$$

and the assertion follows. □

Combining (7.7), Lemma 7.4.1, Corollary 7.4.3 and Lemma 7.4.4, we obtain the following final formula.

7.4.5 Corollary. *If (M, g) is Einstein, then on $C^\infty(G, \operatorname{Sym}^p \mathfrak{m})^H$,*

$$\Delta_{\mathrm{L}} = \frac{3}{2}\operatorname{Cas}^{\mathfrak{g}}_{\ell} + \operatorname{pr}_{\operatorname{Sym}^p \mathfrak{m}}\left(\operatorname{Cas}^{\mathfrak{g}}_{\operatorname{Sym}^p \mathfrak{g}} - \frac{1}{2}\operatorname{Cas}^{\mathfrak{g}}_{\ell\otimes\operatorname{Ad}^{\otimes p}}\right) - \frac{3}{2}\operatorname{Cas}^{\mathfrak{h}}_{\operatorname{Sym}^p \mathfrak{m}} - pE + \frac{p}{4}.$$

This exact formula for Δ_{L} is quite powerful provided the necessary representation-theoretic data is available. However, given a fixed Fourier mode $\gamma \in \hat{G}$, it is in general difficult to explicitly describe the two operators $\operatorname{pr}_{\operatorname{Sym}^p \mathfrak{m}} \operatorname{Cas}^{\mathfrak{g}}_{\operatorname{Sym}^p \mathfrak{g}}$ and $\operatorname{pr}_{\operatorname{Sym}^p \mathfrak{m}} \operatorname{Cas}^{\mathfrak{g}}_{\ell\otimes\operatorname{Ad}^{\otimes p}}$ on the space $\operatorname{Hom}_H(V_\gamma, \operatorname{Sym}^p \mathfrak{m})$ of matrix coefficients. As an example, the first of the two is treated in [SSW22] on the generalized Wallach space $\mathrm{E}_7/\operatorname{PSO}(8)$, where it is possible to exploit additional symmetries. Indeed, for the general setting of normal homogeneous spaces this seems currently out of reach.

Nevertheless it is possible to obtain at least some estimates on $\Delta_{\mathrm{L}}\big|_\gamma$ in terms of the Fourier mode γ. We will do this in two ways. The first (*crude*) estimate relies only on bounds for the fibrewise term $\mathcal{A}^*\mathcal{A}$, as well as the Casimir eigenvalue $\operatorname{Cas}^{\mathfrak{g}}_\gamma$ which can be quickly computed by means of Freudenthal's formula. This has the striking advantage that the fibrewise data needs only be computed once. The second (*refined*) estimate is a direct consequence of the formula in Corollary 7.4.5. It is sharper, but the problematic terms mentioned above need to be handled separately for each Fourier mode.

For the stability analysis of a given space (M, g), both estimates work together effectively: the crude one rules out all but finitely many Fourier modes as candidates for instabilities, so that it remains to apply the refined one to each of the remaining Fourier modes. This synergy will be drawn on by the algorithm described in §7.7.

Let $\lambda_{\min}[L]$ (resp. $\lambda_{\max}[L]$) denote the minimal (resp. maximal) eigenvalue of a self-adjoint linear operator L on a finite-dimensional vector space.

7.4.6 Theorem (Crude Estimate)**.** *For any $\gamma \in \hat{G}$,*

$$\Delta_{\mathrm{L}}\Big|_\gamma \geq \operatorname{Cas}^{\mathfrak{g}}_\gamma + \frac{1}{2}\lambda_{\min}[\mathcal{A}^*\mathcal{A}] - \sqrt{\lambda_{\max}[\mathcal{A}^*\mathcal{A}] \cdot (\operatorname{Cas}^{\mathfrak{g}}_\gamma - \lambda_{\min}[\operatorname{Cas}^{\mathfrak{h}}_{\operatorname{Sym}^p \mathfrak{m}}])}$$

on symmetric p-tensors.

Proof. By (7.7) and Lemma 7.4.1 we can write

$$\Delta_{\mathrm{L}}\Big|_{\gamma} = \mathrm{Cas}^{\mathfrak{g}}_{\gamma} + \mathcal{A}^*\bar{\nabla}\Big|_{\gamma} + \frac{1}{2}\mathcal{A}^*\mathcal{A} \geq \mathrm{Cas}^{\mathfrak{g}}_{\gamma} + \lambda_{\min}[\mathcal{A}^*\bar{\nabla}\Big|_{\gamma}] + \frac{1}{2}\lambda_{\min}[\mathcal{A}^*\mathcal{A}].$$

Let now $F \in \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^p\,\mathfrak{m})$. Then

$$\begin{aligned}
\left|\Big(\mathcal{A}^*\bar{\nabla}F, F\Big)_{L^2}\right| &= \left|\Big(\bar{\nabla}F, \mathcal{A}F\Big)_{L^2}\right| \leq \|\bar{\nabla}F\|_{L^2} \cdot \|\mathcal{A}F\|_{L^2},\\
\|\bar{\nabla}F\|^2_{L^2} &= \Big(\bar{\nabla}^*\bar{\nabla}F, F\Big)_{L^2} = \Big((\mathrm{Cas}^{\mathfrak{g}}_{\gamma} - \mathrm{Cas}^{\mathfrak{h}}_{\mathrm{Sym}^p\,\mathfrak{m}})F, F\Big)_{L^2}\\
&\leq (\mathrm{Cas}^{\mathfrak{g}}_{\gamma} - \lambda_{\min}[\mathrm{Cas}^{\mathfrak{h}}_{\mathrm{Sym}^p\,\mathfrak{m}}]) \cdot \|F\|^2_{L^2},\\
\|\mathcal{A}F\|^2_{L^2} &= (\mathcal{A}^*\mathcal{A}F, F)_{L^2} \leq \lambda_{\max}[\mathcal{A}^*\mathcal{A}] \cdot \|F\|^2_{L^2}.
\end{aligned}$$

Thus, the operator norm of the self-adjoint operator $\mathcal{A}^*\bar{\nabla}\Big|_{\gamma}$ is bounded above by

$$\|\mathcal{A}^*\bar{\nabla}\Big|_{\gamma}\|^2 \leq \lambda_{\max}[\mathcal{A}^*\mathcal{A}] \cdot (\mathrm{Cas}^{\mathfrak{g}}_{\gamma} - \lambda_{\min}[\mathrm{Cas}^{\mathfrak{h}}_{\mathrm{Sym}^p\,\mathfrak{m}}])$$

and together with $\lambda_{\min}[\mathcal{A}^*\bar{\nabla}\Big|_{\gamma}] \geq -\|\mathcal{A}^*\bar{\nabla}\Big|_{\gamma}\|$ the assertion follows. □

7.4.7 Theorem (Refined Estimate)**.** *Suppose (M, g) is Einstein and $\gamma \in \hat{G}$ is fixed. Let $\mathcal{V} = \mathrm{span}\{\mathrm{im}\,F \mid F \in \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^p\,\mathfrak{m})\} \subset \mathrm{Sym}^p\,\mathfrak{m}$, let $\mathcal{W} \subset \mathrm{Sym}^p\,\mathfrak{g}$ denote the smallest G-invariant subspace containing $\mathcal{V}$, and likewise $\mathcal{U} \subset \mathrm{Hom}(V_\gamma, \mathrm{Sym}^p\,\mathfrak{g})$ the smallest G-invariant subspace containing $\mathrm{Hom}(V_\gamma, \mathrm{Sym}^p\,\mathfrak{m})^H$. Then*

$$\begin{aligned}
\Delta_{\mathrm{L}}\Big|_{\gamma} \geq \frac{3}{2}\mathrm{Cas}^{\mathfrak{g}}_{\gamma} &- \frac{1}{2}\lambda_{\max}\Big[\mathrm{Cas}^{\mathfrak{g}}_{V_\gamma\otimes\mathrm{Sym}^p\,\mathfrak{g}}\Big|_{\mathcal{U}}\Big]\\
&+ \lambda_{\min}\Big[\mathrm{Cas}^{\mathfrak{g}}_{\mathrm{Sym}^p\,\mathfrak{g}}\Big|_{\mathcal{W}}\Big] - \frac{3}{2}\lambda_{\max}\Big[\mathrm{Cas}^{\mathfrak{h}}_{\mathrm{Sym}^p\,\mathfrak{m}}\Big|_{\mathcal{V}}\Big] - pE + \frac{p}{4}.
\end{aligned}$$

on symmetric p-tensors.

Proof. This is a direct consequence of Corollary 7.4.5 if we note that $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are by construction the smallest possible subspaces on which the eigenvalues of the respective Casimir operators are of interest. Moreoever, since Δ_{L} is self-adjoint, it suffices to estimate the expression $(\Delta_{\mathrm{L}}F, F)_{L^2}$ for $F \in \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^p\,\mathfrak{m})$, whence the orthogonal projections occurring in the formula of Corollary 7.4.5 can be dropped. □

7.5 Computation of Casimir eigenvalues

In the previous section the Lichnerowicz Laplacian and related quantities were expressed solely in terms of Casimir operators. For the actual computation of their eigenvalues, a few remarks are in order.

We briefly lay out our setting of interest: let $\mathfrak{g}$ be a compact simple Lie algebra, equipped with the standard inner product $-B_{\mathfrak{g}}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be some subalgebra. In general $\mathfrak{h}$ splits as a direct sum into

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k \oplus \mathfrak{z},$$

where $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$ are simple and $\mathfrak{z} = \mathfrak{z}(\mathfrak{h})$ is the central part of $\mathfrak{h}$. We denote with (ω_i) a basis of fundamental weights of $\mathfrak{g}$.

Any irreducible (complex) $\mathfrak{h}$-module V has the form

$$V = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_k} \otimes \mathbb{C}_{\lambda_{\mathfrak{z}}},$$

where V_{λ_i} are the highest weight modules to the weights λ_i of $\mathfrak{h}_i$, and $\mathbb{C}_{\lambda_{\mathfrak{z}}}$ is the $\mathfrak{z}$-module associated to the weight $\lambda_{\mathfrak{z}} \in \mathfrak{z}^*$. We collect all those into a "highest weight" $\lambda = (\lambda_1, \ldots, \lambda_k, \lambda_{\mathfrak{z}})$. The Casimir constant on V is then simply the sum

$$\operatorname{Cas}^{\mathfrak{h}}_{\lambda} = \operatorname{Cas}^{\mathfrak{h}_1}_{\lambda_1} + \ldots + \operatorname{Cas}^{\mathfrak{h}_k}_{\lambda_k} + \operatorname{Cas}^{\mathfrak{z}}_{\lambda_{\mathfrak{z}}}. \tag{7.8}$$

The inner product on $\mathfrak{h}$ (and thus on its components) shall be the restriction of $-B_{\mathfrak{g}}$. We discuss the simple and abelian components separately.

7.5.1 The Casimir operator on simple subalgebras. Let $\lambda = \sum_i a_i \omega_i$ be a weight of a simple Lie algebra $\mathfrak{g}$ and let $\mathbf{a} = (a_1, \ldots, a_r)^\top$ be its coefficient vector. If $C_{\mathfrak{g}}$ denotes the Cartan matrix of $\mathfrak{g}$, then

$$\langle \lambda, \lambda \rangle = \mathbf{a}^\top C_{\mathfrak{g}}^{-1} \mathbf{a}$$

defines an inner product which is proportional to the one induced by $-B_{\mathfrak{g}}$. To find the proportionality constant, we utilize the normalization condition (7.5) for the adjoint representation of $\mathfrak{g}$. The standard Casimir constants can thus be computed with Freudenthal's formula (7.3) using just the inner product $\langle \cdot, \cdot \rangle$:

$$\operatorname{Cas}^{\mathfrak{g}, -B_{\mathfrak{g}}}_{\lambda} = \frac{\langle \lambda, \lambda + 2\delta_{\mathfrak{g}} \rangle}{\langle \lambda_{\mathrm{ad}}, \lambda_{\mathrm{ad}} + 2\delta_{\mathfrak{g}} \rangle} = \frac{\mathbf{a}^\top C_{\mathfrak{g}}^{-1} (\mathbf{a} + \mathbf{2})}{\mathbf{a}_{\mathrm{ad}}^\top C_{\mathfrak{g}}^{-1} (\mathbf{a}_{\mathrm{ad}} + \mathbf{2})},$$

where λ_{ad} denotes the highest root of $\mathfrak{g}$. We recall also that $\delta_{\mathfrak{g}} = \omega_1 + \ldots + \omega_r$, so its coefficient vector is $\mathbf{1} = (1, \ldots, 1)^\top$.

Let now $\mathfrak{h} \subset \mathfrak{g}$ be a simple subalgebra. The Killing forms of $\mathfrak{g}$ and $\mathfrak{h}$ (and thus the Casimir operators of $\mathfrak{h}$ defined by them) differ by an integer factor $[\mathfrak{g} : \mathfrak{h}]$, i.e.

$$B_{\mathfrak{g}} = [\mathfrak{g} : \mathfrak{h}] B_{\mathfrak{h}}, \qquad \operatorname{Cas}^{\mathfrak{h}, -B_{\mathfrak{g}}} = [\mathfrak{g} : \mathfrak{h}]^{-1} \operatorname{Cas}^{\mathfrak{h}, -B_{\mathfrak{h}}},$$

called the *index* of $\mathfrak{h}$ in $\mathfrak{g}$. In order to compute the index, consider the adjoint representation of $\mathfrak{g}$ restricted to $\mathfrak{h}$. A simple calculation then shows that

$$\operatorname{tr} \mathrm{Cas}_{\mathfrak{g}}^{\mathfrak{h},-B_{\mathfrak{g}}} = \dim \mathfrak{h}.$$

Thus if $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_l$ is the irreducible decomposition of the isotropy representation of $\mathfrak{g}/\mathfrak{h}$, we have

$$\dim \mathfrak{h} = [\mathfrak{g} : \mathfrak{h}]^{-1} \cdot \left(\dim \mathfrak{h} + \sum_{j=1}^{l} \dim \mathfrak{m}_j \cdot \mathrm{Cas}_{\mathfrak{m}_j}^{\mathfrak{h},-B_{\mathfrak{h}}} \right),$$

from which the quantity $[\mathfrak{g} : \mathfrak{h}]$ is easily computable.

7.5.2 The Casimir operator on abelian subalgebras. Let now $\mathfrak{h} \subset \mathfrak{g}$ be abelian and let $\mathfrak{t}_{\mathfrak{g}} \subset \mathfrak{g}$ be a maximal abelian subalgebra containing $\mathfrak{h}$. Customarily, the inclusion $\iota : \mathfrak{h} \hookrightarrow \mathfrak{t}_{\mathfrak{g}} \subset \mathfrak{g}$ is characterized by a *restriction matrix*; that is, a matrix R representing the adjoint map $\iota^* : \mathfrak{t}_{\mathfrak{g}}^* \to \mathfrak{h}^*$, where $\mathfrak{t}_{\mathfrak{g}}^*$ carries the basis of fundamental weights of $\mathfrak{g}$, and $\mathfrak{h}^*$ some arbitrary basis of its integral lattice.

Let us again denote with $\mathbf{a}$ the coefficient vector of a weight $\lambda \in \mathfrak{h}^*$ of $\mathfrak{h}$ with respect to the chosen basis of $\mathfrak{h}^*$. The inner product on $\mathfrak{h}^*$ defined by

$$\langle \lambda, \lambda \rangle = \mathbf{a}^\top (R C_{\mathfrak{g}} R^\top)^{-1} \mathbf{a}$$

is then again proportional to the one coming from $-B_{\mathfrak{g}}$. Repeating the trace argument above with the (complexified) isotropy representation $\mathfrak{m} = \mathbb{C}_{\lambda_1} \oplus \ldots \oplus \mathbb{C}_{\lambda_l}$ of $\mathfrak{g}/\mathfrak{h}$, the Casimir constant of λ can now by (7.4) be computed as

$$\mathrm{Cas}_{\lambda}^{\mathfrak{h},-B_{\mathfrak{g}}} = c_{\mathfrak{h}} \cdot \langle \lambda, \lambda \rangle = c_{\mathfrak{h}} \cdot \mathbf{a}^\top (R C_{\mathfrak{g}} R^\top)^{-1} \mathbf{a},$$

where the proportionality constant $c_{\mathfrak{h}}$ is obtained by

$$\dim \mathfrak{h} = c_{\mathfrak{h}} \cdot \sum_{j=1}^{l} \mathbf{a}_j^\top (R C_{\mathfrak{g}} R^\top)^{-1} \mathbf{a}_j.$$

7.5.3 Computation of the Einstein constant. Returning to the general setting of a compact simple Lie group G with a closed subgroup H such that the standard metric on the homogeneous space G/H is Einstein, the computation of the Einstein constant E (and the checking of the Einstein condition) is straightforward once the necessary data is assembled. Given an irreducible decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_l$ of the isotropy representation and a restriction matrix characterizing the embedding of H in G, one computes the Casimir constants $\mathrm{Cas}^{\mathfrak{h},-B_{\mathfrak{g}}}$ on each summand $\mathfrak{m}_j$ by means of (7.8) and the preceding two paragraphs. We recall that the standard metric on G/H is Einstein if and only if $\mathrm{Cas}^{\mathfrak{h},-B_{\mathfrak{g}}}_{\mathfrak{m}_j}$, $j=1,\dots,l$, all act by multiplication with the same constant c, in which case the Einstein constant is calculated from $c = 2E - \frac{1}{2}$.

7.6 tt-tensors and Killing vector fields

In §7.4 we obtained general estimates for the Lichnerowicz Laplacian on $\mathscr{S}^p(M)$ if (M,g) is a normal homogeneous Einstein manifold. In order to analyze the stability of (M,g) we thus specialize to $p=2$. However, only the spectrum of Δ_{L} on the subspace $\mathscr{S}^2_{\mathrm{tt}}(M)$ is of relevance for the stability discussion. Thus we ought to address the issue of distinguishing the tt-tensors among $\mathscr{S}^2(M)$.

Curiously, tt-tensors are closely related to (conformal) Killing vector fields. For a compact manifold (M^n,g), let

$$\delta^* :\ \mathscr{S}^p(M) \to \mathscr{S}^{p+1}(M) : \quad \delta^* h = \sum_i e^i \odot \nabla_{e_i} h$$

denote the formal adjoint of the divergence operator, also called the *Killing operator*. In the case $p=1$, this reduces to

$$\delta^*\alpha = L_{\alpha^\sharp} g, \qquad \alpha \in \Omega^1(M),$$

so that $\ker \delta^*\big|_{\Omega^1}$ is precisely dual to the space of Killing vector fields. Taking the trace-free part we obtain a differential operator

$$\theta :\ \Omega^1(M) \to \mathscr{S}^2_0(M) : \quad \theta\alpha = \delta^*\alpha + \frac{2}{n}\delta\alpha \cdot g$$

whose kernel is dual to the space of *conformal Killing vector fields*. The relation hinted at above is now made manifest in the short exact sequence

$$0 \longrightarrow \ker\theta \xrightarrow{\subset} \Omega^1(M) \xrightarrow{\theta} \mathscr{S}^2_0(M) \xrightarrow{P} \mathscr{S}^2_{\mathrm{tt}}(M) \longrightarrow 0 \tag{7.9}$$

(cf. [Sch22b, Lem. 4.1, Rem. 4.6]), where P shall be the L^2-orthogonal projection onto $\mathscr{S}^2_{\mathrm{tt}}(M)$. Owing to the fact that Δ_{L} commutes with every arrow in (7.9), one may obtain a similar sequence and thus a dimension formula pertaining to the eigenspaces of Δ_{L} on $\Omega^1(M)$, $\mathscr{S}^2_0(M)$ and $\mathscr{S}^2_{\mathrm{tt}}(M)$. This has indeed been utilized in the stability analysis of the irreducible symmetric spaces of compact type [Sch22b; SW22].

Returning to the compact, Riemannian homogeneous setting $M = G/H$, we observe that every arrow of (7.9) is G-equivariant. Thus, introducing the linear operators

$$\begin{aligned} \theta\Big|_\gamma &: \ \mathrm{Hom}_H(V_\gamma, \mathfrak{m}) \longrightarrow \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}), \\ \delta\Big|_\gamma &: \ \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}) \longrightarrow \mathrm{Hom}_H(V_\gamma, \mathfrak{m}), \end{aligned}$$

(in the notation of §7.3.4) we obtain a short exact sequence

$$0 \longrightarrow \ker\theta\Big|_\gamma \longrightarrow \mathrm{Hom}_H(V_\gamma, \mathfrak{m}) \longrightarrow \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}) \longrightarrow \ker\delta\Big|_\gamma \longrightarrow 0 \tag{7.10}$$

for each Fourier mode $\gamma \in \hat{G}$, from which the dimension formula

$$\dim\ker\delta\Big|_\gamma = \dim\mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}) - \dim\mathrm{Hom}_H(V_\gamma, \mathfrak{m}) + \dim\ker\theta\Big|_\gamma$$

follows.

What to make of this? The dimensions of $\mathrm{Hom}_H(V_\gamma, \mathfrak{m})$ and $\mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m})$ may easily be computed using representation theory. We now recall the following well-known fact: If (M, g) is an Einstein manifold not isometric to a round sphere, then every conformal Killing vector field is Killing, i.e. $\ker\theta = \ker\delta^*$ [Sch22b, Lem. 4.2]. Recall also that Killing vector fields are the infinitesimal generators of isometries. Provided G acts almost effectively on M, a lower dimension bound on $\ker\delta^*$ is thus given by the inclusion

$$\mathfrak{g} \hookrightarrow \mathfrak{iso}(M, g) = \ker\delta^* : \ X \mapsto \tilde{X}, \quad \tilde{X}_p = \frac{d}{dt}\Big|_{t=0} \exp(tX).p$$

mapping each Lie algebra element to the fundamental vector field generated by it. Moreover it is not hard to show that these fundamental vector fields do under left-translation in fact transform as the adjoint representation of $\mathfrak{g}$, that is, they are of Fourier type λ_{ad}. The corresponding matrix coefficient in $\mathrm{Hom}_H(\mathfrak{g}, \mathfrak{m})$ is simply the projection $\mathrm{pr}_{\mathfrak{m}}$.

In general, $\mathfrak{iso}(M, g)$ might be larger than $\mathfrak{g}$, so Killing vector fields may not be confined to the Fourier mode λ_{ad} alone. Strikingly, in the isotropy irreducible case, a result due to Wolf tells us that this does not happen in practice:

7.6.1 Proposition ([Wol68], Thm. 17.1)**.** *Let $M = G/H$ be a non-Euclidean, simply connected, isotropy irreducible space with G connected and effective, K compact, and with a G-invariant Riemannian metric g.*

- *If $G/H = \mathrm{G}_2/\mathrm{SU}(3)$, then (M, g) is the round S^6, so $\mathrm{Iso}(M, g)^0 = \mathrm{SO}(7)$.*
- *If $G/H = \mathrm{Spin}(7)/\mathrm{G}_2$, then (M, g) is the round S^7, so $\mathrm{Iso}(M, g)^0 = \mathrm{SO}(8)$.*
- *In every other case, $\mathrm{Iso}(M, g)^0 = G$.*

Even more welcomely, Wang–Ziller extended this statement to the wider class of spaces that we are interested in.

7.6.2 Proposition ([WZ85], Thm. 5.1)**.** *Let $M = G/H$ be a compact, simply connected, isotropy reducible homogeneous space with G compact, connected, simple and effective and a normal Einstein metric g. Then $\mathrm{Iso}(M, g)^0 = G$.*

Having established that Killing vector fields are exclusively of Fourier type λ_{ad} if (M, g) is not isometric to a round sphere, it follows that $\ker\theta\big|_\gamma = \ker\delta^*\big|_\gamma = 0$ if $\gamma \neq \lambda_{\mathrm{ad}}$, i.e. $\theta\big|_\gamma : \mathrm{Hom}_H(V_\gamma, \mathfrak{m}) \to \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m})$ is injective. We can thus formulate a corollary.

7.6.3 Corollary. *If $M = G/H$ is a compact, simply connected homogeneous space with G simple and acting almost effectively, equipped with a normal Einstein metric g such that (M, g) is not isometric to a round sphere, then $\ker\theta = \ker\delta^* \cong \mathfrak{g}$ as a G-module, and $\ker\theta\big|_\gamma = 0$ if $\gamma \neq \lambda_{\mathrm{ad}}$.*

When combined with (7.10), we obtain a simple criterion for when a Fourier mode contains no tt-tensors, which rules them out for the stability discussion.

7.6.4 Corollary. *Under the same assumptions as in Corollary 7.6.3, a Fourier mode $\gamma \in \hat{G}$ contains no tt-tensors (i.e. $\ker\delta\big|_\gamma = 0$) if and only if*

$$\dim \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m}) - \dim \mathrm{Hom}_H(V_\gamma, \mathfrak{m}) = \begin{cases} 0, & \gamma \neq \lambda_{\mathrm{ad}}, \\ -1, & \gamma = \lambda_{\mathrm{ad}}. \end{cases}$$

7.7 Algorithm for obtaining lower bounds on the Lichnerowicz Laplacian

We lay out an overview of the algorithm used in our computations, without any explicit regard to implementation details. The necessary steps for the calculation of the Casimir constants have been subsumed in §7.5. In any of the occurring direct sum decompositions, multiplicities of irreducible summands can be disregarded since they are irrelevant for the computation. The algorithm has been implemented using SageMath (version 9.2) and its interface to the software package LiE.

7.7.1 Algorithm (Lower Bounds for Δ_{L})**.** Let $M = G/H$ be a homogeneous space with G compact and simple such that the standard metric g is Einstein. Assume that (M, g) is not isometric to a round sphere (required for Step 9b).

1. Branch the adjoint representation on $\mathfrak{g}$ to $\mathfrak{h}$ to find the isotropy representation $\mathfrak{m}$.

2. Compute $\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}$ and Einstein constant E.

3. Decompose $\mathrm{Sym}^2_0\,\mathfrak{m} = \bigoplus_{i\in I}\mathfrak{v}_i$ into $\mathfrak{h}$-isotypes and $\mathrm{Sym}^2_0\,\mathfrak{g} = \bigoplus_{j\in J}\mathfrak{w}_j$ into $\mathfrak{g}$-isotypes.

4. For each $j \in J$:

   a) Compute $\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{w}_j}$.

   b) Branch $\mathfrak{w}_j$ to $\mathfrak{h}$.

5. For each $i \in I$:

   a) Compute $\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{v}_i}$.

   b) Find $J_i = \{j \in J \mid \mathrm{Hom}_H(\mathfrak{v}_i, \mathfrak{w}_j) \neq 0\}$.

   c) Find minimum/maximum of $\{\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{w}_j} \mid j \in J_i\}$. These are lower/upper bounds for $\mathrm{Cas}^{\mathfrak{g}}_{\mathrm{Sym}^2\mathfrak{g}}$ on the smallest G-invariant subspace of $\mathrm{Sym}^2_0\,\mathfrak{g}$ containing $\mathfrak{v}_i$.

   d) Combine to find bounds for $\mathcal{A}^*\mathcal{A}$ and $q(R)$ using [SSW22, Cor. 3.2] and Corollary 7.4.3.

   e) Check if $q(R) > E$, in which case the $\mathfrak{v}_i$ cannot contribute to instability.

6. Let $\mathcal{P} = \bigoplus\{\mathfrak{v}_i \mid q(R) \not> E$ by the above bounds$\}$. If $\mathcal{P} = 0$, then $q(R) > E$ on $\mathrm{Sym}^2_0\,\mathfrak{m}$ and hence (M, g) is stable by (7.1).

7. Combine the bounds for $\mathcal{A}^*\mathcal{A}$ with the crude estimate from Theorem 7.4.6 and find $C > 0$ such that $\Delta_{\mathrm{L}}\big|_{\gamma} > 2E$ if $\mathrm{Cas}^{\mathfrak{g}}_{\gamma} > C$.

8. Find $\hat{G}_C = \{\gamma \in \hat{G} \mid \mathrm{Cas}^{\mathfrak{g}}_{\gamma} \leq C\}$.

9. For each $\gamma \in \hat{G}_C$:

   a) Check whether $\mathrm{Hom}_H(V_\gamma, \mathcal{P}) = 0$. If so, then $q(R)h > E$ for all $h \in \mathscr{S}^2_0(M)$ of Fourier type γ. Thus γ cannot contribute to instability by (7.1).

   b) Check whether $\dim \mathrm{Hom}_H(V_\gamma, \mathfrak{m}) = \dim \mathrm{Hom}_H(V_\gamma, \mathrm{Sym}^2_0\,\mathfrak{m})$ ($+1$ if $\gamma = \lambda_{\mathrm{ad}}$). If so, then γ contains no tt-tensors by Corollary 7.6.4 and thus cannot contribute to instability.

   c) Find $\mathcal{V} = \bigoplus\{\mathfrak{v}_i \mid \mathrm{Hom}_H(\mathfrak{v}_i, V_\gamma) \neq 0\}$ (the relevant part of $\mathrm{Sym}^2_0\,\mathfrak{m}$) and compute $\mathrm{Cas}^{\mathfrak{h}}$ there to find $\lambda_{\max}[\mathrm{Cas}^{\mathfrak{h}}_{\mathrm{Sym}^2\mathfrak{m}}\big|_{\mathcal{V}}]$.

   d) Find $\mathcal{W} = \bigoplus\{\mathfrak{w}_j \mid \mathrm{Hom}_H(\mathfrak{w}_j, \mathcal{V}) \neq 0\}$ (the relevant part of $\mathrm{Sym}^2_0\,\mathfrak{g}$) and compute $\mathrm{Cas}^{\mathfrak{g}}$ there to find $\lambda_{\min}[\mathrm{Cas}^{\mathfrak{g}}_{\mathrm{Sym}^2\mathfrak{g}}\big|_{\mathcal{W}}]$.

   e) Compute the tensor product of $\mathfrak{g}$-modules $V_\gamma \otimes \mathcal{W} = \bigoplus_{j\in J_\gamma}\mathfrak{u}_j$.

   f) For each $j \in J_\gamma$:

i. Compute $\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{u}_j}$.

ii. Branch $\mathfrak{u}_j$ to $\mathfrak{h}$ and check whether $\mathfrak{u}_j^H = 0$.

g) Find $\lambda_{\max}[\mathrm{Cas}^{\mathfrak{g}}_{V_\gamma \otimes \mathrm{Sym}^2 \mathfrak{g}} \big|_{\mathcal{U}}] = \max\{\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{u}_j} \mid \mathfrak{u}_j^{\mathfrak{h}} \neq 0\}$.

h) Combine Casimir bounds with the refined estimate from Theorem 7.4.7 to find a lower bound for $\Delta_{\mathrm{L}}\big|_\gamma$.

7.8 Results and discussion

7.8.1 Setup and remarks

We begin with listing our spaces of interest, namely

1. the compact, simply connected *isotropy irreducible* homogeneous spaces G/H which are not symmetric, as classified by Wolf [Wol68], consisting of 10 infinite families I–X (see Table 7.1) and 13 exceptions[3] (see Tables 7.3 and 7.4),

2. the compact, simply connected homogeneous spaces G/H with G simple where the standard metric is Einstein and which are *isotropy reducible*, as classified by Wang and Ziller [WZ85], consisting of 9 infinite families XI–XIX (see Table 7.2) and 22 exceptions (see Table 7.6).

Throughout what follows the spaces in question will be labeled only by pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$. There is a unique simply connected homogeneous manifold $M = G/H$ corresponding to each pair $(\mathfrak{g}, \mathfrak{h})$, although G and H need of course not be unique. If $\mathfrak{g}$ is classical, the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ will usually be defined via a *defining representation* of $\mathfrak{h}$ which can be of unitary, symplectic or orthogonal type and thus yields an embedding into $\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{sp}(n)$ or $\mathfrak{so}(n)$, respectively. In this case it suffices to specify the highest weight of the defining representation, which we express in the usual basis of fundamental weights (η_i). For semisimple $\mathfrak{h}$, this basis will be the union of bases (η_i), (η_i'), etc. corresponding to each simple factor.

For isotropy reducible spaces, the definition of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ tends to be a bit more involved. For the definitions of the families XI–XIX we refer the reader to [LL23] where they are discussed in proper detail.

The family XIX deserves special mention. It is defined as $\mathrm{SO}(\mathfrak{p})/H$ where K/H is a (reducible) symmetric space as in [WZ85, Ex. 3]. Here $\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_l$ denotes the isotropy representation of K/H. This construction actually gives rise to most of the standard homogeneous Einstein manifolds of the form $\mathrm{SO}(n)/H$ (except for $\mathrm{Spin}(8)/\mathrm{G}_2$). In particular it already completely covers the families XV–XVIII. In order to divide up

[3] We do not list the space $\mathfrak{so}(20)/\mathfrak{su}(4)$ appearing in [Wol68] as it is a member of family X ($n = 6$).

No.	$\mathfrak{g}$	$\mathfrak{h}$	Condition	Defining rep.	E
I	$\mathfrak{su}(\frac{n(n-1)}{2})$	$\mathfrak{su}(n)$	$n \geq 5$	η_2	$\frac{1}{4} + \frac{2}{n(n-2)}$
II	$\mathfrak{su}(\frac{n(n+1)}{2})$	$\mathfrak{su}(n)$	$n \geq 3$	$2\eta_1$	$\frac{1}{4} + \frac{2}{n(n+2)}$
III	$\mathfrak{su}(pq)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q)$	$2 \leq p \leq q$, $p + q \neq 4$	$\eta_1 + \eta_1'$	$\frac{1}{4} + \frac{p^2+q^2}{2p^2q^2}$
IV	$\mathfrak{sp}(n)$	$\mathfrak{sp}(1) \oplus \mathfrak{so}(n)$	$n \geq 3$	$\eta_1 + \eta_1'$	$\frac{3}{8} + \frac{n+16}{8n(2n-1)}$
V	$\mathfrak{so}(n^2-1)$	$\mathfrak{su}(n)$	$n \geq 3$	$\eta_1 + \eta_{n-1}$	$\frac{1}{4} + \frac{1}{n^2-3}$
VI	$\mathfrak{so}((n-1)(2n+1))$	$\mathfrak{sp}(n)$	$n \geq 3$	η_2	$\frac{1}{4} + \frac{1}{(n-1)(n+1)(2n-3)}$
VII	$\mathfrak{so}(2n^2+n)$	$\mathfrak{sp}(n)$	$n \geq 2$	$2\eta_1$	$\frac{1}{4} + \frac{1}{2n^2+n-2}$
VIII	$\mathfrak{so}(4n)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$n \geq 2$	$\eta_1 + \eta_1'$	$\frac{3}{8} + \frac{n+4}{8n(2n-1)}$
IX	$\mathfrak{so}(\frac{n(n-1)}{2})$	$\mathfrak{so}(n)$	$n \geq 7$	η_2	$\frac{1}{4} + \frac{2}{n^2-n-4}$
X	$\mathfrak{so}(\frac{(n-1)(n+2)}{2})$	$\mathfrak{so}(n)$	$n \geq 5$	$2\eta_1$	$\frac{1}{4} + \frac{2n}{(n-2)(n+2)(n+3)}$

Table 7.1: The 10 families of *isotropy irreducible* spaces.

No.	$\mathfrak{g}$	$\mathfrak{h}$	Condition	E
XIa	$\mathfrak{su}(n)$	$\mathbb{R}^{n-1}$	$n \geq 3$	$\frac{1}{4} + \frac{1}{2n}$
XIb	$\mathfrak{su}(kn)$	$k\mathfrak{su}(n) \oplus (k-1)\mathbb{R}$	$k \geq 3$, $n \geq 2$	$\frac{1}{4} + \frac{1}{2n}$
XII	$\mathfrak{su}(l+pq)$	$\mathfrak{su}(l) \oplus \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus 2\mathbb{R}$	$2 \leq p \leq q$, $lpq = p^2 + q^2 + 1$	$\frac{1}{4} + \frac{p^2+q^2}{2(p^2+1)(q^2+1)}$
XIII	$\mathfrak{sp}(kn)$	$k\mathfrak{sp}(n)$	$k \geq 3$, $n \geq 1$	$\frac{1}{4} + \frac{2n+1}{4(kn+1)}$
XIV	$\mathfrak{sp}(3n-1)$	$\mathfrak{su}(2n-1) \oplus \mathfrak{sp}(n) \oplus \mathbb{R}$	$n \geq 1$	$\frac{5}{12}$
XV	$\mathfrak{so}(4n^2)$	$2\mathfrak{sp}(n)$	$n \geq 2$	$\frac{1}{4} + \frac{2n+1}{2n(2n^2-1)}$
XVI	$\mathfrak{so}(n^2)$	$2\mathfrak{so}(n)$	$n \geq 3$	$\frac{1}{4} + \frac{n-1}{n(n^2-2)}$
XVIIa	$\mathfrak{so}(2n)$	$\mathbb{R}^n$	$n \geq 3$	$\frac{1}{4} + \frac{1}{4(n-1)}$
XVIIb	$\mathfrak{so}(kn)$	$k\mathfrak{so}(n)$	$k, n \geq 3$	$\frac{1}{4} + \frac{n-1}{2(kn-2)}$
XVIII	$\mathfrak{so}(3n+2)$	$\mathfrak{su}(n+1) \oplus \mathfrak{so}(n) \oplus \mathbb{R}$	$n \geq 3$	$\frac{5}{12}$
XIX	$\mathfrak{so}(n)$	$\mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_l$	see [LL23, §7]	$\frac{1}{4} + \frac{\dim \mathfrak{h}_i}{(n-2)\dim \mathfrak{p}_i}$

Table 7.2: The 8 families of *isotropy reducible* spaces.

the spaces in question more evenly we impose the same contraints as given in [LL23, §7] and collect everything not listed among the families XV–XVIII or the exceptions into our "band of outcasts" XIX.

Tables 7.1 and 7.2 also list the Einstein constants of the families I–XIX. They can be derived *a priori* using the results of Wang and Ziller. For each isotropy irreducible space G/H the Casimir constant of its isotropy representation is in their notation given as $c = E(\chi)/\alpha_G$, where α_G and $E(\chi)$ are listed in the tables on [WZ85, pp. 583, 588]. The Einstein constant is then $E = \frac{1}{4} + \frac{c}{2}$. Similarly the Einstein constants of the isotropy reducible families are derived in [LL23].

Tables 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9 are to be read as follows. The column *Potential instabilities* lists all Fourier modes $\gamma \in \hat{G}$ for which Alg. 7.7.1 does *not* yield the estimate $\Delta_{\mathrm{L}}\big|_{\gamma} > 2E$. If the weaker estimate $\Delta_{\mathrm{L}}\big|_{\gamma} \geq 2E$ holds, the Fourier mode γ is printed in blue. Each γ will be expressed in the basis (ω_i) of fundamental weights of $\mathfrak{g}$.

The abbreviations in the column *Notes* will stand for the following:

- **SQ**: stable by Step 6 of Alg. 7.7.1. That is, $q(R) > E$ is fulfilled, which is sufficient for stability by (7.1).
- **SF**: stable by Step 9 of Alg. 7.7.1, i.e. after applying the new estimates (Theorem 7.4.6 and 7.4.7).
- **SF$_0$**: semistable by Step 9 of Alg. 7.7.1.

We remark that the spaces $\mathfrak{so}(7)/\mathfrak{g}_2$ and $\mathfrak{g}_2/\mathfrak{su}(3)$ from Tables 7.3 and 7.4 are round spheres and thus already known to be stable. Moreover the Berger space $\mathfrak{sp}(2)/\mathfrak{su}(2)$ can also be written as $\mathfrak{so}(5)/\mathfrak{so}(3)$ (the highest weight of defining representation is then $4\eta_1$) and the Fourier mode ω_2 (expressed as weight of $\mathfrak{sp}(2)$) is known to be destabilizing [SWW22, §5]. The stability of the space $\mathfrak{e}_7/\mathfrak{so}(8)$ in Table 7.6 was shown recently [SSW22].

7.8.2 Discussion of results

We begin with discussing the isotropy irreducible case. Tables 7.3 and 7.4 show the obtained results for the exceptional spaces. In some cases (mostly those with $\mathfrak{g}$ of type E), the curvature estimate $q(R) > E$ is already sufficient to prove stability. This phenomenon persists for the isotropy reducible spaces, cf. Table 7.6. We note that the spaces

$$\frac{\mathfrak{so}(8)}{\mathfrak{g}_2}, \quad \frac{\mathfrak{so}(26)}{\mathfrak{sp}(1) \oplus \mathfrak{sp}(5) \oplus \mathfrak{so}(6)}, \quad \frac{\mathfrak{f}_4}{\mathfrak{so}(8)}, \quad \frac{\mathfrak{e}_6}{\mathfrak{so}(8) \oplus \mathbb{R}^2}, \quad \frac{\mathfrak{e}_7}{3\mathfrak{su}(2) \oplus \mathfrak{so}(8)}$$

were shown to be G-unstable in [LW22b; LL23] – this corresponds to the Fourier mode listed as "0". Remarkably, combining our analysis with the G-stability results of [LW22b;

LL23], which rule out the "0" mode, leads to (semi-)stability for the spaces

$$\frac{\mathfrak{e}_6}{3\mathfrak{su}(2)}, \frac{\mathfrak{e}_7}{7\mathfrak{su}(2)}, \frac{\mathfrak{e}_8}{2\mathfrak{su}(5)}, \frac{\mathfrak{e}_8}{2\mathfrak{so}(8)}.$$

Notable is also the space $\frac{\mathfrak{e}_7}{3\mathfrak{su}(2)\oplus\mathfrak{so}(8)}$, shown to be G-unstable in [LL23] (with a G-coindex of 2). According to our analysis, "0" is the only potential instability – so the coindex coincides with the G-coindex here.

Considering the isotropy irreducible families (Table 7.5), we observe varying behavior with respect to stability. Within the scope of our computational capacity, we could show stability for the following spaces:

from I:	$\mathfrak{su}(\frac{n(n-1)}{2})/\mathfrak{su}(n)$,	$8 \leq n \leq 11$,
from II:	$\mathfrak{su}(\frac{n(n+1)}{2})/\mathfrak{su}(n)$,	$6 \leq n \leq 8$,
from III:	$\mathfrak{su}(pq)/(\mathfrak{su}(p) \oplus \mathfrak{su}(q))$,	$13 \leq pq \leq 36$,
from VII:	$\mathfrak{so}(2n^2+n)/\mathfrak{sp}(n)$,	$3 \leq n \leq 6$,
from IX:	$\mathfrak{so}(\frac{n(n-1)}{2})/\mathfrak{so}(n)$,	$7 \leq n \leq 13$.

We turn next to the isotropy reducible families (Tables 7.7, 7.8 and 7.9). These were extensively studied in [Lau22; LW22b; LL23], where the G-instability of XI–XIV, XVIIb, XVIII and XIX was already proved. This leaves open the cases XVI (G-stable) and XVIIa (G-semistable) as well as the family XV where the G-stability type is still unknown. We managed to show stability for the following examples:

from XV:	$\mathfrak{so}(4n^2)/(\mathfrak{sp}(n) \oplus \mathfrak{sp}(n))$,	$3 \leq n \leq 5$,
from XVI:	$\mathfrak{so}(n^2)/(\mathfrak{so}(n) \oplus \mathfrak{so}(n))$,	$4 \leq n \leq 10$,

as well as semistability of $\mathfrak{so}(2n)/\mathbb{R}^n$ from XVIIa if $n = 6, 7$, which follows from the G-semistability result of [LL23].

The effectiveness of Alg. 7.7.1, Step 9b in ruling out instabilities is underwhelming in light of how useful the same method is on symmetric spaces [Sch22b; SW22], only eliminating the Fourier modes ω_2 on $\frac{\mathfrak{so}(8)}{\mathfrak{su}(3)}$ (from the family V), ω_2 on $\frac{\mathfrak{so}(18)}{\mathfrak{so}(4)\oplus\mathfrak{sp}(3)}$ (from the family XIX, defined by the isotropy representation of the symmetric space $S^4 \times \frac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$), and ω_2 on $\frac{\mathfrak{so}(26)}{\mathfrak{sp}(1)\oplus\mathfrak{sp}(5)\oplus\mathfrak{so}(6)}$ (exceptional). It shows however that the absence of tt-tensors in a given Fourier mode is quite a rare phenomenon.

In general we observe a trend towards stability in the infinite families as the rank increases. Some cases (I, II, III, VII, IX, XV, XVI, XVIIa) seem to become and stay stable at some point. For others (IV, V, VI, VIII, X) there seem to be some Fourier modes that always harbor potential instabilities.

7.8.3 Outlook

In order to decide the stability of the cases with remaining potential instabilities, it would be sufficient to compute the Lichnerowicz Laplacian seperately on each of the potentially destabilizing Fourier modes. In particular this requires the problems mentioned in §7.4 to be overcome. Moreover, in order to tackle the stability analysis on the countable families in their entirety, a systematic approach to (at least) estimating the Lichnerowicz Laplacian on each family would be needed.

Another matter entirely is the question of how our approach may be generalized even further to non-normal metrics – say, metrics that are "almost normal" in the sense that they can be written as

$$g = \alpha_1 Q\Big|_{\mathfrak{m}_1} + \alpha_2 Q\Big|_{\mathfrak{m}_2}, \qquad \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2, \qquad \alpha_1, \alpha_2 > 0$$

for some $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$. This is the case for metrics in the canonical variation of a homogeneous fibration $H/K \hookrightarrow G/K \twoheadrightarrow G/H$ with normal fiber and base. Special symmetries of the form

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k}, \qquad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k} \oplus \mathfrak{m}_1$$

that may simplify computations are available if we suppose that fiber and base are symmetric, but also for Kähler–Einstein metrics on generalized flag manifolds with $b_2(M) = 1$ and two isotropy summands.

In the homogeneous fibration setting there is a natural first choice of tt-tensors to investigate, namely those of the form $h = \beta_1 Q\Big|_{\mathfrak{m}_1} + \beta_2 Q\Big|_{\mathfrak{m}_2}$ with $\beta_1, \beta_2 \in \mathbb{R}$ chosen such that $\operatorname{tr}_g h = 0$. These are an instance of *Killing tensors* (that is, they are annihilated by the Killing operator δ^*). Trace-free Killing tensors have the advantage that they realize the equality in (7.1), that is

$$\Delta_L h = 2q(R)h, \tag{7.11}$$

and the fibrewise term $q(R)$ is in general easier to handle. Moreover, Killing tensors have often been sources of instability – see [WW21] for tensors of the particular form above on fiber bundles, and [SWW22] for an exploitation of (7.11) to show instability of the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)$.

We aim to return to all of these issues in future work.

$\mathfrak{g}$	$\mathfrak{h}$	Defining rep.	E	Potential instabilities	Notes
$\mathfrak{su}(16)$	$\mathfrak{so}(10)$	η_4	$\frac{11}{32}$	$\omega_1 + \omega_{15}$	
$\mathfrak{su}(27)$	$\mathfrak{e}_6$	η_1	$\frac{11}{36}$	$\omega_1 + \omega_{26}$	$\mathbf{SF}_0$
$\mathfrak{so}(7)$	$\mathfrak{g}_2$	η_1	$\frac{9}{20}$	–	**SQ**
$\mathfrak{so}(133)$	$\mathfrak{e}_7$	η_3	$\frac{135}{524}$	–	**SF**
$\mathfrak{sp}(2)$	$\mathfrak{su}(2)$	$3\eta_1$	$\frac{9}{20}$	$\omega_2, 2\omega_2, 2\omega_1 + \omega_2, 4\omega_1$	
$\mathfrak{sp}(7)$	$\mathfrak{sp}(3)$	η_3	$\frac{29}{80}$	ω_2	
$\mathfrak{sp}(10)$	$\mathfrak{su}(6)$	η_3	$\frac{15}{44}$	ω_2	
$\mathfrak{sp}(16)$	$\mathfrak{so}(12)$	η_5	$\frac{43}{136}$	ω_2	
$\mathfrak{sp}(28)$	$\mathfrak{e}_7$	η_7	$\frac{17}{58}$	–	**SF**
$\mathfrak{so}(14)$	$\mathfrak{g}_2$	η_2	$\frac{1}{3}$	$2\omega_1$	
$\mathfrak{so}(16)$	$\mathfrak{so}(9)$	η_4	$\frac{23}{56}$	$2\omega_1, \omega_4$	
$\mathfrak{so}(26)$	$\mathfrak{f}_4$	η_4	$\frac{1}{3}$	$\omega_1, 2\omega_1$	
$\mathfrak{so}(42)$	$\mathfrak{sp}(4)$	η_4	$\frac{19}{70}$	–	**SF**
$\mathfrak{so}(52)$	$\mathfrak{f}_4$	η_1	$\frac{27}{100}$	–	**SF**
$\mathfrak{so}(70)$	$\mathfrak{su}(8)$	η_4	$\frac{179}{680}$	–	**SF**
$\mathfrak{so}(78)$	$\mathfrak{e}_6$	η_2	$\frac{5}{19}$	–	**SF**
$\mathfrak{so}(128)$	$\mathfrak{so}(16)$	η_7	$\frac{173}{672}$	–	**SF**
$\mathfrak{so}(248)$	$\mathfrak{e}_8$	η_8	$\frac{125}{492}$	–	**SF**

Table 7.3: Results for the isotropy irreducible exceptions with $\mathfrak{g}$ classical.

$\mathfrak{g}$	$\mathfrak{h}$	E	Potential instabilities	Notes
$\mathfrak{e}_6$	$\mathfrak{su}(3)$	$\frac{11}{36}$	–	**SQ**
$\mathfrak{e}_6$	$3\mathfrak{su}(3)$	$\frac{5}{12}$	$\omega_2, \omega_1 + \omega_6$	
$\mathfrak{e}_6$	$\mathfrak{g}_2$	$\frac{25}{72}$	–	**SQ**
$\mathfrak{e}_6$	$\mathfrak{su}(3) \oplus \mathfrak{g}_2$	$\frac{19}{48}$	$\omega_2, \omega_1 + \omega_6$	
$\mathfrak{e}_7$	$\mathfrak{su}(3)$	$\frac{71}{252}$	–	**SQ**
$\mathfrak{e}_7$	$\mathfrak{su}(6) \oplus \mathfrak{su}(3)$	$\frac{5}{12}$	ω_1, ω_6	
$\mathfrak{e}_7$	$\mathfrak{g}_2 \oplus \mathfrak{sp}(3)$	$\frac{7}{18}$	–	**SF**
$\mathfrak{e}_7$	$\mathfrak{su}(2) \oplus \mathfrak{f}_4$	$\frac{47}{108}$	ω_6	
$\mathfrak{e}_8$	$\mathfrak{su}(9)$	$\frac{5}{12}$	–	**SQ**
$\mathfrak{e}_8$	$\mathfrak{e}_6 \oplus \mathfrak{su}(3)$	$\frac{5}{12}$	–	**SF**
$\mathfrak{e}_8$	$\mathfrak{g}_2 \oplus \mathfrak{f}_4$	$\frac{23}{60}$	–	**SF**
$\mathfrak{f}_4$	$2\mathfrak{su}(3)$	$\frac{5}{12}$	$\omega_4, \omega_1, \omega_3, 2\omega_4$	
$\mathfrak{f}_4$	$\mathfrak{su}(2) \oplus \mathfrak{g}_2$	$\frac{29}{72}$	$\omega_4, 2\omega_4, \omega_1 + \omega_4$	
$\mathfrak{g}_2$	$\mathfrak{su}(2)$	$\frac{43}{112}$	$2\omega_1, \omega_1 + \omega_2, 2\omega_2$	
$\mathfrak{g}_2$	$\mathfrak{su}(3)$	$\frac{5}{12}$	–	**SQ**

Table 7.4: Results for the isotropy irreducible exceptions with $\mathfrak{g}$ exceptional. The embedding $\mathfrak{h} \subset \mathfrak{g}$ is always characterized by $\mathfrak{h}$ being a maximal subalgebra.

Family	Param.	$r = \operatorname{rk} \mathfrak{g}$	Potential instabilities	Notes
I	$n = 5, 6, 7$	$9, 14, 20$	$\omega_1 + \omega_r$	
	$n = 8 \ldots 11$	$27, 35, 44, 54$	–	**SF**
II	$n = 3$	5	$\omega_1 + \omega_5$, $\omega_1 + \omega_2$, $\omega_2 + \omega_4$, $3\omega_1$	*
	$n = 4, 5$	9	$\omega_1 + \omega_r$	
	$n = 6, 7, 8$	$20, 27, 35$	–	**SF**
III	$(p, q) = (2, 3)$	5	$\omega_1 + \omega_5$, $\omega_2 + \omega_4$, $2\omega_1 + \omega_4$, $2\omega_1 + 2\omega_5$	*
	$(p, q) = (2, 4)$	7	$\omega_1 + \omega_7$, ω_4, $\omega_1 + \omega_3$, $\omega_2 + \omega_6$, $2\omega_2$	*
	$(p, q) = (3, 3)$	8	$\omega_1 + \omega_8$, ω_3, $\omega_1 + \omega_2$	*
	$(p, q) = (2, 5)$	9	$\omega_1 + \omega_9$, $\omega_2 + \omega_8$	
	$12 \leq pq \leq 22$; or $(2, q)$ with $12 \leq q \leq 20$; or $(3, q)$ with $8 \leq q \leq 11$	$pq - 1$	$\omega_1 + \omega_r$	
	$(3, q)$ with $12 \leq q \leq 16$; or (p, q) with $p \geq 4$ and $24 \leq pq \leq 49$	$pq - 1$	–	**SF**
IV	$n = 3$	3	ω_2, $\omega_1 + \omega_3$, $2\omega_2$, $2\omega_1 + \omega_2$, $4\omega_1$, $\omega_1 + \omega_2 + \omega_3$, $3\omega_1 + \omega_3$	
	$n = 4$	4	ω_2, $\omega_1 + \omega_3$, $2\omega_2$, $2\omega_1 + \omega_2$, $4\omega_1$, $\omega_2 + \omega_4$, $2\omega_1 + \omega_4$	
	$n = 5$	5	ω_2, $\omega_1 + \omega_3$, $2\omega_2$, $4\omega_1$	
	$n = 6$	6	ω_2, $2\omega_2$	
	$n = 7 \ldots 100$	n	ω_2	
V	$n = 3$	4	ω_1, $\omega_1 + \omega_3$, $2\omega_1$, $\omega_1 + \omega_2$, $\omega_1 + \omega_3 + \omega_4$, $2\omega_1 + \omega_3$, $2\omega_2$	**
	$n = 4$	7	ω_1, $2\omega_1$, ω_2	
	$n = 5 \ldots 18$	$\lfloor \frac{n^2-1}{2} \rfloor$	ω_1	
VI	$n = 3$	7	ω_1, $2\omega_1$	
	$n = 4 \ldots 11$	$\lfloor \frac{(n-1)(2n+1)}{2} \rfloor$	ω_1	
VII	$n = 2$	5	$2\omega_1$, ω_3, $\omega_4 + \omega_5$, $\omega_1 + \omega_2$	
	$n = 3 \ldots 13$	$\lfloor \frac{2n^2+n}{2} \rfloor$	–	**SF**
VIII	$n = 2$	4	$2\omega_4$, $\omega_2 + \omega_4$, $2\omega_2$	
	$n = 3$	6	$2\omega_1$, ω_4, $2\omega_2$	
	$n = 4$	8	$2\omega_1$, $\omega_1 + \omega_7$, ω_4	
	$n = 5 \ldots 75$	$2n$	$2\omega_1$	
IX	$n = 7 \ldots 27$	$\lfloor \frac{n(n-1)}{4} \rfloor$	–	**SF**
X	$n = 5 \ldots 22$	$\lfloor \frac{(n-1)(n+2)}{4} \rfloor$	ω_1	

Table 7.5: Some results for the isotropy irreducible families I–X.
* To obtain all potential instabilities from the listed ones, take closure under the duality automorphism of A_r which sends $\omega_k \mapsto \omega_{r+1-k}$.
** To obtain all potential instabilities from the listed ones, take closure under the automorphisms of D_4 which permute ω_1, ω_3 and ω_4.

$\mathfrak{g}$	$\mathfrak{h}$	Embedding	E	Potential instabilities	Notes
$\mathfrak{so}(8)$	$\mathfrak{g}_2$	$\mathfrak{g}_2 \overset{\eta_1}{\hookrightarrow} \mathfrak{so}(7) \subset \mathfrak{so}(8)$	$\frac{5}{12}$	$0,\ \omega_1,\ \omega_1+\omega_3,\ 2\omega_1,\ \omega_1+\omega_2,$ $\omega_1+\omega_3+\omega_4,\ 2\omega_1+\omega_3$	*
$\mathfrak{so}(26)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(5) \oplus \mathfrak{so}(6)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(5) \overset{\eta_1+\eta_1'}{\hookrightarrow} \mathfrak{so}(20)$	$\frac{29}{80}$	$0,\ 2\omega_1$	
$\mathfrak{f}_4$	$\mathfrak{so}(8)$	$\mathfrak{so}(8) \subset \mathfrak{so}(9) \overset{\text{max.}}{\subset} \mathfrak{f}_4$	$\frac{4}{9}$	$0,\ \omega_4,\ \omega_3,\ 2\omega_4,\ \omega_3+\omega_4$	
$\mathfrak{e}_6$	$3\mathfrak{su}(2)$	$\mathfrak{su}(2) \overset{2\eta_1}{\hookrightarrow} \mathfrak{su}(3)$, $3\mathfrak{su}(3) \overset{\text{max.}}{\subset} \mathfrak{e}_6$	$\frac{5}{16}$	0	[LL23]⇒$\mathbf{SF}_0$
$\mathfrak{e}_6$	$\mathfrak{su}(2) \oplus \mathfrak{so}(6)$	$\mathfrak{so}(6) \subset \mathfrak{su}(6)$, $\mathfrak{su}(2) \oplus \mathfrak{su}(6) \overset{\text{max.}}{\subset} \mathfrak{e}_6$	$\frac{3}{8}$	$0,\ \omega_2$	
$\mathfrak{e}_6$	$\mathfrak{so}(8) \oplus \mathbb{R}^2$	$\mathfrak{so}(8) \oplus \mathbb{R} \subset \mathfrak{so}(10)$, $\mathfrak{so}(10) \oplus \mathbb{R} \overset{\text{max.}}{\subset} \mathfrak{e}_6$	$\frac{5}{12}$	$0,\ \omega_2,\ \omega_1+\omega_6,\ \omega_4$	
$\mathfrak{e}_6$	$\mathbb{R}^6$	max. torus	$\frac{7}{24}$	–	**SQ**
$\mathfrak{e}_7$	$7\mathfrak{su}(2)$	$3\mathfrak{so}(4) \subset \mathfrak{su}(12)$, $\mathfrak{su}(12) \oplus \mathfrak{su}(2) \overset{\text{max.}}{\subset} \mathfrak{e}_7$	$\frac{1}{3}$	0	[LL23]⇒**SF**
$\mathfrak{e}_7$	$\mathfrak{so}(8)$	$\mathfrak{so}(8) \subset \mathfrak{su}(8) \overset{\text{max.}}{\subset} \mathfrak{e}_7$	$\frac{13}{36}$	–	**SQ**
$\mathfrak{e}_7$	$3\mathfrak{su}(2) \oplus \mathfrak{so}(8)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \subset \mathfrak{su}(12)$, $\mathfrak{su}(12) \oplus \mathfrak{su}(2) \overset{\text{max.}}{\subset} \mathfrak{e}_7$	$\frac{7}{18}$	0	
$\mathfrak{e}_7$	$\mathbb{R}^7$	max. torus	$\frac{5}{18}$	–	**SQ**
$\mathfrak{e}_8$	$8\mathfrak{su}(2)$	$4\mathfrak{so}(4) \subset \mathfrak{so}(16) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{3}{10}$	–	**SQ**
$\mathfrak{e}_8$	$4\mathfrak{su}(3)$	$3\mathfrak{su}(3) \overset{\text{max.}}{\subset} \mathfrak{e}_6$, $\mathfrak{e}_6 \oplus \mathfrak{su}(2) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{19}{60}$	–	**SQ**
$\mathfrak{e}_8$	$4\mathfrak{su}(2)$	$\mathfrak{su}(2) \overset{2\eta_1}{\hookrightarrow} \mathfrak{su}(3)$, $4\mathfrak{su}(3) \subset \mathfrak{e}_8$ as above	$\frac{11}{40}$	–	**SQ**
$\mathfrak{e}_8$	$2\mathfrak{su}(3)$	$2\mathfrak{su}(3) \overset{\eta_1+\eta_1'}{\hookrightarrow} \mathfrak{su}(9)$, $\mathfrak{su}(9) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{17}{60}$	–	**SQ**
$\mathfrak{e}_8$	$2\mathfrak{su}(5)$	max. subalgebra	$\frac{7}{20}$	0	[LL23]⇒**SF**
$\mathfrak{e}_8$	$\mathfrak{so}(9)$	$\mathfrak{so}(9) \subset \mathfrak{su}(9) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{13}{40}$	–	**SQ**
$\mathfrak{e}_8$	$\mathfrak{so}(9)$	$\mathfrak{so}(9) \overset{\eta_4}{\hookrightarrow} \mathfrak{so}(16) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{13}{40}$	–	**SQ**
$\mathfrak{e}_8$	$2\mathfrak{so}(8)$	$2\mathfrak{so}(8) \subset \mathfrak{so}(16) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{11}{30}$	0	[LW22b]⇒**SF**
$\mathfrak{e}_8$	$\mathfrak{so}(5)$	max. subalgebra	$\frac{13}{48}$	–	**SQ**
$\mathfrak{e}_8$	$2\mathfrak{sp}(2)$	$2\mathfrak{sp}(2) \overset{\eta_1+\eta_1'}{\hookrightarrow} \mathfrak{so}(16) \overset{\text{max.}}{\subset} \mathfrak{e}_8$	$\frac{7}{24}$	–	**SQ**
$\mathfrak{e}_8$	$\mathbb{R}^8$	max. torus	$\frac{4}{15}$	–	**SQ**

Table 7.6: Results for the isotropy reducible exceptions.

* To obtain all potential instabilities/IED from the listed ones, take closure under the automorphisms of D_4 which permute ω_1, ω_3 and ω_4.

Family	Param.	$r = \operatorname{rk}\mathfrak{g}$	Potential instabilities	Notes
XIa	$n=3$	2	$0,\ \omega_1+\omega_2,\ 3\omega_1,\ 2\omega_1+2\omega_2$	*
	$n=4$	3	$0,\ \omega_1+\omega_3,\ 2\omega_2,\ 2\omega_1+\omega_2,$ $2\omega_1+2\omega_3$	*
	$n=5$	4	$0,\ \omega_1+\omega_4,\ \omega_2+\omega_3,\ 2\omega_1+\omega_3$	*
	$n=6,7,8,9$	$5,6,7,8$	$0,\ \omega_1+\omega_r$	
XIb	$(k,n)=(3,2)$	5	$0,\ \omega_1+\omega_5,\ \omega_2+\omega_4,\ 2\omega_3,$ $2\omega_1+\omega_4,\ 2\omega_1+2\omega_5$	*
	$(k,n)=(4,2)$	7	$0,\ \omega_1+\omega_7,\ \omega_2+\omega_6$	
	$(k,n)=(3,3)$	8	$0,\ \omega_1+\omega_8,\ \omega_2+\omega_7,\ 2\omega_1+\omega_7,$ $2\omega_1+2\omega_8$	*
	$(k,n)=(5,2)$	9	$0,\ \omega_1+\omega_9$	
	$(k,n)=(3,4)$	11	$0,\ \omega_1+\omega_{11},\ \omega_2+\omega_{10},$ $2\omega_1+\omega_{10}$	*
	$(k,n)=(4,3),(6,2)$	11	$0,\ \omega_1+\omega_{11}$	
XII	$(p,q,l)=(2,5,3)$	12	$0,\ \omega_1+\omega_{12}$	
XIII	$(k,n)=(3,1)$	3	$0,\ \omega_2,\ 2\omega_1,\ \omega_1+\omega_3,\ 2\omega_2,$ $2\omega_1+\omega_2$	
	$(k,n)=(4,1)$	4	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ 2\omega_2$	
	$(k,n)=(5,1)$	5	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3$	
	$(k,n)=(6,1)$	6	$0,\ \omega_2,\ 2\omega_1,\ \omega_4$	
	$(k,n)=(3,2)$	6	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ \omega_6,$ $2\omega_2$	
	$(k,n)=(4,2)$	8	$0,\ \omega_2,\ 2\omega_1,\ \omega_4$	
	$(k,n)=(3,3)$	9	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ 2\omega_2$	
	$(k,n)=(10,1)$	10	$0,\ \omega_2,\ 2\omega_1$	
	$(k,n)=(11,1)$	11	$0,\ \omega_2$	
	$(k,n)=(3,4)$	12	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3$	
	$(k,n)=(7,1),\ (8,1),\ (9,1),\ (5,2),$ $(6,2),\ (4,3),\ (7,2),\ (5,3),\ (3,5),\ (8,2),$ $(4,4),\ (6,3),\ (3,6)$	kn	$0,\ \omega_2,\ 2\omega_1$	
XIV	$n=1$	2	$0,\ \omega_2,\ 2\omega_1,\ 2\omega_2,\ 2\omega_1+\omega_2,$ $4\omega_1,\ 3\omega_2,\ 2\omega_1+2\omega_2$	
	$n=2$	5	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ 2\omega_2,$ $2\omega_1+\omega_2,\ \omega_1+\omega_5,\ 4\omega_1,$ $\omega_2+\omega_4,\ 2\omega_1+\omega_4$	
	$n=3$	8	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ 2\omega_2,$ $2\omega_1+\omega_2,\ 4\omega_1,\ \omega_6$	
	$n=4$	11	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ 2\omega_2,$ $2\omega_1+\omega_2,\ 4\omega_1$	
	$n=5$	14	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3,\ 2\omega_2,$ $2\omega_1+\omega_2$	
	$n=6$	17	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1+\omega_3$	
	$n=7$	20	$0,\ \omega_2,\ 2\omega_1,\ \omega_4$	
	$n=8,9,10$	$23,26,29$	$0,\ \omega_2,\ 2\omega_1$	

Table 7.7: Some results for the isotropy reducible families XI–XIV.

* To obtain all potential instabilities from the listed ones, take closure under the duality automorphism of A_r which sends $\omega_k \mapsto \omega_{r+1-k}$.

Family	Param.	$r = \mathrm{rk}\,\mathfrak{g}$	Potential instabilities	Notes
XV	$n = 2$	8	$0,\ 2\omega_1$	
	$n = 3 \ldots 9$	$2n^2$	–	**SF**
XVI	$n = 3$	4	$0,\ \omega_1,\ 2\omega_1,\ \omega_3,\ 2\omega_4$	
	$n = 4 \ldots 16$	$\lfloor \frac{n^2}{2} \rfloor$	–	**SF**
XVIIa	$n = 4$	4	$0,\ \omega_2,\ 2\omega_1,\ 2\omega_3,\ 2\omega_4$	
	$n = 5$	5	$0,\ \omega_2,\ 2\omega_1$	
	$n = 6, 7$	$6, 7$	0	[LL23]⇒$\mathbf{SF}_0$
XVIIb	$(k, n) = (3, 3)$	4	$0,\ \omega_2,\ 2\omega_1,\ \omega_3,\ 2\omega_4,\ \omega_1 + \omega_2$	
	$(k, n) = (3, 4)$	6	$0,\ \omega_2,\ 2\omega_1,\ \omega_4$	
	$(k, n) = (6, 3)$	9	$0,\ \omega_2$	
	$(k, n) = (6, 4)$	12	$0,\ \omega_2,\ 2\omega_1$	
	$(k, n) = (4, 3),\ (5, 3),\ (4, 4),\ (5, 4)$; or $(k, 5)$ with $3 \leq k \leq 6$; or (k, n) with $n \geq 6$ and $kn \leq 40$	$\lfloor \frac{kn}{2} \rfloor$	$0,\ \omega_2,\ 2\omega_1$	
	$(k, n) = (7, 3),\ (8, 3),\ (9, 3),\ (7, 4),\ (10, 3),\ (7, 5)$	$\lfloor \frac{kn}{2} \rfloor$	0	
XVIII	$n = 3$	5	$0,\ \omega_1,\ \omega_2,\ 2\omega_1,\ \omega_3,\ \omega_4,\ \omega_1 + \omega_2,\ 2\omega_5,\ \omega_1 + \omega_3,\ 2\omega_2,\ \omega_1 + \omega_4$	
	$n = 4$	7	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1 + \omega_3,\ 2\omega_2$	
	$n = 5$	8	$0,\ \omega_2,\ 2\omega_1,\ \omega_4,\ \omega_1 + \omega_3$	
	$n = 6$	10	$0,\ \omega_2,\ 2\omega_1,\ \omega_4$	
	$n = 7 \ldots 19$	$\lfloor \frac{3n}{2} \rfloor + 1$	$0,\ \omega_2,\ 2\omega_1$	

Table 7.8: Some results for the isotropy reducible families XV–XVIII.

Family	$K/H = K_1/H_1 \times \ldots \times K_l/H_l$	$r = \mathrm{rk}\,\mathfrak{g}$	Potential instabilities
XIX	$\frac{\mathrm{SU}(3)^2}{\mathrm{SO}(3)^2}$, $S^3 \times \mathrm{SO}(5)$, $S^4 \times \frac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$	$5, 6, 9$	0, ω_1, $2\omega_1$
	$S^3 \times \mathrm{SU}(3)$	5	0, ω_1, ω_2, $2\omega_1$, ω_3, $\omega_1 + \omega_2$
	$(S^3)^2 \times \mathrm{SU}(3)$	7	0, ω_1, ω_2, $2\omega_1$
	$\mathrm{SU}(3)^2$, $(S^3)^3 \times \mathrm{SU}(3)$	8	0, ω_1, ω_2, $2\omega_1$
	$S^3 \times \mathrm{G}_2$	8	0, $2\omega_1$
	$S^3 \times \mathrm{SU}(4)$, $\mathrm{SU}(3) \times \mathrm{SO}(5)$	9	0, ω_1, ω_2
	$(S^4)^2 \times \frac{\mathrm{SU}(6)}{\mathrm{Sp}(3)}$	11	0, ω_1, $2\omega_1$, ω_2
	$(S^3)^k \times \mathrm{SO}(5)$ with $3 \leq k \leq 5$; or $(S^3)^k \times \mathrm{G}_2$ with $2 \leq k \leq 4$; or $(S^3)^k \times \mathrm{SO}(5)^2$ with $0 \leq k \leq 2$; or $S^3 \times \mathrm{SO}(7)$, $S^3 \times \mathrm{Sp}(3)$, $\mathrm{SO}(5) \times \mathrm{G}_2$	$\lfloor \frac{\dim K/H}{2} \rfloor$	0
	all other with $\dim K/H \leq 26$	$\lfloor \frac{\dim K/H}{2} \rfloor$	0, ω_1

Table 7.9: Some results for the isotropy reducible family XIX.
K/H denotes the symmetric space used in the construction of $M = \mathrm{SO}(\mathfrak{p})/H$. Concerning the individual symmetric factors K_i/H_i we insist that Lie groups are presented as $\frac{H_i \times H_i}{H_i}$ and spheres as $S^k = \frac{\mathrm{SO}(k+1)}{\mathrm{SO}(k)}$.

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