Analysis of Hyperbolic Conservation Laws with Random Discontinuous Flux Functions and their Efficient Simulation

Lukas Brencher

2022

Analysis of Hyperbolic Conservation Laws with Random Discontinuous Flux Functions and their Efficient Simulation

Von der Fakultät Mathematik und Physik sowie dem Stuttgarter Zentrum für Simulationswissenschaften der Universität Stuttgart
zur Erlangung der Würde eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

vorgelegt von

Lukas Brencher

aus Kassel

Hauptberichterin:	Prof. Dr. Andrea Barth
Mitberichter:	Prof. Dr. Nils Henrik Risebro
	Prof. Dr. Christian Rohde

Tag der mündlichen Prüfung: 08.12.2022

Institut für Angewandte Analysis und Numerische Simulation

Universität Stuttgart

2022

This thesis is dedicated to my parents

Sigrid Brencher
and
Gerhard Brencher

who have given me invaluable encouragement and educational opportunities,

and to my wife

Larissa Maxine Brencher

for being my emotional anchor throughout this adventure.

Acknowledgements

First and foremost, I would like to express my deepest appreciation to my advisor, *Prof. Dr. Andrea Barth* for her guidance, critical feedback and numerous meetings helping me to make (some) sense of the confusion. It is a great pleasure to work with her and I am grateful for making this journey possible.

Special thanks should also go to the committee members, especially *Prof. Dr. Nils Henrik Risebro* and *Prof. Dr. Christian Rohde*, for their willingness to co-examine this dissertation.

I am also grateful to all my present and former colleagues in the research group of Computational Methods for Uncertainty Quantification at the University of Stuttgart for countless discussions throughout the last years. In particular, I would like to thank *Oliver König* for his suggestions on how to improve this thesis and *Samuel Burbulla* for helping me to get started with DUNE. I would like to extend my sincere thanks to *Dr. Anna Rörich* for innumerable running sessions that were a great balance for lots of frustration.

During my time as a research assistant, I have been partly funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC 2075 – 390740016. I acknowledge the support by the Stuttgart Center for Simulation Science (SimTech) at the University of Stuttgart.

Finally, I would like to thank all my colleagues at the institute for applied analysis and numerical simulation in Stuttgart for creating a pleasant atmosphere.

Contents

I Random conservation laws with a locally finite number of flux discontinuities

II Random conservation laws with infinitely many flux discontinuities

Abstract

Partial differential equations constitute a powerful tool to describe many natural phenomena. In a variety of real-world applications, occurring problems are modeled through hyperbolic conservation laws. Such models often involve experimental data to characterize physical properties, such as porosity or heat conductivity. To account for insufficient measurements or underlying uncertainties in this data, random coefficients may be incorporated. Additionally, these random fields may contain discontinuities to represent, e.g., heterogeneities or fractures of a (porous) medium. In oder to tackle these challenges, the key focus of this manuscript lies on scalar conservation laws with a *random discontinuous flux function* and the corresponding uncertainty quantification. This dissertation addresses the question of well-posedness of the resulting random problem as well as the numerical simulation of the corresponding solutions.

As a first contribution, suitable admissibility criteria for the resulting scalar conservation laws with a random discontinuous flux function are introduced and well-posedness is established. While the question of pathwise existence and uniqueness can be answered by means of the underlying deterministic setting, establishing strong measurability of the random solution requires special treatment. Standard techniques for showing strong measurability of solutions utilize continuous dependence results or leverage the deterministic existence proof. However, these procedures are not applicable in the random discontinuous flux setting or require very restrictive assumptions, such as *strong measurability* of the flux function. With general randomized positions of the flux discontinuities, such an assumption is out of reach and *merely measurable* flux functions can be expected at best. To establish strong measurability of random solutions for such merely measurable flux functions, a novel proof strategy is presented.

Moreover, the numerical approximation of solutions to the random discontinuous-flux conservation law is addressed. For these simulations and as an example of the developed theory, a *Lévy-type random field* is employed in the flux function. This coefficient is constructed via a (continuous) Gaussian part and a spatially discontinuous jump field. Consequently, the Lévy-type random field enables a more realistic modeling of, e.g., heterogeneities or fractures in a (porous) medium, as compared to state-of-the-art continuous coefficients. Numerical simulations demonstrate the ability of sample-adapted discretization schemes to approximate pathwise solutions of the resulting random discontinuous-flux conservation law. In particular, a novel *jump-adapted wave-cell meshing technique* is introduced, which reduces the samplewise variance of finite volume approximations by accounting for standing-wave profiles caused by the flux discontinuities. For estimating statistical moments of the solution, these pathwise approximations are combined with a fast and precise multilevel Monte Carlo method.

Zusammenfassung

Partielle Differentialgleichungen stellen ein wichtiges Mittel zur Beschreibung vieler natürlicher Phänomene dar. In einer Vielzahl von praktischen Anwendungen werden auftretende Probleme durch hyperbolische Erhaltungsgleichungen modelliert. Solche Modelle beinhalten häufig experimentelle Daten, um physikalische Eigenschaften wie beispielsweise Porösität oder Wärmeleitfähigkeit zu charaktierisieren. Um zugrundeliegenden Unsicherheiten oder unzureichenden Messungen in diesen Daten Rechnung zu tragen, können randomisierte Koeffizienten in die Modelle einbezogen werden. Um Heterogenitäten oder Risse in einem (porösen) Medium zu berücksichtigen, sollten diese Zufallsfelder Unstetigkeiten beinhalten dürfen. Motiviert durch diese Herausforderungen, liegt das Hauptaugenmerk dieses Manuskripts auf skalaren Erhaltungsgleichungen mit *zufälligen unstetigen Flussfunktionen* und der entsprechenden Unsicherheitsquantifizierung. Kurzgefasst befasst sich diese Dissertation mit der Frage der Wohlgestelltheit des resultierenden randomisierten Problems sowie mit der numerischen Simulation der entsprechenden Lösungen.

Zuerst werden geeignete Auswahlkriterien für die Lösung der resultierenden skalaren Erhaltungsgleichungen mit randomisierter unstetiger Flussfunktion eingeführt und die Wohlgestelltheit nachgewiesen. Während die Frage der pfadweisen Existenz und Eindeutigkeit mit Hilfe der zugrundeliegenden deterministischen Probleme und Lösungsansätze beantwortet werden kann, erfordert der Beweis starker Messbarkeit der zufälligen Lösung eine besondere Betrachtung. Standardmethoden für den Nachweis starker Messbarkeit von Lösungen nutzen Ergebnisse zu stetiger Abhängigkeit oder machen sich den deterministischen Existenzbeweis zu Nutze. Diese Verfahren sind bei zufälligen unstetigen Flüssen jedoch nicht anwendbar oder erfordern sehr restriktive Annahmen, wie etwa die *starke Messbarkeit* der Flussfunktion. Allerdings ist eine solche Annahme bei allgemeinen Flussfunktionen mit zufälliger Position der Unstetigkeiten unerreichbar und es können bestenfalls *messbare* Flussfunktionen erwartet werden. Um die starke Messbarkeit von zufälligen Lösungen für solche Flussfunktionen, die lediglich messbar sind, nachzuweisen, wird eine neuartige Beweisstrategie vorgestellt.

Darüber hinaus behandelt diese Arbeit die numerische Approximation von Lösungen zu zufälligen Erhaltungsgleichungen mit unstetigen Flüssen. Für diese Simulationen und als Beispiel für die entwickelte Lösungstheorie wird ein *Lévy-artiges Zufallsfeld* in der Flussfunktion verwendet. Dieser Koeffizient wird über einen (kontinuierlichen) Gaußschen Teil und ein Sprungfeld mit räumlichen Unstetigkeiten erzeugt. Mit Hilfe dieser Konstruktion ermöglicht das Lévy-artige Zufallsfeld eine realistischere Modellierung von Heterogenitäten oder Rissen in einem (porösen) Medium als dies mit modernen stetigen Koeffizienten möglich wäre. In numerischen Simulationen wird deutlich, dass Diskretisierungsmethoden, die an die spezielle Form der Stichproben angepasst sind, die Fähigkeit haben, die pfadweisen Lösungen der zufälligen Erhaltungsgleichung mit unstetiger Flussfunktion besser als Standardverfahren anzunähern. Inbesondere wird ein neuartiges *Sprung-adaptiertes Wave-cell meshing Verfahren* eingeführt, das

die sampleweise Varianz der Finite-Volumen-Approximation reduziert, indem stehende Wellenprofile berücksichtigt werden, die von den Unstetigkeiten der Flussfunktion verursacht werden. Zur Schätzung der statistischen Momente der Lösung werden diese pfadweisen Approximationen mit einer schnellen und präzisen multilevel Monte Carlo Methode kombiniert.

1 Introduction

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." [97]

This quote by Albert Einstein is particularly relevant for the field of mathematical modeling, which aims at developing theoretical frameworks to describe natural phenomena. Such mathematical models are employed in a wide range of disciplines such as natural sciences, engineering, computer science or even social sciences. In the spirit of Einstein [97], a "*law of mathematics*" should include uncertainty to be able to "*refer to reality*". This is typically achieved by randomizing model parameters. Discussing the consequences of such randomizations is a key area in the fields of stochastic analysis, uncertainty quantification and statistics. To describe phenomena considered in the natural sciences, it is essential to take the fundamental laws of nature into account, such as conservation laws. These state that some quantity (e.g., mass or energy) in an isolated system[1] does not change over time. Mathematically, conservation laws are described as dynamical systems. Specifically, they are formulated as (partial) differential equations.

In a variety of fields, conservation laws are employed as mathematical models. These include hydrodynamics (see, e.g., [61, 100]), nonlinear elasticity (e.g., [15, 65]), semiconductor simulation (e.g., [125, 195]) or kidney physiology (see, e.g., [232]). Nowadays, models are improved by including experimental data about physical properties, such as porosity or conductivity of a medium. Starting from the classical formulation of a conservation law, there are two natural extensions:

(i) The underlying material might be heterogeneous or may contain fractures. To represent these situations in the model, the flux function of the conservation law should be allowed to depend (possibly discontinuously) on the spatial and temporal variables. Such an approach has been considered in many applications, such as two-phase flow [9, 14, 122, 149, 154], traffic simulations [11, 53, 69] or sedimentation processes [49–51, 90, 91].

[1] An isolated system does not interact with its surroundings. While such a system is an idealization, it is frequently postulated to clarify the nature of physical laws.

(ii) Most of the time experimental data is only available at specific locations in the domain and outside of these points data is unknown. Additionally, the data might not be accurate due to insufficient measurements. Both of these phenomena can be captured in a mathematical model by incorporating uncertainty via random parameters. This approach to account for uncertain behavior has been carried out by including randomized flux functions [41, 42, 189, 190] or by considering stochastic source terms [139, 159, 163].

This thesis combines these two extensions by considering random conservation laws, whose flux function depends discontinuously on the spatial variable. Thereby, this work focusses on two objectives: Building an appropriate theoretical framework for the constructed mathematical model is combined with the development and improvement of numerical simulation methods that enable describing the solution via suitable approximations.

From a mathematical point of view, obtaining a proper model implies that the formulated problem is well-posed. In a deterministic framework, such well-posedness infers the existence of a unique solution. For random problems as they are considered in this work, it is also necessary to ensure that the stochastic solution is *strongly measurable* as a random variable. With this strong measurability, the solution can be described via its stochastic moments, such as expectation and variance.
The technical complexity in establishing the three properties of existence, uniqueness and strong measurability differ strongly: For most cases, the question of existence and uniqueness can be reduced to the deterministic setting by fixing a random parameter, such that the arguments leading to deterministic existence and uniqueness can be applied. Unfortunately, the combination of stochasticity and discontinuous flux functions prohibits the general usage of classical arguments, which can only be applied under extremly restrictive assumptions. To overcome this obstacle, a novel proof strategy is invented which ensures strong measurability of the random solution, even if the underlying flux functions are merely measurable.

Once a suitable theoretical framework is established, the numerical approximation of random solutions can be considered. Hereby, pathwise approximations should converge to the correct theoretical solution. This convergence usually depends on the specific parameter sample and as a consequence is stochastic. To improve these pathwise approximations, sample-adapted approaches are designed that account for the structure of each sample. In particular, such an approach can lead to a significant reduction of the samplewise variance.
As soon as pathwise approximations are available, stochastic moments of the random solution can be computed to describe the statistical behavior of the solution. For this purpose, methods for uncertainty quantification can be employed, such as (multilevel) Monte Carlo methods or stochastic Galerkin approaches. Combining these algorithms with the novel sample-adapted approximations enables fast and precise computations of moments of the random solution.

1.1 Deterministic conservation laws with discontinuous flux functions

Since the middle of the last century, hyperbolic conservation laws have aroused the interest of many researchers. Generally, these conservation laws formulate the problem of finding a unique solution u to the equation

$$\partial_t u + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t, \boldsymbol{x}, u) = 0 \tag{1.1}$$

on some space-time domain $\mathbb{X}_\mathbb{T}$ and equipped with an initial condition u_0. Here, $t \in \mathbb{T} \coloneqq [0,T]$ denotes the time variable and $0 < T < \infty$ is referred to as the final time. Further, $\boldsymbol{x}$ is a spatial point of some domain $\mathbb{X}$ and $\mathfrak{f}$ describes the flux of the unknown u through the domain $\mathbb{X}$.

It is well known that such scalar conservation laws may lead to solutions that develop spatial discontinuities in finite time. This phenomenon is caused by appearing shock-waves and can even be observed in case of smooth flux functions and initial conditions. In general, weak solutions to these equations are not unique. Therefore, an additional selection criterion is necessary that selects the physically meaningful solution. For the case of sufficiently smooth flux functions $\mathfrak{f}$, such a selection principle was first introduced by Kružkov [173] in 1970. Considering the special case of $\mathbb{X} = \mathbb{R}^d$ and $\mathbb{T} = \mathbb{R}_{>0}$, he demanded that a solution u satisfies the entropy inequality

$$\begin{aligned}\partial_t \big|u(t,\boldsymbol{x}) - k\big| + \operatorname{div}_{\boldsymbol{x}} \Big(\operatorname{sign}\big(u(t,\boldsymbol{x}) - k\big)\big(\mathfrak{f}\big(t,\boldsymbol{x},u(t,\boldsymbol{x})\big) - \mathfrak{f}(t,\boldsymbol{x},k)\big)\Big) \\ + \operatorname{sign}\big(u(t,\boldsymbol{x}) - k\big) \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}\big(t,\boldsymbol{x},k\big) \leq 0\end{aligned} \tag{1.2}$$

for every constant $k \in \mathbb{R}$ in the sense of distributions. That means that for any smooth nonnegative test function $\psi \in \mathcal{D} \coloneqq \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}_{>0} \times \mathbb{R}^d; \mathbb{R})$, the following inequality is satisfied:

$$\begin{aligned}\int_{\mathbb{X}} \big|u_0(\boldsymbol{x}) - k\big|\psi(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{X}_\mathbb{T}} \big|u(t,\boldsymbol{x}) - k\big|\partial_t\psi(t,\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x} \\ + \int_{\mathbb{X}_\mathbb{T}} \Big(\operatorname{sign}\big(u(t,\boldsymbol{x}) - k\big)\big(\mathfrak{f}(t,\boldsymbol{x},u(t,\boldsymbol{x})) - \mathfrak{f}(t,\boldsymbol{x},k)\big)\Big) \cdot \nabla\psi(t,\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x} \\ + \int_{\mathbb{X}_\mathbb{T}} \operatorname{sign}\big(u(t,\boldsymbol{x}) - k\big) \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k)\psi(t,\boldsymbol{x}) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{x} \geq 0 \,,\end{aligned}$$

where we have set $\mathbb{X}_\mathbb{T} \coloneqq \mathbb{R}_{>0} \times \mathbb{R}^d$ to denote the space-time domain. Additionally to this inequality, the solution u has to converge to the initial condition u_0 in the $\mathcal{L}^1$ sense, which means that we have

$$\lim_{t \searrow 0} \int_{\mathbb{R}^d} |u(t,\boldsymbol{x}) - u_0(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} = 0 \,.$$

In the work of Kružkov [173], the flux $\mathfrak{f}$ is assumed to be Lipschitz continuous in all variables. It provides the framework of *Kružkov entropy solutions* that can be represented via *entropy solution semigroups* having the $\mathcal{L}^1$-contraction property. The modern theory for Equation (1.1) goes back to the ideas of Hopf [142], Oleïnik [224, 225], Vol'pert [278] and Kružkov [173]. For details on these approaches, we refer to the fundamental monographs [43, 80, 181, 257].

Many classical models naturally lead to Lipschitz continuous flux functions, for example the Lighthill-Whitham-Richards (LWR) model for vehicular traffic flow [81, 187, 243] or the Buckley-Leverett model for two-phase flow in porous media [37, 44, 82]. However, in order to develop more accurate models one may want to consider flux functions that are discontinuous. This line of research has attracted a lot of attention during the last two decades. In general, the discontinuity can appear in the space-time dependency or in the unknown of the flux (or a combination thereof). In the following, we discuss the ideas and advances of both cases to summarize the current state of research in this field. Nevertheless, this dissertation limits its scope to a discontinuous dependence regarding the spatial dependency[2].

[2] In the first part of this work, the spatial discontinuity is allowed to depend on time, leading to spatio-temporal interfaces. We refer to Part I for the formal setting and the discussion of various discontinuity geometries.

1.1.1 Flux functions discontinuous in the solution

Flux functions that depend discontinuously on the solution can be applied in a variety of fields to improve models which have the goal of capturing real-world behavior. For example, in the LWR model, the flux describes the velocity of traffic flow. Here, discontinuities at specific vehicle density can model the transition between free and congested traffic areas [48, 271]. Flux functions, which are discontinuous in the solution, also occur in models with implicit constitutive relations[3], e.g., if one wants to model problems of elasticity with discontinuous relations between the Cauchy stress tensor and the deformation gradient [45, 126].

If flux functions are discontinuous in the solution, the phenomenon of *zero waves* arises, which are waves traveling at infinite speed. The first to consider such problems was Gimse [120] in 1993. He investigated the occurrence of these zero waves and solved the corresponding Riemann problem via a front tracking algorithm. Later, Carillo [57] extended this work and proved existence and uniqueness of solutions using comparison principles and a modified entropy inequality. Another approach to existence was proposed by Dias and Figueira [87–89], who considered a mollified problem and showed that solutions to this regularized problem converge to solutions of the original problem. They also defined a notion of weak entropy solutions and developed a numerical method for solving the occurring Riemann problem. This approach was later applied by Wiens et al. [285] to a piecewise linear flux function with a single discontinuity. Another approach that considers the convergence of approximate solutions in the case of continuous flux functions, was presented by Lu et al. [192]. Based on the vanishing viscosity approach [38], Coclite et al. [70] showed the existence of solutions, where the main idea is to establish convergence of approximate solutions via an estimate for functions having bounded variation ($\mathcal{BV}$) and applying Murat's compact embedding [217].

A different approach to well-posedness was introduced by Bulíček et al. [45]. They extended the concept of weak entropy solutions and measure-valued solutions, such that it covers flux functions which are discontinuous in the solution, but reduces to the standard notions if the flux is continuous. This idea was extended in [46] to the case of flux functions discontinuous in both the unknown and the spatial variable. In a similar setting, Gwiazda et al. [126] showed existence and uniqueness of solutions by formulating the problem in the framework of multi-valued mappings. The assumptions for this approach were relaxed in [47] by Bulíček et al., who also derived an equivalent description of the solution via a kinetic formulation.
From a methodical point of view, the ideas for proving well-posedness have a remarkable common ground: They are based on developing techniques, such that the original ideas of Kružkov entropy solutions [173] or of measure-valued solutions [95, 265] can be exploited. The usual approach in the literature is based on a monotone, possibly multi-valued change of the unknown, see e.g. [45, 46, 126].

Additionally to the theoretical perspectives, the field of conservation laws with discontinuous flux functions has attracted a lot of attention for designing suitable numerical approximation methods. Since the zero waves travel with infinite speed, employing explicit schemes seems hopeless, because these waves contain information about the flux, but are transported instantaneously. Martin and Vovelle

[3] The implicit constitutive theory aims at describing relations between the stress and kinematic equations in an implicit manner to model, e.g., viscoelastic phenomena. This is contrary to classical constitutive models such as the Navier-Stokes fluid model [73, 176, 220, 264, 267] or the Hookean model for solid materials [141, 213, 250]. For details on implicit constitutive theory, we refer to the works of Rajagopal [238–240].

[196] designed a finite volume scheme and showed convergence to the unique entropy solution. Recently, Towers [271] presented a finite difference scheme based on a flux splitting.

1.1.2 Flux functions with space-time discontinuities

When modeling heterogeneous material properties in the scalar conservation law given by Equation (1.1), one is interested in spatio-temporal discontinuities of the flux function. Via such discontinuities sudden changes in the heat conductivity or permeability at interfaces or fractures may be described. This line of research has aroused more attention of researchers than discontinuities in the solution, since the challenges in developing a well-posedness theory change drastically. For flux functions containing a discontinuity in the spatial dependency, the Kružkov entropy Condition (1.2) is not applicable, since the last term of Inequality (1.2) is not properly defined anymore. As a result, new admissibility conditions are necessary, which select the sought notion of solution. Therefore, formulating appropriate selection criteria becomes part of modeling the considered physical problem. Consequently, the selected solution to the same equation may vary for disparate applications[4]. Here, an important result was established by Adimurthi et. al. [3] in 2005, which states the following:

$$\begin{gathered} \textit{There exist infinitely many different,} \\ \textit{but mathematically equally consistent notions of solutions.} \end{gathered} \tag{1.3}$$

Each of these notions form an $\mathcal{L}^1$-contractive *entropy solution semigroup*, which can be characterized by an individual admissibility condition. In the literature, mainly two types of selection criteria are distinguished:

(i) Admissibility conditions that invoke the classical Kružkov entropy condition (1.2) outside of the discontinuities and impose an additional penalty term along the discontinuity interfaces.

(ii) Admissibility conditions that *adapt* the classical Kružkov entropy condition (1.2) in a suitable way, such that it defines a meaningful criterion in the presence of discontinuities. This approach leads to so-called *adapted entropy conditions*.

For both types of selection criterion there exists a variety of admissibility conditions. We summarize the major ideas of both approaches in the following. Let us stress that the distinction between these two types remains important throughout this manuscript.

Admissibility conditions introducing an interface penalty term

The main and common idea of admissibility conditions based on interface penalty terms is to enforce the Rankine-Hugoniot condition[5] [146, 147, 241] across every flux discontinuity. This condition ensures continuity of the flux at the heterogeneities by requiring the following: The left trace of the flux function evaluated at the solution value left of the discontinuity has to be equal to the right trace with

[4] An extensive example for such a situation, in which two different admissibility conditions for the same equation lead to different solutions, is given in [234, Section 1.2].

[5] For the rigorous definition of the Rankine-Hugoniot condition, we refer the reader to the presentation in Section 3.2 and in particular to Equation (3.9) in Definition 3.14.

corresponding right solution value. In order to define the appropriate traces, each flux discontinuity needs to be considered separately. In particular, this implies that the flux discontinuities occur at isolated points (for one space dimension) or at locally finite interfaces (for multiple space dimensions or space-time discontinuities).

Probably the most considered admissibility condition approach is based on vanishing viscosity limit solutions for conservation laws. This line of research started with Gimse and Risebro [120–122] in 1991. At the beginning of this century, Vasseur [276] and Seguin and Vovelle [255] started the study of this type of admissibility condition for scalar conservation laws with discontinuous coefficients. At the same time, Karlsen et al. [158] studied the $\mathcal{L}^1$-stability for solutions of degenerate parabolic convection-diffusion equations with discontinuous coefficients. In addition, the authors extend the classical Kružkov condition in [155] to coefficients, whose discontinuities are located in the space-time plane. In 2006, Bachmann and Vovelle [20] proved the existence and uniqueness of entropy solutions to discontinuous-flux scalar conservation laws via a kinetic formulation. This bypasses additional assumptions on convexity or genuine nonlinearity, which were required before. The question of existence and uniqueness of entropy solutions for conservation laws was also considered in a bounded domain and was positively answered by Jimenez and Lévi in [150, 152]. Here, we also want to mention the work [9], in which different vanishing capillarity limits of the one-dimensional Buckley-Leverett equation were considered based on various choices of the physically relevant vanishing capillarity.

The work on vanishing viscosity solutions has also been extended to multi-dimensional scalar conservation laws with discontinuous flux functions. Here, we want to highlight the early works of Andreianov et al. [12] and Panov [230]. In [151], the vanishing viscosity method was used to prove existence of entropy solutions in multi-dimensional bounded domains. For arguing the uniqueness, the *doubling of variables* approach was combined with a pointwise reasoning along the discontinuity curve. In 2013, the setting of multi-dimensional discontinuous-flux conservation laws was extended to the case of discontinuities in the spatial variable and in the solution by Bulíček [46].
The vanishing viscosity method is closely related to the work of Diehl, who selected unique solutions via a Γ-*condition* [90, 91]. This condition was also derived from investigating viscosity profiles and their stability [92, 94].

Another important admissibility criterion is the Karlsen-Risebro-Towers condition. This formulation was initiated by Klingenberg and Risebro [166], who employed a wave entropy condition to select a unique entropy solution. The stability of entropy solutions with this type of admissibility condition was investigated by Klausen and Risebro [165]. Based on this idea, Towers [268] defined a notion of entropy solutions, in which the entropy condition is embedded in the weak formulation instead of local restrictions. While the flux in [268] is required to be convex, this assumption was dropped in [269]. Recently, Towers [270] extended this framework to the case of time and space discontinuities. For fluxes of bounded variation, Piccoli and Tournus [234] proved a general existence result under the assumption of concavity of the flux. This assumption on concavity of the flux can be omitted due to the recent result of Towers in [272].

The above ideas have been generalized by Panov [231], who introduced, for every $k \in \mathbb{R}$, the entropy inequality

$$\begin{aligned}\partial_t |u(t,\boldsymbol{x}) - k| + \operatorname{div}_{\boldsymbol{x}} \Big(\operatorname{sign}\big(u(t,\boldsymbol{x}) - k\big)\big(\mathfrak{f}(t,\boldsymbol{x},u(t,\boldsymbol{x})) - \mathfrak{f}(t,\boldsymbol{x},k)\big)\Big) \\ \leq -\operatorname{sign}(u(t,\boldsymbol{x}) - k)\,\big(\operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k)\big)^{ac} + \big|\big(\operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k)\big)^{s}\big|\end{aligned} \tag{1.4}$$

in the sense of distributions. Here, the Jordan decomposition [7, 72, 127] of a Radon measure has been used to split $\operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k)$ up into its absolutely continuous and its singular part:

$$\operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k) = \big(\operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k)\big)^{ac} + \big|\big(\operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(t,\boldsymbol{x},k)\big)^{s}\big| \,.$$

With this condition, Panov [231] was able to proof existence of solutions via the vanishing viscosity method. However, in the entropy inequality (1.4), the contribution of the flux discontinuities are estimated quite roughly. This is a major obstacle for proving uniqueness of solutions, which—to the best of the author's knowledge—is still an open problem.

A more sophisticated way of accounting for the flux discontinuities was proposed by Andreianov and Mitrovic [10] based on the unpublished idea of Mitrovic [212], who replaced the entropy Inequality (1.4) by

$$\begin{aligned}\partial_t|u(t,\boldsymbol{x})-k| + \operatorname{div}_{\boldsymbol{x}}\Big(\operatorname{sign}\big(u(t,\boldsymbol{x})-k\big)\big(\mathfrak{f}(t,\boldsymbol{x},u(t,\boldsymbol{x}))-\mathfrak{f}(t,\boldsymbol{x},k)\big)\Big)\\ \leq -\operatorname{sign}(\mathfrak{p}_u-k)\operatorname{div}_{\boldsymbol{x}}\mathfrak{f}(t,\boldsymbol{x},k)\,.\end{aligned} \tag{1.5}$$

Here, $\mathfrak{p}_u$ is a globally defined Borel-measurable function that satisfies $\mathfrak{p}_u = u$ almost everywhere with respect to the Lebesgue measure of the space-time domain. Note, for general flux functions one needs to make sense of the last term in (1.5). The required tools have been developed in [75] using advanced tools of analysis of either functions having bounded variation ($\mathcal{BV}$) or special functions of bounded variation ($\mathcal{SBV}$)[6]. For details on the $\mathcal{BV}$ and $\mathcal{SBV}$ framework, we refer the reader to the original work of Vol'pert [278] or to the monographs of Halmos [127], Ambrosio et al. [7] or Evans and Gariepy [99]. However, we will not discuss this approach any further in this dissertation.

For proving well-posedness, all of the above approaches need to guarantee uniform $\mathcal{L}^\infty$-bounds on approximate solutions. The easiest assumption to ensure this is by requiring, e.g.,

$$\mathfrak{f}(\cdot,\cdot,0) = \mathbf{0}_{\mathbb{R}^d} = \mathfrak{f}(\cdot,\cdot,1) \qquad 0 \leq u_0(\cdot) \leq 1. \tag{1.6}$$

Such an assumption is natural for models describing a relative density, such as road traffic or porous medium models. However, in general, this is a main restriction on the flux function.

To conclude this presentation of selection criteria, which are based on describing conditions at the discontinuity interface, we want to mention the work of Andreianov et al. [13]. The purpose of [13] is to develop a framework which *unifies* the above approaches via the notion of so-called *admissibility germs*. This theory combines the admissibility approaches for conservation laws with discontinuous flux functions, but also includes earlier selection criteria developed for continuous flux functions. For details, we refer the reader to Section 3.2, where these germs are employed for specifying admissible solutions to random conservation laws.

Admissibility conditions adapting the classical Kružkov condition

Requiring every discontinuity point of the flux function to occur on isolated points clearly is a major restriction. The additional requirement of a confinement assumption in the sense of the Estimation (1.6)

[6] There has been an increasing activity in analyzing scalar conservation laws with tools from the $\mathcal{BV}$ and $\mathcal{SBV}$ framework, see [74, 76]. In general, the results, techniques and assumptions are complementary to the methods presented above: The flux $\mathfrak{f}$ may be less regular, but the solution u is more regular than in the discussed approaches.

can be an even greater limitation for possible real-world applications. A framework that overcomes both drawbacks was proposed by Audusse and Perthame [18] in 2005, based on the ideas of Baiti and Jenssen [23] from 1997. Here, the framework is formulated without additional interface terms and as a result the existence of traces is not required. This makes it possible to consider an infinite number of discontinuity points in the flux function. However, the general framework was derived for one-dimensional problems and extending it to multiple dimensions is still an open problem. Moreover, the considered discontinuities are assumed to be stationary, i.e., independent of the temporal variable.

The basic idea of the *adapted entropy* framework is to replace the constants $k \in \mathbb{R}$ in (1.2) with a family of functions $k_\alpha(x)$, $\alpha \in \mathbb{R}$. Here, the function $k_\alpha(x)$ is the unique solution of the steady-state problem

$$\mathfrak{f}(x, k_\alpha(x)) = \alpha \qquad \text{for a.e. } x \in \mathbb{R}\,.$$

One approach to extend this framework to the multi-dimensional case was proposed by Panov [230] in 2009. He considered the special class of flux functions satisfying

$$\mathfrak{f}(\boldsymbol{x}, u) = \boldsymbol{g}\big(\mathfrak{Z}(\boldsymbol{x}, u)\big), \tag{1.7}$$

where $\boldsymbol{g}(\mathfrak{Z}) \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ is a continuous function and $\mathfrak{Z}(\boldsymbol{x}, u)$ is a Carathéodory function, which means that $\mathfrak{Z}$ is measurable in $\boldsymbol{x}$ and continuous in u. For this class of flux functions, Panov showed existence and uniqueness of solutions even in the multi-dimensional setting. Additionally, he proved for the one-dimensional case that a change of unknowns yields equivalence between the Audusse-Perthame adapted entropy solution and the classical Kružkov entropy solution.

For flux functions of Panov-type (1.7), Ghoshal et al. [116] proved uniqueness for adapted entropy solutions via a different proof strategy. This uniqueness proof is based on adapting the rather complicated version of *doubling of variables* of [18] to the multi-dimensional setting. The existence in [116] is proved via the convergence of Godunov-type finite volume schemes, cf. Section 1.4. In the one-dimensional setting, Ghoshal et al. [115] extended the existence, uniqueness and $\mathcal{BV}$ regularity results to discontinuous fluxes with possibly flat regions. Furthermore, for hyperbolic conservation laws with spatial discontinuities, Ghoshal and Jana [113] investigated the optimal jump set of entropy solutions and some qualitative properties of these solutions.

1.2 Stochastic conservation laws

As mentioned in the beginning of this chapter, realistic models should contain information about underlying uncertainties. These may arise via uncertain initial or boundary conditions, random system parameters or stochastic external forcing. Including these uncertainties in scalar conservation laws has attracted the attention of many researchers in the last two decades. To summarize the work in this area, we first consider conservation laws with stochastic forcing in Section 1.2.1. Afterwards, in Section 1.2.2, we present the results on stochastic or random flux functions.

1.2.1 Conservation laws with stochastic forcing

In the area of stochastic source terms, the concept of stochastic weak and entropy solutions was initially introduced by Holden and Risebro [139] for nonlinear hyperbolic problems. For additive noise

terms, this concept was further developed by Kim [163], who applied a change of variables to use the classical Kružkov entropy condition for showing well-posedness. We also refer to Vallet and Wittbold [275] for this line of research.

However, the approach via a change of variables is not applicable if the noise is multiplicative. Here, Feng and Nualart [103] overcame this issue by introducing an additional, rather technical condition. In their work [103], Feng and Nualart consider multiplicative Gaussian noise, which is almost surely continuous. In case of the random force being driven by a Brownian motion, Chen et al. [62] investigated vanishing viscosity approximations of stochastic balance laws and established existence of entropy solutions as well as continuous dependence results. For general noise driven by a geometric p-rough path, Friz and Gess [107] proved existence and uniqueness. Furthermore, they obtained stability of the solutions with respect to the driving noise. In 2012, Bauzet et al. [35] provided a framework that bypasses the additional condition from Feng and Nualart [103]. To do so, they establish existence and uniqueness of entropy solutions via the entropy formulation of Kružkov and the notion of measure-valued solutions. Recently, this approach was generalized by Karlsen and Storrøsten [159], who included Malliavin differentiable random variables in the Kružkov entropy condition. In this context, they also provide existence and uniqueness results.

Instead of considering hyperbolic conservation laws, a number of works studied the related case of degenerate parabolic partial differential equations in the stochastic setting. In this field, Debussche and Vovelle [83, 84] proved the well-posedness for periodic multi-dimensional scalar first-order conservation laws with stochastic forcing. To show this, the authors leverage a kinetic formulation of the problem. This approach was extended by Hofmanová in [137] to scalar semilinear equations. Here, Hofmanová adapted the kinetic formulation to prove a comparison principle and employed the vanishing viscosity method for showing existence of solutions. Both above works were generalized in [86], where the authors developed a well-posedness theory that includes the $\mathcal{L}^1$-contraction property and simplifies the proof of the previous works. These results were further generalized by Gess and Hofmanová in [111], who also established regularity results via averaging techniques. At this point, let us mention that Hofmanová also investigated the approximation of the kinetic solution via *Bathnagar-Gross-Krook* approximations in [138]. Instead of employing a kinetic formulation for degenerate parabolic equations, Bauzet et al. [36] showed existence and uniqueness of a solution by using an adapted entropy formulation.

As a last field of research on conservation laws with stochastic forcing, we want to mention results on invariant measures. The work in this area goes back to the important paper of Weinan et al. [282], who analyzed the invariant measure for the periodic inviscid Burgers' equation with stochastic forcing in one space dimension. This work has been extended to higher dimensions in [148] and to the case of fractional noise in [253]. Recently, in 2015, Debussche and Vovelle [85] studied the invariant measure for periodic scalar first-order conservation laws with stochastic forcing in any space dimension.

1.2.2 Conservation laws with random flux functions

To the best of the author's knowledge, Wehr and Xin [281] were the first to consider conservation laws with a random flux function. Specifically, they investigated the long-time asymptotics of the front speed in the Burgers' equation. However, general well-posedness theory for random flux functions has only gained attention in the last years, starting with the work of Mishra et al. [211], who defined a

notion of random entropy solution. Here, the authors work with random data that allows to employ pathwise existence and uniqueness results from the deterministic theory via continuous dependence results.

Random flux functions were also considered by Lions et al. [189] for the case of rough (stochastic) fluxes. This concept was extended in [190] to the case of space-dependent rough fluxes. Recently, Hoel et al. [134] investigated the spatial regularity of such solutions and derived fine properties of the stochastic solution map for fixed time. The combination of such rough stochastic flux functions with stochastic forcing was considered by Hofmanová [136], who proved well-posedness of the corresponding kinetic formulation and its solution. In 2017, Gess and Souganidis [112] investigated the long-time behavior and regularity of pathwise entropy solutions. Therefore, they considered spatially homogeneous, but random-in-time flux functions. Recently, Müller and Bock [215] applied the setting of stochastic flux functions to the Lighthill-Witham-Richards traffic model.

1.3 Extension to random discontinuous flux functions

The existing (theoretical) results of the previous subsections focus either on the stochasticity or on the flux discontinuities. Therefore, this dissertation aims at extending these results to flux functions containing stochastic discontinuities. Traditionally, the strong measurability of solutions in the case of stochastic flux functions is proven via continuous dependence results. However, this line of argumentation requires the stochastic flux function to be strongly measurable, which imposes a major restriction for possible fluxes[7]. In this setting of strongly measurable stochastic fluxes, Badwaik et al. [22] proposed a multilevel Monte Carlo finite volume method to approximate moments of random scalar conservation laws.

To overcome this restriction in the measurability proof, we employ advanced analytical tools to develop a novel proof strategy. This approach via set-valued mappings and so-called *entropy functionals* allows the flux function to be merely measurable, while guaranteeing the strong measurability of the underlying entropy solution. To round off the investigation of random discontinuous flux functions, numerical simulations are conducted that approximate the random entropy solutions. Here, a special class of random discontinuous flux functions is considered, in which the randomness and the discontinuous dependence is introduced via a stochastic jump coefficient. This random coefficient is designed to allow very flexible modeling [27, 30] and consists of a continuous (Gaussian) part and a discontinuous (jump) part. Such a construction is inspired by the Lévy-Khintchine formula [16, 161], which states that every Lévy process can be uniquely characterized via the composition of three independent components, namely, a drift term, a Brownian motion and a pure jump process.

In the numerical experiments, we investigate the strong convergence of approximations to the unique entropy solution. Additionally, we present novel discretization techniques that reduce the variance of approximations and thus can significantly improve the strong convergence rate. Furthermore, these pathwise methods are combined with (multilevel) Monte Carlo methods to enable fast and precise uncertainty quantification.

[7] In general, requiring strong measurability of the stochastic discontinuous flux function prohibits the use of random discontinuities. Therefore, the class of permitted flux functions is much smaller compared to merely measurable flux functions. For completeness, a counterexample demonstrating the lack of strong measurability for a flux functions having stochastic discontinuities is presented in Appendix B.

1.4 Numerical simulation of (random) conservation laws with discontinuous flux function

As mentioned in the last section, we conduct numerical simulations on the analyzed random conservation laws. Therefore, this section summarizes the current state-of-the-art developments of numerical methods for discontinuous-flux conservation laws in Section 1.4.1. Afterwards, in Section 1.4.2, modern methods for uncertainty quantification are presented, which allow the computation of statistics of a quantity of interest. Here, such a quantity of interest can be, e.g., moments of the stochastic solution or a functional applied to it. Finally, Section 1.4.3 summarizes approaches for combining these two types of numerical methods for the approximation of stochastic entropy conditions.

1.4.1 Numerical approximation of deterministic conservation laws with discontinuous flux function

The numerical investigation of conservation laws with discontinuous flux functions started three decades ago with the early works of Gimse [120–122], who used the *front tracking method* introduced by Dafermos [79]. The idea of the front tracking method is an explicit treatment of appearing interfaces in the solution. Therefore, a piecewise constant approximation of the solution is considered, for which the resulting independent Riemann problems can be solved exactly. The solutions to those Riemann problems consist of a series of shocks, each traveling with constant speed. Therefore, the Riemann problems have to be solved with updated values, if two shock fronts meet. Consequently, the front tracking approximation yields a unique entropy solution for all times. This approach has also been successfully applied to prove existence of solutions, see, e.g., [23, 49, 68, 115, 166, 167, 207, 234]. For an extensive overview on the front tracking method, we refer to the monograph of Holden and Risebro [140]. Recently, Ruf [249] was able to prove a first-order convergence rate of front tracking approximations for the case of discontinuous-flux conservation laws. Such a first-order convergence had already been proved by Lucier [193] in $\mathcal{L}^1$ and Solem [261] in the p-Wasserstein distance for the case of flux functions with no spatial dependency.

Another numerical method that has been proposed and analyzed for the approximation of discontinuous-flux conservation laws is the *finite volume method*. Here, the basic idea is to integrate the governing equation over a small volume, the so-called *control volume*, and apply the divergence theorem to obtain integrals over the surface of these volumes. This approach yields a piecewise constant approximation with discontinuities along the volume surfaces. To describe the evolution of this approximation over time, a *numerical flux* is employed that describes the *transport of information* between two adjoint control volumes. An important property of the finite volume method is its local conservativity.

Finite volume schemes with an upwind-type flux (e.g., Godunov or Engquist-Osher scheme) have been successfully applied to conservation laws with discontinuous flux functions, see, e.g. [4, 54, 157, 283]. For this type of fluxes Badwaik and Ruf [21] showed an optimal convergence rate of $1/2$ for the finite volume method. As for the front tracking method, various finite volume methods have been used to prove the existence of entropy solutions, see, e.g., [114, 268, 269, 272]. Let us also mention the works of Karlsen [155, 156] and Mishra [204, 206], who investigated the convergence of finite volume schemes for various numerical fluxes and entropy conditions. Note that the above list is far from being extensive. For a comprehensive discussion on the various numerical approaches to conservation laws

with a possibly discontinuous flux, we refer the reader to the review paper of Mishra [205] on finite volume methods or to numerous monographs on the topic, such as Kröner [170], LeVeque [182, 183], Hirsch [133] or Hesthaven [131].

1.4.2 Computational methods in uncertainty quantification

Methods for uncertainty quantification are used to describe the statistics of the considered stochastic model. Oftentimes, the problem is formulated in a way such that the sought quantity is the expected value of a suitable chosen random variable. In general, one can discriminate two types of approaches to uncertainty quantification: statistical and non-statistical methods.

Non-statistical methods for uncertainty quantification

The idea of non-statistical methods is to approximate the random space to transform the stochastic problem into a higher-dimensional deterministic one. A popular approach in this direction is the *stochastic collocation* method [259, 260, 287], which uses a polynomial interpolation of the stochastic space. Typically, Lagrange polynomials are employed and the interpolation is chosen to satisfy the stochastic model at prescribed collocation points. As a result, a set of collocated deterministic problems has to be solved. The stochastic collocation method has successfully been applied to random partial differential equations, especially if the uncertainty is introduced via stochastic input data [19, 221, 288, 289].

Another idea for non-statistical approaches is the *stochastic Galerkin* method [177, 178]. Here, the problem is considered in a weak formulation with respect to the random variables. This approach is based on the *polynomial chaos expansion* [63, 77, 223, 256, 284, 289] and the underlying polynomials are chosen to be orthonormal with respect to some inner product on the stochastic space. Using these polynomials in the weak formulation yields a highly coupled deterministic problem, which has to be solved to approximate the stochastic problem. The stochastic Galerkin method has been applied to stochastic partial differntial equations, see, e.g., [33, 96, 203] and the references therein.

The above non-statistical approaches aim at exploring structural properties of the underlying problem. Let us mention that these methods are superior to statistical algorithms, if the stochastic dependence of the problem is sufficiently smooth [71, 254]. However, for the setting of this dissertation, they have two major drawbacks:

(i) Both methods suffer the *curse of dimensionality*, which leads to an enormous computational effort for high-dimensional stochastic spaces. However, such high-dimensional spaces are necessary to accurately describe many important random fields.

(ii) For discontinuous coefficients or random fields, it is still an open question to find a suitable basis that can represent these stochastic objects.

Statistical methods for uncertainty quantification

The most intuitive and straightforward approach for uncertainty is the *Monte Carlo* method, which is based on ideas of Ulam [202, 274], von Neumann [279] and Fermi [104]. We refer to [201] for the historical development and to [105, 124, 128, 169, 216, 247] for a general introduction of this method. It is based on the strong law of large numbers and consist of drawing a large number of independent samples

from the random variable's distribution and computing the arithmetic mean over all these samples to approximate the expected value. This approach has the advantage that it does not depend on the dimension of the stochastic space and thus does not suffer the curse of dimensionality. Unfortunately, this property comes at the price of a slow convergence rate of order $1/2$. As a result, the Monte Carlo method is not feasible for problems in which the computation of one sample is very expensive. To overcome this obstacle, a variety of techniques has been developed to increase the efficiency of the Monte Carlo method.

An effective approach is the *multilevel Monte Carlo (MLMC)* method, which has been invented by Heinrich [130] and further developed by Giles [118, 119]. The main idea is to consider a hierarchy of discretizations and corresponding approximated random variables. A large number of samples with low accuracy is sampled, which is computationally inexpensive and reduces the statistical error. The induced approximation error resulting from coarse discretizations is then corrected by sampling the difference between two adjoined discretizations of the same sample. Here, the number of computed samples decreases with finer discretization size.
The multilevel Monte Carlo method has successfully been used to significantly reduce computational time. In the last decade, this method has been applied to a variety of problems of uncertainty quantification [1, 26, 31, 32, 34, 60, 78, 266]. Among these applications, we want to highlight the works of Barth and Stein [27, 263], in which the MLMC method has successfully been applied to partial differential equations involving discontinuous Lévy-type coefficients. Since the MLMC method is able to handle high-dimensional stochastic domains and improves the computational time significantly compared to the standard Monte Carlo approach, it is the method of choice for approximating moments of the solution to conservation laws in this dissertation. We refer to Section 2.5.1 for details on this algorithm.

1.4.3 Numerical computation of random entropy solutions

The methods for uncertainty quantification, presented in the last section, have also been applied to conservation laws to compute random entropy solutions or moments thereof. In an early work, Poëtte et al. [235] used the stochastic Galerkin method for a stochastic system of conservation laws. In [273], the authors developed an adaptive stochastic Galerkin method to approximate scalar conservation laws with uncertain input data. Here, a finite volume algorithm was employed to solve the resulting deterministic system. In 2014, Bürger et al. [55] developed a hybrid adaptive stochastic Galerkin method, which yields a partially decoupled system.

The multilevel Monte Carlo (MLMC) method has been successfully combined with a variety of deterministic methods for conservation laws, the most popular being the finite volume method. Here, Mishra and Schwab [208] developed a sparse MLMC finite volume approach for random initial data. This method was extended to systems [209] and applied to the shallow-water equations [210]. In 2018, Barth and Kröker [25] employed an MLMC finite volume method to conservation laws with spatial noise. We also refer to [168] for a multilevel Monte Carlo method for random scalar degenerate convection-diffusion equations based on a finite difference method. In [211], an MLMC finite volume method was used to approximate scalar conservation laws with random (spatially homogeneous) flux functions. Recently, Badwaik et al. [22] considered the MLMC method for random conservation laws with strongly measurable discontinuous flux functions[8].

[8] Recall that this strong measurability assumption is a major restriction (cf., Footnote 7 and Appendix B).

For other deterministic methods being combined with the multilevel Monte Carlo approach, we refer to [244] for an MLMC front tracking method for random conservation laws. Here, the flux function was considered to be stochastic, but without spatial dependency. In [29], the MLMC approach was combined with a Discontinuous Galerkin ansatz for semilinear hyperbolic problems with Lévy noise. For a general overview on methods for uncertainty quantification applied to conservation laws, we refer to the monograph [153] and to the review paper [2].

1.5 Structure and notation

As previously mentioned, one goal of this dissertation is to develop a theoretic framework for random conservation laws with discontinuous flux functions. Once such a framework is established, the second goal is to employ and develop efficient numerical methods to approximate the random solution as well as enabling its uncertainty quantification. The main body of this dissertation consist of two parts, which differ by the considered admissibility conditions:

(i) Part I establishes the theoretical framework and numerical methods for admissibility conditions with locally finite discontinuities (and specific flux interface conditions).

(ii) In the second part, the theoretical framework and numerical methods for admissibility conditions with an infinite number of discontinuities is developed.

In both parts, first the theoretical results are established and afterwards numerical simulations are conducted that investigate the behavior of the methods as well as demonstrate their ability of significantly increasing the efficiency of simulations.

Before we start with the main body of this thesis, Chapter 2 familiarizes the reader with important concepts. While the focus is on fundamental theoretical results that are necessary for proving well-posedness of random scalar conservation laws, this chapter also introduces the approximation schemes that are used in the numerical experiments throughout this manuscript.

Part I: Random scalar conservation laws with a locally finite number of flux discontinuities

In the first Part of the thesis, we consider random scalar conservation laws that have a sole flux discontinuity. This simple geometry of the flux interface divides the space-time domain into two parts and is a suitable model problem for introducing the main concepts for admissibility of solutions in Chapter 3. We start the discussion with defining and investigating the flux discontinuities and properties thereof. Based on these constructions, admissibility criteria for solutions are developed, which are utilized for showing well-posedness of solutions afterwards. Chapter 3 is concluded with several examples for abstract theoretical objects that are introduced for the theoretical investigations.

In Chapter 4, an advanced type of flux discontinuity geometries is considered. In particular, the discussed compound flux discontinuities allow for multiple, possibly crossing flux interfaces. After discussing relevant definitions and properties of these geometries, the tools for showing admissibility and well-posedness are extended to this generalized flux discontinuity. The chapter is concluded by proving

well-posedness of entropy solutions to the random scalar conservation law and discussing the existence of statistical moments of these solutions.

The first part on random scalar conservation laws with a locally finite number of flux discontinuities is concluded by conducting numerical simulations in Chapter 5. In this section, we introduce a specific problem setting of two-phase flow in heterogeneous medium. First, the theoretical assumptions that lead to well-posedness of the problem are verified for the presented simulation setting. Afterwards, numerical experiments are conducted that investigate the strong convergence behavior of approximations.

Part II: Random scalar conservation laws with infinitely many flux discontinuities

The second part, Part II, of this manuscript is devoted to the development of an adapted entropy framework for random scalar conservation laws that allows for possibly infinitely many flux discontinuities. As a first investigation, Chapter 6 derives a well-posedness theory for one-dimensional random scalar conservation laws based on the Audusse-Perthame adapted entropy formulation for existence and uniqueness. Afterwards, strong measurability of such adapted entropy solutions is shown as well as the existence of moments.

In Chapter 7 this theory is extended to multi-dimensional random scalar conservation laws, where the flux function is of Panov-type. As in Chapter 6, the strong measurability of adapted entropy solutions is shown after the pathwise existence and uniqueness is established. The chapter concludes by investigating the existence of moments of such solutions.

In the final chapter of this second part, numerical simulations are performed, which rely on the well-posedness results of Chapter 6. In particular, a novel pathwise discretization scheme is introduced based on a specific class of stochastic coefficients that allow for very flexible modeling. The behavior of this discretization method is investigated and the strong convergence is analyzed. Finally, the statistical moments of random entropy solutions are estimated by the multilevel Monte Carlo method.

The dissertation is concluded by some remarks and an outlook on future research perspectives in Chapter 9. Furthermore, the appendices in Part III provide some additional material and background information for the reader's convenience.

Notational conventions

For the remainder of this dissertation, we introduce some notational conventions: In general, matrices are denoted by capitalized bold letters, vectors by lowercase bold characters and scalars by lightface letters. The same convention holds for matrix-, vector- and scalar-valued functions. Additionally, the n-th entry of a vector $\boldsymbol{x}$ is denoted by x_n. Analogously, we write $\boldsymbol{x}_{k:n}$ to denote the sub-vector consisting of the k-th to n-th entry of the vector $\boldsymbol{x}$. For the reader's convenience, the backmatter of this dissertation contains a list of symbols and their explanation.

Preliminaries and fundamental results

2

The purpose of this chapter is to familiarize the reader with preliminary results and concepts that build the foundation of investigating random scalar conservation laws in the remainder of this manuscript. As already mentioned in Chapter 1, the novelty of the considered equations is the combination of stochasticity with discontinuous flux functions, while these fluxes do not need to be *strongly* measurable. While transferring pathwise existence and uniqueness results to the randomized setting is straightforward, showing strong measurability of solutions poses a major challenge, since well-known standard results cannot be applied. To introduce the reader to the concepts of measurability, Section 2.1 briefly discusses basic ideas of measure theory.

In Section 2.2, we give an introduction to set-valued mappings and extend the notion of (strong) measurability to this generalized setting. Furthermore, a metric hyperspace is discussed that allows to describe set-valued mappings as a single-valued mapping into this hyperspace. Section 2.2 is concluded by discussing measurability of set-valued mappings in terms of this metric hyperspace.

With these preliminaries at hand, Section 2.3 aims at familiarizing the reader with random variables and related concepts. Here, we discuss weakly measurable as well as Bochner-integrable random variables. The latter have the advantage that they can be characterized by their moments, such as expectation and variance. While this description is beneficial for describing solutions to random conservation laws, it requires these solutions to be strongly measurable, which is a major obstacle in showing well-posedness of the considered problems. We conclude the discussion by introducing the reader to covariance operators and spectral expansions of random fields in Section 2.3.3.

As motivated in Chapter 1, this manuscript considers random scalar conservation laws with discontinuous flux functions. While the theoretical investigations are based on general discontinuous fluxes, the numerical investigations rely on *Lévy-type random fields*. The choice for this type of random fields is motivated and introduced in Section 2.4, which is concluded by discussing the numerical approximation of these random fields.

Finally, Section 2.5 discusses numerical approximation techniques for random scalar conservation laws. In particular, Section 2.5.1 introduces the concepts of (multilevel) Monte Carlo methods, which are sampling-based approaches to uncertainty quantification. Afterwards, a possible discretization technique for approximating pathwise solutions is introduced by discussing finite volume methods.

2.1 Measure theory

This section aims at familiarizing the reader with the basic concepts of measure theory. An important object for defining these concepts are σ-algebras. For any nonempty set T a σ-algebra $\mathcal{A}$ is a collection of subsets of the set T, which is closed under complement as well as closed under countable unions and contains T itself. Before we continue, let us introduce the *Borel σ-algebra*, which is an important example of a σ-algebra.

Example 2.1 (Borel σ-algebra): *Let E be a topological space. Then, the* Borel σ-algebra over E, *denoted by $\mathcal{B}(E)$, is the σ-algebra generated by the open sets of E. This means that $\mathcal{B}(E)$ is the smallest σ-algebra over E that contains all open sets of E.* ◆

The notion of (strong) measurability of a mapping heavily relies on the concept of such σ-algebras as the following definition demonstrates.

Definition 2.2 ((Strongly) Measurable function):
Let (Ω, Σ) and $(T, \mathcal{A})$ be measurable spaces, meaning that Ω and T are nonempty sets with corresponding σ-algebras Σ and $\mathcal{A}$, respectively. A function $f : \Omega \to T$ is called $(\Sigma - \mathcal{A})$-measurable, *if for every set $E \in \mathcal{A}$, the preimage of E under the function f is in Σ, i.e.,*

$$f^{-1}(E) := \{\omega \in \Omega \mid f(\omega) \in E\} \in \Sigma.$$

Moreover, if the image $f(\Omega)$ of f is separable, we call f strongly $(\Sigma - \mathcal{A})$-measurable. ◆

As an important property of measurable functions, the following lemma states that the composition of a measurable function with a Borel-measurable function[9] is also measurable. For details, the reader is referred to [102, property (4) on page 73].

Lemma 2.3 (Composition with Borel-measurable function is measurable):
Let (Ω, Σ) and $(T, \mathcal{A})$ be measurable spaces and let E be a topological space equipped with the Borel σ-algebra $\mathcal{B}(E)$. Furthermore, let $f : \Omega \to T$ be a $(\Sigma - \mathcal{A})$-measurable function and let $g : T \to E$ be a Borel-measurable function. Then, the composition $g \circ f$ is a $\big(\Sigma - \mathcal{B}(E)\big)$-measurable function. ◆

In the remainder of this section, we introduce two important measures and discuss their relation. To do this, we restrict ourselves to the case of the d-dimensional real space $\mathbb{R}^d$. The following two definitions introduce the *Lebesgue* and *Hausdorff* measure, respectively.

Definition 2.4 (Lebesgue Measure):
Let the d-dimensional real space $\mathbb{R}^d$ be equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and let $\mathcal{I}$ denote the set of all half-open intervals in $\mathbb{R}^d$. Then, the function $\mathcal{L}^d$ satisfying

$$\mathcal{L}^d\big([a,b)\big) := \prod_{i=1}^{d}(b_i - a_i)\,, \qquad \text{for all } [a,b) \in \mathcal{I}\,,$$

can be uniquely extended to a measure on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$, which is called the Lebesgue measure. ◆

[9] A function $g : T \to E$ is called *Borel-measurable* if it is $\mathcal{B}(T) - \mathcal{B}(E)$ measurable, where $\mathcal{B}(\cdot)$ denotes the Borel σ-algebra.

Definition 2.5 (Hausdorff Measure, [106, pp. 349–350]):
Let the d-dimensional real space $\mathbb{R}^d$ be equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. For any set $E \subset \mathbb{R}^d$ and two scalar values $s \geq 0$ and $\delta > 0$, we define

$$\mathcal{H}^s_\delta(E) := \inf\left\{ \sum_{j=1}^{\infty} (\operatorname{diam} U_j)^s \mid E \subset \bigcup_{j=1}^{\infty} U_j \text{ and } \operatorname{diam} U_j \leq \delta \right\},$$

with the convention that $\inf \emptyset = \infty$. Here, as δ decreases, the infimum is taken over a smaller family of countable coverings of E by sets U_j, which satisfy $\operatorname{diam} U_j \leq \delta$. The limit

$$\mathcal{H}^s(E) = \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(E),$$

*is called the s-*dimensional Hausdorff measure *of E.* ◆

We conclude this introduction to measure theory with the following theorem stating that the d-dimensional Hausdorff and Lebesgue measure coincide on $\mathbb{R}^d$. The proof of this standard result is omitted. For the details of the argumentation we refer to the monographs of Folland [106, Proposition 11.20] or Evans and Gariepy [99, Theorem 2 in Section 2.2].

Theorem 2.6 (Equivalence of Lebesgue and Hausdorff measure):
Let the d-dimensional real space $\mathbb{R}^d$ be equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Then, there exists a constant $C_d^{\mathcal{H}} > 0$ that depends solely on the dimension $d \in \mathbb{N}$, such that $C_d^{\mathcal{H}} \mathcal{H}^d$ is equivalent to the d-dimensional Lebesgue measure $\mathcal{L}^d$ on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$. ◆

2.2 Set-valued mappings

In this section, we give an introduction to the theory of set-valued mappings. In the literature, many different terms like *set-valued maps* [17, 110], *correspondences* [5], *multi-valued maps* [143, 245] or *multifunctions* [58] have been used to describe mappings that associate a point to a corresponding set. These set-valued maps are an important tool to justify the (strong) measurability of solutions in the well-posedness investigation of this manuscript. Additionally, the notion of correspondences can be exploited to verify important properties of *admissibility germs* and associated *remainder functions*[10]. Note that we will use the terms *set-valued map* and *correspondence* interchangeably throughout this thesis. We start with formally defining set-valued maps and the corresponding notion of measurability.

Definition 2.7 (Set-valued mappings):
Let $(T, \mathcal{A})$ be a measurable space, i.e., let T be an arbitrary nonempty set and let $\mathcal{A}$ be a σ-algebra of subsets of T. Furthermore, let Y be a topological space. If for every $t \in T$, there exists a corresponding $\Xi(t) \subset Y$, then $\Xi(\cdot)$ is called a set-valued map *from T to Y and is denoted by $\Xi : T \rightrightarrows Y$. Furthermore, the set-valued mapping $\Xi : T \rightrightarrows Y$ is* measurable, *if for every open set $O \subset Y$, the set $\Xi^{-1}(O) \subset T$ is measurable, i.e., $\Xi^{-1}(O) \in \mathcal{A}$. In particular, the set $\operatorname{dom} \Xi := \Xi^{-1}(Y)$ must be measurable.* ◆

[10] These germs and associated remainder functions are important tools to describe the admissibility of solutions in the first part of the manuscript.

As we already mentioned, an important feature is measurability. Therefore, we subsequently collect some results describing the measurability of set-valued mappings. We start by stating that measurability of a correspondence is equivalent to the measurability of its closure.

Proposition 2.8 (Measurability of closure, [110, Proposition 6.1.9])**:**
Let $(T, \mathcal{A})$ be a measurable space and let Y be a topological space. A correspondence $\Xi : T \rightrightarrows Y$ is measurable if and only if its closure, denoted by $\mathrm{cl}\big(\Xi(\cdot)\big)$, is measurable. ◆

For the next result, we consider correspondences with a given structure. In particular, we establish that the area under the graph of a Carathéodory function defines a measurable set-valued mapping. We precise this in the following lemma.

Lemma 2.9 (Measurable correspondence via Carathéodory function, [110, Proposition 6.3.4])**:**
Let a measurable space $(T, \mathcal{A})$ be given and let Y be a separable metric space. Furthermore, let the function $g : T \times Y \to \mathbb{R}$ be Carathéodory, i.e., measurable in $t \in T$ and continuous in $y \in Y$. Then, the set-valued mapping $\Xi : T \rightrightarrows Y$ given by

$$\Xi(t) = \big\{ y \in Y \mid g(t, y) \leq 0 \big\} , \qquad \text{for } t \in T$$

is measurable. ◆

As an immediate consequence, the previous result implies that any set-valued mapping is measurable if it can be written as the zero-level set of a Carathéodory function. We formalize this corollary in the following statement.

Corollary 2.10 (Measurable correspondence via zero-level set, [110, Proposition 6.3.9])**:**
Let $(T, \mathcal{A})$ be a measurable space and let Y be a separable metric space. Furthermore, let $g : T \times Y \to \mathbb{R}$ be a Carathéodory function. Then, the set-valued mapping $\Xi : T \rightrightarrows Y$ given by

$$\Xi(t) = \big\{ y \in Y \mid g(t, y) = 0 \big\} , \qquad \text{for } t \in T ,$$

is measurable. ◆

2.2.1 The metric hyperspace $\mathcal{CS}(\mathbb{R}^d)$

An important class of set-valued maps are closed-valued correspondences. For these, we can construct a *hyperspace*[11], which allows us to describe the set-valued maps as single-valued functions into this hyperspace. Such a hyperspace can be constructed to be a complete metric space for which we can derive measurability results. However, for such a metric space, we need a notion of set convergence, which does not discriminate between a set and its closure. The development and corresponding results of this setting are based on the monograph of [246], where also details and motivations of the constructions are discussed extensively.

[11] The term *hyperspace* is adopted from the discussion by Rockafellar and Wets in [246].

The main tool for constructing this hyperspace is the space of all nonempty, closed subsets of $\mathbb{R}^d$, which we denote by $\mathcal{CS}(\mathbb{R}^d)$. Throughout this section, we will see that this space can form a complete metric space, once it is equipped with a suitable metric. As a beginning of the discussion, we start by defining two (pseudo-) distances for the space $\mathcal{CS}(\mathbb{R}^d)$.

Definition 2.11 (ρ-(pseudo-)distance & Pompeiu-Hausdorff distance)**:**
For any value $\rho \in \mathbb{R}_{\geq 0}$ we define the pseudo metric

$$\mathbb{d}_\rho : \mathcal{CS}(\mathbb{R}^d) \times \mathcal{CS}(\mathbb{R}^d) \to \mathbb{R}_{\geq 0} \qquad \mathbb{d}_\rho(C, D) \coloneqq \max_{|\boldsymbol{x}| \leq \rho} \left| \operatorname{dist}_C(\boldsymbol{x}) - \operatorname{dist}_D(\boldsymbol{x}) \right| ,$$

*where $\operatorname{dist}_C(\boldsymbol{x})$ denotes the Euclidean distance of the set $C \subset \mathbb{R}^d$ and the point $\boldsymbol{x} \in \mathbb{R}^d$. We call $\mathbb{d}_\rho(C, D)$ the ρ-*distance *between C and D. Taking the limit $\rho \to \infty$ leads to the metric*

$$\mathbb{d}_\infty(C, D) \coloneqq \sup_{\boldsymbol{x} \in \mathbb{R}^d} |\operatorname{dist}_C(\boldsymbol{x}) - \operatorname{dist}_D(\boldsymbol{x})| ,$$

which is called the Pompeiu-Hausdorff distance *between C and D.* ◆

Even though the above family of (pseudo-)distances enables us to measure the distance between two closed sets, it cannot be used as a metric on $\mathcal{CS}(\mathbb{R}^d)$, if we want the resulting space to be complete. However, we can use this family to define a new metric, which is called the *(integrated) set distance*.

Definition 2.12 ((Integrated) Set distance)**:**
For two sets $C, D \in \mathcal{CS}(\mathbb{R}^d)$, the function

$$\mathbb{d} : \mathcal{CS}(\mathbb{R}^d) \times \mathcal{CS}(\mathbb{R}^d) \to \mathbb{R}_{\geq 0} \qquad \mathbb{d}(C, D) \coloneqq \int_0^\infty \mathbb{d}_\rho(C, D) e^{-\rho} \, \mathrm{d}\rho ,$$

is called the (integrated) set distance *between the sets C and D.* ◆

The motivation for defining this (integrated) set distance $\mathbb{d}$ was to define a metric, such that the space $\mathcal{CS}(\mathbb{R}^d)$ equipped with the metric $\mathbb{d}$ is complete. Luckily, this is precisely the statement of the subsequent theorem.

Theorem 2.13 (Complete metric space, [246, Theorem 4.42])**:**
The (integrated) set distance defines a metric on the hyperspace $\mathcal{CS}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$ and the resulting metric space $\left(\mathcal{CS}(\mathbb{R}^d), \mathbb{d}\right)$ is complete. ◆

While the (integrated) set distance $\mathbb{d}$ allows us to equip the hyperspace $\mathcal{CS}(\mathbb{R}^d)$ with a metric such that the resulting space is complete, handling the construction via the integration of the ρ-(pseudo-)distances can be tidious to handle. However, the following result shows that the Pompeiu-Hausdorff distance defines an upper bound on the (integrated) set distance $\mathbb{d}$.

Lemma 2.14 (Distance estimate)**:**
The (integrated) set distance $\mathbb{d}$ satisfies the estimate

$$\mathbb{d}(C, D) \leq \mathbb{d}_\infty(C, D) , \tag{2.1}$$

for any two sets $C, D \in \mathcal{CS}(\mathbb{R}^d)$. ◆

Proof. By Definition 2.11 of the ρ-(pseudo-)distance, for all values $\rho \in \mathbb{R}_{\geq 0}$ the estimation

$$\mathbb{d}_\rho(C, D) \leq \mathbb{d}_\infty(C, D)$$

holds. Thus, the assertion follows by noting that the function $e^{-\rho}$ satisfies $\int_0^\infty e^{-\rho} \, \mathrm{d}\rho = 1$. ■

The hyperspace $\mathcal{CS}(\mathbb{R}^d)$ allows us to define a minimal selector $\mathcal{s}_g$ of a function g. The idea of such a selection is that $\mathcal{s}_g$ returns the minimum value of the function g over a closed set $A \in \mathcal{CS}(\mathbb{R})$. The following result now states that this minimal selector is locally Lipschitz continuous provided that the underlying function g is also locally Lipschitz continuous.

Lemma 2.15 (Minimization over closed bounded set is locally Lipschitz continuous)**:**
Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function that takes values in a bounded interval $\mathbb{U} \subset \mathbb{R}$ and consider the function $\mathcal{s}_g : \mathcal{CS}(\mathbb{R}) \to \mathbb{R}$ defined as

$$\mathcal{s}_g(A) \coloneqq \inf_{a \in A} g(a) \, ,$$

where the space $\mathcal{CS}(\mathbb{R})$ of closed subsets of the real numbers is equipped with the (integrated) set distance $\mathbb{d}$. Then, the function $\mathcal{s}_g$ is locally Lipschitz continuous. ◆

Proof. Let two sets $A, B \in \mathcal{CS}(\mathbb{R})$ be given. The definition of $\mathcal{s}_g$ allows us to rewrite the difference $|\mathcal{s}_g(A) - \mathcal{s}_g(B)|$ as

$$\big|\mathcal{s}_g(A) - \mathcal{s}_g(B)\big| \coloneqq \Big| \inf_{a \in A} g(a) - \inf_{b \in B} g(b) \Big| = \Big| \inf_{a \in A} g(a) + \sup_{b \in B} \big(- g(b) \big) \Big| \, .$$

Here, the last step leverages the identity $-\inf_{b \in B} g(b) = \sup_{b \in B} \big(-g(b)\big)$. Inserting the term $g(0) - g(0)$ and applying the triangle inequality, we can estimate the above term via

$$\big|\mathcal{s}_g(A) - \mathcal{s}_g(B)\big| \leq \Big| \inf_{a \in A} \big(g(a) - g(0)\big) \Big| + \Big| \sup_{b \in B} \big(g(0) - g(b)\big) \Big|$$

Based on this estimation, we can utilize the local Lipschitz continuity of g to obtain

$$\big|\mathcal{s}_g(A) - \mathcal{s}_g(B)\big| \leq L_g^A \Big| \inf_{a \in A} |a - 0| \Big| + L_g^B \Big| \sup_{b \in B} |0 - b| \Big| \, ,$$

where L_g^A and L_g^B denote the Lipschitz constants of g over the set A and B, respectively. Define L_g as the maximum of both Lipschitz constants, i.e., $L_g \coloneqq \max\{L_g^A, L_g^B\}$. Then, using again the identity $-\inf_{b \in B} g(b) = \sup_{b \in B} \big(- g(b)\big)$ leads to the estimation

$$\big|\mathcal{s}_g(A) - \mathcal{s}_g(B)\big| \leq L_g \Big| \inf_{a \in A} |a - 0| \Big| + L_g \Big| \inf_{b \in B} |b - 0| \Big| = \Big| L_g \big(\operatorname{dist}_A(0) + \operatorname{dist}_B(0) \big) \Big| \, .$$

Here, $\operatorname{dist}_A(0)$ denotes the Euclidean distance between the set A and the point 0. Inserting the terms $\operatorname{dist}_B(0) - \operatorname{dist}_B(0)$ and $\operatorname{dist}_A(0) - \operatorname{dist}_A(0)$, we can again employ the triangle inequality to obtain

$$\begin{aligned} \big|\mathcal{s}_g(A) - \mathcal{s}_g(B)\big| &\leq \Big| L_g \Big(\operatorname{dist}_A(0) - \operatorname{dist}_B(0) + \operatorname{dist}_B(0) - \operatorname{dist}_A(0) + \operatorname{dist}_A(0) - \operatorname{dist}_B(0) \Big) \Big| \\ &\leq 3 L_g \big| \operatorname{dist}_A(0) - \operatorname{dist}_B(0) \big| \, . \end{aligned}$$

However, by [246, Lemma 4.41], the above term can be bounded by the set distance $\mathbb{d}$ leading to

$$\big|\mathcal{s}_g(A) - \mathcal{s}_g(B)\big| \leq 3 L_g \mathbb{d}(A, B) \, ,$$

which concludes the proof of $\mathcal{s}_g$ being locally Lipschitz continuous. ■

2.2.2 Measurability in hyperspace terms

In the previous section, we have seen how to identify a correspondence with a single-valued map into the hyperspace $\big(\mathcal{CS}(\mathbb{R}^d), \mathbb{d}\big)$. Now, we use this representation to derive measurability results of set-valued maps, whose values are contained in this hyperspace. Here, measurability in the hyperspace has the standard interpretation of single-valued maps via the preimage of Borel subsets. The Borel σ-algebra on $\mathcal{CS}(\mathbb{R}^d)$ is the one generated by the Painlevé-Kuratowski set topology, or equivalently, the one induced by the (integrated) set distance $\mathbb{d}$.

Theorem 2.16 (Measurability in hyperspace terms, [246, Theorem 14.4])**:**
A closed-valued correspondence $\Xi : T \rightrightarrows \mathbb{R}^d$ is measurable if and only if it is measurable when viewed as a single-valued mapping from $\operatorname{dom}\Xi$ into the metric hyperspace $\big(\mathcal{CS}(\mathbb{R}^d), \mathbb{d}\big)$. ◆

As a last tool that we will frequently employ, we define a *set-dependent indicator function*. In contrast to the classical definition of an indicator function, the set-dependent version has an additional argument, which allows us to specify the (closed) set on which the function is nonzero. We precise this construction in the subsequent definition. Afterwards, we conclude this section by proving that the set-dependent indicator function $\mathbb{1}$ is separately measurable.

Definition 2.17 (Set-dependent indicator function)**:**
Let $\mathcal{CS}(\mathbb{R}^d)$ be the hyperspace of closed nonempty subsets of $\mathbb{R}^d$ equipped with the (integrated) set distance $\mathbb{d}$. Then, the function $\mathbb{1} : \mathcal{CS}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ given by

$$\mathbb{1}(Z, \boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in Z\,, \\ 0 & \boldsymbol{x} \notin Z\,, \end{cases}$$

is called the set-dependent indicator function. ◆

Lemma 2.18 (Measurability of set-dependent indicator function)**:**
The set-dependent indicator function $\mathbb{1} : \mathcal{CS}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ as defined in Definition 2.17 is separately measurable. ◆

Proof. To show the measurability in the first argument, let a set $Z \in \mathcal{CS}(\mathbb{R}^d)$ be fixed. By definition, the set Z is closed and thus measurable. This is already sufficient to have measurability with respect to the spatial coordinate $\boldsymbol{x} \in \mathbb{R}^d$, since the indicator function of a measurable set is also measurable [248, Proposition 1.9 (d)].

Now, let $\boldsymbol{x} \in \mathbb{R}^d$ be fixed and let $A \in \mathcal{B}(\mathbb{R})$ be arbitrary, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. We denote by $\mathbb{1}_{\boldsymbol{x}}^{-1}[A]$ the preimage of A under $\mathbb{1}(\cdot, \boldsymbol{x})$, i.e.,

$$\mathbb{1}_{\boldsymbol{x}}^{-1}[A] := \Big\{ Z \in \mathcal{CS}(\mathbb{R}^d) : \mathbb{1}(Z, \boldsymbol{x}) \in A \Big\}.$$

Additionally, let $\mathfrak{Z}_{\boldsymbol{x}} \subset \mathcal{CS}(\mathbb{R}^d)$ be the set of closed nonempty subsets of $\mathbb{R}^d$ containing $\boldsymbol{x}$, i.e.,

$$\mathfrak{Z}_{\boldsymbol{x}} = \Big\{ Z \in \mathcal{CS}(\mathbb{R}^d) : \mathbb{1}(Z, \boldsymbol{x}) = 1 \Big\}.$$

Based on these two definitions of the preimage $\mathbb{1}_{\boldsymbol{x}}^{-1}[A]$ and the set $\mathfrak{Z}_{\boldsymbol{x}}$ and for the spatial variable $\boldsymbol{x} \in \mathbb{R}^d$

still being fixed, we obtain an explicit representation of $\mathbb{1}_{\boldsymbol{x}}^{-1}[A]$ given by

$$\mathbb{1}_{\boldsymbol{x}}^{-1}[A] = \begin{cases} \mathcal{CS}(\mathbb{R}^d) & 0,1 \in A\,, \\ \mathfrak{Z}_{\boldsymbol{x}} & 1 \in A, 0 \notin A\,, \\ \mathfrak{Z}_{\boldsymbol{x}}^c & 1 \notin A, 0 \in A\,, \\ \emptyset & 0,1 \notin A\,. \end{cases}$$

If all these four sets are measurable, then the mapping $Z \mapsto \mathbb{1}(Z, \boldsymbol{x})$ is Borel-measurable. Thus, it remains to show that $\mathfrak{Z}_{\boldsymbol{x}}$ and $\mathfrak{Z}_{\boldsymbol{x}}^c$ are measurable.

We now show that the set $\mathfrak{Z}_{\boldsymbol{x}}$ is closed and thus, $\mathfrak{Z}_{\boldsymbol{x}}$ and $\mathfrak{Z}_{\boldsymbol{x}}^c$ are $\mathcal{B}(\mathcal{CS}(\mathbb{R}^d))$-measurable, where $\mathcal{B}(\mathcal{CS}(\mathbb{R}^d))$ denotes the Borel σ-algebra over $\mathcal{CS}(\mathbb{R}^d)$. Let $P \in \mathfrak{Z}_{\boldsymbol{x}}^c$ be arbitrary but fixed. Then, by definition, P is closed with $\boldsymbol{x} \notin P$ and in particular, there exists a constant $\alpha > 0$ such that $\mathbb{d}(P, \boldsymbol{x}) > \alpha$. Now, define the constant $0 < \varepsilon := \alpha/2$ and the set $Q := P - \varepsilon \boldsymbol{d}$ for some arbitrary $\boldsymbol{d} \in \mathbb{R}^d$ with $\|\boldsymbol{d}\| = 1$. By construction, it holds that

$$\mathbb{d}_\infty(P, Q) := \sup_{\boldsymbol{x} \in \mathbb{R}^d} |\mathrm{dist}_P(\boldsymbol{x}) - \mathrm{dist}_Q(\boldsymbol{x})| = \varepsilon,$$

and by the estimate (2.1) it holds that

$$\mathbb{d}(P, Q) \leq \mathbb{d}_\infty(P, Q) = \varepsilon.$$

Furthermore, since P is closed by definition, it follows that Q is closed and, additionally, we have by construction that $\boldsymbol{x} \notin Q$. Thus, we get $Q \in \mathfrak{Z}_{\boldsymbol{x}}^c$ and therefore, $\mathfrak{Z}_{\boldsymbol{x}}^c$ is open. Consequently, $\mathfrak{Z}_{\boldsymbol{x}}$ is closed and therefore, $\mathfrak{Z}_{\boldsymbol{x}}$ and $\mathfrak{Z}_{\boldsymbol{x}}^c$ are $\mathcal{B}(\mathcal{CS}(\mathbb{R}^d))$-measurable, which concludes the proof of showing that the set-dependent indicator function $\mathbb{1}$ is separately measurable. ■

2.3 Random variables

The purpose of this section is to familiarize the reader with random variables in general Banach and Hilbert spaces. The presented frameworks can be seen as the theoretical basis for introducing Lévy-type random fields in the next section. We start in Section 2.3.1 by discussing *weakly measurable* random variables. Afterwards, *Bochner-integrable* random variables are introduced, which can be characterized via their moments. We conclude this introduction of random variables by discussing covariance operators and spectral expansions in Section 2.3.3. Before we start, the notion of a random variable is introduced in the subsequent definition.

Definition 2.19 (Random variable)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(T, \mathcal{A})$ be a measurable space. Then, a $(\Sigma - \mathcal{A})$-measurable function $X : \Omega \to T$ is called a $(T, \mathcal{A})$-valued random variable. ◆

To describe random variables, oftentimes their statistical properties are employed, such as expectation, variance or higher moments. Let us stress that these moments do not need to exist.

2.3.1 Weakly measurable random variables

The Definition 2.19 of a random variable is the common one when considering uncertain parameters or coefficients. From a theoretical point of view, this definition can be generalized by relaxing the requirement of measurability. This leads to the notion of *weakly measurable random variables*, which may take values in an arbitrary Banach space B. For such random variables to be weakly measurable it is sufficient, if the composition with any element of the dual space is measurable. We precise this construction with the following definition.

Definition 2.20 (Weakly measurable random variable)**:**
Let a probability space $(\Omega, \Sigma, \mathbb{P})$ be given and let $(B, \|\cdot\|_B)$ be a Banach space with dual B'. A function $X : \Omega \to B$ is weakly measurable, if for every element of the dual space $x' \in B'$, the function $Xx' : \Omega \to \mathbb{R}$ defined by

$$Xx'(\omega) \coloneqq \langle X(\omega), x' \rangle \ ,$$

is measurable. Then, we call $X : \Omega \to B$ a weakly measurable random variable. ◆

To describe the statistical moments of weakly measurable random variables, a suitable way of integrating the random variable X over sample space Ω against the probability measure $\mathbb{P}$ needs to be defined. Here, the notion of the *Pettis integral* and the *Dunford integral* can be employed. Both integrals are defined in the subsequent definition.

Definition 2.21 (Pettis & Dunford integrals)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a Banach space. Furthermore, let $X : \Omega \to B$ be a weakly measurable random variable. Then, the Pettis integral *of X is the unique element $x \in B$ satisfying*

$$\langle x, x' \rangle = \int_\Omega \langle X, x' \rangle \ \mathrm{d}\mathbb{P} \ ,$$

for every dual element $x' \in B'$. The Dunford integral *of X is the unique element x'' in the double dual space B'' of B satisfying*

$$\langle x', x'' \rangle = \int_\Omega \langle X, x' \rangle \ \mathrm{d}\mathbb{P} \ ,$$

for every dual element $x' \in B'$. ◆

Based on this Pettis-integrability, one can define the expectation of a weakly measurable random variable. Before we continue, let us introduce some notation: For a probability space $(\Omega, \Sigma, \mathbb{P})$ and a Banach space B, consider a weakly measurable random variable $X : \Omega \to B$. We write $X \in \mathcal{L}^p_w(\Omega; B)$, with $p \geq 1$, if the mapping $\langle X, x' \rangle$ satisfies $\langle X, x' \rangle \in \mathcal{L}^p(\Omega; \mathbb{R})$ for all dual elements $x' \in B'$. Therewith, we are ready to define the expectation and covariance operator of weakly measurable random variables.

Definition 2.22 (Expectation & covariance of weakly measurable random variables)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a Banach space. Furthermore, let $X \in \mathcal{L}^p_w(\Omega; B)$ be a weakly measurable random variable. Then, the element $x \in B$ is called the expectation of X*, if the relation*

$$\int_\Omega \langle X, x' \rangle \ \mathrm{d}\mathbb{P} = \langle x, x' \rangle$$

is satisfied for every dual element $x' \in B'$. We write $\mathbb{E}_P(X) \coloneqq x$ to denote the Pettis integral of the random variable X. Additionally, the continuous linear operator $Q : B' \to B$ satisfying

$$\langle Qx_1', x_2' \rangle = \int_\Omega \langle X - \mathbb{E}_P(X), x_2' \rangle \langle X - \mathbb{E}_P(X), x_1' \rangle \, \mathrm{d}\mathbb{P},$$

is called covariance operator. ◆

2.3.2 Bochner-integrable random variables

While the notion of weakly measurable random variables is very general and can be applied to a tremendeous amount of problems, it also introduces some inconveniences. In practical applications, it is often important to compute or approximate the expectation and higher moments of the random variable. Unfortunately, to the best of the author's knowledge, numerically approximating the Pettis integral is not possible or unfeasible due to its construction. Nevertheless, restricting the situation to Bochner-integrable random variables leads to a feasible notion of statistical moments, which can be approximated numerically. This restriction requires the random variable to be strongly measurable, which means that it is a random variable in the sense of Definition 2.19 with a separable image. Based on such strongly measurable random variables, we can introduce *Lebesgue-Bochner spaces* in the next definition.

Definition 2.23 (Lebesgue-Bochner space)**:**
Let (Ω, Σ, μ) be a measure space and let $(B, \|\cdot\|_B)$ be a Banach space. For any value $p \in [1, \infty)$, we define the set

$$\mathbb{L}^p(\Omega; B) \coloneqq \left\{ X : \Omega \to B \text{ is strongly measurable and } \int_\Omega \|X(\omega)\|_B^p \, \mathrm{d}\mu(\omega) < +\infty \right\}$$

and the seminorm

$$\|X\|_{\mathbb{L}^p(\Omega;B)} \coloneqq \left(\int_\Omega \|X\|_B^p \, \mathrm{d}\mu \right)^{1/p}.$$

Furthermore, define the set

$$\mathcal{N}^p \coloneqq \{ X \in \mathbb{L}^p \mid X = 0 \ \mu\text{-almost everywhere} \}.$$

Then, the Lebesgue-Bochner space *is defined as the function space $\mathcal{L}^p(\Omega; B) \coloneqq \mathbb{L}^p \setminus \mathcal{N}^p$ with the norm $\|[X]\|_{\mathcal{L}^p(\Omega;B)} \coloneqq \|X\|_{\mathbb{L}^p(\Omega;B)}$, for some representative X of the equivalence class $[X]$.* ◆

Let us stress that, in the remainder of this thesis, we do not explicitly distinguish between equivalence classes and functions. With this definition of Lebesgue-Bochner spaces, we have also obtained an integral notion for defining the statistical moments of (strongly measurable) random variables. We specify the expectation as well as higher moments of a Bochner-integrable random variable in the following definition.

Definition 2.24 (Moments of Bochner-integrable random variables)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a Banach space. Furthermore, let $X : \Omega \to B$ be a B-valued random variable.

(i) *If the random variable X satisfies $X \in \mathcal{L}^1(\Omega; B)$, the* expectation of X *is defined as the Lebesgue-Bochner integral*

$$\mathbb{E}(X) \coloneqq \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega) .$$

(ii) *For some value $n \in \mathbb{N}$, let the random variable X satisfy $X \in \mathcal{L}^n(\Omega; B)$. Then, for any value $1 \leq k \leq n$, the value $\mathbb{M}_k(X)$, given by*

$$\mathbb{M}_k(X) \coloneqq \mathbb{E}\left(X^k\right) ,$$

is called the k-th moment of X. ◆

2.3.3 Covariance operators and spectral expansions

Based on the Bochner-integrable random variables of the previous section, an extensive theory has been developed for random variables with finite second moments. In this so-called $\mathcal{L}^2$-theory for random fields, it is natural to consider covariance operators. These also allow to derive spectral expansions of the random fields. Throughout this section on covariance operators and the associated spectral expansions, we restrict ourselves to the discussion of H-valued random variables, where $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space.

To start the discussion of covariance operators and spectral expansions based on the $\mathcal{L}^2$-theory, we first specify the second moment of an H-valued random variable. In particular, the following definition introduces the variance and covariance of a random variable X.

Definition 2.25 (Covariance & variance of random variable):
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space. Furthermore, let $X : \Omega \to H$ be an H-valued random variable satisfying $X \in \mathcal{L}^2(\Omega; H)$. Then, the covariance of X *is defined by*

$$\mathbb{C}\mathrm{ov}(X) \coloneqq \mathbb{E}\left(\big(X - \mathbb{E}(X)\big) \otimes \big(X - \mathbb{E}(X)\big)\right)$$

as an element of the tensor product space $H \otimes H$. Additionally, the term

$$\begin{aligned}\mathbb{V}(X) &\coloneqq \mathbb{E}\left(\left\|\big(X - \mathbb{E}(X)\big) \otimes \big(X - \mathbb{E}(X)\big)\right\|_{H\otimes H}\right) \\ &= \mathbb{E}\left(\left\|X - \mathbb{E}(X)\right\|_H^2\right) = \left\|X - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;H)}^2\end{aligned}$$

is called the variance of X. ◆

Let us stress that the covariance of X is well defined as an element of $H \otimes H$ as soon as X satisfies $X \in \mathcal{L}^2(\Omega; H)$. This follows directly from the estimation

$$\begin{aligned}\left\|\mathbb{C}\mathrm{ov}(X)\right\|_{H\otimes H} &\leq \mathbb{E}\left(\left\|\big(X - \mathbb{E}(X)\big) \otimes \big(X - \mathbb{E}(X)\big)\right\|_{H\otimes H}\right) \\ &= \left\|X - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;H)}^2 < +\infty .\end{aligned}$$

It is also possible to interpret the covariance of an H-valued random variable as an operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. Therefore, let $L(H)$ denote the set of all linear operators on H and let $L_N^+(H)$ be the set of all nonnegative, symmetric, nuclear operators on H. Then, the covariance can be defined via the unique *covariance operator* $Q \in L_N^+(H)$ satisfying

$$\langle \mathbb{C}\mathrm{ov}(X), \psi \otimes \phi \rangle_{H\otimes H} = \langle Q\psi, \phi \rangle_{H\otimes H}$$

for all $\psi, \phi \in H$. The following result guarantees the existence of a unique covariance operator $Q \in L_N^+(H)$ for a square-integrable H-valued random variable X.

Lemma 2.26 (Existence of unique covariance operator, [262, Lemma 3.1.2])**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. Then, for any square-integrable H-valued random variable, i.e., $X \in \mathcal{L}^2(\Omega; H)$, there exists a unique covariance operator $Q \in L_N^+(H)$. ◆

Due to the Hilbert-Schmidt theorem [242, Theorem VI.16], for any covariance operator $Q \in L_N^+(H)$, there exists an orthonormal basis $(\boldsymbol{v}_i, i \in \mathbb{N})$ of H satisfying

$$Q\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \,.$$

Here, $(\lambda_i, i \in \mathbb{N})$ is a decaying sequence of nonnegative eigenvalues with zero being the only accumulation point. Furthermore, Q being a nuclear operator implies that it is also *trace-class operator*, which means that we have

$$\mathrm{Tr}(Q) \coloneqq \sum_{i\in\mathbb{N}} \langle Q\boldsymbol{v}_i, \boldsymbol{v}_i \rangle_H = \sum_{i\in\mathbb{N}} \lambda_i < +\infty \,.$$

Due to the covariance operator $Q \in L_N^+(H)$ being a trace-class operator, the square-integrable H-valued random variable X admits the following spectral representation.

Theorem 2.27 (Karhunen-Loéve expansion)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. Furthermore, let $X \in \mathcal{L}^2(\Omega; H)$ be an H-valued random variable with trace-class covariance operator $Q \in L_N^+(H)$. Denote by $\big((\lambda_i, \boldsymbol{v}_i), i \in \mathbb{N}\big)$ the sequence of eigenpairs of the covariance operator Q with decaying eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Then, the random variable X admits the Karhunen-Loéve expansion

$$X = \mathbb{E}(X) + \sum_{i\in\mathbb{N}} \sqrt{\lambda_i}\boldsymbol{v}_i Z_i \,. \tag{2.2}$$

Here, $(Z_i, i \in \mathbb{N})$ is a sequence of uncorrelated, real-valued centered random variables. ◆

2.4 Lévy-type random fields and their numerical approximation

When modeling uncertainty, the classical approach in the literature employs continuous random fields, see, e.g., [59, 66, 71, 222, 266]. For many applications, such as subsurface flows through heterogeneous porous media, it might be necessary to incorporate spatial and/or temporal discontinuities to model sudden changes of heat conductivity or permeability within the heterogeneous porous medium. However,

such an extension is not possible using continuous random fields. To overcome this problem, one may employ *Lévy-type random fields*, which pose a possible extension of Lévy processes to higher-dimensional parameter spaces.

The fundamental concept of Lévy processes was introduced by Lévy and Khintchine in various works, such as [162, 184–186]. This general form of stochastic processes has been the foundation for extensive research throughout the last centuries as the standard works by Applebaum [16] or Sato [161] prove. Unfortunately, extending the general form of Lévy processes to higher-dimensional parameter spaces while still being able to simulate the resulting random fields is still an open problem and active field of current research. Nevertheless, by sacrificing some of the generality one can define specific instances of Lévy-type random fields. One possibility is to restrict the setting to random fields with a certain structure, such as subordinated random fields. Such an extension has been addressed for example in the works of Merkle and Barth [197–199].

Another extension of Lévy processes to higher-dimensional parameter spaces is based on the Lévy-Khintchine decomposition [16, 161]. This theoretical result states that every Lévy process may be uniquely characterized via a continuous and a discontinuous part. In particular, the Lévy-Khintchine formula states that every Lévy process can be seen as the composition of three independent components: A deterministic drift term, a Brownian motion and a pure jump process. For a precise statement of this decomposition, we refer to [233, Theorem 4.23]. It is exactly this decomposition of Lévy processes that motivates the following definition of Lévy-type random fields.

Definition 2.28 (Lévy-type random fields, [27, Definition 3.1])**:**
Let $(\Omega, \Sigma, \mathbb{P})$ *be a complete probability space and let* $\mathbb{R}^d$ *be the spatial domain. Then, a* Lévy-type random field $\mathfrak{a}$ *is defined as a function*

$$\mathfrak{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}_{>0} , \qquad (\omega, \boldsymbol{x}) \mapsto \overline{\mathfrak{a}}(\boldsymbol{x}) + \Phi\big(\mathcal{G}(\omega, \boldsymbol{x})\big) + \mathfrak{P}(\omega, \boldsymbol{x}) , \tag{2.3}$$

where

- $\overline{\mathfrak{a}} \in \mathcal{C}(\mathbb{R}^d; \mathbb{R}_{\geq 0})$ *is a deterministic, uniformly bounded mean function.*
- $\Phi \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}_{>0})$ *is a continuously differentiable, positive mapping.*
- $\mathcal{G} \in \mathcal{L}^2\big(\Omega; \mathcal{L}^2(\mathbb{R}^d; \mathbb{R})\big)$ *is a zero-mean Gaussian random field associated to a nonnegative, symmetric trace-class (covariance) operator* $Q : \mathcal{L}^2(\mathbb{R}^d; \mathbb{R}) \to \mathcal{L}^2(\mathbb{R}^d; \mathbb{R})$.
- $\mathfrak{T} : \Omega \to \mathcal{B}(\mathbb{R}^d)$, $\omega \mapsto \{\mathfrak{T}_1, \ldots, \mathfrak{T}_\tau\}$ *is a random partition of the domain* $\mathbb{R}^d$ *in the sense that* $\{\mathfrak{T}_i\}_{i=1}^{\tau}$ *is a family of disjoint open subsets of* $\mathbb{R}^d$ *satisfying* $\mathbb{R}^d = \cup_{i=1}^{\tau} \mathfrak{T}_i$. *The number of elements in* $\mathfrak{T}$ *is given by an integrable random variable* $\tau : \Omega \to \mathbb{N}$ *on the probability space* $(\Omega, \Sigma, \mathbb{P})$.
- *A finite measure* Λ *on the measurable space* $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$ *is associated to the partition* $\mathfrak{T}$ *and controls the distribution of the positions of the random elements* $\mathfrak{T}_i$.
- *For a sequence* $(\mathfrak{p}_i, i \in \mathbb{N})$ *of random variables on the probability space* $(\Omega, \Sigma, \mathbb{P})$ *with arbitrary nonnegative distribution(s), we define the jump field* $\mathfrak{P}$ *as*

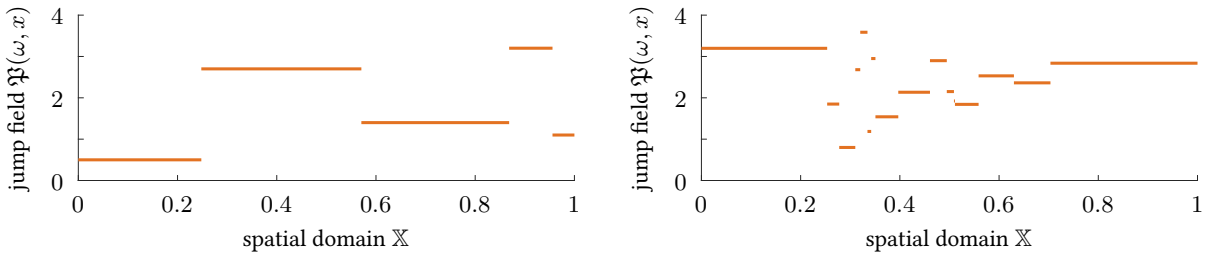
$$\mathfrak{P} : \Omega \times \mathbb{R}^d \to \mathbb{R}_{>0}, \quad (\omega, \boldsymbol{x}) \mapsto \sum_{i=1}^{\tau} \mathbb{1}_{\mathfrak{T}_i}(\boldsymbol{x}) \mathfrak{p}_i(\omega) ,$$

where the sequence $(\mathfrak{p}_i, i \in \mathbb{N}_0)$ *is independent of the number* τ *of elements in the partition* $\mathfrak{T}$, *but not necessarily pairwise independent and identically distributed.* ◆

While inspecting the construction of Lévy-type random fields in Definition 2.28 via Equation (2.3) it immediately becomes apparent that the ingredients are inspired by the Lévy-Khintchine decomposition of Lévy processes. Nevertheless, some comments on the construction of the jump field $\mathfrak{P}$ are in order.

Remark 2.29 (Jump measure & independence): *Let us stress that we do not require the Gaussian random field $\mathcal{G}$ and the jump field $\mathfrak{P}$ to be stochastically independent. On a further note, observe that the jump measure Λ on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$, which is associated to the partition $\mathfrak{T}$ of $\mathbb{R}^d$, does not only affect the average number of partition elements. Additionally, this measure Λ also controls the size of the partition elements $\mathfrak{T}_i$ and can be utilized to concentrate the discontinuities of the jump-advection coefficient $\mathfrak{a}$ to specific areas of the domain $\mathbb{R}^d$.* ◆

Before we continue with a discussion on how to numerically approximate the constructed Lévy-type random fields, let us illustrate the influence of the jump measure Λ on the resulting random field. Therefore, Figure 2.1 depicts a realization of a random field for which the underlying jump measure corresponds to a uniform and a normal distribution in Figure 2.1a and 2.1b, respectively.



(a) Jump field $\mathfrak{P}$ with a partition $\mathfrak{T}$ controlled by a jump measure Λ corresponding to a uniform distribution.

(b) Jump field $\mathfrak{P}$ with a partition $\mathfrak{T}$ controlled by a jump measure Λ corresponding to a normal distribution with mean $x = 1/2$.

Figure 2.1: Illustration of the jump measure Λ on the discontinuous jump-advection coefficient.

While the jump positions in Figure 2.1a are evenly distributed over the spatial domain $\mathbb{X}$, they are concentrated around the center of the domain in Figure 2.1b. This is not surprising, as the jump measure underlying the random field shown in Figure 2.1b corresponds to a normal distribution with mean $x = 1/2$, which results in a higher density around that mean. Consequently, this illustration shows how the jump field $\mathfrak{P}$ can be influenced by the jump measure Λ.

Based on the form of the Lévy-type random field given by Equation (2.3), a numerical approximation can be obtained by approximating both the continuous (Gaussian) random field and the pure jump field, separately. Depending on the specific construction of the jump field $\mathfrak{P}$ it might be possible to evaluate the jump field exactly. All jump fields considered in this manuscript admit such an exact evaluation. However, if it is not possible to exactly evaluate the jump field $\mathfrak{P}$, an approximation can be obtained via Fourier inversion [28, 117, 145].

In general, there is no hope for an exact evaluation of the Gaussian random field and a numerical approximation technique needs to be employed. Luckily, since the covariance operator Q is trace-class, the Gaussian random field admits the spectral representation via the Karhunen-Loéve expansion defined in Theorem 2.27. This representation given by Equation (2.2) allows us to approximate the random variable (or random field) by truncating the series after $N_{\mathrm{KL}} \in \mathbb{N}$ terms. The subsequent theorem states that such an approximation converges to the random variable. Additionally, an upper bound on the truncation error is obtained. For the proof of this result, the reader is referred to [262, Theorem 3.1.6].

Theorem 2.30 (Truncated Karhunen-Loéve expansion)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. Furthermore, let $X \in \mathcal{L}^2(\Omega; H)$ be an H-valued random variable with trace-class covariance operator $Q \in L_N^+(H)$ and denote by $\big((\lambda_i, \boldsymbol{v}_i), i \in \mathbb{N}\big)$ the sequence of eigenpairs of the covariance operator Q with decaying eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Then, the Karhunen-Loéve expansion given by Equation (2.2) converges in $\mathcal{L}^2(\Omega; H)$ and the truncation error is bounded by

$$\left\| \sum_{i>N_{\mathrm{KL}}}^{\infty} \sqrt{\lambda_i} \boldsymbol{v}_i Z_i \right\|_{\mathcal{L}^2(\Omega;H)}^2 = \mathbb{E}\left(\left\| \sum_{i>N_{\mathrm{KL}}}^{\infty} \sqrt{\lambda_i} \boldsymbol{v}_i Z_i \right\|_H^2 \right) = \sum_{i>N_{\mathrm{KL}}} \lambda_i \,,$$

for some truncation index $N_{\mathrm{KL}} \in \mathbb{N}$. ◆

Let us stress that approximating the Gaussian random field via the truncated Karhunen-Loéve expansion requires the eigenbasis of the covariance operator Q. In many situations such an eigenbasis might not be available in an explicit or analytical form. However, for a given covariance operator or discrete evaluations thereof, Nyström's method provides a solution to obtain the eigenbasis. Nyström's method was originally introduced by Baker [24] and Press et al. [237]. For an extensive discussion, the reader is referred to the monograph [286].

2.5 Numerical approximation techniques for random scalar conservation laws

The purpose of this section is to introduce the numerical methods for approximating random conservation laws that are employed throughout this manuscript. Therefore, the discussion starts by introducing (multilevel) Monte Carlo methods for approximating moments of random variables in Section 2.5.1. Afterwards, in Section 2.5.2, the finite volume method is discussed, which is suitable to approximate pathwise solutions to random scalar conservation laws.

Let us point out that this section does not aim at giving an overview of existing algorithms, but rather explain the employed methods in more detail. For an overview of numerical approximation techniques for random scalar conservation laws, the reader is referred to Section 1.4 and the references therein. In this introductory Section 1.4 also advantages and possible shortcomings are discussed.

2.5.1 (Multilevel) Monte Carlo methods

In this section, we introduce the ideas of the Monte Carlo method and its multilevel version. The main concepts of Monte Carlo methods are based on ideas that Ulam [202, 274], von Neumann [279] and Fermi [104] formulated in the middle of the last century. For the historical development of the Monte Carlo method, the reader is referred to [201].

Monte Carlo method

The idea of the sampling-based Monte Carlo method is straightforward: By drawing a large number of independent samples from the distribution of a random variable, the arithmetic mean of these samples approximates the expected value of the random variable. This idea motivates the following definition of the *Monte Carlo estimator* of the expectation of a random variable.

Definition 2.31 (Monte Carlo estimator):
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $(B, \|\cdot\|_B)$ a Banach space and let $X : \Omega \to B$ be a random variable that satisfies $\mathbb{E}(\|X\|_B) < +\infty$. Then, the Monte Carlo estimator of $\mathbb{E}(X)$ *is defined as*

$$E_M(X) \coloneqq \frac{1}{M} \sum_{i=1}^{M} X^{(i)} ,$$

where $\{X^{(i)}\}_{i=1}^{M}$, with $M \in \mathbb{N}$, are independent and identically distributed (i.i.d.) copies of the random variable X. ◆

Based on this construction, we can deduce several observations. First, the Monte Carlo estimator is unbiased, which means that it satisfies $\mathbb{E}(E_M(X)) = \mathbb{E}(X)$. Additionally, the strong law of large numbers [64, 135] ensures convergence to the expectation $\mathbb{P}$-almost surely, which means that we have

$$\lim_{M\to\infty} E_M(X) = \mathbb{E}(X) , \qquad \mathbb{P}\text{-almost surely.}$$

To investigate the convergence behavior of the Monte Carlo estimator E_M further, we need to take the *Rademacher type of the Banach space* into account.

Definition 2.32 (Rademacher type of Banach space, [168, Definition 2.2]):
Let $(Z_i, i \in \mathbb{N})$ be a sequence of Bernoulli-Rademacher random variables. For $1 \leq p_B \leq \infty$, a Banach space B is said to be of Rademacher type p_B *if there exists a* type constant $C_B > 0$ *such that for all finite sequences $\{X^{(i)}\}_{i=1}^{M} \subset B$, with $M \in \mathbb{N}$, the estimate*

$$\left\| \sum_{i=1}^{M} Z_i X^{(i)} \right\|_B \leq C_B \left(\sum_{i=1}^{M} \left\| X^{(i)} \right\|_B^{p_B} \right)^{1/p_B}$$

is satisfied. ◆

Before we continue, the following remark summarizes some important results regarding the Rademacher type of a Banach space.

Remark 2.33 (Rademacher type of Banach space, [168, Remark 2.3]):

- *The triangle inequality implies that every Banach space has Rademacher type* 1.
- *Hilbert spaces (and finite-dimensional spaces) have Rademacher type* 2 *and the type constant C_B depends on the dimension.*
- *For $1 \leq p < \infty$, $\mathcal{L}^p$-spaces have Rademacher type $p_B = \min\{2, p\}$.* ◆

With this definition of a Rademacher type p_B of a general Banach space, it is now possible to obtain the following result regarding the convergence of a Monte Carlo estimator. For the technical details on convergence rates for the Monte Carlo method in Banach spaces, the reader is referred to the recent work of Kirchner and Schwab [164].

Theorem 2.34 (Convergence of Monte Carlo estimator, [168, Proposition 2.4])**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a Banach space of Rademacher type p_B with type constant C_B. Then, for every finite sequence $\{X^{(i)}\}_{i=1}^M$ of independent random variables in $\mathcal{L}^{p_B}(\Omega; B)$ with zero mean, the estimation

$$\mathbb{E}\left(\left\|\sum_{i=1}^{M} X^{(i)}\right\|_B^{p_B}\right) \leq (2C_B)^{p_B} \sum_{i=1}^{M} \mathbb{E}\left(\left\|X^{(i)}\right\|_B^{p_B}\right) \tag{2.4}$$

is satisfied. In the particular case of B being a Hilbert space and X satisfying $X \in \mathcal{L}^2(\Omega; B)$, the convergence of the Monte Carlo estimator E_M can be described via the root-mean-squared-error $\mathcal{E}_{\mathrm{RMS}}$*, which is given by*

$$\mathcal{E}_{\mathrm{RMS}} := \mathbb{E}\left(\left\|E_M(X) - \mathbb{E}(X)\right\|_B^2\right)^{1/2} = \frac{\mathbb{V}(X)^{1/2}}{\sqrt{M}},$$

where $\mathbb{V}(X)$ denotes the variance of X. ◆

By the latter error estimate, we obtain mean-square convergence of the Monte Carlo method as soon as X is square-integrable. Furthermore, Monte Carlo methods are implemented very easily and due to the independence of the samples, parallelization is straightforward. However, the slow convergence of $\mathcal{O}\left(M^{1/2}\right)$ is a major obstacle and therefore employing a Monte Carlo estimator can be unfeasible if the simulation of X is computationally expensive. For example, this can be the case, if the random variable X is (based on) the solution of a partial differential equation.

Multilevel Monte Carlo method

To overcome this obstacle of the slow convergence rate, the multilevel Monte Carlo method has been developed by Giles [118, 119] based on the ideas of Heinrich [130]. Instead of a single random variable X, consider now a hierarchy of approximations of X, denoted by $X_0, \ldots, X_L$, for $L \in \mathbb{N}$. With this hierarchy of approximations, we can write the approximation with the highest accuracy as the telescoping sum

$$X_L = X_0 + \sum_{l=1}^{L} X_l - X_{l-1}. \tag{2.5}$$

Based on this representation, the idea of the multilevel Monte Carlo (MLMC) method is now to decouple the computational effort in approximating the expected value of X: On a coarse discretization level, many (computationally inexpensive) samples can be computed to obtain an accurate approximation of the underlying stochastic at the price of a low spatial resolution. This low spatial resolution can be corrected by sampling the difference $X_l - X_{l-1}$ of two approximations of the random variable with the standard Monte Carlo method. Under the assumption that the variance $\mathbb{V}(X_l - X_{l-1})$ decays sufficiently fast, this leads to a fast decreasing level-dependent number of samples $M_0 > \cdots > M_L$. This (spatial) accuracy versus number of (stochastic) samples trade-off leads to the following *multilevel Monte Carlo estimator.*

Definition 2.35 (Multilevel Monte Carlo estimator)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a Banach space of Rademacher type p_B with type constant C_B. Furthermore, let $X \in \mathcal{L}^2(\Omega; B)$ be a random variable with finite second moment, i.e., $\mathbb{E}\left(\|X\|_B^2\right) < +\infty$. Additionally, for $L \in \mathbb{N}$, let $X_0, \dots, X_L$ be a hierarchy of approximations of X and let $M_0 > \cdots > M_L$ be a decreasing sequence of sample numbers. Then, the multilevel Monte Carlo (MLMC) estimator *of $\mathbb{E}(X_L)$ is defined as*

$$E^L(X_L) \coloneqq E_{M_0}(X_0) + \sum_{l=1}^{L} E_{M_l}(X_l - X_{l-1}) = \frac{1}{M_0}\sum_{i=1}^{M_0} X_0^{(i)} + \sum_{l=1}^{L}\frac{1}{M_l}\sum_{i=1}^{M_l}\left(X_l^{(i)} - X_{l-1}^{(i)}\right), \tag{2.6}$$

where $\left\{X_l^{(i)}\right\}_{i=1}^{M_l}$, with $M_l \in \mathbb{N}$, are i.i.d. copies of the random variable X_l. ◆

Let us emphasize that the correction term $X_l^{(i)} - X_{l-1}^{(i)}$ in Equation (2.6) has to be sampled from the same set of random variables. Additionally, note that the correction terms themselves are again independent random variables. To conclude the fundamental discussions of (multilevel) Monte Carlo methods in this section, the convergence of the MLMC estimator is investigated. Therefore, the subsequent theorem provides an estimation of the root-mean-squared-error $\mathcal{E}_{\mathrm{RMS}}$ that describes the convergence of the MLMC estimator. While the stochastic error can be bounded similar to the convergence of the standard Monte Carlo estimator, the additional error induced by the spatial discretization has to be accounted for.

Theorem 2.36 (Convergence of Multilevel Monte Carlo estimator)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a Banach space of Rademacher type p_B with type constant C_B. Furthermore, let $(X_l, l \in \mathbb{N}_0) \subset \mathcal{L}^{p_B}(\Omega; B)$ be a sequence of random variables converging to $X \in \mathcal{L}^{p_B}(\Omega; B)$. Additionally, let the approximations X_l satisfy $\|X - X_l\|_{\mathcal{L}^{p_B}(\Omega;B)} \leq C_{\mathrm{MLMC}}\Delta_{x,\max}^l$, for some constant $C_{\mathrm{MLMC}} > 0$ and a decreasing sequence $(\Delta_{x,\max}^l, l \in \mathbb{N}_0)$ of refinement parameters. Then, for a number $L \in \mathbb{N}$ of levels and a decreasing sequence $M_0 > \cdots > M_L$ of sample numbers, the multilevel Monte Carlo estimator $E^L(X_L)$ satisfies the estimation

$$\left\|E^L(X_L) - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;B)} \leq C_{\mathrm{MLMC}}\Delta_{x,\max}^L + C_B\sum_{l=0}^{L} M_l^{1-p_B}\,\mathbb{V}(X_l - X_{l-1})^{1/p_B}$$

where $\mathbb{V}(X_l - X_{l-1})$ denotes the variance of the difference of two consecutive approximations to the random variable X. ◆

Proof. As a first step, applying the triangle inequality to the root-mean-squared-error of the multilevel Monte Carlo estimator $E^L(X_L)$ yields

$$\left\|E^L(X_L) - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;B)} \leq \left\|E^L(X_L) - \mathbb{E}(X_L)\right\|_{\mathcal{L}^2(\Omega;B)} + \left\|\mathbb{E}(X_L) - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;B)}.$$

The definition of the $\mathcal{L}^2(\Omega; B)$-norm allows us to estimate the second term as

$$\left\|\mathbb{E}(X_L) - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;B)} = \left\|\int_\Omega X_L - X\,\mathrm{d}\mathbb{P}\right\|_B \leq \int_\Omega \|X_L - X\|_B\,\mathrm{d}\mathbb{P} = \mathbb{E}\left(\|X_L - X\|_B\right).$$

Now, due to the approximation assumption, this can be further estimated as

$$\left\|\mathbb{E}(X_L) - \mathbb{E}(X)\right\|_{\mathcal{L}^2(\Omega;B)} \leq \mathbb{E}\left(\|X_L - X\|_B\right) \leq C_{\mathrm{MLMC}}\Delta_{x,\max}^L.$$

It remains to estimate the term $\left\|E^L(X_L) - \mathbb{E}\left(X_L\right)\right\|_{\mathcal{L}^2(\Omega;B)}$. To do this, we define $X_{-1} \coloneqq 0$ and $\Delta X_l \coloneqq X_l - X_{l-1}$. With these definitions and the telescoping sum (2.5) of the random variable X_L, we have

$$\begin{aligned}\left\|E^L(X_L) - \mathbb{E}\left(X_L\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} &= \left\|\sum_{l=0}^{L} E_{M_l}(\Delta X_l) - \mathbb{E}\left(\sum_{l=0}^{L}\Delta X_l\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} \\ &= \left\|\sum_{l=0}^{L}\left(E_{M_l}(\Delta X_l) - \mathbb{E}\left(\Delta X_l\right)\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} \\ &= \left\|\sum_{l=0}^{L}\sum_{i=1}^{M_l}\left(\frac{\Delta X_l^{(i)} - \mathbb{E}\left(\Delta X_l\right)}{M_l}\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} .\end{aligned}$$

For fixed $l = 0, \dots, L$, defining

$$Z_l^{(i)} \coloneqq \frac{\Delta X_l^{(i)} - \mathbb{E}\left(\Delta X_l\right)}{M_l}, \qquad \text{for } i = 1, \dots, M_l \,,$$

leads to a finite sequence of independent random variables with zero-mean. Consequently, we can apply the estimate (2.4) on the convergence of the Monte Carlo estimator (see also [179, Proposition 9.11] or [168, Corollary 2.5]). Hence, we obtain

$$\begin{aligned}\left\|E^L(X_L) - \mathbb{E}\left(X_L\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} &\leq C_B \sum_{l=0}^{L}\sum_{i=1}^{M_l}\left\|Z_l^{(i)}\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} \\ &= C_B \sum_{l=0}^{L} M_l \left\|Z_l^{(1)}\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} \\ &= C_B \sum_{l=0}^{L} M_l \left\|\frac{\Delta X_l^{(1)} - \mathbb{E}\left(\Delta X_l\right)}{M_l}\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} \\ &= C_B \sum_{l=0}^{L} M_l^{1-p_B} \left\|\Delta X_l^{(1)} - \mathbb{E}\left(\Delta X_l\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} ,\end{aligned}$$

where C_B is a constant depending on the Rademacher type p_B of the Banach space B.

Based on this estimation, we can conclude that

$$\left\|E^L(X_L) - \mathbb{E}\left(X_L\right)\right\|^{p_B}_{\mathcal{L}^2(\Omega;B)} \leq C_B \sum_{l=0}^{L} M_l^{1-p_B} \mathbb{V}\left(\Delta X_l\right)^{p_B} \,,$$

which completes the proof. ∎

For the precise statement of computational gain of the multilevel Monte Carlo method, we refer to the famous complexity theorem in [118]. Roughly speaking, the multilevel Monte Carlo methods have successfully been used to reduce the computational time by several orders of magnitude compared to the classical Monte Carlo method.

2.5.2 Finite volume methods

In this last section on numerical approximation techniques for random conservation laws, the finite volume method is introduced. Since the last century, finite volume methods have been extensively studied by scientists and have been used for the numerical simulation of various problems in engineering. As an important feature, finite volume schemes contain local conservativity of the fluxes within the considered problem. This property is an important physical characteristic in the description of phenomena.

The basic idea of finite volume schemes is to track the numerical fluxes of the unknown through the considered domain. This is achieved by observing so-called *control volumes*, which are the cells that result from discretizing the computational domain. For each such control volume, the flux across its boundary is then obtained via an integral formulation of the problem.

Discretization of the space-time domain

To start the discussion of finite volume methods, we describe the discretization of the considered space-time domain $\mathbb{X}_\mathbb{T}$ is a rather extensive way. Consider a triangulation $\mathbb{X}_\Delta$ of the spatial domain $\mathbb{X} \subset \mathbb{R}^d$. We use the term *triangulation* for a partition of the domain into a finite set of polyhedra $\mathcal{X}_i \subset \mathbb{X}$, which are disjoint, open and convex. For each such *control volume* $\mathcal{X}$, we denote its boundary as $\Gamma_\mathcal{X}$, which is itself a finite union of closed, plane faces. As important characteristics of the control volumes $\mathcal{X}_i \in \mathbb{X}_\Delta$, we write $|\mathcal{X}_i|$ to denote the volume of the i-th cell and $\Delta\mathcal{X}_i$ denotes its diameter. Furthermore, for any control volume $\mathcal{X} \in \mathbb{X}_\Delta$, we define the set $N(\mathcal{X})$ of neighboring cells as

$$N(\mathcal{X}) \coloneqq \left\{ \mathcal{Y} \in \mathbb{X}_\Delta \mid \mathcal{Y} \neq \mathcal{X} \text{ and } \mathcal{H}^{d-1}\big(\mathrm{cl}(\mathcal{X}) \cap \mathrm{cl}(\mathcal{Y})\big) > 0 \right\} .$$

Here, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. It remains to discretize the time interval $\mathbb{T} = [0, T] \subset \mathbb{R}_{>0}$, where $0 < T < \infty$ is some final time for the simulation. Therefore, let $\mathbb{T}_\Delta = \{t^m\}_{m \in \mathbb{N}} \subset \mathbb{T}$ be a sequence of points in the time interval $\mathbb{T}$. For stability reasons, we may require the time step sizes $\Delta_t^m = t^m - t^{m-1}$ to satisfy the Courant-Friedrichs-Lewy (CFL) condition, which is given as

$$\frac{\Delta_t^m}{\min\{\Delta\mathcal{X}_i \mid \mathcal{X}_i \in \mathbb{X}_\Delta\}} C_{\mathrm{CFL}}^m < 1 . \tag{2.7}$$

Here, the constant $C_{\mathrm{CFL}}^m > 0$ depends on the maximum wave speed at the m-th time step.

Weak formulation of conservation law and numerical scheme

Based on the aforementioned discretization, we have now everything at hand to introduce and discuss finite volume methods. Recall that the goal of the finite volume method is to obtain a discretization to enable numerically simulating a scalar conservation law of the form

$$\begin{aligned} \partial_t u + \mathrm{div}_{\boldsymbol{x}} \mathbf{f}(t, \boldsymbol{x}, u) = 0 \qquad & \text{in } \mathbb{X} \times \mathbb{T} , \\ u(0, \boldsymbol{x}) = u_0(\boldsymbol{x}) \qquad & \text{on } \mathbb{X} \times \{0\} . \end{aligned} \tag{2.8}$$

Here u_0 is some initial condition to the considered problem. One could write Equation (2.8) by means of a space-time divergence in which case deriving the finite volume scheme would require a space-time

control volume. Another possibility, which we consider in this thesis, is based on only integrating over a spatial control volume. Thus, to approximate the time derivative, a finite difference scheme is employed. With the Euler time discretization, this results in the approximation

$$\partial_t u \approx \frac{u^{m+1} - u^m}{\Delta_t^m} \, .$$

Here, u^m denotes an approximate value of the solution u at time t^m. The fundamental idea of the finite volume method is to integrate the scalar conservation law given by Equation (2.8) over each cell $\mathcal{X} \in \mathbb{X}_\Delta$ of the triangulation $\mathbb{X}_\Delta$. Using the explicit (forward) Euler time discretization[12], this leads to the following weak formulation of the conservation law:

$$\int_{\mathcal{X}} \frac{u^{m+1}(\boldsymbol{x}) - u^m(\boldsymbol{x})}{\Delta_t^m} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_{\mathcal{X}}} \mathfrak{f}\big(t^m, \boldsymbol{x}, u^m(\boldsymbol{x})\big) \cdot \boldsymbol{n}_{\mathcal{X}}(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{d-1} = 0 \, .$$

Here, $\boldsymbol{n}_{\mathcal{X}}(\boldsymbol{x})$ denotes the outer unit normal vector of cell $\mathcal{X}$ at point $\boldsymbol{x} \in \Gamma_{\mathcal{X}}$ and $\mathrm{d}\mathcal{H}^{d-1}$ denotes the differential for integrating against the $(d-1)$-dimensional Hausdorff measure. The remaining step in defining the finite volume method is to approximate the flux $\mathfrak{f}\big(t^m, \boldsymbol{x}, u^m(\boldsymbol{x})\big) \cdot \boldsymbol{n}_{\mathcal{X}}(\boldsymbol{x})$ across the boundary $\Gamma_{\mathcal{X}}$ for each cell $\mathcal{X}$. We write $G^m_{\mathcal{X},\mathcal{Y}}(t^m, \mathcal{X}_\mathrm{c}, \mathcal{Y}_\mathrm{c}, U^m_{\mathcal{X}}, U^m_{\mathcal{Y}})$ to denote an approximation of the flux between the control volumes $\mathcal{X}$ and $\mathcal{Y}$. This approximation depends on approximate values $U^m_{\mathcal{X}}, U^m_{\mathcal{Y}}$ of the solution u at time t^m on cells $\mathcal{X}, \mathcal{Y} \in \mathbb{X}_\Delta$ as well as on the center points $\mathcal{X}_\mathrm{c}, \mathcal{Y}_\mathrm{c}$ of the cell $\mathcal{X}$ and $\mathcal{Y}$, respectively. Usually, the values $U^m_{\mathcal{X}}$ are approximations to the cell average of the solution u, i.e.,

$$U^m_{\mathcal{X}} \approx \frac{1}{|\mathcal{X}|} \int_{\mathcal{X}} u(t^m, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, .$$

Combining the presented approximations of the solution u and the flux function $\mathfrak{f}$, this leads to the numerical approximation scheme

$$U^{m+1}_{\mathcal{X}} = U^m_{\mathcal{X}} - \frac{\Delta_t^m}{|\mathcal{X}|} \sum_{\mathcal{Y} \in N(\mathcal{X})} G^m_{\mathcal{X},\mathcal{Y}}\big(t^m, \mathcal{X}_\mathrm{c}, \mathcal{Y}_\mathrm{c}, U^m_{\mathcal{X}}, U^m_{\mathcal{Y}}\big) \, .$$

Before we conclude the discussion of finite volume approximations, let us stress two important properties of the finite volume method:

(i) The scheme is conservative in the sense that the numerical flux $G^m_{\mathcal{X},\mathcal{Y}}$ satisfies

$$G^m_{\mathcal{X},\mathcal{Y}}\big(t^m, \mathcal{X}_\mathrm{c}, \mathcal{Y}_\mathrm{c}, U^m_{\mathcal{X}}, U^m_{\mathcal{Y}}\big) = G^m_{\mathcal{Y},\mathcal{X}}\big(t^m, \mathcal{Y}_\mathrm{c}, \mathcal{X}_\mathrm{c}, U^m_{\mathcal{Y}}, U^m_{\mathcal{X}}\big) \, ,$$

for all cells $\mathcal{X}, \mathcal{Y} \in \mathbb{X}_\Delta$ in the triangulation $\mathbb{X}_\Delta$ and for all time steps $m \in \mathbb{N}$.

(ii) The numerical flux $G^m_{\mathcal{X},\mathcal{Y}}$ of the finite volume method is consistent with the flux given by $\mathfrak{f}(t^m, \boldsymbol{x}, u) \cdot \boldsymbol{n}_{\mathcal{X}}$.

It is a standard result that combining these two properties with suitable stability properties results in convergence of the presented finite volume method (see, e.g., [101] or [183]).

[12] Alternatively, the implicit (backward) Euler time discretization can be employed, in which case the term $\mathfrak{f}\big(t^m, \boldsymbol{x}, u^m(\boldsymbol{x})\big)$ is replaced by $\mathfrak{f}\big(t^{m+1}, \boldsymbol{x}, u^{m+1}(\boldsymbol{x})\big)$. In general, any stability preserving Runge-Kutta time integration scheme can be employed. For the details, we refer to the monographs [101, 131, 183].

Numerical flux functions

Before we conclude the discussion of finite volume methods we have a closer look into the approximations of the flux function $\mathfrak{f}$. For the ease of presentation, we subsequently restrict ourselves to the one-dimensional setting. A standard procedure to derive numerical fluxes is based on considering Riemann problems. These are one-dimensional initial value problems based on the scalar conservation law of Equation (2.8), where the initial condition u_0 is of the form

$$u_0(x) \coloneqq \begin{cases} u_\mathrm{L} & \text{for } x < 0 \,, \\ u_\mathrm{R} & \text{for } x > 0 \,, \end{cases}$$

for some tuple $(u_\mathrm{L}, u_\mathrm{R}) \in \mathbb{R}^2$. Here, we tacitly assume $u_\mathrm{L} \neq u_\mathrm{R}$. For deriving a numerical flux, such a Riemann problem is solved or approximated for any edge of the cell $\mathcal{X} \in \mathbb{X}_\Delta$. One important numerical flux based on this approach is the *Godunov flux*, which—for the case of spatially independent flux functions—is given by

$$G^{\mathrm{God}}_{\mathcal{X},\mathcal{Y}}(U_\mathcal{X}, U_\mathcal{Y}) \coloneqq \begin{cases} \min\limits_{U_\mathcal{X} \leq \theta \leq U_\mathcal{Y}} \mathfrak{f}(\theta) \cdot \boldsymbol{n}_{\mathcal{X},\mathcal{Y}} & \text{if } U_\mathcal{X} \leq U_\mathcal{Y} \,, \\ \max\limits_{U_\mathcal{X} \leq \theta \leq U_\mathcal{Y}} \mathfrak{f}(\theta) \cdot \boldsymbol{n}_{\mathcal{X},\mathcal{Y}} & \text{if } U_\mathcal{X} \geq U_\mathcal{Y} \,. \end{cases} \tag{2.9}$$

Here, $\boldsymbol{n}_{\mathcal{X},\mathcal{Y}}$ denotes the outward unit normal vector pointing from cell $\mathcal{X}$ to cell $\mathcal{Y}$. For more examplary fluxes based on this approach, such as the Roe or Harten-Lax-van Leer fluxes, the reader is referred to the extensive discussion in [183].

An important property of numerical flux functions is monotonicity. A numerical flux $G_{\mathcal{X},\mathcal{Y}} : \mathbb{R}^2 \to \mathbb{R}$ is called monotone, provided that

(i) the function $G_{\mathcal{X},\mathcal{Y}}(\cdot, \xi)$ is monotone increasing for all scalar values $\xi \in \mathbb{R}$ and

(ii) the function $G_{\mathcal{X},\mathcal{Y}}(\xi, \cdot)$ is monotone decreasing for all scalar values $\xi \in \mathbb{R}$.

Examples for monotone fluxes are the Godunov flux (compare Equation (2.9)) or the Lax-Friedrichs flux.

Approximation error and (theoretical) convergence results

To conclude the discussion of prerequisites of finite volume methods we give an overview of existing results regarding the convergence rates of finite volume schemes. As before, this discussion is limited to the case of deterministic conservation laws and flux functions.

We start by summarizing convergence results for the $\mathcal{L}^1$-theory of conservation laws. Due to the result of Harten et al. [129] the convergence rate of monotone finite volume approximations in the $\mathcal{L}^1$ space is restricted to 1, since finite volume schemes are at most first-order accurate. For this result, the authors considered spatially independent flux functions. An extension to fluxes with a discontinuous spatial dependency was recently introduced by Badwaik and Ruf [21]. In this work, the authors derive a $\mathcal{L}^1$ convergence rate of $1/2$ for conservation laws with discontinuous flux functions in the case where a monotone finite volume scheme is considered and the solution u dependency of the flux function is strictly monotone. In a sequel work, Badwaik et al. [22] state that the $\mathcal{L}^1$ convergence rate of $1/2$

"cannot be improved without further assumptions on the initial datum". Here, possible assumptions are regarding the regularity of the initial datum and the statement is based on a result of Sabac [251] from 1997.

As a last consideration of this section, the convergence behavior and results in the $\mathcal{L}^\infty(\mathbb{T};\mathcal{L}^1)$ space are discussed. The first theoretical result goes back to Kuznetsov [174], who proved a convergence of order $1/2$ in $\mathcal{L}^\infty(\mathbb{T};\mathcal{L}^1)$ for spatially independent flux functions when a structured grid is used for the finite volume discretization. This result has been extended to a convergence rate of $1/4$ on unstructured grids by Cockburn et al. [67], Vila [277] and Eymard et al. [101]. In contrast to this theoretical result, error rates that are observable in numerical simulations may show a different behavior. For example, Merlet and Vovelle [200] find that "[...] numerical tests give an order $h^{1/2}$ for structured as well as unstructured meshes". To conclude the discussion of convergence results, let us not that–to the best of our knowledge–no theoretical results for $\mathcal{L}^\infty$ convergence rates have been established for the case where the flux function contains spatial discontinuities.

I

Random conservation laws with a locally finite number of flux discontinuities

Random conservation laws with a sole flux discontinuity

3

The purpose of this chapter is to develop a well-posedness theory for random scalar conservation laws with a discontinuous flux function. While we allow the conservation law to be defined in multiple space dimensions, we restrict ourselves to the rather simple case of a *sole flux discontinuity*. This basic geometrical form of the jump interface of the flux function allows us to discuss the admissibility of entropy solutions and the well-posedness of such, whilst avoiding heavy technical arguments.

Throughout this chapter, let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. Furthermore, for a time interval $\mathbb{T} \coloneqq [0, T]$, with $0 < T < \infty$, and a spatial domain $\mathbb{X} \coloneqq \mathbb{R}^d$, with $d \in \mathbb{N}$, we denote the corresponding space-time domain as $\mathbb{X}_\mathbb{T} \coloneqq \mathbb{T} \times \mathbb{X}$. Then, for unknown $u \coloneqq u(\omega, t, \boldsymbol{x})$, the random scalar conservation law is given by

$$\begin{aligned} \partial_t u + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, t, \boldsymbol{x}, u) = 0 \qquad & \text{in } \Omega \times \mathbb{T} \times \mathbb{X}\,, \\ u(\omega, 0, \boldsymbol{x}) = u_0(\omega, \boldsymbol{x}) \quad & \text{on } \Omega \times \{0\} \times \mathbb{X}\,, \end{aligned} \tag{3.1}$$

where $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, is a stochastic initial condition. Furthermore, the random flux function $\mathfrak{f}$ is assumed to depend discontinuously on the spatial variable. This discontinuity $\mathfrak{D}(\omega) \subset \mathbb{X}_\mathbb{T}$, which is discussed in detail in Section 3.1, is assumed to be a random hypersurface that divides the space-time domain into two parts.

For the well-posedness of random entropy solutions, three ingredients need to be established, namely the existence of a unique solution as well as the measurability with respect to the stochastic variable of such a solution. To achieve this, we start by discussing *random sole discontinuity hypersurfaces* in Section 3.1. These (potentially curved) hypersurfaces describe the flux discontinuities that are considered throughout this chapter. Afterwards, in Section 3.2, two different approaches and techniques for the definition and admissibility of entropy solutions are discussed. The first one is based on a local consideration of admissible solutions, while the second approach employs a global entropy inequality to describe the admissibility of solutions. Both approaches will be leveraged in various ways to argue the well-posedness of the random scalar conservation law given by Equation (3.1) in Section 3.3. Here, we first reduce the question of existence and uniqueness of pathwise entropy solutions to the corresponding deterministic case. Thereafter, the measurability of entropy solutions is derived, followed by an investigation of the existence of moments of the solution.

3.1 Sole flux discontinuities

A rather simple, though very important case of discontinuity geometries is given by *sole discontinuities*, which consist of a single discontinuity hypersurface that divides the space-time domain $\mathbb{X}_\mathbb{T}$ into two parts. Even though the jump manifold is allowed to be curved, this type of discontinuity is a *model geometry* that simplifies the presentation and is suitable to describe the main ideas and difficulties. The section starts with the definition of such discontinuities and discusses some direct consequences thereof in Section 3.1.1. Afterwards, in Section 3.1.2, the measurability of the hypersurface is investigated. Then, we have a brief look at the normal vector field of the discontinuity manifold in Section 3.1.3, before concluding this section by looking at the resulting left and right domain parts and relevant properties in Section 3.1.4.

3.1.1 Sole flux discontinuities and their parametrization

We start by defining the notion of a *sole flux discontinuity (hypersurface)*, which is given as an equation-defined manifold of the space-time domain $\mathbb{X}_\mathbb{T}$.

Definition 3.1 (Sole flux discontinuity)**:**
Let $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$ be a hypersurface in the space-time domain $\mathbb{X}_\mathbb{T}$. If there exists a continuously differentiable function $\Phi^\mathfrak{D} \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^{d-1}; \mathbb{R})$, such that $\mathfrak{D}$ is given by the graph of $\Phi^\mathfrak{D}$, i.e.,

$$\mathfrak{D} = \operatorname{Gr} \Phi^\mathfrak{D} := \left\{ (t, \boldsymbol{x}) \in \mathbb{X}_\mathbb{T} \mid x_1 = \Phi^\mathfrak{D}(t, \boldsymbol{x}_{2:d}) \right\}, \tag{3.2}$$

then $\mathfrak{D}$ is called a sole discontinuity (hypersurface)*. Here, $\boldsymbol{x} = (x_1, \dots, x_d)$ denotes the spatial variable and we write $\boldsymbol{x}_{2:d}$ for the vector $(x_2, \dots, x_d)$. Furthermore, if $\mathfrak{D}$ depends on $\omega \in \Omega$ and $\mathfrak{D}(\omega)$ is a sole discontinuity for every $\omega \in \Omega$, then $\mathfrak{D}$ is called a* random sole discontinuity (hypersurface)*. This dependency is formalized by writing $\mathfrak{D}$ as a correspondence $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$.* ◆

By this construction of the jump interface, we immediately obtain the following characterization of a sole discontinuity as a parametrized submanifold of the space-time domain $\mathbb{X}_\mathbb{T}$.

Lemma 3.2 (Parametrization of (random) sole discontinuity)**:**
For a random sole discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$, the function $\boldsymbol{P}_\mathfrak{D} : \Omega \times \mathbb{T} \times \mathbb{R}^{d-1} \to \mathbb{X}_\mathbb{T}$ given by

$$\boldsymbol{P}_\mathfrak{D}(\omega, t, \boldsymbol{y}) := \left(t, \Phi^\mathfrak{D}(\omega, t, \boldsymbol{y}), \boldsymbol{y}\right), \tag{3.3}$$

defines a parametrization of $\mathfrak{D}$. ◆

Another property of the jump interface is its closedness, which we directly obtain from the definition. This property is important, since it implies measurability of the discontinuity hypersurface as a subset of $\left(\mathbb{X}_\mathbb{T}, \mathcal{B}(\mathbb{X}_\mathbb{T})\right)$, where $\mathcal{B}(\mathbb{X}_\mathbb{T})$ denotes the Borel σ-algebra of the space-time domain $\mathbb{X}_\mathbb{T}$.

Corollary 3.3 (Sole discontinuity is closed)**:**
If $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$ is a sole discontinuity hypersurface, then $\mathfrak{D}$ is closed as a subset of $\mathbb{X}_\mathbb{T}$. Additionally, if $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ is a random sole discontinuity, then $\mathfrak{D}$ is a closed-valued correspondence. ◆

Proof. By Definition 3.1 and Equation (3.2), the discontinuity $\mathfrak{D}$ is given as the graph of a continuous function $\Phi^{\mathfrak{D}} \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^{d-1}; \mathbb{R})$. Now, since $\Phi^{\mathfrak{D}}$ is continuous and both $\mathbb{T} \times \mathbb{R}^{d-1}$ and $\mathbb{R}$ are Hausdorff spaces, the graph of $\Phi^{\mathfrak{D}}$ is closed, which proves the first assertion. If $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ is a random sole discontinuity, then by Definition 3.1 the preceeding argumentation holds for every $\omega \in \Omega$, which concludes the proof. ■

Additionally to implying measurability, the closedness of $\mathfrak{D}$ allows us to represent random sole discontinuities $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ as single-valued mappings, which is useful for a variety of proofs.

3.1.2 Measurability of random sole discontinuities

In the last section, the (spatio-temporal) measurability of sole discontinuities has been established. If such a manifold is randomized, we additionally need stochastic measurability, i.e., measurability with respect to $\omega \in \Omega$. Unfortunately, this property cannot be derived from the definition. Therefore, we impose the following assumption on random sole discontinuities.

Assumption 3.4 (Stochastic measurability of random sole discontinuities)**:**
We assume that the random sole discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ is a measurable correspondence. ◆

If we combine this stochastic measurability assumption with the Definition 3.1 of random sole discontinuities, for every $\omega \in \Omega$ we can derive a representation of $\mathfrak{D}(\omega)$ via the level set of a Carathéodory function. We show this in the following Lemma.

Lemma 3.5 (Sole discontinuity as level set)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{T} \times \mathbb{X}$ be a random sole discontinuity. Then, there exists a function $\Psi^{\mathfrak{D}} : \Omega \times \mathbb{T} \times \mathbb{X} \to \mathbb{R}$ such that, for every stochastic parameter $\omega \in \Omega$, the discontinuity $\mathfrak{D}(\omega)$ is given as the zero-level set

$$\mathfrak{D}(\omega) = \left\{ (t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X} \mid \Psi^{\mathfrak{D}}(\omega, t, \boldsymbol{x}) = 0 \right\} \tag{3.4}$$

of the continuously differentiable function $\Psi^{\mathfrak{D}}(\omega, \cdot, \cdot)$. Furthermore, if $\mathfrak{D}$ satisfies the stochastic measurability Assumption 3.4, then the function $\Psi^{\mathfrak{D}}$ is Carathéodory in the sense that

(i) for fixed $(t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X}$, the mapping $\omega \mapsto \Psi^{\mathfrak{D}}(\omega, t, \boldsymbol{x})$ is measurable.

(ii) for fixed $\omega \in \Omega$, the mapping $(t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X} \mapsto \Psi^{\mathfrak{D}}(\omega, t, \boldsymbol{x})$ is continuous. ◆

Proof. Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{T} \times \mathbb{X}$ be a random sole discontinuity. Then, by Definition 3.1 of a random sole discontinuity, for every $\omega \in \Omega$, there exists a differentiable function $\Phi^{\mathfrak{D}}(\omega, \cdot, \cdot) \in \mathcal{C}^1(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ such that Equation (3.2) holds. Defining now the function $\Psi^{\mathfrak{D}} : \Omega \times \mathbb{T} \times \mathbb{X} \to \mathbb{R}$ via

$$\Psi^{\mathfrak{D}}(\omega, t, \boldsymbol{x}) := \Phi^{\mathfrak{D}}(\omega, t, \boldsymbol{x}_{2:d}) - x_1 \tag{3.5}$$

we have proven the existence of a function $\Psi^{\mathfrak{D}}$. Furthermore, from $\Phi^{\mathfrak{D}}(\omega, \cdot, \cdot)$ being continuously differentiable, we immediately obtain that the function $\Psi^{\mathfrak{D}}(\omega, \cdot, \cdot) \in \mathcal{C}^1(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is continuously differentiable (and thus especially continuous). If additionally $\mathfrak{D} : \Omega \rightrightarrows \mathbb{T} \times \mathbb{X}$ is measurable by the stochastic measurability Assumption 3.4, then Equation (3.4) and Lemma 2.9 imply that $\Psi^{\mathfrak{D}}(\cdot, t, \boldsymbol{x})$ is measurable. This completes the proof. ■

To conclude the measurability investigation of random sole jump interfaces, we show that the parametrization $\boldsymbol{P}_{\mathfrak{D}}$, defined via Equation (3.3), is jointly measurable.

Lemma 3.6 (Parametrization is jointly measurable)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity that satisfies the measurability Assumption 3.4. Then, the corresponding parametrization $\boldsymbol{P}_{\mathfrak{D}} : \Omega \times \mathbb{T} \times \mathbb{R}^{d-1} \to \mathbb{X}_{\mathbb{T}}$ is jointly measurable. ◆

Proof. Since the measurability Assumption 3.4 of the sole discontinuity $\mathfrak{D}$ is satisfied, we can apply Lemma 3.5 to obtain that there exists a Carathéodory function $\Psi^{\mathfrak{D}}$ that is defined by Equation (3.5). However, Equation (3.5) directly implies that $\Phi^{\mathfrak{D}}$ is Carathéodory, and with [5, Lemma 4.51] also jointly measurable. Now, the joint measurability of the parametrization follows immediately from its definition via Equation (3.3) and [5, Lemma 4.49]. ■

3.1.3 Normal vector field of random sole discontinuities

After deriving the measurability properties of the sole discontinuity and its parametrization, we also need to investigate the normal vector field of these hypersurfaces. To do this, we can employ the representation of $\mathfrak{D}$ via the level set of a Carathéodory function. This specific form of the discontinuity allows us to define a normal vector field of $\mathfrak{D}$ via the *Gaussian mapping*, as the following result shows.

Lemma 3.7 (Normal vector field of sole discontinuity hypersurface)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity that satisfies the stochastic measurability Assumption 3.4. Furthermore, let $\Psi^{\mathfrak{D}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{R}$ be a Carathéodory function, such that Equation (3.4) *is satisfied. Then, for each $\omega \in \Omega$, the unit vector field defined by the Gaussian mapping*

$$\mathfrak{n}_{\mathfrak{D}}(\omega, \mathfrak{d}) \coloneqq \frac{(\nabla_{\mathfrak{x}} \Psi^{\mathfrak{D}})(\omega, \mathfrak{d})}{|(\nabla_{\mathfrak{x}} \Psi^{\mathfrak{D}})(\omega, \mathfrak{d})|}, \qquad \mathfrak{d} \in \mathfrak{D}(\omega) \tag{3.6}$$

is normal to the discontinuity hypersurface $\mathfrak{D}(\omega)$ and $\nabla_{\mathfrak{x}} \Psi^{\mathfrak{D}}$ denotes the spatio-temporal gradient of the function $\Psi^{\mathfrak{D}}$. ◆

Proof. Recall that the discontinuity manifold $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ is equation-defined by Equation (3.4) via the zero-level set of the Carathéodory function $\Psi^{\mathfrak{D}}$. For each random parameter $\omega \in \Omega$, [236, Theorem 50] implies that the gradient $(\nabla_{\mathfrak{x}} \Psi^{\mathfrak{D}})(\omega, \mathfrak{d})$ spans the normal space of the flux discontinuity manifold $\mathfrak{D}(\omega)$ at the point $\mathfrak{d} \in \mathfrak{D}(\omega)$. Combining this result with the normalization in Equation (3.6), we obtain that $\mathfrak{n}_{\mathfrak{D}}(\omega, \mathfrak{d})$ defines a unit normal vector of the discontinuity hypersurface $\mathfrak{D}(\omega)$ at $\mathfrak{d} \in \mathfrak{D}(\omega)$, which concludes the proof. ■

When discussing the admissibility of solutions to Problem (3.1), we want to consider the normal components of the flux corresponding to the normal unit vector field $\mathfrak{n}_{\mathfrak{D}}$ of the discontinuity $\mathfrak{D}$. While we have successfully argued the existence and form of this normal unit vector field of the random sole discontinuity, let us also discuss the existence of an *extension* of this normal field, which is Carathéodory. Such an extension allows us to formulate the random normal field $\mathfrak{n}_{\mathfrak{D}}(\omega, \cdot)$ as a mapping $\widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \cdot)$, whose domain $\mathbb{X}_{\mathbb{T}}$ is independent of the stochastic parameter $\omega \in \Omega$.

Lemma 3.8 (Extension of normal unit vector field)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity hypersurface that satisfies the stochastic measurability Assumption 3.4. Then, the unit vector field

$$\widehat{\mathfrak{n}_\mathfrak{D}}(\omega, \mathfrak{x}) := \frac{(\nabla_\mathfrak{x} \Psi^\mathfrak{D})(\omega, \mathfrak{x})}{|(\nabla_\mathfrak{x} \Psi^\mathfrak{D})(\omega, \mathfrak{x})|}, \quad \omega \in \Omega, \ \mathfrak{x} \in \mathbb{X}_\mathbb{T}, \tag{3.7}$$

defines an extension of the normal vector field $\mathfrak{n}_\mathfrak{D}$ to the whole space-time domain $\mathbb{X}_\mathbb{T}$. Furthermore, the mapping $\widehat{\mathfrak{n}_\mathfrak{D}} : \Omega \times \mathbb{X}_\mathbb{T} \to \mathbb{X}_\mathbb{T}$ is Carathéodory. ◆

Proof. The definition of $\Psi^\mathfrak{D}$ in Equation (3.5), yields $\left|(\nabla_\mathfrak{x} \Psi^\mathfrak{D})(\omega, \mathfrak{x})\right| \geq 1$ for every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Therefore, the definition of $\widehat{\mathfrak{n}_\mathfrak{D}}$ via Equation (3.7) is meaningful for every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Further, it is obvious that $\widehat{\mathfrak{n}_\mathfrak{D}}$ satisfies $\widehat{\mathfrak{n}_\mathfrak{D}}\big|_{\mathfrak{D}(\omega)} \equiv \mathfrak{n}_\mathfrak{D}$ and thus defines an extension of $\mathfrak{n}_\mathfrak{D}$.
Now, by Definition 3.1 and Lemma 3.5, the function $\Psi^\mathfrak{D}$ is measurable in $\omega \in \Omega$ and continuously differentiable in $\mathfrak{x}$. This means that $(\nabla_\mathfrak{x} \Psi^\mathfrak{D})(\omega, \mathfrak{x})$ is Carathéodory and since $\left|(\nabla_\mathfrak{x} \Psi^\mathfrak{D})(\omega, \mathfrak{x})\right| \geq 1$, the extension $\widehat{\mathfrak{n}_\mathfrak{D}} : \Omega \times \mathbb{X}_\mathbb{T} \to \mathbb{X}_\mathbb{T}$ of the unit normal vector field is also Carathéodory. ■

3.1.4 Left and right space-time domain part

To conclude the discussion of sole discontinuities, we look at the left and right space-time domain parts created by the manifold. To do so, we first define what we mean by *left* and *right* domain part.

Definition 3.9 (Domain parts)**:**
Let $\mathfrak{D}$ be a sole discontinuity given by Equation (3.2)*. Then, the interface $\mathfrak{D}$ splits the space-time domain $\mathbb{X}_\mathbb{T}$ into a* left and right domain part*, denoted $\mathbb{X}_\mathbb{T}^{l,r}$, which are given by*

$$\begin{aligned} \mathbb{X}_\mathbb{T}^l &:= \left\{ (t, \boldsymbol{x}) \in \mathbb{X}_\mathbb{T} \mid x_1 < \Phi^\mathfrak{D}(t, \boldsymbol{x}_{2:d}) \right\} \\ \mathbb{X}_\mathbb{T}^r &:= \left\{ (t, \boldsymbol{x}) \in \mathbb{X}_\mathbb{T} \mid x_1 > \Phi^\mathfrak{D}(t, \boldsymbol{x}_{2:d}) \right\}. \end{aligned} \tag{3.8}$$

Here, $\Phi^\mathfrak{D} \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^{d-1}; \mathbb{R})$ is implicitly defined by $\mathfrak{D}$ via Equation (3.2)*. If $\mathfrak{D}$ is a random sole discontinuity, we write $\mathbb{X}_\mathbb{T}^{l,r} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$.* ◆

As a direct consequence of this definition, we obtain that the set-valued mappings $\mathbb{X}_\mathbb{T}^{l,r} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ are stochastically measurable as soon as the discontinuity hypersurface $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ is measurable w.r.t. the random parameter $\omega \in \Omega$. We show this in the following corollary.

Corollary 3.10 (Stochastic measurability of domain parts)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a sole discontinuity satisfying the measurability Assumption 3.4. Then, the left and right domain parts $\mathbb{X}_\mathbb{T}^{l,r} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ corresponding to $\mathfrak{D}$ are measurable set-valued mappings. ◆

Proof. By Lemma 3.5 there exists a Carathéodory function $\Psi^\mathfrak{D} : \Omega \times \mathbb{T} \times \mathbb{X} \to \mathbb{R}$ such that, for for every random parameter $\omega \in \Omega$, the left and right domain parts $\mathbb{X}_\mathbb{T}^{l,r}(\omega)$ can be written as the area

above and below the graph of $\Psi^{\mathfrak{D}}$, respectively. This leads to the representation

$$\mathbb{X}_{\mathbb{T}}^{l}(\omega) \coloneqq \left\{(t,\boldsymbol{x}) \in \mathbb{X}_{\mathbb{T}} \mid \Psi^{\mathfrak{D}}(\omega,t,\boldsymbol{x}) > 0\right\}, \quad \omega \in \Omega\,,$$
$$\mathbb{X}_{\mathbb{T}}^{r}(\omega) \coloneqq \left\{(t,\boldsymbol{x}) \in \mathbb{X}_{\mathbb{T}} \mid \Psi^{\mathfrak{D}}(\omega,t,\boldsymbol{x}) < 0\right\}, \quad \omega \in \Omega\,.$$

Similar, we can describe the closure of $\mathbb{X}_{\mathbb{T}}^{l}(\omega)$ and $\mathbb{X}_{\mathbb{T}}^{r}(\omega)$ by again utilizing the function $\Psi^{\mathfrak{D}}$:

$$\mathrm{cl}\big(\mathbb{X}_{\mathbb{T}}^{l}(\omega)\big) \coloneqq \left\{(t,\boldsymbol{x}) \in \mathbb{X}_{\mathbb{T}} \mid \Psi^{\mathfrak{D}}(\omega,t,\boldsymbol{x}) \geq 0\right\}, \quad \omega \in \Omega\,,$$
$$\mathrm{cl}\big(\mathbb{X}_{\mathbb{T}}^{r}(\omega)\big) \coloneqq \left\{(t,\boldsymbol{x}) \in \mathbb{X}_{\mathbb{T}} \mid \Psi^{\mathfrak{D}}(\omega,t,\boldsymbol{x}) \leq 0\right\}, \quad \omega \in \Omega\,.$$

These closures are measurable correspondences by Lemma 2.9. However, by Proposition 2.8, a set-valued map is measurable if and only if its closure is measurable. Consequently, we have proven that the left and right domain parts $\mathbb{X}_{\mathbb{T}}^{l,r} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ are measurable correspondences. ■

To describe the admissibility of entropy solutions to the random scalar discontinuous-flux conservation law given by Problem (3.1), functions need to be defined on the domain parts $\mathbb{X}_{\mathbb{T}}^{l,r}(\omega)$. In order to create such functions, which depend on the different domain parts, we introduce the corresponding indicator functions of these domain parts as follows.

Definition 3.11 (Domain part indicator function)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity manifold and let $\mathbb{X}_{\mathbb{T}}^{l,r} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ denote the corresponding left and right space-time domain parts. Then, the function $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^{l,r}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{R}$ defined via

$$\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^{l,r}}(\omega,\mathfrak{x}) \coloneqq \mathbb{1}_{\mathbb{X}_{\mathbb{T}}^{l,r}(\omega)}(\mathfrak{x}) = \begin{cases} 1 & \mathfrak{x} \in \mathbb{X}_{\mathbb{T}}^{l,r}(\omega)\,, \\ 0 & \mathfrak{x} \notin \mathbb{X}_{\mathbb{T}}^{l,r}(\omega)\,, \end{cases}$$

is called the (random) domain part indicator function. ◆

These particular indicator functions are separately measurable, if the discontinuity interface is stochastically measurable. Proving this in the following statement concludes this section on random sole discontinuity hypersurfaces.

Lemma 3.12 (Measurability of (random) domain part indicator function)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity that satisfies the measurability Assumption 3.4. Furthermore, let $\mathbb{X}_{\mathbb{T}}^{l,r} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ denote the corresponding left and right domain parts. Then, the indicator function $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^{l,r}}$ of the domain part $\mathbb{X}_{\mathbb{T}}^{l,r}$ is separately measurable. ◆

Proof. The proof is divided into two steps: First, the measurability with respect to the space-time variable $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ is shown and afterwards, we prove the measurability with respect to $\omega \in \Omega$.

(i) Let the stochastic variable $\omega \in \Omega$ be fixed. Recall that by Definition 3.1 of a sole flux discontinuity, the function $\Phi^{\mathfrak{D}}(\omega,\cdot,\cdot) \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^{d-1};\mathbb{R})$ is continuously differentiable. Furthermore, by construction of the domain parts $\mathbb{X}_{\mathbb{T}}^{l,r}(\omega)$ via Equation (3.8), the sets $\mathbb{X}_{\mathbb{T}}^{l,r}(\omega)$ are open and therefore, also Borel-measurable.

Since Borel subsets of $\mathbb{X}_\mathbb{T}$ are Lebesgue measurable (see, e.g., [72, Proposition 1.3.8]), we obtain the measurability of $\mathbb{X}_\mathbb{T}^{l,r}(\omega)$ as subsets of $\mathbb{X}_\mathbb{T}$. This immediately implies the measurability of $\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}}$ with respect to $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, since the indicator function of a measurable set is a measurable function (see, e.g., [248, Proposition 1.9 (d)]).

(ii) Let $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ be fixed. Note, we can rewrite the indicator function as

$$\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}(\omega)}(\mathfrak{x}) = \mathbb{1}_{\mathbb{X}_\mathbb{T}}(\mathfrak{x}) - \mathbb{1}_{\mathbb{X}_\mathbb{T}\setminus\mathbb{X}_\mathbb{T}^{l,r}(\omega)}(\mathfrak{x}) = \mathbb{1}_{\mathbb{X}_\mathbb{T}}(\mathfrak{x}) - \mathbb{1}_{\mathrm{cl}\left(\mathbb{X}_\mathbb{T}^{r,l}(\omega)\right)}(\mathfrak{x}) \,.$$

Here, $\mathrm{cl}\big(\mathbb{X}_\mathbb{T}^{r,l}(\omega)\big)$ denotes the closure of $\mathbb{X}_\mathbb{T}^{r,l}(\omega)$ and $\mathbb{1}_{\mathbb{X}_\mathbb{T}}$ denotes the indicator of the whole space-time domain $\mathbb{X}_\mathbb{T}$. However, $\mathbb{1}_{\mathbb{X}_\mathbb{T}}$ is measurable in $\omega \in \Omega$, since it is independent of it. Furthermore, the indicator function $\mathbb{1}_{\mathrm{cl}\left(\mathbb{X}_\mathbb{T}^{r,l}(\omega)\right)}$ is measurable:
$\mathrm{cl}\big(\mathbb{X}_\mathbb{T}^{r,l}(\omega)\big)$ is a closed nonempty subset of $\mathbb{X}_\mathbb{T}$, so we have $\mathrm{cl}\big(\mathbb{X}_\mathbb{T}^{r,l}(\omega)\big) \in \mathcal{CS}(\mathbb{X}_\mathbb{T})$, where $\mathcal{CS}(\mathbb{X}_\mathbb{T})$ is the hyperspace of nonempty closed subsets of $\mathbb{X}_\mathbb{T}$ as introduced in Section 2.2.1. Additionally, by Corollary 3.10 the correspondences $\mathbb{X}_\mathbb{T}^{l,r} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ are measurable and by Proposition 2.8 a set-valued map is measurable if and only if its closure is measurable. Therefore, the closure $\mathrm{cl}\big(\mathbb{X}_\mathbb{T}^{r,l}(\omega)\big)$ is measurable in $\omega \in \Omega$. The measurability of the random domain part indicator function $\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}}$ follows now immediately via the measurability of the set-dependent indicator function, which was shown in Lemma 2.18, and by noting that the composition of two Borel-measurable functions is again Borel-measurable.

Combining the above results, we have shown the separate measurability of the domain part indicator functions $\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}} : \Omega \times \mathbb{X}_\mathbb{T} \to \mathbb{R}$. ■

3.2 Random admissibility conditions and random $\mathfrak{G}$-entropy solutions

In this section, we introduce and discuss admissibility criteria for random scalar conservation laws having a sole flux discontinuity. Such conditions are necessary to select a unique weak solution of Problem (3.1), which coincides with the modeled physical phenomenon. As we have mentioned in Result (1.3) in the introduction, there may exist infinitely many notions of solutions to the discontinuous-flux Problem (3.1) that are mathematically equally consistent, which was shown in [3]. In general, two types of admissibility conditions can be distinguished:

The first type of criterion is based on locally identifying *strong one-sided traces* of admissible solutions that enforce the Rankine-Hugoniot condition across the discontinuity interface, which guarantees local conservativity of solutions. The main tool for defining solutions in this local manner, so-called *admissibility germs*, are introduced in Section 3.2.1, where also relevant properties of germs are derived. The corresponding type of admissibility criterion is discussed in Section 3.2.2. Alternatively, one can modify the Kružkov entropy Condition (1.2) by incorporating the contribution of the discontinuity interface into the Inequality (1.2). This leads to global admissibility criteria, which are discussed in Section 3.2.3. We conclude this section by showing equivalence of these two definitions for the case of $\mathcal{L}^1$-dissipative admissibility germs and highlighting advantages and shortcomings of the two presented ways to express admissibility in Section 3.2.4

Before we dive into these various aspects of admissibility conditions and selection criteria for entropy solutions, let us introduce an assumption on the stochastic discontinuous flux function $\mathfrak{f}$ that we impose throughout this section and the remainder of this chapter.

Assumption 3.13 (Flux function with sole discontinuity)**:**
A flux function $\mathfrak{f}$ *having a sole discontinuity hypersurface* $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ *satisfies the following assumptions for every random parameter* $\omega \in \Omega$*:*

(F-1) The flux function $\mathfrak{f} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \to \mathbb{X}$ *has the form*

$$(\omega, \mathfrak{x}, \upsilon) \mapsto \begin{cases} \mathfrak{f}^l(\omega, \mathfrak{x}, \upsilon) & \text{for } \mathfrak{x} \in \mathbb{X}_\mathbb{T}^l(\omega) \,, \\ \mathfrak{f}^r(\omega, \mathfrak{x}, \upsilon) & \text{for } \mathfrak{x} \in \mathbb{X}_\mathbb{T}^r(\omega) \,, \end{cases}$$

where $\mathfrak{f}^{l,r}$ *are the restrictions of* $\mathfrak{f}$ *to the random left and right space-time domain part* $\mathbb{X}_\mathbb{T}^{l,r}(\omega)$*, respectively.*

(F-2) For fixed space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$*, the left and right flux functions* $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ *are* locally Lipschitz *continuous.*

(F-3) For fixed scalar value $\upsilon \in \mathbb{R}$*, the left and right flux functions* $\mathfrak{f}^{l,r}(\omega, \cdot, \upsilon)$ *are* globally Lipschitz *continuous.* ◆

Let us briefly comment on the locally Lipschitz Assumption (F-2). In the deterministic setting, one can relax the locally Lipschitz assumption. For one-dimensional deterministic problems, mere continuity would be sufficient, while an additional assumption on the character of continuity needs to be imposed for multi-dimensional problems, see [171, 172]. However, in the randomized setting, the locally Lipschitz Assumption (F-2) is crucial to proving the existence of moments of the solution.

3.2.1 Admissibility germs and their properties

In this section, we introduce the theory of so-called *admissibility germs* that describe admissible solutions by explicitly prescribing possible jumps across the flux discontinuity. This unifying framework was initially introduced by Andreianov et al. [13] for a deterministic one-dimensional model problem. Many of these deterministic ideas can be generalized to the stochastic setting in a straightforward manner. In this section, we recapitulate the main definitions and properties, while focusing on their extension to our random scalar multi-dimensional conservation law with a sole flux discontinuity, which is given by Equation (3.1). We start the discussion with the following basic definition of an admissibility germ, which is a main tool for locally describing the permissibility of solutions.

Definition 3.14 (Admissibility germ)**:**
Let a couple of continuous functions $g^{l,r} \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ *be given. Any set* $\mathfrak{G} \subset \mathbb{R}^2$ *is called an* admissibility germ *associated to the couple* (g^l, g^r)*, if the Rankine-Hugoniot condition*

$$g^l(\mathfrak{u}^l) = g^r(\mathfrak{u}^r) \tag{3.9}$$

is satisfied for every pair $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathfrak{G}$*.* ◆

For the task of finding admissible solutions, the Rankine-Hugoniot Condition (3.9) ensures local conservativity across jump discontinuities. As we are dealing with random sole jump interfaces of the flux, we extend this notion of admissibility germs to *random families of admissibility germs* in the next definition.

Definition 3.15 (Random family of admissibility germs)**:**
Let $\omega \in \Omega$ and $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ be fixed. Further, let $g^{l,r}(\omega, \mathfrak{x}, \cdot)$ denote the normal components of the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$, i.e.,

$$g^l(\omega, \mathfrak{x}, \cdot) \coloneqq \mathfrak{f}^l(\omega, \mathfrak{x}, \cdot) \cdot \widehat{\mathfrak{n}_\mathfrak{D}}(\omega, \mathfrak{x}) \,, \qquad g^r(\omega, \mathfrak{x}, \cdot) \coloneqq \mathfrak{f}^r(\omega, \mathfrak{x}, \cdot) \cdot \widehat{\mathfrak{n}_\mathfrak{D}}(\omega, \mathfrak{x}) \,, \tag{3.10}$$

where, $\widehat{\mathfrak{n}_\mathfrak{D}}$ is the normal field of the random sole discontinuity extended to the whole space-time domain $\mathbb{X}_\mathbb{T}$, as defined in Equation (3.7)*.*
Based on these normal components, any family of admissibility germs $\mathfrak{G}(\omega, \mathfrak{x})$ associated to the pair of continuous functions $\big(g^l(\omega, \mathfrak{x}, \cdot), g^r(\omega, \mathfrak{x}, \cdot)\big)$ is called a random family of admissibility germs*. The dependency on the random parameter $\omega \in \Omega$ and the space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ is formalized by writing the random family of admissibility germs as a set-valued mapping $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$.* ◆

To establish a well-posedness theory with these random families of admissibility germs, we need to specify the dependency of $\mathfrak{G}$ on the stochastic parameter $\omega \in \Omega$ and the space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Therefore, we impose the following joint measurability assumption on the random family of admissibility germs.

Assumption 3.16 (Joint measurability of random family of germs)**:**
We assume that any random family of germs $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ as defined in Definition 3.15 is jointly measurable. ◆

At a first glance, this joint measurability assumption might seem rather restrictive. However, measurability of a family of germs is closely related to the existence of $\mathfrak{G}$-entropy solutions as the example in [8, Section 1.3 and Appendix] demonstrates for a deterministic discontinuous-flux setting. A similar example and argumentation can be used for the joint measurability Assumption 3.16 in the stochastic setting at hand. Additionally, in Section 3.4, we verify this joint measurability property for the important example of a random family of vanishing viscosity admissibility germs.

An important class of admissibility germs are those, which satisfy a $\mathcal{L}^1$-dissipativity condition. Such $\mathcal{L}^1$-dissipativity is inspired by interpreting the solutions of Riemann problems as trajectories of an $\mathcal{L}^1$-dissipative solver for the discontinuous-flux conservation law – we refer to [13] for the details of this interpretation. We specify this condition as well as the corresponding type of admissibility germ in the next definition.

Definition 3.17 ($\mathcal{L}^1$-dissipative admissibility germ)**:**
Let $\mathfrak{G} \subset \mathbb{R}^2$ be an admissibility germ associated to a couple of continuous functions (g^l, g^r). If the $\mathcal{L}^1$-dissipativity condition

$$\mathfrak{q}^l(\mathfrak{u}^l, \mathfrak{v}^l) \geq \mathfrak{q}^r(\mathfrak{u}^r, \mathfrak{v}^r) \tag{3.11}$$

*holds for all $(\mathfrak{u}^l, \mathfrak{u}^r), (\mathfrak{v}^l, \mathfrak{v}^r) \in \mathfrak{G}$, we call $\mathfrak{G}$ an $\mathcal{L}^1$-*dissipative (admissibility) germ*, or $\mathcal{L}^1D$ germ for short. Here, $\mathfrak{q}^{l,r}$ denotes the Kružkov entropy flux*

$$\mathfrak{q}(\mathfrak{u}, \mathfrak{v}) \coloneqq \operatorname{sign}(\mathfrak{u} - \mathfrak{v})\big(g(\mathfrak{u}) - g(\mathfrak{v})\big)$$

*associated to the functions $g^{l,r}$. A random family of germs $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is a $\mathcal{L}^1$-*dissipative family of germs*, if $\mathfrak{G}(\omega, \mathfrak{x})$ is a $\mathcal{L}^1D$ admissibility germ for every $\omega \in \Omega$ and $\mathcal{H}^d$-almost every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$.* ◆

Building upon this definition, we are also interested in identifying all couples $(\hat{u}^l, \hat{u}^r) \in \mathbb{R}^2$, such that the $\mathcal{L}^1$-dissipativity is satisfied for all pairs $(u^l, u^r) \in \mathfrak{G}$ contained in a fixed germ $\mathfrak{G}$. This leads to the notion of a *dual germ*.

Definition 3.18 (Dual germ of $\mathfrak{G}$)**:**
Let $\mathfrak{G} \subset \mathbb{R}^2$ be an admissibility germ. The dual germ *of $\mathfrak{G}$, denoted $\mathfrak{G}^*$, is the set of all tuples $(\hat{u}^l, \hat{u}^r) \in \mathbb{R}^2$ satisfying the following two conditions:*

(i) The $\mathcal{L}^1$-dissipativity condition (3.11) *is satisfied for all pairs $(u^l, u^r) \in \mathfrak{G}$.*

(ii) The tuple $(\hat{u}^l, \hat{u}^r)$ satisfies the Rankine-Hugoniot condition (3.9).

If $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is a random family of germs, then $\mathfrak{G}^* : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is the family of dual germs, *if $\mathfrak{G}^*(\omega, \mathfrak{x})$ is the dual germ of $\mathfrak{G}(\omega, \mathfrak{x})$ for every $\omega \in \Omega$ and every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$.* ◆

To show the uniqueness of $\mathfrak{G}$-entropy solutions, we need the notion of a *definite* germ. However, this definiteness property requires the concept of a *maximal* germ and an *extension* of a germ. We introduce these aforementioned features in the next definition, which concludes this section on (random) admissibility germs and their properties.

Definition 3.19 (Extension of germs and properties maximal and definite)**:**
Let $\mathfrak{G} \subset \mathbb{R}^2$ be an admissibility germ. We define the following properties of the germ $\mathfrak{G}$:

- Extension of $\mathfrak{G}$*: Any set $\widehat{\mathfrak{G}} \subset \mathbb{R}^2$ is called extension of $\mathfrak{G}$, if it satisfies $\mathfrak{G} \subset \widehat{\mathfrak{G}}$ and every $(u^l, u^r) \in \widehat{\mathfrak{G}}$ satisfies the Rankine-Hugoniot condition* (3.9)*, i.e., if $\widehat{\mathfrak{G}}$ is itself a germ. If both $\mathfrak{G}$ and $\widehat{\mathfrak{G}}$ are a $\mathcal{L}^1D$ germs, then $\widehat{\mathfrak{G}}$ is called a $\mathcal{L}^1D$* extension of $\mathfrak{G}$.
- Maximal germ*: If an admissibility germ $\mathfrak{G}$ does not admit a nontrivial extension, then $\mathfrak{G}$ is called a* maximal germ.
- Definite germ*: If a germ $\mathfrak{G}$ possesses a unique maximal $\mathcal{L}^1D$ extension, then $\mathfrak{G}$ is called a* definite germ.

If $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is a random family of germs, then $\widehat{\mathfrak{G}} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is an extension of $\mathfrak{G}$, if $\widehat{\mathfrak{G}}(\omega, \mathfrak{x})$ is an extension of $\mathfrak{G}(\omega, \mathfrak{x})$ for every $\omega \in \Omega$ and every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Furthermore, $\mathfrak{G}$ is called a random family of maximal/definite germs*, if $\mathfrak{G}(\omega, \mathfrak{x})$ is maximal/definite for every $\omega \in \Omega$ and every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$.* ◆

As was shown in [13, Proposition 3.4], definiteness of an admissibility germ $\mathfrak{G}$ implies that the corresponding dual germ $\mathfrak{G}^*$ is the unique maximal $\mathcal{L}^1$D extension of $\mathfrak{G}$. Furthermore, it follows that $\mathfrak{G}$ being definite is a necessary and sufficient condition for $\mathfrak{G}^*$ to be $\mathcal{L}^1$-dissipative.

3.2.2 $\mathfrak{G}$-entropy solutions via admissibility germs

After the preliminary discussion of admissibility germs, we are almost ready to define admissible solutions. The main idea of these germ-based admissibility criteria is the following: Away from the flux discontinuity, i.e., locally in each domain part $\mathbb{X}_\mathbb{T}^{l,r}$, the solution should be a Kružkov entropy solution in the sense of Condition (1.2). Additionally, along the discontinuity $\mathfrak{D}$, the solution should satisfy the local conservativity induced by the Rankine-Hugoniot condition. To evaluate the Rankine-Hugoniot condition with the values of a solution across the discontinuity hypersurface, we utilize strong one-sided traces of such a solution, which are defined in the subsequent definition.

Definition 3.20 (Strong one-sided traces)**:**
Let $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$ be a sole discontinuity manifold, which is described via the graph of a continuous function $\Phi^\mathfrak{D} \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^{d-1}; \mathbb{R})$ as in Equation (3.2)*. A function $g \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ admits* strong[13] right- and left-sided traces $\gamma^{r,l} g$ *on $\mathfrak{D}$, if for all test functions $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T})$ it holds that*

$$\lim_{h \to 0} \frac{1}{h} \int_\mathbb{T} \int_0^h \int_{\mathbb{R}^{d-1}} \left| g\big(\mathfrak{d} \pm y \boldsymbol{e}_2\big) - (\gamma^r g)(\mathfrak{d}) \right| \psi(\mathfrak{d}) \, \mathrm{d}\boldsymbol{x}_{2:d} \, \mathrm{d}y \, \mathrm{d}t = 0 \, .$$

Here, $\mathfrak{d} = \big(t, \Phi^\mathfrak{D}(t, \boldsymbol{x}_{2:d}), \boldsymbol{x}_{2:d}\big) \in \mathfrak{D} \subset \mathbb{X}_\mathbb{T} \subset \mathbb{R}^{d+1}$ denotes a point on the discontinuity hypersurface $\mathfrak{D}$ and $\boldsymbol{e}_2 \in \mathbb{R}^{d+1}$ denotes the second canonical basis vector of $\mathbb{R}^{d+1}$. Similarly, we can define the strong initial trace $\gamma^0 g$ *on the set $\{(t, \boldsymbol{x}) \in \mathbb{X}_\mathbb{T} \mid t = 0\}$.* ◆

Unfortunately, we cannot ensure the existence of strong one-sided traces of a solution for our general setting. Therefore, we impose the following assumption that overcomes this problem.[14]

Assumption 3.21 (Genuine nonlinearity of flux function)**:**
Let a random parameter $\omega \in \Omega$, a time $t \in \mathbb{T}$ and a space location $\boldsymbol{x} \in \mathbb{X}$ be fixed. We assume that the left and right flux functions $\mathfrak{f}^{l,r}(\omega, t, \boldsymbol{x}, \cdot)$ are genuinely nonlinear in the sense that

$$\mathcal{L}^1\left(\hat{v} \in \mathbb{R} \mid \frac{\partial}{\partial v} \mathfrak{f}^{l,r}(\omega, t, \boldsymbol{x}, \hat{v}) = 0 \right) = 0,$$

where $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure. In other words, the assumption postulates that the left and right fluxes $\mathfrak{f}^{l,r}(\omega, t, \boldsymbol{x}, \cdot)$ are not constant on any nontrivial interval $I \subset \mathbb{R}$. ◆

The above genuine nonlinearity assumption ensures the existence of strong traces of the solution. By virtue of the following remark, this also implies that the flux functions $\mathfrak{f}^{l,r}$ and the Kružkov entropy fluxes $\mathfrak{q}^{l,r}$ admit strong one-sided traces.

Remark 3.22 (Composition admits traces)**:** *Let $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$ be a sole discontinuity hypersurface. Furthermore, let a function $h : \mathbb{X}_\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be continuous and let $g \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ admit strong one-sided traces $\gamma^{l,r} g$ on $\mathfrak{D}$. Then, the composition $h \circ g := h(\cdot, g(\cdot))$ admits strong one-sided traces on $\mathfrak{D}$. Additionally, the equality*

$$\Big(\gamma^{l,r}(h \circ g)\Big)(\mathfrak{d}) = h\Big(\mathfrak{d}, (\gamma^{l,r} g)(\mathfrak{d})\Big)$$

holds for $\mathcal{H}^d$-almost every $\mathfrak{d} \in \mathfrak{D}$. ◆

Equipped with these strong one-sided traces and the admissibility germs presented in the previous section, we have all necessary tools at hand to define pathwise admissible $\mathfrak{G}$-entropy solutions to the random scalar conservation law given by Equation (3.1).

[13] We use the word "strong" to indicate that these traces should be understood in the $\mathcal{L}^1_{\mathrm{loc}}$ topology.

[14] The existence of traces of a solution is not necessary in the general case. One rather works with strong traces of the normal components of the flux function and the normal components of the corresponding Kružkov entropy flux. We refer to Panov [229] for the appropriate notion of traces. Additionally, we refer to [13] for a formulation that avoids the existence of traces $\gamma^{l,r} u$ of the solution u.

Definition 3.23 (Pathwise $\mathfrak{G}$-entropy solution via germ formulation)**:**
Let the following conditions be fulfilled:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity as defined in Definition 3.1.*
- *Let $\mathfrak{f}$ be a flux function that satisfies the sole-flux-discontinuity Assumption 3.13 and the genuine nonlinearity Assumption 3.21.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs satisfying the joint measurability Assumption 3.16 and let $\mathfrak{G}^* : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be the family of dual germs of $\mathfrak{G}$ according to Definition 3.18.*

Then, for fixed $\omega \in \Omega$, a function $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is called a $\mathfrak{G}$-entropy solution to Problem (3.1)*, if the following conditions are satisfied:*

(i) The restriction of $u(\omega, \cdot, \cdot)$ to the domain parts $\mathbb{X}_\mathbb{T}^{l,r}(\omega)$ is a Kružkov entropy solution of Equation (3.1) *with flux functions $\mathfrak{f}^{l,r}$, respectively, in the sense of Condition* (1.2)*.*

(ii) For $\mathcal{H}^d$-almost every $\mathfrak{d} \in \mathfrak{D}(\omega)$, the couple of strong traces $(\gamma^l u, \gamma^r u)$ of $u(\omega, \cdot, \cdot)$ on the discontinuity $\mathfrak{D}(\omega)$ belongs to the dual germ $\mathfrak{G}^*(\omega, \mathfrak{d})$.

(iii) $\mathcal{H}^d$-almost everywhere on $\{0\} \times \mathbb{X}$, the initial trace $\gamma^0 u$ of $u(\omega, \cdot, \cdot)$ is equal to the initial condition $u_0(\omega, \cdot)$. ◆

Let us briefly comment on why this definition makes sense: Condition (i) implies the existence of the initial trace $\gamma^0 u$ due to the result of Panov [228]. Since the discontinuity is given by the graph of a continuously differentiable function and the fluxes $\mathfrak{f}^{l,r}$ are genuine nonlinear, the boundary traces $\gamma^{l,r} u$ of u on $\mathfrak{D}(\omega)$ exist (see Panov [229]).

3.2.3 Global admissibility via adapted entropy conditions

In the previous section, the admissibility of pathwise $\mathfrak{G}$-entropy solutions has been described in a local manner via admissibility germs. The purpose of this section is to derive an equivalent global admissibility criterion, similar to the Kružkov entropy Condition (1.2). While the definition via germs is very useful for proving uniqueness, these kind of global inequalities are better suited for numerical analysis or to perform investigation on the measurability of random solutions.

As a starting point, we adapt the classical notion of Kružkov entropies and Kružkov entropy fluxes to the discontinuous flux setting in the following two definitions.

Definition 3.24 (Adapted Kružkov entropy)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity hypersurface and denote by $\mathbb{X}_\mathbb{T}^{l,r} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ the resulting left and right part of the space-time domain $\mathbb{X}_\mathbb{T}$. Then, for a fixed pair of entropy values $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$, the adapted Kružkov entropy *is defined as*

$$k(\omega, t, \boldsymbol{x}) := k^l \mathbb{1}_{\mathbb{X}_\mathbb{T}^l}(\omega, t, \boldsymbol{x}) + k^r \mathbb{1}_{\mathbb{X}_\mathbb{T}^r}(\omega, t, \boldsymbol{x}) \,, \tag{3.12}$$

where $\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}}$ denotes the indicator function of $\mathbb{X}_\mathbb{T}^{l,r}$ as defined in Definition 3.11. ◆

Definition 3.25 (Kružkov entropy flux)**:**
Let $\mathfrak{f}$ be a flux function that satisfies the sole-flux-discontinuity Assumption 3.13. Then, the Kružkov entropy flux $\mathfrak{q} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ *is defined as*

$$\mathfrak{q}(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon}) := \operatorname{sign}(\upsilon - \tilde{\upsilon})\big(\mathfrak{f}(\omega, \mathfrak{x}, \upsilon) - \mathfrak{f}(\omega, \mathfrak{x}, \tilde{\upsilon})\big) \, . \tag{3.13}$$

The left and right entropy fluxes $\mathfrak{q}^{l,r}$ are defined similarly by replacing $\mathfrak{f}$ in Equation (3.13) *with the left and right flux function $\mathfrak{f}^l$ and $\mathfrak{f}^r$, respectively.* ◆

Now, we can define the main tool in deriving a global admissibility criterion corresponding to a germ $\mathfrak{G}$, which is a *(random) remainder function*. The purpose of such a remainder function is to measure how close a tuple $\boldsymbol{k}$ consisting of values $(k^l, k^r) \in \mathbb{R}^2$ is to a fixed germ $\mathfrak{G}$.

Definition 3.26 (Remainder function of admissibility germ)**:**
Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-disspative germs as defined in Definitions 3.15 and 3.17. A function $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ associated to $\mathfrak{G}$ is called a remainder function *of the family of $\mathcal{L}^1 D$ germs $\mathfrak{G}$, if it satisfies the following conditions:*

(i) For fixed $\omega \in \Omega$ and fixed $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the function $\mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{x}; \cdot)$ is continuous.

(ii) For fixed $\omega \in \Omega$ and $\mathcal{H}^d$ almost every $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the function $\mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{x}; \cdot)$ vanishes for all $\boldsymbol{k} \in \mathfrak{G}(\omega, \mathfrak{x})$ in the sense that

$$\text{for all } \boldsymbol{k} \in \mathfrak{G}(\omega, \mathfrak{x}) : \qquad \lim_{r \searrow 0} \fint_{\mathbb{B}_r(\mathfrak{x}) \cap \mathfrak{D}(\omega)} \mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{z}; \boldsymbol{k}) \, \mathrm{d}\mathfrak{z} = 0 \, ,$$

where $\mathbb{B}_r(\mathfrak{x})$ denotes the closed ball around $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ with radius $r \in \mathbb{R}_{>0}$. Note, the integral $\fint_E$ denotes that the integral $\int_E$ is divided by the volume of its domain E.

(iii) For fixed $\omega \in \Omega$ and $\mathcal{H}^d$ almost every $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the function $\mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{x}; \cdot)$ is an upper bound on the Kružkov entropy flux, i.e., for all $\boldsymbol{k} \in \mathbb{R}^2$ and $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathfrak{G}(\omega, \mathfrak{x})$ it holds that

$$\Big(\mathfrak{q}^r(\omega, \mathfrak{x}, \mathfrak{u}^r, k^r) - \mathfrak{q}^l(\omega, \mathfrak{x}, \mathfrak{u}^l, k^l)\Big) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \leq \mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{x}; \boldsymbol{k}) \, ,$$

where $\mathfrak{q}^{l,r}$ is the left and right entropy flux given by Equation (3.13) *and $\widehat{\mathfrak{n}_{\mathfrak{D}}}$ denotes the extended normal unit vector of $\mathfrak{D}(\omega)$ as defined in Equation* (3.7). ◆

The continuity requirement (i) is most natural, since the purpose of $\mathfrak{R}_{\mathfrak{G}}$ is, roughly speaking, to measure some distance between $\mathfrak{G} \subset \mathbb{R}^2$ and a tuple $\boldsymbol{k} \in \mathbb{R}^2$. Possible choices for $\mathfrak{R}_{\mathfrak{G}}$ that might depend on the flux functions $\mathfrak{f}^{l,r}$ are given in Appendix A. The conditions (ii) and (iii) in Definition 3.26 are closely related to arguing equivalence of the two formulations of admissible solutions. Therefore, the corresponding discussion is postponed to Section 3.2.4.

With this remainder function, all necessary tools for stating an adapted Kružkov entropy condition as a global admissibility criterion are available. Similar to the classical entropy condition, this new criterion is supposed to hold in the sense of distributions, which implies integration (after multiplication with a test function). However, integrating the remainder function is only defined, if the remainder function is measurable. By its construction, the remainder function $\mathfrak{R}_{\mathfrak{G}}$ can only be (jointly) measurable, if the associated admissibility germ $\mathfrak{G}$ is (jointly) measurable. To simplify the well-posedness presentation, we impose the following measurability assumption on the remainder function.

Assumption 3.27 (Measurability of remainder function)**:**
Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs satisfying the joint measurability Assumption 3.16. Then, the remainder function $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ associated to $\mathfrak{G}$ is also jointly measurable. ◆

This assumption is simplifying in the sense that the remainder function can take various forms. In Appendix A, we discuss the measurability of some particular choices for the remainder function. We can now state the definition of a $\mathfrak{G}$-entropy solution via a global admissibility criterion, the *adapted entropy inequality*.

Definition 3.28 (Pathwise $\mathfrak{G}$-entropy solution via entropy inequality)**:**
Let the following requirements be fulfilled:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity.*
- *Let $\mathfrak{f}$ be a flux function satisfying the sole-flux-discontinuity Assumption 3.13.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1 D$ germs associated to the left and right flux function $\mathfrak{f}^{l,r}$ that satisfies the joint measurability Assumption 3.16.*
- *Suppose that $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is the remainder function associated to $\mathfrak{G}$ and satisfies the joint measurability Assumption 3.27.*

Then, for fixed stochastic parameter $\omega \in \Omega$, a function $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is called a $\mathfrak{G}$-entropy solution *to Problem* (3.1)*, if the following two conditions are satisfied:*

(i) The function $u(\omega, \cdot, \cdot)$ is a solution in the sense of distributions, i.e., for any nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, it holds that

$$\int_{\mathbb{T}} \int_{\mathbb{X}} u(\omega, t, \boldsymbol{x}) \partial_t \psi(t, \boldsymbol{x}) + \mathfrak{f}\big(\omega, t, \boldsymbol{x}, u(\omega, t, \boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0 \, .$$

(ii) For all pairs of entropy values $\boldsymbol{k} := (k^l, k^r) \in \mathbb{R}^2$ let $k(\omega, t, \boldsymbol{x})$ denote the corresponding adapted Kružkov entropies given by Equation (3.12)*. For all nonnegative test functions $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ the function $u(\omega, \cdot, \cdot)$ satisfies the* adapted entropy inequality

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\mathbb{X}} |u(\omega, t, \boldsymbol{x}) - k(\omega, t, \boldsymbol{x})| \partial_t \psi(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &\qquad + \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}(\omega, t, \boldsymbol{x}; u(\omega, t, \boldsymbol{x}), k(\omega, t, \boldsymbol{x})) \cdot \nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &- \int_{\mathbb{X}} |u_0(\omega, \boldsymbol{x}) - k(\omega, 0, \boldsymbol{x})| \psi(0, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{\mathfrak{D}(\omega)} \mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{d}; \boldsymbol{k}) \psi(\mathfrak{d}) \, \mathrm{d}\mathfrak{d} \geq 0 \, , \end{aligned} \tag{3.14}$$

where $\mathfrak{R}_{\mathfrak{G}}$ is the remainder function associated to $\mathfrak{G}$ as defined in Definition 3.26. ◆

Before we continue, the following two remarks reflect on the two above-mentioned global conditions for pathwise $\mathfrak{G}$-entropy solutions to the random scalar discontinuous-flux conservation law given by Equation (3.1): First, we comment on the explicit demand of the function $u(\omega, \cdot, \cdot)$ being a weak solution. Afterwards, we discuss, how the integral of the remainder function $\mathfrak{R}_{\mathfrak{G}}$ over the random discontinuity hypersurface $\mathfrak{D}(\omega)$ should be understood.

Remark 3.29 (Requirement of pathwise $\mathfrak{G}$-entropy solution being weak solution, [13]): *The explicit requirement in Definition 3.28 (i) stating that $u(\omega,\cdot,\cdot)$ should be a solution in the sense of distributions, might seem superfluous. Indeed, its only purpose is to guarantee that the Rankine-Hugoniot condition holds across the jump discontinuity of the flux function. While for specific choices of the admissibility germ $\mathfrak{G}$ the function $u(\omega,\cdot,\cdot)$ might automatically be a weak solution, this does not hold in the general case.* ◆

Remark 3.30 (Integral over discontinuity hypersurface): *The domain of the last integral in the adapted entropy Inequality* (3.14) *is the random discontinuity hypersurface $\mathfrak{D}(\omega)$. Therefore, the integral should be understood in the sense of* integration on manifolds. *In particular, we emphasize the following insights and concepts regarding the integration over the discontinuity hypersurface $\mathfrak{D}(\omega)$:*

(i) If the integrand $\mathbb{I}_{\mathfrak{R}}(\omega,\mathfrak{x};\boldsymbol{k}) := \mathfrak{R}_{\mathfrak{G}}(\omega,\mathfrak{x};\boldsymbol{k})\psi(\mathfrak{x})$ is continuous and compactly supported with respect to the space-time variable $\mathfrak{x}\in\mathbb{X}_{\mathbb{T}}$, then $\mathrm{d}\mathfrak{d}$ corresponds to the Riemannian volume (differential) form of the discontinuity hypersurface $\mathfrak{D}(\omega)$. This question reduces to whether the mapping $\mathfrak{x}\mapsto\mathbb{I}_{\mathfrak{R}}(\omega,\mathfrak{x};\boldsymbol{k})$ is continuous, as the compact support property is satisfied by $\psi\in\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{X}_{\mathbb{T}};\mathbb{R})$. If the integrand $\mathbb{I}_{\mathfrak{R}}$ is not continuous, then $\mathrm{d}\mathfrak{d}$ needs to be interpreted as the Riemannian volume density of $\mathfrak{D}(\omega)$, where the Riemannian structure is guaranteed by $\mathfrak{D}(\omega)$ being a d-dimensional submanifold of the space-time domain $\mathbb{X}_{\mathbb{T}}=\mathbb{T}\times\mathbb{R}^d$. [15]

(ii) This surface integral *of the discontinuity hypersurface $\mathfrak{D}$ is equivalent to integration against the d-dimensional Hausdorff measure $\mathcal{H}^d$ (since $\mathfrak{D}(\omega)$ is a hypersurface of the $(d+1)$-dimensional space-time domain $\mathbb{X}_{\mathbb{T}}$).* [16] ◆

3.2.4 Equivalency of definitions and discourse of selection criteria

In the previous sections, we introduced two notions of pathwise $\mathfrak{G}$-entropy solutions to the random scalar discontinuous-flux conservation law given by Equation (3.1). As using the same name for both definitions already indicates, the notions of solutions are indeed equivalent. This leads to the following theorem on the equivalency of the two definitions of $\mathfrak{G}$-entropy solutions.

Theorem 3.31 (Equivalency of definitions):
Suppose the following conditions are satisfied:

- *Let $\mathfrak{D}:\Omega\rightrightarrows\mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity.*
- *Let the flux $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 and the genuine nonlinearity Assumption 3.21.*
- *Let $\mathfrak{G}:\Omega\times\mathbb{X}_{\mathbb{T}}\rightrightarrows\mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs associated to the left and right flux functions $\mathfrak{f}^{l,r}$.*

[15] Depending on the depth of information, a discussion on this topic of integration on manifolds can be found in many books, see, e.g., the introductory text [106, Section 11.4], the monographs [191, 280] on the theory of integrating differential forms and densities or the general works [99, 102] on geometric measure theory.

[16] This is a well-known consequence of standard results in the field of geometric measure theory. We refer to the monographs [99, 102, 194] for details on this approach. Let us also mention the book [180], which discusses this equivalency of the surface integral to the Hausdorff measure from a perspective of (smooth) manifolds.

- *Let $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be the remainder function associated to $\mathfrak{G}$ and let $\mathfrak{R}_{\mathfrak{G}}$ satisfy the joint measurability Assumption 3.27.*

Then, for every $\omega \in \Omega$, the definitions of pathwise $\mathfrak{G}$-entropy solutions via Definition 3.23 based on the germ formulation and Definition 3.28 using the adapted entropy inequality are equivalent. ◆

The equivalence of Definitions 3.23 and 3.28 has been proven for general $\mathcal{L}^1$-dissipative germs by Andreianov et al. in [13, Theorem 3.18] for a one-dimensional deterministic model problem. This result can also be established in the multi-dimensional sole-flux-discontinuity setting in a similar fashion. We refer to [12, Proposition 9], where the equivalency was established for the particular choice of the *vanishing viscosity germ*, which will also be introduced in Section 3.3.1. The proof presented in [12, Proposition 9] does not depend on the specific form of this vanishing viscosity germ. We give the main ideas of the proof, since these details provide insight on the conditions (ii) and (iii) of Definition 3.26 of the remainder function.

Proof of Theorem 3.31 (main ideas). Let the presumptions of Theorem 3.31 be satisfied and let the stochastic parameter $\omega \in \Omega$ be fixed. By a truncation argument, the equivalence reduces to showing that Definition 3.23 (ii) is equal to all couples $(k^l, k^r) \in \mathbb{R}^2$ satisfying the inequality

$$\liminf_{h \searrow 0} \int_{\mathbb{T}} \int_{\mathbb{X}} \psi(t,\boldsymbol{x}) \mathbf{q}\big(\omega, t, \boldsymbol{x}, u(\omega,t,\boldsymbol{x}), k(\omega,t,\boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi_h(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{\mathfrak{D}(\omega)} \mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{d}; \boldsymbol{k}) \psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d} \geq 0 \,.$$

Here, h denotes the truncation variable of the test function ψ. By the existence of strong traces $\gamma^{l,r}u$, this inequality can be reformulated to

$$\int_{\mathfrak{D}(\omega)} \Big(\mathbf{q}^l\big(\omega, \mathfrak{d}, \gamma^l u(\omega,\mathfrak{d}), k(\omega,\mathfrak{d})\big) - \mathbf{q}^r\big(\omega, \mathfrak{d}, \gamma^r u(\omega,\mathfrak{d}), k(\omega,\mathfrak{d})\big) + \mathfrak{R}_{\mathfrak{G}}(\omega,\mathfrak{d};\boldsymbol{k}) \Big) \psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d} \geq 0 \,, \quad (3.15)$$

which has to hold for all pairs $(k^l, k^r) \in \mathbb{R}^2$. Based on this inequality, we can now prove equivalency of the two definitions:

- Assume that Definition 3.23 (ii) holds. One immediately observes that the Inequality (3.15) is satisfied as soon as the remainder function $\mathfrak{R}_{\mathfrak{G}}$ satisfies Definition 3.26 (iii). This is due to the $\mathcal{L}^1$-dissipativity property of the germ $\mathfrak{G}$. Possible choices for the remainder function were introduced in [13], which are also presented in Appendix A where also the measurability Assumption 3.27 is verified. Let us also refer to the appendix of [8], where the question of measurability is answered for a general family of admissibility germs via a so-called *topology of germs*.
- Assume that a remainder function $\mathfrak{R}_{\mathfrak{G}}$ is given that satisfies the conditions of Definition 3.26 as well as the joint measurability Assumption 3.27. For test functions, which satisfy that $\psi\big|_{\mathfrak{D}(\omega)}$ is concentrated at a Lebesgue point $\mathfrak{d}$ of $\mathfrak{D}(\omega)$, Definition 3.26 (ii) implies

  $$\mathbf{q}^l\big(\omega, \mathfrak{d}, \gamma^l u(\omega,\mathfrak{d}), k(\omega,\mathfrak{d})\big) - \mathbf{q}^r\big(\omega, \mathfrak{d}, \gamma^r u(\omega,\mathfrak{d}), k(\omega,\mathfrak{d})\big) \geq 0 \,,$$

  for all $(k^l, k^r) \in \mathfrak{G}(\omega, \mathfrak{d})$. However, by the $\mathcal{L}^1$-dissipativity condition of the family of germs $\mathfrak{G}$, we can conclude that $\big(\gamma^l u(\omega,\mathfrak{d}), \gamma^r u(\omega,\mathfrak{d})\big)$ belongs to the dual germ $\mathfrak{G}^*(\omega,\mathfrak{d})$.

The above considerations have shown that one can deduce the Definitions 3.23 and 3.28 of $\mathfrak{G}$-entropy solutions from one another, which proves the assertion. ■

We conclude this section by comparing the two formulations of $\mathfrak{G}$-entropy solutions and highlighting advantages and drawbacks implied by the definitions. The preferred choice of the formulation is highly dependent on the given situation as the following argumentation demonstrates. Let us emphasize that the arguments carry over from the deterministic theory and we refer to [8] for details.

Identifying an admissibility germ $\mathfrak{G}$ that encodes specific modeling assumptions at the discontinuity interface is rather challenging. In contrast, formulating an entropy inequality might be simpler, if some knowledge about the germ is available. Here, information of a very small subset of this germ might suffice to successfully formulate the appropriate entropy inequality. However, this advantage vanishes, if the underlying germ depends on some parameter of the discontinuity, such as time or space.

Not only the formulation of admissibility criteria requires a different amount of work for both definitions. As we see in the next section, both notions of $\mathfrak{G}$-entropy solutions are differently well-suited to perform well-posedness analysis. Apparently, once a (definite) admissibility germ is identified, showing uniqueness of solutions appears natural. However, the approach via entropy inequalities is better suited for investigating the stochastic measurability of solutions and also for showing existence of such solutions via convergence of approximation schemes or conducting numerical analysis.

3.3 Well-posedness of random entropy solutions

In this section, the well-posedness of $\mathfrak{G}$-entropy solutions to the random scalar conservation law given by Equation (3.1) is shown. In contrast to deterministic well-posedness investigations, not only the (pathwise) existence and uniqueness of $\mathfrak{G}$-entropy solutions to Problem (3.1) needs to be argued. Additionally, strong measurability with respect to the stochastic parameter $\omega \in \Omega$ needs to be established, as we aim at interpreting the solution u as a $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$-valued, Bochner-integrable random variable $u : \Omega \to \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$.[17] Once this strong measurability of the solution has been established, the existence of moments of random $\mathfrak{G}$-entropy solutions can be investigated. These allow us to describe the statistical properties of the solution u, when interpreted as a Bochner-integrable random variable.

We start this well-posedness investigation with a discussion of pathwise existence and uniqueness of $\mathfrak{G}$-entropy solutions in Section 3.3.1. Afterwards, in Section 3.3.2, we introduce so-called *entropy functionals* and discuss their properties. These functionals provide an important tool for establishing the strong measurability of $\mathfrak{G}$-entropy solutions with respect to the stochastic parameter $\omega \in \Omega$ in Section 3.3.3. This section on well-posedness is concluded by investigating the existence of moments of $\mathfrak{G}$-entropy solutions in Section 3.3.4.

3.3.1 Pathwise existence and uniqueness of $\mathfrak{G}$-entropy solutions

In the beginning of this section, the uniqueness of $\mathfrak{G}$-entropy solutions is investigated. Therefore, we utilize the definition of solutions via the underlying admissibility germs. This strategy allows to

[17] The space $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$ is not separable. Therefore, a random variable $u : \Omega \to \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$ can only be strongly measurable, if u only takes values in a separable subspace $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$. If such a separable subspace cannot be established, one has to work in the more general setting introduced in Section 2.3.1. However, such an investigation based on weakly measurable solutions u exceeds the scope of this thesis.

formulate a general uniqueness result as soon as the germ $\mathfrak{G}$ is definite. Unfortunately, such a general result cannot be established for the existence of $\mathfrak{G}$-entropy solutions. However, for a one-dimensional model problem, we state a general result from [13] that provides the existence of a unique solution, as soon as the underlying germ is *complete*[18]. We conclude this pathwise existence and uniqueness investigation by discussing the existence of solutions to the multi-dimensional problem for the particular choice of the important (randomized) family of *vanishing viscosity germs*.

Pathwise uniqueness of $\mathfrak{G}$-entropy solutions

Let us start the well-posedness investigation by showing the pathwise uniqueness of $\mathfrak{G}$-entropy solutions. Actually, for the presented sole-flux-discontinuity setting of the previous sections, we are able to show the validity of the *Kato inequality*. From this result, it is a standard argumentation that the locally Lipschitz Property (F-2) of the flux function $\mathfrak{f}$ can be used to deduce the $\mathcal{L}^1$-contraction property as well as uniqueness of solutions. We precise this statement in the following theorem.

Theorem 3.32 (Pathwise uniqueness of $\mathfrak{G}$-entropy solutions)**:**
Let the following requirements be fulfilled:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface.*
- *Let $\mathfrak{f}$ be a flux function satisfying the sole-flux-discontinuity Assumption 3.13 and the genuine nonlinearity Assumption 3.21.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of definite admissibility germs.*

Now, for fixed $\omega \in \Omega$, let $u(\omega,\cdot,\cdot), \tilde{u}(\omega,\cdot,\cdot) \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be two pathwise $\mathfrak{G}$-entropy solutions to Problem (3.1) corresponding to the two initial conditions $u_0(\omega,\cdot), \tilde{u_0}(\omega,\cdot) \in \mathcal{L}^\infty(\mathbb{X})$, respectively. Then, the two solutions $u, \tilde{u}$ satisfy the Kato inequality

$$\begin{aligned} -\int_{\mathbb{T}}\int_{\mathbb{X}} |u(\omega,t,\boldsymbol{x}) - \tilde{u}(\omega,t,\boldsymbol{x})| \partial_t \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ - \int_{\mathbb{T}}\int_{\mathbb{X}} \mathbf{q}\big(\omega,t,\boldsymbol{x},u(\omega,t,\boldsymbol{x}),\tilde{u}(\omega,t,\boldsymbol{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ \leq \int_{\mathbb{X}} |u_0(\omega,\boldsymbol{x}) - \tilde{u_0}(\omega,\boldsymbol{x})|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{3.16}$$

for every nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Additionally, if the pointwise difference of the two initial conditions is integrable, i.e., $|u_0(\omega,\boldsymbol{x}) - \tilde{u_0}(\omega,\boldsymbol{x})| \in \mathcal{L}^1(\mathbb{X})$, the $\mathcal{L}^1$-contraction property

$$\int_{\mathbb{X}} |u(\omega,t,\boldsymbol{x}) - \tilde{u}(\omega,t,\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{X}} |u_0(\omega,\boldsymbol{x}) - \tilde{u_0}(\omega,\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \tag{3.17}$$

holds for almost every time $t \in \mathbb{T}$. In particular, the pathwise $\mathfrak{G}$-entropy solution to the random scalar discontinuous-flux conservation law given by Problem (3.1) is unique, if it exists. ◆

[18] This property of an admissibility germ $\mathfrak{G}$ is defined in Definition 3.33 and is connected to the related Riemann problems of the discontinuous-flux conservation law.

Proof (main ideas). The proof is based on the Definition 3.23 of a $\mathfrak{G}$-entropy solution via the traces being in the corresponding dual germ $\mathfrak{G}^*$. For a fixed random parameter $\omega \in \Omega$, the conditions (i) and (iii) of Definition 3.23 and the Kružkov *doubling of variables* approach give the Kato Inequality (3.16) for nonnegative test functions $\psi \in \mathcal{D}\big(\mathbb{X}_\mathbb{T} \setminus \mathfrak{D}(\omega); \mathbb{R}\big)$. We refer to Panov [228] for the details of this argumentation. Now, using a truncated test function

$$\psi_h(t,\boldsymbol{x}) \coloneqq \psi(t,\boldsymbol{x}) \min\left\{1, \frac{\big|x_1 - \Phi^{\mathfrak{D}}(\omega,t,\boldsymbol{x}_{2:d})\big|}{h}\right\} \qquad h>0\,,$$

similar to the proof of Theorem 3.31, it suffices to show that

$$\liminf_{h\searrow 0} \int_\mathbb{T}\int_\mathbb{X} \psi(t,\boldsymbol{x}) \mathfrak{q}\big(\omega,t,\boldsymbol{x},u(\omega,t,\boldsymbol{x}),\tilde{u}(\omega,t,\boldsymbol{x})\big)\cdot \nabla_{\boldsymbol{x}}\,\psi_h(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \geq 0\,. \tag{3.18}$$

We refer to [12, Section 3] and [13, Proof of Theorem 3.11] for the details on this step. Since the functions $u(\omega,\cdot,\cdot), \tilde{u}(\omega,\cdot,\cdot)$ are $\mathfrak{G}$-entropy solutions, we obtain the existence of strong traces $\gamma^{l,r}u$ and $\gamma^{l,r}\tilde{u}$. With these traces and the definition of ψ_h, the Inequality (3.18) can be rewritten to the condition

$$\mathfrak{q}^l(\omega,\mathfrak{x},\gamma^l u,\gamma^l\tilde{u}) \geq \mathfrak{q}^r(\omega,\mathfrak{x},\gamma^r u,\gamma^r\tilde{u}) \qquad \mathcal{H}^d-\text{a.e. on } \mathfrak{D}(\omega)\,.$$

Since $\mathfrak{G}$ is a family of definite admissibility germs, it follows directly that the corresponding dual germs $\mathfrak{G}^*$ form a family of $\mathcal{L}^1$-dissipative germs. This completes the proof of the Kato inequality (3.16), since the $\mathcal{L}^1$-dissipativity condition of the dual germ $\mathfrak{G}^*$ is implied by Definition 3.23 (ii). Since the flux function $\mathfrak{f}$ is assumed to be locally Lipschitz by Assumption (F-2), it is a standard result that the Kato inequality implies the $\mathcal{L}^1$-contraction property (3.17) and uniqueness of the pathwise $\mathfrak{G}$-entropy solution to Problem (3.1).[19] ∎

Pathwise existence of $\mathfrak{G}$-entropy solutions in one space dimension

With the above discussion, we have established the uniqueness of $\mathfrak{G}$-entropy solutions, as soon as the underlying admissibility germ is definite. Unfortunately, such a general result is not possible for ensuring existence of solutions in the multi-dimensional sole-flux-discontinuity setting. However, a general existence result can be established for the one-dimensional model problem

$$\begin{aligned} \partial_t u + \operatorname{div}_x \mathfrak{f}(\omega,x,u) &= 0 && \text{in } \Omega\times\mathbb{X}_\mathbb{T} \\ u(\omega,0,x) &= u_0(\omega,x) && \text{on } \Omega\times\{0\}\times\mathbb{R}\,. \end{aligned} \tag{3.19}$$

Here, the flux function $\mathfrak{f}:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is assumed to be defined via the mapping

$$(\omega,x,\upsilon) \mapsto \begin{cases} \mathfrak{f}^l(\omega,\upsilon) & x<\mathfrak{d}(\omega)\,, \\ \mathfrak{f}^r(\omega,\upsilon) & x>\mathfrak{d}(\omega)\,. \end{cases} \tag{3.20}$$

Before we are able to state the existence theorem for $\mathfrak{G}$-entropy solutions on Problem (3.19), (3.20), we need to define the notion of a *complete* germ.

[19] The local Lipschitz assumption on the flux $\mathfrak{f}$ is crucial for deducing the $\mathcal{L}^1$-contraction and uniqueness in several spatial dimensions. Indeed, mere continuity of the flux is not sufficient as the counterexample in [171] demonstrates. This issue of non-uniqueness for merely continuous flux functions is closely related to the property of *infinite speed of propagation*. We refer to [171, 172, 227] for details on this topic.

Definition 3.33 (Complete germs):
Let $\mathfrak{G} \subset \mathbb{R}^2$ be a germ. Then $\mathfrak{G}$ is called complete, *if all Riemann problems of Equation* (3.19) *admit a self-similar solution*[20] *u, such that the pair of left and right traces*[21] *$(\gamma^l u, \gamma^r u)$ are contained in $\mathfrak{G}$. A random germ $\mathfrak{G} : \Omega \rightrightarrows \mathbb{R}^2$ is called complete, if $\mathfrak{G}(\omega)$ is complete for every $\omega \in \Omega$.* ◆

Completeness of the germ $\mathfrak{G}$ implies the availability of uniform $\mathcal{L}^\infty$-bounds on the solution (see [13, Section 3.4] for details). However, such an assumption is rather restrictive and can be seen as the main limitation of the presented theory. Nevertheless, with the help of complete germs, we can now state the following existence result.

Theorem 3.34 (Pathwise existence of $\mathfrak{G}$-entropy solutions in one space dimension):
Let $\mathfrak{G} : \Omega \rightrightarrows \mathbb{R}^2$ be a random complete $\mathcal{L}^1$-dissipative germ associated to a pair $(\mathfrak{f}^l, \mathfrak{f}^r)$ of flux functions such that $\mathfrak{f}$ is given by Equation (3.20) *and that $\mathfrak{f}$ satisfies Assumption (F-2). Then, for any initial function $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X})\big)$ and fixed $\omega \in \Omega$, there exists a unique $\mathfrak{G}$-entropy solution to the random scalar conservation law given by Equation* (3.19). ◆

Proof (main ideas). The idea of the proof is showing the convergence of approximate solutions to Problem (3.19) for fixed stochastic parameter $\omega \in \Omega$. These approximations are obtained by a monotone three-point finite volume scheme (compare Section 2.5.2), which is adapted to the flux discontinuity by employing the Godunov scheme (2.9) at the jump interface. Due to the completeness of the admissibility germ $\mathfrak{G}(\omega)$, uniform $\mathcal{L}^\infty$-bounds on the solution are available, similar to the result of [13, Proposition 3.20]. To argue the compactness of the family of approximate solutions, the $\mathcal{BV}_{\text{loc}}$ estimate from Bürger et al. [52, 54] is used for compactly supported initial conditions $u_0(\omega, \cdot) \in \mathcal{BV}(\mathbb{R}; \mathbb{R})$. This estimate establishes convergence of the constructed approximations.

Afterwards, the $\mathcal{BV}_{\text{loc}}$ estimates are used to argue that the limit of these approximations is indeed the unique $\mathfrak{G}$-entropy solution. To conclude this result, one can deduce that the constructed scheme yields an approximation, such that per definition its traces belong to the admissibility germ $\mathfrak{G}$. This concludes the pathwise existence proof for compactly supported initial conditions $u_0(\omega, \cdot) \in \mathcal{BV}(\mathbb{R}; \mathbb{R})$. The generalization to arbitrary initial conditions $u_0(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{R}; \mathbb{R})$ can then be argued via the $\mathcal{L}^1$-contraction principle: A truncation and regularization of the initial condition u_0 can be used to construct a strongly compact sequence of approximations. Afterwards, Definition 3.28 can be leveraged for passing to the limit. ■

Pathwise existence of $\mathfrak{G}$-entropy solutions in multiple space dimensions

To the best of the author's knowledge, a general existence result analogous to Theorem 3.34 is not available for multiple spatial dimensions. However, the existence of pathwise $\mathfrak{G}$-entropy solutions can be established for the *vanishing viscosity germ* $\mathfrak{G}_{\text{VV}}$. This germ describes the behavior of $\mathfrak{G}_{\text{VV}}$-entropy solutions as the limit of solutions to the vanishing viscosity problem

$$\partial_t u^\eta + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, t, \boldsymbol{x}, u^\eta) = \eta \, \Delta \, u^\eta \, . \tag{3.21}$$

[20] A solution u is called *self-similar* if it only depends on the ratio $\xi := x/t$.

[21] Due to the specific form of the random model Problem (3.19), the traces $\gamma^{l,r} u$ are just the limits of u as x approaches the random flux discontinuity $\mathfrak{d}(\omega)$ from below and above, respectively.

As a starting point to argue the existence of vanishing viscosity solutions, a formal definition of the underlying vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}$ is necessary.

Definition 3.35 (Vanishing viscosity germ)**:**
Let $\mathfrak{G} \subset \mathbb{R}^2$ be an admissibility germ as defined in Definition 3.14, associated to the pair of continuous functions (g^l, g^r). The germ is called vanishing viscosity germ*, denoted $\mathfrak{G}_{\mathrm{VV}}$, if it consists of all pairs $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R}$ satisfying one of the following conditions*

- $\mathfrak{u}^l = \mathfrak{u}^r$,
- $\mathfrak{u}^l < \mathfrak{u}^r$ *and there exists a* $\mathfrak{u}^o \in [\mathfrak{u}^l, \mathfrak{u}^r]$ *such that* $\begin{cases} g^l(\rho) \geq \mathfrak{s} & \text{for all } \rho \in [\mathfrak{u}^l, \mathfrak{u}^o] \,, \\ g^r(\rho) \geq \mathfrak{s} & \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^r] \,, \end{cases}$
- $\mathfrak{u}^l > \mathfrak{u}^r$ *and there exists a* $\mathfrak{u}^o \in [\mathfrak{u}^r, \mathfrak{u}^l]$ *such that* $\begin{cases} g^l(\rho) \leq \mathfrak{s} & \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^l] \,, \\ g^r(\rho) \leq \mathfrak{s} & \text{for all } \rho \in [\mathfrak{u}^r, \mathfrak{u}^o] \,, \end{cases}$

where $\mathfrak{s} := g^l(\mathfrak{u}^l) = g^r(\mathfrak{u}^r)$ is defined via the Rankine-Hugoniot Condition (3.9)*. Analogous to the construction in Definition 3.15, we call $\mathfrak{G}_{\mathrm{VV}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ a* random family of vanishing viscosity germs*, if for every stochastic parameter $\omega \in \Omega$ and every space-time variable $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, it holds that $\mathfrak{G}_{\mathrm{VV}}(\omega, \mathfrak{x})$ is the vanishing viscosity germ associated to the normal components of the flux functions $g^{l,r}(\omega, \mathfrak{x}, \cdot)$ as defined in Equation* (3.10)*.* ◆

To prove the pathwise existence, the corresponding deterministic proof presented in [12, Section 4] should be exploited. Here, to obtain the existence, the Kato Inequality (3.16) is utilized for solutions of the inhomogeneous vanishing viscosity Equation (3.21). However, to conclude the proof, a blow-up technique has to be employed, which requires the following additional regularity assumption on the discontinuity hypersurface.

Assumption 3.36 (Regularity of discontinuity hypersurface)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{R}^2$ be a random sole flux discontinuity as defined in Definition 3.1. Then, for every $\omega \in \Omega$, the interface $\mathfrak{D}$ is assumed to satisfy one of the following regularity assumptions:

(i) The function $\Phi^{\mathfrak{D}}$, defined by Equation (3.2)*, satisfies $\Phi^{\mathfrak{D}}(\omega, \cdot, \cdot) \in \mathcal{C}^2(\mathbb{T} \times \mathbb{R}^{d-1}; \mathbb{R})$.*

(ii) The function $\Phi^{\mathfrak{D}}$ is (globally) Lipschitz continuous with respect to $(t, \boldsymbol{x}_{2:d}) \in \mathbb{T} \times \mathbb{R}^{d-1}$ and additionally $\Delta_{\boldsymbol{x}_{2:d}} \Phi^{\mathfrak{D}}(\omega, \cdot, \cdot)$ is a Radon measure on $\mathbb{T} \times \mathbb{R}^{d-1}$. ◆

Theorem 3.37 (Pathwise existence of $\mathfrak{G}_{\mathrm{VV}}$-entropy solutions)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface that satisfies the additional regularity Assumption 3.36. Furthermore, let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$ be a random initial condition and let the flux function $\mathfrak{f}$ of Problem (3.1) *satisfy the sole-flux-discontinuity Assumption 3.13 and the genuine nonlinearity Assumption 3.21. For fixed stochastic parameter $\omega \in \Omega$, let $\{u^\eta\}_{\eta>0}$ be a sequence of solutions to the random vanishing viscosity problem*

$$\partial_t u^\eta + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, \mathfrak{x}, u^\eta) = \eta \, \Delta \, u^\eta \tag{3.22}$$

that is bounded in $\mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ and satisfies $u^\eta\big|_{t=0} = u_0^\eta$ and $u_0^\eta \to u_0$ in $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$.
Then, for vanishing viscosity $\eta \searrow 0$, the sequence of solutions $\{u^\eta\}_{\eta>0}$ converges almost everywhere on $\mathbb{X}_{\mathbb{T}}$ to the unique pathwise $\mathfrak{G}_{\mathrm{VV}}(\omega)$-entropy solution of Problem (3.1)*.* ◆

Proof (main ideas). Before we summarize the main ideas of the existence proof, let us mention that, for every $\omega \in \Omega$ and every $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}(\omega, \mathfrak{x})$ is a maximal $\mathcal{L}^1$-dissipative germ. This is proven in [12, Proposition 7] and implies that a $\mathfrak{G}_{\mathrm{VV}}$-entropy solution is unique, if it exists.

We summarize the main ideas of the pathwise existence proof. For fixed $\omega \in \Omega$, the statement reduces to the deterministic result of [12, Theorem 5 (ii)]: Due to the uniform $\mathcal{L}^\infty$-bounds of the viscous approximations u^η and the genuine nonlinearity Assumption 3.21 on the flux function, the precompactness results of Lions et al. [188] and Panov [231] can be applied on $\mathbb{X}_{\mathbb{T}} \setminus \mathfrak{D}(\omega)$. This ensures the convergence of $\{u^\eta\}_{\eta>0}$ to some function u that is a Kružkov entropy solution away from the discontinuity hypersurface and satisfies the Rankine-Hugoniot condition on the discontinuity $\mathfrak{D}(\omega)$. The existence of strong traces of the solution u is justified almost everywhere on $\mathfrak{D}(\omega)$ with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$ via the Kato Inequality (3.16). To conclude the proof, a blow-up technique is applied to an inhomogeneous version of the random vanishing viscosity Problem (3.22). ■

As in the one-dimensional setting, the major restriction of this theorem is the existence of uniform $\mathcal{L}^\infty$-bounds on the solutions of the vanishing viscosity Problem (3.22). Such $\mathcal{L}^\infty$-bounds can be ensured via a variety of assumptions. We conclude this section by presenting an exemplary assumption, which has applications in porous medium simulations and road traffic models: In these fields of porous medium or road traffic investigations, the solution u oftentimes is understood as a relative density (e.g., of liquid saturation or traffic volume). Therefore, it is natural to assume the solution u to be confined to the interval $[0, 1]$. Such a confinement assumption can be established in a (slightly) more general way.

Assumption 3.38 (Confinement assumption)**:**
We assume that, for every $\omega \in \Omega$, there exists an interval $\mathbb{U} = [\underline{\mathfrak{u}}, \overline{\mathfrak{u}}] \subset \mathbb{R}$ that might depend on $\omega \in \Omega$, such that the following two conditions are satisfied:

(a) For a.e. $t \in \mathbb{T}$, the flux function $\mathfrak{f}$ satisfies

$$\mathfrak{f}(\omega, t, \cdot, \underline{\mathfrak{u}}) \equiv \text{const.} \qquad \textit{and} \qquad \mathfrak{f}(\omega, t, \cdot, \overline{\mathfrak{u}}) \equiv \text{const.}\,,$$

which means that $\mathfrak{f}$ is constant at the boundaries of the interval $\mathbb{U}$.

(b) The initial condition satisfies $\underline{\mathfrak{u}} \leq u_0(\omega, \cdot) \leq \overline{\mathfrak{u}}$. ◆

From a theoretical perspective, the confinement Assumption 3.38 ensures that, for fixed stochastic parameter $\omega \in \Omega$, the functions $u(\omega, t, \boldsymbol{x}) \equiv \underline{\mathfrak{u}}$ and $u(\omega, t, \boldsymbol{x}) \equiv \overline{\mathfrak{u}}$ are a sub- and supersolution, respectively, of the random scalar discontinuous-flux conservation law given by Equation (3.1). Consequently, the solution u as well as appropriately constructed approximations are confined to the interval $\mathbb{U}$.

3.3.2 Random entropy functionals

In this section, we define the notion of *random entropy functionals*. Roughly speaking, these functionals are a tool to evaluate, whether a function $v \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ satisfies the adapted $\mathfrak{G}$-entropy inequality for a fixed adapted entropy k and a fixed test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. We specify this approach with the subsequent definition.

Definition 3.39 (Random $\mathfrak{G}$-entropy functional)**:**
Let the following requirements be fulfilled:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity hypersurface.*
- *Let $\mathfrak{f}$ be a flux function that satisfies the sole-discontinuity flux Assumption 3.13 and the genuine nonlinearity Assumption 3.21.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of admissibility germs associated to the flux function $\mathfrak{f}$ that satisfies the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_\mathfrak{G}$ be a remainder function associated to $\mathfrak{G}$ satisfying the joint measurability Assumption 3.27.*

Then, for a fixed pair of Kružkov entropy values $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a fixed nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, we define the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ *associated to Problem* (3.1) *as a mapping $\mathbb{J}_\psi^{\boldsymbol{k}} : \Omega \times \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R}) \to \mathbb{R}$ given by*

$$(\omega, \upsilon) \mapsto \int_\mathbb{T} \int_\mathbb{X} \big|\upsilon(t,\boldsymbol{x}) - k(\omega,t,\boldsymbol{x})\big| \partial_t \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.23a}$$

$$+ \int_\mathbb{T} \int_\mathbb{X} \mathbf{q}\big(\omega,t,\boldsymbol{x};\upsilon(t,\boldsymbol{x}),k(\omega,t,\boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.23b}$$

$$- \int_\mathbb{X} |u_0(\omega,\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})| \psi(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \tag{3.23c}$$

$$+ \int_{\mathfrak{D}(\omega)} \mathfrak{R}_\mathfrak{G}(\omega,\mathfrak{d};\boldsymbol{k}) \psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d}\,. \tag{3.23d}$$

Here, k denotes the adapted Kružkov entropy defined via Equation (3.12) *corresponding to $\boldsymbol{k} \in \mathbb{R}^2$. Furthermore, $\mathbf{q}$ is the Kružkov entropy flux given by Equation* (3.13). ◆

The remainder of this section is devoted to establishing two important properties of the $\mathfrak{G}$-entropy functional:

(i) In Theorem 3.47 we show that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ is Carathéodory, which means that it is measurable in $\omega \in \Omega$ and continuous in $\upsilon \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$.

(ii) The random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the adapted entropy constants $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$, which is shown in Theorem 3.47.

Prerequisites

Before these properties of the $\mathfrak{G}$-entropy functional can be shown, some preparatory results need to established. We start by arguing that the adapted Kružkov entropy k, given by Equation (3.12), is measurable with respect to the stochastic parameter $\omega \in \Omega$, as soon as the sole discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ is measurable as a set-valued mapping.

Proposition 3.40 (Stochastic measurability of adapted Kružkov entropy)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity hypersurface satisfying the measurability Assumption 3.4. Then, for a fixed pair of entropy values $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a fixed space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, the

adapted Kružkov entropy k defined in Equation (3.12) *as*

$$k(\omega, \mathfrak{x}) := k^l \mathbb{1}_{\mathbb{X}_\mathbb{T}^l}(\omega, \mathfrak{x}) + k^r \mathbb{1}_{\mathbb{X}_\mathbb{T}^r}(\omega, \mathfrak{x}) \,,$$

is measurable in the sense that the mapping $\omega \mapsto k(\omega, \mathfrak{x})$ is measurable. ◆

Proof. By hypothesis, the sole discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ satisfies the measurability Assumption 3.4. Thereby, we can apply Lemma 3.12 to obtain the stochastic measurability of the domain part indicator functions $\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}}(\omega, \mathfrak{x})$. Now, the assertion follows from the construction of k in Equation (3.12). ■

So far, we cannot make any statement on the (stochastic) measurability of the $\mathfrak{G}$-entropy functional because we do not have any knowledge about the measurability of the flux function $\mathfrak{f}$. To overcome this issue, the following stochastic measurability assumption is imposed.

Assumption 3.41 (Stochastic measurability of flux function)**:**
Let $\mathfrak{f}$ be a flux function that satisfies the sole-discontinuity-flux Assumption 3.13. We assume that the flux function $\mathfrak{f}$ is stochastically measurable in the sense that, for fixed $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ and $\upsilon \in \mathbb{R}$, the mappings $\omega \mapsto \mathfrak{f}^l(\omega, \mathfrak{x}, \upsilon)$ and $\omega \mapsto \mathfrak{f}^r(\omega, \mathfrak{x}, \upsilon)$ are measurable. ◆

With this stochastic measurability assumption on the flux function, we are able to show that the left and right Kružkov entropy fluxes $\mathfrak{q}^{l,r}$ are also measurable. Actually, we can prove an even stronger result, as the next corollary shows.

Corollary 3.42 (Left/right Kružkov entropy fluxes are Carathéodory)**:**
Let $\mathfrak{f}$ be a flux function that satisfies the sole-flux-discontinuity Assumption 3.13 and the stochastic measurability Assumption 3.41. Then, the left and right Kružkov entropy fluxes $\mathfrak{q}^{l,r} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ as defined in Definition 3.25 via

$$\mathfrak{q}^{l,r}(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon}) := \operatorname{sign}(\upsilon - \tilde{\upsilon}) \big(\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \upsilon) - \mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \tilde{\upsilon}) \big) \tag{3.24}$$

are Carathéodory in the sense that they are measurable w.r.t. the stochastic parameter $\omega \in \Omega$ and continuous in the remaining arguments. ◆

Proof. Since the flux functions $\mathfrak{f}^{l,r}$ are measurable in the stochastic parameter $\omega \in \Omega$ by the measurability Assumption 3.41, we directly obtain that the mappings $\omega \mapsto \mathfrak{q}^{l,r}(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon})$ are measurable for fixed space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ and fixed scalar values $\upsilon, \tilde{\upsilon} \in \mathbb{R}$. Furthermore, the continuous dependence of $\mathfrak{q}^{l,r}$ with respect to the space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ follows immediately from the global Lipschitz Assumption (F-3) on the flux functions $\mathfrak{f}^{l,r}$.

It remains to show the continuity in $\upsilon \in \mathbb{R}$ and $\tilde{\upsilon} \in \mathbb{R}$. However, the local Lipschitz Assumption (F-2) on $\mathfrak{f}^{l,r}$ and Equation (3.24) yield the estimation

$$\big\| \mathfrak{q}^{l,r}(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon}) \big\|_d \leq \Big| \operatorname{sign}(\upsilon - \tilde{\upsilon}) L_{\mathfrak{f}^{l,r}} |\upsilon - \tilde{\upsilon}| \Big| \leq L_{\mathfrak{f}^{l,r}} |\upsilon - \tilde{\upsilon}| \,.$$

Here, $L_{\mathfrak{f}^{l,r}}$ denotes the Lipschitz constants of $\mathfrak{f}^{l,r}$. Noting that $\mathfrak{q}^{l,r}$ vanishes for $\upsilon = \tilde{\upsilon}$ concludes the proof of $\mathfrak{q}^{l,r}$ being Carathéodory. ■

$\mathfrak{G}$-entropy functional is Carathéodory

On the way to showing that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ is Carathéodory, we argue in the subsequent proposition that the functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends measurably on the stochastic parameter $\omega \in \Omega$.

Proposition 3.43 (Stochastic measurability of random $\mathfrak{G}$-entropy functional)**:**
Suppose the following conditions are fulfilled:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity that satisfies the measurability Assumption 3.4.*
- *Let the flux function $\mathfrak{f}$ satisfy the sole-discontinuity-flux Assumption 3.13 and the stochastic measurability Assumption 3.41.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative germs satisfying the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$ that satisfies the measurability Assumption 3.27.*

Now, let $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. Then, the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ given by Equation (3.23) is stochastically measurable in the sense that, for fixed $\upsilon \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, the mapping $\omega \mapsto \mathbb{J}_\psi^{\boldsymbol{k}}(\omega, \upsilon)$ is measurable. ◆

Proof. To show the measurability of the stochastic entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ with respect to the stochastic parameter $\omega \in \Omega$, we consider the four integrals of the Mapping (3.23) separately:

Stochastic measurability of Integral (3.23a)**.** The adapted Kružkov entropy k is measurable in the sense that the mapping $\omega \mapsto k(\omega, t, \boldsymbol{x})$ is measurable for fixed $(t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X}$ by Proposition 3.40. Since k is the only ingredient in Integral (3.23a) that depends on $\omega \in \Omega$, we obtain that the integrand is measurable. Now, since the test function $\psi \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is compactly supported, we have

$$\int_\mathbb{T} \int_\mathbb{X} |\upsilon(t, \boldsymbol{x}) - k(\omega, t, \boldsymbol{x})| \partial_t \psi(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t = \int_{\operatorname{supp}\psi} |\upsilon(t, \boldsymbol{x}) - k(\omega, t, \boldsymbol{x})| \partial_t \psi(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \,.$$

However, since $\operatorname{supp}\psi$ is compact, taking the integral is a bounded linear operator. Linear operators are bounded if and only if they are continuous and thus, taking the integral over $\operatorname{supp}\psi$ is a continuous operation. Since the composition of a continuous operation with a measurable function is measurable [5, Lemma 4.22], the Integral (3.23a) is measurable.

Stochastic measurability of Integral (3.23b)**.** First, the sole-flux-discontinuity Assumption 3.13 imposed on the flux function $\mathfrak{f}$ admits splitting up the Integral (3.23b) over the space-time domain $\mathbb{X}_\mathbb{T}$ into the left and right domain parts $\mathbb{X}_\mathbb{T}^{l,r}$ as

$$\begin{aligned}\int_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega, \mathfrak{x}; \upsilon(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} = &\int_{\mathbb{X}_\mathbb{T}^l(\omega)} \mathbf{q}^l\big(\omega, \mathfrak{x}; \upsilon(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \\ &+ \int_{\mathbb{X}_\mathbb{T}^r(\omega)} \mathbf{q}^r\big(\omega, \mathfrak{x}; \upsilon(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \,.\end{aligned}$$

However, to avoid the dependence of the integral domain on the stochastic parameter $\omega \in \Omega$, we can further rewrite the Integral (3.23b) as

$$\int_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega, \mathfrak{x}; v(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} = \int_{\mathbb{X}_\mathbb{T}} \Big(\mathbf{q}^l\big(\omega, \mathfrak{x}; v(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \mathbb{1}_{\mathbb{X}_\mathbb{T}^l}(\omega, \mathfrak{x})\Big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} + \int_{\mathbb{X}_\mathbb{T}} \Big(\mathbf{q}^r\big(\omega, \mathfrak{x}; v(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \mathbb{1}_{\mathbb{X}_\mathbb{T}^r}(\omega, \mathfrak{x})\Big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \,,$$

where $\mathbb{1}_{\mathbb{X}_\mathbb{T}^{l,r}}(\omega, \mathfrak{x})$ denotes the indicator functions of the domain parts $\mathbb{X}_\mathbb{T}^{l,r}(\omega)$ as defined in Definition 3.11. Now, by Corollary 3.42, the left and right Kružkov entropy fluxes $\mathbf{q}^{l,r}$ are Carathéodory in the sense that they are measurable in $\omega \in \Omega$ and continuous in all remaining arguments. In particular, by being Carathéodory the entropy fluxes $\mathbf{q}^{l,r}$ are jointly measurable [5, Lemma 4.51]. Consequently, the integrand of Integral (3.23b) is measurable in $\omega \in \Omega$, since the adapted Kružkov entropy is measurable in $\omega \in \Omega$ by Proposition 3.40 and the domain part indicator functions are separately measurable by Lemma 3.12.
As in the previous step, the test function ψ is compactly supported and thus taking the integral is a continuous operation. The measurability of Integral (3.23b) follows as the composition of a continuous operation with a measurable function.

Stochastic measurability of Integral (3.23c)**.** In Proposition 3.40, we have already established that the adapted Kružkov entropy k is measurable as a mapping $\omega \mapsto k(\omega, t, \boldsymbol{x})$. Recall that by the definition of Problem (3.1), the initial condition u_0 satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$. This immediately allows us to conclude the measurability of the integrand of Integral (3.23c) with respect to the stochastic parameter $\omega \in \Omega$. Now, since the support $\operatorname{supp}\psi$ of the test function is compact by hypothesis, the stochastic measurability of the Integral (3.23c) follows via the composition of a continuous operation with a measurable function.

Stochastic measurability of Integral (3.23d)**.** It remains to show the stochastic measurability of Integral (3.23d). According to Remark 3.30 (ii) on the integration over the random sole discontinuity hypersurface, the surface integral is equivalent to integrating against the d-dimensional Hausdorff measure $\mathcal{H}^d$:

$$\int_{\mathfrak{D}(\omega)} \mathfrak{R}_\mathfrak{G}(\omega, \mathfrak{d}; \boldsymbol{k}) \psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d} = \int_{\mathfrak{D}(\omega)} \mathfrak{R}_\mathfrak{G}(\omega, \mathfrak{d}; \boldsymbol{k}) \psi(\mathfrak{d}) \,\mathrm{d}\mathcal{H}^d(\mathfrak{d}) \,.$$

Here, $\mathfrak{R}_\mathfrak{G}$ is the remainder function associated to the random family of $\mathcal{L}^1$D germs $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ as introduced in Definition 3.26. By construction, the integrand $\mathfrak{R}_\mathfrak{G}(\omega, \mathfrak{x}; \boldsymbol{k})\psi(\mathfrak{x})$ is nonnegative, since both $\mathfrak{R}_\mathfrak{G}$ and ψ are nonnegative. By hypothesis, $\mathfrak{G}$ satisfies the joint measurability Assumption 3.16 and $\mathfrak{R}_\mathfrak{G}$ satisfies the joint measurability Assumption 3.27. Combining this with the continuity of ψ, we obtain the measurability of $\mathfrak{R}_\mathfrak{G}(\omega, \mathfrak{x}; \boldsymbol{k})\psi(\mathfrak{x})$ with respect to the space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Since we established that the integrand is measurable and nonnegative, we can apply [106, Theorem 11.25] to write this surface integral as

$$\int_{\mathfrak{D}(\omega)} \mathfrak{R}_\mathfrak{G}(\omega, \mathfrak{d}, \boldsymbol{k})\psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d} = \int_{\mathbb{T}\times\mathbb{R}^{d-1}} \mathfrak{R}_\mathfrak{G}\big(\omega, \boldsymbol{P}_\mathfrak{D}(\omega, \mathfrak{y}); \boldsymbol{k}\big)\psi\big(\boldsymbol{P}_\mathfrak{D}(\omega, \mathfrak{y})\big)\boldsymbol{J}\big(\boldsymbol{D}_\mathfrak{y}\boldsymbol{P}_\mathfrak{D}(\omega, \mathfrak{y})\big) \,\mathrm{d}\mathcal{H}^d(\mathfrak{y}) \,. \quad (3.25)$$

Here, we denote by $\mathfrak{y} = (t, \boldsymbol{y}) \in \mathbb{T} \times \mathbb{R}^{d-1}$ a point of the (lower-dimensional) space $\mathbb{T} \times \mathbb{R}^{d-1}$. The function $\boldsymbol{P}_\mathfrak{D} : \Omega \times \mathbb{T} \times \mathbb{R}^{d-1} \to \mathbb{X}_\mathbb{T}$ is a parametrization of the sole discontinuity hypersurface $\mathfrak{D}$ as

defined in Equation (3.3). Furthermore, $\boldsymbol{D}_{\mathfrak{y}}\boldsymbol{P}_{\mathfrak{D}}$ is the differential of $\boldsymbol{P}_{\mathfrak{D}}$, i.e., it is the linear map from $\mathbb{T}\times\mathbb{R}^{d-1}$ to $\mathbb{T}\times\mathbb{R}^d$ whose matrix is given by

$$\left[\frac{\partial\big(\boldsymbol{P}_{\mathfrak{D}}\big)_{\iota}}{\partial\mathfrak{y}_{\kappa}}(\omega,\mathfrak{y})\right], \qquad 1\leq\iota\leq d+1\,, \quad 1\leq\kappa\leq d\,.$$

Finally, for some linear map T the term $\boldsymbol{J}(T)$ is defined as $\boldsymbol{J}(T)\coloneqq\sqrt{\det(T^{\mathrm{T}}T)}$, where T^{T} denotes the transpose of T. [22]

To conclude the measurability of the integral, we first show that the integrand is measurable: Since $\boldsymbol{P}_{\mathfrak{D}}$ is jointly measurable by Lemma 3.6, we immediately obtain the measurability of $\psi\big(\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y})\big)$ and $\boldsymbol{J}\big(\boldsymbol{D}_{\mathfrak{y}}\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y})\big)$. By the measurability Assumption 3.27, the random remainder function is jointly measurable, since $\mathfrak{G}$ is jointly measurable by hypothesis. Consequently, the composition $\mathfrak{R}_{\mathfrak{G}}\big(\omega,\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y});\boldsymbol{k}\big)$ is measurable in $\omega\in\Omega$. As the product of measurable functions is again measurable, the integrand in Integral (3.23d) is measurable.
It remains to show that measurability of the integrand implies measurability of the Integral (3.23d). Recall that we have $\psi\in\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{X}_{\mathbb{T}};\mathbb{R})$ compactly supported. Therefore, it is sufficient to integrate over the set $\boldsymbol{S}_{\mathfrak{D}}(\omega)=\mathfrak{D}(\omega)\cap\operatorname{supp}\psi$, which is also compact[23]: The boundedness is inferred by $\operatorname{supp}\psi$ being bounded in combination with the definition of $\boldsymbol{S}_{\mathfrak{D}}(\omega)$. Since $\mathfrak{D}(\omega)$ is closed by Corollary 3.3 and $\operatorname{supp}\psi$ is closed by being compact, the closedness of $\boldsymbol{S}_{\mathfrak{D}}$ is obtained as the finite intersection of closed sets. Thereby, we can rewrite the Integral (3.25) as

$$\begin{aligned}\int_{\mathfrak{D}(\omega)}\mathfrak{R}_{\mathfrak{G}}(\omega,\mathfrak{d};\boldsymbol{k})\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d}&=\int_{\mathbb{T}\times\mathbb{R}^{d-1}}\mathfrak{R}_{\mathfrak{G}}\big(\omega,\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y});\boldsymbol{k}\big)\psi\big(\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y})\big)\boldsymbol{J}\big(\boldsymbol{D}_{\mathfrak{y}}\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y})\big)\,\mathrm{d}\mathcal{H}^d(\mathfrak{y})\\&=\int_{\boldsymbol{S}_{\mathfrak{D}}}\mathfrak{R}_{\mathfrak{G}}\big(\omega,\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y});\boldsymbol{k}\big)\psi\big(\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y})\big)\boldsymbol{J}\big(\boldsymbol{D}_{\mathfrak{y}}\boldsymbol{P}_{\mathfrak{D}}(\omega,\mathfrak{y})\big)\,\mathrm{d}\mathcal{H}^d(\mathfrak{y})\,.\end{aligned}$$

As $\boldsymbol{S}_{\mathfrak{D}}$ is compact, integrating over $\boldsymbol{S}_{\mathfrak{D}}$ is a bounded linear operator. Since linear operators are bounded if and only if they are continuous, this is a continuous operation. As the composition of a continuous operator with a measurable function is measurable by [5, Lemma 4.22], we obtain the measurability of the Integral (3.23d) in $\omega\in\Omega$.

Thus, combining the measurability results of the four integrals, we obtain the stochastic measurability of the random entropy functional $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ with respect to $\omega\in\Omega$. ∎

The above proposition has shown the measurability of the random $\mathfrak{G}$-entropy functional $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ with respect to the stochastic parameter $\omega\in\Omega$. To constitute that the functional $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ is Carathéodory, the second ingredient can now be justified: The continuous dependence of $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ on the function $\upsilon\in\mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}};\mathbb{R})$. The corresponding result is established in the subsequent proposition.

[22] Note that $T^{\mathrm{T}}T$ is a positive semidefinite operator and, therefore, its determinant is nonnegative.

[23] In case we have $\mathfrak{D}(\omega)\cap\operatorname{supp}\psi=\emptyset$, the integral vanishes. Anyway, since the empty set $\emptyset$ is compact due to being finite, the argumentation for establishing measurability remains true.

Proposition 3.44 (Continuous dependence of entropy functional on solution function)**:**
Suppose the following conditions are satisfied:

- *Let* $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ *be a random sole discontinuity.*
- *Let the flux function* $\mathfrak{f}$ *satisfy the sole-discontinuity-flux Assumption 3.13.*
- *Let* $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ *be a random family of* $\mathcal{L}^1$*-dissipative germs.*
- *Let* $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ *be a remainder function associated to* $\mathfrak{G}$*.*

Furthermore, let the stochastic parameter $\omega \in \Omega$*, a pair of entropy values* $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ *and a nonnegative test function* $\psi \in \mathcal{D}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ *be fixed. Then, the random* $\mathfrak{G}$*-entropy functional* $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ *depends continuously on the function* $\nu \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$*.* ◆

Proof. Let a stochastic parameter $\omega \in \Omega$, a pair of entropy values $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ be fixed. To show that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ depends continuously on $\nu \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$, we consider the four integrals in Equation (3.23) separately:

The Integrals (3.23c) and (3.23d) are both independent of the function $\nu \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$, which directly proves their continuous dependence property with respect to this function. Considering the Integral (3.23a) observe that its integrand is essentially bounded, i.e., $|\nu(\mathfrak{x}) - k(\omega, \mathfrak{x})| \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$, since both ν and $k(\omega, \cdot)$ are essentially bounded. This implies that the integral is finite because the test function ψ and its time derivative $\partial_t \psi$ are compactly supported. The continuous dependence follows now via the dominated convergence theorem. We refer to [98, Theorem 5.6] for a precise formulation of this argument. Consequently, it remains to show the continuous dependence on the function $\nu \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ of the second Integral (3.23b):

To show the continuous dependence of Integral (3.23b), let two functions $u, \nu \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ be given. The idea of the proof is to show that there exists a constant $C_{\mathbb{J}} > 0$ such that the estimate

$$\begin{aligned} \Big| \int_{\mathbb{X}_{\mathbb{T}}} \mathfrak{q}\big(\omega, \mathfrak{x}; u(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \\ - \int_{\mathbb{X}_{\mathbb{T}}} \mathfrak{q}\big(\omega, \mathfrak{x}; \nu(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| \leq C_{\mathbb{J}} \|u - \nu\|_{\mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})} \end{aligned} \tag{3.26}$$

holds. Recall that the entropy flux $\mathfrak{q} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ depends on the flux function $\mathfrak{f}$ and was introduced in Definition 3.25 as the mapping

$$\mathfrak{q}(\omega, \mathfrak{x}, \nu, \upsilon) \coloneqq \operatorname{sign}(\nu - \upsilon)\big(\mathfrak{f}(\omega, \mathfrak{x}, \nu) - \mathfrak{f}(\omega, \mathfrak{x}, \upsilon)\big) \, .$$

We start our investigation by introducing four sets $\mathbb{D}^{(\cdot)}(\omega) \subset \mathbb{X}_{\mathbb{T}}$, where the superscript will become clear via the definition. These sets allow us to divide the space-time domain $\mathbb{X}_{\mathbb{T}}$ in a certain way that depends on the values of $\operatorname{sign}\big(u(\mathfrak{x}) - k(\omega, \mathfrak{x})\big)$ and $\operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega, \mathfrak{x})\big)$. For every stochastic parameter $\omega \in \Omega$, these four subsets $\mathbb{D}^{(\cdot)}(\omega)$ of the space-time domain $\mathbb{X}_{\mathbb{T}}$ are defined as

$$\begin{aligned} \mathbb{D}^{=}(\omega) &\coloneqq \Big\{ \mathfrak{x} \in \mathbb{X}_{\mathbb{T}} \mid \operatorname{sign}\big(u(\mathfrak{x}) - k(\omega, \mathfrak{x})\big) = \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega, \mathfrak{x})\big) \Big\}, \\ \mathbb{D}^{\pm}(\omega) &\coloneqq \Big\{ \mathfrak{x} \in \mathbb{X}_{\mathbb{T}} \mid \operatorname{sign}\big(u(\mathfrak{x}) - k(\omega, \mathfrak{x})\big) = -\operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega, \mathfrak{x})\big) \Big\}, \end{aligned}$$

$$\mathbb{D}_k^u(\omega) \coloneqq \Big\{ \mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid u(\mathfrak{x}) = k(\omega, \mathfrak{x}) \Big\},$$
$$\mathbb{D}_k^\nu(\omega) \coloneqq \Big\{ \mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid \nu(\mathfrak{x}) = k(\omega, \mathfrak{x}) \Big\}.$$

Let us emphasize that these sets are not necessarily disjoint. However, a closer inspection of the definition immediately reveals that the intersection of the sets $\mathbb{D}^{(\cdot)}(\omega)$ is given for every $\omega \in \Omega$ as

$$\mathbb{D}^=(\omega) \cap \mathbb{D}^\pm(\omega) \cap \mathbb{D}_k^u(\omega) \cap \mathbb{D}_k^\nu(\omega) = \Big\{ \mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid u(\mathfrak{x}) = k(\omega, \mathfrak{x}) = \nu(\mathfrak{x}) \Big\} .$$

Therefore, the integral of the entropy flux $\mathbf{q}$ over these overlapping areas vanishes. This allows us to split the integral over $\mathbb{X}_\mathbb{T}$ into the four integrals over the sets $\mathbb{D}^{(\cdot)}(\omega)$.

To improve the readability of the upcoming estimation, let us introduce a function $\mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}$ that describes the difference of two entropy fluxes whose third argument varies. We define $\mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}$ as the mapping

$$\mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}(\nu, \upsilon) \coloneqq \mathbf{q}\big(\omega, \mathfrak{x}, \nu, k(\omega, \mathfrak{x})\big) - \mathbf{q}\big(\omega, \mathfrak{x}, \upsilon, k(\omega, \mathfrak{x})\big) .$$

To start the estimation of the left-hand side of Inequality (3.26), recall that we fixed a stochastic parameter $\omega \in \Omega$ as well as an entropy pair $\boldsymbol{k} \in \mathbb{R}^2$ and use the linearity of the integral. After splitting up the space-time domain $\mathbb{X}_\mathbb{T}$ into the sets $\mathbb{D}^{(\cdot)}(\omega)$, we can employ the triangle inequality to obtain

$$\begin{aligned}
\Big| \int_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega, \mathfrak{x}; u(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega, \mathfrak{x}; \nu(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| & \\
\leq \Big| \int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| & \quad (3.27a) \\
+ \Big| \int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| & \quad (3.27b) \\
+ \Big| \int_{\mathbb{D}_k^u(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| & \quad (3.27c) \\
+ \Big| \int_{\mathbb{D}_{\boldsymbol{k}}^\nu(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| . & \quad (3.27d)
\end{aligned}$$

For simplicity of the proceeding estimation, the continuous dependence on $u, \nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ is considered for each integral in the above Inequality (3.27) separately:

Continuous dependence of Integral (3.27a). To obtain an estimate for Term (3.27a), first Hölder's inequality is applied. Utilizing the Cauchy-Schwarz-Bunyakovsky inequality afterwards yields

$$\Big| \int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \, \mathrm{d}\mathfrak{x} \Big| \leq \int_{\mathbb{D}^=(\omega)} \Big\| \mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \Big\|_d \Big\| \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \Big\|_d \, \mathrm{d}\mathfrak{x} .$$

Here, $\|\cdot\|_d$ denotes the Euclidean distance norm of the spatial domain $\mathbb{X} = \mathbb{R}^d$. Using the definitions of the function $\mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}$ and the entropy flux $\mathbf{q}$, we can leverage the construction of the set $\mathbb{D}^=(\omega)$ as the set of space-time points $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ that fulfill the condition that $\operatorname{sign}\big(u(\mathfrak{x}) - k(\omega, \mathfrak{x})\big) = \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega, \mathfrak{x})\big)$.

Thereby, we obtain the estimation:

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}^{=}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| &\\ \leq \int\limits_{\mathbb{D}^{=}(\omega)} \big\|\mathfrak{q}\big(\omega,\mathfrak{x},u(\mathfrak{x}),k(\omega,\mathfrak{x})\big)-\mathfrak{q}\big(\omega,\mathfrak{x},\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\big\|_d \,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}&\\ = \int\limits_{\mathbb{D}^{=}(\omega)} \Big\|\mathfrak{f}\big(\omega,\mathfrak{x},u(\mathfrak{x})\big)-\mathfrak{f}\big(\omega,\mathfrak{x},\nu(\mathfrak{x})\big)\Big\|_d \,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.&\end{aligned}$$

By Assumption (F-2) the flux function $\mathfrak{f}$ is locally Lipschitz continuous. Since both u,ν are essentially bounded, we can apply this local Lipschitz property to estimate

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}^{=}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| &\leq \int\limits_{\mathbb{D}^{=}(\omega)} L_{\mathfrak{f}}\Big|u(\mathfrak{x})-\nu(\mathfrak{x})\Big|\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\\ &\leq L_{\mathfrak{f}}\|u-\nu\|_{\mathcal{L}^\infty(\mathbb{X}_{\mathrm{T}};\mathbb{R})}\int\limits_{\mathbb{D}^{=}(\omega)}\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.\end{aligned}$$

Here, the last estimation is possible since we have $|u(\mathfrak{x})-\nu(\mathfrak{x})|\leq\|u-\nu\|_{\mathcal{L}^\infty(\mathbb{X}_{\mathrm{T}};\mathbb{R})}$ for almost every $\mathfrak{x}\in\mathbb{X}_{\mathrm{T}}$. Since the nonnegative test function $\psi\in\mathcal{D}(\mathbb{X}_{\mathrm{T}};\mathbb{R})$ is compactly supported and smooth, its gradient $\nabla_{\boldsymbol{x}}\,\psi$ inherits these properties of being smooth and compactly supported. In particular, this implies that $\nabla_{\boldsymbol{x}}\,\psi$ is bounded. This leads to the estimation

$$\Big| \int\limits_{\mathbb{D}^{=}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq C_\psi L_{\mathfrak{f}}\Big\|u-\nu\Big\|_{\mathcal{L}^\infty(\mathbb{X}_{\mathrm{T}};\mathbb{R})}\,,$$

which concludes the proof of showing continuous dependence of the Integral (3.27a) on the functions $u,\nu\in\mathcal{L}^\infty(\mathbb{X}_{\mathrm{T}};\mathbb{R})$.

Continuous dependence of Integral (3.27b). To show that the Integral (3.27b) depends continuously on the functions $u,\nu\in\mathcal{L}^\infty(\mathbb{X}_{\mathrm{T}};\mathbb{R})$, we start by applying first Hölder's and afterwards, the Cauchy-Schwarz-Bunyakovsky inequality to obtain

$$\Big| \int\limits_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq \int\limits_{\mathbb{D}^{\pm}(\omega)} \Big\|\mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\Big\|_d\,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

We continue by inserting the definitions of the function $\mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}$ and the entropy flux $\mathfrak{q}$. Afterwards, the construction of the set $\mathbb{D}^{\pm}(\omega)$ consisting of those space-time points $\mathfrak{x}\in\mathbb{X}_{\mathrm{T}}$, which satisfy the relation $\operatorname{sign}\big(u(\mathfrak{x})-k(\omega,\mathfrak{x})\big)=-\operatorname{sign}\big(\nu(\mathfrak{x})-k(\omega,\mathfrak{x})\big)$, permits to perform the estimation:

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),&\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big|\\ &\leq \int\limits_{\mathbb{D}^{\pm}(\omega)} \Big\|\mathfrak{q}\big(\omega,\mathfrak{x},u(\mathfrak{x}),k(\omega,\mathfrak{x})\big)-\mathfrak{q}\big(\omega,\mathfrak{x},\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\Big\|_d\,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\\ &\leq \int\limits_{\mathbb{D}^{\pm}(\omega)} \Big\|\mathfrak{f}\big(\omega,\mathfrak{x},u(\mathfrak{x})\big)+\mathfrak{f}\big(\omega,\mathfrak{x},\nu(\mathfrak{x})\big)-2\mathfrak{f}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)\Big\|_d\,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.\end{aligned}$$

Recall that the flux function $\mathfrak{f}$ is locally Lipschitz continuous by Assumption (F-2). Proceeding the estimation by first applying the triangle inequality and employing the local Lipschitz property thereafter, we obtain

$$\Big| \int\limits_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq L_{\mathfrak{f}} \int\limits_{\mathbb{D}^{\pm}(\omega)} \Big(\Big|u(\mathfrak{x})-k(\omega,\mathfrak{x})\Big|+\Big|\nu(\mathfrak{x})-k(\omega,\mathfrak{x})\Big|\Big)\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

For any two scalar values $\xi_1,\xi_2\in\mathbb{R}$, the identity $|\xi_1-\xi_2|=\operatorname{sign}(\xi_1-\xi_2)(\xi_1-\xi_2)$ holds. Combining this identity with the construction of the integral domain $\mathbb{D}^{\pm}(\omega)$ allows us to further estimate

$$\Big| \int\limits_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq L_{\mathfrak{f}} \int\limits_{\mathbb{D}^{\pm}(\omega)} \Big|\operatorname{sign}\big(u(\mathfrak{x})-k(\omega,\mathfrak{x})\big)\big(u(\mathfrak{x})-\nu(\mathfrak{x})\big)\Big|\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

Observe that for any two scalar values $\nu,\upsilon\in\mathbb{R}$ the estimate $\big|\operatorname{sign}(\nu-\upsilon)\big|\leq 1$ holds by construction. Furthermore, the estimation $|u(\mathfrak{x})-\nu(\mathfrak{x})|\leq\|u-\nu\|_{\mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}};\mathbb{R})}$ holds for almost every $\mathfrak{x}\in\mathbb{X}_{\mathbb{T}}$. Since the test function ψ is smooth and compactly supported, we obtain boundedness of its gradient $\nabla_{\boldsymbol{x}}\,\psi$. Thereby, we can conclude

$$\Big| \int\limits_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq C_\psi L_{\mathfrak{f}}\Big\|u-\nu\Big\|_{\mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}};\mathbb{R})}\,.$$

Consequently, we have shown the continuous dependence of Integral (3.27b) on $u,\nu\in\mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}};\mathbb{R})$.

Continuous dependence of Integral (3.27c)**.** For showing the continuous dependence of Integral (3.27c), we start again by applying Hölder's inequality and the Cauchy-Schwarz-Bunyakovsky inequality.

$$\Big| \int\limits_{\mathbb{D}^{u}_{k}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq \int\limits_{\mathbb{D}^{u}_{k}(\omega)} \Big\|\mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\Big\|_d\,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

Using the definitions of the function $\mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}$ and the entropy flux $\mathfrak{q}$ and then leveraging the construction of the set $\mathbb{D}^{u}_{k}(\omega)$ this leads to the estimation

$$\begin{aligned}
&\Big| \int\limits_{\mathbb{D}^{u}_{k}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \\
&\qquad\leq \int\limits_{\mathbb{D}^{u}_{k}(\omega)} \big\|\mathfrak{q}\big(\omega,\mathfrak{x},u(\mathfrak{x}),k(\omega,\mathfrak{x})\big)-\mathfrak{q}\big(\omega,\mathfrak{x},\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\big\|_d\,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x} \\
&\qquad= \int\limits_{\mathbb{D}^{u}_{k}(\omega)} \Big\|\operatorname{sign}\big(\nu(\mathfrak{x})-k(\omega,\mathfrak{x})\big)\Big(\mathfrak{f}\big(\omega,\mathfrak{x},\nu(\mathfrak{x})\big)-\mathfrak{f}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)\Big)\Big\|_d\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

By construction, it holds that $|\operatorname{sign}(\nu-\upsilon)|\leq 1$ for two scalar values $\nu,\upsilon\in\mathbb{R}$. Recall that the flux function $\mathfrak{f}$ is locally Lipschitz by Assumption (F-2). Thus, we obtain

$$\Big| \int\limits_{\mathbb{D}^{u}_{k}(\omega)} \mathfrak{Q}^{\boldsymbol{k}}_{\omega,\mathfrak{x}}\big(u(\mathfrak{x}),\nu(\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq L_{\mathfrak{f}} \int\limits_{\mathbb{D}^{u}_{k}(\omega)} |\nu(\mathfrak{x})-k(\omega,\mathfrak{x})|\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

By definition of the set $\mathbb{D}_k^u(\omega)$, we have $u(\mathfrak{x}) = k(\omega, \mathfrak{x})$ for every spatio-temporal point $\mathfrak{x} \in \mathbb{D}_k^u(\omega)$. Since $|u(\mathfrak{x}) - \nu(\mathfrak{x})| \leq \|u - \nu\|_{\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})}$ holds for almost every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, we can conclude

$$\Big| \int\limits_{\mathbb{D}_k^u(\omega)} \boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq C_\psi L_{\mathfrak{f}} \Big\| u - \nu \Big\|_{\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})} .$$

Here, we exploited that the gradient of the test function $\nabla_{\boldsymbol{x}}\, \psi$ is bounded, since the test function ψ is compactly supported and smooth. Consequently, we have shown that the Integral (3.27c) depends continuously on the functions $u, \nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$.

Continuous dependence of Integral (3.27d)**.** Arguing the continuous dependence of Integral (3.27d) is analogous to showing that the Integral (3.27c) depends continuously on $u, \nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$. Therefore, we first employ Hölder's inequality and the Cauchy-Schwarz-Bunyakovsky inequality, and obtain

$$\Big| \int\limits_{\mathbb{D}_k^\nu(\omega)} \boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq \int\limits_{\mathbb{D}_k^\nu(\omega)} \Big\| \boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \Big\|_d \, \Big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \Big\|_d \, \mathrm{d}\mathfrak{x} .$$

We continue by using the definitions of the function $\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}$ and the entropy flux $\mathfrak{q}$. Furthermore, the construction of the set $\mathbb{D}_k^u(\omega)$ and the local Lipschitz property of the flux function $\mathfrak{f}$ induced by Assumption (F-2) lead to

$$\Big| \int\limits_{\mathbb{D}_k^\nu(\omega)} \boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq L_{\mathfrak{f}} \int\limits_{\mathbb{D}_k^\nu(\omega)} |u(\mathfrak{x}) - k(\omega, \mathfrak{x})| \, \Big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \Big\|_d \, \mathrm{d}\mathfrak{x} .$$

By construction of the set $\mathbb{D}_k^u(\omega)$, the equality $\nu(\mathfrak{x}) = k(\omega, \mathfrak{x})$ holds for every space-time variable $\mathfrak{x} \in \mathbb{D}_k^u(\omega)$. Inserting this into the above inequality and using the smoothness and the compact support of the test function ψ, we can deduce the estimate

$$\Big| \int\limits_{\mathbb{D}_k^\nu(\omega)} \boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}\big(u(\mathfrak{x}), \nu(\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq C_\psi L_{\mathfrak{f}} \|u - \nu\|_{\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})} .$$

This concludes the proof of showing the continuous dependence of the Integral (3.27d) on the functions $u, \nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$.

Combining the above results, we have shown that the estimate

$$\Big| \int\limits_{\mathbb{X}_\mathbb{T}} \mathfrak{q}\big(\omega, \mathfrak{x}; u(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} - \int\limits_{\mathbb{X}_\mathbb{T}} \mathfrak{q}\big(\omega, \mathfrak{x}; \nu(\mathfrak{x}), k(\omega, \mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq 4 C_\psi L_{\mathfrak{f}} \Big\| u - \nu \Big\|_{\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})}$$

holds, which proves that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$. ■

By virtue of the previous two propositions, we have established the following theorem, which states that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ is Carathéodory, meaning that it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous with respect to the function $\upsilon \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$.

Theorem 3.45 (Entropy functional is Carathéodory)**:**
Let the following conditions be satisfied:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity satisfying the measurability Assumption 3.4.*
- *Let $\mathfrak{f}$ be a flux function satisfying the sole-flux-discontinuity Assumption 3.13 and the stochastic measurability Assumption 3.41.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative germs satisfying the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$ that satisfies the measurability Assumption 3.27.*

Furthermore, let $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. Then, the random $\mathfrak{G}$-entropy functional $\mathbb{J}_{\psi}^{\boldsymbol{k}}$ is Carathéodory, which means that it depends measurably on $\omega \in \Omega$ and is continuous in $\nu \in \mathcal{L}^{\infty}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. ◆

Continuous dependence of $\mathfrak{G}$-entropy functional on adapted Kružkov entropy

We have now established that $\mathfrak{G}$-entropy functionals are Carathéodory. Therefore, we can turn to the second property that we want to establish: the continuous dependence on the pair of entropy values $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and the corresponding adapted Kružkov entropy. Before we are able to prove such a continuous dependence result, we need the following integrability assumption on the (random) remainder function.

Assumption 3.46 (Integrability of remainder function)**:**
Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative germs and let $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$. We assume that $\mathfrak{R}_{\mathfrak{G}}$ satisfies that for each compact set $K \subset \mathbb{R}^2$ and fixed $\omega \in \Omega$, the function

$$\mathcal{M}_K^{\mathfrak{R}}(\omega, \mathfrak{x}) \coloneqq \sup_{\boldsymbol{k} \in K} |\mathfrak{R}_{\mathfrak{G}}(\omega, \mathfrak{x}, \boldsymbol{k})|$$

is locally Lebesgue integrable, i.e., $\mathcal{M}_K^{\mathfrak{R}}(\omega, \cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$. ◆

Analogously to the measurability Assumption 3.27 of the remainder function, the above integrability assumption is verified for particular choices of the remainder function in Appendix A. With this assumption, we can now prove the continuous dependence result for random $\mathfrak{G}$-entropy functionals with respect to the pair of entropy values $\boldsymbol{k} \in \mathbb{R}^2$, which concludes this section.

Theorem 3.47 (Continuous dependence of entropy functional on entropy pair)**:**
Suppose the following conditions are satisfied:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a sole discontinuity hypersurface.*
- *Let $\mathfrak{f}$ be a flux function satisfying the sole-flux-discontinuity Assumption 3.13.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs satisfying the measurability Assumption 3.16.*

- *Let $\mathfrak{R}_{\mathfrak{G}}$ be a remainder function associated to $\mathfrak{G}$, which satisfies the measurability Assumption 3.27 and the integrability Assumption 3.46.*

Then, for fixed stochastic parameter $\omega \in \Omega$, fixed test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$ and a fixed function $\nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$, the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on $\boldsymbol{k} \in \mathbb{R}^2$. ◆

Proof. Let a stochastic parameter $\omega \in \Omega$, a nonnegative test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$ and a function $\nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$ be fixed. We prove the continuous dependence of the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ on the entropy pair $\boldsymbol{k} \in \mathbb{R}^2$ by considering each integral of the mapping (3.23) separately. To do this, let $\boldsymbol{k}, \boldsymbol{c} \in \mathbb{R}^2$ be fixed and denote by k, c the corresponding adapted Kružkov entropies.

Continuous dependence of Integral (3.23a)**.** Since the integral operator is linear, we can first apply Hölder's inequality and afterwards utilize the Cauchy-Schwarz-Bunyakovsky inequality, to obtain

$$\begin{aligned}\Big| \int_{\mathbb{X}_\mathbb{T}} |\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})| \partial_t \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big| \partial_t \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \Big| \\ \leq \int_{\mathbb{X}_\mathbb{T}} \Big| \Big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\Big| - |\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})| \Big| \big|\partial_t \psi(\mathfrak{x})\big| \,\mathrm{d}\mathfrak{x} \,.\end{aligned}$$

Observe that the adapted Kružkov entropies k, c satisfy $|k(\omega,\mathfrak{x}) - c(\omega,\mathfrak{x})| \leq \|\boldsymbol{k} - \boldsymbol{c}\|_2$ for every space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Here, $\|\cdot\|_2$ denotes the Euclidean distance in $\mathbb{R}^2$. Employing this estimate after applying the reverse triangle inequality, we obtain

$$\begin{aligned}\Big| \int_{\mathbb{X}_\mathbb{T}} |\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})| \partial_t \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} |\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})| \partial_t \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \Big| &\leq \int_{\mathbb{X}_\mathbb{T}} \big|c(\omega,\mathfrak{x}) - k(\omega,\mathfrak{x})\big| |\partial_t \psi(\mathfrak{x})| \,\mathrm{d}\mathfrak{x} \\ &\leq \|\boldsymbol{k} - \boldsymbol{c}\|_2 \int_{\mathbb{X}_\mathbb{T}} |\partial_t \psi(\mathfrak{x})| \,\mathrm{d}\mathfrak{x} \,.\end{aligned}$$

Recall that the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$ is compactly supported and smooth, which implies that its time derivative $\partial_t \psi$ is bounded. Therefore, we can conclude

$$\Big| \int_{\mathbb{X}_\mathbb{T}} |\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})| \partial_t \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} |\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})| \partial_t \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \Big| \leq C_\psi \|\boldsymbol{k} - \boldsymbol{c}\|_2 \,.$$

This shows that the Integral (3.23a) depends continuously on the entropy pair $\boldsymbol{k} \in \mathbb{R}^2$.

Continuous dependence of Integral (3.23b)**.** Showing continuous dependence of the Integral (3.23b) on the entropy pair $\boldsymbol{k} \in \mathbb{R}^2$ is similar to arguing its continuous dependence on the function $\nu \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$. We aim at establishing

$$\Big| \int_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \Big| \leq C_{\mathbb{J}} \|\boldsymbol{k} - \boldsymbol{c}\|_2 \,. \quad (3.28)$$

We introduce four sets $\mathbb{D}^{(\cdot)}(\omega)$ that depend on the terms $\operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big)$ and $\operatorname{sign}\big(\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big)$. For every random parameter $\omega \in \Omega$, we define

$$\begin{aligned}\mathbb{D}^{=}(\omega) &:= \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big) = \operatorname{sign}\big(\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big)\big\}, \\ \mathbb{D}^{\pm}(\omega) &:= \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big) = -\operatorname{sign}\big(\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big)\big\},\end{aligned}$$

$$\mathbb{D}_k^\nu(\omega) \coloneqq \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid \nu(\mathfrak{x}) = k(\omega,\mathfrak{x})\big\},$$
$$\mathbb{D}_c^\nu(\omega) \coloneqq \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid \nu(\mathfrak{x}) = c(\omega,\mathfrak{x})\big\}.$$

With these sets, we want to split up the integration over the space-time domain $\mathbb{X}_\mathbb{T}$. Even though these sets are not necessarily disjoint, for every random parameter $\omega \in \Omega$, their intersection is given as

$$\mathbb{D}^= \cap \mathbb{D}^\pm \cap \mathbb{D}_k^\nu \cap \mathbb{D}_c^\nu = \Big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T} \mid k(\omega,\mathfrak{x}) = \nu(\mathfrak{x}) = c(\omega,\mathfrak{x})\Big\}\,.$$

This characterization allows us to divide the integral over the space-time domain $\mathbb{X}_\mathbb{T}$ into four integrals over the sets $\mathbb{D}^{(\cdot)}(\omega)$, since the integrand in (3.28) vanishes on their intersection. For increased readability, we introduce the function $\mathfrak{Q}_{\omega,\mathfrak{x}}^\nu$ that describes the difference of two entropy fluxes via the mapping

$$\mathfrak{Q}_{\omega,\mathfrak{x}}^\nu(k,c) \coloneqq \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big) - \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\,.$$

We can now start with the estimation of the left-hand side of Inequality (3.28). The linearity of the integral operation and the sets $\mathbb{D}^{(\cdot)}(\omega)$ allow us to employ the triangle inequality to obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{X}_\mathbb{T}} \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \\
&\qquad\le \Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| && \text{(3.29a)}\\
&\qquad\qquad + \Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| && \text{(3.29b)}\\
&\qquad\qquad + \Big|\int_{\mathbb{D}_k^\nu(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| && \text{(3.29c)}\\
&\qquad\qquad + \Big|\int_{\mathbb{D}_c^\nu(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big|\,. && \text{(3.29d)}
\end{aligned}$$

To proceed with the estimation that shows the continuous dependence of Integral (3.23b) on $\boldsymbol{k} \in \mathbb{R}^2$ via Inequality (3.28), we consider the four terms in Inequality (3.29) separately:

Continuous dependence of Integral (3.29a). To show the continuous dependence of the Integral (3.29a) on the entropy value pair $\boldsymbol{k} \in \mathbb{R}^2$, we first apply Hölder's inequality followed by the Cauchy-Schwarz-Bunyakovsky inequality. This leads to the following estimation:

$$\Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \le \int_{\mathbb{D}^=(\omega)} \Big\|\mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_d \,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_d\;\mathrm{d}\mathfrak{x}\,.$$

We can now employ the definitions of the function $\mathfrak{Q}_{\omega,\mathfrak{x}}^\nu$ and the entropy flux $\mathfrak{q}$. Additionally, utilizing the definition of the set $\mathbb{D}^=(\omega)$ we can further estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,\mathfrak{x}}^\nu\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \\
&\qquad\le \int_{\mathbb{D}^=(\omega)} \Big\|\mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big) - \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_d \,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_d\;\mathrm{d}\mathfrak{x} \\
&\qquad\le \int_{\mathbb{D}^=(\omega)} \Big\|\mathfrak{f}\big(\omega,\mathfrak{x},c(\omega,\mathfrak{x})\big) - \mathfrak{f}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)\Big\|_d \,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_d\;\mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

By Assumption (F-2) the flux function $\mathfrak{f}$ is locally Lipschitz continuous. Since the adapted Kružkov en-

tropies k, c satisfy $|k(\omega,\mathfrak{x}) - c(\omega,\mathfrak{x})| \leq \|\boldsymbol{k} - \boldsymbol{c}\|_2$ for every space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, we can conclude

$$\Big| \int\limits_{\mathbb{D}^=(\omega)} \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq L_{\mathfrak{f}} \Big\| \boldsymbol{c} - \boldsymbol{k} \Big\|_2 \int\limits_{\mathbb{D}^=(\omega)} \big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \big\|_d \, \mathrm{d}\mathfrak{x}\,.$$

Since the test function ψ is compactly supported and smooth, its gradient $\nabla_{\boldsymbol{x}}\, \psi$ is bounded and we obtain

$$\Big| \int\limits_{\mathbb{D}^=(\omega)} \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq C_\psi L_{\mathfrak{f}} \Big\| \boldsymbol{c} - \boldsymbol{k} \Big\|_2\,,$$

which concludes the proof of Integral (3.29a) depending continuously on the entropy pair $\boldsymbol{k} \in \mathbb{R}^2$.

Continuous dependence of Integral (3.29b). To show that the Integral (3.29b) depends continuously on the pair of entropy values $\boldsymbol{k} \in \mathbb{R}^2$, we start by applying first Hölder's inequality. Employing the Cauchy-Schwarz-Bunyakovsky inequality afterwards, we obtain the estimation

$$\Big| \int\limits_{\mathbb{D}^\pm(\omega)} \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq \int\limits_{\mathbb{D}^\pm(\omega)} \Big\| \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \Big\|_d \, \Big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \Big\|_d \, \mathrm{d}\mathfrak{x}\,.$$

Now, we can continue by exploiting the definitions of the function $\mathfrak{A}^\nu_{\omega,\mathfrak{x}}$ and the entropy flux $\mathfrak{q}$. Additionally, using the construction of $\mathbb{D}^\pm(\omega)$ we can estimate

$$\begin{aligned}
&\Big| \int\limits_{\mathbb{D}^\pm(\omega)} \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \\
&\qquad \leq \int\limits_{\mathbb{D}^\pm(\omega)} \Big\| \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big) - \mathfrak{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big) \Big\|_d \, \big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \big\|_d \, \mathrm{d}\mathfrak{x} \\
&\qquad \leq \int\limits_{\mathbb{D}^\pm(\omega)} \Big\| 2\mathfrak{f}\big(\omega,\mathfrak{x},\nu(\mathfrak{x})\big) - \mathfrak{f}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big) - \mathfrak{f}\big(\omega,\mathfrak{x},c(\omega,\mathfrak{x})\big) \Big\|_d \, \big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \big\|_d \, \mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

Recall that by Assumption (F-2) the flux function is locally Lipschitz continuous. By employing the triangle inequality first and the local Lipschitz property afterwards, this yields

$$\Big| \int\limits_{\mathbb{D}^\pm(\omega)} \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \leq L_{\mathfrak{f}} \int\limits_{\mathbb{D}^\pm(\omega)} \Big(\big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big| + \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big| \Big) \, \big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \big\|_d \, \mathrm{d}\mathfrak{x}\,.$$

For any two scalar values $\xi_1, \xi_2 \in \mathbb{R}$, the identity $|\xi_1 - \xi_2| = \operatorname{sign}(\xi_1 - \xi_2)(\xi_1 - \xi_2)$ holds. Combining this identity with the construction of the set $\mathbb{D}^\pm(\omega)$ allows us to further estimate

$$\begin{aligned}
&\Big| \int\limits_{\mathbb{D}^\pm(\omega)} \mathfrak{A}^\nu_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}), c(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x})\, \mathrm{d}\mathfrak{x} \Big| \\
&\qquad \leq L_{\mathfrak{f}} \int\limits_{\mathbb{D}^\pm(\omega)} \Big| \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big)\big(c(\omega,\mathfrak{x}) - k(\omega,\mathfrak{x})\big) \Big| \, \Big\| \nabla_{\boldsymbol{x}}\, \psi(\mathfrak{x}) \Big\|_d \, \mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

Recall that the adapted Kružkov entropies k, c satisfy $|k(\omega,\mathfrak{x}) - c(\omega,\mathfrak{x})| \leq \|\boldsymbol{k} - \boldsymbol{c}\|_2$ for every space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Furthermore, the gradient of the test function $\nabla_{\boldsymbol{x}}\, \psi$ is compactly supported and

bounded. Consequently, the following estimation holds

$$\Big| \int\limits_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq C_{\psi}L_{\mathfrak{f}}\|\boldsymbol{c}-\boldsymbol{k}\|_2\,.$$

Therefore, we have shown that the Integral (3.29b) depends continuously on $\boldsymbol{k}\in\mathbb{R}^2$.

Continuous dependence of Integral (3.29c). To estimate the Integral (3.29c) we start by applying Hölder's inequality. Then, the Cauchy-Schwarz-Bunyakovsky inequality yields

$$\Big| \int\limits_{\mathbb{D}^{\nu}_{k}(\omega)} \mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq \int\limits_{\mathbb{D}^{\nu}_{k}(\omega)} \Big\|\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_d \,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

Inserting the definitions of $\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}$ and $\mathbf{q}$ as well as leveraging that $\nu(\mathfrak{x})=k(\omega,\mathfrak{x})$ for every space-time point $\mathfrak{x}\in\mathbb{D}^{\nu}_{k}(\omega)$ by the definition of the set $\mathbb{D}^{\nu}_{k}(\omega)$ leads to

$$\Big| \int\limits_{\mathbb{D}^{\nu}_{k}(\omega)} \mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq \int\limits_{\mathbb{D}^{\nu}_{k}(\omega)} \Big\|\mathfrak{f}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)-\mathfrak{f}\big(\omega,\mathfrak{x},c(\omega,\mathfrak{x})\big)\Big\|_d \,\Big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\Big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

By Assumption (F-2) the flux function is locally Lipschitz continuous. Furthermore, the adapted Kružkov entropies k,c satisfy $|k(\omega,\mathfrak{x})-c(\omega,\mathfrak{x})|\leq\|\boldsymbol{k}-\boldsymbol{c}\|_2$ for every space-time point $\mathfrak{x}\in\mathbb{X}_{\mathbb{T}}$. With recalling that the gradient of the test function $\nabla_{\boldsymbol{x}}\,\psi$ is bounded, we obtain

$$\Big| \int\limits_{\mathbb{D}^{\nu}_{k}(\omega)} \mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq C_{\psi}L_{\mathfrak{f}}\Big\|\boldsymbol{k}-\boldsymbol{c}\Big\|_2\,.$$

Thus, we have proven that the Integral (3.29c) depends continuously on $\boldsymbol{k}\in\mathbb{R}^2$.

Continuous dependence of Integral (3.29d). Proving the continuous dependence of the Integral (3.29d) on the pair of entropy values $\boldsymbol{k}\in\mathbb{R}^2$ is completely analogous to the estimation of the Integral (3.29c). Ultimately, this calculation yields

$$\Big| \int\limits_{\mathbb{D}^{\nu}_{c}(\omega)} \mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq C_{\psi}L_{\mathfrak{f}}\Big\|\boldsymbol{c}-\boldsymbol{k}\Big\|_2\,,$$

which implies the sought continuous dependence result.

Combining the estimations of the four Integrals (3.29a), (3.29b), (3.29c) and (3.29d) shows that the Integral (3.23b) depends continuously on the pair of entropy values $\boldsymbol{k}\in\mathbb{R}^2$.

Continuous dependence of Integral (3.23c). For the two adapted Kružkov entropies k,c, employing the linearity of the integral operation and Hölder's inequality yields

$$\begin{aligned}\Big| \int_{\mathbb{X}} \big|u_0(\omega,\boldsymbol{x})-k(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{X}} \big|u_0(\omega,\boldsymbol{x})-c(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \\ \leq \int_{\mathbb{X}} \Big|\big|u_0(\omega,\boldsymbol{x})-k(\omega,0,\boldsymbol{x})\big|-\big|u_0(\omega,\boldsymbol{x})-c(\omega,0,\boldsymbol{x})\big|\Big|\big|\psi(0,\boldsymbol{x})\big|\,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

We continue by applying the reverse triangle inequality. Recall that two entropy pairs $\boldsymbol{k},\boldsymbol{c}\in\mathbb{R}^2$ and the corresponding adapted Kružkov entropies k,c satisfy $|k(\omega,\mathfrak{x})-c(\omega,\mathfrak{x})|\leq\|\boldsymbol{k}-\boldsymbol{c}\|_2$ for every point

$\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. This leads to the estimate

$$\Big| \int_\mathbb{X} \big|u_0(\omega,\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\mathbb{X} \big|u_0(\omega,\boldsymbol{x}) - c(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \\ \leq \|\boldsymbol{c}-\boldsymbol{k}\|_2 \int_\mathbb{X} \big|\psi(0,\boldsymbol{x})\big|\,\mathrm{d}\boldsymbol{x}\,.$$

Since the test function ψ is smooth and compactly supported it is in particular bounded. This permits to conclude the estimation

$$\Big| \int_\mathbb{X} \big|u_0(\omega,\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\mathbb{X} \big|u_0(\omega,\boldsymbol{x}) - c(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \leq C_\psi \Big\|\boldsymbol{c}-\boldsymbol{k}\Big\|_2\,,$$

for some constant $C_\psi > 0$ that solely depends on the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$. This shows the continuous dependence of the Integral (3.23c) on the entropy pair $\boldsymbol{k} \in \mathbb{R}^2$.

Continuous dependence of Integral (3.23d). We now prove that the Integral (3.23d) depends continuously on the pair of entropy values $\boldsymbol{k} \in \mathbb{R}^2$. Recall that the remainder function $\mathfrak{R}_\mathfrak{G}$ is continuous in $\boldsymbol{k} \in \mathbb{R}^2$ by its construction in Definition 3.26. Furthermore, by the integrability Assumption 3.46 on the remainder function $\mathfrak{R}_\mathfrak{G}$, for every compact set $K \subset \mathbb{R}^2$ and every stochastic parameter $\omega \in \Omega$, there exists a function $\mathcal{M}_K^\mathfrak{R}$ defined by

$$\mathcal{M}_K^\mathfrak{R}(\omega,\mathfrak{x}) \coloneqq \sup_{\boldsymbol{k}\in K} |\mathfrak{R}_\mathfrak{G}(\omega,\mathfrak{x};\boldsymbol{k})|\,,$$

which is locally integrable in the sense that $\mathcal{M}_K^\mathfrak{R}(\omega,\cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_\mathbb{T};\mathbb{R})$. Now, for any compact set $K \subset \mathbb{R}^2$, this majorant of the remainder function $\mathfrak{R}_\mathfrak{G}$ provides us with the upper bound

$$\int_{\mathfrak{D}(\omega)} \mathfrak{R}_\mathfrak{G}(\omega,\mathfrak{d};\boldsymbol{k})\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d} \leq \int_{\mathfrak{D}(\omega)} \mathcal{M}_K^\mathfrak{R}(\omega,\mathfrak{d})\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d}\,. \tag{3.30}$$

Since the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$ is compactly supported and the function $\mathcal{M}_K^\mathfrak{R}(\omega,\cdot)$ is locally integrable, the Integral (3.30) is finite. Therefore, we can apply the dominated convergence theorem to obtain that the Integral (3.23d) depends continuously on the pair of entropy values $\boldsymbol{k} \in \mathbb{R}^2$, which completes the proof. For more details on this argumentation, we refer to [98, Theorem 5.6].

Combining the continuous dependence results of the four integrals in Mapping (3.23), we have shown that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the entropy pair $\boldsymbol{k} \in \mathbb{R}^2$. ■

3.3.3 Measurability of random entropy solutions

After this intermittent discussion of random $\mathfrak{G}$-entropy functionals, we can turn to investigating the (strong) measurability of random $\mathfrak{G}$-entropy solutions. For showing the measurability, we demand the pathwise existence and uniqueness of such solutions. However, as we have seen in our discussion of pathwise existence and uniqueness in Section 3.3.1, arguing existence of solutions might require various assumptions on the problem at hand, such as additional regularity assumptions on the flux discontinuity $\mathfrak{D}$ or the existence of uniform $\mathcal{L}^\infty$-bounds on approximate solutions. At this point, to simplify the presentation of the measurability study, we formulate the following assumption on the pathwise existence and uniqueness of $\mathfrak{G}$-entropy solutions. Recall that showing strong measurability requires the pathwise solution u to take values in a separable subspace $\mathcal{S}$ of the nonseparable space $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$.

Assumption 3.48 (Pathwise existence of unique $\mathfrak{G}$-entropy solution)**:**
For every stochastic parameter $\omega \in \Omega$, there exists a unique $\mathfrak{G}$-entropy solution to the random scalar discontinuous-flux conservation law given by Equation (3.1) *in the sense of Definition 3.23 or Definition 3.28. Furthermore, this pathwise $\mathfrak{G}$-entropy solution $u(\omega, \cdot, \cdot)$ is assumed to be separably-valued, i.e., the function $u(\omega, \cdot, \cdot)$ only takes values in a separable subspace $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$.* ◆

The above assumption on the random $\mathfrak{G}$-entropy solution u being separably-valued might seem very restrictive. However, the following example demonstrates two situations, in which separability appears most naturally and as a direct consequence of the considered problem.

Example 3.49 (Separable subspaces of $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$)**:** *The following two situations demonstrate examples of separable subspaces of $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ that occur under mild assumptions on the flux discontinuity or the initial data:*

(i) So far, we have considered random initial data that satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, for $1 \leq p \leq \infty$. If we require $1 \leq p < \infty$, the $\mathcal{L}^1$-contraction (3.17) *implies that the pathwise solution $u(\omega, \cdot)$ satisfies $u(\omega, \cdot) \in \mathcal{L}^p(\mathbb{X}_\mathbb{T}; \mathbb{R})$. Since $u(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ holds by definition, we obtain a separable subspace $\mathcal{S}$ of $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ by setting $\mathcal{S} = \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R}) \cap \mathcal{L}^p(\mathbb{X}_\mathbb{T}; \mathbb{R})$. The separability of $\mathcal{S}$ follows from the separability of the space $\mathcal{L}^p(\mathbb{X}_\mathbb{T}; \mathbb{R})$ for $1 \leq p < \infty$.*

(ii) If the sole discontinuity hypersurface $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ is stationary (i.e., independent of the temporal variable), it holds that the pathwise $\mathfrak{G}$-entropy solution $u(\omega, \cdot)$ satisfies $u(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})\big)$. Here, $\mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})\big)$ denotes the space of continuous functions from the time interval $\mathbb{T}$ into the function space $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})$ of locally integrable functions. Note, the space $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})$ is separable, since the space $\mathcal{L}^1(\mathbb{X}; \mathbb{R})$ of integrable functions is separable and a dense subspace of $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})$. Furthermore, assuming that the time interval $\mathbb{T}$ has the form $[0, T] \subset \mathbb{R}$, the space $\mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})\big)$ is separable because the time interval $\mathbb{T}$ is compact and $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})$ is separable (see, e.g., [160, Theorem 4.19]). Therefore, defining $\mathcal{S} = \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})\big)$, we obtain a separable subspace of $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$. ◆

We are now able to state the main result that guarantees strong measurability of $\mathfrak{G}$-entropy solutions to the random scalar discontinuous-flux conservation law given by Equation (3.1). This result allows us to interpret the solution u as a $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$-valued, Bochner-integrable random variable.

Theorem 3.50 (Measurability of random $\mathfrak{G}$-entropy solutions)**:**
Suppose the following conditions are satisfied:

- *Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$ be a random initial condition to the scalar conservation law given by Problem* (3.1).
- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity satisfying the measurability Assumption 3.4.*
- *Let the flux function $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 and the measurability Assumption 3.41.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a family of $\mathcal{L}^1D$ germs satisfying the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$ that satisfies the measurability Assumption 3.27 and the integrability Assumption 3.46.*

- *Let the pathwise existence and uniqueness Assumption 3.48 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, the pathwise $\mathfrak{G}$-entropy solution u to Problem (3.1) is strongly measurable in the sense that the mapping $u : \Omega \to \mathcal{S}$ is strongly measurable. ◆

Proof. The proof is divided into three steps: First, we construct a modified entropy functional that inherits the properties of being Carathéodory and continuously depending on the entropy constants from the standard adapted $\mathfrak{G}$-entropy functional. Afterwards, we prove that this modified entropy functional can represent a particular version of the adapted entropy Condition (3.14) and that a corresponding set-valued map is measurable. In the third step, we connect the constructed set-valued map to the $\mathfrak{G}$-entropy solution of Problem (3.1) to show its measurability. A technical justification of an argument in step three is postponed for readability purposes. This result is argued directly after this proof.

Step 1: Random modified entropy functional. We start by defining a sequence of function spaces consisting of smooth nonnegative functions with compact support. Therefore, for $N \in \mathbb{N}$, define

$$\mathcal{D}_N := \Big\{ \psi \in \mathcal{C}^\infty(\mathbb{R}_{>0} \times \mathbb{R}^d; \mathbb{R}_{\geq 0}) \mid \operatorname{supp}(\psi) \subseteq \big\{ (t, \boldsymbol{x}) \in \mathbb{R}_{>0} \times \mathbb{R}^d \mid t \in [0, N] \text{ and } \boldsymbol{x} \in \mathbb{B}_N(\mathbf{0}_{\mathbb{R}^d}) \big\} \Big\}.$$

Here, $\mathbb{B}_N(\boldsymbol{x})$ denotes the closed ball with radius N around $\boldsymbol{x}$ and $\mathbf{0}_{\mathbb{R}^d}$ is the d-dimensional zero-element. Each space $\mathcal{D}_N$ is a subspace of $\mathcal{C}^\infty(\mathbb{R}_{>0} \times \mathbb{R}^d; \mathbb{R}_{\geq 0})$ by construction. Thereby, since $\mathcal{C}^\infty$ has a countable basis, $\mathcal{D}_N$ also has a countable basis. Consequently, for every nonnegative test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T}; \mathbb{R}) = \mathcal{C}_c^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$, there exists a constant $R_\psi \in \mathbb{N}$, such that $\psi \in \mathcal{D}_{R_\psi}$.

Now, fix a number $N \in \mathbb{N}$ and let $(\psi_N^i, i \in \mathbb{N}) \subset \mathcal{D}_N$ be a basis of the space $\mathcal{D}_N$. Then, for a fixed pair of constants $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a fixed number $i \in \mathbb{N}$, we define the *random modified entropy functional* $\mathbb{J}_{i,N}^{\boldsymbol{k}} : \Omega \times \mathcal{S} \to \mathbb{R}$ via the mapping

$$\begin{aligned}(\omega, \nu) \mapsto &\int_\mathbb{T} \int_\mathbb{X} \big| \nu(t, \boldsymbol{x}) - k(\omega, t, \boldsymbol{x}) \big| \partial_t \psi_N^i(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &+ \int_\mathbb{T} \int_\mathbb{X} \mathbf{q}\big(\omega, t, \boldsymbol{x}; \nu(t, \boldsymbol{x}), k(\omega, t, \boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi_N^i(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &- \int_\mathbb{X} \big| u_0(\omega, \boldsymbol{x}) - k(\omega, t, \boldsymbol{x}) \big| \psi_N^i(0, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{\mathfrak{D}(\omega)} \mathfrak{R}_\mathfrak{G}(\omega, \mathfrak{d}; \boldsymbol{k}) \psi_N^i(\mathfrak{d}) \, \mathrm{d}\mathfrak{d} \, .\end{aligned}$$

Here, k is the adapted Kružkov entropy given by Equation (3.12) corresponding to $\boldsymbol{k}$, $\mathbf{q}$ is the entropy flux defined via Equation (3.13) and $\mathfrak{R}_\mathfrak{G}$ is the remainder function associated to the random family of germs $\mathfrak{G}$ as introduced in Definition 3.26. Note, we can apply Theorem 3.45 to obtain that the modified entropy functional $\mathbb{J}_{i,N}^{\boldsymbol{k}}$ is Carathéodory.

Step 2: Measurable set-valued map. For $\boldsymbol{k} \in \mathbb{R}^2$ and $i, N \in \mathbb{N}$ still being fixed, we define the set-valued mapping

$$\Xi_{i,N}^{\boldsymbol{k}} : \Omega \rightrightarrows \mathcal{S} \qquad \omega \mapsto \Big\{ \nu \in \mathcal{S} \mid \mathbb{J}_{i,N}^{\boldsymbol{k}}(\omega, \nu) \geq 0 \Big\} \, .$$

This multifunction selects all functions $\nu \in \mathcal{S}$ satisfying the adapted entropy Condition (3.14) for a fixed couple of entropy values $\boldsymbol{k} = (k^l, k^r) \in \mathbb{R}^2$ and a fixed test function $\psi_N^i \in \mathcal{D}_N$. Since, by Assumption 3.48, the subspace $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ is separable, we can apply Lemma 2.9 to obtain that the set-valued map $\Xi_{i,N}^{\boldsymbol{k}}$ is measurable.

Step 3: Singleton containing solution. To omit functions $\nu \in \mathcal{S}$ that only satisfy the adapted entropy Condition (3.14) for some fixed test function ψ_N^i, we define the set-valued map

$$\Xi^{\boldsymbol{k}} : \Omega \rightrightarrows \mathcal{S} \qquad \Xi^{\boldsymbol{k}}(\omega) \coloneqq \bigcap_{i\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \Xi_{i,N}^{\boldsymbol{k}}(\omega) \qquad \text{for } \omega \in \Omega \,.$$

This correspondence is measurable, since the countable intersection of measurable maps is again measurable. Furthermore, $\Xi^{\boldsymbol{k}}$ now contains all the functions that satisfy the entropy condition for fixed $\boldsymbol{k}$, but for all test functions $\psi \in \mathcal{D}$. Now, we can define

$$\Xi : \Omega \rightrightarrows \mathcal{S} \qquad \Xi(\omega) \coloneqq \bigcap_{\boldsymbol{k}\in\mathbb{Q}^2} \Xi^{\boldsymbol{k}}(\omega) \qquad \text{for } \omega \in \Omega \,.$$

Again, this set-valued map is measurable via the countable intersection of measurable maps. The measurable mapping Ξ now contains all those functions $\nu \in \mathcal{S}$ satisfying the stochastic entropy condition (without any restrictions due to fixed variables or functions). While the adapted entropy condition (3.14) is formulated to hold for every entropy $\boldsymbol{k} \in \mathbb{R}^2$, it is sufficient to only consider $\boldsymbol{k} \in \mathbb{Q}^2$. For readability purposes, the verification of this claim is postponed until after this proof.

To conclude the proof, we note that by Assumption 3.48 there exists a unique $\mathfrak{G}$-entropy solution for every $\omega \in \Omega$. Therefore, the correspondence Ξ only contains this pathwise $\mathfrak{G}$-entropy solution of Problem (3.1), which is measurable as a map $u : \Omega \to \mathcal{S}$ due to the measurability of Ξ. ■

Sufficiency of intersecting over rationals. It remains to show that it is sufficient to intersect over $\boldsymbol{k} \in \mathbb{Q}^2$ to select functions that satisfy the stochastic entropy condition for every $\boldsymbol{k} \in \mathbb{R}^2$. We show this via a contradiction:

Assume that intersecting over $\mathbb{Q}^2$ is not sufficient to select the adapted entropy solution, which is unique by Assumption 3.48. Then, for $\boldsymbol{k} \in \mathbb{Q}^2$, the set $\Xi(\omega)$ does contain the stochastic adapted entropy solution u and another function ν. By construction, both functions satisfy the adapted entropy Condition (3.14) for $\boldsymbol{k} \in \mathbb{Q}^2$. However, due to the uniqueness Assumption 3.48, for $\boldsymbol{c} \in (\mathbb{R} \setminus \mathbb{Q})^2$, the image of $\Xi(\omega)$ only contains the adapted entropy solution. By Definition 3.39 of the random $\mathfrak{G}$-entropy functional, this means that there exist $j, M \in \mathbb{N}$ such that

$$\mathbb{J}_{i,N}^{\boldsymbol{k}}(\omega, \nu) \geq 0 \quad \text{for all } i, N \in \mathbb{N} \qquad \mathbb{J}_{j,M}^{\boldsymbol{c}}(\omega, \nu) < 0 \quad \text{for } j, M \in \mathbb{N} \,.$$

Due to Theorem 3.47 the random $\mathfrak{G}$-entropy functional $\mathbb{J}_{j,M}^{\boldsymbol{c}}$ is continuous with respect to the pair of entropy values $\boldsymbol{c}$. Thus, for $\varepsilon > 0$, there exists an ε-neighborhood $\mathbb{B}_\varepsilon$ around the value $\mathbb{J}_{j,M}^{\boldsymbol{c}}(\omega, \nu)$ such that for every $\xi \in \mathbb{B}_\varepsilon$ it holds that $\xi < 0$. Again using the continuous dependence of $\mathbb{J}_{j,M}^{\boldsymbol{c}}$ in $\boldsymbol{c}$ and noting that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, there exists a $\delta > 0$ and $\tilde{\boldsymbol{c}} \in \mathbb{Q}^2$ such that $\|\boldsymbol{c} - \tilde{\boldsymbol{c}}\|_2 < \delta$ and $\mathbb{J}_{j,M}^{\tilde{\boldsymbol{c}}}(\omega, \nu) \in \mathbb{B}_\varepsilon$. Here, $\|\cdot\|_2$ denotes the Euclidean distance of $\mathbb{R}^2$.

However, $\mathbb{J}_{j,M}^{\tilde{\boldsymbol{c}}}(\omega, \nu) \in \mathbb{B}_\varepsilon$ implies that $\mathbb{J}_{j,M}^{\tilde{\boldsymbol{c}}}(\omega, \nu) < 0$, which contradicts to $\mathbb{J}_{i,N}^{\boldsymbol{k}}(\omega, \nu) \geq 0$ for all numbers $i, N \in \mathbb{N}$ and all entropy pairs $\boldsymbol{k} \in \mathbb{Q}^2$. Consequently, it is sufficient to intersect over $\mathbb{Q}^2$ instead of $\mathbb{R}^2$ to select the unique adapted entropy solution. ■

3.3.4 Existence of moments of random entropy solutions

By the strong measurability result of the previous section, the random $\mathfrak{G}$-entropy solution can be interpreted as an $\mathcal{S}$-valued, Bochner-integrable random variable. Thereby, its statistical properties can be described via stochastic moments, if these exist. The subsequent investigation describe conditions, such that the q-th moment of the solution exists, with $1 \leq q < \infty$. Here, the main ingredient of this consideration is the $\mathcal{L}^1$-contraction principle (3.17) of the random $\mathfrak{G}$-entropy solutions. Therefore, throughout this section, we require the (random) initial condition u_0 to be integrable. Additionally, we need to impose the following technical assumption on the flux function. Afterwards, we state the main theorem of this section.

Assumption 3.51 (Zero mass creation)**:**
We assume that the flux function f is chosen such that the zero initial condition $u_0 \equiv 0$ leads to the vanishing $\mathfrak{G}$-entropy solution $u \equiv 0$. ◆

Theorem 3.52 (Existence of moments of $\mathfrak{G}$-entropy solution)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^1(\mathbb{X})\big)$, with $1 \leq q < \infty$, be a random initial condition to Problem (3.1)*. Furthermore, let the following conditions be satisfied:*

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface that satisfies the measurability Assumption 3.4.*
- *Let the flux function $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 and the measurability Assumption 3.41 as well as the zero-mass-creation Assumption 3.51.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a family of $\mathcal{L}^1$-dissipative germs satisfying the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$ that satisfies the joint measurability Assumption 3.27 and the integrability Assumption 3.46.*
- *Let the pathwise existence and uniqueness Assumption 3.48 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, at almost every $t \in \mathbb{T}$, the stochastic $\mathfrak{G}$-entropy solution u admits moments up to order q. In particular, the random $\mathfrak{G}$-entropy solution satisfies the estimation

$$\|u(\omega, t, \boldsymbol{x})\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)} \leq \|u_0(\omega, \boldsymbol{x})\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)} \tag{3.31}$$

for almost every time $t \in \mathbb{T}$. ◆

Proof. We only prove the Estimation (3.31), which directly implies the existence of moments up to order $q \in [1, \infty)$ because the initial condition u_0 satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\big)$ by hypothesis. First, note that the term $\|u(\omega, t, \boldsymbol{x})\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)}$ makes sense. For every random parameter $\omega \in \Omega$, a pathwise unique $\mathfrak{G}$-entropy solution exists and takes values only in a separable space $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ due to Assumption 3.48. Furthermore, the mapping $u : \Omega \to \mathcal{S}$ is strongly measurable by the result

of Theorem 3.50. Recall the $\mathcal{L}^1$-contraction Property (3.17) of the pathwise $\mathfrak{G}$-entropy solution $u(\omega, \cdot)$, which was given by

$$\int_{\mathbb{X}} |u(\omega, t, \boldsymbol{x}) - \tilde{u}(\omega, t, \boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{X}} |u_0(\omega, \boldsymbol{x}) - \tilde{u_0}(\omega, \boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \, .$$

Due to the zero-mass creation Assumption 3.51, we can choose $\tilde{u_0} \equiv 0$ and immediately obtain $\tilde{u} \equiv 0$. Based on this estimation, for $1 \leq q < \infty$, we can compute

$$\left\| u(\omega, t, \boldsymbol{x}) \right\|^q_{\mathcal{L}^q\left(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\right)} = \int_\Omega \left\| u(\omega, t, \boldsymbol{x}) \right\|^q_{\mathcal{L}^1(\mathbb{X}; \mathbb{R})} \, \mathrm{d}\mathbb{P} \leq \int_\Omega \left\| u_0(\omega, \boldsymbol{x}) \right\|^q_{\mathcal{L}^1(\mathbb{X}; \mathbb{R})} \, \mathrm{d}\mathbb{P} \, .$$

The assertion of the theorem follows by taking the q-th root and noting that these integrals exists due to the presumption that the initial condition u_0 satisfies $u_0 \in \mathcal{L}^q(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R}))$. ■

Roughly speaking, the above theorem states that a random $\mathfrak{G}$-entropy solution inherits the moments of the underlying stochastic initial condition u_0. Let us stress that this result does not require the random initial condition and the stochastic discontinuity interface to be independent. The above statement allows us to deduce the following special case, if the initial condition is deterministic.

Corollary 3.53 (Existence of moments with deterministic initial conditions)**:**
Let $u_0 \in \mathcal{L}^1(\mathbb{X}; \mathbb{R})$ be a deterministic initial condition to Problem (3.1)*. Furthermore, as in Theorem 3.52, let the following conditions be satisfied:*

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity that satisfies the measurability Assumption 3.4.*
- *Let the flux function $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 and the measurability Assumption 3.41 as well as the zero-mass-creation Assumption 3.51.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a family of $\mathcal{L}^1$-dissipative germs satisfying the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$ that satisfies the measurability Assumption 3.27 and the integrability Assumption 3.46.*
- *Let the pathwise existence and uniqueness Assumption 3.48 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, at almost every $t \in \mathbb{T}$, the random $\mathfrak{G}$-entropy solution u admits all moments of order $1 \leq q < \infty$. ◆

Proof. From Theorem 3.52, we know that for $1 \leq q < \infty$ the estimation

$$\|u(\omega, t, \boldsymbol{x})\|^q_{\mathcal{L}^q\left(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\right)} \leq \int_\Omega \|u_0\|^q_{\mathcal{L}^1(\mathbb{X}; \mathbb{R})} \, \mathrm{d}\mathbb{P} = \|u_0\|^q_{\mathcal{L}^1(\mathbb{X}; \mathbb{R})}$$

holds. Thus, the assertion follows by taking the q-th root and noting that the initial condition satisfies $u_0 \in \mathcal{L}^1(\mathbb{X}; \mathbb{R})$ by hypothesis. ■

Let us conclude this examination of stochastic moments with another specific existence result, which focuses on stationary flux discontinuities. Time independency (i.e., stationarity) of sole discontinuity

hypersurfaces has already been described in Example 3.49 (ii). In particular, by the result of [228], each pathwise $\mathfrak{G}$-entropy solution $u(\omega,\cdot)$ has a representative in the space $\mathcal{C}\big(\mathbb{T};\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})\big)$ of continuous functions mapping from the time interval $\mathbb{T}$ into the function space $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})$ of locally integrable functions. This motivates the following theorem.

Theorem 3.54 (Existence of moments for stationary flux discontinuities)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$, be a random initial condition to Problem (3.1)*. Furthermore, as in the existence-of-moments Theorem 3.52 for general flux functions, let the following conditions be satisfied:*

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface that satisfies the measurability Assumption 3.4.*
- *Let the flux function $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 and the measurability Assumption 3.41 as well as the zero-mass-creation Assumption 3.51.*
- *Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a family of $\mathcal{L}^1$-dissipative germs satisfying the joint measurability Assumption 3.16.*
- *Let $\mathfrak{R}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a remainder function associated to $\mathfrak{G}$ that satisfies the joint measurability Assumption 3.27 and the integrability Assumption 3.46.*
- *Let the pathwise existence and uniqueness Assumption 3.48 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, if the sole discontinuity $\mathfrak{D}(\omega)$ is stationary for every $\omega \in \Omega$, the following estimation holds:

$$\Big\|u(\omega,t,\boldsymbol{x})\Big\|_{\mathcal{L}^q\left(\Omega;\mathcal{C}\left(\mathbb{T};\mathcal{L}^1(\mathbb{X})\right)\right)} \leq \Big\|u_0\Big\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)}$$

In particular, the random $\mathfrak{G}$-entropy solution u admits all moments up to order $1 \leq q < \infty$. ◆

Proof. Recall that by Assumption 3.48 and Example 3.49 (ii), the solution u satisfies for every $\omega \in \Omega$ that $u(\omega,\cdot,\cdot) \in \mathcal{S} = \mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}}) \cap \mathcal{C}\big(\mathbb{T};\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X})$. Thereby, we can estimate

$$\begin{aligned}
\Big\|u(\omega,t,\boldsymbol{x})\Big\|^q_{\mathcal{L}^q\left(\Omega;\mathcal{C}\left(\mathbb{T};\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)\right)} &= \int_\Omega \sup_{t\in\mathbb{T}} \|u(\omega,t,\cdot)\|^q_{\mathcal{L}^1(\mathbb{X};\mathbb{R})}\,\mathrm{d}\mathbb{P} \\
&\leq \int_\Omega \Big\|u_0(\omega,\cdot)\Big\|^q_{\mathcal{L}^1(\mathbb{X};\mathbb{R})}\,\mathrm{d}\mathbb{P} \\
&\leq \|u_0\|^q_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)}\,.
\end{aligned}$$

Here, we leveraged the $\mathcal{L}^1$-contraction property 3.17 of the $\mathfrak{G}$-entropy solution u in a similar manner as in Theorem 3.52. The latter norm in the estimation is finite by the hypothesis that $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\big)$ and therefore, the assertion follows by taking the q-th root. ■

3.4 Examples of (random) admissibility germs and remainder functions

In this section, we provide examples of random admissibility germs and remainder functions. For the presented families of random admissibility germs, we verify the joint measurability Assumption 3.16, which is necessary to prove that the corresponding $\mathfrak{G}$-entropy solution is strongly measurable. Similarly, for the presented random remainder functions, the joint measurability Assumption 3.27 and the integrability Assumption 3.46 are verified.

As a first example, Section 3.4.1 discusses the *Rankine-Hugoniot admissibility germ*, which is a very fundamental germ containing all value pairs that satisfy the Rankine-Hugoniot condition. This property readily implies that any other admissibility germ is a subset of the Rankine-Hugoniot admissibility germ. Afterwards, in Section 3.4.2, we consider the vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}$, which was already introduced and discussed in Section 3.3.1. We conclude this Section by discussing the simplest choice for the random remainder function in Section 3.4.3, which requires an additional assumption of the flux function. However, Appendix A discusses various other types of random remainder functions, which avoid this additional assumption and have a more general form.

3.4.1 Rankine-Hugoniot admissibility germ

The first admissibility germ discussed is very essential. It consists of all pairs $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ that satisfy the Rankine-Hugoniot condition. Hence, it is called the *(random) Rankine-Hugoniot germ*. Note that by the Definition 3.14 of an admissibility germ $\mathfrak{G}$, every couple of values $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathfrak{G} \subset \mathbb{R}^2$ needs to satisfy this Rankine-Hugoniot condition. Thus, any admissibility germ is a subset of the Rankine-Hugoniot germ $\mathfrak{G}_{\mathrm{RH}}$, which is defined subsequently.

Definition 3.55 (Random Rankine-Hugoniot germ)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface (cf., Definition 3.1) that satisfies the stochastic measurability Assumption 3.4. Furthermore, let $\widehat{\mathfrak{n}_{\mathfrak{D}}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{X}_{\mathbb{T}}$ denote the extension of the normal field $\mathfrak{n}_{\mathfrak{D}}$ of the sole discontinuity $\mathfrak{D}$ as defined in Equation (3.7)*. Then, the family of sets $\mathfrak{G}_{\mathrm{RH}}(\omega, \mathfrak{x}) \subset \mathbb{R}^2$ containing all pairs $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ satisfying the Rankine-Hugoniot condition*

$$\mathfrak{s}(\omega, \mathfrak{x}) := \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) = \mathfrak{f}^r(\omega, \mathfrak{x}, \mathfrak{u}^r) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \tag{3.32}$$

is called a random family of Rankine-Hugoniot germs. ◆

An important property of random families of admissibility germs is their joint measurability, which we ensured via Assumption 3.27. While such measurability is crucial to show that the random entropy solution is strongly measurable, it is also related to showing existence of solutions. For the details on this relation, we refer to the deterministic work [8], where the connection of measurability and existence of solutions is extensively discussed. In the subsequent proposition, we show that a random family of Rankine-Hugoniot germs as defined in Definition 3.55 is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$.

Proposition 3.56 (Joint measurability of random Rankine-Hugoniot germs):
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole flux discontinuity that satisfies the stochastic measurability Assumption 3.4. Furthermore, let $\mathfrak{G}_{\mathrm{RH}} \subset \mathbb{R}^2$ be a random family of Rankine-Hugoniot germs as defined in Definition 3.55 and let the flux function $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 as well as the stochastic measurability Assumption 3.41. Then, $\mathfrak{G}_{\mathrm{RH}}$ can be written as a set-valued mapping $\mathfrak{G}_{\mathrm{RH}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$, which is jointly measurable. ◆

Proof. First, recall that by Lemma 3.8 the extended normal vector field $\widehat{\mathfrak{n}_{\mathfrak{D}}}$ is Carathéodory as a mapping $\widehat{\mathfrak{n}_{\mathfrak{D}}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{X}_{\mathbb{T}}$. This implies that it is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ due to [5, Lemma 4.51]. Now, let the function $\Pi_{\mathrm{RH}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\begin{aligned}\Pi_{\mathrm{RH}}\big(\omega, \mathfrak{x}, (\mathfrak{u}^l, \mathfrak{u}^r)\big) &:= \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) - \mathfrak{f}^r(\omega, \mathfrak{x}, \mathfrak{u}^r) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \\ &= \big(\mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l) - \mathfrak{f}^r(\omega, \mathfrak{x}, \mathfrak{u}^r)\big) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \, .\end{aligned}$$

This function vanishes as soon as the Rankine-Hugoniot Condition (3.32) is satisfied. Therefore, we aim at writing the germ $\mathfrak{G}_{\mathrm{RH}}$ as the zero-level set of the function Π_{RH}. Before we do so, some properties of Π_{RH} need to be established, such that joint measurability of $\mathfrak{G}_{\mathrm{RH}}$ can be deduced. By the stochastic measurability Assumption 3.41 and the sole-flux-discontinuity Assumption 3.13, the functions $\mathfrak{f}^{l,r}$ are measurable in $\omega \in \Omega$ and continuous in the remaining arguments. Thus, by [5, Lemma 4.51], they are jointly measurable. This immediately implies that the function Π_{RH} is Carathéodory in the sense that it is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ and continuous in $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R}^2$, since the normal vector field extension $\widehat{\mathfrak{n}_{\mathfrak{D}}}$ is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ and the fluxes $\mathfrak{f}^{l,r}$ are also jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ and continuous in $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R}^2$. Exploiting the construction of Π_{RH}, we can write the random family of Rankine-Hugoniot germs as a correspondence $\mathfrak{G}_{\mathrm{RH}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ defined as

$$\mathfrak{G}_{\mathrm{RH}}(\omega, \mathfrak{x}) := \Big\{ (\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R}^2 \mid \Pi_{\mathrm{RH}}\big(\omega, \mathfrak{x}, (\mathfrak{u}^l, \mathfrak{u}^r)\big) = 0 \Big\} \, . \tag{3.33}$$

Since the function Π_{RH} is Carathéodory, Equation (3.33) and Lemma 2.9 prove the assertion. ■

The Rankine-Hugoniot germ is the simplest admissibility germ one can choose, since it does not impose any additional restriction on the jump of the entropy solution u across the flux discontinuity (besides the Rankine-Hugoniot condition). For the one-dimensional model Problem (3.19) with strictly increasing (or strictly descreasing) flux functions $\mathfrak{f}^{l,r}$, which are also uniformly Lipschitz continuous, the Rankine-Hugoniot germ corresponds to the entropy formulation developed by Baiti and Jenssen [23]. For the details on this admissibility condition and an analysis of the corresponding germ, we refer to Andreianov et al. [13, Section 4.4].

3.4.2 Vanishing viscosity germ

The next family of (random) admissibility germs we consider has already been discussed briefly in Section 3.3.1 and is called the *vanishing viscosity germ*, denoted $\mathfrak{G}_{\mathrm{VV}}$. This admissibility germ describes the behavior of entropy solutions as the limit of solutions to a problem containing a small viscosity term. For details on the motivation for considering such vanishing viscosity solutions, we refer to the numerical experiments in Chapter 5 as well as to the monograph [80] or to the work [38]. For the ease of presentation, we start by recalling the definition of a random family of vanishing viscosity germs.

Definition 3.57 (Random vanishing viscosity germ)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity that satisfies the stochastic measurability Assumption 3.4. Furthermore, let $\widehat{\mathfrak{n}_{\mathfrak{D}}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{X}_{\mathbb{T}}$ denote the extension of the normal field $\mathfrak{n}_{\mathfrak{D}}$ of the sole discontinuity $\mathfrak{D}$ as defined in Equation (3.7). Then, the sets $\mathfrak{G}_{\mathrm{VV}}(\omega, \mathfrak{x}) \subset \mathbb{R}^2$ containing all pairs $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ that satisfy the Rankine-Hugoniot condition

$$\mathfrak{s}(\omega, \mathfrak{x}) := \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) = \mathfrak{f}^r(\omega, \mathfrak{x}, \mathfrak{u}^r) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \tag{3.34}$$

and one of the following conditions

(i) $\mathfrak{u}^l = \mathfrak{u}^r$

(ii) $\mathfrak{u}^l < \mathfrak{u}^r$ *and there exists a* $\mathfrak{u}^o \in [\mathfrak{u}^l, \mathfrak{u}^r]$ *such that*

$$\begin{cases} \mathfrak{f}^l(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathfrak{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^l, \mathfrak{u}^o] \,, \\ \qquad\qquad \text{and} & \\ \mathfrak{f}^r(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathfrak{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^r] \,. \end{cases}$$

(iii) $\mathfrak{u}^l > \mathfrak{u}^r$ *and there exists a* $\mathfrak{u}^o \in [\mathfrak{u}^r, \mathfrak{u}^l]$ *such that*

$$\begin{cases} \mathfrak{f}^l(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \leq \mathfrak{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^l] \,, \\ \qquad\qquad \text{and} & \\ \mathfrak{f}^r(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \leq \mathfrak{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^r, \mathfrak{u}^o] \,. \end{cases}$$

are called a random family of vanishing viscosity germs. ◆

There exists an equivalent formulation for admissible vanishing viscosity solutions via the Γ-condition of Diehl [93]. This Γ-condition was originally derived via a standing-waves approach[24] for the vanishing viscosity method including smoothing [90–92, 94]. The equivalence of the vanishing viscosity germ and the Γ-condition was argued in [12] for the deterministic setting. We summarize this equivalency in the subsequent remark, where we formulate the Γ-condition in a way that reminds of Oleïnik-type admissibility conditions.

Remark 3.58 (Γ-condition represents vanishing viscosity solutions)**:** *The equivalency of the Γ-condition to the vanishing viscosity germ becomes clear when using the formulation of [93]:*

For $\xi_1, \xi_2 \in \mathbb{R}$, denote by $\mathrm{ch}(\xi_1, \xi_2)$ the convex hull of ξ_1 and ξ_2, given by $\big[\min\{\xi_1, \xi_2\}, \max\{\xi_1, \xi_2\}\big]$. A couple $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R}^2$ satisfies the Γ-condition, if the Rankine-Hugoniot Condition (3.34) is satisfied and there exists a $\mathfrak{u}^o \in \mathrm{ch}(\mathfrak{u}^l, \mathfrak{u}^r)$ such that

- $(\mathfrak{u}^r - \mathfrak{u}^o)\big(\mathfrak{f}^r(\omega, \mathfrak{x}, \rho) - \mathfrak{f}^r(\omega, \mathfrak{x}, \mathfrak{u}^r)\big) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq 0$ *for all* $\rho \in \mathrm{ch}(\mathfrak{u}^r, \mathfrak{u}^o)$,
- $(\mathfrak{u}^o - \mathfrak{u}^l)\big(\mathfrak{f}^l(\omega, \mathfrak{x}, \rho) - \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l)\big) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq 0$ *for all* $\rho \in \mathrm{ch}(\mathfrak{u}^l, \mathfrak{u}^o)$,

where $\widehat{\mathfrak{n}_{\mathfrak{D}}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{X}_{\mathbb{T}}$ is the extended normal vector field of the random discontinuity hypersurface $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$. ◆

[24] For details on the standing-waves approach, we refer to [80, 140] or to the early work [109].

The remainder of this section is devoted to establishing that the family of random vanishing viscosity germs is jointly measurable. As a preparational result, the following proposition enables us to represent the Condition (ii) of Definition 3.57 as a jointly measurable correspondence.

Proposition 3.59 (Vanishing viscosity condition via jointly measurable correspondence)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity that satisfies the stochastic measurability Assumption 3.4. Furthermore, let $\widehat{\mathfrak{n}_{\mathfrak{D}}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{X}_{\mathbb{T}}$ denote the extension of the normal field $\mathfrak{n}_{\mathfrak{D}}$ of the sole discontinuity $\mathfrak{D}$ as defined in Equation (3.7)*. Additionally, let the flux $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 as well as the stochastic measurability Assumption 3.41 and the confinement Assumption 3.38. Then, there exists a set-valued mapping $\Theta_{\mathrm{VV}}^{l<r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ such that every couple $(\mathfrak{u}^l, \mathfrak{u}^r) \in \Theta_{\mathrm{VV}}^{l<r}$ satisfies Condition (ii) of Definition 3.57.* ◆

Proof. The proof is divided into four steps: First, we construct a representation of the condition part $\mathfrak{f}^l(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x})$ for all $\rho \in [\mathfrak{u}^l, \mathfrak{u}^o]$ via a correspondence. Afterwards, we show that the derived set-valued mapping is Carathéodory. This allows us to formulate the whole part of Condition (ii) of Definition 3.57 regarding the existence of a value $\mathfrak{u}^o$ via correspondences. In the last step, we conclude the assertion by combining the preceeding investigations.

Flux conditions for arbitrary intervals via correspondences. The goal of this first step is to derive a general formulation of the condition

$$\mathfrak{f}^l(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x}) \qquad \text{for all } \rho \in [\mathfrak{u}^l, \mathfrak{u}^o] \,, \tag{3.35}$$

where $\mathfrak{u}^o \in \mathbb{R}$, with $\mathfrak{u}^l \leq \mathfrak{u}^o$, is a real number. Since the Rankine-Hugoniot Condition (3.34) implies $\mathbb{s}(\omega, \mathfrak{x}) \coloneqq \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x})$, the Inequality (3.35) can be written as

$$\big(\mathfrak{f}^l(\omega, \mathfrak{x}, \rho) - \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l)\big) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq 0 \qquad \text{for all } \rho \in [\mathfrak{u}^l, \mathfrak{u}^o] \,. \tag{3.36}$$

Based on this form of the inequality, we introduce a function $H^l : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathcal{CS}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ via

$$H^l(\omega, \mathfrak{x}, A, \mathfrak{u}^l) \coloneqq \inf_{a \in A} \Big(\big(\mathfrak{f}^l(\omega, \mathfrak{x}, a) - \mathfrak{f}^l(\omega, \mathfrak{x}, \mathfrak{u}^l)\big) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \Big) \,.$$

Here, $\mathcal{CS}(\mathbb{R})$ is the hyperspace of closed subsets of the real numbers $\mathbb{R}$ as introduced in Section 2.2.1. Due to the continuity of $\mathfrak{f}^l$, the function H^l is continuous in the last argument. Additionally, Lemma 2.15 implies that the function $H^l(\omega, \mathfrak{x}, \cdot, \mathfrak{u}^l)$ is continuous. Thus, the joint measurability of the normal vector field extension $\widehat{\mathfrak{n}_{\mathfrak{D}}}$ implies that H^l is Carathéodory in the sense that it is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ and continuous in the remaining arguments $A \in \mathcal{CS}(\mathbb{R})$ and $\mathfrak{u}^l \in \mathbb{R}$. Consequently, the function H^l is jointly measurable due to [5, Lemma 4.51].

Let us note that the Inequality (3.36) can also be formulated via the function H^l by requiring

$$H^l\big(\omega, \mathfrak{x}, [\mathfrak{u}^l, \mathfrak{u}^o], \mathfrak{u}^l\big) \geq 0 \,. \tag{3.37}$$

Via a correspondence $\mathbb{I}$ we can describe the set of all values $\mathfrak{u}^o \in \mathbb{R}$ such that Condition (3.37) is satisfied. Specifically, this set-valued mapping $\mathbb{I} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R} \rightrightarrows \mathbb{R}$ is defined as

$$\mathbb{I}(\omega, \mathfrak{x}, \mathfrak{u}^l) \coloneqq \Big\{ \mathfrak{u}^o \in \mathbb{R} \mid \mathfrak{u}^l \leq \mathfrak{u}^o \text{ and } H^l\big(\omega, \mathfrak{x}, [\mathfrak{u}^l, \mathfrak{u}^o], \mathfrak{u}^l\big) \geq 0 \Big\} \,.$$

Via a similar construction, one can define a correspondence $\mathbb{r} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R} \rightrightarrows \mathbb{R}$ to represent the condition

$$\mathfrak{f}^r(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x}) \quad \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^r] \, .$$

This requires a function H^r, which is based on the flux function $\mathfrak{f}^r$. Nevertheless, the procedure is completely analogous to the construction of the set-valued mapping $\mathbb{l}$.

Correspondences for flux conditions are Carathéodory. We continue by showing that the set-valued mapping $\mathbb{l}$ is Carathéodory in the sense that it is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ and continuous with respect to the value $\mathfrak{u}^l \in \mathbb{R}$. For fixed $\mathfrak{u}^l \in \mathbb{R}$, the joint measurability follows from Lemma 2.9, since the function H^l is Carathéodory (with joint measurability in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$). This argument becomes clear by writing $\mathbb{l}$ via the intersection

$$\mathbb{l}(\omega, \mathfrak{x}, \mathfrak{u}^l) \coloneqq \left\{ \mathfrak{u}^o \in \mathbb{R} \mid \mathfrak{u}^l \leq \mathfrak{u}^o \right\} \cap \left\{ \mathfrak{u}^o \in \mathbb{R} \mid H^l\big(\omega, \mathfrak{x}, [\mathfrak{u}^l, \mathfrak{u}^o], \mathfrak{u}^l\big) \geq 0 \right\} ,$$

since the latter set is the only one depending on $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$. For the continuity in $\mathfrak{u}^l \in \mathbb{R}$, let a stochastic parameter $\omega \in \Omega$ and a space-time coordinate $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ be fixed. Now, the function H^l is continuous in $A \in \mathcal{CS}(\mathbb{R})$ and in $\mathfrak{u}^l \in \mathbb{R}$. Therefore, the set-valued mapping $\mathbb{l}$ depends continuously[25] on the value $\mathfrak{u}^l \in \mathbb{R}$ by [246, Example 5.5].

Consequently, we have shown that the correspondence $\mathbb{l}$ is Carathéodory in the sense that it is continuous in $\mathfrak{u}^l \in \mathbb{R}$ and jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$. Via an analogous argumentation one can show that the set-valued mapping $\mathbb{r}$ is also Carathéodory.

Existence condition via correspondences. With the set-valued mappings $\mathbb{l}$ and $\mathbb{r}$ at hand, we are now able to formulate the condition

$$\text{there exists a } \mathfrak{u}^o \in \mathbb{R} \text{ such that } \begin{cases} \mathfrak{f}^l(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^l, \mathfrak{u}^o] \, , \\ \text{and} & \\ \mathfrak{f}^r(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^r] \, , \end{cases}$$

via a jointly measurable correspondence $\Lambda_{l<r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$. Therefore, let us first note that the correspondences $\mathbb{l}$ and $\mathbb{r}$ are closed-valued by construction. To capture the existence of a scalar value $\mathfrak{u}^o$ that is contained in both of these set-valued mappings, we define a function $\lambda : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ via

$$\lambda\big(\omega, \mathfrak{x}, (\mathfrak{u}^l, \mathfrak{u}^r)\big) \coloneqq \mathbb{O}\big(\mathbb{l}(\omega, \mathfrak{x}, \mathfrak{u}^l), \mathbb{r}(\omega, \mathfrak{x}, \mathfrak{u}^r)\big) \, .$$

Here, the function $\mathbb{O} : \mathcal{CS}(\mathbb{R}) \times \mathcal{CS}(\mathbb{R}) \to \mathbb{R}$ measures the distance between two closed subsets of the real numbers. Specifically, it is defined as

$$\mathbb{O}(A, B) \coloneqq \inf_{a \in A} \big(\operatorname{dist}_B(a) \big) \, ,$$

where dist is the Euclidean distance between the point a and the set B. Recall that the hyperspace $\mathcal{CS}(\mathbb{R})$ is equipped with the set distance $\mathbb{d}$. With this in mind, note that the function $\mathbb{O}$ is continuous

[25] For a detailed discussion on the continuity of set-valued mappings, we refer to the monographs of Aubin and Frankowska [17, Chapter 1], Rockafellar and Wets [246, Section 5.B] or Aliprantis and Border [5, Section 17.2].

in both arguments by Lemma 2.15. Therefore, the function λ is Carathéodory in the sense that it is continuous with respect to $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ and jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$. Based on this Carathéodory function λ, we can construct the set-valued mapping $\Lambda_{l<r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ as the zero-level set of λ via

$$\Lambda_{l<r}(\omega, \mathfrak{x}) \coloneqq \left\{ (\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R} \mid \lambda\big(\omega, \mathfrak{x}, (\mathfrak{u}^l, \mathfrak{u}^r)\big) = 0 \right\}.$$

By construction, this correspondence contains all values $\mathfrak{u}^o \in \mathbb{R}$ such that the conditions

$$\begin{cases} \mathfrak{f}^l(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^l, \mathfrak{u}^o], \\ \qquad\qquad \text{and} & \\ \mathfrak{f}^r(\omega, \mathfrak{x}, \rho) \cdot \widehat{\mathfrak{n}_{\mathfrak{D}}}(\omega, \mathfrak{x}) \geq \mathbb{s}(\omega, \mathfrak{x}) & \text{for all } \rho \in [\mathfrak{u}^o, \mathfrak{u}^r], \end{cases}$$

are satisfied. Furthermore, since the function λ is Carathéodory, Lemma 2.9 implies that the set-valued map $\Lambda_{l<r}$ is (jointly) measurable.

Condition (ii) of Definition 3.57 via jointly measurable correspondence. Let us now conclude the assertion that Condition (ii) of Definition 3.57 can be written as a jointly measurable set-valued mapping $\Theta_{\mathrm{VV}}^{l<r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$. Therefore, define this correspondence as

$$\Theta_{\mathrm{VV}}^{l<r}(\omega, \mathfrak{x}) \coloneqq \Lambda_{l<r}(\omega, \mathfrak{x}) \cap \left\{ (\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R} \mid \mathfrak{u}^l < \mathfrak{u}^r \right\}.$$

Due to the set $\left\{ (\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R} \mid \mathfrak{u}^l < \mathfrak{u}^r \right\}$, the mapping $\Theta_{\mathrm{VV}}^{l<r}$ only contains values satisfying $\mathfrak{u}^l < \mathfrak{u}^r$, which is the first requirement in Condition (ii) of Definition 3.57. Additionally, due to the construction of the correspondence $\Lambda_{l<r}$, it only contains those values $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ that admit a value $\mathfrak{u}^o \in [\mathfrak{u}^l, \mathfrak{u}^r]$ satisfying the conditions on the flux. Since the set-valued mapping $\Lambda_{l<r}$ is jointly measurable by construction and the set

$$\left\{ (\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R} \mid \mathfrak{u}^l < \mathfrak{u}^r \right\}$$

does neither depend on the stochastic parameter $\omega \in \Omega$ nor on the space-time coordinate $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the correspondence $\Theta_{\mathrm{VV}}^{l<r}$ is jointly measurable, which proves the assertion. ■

Let us stress that an analogous result can be established for Condition (iii) of Definition 3.57. With these joint measurability statements on the set-valued mappings $\Theta_{\mathrm{VV}}^{l<r}$ and $\Theta_{\mathrm{VV}}^{r<l}$, we have all ingredients at hand to argue that the vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}$ as defined in Definition 3.57 can be written as a jointly measurable correspondence $\mathfrak{G}_{\mathrm{VV}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$.

Theorem 3.60 (Joint measurability of random vanishing viscosity germs)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface (cf., Definition 3.1) that satisfies the stochastic measurability Assumption 3.4. Furthermore, let $\widehat{\mathfrak{n}_{\mathfrak{D}}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{X}_{\mathbb{T}}$ denote the extension of the normal field $\mathfrak{n}_{\mathfrak{D}}$ of the sole discontinuity $\mathfrak{D}$ as defined in Equation (3.7)*. Additionally, let the flux function $\mathfrak{f}$ satisfy the sole-flux-discontinuity Assumption 3.13 as well as the stochastic measurability Assumption 3.41 and the confinement Assumption 3.38. Then, the random vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}$ can be written as a set-valued mapping $\mathfrak{G}_{\mathrm{VV}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$, which is jointly measurable.* ◆

Proof. Let us start by recalling that a pair $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ belongs to the vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}} \subset \mathbb{R}^2$, if it satisfies the Rankine-Hugoniot Condition (3.34) and one of the conditions (i) – (iii) of Definition 3.57. By Definition 3.55, the set of pairs $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ satisfying the Rankine-Hugoniot

Condition (3.34) corresponds to the Rankine-Hugoniot admissibility germ $\mathfrak{G}_{\mathrm{RH}} \subset \mathbb{R}^2$. Additionally, by Proposition 3.59, the values $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R} \times \mathbb{R}$ satisfying Condition (ii) of Definition 3.57 can be written via a set-valued mapping $\Theta_{\mathrm{VV}}^{l<r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$. Analogously, a correspondence $\Theta_{\mathrm{VV}}^{r<l} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ can be used to describe Condition (iii) of Definition 3.57. Before we continue, we define the constant-valued set-valued mapping $\Theta_{\mathrm{VV}}^{l=r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ as

$$\Theta_{\mathrm{VV}}^{l=r}(\omega, \mathfrak{x}) \coloneqq \left\{ (\mathfrak{v}^l, \mathfrak{v}^r) \in \mathbb{R}^2 \mid \mathfrak{v}^l = \mathfrak{v}^r \right\}.$$

Let us stress that this correspondence is independent of the stochastic parameter $\omega \in \Omega$ and the space-time point $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$. With these set-valued mappings, we have collected all ingredients to write the vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}(\omega, \mathfrak{x}) \subset \mathbb{R}^2$ as the following intersection:

$$\mathfrak{G}_{\mathrm{VV}}(\omega, \mathfrak{x}) = \mathfrak{G}_{\mathrm{RH}}(\omega, \mathfrak{x}) \cap \left(\Theta_{\mathrm{VV}}^{l=r}(\omega, \mathfrak{x}) \cup \Theta_{\mathrm{VV}}^{l<r}(\omega, \mathfrak{x}) \cup \Theta_{\mathrm{VV}}^{r<l}(\omega, \mathfrak{x}) \right). \tag{3.38}$$

By Proposition 3.56, the Rankine-Hugoniot germ $\mathfrak{G}_{\mathrm{RH}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ is jointly measurable. Since the set-valued mapping $\Theta_{\mathrm{VV}}^{l=r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ is constant-valued it is trivially jointly measurable. Additionally, the correspondence $\Theta_{\mathrm{VV}}^{l<r} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ is jointly measurable by Proposition 3.59 and ananalogous argumentation yields the joint measurability of the set-valued mapping $\Theta_{\mathrm{VV}}^{r<l}$.

Consequently, Equation (3.38) implies the joint measurability of the random family of vanishing viscosity germs $\mathfrak{G}_{\mathrm{VV}} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ as the finite intersection of jointly measurable correspondences. This concludes the proof. ■

3.4.3 Remainder function based on Euclidean distance

In this section we discuss a random remainder function, which is based on the Euclidean distance dist in $\mathbb{R}^2$ and is probably the simplest choice for the remainder term. For a more extensive discussion on possible choices of remainder functions, we refer the reader to Appendix A. To employ this choice of the remainder term based on the Euclidean distance, we require the left and right flux function $\mathfrak{f}^{l,r}$ to be globally Lipschitz continuous, which is a stronger restriction than the local Lipschitz continuity imposed by Assumption (F-2). In particular, we assume that the functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ are Lipschitz continuous with a common Lipschitz constant $L_{\mathfrak{f}}$ for all spatio-temporal variables $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$. However, we allow the Lipschitz constant $L_{\mathfrak{f}}$ to depend on the stochastic parameter $\omega \in \Omega$. With these assumptions and prerequisites, we are ready to define the *remainder function based on the Euclidean distance.*

Definition 3.61 (Remainder function via Euclidean distance)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface. Additionally, let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be globally Lipschitz continuous with common Lipschitz constant for each space-time variable $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ that may depend on the stochastic parameter $\omega \in \Omega$. Then, the function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{dist}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} \mathfrak{R}_{\mathfrak{G}}^{\mathrm{dist}}\big(\omega, \mathfrak{x}; (k^l, k^r)\big) &\coloneqq 2 \left\| (\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot))' \right\|_{\mathcal{L}^\infty} \inf_{(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathfrak{G}(\omega; \mathfrak{x})} \left\{ \left| \mathfrak{u}^l - k^l \right| + \left| \mathfrak{u}^r - k^r \right| \right\} \\ &\equiv C_{\mathfrak{f}}(\omega) \operatorname{dist}\Big((k^l, k^r), \mathfrak{G}(\omega; \mathfrak{x}) \Big), \end{aligned}$$

is called the remainder function via the Euclidean distance. *Here, $C_{\mathfrak{f}} \in \mathbb{R}_{>0}$ is a sufficiently large constant that may depend on the stochastic parameter $\omega \in \Omega$ and* dist *denotes the Euclidean distance in $\mathbb{R}^2$.* ◆

With this definition of the remainder function based on the Euclidean distance at hand, we can verify the assumptions on joint measurability and integrability. Therefore, the following proposition states that the function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ is jointly measurable if the underlying family of admissibility germs is also jointly measurable.

Proposition 3.62 (Joint measurability of remainder function via Euclidean distance)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole flux discontinuity. Furthermore, let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs that satisfies the joint measurability Assumption 3.16. Additionally, let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be globally Lipschitz continuous with common Lipschitz constant for each space-time variable $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ which may depend on the stochastic parameter $\omega \in \Omega$.
Then, the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ based on the Euclidean distance, which is associated to the admissibility germs $\mathfrak{G}$, is jointly measurable. ◆

Proof. First, recall that the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ is defined as

$$\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}\big(\omega, \mathfrak{x}; (k^l, k^r)\big) \equiv C_{\mathfrak{f}} \operatorname{dist}\Big((k^l, k^r), \mathfrak{G}(\omega; \mathfrak{x})\Big),$$

where dist denotes the Euclidean distance of $\mathbb{R}^2$. Now, for fixed stochastic parameter $\omega \in \Omega$ and fixed spatio-temporal coordinate $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}\big(\omega, \mathfrak{x}; \cdot\big)$ is continuous due to the continuity of the Euclidean distance function. However, by Assumption 3.16, the admissibility germ $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$. This implies that the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ is Carathéodory in the sense that it is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ and continuous in $(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathbb{R}^2$. Consequently, by [5, Lemma 4.51], the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ is jointly measurable. ■

As a last property of the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ based on the Euclidean distance, we validate the integrability Assumption 3.46. Before we do so, recall that the remainder function is a tool to transform the admissibility condition via germs $\mathfrak{G}$ into an entropy inequality. Therefore, we may assume that for every stochastic parameter $\omega \in \Omega$, there exists a $\mathfrak{G}$-entropy solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. This implies that there exists an interval $\mathbb{U}(\omega) \subset \mathbb{R}$ such that the solution u satisfies $u(\omega, \cdot, \cdot) \in \mathbb{U}(\omega)$ for almost every $(\boldsymbol{x}, t) \in \mathbb{X} \times \mathbb{T}$. This allows us to formulate the following proposition.

Proposition 3.63 (Integrability of remainder function via Euclidean distance)**:**
Let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs such that for every stochastic parameter $\omega \in \Omega$ there exists a solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Then, the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ is locally integrable in the sense of Assumption 3.46, meaning that for each compact set $K \subset \mathbb{R}^2$ and fixed stochastic parameter $\omega \in \Omega$, the function

$$\mathcal{M}^{\mathfrak{R}}_K(\omega, \mathfrak{x}) \coloneqq \sup_{\boldsymbol{k} \in K} \Big| \mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}(\omega, \mathfrak{x}; \boldsymbol{k}) \Big|$$

is locally Lebesgue integrable, i.e., $\mathcal{M}^{\mathfrak{R}}_K(\omega, \cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$. ◆

Proof. By hypothesis, for every stochastic parameter $\omega \in \Omega$, there exists a $\mathfrak{G}$-entropy solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Therefore, there also exists a (possibly random) interval $\mathbb{U}(\omega) \subset \mathbb{R}$ such that the solution u satisfies $u(\omega, \cdot, \cdot) \in \mathbb{U}(\omega)$ for almost every $(\boldsymbol{x}, t) \in \mathbb{X} \times \mathbb{T}$. However, this implies that $\mathfrak{G}(\omega, \mathfrak{x}) \cap \big(\mathbb{U}(\omega) \times \mathbb{U}(\omega)\big) \neq \emptyset$ for almost every spatio-temporal coordinate $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$. Otherwise, this would be a contradiction to the existence of a $\mathfrak{G}$-entropy solution. Now, recall that the remainder

function is defined as

$$\mathfrak{R}_{\mathfrak{G}}^{\mathrm{dist}}\big(\omega, \mathfrak{x}; (k^l, k^r)\big) \equiv C_{\mathfrak{f}} \operatorname{dist}\Big((k^l, k^r), \mathfrak{G}(\omega; \mathfrak{x})\Big) ,$$

where $C_{\mathfrak{f}} \in \mathbb{R}_{>0}$ is a sufficiently large constant. Writing the interval $\mathbb{U}$ as $\mathbb{U}(\omega) = [\underline{\mathfrak{u}}(\omega), \overline{\mathfrak{u}}(\omega)]$ we can exploit the fact that $\mathfrak{G}(\omega, \mathfrak{x}) \cap \mathbb{U}(\omega)^2 \neq \emptyset$ for almost every $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ to estimate

$$\mathfrak{R}_{\mathfrak{G}}^{\mathrm{dist}}(\omega, \mathfrak{x}; \boldsymbol{k}) \leq \sqrt{2} C_{\mathfrak{f}} |\overline{\mathfrak{u}}(\omega) - \underline{\mathfrak{u}}(\omega)| .$$

However, by construction of the majorant $\mathcal{M}_K^{\mathfrak{R}}$, it also holds that

$$\mathcal{M}_K^{\mathfrak{R}}(\omega, \mathfrak{x}) \leq \sqrt{2} C_{\mathfrak{f}} |\overline{\mathfrak{u}}(\omega) - \underline{\mathfrak{u}}(\omega)| .$$

Since the upper bound does not depend on the spatio-temporal coordinate $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the function $\mathcal{M}_K^{\mathfrak{R}}$ satisfies $\mathcal{M}_K^{\mathfrak{R}}(\omega, \cdot) \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R}_{\geq 0})$. Due to the inclusion $\mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R}_{\geq 0}) \subset \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$, this concludes the proof. ■

Random conservation laws with a compound flux discontinuity

4

Undeniably, the case of a sole discontinuity is a simplifying model scenario. For real-world applications, e.g., in physics or engineering problems, it is desirable to consider multiple, possibly curved and intersecting discontinuities of the flux function. Let us illustrate this request with two geometrical examples, which are both inspired by the modeling of a porous/layered medium: As a first example, one might be interested in defining a homogeneous porous medium that has inclusions of some other (homogeneous) porous medium. Alternatively, a (porous) medium might consist of a variety of materials that are ordered in layers. Both examples lead to discontinuous fluxes as soon as the considered flux function depends on material properties such as the conductivity or porosity.

While the construction of such more general geometries might seem straightforward from a geometrical perspective, the well-posedness of the random conservation law needs to be ensured in this setting. Therefore, in this chapter, we consider *compound flux discontinuities* that are constructed as the locally finite union of sole discontinuity hypersurfaces. In the deterministic setting such an extension is rather straightforward (up to heavy technical/notational changes). However, the randomization of the flux imposes major difficulties, hence these geometries are a good starting point for generalizing the discontinuity.

Similar to the previous chapter, let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. Denote the space-time domain again by $\mathbb{X}_\mathbb{T} \coloneqq \mathbb{T} \times \mathbb{X}$ for a time interval $\mathbb{T} \coloneqq [0, T]$ with $T \in \mathbb{R}_{>0}$ and a spatial domain $\mathbb{X} \coloneqq \mathbb{R}^d$ with dimension $d \in \mathbb{N}$. Then, for an unknown $u \coloneqq u(\omega, t, \boldsymbol{x})$, we consider the random scalar conservation law

$$\begin{aligned} \partial_t u + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, t, \boldsymbol{x}, u) &= 0 && \text{in } \Omega \times \mathbb{T} \times \mathbb{X}\,, \\ u(\omega, 0, \boldsymbol{x}) &= u_0(\omega, \boldsymbol{x}) && \text{on } \Omega \times \{0\} \times \mathbb{R}^d\,. \end{aligned} \tag{4.1}$$

Here, $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, is a random initial condition and the flux function $\mathfrak{f}$ is assumed to depend discontinuously on the spatial variable $\boldsymbol{x} \in \mathbb{X}$.

The chapter is structured as follows: First, in Section 4.1, we define the notion of compound flux discontinuities and investigate its properties. Afterwards, we adapt the admissibility conditions for random $\mathfrak{G}$-entropy solutions to this new setting in Section 4.2. We conclude this chapter by investigating the well-posedness of random $\mathfrak{G}$-entropy solutions in Section 4.3.

4.1 Compound flux discontinuities

In this section, we introduce the geometry of compound flux discontinuities. This type of discontinuity can be seen as the straightforward generalization of sole discontinuities, since it is defined as the locally finite union of such. We precise this construction subsequently.

Definition 4.1 (Compound flux discontinuity)**:**
Let $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$ be a locally finite union of sole discontinuity hypersurfaces $\mathfrak{D}_\mathfrak{i} \subset \mathbb{X}_\mathbb{T}$, where each $\mathfrak{D}_\mathfrak{i}$ is defined as in Definition 3.1. We call such union $\mathfrak{D}$ a compound flux discontinuity *and we symbolize the number of discontinuity parts $\mathfrak{D}_\mathfrak{i}$ contained in $\mathfrak{D}$ by writing $\mathfrak{N}_\mathfrak{D} \in \big(\mathbb{N} \cup \{\infty\}\big)$.*
Furthermore, if $\mathfrak{N}_\mathfrak{D}$ or any of the sole discontinuities $\mathfrak{D}_\mathfrak{i}$ is random, $\mathfrak{D}$ is called a random compound flux discontinuity *and we indicate this dependency on the stochastic parameter $\omega \in \Omega$ by writing $\mathfrak{D}$ as a set-valued mapping $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$.* ◆

To be able to better describe the (random) sole discontinuities $\mathfrak{D}_\mathfrak{i}(\omega)$ contained in the (random) compound flux discontinuity $\mathfrak{D}(\omega)$, we introduce the following *(random) index set* $\mathcal{I}_\mathfrak{D}(\omega)$ that depends on the number $\mathfrak{N}_\mathfrak{D} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$ of sole discontinuities:

$$\mathcal{I}_\mathfrak{D}(\omega) = \begin{cases} \big\{\mathfrak{i} \in \mathbb{N} \mid 1 \leq \mathfrak{i} \leq \mathfrak{N}_\mathfrak{D}(\omega)\big\} & \text{if } \mathfrak{N}_\mathfrak{D}(\omega) < \infty\,, \\ \mathbb{N} & \text{if } \mathfrak{N}_\mathfrak{D}(\omega) = \infty\,. \end{cases}$$

Note that this index set is random if and only if the number $\mathfrak{N}_\mathfrak{D}$ of sole discontinuities is random. By means of this index set $\mathcal{I}_\mathfrak{D}(\omega)$ the following set of intersection points of the sole discontinuities $\mathfrak{D}_\mathfrak{i}$ can be defined.

Definition 4.2 (Intersection points of sole discontinuities)**:**
For a compound flux discontinuity $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$, we define the set of intersection points $\mathfrak{I}_\mathfrak{D} \subset \mathfrak{D}$ *of the contained sole discontinuities $\mathfrak{D}_\mathfrak{i}$ as*

$$\mathfrak{I}_\mathfrak{D} \coloneqq \Big\{\mathfrak{d} \in \mathfrak{D} \mid \textit{there exist } i, j \in \mathcal{I}_\mathfrak{D}(\omega) : \mathfrak{d} \in \mathfrak{D}_i \cap \mathfrak{D}_j\Big\}\,.$$

If the compound flux discontinuity $\mathfrak{D}$ is random, then the corresponding set of intersection points $\mathfrak{I}_\mathfrak{D}$ is also random. In this case, we write $\mathfrak{I}_\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ to indicate the dependence on $\omega \in \Omega$. ◆

Note that we do not require the set of intersection points $\mathfrak{I}_\mathfrak{D}$ to be a set of $\mathcal{H}^d$-measure zero. This means that the sole discontinuities $\mathfrak{D}_\mathfrak{i}$ are allowed to overlap on sets with positive $\mathcal{H}^d$-measure.

We are now ready to investigate the measurability of random compound flux discontinuities. Their spatio-temporal measurability is directly implied by the construction in Definition 4.1: Since any sole discontinuity $\mathfrak{D}_\mathfrak{i}$ is a closed subset of the space-time domain $\mathbb{X}_\mathbb{T}$, each $\mathfrak{D}_\mathfrak{i}$ is particularly measurable. By definition, the compound flux discontinuity $\mathfrak{D}$ is constructed as the locally finite union of these measurable sole discontinuities $\mathfrak{D}_\mathfrak{i}$, which implies that it is also measurable as a subset $\mathfrak{D} \subset \mathbb{X}_\mathbb{T}$. In particular, $\mathfrak{D}$ is also a closed subset of $\mathbb{X}_\mathbb{T}$.

If the compound flux discontinuity $\mathfrak{D}$ is random, we aim at guaranteeing its stochastic measurability. Unfortunately, but also not surprisingly, there is no chance of stating any stochastic measurability result

based solely on the Definition 4.1 of compound flux discontinuities. Such a statement was already out of reach for the sole discontinuity setting. Hence, we impose the following assumption on the stochastic measurability of random compound discontinuities to overcome this lack of information.

Assumption 4.3 (Stochastic measurability of random compound flux discontinuities)**:**
We assume that a random compound flux discontinuity $\mathfrak{D}$ is measurable in the sense that it is measurable as a set-valued mapping $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$. ◆

Let us mention some direct consequences of this stochastic measurability Assumption 4.3: First, the random number $\mathfrak{N}_{\mathfrak{D}} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$ of (random) sole discontinuities $\mathfrak{D}_{\mathfrak{i}}$ contained in the compound discontinuity $\mathfrak{D}(\omega)$ is measurable. Note that this also implies the measurability of the corresponding index set $\mathcal{I}_{\mathfrak{D}} : \Omega \rightrightarrows \mathbb{N}$. Furthermore, we can deduce that, for every index $\mathfrak{i} \in \mathcal{I}_{\mathfrak{D}}(\omega)$, the corresponding random sole discontinuity $\mathfrak{D}_{\mathfrak{i}} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ is measurable. This follows from combining the stochastic measurability Assumption 4.3 with the Definition 4.1 of a compound discontinuity.

4.1.1 Partitioning of compound flux discontinuities

The core idea of extending the admissibility of random $\mathfrak{G}$-entropy solutions is in locally reducing the situation to the underlying sole discontinuity setting. If the sole discontinuities $\mathfrak{D}_{\mathfrak{i}}$ contained in the compound flux discontinuity $\mathfrak{D}$ are disjoint, such an extension is straightforward. However, in case the set of intersection points $\mathfrak{I}_{\mathfrak{D}}$ is nonempty, we require some additional knowledge on the compound flux discontinuity at hand. In particular, we need to demand the existence of a *partition up to a null set* of the discontinuity. Additionally, such a partition up to a null set should satisfy some properties to be specified throughout this subsection. We start by formally defining what we mean by a *partition up to a null set.*

Assumption 4.4 (Compound discontinuity admits partition up to a null set)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity. For every stochastic parameter $\omega \in \Omega$, the corresponding compound flux discontinuity $\mathfrak{D}(\omega)$ admits a partition up to a null set $\mathfrak{C}_{\mathfrak{D}}(\omega) = \big\{\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega)\big\}_{\kappa\in\mathbb{N}}$ *in the sense that every partition part $\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega) \subset \mathfrak{D}(\omega)$ is open and connected. Furthermore, for two indices $\kappa, \iota \in \mathbb{N}$, the set of partition parts $\big\{\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega)\big\}_{\kappa\in\mathbb{N}}$ is assumed to satisfy*

$$\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega) \cap \mathfrak{C}^{\iota}_{\mathfrak{D}}(\omega) = \emptyset \qquad \text{for } \kappa \neq \iota \,.$$

Additionally, by $\mathfrak{C}_{\mathfrak{D}}(\omega)$ being a partition up to a null set *of the compound discontinuity hypersurface $\mathfrak{D}(\omega)$, we mean that $\mathfrak{C}_{\mathfrak{D}}(\omega)$ satisfies*

$$\mathcal{H}^d\big(\mathfrak{D}(\omega) \setminus \mathfrak{C}_{\mathfrak{D}}(\omega)\big) = \mathcal{H}^d\big(\mathfrak{I}^o_{\mathfrak{D}}(\omega)\big) = 0 \,.$$

Here, the set $\mathfrak{I}^o_{\mathfrak{D}}(\omega) \subset \mathfrak{I}_{\mathfrak{D}}(\omega)$ is a zero-$\mathcal{H}^d$-measure subset of the (random) set of intersection points $\mathfrak{I}_{\mathfrak{D}} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$. ◆

Let us stress two implications of this assumption: First, such a partition may not be unique. Secondly, a partition up to a null set always consists of countable many parts, even though the underlying compound flux discontinuity might only have finitely many sole discontinuities. Since partitions up to a null set are

the only type of partitions considered in this chapter, we may omit the term *up to a null set* if it benefits the readability. Definition 4.1 of compound flux discontinuities implies local finiteness of the contained sole discontinuity hypersurfaces. This property should also be inherited by the considered partition up to a null set. To ensure this, we impose the following local finiteness assumption on the partition $\mathfrak{C}_{\mathfrak{D}}$.

Assumption 4.5 (Partition is locally finite)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity that, for every $\omega \in \Omega$, admits a partition $\mathfrak{C}_{\mathfrak{D}}(\omega)$ up to a null set by Assumption 4.4. We assume that this partition $\mathfrak{C}_{\mathfrak{D}}(\omega)$ is locally finite in the sense that any compact set $K \subset \mathbb{X}_{\mathbb{T}}$ intersects with at most finitely many parts $\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega)$ of the partition. ◆

The construction of the compound flux discontinuity $\mathfrak{D}$ allows the contained sole discontinuity hypersurfaces $\mathfrak{D}_{\mathfrak{i}}$ to overlap on sets with positive d-dimensional Hausdorff measure $\mathcal{H}^d$. Furthermore, by Assumption 4.4 any compound flux discontinuity $\mathfrak{D}$ admits a partition $\mathfrak{C}_{\mathfrak{D}}$ whose parts $\mathfrak{C}^{\kappa}_{\mathfrak{D}}$ satisfy $\mathfrak{C}^{\kappa}_{\mathfrak{D}} \subset \mathfrak{D}$. This implies that for fixed $\omega \in \Omega$ it holds that for any index $\kappa \in \mathbb{N}$ there exists at least one index $\mathfrak{i} \in \mathbb{N}$ such that $\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega) \subset \mathfrak{D}_{\mathfrak{i}}(\omega)$ is satisfied. If the partition part $\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega)$ is contained in the intersection points $\mathfrak{I}_{\mathfrak{D}}(\omega)$ there may exist finitely many indices $\mathfrak{i} \in \mathbb{N}$ satisfying the above condition. Therefore, for every $\omega \in \Omega$, we introduce the following set-valued mapping $\mathcal{S}_\omega : \mathbb{N} \rightrightarrows \mathbb{N}$ that selects all such indices $\mathfrak{i} \in \mathbb{N}$ via the condition

$$\mathcal{S}_\omega(\kappa) \subset \mathbb{N} \qquad \text{such that for every } \mathfrak{i} \in \mathcal{S}_\omega(\kappa) \text{ it holds that} \qquad \mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega) \subset \mathfrak{D}_{\mathfrak{i}}(\omega)\,. \tag{4.2}$$

For reducing the compound flux discontinuity setting to the situation with sole discontinuities, for any index $\kappa \in \mathbb{N}$ we want to be able to select a sole discontinuity $\mathfrak{D}_{\mathfrak{i}}(\omega)$ containing the partition part $\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega)$. In case the mapping $\mathcal{S}_\omega$ is singleton-valued, this is straightforward. However, if we have overlappings in the compound flux discontinuity $\mathfrak{D}(\omega)$ we need to be able to find a (measurable) *selector* that provides us with a unique index $\mathfrak{i} \in \mathcal{I}_{\mathfrak{D}}$ for every partition index $\kappa \in \mathbb{N}$. Luckily, the following theorem states that we can find such a selector.

Theorem 4.6 (Existence of measurable selector)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a compound flux discontinuity that satisfies Assumption 4.4 on the existence of a partition up to a null set and the stochastic measurability Assumption 4.3. Furthermore, for any random parameter $\omega \in \Omega$ let the set-valued mapping $\mathcal{S}_\omega : \mathbb{N} \rightrightarrows \mathbb{N}$ be implicitly defined via Condition (4.2)*. Then, if we equip the natural numbers $\mathbb{N}$ with the discrete metric*[26]*, the correspondence $\mathcal{S}_\omega$ admits a measurable selector, which we denote again by $\mathcal{S}_\omega : \mathbb{N} \to \mathbb{N}$.* ◆

Proof. The idea of the proof is to apply the Kuratowski-Ryll-Nardzewski selection theorem [5, Theorem 18.13] to prove the existence of a measurable selector. Therefore, we have to ensure that the correspondence $\mathcal{S}_\omega : \mathbb{N} \rightrightarrows \mathbb{N}$ is measurable, has nonempty closed values and maps from a measurable space into a Polish space:

Let a stochastic parameter $\omega \in \Omega$ be fixed and let the correspondence $\mathcal{S}_\omega : \mathbb{N} \rightrightarrows \mathbb{N}$ be implicitly defined via Condition (4.2). By the construction of the partition $\mathfrak{C}_{\mathfrak{D}}$ in Assumption 4.4, for any index $\kappa \in \mathbb{N}$ there exists at least one index $\mathfrak{i} \in \mathbb{N}$ such that $\mathfrak{C}^{\kappa}_{\mathfrak{D}}(\omega) \subset \mathfrak{D}_{\mathfrak{i}}(\omega)$. This readily shows that the mapping

[26] The discrete metric is defined by assigning the distance 1 to any two points, which are distinct. For details, the reader is referred to [108, Equation (1.7)] or [290, Example 2 on page 2].

$\mathcal{s}_\omega$ is nonempty-valued. Furthermore, the natural numbers $\mathbb{N}$ equipped with the discrete metric form a Polish space, i.e., a separable completely metrizable topological space. Furthermore, in a discrete metric space any subspace is both open and closed, which implies that the set-valued mapping $\mathcal{s}_\omega : \mathbb{N} \rightrightarrows \mathbb{N}$ is closed-valued.

It remains to argue that the correspondence $\mathcal{s}_\omega : \mathbb{N} \rightrightarrows \mathbb{N}$ is measurable. Let $\mathcal{B}(\mathbb{N})$ denote the Borel σ-algebra of the set of natural numbers $\mathbb{N}$ [27]. For any open set $O \in \mathcal{B}(\mathbb{N})$ we want to show that the preimage $\mathcal{s}_\omega^{-1}(O)$ is also contained in the Borel σ-algebra $\mathcal{B}(\mathbb{N})$. Recall that we have fixed the parameter $\omega \in \Omega$, which implies that the number $\mathfrak{N}_\mathfrak{D}(\omega)$ of sole discontinuities in $\mathfrak{D}(\omega)$ is also fixed. We distinguish two cases for the arbitrary open set $O \in \mathcal{B}(\mathbb{N})$:

- First, assume that O does not contain any index of the index set $\mathcal{I}_\mathfrak{D}(\omega)$, i.e., $O \cap \mathcal{I}_\mathfrak{D}(\omega) = \emptyset$. However, by construction of the correspondence $\mathcal{s}_\omega$, we have for any random parameter $\omega \in \Omega$ that $\mathcal{s}_\omega(\mathbb{N}) = \mathcal{I}_\mathfrak{D}(\omega)$. That means that the preimage $\mathcal{s}_\omega^{-1}(O) = \emptyset$ is empty, but by definition of the Borel σ-algebra $\mathcal{B}(\mathbb{N})$, the empty set $\emptyset$ is contained in $\mathcal{B}(\mathbb{N})$, i.e., $\emptyset \in \mathcal{B}(\mathbb{N})$.
- Now, assume that $O \cap \mathcal{I}_\mathfrak{D}(\omega) \neq \emptyset$. This automatically implies by the construction of $\mathcal{s}_\omega$ that the preimage $\mathcal{s}_\omega^{-1}$ is a nonempty subset of $\mathbb{N}$. However, since we equipped $\mathbb{N}$ with the discrete metric, *any* subset of $\mathbb{N}$ is open. This concludes the measurability proof of the correspondence $\mathcal{s}_\omega$ since the Borel σ-algebra consists of all open subsets of $\mathbb{N}$, which implies $\mathcal{s}_\omega^{-1}(O) \in \mathcal{B}(\mathbb{N})$.

Thereby, we can conclude that all presumptions of the Kuratowski-Ryll-Nardzewski selection theorem [5, Theorem 18.13] are satisfied, which proves that there exists a measurable selector of $\mathcal{s}_\omega$. ■

In the remainder of this chapter, we will write $\mathcal{s}_\omega$ to denote the selector of the corresponding set-valued map. By virtue of this selection function, we can now introduce the normal unit vector fields of the partition parts $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$ as well as the extension of those vector fields to the whole space-time domain. The main idea of this construction is that for every index $\kappa \in \mathbb{N}$, we can select a sole discontinuity hypersurface $\mathfrak{D}_{\mathcal{s}_\omega(\kappa)}(\omega)$, which admits a normal unit vector field $\mathbf{n}_{\mathfrak{D}_{\mathcal{s}_\omega}}$ by Lemma 3.7. Restricting this normal vector field $\mathbf{n}_{\mathfrak{D}_{\mathcal{s}_\omega}}$ to the partition part $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$ yields the unit vector field of $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$, which we denote by $\mathbf{n}_{\mathfrak{C}_\mathfrak{D}^\kappa}$. Similarly, the extension of this vector field to the whole space-time domain $\mathbb{X}_\mathbb{T}$, denoted by $\widehat{\mathbf{n}}_{\mathfrak{C}_\mathfrak{D}^\kappa}$, is achieved by identifying $\mathbf{n}_{\mathfrak{D}_{\mathcal{s}_\omega}}$ with $\widehat{\mathbf{n}}_{\mathfrak{D}_{\mathcal{s}_\omega}}$. We formalize this in the following definition.

Definition 4.7 ((Extension of) normal unit vector field of partition part)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies Assumption 4.4 on the existence of a partition $\mathfrak{C}_\mathfrak{D}(\omega)$ as well as the stochastic measurability Assumption 4.3. Furthermore, for every random parameter $\omega \in \Omega$, let $\mathcal{s}_\omega : \mathbb{N} \to \mathbb{N}$ be a selector of the sole discontinuities containing a partition part $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$. Then, for any index $\kappa \in \mathbb{N}$, the normal unit vector field $\mathbf{n}_{\mathfrak{C}_\mathfrak{D}^\kappa}$ *of the partition part $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$ is defined as the restriction of the normal field $\mathbf{n}_{\mathfrak{D}_{\mathcal{s}_\omega}}$ to the partition part $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$. Furthermore, the extension of the vector field $\mathbf{n}_{\mathfrak{C}_\mathfrak{D}^\kappa}$ to the whole space-time domain $\mathbb{X}_\mathbb{T}$ is defined as $\widehat{\mathbf{n}}_{\mathfrak{C}_\mathfrak{D}^\kappa} = \widehat{\mathbf{n}}_{\mathfrak{D}_{\mathcal{s}_\omega}}$, where $\widehat{\mathbf{n}}_{\mathfrak{D}_{\mathcal{s}_\omega}}$ is the extension of $\mathbf{n}_{\mathfrak{D}_{\mathcal{s}_\omega}}$ as constructed in Lemma 3.8.* ◆

We conclude this section on the partitioning of compound flux discontinuities by investigating possible parametrizations of the partition parts $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$. Any sole discontinuity admits a parametrization $\boldsymbol{P}_\mathfrak{D}$ as

[27] Since the natural numbers $\mathbb{N}$ are equipped with the discrete metric, every subset of $\mathbb{N}$ is open. In particular, because $\mathbb{N}$ is countable, the Borel σ-algebra $\mathcal{B}(\mathbb{N})$ and the power set $\mathcal{P}(\mathbb{N})$ coincide. To emphasize that each subset of $\mathbb{N}$ is open, only $\mathcal{B}(\mathbb{N})$ is used in the subsequent discussion.

constructed in Lemma 3.2. Therefore, the existence of a parametrization of $\mathfrak{C}_{\mathfrak{D}}^{\kappa}(\omega)$ is straightforward: We can restrict the parametrization of $\mathfrak{D}_{s_\omega(\kappa)}(\omega)$. However, this construction does not allow us to easily verify important properties such as joint measurability. To overcome this issue, we impose the following simplifying assumption.

Assumption 4.8 (Parametrization of partition)**:**
For any $\kappa \in \mathbb{N}$, we assume that the partition part $\mathfrak{C}_{\mathfrak{D}}^{\kappa}$ admits a parametrization $\boldsymbol{P}_{\mathfrak{C}_{\mathfrak{D}}^{\kappa}} : \Omega \times \mathbb{T} \times \mathbb{R}^{d-1} \to \mathbb{X}_{\mathbb{T}}$, which is separately measurable in the sense that it is measurable in the random parameter $\omega \in \Omega$ for fixed points $(t, \boldsymbol{y}) \in \mathbb{T} \times \mathbb{R}^{d-1}$, and measurable with respect to the point $(t, \boldsymbol{y}) \in \mathbb{T} \times \mathbb{R}^{d-1}$ for fixed stochastic parameter $\omega \in \Omega$. ◆

Since we have already argued the existence of a parametrization, the above assumption can be broken down into two parts, namely that the domain of $\boldsymbol{P}_{\mathfrak{C}_{\mathfrak{D}}^{\kappa}}$ can be written as $\Omega \times \mathbb{T} \times \mathbb{R}^{d-1}$ and the separate measurability of the parametrization.

4.1.2 Investigation of the resulting space-time domain parts

As a last investigation of this section on random compound flux discontinuities, we can now turn to the (random) space-time domain parts resulting from a stochastic compound flux discontinuity. Here, by *random space-time domain part* we mean connected open subsets $\mathbb{X}_{\mathbb{T}}^{s}(\omega) \subset \mathbb{X}_{\mathbb{T}}$ of the space-time domain $\mathbb{X}_{\mathbb{T}}$ satisfying the following two conditions:
First, the random space-time domain part $\mathbb{X}_{\mathbb{T}}^{s}(\omega)$ does not contain any points $\mathfrak{d} \in \mathfrak{D}(\omega)$ of the random compound flux discontinuity $\mathfrak{D}(\omega)$. Second, the boundary of the space-time domain part $\mathbb{X}_{\mathbb{T}}^{s}(\omega)$ is a subset of the random compound flux discontinuity $\mathfrak{D}(\omega)$.

The purpose of discussing these random domain parts $\mathbb{X}_{\mathbb{T}}^{s}(\omega)$ is to establish two properties that are necessary for the well-posedness investigation of random entropy solutions in Section 4.3:

(i) First, the random space-time domain parts $\mathbb{X}_{\mathbb{T}}^{s}(\omega)$ are measurable when considered as a correspondence $\mathbb{X}_{\mathbb{T}}^{s} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$. The corresponding result is shown in Corollary 4.12.

(ii) Second, in Lemma 4.13 we prove that the indicator functions $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^{s}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{R}$ of the random space-time domain parts $\mathbb{X}_{\mathbb{T}}^{s}$ are separately measurable.

Contrary to the random sole discontinuity setting of Chapter 3, in which we always obtained two space-time domain parts, a general a-priori statement on the random number of resulting domain parts is not easy to attain for random compound flux discontinuities. Therefore, denote by $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$ the (random) number of space-time domain parts $\mathbb{X}_{\mathbb{T}}^{s}(\omega) \subset \mathbb{X}_{\mathbb{T}}$ resulting from a random compound discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$. Unfortunately, we cannot make any statement on the measurability of the random variable $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$ without imposing the following measurability assumption:

Assumption 4.9 (Measurability of number of domain parts)**:**
For a random compound flux discontinuity hypersurface $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$, the corresponding random number $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)$ of resulting space-time domain parts $\mathbb{X}_{\mathbb{T}}^{s}(\omega) \subset \mathbb{X}_{\mathbb{T}}$ is measurable, when viewed as a mapping $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$. ◆

When we introduced random compound flux discontinuities in the beginning of this section, we also defined an index set $\mathcal{I}_{\mathfrak{D}}(\omega) \subset \mathbb{N}$ containing all indices $\mathfrak{i} \in \mathbb{N}$ of the sole discontinuities $\mathfrak{D}_{\mathfrak{i}}(\omega)$ contained in $\mathfrak{D}(\omega)$. Analogously, we can now define the index set $\mathcal{I}_{\mathbb{X}_{\mathbb{T}}} : \Omega \rightrightarrows \mathbb{N}$ that describes the indices of the domain parts resulting from a random compound discontinuity $\mathfrak{D}(\omega)$ via

$$\mathcal{I}_{\mathbb{X}_{\mathbb{T}}}(\omega) = \begin{cases} \{\mathfrak{i} \in \mathbb{N} \mid 1 \leq \mathfrak{i} \leq \mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)\} & \text{if } \mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega) < \infty \,, \\ \mathbb{N} & \text{if } \mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega) = \infty \,. \end{cases}$$

This index set $\mathcal{I}_{\mathbb{X}_{\mathbb{T}}}$ allows us to describe the number of space-time domain parts and to formalize the corresponding family as $\left\{\mathbb{X}_{\mathbb{T}}^s\right\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}$. However, to establish measurability of the random space-time parts, we need to impose the following local finiteness assumption on this family of domain parts.

Assumption 4.10 (Space-time domain parts are locally finite)**:**
For a random compound flux discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$, let $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}} : \Omega \to \left(\mathbb{N} \cup \{\infty\}\right)$ be the corresponding number of connected space-time domain parts $\mathbb{X}_{\mathbb{T}}^s(\omega) \subset \mathbb{X}_{\mathbb{T}}$, with $s \in \mathcal{I}_{\mathbb{X}_{\mathbb{T}}}(\omega)$. Then, we assume that the family of domain parts $\left\{\mathbb{X}_{\mathbb{T}}^s\right\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}$ is locally finite in the sense that each compact set $K \subset \mathbb{X}_{\mathbb{T}}$ intersects only with a finite number of domain parts $\mathbb{X}_{\mathbb{T}}^s(\omega)$. ◆

So far, the construction of random compound flux discontinuities $\mathfrak{D}$ and associated variables and properties such as $\mathcal{I}_{\mathfrak{D}}$ has been rather straightforward. However, the stochasticity of the index sets $\mathcal{I}_{\mathfrak{D}}$ and $\mathcal{I}_{\mathbb{X}_{\mathbb{T}}}$ turn out to be quiet cumbersome for measurability investigations. To overcome this obstacle, we make the following convention on the random domain parts:

$$\text{For fixed } s \in \mathbb{N} \text{ it holds that} \qquad \mathbb{X}_{\mathbb{T}}^s(\omega) = \emptyset \qquad \text{if and only if} \qquad s > \mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega) \,. \tag{4.3}$$

Another important tool that we will frequently use are the left and right domain parts corresponding to the $\mathfrak{i}$-th sole discontinuity $\mathfrak{D}_{\mathfrak{i}}(\omega)$. Roughly speaking, these sets are the left and right domain part, if we would only consider the $\mathfrak{i}$-th sole discontinuity. We formalize this construction with the next definition.

Definition 4.11 (Domain parts of sole discontinuities)**:**
For any index $\mathfrak{i} \in \mathbb{N}$, the random left and right domain parts $\mathbb{C}_{\mathfrak{i}}^{l,r} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ *are defined via*

$$\begin{aligned} \mathbb{C}_{\mathfrak{i}}^{l}(\omega) &\coloneqq \left\{(t, \boldsymbol{x}) \in \mathbb{X}_{\mathbb{T}} \mid x_1 < \Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega, t, \boldsymbol{x}_{2:d})\right\} , \\ \mathbb{C}_{\mathfrak{i}}^{r}(\omega) &\coloneqq \left\{(t, \boldsymbol{x}) \in \mathbb{X}_{\mathbb{T}} \mid x_1 > \Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega, t, \boldsymbol{x}_{2:d})\right\} , \end{aligned} \tag{4.4}$$

where $\Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega, \cdot, \cdot) \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^{d-1}; \mathbb{R})$ is the function, whose graph defines the $\mathfrak{i}$-th sole discontinuity $\mathfrak{D}_{\mathfrak{i}}(\omega)$ of the random compound flux discontinuity $\mathfrak{D}(\omega)$. ◆

With the help of these left and right domain parts corresponding to the $\mathfrak{i}$-th random sole discontinuity hypersurface, we are now able to state the following stochastic measurability result of the domain parts $\mathbb{X}_{\mathbb{T}}^s : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$. As for the discontinuity and the domain parts for random sole discontinuities, we formulate this measurability via correspondences.

Corollary 4.12 (Stochastic measurability of domain parts)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity that satisfies the stochastic measurability Assumption 4.3 and the local finiteness Assumption 4.10 on the resulting domain parts $\mathbb{X}_{\mathbb{T}}^s$. Furthermore,

let the random number of space-time parts $\mathfrak{N}_{\mathbb{X}_\mathbb{T}} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$ satisfy the measurability Assumption 4.9. Then, for any index $s \in \mathbb{N}$, the corresponding space-time domain part $\mathbb{X}_\mathbb{T}^s : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ is a measurable set-valued mapping. ◆

Proof. Let an index $s \in \mathbb{N}$ be fixed. Showing the measurability of the space-time domain part $\mathbb{X}_\mathbb{T}^s : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ consists of two major steps: First, we divide the stochastic domain Ω into two parts consisting of those $\omega \in \Omega$ leading to empty-valued $\mathbb{X}_\mathbb{T}^s(\omega)$ and the corresponding complement leading to nonempty-valued $\mathbb{X}_\mathbb{T}^s(\omega)$. After showing the measurability of these sets, we use this partitioning of Ω to argue the measurability of the domain parts $\mathbb{X}_\mathbb{T}^s$.

Division of Ω. As a first step, we define a family of sets of parameters $\omega \in \Omega$ that lead to exactly $\mathfrak{m} \in \big(\mathbb{N} \cup \{\infty\}\big)$ space-time domain parts as

$$\Omega_{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}}^{\mathfrak{m}} \coloneqq \{\omega \in \Omega \mid \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega) = \mathfrak{m}\} \, .$$

Due to Assumption 4.9, the number $\mathfrak{N}_{\mathbb{X}_\mathbb{T}} : \Omega \to \big(\mathbb{N} \cup \{\infty\}\big)$ of domain parts is measurable. Therefore, also the sets $\Omega_{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}}^{\mathfrak{m}}$ are measurable. With the help of these sets, we can define the set

$$\Omega^s \coloneqq \{\omega \in \Omega \mid \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega) \leq s\} = \bigcup_{\mathfrak{m}=1}^{s} \Omega_{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}}^{\mathfrak{m}} \, ,$$

which consists of all parameters $\omega \in \Omega$ leading to at most $s \in \mathbb{N}$ domain parts. This set is measurable as it is a countable union of measurable sets. From this measurability, we can immediately deduce that the complement of $\Omega_{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}}^{s}$, given by

$$\Big(\Omega_{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}}^{s}\Big)^{\mathsf{c}} = \{\omega \in \Omega \mid \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega) > s\} =: \Omega_{\emptyset}^{s} \, ,$$

is also measurable and consists of all those parameters $\omega \in \Omega$ that lead to more than $s \in \mathbb{N}$ space-time domain parts.

Measurability of $\mathbb{X}_\mathbb{T}^s$. With these two (measurable) sets, we can investigate the measurability of the s-th domain part $\mathbb{X}_\mathbb{T}^s : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$. Therefore, we write $\mathbb{X}_\mathbb{T}^s(\omega)$ via the following piecewise definition

$$\mathbb{X}_\mathbb{T}^s(\omega) = \begin{cases} \mathbb{F}(\omega) & \text{for } \omega \in \Omega^s \, , \\ \mathbb{G}(\omega) & \text{for } \omega \in \Omega_{\emptyset}^{s} \, . \end{cases} \tag{4.5}$$

Both mappings $\mathbb{F}$ and $\mathbb{G}$ are correspondences defined implicitly via $\mathbb{X}_\mathbb{T}^s$. Equivalently to the piecewise definition in Equation (4.5), we can write the space-time domain part $\mathbb{X}_\mathbb{T}^s(\omega)$ via the indicator functions of the sets Ω^s and $\Omega_{\emptyset}^{s}$, which yields

$$\mathbb{X}_\mathbb{T}^s(\omega) = \mathbb{F}(\omega)\mathbb{1}_{\Omega^s}(\omega) + \mathbb{G}(\omega)\mathbb{1}_{\Omega_{\emptyset}^{s}}(\omega) \, .$$

Both indicator functions are measurable due to the measurability of the two sets Ω^s and $\Omega_{\emptyset}^{s}$, cf. [248, Proposition 1.9 (d)]. From this formulation, we can deduce that $\mathbb{X}_\mathbb{T}^s$ is measurable if and only if both $\mathbb{F}$ and $\mathbb{G}$ are measurable. In case the stochastic parameter ω satisfies $\omega \in \Omega_{\emptyset}^{s}$, Convention (4.3) yields $\mathbb{X}_\mathbb{T}^s(\omega) = \emptyset$. This directly implies that $\mathbb{G}$ is the constant-valued mapping $\mathbb{G} \equiv \emptyset$, whose image is the empty set, due to its construction in Equation (4.5). Consequently, the correspondence $\mathbb{G}$ is measurable, since constant-valued maps are measurable by construction.

It remains to show the measurability of the set-valued mapping $\mathbb{F}$. By construction of the set Ω^s, it holds that $\mathbb{X}_{\mathbb{T}}^s(\omega) \neq \emptyset$, whenever $\omega \in \Omega^s$. This immediately implies that the mapping $\mathbb{F}$ is nonempty, i.e., $\mathbb{F}(\omega) \neq \emptyset$. Furthermore, for every index $\mathfrak{i} \in \mathcal{I}_{\mathfrak{D}}(\omega)$, we have the left and right domain parts $\mathbb{C}_{\mathfrak{i}}^{l,r} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ of the $\mathfrak{i}$-th random sole discontinuity $\mathfrak{D}_{\mathfrak{i}}(\omega)$ given by

$$\begin{aligned} \mathbb{C}_{\mathfrak{i}}^{l}(\omega) &:= \left\{ (t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X} \mid x_1 < \Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega, t, \boldsymbol{x}_{2:d}) \right\}, \\ \mathbb{C}_{\mathfrak{i}}^{r}(\omega) &:= \left\{ (t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X} \mid x_1 > \Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega, t, \boldsymbol{x}_{2:d}) \right\}. \end{aligned} \tag{4.6}$$

By construction it holds for every parameter $\omega \in \Omega$ that either $\mathbb{X}_{\mathbb{T}}^s(\omega) \subset \mathbb{C}_{\mathfrak{i}}^l(\omega)$ or $\mathbb{X}_{\mathbb{T}}^s(\omega) \subset \mathbb{C}_{\mathfrak{i}}^r(\omega)$ is satisfied. Denoting the set $\mathbb{C}_{\mathfrak{i}}^{l,r}(\omega)$ containing $\mathbb{X}_{\mathbb{T}}^s(\omega)$ by $\mathbb{C}_{\mathfrak{i}}(\omega)$, we obtain the measurability of $\mathbb{C}_{\mathfrak{i}} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ via the definitions in Equation (4.6). Here, measurability of the functions $\Phi_{\mathfrak{i}}^{\mathfrak{D}}$ is implied by the measurability Assumption 4.3.

If the index $\mathfrak{i}$ is not contained in the index set $\mathcal{I}_{\mathfrak{D}}(\omega)$ we define $\mathbb{C}_{\mathfrak{i}}(\omega)$ as the whole space-time domain, i.e., $\mathbb{C}_{\mathfrak{i}}(\omega) = \mathbb{X}_{\mathbb{T}}$. This allows us to reformulate the space-time part $\mathbb{X}_{\mathbb{T}}^s$ via the sets $\mathbb{C}_{\mathfrak{i}}(\omega)$, for $\mathfrak{i} \in \mathbb{N}$, as

$$\mathbb{X}_{\mathbb{T}}^s(\omega) = \bigcap_{\mathfrak{i} \in \mathbb{N}} \mathbb{C}_{\mathfrak{i}}(\omega) \,. \tag{4.7}$$

Since the countable intersection of measurable sets is again measurable, we have proven the assertion, which states that the domain part $\mathbb{X}_{\mathbb{T}}^s : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ is a measurable set-valued mapping. ■

For the space-time parts resulting from a random compound flux discontinuity, we can define the corresponding domain part indicator functions completely analogous to the sole flux discontinuity setting of Chapter 3. We refer to Definition 3.11 for the details. To conclude this section on compound flux discontinuities, we show that these random domain part indicator functions $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^s} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{R}$ are separately measurable.

Lemma 4.13 (Separate measurability of domain part indicator functions)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity that satisfies the measurability Assumption 4.3 and the local finiteness Assumption 4.10 on the resulting domain parts $\mathbb{X}_{\mathbb{T}}^s$. Furthermore, let the number $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}} : \Omega \to \left(\mathbb{N} \cup \{\infty\}\right)$ of space-time parts satisfy the measurability Assumption 4.9. Then, for any index $s \in \mathbb{N}$, the indicator function $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^s} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{R}$ of the domain part $\mathbb{X}_{\mathbb{T}}^s(\omega)$ is separately measurable. ◆

Proof. We show the spatio-temporal and the stochastic measurability of the domain part indicator function $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^s}$ separately:

(i) Let the stochastic parameter $\omega \in \Omega$ and an index $s \in \mathbb{N}$ be fixed. We argue the spatio-temporal measurability of $\mathbb{1}_{\mathbb{X}_{\mathbb{T}}^s}$ by showing that $\mathbb{X}_{\mathbb{T}}^s(\omega)$ is an open set and therefore measurable:
In the case that we have $\omega \in \Omega_{\emptyset}^s$, the domain part $\mathbb{X}_{\mathbb{T}}^s(\omega)$ is empty, i.e., $\mathbb{X}_{\mathbb{T}}^s(\omega) = \emptyset$. However, by definition, the empty set $\emptyset$ is open. Therefore, assume that the stochastic parameter ω satisfies $\omega \in \Omega^s$. By the construction of $\mathbb{X}_{\mathbb{T}}^s(\omega)$ in Equation (4.7), the set $\mathbb{X}_{\mathbb{T}}^s(\omega)$ corresponds to the countable intersection of the sets $\mathbb{C}_{\mathfrak{i}}(\omega)$. However, by their construction in Equation (4.4), the sets $\mathbb{C}_{\mathfrak{i}}(\omega)$ are open. This implies that $\mathbb{X}_{\mathbb{T}}^s(\omega)$ is open as well. Furthermore, open sets are Borel-measurable and the indicator function of a measurable set is again measurable [248, Proposition 1.9 (d)].

(ii) Let $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ and $s \in \mathbb{N}$ be fixed. Now, rewrite the indicator function of $\mathbb{X}_\mathbb{T}^s(\omega)$ as

$$\mathbb{1}_{\mathbb{X}_\mathbb{T}^s}(\omega, \mathfrak{x}) = \mathbb{1}_{\mathbb{X}_\mathbb{T}}(\mathfrak{x}) - \mathbb{1}_{\mathbb{X}_\mathbb{T} \setminus \mathbb{X}_\mathbb{T}^s}(\omega, \mathfrak{x}) \, .$$

Since the space-time domain $\mathbb{X}_\mathbb{T}$ is independent of the stochastic parameter $\omega \in \Omega$, it is trivially measurable. Furthermore, by the discussion of (i), the space-time part $\mathbb{X}_\mathbb{T}^s(\omega)$ is open and therefore $\mathbb{X}_\mathbb{T} \setminus \mathbb{X}_\mathbb{T}^s(\omega)$ is closed as its complement. With this result, we can interpret the indicator function $\mathbb{1}_{\mathbb{X}_\mathbb{T} \setminus \mathbb{X}_\mathbb{T}^s}$ as the composition of a measurable closed-valued correspondence and the set-dependent indicator function. By Lemma 2.18, the set-dependent indicator function is separately measurable. Noting that the composition of two Borel-measurable functions is again measurable, we can conclude stochastic measurability of the indicator function $\mathbb{1}_{\mathbb{X}_\mathbb{T}^s}$ of the space-time part $\mathbb{X}_\mathbb{T}^s$.

Combining the results of (i) and (ii), we conclude the assertion. ■

4.2 Admissibility conditions and $\mathfrak{G}$-entropy solutions

With the discussion of compound flux discontinuities and their properties in the previous section, we can now proceed by extending the admissibility conditions for random entropy solutions to the scalar discontinuous-flux conservation law given by Equation (4.1). Therefore, we start in Section 4.2.1 by generalizing the notion of admissibility germs to this advanced setting of compound flux discontinuities and define the associated notion of a random entropy solution. Afterwards, in Section 4.2.2, we discuss the corresponding adapted Kružkov entropy conditions as well as defining the random $\mathfrak{G}$-entropy solutions via these inequalities. We conclude this Section by discussing stochastic measurability of this Kružkov entropy and Kružkov entropy flux for the random compound discontinuity setting in Section 4.2.3.

As in the investigation of admissible solutions for the sole discontinuity setting, we need to impose some assumptions on the random discontinuous flux function $\mathfrak{f}$ that we require to hold throughout this section and in the remainder of this chapter.

Assumption 4.14 (Flux function with compound discontinuity)**:**
A flux function $\mathfrak{f}$ having a (random) compound discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ satisfies the following assumptions for every stochastic parameter $\omega \in \Omega$:

(C-1) The flux function $\mathfrak{f} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \to \mathbb{X}$ has the form

$$(\omega, \mathfrak{x}, \upsilon) \mapsto \sum_{s \in \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)} \mathfrak{f}^s(\omega, \mathfrak{x}, \upsilon) \mathbb{1}_{\mathbb{X}_\mathbb{T}^s}(\omega, \mathfrak{x}) \, ,$$

where $\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ is the number of space-time domain parts $\mathbb{X}_\mathbb{T}^s$ resulting from the random compound flux discontinuity $\mathfrak{D}$ and for each index $s \in \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ the function $\mathfrak{f}^s$ is globally defined as a function $\mathfrak{f}^s : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \to \mathbb{X}$.

(C-2) For each $s \in \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ and fixed $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, the flux $\mathfrak{f}^s(\omega, \mathfrak{x}, \cdot)$ is locally Lipschitz continuous.

(C-3) For each index $s \in \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ and fixed scalar value $\upsilon \in \mathbb{R}$, the flux function $\mathfrak{f}^s(\omega, \cdot, \upsilon)$ is globally Lipschitz continuous on the whole space-time domain $\mathbb{X}_\mathbb{T}$. ◆

4.2.1 $\mathfrak{G}$-entropy solutions via admissibility germs

Roughly speaking, the idea of extending the admissibility of solutions to the compound flux discontinuity case is to locally reduce the problem to the underlying sole discontinuity theory. Hence, we need to be able to *select* the pair of flux functions $\mathfrak{f}^{l,r}$ from the family of functions $\{\mathfrak{f}^s\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$ that locally represent this associated sole discontinuity problem. The main tool of doing this selection are *selection functions*, which we introduce with the following definition.

Definition 4.15 (Selection functions for flux function)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound discontinuity that satisfies Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D}$. Associated to $\mathfrak{C}_\mathfrak{D}$, we define the selection functions $\mathfrak{s}_\mathfrak{f}^{l,r}$ *as the following mappings:*

$$\mathfrak{s}_\mathfrak{f}^{l,r} : \Omega \times \mathbb{N} \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \to \mathbb{R} \,,$$

where $\mathfrak{s}_\mathfrak{f}^{l,r}$ satisfies the condition that $\mathfrak{s}_\mathfrak{f}^{l,r}(\omega, \kappa, \mathfrak{x}, \upsilon) = \mathfrak{f}^{\mathfrak{k},\mathfrak{m}}(\omega, \mathfrak{x}, \upsilon)$ for some indices $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}$ such that $\mathrm{cl}\big(\mathbb{X}_\mathbb{T}^\mathfrak{k}\big) \cap \mathrm{cl}\big(\mathbb{X}_\mathbb{T}^\mathfrak{m}\big) \cap \mathfrak{C}_\mathfrak{D}^\kappa \neq \emptyset$. For simplicity, we always identify the selection functions $\mathfrak{s}_\mathfrak{f}^{l,r}$ with the flux functions $\mathfrak{f}^{\mathfrak{k},\mathfrak{m}}$, such that the corresponding domain parts satisfy $\mathbb{X}_\mathbb{T}^\mathfrak{k} \subset \mathbb{C}_\kappa^l$ and $\mathbb{X}_\mathbb{T}^\mathfrak{m} \subset \mathbb{C}_\kappa^r$, respectively. Here, $\mathbb{C}_\kappa^{l,r}$ are the left and right domain part of the κ-th sole discontinuity as defined in Definition 4.11. ◆

These selection functions allow us to extend the framework of random admissibility germs. While the core definition of an admissibility germ remains the same, we extend the family of germs to not only depend on $\omega \in \Omega$ and $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, but also on a given part $\kappa \in \mathbb{N}$ of the partition up to a null set $\mathfrak{C}_\mathfrak{D}(\omega)$ of the random compound flux discontinuity $\mathfrak{D}(\omega)$. This idea leads to the following definition of a family (of families) of admissibility germs.

Definition 4.16 (Families of random admissibility germs)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D}$. Furthermore, let this partition $\mathfrak{C}_\mathfrak{D}$ satisfy the local finiteness Assumption 4.5. Then, for every index $\kappa \in \mathbb{N}$, we can define the random family of admissibility germs $\mathfrak{G}_\kappa : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ *associated to the sole discontinuity hypersurface $\mathfrak{D}_{\mathfrak{s}_\omega(\kappa)}(\omega)$ and the normal components of the functions $\mathfrak{s}_\mathfrak{f}^{l,r}(\omega, \kappa, \mathfrak{x}, \cdot)$, which are given by $\mathfrak{s}_\mathfrak{f}^{l,r}(\omega, \kappa, \mathfrak{x}, \cdot) \cdot \widehat{\mathfrak{n}}_{\mathfrak{C}_\mathfrak{D}^\kappa}$. Here, $\widehat{\mathfrak{n}}_{\mathfrak{C}_\mathfrak{D}^\kappa}$ is the extension of the normal unit vector to the partition part $\mathfrak{C}_\mathfrak{D}^\kappa$ as constructed in Definition 4.7.*
Formally, the family $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ is a family of random families of germs. However, for simplicity we call $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ a family of random admissibility germs. ◆

Let us stress that for each index $\kappa \in \mathbb{N}$, each stochastic parameter $\omega \in \Omega$ and every space-time coordinate $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, we obtain a standard admissibility germ associated to two continuous functions as introduced in Definition 3.14 in Chapter 3. For this germ, we can define the same properties of $\mathcal{L}^1$-dissipativity, maximality, definiteness and completeness as well as corresponding extensions and its dual germ. However, as in the sole discontinuity case, we need the following assumption on joint measurability of each germ $\mathfrak{G}_\kappa$ in this family of germs.

Assumption 4.17 (Joint measurability of family of germs)**:**
Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of random admissibility germs $\mathfrak{G}$ as defined in Definition 4.16. We assume that this family $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ is jointly measurable in the sense that, for every index $\kappa \in \mathbb{N}$, the family of germs $\mathfrak{G}_\kappa : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is jointly measurable. ◆

We have now introduced every ingredient that we need to formalize the notion of random $\mathfrak{G}$-entropy solutions via its traces and the underlying family of random admissibility germs. However, to ensure that these traces exist, we need the following genuine nonlinearity assumption on the flux function, which is completely analogous to the sole discontinuity setting. Afterwards, we define random $\mathfrak{G}$-entropy solutions in the formulation via germs.

Assumption 4.18 (Genuine nonlinearity of flux functions)**:**
Let a stochastic parameter $\omega \in \Omega$ and a space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ be fixed. Then, for every index $s \in \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$, we assume that the corresponding flux function $\mathfrak{f}^s(\omega, \mathfrak{x}, \cdot)$ is genuinely nonlinear in the sense that the function $\mathfrak{f}^s(\omega, \mathfrak{x}, \cdot)$ is not constant on any nontrivial interval $I \subset \mathbb{R}$. ◆

Definition 4.19 (Pathwise $\mathfrak{G}$-entropy solution via germ formulation)**:**
Let the following conditions be satisfied:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a compound discontinuity satisfying Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D}$, which itself satisfies the locally finiteness Assumption 4.5.*
- *Let $\mathfrak{f}$ be a flux function satisfying the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18.*
- *Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a family of random $\mathcal{L}^1 D$ admissibility germs that satisfies the joint measurability Assumption 4.17 and let $\{\mathfrak{G}^*_\kappa\}_{\kappa\in\mathbb{N}} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ denote the corresponding family of dual germs.*

Then, for fixed stochastic parameter $\omega \in \Omega$, a function $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is called a random $\mathfrak{G}$-entropy solution *to Problem* (4.1)*, if the following conditions are satisfied:*

(i) For every $s \in \mathcal{I}_{\mathbb{X}_\mathbb{T}}(\omega)$, the restriction of $u(\omega, \cdot, \cdot)$ to the domain parts $\mathbb{X}^s_\mathbb{T}(\omega)$ is a Kružkov entropy solution of Equation (4.1) *with flux function $\mathfrak{f}^s$ in the sense of the Kružkov entropy condition* (1.2)*.*

(ii) For every index $\kappa \in \mathbb{N}$ it holds that for $\mathcal{H}^d$-almost every point $\mathfrak{d} \in \mathfrak{C}^\kappa_\mathfrak{D}(\omega)$, the couple of strong traces $(\gamma^l u, \gamma^r u)$ of $u(\omega, \cdot, \cdot)$ on the discontinuity part $\mathfrak{C}^\kappa_\mathfrak{D}(\omega)$ belongs to the dual germ $\mathfrak{G}^*_\kappa(\omega, \mathfrak{d})$.

(iii) $\mathcal{H}^d$-almost everywhere on $\{0\} \times \mathbb{X}$, the initial trace $\gamma^0 u$ of $u(\omega, \cdot, \cdot)$ is equal to the initial condition $u_0(\omega, \cdot)$. ◆

4.2.2 $\mathfrak{G}$-entropy solutions via adapted entropy inequalities

Now that we have defined random $\mathfrak{G}$-entropy solutions in a local manner via admissibility germs, we need to establish the corresponding global admissibility criterion via adapted Kružkov entropy inequalities. While the main ideas are similar to the sole flux setting, the extension to compound flux discontinuities requires vary generalizations that may not seem intuitive. However, to derive the adapted Kružkov entropy inequalities, we start with a straightforward generalization of adapted Kružkov entropies and the Kružkov entropy flux. This leads to the two subsequent definitions.

Definition 4.20 (Adapted Kružkov entropy)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity and let the stochastic parameter $\omega \in \Omega$ be fixed. Furthermore, let $\{\mathbb{X}^s_\mathbb{T}\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$ be a family of resulting space-time parts, where $\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ denotes the random number of domain parts $\mathbb{X}^s_\mathbb{T}$. Then, for a fixed sequence $\boldsymbol{k} = \{k^s\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)} \subset \mathbb{R}$, the adapted

Kružkov entropy *is defined as*

$$k(\omega, \mathfrak{x}) \coloneqq \sum_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)} k^s \mathbb{1}_{\mathbb{X}_\mathbb{T}^s}(\omega, \mathfrak{x}) \,, \tag{4.8}$$

where $\mathbb{1}_{\mathbb{X}_\mathbb{T}^s} : \Omega \times \mathbb{X}_\mathbb{T} \to \mathbb{R}$ *denotes the indicator function of the* s*-th domain part* $\mathbb{X}_\mathbb{T}^s$. ◆

Definition 4.21 (Kružkov entropy flux)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ *be a random compound flux discontinuity and let a stochastic parameter* $\omega \in \Omega$ *be fixed. Deducing from this, let* $\{\mathbb{X}_\mathbb{T}^s\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$ *be the family of resulting domain parts, where* $\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ *denotes the number of space-time parts* $\mathbb{X}_\mathbb{T}^s$*. Furthermore, let* $\mathfrak{f}$ *be a flux function that satisfies the compound-flux-discontinuity Assumption 4.14. Then, the* Kružkov entropy flux *is defined as*

$$\mathfrak{q}(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon}) \coloneqq \sum_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)} \mathfrak{q}^s(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon}) \mathbb{1}_{\mathbb{X}_\mathbb{T}^s}(\omega, \mathfrak{x}) \,,$$

where each entropy flux function $\mathfrak{q}^s$ *is defined as in the sole discontinuity case, i.e.,*

$$\mathfrak{q}^s(\omega, \mathfrak{x}, \upsilon, \tilde{\upsilon}) = \operatorname{sign}(\upsilon - \tilde{\upsilon}) \big(\mathfrak{f}^s(\omega, \mathfrak{x}, \upsilon) - \mathfrak{f}^s(\omega, \mathfrak{x}, \tilde{\upsilon}) \big) \,. \tag{4.9}$$

Here, $\mathfrak{f}^s : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \to \mathbb{X}$ *is the* s*-th flux function as defined in the compound-flux-discontinuity Assumption 4.14.* ◆

For deriving the global admissibility criterion in the sole discontinuity setting, we employed remainder functions that were associated to an admissibility germ $\mathfrak{G}$. These functions were designed to measure the distance of an arbitrary point $\boldsymbol{c} \in \mathbb{R}^2$ to this germ $\mathfrak{G} \subset \mathbb{R}^2$ in a certain way that might depend on the flux function $\mathfrak{f}$. For compound flux discontinuities, we already introduced a family of germs $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ associated to a partition $\mathfrak{C}_\mathfrak{D}(\omega)$ of the discontinuity $\mathfrak{D}(\omega)$. Therefore, we need to extend the notion of this remainder function to a *family of remainder functions* associated to this family of admissibility germs $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$. We do this in the next definition.

Definition 4.22 (Family of remainder functions)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ *be a random compound discontinuity satisfying Assumption 4.4 on the existence of a partition up to a null set* $\mathfrak{C}_\mathfrak{D}$*, which itself satisfies the locally finiteness Assumption 4.5. Furthermore, let* $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ *be a random family of* $\mathcal{L}^1$*-dissipative germs. A random family of functions* $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa \in \mathbb{N}}$*, with* $\mathfrak{R}_\mathfrak{G}^\kappa : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$*, is called a* family of remainder functions*, if for every stochastic parameter* $\omega \in \Omega$ *and every index* $\kappa \in \mathbb{N}$ *the function* $\mathfrak{R}_\mathfrak{G}^\kappa$ *is a remainder function of the (family of) admissibility germ(s)* $\mathfrak{G}_\kappa$ *in the sense of Definition 3.26.* ◆

In the sole discontinuity setting, we assumed that the remainder function is jointly measurable as soon as the associated admissibility germ is jointly measurable. This assumption was a simplifying one, since the remainder function can take various forms that may depend on the flux function $\mathfrak{f}$. Again, we refer to Appendix A for a discussion of the measurability for some choices of the remainder function. For a compound discontinuity, we impose a similar measurability assumption on the family of remainder functions. Let us stress that the joint measurability Assumption 4.17 on the family of germs $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ is crucial for this assumption.

Assumption 4.23 (Joint measurability of remainder functions)**:**
Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of random admissibility germs, which satisfies the joint measurability Assumption 4.17. Then, we assume that for every index $\kappa \in \mathbb{N}$, the remainder function $\mathfrak{R}^\kappa_{\mathfrak{G}}$ is jointly measurable. ◆

With the family of remainder functions $\{\mathfrak{R}^\kappa_{\mathfrak{G}}\}_{\kappa\in\mathbb{N}}$, we have almost every ingredient to define the global admissibility criterion via adapted Kružkov entropy inequalities. However, recall that for a compound discontinuity, the adapted Kružkov entropy is defined as a sequence $\boldsymbol{k} = \{k^s\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$. To formalize the definition of the Kružkov entropy sequence $\boldsymbol{k} = \{k^s\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)} \subset \mathbb{R}$ in the subsequent discussion, let $\mathbb{R}^{\leq\mathbb{N}}$ denote the space of all finite or infinite real-valued sequences. Obviously, the Kružkov entropy sequence $\boldsymbol{k}$ always satisfies $\boldsymbol{k} \in \mathbb{R}^{\leq\mathbb{N}}$.

To evaluate the remainder function $\mathfrak{R}^\kappa_{\mathfrak{G}}$ for a given index $\kappa \in \mathbb{N}$, we need to be able to select a pair of entropy values of this sequence. Therefore, similarly to selecting the correct flux functions $\mathfrak{f}^{\mathfrak{k},\mathfrak{m}}$ for the admissibility germ at hand, we define a *selection function* for the Kružkov entropy sequence.

Definition 4.24 (Selection function for Kružkov entropy)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D}(\omega)$. Furthermore, let $\{\mathbb{X}^s_\mathbb{T}\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$ be a family of space-time domain parts resulting from the discontinuity $\mathfrak{D}$. For a sequence of Kružkov entropy values $\boldsymbol{k} \in \mathbb{R}^{\leq\mathbb{N}}$ associated to the family $\{\mathbb{X}^s_\mathbb{T}\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$ of domain parts, we define the selection function *$\mathcal{s}_{\boldsymbol{k}}$ for the Kružkov entropy as the mapping*

$$\mathcal{s}_{\boldsymbol{k}} : \Omega \times \mathbb{N} \times \mathbb{R}^{\leq\mathbb{N}} \to \mathbb{R}^2 \qquad \mathcal{s}_{\boldsymbol{k}}(\omega, \kappa, \boldsymbol{k}) = (k^{\mathfrak{k}}, k^{\mathfrak{m}}) \,.$$

Here, the indices $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}$ are chosen such that we have $k^{\mathfrak{k}}, k^{\mathfrak{m}} \in \boldsymbol{k}$ and such that the space-time parts $\mathbb{X}^{\mathfrak{k},\mathfrak{m}}_\mathbb{T}(\omega)$ satisfy the condition

$$\mathrm{cl}\big(\mathbb{X}^{\mathfrak{k}}_\mathbb{T}(\omega)\big) \cap \mathrm{cl}\big(\mathbb{X}^{\mathfrak{m}}_\mathbb{T}(\omega)\big) \cap \mathfrak{C}^\kappa_\mathfrak{D}(\omega) \neq \emptyset \,,$$

as well as $\mathbb{X}^{\mathfrak{k}}_\mathbb{T}(\omega) \subset \mathbb{C}^l_{\mathcal{s}_\omega(\kappa)}(\omega)$ and $\mathbb{X}^{\mathfrak{m}}_\mathbb{T}(\omega) \subset \mathbb{C}^r_{\mathcal{s}_\omega(\kappa)}(\omega)$, where $\mathbb{C}^{l,r}_{\mathcal{s}_\omega(\kappa)}(\omega)$ are the left and right domain parts of the $\mathcal{s}_\omega(\kappa)$-th sole discontinuity. ◆

With this selection function for the Kružkov entropy sequence, we can define the notion of random $\mathfrak{G}$-entropy solutions via a globally defined adapted Kružkov entropy inequality. While the idea is similar to the sole discontinuity case, the contribution of the remainder function needs special treatment: Instead of integrating one remainder function over the whole (sole) discontinuity, a family of functions needs to be integrated. This is achieved by integrating this family of remainder functions over the corresponding parts of the partition up to a null set of the random compound discontinuity.

Definition 4.25 (Pathwise $\mathfrak{G}$-entropy solution via entropy inequality)**:**
Let the following conditions be satisfied:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound discontinuity that satisfies Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D}(\omega)$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let $\mathfrak{f} : \Omega \times \mathbb{T} \times \mathbb{X} \times \mathbb{R} \to \mathbb{X}$ be a flux function satisfying the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18.*

- *Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs satisfying the joint measurability Assumption 4.17.*
- *Associated to the family of admissibility germs $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$, let $\{\mathfrak{R}^\kappa_{\mathfrak{G}}\}_{\kappa\in\mathbb{N}}$ be a family of remainder functions, which satisfies the joint measurability Assumption 4.23.*

Then, for fixed stochastic parameter $\omega\in\Omega$, a function $u(\omega,\cdot,\cdot)\in\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ is called a $\mathfrak{G}$-entropy solution *to the random scalar conservation law given by Equation* (4.1)*, if the following conditions are satisfied:*

(i) The function $u(\omega,\cdot,\cdot)$ is a solution in the sense of distributions, i.e., for any nonnegative test function $\psi\in\mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$, it holds that

$$\int_{\mathbb{T}}\int_{\mathbb{X}} u(\omega,t,\boldsymbol{x})\partial_t\psi(t,\boldsymbol{x})+\mathfrak{f}\big(\omega,t,\boldsymbol{x},u(\omega,t,\boldsymbol{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0\,.$$

(ii) For all sequences $\boldsymbol{k}\in\mathbb{R}^{\leq\mathbb{N}}$ with the corresponding adapted Kružkov entropy $k:\Omega\times\mathbb{T}\times\mathbb{X}\to\mathbb{R}$ as defined in Definition 4.20 and for all nonnegative test functions $\psi\in\mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$, the function $u(\omega,\cdot,\cdot)$ satisfies the adapted entropy inequality

$$\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathbb{X}}\big|u(\omega,t,\boldsymbol{x})-k(\omega,t,\boldsymbol{x})\big|\partial_t\psi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad+\int_{\mathbb{T}}\int_{\mathbb{X}}\mathfrak{q}\big(\omega,t,\boldsymbol{x};u(\omega,t,\boldsymbol{x}),k(\omega,t,\boldsymbol{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad-\int_{\mathbb{X}}|u_0(\omega,\boldsymbol{x})-k(\omega,0,\boldsymbol{x})|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&\quad+\sum_{\kappa\in\mathbb{N}}\int_{\mathfrak{C}^\kappa_{\mathfrak{D}}(\omega)}\mathfrak{R}^\kappa_{\mathfrak{G}}\big(\omega,\mathfrak{d};s_{\boldsymbol{k}}(\omega,\kappa,\boldsymbol{k})\big)\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d}\geq 0\,.
\end{aligned}\tag{4.10}$$

Here, $\mathfrak{C}^\kappa_{\mathfrak{D}}(\omega)$ is the κ-th part of the partition up to a null set $\mathfrak{C}_{\mathfrak{D}}$ of the compound flux discontinuity $\mathfrak{D}(\omega)$ and $s_{\boldsymbol{k}}$ denotes the selection function for the Kružkov entropy sequence $\boldsymbol{k}$ as specified in Definition 4.24. ◆

Recall that the requirement of $u(\omega,\cdot,\cdot)$ being a solution in the sense of distributions is imposed to ensure that the Rankine-Hugoniot condition is satisfied across the discontinuity of the flux function. Also recollect that we can identify the surface integral of $\mathfrak{C}^\kappa_{\mathfrak{D}}(\omega)$ with the d-dimensional Hausdorff measure $\mathcal{H}^d$. For the details on these statements, we refer to the Remarks 3.29 and 3.30 in Chapter 3.

4.2.3 Measurability of adapted Kružkov entropy and entropy flux

To conclude this section on the admissibility conditions of random $\mathfrak{G}$-entropy solutions, we investigate the stochastic measurability of the adapted Kružkov entropy k and the Kružkov entropy flux $\mathfrak{q}$. These measurability results will be important for the discussion of well-posedness of random $\mathfrak{G}$-entropy solutions in the next section. With the definitions and assumptions already made during this chapter, we are able to state the following result on the stochastic measurability of the adapted Kružkov entropy.

Proposition 4.26 (Measurability of adapted Kružkov entropy)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies the measurability Assumption 4.3. Furthermore, let the family $\mathbb{X}_\mathbb{T}^s(\omega) \subset \mathbb{X}_\mathbb{T}$, with $s \in \mathcal{I}_{\mathbb{X}_\mathbb{T}}(\omega)$ of domain parts satisfy the local finiteness Assumption 4.10. Then, for a fixed space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ and a fixed entropy sequence $\boldsymbol{k} = \{k^s\}_{s=1}^\infty \subset \mathbb{R}$, the adapted Kružkov entropy k defined via Equation (4.8) *is stochastically measurable in the sense that the mapping $\omega \mapsto k(\omega, \mathfrak{x})$ is measurable.* ◆

Proof. By hypothesis, the random compound flux discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ satisfies the measurability Assumption 4.3 and the local finiteness Assumption 4.10 on the resulting space-time parts $\{\mathbb{X}_\mathbb{T}^s\}_{s\in\mathbb{N}}$. Thus, we can apply Lemma 4.13 to obtain that the indicator functions $\mathbb{1}_{\mathbb{X}_\mathbb{T}^s}$ of these domain parts are separately measurable. In particular, for a fixed space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, the indicator functions $\mathbb{1}_{\mathbb{X}_\mathbb{T}^s}$, with index $s \in \mathbb{N}$, are random variables. Recall that $s > \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$ makes sense due to Convention (4.3) stating that $\mathbb{X}_\mathbb{T}^s = \emptyset$ if and only if $s > \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$. However, this proves the assertion due to the construction of the adapted Kružkov entropy k in Equation (4.8) and the fact that the sum over countably many random variables is measurable. ■

Unfortunately, without any knowledge on the stochastic measurability of the flux function $\mathfrak{f}$, we have no hope for arguing the measurability of the Kružkov entropy flux. Before we impose an assumption that allows us to overcome this obstacle, we formulate the following convention on the flux functions $\mathfrak{f}^s$:

$$\text{For fixed } s \in \mathbb{N} \text{ it holds that} \quad \mathfrak{f}^s \equiv \mathbf{0}_{\mathbb{R}^d} \quad \text{if and only if} \quad s > \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)\,.$$

Note, this convention does not affect any of the previous definitions and statements. Rather, it allows us to impose the following measurability assumption on the flux function $\mathfrak{f}$.

Assumption 4.27 (Stochastic measurability of flux function)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity and let the resulting family $\{\mathbb{X}_\mathbb{T}^s\}_{s\in\mathbb{N}}$ of space-time parts satisfy the local finiteness Assumption 4.10. Furthermore, let $\mathfrak{f}$ be a flux function that satisfies the compound-flux-discontinuity Assumption 4.14. Then, we assume that the flux function $\mathfrak{f}$ is stochastically measurable in the sense that, for fixed space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$ and fixed scalar value $\upsilon \in \mathbb{R}$, the mapping $\omega \mapsto \mathfrak{f}^s(\omega, \mathfrak{x}, \upsilon)$ is measurable for every index $s \in \mathbb{N}$. ◆

With this assumption available, we are now able to show that the Kružkov entropy flux is measurably dependent on the stochastic parameter $\omega \in \Omega$. Actually, we can include this in a stronger result stating that the Kružkov entropy flux is Carathéodory in the sense that it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous in all remaining arguments.

Proposition 4.28 (Kružkov entropy flux is Carathéodory)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity such that the family $\{\mathbb{X}_\mathbb{T}^s\}_{s\in\mathbb{N}}$ of resulting domain parts satisfies the local finiteness Assumption 4.10. Furthermore, let the flux function $\mathfrak{f}$ satisfy the compound-flux-discontinuity Assumption 4.14 and the stochastic measurability Assumption 4.27. Then, for any index $s \in \mathbb{N}$, the Kružkov entropy flux function $\mathfrak{q}^s : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is Carathéodory in the sense that it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous in all of the remaining arguments. ◆

Proof. Let an index $s \in \mathbb{N}$ and a space-time point $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$ be fixed as well as two scalar values $v, \tilde{v} \in \mathbb{R}$. Now recall the construction of the Kružkov entropy flux function $\mathfrak{q}^s$ in Equation (4.9) as

$$\mathfrak{q}^s(\omega, \mathfrak{x}, v, \tilde{v}) = \operatorname{sign}(v - \tilde{v})\big(\mathfrak{f}^s(\omega, \mathfrak{x}, v) - \mathfrak{f}^s(\omega, \mathfrak{x}, \tilde{v})\big) \, .$$

Therefore, the stochastic measurability of the Kružkov entropy flux function $\mathfrak{q}^s$ reduces to the question, whether $\mathfrak{f}^s$ is measurable with respect to $\omega \in \Omega$. However, the flux $\mathfrak{f}^s$ is measurable in the random parameter $\omega \in \Omega$ due to Assumption 4.27. The continuity results for the remaining arguments of the Kružkov entropy fluxes $\mathfrak{q}^s$ are proven analogous to the sole-flux discontinuity case in Corollary 3.42, which concludes the proof of $\mathfrak{q}^s$ being Carathéodory. ■

4.3 Well-posedness of random $\mathfrak{G}$-entropy solutions

In this section, we discuss the well-posedness of random $\mathfrak{G}$-entropy solutions. First, in Section 4.3.1, we investigate the pathwise existence and uniqueness of random $\mathfrak{G}$-entropy solutions. As in the sole discontinuity setting, pathwise uniqueness can be shown for (a family of) general definite admissibility germs. For the pathwise existence result, we restrict ourselves to the case of vanishing viscosity solutions. Afterwards, in Section 4.3.2, we discuss the entropy functionals for the compound flux discontinuity setting, which we employ in Section 4.3.3 to prove the measurability of random $\mathfrak{G}$-entropy solutions. We conclude this section on well-posedness of random $\mathfrak{G}$-entropy solutions by discussing the existence of moments thereof in Section 4.3.4.

4.3.1 Pathwise existence and uniqueness of $\mathfrak{G}$-entropy solutions

We start the investigation of well-posedness of random $\mathfrak{G}$-entropy solutions by discussing their pathwise uniqueness. Luckily, for a family $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ of definite admissibility germs, we are able to extend the result of the sole flux discontinuity in a straightforward way. That is, we can establish the Kato inequality, which provides the properties of $\mathcal{L}^1$-contraction and uniqueness of $\mathfrak{G}$-entropy solutions. The proof of this statement is completely analogous to the corresponding pathwise uniqueness Theorem 3.32 in the sole flux discontinuity setting. Therefore, it is omitted in the subsequent presentation.

Theorem 4.29 (Pathwise uniqueness of $\mathfrak{G}$-entropy solutions)**:**
Let the following conditions be satisfied:

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity, such that the family $\{\mathbb{X}_{\mathbb{T}}^s\}_{s\in\mathbb{N}}$ of space-time parts satisfies the local finiteness Assumption 4.10.*
- *Let $\mathfrak{f}$ be a flux function that satisfies the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18.*
- *Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of definite admissibility germs.*

Then, for fixed $\omega \in \Omega$, let the functions $u(\omega, \cdot, \cdot), \tilde{u}(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be two pathwise $\mathfrak{G}$-entropy solutions to Problem (4.1), with initial conditions $u_0(\omega, \cdot), \tilde{u_0}(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{X}; \mathbb{R})$, respectively. Then, for all

nonnegative test functions $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$*, the following Kato inequality*

$$
\begin{aligned}
-\int_{\mathbb{T}}\int_{\mathbb{X}} \left|u(\omega,t,\boldsymbol{x}) - \tilde{u}(\omega,t,\boldsymbol{x})\right| \partial_t \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
- \int_{\mathbb{T}}\int_{\mathbb{X}} \mathbf{q}\big(\omega,t,\boldsymbol{x},u(\omega,t,\boldsymbol{x}),\tilde{u}(\omega,t,\boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
\leq \int_{\mathbb{X}} \left|u_0(\omega,\boldsymbol{x}) - \tilde{u_0}(\omega,\boldsymbol{x})\right| \psi(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}
\end{aligned}
$$

is satisfied. Additionally, if the initial conditions satisfy $\left|u_0(\omega,\boldsymbol{x}) - \tilde{u_0}(\omega,\boldsymbol{x})\right| \in \mathcal{L}^1(\mathbb{X};\mathbb{R})$*, the* $\mathcal{L}^1$*-contraction property*

$$
\int_{\mathbb{X}} \left|u(\omega,t,\boldsymbol{x}) - \tilde{u}(\omega,t,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{X}} \left|u_0(\omega,\boldsymbol{x}) - \tilde{u_0}(\omega,\boldsymbol{x})\right| \mathrm{d}\boldsymbol{x} \tag{4.11}
$$

holds for almost every time $t \in \mathbb{T}$*. In particular, the pathwise* $\mathfrak{G}$*-entropy solution to Problem* (4.1) *is unique, if it exists.* ◆

We can now turn to arguing pathwise existence of random $\mathfrak{G}$-entropy solutions. As in the multi-dimensional sole discontinuity setting, we restrict ourselves to showing the existence of vanishing viscosity solutions[28]. Unfortunately, Definition 4.1 of a (random) compound flux discontinuity does not guarantee sufficient regularity of the discontinuity for the pathwise existence proof to succeed. Therefore, we impose the following regularity assumption on the compound flux discontinuity, which is the straightforward generalization of regularity Assumption 3.36 on sole discontinuity hypersurfaces.

Assumption 4.30 (Regularity of compound flux discontinuity)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{R}^2$ *be a random compound flux discontinuity, which is defined as a locally finite union of sole discontinuities* $\mathfrak{D}_{\mathfrak{i}} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ *according to Definition 4.1. Furthermore, let* $\mathfrak{N}_{\mathfrak{D}}(\omega)$ *denote the number of sole discontinuities* $\mathfrak{D}_{\mathfrak{i}}(\omega)$ *contained in* $\mathfrak{D}(\omega)$*. Then, for every stochastic parameter* $\omega \in \Omega$*, the compound flux discontinuity* $\mathfrak{D}(\omega)$ *satisfies one of the following regularity assumptions:*

(i) For every index $\mathfrak{i} \in \mathfrak{N}_{\mathfrak{D}}(\omega)$*, the function* $\Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega,\cdot,\cdot)$*, whose graph defines* $\mathfrak{D}_{\mathfrak{i}}(\omega)$ *and which is defined by Equation* (3.2)*, satisfies* $\Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega,\cdot,\cdot) \in \mathcal{C}^2(\mathbb{T} \times \mathbb{R}^{d-1};\mathbb{R})$*.*

(ii) For every $\mathfrak{i} \in \mathfrak{N}_{\mathfrak{D}}(\omega)$*, the function* $\Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega,\cdot,\cdot)$*, whose graph defines* $\mathfrak{D}_{\mathfrak{i}}(\omega)$*, is (globally) Lipschitz continuous with respect to* $(t,\boldsymbol{x}_{2:d}) \in \mathbb{T} \times \mathbb{R}^{d-1}$ *and additionally* $\Delta_{\boldsymbol{x}_{2:d}} \Phi_{\mathfrak{i}}^{\mathfrak{D}}(\omega,\cdot,\cdot)$ *is a Radon measure on the lower-dimensional space-time domain* $\mathbb{T} \times \mathbb{R}^{d-1}$*.* ◆

With this regularity assumption at hand, we can formulate the pathwise existence result for random $\mathfrak{G}_{\mathrm{VV}}$-entropy solutions. Since the argumentation is completely similar to proving the pathwise existence Theorem 3.37 for sole flux discontinuities, we omit the proof. Stating this result also concludes the investigation of pathwise existence and uniqueness of random $\mathfrak{G}$-entropy solutions to the scalar conservation law given by Equation (4.1).

Theorem 4.31 (Pathwise existence of $\mathfrak{G}_{\mathrm{VV}}$-entropy solutions)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ *be a random compound flux discontinuity that satisfies the regularity Assumption*

[28] For a compound flux discontinuity, the vanishing viscosity problem is defined completely similar to the one with a sole discontinuity. We refer to Equation (3.21) and Definition 3.35 for the details.

4.30. Furthermore, let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$ *be a random initial condition to Problem* (4.1) *and let the flux function* $\mathfrak{f}$ *satisfy the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18. For a fixed stochastic parameter* $\omega \in \Omega$*, let* $\{u^\eta\}_{\eta>0}$ *be a sequence of solutions to the random vanishing viscosity problem*

$$\partial_t u^\eta + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, \mathfrak{x}, u^\eta) = \eta \, \Delta \, u^\eta \,, \tag{4.12}$$

which is bounded in $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ *and satisfies* $u^\eta\big|_{t=0} = u_0^\eta$ *and* $u_0^\eta \to u_0$ *in* $\mathcal{L}^1_{\mathrm{loc}}$*.*
Then, for vanishing viscosity $\eta \searrow 0$*, the sequence* $\{u^\eta\}_{\eta>0}$ *of solutions converges almost everywhere on* $\mathbb{X}_\mathbb{T}$ *to the unique pathwise* $\mathfrak{G}_{\mathrm{VV}}(\omega)$*-entropy solution of the scalar conservation law given by Equation* (4.1)*.* ◆

4.3.2 Random $\mathfrak{G}$-entropy functionals

In this section, we discuss the random $\mathfrak{G}$-entropy functionals for the scalar conservation law given by Equation (4.1), which has a random compound flux discontinuity. Recall that these functionals are a tool of major importance for showing the measurability of random $\mathfrak{G}$-entropy solutions, since they allow us to measure, whether a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is satisfying the Kružkov entropy condition.

The underlying ideas and the particular construction of these functionals were extensively discussed in Section 3.3.2 for the sole flux discontinuity setting. While the ideas and the conceptional construction of random $\mathfrak{G}$-entropy functionals are completely similar in the sole discontinuity setting and the compound flux discontinuity case, the particular definition needs to be adapted to the generalized adapted Kružkov entropy inequality (4.8).

Definition 4.32 (Random $\mathfrak{G}$-entropy functional)**:**
Let the following requirements be satisfied:

- *Let* $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$*, with* $1 \leq q < \infty$ *and* $1 \leq p \leq \infty$*, be a random initial condition to Problem* (4.1)*.*
- *Let* $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ *be a random compound flux discontinuity that satisfies Assumption 4.4 on the existence of a partition* $\mathfrak{C}_\mathfrak{D} = \{\mathfrak{C}_\mathfrak{D}^\kappa\}_{\kappa \in \mathbb{N}}$*, which itself satisfies the local finiteness Assumption 4.5.*
- *Let* $\{\mathbb{X}_\mathbb{T}^s\}_{s \in \mathbb{N}}$ *be a family of space-time parts that satisfy the local finiteness Assumption 4.10.*
- *Let* $\mathfrak{f}$ *be a flux function that satisfies the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18.*
- *Let* $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ *be a family of random admissibility germs associated to the flux function* $\mathfrak{f}$*, which satisfies the joint measurability Assumption 4.17.*
- *Let* $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa \in \mathbb{N}}$ *be a remainder function associated to the family of germs* $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ *that satisfies the joint measurability Assumption 4.23.*

Furthermore, let a Kružkov entropy sequence $\boldsymbol{k} \in \mathbb{R}^{\leq \mathbb{N}}$ *and a nonnegative test function* $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ *be fixed. Then, we define the* random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ *associated to Problem* (4.1) *as a mapping*

$\mathbb{J}_\psi^{\boldsymbol{k}} : \Omega \times \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R}) \to \mathbb{R}$ *given by*

$$(\omega,\nu) \mapsto \int_{\mathbb{T}}\int_{\mathbb{X}} \big|\nu(t,\boldsymbol{x}) - k(\omega,t,\boldsymbol{x})\big|\partial_t\psi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{4.13a}$$

$$+ \int_{\mathbb{T}}\int_{\mathbb{X}} \mathbf{q}\big(\omega,t,\boldsymbol{x};\nu(t,\boldsymbol{x}),k(\omega,t,\boldsymbol{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{4.13b}$$

$$- \int_{\mathbb{X}} \big|u_0(\omega,\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{4.13c}$$

$$+ \sum_{\kappa\in\mathbb{N}} \int_{\mathfrak{C}_{\mathfrak{D}}^\kappa(\omega)} \mathfrak{R}_{\mathfrak{G}}^\kappa\big(\omega,\mathfrak{d};s_{\boldsymbol{k}}(\omega,\kappa,\boldsymbol{k})\big)\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d}\,. \tag{4.13d}$$

Here, k denotes the adapted Kružkov entropy associated to the entropy sequence $\boldsymbol{k}$ as defined in Definition 4.20 and $\mathbf{q}$ denotes the Kružkov entropy flux introduced in Definition 4.21. Furthermore, $s_{\boldsymbol{k}}$ is the selection function for the Kružkov entropy sequence as in Definition 4.24. ◆

The remainder of this section is devoted to showing two results: First, we show that the entropy functional is Carathéodory in the sense that it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous in the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Afterwards, we show that the entropy functional depends continuously on the Kružkov entropy sequence $\boldsymbol{k} \in \mathbb{R}^{\leq\mathbb{N}}$.

Theorem 4.33 (Entropy functional is Carathéodory)**:**
Let the following conditions be satisfied:

- *Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem* (4.1).
- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity that satisfies the stochastic measurability Assumption 4.3 and Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_{\mathfrak{D}} = \{\mathfrak{C}_{\mathfrak{D}}^\kappa\}_{\kappa\in\mathbb{N}}$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let the family $\{\mathbb{X}_{\mathbb{T}}^s\}_{s\in\mathbb{N}}$ of space-time parts satisfy the local finiteness Assumption 4.10.*
- *Let $\mathfrak{f}$ be a flux function that satisfies the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the stochastic measurability Assumption 4.27.*
- *Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of $\mathcal{L}^1D$ germs satisfying the joint measurability Assumption 4.17.*
- *Let $\{\mathfrak{R}_{\mathfrak{G}}^\kappa\}_{\kappa\in\mathbb{N}}$ be a remainder function associated to the family of germs $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ that satisfies the joint measurability Assumption 4.23.*

Furthermore, let a Kružkov entropy sequence $\boldsymbol{k} \in \mathbb{R}^{\leq\mathbb{N}}$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. Then, the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ is Carathéodory, i.e., it is measurable in $\omega \in \Omega$ and continuous in the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. ◆

Proof. Many arguments of proving that the random $\mathfrak{G}$-entropy functional is Carathéodory are analogous to the similar result for a sole flux discontinuity, which was stated and proven in Theorem 3.45. To account for this similarity of the proofs, we do not provide all the details, but highlight the

main differences of the two proofs. Therefore, the stochastic measurability of $\mathbb{J}_\psi^{\boldsymbol{k}}$ and the continuous dependence of the random $\mathfrak{G}$-entropy functional on $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ are considered separately.

Continuity in $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. Let $\omega \in \Omega$ be fixed. Proving that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is completely analogous to the sole discontinuity case: The continuity of the Terms (4.13a), (4.13c) and (4.13d) is obvious and the continuous dependence of Integral (4.13b) follows from a case study analogous to Proposition 3.44. For the details, we refer to the proof of Proposition 3.44.

Measurability in $\omega \in \Omega$. Let a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. Recall that the adapted Kružkov entropy k associated to the sequence $\boldsymbol{k} \in \mathbb{R}^{\leq \mathbb{N}}$ is measurable in $\omega \in \Omega$ by Proposition 4.26. Thereby, the integrand of the Integral (4.13a) is stochastically measurable. Since the test function ψ is compactly supported, we can restrict the integral to $\operatorname{supp} \psi$. Due to this compactness of the support $\operatorname{supp} \psi$, taking the integral is a bounded linear operator. Thus, the integration is a continuous operation. Consequently, this shows the stochastic measurability of Integral (4.13a), since the composition of a continuous operation and a measurable function is again measurable [5, Lemma 4.22].

A similar argument yields the stochastic measurability of Integral (4.13c): Since the initial condition u_0 is stochastically measurable by the hypothesis $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$ and the adapted Kružkov entropy k associated to the sequence $\boldsymbol{k} \in \mathbb{R}^{\leq \mathbb{N}}$ is measurable by Proposition 4.26, the integrand of Term (4.13c) is measurable in the stochastic parameter $\omega \in \Omega$. Restricting the integral domain to the compact support of the test function ψ proves the measurability of Integral (4.13c).

Stochastic measurability of (4.13b). To show the stochastic measurability of the Integral (4.13b), recall that by Convention (4.3) we have defined the space-time parts $\mathbb{X}_\mathbb{T}^s(\omega)$ to satisfy $\mathbb{X}_\mathbb{T}^s(\omega) = \emptyset$ if the index $s \in \mathbb{N}$ exceeds the (random) number of domain parts, i.e., $s > \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$. This allows us to reformulate Term (4.13b) as

$$\begin{aligned} \int_\mathbb{T} \int_\mathbb{X} \mathbf{q}\big(\omega, t, \boldsymbol{x}; \nu(t,\boldsymbol{x}), k(\omega,t,\boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \\ = \sum_{s \in \mathbb{N}} \int_{\mathbb{X}_\mathbb{T}^s(\omega)} \mathbf{q}^s\big(\omega, \mathfrak{x}, \nu(\mathfrak{x}), k(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \,. \end{aligned} \tag{4.14}$$

Here, we divided the integral over the space-time domain $\mathbb{X}_\mathbb{T} = \mathbb{T} \times \mathbb{X}$ into a sum over the domain parts $\mathbb{X}_\mathbb{T}^s(\omega)$ resulting from the random compound flux discontinuity $\mathfrak{D}(\omega)$. Additionally, we replaced the tuple $(t, \boldsymbol{x}) \in \mathbb{X}_\mathbb{T}$ by the generic space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. However, each integral in the sum of Equation (4.14) can be written as

$$\int_{\mathbb{X}_\mathbb{T}^s(\omega)} \mathbf{q}^s\big(\omega, \mathfrak{x}, \nu(\mathfrak{x}), k(\omega,\mathfrak{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} = \int_{\mathbb{X}_\mathbb{T}} \Big(\mathbf{q}^s\big(\omega, \mathfrak{x}, \nu(\mathfrak{x}), k(\omega,\mathfrak{x})\big) \mathbb{1}_{\mathbb{X}_\mathbb{T}^s}(\omega,\mathfrak{x})\Big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \,, \tag{4.15}$$

where $\mathbb{1}_{\mathbb{X}_\mathbb{T}^s}$ denotes the indicator function of the space-time part $\mathbb{X}_\mathbb{T}^s$. By Lemma 4.13, the indicator function $\mathbb{1}_{\mathbb{X}_\mathbb{T}^s}$ is separately measurable. Recall that by Proposition 4.28 the entropy fluxes $\mathbf{q}^s$ are Carathéodory in the sense that they are measurable in the stochastic parameter $\omega \in \Omega$ and continuous in all remaining arguments. In particular, this means that the entropy flux functions $\mathbf{q}^s$ are jointly measurable by [5, Lemma 4.51]. Since the adapted Kružkov entropy k is stochastically measurable by Proposition 4.26, the integrand of Equation (4.15) is measurable. By restricting the integral domain on the right-hand side of

Equation (4.15) to the compact support of the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T}; \mathbb{R})$, we can write the Integral (4.13b) as

$$\int_{\mathbb{T}} \int_{\mathbb{X}} \mathbf{q}\big(\omega, t, \boldsymbol{x}; \nu(t,\boldsymbol{x}), k(\omega,t,\boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t$$
$$= \sum_{s\in\mathbb{N}} \int_{\operatorname{supp}\psi} \Big(\mathbf{q}^s\big(\omega, \mathfrak{x}, \nu(\mathfrak{x}), k(\omega,\mathfrak{x})\big) \mathbb{1}_{\mathbb{X}^s_\mathbb{T}}(\omega,\mathfrak{x})\Big) \cdot \nabla_{\boldsymbol{x}} \psi(\mathfrak{x}) \,\mathrm{d}\mathfrak{x} \,.$$

Since the support of the test function ψ is compact, each integral is measurable as the composition of a continuous operation with a measurable function by [5, Lemma 4.22]. Leveraging the compact support of the test function ψ, the local finiteness Assumption 4.10 on the domain parts guarantees that the sum contains finitely many nonzero values. This proves that the Integral (4.13b) is stochastically measurable because the finite sum of random variables is again a random variable and thus measurable.

Stochastic measurability of (4.13d). It remains to prove the stochastic measurability of the sum of integrals in Term (4.13d). By Remark 3.30, we can identify the differential $\mathrm{d}\mathfrak{d}$ with the d-dimensional Hausdorff measure $\mathcal{H}^d$, which yields

$$\sum_{\kappa\in\mathbb{N}} \int_{\mathfrak{C}^\kappa_\mathfrak{D}(\omega)} \mathfrak{R}^\kappa_\mathfrak{G}\big(\omega, \mathfrak{d}; \mathcal{s}_{\boldsymbol{k}}(\omega,\kappa,\boldsymbol{k})\big)\psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d} = \sum_{\kappa\in\mathbb{N}} \int_{\mathfrak{C}^\kappa_\mathfrak{D}(\omega)} \mathfrak{R}^\kappa_\mathfrak{G}\big(\omega, \mathfrak{d}; \mathcal{s}_{\boldsymbol{k}}(\omega,\kappa,\boldsymbol{k})\big)\psi(\mathfrak{d}) \,\mathrm{d}\mathcal{H}^d(\mathfrak{d}) \,. \tag{4.16}$$

Here, the integral domain $\mathfrak{C}^\kappa_\mathfrak{D}(\omega)$ is the κ-th part of the partition up to a null set $\mathfrak{C}_\mathfrak{D}(\omega)$ of the random compound flux discontinuity $\mathfrak{D}(\omega)$. This partition $\mathfrak{C}_\mathfrak{D}$ exists due to Assumption 4.4.

By the measurability Assumption 4.17, the family $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ of admissibility germs is jointly measurable in the sense that each germ $\mathfrak{G}_\kappa : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is jointly measurable. Therefore, also each remainder function $\mathfrak{R}^\kappa_\mathfrak{G}$ is jointly measurable by Assumption 4.23. Furthermore, the Assumption 4.8 on the existence of a parametrization of the partition up to a null set implies that, for every index $\kappa \in \mathbb{N}$, the partition part $\mathfrak{C}^\kappa_\mathfrak{D}$ is separately measurable. On the other hand, this stochastic measurability of the partition part $\mathfrak{C}^\kappa_\mathfrak{D}$ implies by the construction of the selection function $\mathcal{s}_{\boldsymbol{k}}$ of the Kružkov entropy sequence in Definition 4.24 that the function $\mathcal{s}_{\boldsymbol{k}}$ is measurable in the stochastic parameter $\omega \in \Omega$. Consequently, we have established the stochastic measurability of the integrands in Equation (4.16).

Now, for the partition up to a null set $\mathfrak{C}_\mathfrak{D}$, Assumption 4.8 guarantees the existence of a parametrization $\boldsymbol{P}_{\mathfrak{C}^\kappa_\mathfrak{D}} : \Omega \times \mathbb{T} \times \mathbb{R}^{d-1} \to \mathbb{T} \times \mathbb{X}$ of the partition part $\mathfrak{C}^\kappa_\mathfrak{D}$. Since the selection function $\mathcal{s}_{\boldsymbol{k}}$ does not depend on the spatio-temporal point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, we can use an analogous argumentation as in the proof of Proposition 3.43 to show that each integral is stochastically measurable.
Finally, due to the locally finiteness Assumption 4.5 on the partition $\mathfrak{C}_\mathfrak{D}$, the sum over $\kappa \in \mathbb{N}$ consists of only finitely many terms, which are nonzero. This follows directly from the fact that only finitely many partition parts $\mathfrak{C}^\kappa_\mathfrak{D}$ intersect with the compact support of the test function ψ. However, a finite sum of random variables is again a random variable and thus in particular measurable, which concludes the stochastic measurability of Term (4.13d).

Combining the measurability result of the random $\mathfrak{G}$-entropy functional $\mathbb{J}^{\boldsymbol{k}}_\psi$ with respect to the stochastic parameter $\omega \in \Omega$ and its continuous dependence on the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, we have proven the assertion that the random $\mathfrak{G}$-entropy functional $\mathbb{J}^{\boldsymbol{k}}_\psi$ is Carathéodory. ■

With the preceeding result that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ is Carathéodory, it remains to show the continuous dependence of $\mathbb{J}_\psi^{\boldsymbol{k}}$ on the Kružkov entropy sequence $\boldsymbol{k} \in \mathbb{R}^{\leq \mathbb{N}}$ to complete this section on entropy functionals. Before we move on to the corresponding result, recall that the pathwise existence result of $\mathfrak{G}_{\mathrm{VV}}$-entropy solutions requires the sequence $\{u^\eta\}_{\eta>0}$ of solutions to the vanishing viscosity problem (4.12) to be uniformly bounded in $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$. Also, if a random $\mathfrak{G}$-entropy solution is confined to an interval $\mathbb{U} = [\underline{\mathfrak{u}}, \overline{\mathfrak{u}}] \subset \mathbb{R}$, it is sufficient to consider the Kružkov entropy $k \in \mathbb{R}$ to be an element of this interval, i.e., it suffices to have $k \in \mathbb{U}$. This can easily be seen, since the two functions $u \equiv \underline{\mathfrak{u}}$ and $u \equiv \overline{\mathfrak{u}}$ are sub- and supersolutions to the scalar conservation law, Problem (4.1), respectively. Therefore, analogously to considering $\boldsymbol{k} \in \mathbb{U}^2$ in the sole discontinuity case, it appears most naturally to require the sequence $\boldsymbol{k} \in \mathbb{R}^{\leq \mathbb{N}}$ of Kružkov entropy values to be bounded in the sense that it satisfies $\boldsymbol{k} \in \ell^\infty(\mathbb{U})$. To achieve this boundedness of the Kružkov entropy sequence, we impose the following assumption.

Assumption 4.34 (Boundedness of entropy sequence)**:**
We assume that one of the following two conditions is satisfied, which both lead to the Kružkov entropy sequence being a bounded sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$:

(i) For every stochastic parameter $\omega \in \Omega$, the random compound flux discontinuity $\mathfrak{D}(\omega)$ divides the space-time domain $\mathbb{X}_\mathbb{T}$ in finitely many domain parts $\mathbb{X}_\mathbb{T}^s(\omega)$.

(ii) For every stochastic parameter $\omega \in \Omega$, the random $\mathfrak{G}$-entropy solution is confined to a deterministic interval $\mathbb{U} \subset \mathbb{R}$ (e.g., via the confinement Assumption 3.38). ◆

For the first condition, we can identify the finite entropy sequence $\boldsymbol{k} = \{k^{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)}$ with the sequence $\boldsymbol{k} = \{k^{\mathfrak{i}}\}_{\mathfrak{i}\in\mathbb{N}}$ such that $k^{\mathfrak{i}} = 0$, if $\mathfrak{i} > \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$. Thereby, we obtain $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. For the second condition, the boundedness of the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ is obvious. Another nice consequence of this boundedness Assumption 4.34 is the availability of the continuous norm $\|\cdot\|_{\ell^\infty(\mathbb{R})}$ of the sequence space $\ell^\infty(\mathbb{R})$ rather than having to work with the (product) topology of the space $\mathbb{R}^{\leq\mathbb{N}}$, which does not admit any continuous norm.

As in the sole discontinuity setting, we also need to impose an integrability assumption on the family $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa\in\mathbb{N}}$, since otherwise there is no hope for showing continuous dependence of the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. Recall that the remainder function $\mathfrak{R}_\mathfrak{G}$ can take various forms, which might depend on the underlying flux function $\mathfrak{f}$. Therefore, this assumption is a simplifying one, since we do not impose a specific form of the remainder function $\mathfrak{R}_\mathfrak{G}$. However, the assumption is verified for particular choices of the remainder function in Appendix A.

Assumption 4.35 (Integrability of family of remainder functions)**:**
Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of random admissibility germs and let $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa\in\mathbb{N}}$ be the associated family of remainder functions. For any index $\kappa \in \mathbb{N}$, we assume that the function $\mathfrak{R}_\mathfrak{G}^\kappa$ satisfies that for each compact set $K \subset \mathbb{R}^2$ and fixed stochastic parameter $\omega \in \Omega$, the function

$$\mathcal{M}_{\kappa,K}^{\mathfrak{R}}(\omega, \mathfrak{x}) := \sup_{\boldsymbol{k}\in K} |\mathfrak{R}_\mathfrak{G}^\kappa(\omega, \mathfrak{x}; \boldsymbol{k})|$$

is locally Lebesgue integrable in the sense that $\mathcal{M}_{\kappa,K}^{\mathfrak{R}}(\omega, \cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_\mathbb{T}; \mathbb{R})$. ◆

With this boundedness Assumption 4.34 and the integrability Assumption 4.35 at hand, we can now argue that the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. Showing this result in the subsequent theorem concludes the section on random $\mathfrak{G}$-entropy functionals.

Theorem 4.36 (Entropy functional depends continuously on entropy sequence)**:**
Let the following requirements be satisfied:

- *Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem* (4.1).
- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies Assumption 4.4 on the existence of a partition $\mathfrak{C}_\mathfrak{D} = \{\mathfrak{C}_\mathfrak{D}^\kappa\}_{\kappa\in\mathbb{N}}$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let the family $\{\mathbb{X}_\mathbb{T}^s\}_{s\in\mathbb{N}}$ of domain parts satisfy the local finiteness Assumption 4.10.*
- *Let the boundedness Assumption 4.34 on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ be satisfied.*
- *Let the flux function $\mathfrak{f}$ satisfy the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the measurability Assumption 4.27.*
- *Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of $\mathcal{L}^1D$ germs satisfying the joint measurability Assumption 4.17.*
- *Let $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa\in\mathbb{N}}$ be a family of remainder functions associated to the germs $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ that satisfies the joint measurability Assumption 4.23 and the integrability Assumption 4.35.*

Furthermore, let the parameter $\omega \in \Omega$, a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ and a function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. Then, the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. ◆

Proof. Let the stochastic parameter $\omega \in \Omega$, a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ and a function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. We show the continuous dependence of the random $\mathfrak{G}$-entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ by considering each term of the mapping (4.13) separately. Therefore, let $\boldsymbol{k}, \boldsymbol{c} \in \ell^\infty(\mathbb{R})$ be two Kružkov entropy sequences and denote by k, c the corresponding adapted Kružkov entropies as defined in Definition 4.20.

Continuous dependence of Integral (4.13a)**.** For the first integral of the random $\mathfrak{G}$-entropy functional definition, given by Term (4.13a), we can do the following estimation:

$$\begin{aligned}
&\Big| \int_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big|\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big|\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \\
&\qquad = \Big| \int_{\mathbb{X}_\mathbb{T}} \Big(\big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big| - \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big|\Big)\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \\
&\qquad \leq \int_{\mathbb{X}_\mathbb{T}} \Big|\big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big| - \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big|\Big|\big|\partial_t\psi(\mathfrak{x})\big|\,\mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

Applying the reverse triangle inequality to the integrand of this estimation and noting that the adapted Kružkov entropies k, c satisfy $|k(\omega,\mathfrak{x}) - c(\omega,\mathfrak{x})| \leq \|\boldsymbol{k} - \boldsymbol{c}\|_\infty$ for every space-time point $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, we

can further estimate

$$\Big| \int\limits_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big|\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x} - \int\limits_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big|\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq \int\limits_{\mathbb{X}_\mathbb{T}} \big|c(\omega,\mathfrak{x}) - k(\omega,\mathfrak{x})\big||\partial_t\psi(\mathfrak{x})|\,\mathrm{d}\mathfrak{x}$$
$$\leq \int\limits_{\mathbb{X}_\mathbb{T}} \|\boldsymbol{k} - \boldsymbol{c}\|_\infty |\partial_t\psi(\mathfrak{x})|\,\mathrm{d}\mathfrak{x}\,.$$

Since the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$ is smooth and compactly supported, its time derivative $\partial_t\psi$ is bounded. Therefore, we can conclude

$$\Big| \int_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big|\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x} - \int_{\mathbb{X}_\mathbb{T}} \big|\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big|\partial_t\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big| \leq C_\psi \|\boldsymbol{k} - \boldsymbol{c}\|_\infty\,,$$

which proves the continuous dependence of Integral (4.13a) on the sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$.

Continuous dependence of Integral (4.13b). We continue by showing the continuous dependence of Integral (4.13b) on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. Due to the locally finiteness Assumption 4.10 on the domain parts $\big\{\mathbb{X}_\mathbb{T}^s(\omega)\big\}_{s\in\mathbb{N}}$, this family is countable. Recall that Convention (4.3) yields $\mathbb{X}_\mathbb{T}^s(\omega) = \emptyset$, if $s > \mathfrak{N}_{\mathbb{X}_\mathbb{T}}(\omega)$. This allows us to split the integral over the whole space-time domain $\mathbb{X}_\mathbb{T}$ into the individual integrals over each space-time part $\mathbb{X}_\mathbb{T}^s(\omega)$:

$$\int\limits_{\mathbb{X}_\mathbb{T}} \mathbf{q}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x} = \sum_{s\in\mathbb{N}} \int\limits_{\mathbb{X}_\mathbb{T}^s(\omega)} \mathbf{q}^s\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\,.$$

With this representation of the integral, we can consider the difference between the above term corresponding to the two entropy sequences $\boldsymbol{k},\boldsymbol{c} \in \ell^\infty(\mathbb{R})$ by applying first Hölder's inequality and the Cauchy-Schwarz-Bunyakovsky inequality thereafter to obtain

$$\Big|\sum_{s\in\mathbb{N}} \int\limits_{\mathbb{X}_\mathbb{T}^s(\omega)} \mathbf{q}^s\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x} - \sum_{s\in\mathbb{N}} \int\limits_{\mathbb{X}_\mathbb{T}^s(\omega)} \mathbf{q}^s\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big|$$
$$\leq \sum_{s\in\mathbb{N}} \int\limits_{\mathbb{X}_\mathbb{T}^s(\omega)} \Big\|\mathbf{q}^s\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big) - \mathbf{q}^s\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_d \,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_d\,\mathrm{d}\mathfrak{x}\,.$$

For any stochastic parameter $\omega \in \Omega$ and any space-time part index $s \in \mathbb{N}$, we can now introduce four sets ${}^s\mathbb{D}^{(\cdot)}(\omega)$ that depend on the terms $\operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big)$ and $\operatorname{sign}\big(\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big)$ for space-time variable $\mathfrak{x} \in \mathbb{X}_\mathbb{T}^s(\omega)$ as

$$\begin{aligned}
{}^s\mathbb{D}^{=}(\omega) &\coloneqq \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T}^s(\omega) \mid \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big) = \operatorname{sign}\big(\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big)\big\},\\
{}^s\mathbb{D}^{\pm}(\omega) &\coloneqq \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T}^s(\omega) \mid \operatorname{sign}\big(\nu(\mathfrak{x}) - k(\omega,\mathfrak{x})\big) = -\operatorname{sign}\big(\nu(\mathfrak{x}) - c(\omega,\mathfrak{x})\big)\big\},\\
{}^s\mathbb{D}_k^{\nu}(\omega) &\coloneqq \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T}^s(\omega) \mid \nu(\mathfrak{x}) = k(\omega,\mathfrak{x})\big\},\\
{}^s\mathbb{D}_c^{\nu}(\omega) &\coloneqq \big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T}^s(\omega) \mid \nu(\mathfrak{x}) = c(\omega,\mathfrak{x})\big\}.
\end{aligned}$$

With these sets, we are able to split up the integration over the space-time domain parts $\mathbb{X}_\mathbb{T}^s(\omega)$ even further. Even though these sets are not necessarily disjoint, note that the entropy flux function $\mathbf{q}^s$ vanishes on the intersection, which is given by

$${}^s\mathbb{D}^{=}(\omega) \cap {}^s\mathbb{D}^{\pm}(\omega) \cap {}^s\mathbb{D}_k^{\nu}(\omega) \cap {}^s\mathbb{D}_c^{\nu}(\omega) = \Big\{\mathfrak{x} \in \mathbb{X}_\mathbb{T}^s(\omega) \mid k(\omega,\mathfrak{x}) = \nu(\mathfrak{x}) = c(\omega,\mathfrak{x})\Big\}\,.$$

We continue our investigation by splitting up the integral over the space-time domain parts by employing

the four sets, we just introduced. As a result, we obtain the following estimation

$$\begin{aligned}
&\Big|\sum_{s\in\mathbb{N}}\int_{\mathbb{X}_{\mathbb{T}}^{s}(\omega)}\boldsymbol{\mathfrak{q}}^{s}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}-\sum_{s\in\mathbb{N}}\int_{\mathbb{X}_{\mathbb{T}}^{s}(\omega)}\boldsymbol{\mathfrak{q}}^{s}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\cdot\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\,\mathrm{d}\mathfrak{x}\Big|\\
&\qquad\leq\sum_{s\in\mathbb{N}}\Big(\int_{{}^{s}\mathbb{D}^{=}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x} &&\text{(4.17a)}\\
&\qquad\quad+\int_{{}^{s}\mathbb{D}^{\pm}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x} &&\text{(4.17b)}\\
&\qquad\quad+\int_{{}^{s}\mathbb{D}_{k}^{\nu}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x} &&\text{(4.17c)}\\
&\qquad\quad+\int_{{}^{s}\mathbb{D}_{c}^{\nu}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}\Big)\,. &&\text{(4.17d)}
\end{aligned}$$

As in the proof of Theorem 3.47, we employ a function ${}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}$ that describes the difference of the two entropy flux functions via the mapping

$${}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}(k,c)\coloneqq\boldsymbol{\mathfrak{q}}^{s}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),k(\omega,\mathfrak{x})\big)-\boldsymbol{\mathfrak{q}}^{s}\big(\omega,\mathfrak{x};\nu(\mathfrak{x}),c(\omega,\mathfrak{x})\big)\,.$$

To proceed with the estimation, the four integrals in Inequality (4.17) are considered separately:

Continuous dependence of Integral (4.17a). With the definitions of the function ${}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}$ and the entropy flux function $\boldsymbol{\mathfrak{q}}^{s}$, the definition of the integral set ${}^{s}\mathbb{D}^{=}(\omega)$ leads to the estimation

$$\begin{aligned}
&\int_{{}^{s}\mathbb{D}^{=}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}\\
&\qquad\leq\int_{{}^{s}\mathbb{D}^{=}(\omega)}\Big\|\mathfrak{f}^{s}\big(\omega,\mathfrak{x},c(\omega,\mathfrak{x})\big)-\mathfrak{f}^{s}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

Due to Assumption (C-2), the flux function $\mathfrak{f}^{s}$ is locally Lipschitz continuous. Furthermore, for any space-time point $\mathfrak{x}\in\mathbb{X}_{\mathbb{T}}$, the adapted Kružkov entropies k,c satisfy $|k(\omega,\mathfrak{x})-c(\omega,\mathfrak{x})|\leq\|\boldsymbol{k}-\boldsymbol{c}\|_{\infty}$. This already concludes the continuous dependence of Integral (4.17a) on the Kružkov entropy sequence $\boldsymbol{k}\in\ell^{\infty}(\mathbb{R})$, since we obtain the following estimation:

$$\begin{aligned}
\int_{{}^{s}\mathbb{D}^{=}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}&\leq L_{\mathfrak{f}^{s}}\|\boldsymbol{k}-\boldsymbol{c}\|_{\infty}\int_{{}^{s}\mathbb{D}^{=}(\omega)}\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}\\
&\leq C_{\psi}L_{\mathfrak{f}^{s}}\|\boldsymbol{k}-\boldsymbol{c}\|_{\infty}\,.
\end{aligned}$$

Here, the last estimation is possible, since the test function ψ is smooth and compactly supported, which implies that its gradient $\nabla_{\boldsymbol{x}}\,\psi$ is bounded.

Continuous dependence of Integral (4.17b). To show the continuous dependence of Integral (4.17b), we exploit the definitions of the function ${}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}$ and the entropy flux function $\boldsymbol{\mathfrak{q}}^{s}$. Thereby, with the definition of the integral set ${}^{s}\mathbb{D}^{\pm}(\omega)$ we can establish the estimation

$$\begin{aligned}
&\int_{{}^{s}\mathbb{D}^{\pm}(\omega)}\Big\|{}^{s}\boldsymbol{\mathfrak{Q}}_{\omega,\mathfrak{x}}^{\nu}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}\\
&\qquad\leq\int_{{}^{s}\mathbb{D}^{\pm}(\omega)}\Big\|2\mathfrak{f}^{s}\big(\omega,\mathfrak{x},\nu(\mathfrak{x})\big)-\mathfrak{f}^{s}\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)-\mathfrak{f}^{s}\big(\omega,\mathfrak{x},c(\omega,\mathfrak{x})\big)\Big\|_{d}\,\big\|\nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\big\|_{d}\,\mathrm{d}\mathfrak{x}\,.
\end{aligned}$$

With the triangle inequality and the local Lipschitz property of the flux function $\mathfrak{f}^s$, which is ensured by Assumption (C-2), this can be further estimated to obtain

$$\int_{{}^s\mathbb{D}^{\pm}(\omega)} \left\| {}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\right\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x} \\ \leq L_{\mathfrak{f}^s} \int_{{}^s\mathbb{D}^{\pm}(\omega)} \Big(\big|\nu(\mathfrak{x})-k(\omega,\mathfrak{x})\big| + \big|\nu(\mathfrak{x})-c(\omega,\mathfrak{x})\big|\Big) \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x}\,.$$

For any two scalar values $\xi_1,\xi_2\in\mathbb{R}$, the identity $|\xi_1-\xi_2| = \operatorname{sign}(\xi_1-\xi_2)(\xi_1-\xi_2)$ holds. Combining this identity with the construction of the set ${}^s\mathbb{D}^{\pm}(\omega)$ allows us to further estimate

$$\int_{{}^s\mathbb{D}^{\pm}(\omega)} \left\| {}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\right\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x} \\ \leq L_{\mathfrak{f}} \int_{{}^s\mathbb{D}^{\pm}(\omega)} \Big|\operatorname{sign}\big(\nu(\mathfrak{x})-k(\omega,\mathfrak{x})\big)\big(c(\omega,\mathfrak{x})-k(\omega,\mathfrak{x})\big)\Big| \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x}\,.$$

Recall that the adapted Kružkov entropies k,c satisfy $|k(\omega,\mathfrak{x})-c(\omega,\mathfrak{x})|\leq \|\boldsymbol{k}-\boldsymbol{c}\|_\infty$ for every space-time point $\mathfrak{x}\in\mathbb{X}_{\mathbb{T}}$. Furthermore, the gradient of the test function $\nabla_{\boldsymbol{x}}\,\psi$ is compactly supported and bounded. Thus, we obtain the estimation

$$\int_{{}^s\mathbb{D}^{\pm}(\omega)} \left\| {}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\right\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x} \leq C_\psi L_{\mathfrak{f}^s}\|\boldsymbol{k}-\boldsymbol{c}\|_\infty\,,$$

which means that the Integral (4.17b) depends continuously on the sequence $\boldsymbol{k}\in\ell^\infty(\mathbb{R})$.

Continuous dependence of Integral (4.17c). To show the continuous dependence of Integral (4.17c), we start again by exploiting the definitions of ${}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}$ and $\mathfrak{q}^s$. Combined with the definition of the set ${}^s\mathbb{D}^{\nu}_{k}(\omega)$, which guarantees $\nu(\mathfrak{x})=k(\omega,\mathfrak{x})$ for every point $\mathfrak{x}\in{}^s\mathbb{D}^{\nu}_{k}(\omega)$, this leads to the estimation

$$\int_{{}^s\mathbb{D}^{\nu}_{k}(\omega)} \left\| {}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\right\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x} \\ \leq \int_{{}^s\mathbb{D}^{\nu}_{k}(\omega)} \Big\|\mathfrak{f}^s\big(\omega,\mathfrak{x},k(\omega,\mathfrak{x})\big)-\mathfrak{f}^s\big(\omega,\mathfrak{x},c(\omega,\mathfrak{x})\big)\Big\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x}\,.$$

With the local Lipschitz continuity Assumption (C-2) on the flux function $\mathfrak{f}^s$ and the boundedness of the gradient of the test function $\nabla_{\boldsymbol{x}}\,\psi$, we can conclude

$$\int_{{}^s\mathbb{D}^{\nu}_{k}(\omega)} \left\| {}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\right\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x} \leq C_\psi L_{\mathfrak{f}^s}\|\boldsymbol{k}-\boldsymbol{c}\|_\infty\,.$$

Here, we also used the fact that, for every space-time point $\mathfrak{x}\in\mathbb{X}_{\mathbb{T}}$, the adapted Kružkov entropies k,c satisfy $|k(\omega,\mathfrak{x})-c(\omega,\mathfrak{x})|\leq\|\boldsymbol{k}-\boldsymbol{c}\|_\infty$.

Continuous dependence of Integral (4.17d). Proving the continuous dependence of the Integral (4.17d) on $\boldsymbol{k}\in\ell^\infty(\mathbb{R})$ is analogous to the estimation of Integral (4.17c). Ultimately, this calculation yields

$$\int_{{}^s\mathbb{D}^{\nu}_{c}(\omega)} \left\| {}^s\mathfrak{Q}^{\nu}_{\omega,\mathfrak{x}}\big(k(\omega,\mathfrak{x}),c(\omega,\mathfrak{x})\big)\right\|_d \left\| \nabla_{\boldsymbol{x}}\,\psi(\mathfrak{x})\right\|_d \mathrm{d}\mathfrak{x} \leq C_\psi L_{\mathfrak{f}}\Big\|\boldsymbol{c}-\boldsymbol{k}\Big\|_\infty\,,$$

which implies the sought continuous dependence result.

Combining the estimations of the four Integrals (4.17a), (4.17b), (4.17c) and (4.17d) shows that the integrals in Inequality (4.17) depend continuously on the entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. Furthermore, the summation over the index $s \in \mathbb{N}$ consists of finitely many nonzero terms due to the local finiteness Assumption 4.10 on the family of domain parts $\{\mathbb{X}_\mathbb{T}^s\}_{s\in\mathbb{N}}$ and the compact support of the test function ψ. Therefore, the Integral (4.13b) depends continuously on the entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$.

Continuous dependence of Integral (4.13c)**.** Proving continuous dependence of Integral (4.13c) is analogous to showing continuity of Integral (4.13a). Therefore, we estimate

$$\begin{aligned}
&\Big| \int_\mathbb{X} \big|u_0(\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\mathbb{X} \big|u_0(\boldsymbol{x}) - c(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \\
&\qquad = \Big| \int_\mathbb{X} \Big(\big|u_0(\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big| - \big|u_0(\boldsymbol{x}) - c(\omega,0,\boldsymbol{x})\big|\Big)\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \\
&\qquad \leq \int_\mathbb{X} \Big|\big|u_0(\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big| - \big|u_0(\boldsymbol{x}) - c(\omega,0,\boldsymbol{x})\big|\Big|\big|\psi(0,\boldsymbol{x})\big|\,\mathrm{d}\boldsymbol{x}\,.
\end{aligned}$$

Applying the reverse triangle inequality and noting that the adapted Kružkov entropies k, c satisfy $|k(\omega,\mathfrak{x}) - c(\omega,\mathfrak{x})| \leq \|\boldsymbol{k} - \boldsymbol{c}\|_\infty$ for every $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$, we can further estimate

$$\begin{aligned}
&\Big| \int_\mathbb{X} \big|u_0(\boldsymbol{x}) - k(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\mathbb{X} \big|u_0(\boldsymbol{x}) - c(\omega,0,\boldsymbol{x})\big|\psi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \\
&\qquad = \int_\mathbb{X} \|\boldsymbol{k} - \boldsymbol{c}\|_\infty |\partial_t \psi|\,\mathrm{d}\boldsymbol{x} \\
&\qquad = C_\psi \|\boldsymbol{k} - \boldsymbol{c}\|_\infty\,.
\end{aligned}$$

Here, in the last estimation, we used the boundedness of ψ due to its continuity and compact support. Thus, this proves the continuous dependence of Integral (4.13c).

Continuous dependence of Term (4.13d)**.** At this point, it remains to show continuous dependence of Term (4.13d) on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. Recall that $\mathfrak{C}_\mathfrak{D} = \{\mathfrak{C}_\mathfrak{D}^\kappa\}_{\kappa\in\mathbb{N}}$ denotes a partition up to a null set of the random compound flux discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$. By the integrability Assumption 4.35 on the family of remainder functions $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa\in\mathbb{N}}$, for each $\kappa \in \mathbb{N}$ and every $\omega \in \Omega$, there exists a function $\mathcal{M}_{\kappa,K}^\mathfrak{R} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_\mathbb{T};\mathbb{R})$ dominating $\mathfrak{R}_\mathfrak{G}^\kappa$. This leads to the estimation:

$$\sum_{\kappa\in\mathbb{N}} \int_{\mathfrak{C}_\mathfrak{D}^\kappa(\omega)} \mathfrak{R}_\mathfrak{G}^\kappa\big(\omega,\mathfrak{d};s_{\boldsymbol{k}}(\omega,\kappa,\boldsymbol{k})\big)\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d} \leq \sum_{\kappa\in\mathbb{N}} \int_{\mathfrak{C}_\mathfrak{D}^\kappa(\omega)} \mathcal{M}_{\kappa,K}^\mathfrak{R}(\omega,\mathfrak{d})\psi(\mathfrak{d})\,\mathrm{d}\mathfrak{d}\,. \tag{4.18}$$

Now, each integral on the right-hand side of Estimation (4.18) is finite, since the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$ is compactly supported and each function $\mathcal{M}_{\kappa,K}^\mathfrak{R} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_\mathbb{T};\mathbb{R})$ is locally integrable. Additionally, by the local finiteness Assumption 4.5, the partition up to a null set $\mathfrak{C}_\mathfrak{D}(\omega)$ is locally finite in the sense that any compact set intersects only with finitely many partition parts $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$. Therefore, the summation contains only finitely many partition parts $\mathfrak{C}_\mathfrak{D}^\kappa(\omega)$, which intersect with the compact support of the test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T};\mathbb{R})$. Consequently, the sum is over finitely many terms and therefore also finite. Thereby, we can apply the dominated convergence theorem to deduce that the term (4.13d) depends continuously on the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$.

Combining the previous investigations, we have shown that the entropy functional $\mathbb{J}_\psi^{\boldsymbol{k}}$ depends continuously on the entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. ■

4.3.3 Measurability of random entropy solutions

With the random $\mathfrak{G}$-entropy functionals and their properties at hand, we can now turn to investigating whether random $\mathfrak{G}$-entropy solutions are (strongly) measurable. To show this measurability, it is natural to demand the pathwise existence and uniqueness of such solutions. Analogously to the discussion of measurability in Chapter 3, we want to avoid specific assumptions that are only necessary for particular choices of admissibility germs. Also, for being able to show strong measurability, we need the random $\mathfrak{G}$-entropy solutions to take values of a separable subspace $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$ only. To achieve this separability and avoiding assumptions that are not necessary in general, we impose the following assumption on the pathwise existence and uniqueness of $\mathfrak{G}$-entropy solutions.

Assumption 4.37 (Pathwise existence of unique $\mathfrak{G}$-entropy solution)**:**
For every stochastic parameter $\omega \in \Omega$, there exists a unique $\mathfrak{G}$-entropy solution to the random scalar conservation law given by Equation (4.1) *in the sense of Definitions 4.19 and 4.25. Furthermore, the pathwise $\mathfrak{G}$-entropy solution $u(\omega,\cdot,\cdot)$ is assumed to be separably-valued, i.e., the function $u(\omega,\cdot,\cdot)$ only takes values in a separable subspace $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_\mathbb{T};\mathbb{R})$.* ◆

Let us note that this assumption is completely analogous to the one in Chapter 3 for sole flux discontinuities. While assuming that the random $\mathfrak{G}$-entropy solution is separably-valued might seem restrictive, such separability is most natural in many situations. We refer to Example 3.49 for details.

We have now gathered every ingredient that we need to show strong measurability of the random $\mathfrak{G}$-entropy solution to Problem (4.1). This result allows us to interpret the $\mathfrak{G}$-entropy solution u as an $\mathcal{S}$-valued Bochner-integrable random variable.

Theorem 4.38 (Measurability of random $\mathfrak{G}$-entropy solutions)**:**
Suppose the following requirements are satisfied:

- *Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$ be a random initial condition to Problem* (4.1).
- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies the measurability Assumption 4.3 and the Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D} = \{\mathfrak{C}_\mathfrak{D}^\kappa\}_{\kappa\in\mathbb{N}}$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let the family $\{\mathbb{X}_\mathbb{T}^s\}_{s\in\mathbb{N}}$ of domain parts satisfy the local finiteness Assumption 4.10.*
- *Let the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ satisfy the boundedness Assumption 4.34.*
- *Let the flux function $\mathfrak{f}$ satisfy the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the measurability Assumption 4.27.*
- *Let $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ be a family of $\mathcal{L}^1D$ germs satisfying the joint measurability Assumption 4.17.*
- *Let $\{\mathfrak{R}_\mathfrak{G}^\kappa\}_{\kappa\in\mathbb{N}}$ be a family of remainder functions associated to $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ that satisfies the joint measurability Assumption 4.23 and the integrability Assumption 4.35.*
- *Let the pathwise existence and uniqueness Assumption 4.37 of a $\mathcal{S}$-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, the pathwise $\mathfrak{G}$-entropy solution u to Problem (4.1) *is strongly measurable in the sense that the mapping $u : \Omega \to \mathcal{S}$ is strongly measurable.* ◆

Proof. The proof ideas are analogous to the measurability result of Theorem 3.50. Therefore, we summarize the main ideas and highlight the differences in the argumentation.

Let $\mathcal{D}_N$ be the separable subspace of $\mathcal{C}^\infty(\mathbb{R}_{>0} \times \mathbb{R}^d; \mathbb{R}_{\geq 0})$ consisting of functions ψ, whose support is contained in $[0, N] \times \mathbb{B}_N(\mathbf{0}_{\mathbb{R}^d})$. Then, with a basis $(\psi_N^i, i \in \mathbb{N})$ of $\mathcal{D}_N$, for fixed indices $i, N \in \mathbb{N}$ and a fixed Kružkov entropy sequence $\boldsymbol{k} = \{k^s\}_{s=1}^\infty \subset \mathbb{R}$, we can defined the *modified entropy functional* $\mathbb{J}_{i,N}^{\boldsymbol{k}} : \Omega \times \mathcal{S} \to \mathbb{R}$ via the mapping

$$
\begin{aligned}
(\omega, \nu) \mapsto & \int_{\mathbb{T}} \int_{\mathbb{X}} \big|\nu(t,\boldsymbol{x}) - k(\omega,t,\boldsymbol{x})\big| \partial_t \psi_N^i(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
& + \int_{\mathbb{T}} \int_{\mathbb{X}} \mathbf{q}\big(\omega,t,\boldsymbol{x};\nu(t,\boldsymbol{x}),k(\omega,t,\boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi_N^i(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
& - \int_{\mathbb{X}} \big|u_0(\omega,\boldsymbol{x}) - k(\omega,t,\boldsymbol{x})\big| \psi_N^i(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
& + \sum_{\kappa \in \mathbb{N}} \int_{\mathfrak{C}_{\mathfrak{D}}^\kappa(\omega)} \mathfrak{R}_{\mathfrak{G}}^\kappa\big(\omega,\mathfrak{d};s_{\boldsymbol{k}}(\omega,\kappa,\boldsymbol{k})\big)\psi(\mathfrak{d}) \,\mathrm{d}\mathfrak{d}\,.
\end{aligned}
$$

Here, $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R})$ is the separable subspace that by Assumption 4.37 contains the solution u. Additionally, k and $\mathbf{q}$ denote the adapted Kružkov entropy corresponding to the sequence $\boldsymbol{k}$ and the Kružkov entropy flux, respectively. Furthermore, $\{\mathfrak{R}_{\mathfrak{G}}^\kappa\}_{\kappa\in\mathbb{N}}$ is the family of remainder functions associated to the family of germs $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ and $\mathfrak{C}_{\mathfrak{D}}(\omega) = \{\mathfrak{C}_{\mathfrak{D}}^\kappa(\omega)\}_{\kappa\in\mathbb{N}}$ is the partition up to a null set of the random compound flux discontinuity $\mathfrak{D}$. Now, the modified entropy functional $\mathbb{J}_{i,N}^{\boldsymbol{k}}$ is Carathéodory by Theorem 4.33.

For the Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ and the indices $i, N \in \mathbb{N}$ still being fixed, we can now define the set-valued mapping

$$
\Xi_{i,N}^{\boldsymbol{k}} : \Omega \rightrightarrows \mathcal{S} \qquad \omega \mapsto \Big\{\nu \in \mathcal{S} \mid \mathbb{J}_{i,N}^{\boldsymbol{k}}(\omega,\nu) \geq 0\Big\}\,.
$$

As in the measurability Theorem 3.50, this correspondence $\Xi_{i,N}^{\boldsymbol{k}}$ selects all functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy Condition (4.10) for a fixed entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ and fixed indices $i, N \in \mathbb{N}$. Furthermore, the correspondence $\Xi_{i,N}^{\boldsymbol{k}}$ is measurable by Lemma 2.9. Intersecting now over the indices $i \in \mathbb{N}$ and $N \in \mathbb{N}$, we obtain a set-valued map $\Xi^{\boldsymbol{k}} : \Omega \rightrightarrows \mathcal{S}$ that contains all functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy Inequality (4.10) for a fixed entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$, but for all test functions $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T}; \mathbb{R})$:

$$
\Xi^{\boldsymbol{k}} : \Omega \rightrightarrows \mathcal{S} \qquad \Xi^{\boldsymbol{k}}(\omega) \coloneqq \bigcap_{i\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \Xi_{i,N}^{\boldsymbol{k}}(\omega) \qquad \omega \in \Omega\,.
$$

Note, the intersection is nonempty, since we assumed the pathwise existence of a unique $\mathcal{S}$-valued $\mathfrak{G}$-entropy solution via Assumption 4.37, which satisfies Inequality (4.10) for all entropy sequences $\boldsymbol{k} \in \ell^\infty(\mathbb{R}) \subset \mathbb{R}^{\leq\mathbb{N}}$ by Definition 4.25. The correspondence $\Xi^{\boldsymbol{k}}$ is measurable, since it is constructed via the countable intersection of measurable set-valued mappings $\Xi_{i,N}^{\boldsymbol{k}}$.

While the proof of Theorem 3.50 continues by intersecting over entropy pairs $\boldsymbol{k} \in \mathbb{Q}^2$ and using the density of the rational numbers $\mathbb{Q}$ in the real numbers $\mathbb{R}$, this is not possible in our current setting: Neither the space of rational sequences $\mathbb{Q}^{\leq\mathbb{N}}$ nor the space of bounded rational sequences $\ell^\infty(\mathbb{Q})$ are countable. Therefore, we cannot intersect over the Kružkov entropy sequences to exclude functions $\nu \in \mathcal{S}$ satisfying Inequality (4.10) only for particular $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$. Luckily, Assumption 4.34 provides us with two possibilities to conclude the proof:

Finitely many domain parts. If we assume that the random compound discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ divides the space-time domain in only finitely many domain parts $\mathbb{X}_{\mathbb{T}}^s(\omega)$, we have identified the finite sequence $\boldsymbol{k} = \{k^s\}_{s=1}^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}$ with the sequence $\boldsymbol{k} = \left\{k^1, \dots, k^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}, 0, 0, \dots\right\}$. Here, $\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}} : \Omega \to \mathbb{N}$ denotes the random number of space-time domain parts. However, for each stochastic parameter $\omega \in \Omega$, the set $\mathbb{Q}^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}$ is countable as a finite Cartesian product of countable sets. This allows us to construct the correspondence

$$\Xi : \Omega \rightrightarrows \mathcal{S} \qquad \Xi(\omega) \coloneqq \bigcap_{\boldsymbol{k} \in \mathbb{Q}^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}} \Xi^{\boldsymbol{k}}(\omega) \qquad \omega \in \Omega \,,$$

which is measurable as a countable intersection of measurable maps. As in the above argumentation, the pathwise existence and uniqueness Assumption 4.37 guarantees that the set-valued map Ξ is nonempty for every stochastic parameter $\omega \in \Omega$. This is due to the solution $u \in \mathcal{S}$ satisfying the adapted entropy Inequality (4.10) for every Kružkov entropy sequence $\boldsymbol{k} \in \mathbb{R}^{\leq \mathbb{N}}$. Furthermore, this correspondence Ξ contains all functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy Inequality (4.10) for any test function $\psi \in \mathcal{D}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ and any $\boldsymbol{k} \in \mathbb{Q}^{\mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)}$.

Focusing on the adapted entropy Inequality (4.10), note that the values k^s do not contribute for indices $s > \mathfrak{N}_{\mathbb{X}_{\mathbb{T}}}(\omega)$: Due to the construction of the adapted Kružkov entropy k, the values k^s are never taken and the selection function $\mathcal{S}_{\boldsymbol{k}}$ does not select these values due to the condition

$$\operatorname{cl}\big(\mathbb{X}_{\mathbb{T}}^{\mathfrak{k}}(\omega)\big) \cap \operatorname{cl}\big(\mathbb{X}_{\mathbb{T}}^{\mathfrak{m}}(\omega)\big) \cap \mathfrak{C}_{\mathfrak{D}}^{\kappa}(\omega) \neq \emptyset \,.$$

Consequently, the measurable correspondence Ξ contains all functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy Inequality (4.10) for any nonnegative test function $\psi \in \mathcal{D}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$ and any Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{Q})$.

Confinement assumption on solution. If we assume that the solution u is confined to an interval $\mathbb{U} = [\underline{\mathfrak{u}}, \overline{\mathfrak{u}}] \subset \mathbb{R}$, it is sufficient to have $\boldsymbol{k} \in \ell^\infty(\mathbb{U})$, since the functions $u(\omega, \mathfrak{x}) \equiv \underline{\mathfrak{u}}$ and $u(\omega, \mathfrak{x}) \equiv \overline{\mathfrak{u}}$ are a sub- and supersolution, respectively. Now, for a finite number $\mathfrak{i} \in \mathbb{N}$, define the set $\boldsymbol{k}_{\mathfrak{i}:\infty}$ consisting of sequences $\boldsymbol{c} \in \ell^\infty(\mathbb{U})$, which are identical to the fixed Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{U})$ from the $(\mathfrak{i}+1)$-th value onwards. For the first $\mathfrak{i}$ values, we assume the sequence $\boldsymbol{c} \in \boldsymbol{k}_{\mathfrak{i}:\infty}$ to be rational. More concisely, we define the set $\boldsymbol{k}_{\mathfrak{i}:\infty}$ as

$$\boldsymbol{k}_{\mathfrak{i}:\infty} \coloneqq \left\{\boldsymbol{c} \in \ell^\infty(\mathbb{U}) \mid c^j \in \mathbb{U} \cap \mathbb{Q} \text{ for } j \in [1, \dots, \mathfrak{i}] \text{ and } c^j = k^j \text{ for } j > \mathfrak{i}\right\} .$$

Since for any number $\mathfrak{i} \in \mathbb{N}$ the sets $\mathbb{Q}^{\mathfrak{i}}$ and $\left\{k^{\mathfrak{i}+1}, k^{\mathfrak{i}+2}, k^{\mathfrak{i}+3}, \dots\right\}$ are countable, we obtain countability of the set $\boldsymbol{k}_{\mathfrak{i}:\infty}$. This allows us to define the set-valued mapping

$$\Xi_{\mathfrak{i}}^{\boldsymbol{k}} : \Omega \rightrightarrows \mathcal{S} \qquad \Xi_{\mathfrak{i}}^{\boldsymbol{k}}(\omega) \coloneqq \bigcap_{\boldsymbol{c} \in \boldsymbol{k}_{\mathfrak{i}:\infty}} \Xi^{\boldsymbol{c}}(\omega) \qquad \omega \in \Omega \,,$$

which is measurable as the countable intersection of measurable maps. Furthermore, the correspondence $\Xi_{\mathfrak{i}}^{\boldsymbol{k}}$ contains all those functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy Inequality (4.10) for all Kružkov entropy sequences $\boldsymbol{c} \in \boldsymbol{k}_{\mathfrak{i}:\infty}$. To also exclude functions $\nu \in \mathcal{S}$ from the correspondence $\Xi_{\mathfrak{i}}^{\boldsymbol{k}}$, that only satisfy the adapted entropy Inequality (4.10) for some values $\mathfrak{i} \in \mathbb{N}$, we define the following set-valued mapping:

$$\Xi : \Omega \rightrightarrows \mathcal{S} \qquad \Xi(\omega) \coloneqq \bigcap_{\mathfrak{i} \in \mathbb{N}} \Xi_{\mathfrak{i}}^{\boldsymbol{k}}(\omega) \qquad \omega \in \Omega \,.$$

This set-valued map is measurable as the countable intersection of measurable maps and contains all functions $\nu \in \mathcal{S}$ satisfying the adapted entropy Inequality (4.10) for any nonnegative test function ψ and any entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{U} \cap \mathbb{Q})$.

Both above argumentations lead to a measurable correspondence Ξ that contains all functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy inequality for suitable rational-valued Kružkov entropy sequences. Note that it is sufficient to only consider rational-valued entropy sequences $\boldsymbol{k} \in \ell^\infty(\mathbb{Q})$ for the finite space-time domain case and $\boldsymbol{k} \in \ell^\infty(\mathbb{U} \cap \mathbb{Q})$ for the confined solution case: The verification of this claim is completely analogous to the corresponding justification in the sole discontinuity setting.

In both cases, this establishes that the correspondence Ξ contains all functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy Inequality (4.10) for any test function $\psi \in \mathcal{D}(\mathbb{X}_\mathbb{T}; \mathbb{R})$ and any Kružkov entropy sequence $\boldsymbol{k} \in \ell^\infty(\mathbb{Q})$ and $\boldsymbol{k} \in \ell^\infty(\mathbb{U} \cap \mathbb{Q})$, respectively. However, this is just the admissibility condition for $\mathfrak{G}$-entropy solutions, which exist and are unique by Assumption 4.37. Consequently, the set-valued mapping $\Xi : \Omega \rightrightarrows \mathcal{S}$ is a singleton containing only the random $\mathfrak{G}$-entropy solution, which is measurable due to the measurability of the correspondence Ξ. This proves the assertion. ■

4.3.4 Existence of moments of random entropy solutions

With the strong measurability result on random $\mathfrak{G}$-entropy solutions to the scalar discontinuous-flux conservation law, we have established the well-posedness of Problem (4.1). As an additional consequence, we are able to interpret the solution as an $\mathcal{S}$-valued, Bochner-integrable random variable. This allows us to describe its statistical properties via the existence of stochastic moments. Therefore, we conclude this Chapter on compound flux discontinuities with a discussion under which conditions the moment of order $1 \leq q < \infty$ exists. Since the main tool for arguing this existence is the $\mathcal{L}^1$-contraction principle (4.11) of random $\mathfrak{G}$-entropy solutions, we will assume throughout this section that the (random) initial condition u_0 is integrable. This leads to the following main theorem of this section, which is proven completely analogous to the sole discontinuity setting. Due to the similarity of the result to Theorem 3.52, we omit its proof. Recall that Assumption 3.51 restricted the allowed flux functions to ensure that the zero initial condition $u_0 \equiv 0$ leads to a vanishing $\mathfrak{G}$-entropy solution $u \equiv 0$.

Theorem 4.39 (Existence of moments of $\mathfrak{G}$-entropy solution)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$, be a random initial condition to the scalar conservation law, Problem (4.1)*. Furthermore, let the following conditions be satisfied:*

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random compound flux discontinuity that satisfies the measurability Assumption 4.3 and Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_\mathfrak{D} = \{\mathfrak{C}_\mathfrak{D}^\kappa\}_{\kappa \in \mathbb{N}}$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let the family $\{\mathbb{X}_\mathbb{T}^s\}_{s \in \mathbb{N}}$ of domain parts satisfy the local finiteness Assumption 4.10.*
- *Let Assumption 4.34 on the boundedness of Kružkov entropy sequences $\boldsymbol{k} \in \ell^\infty(\mathbb{R})$ be satisfied.*
- *Let the flux function $\mathfrak{f}$ satisfy the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the measurability Assumption 4.27 as well as the zero-mass-creation Assumption 3.51.*
- *Let $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ be a family of $\mathcal{L}^1 D$ germs satisfying the joint measurability Assumption 4.17.*

- *Let $\{\mathfrak{R}_{\mathfrak{G}}^{\kappa}\}_{\kappa\in\mathbb{N}}$ be a family of remainder functions associated to the germs $\{\mathfrak{G}_{\kappa}\}_{\kappa\in\mathbb{N}}$ satisfying the measurability Assumption 4.23 and the integrability Assumption 4.35.*
- *Let the pathwise existence and uniqueness Assumption 4.37 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, at almost every time $t \in \mathbb{T}$, the random $\mathfrak{G}$-entropy solution u admits stochastic moments up to order $1 \leq q < \infty$. In particular, the estimation

$$\|u(\omega, t, \boldsymbol{x})\|_{\mathcal{L}^q(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R}))} \leq \|u_0\|_{\mathcal{L}^q(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R}))}$$

is satisfied for almost every time $t \in \mathbb{T}$. ◆

The above theorem shows that random $\mathfrak{G}$-entropy solutions inherit the existence of moments from the underlying random initial condition. Since the result and argumentation are completely analogous to the sole flux discontinuity setting of Chapter 3, we also obtain special results for the case of the initial condition being deterministic and the compound flux discontinuities being stationary. In the latter case, the random $\mathfrak{G}$-entropy solution u has a representative in the function space $\mathcal{C}\big(\mathbb{T};\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})\big)$. Due to the similarity of the argumentation, we omit the proofs for the subsequent special results for the existence of stochastic moments, which also conclude this chapter on compound flux discontinuities.

Corollary 4.40 (Existence of moments for deterministic initial conditions)**:**
Let $u_0 \in \mathcal{L}^1(\mathbb{X})$ be a deterministic initial condition to Problem (4.1)*. Furthermore, as in Theorem 4.39, let the following conditions be satisfied:*

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity that satisfies the measurability Assumption 4.3 and Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_{\mathfrak{D}} = \{\mathfrak{C}_{\mathfrak{D}}^{\kappa}\}_{\kappa\in\mathbb{N}}$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let the family $\{\mathbb{X}_{\mathbb{T}}^{s}\}_{s\in\mathbb{N}}$ of space-time parts satisfy the local finiteness Assumption 4.10.*
- *Let Assumption 4.34 on the boundedness of Kružkov entropy sequences $\boldsymbol{k} \in \ell^{\infty}(\mathbb{R})$ be satisfied.*
- *Let the flux function $\mathfrak{f}$ satisfy the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the measurability Assumption 4.27 as well as the zero-mass-creation Assumption 3.51.*
- *Let $\{\mathfrak{G}_{\kappa}\}_{\kappa\in\mathbb{N}}$ be a family of $\mathcal{L}^1D$ germs, which satisfies the joint measurability Assumption 4.17.*
- *Let $\{\mathfrak{R}_{\mathfrak{G}}^{\kappa}\}_{\kappa\in\mathbb{N}}$ be a family of remainder functions associated to the germs $\{\mathfrak{G}_{\kappa}\}_{\kappa\in\mathbb{N}}$ satisfying the measurability Assumption 4.23 and the integrability Assumption 4.35.*
- *Let the pathwise existence and uniqueness Assumption 4.37 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Then, at almost every time $t \in \mathbb{T}$, the random $\mathfrak{G}$-entropy solution u admits all stochastic moments of order $1 \leq q < \infty$. ◆

Theorem 4.41 (Existence of moments for stationary flux discontinuities)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^1(\mathbb{X})\big)$, with $1 \leq q < \infty$, be a stochastic initial condition to the random scalar conservation law given by Equation (4.1)*. Furthermore, as in Theorem 4.39, let the following requirements be satisfied:*

- *Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random compound flux discontinuity that satisfies the measurability Assumption 4.3 and Assumption 4.4 on the existence of a partition up to a null set $\mathfrak{C}_{\mathfrak{D}} = \{\mathfrak{C}_{\mathfrak{D}}^{\kappa}\}_{\kappa\in\mathbb{N}}$, which itself satisfies the local finiteness Assumption 4.5.*
- *Let the family $\{\mathbb{X}_{\mathbb{T}}^{s}\}_{s\in\mathbb{N}}$ of domain parts satisfy the local finiteness Assumption 4.10.*
- *Let Assumption 4.34 on the boundedness of Kružkov entropy sequences $\boldsymbol{k} \in \ell^{\infty}(\mathbb{R})$ be satisfied.*
- *Let the flux function $\mathfrak{f}$ satisfy the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the measurability Assumption 4.27 as well as the zero-mass-creation Assumption 3.51.*
- *Let $\{\mathfrak{G}_{\kappa}\}_{\kappa\in\mathbb{N}}$ be a family of $\mathcal{L}^1 D$ germs, which satisfies the joint measurability Assumption 4.17.*
- *Let $\{\mathfrak{R}_{\mathfrak{G}}^{\kappa}\}_{\kappa\in\mathbb{N}}$ be a family of remainder functions associated to the germs $\{\mathfrak{G}_{\kappa}\}_{\kappa\in\mathbb{N}}$ satisfying the measurability Assumption 4.23 and the integrability Assumption 4.35.*
- *Let the pathwise existence and uniqueness Assumption 4.37 of a separably-valued $\mathfrak{G}$-entropy solution u be satisfied.*

Furthermore, let the compound flux discontinuity $\mathfrak{D}$ be stationary for every stochastic parameter $\omega \in \Omega$, i.e., $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}$. Then, the estimation

$$\|u(\omega, t, \boldsymbol{x})\|_{\mathcal{L}^q\left(\Omega;\mathcal{C}(\mathbb{T};\mathcal{L}^1(\mathbb{X}))\right)} \leq \|u_0\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)}$$

holds and in particular, the random $\mathfrak{G}$-entropy solution u admits all stochastic moments up to order $1 \leq q < \infty$. ◆

Numerical simulation of random conservation laws with a locally finite number of flux discontinuities

5

Throughout this chapter, a problem setting is investigated, which is motivated by two-phase flow in a heterogeneous porous medium. Considering two phases, for example *oil* and *water*, we are interested in simulating the time evolution of the oil saturation of the porous medium. To precisely formulate the problem, let $\mathbb{T} = [0, T] \subset \mathbb{R}_{\geq 0}$ be a time interval and write $\mathbb{X} \subset \mathbb{R}^d$ for the considered spatial domain. Denoting the unknown saturation by $u := u(\omega, t, \boldsymbol{x}) \in [0, 1]$, this problem can be modeled by the scalar conservation law

$$\begin{aligned} \partial_t u + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, \boldsymbol{x}, u) = 0 \qquad & \text{in } \Omega \times \mathbb{T} \times \mathbb{X}\,, \\ u(\omega, 0, \boldsymbol{x}) = u_0(\omega, \boldsymbol{x}) \quad & \text{on } \Omega \times \{0\} \times \mathbb{X}\,. \end{aligned} \tag{5.1}$$

Here, the initial condition u_0 describes the (possibly random) saturation at the beginning of the considered time interval (at time $t = 0$). By construction, the initial condition u_0 satisfies $u_0(\boldsymbol{x}) \in [0, 1]$ for every spatial point $\boldsymbol{x} \in \mathbb{X}$. Additionally, we assume that the random initial saturation satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$ with $1 \leq q < \infty$ and $1 \leq p \leq \infty$. As the saturation problem occurrs in a heterogeneous medium, the flux function $\mathfrak{f}$ should depend discontinuously on the spatial variable $\boldsymbol{x} \in \mathbb{X}$ and is given by

$$\mathfrak{f}(\omega, \boldsymbol{x}, u) = \frac{\Bbbk_o(u)}{\Bbbk_o(u) + \Bbbk_w(u)} \big(\mathbf{1} - \mathfrak{a}(\omega, \boldsymbol{x}) \Bbbk_w(u)\big)\,. \tag{5.2}$$

Here, the random jump-advection coefficient $\mathfrak{a}$ models the absolute permeability of the porous medium and the functions $\Bbbk_o$ and $\Bbbk_w$ correspond to the relative permeabilities of the two phases (phase mobilities). As a simple model, these relative permeabilities are given as

$$\Bbbk_o(u) = u^2 \qquad \text{and} \qquad \Bbbk_w(u) = (1 - u)^2\,.$$

The considered porous medium is typically heterogeneous and consists of various layers of homogeneous materials. Additionally, the medium may contain fractures or inclusions of other media. Therefore, the coefficient $\mathfrak{a}$ is modeled as a discontinuous random field. The formulated Problem (5.1), (5.2) is inspired by the Buckley-Leverett equation [9, 44, 154], which models the displacement of two immiscible fluids in porous media. This type of equation is employed for simulating two-phase flow problems, such as in reservoir engineering [123, 140].

As a first step, in Section 5.1 we establish that the given problem setting corresponds to the theory discussed in the previous chapters. Afterwards, in Section 5.2, we investigate the pathwise convergence of entropy solutions to Problem (5.1), (5.2) to illustrate the influence of the random jump coefficient $\mathfrak{a}$ on the strong convergence rate.

Note that we do not investigate approximations of stochastic moments of the entropy solution u in this chapter due to multiple reasons: The strategy for approximating the moments by means of (multilevel) Monte Carlo methods has already been applied to scalar discontinuous-flux conservation laws in a variety of problem settings. In particular, we want to emphasize the works of Müller et al. [214], which discusses the multilevel Monte Carlo method for a two-phase flow in random heterogeneous media for the Buckley-Leverett transport problem. Additionally, we want to highlight the recent work of Badwaik et al. [22], where the one-dimensional saturation Problem (5.1), (5.2) is discussed. Since the ideas of the multilevel Monte Carlo method remain valid for extending the one-dimensional setting to two spatial dimensions, we omit the numerical investigation at this point.

However, we refer to Chapter 8, where we apply the MLMC method to a discontinuous-flux Burgers' equation. There, we demonstrate the ability of pathwise discretization schemes to improve the convergence behavior of the sample-based multilevel Monte Carlo method. Since the convergence is influenced via the accuracy of pathwise approximations, Section 5.2 indicates how the MLMC method performs for the discussed two-phase flow saturation problem.

5.1 Two-phase flow in heterogeneous porous medium

The purpose of this section is to verify the theoretical well-posedness of the considered problem. In particular, for a specific type of random jump coefficient $\mathfrak{a}$, we verify the assumptions that lead to existence and uniqueness of a $\mathfrak{G}$-entropy solution as well as to its stochastic measurability. Throughout this section and the remainder of this chapter, vanishing viscosity solutions are considered, which correspond to a random family of vanishing viscosity germs as defined in Definition 3.35. Let us briefly motivate this choice: To a certain extent, every material in nature has viscous effects and conducts heat. These properties correspond to viscosity terms in a considered equation and, e.g., may be caused by capillary effects. We refer to [9] for a derivation of the vanishing viscosity approach resulting from (vanishing) capillary pressure. In various applications, heat conductivity or viscous behavior may be negligible, even though it is not completely absent. For a more detailed motivation for vanishing viscosity solutions, we refer to [80, Section 4.6].

To verify the theoretical well-posedness of the two-phase flow in heterogeneous media, we first ensure the pathwise existence and uniqueness in Section 5.1.1. Afterwards, in Section 5.1.2, it is argued that the considered problem satisfies all assumptions leading to strong measurability of vanishing viscosity solutions.

Before we start with the verification, some properties of the considered problem need to be specified. For the numerical investigations, we restrict ourselves to the two-dimensional problem setting. The partition of the spatial domain $\mathbb{X} = \mathbb{R}^2$ resulting from the random jump coefficient is constructed via a random number of affine functions. More precisely, let $\mathfrak{N}_{\mathfrak{D}} : \Omega \to \mathbb{N}$ be a random variable describing the number of affine functions contained in the compound flux discontinuity $\mathfrak{D}$. The random partition $\mathfrak{T}$ of the jump coefficient $\mathfrak{a}$ is constructed via random affine functions, each determined via a uniformly distributed x_2-axis interception $\mathfrak{y} \sim \mathcal{U}(\mathbb{R})$ and a slope described via a random angle $\mathfrak{a} \sim \mathcal{U}\big([0, 2\pi] \setminus \{\pi/2, 3\pi/2\}\big)$ to

the x_1-axis[29]. The random compound flux discontinuity $\mathfrak{D}$ is than given as the union of the graphs of these affine function. Based on the resulting partition $\mathfrak{T}$ of the spatial domain, the jump coefficient is constructed by assigning a uniformly distributed random variable $\mathfrak{a}_i \sim \mathcal{U}\big([0.1, 0.9]\big)$ to the associated partition element $\mathfrak{T}_i$.

5.1.1 Pathwise existence and uniqueness of vanishing viscosity solutions

Based on the previously introduced constructions of the random jump coefficient $\mathfrak{a}$ and the corresponding discontinuity interfaces, we can now investigate the pathwise existence and uniqueness of vanishing viscosity solutions to the two-phase flow Problem (5.1) in a heterogeneous medium. With this goal in mind, the purpose of this section is to verify the assumptions leading to pathwise existence and uniqueness of these solutions. The corresponding result is presented in the subsequent theorem.

Theorem 5.1 (Pathwise existence of unique solution to two-phase flow problem)**:**
For the considered random two-phase flow in heterogeneous media, there exists a unique pathwise vanishing viscosity solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. ◆

Proof. The statement follows from applying the pathwise existence and uniqueness Theorems 4.29 and 4.31. To do this, the following requirements need to be guaranteed:

(i) For the random compound flux discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$, the family $\big\{\mathbb{X}_\mathbb{T}^s\big\}_{s\in\mathbb{N}}$ of space-time domain parts needs to be locally finite.

(ii) Additionally, the compound flux discontinuity $\mathfrak{D}$ fulfills the regularity Assumption 4.30 and Assumption 4.4 on the existence of a partition up to a null set is satisfied.

(iii) The flux function $\mathfrak{f}$ needs to satisfy the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18.

(iv) Each germ in the family $\{\mathfrak{G}_\kappa\}_{\kappa\in\mathbb{N}}$ of admissibility germs is definite.

We justify each of the above requirements separately.

Condition (i). By construction of the random jump coefficient $\mathfrak{a}$, the discontinuity hypersurface $\mathfrak{D}$ is given as the finite union of affine functions, for each stochastic parameter $\omega \in \Omega$. Since affine functions are sole discontinuity hypersurfaces in the sense of Definition 3.1, the jump interface $\mathfrak{D}$ is a random compound discontinuity. As the number of intersection points of finitely many affine functions is also finite, the compound discontinuity divides the domain into finitely many connected space-time parts. Consequently, Condition (i) is satisfied.

Condition (ii). The regularity Assumption 4.30 is automatically satisfied by considering a compound discontinuity consisting of finitely many affine functions, since each affine function is twice continuously differentiable. As already established, the number of intersection points in the compound discontinuity is finite. However, this immediately yields the existence of a partition up to a null set of the jump interface $\mathfrak{D}$ by restricting the affine functions to the subintervals between the intersection points. Therefore, we have established Condition (ii).

[29] Due to the construction of sole discontinuity hypersurfaces in Definition 3.1, we need to exclude the possibility of discontinuities given by straight lines parallel to the x_1-axis. However, if one needs to consider this particular case, a rotation of the domain and problem is sufficient to admit this special setting.

Condition (iii). To verify Condition (iii), we need to ensure that the compound-flux-discontinuity Assumption 4.14 and the genuine nonlinearity Assumption 4.18 are satisfied. For the first one, note that Assumption (C-1) is trivially satisfied by the definition of the flux function in Equation (5.2) in combination with the piecewise constant construction of the random jump coefficient $\mathfrak{a}$. Additionally, by the choice of the functions $\mathbb{k}_o$ and $\mathbb{k}_w$, the flux function $\mathfrak{f}$ is locally Lipschitz continuous in $u \in [0, 1]$. Finally, since the jump coefficient $\mathfrak{a}$ is piecewise constant by constrution, the flux functions $\mathfrak{f}^s$ are obviously globally Lipschitz continuous. Thus, we have established the compound-flux-discontinuity Assumption 4.14. To conclude the justification of Condition (iii), note that the choice of $\mathbb{k}_o$ and $\mathbb{k}_w$ also imply that the flux $\mathfrak{f}$ is genuinely nonlinear.

Condition (iv). Due to the existence of a partition up to a null set $\mathfrak{C}_{\mathfrak{D}}$, we can identify a random family of vanishing viscosity germs. However, since each of these admissibility germs corresponds to the standard notion of a vanishing viscosity germ in the sole discontinuity case (see Definition 3.35 and Section 3.3.1), the definiteness of $\mathfrak{G}_\kappa$ was already established in the fundamental works of Andreianov et al. [12, 13]. ■

5.1.2 Stochastic measurability of vanishing viscosity solutions

To conclude the justification that Problem (5.1) modeling two-phase flow in heterogeneous medium is well-posed, it remains to verify that the random vanishing viscosity solutions are strongly measurable. Therefore, the various assumptions regarding measurability need to be established, which is formalized by the following theorem.

Theorem 5.2 (Strong measurability of solution to two-phase flow problem)**:**
The vanishing viscosity solution u to the two-phase flow in heterogeneous porous media Problem (5.1) *is strongly measurable as a mapping $u : \Omega \to \mathcal{S}$, where $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is a separable subspace of $\mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$.* ◆

Proof. The strong measurability of the vanishing viscosity solution u is established by Theorem 4.38. To be able to apply this theorem, we need to verify the following requirements:

(i) The initial condition u_0 needs to satisfy $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$.

(ii) The random compound flux discontinuity $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ has to fulfill the measurability Assumption 4.3, Assumption 4.4 on the existence of a partition up to a null set and the local finiteness Assumption 4.5. Additionally, the resulting space-time domain parts have to satisfy the local finiteness Assumption 4.10.

(iii) The Kružkov entropy sequence needs to satisfy the boundedness Assumption 4.34.

(iv) The discontinuous flux function should fulfill the compound-flux-discontinuity Assumption 4.14, the genuine nonlinearity Assumption 4.18 and the stochastic measurability Assumption 4.27.

(v) The family $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$ of $\mathcal{L}^1$D germs needs to satisfy the joint measurability Assumption 4.17.

(vi) The family of remainder functions $\{\mathfrak{R}^\kappa_{\mathfrak{G}}\}_{\kappa \in \mathbb{N}}$ satisfies the joint measurability Assumption 4.23 as well as the integrability Assumption 4.35.

(vii) The random solution $u(\omega, \cdot, \cdot)$ needs to take values in a separable subspace $\mathcal{S} \subset \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$.

Note, since the jump coefficient $\mathfrak{a}$ does not depend on the time variable, the compound flux discontinuity $\mathfrak{D}$ is stationary. As a result, the random entropy solution $u(\omega, \cdot, \cdot)$ only takes values in the space $\mathcal{S} = \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})\big)$, which is separable by the argumentation in Example 3.49 (ii). Therefore, Condition (vii) is satisfied. The remaining conditions are argued separately:

Condition (i). The first requirement states that the random initial saturation u_0 should satisfy $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$. However, this is precisely the assumption that we imposed on the initial condition u_0 in the description of the two-phase flow problem through heterogeneous media.

Condition (ii). The random compound flux discontinuity $\mathfrak{D}$ is constructed via a finite (random) number of affine functions. The existence of a partition up to a null set and its local finiteness were already argued in the proof of Theorem 5.1. Here, also the local finiteness of the resulting space-time domain parts was established. Therefore, it remains to show the stochastic measurability of the random compound flux discontinuity $\mathfrak{D}$. First, note that the jump interface is constructed via random affine functions, which are defined via a uniformly distributed interception point of the x_2-axis and a random angle to the x_1-axis, which is also uniformly distributed. Thus, each random affine function is measurable and consequently the random compound discontinuity is also measurable as the finite intersection of measurable functions. As a result, Condition (ii) is satisfied.

Condition (iii). Due to the problem formulation, the solution is confined to the interval $\mathbb{U} = [0, 1]$, analogously to the confinement Assumption 3.38. As a result, it is sufficient to restrict the entropy values to this interval $\mathbb{U} = [0, 1]$, which is bounded. Therefore, Condition (iii) is satisfied.

Condition (iv). To establish Condition (iv), it is only necessary to show the measurability of the flux function $\mathfrak{f}$, since the other requirements were already argued in Theorem 5.1. However, this immediately follows from the measurability of the compound flux discontinuity $\mathfrak{D}$, as the random jump coefficient of the flux function is constructed to be piecewise constant.

Condition (v). To verify the joint measurability Condition (v) on the family of admissibility germs $\{\mathfrak{G}_\kappa\}_{\kappa \in \mathbb{N}}$, we heavily rely on the discussion of the random vanishing viscosity germs in Section 3.4.2. For each fixed $\kappa \in \mathbb{N}$, we can identify the admissibility germ $\mathfrak{G}_\kappa$ with an admissibility germ corresponding to a sole flux discontinuity. However, Theorem 3.60 states that the random families of vanishing viscosity germs $\mathfrak{G}_{\mathrm{VV}}$ is jointly measurable. Since the two-phase flow Problem (5.1) satisfies all the requirements of Theorem 3.60, the joint measurability requirement on the family of admissibility germs is satisfied.

Condition (vi). Before we start with the verification of Condition (vi), let us stress that the flux function $\mathfrak{f}$ of the two-phase flow Problem (5.1) given by Equation (5.2) satisfies all requirements to employ the remainder function $\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ based on the Euclidean distance. For the details on this remainder function, we refer to Section 3.4.3. However, since by Condition (v) the random vanishing viscosity germs are jointly measurable, we immediately obtain the joint measurability of this remainder function by Proposition 3.62. Furthermore, the integrability Assumption 4.35 follows immediately from Proposition 3.63, since the result carries over to the compound discontinuity setting in a straightforward manner.

As a result of the previous investigations, we have verified all requirements of Theorem 4.38. Consequently, the vanishing viscosity solution to the two-phase flow through heterogeneous medium Problem (5.1) is strongly measurable. ■

5.2 Pathwise convergence study

In this section, the pathwise convergence behavior of finite volume approximations to the entropy solution of the two-dimensional saturation Problem (5.1), (5.2) is investigated. Before we discuss two different approaches for creating meshes of the spatial domain $\mathbb{X}$, we precise the considered simulation setting.

Throughout this section, we consider the spatial domain $\mathbb{X} = [0, 2] \times [0, 1/2] \subset \mathbb{R}^2$ and the time interval $\mathbb{T} = [0, 2]$. On the upper and lower boundary, i.e., on $\partial\mathbb{X} = [0, 2] \times \{0\}$ and $\partial\mathbb{X} = [0, 2] \times \{1/2\}$, we impose *perfect slip boundary conditions*. While this means that no flux can move through the boundary, it also idealizes that the flux velocity is not affected by the boundary, since friction forces are neglected. On the left boundary, an inflow boundary condition is imposed, which is set to $u(\omega, t, \boldsymbol{x}) \equiv 1$ for $\boldsymbol{x} \in \{0\} \times [0, 1/2]$. Finally, we assume an outflow boundary condition on $\boldsymbol{x} \in \{2\} \times [0, 1/2]$. An illustration of this simulation setup is given in Figure 5.1.

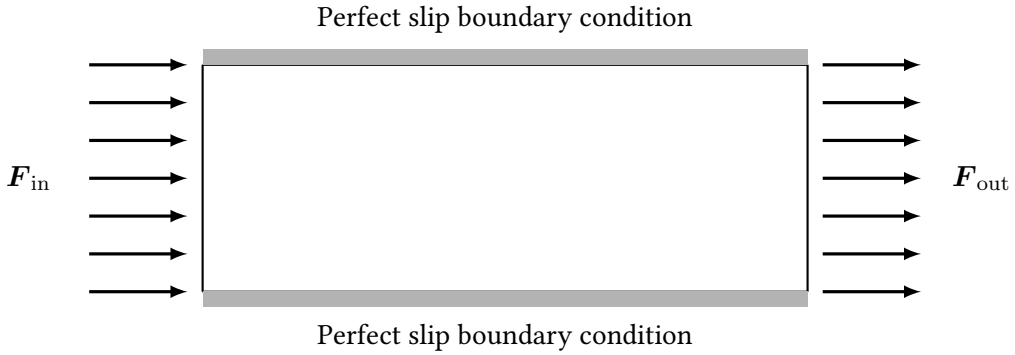


Figure 5.1: Illustration of the problem setting for two-phase flow simulations in heterogeneous porous media.

Due to the presented simulation setting, the situation is simplified to a quasi one-dimensional problem. Instead of considering the flux function given by Equation (5.2), it is sufficient to employ a flux function $\mathfrak{f}(\omega, \boldsymbol{x}, u) = (\mathfrak{f}_1(\omega, \boldsymbol{x}, u), 0)^\top$. Here, the scalar flux function $\mathfrak{f}_1(\omega, \boldsymbol{x}, u)$ corresponds to the one-dimensional version of Equation (5.2), i.e.,

$$\mathfrak{f}_1(\omega, \boldsymbol{x}, u) \coloneqq \frac{\mathbb{k}_o(u)}{\mathbb{k}_o(u) + \mathbb{k}_w(u)} \big(1 - \mathfrak{a}(\omega, \boldsymbol{x}) \mathbb{k}_w(u)\big) \,.$$

Note, instead of employing a vector-valued jump coefficient $\mathfrak{a}$, a scalar discontinuous random field $\mathfrak{a}$ is used. In the subsequent pathwise convergence study, two different choices for this discontinuous jump-advection coefficient $\mathfrak{a}$ are considered.

Prior to this convergence investigation, the employed discretization schemes have to be discussed. In particular, we want to focus on the spatial meshing algorithm employed. Here, we distinguish two strategies: First, we employ a standard triangulation algorithm, which creates a mesh of the spatial domain $\mathbb{X}$ satisfying some maximum step size restriction $\Delta_{\boldsymbol{x}}^i \leq \Delta_{x,\max}^l$, where $\Delta_{x,\max}^l$ denotes the maximum allowed spatial step size measured by the largest edge length of the triangulation.

Throughout this section, we compare this standard meshing algorithm with the alternative approach of creating sample-adapted grids for the pathwise discretization. In this approach, for each $\omega \in \Omega$, the triangulation is adapted to the discontinuities of the jump advection coefficient $\mathfrak{a}$. Consequently, the resulting triangulation is random. The particular algorithm, how such a sample-adapted triangulation may be obtained depends on the geometrical structure of the coefficient $\mathfrak{a}$ and is discussed in each section individually. For propagating the solution through the time interval $\mathbb{T}$, we employ a forward Euler time discretization scheme. To ensure the stability of this time marching algorithm, the time step size is chosen such that it suffices the CFL condition given by Inequality (2.7).

To compute a solution to Problem (5.1), (5.2), the entropy solution u is approximated with a finite volume scheme based on a hierarchy of grids. For each discretization level of the spatial grid, a step size restriction $\Delta^l_{x,\max} = \Delta^0_{x,\max} \cdot 0.65^l$ is imposed, where the initial spatial step size $\Delta^0_{x,\max}$ is chosen as $\Delta^0_{x,\max} = 1/2$. While a refinement parameter of 0.5 would be the usual choice, it is set to 0.65 in the presented experiments to reduce the computational complexity of simulations with fine spatial resolutions. Throughout this section, we consider the strong error $\mathcal{E}$ of the finite volume approximation u_Δ, which is given by

$$\mathcal{E}(\omega) \coloneqq \left\| u^{\mathrm{ref}}_\Delta(\omega, \cdot, T) - u_\Delta(\omega, \cdot, T) \right\|_{\mathcal{L}^\star(\mathbb{X};\mathbb{R})} ,$$

where $\mathcal{L}^\star(\mathbb{X};\mathbb{R})$ is either $\mathcal{L}^1(\mathbb{X};\mathbb{R})$, $\mathcal{L}^2(\mathbb{X};\mathbb{R})$ or $\mathcal{L}^\infty(\mathbb{X};\mathbb{R})$. Here, u^{ref}_Δ denotes a reference solution which is computed by the jump-adapted finite volume discretization with maximum spatial step size $\Delta^{\mathrm{ref}}_{x,\max} = \Delta^7_{x,\max} = \Delta^0_{x,\max} \cdot 0.65^7$. For the numerical flux of the approximation scheme, we always employ the Godunov scheme (2.9).

The implementation of the simulation utilizes the Distributed and Unified Numerics Environment (DUNE) software framework [39, 252]. Additionally, for creating the (jump-adapted) meshes, the DUNE-ALUGrid module by Alkämper et al. [6] has been used.

5.2.1 Piecewise homogeneous media

In this first experiment, the random jump coefficient $\mathfrak{a}$ is chosen as a Lévy-type random field as defined in Definition 2.28. As the focus of the numerical investigation is on the jump discontinuities of $\mathfrak{a}$, we omit the Gaussian random field included in Equation (2.3). More precisely, we consider a random jump coefficient $\mathfrak{a}$, which is given as

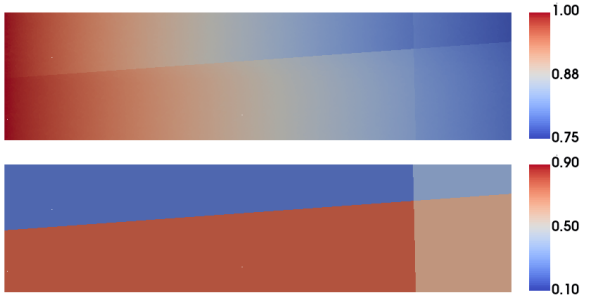
$$\mathfrak{a} : \Omega \times \mathbb{X} \to \mathbb{R}_{>0} \qquad (\omega, \boldsymbol{x}) \mapsto \sum_{i=1}^{\tau} \mathbb{1}_{\mathfrak{T}_i}(\boldsymbol{x}) \mathfrak{p}_i(\omega) .$$

Here, the partition $\mathfrak{T} : \Omega \to \mathcal{B}(\mathbb{X})$ of the spatial domain $\mathbb{X}$ is generated by random lines through the domain. In particular, the porous medium is modeled as a composite material consisting of four homogeneous media resulting from one vertical and one horizontal line through the domain. To construct these lines, we sample four independent, uniformly distributed random variables $\mathfrak{d}_{1,2} \sim \mathcal{U}([0.15, 0.35])$ and $\mathfrak{d}_{3,4} \sim \mathcal{U}([0.15, 1.85])$. Here, the first two random variables $\mathfrak{d}_{1,2}$ represent the start and end point of the horizontal line, given by $(0, \mathfrak{d}_1)$ and $(2, \mathfrak{d}_2)$, respectively. Analogously, the random variables $\mathfrak{d}_{3,4}$ yield the start and end point of the vertical line, given by $(\mathfrak{d}_3, 0)$ and $(\mathfrak{d}_4, 1/2)$. Consequently, we obtain a partition of the spatial domain $\mathbb{X}$ consisting of four subdomains. As a last step of constructing the random jump coefficient $\mathfrak{a}$, we associated a jump height $\mathfrak{p}_i$ to each of the subdomains $\mathfrak{T}_i$. Here, every jump height $\mathfrak{p}_i$ of the sequence $(\mathfrak{p}_i, i \in \mathbb{N})$ is independent and identically distributed with a uniform

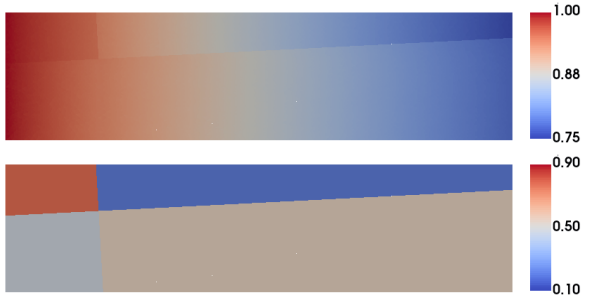
$\mathcal{U}([0.1, 0.9])$ distribution. Since the jump coefficient $\mathfrak{a}$ is constant on every partition element of the spatial domain $\mathbb{X}$, we call this discontinuous random field a *piecewise homogeneous medium*.

Creating a sample-adapted triangulation for these piecewise homogeneous medium coefficients is rather simple. Meshing each element of the partition $\mathfrak{T}$ individually leads to a spatial grid that fulfills the adaptivity constraint. For the subsequent experiments, we also impose a restriction of conformitivity of the resulting mesh. This is obtained by requiring that adjacent subdomain grids share the same nodes on the lines of discontinuity of the random jump coefficient $\mathfrak{a}$.

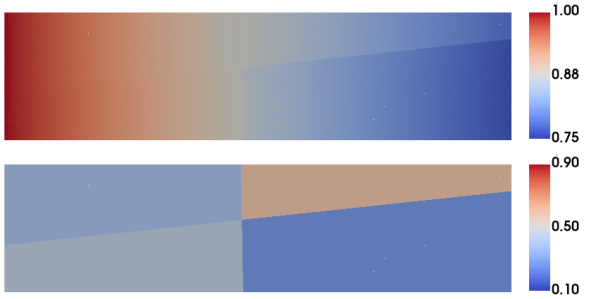
Before we turn to the strong convergence investigation, let us illustrate the resulting piecewise homogeneous medium jump coefficients and how they affect the entropy solution u of the two-phase flow saturation Problem (5.1), (5.2). The effect of the random jump-advection coefficient $\mathfrak{a}$ on the random entropy solution u at the final time $T = 2$ is depicted in Figure 5.2.



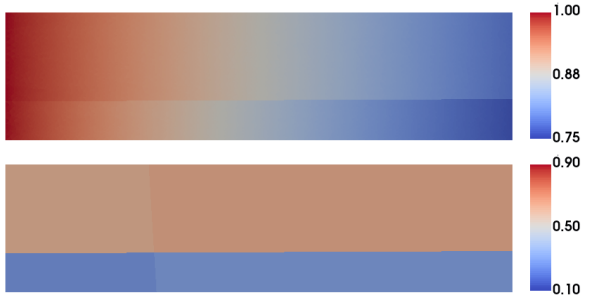
(a) Jump coefficient $\mathfrak{a}$ with a wide range of values and visible influence of the discontinuities on the entropy solution u.



(b) Vertical jump appears closer to the left boundary resulting in a reduced effect on the solution u.



(c) Reduced range of jump heights $\mathbb{p}_i$ still leading to a significant (horizontal) discontinuity in the entropy solution u.



(d) Similar jump heights $\mathbb{p}_i$ in the top and bottom part of the domain resulting in a visible (horizontal) discontinuity in the solution u.

Figure 5.2: Illustration of random entropy solutions u (top) at the final time $T = 2$ with underlying random jump coefficients $\mathfrak{a}$, which are modeled as piecewise homogeneous media (bottom).

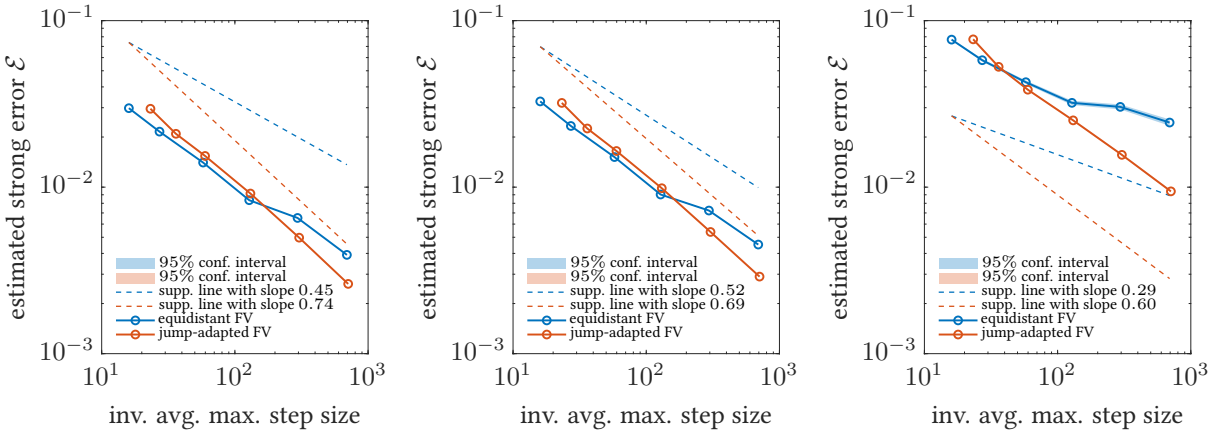
As one can see in the second row of Figure 5.2a, the piecewise homogeneous medium consists of four subdomains on which the random field $\mathfrak{a}$ is constant. The corresponding entropy solution has a high saturation at the left boundary of the domain and decreases towards the right boundary. One immediately notices that the discontinuities in the coefficient lead to jumps in the solutions. Turning to Figure 5.2b, the vertical interface in the absolute permeability shifts towards the left domain boundary. However, this does not change the characteristics of the random entropy solution, which still shows a decreasing saturation over the domain. Nevertheless, the height of the jump-advection coefficient $\mathfrak{a}$ affects how fast the saturation propagates. In Figure 5.2c the values of the absolute permeability $\mathfrak{a}$ are

contained in a reduced range. Thus, the jump sizes (jumping from a high jump height $\mathfrak{p}_i$ to a low one or the other way around) are lower and result in smaller discontinuities in the entropy solution u. Finally, Figure 5.2d demonstrates that neighboring jump heights $\mathfrak{p}_i$ may be similar to each other and are not ordered in any way. As a consequence, the vertical discontinuity barely has any effect on the entropy solution u, while the horizontal jump is still noticable since it has a greater impact on the temporal behavior of the solution. A similar phenomenon can also be observed in the left part of the domain in Figure 5.2c, where the horizontal discontinuity is barely visible.

To conclude the above observations, it appears that horizontal discontinuities in the piecewise homogeneous medium lead to more profound jump discontinuities in the entropy solution u. This behavior can be explained in a straightforward manner: The jump coefficient $\mathfrak{a}$ affects the velocity at which the saturation spreads. While a vertical discontinuity influences this speed, once the saturation reaches the jump, a horizontal discontinuity affects the velocity of the fluids throughout the entire simulation. Consequently, the effects on the solution are greater for horizontal jump discontinuities in the piecewise homogeneous medium.

Based on this qualititive behavior of the solution, we can now turn to the pathwise convergence investigation. Therefore, Figure 5.3 depicts the strong $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error of finite volume approximations u_Δ of the solution u. Note, while the notation of the $\mathcal{L}^\infty$-error is oftentimes used to denote the error in $\mathcal{L}^\infty\big(\mathbb{T};\mathcal{L}^1(\mathbb{X})\big)$, we consider the $\mathcal{L}^\infty(\mathbb{X})$-error.

As the jump coefficient $\mathfrak{a}$ is random, so are these strong errors (and in case of jump-adapted triangulations also the spatial mesh). Therefore, the pathwise error is estimated by a Monte Carlo estimator using 50 samples, which is sufficient as the 95% confidence intervals in Figure 5.3 show.



(a) Pathwise $\mathcal{L}^1$-error. (b) Pathwise $\mathcal{L}^2$-error. (c) Pathwise $\mathcal{L}^\infty$-error.

Figure 5.3: Pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error of finite volume (FV) approximations of the random entropy solution u for a piecewise homogeneous medium. Each error is estimated via a Monte Carlo estimator using 50 samples.

As one can see in Figures 5.3a and 5.3b, the pathwise $\mathcal{L}^1$- and $\mathcal{L}^2$-error converge at almost the same rate. While the standard non-adapted triangulation leads to a convergence rate of approximately $1/2$, the jump-adapted meshing converges at a faster rate of approximately 0.7. Let us also stress the kink in the error plot of the non-adapted discretizations. This behavior is caused by the evaluation of the jump coefficient in the cells close to the discontinuities. Since we cannot ensure a constant evaluation on the correct side of the discontinuity, the varying values at the jump interfaces lead to an increased

approximation error. While the 95% confidence interval of the error can barely be detected for the strong $\mathcal{L}^1$- and $\mathcal{L}^2$-error, it is significantly wider for the pathwise $\mathcal{L}^\infty$-error, especially for the non-adapted grid. This is by far not surprising: The effect of inaccurately approximating the jump coefficient $\mathfrak{a}$ at the discontinuity interfaces has a more intense affect in this $\mathcal{L}^\infty$-error norm. In fact, this behavior also influences the convergence rate of the finite volume approximations. The approximated solutions based on jump-adapted discretizations converge at a rate of 0.6 and thus are not significantly affected by this error measure. However, with an experimental convergence rate of merely 0.3, the approximations based on a standard triangulation converge only half as fast as in the $\mathcal{L}^1$- and $\mathcal{L}^2$-error norm.

5.2.2 Heterogeneous media

In this second experiment, we consider a similar jump coefficient $\mathfrak{a}$ as in the previous section. Due to an increased number of jumps, the considered coefficient can be interpreted as modeling a *heterogeneous medium*. This higher number of jump discontinuities contained in the random advection coefficient $\mathfrak{a}$ is the main difference compared to the piecewise homogeneous medium of the previous section. Instead of considering one vertical and one horizontal jump interface, the number of discontinuity lines in the heterogeneous medium is stochastic. In particular, the number of lines partitioning the spatial domain $\mathbb{X}$ is given via two independent Poisson-distributed random variables $\tau_{\mathrm{hor}} \sim \mathrm{Poi}(2)$ and $\tau_{\mathrm{vert}} \sim \mathrm{Poi}(4)$. Here, τ_{hor} denotes the random number of horizontal lines and τ_{vert} defines the amount of vertical discontinuities. Here, the distribution $\tau_{\mathrm{hor}} \sim \mathrm{Poi}(2)$ leads to an expected value of 2 horizontal jumps, where the value 2 is chosen arbitrarily to consider more than one horizontal jumps. Analogously, the distribution $\tau_{\mathrm{vert}} \sim \mathrm{Poi}(4)$ leads to an average number of 4 vertical jumps. Also, the expected number of vertical jumps is higher than the average amount of horizontal jumps to account for the different dimensions of the spatial domain $\mathbb{X}$. For the heterogeneous medium random field, the designations *horizontal* and *vertical* for describing the discontinuity lines are completely artificial.

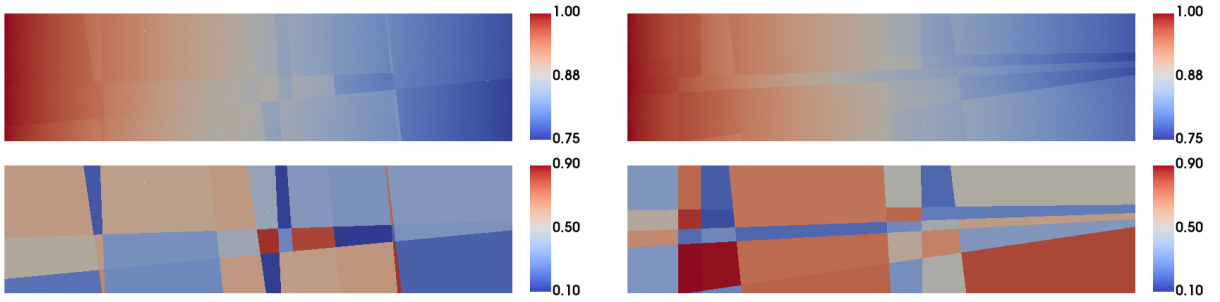
Each line of discontinuity in the jump-advection coefficient $\mathfrak{a}$ consists of a starting point lying on the boundary $\partial\mathbb{X}$ of the spatial domain. Additionally, an angle is associated to each line, defining the angle at which the line is moving into the spatial domain. More precisely, for a random number $\tau_{\mathrm{hor}} \sim \mathrm{Poi}(2)$ of *horizontal* lines, we sample τ_{hor} points on the left boundary, i.e., $\mathfrak{d}_1, \dots, \mathfrak{d}_{\tau_{\mathrm{hor}}} \sim \mathcal{U}\big(\{0\} \times [0, 1/2]\big)$. Each spatial point $\mathfrak{d}_j \in \{0\} \times [0, 1/2]$ defines a starting point of a jump interface on the left domain boundary. Furthermore, for each line, we sample a random angle $\mathfrak{a}_j^{\mathrm{hor}} \in \mathcal{U}\big([0, \pi]\big)$. Connecting each starting point $\mathfrak{d}_j$ with the corresponding point at which the line exits the domain, provides a *horizontal* random partition on the spatial domain $\mathbb{X}$. Note, contrary to the previous piecewise homogeneous random field, this exit point does not necessarily lie on the opposite domain boundary.

To obtain a *vertical* random partition of the spatial domain $\mathbb{X}$, we proceed similarly. For the random number $\tau_{\mathrm{vert}} \sim \mathrm{Poi}(4)$ of vertical flux interfaces, we sample τ_{vert} points on the bottom boundary, i.e., $\mathfrak{d}_{\tau_{\mathrm{hor}}+1}, \dots, \mathfrak{d}_{\tau_{\mathrm{hor}}+\tau_{\mathrm{vert}}} \sim \mathcal{U}\big([0, 2] \times \{0\}\big)$. For each such starting point on the bottom boundary, we sample a random angle $\mathfrak{a}_j^{\mathrm{vert}} \sim \mathcal{U}\big([\pi/2, 3\pi/2]\big)$. Connecting again each starting point $\mathfrak{d}_{\tau_{\mathrm{hor}}+1}, \dots, \mathfrak{d}_{\tau_{\mathrm{hor}}+\tau_{\mathrm{vert}}}$ with the corresponding domain exit point provides a *vertical* random partitioning of the spatial domain $\mathbb{X}$. Let us stress that the discontinuity lines of the heterogeneous medium random field $\mathfrak{a}$ may intersect. Additionally, we do not ensure any distance of the starting or exit points to the corners of the domain. Both of these properties may affect the meshing algorithm in the sense that the jump-adapted triangulations may cause stability problems for the simulation.

With such a random partitioning of the spatial domain $\mathbb{X}$ at hand, it remains to discuss the jump heights $\mathfrak{p}_i$. Here, as in the previous section, every jump height $\mathfrak{p}_i$ of the sequence $(\mathfrak{p}_i, i \in \mathbb{N})$ is independent and identically distributed with a uniform $\mathcal{U}([0.1, 0.9])$ distribution. Therefore, the jump coefficient is again piecewise constant on each part of the resulting partitioning.

Creating a sample-adapted triangulation for these heterogeneous medium random fields can be achieved by the same procedure as for the piecewise homogeneous medium of the previous section. That is, meshing each element of the partition $\mathfrak{T}$ individually is sufficient to obtain a jump-adapted grid. As in the last experiment, we additionally require the resulting mesh to be conforming, which can be ensured by requiring that neighboring subdomain grids share the same nodes on the discontinuity lines of the random field $\mathfrak{a}$.

With this construction in mind, we can investigate the resulting samples of the heterogeneous medium coefficient $\mathfrak{a}$ and its influence on the random entropy solution u. To do so, Figure 5.4 depicts four samples of the random field $\mathfrak{a}$ and the corresponding entropy solutions.



(a) Jump-advection coefficient $\mathfrak{a}$ having very small partition elements and the corresponding solution u.

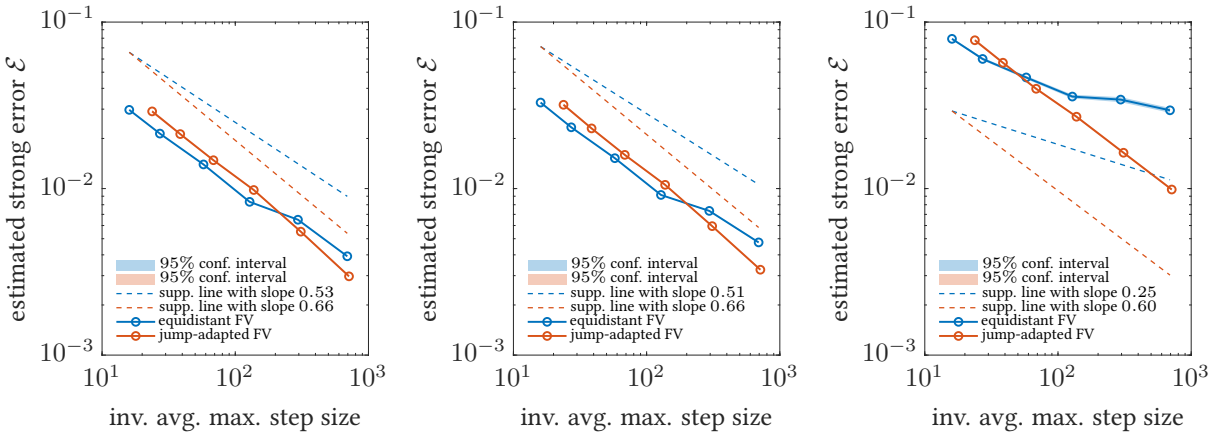
(b) Jump-advection coefficient $\mathfrak{a}$ with a diagonal discontinuity line and the corresponding solution u.

Figure 5.4: Illustration of entropy solutions u (top) with underlying random jump coefficients $\mathfrak{a}$ (bottom) modeled as a heterogeneous medium.

As a first observation, note that the heterogeneous medium coefficient $\mathfrak{a}$ may contain very small partition elements, which is particularly well visible in Figure 5.4a. Additionally, Figure 5.4b illustrates that diagonal discontinuity lines can appear in the partition. To see this, we point to the diagonal line from approximately $(0.3, 0)$ to $(2, 1/4)$. Apparently, the discontinuities in the random jump-advection coefficient $\mathfrak{a}$ lead to jumps in the entropy solution u, which is analogously to the piecewise homogeneous medium setting of the previous section.

Based on these observations and the significantly higher number of domain partitions, one expects lower convergence rates for the finite volume approximations u_Δ. The strong (pathwise) error of these approximations is presented in Figure 5.5 for the $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error. As in the previous section, the strong error depends on the stochastic parameter $\omega \in \Omega$ and consequently needs to be approximated by a Monte Carlo estimator. Here, we used again 50 realizations of the random entropy solution for the corresponding estimation and the 95% confidence intervals in Figure 5.5 show that this number is sufficient to yield an accurate approximation.

As one can see for the $\mathcal{L}^1$- and $\mathcal{L}^2$-error, the convergence rate of the standard triangulation is roughly $1/2$, whereas the convergence of the jump-adapted discretization is approximately $2/3$. This result is rather ambivalent: On the one hand, it is by far not surprising that the sample-adapted discretization



(a) Pathwise $\mathcal{L}^1$-error. (b) Pathwise $\mathcal{L}^2$-error. (c) Pathwise $\mathcal{L}^\infty$-error.

Figure 5.5: Pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error of finite volume (FV) approximations of the random entropy solution u for a heterogeneous medium. Each error is estimated via a Monte Carlo estimator using 50 samples.

approach leads to a better convergence rate. On the other hand, it is rather irritating that the non-adapted triangulation yields a similar order of convergence as for the piecewise homogeneous medium, whereas the convergence rate of the jump-adapted discretization is slightly reduced. Considering the $\mathcal{L}^\infty$-error, the situation is slightly different. Here, the jump-adapted meshing is able to achieve the same convergence rate of approximately 0.6 and the standard triangulation leads to a reduced order of convergence of roughly $1/4$. Focussing on the variance of the random pathwise error, the qualitative behavior of the approximations is similar as for the piecewise homogeneous medium of the previous section. Only for the $\mathcal{L}^\infty$-error with the standard triangulation can one notice the 95% confidence interval. For all other approximations, the variation in the error is too low to be noticable.

II

Random conservation laws with infinitely many flux discontinuities

One-dimensional random conservation laws with infinitely many flux discontinuities

6

In the previous part of this thesis, a well-posedness theory to scalar conservation laws with a random discontinuous flux function was discussed in the unifying framework based on admissibility germs. A major restriction of the presented theory are the assumptions that lead to uniform $\mathcal{L}^\infty$-bounds on entropy solutions, such as the confinement Assumption 3.38. Two examples of important conservation laws, which generally do not satisfy such an assumption, are given by the classical *linear transport equation* or the *Burgers-Hopf's equation*. In both cases the spatial discontinuities may be incorporated via a discontinuous advection coefficient.

In this chapter, we introduce a well-posedness theory for scalar conservation laws with random discontinuous flux functions, which does not require such a confinement assumption. The ideas of the presented approach were originally introduced by Baiti and Jenssen [23] in 1997 and later formalized by Audusse and Perthame [18] in 2005.

To the best of the author's knowledge, the general well-posedness theory of scalar conservation laws presented in [18] is only available for one spatial dimension. Therefore, as in the previous chapters, let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. Furthermore, for some final time $0 < T < \infty$, let a time interval $\mathbb{T} \coloneqq [0, T]$ and a spatial domain $\mathbb{X} \coloneqq \mathbb{R}$ be given. Then, for unknown $u \coloneqq u(\omega, t, x)$, the following random scalar conservation law is considered:

$$\begin{aligned} \partial_t u + \operatorname{div}_x \mathfrak{f}(\omega, x, u) &= 0 && \text{in } \Omega \times \mathbb{T} \times \mathbb{X}\,, \\ u(\omega, 0, x) &= u_0(\omega, x) && \text{on } \Omega \times \{0\} \times \mathbb{X}\,. \end{aligned} \tag{6.1}$$

Here, $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, is a random initial condition. Furthermore, the flux $\mathfrak{f}$ is assumed to depend discontinuously on $x \in \mathbb{X}$. Let us stress that — contrary to the previous part — the flux function does not depend on the time $t \in \mathbb{T}$.

Recall that the classical approach to well-posedness is based on the Kružkov entropy Condition (1.2), which selects a unique weak solution to Problem (6.1) in a physically meaningful way. However, for flux functions that involve spatial discontinuities, the Condition (1.2) is not properly defined anymore. The idea of Baiti and Jenssen [23] and Audusse and Perthame [18] is to *adapt* this entropy condition in such a way that the troublesome third term in Inequality (1.2) vanishes. In particular, the Kružkov entropy values $k \in \mathbb{R}$ are replaced by the solution of the steady-state equation corresponding to parameter $\alpha \in \mathbb{R}$.

Since this approach does not involve interface conditions at the flux discontinuities, we do not need to formulate admissibility germs. Furthermore, the existence of strong one-sided traces of the solution does not have to be ensured at the flux discontinuities. These weaker restrictions enable the flux function to have infinitely many points of discontinuity, which may even have accumulation points, since the discontinuity points do not have to be distinguished separately. Nevertheless, the Audusse-Perthame adapted entropy condition for admissible solutions can be formulated via a corresponding admissibility germ, if the additional assumptions of the previous part are satisfied. For a extensive discussion of the Audusse-Perthame selection criterion via the approach of admissibility germs, the reader is reffered to [13, Section 4.6].

This chapter starts in Section 6.1 with a discussion of Audusse-Perthame flux functions and solutions to the random steady-state equation of Problem (6.1). Afterwards, in Section 6.2 we introduce the notion of adapted entropy solutions and the corresponding admissibility criterion based on an adapted Kružkov entropy condition. Additionally, we introduce in Section 6.2 the notion of random adapted entropy functionals and discuss their properties. These functionals and their properties are important for showing the strong measurability of adapted entropy solutions in Section 6.3, which discusses the well-posedness framework of Audusse-Perthame for random conservation laws with discontinuous flux functions. This includes a pathwise existence and uniqueness investigation and a discussion of the existence of moments of random adapted entropy solutions.

6.1 Audusse-Perthame flux functions & random steady-state solutions

In this section, we discuss the randomization of flux functions $\mathfrak{f}$ in the Audusse-Perthame framework. In particular, these discontinuous fluxes satisfy conditions that ensure the existence of solutions to the steady-state problem of the scalar conservation law. After the discussion of the flux functions in Section 6.1.1, we discuss the steady-state solutions and their properties in Section 6.1.2.

6.1.1 Audusse-Perthame flux functions

We start by introducing random discontinuous flux function that correspond to the Audusse-Perthame framework. As already mention in the introduction of this chapter, the flux functions are allowed have infinitely many points of discontinuity, which may have accumulation points. However, these discontinuities must form a closed set of $\mathcal{H}^1$-measure zero. Furthermore, the flux functions need to satisfy some assumptions that guarantee the pathwise existence of random steady-state solutions. We precise these pathwise conditions on the flux function in the proceeding assumption.

Assumption 6.1 (Audusse-Perthame flux function)**:**
For every stochastic parameter $\omega \in \Omega$, the random discontinuous flux function $\mathfrak{f}$ satisfies the following conditions:

(A-1) The flux function $\mathfrak{f}(\omega, \cdot, \cdot)$ is continuous at all points of $\big(\mathbb{R} \setminus \mathfrak{D}(\omega)\big) \times \mathbb{R}$, where the set $\mathfrak{D}(\omega) \subset \mathbb{R}$ is a closed set of $\mathcal{H}^1$-measure zero that contains the spatial discontinuity points of the flux function $\mathfrak{f}(\omega, \cdot, \rho)$ for some scalar value $\rho \in \mathbb{R}$.

(A-2) *There exist two continuous functions $\mathfrak{f}_-(\omega,\cdot),\mathfrak{f}_+(\omega,\cdot)\in C(\mathbb{R};\mathbb{R})$ such that for all spatial points $x\in\mathbb{R}$ and all scalar values $\rho\in\mathbb{R}$ the condition $\mathfrak{f}_-(\omega,\rho)\leq\big|\mathfrak{f}(\omega,x,\rho)\big|\leq\mathfrak{f}_+(\omega,\rho)$ holds, where $\mathfrak{f}_-(\omega,\cdot)$ is a first decreasing then increasing function that satisfies:*

(i) *The function $\mathfrak{f}_-(\omega,\cdot)$ is nonnegative in the sense that it satisfies $\mathfrak{f}_-(\omega,\rho)\geq 0$ for any scalar value $\rho\in\mathbb{R}$.*

(ii) *The function $\mathfrak{f}_-(\omega,\cdot)$ is unbounded, i.e., it satisfies $\lim_{\rho\to\pm\infty}|\mathfrak{f}_-(\omega,\rho)|=+\infty$.*

(A-3) *There exists a function $u_{\mathfrak{m}}(\omega,x):\Omega\times\mathbb{R}\to\mathbb{R}$ and a constant $\mathfrak{m}_0\in\mathbb{R}$ such that for any spatial point $x\in\mathbb{R}\setminus\mathfrak{D}(\omega)$, the flux function $\mathfrak{f}(\omega,x,\cdot)$ is locally Lipschitz continuous and one-to-one from $\big(-\infty,u_{\mathfrak{m}}(\omega,x)\big]$ and $\big[u_{\mathfrak{m}}(\omega,x),+\infty\big)$ to $[\mathfrak{m}_0,+\infty)$ (or $(-\infty,\mathfrak{m}_0]$). Furthermore, the flux function $\mathfrak{f}$ satisfies $\mathfrak{f}\big(\omega,x,u_{\mathfrak{m}}(\omega,x)\big)=\mathfrak{m}_0$ with a common Lipschitz constant L_I for all spatial points $x\in\mathbb{R}\setminus\mathfrak{D}(\omega)$ and all scalar values $\rho\in I$, where $I\subset\mathbb{R}$ is any bounded interval.*

Alternatively, instead of Assumption (A-3), we may consider the assumption

(A-3') *For all spatial points $x\in\mathbb{R}\setminus\mathfrak{D}(\omega)$, the flux function $\mathfrak{f}(\omega,x,\cdot)$ is locally Lipschitz continuous and one-to-one from $\mathbb{R}$ to $\mathbb{R}$ with a common Lipschitz constant L_I for all points $x\in\mathbb{R}\setminus\mathfrak{D}(\omega)$ and all values $\rho\in I$, where $I\subset\mathbb{R}$ is any bounded interval.* ◆

At a first sight, these assumptions may look rather complicated. The following example demonstrates an important type of random discontinuous flux functions, and discusses conditions for the Audusse-Perthame flux Assumption 6.1 to be satisfied in a more specific way. Furthermore, the example also illustrates that the linear transport equation and the Burgers' equation are covered by this theory.

Example 6.2 (Multiplicative flux function): *An important type of random discontinuous flux functions are those functions, which admit a multiplicative formulation in the sense that their (random) spatial dependence can be separated from the part depending on the solution. More specifically, those flux functions $\mathfrak{f}$ are given by*

$$\mathfrak{f}(\omega,x,\xi)=\mathfrak{a}(\omega,x)f(\xi)\,.$$

Here, $\mathfrak{a}$ is a random jump coefficient. Based on this form of the flux function, we now discuss conditions such that the Audusse-Perthame flux Assumption 6.1 is satisfied:

- *To satisfy Assumption (A-1), the random jump discontinuities of $\mathfrak{a}(\omega,\cdot)$ need to form a set $\mathfrak{D}(\omega)$, which might depend on the stochastic parameter $\omega\in\Omega$. Furthermore, this set of discontinuities has to be a closed set of $\mathcal{H}^1$-measure zero.*
- *For Assumption (A-2) to hold, it is sufficient that the flux function f is unbounded in the sense that it satisfies $\lim_{\rho\to\pm\infty}|f(\rho)|=+\infty$ and the random jump coefficient $\mathfrak{a}$ has positive spatially bounded paths in the sense that it satisfies*

  $$0<\mathfrak{a}_-(\omega)\leq\mathfrak{a}(\omega,x)\leq\mathfrak{a}_+(\omega)<\infty\,.$$

  Here, $\mathfrak{a}_-$ and $\mathfrak{a}_+$ denote some constants that might depend on $\omega\in\Omega$.
- *For Assumption (A-3') to hold, it is sufficient that the flux function f is locally Lipschitz continuous and strictly monotone. One example for this case is the linear transport equation, in which case the flux function f satisfies $f(\xi)=\xi$.*

- *For Assumption (A-3) to hold, the flux f should be locally Lipschitz and either convex or concave with $f(u_{\mathbb{m}}) = 0$ for some value $u_{\mathbb{m}}$. The latter also implies $\mathbb{m}_0 = 0$. One example for this is the Burgers' equation, in which the flux f is given as $f(\xi) = \xi^2/2$.* ◆

The Audusse-Perthame flux Assumption 6.1 on the flux function $\mathfrak{f}$ immediately provides us with additional information regarding the discontinuous-flux Problem (6.1). A particular consequence is the pathwise boundedness result on the random flux function $\mathfrak{f}$ for some given bounded data $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. We formalize this result in the following corollary.

Corollary 6.3 (Pathwise boundedness of flux function)**:**
Let a stochastic parameter $\omega \in \Omega$ be fixed and let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1. Furthermore, let a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be given and denote by $\mathcal{W}$ the interval $\mathcal{W} \coloneqq \big[-\|\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}, \|\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})} \big] \subset \mathbb{R}$. Then, the flux function $\mathfrak{f}$ admits a pathwise upper bound in the sense that the estimate

$$\Big| \mathfrak{f}\big(\omega, x, \nu(t,x)\big) \Big| \leq \max_{v \in \mathcal{W}} \mathbb{f}_+(\omega, v) =: \mathcal{M}_{\mathfrak{f}}(\omega) < \infty \tag{6.2}$$

holds for $\mathcal{H}^1$-almost every spatial point $x \in \mathbb{R}$ and $\mathcal{H}^1$-almost every time $t \in \mathbb{T}$. ◆

Proof. Let a stochastic parameter $\omega \in \Omega$ and a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. By Assumption (A-2), there exists a continuous function $\mathbb{f}_+(\omega, \cdot) \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ such that

$$\Big| \mathfrak{f}\big(\omega, x, \nu(t,x)\big) \Big| \leq \mathbb{f}_+\big(\omega, \nu(t,x)\big)$$

holds for $\mathcal{H}^1$-almost every $x \in \mathbb{R}$ and $\mathcal{H}^1$-almost every $t \in \mathbb{T}$. This can further be estimated as

$$\Big| \mathfrak{f}\big(\omega, x, \nu(t,x)\big) \Big| \leq \mathbb{f}_+\big(\omega, \nu(t,x)\big) \leq \sup_{v \in \mathcal{W}} \mathbb{f}_+(\omega, v) \,, \tag{6.3}$$

where the interval $\mathcal{W}$ is given by $\mathcal{W} \coloneqq \big[-\|\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}, \|\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})} \big] \subset \mathbb{R}$. Obviously, the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ satisfies $\nu(t, \boldsymbol{x}) \in \mathcal{W}$ for $\mathcal{H}^1$-almost every spatial point x and $\mathcal{H}^1$-almost every time $t \in \mathbb{T}$, which justifies the Estimation (6.3).

By construction, the interval $\mathcal{W} \subset \mathbb{R}$ is compact. Since continuous real-valued functions attain a maximum on compact intervals, the supremum is equivalent to this maximum. Denoting this pathwise maximum of $\mathbb{f}_+(\omega, \cdot)$ over the interval $\mathcal{W}$ by $\mathcal{M}_{\mathfrak{f}}(\omega)$, the same argumentation justifies $\mathcal{M}_{\mathfrak{f}}(\omega) < \infty$. This proves the assertion. ■

The idea of Audusse and Perthame is to replace the Kružkov entropy values $k \in \mathbb{R}$ in Inequality (1.2) by the solutions to the steady-state problem corresponding to Equation (6.1). The following result on the inverse flux function is crucial to ensure the existence of such steady-state solutions. As for the previous corollary, this result is an immediate consequence of the Audusse-Perthame flux Assumption 6.1.

Corollary 6.4 (Existence of inverse flux function)**:**
Let a stochastic parameter $\omega \in \Omega$ be fixed and let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1. Then, for every spatial point $x \in \mathbb{R} \setminus \mathfrak{D}(\omega)$, the flux function $\mathfrak{f}(\omega, x, \cdot)$ admits an inverse function $\mathfrak{f}_\pm^{-1}(\omega, x, \cdot)$, which is continuous. ◆

Proof. Let a stochastic parameter $\omega \in \Omega$ be fixed and let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1. We distinguish the two cases of Assumption (A-3) and (A-3') separately:

(i) *Assumption (A-3')*: We start with the situation that the flux function $\mathfrak{f}$ satisfies Assumption (A-3'), which guarantees that for every spatial point $x \in \mathbb{R} \setminus \mathfrak{D}(\omega)$, the function $\mathfrak{f}(\omega, x, \cdot)$ is locally Lipschitz continuous and one-to-one from $\mathbb{R}$ to $\mathbb{R}$. Thereby, the flux function $\mathfrak{f}(\omega, x, \cdot)$ is strictly monotone. This allows us to apply the inverse function theorem for strictly monotone functions [132, Theorem 37.1] to obtain that the flux $\mathfrak{f}(\omega, x, \cdot)$ admits an inverse function $\mathfrak{f}^{-1}(\omega, x, \cdot)$ mapping from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, the inverse flux function $\mathfrak{f}^{-1}(\omega, x, \cdot)$ is strictly monotone and continuous.

(ii) *Assumption (A-3)*: In case the flux function $\mathfrak{f}$ satisfies Assumption (A-3), for any spatial point $x \in \mathbb{R} \setminus \mathfrak{D}(\omega)$, the flux function $\mathfrak{f}(\omega, x, \cdot)$ is locally Lipschitz continuous and one-to-one from $\big(-\infty, u_{\mathfrak{m}}(\omega, x)\big]$ and $\big[u_{\mathfrak{m}}(\omega, x), +\infty\big)$ to $[\mathfrak{m}_0, +\infty)$ (or $(-\infty, \mathfrak{m}_0]$). This implies that the flux function $\mathfrak{f}(\omega, x, \cdot)$ is strictly monotone mapping from $\big(-\infty, u_{\mathfrak{m}}(\omega, x)\big]$ and $\big[u_{\mathfrak{m}}(\omega, x), +\infty\big)$ to $[\mathfrak{m}_0, +\infty)$ (or $(-\infty, \mathfrak{m}_0]$). Therefore, the inverse function theorem for strictly monotone functions [132, Theorem 37.1] guarantees the existence of a strictly monotone and continuous inverse function $\mathfrak{f}_-^{-1}(\omega, x, \cdot)$ mapping from $\big(-\infty, u_{\mathfrak{m}}(\omega, x)\big]$ to $[\mathfrak{m}_0, +\infty)$ (or $(-\infty, \mathfrak{m}_0]$). An analogous argumentation yields the existence of a strictly monotone and continuous inverse function $\mathfrak{f}_+^{-1}(\omega, x, \cdot)$ mapping from $\big[u_{\mathfrak{m}}(\omega, x), +\infty\big)$ to $[\mathfrak{m}_0, +\infty)$ (or $(-\infty, \mathfrak{m}_0]$).

To unify the notation for those different cases, we denote the inverse function of the flux $\mathfrak{f}(\omega, x, \cdot)$ by generically writing $\mathfrak{f}_\pm^{-1}(\omega, x, \cdot)$, unless we want to emphasize the particular instance of $\mathfrak{f}_+^{-1}, \mathfrak{f}_-^{-1}$ or $\mathfrak{f}^{-1}$. Therefore, we have proven the assertion on the existence of a strictly monotone and continuous inverse flux function $\mathfrak{f}_\pm^{-1}(\omega, x, \cdot)$. ■

To ensure the well-posedness of adapted entropy solutions to the discontinuous-flux conservation law of Equation (6.1), we need to establish existence of a unique solution as well as its strong measurability. However, to obtain such a strong measurability result, we need to guarantee that the flux function $\mathfrak{f}$ is itself measurable (but not necessarily *strongly* measurable). Thus, we impose the following measurability assumption on Audusse-Perthame flux functions.

Assumption 6.5 (Measurability of Audusse-Perthame flux function)**:**
Let $\mathfrak{f}$ be a flux function that satisfies the Audusse-Perthame flux Assumption 6.1. Additionally, we assume that the flux function $\mathfrak{f}$ satisfies the following stochastic measurability assumptions:

(A-4) For $\mathcal{H}^1$-almost every spatial point $x \in \mathbb{R}$ and every scalar value υ, the flux function $\mathfrak{f}$ is stochastically measurable in the sense that the mapping $\omega \mapsto \mathfrak{f}(\omega, x, \upsilon)$ is measurable.

(A-5) For almost every point $x \in \mathbb{R}$ and every value υ, the inverse flux function $\mathfrak{f}_\pm^{-1}$ is stochastically measurable in the sense that the mapping $\omega \mapsto \mathfrak{f}_\pm^{-1}(\omega, x, \upsilon)$ is measurable.

(A-6) For every $\omega \in \Omega$, the spatial paths of the inverse flux function $\mathfrak{f}_\pm^{-1}$ are bounded in the sense that there exists a function $\mathbb{w}(\omega, \cdot) \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ such that the condition $\big|\mathfrak{f}_\pm^{-1}(\omega, x, \rho)\big| \leq \mathbb{w}(\omega, \rho)$ is satisfied for all scalar values $\rho \in \mathbb{R}$. ◆

To conclude the discussion of Audusse-Perthame flux functions, we demonstrate this measurability assumption with the subsequent example. Therefore, we revise the multiplicative flux functions already discussed in Example 6.2.

Example 6.6 (Measurability of multiplicative flux function): *Recall the construction of multiplicative flux functions in Example 6.2. For a random jump coefficient* $\mathfrak{a}$*, these were given by*

$$\mathfrak{f}(\omega, x, \xi) = \mathfrak{a}(\omega, x) f(\xi) \, .$$

As a result, the measurability Assumption 6.5 transforms to the following conditions:

- *To satisfy Assumption (A-4), for almost every spatial point* $x \in \mathbb{R}$*, the random jump coefficient* $\mathfrak{a}$ *is measurable as a mapping* $\omega \mapsto \mathfrak{a}(\omega, x)$*.*
- *To ensure the conditions implied by Assumption (A-5), note that the inverse flux function is given by*

$$\mathfrak{f}_{\pm}^{-1}(\omega, x, \xi) = \frac{1}{\mathfrak{a}(\omega, x)} f^{-1}(\xi) \, .$$

  Since the function f^{-1} *is continuous by Corollary 6.4, it is in particular measurable. Thus, the flux function* $\mathfrak{f}$ *is measurable as the composition of two measurable functions. Consequently, for multiplicative flux functions* $\mathfrak{f}$ *satisfying the Audusse-Perthame flux Assumption 6.1 and Assumption (A-4), the Assumption (A-5) is automatically satisfied.*
- *Finally, recall that the random jump coefficient* $\mathfrak{a}$ *has spatially bounded paths in the sense that the estimation* $0 < \mathfrak{a}_-(\omega) \leq \mathfrak{a}(\omega, x) \leq \mathfrak{a}_+(\omega) < \infty$ *holds for every* $\omega \in \Omega$*. Therefore, we can define a function* $\mathfrak{w}$ *as* $\mathfrak{w}(\omega, \rho) := {}^{1}/_{\mathfrak{a}_-(\omega)} f^{-1}(\rho)$ *to satisfy the spatial boundedness Assumption (A-6) on the inverse flux function.* ◆

6.1.2 Random steady-state solutions

As already indicated in the introduction of this chapter, the idea of the Audusse-Perthame admissibility criterion is to replace the Kružkov entropy values $k \in \mathbb{R}$ in the Condition (1.2) by the solutions of the steady-state problem corresponding to the initial value Problem (6.1). With the preliminary investigations of the flux functions satisfying the Audusse-Perthame flux Assumption 6.1, we are now ready to investigate these steady-state solutions and their properties. Therefore, let the following steady-state equation be given:

$$\mathfrak{f}\big(\omega, x, k_\alpha(\omega, x)\big) = \alpha \qquad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R} \, . \tag{6.4}$$

By the existence result of Corollary 6.4, we already know that the flux function $\mathfrak{f}(\omega, x, \cdot)$ admits a continuous and strictly monotone inverse. Thereby, we obtain the existence of a steady-state solution k_α or two steady-state solutions $k_\alpha^{\pm}$, depending on whether Assumption (A-3') or Assumption (A-3) is satisfied. As for the inverse flux function, we will always use the superscript $\pm$ to unify the notation and presentation. For the details we refer to the proof of Corollary 6.4. With these steady-state solutions at hand, we can now investigate their properties. First, let us stress that Assumption (A-2) implies that for each parameter $\alpha \in \mathbb{R}$, the corresponding steady-state solution satisfies $k_\alpha^{\pm} \in \mathcal{L}^\infty(\mathbb{X}; \mathbb{R})$. In case Assumption (A-3) is satisfied, the domain of the steady-state parameter α is either $[\mathfrak{m}_0, \infty)$ or $(-\infty, \mathfrak{m}_0]$, depending on the flux function $\mathfrak{f}$. In the sequel, we will always consider this case. Furthermore, the next proposition shows that steady-state solutions $k_\alpha^{\pm}$ depend continuously on the parameter α.

Proposition 6.7 (Continuous dependence of steady-state solutions):
Let $\omega \in \Omega$ *be fixed. Furthermore, let the function* $\mathfrak{f}$ *satisfy the Audusse-Perthame flux Assumption 6.1. Then, for every point* $x \in \mathbb{R} \setminus \mathfrak{D}(\omega)$*, the solutions* $k_\alpha^{\pm}(\omega, x)$ *of the random steady-state Equation* (6.4) *depend continuously on the parameter* $\alpha \in [\mathfrak{m}_0, \infty)$ *(or* $\alpha \in (-\infty, \mathfrak{m}_0]$*).* ◆

Proof. Let $\omega \in \Omega$ be fixed. Furthermore, let two parameters $\alpha, \beta \in [\mathfrak{m}_0, \infty)$ (or $\alpha, \beta \in (-\infty, \mathfrak{m}_0]$) be given and consider the two corresponding random steady-state equations

$$\mathfrak{f}\big(\omega, x, k_\alpha^\pm(\omega, x)\big) = \alpha \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,, \tag{6.5a}$$
$$\mathfrak{f}\big(\omega, x, k_\beta^\pm(\omega, x)\big) = \beta \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,. \tag{6.5b}$$

By Corollary 6.4, for every stochastic parameter $\omega \in \Omega$ and every spatial point $x \in \mathbb{R} \setminus \mathfrak{D}(\omega)$, the flux function $\mathfrak{f}$ admits an inverse function $\mathfrak{f}_\pm^{-1}(\omega, x, \cdot)$. This implies the existence of the two steady-state solutions $k_\alpha^\pm, k_\beta^\pm$ to the steady-state Equations (6.5a) and (6.5b), respectively. Furthermore, these steady-state solutions are given by

$$\begin{aligned} k_\alpha^\pm(\omega, x) &= \mathfrak{f}_\pm^{-1}(\omega, x, \alpha) \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,, \\ k_\beta^\pm(\omega, x) &= \mathfrak{f}_\pm^{-1}(\omega, x, \beta) \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,. \end{aligned} \tag{6.6}$$

However, Corollary 6.4 also states that the inverse flux function $\mathfrak{f}_\pm^{-1}(\omega, x, \cdot)$ is continuous. Therefore, the construction of the steady-state solutions in Equation (6.6) immediately leads to the continuous dependence of $k_\alpha^\pm(\omega, x)$ on $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$), which proves the assertion. ■

Another important property of the random steady-state solutions $k_\alpha^\pm$ is their stochastic measurability. This property is directly inherited from the measurability Assumption 6.5 on the flux function. The corresponding result is justified in the subsequent Proposition, which also concludes this section on Audusse-Perthame flux functions and random steady-state solutions.

Proposition 6.8 (Measurability of steady-state solutions)**:**
Let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1 as well as the measurability Assumption 6.5. Then, for any steady-state parameter $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$), the solutions $k_\alpha^\pm$ to the random steady-state Equation (6.4) *are stochastically measurable in the sense that, for $\mathcal{H}^1$-almost every spatial point $x \in \mathbb{R}$, the mapping $\omega \mapsto k_\alpha^\pm(\omega, x)$ is measurable.* ◆

Proof. Since the flux function $\mathfrak{f}$ satisfies the Audusse-Perthame flux Assumption 6.1, we can apply Corollary 6.4 to obtain the existence of the steady-state solutions $k_\alpha^\pm$. Furthermore, these steady-state solutions are given by

$$k_\alpha^\pm(\omega, x) = \mathfrak{f}_\pm^{-1}(\omega, x, \alpha) \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,. \tag{6.7}$$

However, Assumption (A-5) implies that for almost every point $x \in \mathbb{R}$, the inverse flux function $\mathfrak{f}_\pm^{-1}$ is measurable in the sense that the mapping $\omega \mapsto \mathfrak{f}_\pm^{-1}(\omega, x, v)$ is measurable. Thus, the assertion follows via the construction of the steady-state solutions $k_\alpha^\pm$ in Equation (6.7). ■

6.2 Random adapted entropy solutions and functionals

With the random steady-state solutions of the previous section at hand, we can now turn to defining the notion of an adapted entropy solution. Closely related to the adapted entropy condition selecting the admissible solution to Problem (6.1) is the notion of an *adapted entropy functional*. These functionals provide a tool for describing the admissibility of functions $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ for fixed ingredients of the adapted entropy condition.

6.2.1 Notion of random adapted entropy solutions and functionals

We start our discussion of random adapted entropy solutions and random adapted entropy functionals by introducing the main definitions. To simplify the presentation, we start by formally defining the *adapted entropy flux* in the following definition.

Definition 6.9 (Adapted entropy flux)**:**
Let the flux function $\mathfrak{f}$ *satisfy the Audusse-Perthame flux Assumption 6.1. Then, we define the* adapted Kružkov entropy flux *as the mapping* $\mathfrak{q} : \Omega \times \mathbb{X} \times \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ *given by*

$$\mathfrak{q}(\omega, x, v, \tilde{v}) := \operatorname{sign}(v - \tilde{v})\big(\mathfrak{f}(\omega, x, v) - \mathfrak{f}(\omega, x, \tilde{v})\big) \,, \tag{6.8}$$

for some scalar values $v, \tilde{v} \in \mathbb{R}$. ◆

With this adapted entropy flux and the investigation of the previous section at hand, we are now able to define the notion of *random adapted entropy solutions*. Here, the crucial idea for admissibility is the usage of the steady-state solutions $k_\alpha^\pm$ as *adapted* Kružkov entropies in the entropy inequality. We formalize this in the next definition.

Definition 6.10 (Adapted entropy solution)**:**
Let the stochastic parameter $\omega \in \Omega$ *be fixed and let the flux function* $\mathfrak{f}$ *satisfy the Audusse-Perthame flux Assumption 6.1. A function* $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})\big)$ *is called an* adapted entropy solution *to the scalar discontinuous-flux conservation law given by Equation* (6.1) *on* $\mathbb{T} \times \mathbb{R}$*, provided that for each steady-state parameter* $\alpha \in [\mathbb{m}_0, \infty)$ *(or* $\alpha \in (-\infty, \mathbb{m}_0]$*) and the corresponding two steady-state solutions* $k_\alpha^\pm$ *of Equation* (6.4)*, the adapted entropy inequality*

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\mathbb{X}} \big|u(\omega, t, x) - k_\alpha^\pm(\omega, x)\big| \partial_t \psi(t, x) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}\big(\omega, x, u(\omega, t, x), k_\alpha^\pm(\omega, x)\big) \partial_x \psi(t, x) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{\mathbb{X}} \big|u_0(\omega, x) - k_\alpha^\pm(\omega, x)\big| \psi(0, x) \,\mathrm{d}x \;\geq\; 0 \end{aligned} \tag{6.9}$$

holds for every nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. ◆

The adapted entropy Inequality (6.9) is also the main ingredient for defining *adapted entropy functionals*. These functionals evaluate for fixed parameters of Condition (6.9), whether a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ is admissible for this particular parameter configuration. Furthermore, these functionals are of major importance for showing strong measurability of random adapted entropy solutions in Section 6.3. The following definition specifies this construction.

Definition 6.11 (Adapted entropy functional)**:**
Let the flux function $\mathfrak{f}$ *satisfy the Audusse-Perthame flux Assumption 6.1. Furthermore, let a steady-state parameter* $\alpha \in [\mathbb{m}_0, \infty)$ *(or* $\alpha \in (-\infty, \mathbb{m}_0]$*) and a nonnegative test function* $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ *be fixed. Then, we define the* random adapted entropy functional $\mathbb{J}_\psi^\alpha$ *associated to Problem* (6.1) *as the mapping*

$\mathbb{J}_\psi^\alpha : \Omega \times \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R}) \to \mathbb{R}$ *given by*

$$(\omega,\nu) \mapsto \int_{\mathbb{T}}\int_{\mathbb{X}} \big|\nu(t,x) - k_\alpha^\pm(\omega,x)\big|\partial_t\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{6.10a}$$

$$+ \int_{\mathbb{T}}\int_{\mathbb{X}} \mathfrak{q}\big(\omega,x,\nu(t,x),k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{6.10b}$$

$$+ \int_{\mathbb{X}} \big|u_0(\omega,x) - k_\alpha^\pm(\omega,x)\big|\psi(0,x)\,\mathrm{d}x\,. \tag{6.10c}$$

Here, $k_\alpha^\pm$ denotes the two steady-state solutions of Equation (6.4) *corresponding to the steady-state parameter $\alpha \in [\mathfrak{m}_0,\infty)$ (or $\alpha \in (-\infty,\mathfrak{m}_0]$).* ◆

6.2.2 Random adapted entropy functionals are Carathéodory

We continue our discussion on random adapted entropy functionals by investigating their properties. This subsection is devoted to showing that the functional $\mathbb{J}_\psi^\alpha$ is Carathéodory, which means that it is measurable in $\omega \in \Omega$ and continuous $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. The corresponding continuity result is proven in the succeeding proposition.

Proposition 6.12 (Continuous dependence of adapted entropy functional)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \le q < \infty$ and $1 \le p \le \infty$, be a random initial condition to Problem (6.1)*. Furthermore, let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1 and let a steady-state parameter $\alpha \in [\mathfrak{m}_0,\infty)$ (or $\alpha \in (-\infty,\mathfrak{m}_0]$) and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. Then, for fixed stochastic parameter $\omega \in \Omega$, the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$.* ◆

Proof. Let $\omega \in \Omega$ and a steady-state parameter $\alpha \in [\mathfrak{m}_0,\infty)$ (or $\alpha \in (-\infty,\mathfrak{m}_0]$) be fixed. Furthermore, let a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. To show the continuous dependence of the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ on the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$, we consider the three integrals of mapping (6.10) separately:

The continuous dependence of Integral (6.10c) is rather obvious, since it is independent of the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Turning to Integral (6.10a), one can observe that the term $\big|\nu(t,x) - k_\alpha^\pm(\omega,x)\big|$ satisfies $\big|\nu(t,\cdot) - k_\alpha^\pm(\omega,\cdot)\big| \in \mathcal{L}^\infty(\mathbb{X};\mathbb{R})$, since both ν and $k_\alpha^\pm(\omega,\cdot)$ are essentially bounded. Furthermore, the test function ψ and its time derivative $\partial_t\psi$ are compactly supported and smooth by hypothesis. This implies that the integral is finite. The continuous dependence of Integral (6.10a) follows via the dominated convergence theorem. We refer to [98, Theorem 5.6] for details on this argumentation.

It remains to show the continuous dependence of Integral (6.10b). Therefore, let two essentially bounded functions $u,\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be given. The idea of the proof is to establish the estimation

$$\Big|\int_{\mathbb{T}}\int_{\mathbb{X}} \mathfrak{q}\big(\omega,x,u(t,x),k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}}\int_{\mathbb{X}} \mathfrak{q}\big(\omega,x,\nu(t,x),k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \le C_{\mathbb{J}}\|u-\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}\,.$$

Recall that the adapted Kružkov entropy flux $\mathfrak{q} : \Omega\times\mathbb{X}\times\mathbb{R}\times\mathbb{R} \to \mathbb{X}$ depends on the discontinuous

flux function $\mathfrak{f}$ and was introduced in Definition 6.9 as the mapping

$$\mathfrak{q}(\omega, x, v, \tilde{v}) \coloneqq \operatorname{sign}(v - \tilde{v})\big(\mathfrak{f}(\omega, x, v) - \mathfrak{f}(\omega, x, \tilde{v})\big) \,,$$

for some scalar values $v, \tilde{v} \in \mathbb{R}$. To facilitate the computations involving the sign function, we introduce four random sets $\mathbb{D}^{(\cdot)}(\omega) \subset \mathbb{T} \times \mathbb{X}$, which depend on the values of the terms $\operatorname{sign}\big(u(t,x) - k_\alpha^\pm(\omega,x)\big)$ and $\operatorname{sign}\big(\nu(t,x) - k_\alpha^\pm(\omega,x)\big)$. Specifically, for every stochastic parameter $\omega \in \Omega$, we define

$$\begin{aligned}
\mathbb{D}^{=}(\omega) &\coloneqq \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid \operatorname{sign}\big(u(t,x) - k_\alpha^\pm(\omega,x)\big) = \operatorname{sign}\big(\nu(t,x) - k_\alpha^\pm(\omega,x)\big)\Big\},\\
\mathbb{D}^{\pm}(\omega) &\coloneqq \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid \operatorname{sign}\big(u(t,x) - k_\alpha^\pm(\omega,x)\big) = -\operatorname{sign}\big(\nu(t,x) - k_\alpha^\pm(\omega,x)\big)\Big\},\\
\mathbb{D}_k^{u}(\omega) &\coloneqq \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid u(t,x) = k_\alpha^\pm(\omega,x)\Big\},\\
\mathbb{D}_k^{\nu}(\omega) &\coloneqq \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid \nu(t,x) = k_\alpha^\pm(\omega,x)\Big\}.
\end{aligned}$$

By the construction of these sets, for every stochastic parameter $\omega \in \Omega$, their union corresponds to $\mathbb{D}^{=}(\omega) \cup \mathbb{D}^{\pm}(\omega) \cup \mathbb{D}_k^{u}(\omega) \cup \mathbb{D}_k^{\nu}(\omega) = \mathbb{T} \times \mathbb{X}$. However, these random sets are not necessarily disjoint. Nevertheless, for any stochastic parameter $\omega \in \Omega$, the intersection is given by the set

$$\mathbb{D}^{=}(\omega) \cap \mathbb{D}^{\pm}(\omega) \cap \mathbb{D}_k^{u}(\omega) \cap \mathbb{D}_k^{\nu}(\omega) = \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid u(t,x) = k_\alpha^\pm(\omega,x) = \nu(t,x)\Big\} \,.$$

Thus, the adapted entropy flux $\mathfrak{q}$ vanishes on this intersection. Therefore, we can split the integral over $\mathbb{T} \times \mathbb{X}$ into the sum of the integrals over these four random sets by the linearity of integration. Additionally applying the triangle inequality, we obtain

$$\begin{aligned}
\Big| \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}\big(\omega, x, u(t,x), k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}\big(\omega, x, \nu(t,x), k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
\leq \Big| \int_{\mathbb{D}^{=}(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x), \nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \Big| && \text{(6.11a)}\\
+ \Big| \int_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x), \nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \Big| && \text{(6.11b)}\\
+ \Big| \int_{\mathbb{D}_k^{u}(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x), \nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \Big| && \text{(6.11c)}\\
+ \Big| \int_{\mathbb{D}_k^{\nu}(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x), \nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \Big| \,. && \text{(6.11d)}
\end{aligned}$$

Here, to simplify the readability, we introduce the function $\mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}$, which describes the difference of the two adapted Kružkov entropy flux functions and is given as

$$\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}(v,\tilde{v}) = \mathfrak{q}\big(\omega, x, v, k_\alpha^\pm(\omega,x)\big) - \mathfrak{q}\big(\omega, x, \tilde{v}, k_\alpha^\pm(\omega,x)\big) \,.$$

To proceed with proving continuous dependence of Term (6.10b), we consider each integral in the above Inequality (6.11) separately:

Continuous dependence of Integral (6.11a). For the Integral (6.11a), we start estimating

$$\Big| \int\limits_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int\limits_{\mathbb{D}^=(\omega)} \Big|\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

With the definitions of the function $\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}$ and the entropy flux $\mathfrak{q}$, we can exploit the construction of the random set $\mathbb{D}^=(\omega)$ to obtain

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ \leq \int\limits_{\mathbb{D}^=(\omega)} \Big|\mathfrak{f}\big(\omega,x,u(t,x)\big)-\mathfrak{f}\big(\omega,x,\nu(t,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Recall now, that the flux function $\mathfrak{f}(\omega,x,\cdot)$ is locally Lipschitz continuous by Assumption (A-3) or Assumption (A-3'). Furthermore, by hypothesis there exists an interval $I\subset\mathbb{R}$, such that $u(t,x)$ and $\nu(t,x)$ are contained in I for all $(t,x)\in\mathbb{T}\times\mathbb{X}$, since the functions u,ν satisfy $u,\nu\in\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Consequently, we can continue the estimation via

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| &\leq \int\limits_{\mathbb{D}^=(\omega)} L_I\big|u(t,x)-\nu(t,x)\big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C_\psi\|u-\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}\,.\end{aligned}$$

Here, the last estimation is possible, because the test function ψ is smooth and compactly supported, which implies that also its spatial derivative $\partial_x\psi$ is smooth with compact support. This implies that the derivative is bounded and the integral finite, which concludes the proof of showing the continuous dependence of Integral (6.11a).

Continuous dependence of Integral (6.11b). For Integral (6.11b), we start by estimating

$$\Big| \int\limits_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int\limits_{\mathbb{D}^\pm(\omega)} \Big|\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

Now, the definitions of the function $\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}$ and the entropy flux $\mathfrak{q}$ admit leveraging the construction of the random set $\mathbb{D}^\pm(\omega)$, which yields

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ &\leq \int\limits_{\mathbb{D}^\pm(\omega)} \Big|\mathfrak{q}\big(\omega,x,u(t,x),k_\alpha^\pm(\omega,x)\big)-\mathfrak{q}\big(\omega,x,\nu(t,x),k_\alpha^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int\limits_{\mathbb{D}^\pm(\omega)} \Big|\mathfrak{f}\big(\omega,x,u(t,x)\big)+\mathfrak{f}\big(\omega,x,\nu(t,x)\big)-2\mathfrak{f}\big(\omega,x,k_\alpha^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Recall that the flux $\mathfrak{f}(\omega,x,\cdot)$ is locally Lipschitz continuous by Assumption (A-3) or Assumption (A-3'). By hypothesis there exists an interval $I\subset\mathbb{R}$, such that $u(t,x)$ and $\nu(t,x)$ are contained in I for all $(t,x)\in\mathbb{T}\times\mathbb{X}$, since the functions u,ν satisfy $u,\nu\in\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Additionally, for each $\omega\in\Omega$ and each parameter $\alpha\in[\mathfrak{m}_0,\infty)$ (or $\alpha\in(-\infty,\mathfrak{m}_0]$), Assumption (A-2) ensures that the steady-state

solutions $k_\alpha^\pm$ satisfy $k_\alpha^\pm \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Therefore, by applying the triangle inequality and the local Lipschitz continuity afterwards, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ \leq L_I \int_{\mathbb{D}^\pm(\omega)} \Big(\big|u(t,x)-k_\alpha^\pm(\omega,x)\big|+\big|\nu(t,x)-k_\alpha^\pm(\omega,x)\big|\Big)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

So far, we do not explicitly know that the above integrals are well defined. However, their existence follows via the dominated convergence theorem from the subsequent estimation. For any two values $\xi_1,\xi_2\in\mathbb{R}$, the identity $|\xi_1-\xi_2| = \operatorname{sign}(\xi_1-\xi_2)(\xi_1-\xi_2)$ holds. Combining this identity with the construction of the random set $\mathbb{D}^\pm(\omega)$ allows us to further estimate

$$\begin{aligned}&\Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ &\qquad\leq L_I \int_{\mathbb{D}^\pm(\omega)} \operatorname{sign}\big(u(t,x)-k_\alpha^\pm(\omega,x)\big)\big(u(t,x)-\nu(t,x)\big)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\leq L_I \int_{\mathbb{D}^\pm(\omega)} \big|u(t,x)-\nu(t,x)\big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Since the test function ψ is smooth and compactly supported, its spatial derivative $\partial_x\psi$ is bounded. Therefore, we obtain the estimate

$$\Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq L_I C_\psi \|u-\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}\,,$$

which shows that the Integral (6.11b) depends continuously on $\nu\in\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$.

Continuous dependence of Integral (6.11c). To show the continuous dependence of Integral (6.11c), we start with the estimation

$$\Big|\int_{\mathbb{D}_k^u(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int_{\mathbb{D}_k^u(\omega)} \Big|\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

With the definition of the function $\mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}$ and the entropy flux $\mathfrak{q}$, leveraging the construction of the random set $\mathbb{D}_k^u(\omega)$ leads to the estimation

$$\begin{aligned}&\Big|\int_{\mathbb{D}_k^u(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ &\qquad\leq \int_{\mathbb{D}_k^u(\omega)} \Big|\operatorname{sign}\big(\nu(t,x)-k_\alpha^\pm(\omega,x)\big)\Big(\mathfrak{f}\big(\omega,x,\nu(t,x)\big)-\alpha\Big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

where we have used the steady-state Equation (6.4) to write $\alpha=\mathfrak{f}\big(\omega,x,k_\alpha^\pm(\omega,x)\big)$. By definition of the sign function, we have that $\big|\operatorname{sign}\big(\nu(t,x)-k_\alpha^\pm(\omega,x)\big)\big|\leq 1$. Furthermore, by construction of the random set $\mathbb{D}_k^u(\omega)$ it holds that

$$\alpha=\mathfrak{f}\big(\omega,x,k_\alpha^\pm(\omega,x)\big)=\mathfrak{f}\big(\omega,x,u(t,x)\big)\qquad\text{for } x\in\mathbb{D}_k^u(\omega)\,.$$

Exploiting these results, we can conclude the continuous dependence of Integral (6.11c) via

$$\begin{aligned}\Big| \int\limits_{\mathbb{D}_k^u(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ \leq \int\limits_{\mathbb{D}_k^u(\omega)} \Big|\mathfrak{f}\big(\omega,x,\nu(t,x)\big)-\mathfrak{f}\big(\omega,x,u(t,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t \\ \leq C_\psi L_I \|\nu-u\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}\,.\end{aligned}$$

Here, we first used the local Lipschitz continuity of the flux function $\mathfrak{f}$, which is guaranteed by Assumption (A-3) or Assumption (A-3') for an interval $I \subset \mathbb{R}$ containing both functions $u, \nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ for every $(t,x) \in \mathbb{T}\times\mathbb{X}$. Finally, we applied the boundedness of the spatial gradient $\partial_x\psi$ of the test function ψ and its compact support to conclude the above continuous dependence estimate on Integral (6.11c).

Continuous dependence of Integral (6.11d). It remains to argue the continuous dependence of the Integral (6.11d). However, the argumentation is completely analogous to the preceeding continuous dependence proof of Integral (6.11c). Consequently, we end up with

$$\Big| \int\limits_{\mathbb{D}_k^\nu(\omega)} \mathfrak{Q}_{\omega,x}^{k_\alpha^\pm}\big(u(t,x),\nu(t,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq C_\psi L_I \|u-\nu\|_{\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})}\,.$$

This shows the continuous dependence of Integral (6.11d) on $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$.

Combining the continuous dependence results of the Integrals (6.11a), (6.11b), (6.11c) and (6.11d) we can conclude that the Integral (6.10b) of the adapted entropy functional $\mathbb{J}_\psi^\alpha$ depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. This concludes the proof of showing continuous dependence of the adapted entropy functional $\mathbb{J}_\psi^\alpha$. ■

With the last proposition, we have argued that the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Since we aim at showing that the functional $\mathbb{J}_\psi^\alpha$ is Carathéodory, it remains to show that it is measurable with respect to the stochastic parameter $\omega \in \Omega$. The corresponding result is proven in the following proposition.

Proposition 6.13 (Stochastic measurability of adapted entropy functional)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (6.1). Furthermore, let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1 as well as the measurability Assumption 6.5. Moreover, let a steady-state parameter $\alpha \in [\mathfrak{m}_0,\infty)$ (or $\alpha \in (-\infty,\mathfrak{m}_0]$) and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. Then, for a fixed function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$, the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ is stochastically measurable in the sense that the mapping $\omega \mapsto \mathbb{J}_\psi^\alpha(\omega,\nu)$ is measurable. ◆

Proof. Let a parameter $\alpha \in [\mathfrak{m}_0,\infty)$ (or $\alpha \in (-\infty,\mathfrak{m}_0]$) be fixed and let $k_\alpha^\pm$ denote the solutions to the corresponding steady-state Equation (6.4). Furthermore, let a function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. We prove the stochastic measurability of the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ in two steps: First, we establish the measurability of the integrands of the entropy functional Mapping (6.10) with respect to $\omega \in \Omega$. Afterwards, we exploit that information to deduce that the entropy functional itself is stochastically measurable, i.e., measurable with respect to $\omega \in \Omega$.

Measurability of integrands. The integrand of Integral (6.10a) is stochastically measurable as soon as the steady-state solutions $k_\alpha^\pm$ are measurable, since the other ingredients are independent of $\omega \in \Omega$. However, this is exactly the statement of Proposition 6.8.

Using Proposition 6.8 to obtain the stochastic measurability of the steady-state solutions $k_\alpha^\pm$ again, it remains to argue that the initial condition u_0 is measurable to obtain measurability of the integrand of Integral (6.10c). Anyway, the random initial condition u_0 satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, by hypothesis, which implies its measurability with respect to $\omega \in \Omega$.

It remains to argue the measurability of the integrand of Integral (6.10b). Here, we need to establish the measurability of the term $\mathfrak{q}\big(\omega, x, \nu(t,x), k_\alpha^\pm(\omega, x)\big)$ with respect to $\omega \in \Omega$, since the test function ψ is independent of it. Using the definition of the entropy flux $\mathfrak{q}$ and the fact that the steady-state solutions satisfy Equation (6.4), we can rewrite this term as

$$\begin{aligned}\mathfrak{q}\big(\omega, x, \nu(t,x), k_\alpha^\pm(\omega, x)\big) &= \operatorname{sign}\big(\nu(t,x) - k_\alpha^\pm(\omega,x)\big)\Big(\mathfrak{f}\big(\omega, x, \nu(t,x)\big) - \mathfrak{f}\big(\omega, x, k_\alpha^\pm(\omega,x)\big)\Big) \\ &= \operatorname{sign}\big(\nu(t,x) - k_\alpha^\pm(\omega,x)\big)\Big(\mathfrak{f}\big(\omega, x, \nu(t,x)\big) - \alpha\Big)\,.\end{aligned}$$

From this formulation, we can readily deduce the stochastic measurability of the integrand of Integral (6.10b): First, the steady-state solutions $k_\alpha^\pm$ are measurable by Proposition 6.8 and the composition of two Borel measurable functions is again measurable, which implies measurability of the sign term. Secondly, the flux function $\mathfrak{f}\big(\omega, x, \nu(t,x)\big)$ is measurable by Assumption (A-4) and $\alpha \in [\mathbb{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathbb{m}_0]$) is independent of $\omega \in \Omega$. Noting that the pointwise product of measurable functions is again measurable, we have shown the stochastic measurability of the integrand of Integral (6.10b).

Measurability of integrals. We now argue that measurability of the integrands implies that the entropy functional $\mathbb{J}_\psi^\alpha$ is stochastically measurable. Therefore, note that the test function ψ satisfies $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ by hypothesis. This implies that ψ has a compact support and we can rewrite the entropy functional $\mathbb{J}_\psi^\alpha$ as the mapping

$$\begin{aligned}(\omega, \nu) \mapsto \int_{\operatorname{supp}\psi} &\big|\nu(t,x) - k_\alpha^\pm(\omega,x)\big|\partial_t\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\operatorname{supp}\psi} \mathfrak{q}\big(\omega, x, \nu(t,x), k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\operatorname{supp}\psi_0} \big|u_0(\omega,x) - k_\alpha^\pm(\omega,x)\big|\psi(0,x)\,\mathrm{d}x\,.\end{aligned} \tag{6.12}$$

Here, we introduced the function $\psi_0 \coloneqq \psi(0,\cdot)$. Since the support of the test function ψ (and also of ψ_0) is compact, taking the integral over $\operatorname{supp}\psi$ and $\operatorname{supp}\psi_0$ is a bounded linear operator. This means that the integration is a continuous operation, because linear operators are bounded if and only if they are continuous. Moreover, each integral in the mapping (6.12) is the composition of a continuous operator with a measurable function, which implies measurability by [5, Lemma 4.22]. Consequently, we have shown that the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ is stochastically measurable, which concludes the proof. ∎

With the preceeding two propositions, we have established that the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ is Carathéodory, i.e., it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous in the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. We summarize this result in the following theorem, whose proof is omitted to avoid redundancy to the previous propositions.

Theorem 6.14 (Adapted entropy functional is Carathéodory)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (6.1). Furthermore, let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1 as well as the measurability Assumption 6.5. Moreover, let a steady-state parameter $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$) and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. Then, the random adapted entropy functional $\mathbb{J}^\alpha_\psi$ is Carathéodory, i.e., it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous with respect to the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. ◆

6.2.3 Continuous dependence of entropy functional on steady-state parameter

In this subsection, we want to establish another continuous dependence result on the random adapted entropy functional $\mathbb{J}^\alpha_\psi$. Specifically, we aim at showing that $\mathbb{J}^\alpha_\psi$ depends continuously on the steady-state parameter α. This continuity property is important for arguing strong measurability of random adapted entropy solutions in the proceeding section, which discusses the well-posedness of solutions. The sought continuous dependence result on the functional $\mathbb{J}^\alpha_\psi$ is justified in the next theorem.

Theorem 6.15 (Continuous dependence of functional on steady-state parameter)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (6.1). Furthermore, let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1 and let a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. Then, for a fixed stochastic parameter $\omega \in \Omega$ and a fixed function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, the random adapted entropy functional $\mathbb{J}^\alpha_\psi$ depends continuously on the steady-state parameter $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$). ◆

Proof. Let the stochastic parameter $\omega \in \Omega$, a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ and a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. To prove the continuous dependence of the functional $\mathbb{J}^\alpha_\psi$ on the steady-state parameter $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$), we consider each integral of mapping (6.10) separately:

By Proposition 6.7, the solutions $k^\pm_\alpha$ of the steady-state Equation (6.4) depend continuously on the steady-state parameter $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$). This readily implies that the integrands of Integral (6.10a) and Integral (6.10c) depend continuously on the steady-state parameter α. Furthermore, the test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ and its temporal derivative $\partial_t \psi$ are compactly supported by hypothesis. Thereby, both Integrals (6.10a) and (6.10c) are finite and the continuous dependence can be deduced via dominated convergence. For details on this argument, we refer to [98, Theorem 5.6].

It remains to show the continuous dependence of Integral (6.10b) on the steady-state parameter α. Therefore, let two parameters $\alpha, \beta \in [\mathfrak{m}_0, \infty)$ (or $\alpha, \beta \in (-\infty, \mathfrak{m}_0]$) be given. Analogously to the proof of Proposition 6.12, we have to deal with the varying values of the sign terms in the adapted entropy flux $\mathfrak{q}$. Therefore, for any stochastic parameter $\omega \in \Omega$, we define the following four random sets $\mathbb{D}^{(\cdot)}(\omega) \subset \mathbb{T} \times \mathbb{X}$:

$$\mathbb{D}^{=}(\omega) := \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid \operatorname{sign}\big(\nu(t,x) - k^\pm_\alpha(\omega,x)\big) = \operatorname{sign}\big(\nu(t,x) - k^\pm_\beta(\omega,x)\big)\Big\},$$
$$\mathbb{D}^{\pm}(\omega) := \Big\{(t,x) \in \mathbb{T} \times \mathbb{X} \mid \operatorname{sign}\big(\nu(t,x) - k^\pm_\alpha(\omega,x)\big) = -\operatorname{sign}\big(\nu(t,x) - k^\pm_\beta(\omega,x)\big)\Big\},$$

$$\mathbb{D}_\alpha^\nu(\omega) \coloneqq \Big\{(t,x) \in \mathbb{T}\times\mathbb{X} \mid \nu(t,x) = k_\alpha^\pm(\omega,x)\Big\},$$
$$\mathbb{D}_\beta^\nu(\omega) \coloneqq \Big\{(t,x) \in \mathbb{T}\times\mathbb{X} \mid \nu(t,x) = k_\beta^\pm(\omega,x)\Big\}.$$

Note, the union of these sets always results in the whole space-time domain $\mathbb{T}\times\mathbb{X}$. However, the random sets $\mathbb{D}^{(\cdot)}(\omega)$ are not necessarily disjoint, but the entropy flux vanishes on the intersection by construction. Therefore, we can exploit these sets to split up the integration. With the triangle inequality, we obtain the estimation

$$\begin{aligned}
\Big|\int_{\mathbb{T}}\int_{\mathbb{X}} \mathfrak{q}\big(\omega,x,\nu(t,x),k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}}\int_{\mathbb{X}} \mathfrak{q}\big(\omega,x,\nu(t,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
\leq \Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| && \text{(6.13a)}\\
+ \Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| && \text{(6.13b)}\\
+ \Big|\int_{\mathbb{D}_\alpha^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| && \text{(6.13c)}\\
+ \Big|\int_{\mathbb{D}_\beta^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big|\,. && \text{(6.13d)}
\end{aligned}$$

Here, the function $\mathfrak{Q}_{\omega,x}^\nu$ denotes the difference of two adapted entropy fluxes in the sense that

$$\mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big) \coloneqq \mathfrak{q}\big(\omega,x,\nu(t,x),k_\alpha^\pm(\omega,x)\big) - \mathfrak{q}\big(\omega,x,\nu(t,x),k_\beta^\pm(\omega,x)\big)\,.$$

With these sets at hand, we can proceed with the estimation by considering each integral in the above Inequality (6.13) separately:

Continuous dependence on Integral (6.13a). We start by estimating

$$\Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int_{\mathbb{D}^=(\omega)} \Big|\mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

Inserting the definitions of the function $\mathfrak{Q}_{\omega,x}^\nu$ and of the entropy flux $\mathfrak{q}$, we can leverage the construction of the random set $\mathbb{D}^=(\omega)$ to further estimate

$$\begin{aligned}
\Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
\leq \int_{\mathbb{D}^=(\omega)} \Big|\mathfrak{f}\big(\omega,x,k_\beta^\pm(\omega,x)\big) - \mathfrak{f}\big(\omega,x,k_\alpha^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Now, the functions $k_\alpha^\pm$ and $k_\beta^\pm$ satisfy the steady-state Equation (6.4) by construction. Furthermore, the test function ψ is smooth and compactly supported by hypothesis, which implies that its spatial derivative is bounded. This leads to the estimate

$$\Big|\int_{\mathbb{D}^=(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq C_\psi\big|\beta-\alpha\big|\,,$$

which proves the continuous dependence of Integral (6.13a) on the parameters $\alpha,\beta \in [\mathfrak{m}_0,\infty)$ (or $\alpha,\beta \in (-\infty,\mathfrak{m}_0]$).

Continuous dependence on Integral (6.13b). For the continuous dependence proof of Integral (6.13b), we start again by estimating

$$\Big| \int_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\nu}_{\omega,x}\big(k^{\pm}_{\alpha}(\omega,x),k^{\pm}_{\beta}(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int_{\mathbb{D}^{\pm}(\omega)} \Big|\mathfrak{Q}^{\nu}_{\omega,x}\big(k^{\pm}_{\alpha}(\omega,x),k^{\pm}_{\beta}(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

With the definition of the function $\mathfrak{Q}^{\nu}_{\omega,x}$ and the entropy flux $\mathfrak{q}$, exploiting the construction of the random integral set $\mathbb{D}^{\pm}(\omega)$ allows us to further estimate

$$\Big| \int_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\nu}_{\omega,x}\big(k^{\pm}_{\alpha}(\omega,x),k^{\pm}_{\beta}(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int_{\mathbb{D}^{\pm}(\omega)} \Big|2\mathfrak{f}\big(\omega,x,\nu(t,x)\big)-\mathfrak{f}\big(\omega,x,k^{\pm}_{\beta}(\omega,x)\big)-\mathfrak{f}\big(\omega,x,k^{\pm}_{\alpha}(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

To continue with the estimation, we employ the triangle inequality and use the fact that the sign function is bounded from above by 1. Furthermore, the functions $\nu, k^{\pm}_{\alpha}, k^{\pm}_{\beta}$ satisfy $\nu \in \mathcal{L}^{\infty}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ and $k^{\pm}_{\alpha}(\omega,\cdot), k^{\pm}_{\beta}(\omega,\cdot) \in \mathcal{L}^{\infty}(\mathbb{X};\mathbb{R})$, respectively. Therefore, by the local Lipschitz continuity Assumption (A-3) (or Assumption (A-3')), there exists an interval $I \subset \mathbb{R}$, which contains $\nu, k^{\pm}_{\alpha}$ and $k^{\pm}_{\beta}$. On this interval, the flux function is Lipschitz continuous, which leads to the estimation

$$\Big| \int_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\nu}_{\omega,x}\big(k^{\pm}_{\alpha}(\omega,x),k^{\pm}_{\beta}(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq L_I \int_{\mathbb{D}^{\pm}(\omega)} \Big(\big|\nu(t,x)-k^{\pm}_{\alpha}(\omega,x)\big|+\big|\nu(t,x)-k^{\pm}_{\beta}(\omega,x)\big|\Big)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

Recall that, for any two scalar values $\xi_1, \xi_2 \in \mathbb{R}$, the identity $|\xi_1-\xi_2| = \operatorname{sign}(\xi_1-\xi_2)(\xi_1-\xi_2)$ holds. Inserting this identity into the above inequality yields

$$\begin{aligned}\Big| \int_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\nu}_{\omega,x}\big(k^{\pm}_{\alpha}(\omega,x),k^{\pm}_{\beta}(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| &\leq L_I \int_{\mathbb{D}^{\pm}(\omega)} \Big(\operatorname{sign}\big(\nu(t,x)-k^{\pm}_{\alpha}(\omega,x)\big)\big(\nu(t,x)-k^{\pm}_{\alpha}(\omega,x)\big)\\ &\qquad +\operatorname{sign}\big(\nu(t,x)-k^{\pm}_{\beta}(\omega,x)\big)\big(\nu(t,x)-k^{\pm}_{\beta}(\omega,x)\big)\Big)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

With this estimation, we can leverage the construction of the random set $\mathbb{D}^{\pm}(\omega)$ again. Also using the fact that the sign function is bounded by 1 from above, we obtain the estimate

$$\begin{aligned}\Big| \int_{\mathbb{D}^{\pm}(\omega)} \mathfrak{Q}^{\nu}_{\omega,x}\big(k^{\pm}_{\alpha}(\omega,x),k^{\pm}_{\beta}(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| &\leq L_I \int_{\mathbb{D}^{\pm}(\omega)} \operatorname{sign}\big(\nu(t,x)-k^{\pm}_{\alpha}(\omega,x)\big)\big(k^{\pm}_{\beta}(\omega,x)-k^{\pm}_{\alpha}(\omega,x)\big)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\\ &\leq L_I \int_{\mathbb{D}^{\pm}(\omega)} \big(k^{\pm}_{\beta}(\omega,x)-k^{\pm}_{\alpha}(\omega,x)\big)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Since the steady-state solutions $k_\alpha^\pm, k_\beta^\pm$ are implicitly defined via the steady-state Equation (6.4), it follows that they are given as the functions

$$k_\alpha^\pm(\omega,x) = \mathfrak{f}_\pm^{-1}(\omega,x,\alpha) \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,,$$
$$k_\beta^\pm(\omega,x) = \mathfrak{f}_\pm^{-1}(\omega,x,\beta) \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \mathbb{R}\,.$$

Here, the function $\mathfrak{f}_\pm^{-1}(\omega,x,\cdot)$ denotes the inverse of the flux function $\mathfrak{f}(\omega,x,\cdot)$, which exists by Corollary 6.4. Furthermore, by the boundedness Assumption (A-6) on the inverse flux function, using the linearity of the integral leads to the estimation

$$\begin{aligned}\Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ \leq L_I \int_{\mathbb{D}^\pm(\omega)} \mathbb{w}(\omega,\beta)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t + L_I \int_{\mathbb{D}^\pm(\omega)} \mathbb{w}(\omega,\alpha)\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

By hypothesis, the test function ψ is smooth and compactly supported, which implies that its spatial derivative is bounded. Therefore, we can further estimate

$$\Big|\int_{\mathbb{D}^\pm(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq L_I C_\psi \mathbb{w}(\omega,\beta) + L_I C_\psi \mathbb{w}(\omega,\alpha)\,.$$

Here, the function $\mathbb{w}(\omega,\cdot)$ is continuous by Assumption (A-6), which proves the continuous dependence of Integral (6.13b).

Continuous dependence on Integral (6.13c). We start with the estimation

$$\Big|\int_{\mathbb{D}_\alpha^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \leq \int_{\mathbb{D}_\alpha^\nu(\omega)} \Big|\mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.$$

Exploiting the construction of the random set $\mathbb{D}_k^u(\omega)$ after inserting the definitions of the function $\mathfrak{Q}_{\omega,x}^\nu$ and the entropy flux $\mathfrak{q}$ yields

$$\begin{aligned}&\Big|\int_{\mathbb{D}_\alpha^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ &\quad\leq \int_{\mathbb{D}_\alpha^\nu(\omega)} \Big|\mathfrak{q}\big(\omega,x,\nu(t,x),k_\alpha^\pm(\omega,x)\big) - \mathfrak{q}\big(\omega,x,\nu(t,x),k_\beta^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leq \int_{\mathbb{D}_\alpha^\nu(\omega)} \Big|\operatorname{sign}\big(\nu(t,x)-k_\beta^\pm(\omega,x)\big)\cdot\Big(\mathfrak{f}\big(\omega,x,\nu(t,x)\big)-\mathfrak{f}\big(\omega,x,k_\beta^\pm(\omega,x)\big)\Big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Since the sign function is bounded by 1 from above, we can leverage the construction of the random set $\mathbb{D}_k^u(\omega)$ again to obtain

$$\begin{aligned}\Big|\int_{\mathbb{D}_\alpha^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x),k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ \leq \int_{\mathbb{D}_\alpha^\nu(\omega)} \Big|\mathfrak{f}\big(\omega,x,k_\alpha^\pm(\omega,x)\big) - \mathfrak{f}\big(\omega,x,k_\beta^\pm(\omega,x)\big)\Big|\big|\partial_x\psi(t,x)\big|\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

By their construction, the functions $k_\alpha^\pm, k_\beta^\pm$ satisfy the steady-state Equation (6.4). This already concludes

the continuous dependence estimate for Integral (6.13c), as we can estimate

$$\left| \int\limits_{\mathbb{D}_\alpha^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\alpha^\pm(\omega,x), k_\beta^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \right| \leq C_\psi\big|\alpha-\beta\big| \,.$$

Here, the constant $C_\psi > 0$ results from the integral being finite, since the spatial derivative of the test function is bounded and compactly supported by hypothesis $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$.

Continuous dependence on Integral (6.13d). Proving the continuous dependence of Integral (6.13d) on the steady-state parameters α, β is completely analogous to arguing that Integral (6.13c) depends continuously on these parameters. As a result, we obtain the estimation

$$\left| \int\limits_{\mathbb{D}_\beta^\nu(\omega)} \mathfrak{Q}_{\omega,x}^\nu\big(k_\beta^\pm(\omega,x), k_\alpha^\pm(\omega,x)\big)\partial_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \right| \leq C_\psi\big|\alpha-\beta\big| \,,$$

which concludes this step of the proof.

Combining the continuous dependence results of the Integrals (6.13a), (6.13b), (6.13c) and (6.13d) concludes the proof of showing that Integral (6.13) depends continuously on the steady-state parameter. Therefore, we have shown the assertion stating that the random adapted entropy functional $\mathbb{J}_\psi^\alpha$ depends continuously on the steady-state parameter $\alpha \in [\mathfrak{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathfrak{m}_0]$). ■

6.3 Well-posedness of random adapted entropy solutions

In this section, we investigate the well-posedness of random adapted entropy solutions to the scalar discontinuous-flux conservation law given by Problem (6.1). Since the conservation law is random, this well-posedness does not only include (pathwise) existence and uniqueness of a solution, but also its strong measurability with respect to the stochastic parameter $\omega \in \Omega$. This strong measurability allows us to interpret the adapted entropy solution u as a $\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$-valued, Bochner-integrable random variable $u : \Omega \to \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Furthermore, such an interpretation admits investigating the moments of the random adapted entropy solution to describe its statistical properties.

To investigate the well-posedness of random adapted entropy solutions, we first discuss the pathwise existence and uniqueness of such solutions in Section 6.3.1. Afterwards, in Section 6.3.2, we establish the strong measurability of these solutions. We conclude this section by investigating the existence of stochastic moments of adapted entropy solutions in Section 6.3.3.

6.3.1 Pathwise existence and uniqueness of random entropy solutions

We start the well-posedness analysis by discussing the pathwise existence and uniqueness of random adapted entropy solutions. Therefore, let us stress that our formulation of the Audusse-Perthame flux Assumption 6.1 reduces to the deterministic framework, once $\omega \in \Omega$ is fixed. Consequently, the pathwise existence and uniqueness is argued similar as the deterministic results, where the first was justified in [61] and the latter was established in [18]. Due to this similarity, only the main ideas for proving the following theorem are given.

Theorem 6.16 (Pathwise existence & uniqueness of adapted entropy solutions)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (6.1)*. Furthermore, let the flux function $\mathfrak{f}$ satisfy the Audusse-Perthame flux Assumption 6.1. Then, for every stochastic parameter $\omega \in \Omega$, there exists a unique pathwise adapted entropy solution $u(\omega,\cdot,\cdot) \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R}) \cap \mathcal{C}\big(\mathbb{T};\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R})\big)$ of the initial value Problem* (6.1)*. Furthermore, two pathwise adapted entropy solutions $u(\omega,\cdot,\cdot), \tilde{u}(\omega,\cdot,\cdot)$ corresponding to initial conditions $u_0, \tilde{u_0}$ satisfy for almost every time $t \in \mathbb{T}$:*

$$\int_a^b |u(\omega,t,x) - \tilde{u}(\omega,t,x)| \,\mathrm{d}x \leq \int_{a-\mathcal{M}_{\mathfrak{f}}(\omega)t}^{b+\mathcal{M}_{\mathfrak{f}}(\omega)t} |u_0(\omega,x) - \tilde{u_0}(\omega,x)| \,\mathrm{d}x \,. \tag{6.14}$$

Here, the random number $\mathcal{M}_{\mathfrak{f}}(\omega)$ is the pathwise bound of the flux function resulting from Corollary 6.3, cf. Equation (6.2)*. Furthermore, the values $a, b \in \mathbb{R}$ are arbitrary with $a < b$.* ◆

Proof (main ideas). Let the stochastic parameter $\omega \in \Omega$ be fixed. The existence of a pathwise adapted entropy solution is proved similarly as for the deterministic case and the corresponding result was established in [61]. The proof is based on the reduction of measure-valued solutions to adapted entropy solutions in $\mathcal{L}^\infty$ and the consideration of a mollified version of the problem.

For the uniqueness argument, which was shown in [18], the use of adapted Kružkov entropies allows the argumentation via the *doubling of variables* procedure of Kružkov [173]. Then, the $\mathcal{L}^1$-contraction principle (6.14) is an immediate consequence of the uniqueness result. ■

6.3.2 Strong measurability of random adapted entropy solutions

To complete the well-posedness investigation, this section establishes strong measurability of random adapted entropy solutions $u : \Omega \to \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Recall that strong measurability requires that the image of $u(\Omega)$ is separable. While the function space $\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ is not separable, we can leverage the stationarity of the flux function $\mathfrak{f}$. Recall that Panov [228] has shown that each entropy solution $u(\omega,\cdot,\cdot) \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ has a representative in the function space $\mathcal{C}\big(\mathbb{T};\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})\big)$. Consequently, the solution only takes values in the space

$$\mathcal{S} \coloneqq \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R}) \cap \mathcal{C}\big(\mathbb{T};\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})\big) \,,$$

which is separable, since the space $\mathcal{L}^1(\mathbb{X};\mathbb{R})$ of integrable functions is separable and a dense subset of $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})$. Then, the separability follows from the time interval $\mathbb{T} = [0,T]$ being compact. For the details, we refer to Example 3.49 (ii). Using this separable space $\mathcal{S}$ containing the random adapted entropy solution, we have now all ingredients necessary to prove the strong measurability of these solutions.

Theorem 6.17 (Measurability of adapted entropy solutions)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (6.1)*. Furthermore, let $\mathfrak{f}$ be a flux function that satisfies the Audusse-Perthame flux Assumption 6.1 and the measurability Assumption 6.5. Then, the random adapted entropy solution u is strongly measurable in the sense that the mapping $u : \Omega \to \mathcal{S}$ with $\omega \mapsto u(\omega,\cdot,\cdot)$ is strongly measurable.* ◆

Proof. The proof is divided into three steps: First, we construct a modified entropy functional that inherits the properties of being Carathéodory and continuously depending on the steady-state parameters from the standard adapted entropy functional. Afterwards, we prove that this modified entropy functional can represent the adapted entropy Condition (6.9) and that a corresponding set-valued map is measurable. In the third step, we connect the constructed set-valued map to the random adapted entropy solution to Problem (6.1) to show its measurability. A technical justification of an argument in step three is postponed for readability purposes. This result is argued directly after this proof.

Step 1: Random modified entropy functional. For $N \in \mathbb{N}$, define a sequence of function spaces consisting of smooth nonnegative functions with compact support via

$$\mathcal{D}_N \coloneqq \Big\{ \psi \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R}_{\geq 0}) \mid \operatorname{supp}(\psi) \subseteq \big\{ (t,x) \in \mathbb{T} \times \mathbb{X} \mid t \in [0,N] \text{ and } x \in \mathbb{B}_N(0) \big\} \Big\}.$$

Here, $\mathbb{B}_N(x)$ denotes the closed ball with radius N around x. Each trunctated space $\mathcal{D}_N$ is a subspace of $\mathcal{C}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R}_{\geq 0})$ by construction. Thereby, since $\mathcal{C}^\infty$ has a countable basis, $\mathcal{D}_N$ also has a countable basis. Consequently, for every nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R}) = \mathcal{C}_c^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, there exists a constant $R_\psi \in \mathbb{N}$, such that $\psi \in \mathcal{D}_{R_\psi}$.

Now, fix a number $N \in \mathbb{N}$ and let $(\psi_N^i, i \in \mathbb{N}) \subset \mathcal{D}_N$ be a basis of the space $\mathcal{D}_N$. Then, for a fixed steady-state parameter $\alpha \in [\mathbb{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathbb{m}_0]$) and a fixed number $i \in \mathbb{N}$, we define the *random modified entropy functional* $\mathbb{J}_{i,N}^\alpha : \Omega \times \mathcal{S} \to \mathbb{R}$ via the mapping

$$\begin{aligned}(\omega, \nu) \mapsto \int_{\mathbb{T}} \int_{\mathbb{X}} \big| \nu(t,x) - k_\alpha^\pm(\omega,x) \big| \partial_t \psi_N^i(t,x) \,\mathrm{d}x \,\mathrm{d}t \\ + \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}\big(\omega, x; \nu(t,x), k_\alpha^\pm(\omega,x)\big) \partial_x \psi_N^i(t,x) \,\mathrm{d}x \,\mathrm{d}t \\ - \int_{\mathbb{X}} \big| u_0(\omega,x) - k_\alpha^\pm(\omega,x) \big| \psi_N^i(0,x) \,\mathrm{d}x \,.\end{aligned}$$

Here, the functions $k_\alpha^\pm$ are the solutions to the steady-state Equation (6.4) with steady-state parameter α and $\mathfrak{q}$ is the Kružkov entropy flux defined via Equation (6.8). Note, we can apply Theorem 6.14 to obtain that the random modified entropy functional $\mathbb{J}_{i,N}^\alpha$ is Carathéodory.

Step 2: Measurable set-valued map. For the parameter $\alpha \in [\mathbb{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathbb{m}_0]$) and $i, N \in \mathbb{N}$ still being fixed, we define the set-valued mapping

$$\Xi_{i,N}^\alpha : \Omega \rightrightarrows \mathcal{S} \qquad \omega \mapsto \big\{ \nu \in \mathcal{S} \mid \mathbb{J}_{i,N}^\alpha(\omega, \nu) \geq 0 \big\} \,.$$

This multifunction selects all functions $\nu \in \mathcal{S}$ satisfying the adapted entropy Condition (6.9) for a fixed steady-state parameter $\alpha \in [\mathbb{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathbb{m}_0]$) and a fixed test function $\psi_N^i \in \mathcal{D}_N$. Since the function space $\mathcal{S} = \mathcal{L}^\infty(\mathbb{R} \times \mathbb{T}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}_{\mathrm{loc}}^1(\mathbb{R})\big)$ is separable, we can apply Lemma 2.9 to obtain that the set-valued map $\Xi_{i,N}^\alpha$ is measurable.

Step 3: Singleton containing solution. To omit functions $\nu \in \mathcal{S}$ that only satisfy the adapted entropy Condition (6.9) for some fixed test function ψ_N^i, we define the set-valued map

$$\Xi^\alpha : \Omega \rightrightarrows \mathcal{S} \qquad \Xi^\alpha(\omega) \coloneqq \bigcap_{i \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \Xi_{i,N}^\alpha(\omega) \qquad \text{for } \omega \in \Omega \,.$$

This correspondence is measurable, since the countable intersection of measurable maps is again measurable. Furthermore, Ξ^α now contains all the functions $\nu \in \mathcal{S}$ that satisfy the adapted entropy

condition for fixed steady-state parameter $\alpha \in [\mathbb{m}_0, \infty)$ (or $\alpha \in (-\infty, \mathbb{m}_0]$) and $i, N \in \mathbb{N}$, but for all test functions $\psi \in \mathcal{D}$. Now, we can define the set-valued mapping

$$\Xi : \Omega \rightrightarrows \mathcal{S} \qquad \Xi(\omega) \coloneqq \bigcap_{\alpha \in \mathbb{A} \cap \mathbb{Q}} \Xi^{\alpha}(\omega) \qquad \text{for } \omega \in \Omega \,.$$

Here, the set $\mathbb{A}$ corresponds to $\mathbb{R}$ in the case of Assumption (A-3') or to either the set $[\mathbb{m}_0, \infty)$ (or $(-\infty, \mathbb{m}_0]$) in case of Assumption (A-3). Again, this set-valued map Ξ is measurable via the countable intersection of measurable maps. Furthermore, the measurable mapping Ξ now contains all those functions $\nu \in \mathcal{S}$ satisfying the random adapted entropy condition (without any restrictions due to fixed variables or functions). While the adapted entropy condition (6.9) is formulated to hold for every parameter $\alpha \in \mathbb{A}$, it is sufficient to only consider $\alpha \in \mathbb{A} \cap \mathbb{Q}$. For readability purposes, the verification of this claim is postponed until after this proof.

To conclude the proof, we note that by Theorem 6.16 there exists a unique random adapted entropy solution for every stochastic parameter $\omega \in \Omega$. Therefore, the correspondence Ξ only contains this pathwise adapted entropy solution of Problem (6.1), which is measurable as a map $u : \Omega \to \mathcal{S}$ due to the measurability of Ξ. Since the function space $\mathcal{S}$ is separable by construction, this proves the assertion. ■

Sufficiency of intersecting over rationals. It remains to justify that it is sufficient to intersect over the values $\alpha \in \mathbb{A} \cap \mathbb{Q}$ to select functions that satisfy the random adapted entropy Condition (6.9) for every steady-state parameter $\alpha \in \mathbb{A}$. We show this via a contradiction:

Assume that intersecting over $\mathbb{A} \cap \mathbb{Q}$ is not sufficient to select the adapted entropy solution, which is unique by Theorem 6.16. Then, for $\alpha \in \mathbb{A} \cap \mathbb{Q}$, the set $\Xi(\omega)$ does contain the random adapted entropy solution u and another function ν. By construction, both functions satisfy the adapted entropy Condition (6.9) for the steady-state parameter $\alpha \in \mathbb{A} \cap \mathbb{Q}$. However, due to the uniqueness result of Theorem 6.16, for $\beta \in \mathbb{A} \setminus \mathbb{Q}$, the image of $\Xi(\omega)$ only contains the adapted entropy solution u. By Definition 6.11 of the random adapted entropy functional, this means that there exist indices $j, M \in \mathbb{N}$ such that

$$\mathbb{J}^{\alpha}_{i,N}(\omega, \nu) \geq 0 \quad \text{for all } i, N \in \mathbb{N} \qquad \mathbb{J}^{\beta}_{j,M}(\omega, \nu) < 0 \quad \text{for } j, M \in \mathbb{N} \,.$$

Due to Theorem 6.15 the random adapted entropy functional $\mathbb{J}^{\beta}_{j,M}$ is continuous with respect to the steady-state parameter $\beta \in \mathbb{A}$. Thus, for $\varepsilon > 0$, there exists an ε-neighborhood $\mathbb{B}_{\varepsilon}$ around the value $\mathbb{J}^{\beta}_{j,M}(\omega, \nu)$ such that for every $\xi \in \mathbb{B}_{\varepsilon}$ it holds that $\xi < 0$. Again using the continuous dependence of $\mathbb{J}^{\beta}_{j,M}$ in β and noting that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, there exists a $\delta > 0$ and $\beta' \in \mathbb{A} \cap \mathbb{Q}$ such that $|\beta - \beta'| < \delta$ and $\mathbb{J}^{\beta'}_{j,M}(\omega, \nu) \in \mathbb{B}_{\varepsilon}$.

The fact that $\mathbb{J}^{\beta'}_{j,M}(\omega, \nu) \in \mathbb{B}_{\varepsilon}$ implies $\mathbb{J}^{\beta'}_{j,M}(\omega, \nu) < 0$, which is a contradiction to $\mathbb{J}^{\alpha}_{i,N}(\omega, \nu) \geq 0$ for all $i, N \in \mathbb{N}$ and all parameters $\alpha \in \mathbb{A} \cap \mathbb{Q}$. Consequently, it is sufficient to intersect over $\mathbb{A} \cap \mathbb{Q}$ instead of $\mathbb{A}$ to select the unique adapted entropy solution. ■

6.3.3 Existence of moments of random adapted entropy solutions

We conclude this section on the well-posedness of random adapted entropy solutions by investigating the existence of stochastic moments of the solution. These moments are a tool to describe the random solution via its statistical properties. The main ingredient in discussing the existence of moments is the

$\mathcal{L}^1$-contraction Property (6.14) of the pathwise adapted entropy solutions. To employ this property, we assume that the initial condition u_0 is integrable. Furthermore, we have to ensure that a zero initial condition $u_0 \equiv 0$ leads to the zero adapted entropy solution $u \equiv 0$. Therefore, we impose the following technical assumption on the Audusse-Perthame flux function.

Assumption 6.18 (Zero mass creation)**:**
We assume that the Audusse-Perthame flux function f is chosen such that the zero initial condition $u_0 \equiv 0$ leads to the vanishing $\mathfrak{G}$-entropy solution $u \equiv 0$. ◆

Note, the family of multiplicative flux functions from Example 6.2 automatically satisfy the above assumption. The subsequent theorem states a general existence result of moments of the adapted entropy solution for arbitrary random integrable initial conditions.

Theorem 6.19 (Existence of moments of adapted entropy solutions)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$, be a random initial condition to Problem (6.1)*. Furthermore, let $\mathfrak{f}$ be a flux function that satisfies the Audusse-Perthame flux Assumption 6.1 and the measurability Assumption 6.5 as well as the zero-mass-creation Assumption 6.18. Then, at almost every time $t \in \mathbb{T}$, the random adapted entropy solution u admits moments up to order q. In particular, the adapted entropy solution satisfies*

$$\|u(\omega, t, x)\|_{\mathcal{L}^q\left(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\right)} \leq \|u_0(\omega, x)\|_{\mathcal{L}^q\left(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\right)} \tag{6.15}$$

for almost every time $t \in \mathbb{T}$. ◆

Proof. Once the estimation (6.15) is established, the existence of moments up to order $q \in [1, \infty)$ follows directly from the hypothesis that the initial condition u_0 satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\big)$. For every stochastic parameter $\omega \in \Omega$, a pathwise adapted entropy solution $u(\omega, \cdot, \cdot)$ exists and is unique due to Theorem 6.16. Furthermore, this adapted entropy solution $u(\omega, \cdot, \cdot)$ is strongly measurable by Theorem 6.17. Thus, for any $1 \leq q < \infty$, we can compute

$$\Big\|u(\omega, t, x)\Big\|^q_{\mathcal{L}^q\left(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\right)} = \int_\Omega \Big\|u(\omega, t, x)\Big\|^q_{\mathcal{L}^1(\mathbb{X}; \mathbb{R})} \, \mathrm{d}\mathbb{P} \leq \int_\Omega \Big\|u_0(\omega, x)\Big\|^q_{\mathcal{L}^1(\mathbb{X}; \mathbb{R})} \, \mathrm{d}\mathbb{P} \, .$$

Here, we used the pathwise $\mathcal{L}^1$-contraction property (6.14) of the random adapted entropy solution and chose $\tilde{u_0} \equiv 0$. Due to the zero-mass-creation Assumption 6.18, this implies that $\tilde{u}$ vanishes in the pathwise $\mathcal{L}^1$-contraction property (6.14). Based on this estimation, the assertion follows by taking the q-th root and noting that these integrals exist due to the presumption $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^1(\mathbb{X}; \mathbb{R})\big)$ on the initial condition. ■

Roughly speaking, the above result states that the random adapted entropy solution inherits the existence of moments from the underlying (random) initial condition u_0. Therefore, as an immediate consequence, we obtain the existence of all moments of order $1 \leq q < \infty$, if the initial condition is deterministic. We precise this statement in the following corollary, which also concludes the investigation of stochastic moments of the adapted entropy solution u.

Corollary 6.20 (Existence of moments for deterministic initial conditions)**:**
Let $u_0 \in \mathcal{L}^1(\mathbb{X}; \mathbb{R})$ be a deterministic initial condition to the scalar discontinuous-flux conservation law given by Equation (6.1)*. Furthermore, let $\mathfrak{f}$ be a flux function that satisfies the Audusse-Perthame flux*

Assumption 6.1 and the measurability Assumption 6.5 as well as the zero-mass-creation Assumption 6.18. Then, for almost every time $t \in \mathbb{T}$, the random adapted entropy solution u admits all moments of order $1 \leq q < \infty$. ◆

Proof. For every stochastic parameter $\omega \in \Omega$, a pathwise adapted entropy solution $u(\omega, \cdot, \cdot)$ exists and is unique due to Theorem 6.16. Furthermore, this random adapted entropy solution $u(\omega, \cdot, \cdot)$ is strongly measurable by Theorem 6.17. Utilizing the result of Theorem 6.19 on the existence of moments of random adapted entropy solutions, we obtain the estimation

$$\|u(\omega, t, x)\|_{\mathcal{L}^q(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R}))} \leq \int_\Omega \|u_0(x)\|^q_{\mathcal{L}^1(\mathbb{X};\mathbb{R})} \, \mathrm{d}\mathbb{P} = \|u_0(x)\|^q_{\mathcal{L}^1(\mathbb{X};\mathbb{R})}$$

for almost every time $t \in \mathbb{T}$. Thus, the assertion follows by taking the q-th root, since the initial condition u_0 satisfies $u_0 \in \mathcal{L}^1(\mathbb{X};\mathbb{R})$ by hypothesis. ■

Multi-dimensional random conservation laws with infinitely many flux discontinuities

7

In the previous chapter, we established a well-posedness theory for random scalar discontinuous-flux conservation laws in one spatial dimension. This well-posedness theory avoids the rather restrictive assumption that ensure uniform $\mathcal{L}^\infty$-bounds on the entropy solution, which was required in Part I. Additionally, the presented framework allows the flux function to have infinitely many spatial discontinuities that may have an accumulation point. However, the restriction to one spatial dimension is another restraining limitation for real-world applications.

In this chapter, we extend the framework of Chapter 6 to multiple space dimensions. Since such an extension is not straightforward, generalizing the theory of random scalar discontinuous-flux conservation laws to mutliple space dimensions requires some restrictions on the considered type of flux functions. The presented extension is based on the deterministic work of Panov [230], who developed a well-posedness theory for so-called *Panov-type* flux functions.

As in the previous chapters, let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. Furthermore, let a time interval $\mathbb{T} \coloneqq [0, T]$ with a final time $0 < T < \infty$ and a spatial domain $\mathbb{X} \coloneqq \mathbb{R}^d$ be given. Then, for unknown $u \coloneqq u(\omega, t, \boldsymbol{x})$, the random scalar conservation law reads

$$\begin{aligned} \partial_t u + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, \boldsymbol{x}, u) &= 0 && \text{in } \Omega \times \mathbb{T} \times \mathbb{X}\,, \\ u(\omega, 0, \boldsymbol{x}) &= u_0(\omega, \boldsymbol{x}) && \text{on } \Omega \times \{0\} \times \mathbb{X}\,. \end{aligned} \tag{7.1}$$

Here, $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, is a random initial condition. As in chapter 6, the flux function $\mathfrak{f}$ is assumed to depend discontinuously on the spatial coordinate $\boldsymbol{x} \in \mathbb{X}$ and does not depend on the time $t \in \mathbb{T}$. Additionally, we assume the flux function $\mathfrak{f}$ to be a *Panov-type flux function*, which means that it has the form

$$\mathfrak{f}(\omega, \boldsymbol{x}, v) = \boldsymbol{g}\big(\omega, \mathfrak{Z}(\omega, \boldsymbol{x}, v)\big)\,. \tag{7.2}$$

Here, the random function $\boldsymbol{g}(\omega, \cdot)$ is assumed to be continuous and the random function $\mathfrak{Z}(\omega, \cdot, \cdot)$ is assumed to be Carathéodory for each stochastic parameter $\omega \in \Omega$. A more detailed discussion of this type of flux functions and the corresponding steady-state equation is presented in Section 7.1. Afterwards, in Section 7.2, the notion of random entropy solutions is introduced and the corresponding entropy

functionals are discussed, including their properties. These entropy functionals are exploited in Section 7.3 to argue the strong measurability of random entropy solutions as part of the well-posedness discussion, which includes pathwise existence and uniqueness. The Chapter is concluded by an investigation of the existence of moments of random entropy solutions at the end of Section 7.3.

7.1 Panov-type flux functions and steady-state solutions

In this section, we discuss random Panov-type flux functions, which are given by Equation (7.2). Similar to the idea of Audusse-Perthame flux functions in Chapter 6, these fluxes ensure the existence of solutions to the corresponding steady-state problem of the conservation law. Starting with a general discussion of Panov-type flux functions in Section 7.1.1, the corresponding random steady-state solutions and their properties are discussed in Section 7.1.2.

7.1.1 Panov-type flux functions

We start the discussion by introducing a randomized version of Panov-type flux functions, which are given by Equation (7.2). While this particular form is a potential restriction, it allows to generalize the Audusse-Perthame setting to mutliple space dimensions. In particular, we may have infinitely many flux discontinuities and the entropy solutions need not satisfy any a-priori $\mathcal{L}^\infty$-bound assumption. To precise the conditions on the flux function, we impose the following pathwise assumption on the randomized Panov-type flux function.

Assumption 7.1 (Pathwise assumption for random Panov-type fluxes)**:**
For every stochastic parameter $\omega \in \Omega$ we assume that the random Panov-type flux function $\mathfrak{f}$ given by Equation (7.2) *satisfies the following assumptions:*

(P-1) *The function $\boldsymbol{g}(\omega, \cdot)$ is continuous as a function $\boldsymbol{g}(\omega, \cdot) \in \mathcal{C}(\mathbb{R}; \mathbb{R}^d)$.*

(P-2) *The function $\mathfrak{Z}(\omega, \cdot, \cdot)$ is Carathéodory in the sense that the mapping $\boldsymbol{x} \mapsto \mathfrak{Z}(\omega, \boldsymbol{x}, \upsilon)$ is measurable and the mapping $\upsilon \mapsto \mathfrak{Z}(\omega, \boldsymbol{x}, \upsilon)$ is continuous.*

(P-3) *The function $\mathfrak{Z}(\omega, \boldsymbol{x}, \cdot)$ is strictly increasing.*

(P-4) *There exist two continuous functions $\mathfrak{f}_-(\omega, \cdot), \mathfrak{f}_+(\omega, \cdot) \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ such that the function $\mathfrak{Z}$ is bounded in the sense*

$$\mathfrak{f}_-(\omega, \upsilon) \leq |\mathfrak{Z}(\omega, \boldsymbol{x}, \upsilon)| \leq \mathfrak{f}_+(\omega, \upsilon) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^d$$

and the function $\mathfrak{f}_-(\omega, \cdot)$ satisfies $\lim_{\upsilon \to \infty} \mathfrak{f}_-(\omega, \upsilon) \to \infty$.

(P-5) *The flux function $\mathfrak{f}(\omega, \boldsymbol{x}, \cdot)$ is locally Lipschitz continuous with a Lipschitz constant that is independent of the spatial position $\boldsymbol{x} \in \mathbb{R}^d$.* ◆

At a first glance, the local Lipschitz Assumption (P-5) on the flux may seem rather arbitrary. Indeed, an assumption on the character of continuity of the flux function would be sufficient as the subsequent remark specifies.

Remark 7.2 (Mitigating the local Lipschitz assumption): *The local Lipschitz continuity Assumption (P-5) on the Panov-type flux function $\mathfrak{f}(\omega, \boldsymbol{x}, \cdot)$ can be mitigated to an assumption on the character of continuity of the flux. The simplest of such an assumption is to assume the existence of a modulus of continuity for the flux function. For the details on this mitigation, we refer the reader to [230, Section §3] or [171, 172] for details.* ◆

This weaker assumption is always satisfied for one-dimensional scalar conservation laws and for locally Lipschitz continuous flux functions. We restrict ourselves to the case of local Lipschitz continuity, since it simplifies the subsequent discussion. Additionally, the more general character of continuity assumption would increase the technical complexity of some of the presented proofs without generating any additional insight. Before we conclude this discussion of random flux function of Panov type, we impose the following assumption on the stochastic measurability of the flux. Here, again the form induced by Equation (7.2) is exploited.

Assumption 7.3 (Stochastic measurability of flux function):
Let $\mathfrak{f}$ be a flux function that satisfies the Panov-type flux Assumption 7.1. Additionally, we assume that the flux $\mathfrak{f}$ given by Equation (7.2) *satisfies the following stochastic measurability assumptions:*

(P-6) For every value $\upsilon \in \mathbb{R}$, the mapping $\omega \mapsto \boldsymbol{g}(\omega, \upsilon)$ is measurable.

(P-7) For every value $\upsilon \in \mathbb{R}$ and $\mathcal{H}^d$-almost every spatial point $\boldsymbol{x} \in \mathbb{R}^d$, the mapping $\omega \mapsto \mathfrak{Z}(\omega, \boldsymbol{x}, \upsilon)$ is measurable. ◆

7.1.2 Random steady-state solutions

The idea of Panov [230] is inspired by the works of Baiti and Jenssen [23] and Audusse and Perthame [18] and relies on replacing the Kružkov entropy constant by the solutions to the steady-state problem. With the definitions of the previous Section, this random steady-state problem is given as

$$\mathfrak{Z}\big(\omega, \boldsymbol{x}, k_\alpha(\omega, \boldsymbol{x})\big) = \alpha \qquad \text{for } \mathcal{H}^d - \text{almost every } \boldsymbol{x} \in \mathbb{X}\,. \tag{7.3}$$

By the Panov-type flux Assumption 7.1, the function $\mathfrak{Z}(\omega, \boldsymbol{x}, \cdot)$ is continuous and strictly monotone (increasing). Therefore, it admits an inverse, which is also continuous and strictly monotone increasing. This allows us to formulate the following pathwise existence result for random steady-state solutions. Since the argumentation is completely analogous to the discussion of Corollary 6.4 and Section 6.1.2, we omit the proof of the subsequent corollary.

Corollary 7.4 (Existence of steady-state solutions):
Let the Panov-type flux function $\mathfrak{f}$, defined by Equation (7.2)*, satisfy the pathwise flux Assumption 7.1. Then, for every stochastic parameter $\omega \in \Omega$ and every value $\alpha \in \mathbb{R}$, there exists a unique solution k_α to the random steady-state Problem* (7.3). ◆

The Assumption 7.1 on Panov-type flux functions allows us to immediately deduce some important properties of the random steady-state solutions k_α. Since the justification is analogous to the deterministic setting, which can be found in the introduction of [230], we again omit the proof of the following proposition.

Proposition 7.5 (Properties of steady-state solutions)**:**
For every stochastic parameter $\omega \in \Omega$, the random steady-state solution $k(\omega, \cdot, \alpha) \coloneqq k_\alpha(\omega, \cdot)$ possesses the following properties:

(i) The steady-state solution $k(\omega, \cdot, \cdot)$ is Carathéodory in the sense that $k(\omega, \boldsymbol{x}, \alpha)$ is measurable with respect to the spatial coordinate $\boldsymbol{x} \in \mathbb{R}^d$ and continuous in $\alpha \in \mathbb{R}$.

(ii) For a fixed point $\boldsymbol{x} \in \mathbb{R}^d$, the steady-state solution $k(\omega, \boldsymbol{x}, \cdot)$ is strictly increasing.

(iii) For every scalar value $R_k > 0$, the steady-state solution $k(\omega, \cdot, \cdot)$ is essentially bounded in the sense that the function

$$\mathcal{M}_{R_k}(\omega, \boldsymbol{x}) \coloneqq \max_{|\alpha| \leq R_k} \left| k(\omega, \boldsymbol{x}, \alpha) \right|$$

satisfies $\mathcal{M}_{R_k}(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{R}; \mathbb{R})$. ◆

Even though we can ensure the measurability of the Panov-type flux function $\mathfrak{f}$ via Assumption 7.3, this knowledge is not sufficient to argue the stochastic measurability of the random steady-state solutions. Therefore, we impose the following assumption that guarantees that the random steady-state solutions k_α are measurable with respect to the stochastic parameter $\omega \in \Omega$.

Assumption 7.6 (Stochastic measurability of steady-state solutions)**:**
Let a steady-state parameter $\alpha \in \mathbb{R}$ be fixed. We assume that, for almost every spatial coordinate $\boldsymbol{x} \in \mathbb{R}^d$, the corresponding steady-state solution k_α is stochastically measurable in the sense that the mapping $\omega \mapsto k_\alpha(\omega, \boldsymbol{x})$ is measurable. ◆

7.2 Random entropy solutions and functionals

The next step towards discussing the well-posedness of random scalar conservation laws with discontinuous flux functions of Panov-type is to introduce the underlying notion of random entropy solutions. These solutions are defined via an entropy condition, which can also be formulated via a corresponding entropy functional. In Section 7.2.1, we introduce the entropy inequality to select solutions in the Panov-type flux function setting. Additionally, we exploit this entropy condition to define the corresponding random entropy functionals. Afterwards, Sections 7.2.2 and 7.2.3 are devoted to establish that this random entropy functional is Carathéodory and depends continuously on the steady-state parameter $\alpha \in \mathbb{R}$, respectively.

7.2.1 Notion of random entropy solutions and functionals

We start the discussion of random entropy solutions and random entropy functionals by defining the notion of a pathwise entropy solution for random scalar conservation laws with a Panov-type flux function. Here, the crucial idea for admissibility of solutions is to replace the entropy constant $k \in \mathbb{R}$ of the Kružkov entropy Condition (1.2) by the random steady-state solutions k_α for steady-state parameters $\alpha \in \mathbb{R}$. This leads to the following definition.

Definition 7.7 (Entropy solution for Panov-type flux functions)**:**
Let a $\omega \in \Omega$ be fixed and let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1. A function $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}; \mathbb{R})\big)$ is called an entropy solution *to the scalar conservation law given by Equation* (7.1) *on $\mathbb{T} \times \mathbb{X}$, provided that for each parameter $\alpha \in \mathbb{R}$ and the corresponding steady-state solution k_α of Equation* (7.3)*, the entropy inequality*

$$\begin{aligned}\int_{\mathbb{T}} \int_{\mathbb{X}} \big|u(\omega, t, \boldsymbol{x}) - k_\alpha(\omega, \boldsymbol{x})\big| \partial_t \psi(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \\ + \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}\big(\omega, \boldsymbol{x}, u(\omega, t, \boldsymbol{x}), k_\alpha(\omega, \boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \\ + \int_{\mathbb{X}} \big|u_0(\omega, \boldsymbol{x}) - k_\alpha(\omega, \boldsymbol{x})\big| \psi(0, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \;\geq\; 0\end{aligned} \tag{7.4}$$

is satisfied for every nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. Here, the function

$$\mathfrak{q}(\omega, \boldsymbol{x}, v, \tilde{v}) \coloneqq \operatorname{sign}(v - \tilde{v})\big(\mathfrak{f}(\omega, \boldsymbol{x}, v) - \mathfrak{f}(\omega, \boldsymbol{x}, \tilde{v})\big)$$

denotes the Kružkov entropy flux for two scalar values $v, \tilde{v} \in \mathbb{R}$. ◆

Based on the random entropy Inequality (7.4), we can define random entropy functionals. These are a major tool for arguing strong measurability of random entropy solutions in Section 7.3, since they allow us to evaluate the Condition (7.4) for fixed parameters $\alpha \in \mathbb{R}$ and $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$, but for arbitrary functions $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. The subsequent definition formalizes the construction of these functionals.

Definition 7.8 (Entropy functional for Panov-type flux functions)**:**
Let the flux $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1. Furthermore, let a parameter $\alpha \in \mathbb{R}$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X}; \mathbb{R})$ be fixed. Then, we define the random entropy functional $\mathbb{J}^\alpha_\psi$ *associated to Problem* (7.1) *as a mapping $\mathbb{J}^\alpha_\psi : \Omega \times \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R}) \to \mathbb{R}$ given by*

$$(\omega, \nu) \mapsto \int_{\mathbb{T}} \int_{\mathbb{X}} \big|\nu(t, \boldsymbol{x}) - k_\alpha(\omega, \boldsymbol{x})\big| \partial_t \psi(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \tag{7.5a}$$

$$+ \int_{\mathbb{T}} \int_{\mathbb{X}} \mathfrak{q}\big(\omega, \boldsymbol{x}, \nu(t, \boldsymbol{x}), k_\alpha(\omega, \boldsymbol{x})\big) \cdot \nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \tag{7.5b}$$

$$+ \int_{\mathbb{X}} \big|u_0(\omega, \boldsymbol{x}) - k_\alpha(\omega, \boldsymbol{x})\big| \psi(0, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,. \tag{7.5c}$$

Here, k_α denotes the steady-state solution to Equation (7.3) *corresponding to $\alpha \in \mathbb{R}$.* ◆

7.2.2 Entropy functional is Carathéodory

We continue our discussion of random entropy solutions and functionals by discussing properties of the random entropy functional. Therefore, this subsection establishes that the functional $\mathbb{J}^\alpha_\psi$ is Carathéodory in the sense that it is measurable in the stochastic parameter $\omega \in \Omega$ and continuous in the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R})$. The following proposition shows the corresponding continuity result.

Proposition 7.9 (Continuous dependence of entropy functional)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (7.1)*. Furthermore, let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 and let a steady-state parameter $\alpha \in \mathbb{R}$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X};\mathbb{R})$ be fixed. Then, for fixed stochastic parameter $\omega \in \Omega$, the random entropy functional $\mathbb{J}^\alpha_\psi$ depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X};\mathbb{R})$.* ◆

Proof. Let a stochastic parameter $\omega \in \Omega$, a parameter $\alpha \in \mathbb{R}$ and a non-negative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X};\mathbb{R})$ be fixed. Showing the continuous dependence of the entropy functional $\mathbb{J}^\alpha_\psi$ on the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X};\mathbb{R})$ is similar to the proof of Proposition 6.12.

The continuous dependence of the Integrals (7.5a) and (7.5c) is completely analogous to the argumentation in the proof of Proposition 6.12 and follows via independence and dominated convergence, respectively. For the continuous dependence of Integral (7.5b), the argumentation is based on splitting up the integral domain into sets depending on the value of the term $\operatorname{sign}\big(\nu(t,\boldsymbol{x}) - k_\alpha(\omega,\boldsymbol{x})\big)$, which appears in the Kružkov entropy flux $\mathbf{q}$. This approach, as well as the estimation that follows thereafter, is completely analogous to the discussion of Proposition 6.12. However, while the setting of Chapter 6 is one-dimensional, we need to adapt for the multi-dimensional setting. Here, the only difference in the estimations is the usage of the norm $\|\cdot\|_d$ instead of the absolute value $|\cdot|$. ■

Now that we have established continuous dependence of the entropy functional $\mathbb{J}^\alpha_\psi$ on the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X};\mathbb{R})$, it remains to show that it is stochastically measurable. This result is justified in the subsequent proposition.

Proposition 7.10 (Stochastic measurability of entropy functional)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (7.1)*. Furthermore, let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 as well as the measurability Assumption 7.3. Moreover, let a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X};\mathbb{R})$ and a parameter $\alpha \in \mathbb{R}$ be fixed and let the corresponding steady-state solution k_α satisfy the measurability Assumption 7.6. Then, for a fixed function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X};\mathbb{R})$, the random adapted entropy functional $\mathbb{J}^\alpha_\psi$ is stochastically measurable in the sense that the mapping $\omega \mapsto \mathbb{J}^\alpha_\psi(\omega,\nu)$ is measurable.* ◆

Proof. Let a steady-state parameter $\alpha \in \mathbb{R}$ be fixed and let k_α denote the (random) solution to the corresponding steady-state Equation (7.3). Furthermore, let a function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X};\mathbb{R})$ and a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T} \times \mathbb{X};\mathbb{R})$ be fixed. The proof consists of two steps: First, the stochastic measurability of the integrands of the random entropy functional mapping (7.5) is established. Afterwards, we leverage this information to conclude the measurability of the random entropy functional $\mathbb{J}^\alpha_\psi$ with respect to the stochastic parameter $\omega \in \Omega$.

Measurability of integrands. First, for $\mathcal{H}^d$-almost every spatial point $\boldsymbol{x} \in \mathbb{R}^d$, the steady-state solutions are measurable as a mapping $\omega \mapsto k_\alpha(\omega,\boldsymbol{x})$ by the measurability Assumption 7.6. This immediately implies the stochastic measurability of the integrand of Integral (7.5a), since the function $\nu \in \mathcal{L}^\infty(\mathbb{T} \times \mathbb{X};\mathbb{R})$ is independent of $\omega \in \Omega$. By hypothesis, the initial condition u_0 satisfies $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$. Therefore, it is in particular measurable in $\omega \in \Omega$. Combining the with the measurability of k_α by Assumption 7.6, we obtain that the integrand of Integral (7.5c) is stochastically measurable.

It remains to argue the measurability of the integrand of Integral (7.5b). By definition, the Kružkov entropy flux is given by

$$\mathfrak{q}\big(\omega, \boldsymbol{x}, \nu(t,\boldsymbol{x}), k_\alpha(\omega,\boldsymbol{x})\big) \coloneqq \operatorname{sign}\big(\nu(t,\boldsymbol{x}) - k_\alpha(\omega,\boldsymbol{x})\big)\Big(\mathfrak{f}\big(\omega,\boldsymbol{x},\nu(t,\boldsymbol{x})\big) - \mathfrak{f}\big(\omega,\boldsymbol{x},k_\alpha(\omega,\boldsymbol{x})\big)\Big) .$$

The sign term is measurable in $\omega \in \Omega$ due to the steady-state solution k_α being measurable and the fact that the composition of two Borel-measurable functions is again measurable. The flux function $\mathfrak{f}$ is stochastically measurable by the measurability Assumption 7.3. This follows directly from its definition in Equation (7.2) as $\mathfrak{f}(\omega,\boldsymbol{x},\upsilon) \coloneqq \boldsymbol{g}\big(\omega,\mathfrak{Z}(\omega,\boldsymbol{x},\upsilon)\big)$. By Assumption 7.3, the function $\mathfrak{Z}$ is measurable for fixed value υ and for $\mathcal{H}^d$-almost every spatial point $\boldsymbol{x} \in \mathbb{R}^d$. Since the function $\boldsymbol{g}$ is Carathéodory, it is in particular jointly measurable (see [5, Lemma 4.51]), which implies the stochastic measurability of the flux function $\mathfrak{f}$. This establishes the measurability of the term $\mathfrak{f}\big(\omega,\boldsymbol{x},\nu(t,\boldsymbol{x})\big)$. For the measurability of the term $\mathfrak{f}\big(\omega,\boldsymbol{x},k_\alpha(\omega,\boldsymbol{x})\big)$ note that we can rewrite this term as

$$\mathfrak{f}\big(\omega,\boldsymbol{x},k_\alpha(\omega,\boldsymbol{x})\big) \equiv \alpha ,$$

which gives us the measurability of this term. Now, the difference of two measurable functions is measurable and the pointwise product of measurable functions is again measurable. Therefore, we have established stochastic measurability of the integrand of Integral (7.5b).

Measurability of integrals. Concluding the stochastic measurability of the integrals in the Mapping (7.5) is similar to the reasoning in the proof of Proposition 6.13: Since the test function is compactly supported, taking the integral is a bounded linear operator. Therefore, integration is a continuous operation and stochastic measurability of each integral in the Mapping (7.5) follows as the composition of a continuous operator with a measurable function (cf., [5, Lemma 4.22]). ■

With the two preceeding propositions, we have argued that the random entropy functional $\mathbb{J}_\psi^\alpha$ is Carathéodory, since it is measurable with respect to the stochastic parameter $\omega \in \Omega$ by Proposition 7.10 and depends continuously on the function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ by Proposition 7.9. We summarize this result in the following theorem, which also concludes this section.

Theorem 7.11 (Entropy functional is Carathéodory)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \le q < \infty$ and $1 \le p \le \infty$, be a random initial condition to Problem (7.1). Furthermore, let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 as well as the measurability Assumption 7.3. Moreover, let a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ and a parameter $\alpha \in \mathbb{R}$ be fixed and let the corresponding steady-state solution k_α satisfy the measurability Assumption 7.6. Then, the random entropy functional $\mathbb{J}_\psi^\alpha$ is Carathéodory in the sense that it is measurable in $\omega \in \Omega$ and continuous in $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. ◆

7.2.3 Continuous dependence of entropy functional

To establish a last property of the random entropy functional, this section is devoted to showing that $\mathbb{J}_\psi^\alpha$ depends continuously on the steady-state parameter $\alpha \in \mathbb{R}$. This continuity property will turn out to be crucial for showing that the random entropy solution u is strongly measurable in Section 7.3. The sought continuous dependence result is justified in the next theorem.

Theorem 7.12 (Continuous dependence of functional on steady-state parameter)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (7.1). Furthermore, let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 as well as the measurability Assumption 7.3. Moreover, let a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. Then, for a fixed $\omega \in \Omega$ and fixed $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$, the random entropy functional $\mathbb{J}^\alpha_\psi$ depends continuously on the steady-state parameter $\alpha \in \mathbb{R}$. ◆

Proof. Let a stochastic parameter $\omega \in \Omega$, a nonnegative test function $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{X};\mathbb{R})$ and a function $\nu \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ be fixed. The continuous dependence proof is similar to the argumentation of Theorem 6.15:

By Proposition 7.5, the steady-state solution k_α to the corresponding steady-state Equation (7.3) depends continuously on the parameter $\alpha \in \mathbb{R}$. This immediately implies that the integands of the integrals (7.5a) and (7.5c) depend continuously on the parameter $\alpha \in \mathbb{R}$. Furthermore, since the time derivative $\partial_t\psi$ of the test function is compactly supported by hypothesis, the continuous dependence follows via dominated convergence. For details we refer to the proof of Proposition 6.15 and to [98, Theorem 5.6].

The argumentation of Integral (7.5b) depending continuously on the parameter $\alpha \in \mathbb{R}$ is based on splitting up the integral domain depending on the value of the term $\operatorname{sign}\big(\nu(t,\boldsymbol{x}) - k_\alpha(\omega,\boldsymbol{x})\big)$, which appears in the Kružkov entropy flux $\mathbf{q}$. The main difference to the proof of Theorem 6.15 is that the norm $\|\cdot\|_d$ has to be used instead of the absolute value $|\cdot|$, since we consider the multi-dimensional case. Then, continuous dependence is concluded by a case-by-case study of the resulting integrals. ■

7.3 Well-posedness of random entropy solutions

With the notion of random entropy solutions and functionals at hand, we are now ready to investigate the well-posedness of these solutions to the random scalar conservation law given by Problem (7.1). This well-posedness consists of (pathwise) existence and uniqueness as well as strong measurability of the solution with respect to the stochastic parameter $\omega \in \Omega$. The latter allows us to interpret the solution u as a $\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$-valued, Bochner-integrable random variable $u : \Omega \to \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$. Thus, the statistical properties of the solution u can be described via the existence of moments.

The well-posedness investigation begins with a discussion of pathwise existence and uniqueness in Section 7.3.1. Afterwards, in Section 7.3.2, the strong measurability of these unique entropy solutions is established. To conclude this chapter, Section 7.3.3 discusses the existence of stochastic moments of random entropy solutions.

7.3.1 Pathwise existence and uniqueness of random entropy solutions

In this section, we discuss the pathwise existence and uniqueness of entropy solutions to the random scalar discontinuous-flux conservation law (7.1). For fixed stochastic parameter $\omega \in \Omega$, the problem reduces to the deterministic setting, which allows us to exploit the findings of Panov [230]. Therefore, the subsequent theorem states the pathwise existence and uniqueness result of [230]. Due to the similarity of the proof to the deterministic framework, we only summarize the main ideas of the proof. For the precise argumentation we refer the reader to [230, Theorem 3] for the uniqueness result and to [230, Theorem 4] for the existence of entropy solutions.

Theorem 7.13 (Pathwise existence & uniqueness of entropy solutions)**:**
Let $\omega \in \Omega$ be fixed and let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1. Furthermore, let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (7.1). Then, there exists a unique entropy solution $u(\omega,\cdot,\cdot) \in \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})\big)$ to the random scalar conservation law given by Equation (7.1). ◆

Proof (main ideas). For both results, the notion of *process entropy solutions*[30] is leveraged. For these, a rather evolved version of the Kružkov doubling of variables method can be established to argue uniqueness of solutions. Additionally, this argumentation can be adapted in a similar way as the one-dimensional uniqueness shown in [18], if one wants to establish a more general comparison principle for process entropy sub- and supersolutions to the discontinuous-flux conservation law given by Problem (7.1).

The existence proof is based on the construction of an approximate sequence of process entropy solutions. For this sequence, weak-$\star$ convergence in $\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ is shown. The idea for constructing such a sequence is to consider a convoluted truncation of the flux function, which satisfies the assumptions of Kružkov [173]. Thereby, the existence of Kružkov entropy solutions to the approximated conservation law is ensured. Now, since the corresponding steady-state solutions to this approximated equation are uniformly bounded in $\mathcal{L}^\infty(\mathbb{X};\mathbb{R})$, a subsequence converges to the unique process entropy solution.

Panov [230] shows that process entropy solutions reduce to entropy solutions in the sense of Definition 7.7. Therefore, the (pathwise) existence and uniqueness of entropy solutions to the random discontinuous-flux conservation law given by Equation (7.1) is established. ■

7.3.2 Strong measurability of entropy solutions

With the pathwise existence and uniqueness result of Theorem 7.13 at hand, we are ready to complete the well-posedness analysis by establishing strong measurability of solutions. While, in general, the random entropy solution u is a random function $u : \Omega \to \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$, strong measurability requires the image $u(\Omega)$ to be separable. Even though the space $\mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R})$ is not separable, the following remark gives us separability of the image $u(\Omega)$.

Remark 7.14 (Separability of solution space)**:** *By Theorem 7.13, the pathwise entropy solution $u(\omega,\cdot,\cdot)$ takes values in the function space*

$$\mathcal{S} \coloneqq \mathcal{L}^\infty(\mathbb{T}\times\mathbb{X};\mathbb{R}) \cap \mathcal{C}\big(\mathbb{T}; \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})\big)\,.$$

Recall that this space is separable, since the space $\mathcal{L}^1(\mathbb{X};\mathbb{R})$ of integrable functions is separable and a dense subset of $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X};\mathbb{R})$. Thus, the separability of $\mathcal{S}$ follows from the time interval $\mathbb{T} = [0,T]$ being compact. For details on this argumentation, we refer to Example 3.49 (ii). ◆

Leveraging this separability of the image $u(\Omega)$ as a subset of the separable space $\mathcal{S}$, we have every ingredient available that we need to establish the strong measurability of random entropy solutions u. The corresponding result is proved in the next theorem.

[30] This type of solution was introduced by Panov [230] to shorten the definition of *measure-valued solutions* (or a *bounded measurable process*). However, there exsists a one-to-one correspondence between processes and measure-valued solutions.

Theorem 7.15 (Strong measurability of entropy solutions)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (7.1). Furthermore, let the flux $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 as well as the measurability Assumption 7.3 and let the steady-state solution k_α corresponding to $\alpha \in \mathbb{R}$ satisfy the measurability Assumption 7.6. Then, the random entropy solution u is strongly measurable in the sense that the mapping $u : \Omega \to \mathcal{S}$, with $\omega \mapsto u(\omega, \cdot, \cdot)$ is strongly measurable. ◆

Proof. The proof is similar to the corresponing one-dimensional strong measurability result of Theorem 6.17. We start with the definition of a separable function space containing nonnegative compactly supported functions. For $N \in \mathbb{N}$, define

$$\mathcal{D}_N := \Big\{ \psi \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{X}; \mathbb{R}_{\geq 0}) \mid \operatorname{supp}(\psi) \subseteq \big\{ (t, \boldsymbol{x}) \in \mathbb{T} \times \mathbb{X} \mid t \in [0, N] \text{ and } x \in \mathbb{B}_N(\mathbf{0}_d) \big\} \Big\}.$$

Here, $\mathbb{B}_N(\boldsymbol{x})$ denotes the closed ball with radius N around $\boldsymbol{x} \in \mathbb{R}^d$ and $\mathbf{0}_d$ is the d-dimensional zero vector. Now, for a fixed number $N \in \mathbb{N}$, let $(\psi_N^i, i \in \mathbb{N}) \subset \mathcal{D}_N$ be a basis of the space $\mathcal{D}_N$. Then, for a fixed parameter $\alpha \in \mathbb{R}$ and a fixed index $i \in \mathbb{N}$, we define the random modified entropy functional $\mathbb{J}^\alpha_{i,N} : \Omega \times \mathcal{S} \to \mathbb{R}$. This modified functional is identical to the random entropy functional $\mathbb{J}^\alpha_\psi$, except that the test function ψ is replaced by the basis function ψ_N^i of the space $\mathcal{D}_N$. By construction, this functional is Carathéodory by Theorem 7.11.

With the steady-state parameter $\alpha \in \mathbb{R}$ still being fixed as well as the numbers $i, N \in \mathbb{N}$, we can define the set-valued mapping

$$\Xi^\alpha_{i,N} : \Omega \rightrightarrows \mathcal{S} \qquad \omega \mapsto \big\{ \nu \in \mathcal{S} \mid \mathbb{J}^\alpha_{i,N}(\omega, \nu) \geq 0 \big\} .$$

By Lemma 2.9, this multifunction is measurable, since the space $\mathcal{S}$ is separable. Furthermore, the set-valued mapping $\Xi^\alpha_{i,N}$ contains all functions $\nu \in \mathcal{S}$ that satisfy the entropy Condition (7.4) for a fixed steady-state parameter $\alpha \in \mathbb{R}$ and fixed test function $\psi_N^i \in \mathcal{D}_N$.

To exclude functions $\nu \in \mathcal{S}$ that only satisfy the entropy Condition (7.4) for some fixed test function ψ_N^i, we define the multifunction

$$\Xi^\alpha : \Omega \rightrightarrows \mathcal{S} \qquad \Xi^\alpha(\omega) := \bigcap_{i \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \Xi^\alpha_{i,N}(\omega) \qquad \text{for } \omega \in \Omega .$$

The measurability of this correspondence follows as the countable intersection of measurable maps is again measurable. Further defining the set-valued map

$$\Xi : \Omega \rightrightarrows \mathcal{S} \qquad \Xi(\omega) := \bigcap_{\alpha \in \mathbb{Q}} \Xi^\alpha(\omega) \qquad \text{for } \omega \in \Omega ,$$

the correspondence Ξ contains all functions $\nu \in \mathcal{S}$ that satisfy the entropy Condition (7.4). Again, this correspondence Ξ is measurable as the countable intersection of measurable maps.

Now, by Theorem 7.13 there exists a unique random entropy solution for every stochastic parameter $\omega \in \Omega$. Therefore, this correspondence Ξ is a singleton containing only the pathwise unique entropy solution u. This implies the measurability of the solution u as a mapping $u : \Omega \to \mathcal{S}$, which implies strong measurability due to the separability of $\mathcal{S}$. The proof is concluded by noting that it is sufficient to intersect over the rational numbers $\mathbb{Q}$ instead of the real numbers $\mathbb{R}$. This argument was discussed in detail at the end of the proof of Theorem 6.17 and thus is omitted here. ■

7.3.3 Existence of moments of random entropy solutions

With the previous investigations of this section, we have established the well-posedness of random entropy solutions. In particular, we can interpret the solution u as an $\mathcal{S}$-valued, Bochner-integrable random variable, whose statistical properties can be described via the existence of its stochastic moments. This investigation is the purpose of this section, which concludes the chapter on random scalar conservation laws with Panov-type flux functions. As in the previous chapters, the main ingredients for the existence of stochastic moments is the $\mathcal{L}^1$-contraction property resulting from the uniqueness proof combined with integrability of the random initial condition u_0. Additionally, we need to restrict the possible choices of flux functions according to the following assumption. Afterwards, the main result of this section is presented, which states the existence of moments of random entropy solutions.

Assumption 7.16 (Zero mass creation via flux function)**:**
We assume that the Panov-type flux function f is chosen such that the zero initial condition $u_0 \equiv 0$ leads to the vanishing entropy solution $u \equiv 0$. ◆

Theorem 7.17 (Existence of moments of random entropy solutions)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X};\mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, be a random initial condition to Problem (7.1). Furthermore, let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 as well as the measurability Assumption 7.3 and let the steady-state solution k_α corresponding to a parameter $\alpha \in \mathbb{R}$ satisfy the measurability Assumption 7.6. Additionally, let the flux function satisfy the zero-mass-creation Assumption 7.16. Then, at almost every time $t \in \mathbb{T}$, the random entropy solution u admits moments up to order q. In particular, the entropy solution satisfies

$$\|u(\omega,t,x)\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)} \leq \|u_0(\omega,x)\|_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)}$$

for almost every time $t \in \mathbb{T}$. ◆

Proof. For every stochastic parameter $\omega \in \Omega$, a pathwise unique entropy solution $u(\omega,\cdot,\cdot)$ exists due to the result of Theorem 7.13 and additionally Theorem 7.15 yields its strong measurability. Therefore, for any value $1 \leq q < \infty$, we can estimate

$$\Big\|u(\omega,t,x)\Big\|^q_{\mathcal{L}^q\left(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\right)} = \int_\Omega \Big\|u(\omega,t,x)\Big\|^q_{\mathcal{L}^1(\mathbb{X};\mathbb{R})} \,\mathrm{d}\mathbb{P} \leq \int_\Omega \Big\|u_0(\omega,x)\Big\|^q_{\mathcal{L}^1(\mathbb{X};\mathbb{R})} \,\mathrm{d}\mathbb{P}\,.$$

Here, the estimation follows from the pathwise $\mathcal{L}^1$-contraction property of the random entropy solution $u(\omega,\cdot,\cdot)$, which follows from the uniqueness proof of Panov [230]. Let us stress the usage of Assumption 7.16 in this step, which is required to obtain an estimate solely for the entropy solution u rather than estimating the difference of two entropy solutions u and $\tilde{u}$. The assertion follows by taking the q-th root and noting that the integrals exist due to the initial condition satisfying $u_0 \in \mathcal{L}^q\big(\Omega;\mathcal{L}^1(\mathbb{X};\mathbb{R})\big)$ by hypothesis. ■

As a special consequence of the preceeding theorem, we obtain the existence of all moments of order $1 \leq q < \infty$, as soon as the initial condition is deterministic. We formalize this result in the following corollary, which also concludes this chapter. Since the statement is similar to Corollary 6.20, we omit its proof.

Corollary 7.18 (Existence of moments for deterministic initial conditions)**:**
Let $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$ be a deterministic initial condition to Problem (7.1)*. Furthermore, let the flux function $\mathfrak{f}$ satisfy the Panov-type flux Assumption 7.1 as well as the measurability Assumption 7.3 and let the steady-state solution k_α corresponding to a parameter $\alpha \in \mathbb{R}$ satisfy the measurability Assumption 7.6. Then, at almost every $t \in \mathbb{T}$, the random entropy solution u admits all moments of order $1 \leq q < \infty$.* ◆

Numerical simulation of one-dimensional random conservation laws

8

In this chapter, we discuss how random entropy solutions to discontinuous-flux conservation laws can be approximated. Recall that flux functions of Panov-type correspond to a straightforward extension of one-dimensional flux functions to multiple space dimensions. Therefore, throughout this chapter, we restrict ourselves to simulating scalar conservation laws in one spatial dimension. For details on the numerical approximation of multi-dimensional scalar conservation laws with Panov-type flux functions, we refer to [116, Section 5].

For the numerical experiments in this chapter, we consider the space-time domain $\mathbb{T} \times \mathbb{X} = (0,1)^2$, unless stated otherwise. Then, for an unknown $u := u(\omega, t, x)$, the *random discontinuous-flux Burgers' equation* is defined as

$$\begin{aligned} \partial_t u + \operatorname{div}_x \left(\mathfrak{a}(\omega, x) \frac{u^2}{2} \right) &= 0 && \text{in } \Omega \times \mathbb{T} \times \mathbb{X} \,, \\ u(\omega, 0, x) &= u_0(\omega, x) && \text{on } \Omega \times \{0\} \times \mathbb{X} \,. \end{aligned} \tag{8.1}$$

Here, $u_0 \in \mathcal{L}^q\big(\Omega; \mathcal{L}^p(\mathbb{X}; \mathbb{R})\big)$, with $1 \leq q < \infty$ and $1 \leq p \leq \infty$, is a random initial condition and $\mathfrak{a}(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{X}; \mathbb{R})$ is a random jump-advection coefficient. The Burgers' equation poses a convenient prototype for numerical investigations due to several reasons: First, it serves as a simple model in many applications, such as fluid flow or traffic simulations (see, e.g., [40, 218, 219]). Additionally, the behavior of its solutions is similar to the nonlinear part of the Navier-Sokes equation, which is important for modeling and applications [40, 226]. Another advantage of the Burgers' equation is the fact that it is the simplest nonlinear conservation law, in which shock-waves may appear and thus provides many interesting (numerical) challenges.

Let us stress that the flux function of the random discontinuous-flux Burgers' Equation (8.1) has a multiplicative form as discussed in Examples 6.2 and 6.6. Unless stated otherwise, we equip this initial value problem with periodic boundary conditions, which means that we enforce

$$\mathfrak{a}(\omega, 0) \frac{u(\omega, t, 0)^2}{2} = \mathfrak{a}(\omega, 1) \frac{u(\omega, t, 1)^2}{2} \,.$$

To the best of the author's knowledge, writing a general numerical approximation scheme to simulate the (random) adapted entropy solution in the sense of Audusse-Perthame is still an open problem. While there exist several schemes that enable the computation of such entropy solutions, they all require some

additional assumption. In 2020, Towers [272] proved convergence of a finite difference scheme to the adapted entropy solution of Audusse-Perthame type for the special class of conservation laws, whose flux functions have bounded variation ($\mathcal{BV}$). Recently, Ghoshal et al. [114] showed convergence of a Godunov scheme to the Audusse-Perthame adapted entropy solution for the same case of $\mathcal{BV}$ fluxes. The method of [114] might be suitable to approximate the proposed random discontinuous-flux conservation law for a broad class of stochastic jump-advection coefficients. However, for general Lévy-type random fields we cannot guarantee the flux function to have bounded variation without restricting the possible choice of the covariance operator Q. Nevertheless, Godunov-type methods have been successfully applied to hyperbolic conservation laws with discontinuous flux functions for many other notions of solutions. We refer to [4] for so-called solutions of type (A, B) or to [268, 270] for entropy solutions that satisfy an interface condition at the discontinuities in the flux function. Therefore, throughout this chapter, we also employ a Godunov-type finite volume scheme for the approximation of the random adapted entropy solution.

We start this chapter on the numerical simulation of discontinuous-flux conservation laws by discussing the construction of the considered random jump-advection coefficient $\mathfrak{a}$ in Section 8.1. This coefficient is a particular instance of a Lévy-type random field as introduced in Section 2.4. Afterwards, in Section 8.2, we introduce two samplewise discretization techniques that take the flux discontinuities into account. Thereafter, we investigate the parameter dependency of finite volume approximations of the random entropy solution in Section 8.3. In particular, these results show that the pathwise convergence rate is stochastic and depends on various characteristics of the underlying random jump-advection coefficient. In Section 8.4, we discuss these pathwise convergence rates for particular random fields addressing these characteristics. Via these experiments, we also demonstrate how the samplewise discretization techniques have the ability to significantly improve the convergence behavior of approximations. We conclude this chapter on the numerical simulation of conservation laws by numerically approximating the stochastic moments of random entropy solutions in Section 8.5.

8.1 Random jump-advection coefficient

In this section, we discuss the construction of a Lévy-type random field $\mathfrak{a}$ that we employ in the stochastic discontinuous-flux Burgers' Equation (8.1). While the jump coefficient $\mathfrak{a}$ is inspired by the construction of Lévy-type random fields in Definition 2.28, we need to guarantee that the Assumptions 6.1 and 6.5 are satisfied, such that the well-posedness theory of Chapter 6 is applicable. Recall that, by the discussion and construction of Section 2.4, a Lévy-type random field $\mathfrak{a}$ for one spatial dimension is defined as a mapping

$$\mathfrak{a} : \Omega \times \mathbb{R} \to \mathbb{R}_{>0} , \qquad (\omega, x) \mapsto \overline{\mathfrak{a}}(x) + \Phi\big(\mathcal{G}(\omega, x)\big) + \mathfrak{P}(\omega, x) , \tag{8.2}$$

where the jump field $\mathfrak{P}$ is defined as a piecewise function that depends on a random partition $\mathfrak{T}$ of the domain $\mathbb{R}$, where $\mathfrak{T}$ consists of $\tau : \Omega \to \mathbb{N}$ elements. Furthermore, $\overline{\mathfrak{a}}$ is a deterministic, uniformly bounded mean function and Φ is a continuously differentiable, positive mapping that is applied to the Gaussian random field $\mathcal{G}$ associated to a covariance operator Q.

Unfortunately, a random coefficient $\mathfrak{a}$ as defined in Equation (8.2), does not necessarily satisfy the Audusse-Perthame flux Assumption 6.1, since the term $\Phi\big(\mathcal{G}(\omega, \boldsymbol{x})\big)$ is not pathwise bounded. Thus, it may violate Assumption (A-2). Before we introduce *pathwise bounded Lévy-type random fields* to overcome this obstacle, we need the notion of *bounded Gauss-type random fields*.

Definition 8.1 (Gauss-type random field)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ *be a complete probability space and let* $\mathbb{X}_{\mathcal{G}} \subset \mathbb{R}$ *be a compact subset of the spatial domain* $\mathbb{R}$*. Then, a* bounded Gauss-type random field $\mathcal{G}$ *is defined as a function*

$$\mathcal{G} : \Omega \times \mathbb{R} \to \mathbb{R}_{>0} , \qquad (\omega, x) \mapsto \Phi\big(\mathcal{G}(\omega, x)\big) ,$$

where $\Phi \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}_{>0})$ *is a continuously differentiable, positive mapping. Let* $\mathcal{G}_{\mathbb{R}} \in \mathcal{L}^2\big(\Omega; \mathcal{L}^2(\mathbb{R}; \mathbb{R})\big)$ *be a zero-mean Gaussian random field associated to a nonnegative, symmetric trace-class (covariance) operator* $Q : \mathcal{L}^2(\mathbb{R}; \mathbb{R}) \to \mathcal{L}^2(\mathbb{R}; \mathbb{R})$ *let one of the following two conditions be satisfied:*

- $\mathcal{G} \in \mathcal{L}^2\big(\Omega; \mathcal{L}^2(\mathbb{R}; \mathbb{R})\big)$ *is a* truncated Gaussian random field*, defined as*

$$\mathcal{G}(\omega, x) = \begin{cases} \mathcal{G}_{\mathbb{R}}(\omega, x), & x \in \mathbb{X}_{\mathcal{G}} , \\ \min\big(\mathcal{G}_{\mathbb{R}}(\omega, x), \sup_{x \in \mathbb{X}_{\mathcal{G}}} \mathcal{G}_{\mathbb{R}}(\omega, x)\big), & x \in \mathbb{R} \setminus \mathbb{X}_{\mathcal{G}} . \end{cases}$$

- *The functional* Φ *is bounded and the function* $\mathcal{G}$ *satisfies* $\mathcal{G} = \mathcal{G}_{\mathbb{R}}$*.* ◆

With these Gauss-type random fields, we have all ingredients at hand to introduce *pathwise bounded Lévy-type random fields*. Here, the subsequent definition is inspired by the Definition 2.28 of Lévy-type random fields.

Definition 8.2 (Pathwise bounded Lévy-type random fields)**:**
Let $(\Omega, \Sigma, \mathbb{P})$ *be a complete probability space and let* $\mathbb{X}_{\mathcal{G}} \subset \mathbb{R}$ *be a compact subset of the spatial domain* $\mathbb{R}$*. Then, a* pathwise bounded Lévy-type random field (LTRF) $\mathfrak{a}$ *is defined as a function*

$$\mathfrak{a} : \Omega \times \mathbb{R} \to \mathbb{R}_{>0} , \qquad (\omega, x) \mapsto \overline{\mathfrak{a}}(x) + \mathcal{G}(\omega, x) + \mathfrak{P}(\omega, x) , \tag{8.3}$$

where

- $\overline{\mathfrak{a}} \in \mathcal{C}(\mathbb{R}; \mathbb{R}_{\geq 0})$ *is a deterministic, uniformly bounded mean function.*
- $\mathcal{G}$ *is a Gauss-type random field as introduced in Definition 8.1.*
- $\mathfrak{T} : \Omega \to \mathcal{B}(\mathbb{X}_{\mathcal{G}})$, $\omega \mapsto \{\mathfrak{T}_1, \dots, \mathfrak{T}_\tau\}$ *is a random partition of* $\mathbb{X}_{\mathcal{G}}$ *in the sense that* $\{\mathfrak{T}_i\}_{i=1}^{\tau}$ *is a family of disjoint open subsets of* $\mathbb{X}$ *satisfying* $\mathbb{X} = \cup_{i=1}^{\tau} \mathrm{cl}\big(\mathfrak{T}_i\big)$*. The number of elements in* $\mathfrak{T}$ *is given by an integrable random variable* $\tau : \Omega \to \mathbb{N}$ *on the probability space* $(\Omega, \Sigma, \mathbb{P})$*. Furthermore, for* $\mathbb{X}_l$ *and* $\mathbb{X}_r$ *being the left and right boundary of* $\mathbb{X}$*, respectively, we define the partition elements* $\mathfrak{T}_0 \coloneqq (-\infty, \mathbb{X}_l)$ *and* $\mathfrak{T}_{\tau+1} \coloneqq (\mathbb{X}_r, +\infty)$*.*
- *A measure* Λ *on the measurable space* $\big(\mathbb{X}, \mathcal{B}(\mathbb{X})\big)$ *is associated to the partition* $\mathfrak{T}$ *and controls the positions of the random elements* $\mathfrak{T}_i$*.*
- *Let* $(\mathfrak{p}_i, i \in \mathbb{N}_0)$ *be a sequence of random variables on the probability space* $(\Omega, \Sigma, \mathbb{P})$ *with arbitrary positive distribution(s) satisfying* $\mathfrak{p}_i < \infty$ *for every stochastic parameter* $\omega \in \Omega$*. Then, we define the jump field* $\mathfrak{P}$ *as*

$$\mathfrak{P} : \Omega \times \mathbb{R} \to \mathbb{R}_{>0}, \quad (\omega, x) \mapsto \sum_{i=0}^{\tau+1} \mathbb{1}_{\mathfrak{T}_i}(x) \mathfrak{p}_i(\omega) , \tag{8.4}$$

where the sequence $(\mathfrak{p}_i, i \in \mathbb{N}_0)$ *is independent of the number* τ *of elements in the partition* $\mathfrak{T}$*, but not necessarily pairwise independent and identically distributed.* ◆

Let us stress that we do not require the Gauss-type random field $\mathcal{G}$ and the jump field $\mathfrak{P}$ to be stochastically independent. Furthermore, recall that by Remark 2.29, the measure Λ associated to the partition $\mathfrak{T}$ does not only affect the average number of partition elements $\mathbb{E}(\tau)$, but also the size of the elements $\mathfrak{T}_i$.

We are now ready to show that the random discontinuous-flux Burgers' Equation (8.1) involving pathwise bounded Lévy-type random fields satisfies the Audusse-Perthame flux Assumptions 6.1 and 6.5. The corresponding result is proven in the subsequent corollary.

Corollary 8.3 (Burgers' equation with pathwise bounded LTRF satisfies flux assumptions)**:**
Let the discontinuous flux function $\mathfrak{f}$ be given as $\mathfrak{f}(\omega, x, u) = \mathfrak{a}(\omega, x)u^2/2$, where $\mathfrak{a}$ is a pathwise bounded Lévy-type random field. Then, the flux $\mathfrak{f}$ satisfies the Audusse-Perthame flux Assumptions 6.1 and 6.5. ◆

Proof. Let the discontinuous flux function $\mathfrak{f}$ be given as $\mathfrak{f}(\omega, x, u) = \mathfrak{a}(\omega, x)u^2/2$ with a pathwise bounded Lévy-type random field $\mathfrak{a}$. We show each of the Assumptions 6.1 and 6.5 separately. First, note that the flux function is multiplicative. This special type of flux function was already discussed in Example 6.2 and 6.6, which simplifies the argumentation.

- *Discontinuity set is closed with measure zero (Assumption (A-1)):* Note that the deterministic mean function $\overline{\mathfrak{a}}$ is continuous by hypothesis. Furthermore, we can always consider a continuous modification of the Gauss-type random field $\mathcal{G}$. Now, for every stochastic parameter $\omega \in \Omega$, the number of elements τ in the partition $\mathfrak{T}$ is finite and the jump field $\mathfrak{P}(\omega, \cdot)$ is a piecewise constant function by its construction in Equation (8.4). Thus, $\mathfrak{P}$ has only finitely many discontinuities. Since the domain $\mathbb{X}$ is compact, the set $\mathfrak{D}(\omega)$ is closed and of measure zero.
- *Positive spatially bounded paths (Assumption (A-2)):* By construction of the jump field $\mathfrak{P}$, the jump heights $\mathfrak{p}_i$ satisfy $\mathfrak{p}_i > 0$ for every index $i \in \mathbb{N}_0$. Thus, there exists an $\mathfrak{a}_-(\omega) > 0$ such that the jump field $\mathfrak{P}$ satisfies $\mathfrak{P}(\omega, x) \geq \mathfrak{a}_- > 0$ for every spatial point $x \in \mathbb{R}$. Furthermore, the mean function $\overline{\mathfrak{a}}$ is nonnegative by construction and the functional Φ of the Gauss-type random field $\mathcal{G}$ is positive. Therefore, we can conclude $\mathfrak{a}(\omega, x) \geq \mathfrak{a}_-(\omega) > 0$. Additionally, by construction of the jump field $\mathfrak{P}$, there exists a pathwise upper bound $\mathfrak{P}_+ < +\infty$ satisfying $\mathfrak{P}(\omega, x) \leq \max_{0 \leq i \leq \tau(\omega)+1} \mathfrak{p}_i(\omega) =: \mathfrak{P}_+(\omega)$, since the number τ of partition elements is finite in the sense that $\tau(\omega) < \infty$ for every stochastic parameter $\omega \in \Omega$. Furthermore, the Gauss-type random field $\mathcal{G}(\omega, \cdot)$ is continuous because both the functional Φ and the random field $\mathcal{G}$ are continuous. Thus, there exists an upper bound $\mathcal{G}_+ < \infty$ such that $\mathcal{G}(\omega, x) \leq \mathcal{G}_+$. Finally, the mean function $\overline{\mathfrak{a}}$ is uniformly bounded by construction.
- *Local Lipschitz continuity and monotonicity (Assumption (A-3) or (A-3')):* The random discontinuous-flux Burgers' Equation (8.1) satisfies Assumption (A-3), which was argued in Example 6.2.
- *Measurability of flux and inverse flux (Assumptions (A-4) and (A-5)):* The stochastic measurability of the random jump-advection coefficient $\mathfrak{a}$ follows directly from its construction in Definition 8.2. Thus, by the discussion of Example 6.6, also the inverse flux function is measurable.
- *Bounded spatial paths of inverse flux function (Assumption (A-6)):* We have already established that the jump-advection coefficient has bounded spatial paths. Thus, Assumption (A-6) follows with the discussion of Example 6.6.

Combining all the above arguments, we have argued that the random discontinuous-flux Burgers' Equation (8.1) involving a pathwise bounded Lévy-type random field satisfies the Audusse-Perthame flux Assumption 6.1 and 6.5. ■

8.2 Pathwise discretization techniques

In this section, we introduce sample-adapted discretization techniques for the discontinuous-flux conservation law (8.1). Here, sample-adaptivity should not be confused with classical *adaptive* finite volume methods. These adaptive finite volume methods are based on a-posteriori error estimates and refine the underlying mesh various times for each sample. In contrast to that, the proposed sample-adaptive discretization methods are adapted to the discontinuities of the random jump-advection coefficient $\mathfrak{a}$. Here, adaptivity means that the mesh is aligned *a-priori* to these flux discontinuities. Consequently, the stochasticity of the coefficient leads to a random discretization in the sense that the mesh changes for every $\omega \in \Omega$.

While the framework of Audusse-Perthame solutions allows the flux $\mathfrak{f}$ to have infinitely many discontinuities, this is not the case for the subsequent numerical experiments. By construction, the partition $\mathfrak{T}$ consists of τ elements, where $\tau : \Omega \to \mathbb{N}$ is an integrable random variable. Thus, it makes sense to include the jump discontinuities in the spatial discretization.

8.2.1 Samplewise jump-adapted meshing

Let $\mathbb{X} \subset \mathbb{R}$ be the considered simulation domain. For a fixed stochastic parameter $\omega \in \Omega$, let $\mathfrak{D}(\omega) \subset \mathbb{X}$ denote the set of discontinuity points of the random jump-advection coefficient $\mathfrak{a}(\omega, \cdot)$ in $\mathbb{X}$. Furthermore, let $\mathbb{X}_\Delta \subset \mathbb{X}$ denote the set of all discretization points of the domain $\mathbb{X}$. The main idea of samplewise adapted meshing is the restriction that any admissible spatial mesh $\mathbb{X}_\Delta$ should satisfy $\mathfrak{D}(\omega) \subset \mathbb{X}_\Delta$. Consequently, any samplewise jump-adapted mesh $\mathbb{X}_\Delta$ contains all discontinuity points $\mathfrak{D}(\omega)$ of the random jump-advection coefficient $\mathfrak{a}(\omega, \cdot)$.

For a given discontinuous random field $\mathfrak{a}(\omega, \cdot)$, the simplest method for creating a samplewise jump-adapted mesh is the following: Let $\Delta_{x,\text{bound}} > 0$ be a maximum step size restriction for the sought discretization. To create a samplewise jump-adapted mesh, we start by initializing a discretization as $\mathbb{X}_\Delta = \mathfrak{D}(\omega)$. Thereby, the jump-adaptivity restriction on the spatial mesh is already satisfied by initialization and it remains to guarantee that the discretization complies with the step size constraint. To achieve this, let $\mathfrak{d}_j \in \mathfrak{D}(\omega)$ denote the j-th discontinuity point of the jump coefficient $\mathfrak{a}(\omega, \cdot)$. Now, each interval $[\mathfrak{d}_{j-1}, \mathfrak{d}_j]$ can be considered individually. If this interval violates step size constraint, i.e., $|\mathfrak{d}_j - \mathfrak{d}_{j-1}| > \Delta_{x,\text{bound}}$, equidistant points can be inserted until the condition is fulfilled. The resulting mesh is a piecewise equidistant mesh satisfying both the jump-adaptivity and the step size constraint.

8.2.2 Samplewise jump-adapted wave-cell meshing

The samplewise jump-adapted meshing strategy accounts for the discontinuities in the jump-advection coefficient $\mathfrak{a}(\omega, \cdot)$. Thus, these meshes already leverage information on the underlying problem. Recall that the randomized Burgers' equation (8.1) is the simplest conservation law in which shock-waves may appear. Combined with the Audusse-Perthame admissibility condition (6.9), this leads to standing-wave profiles in the adapted entropy solution, which appear at the stationary flux discontinuities. If the step size restriction $\Delta_{x,\text{bound}} > 0$ allows for large cells in the spatial mesh, these standing-wave profiles might not be approximated sufficiently, if one of the adjoining cells is large. Therefore, the *samplewise jump-adapted wave-cell meshing* aims at overcoming this obstacle for coarse discretizations $\mathbb{X}_\Delta$ or those containing very small and large cells at the same time.

Let a stochastic parameter $\omega \in \Omega$ be fixed and let $\Delta_{x,\text{bound}} > 0$ be a given step size restriction for the discretization. As in the samplewise jump-adapted meshing, we start by initializing the grid with the discontinuity points of the random coefficient $\mathfrak{a}(\omega, \cdot)$, i.e., $\mathbb{X}_\Delta = \mathfrak{D}(\omega)$. Similar to the samplewise jump-adapted meshing, let now $\mathfrak{d}_j \in \mathfrak{D}(\omega)$ denote the j-th discontinuity point of the jump coefficient $\mathfrak{a}(\omega, \cdot)$. Before we refine the mesh $\mathbb{X}_\Delta$, such that it satisfies the step size constraint, so-called *wave-cells* are inserted into the grid. Therefore, let $\Delta\mathfrak{w}$ denote the size of these wave-cells, which is given by

$$\Delta\mathfrak{w} \coloneqq \min_i \Delta_{\boldsymbol{x}}^i \,,$$

where $\Delta_{\boldsymbol{x}}^i$ denotes the size of the i-th cell size of $\mathbb{X}_\Delta = \mathfrak{D}(\omega)$, i.e., $\Delta_{\boldsymbol{x}}^i \coloneqq [\mathfrak{d}_{i-1}, \mathfrak{d}_i]$. We can now create the wave-cells in an iterative manner. For each index i, we compute the left and right wave-cell points $\mathfrak{w}_i^{l,r} = \mathfrak{d}_i \mp \Delta\mathfrak{w}$. If the wave-cell point $\mathfrak{w}_i^l$ satisfies

$$\mathfrak{w}_i^l \in \mathbb{X} \quad \text{and} \quad [\mathfrak{d}_{i-1}, \mathfrak{w}_i^l] > \Delta\mathfrak{w} \,,$$

we add this point $\mathfrak{w}_i^l$ to the current mesh $\mathbb{X}_\Delta$. Analogously, the wave-cell point $\mathfrak{w}_i^r$ is inserted into the discretization $\mathbb{X}_\Delta$, if it satisfies

$$\mathfrak{w}_i^r \in \mathbb{X} \quad \text{and} \quad [\mathfrak{d}_i, \mathfrak{w}_i^r] > \Delta\mathfrak{w}$$

In this way, we refine the cells $\Delta_{\boldsymbol{x}}^i$, which are adjacent to the flux discontinuities. Additionally, via the conditions $[\mathfrak{d}_{i-1}, \mathfrak{w}_i^l] > \Delta\mathfrak{w}$ and $[\mathfrak{d}_i, \mathfrak{w}_i^r] > \Delta\mathfrak{w}$, we ensure that the CFL condition is not affected, since we do not change the minimal step size of the discretization.

As we will see in the subsequent numerical experiments, especially in the pathwise convergence study in Section 8.4, this samplewise jump-adapted wave-cell meshing can significantly improve the approximation of standing-wave profiles occurring in the solution. In particular, this novel meshing approach reduces the samplewise variance of approximations, especially if the coefficient admits small distances between jumps. Let us stress that we do not expect this samplewise jump-adapted wave-cell discretization to converge at a better rate, but rather we aim at reducing the error constant of the convergence estimate.

8.3 Parameter dependency of approximations

The purpose of this section is to investigate the influence of various parameters of a Lévy-type random field $\mathfrak{a}$ on the entropy solution u to the random Burgers' Equation (8.1). Here, we are interested in how the inspected parameters affect the convergence rate of pathwise approximations. Therefore, we consider the strong error, which is given by

$$\mathcal{E}(\omega) \coloneqq \left\| u_\Delta^{\text{ref}}(\omega, \cdot, T) - u_\Delta(\omega, \cdot, T) \right\|_{\mathcal{L}^\star(\mathbb{X};\mathbb{R})} \,, \tag{8.5}$$

where $\mathcal{L}^\star(\mathbb{X};\mathbb{R})$ denotes either $\mathcal{L}^1(\mathbb{X};\mathbb{R})$ or $\mathcal{L}^\infty(\mathbb{X};\mathbb{R})$. Let us stress that the $\mathcal{L}^\infty$-error should not be confused with the error in the Chebyshev norm[31] $\mathcal{L}^\infty\big(\mathbb{T};\mathcal{L}^1(\mathbb{X})\big)$. Instead, we consider the (spatial) $\mathcal{L}^\infty(\mathbb{X})$-error. Furthermore, u_Δ is the computed finite volume approximation of the solution u and u_Δ^{ref}

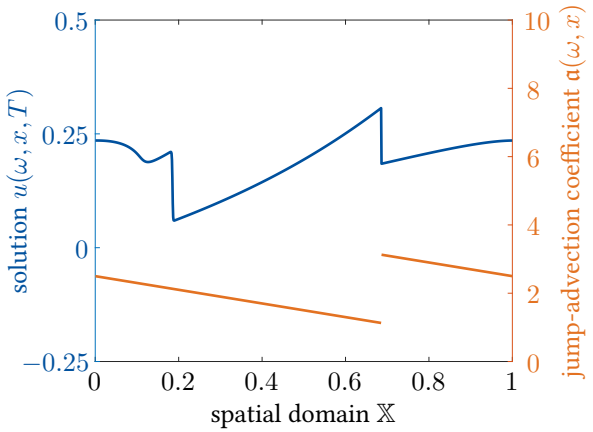
[31] The Chebyshev norm is also known as the "infinity norm" or "supremum norm".

is a reference solution, which is simulated using a jump-adapted discretization with maximum step size $\Delta_{x,\text{bound}}$ set to one fourth of the smallest step size bound considered for the approximations u_Δ. Throughout this section, we only consider an equidistant meshing and the samplewise jump-adapted discretization technique, which was presented in Section 8.2.1. Since we focus on the effect of parameters on the convergence rate, this is sufficient because the pathwise jump-adapted wave-cell discretization aims at reducing the samplewise variance of approximations, which does not affect the convergence rate. Let us stress that the qualitative findings of this parameter study remain true, when we consider the $\mathcal{L}^2$-error instead, even though the particular results are not shown for this error norm.

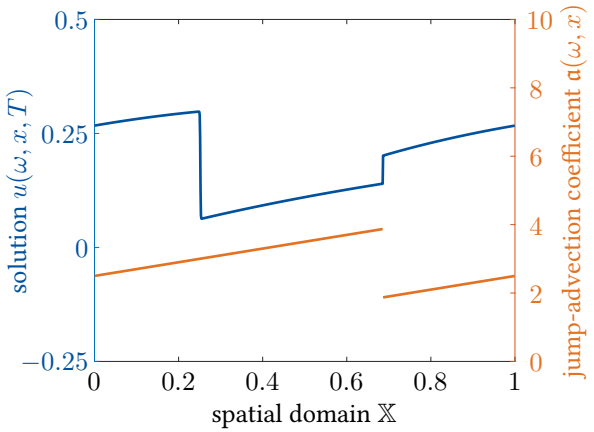
In Section 8.3.1, we start the parameter study by investigating parameters of the jump field $\mathfrak{P}$, such as the number of jumps or how close occurring jumps are located to each other. Afterwards, in Section 8.3.2, two experiments examine the performance of an explicit and implicit Euler time integration scheme for various simulation settings. To conclude this section, the influence of parameters of a Gaussian random field is investigated in Section 8.3.3.

8.3.1 Jump field parameters

We start the parameter study by investigating the influence of discontinuities in the jump-advection coefficient on the approximated solution u_Δ and the convergence rate. Therefore, we employ rather artificial, deterministic jump coefficients $\mathfrak{a}(\omega, x) = \mathfrak{P}(x)$ to illustrate the effect of various parameters. Before we start, however, Figure 8.1 depicts two solutions at the final time $T = 1$, for which the random field contains an upwards and downwards jump, respectively.



(a) Approximated entropy solution u with a jump-advection coefficient jumping up.



(b) Approximated entropy solution u with a jump-advection coefficient jumping down.

Figure 8.1: Illustration of the influence of a discontinuity in the jump-advection coefficient on the computed entropy solution.

Here, the initial condition u_0 was chosen as $u_0 = 0.3\sin(\pi x)$ and the jump-advection coefficient was set to $\mathfrak{a}(\omega, x) = 2(1-x) + \mathfrak{P}_{\text{up}}(x)$ and $\mathfrak{a}(\omega, x) = 2x + \mathfrak{P}_{\text{down}}(x)$, respectively. Here, the jump fields $\mathfrak{P}_{\text{up}}$ and $\mathfrak{P}_{\text{down}}$ are given by

$$\mathfrak{P}_{\text{up}}(x) \coloneqq \begin{cases} \frac{1}{2} & \text{for } 0 \le x \le \frac{2\pi}{10}\,, \\ \frac{5}{2} & \text{for } \frac{2\pi}{10} \le x \le 1\,, \end{cases} \qquad \mathfrak{P}_{\text{down}}(x) \coloneqq \begin{cases} \frac{5}{2} & \text{for } 0 \le x \le \frac{2\pi}{10}\,, \\ \frac{1}{2} & \text{for } \frac{2\pi}{10} \le x \le 1\,. \end{cases}$$

Due to this construction, the discontinuous random field contains a jump of height 2 facing upwards and downwards, respectively. One immediately observes that the jump up leads to a downward-facing discontinuity in the solution and the jump down yields a discontinuity at which the solution increases. Around the point $x \approx 0.2$ in Figure 8.1a and $x \approx 0.25$ in Figure 8.1b the solution contains a shock-wave, which appears due to the chosen initial condition u_0. In contrast to the standing-wave profiles at the discontinuities of the jump-advection coefficient $\mathfrak{a}$, these shock-waves move through the spatial domain.

Distance between jumps

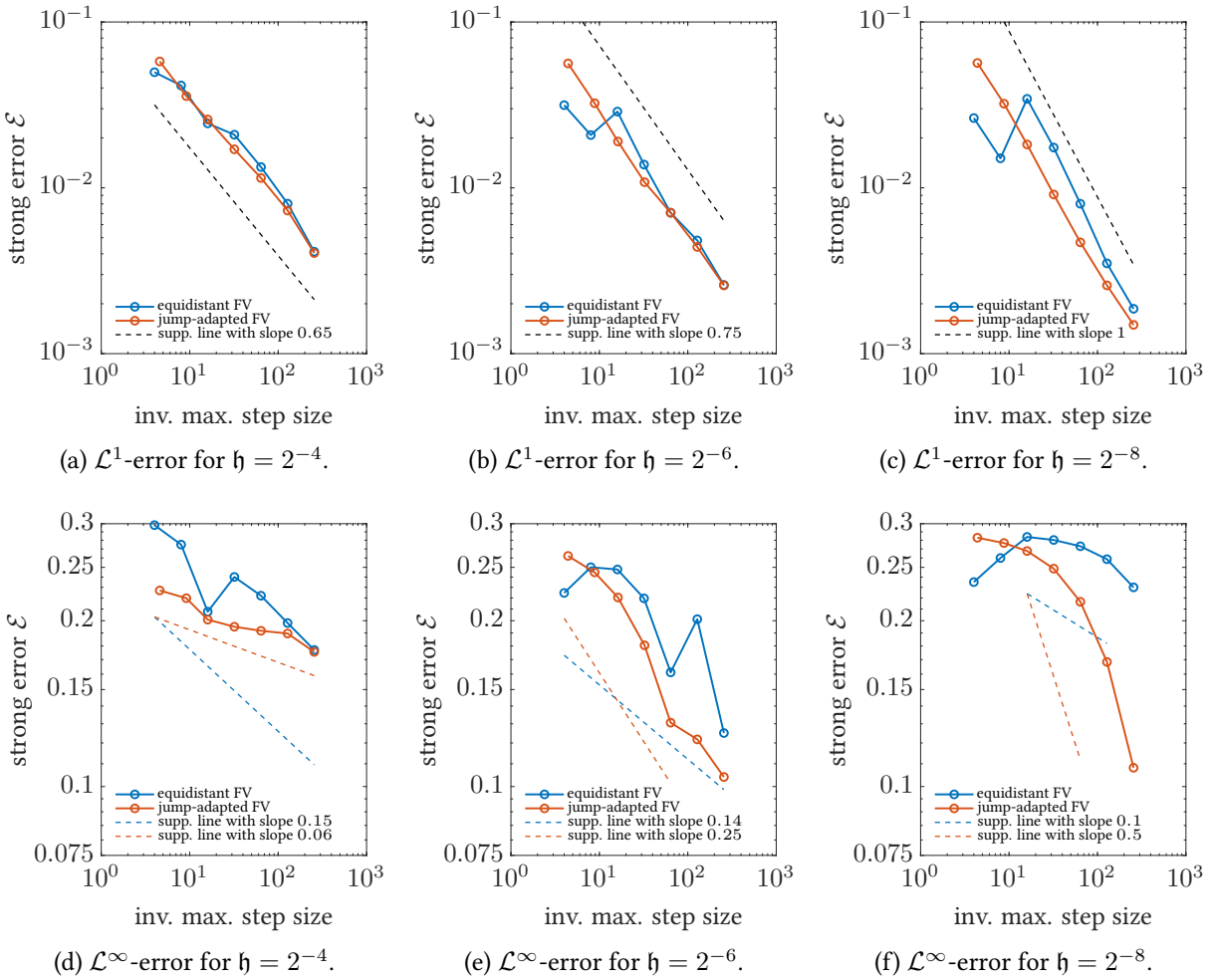
The first investigated parameter is the distance between two jumps that occur in the discontinuous jump-advection coefficient. As the illustration in Figure 8.1 shows, the impact of a discontinuity may vary depending on whether it decreases or increases the value of the jump-advection coefficient. Therefore, we define the two jump coefficients

$$\mathfrak{P}^{\mathrm{up}}(x) = \begin{cases} \frac{1}{2} & \text{for } x \in \mathbb{X} \setminus \mathfrak{H}^{\mathfrak{h}} \,, \\ \frac{3}{2\mathfrak{h}} & \text{for } x \in \mathfrak{H}^{\mathfrak{h}} \,, \end{cases} \qquad \mathfrak{P}^{\mathrm{down}}(x) = \begin{cases} \frac{3}{2} & \text{for } x \in \mathbb{X} \setminus \mathfrak{H}^{\mathfrak{h}} \,, \\ \mathfrak{h} & \text{for } x \in \mathfrak{H}^{\mathfrak{h}} \,. \end{cases}$$

Here, $\mathfrak{H}^{\mathfrak{h}}$ denotes the area on which the coefficient differs from its constant value (due to a jump discontinuity at the beginning and end of $\mathfrak{H}^{\mathfrak{h}}$). To be more specific, we define the considered *jump area* as $\mathfrak{H}^{\mathfrak{h}} := \left(\mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}} - \mathfrak{h}/2, \mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}} + \mathfrak{h}/2\right) \subset \mathbb{X}$, where $\mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}} = 1 - \pi/10$ is the midpoint of $\mathfrak{H}^{\mathfrak{h}}$ and $\mathfrak{h} > 0$ denotes its width. Note, by this construction of the random jump coefficients, we have ensured that the total mass of the coefficient remains constant when varying the width $\mathfrak{h}$ of the jump areas $\mathfrak{H}^{\mathfrak{h}}$. In Figure 8.2 the approximation error of the finite volume method for varying widths $\mathfrak{h}$ is presented. Here, the solution u was computed with an equidistant meshing and an jump-adapted discretization for the jump-advection coefficient being set to $\mathfrak{P}^{\mathrm{up}}$.

Considering the $\mathcal{L}^1$-error of the finite volume discretizations (Figures 8.2a – 8.2c), one can observe similar convergence rates of the equidistant and the jump-adapted finite volume approximation. However, with decreasing jump area width $\mathfrak{h}$, one detects an enlarging kink in the error data of the equidistant discretization. This kink is caused by the resulting mesh not resolving the jump area of the underlying coefficient at all. Consequently, the error increases, once the discretization resolves the jump. Such a behavior cannot be observed for the jump-adapted meshing, which results in a much more reliable convergence behavior. The influence of the jump area width $\mathfrak{h}$ is even more lucid if one considers the $\mathcal{L}^\infty$-error of finite volume approximations. In this case, the jump-adapted discretization is able to improve the convergence rate of finite volume approximations compared to equidistant meshing. A different behavior of the approximation error can be observed for the jump-advection coefficient being set to $\mathfrak{P}^{\mathrm{down}}$, which is visualized in Figure 8.3.

One immediately notices various differences in the convergence behavior. For the $\mathcal{L}^1$-error of finite volume approximations, we observe a pre-asymptotic error decay for coarse spatial discretizations. This behavior is significantly stronger for the equidistant discretization than it is for the jump-adapted one. As in the previous experiment, changing the error measure to $\mathcal{L}^\infty$ magnifies the observings. In addition to this pre-asymptotic behavior, the equidistant meshing shows kinks in the error data, which we already observed in the previous investigation.



(a) $\mathcal{L}^1$-error for $\mathfrak{h} = 2^{-4}$. (b) $\mathcal{L}^1$-error for $\mathfrak{h} = 2^{-6}$. (c) $\mathcal{L}^1$-error for $\mathfrak{h} = 2^{-8}$.

(d) $\mathcal{L}^\infty$-error for $\mathfrak{h} = 2^{-4}$. (e) $\mathcal{L}^\infty$-error for $\mathfrak{h} = 2^{-6}$. (f) $\mathcal{L}^\infty$-error for $\mathfrak{h} = 2^{-8}$.

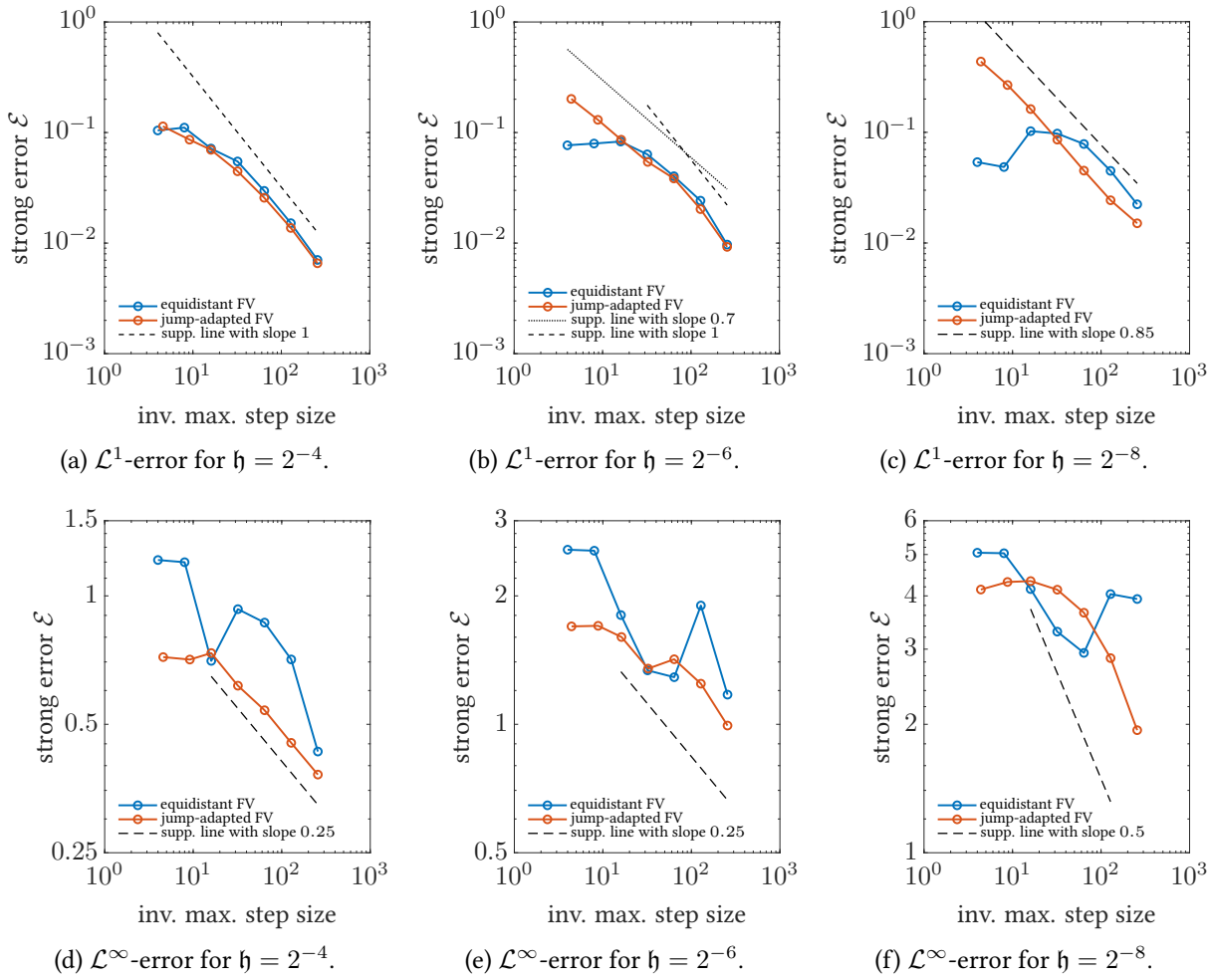
Figure 8.2: $\mathcal{L}^1$- and $\mathcal{L}^\infty$-error of finite volume (FV) approximations of the entropy solution based on equidistant and jump-adapted discretizations. Here, the width $\mathfrak{h}$ of the jump area $\mathfrak{H}^\mathfrak{h}$ in the underlying random coefficient $\mathfrak{P}^{\mathrm{up}}$ is varied.

Number of jumps

We continue the parameter study by investigating, how the number of discontinuities in the jump-advection coefficient affects the convergence rate of finite volume approximations with equidistant or jump-adapted discretizations. Therefore, we consider an *alternating jump field*, whose construction is inspired by Equation (8.4) and given by

$$\mathfrak{P} : \mathbb{X} \to \mathbb{R}_{>0} \,, \qquad x \mapsto \sum_{j=1}^{\tau+1} \mathbb{1}_{\mathfrak{T}_i}(x)\mathfrak{p}_i \,, \qquad \mathfrak{p}_i = \begin{cases} \frac{1}{2} & \text{for } i \text{ odd} \,, \\ \frac{3}{2} & \text{for } i \text{ even} \,. \end{cases}$$

Here, $\tau \in \mathbb{N}$ denotes the number of jumps and $\mathfrak{T}$ denotes the partition of the spatial domain $\mathbb{X} = (0,1)$ resulting from the discontinuities $\mathfrak{d}_j$ of the jump field $\mathfrak{P}$. In particular, we assume that these discontinuity points $\mathfrak{d}_j$ are distributed equidistantly across the domain $\mathbb{X}$, such that each element $\mathfrak{T}_i$ of the partition $\mathfrak{T}$ has equal width. For this setting, Figure 8.4 indicates the convergence rates of the finite volume approximations for various values of τ describing the number of jumps.



(a) $\mathcal{L}^1$-error for $\mathfrak{h} = 2^{-4}$. (b) $\mathcal{L}^1$-error for $\mathfrak{h} = 2^{-6}$. (c) $\mathcal{L}^1$-error for $\mathfrak{h} = 2^{-8}$.

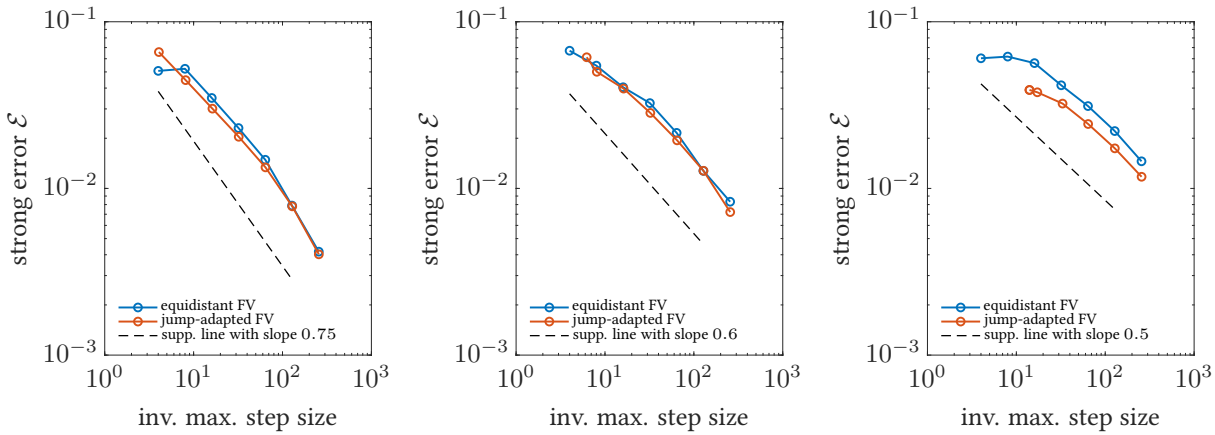
(d) $\mathcal{L}^\infty$-error for $\mathfrak{h} = 2^{-4}$. (e) $\mathcal{L}^\infty$-error for $\mathfrak{h} = 2^{-6}$. (f) $\mathcal{L}^\infty$-error for $\mathfrak{h} = 2^{-8}$.

Figure 8.3: $\mathcal{L}^1$- and $\mathcal{L}^\infty$-errors of finite volume (FV) approximations of the entropy solution based on equidistant and jump-adapted discretizations. Here, the width $\mathfrak{h}$ of the jump area $\mathfrak{H}^{\mathfrak{h}}$ in the underlying random coefficient $\mathfrak{P}^{\text{down}}$ is varied.

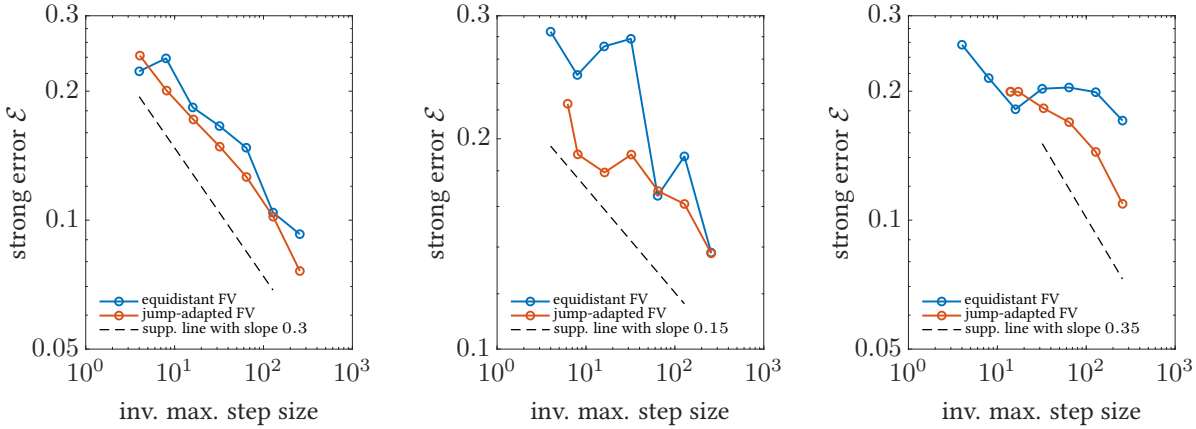
Considering the $\mathcal{L}^1$-error of finite volume approximations (Figures 8.4a – 8.4c) one immediately observes that an increasing number of jumps leads to a lower convergence rate. This consequence is independent on the chosen discretization technique, but jump-adaptivity in the meshing may lead to a better error constant, as can be seen in Figure 8.4c. For the $\mathcal{L}^\infty$-error, we note that the equidistant finite volume approximation leads to kinks in the convergence behavior, which is similar to the previous experiments. Additionally, in the $\mathcal{L}^\infty$-norm, the experiment indicates that the jump-adapted discretization technique may result in a higher convergence rate than the equidistant meshing. Finally, for this error measure, the constant of the convergence behavior with jump-adapted meshing seems to be significantly lower than the constant for an equidistant discretization.

8.3.2 Explicit vs. implicit time integration

The previous investigations on the jump field parameters have shown that it might be advantageous for the approximation error to employ an jump-adapted discretization. By construction, the jump-adapted meshing strategies may lead to discretizations with a large discrepancy between the maximum and



(a) $\mathcal{L}^1$-error for $\tau = 4$ number of jumps. (b) $\mathcal{L}^1$-error for $\tau = 16$ number of jumps. (c) $\mathcal{L}^1$-error for $\tau = 64$ number of jumps.



(d) $\mathcal{L}^\infty$-error for $\tau = 4$ number of jumps. (e) $\mathcal{L}^\infty$-error for $\tau = 16$ number of jumps. (f) $\mathcal{L}^\infty$-error for $\tau = 64$ number of jumps.

Figure 8.4: $\mathcal{L}^1$- and $\mathcal{L}^\infty$-errors of finite volume (FV) approximations of the entropy solution u based on equidistant and jump-adapted discretizations. Here, the number of jumps τ in the underlying jump-advection coefficient $\mathfrak{P}$ is varied.

minimum cell size. While this property directly influences the performance of an explicit Euler time integration scheme via the necessary CFL stability condition, it may also require using a globalized optimizer (e.g., Newton-Raphson algorithm with step size control) in an implicit Euler time propagator. As a consequence, this section investigates how the two mentioned time integrators perform in various settings. Therefore, let the jump-advection coefficient $\mathfrak{a}$ consist only of a deterministic jump field in the sense that $\mathfrak{a}(\omega, x) = \mathfrak{P}(x)$. In particular, we discuss, how the ratio of maximum and minimum step size influence the performance. Additionally, we investigate the affect of the estimated shock-wave speed in the solution on the preformance of these time integrators.

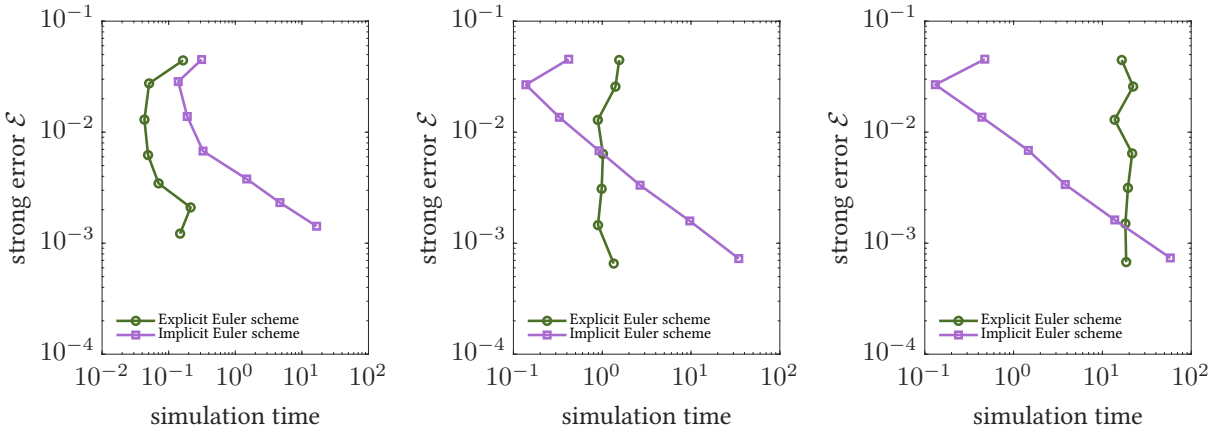
Ratio of maximum step size to minimum step size

We start the performance investigation of the explicit and implicit Euler time integration scheme by discussing the influence of the step size ration $\max(\Delta_x)/\min(\Delta_x)$. For this investigation, we consider the

following jump coefficient, which is inspired by the jump field $\mathfrak{P}^{\mathrm{up}}$ of Section 8.3.1:

$$\mathfrak{P}(x) = \begin{cases} \frac{1}{2} & \text{for } x \in \mathbb{X} \setminus \mathfrak{H}^{\mathfrak{h}} \,, \\ \frac{3}{2\mathfrak{h}} & \text{for } x \in \mathfrak{H}^{\mathfrak{h}} \,. \end{cases}$$

Here, $\mathfrak{H}^{\mathfrak{h}}$ denotes the area of size $\mathfrak{h} > 0$ on which the coefficient differs from its constant value. In particular, this area is given as $\mathfrak{H}^{\mathfrak{h}} := \left(\mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}} - \mathfrak{h}/2, \mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}} + \mathfrak{h}/2\right) \subset \mathbb{X}$, where $\mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}} = \pi/5$ denotes the center point of $\mathfrak{H}^{\mathfrak{h}}$. Since we consider a jump-adapted finite volume discretization, the size $\mathfrak{h}$ of the jump area $\mathfrak{H}^{\mathfrak{h}}$ imposes a restriction on the minimum step size $\min(\Delta_x)$. In Figure 8.5, we present the time-to-error data of the finite volume simulations for both explicit and implicit Euler time integration.



(a) Jump area size $\mathfrak{h} = 10^{-2}$. (b) Jump area size $\mathfrak{h} = 10^{-3}$. (c) Jump area size $\mathfrak{h} = 10^{-4}$.

Figure 8.5: Time-to-error data of the explicit and implicit Euler time propagator for computing a jump-adapted finite volume approximation of the entropy solution u. Here, the strong $\mathcal{L}^1$-error is compared to the required simulation time for different jump area sizes $\mathfrak{h} > 0$.

Here, the error values are computed for multiple restrictions on the maximum spatial step size $\Delta_{x,\mathrm{bound}}$. The time-to-error data of Figure 8.5 shows the result one would expect: The computational time needed by the implicit Euler time propagation increases with decreasing error. This is by far not surprising, since the nonlinear system, which needs to be solved at every time step, increases. However, one can also observe that the overall simulation time does not increase as the minimal step size gets finer. Contrary to this behavior is the explicit Euler time propagation. The amount of time, which is needed to evolve the solution over time, depends highly on the allowed maximum time step size. This behavior is again, what one would expect, since the CFL condition is directly influence by the area size $\mathfrak{h}$, which represents the minimal spatial cell size. Nevertheless, the required simulation time of the explicit Euler scheme does not increase for finer discretizations, if the minimal cell size remains constant.

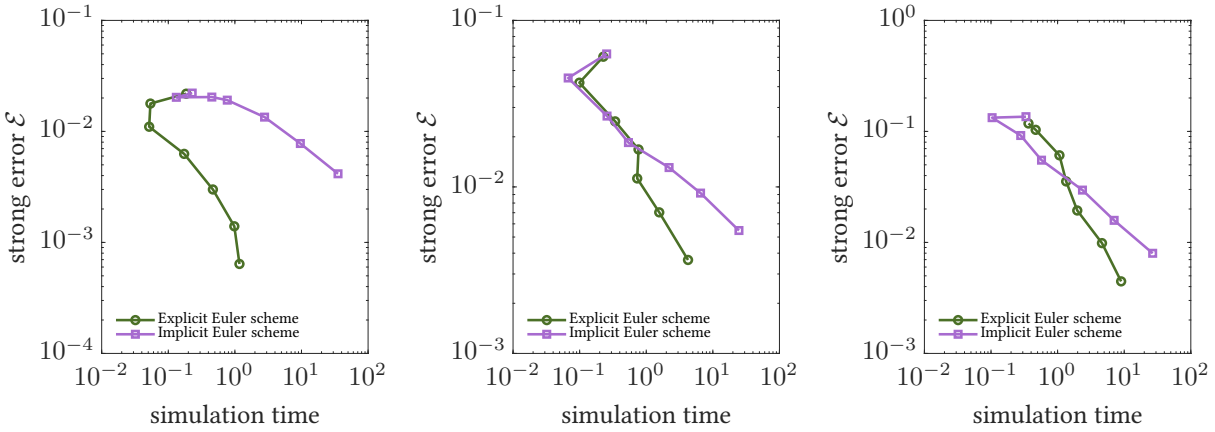
Let us note that this result is barely influenced by different error norms. On the one hand, the error values would change when using a different error measure. On the other hand, the error is evaluated *after* computing the approximation and thus does not affect the simulation time. Consequently, the qualitative findings of this experiment remain true for different norms.

Estimated wave speed

Another property that might affect the performance of the temporal propagators is the estimated speed of appearing shock-waves, since it affects the CFL condition. Since this estimated wave speed is depending in the jump-advection coefficient as well as on the absolute value of current (approximated) solution, we consider an initial condition $u_0(x) = \mathfrak{c}\sin(\pi x)$ for some positive scalar value $\mathfrak{c} \in \mathbb{R}_{>0}$. Additionally, we adapted the jump coefficient $\mathfrak{P}$ to have the form

$$\mathfrak{P}(x) = \begin{cases} \mathfrak{P}_1 & \text{for } x \in \mathbb{X} \setminus \mathfrak{H}^{\mathfrak{h}} \,, \\ \mathfrak{P}_2 & \text{for } x \in \mathfrak{H}^{\mathfrak{h}} \,, \end{cases}$$

for variable jump heights $\mathfrak{P}_{1,2} \in \mathbb{R}_{>0}$. Furthermore, the jump area $\mathfrak{H}^{\mathfrak{h}}$ is fixed in this experiment to $\mathfrak{H}^{\mathfrak{h}} = \left(\mathfrak{H}_c^{\mathfrak{h}} - 1/20, \mathfrak{H}_c^{\mathfrak{h}} + 1/20\right) \subset \mathbb{X}$ with center point $\mathfrak{H}_c^{\mathfrak{h}} = \pi/5$. The time-to-error plots for the two time integrators are given in Figure 8.6.



(a) Jump coefficient $\mathfrak{P}$ with values $\mathfrak{P}_1 = 10.5$ and $\mathfrak{P}_2 = 20$ and an initial condition scaling parameter $\mathfrak{c} = 0.3$.

(b) Jump coefficient $\mathfrak{P}$ with values $\mathfrak{P}_1 = 1$ and $\mathfrak{P}_2 = 50$ and an initial condition scaling parameter $\mathfrak{c} = 0.3$.

(c) Jump coefficient $\mathfrak{P}$ with values $\mathfrak{P}_1 = 1$ and $\mathfrak{P}_2 = 50$ and an initial condition scaling parameter $\mathfrak{c} = 1$.

Figure 8.6: Time-to-error data of the explicit and implicit Euler time propagator for computing a jump-adapted finite volume approximation of the entropy solution u. The $\mathcal{L}^1$-error is compared to the required simulation time for various values $\mathfrak{P}_1, \mathfrak{P}_2$ of the jump field and various scaling parameters $\mathfrak{c}$ of the initial condition, which result in different estimated wave speeds.

The results demonstrate that the implicit Euler scheme is outperforming the explicit Euler time propagation in only one occasion: In case a coarse spatial discretization is combined with a high maximum value of the jump coefficient, as is shown in Figure 8.6c. As soon as the mesh is refined, the performance of the explicit Euler scheme is rapidly improving compared to the performance of the implicit Euler. This is caused by the increasing size of the nonlinear system, which the implicit Euler scheme has to solve at every time step. Due to these performance and convergence results of the explicit and implicit Euler time integration scheme, the proceeding numerical experiments are all conducted with an explicit Euler time propagation.

8.3.3 Parameters of Gaussian random field

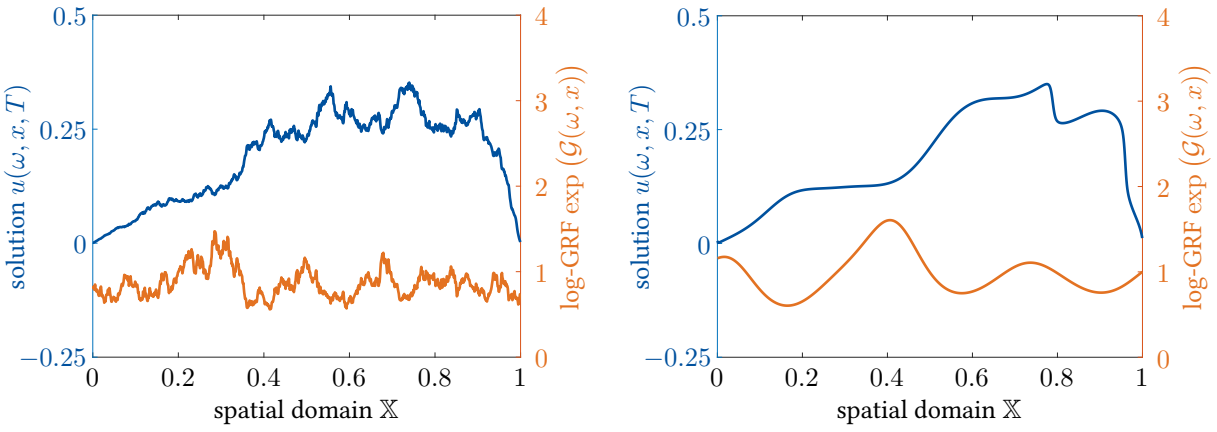
The purpose of this section is to investigate the characterizing parameters of the Gauss-type random field $\mathcal{G}(\omega, x) = \Phi\big(\mathcal{G}(\omega, x)\big)$ as introduced in Definition 8.1. Here, we set the functional Φ to $\Phi(\xi) = \exp(\xi)$, for some value $\xi \in \mathbb{R}$, such that the considered random field is log-Gaussian. To uniquely characterize the underlying Gaussian random field $\mathcal{G}$, we employ a *Matérn covariance operator*, which is given by $Q_{\mathrm{M}} : \mathcal{L}^2(\mathbb{X}; \mathbb{R}) \to \mathcal{L}^2(\mathbb{X}; \mathbb{R})$ defined via

$$[Q_{\mathrm{M}}\, \psi](y) := \int_{\mathbb{X}} \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\sqrt{2\nu}\, \frac{|x-y|}{\rho} \right)^{\nu} K_\nu \left(\sqrt{2\nu}\, \frac{|x-y|}{\rho} \right) \psi(x)\, \mathrm{d}x \,,$$

for all functions $\psi \in \mathcal{L}^2(\mathbb{X}; \mathbb{R})$. Here, the parameter $\nu > 0$ controls the smoothness of the Matérn covariance operator, $\sigma^2 > 0$ denotes its variance and $\rho > 0$ is the correlation length of Q_{M}. Additionally, Γ denotes the Gamma function and K_ν is the modified Bessel function of second kind with ν degrees of freedom. The Gaussian random field $\mathcal{G}$ is approximated via a truncated Karhunen-Loéve expansion introduced in Theorem 2.30. For this expansion, the spectral basis of Q_{M} is itself approximated via Nyström's method [286]. In the subsequent experiments, we focus on the influence of the smoothness parameter ν and the correlation length ρ of the covariance operator and their effects on the resulting Gaussian random field.

Matérn smoothness parameter

To investigate the Matérn smoothness parameter ν and its effect on the approximations and convergence rates, let the variance σ^2 be fixed at $\sigma^2 = 0.1$ and let the correlation length ρ be fixed at $\rho = 0.1$. We start by visualizing solutions to the random Burgers' Equation (8.1) for two underlying log-Gaussian random fields with varying smoothness parameters in Figure 8.7.



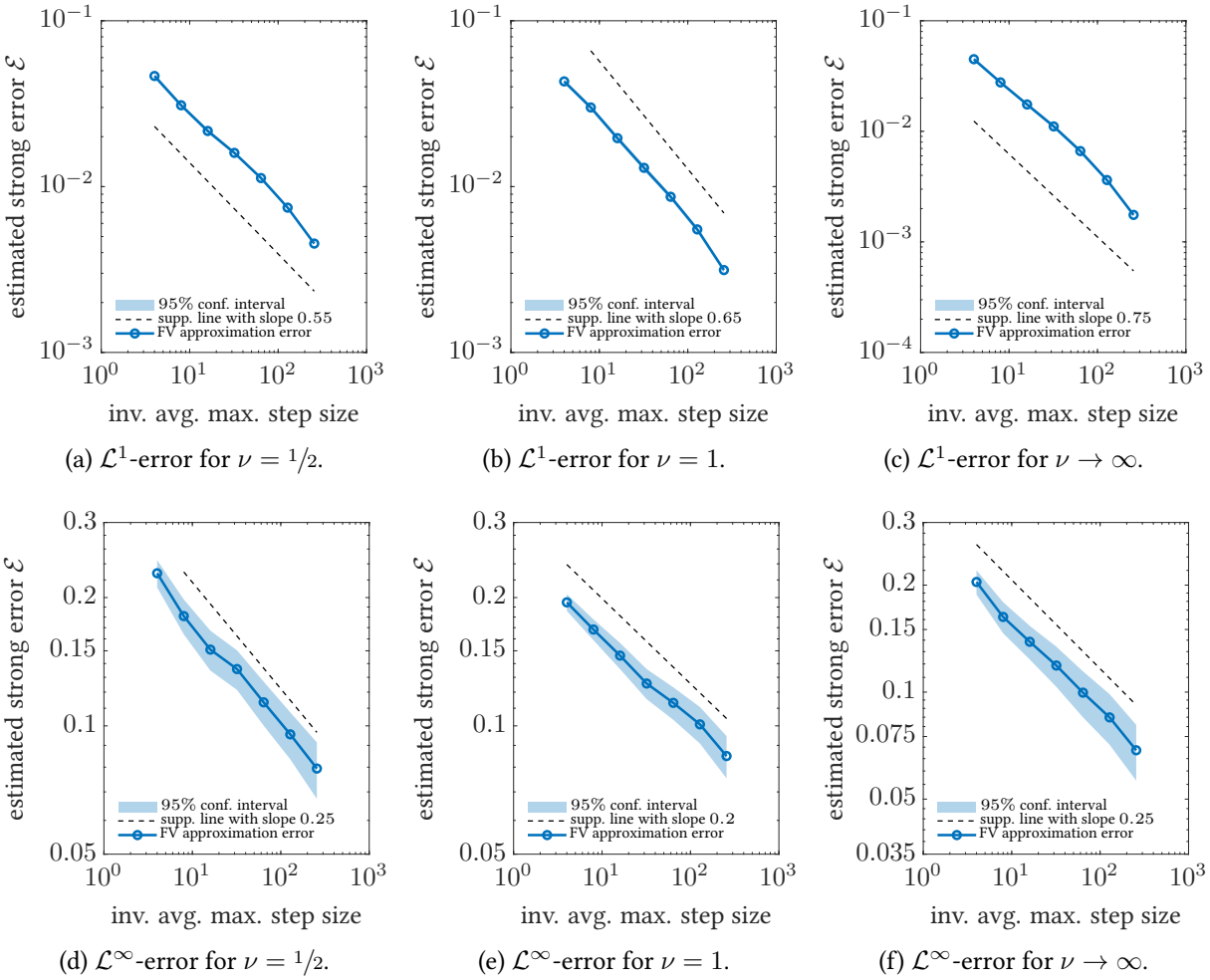
(a) Smoothness parameter $\nu = 1/2$.

(b) Smoothness parameter $\nu \to \infty$.

Figure 8.7: Solutions to the randomized Burgers' equation at the final time $T = 1$ with an underlying log-Gaussian random field (GRF), which is characterized by a Matérn covariance operator with varying smoothness parameter ν.

In Figure 8.7b the log-Gaussian random field is characterized by a Matérn covariance operator with smoothness parameter $\nu \to \infty$. Apparently, this smoothness is inherited by the solution. Here, the piles in the solution appear due to the initial condition $u_0 = 0.3\sin(\pi x)$. Additionally, at approximately $x \approx 0.8$, a shock-wave appears that is moving through the domain. Contrary to Figure 8.7b, the Matérn covariance operator in Figure 8.7a has a smoothness of $\nu = 1/2$. This low smoothness is again enherited to the solution. While we would still expect a shock to appear, such a wave is not identifiable due to the low regularity of the solution.

Recall that the Gaussian random field is stochastic and thus requires an estimation of the random convergence rates. Therefore, we employ a Monte Carlo estimator with 20 samples. The corresponding results are visualized in Figure 8.8.



(a) $\mathcal{L}^1$-error for $\nu = 1/2$. (b) $\mathcal{L}^1$-error for $\nu = 1$. (c) $\mathcal{L}^1$-error for $\nu \to \infty$.

(d) $\mathcal{L}^\infty$-error for $\nu = 1/2$. (e) $\mathcal{L}^\infty$-error for $\nu = 1$. (f) $\mathcal{L}^\infty$-error for $\nu \to \infty$.

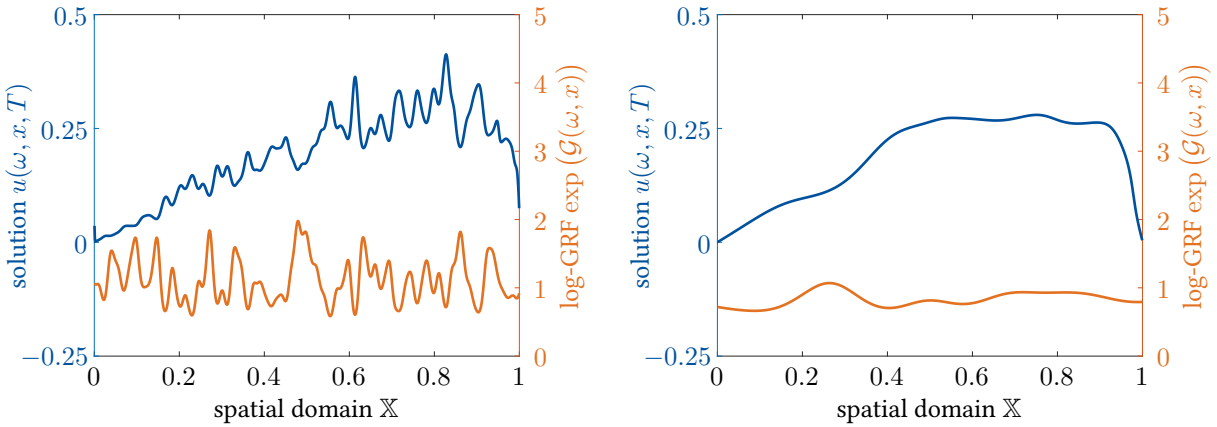
Figure 8.8: Convergence of (equidistant) finite volume (FV) approximations of the entropy solution u with an underlying log-Gaussian random field, which is charactierized by the Matérn covariance operator for varying smoothness parameters ν. Each error estimation is based on a Monte Carlo estimator with 20 samples.

As one can see in Figures 8.8a – 8.8c, the convergence rate for the $\mathcal{L}^1$-error of the finite volume approximation decreases as the smoothness parameter ν gets lower. We also observe that 20 samples are already sufficient to obtain a rather narrow confidence interval. This behavior is different, when

we consider the $\mathcal{L}^\infty$-error of the finite volume approximations. Here, the confidence interval is rather broad. Additionally, for this case the convergence rate of the $\mathcal{L}^\infty$-error does not seem to be affected by the smoothness parameter ν of the Matérn covariance operator.

Correlation length

For investigating the correlation length ρ of the Matérn covariance operator Q_{M}, the variance σ^2 was fixed at $\sigma^2 = 0.1$. Additionally, for this experiment, we fixed the smoothness parameter ν to satisfy $\nu \to \infty$. As in the previous experiment, we start by illustrating the effect of the correlation length ρ on the random entropy solution u. Therefore, Figure 8.9 depicts these solutions for two different values of the correlation length.



(a) Correlation length $\rho = 1/100$.

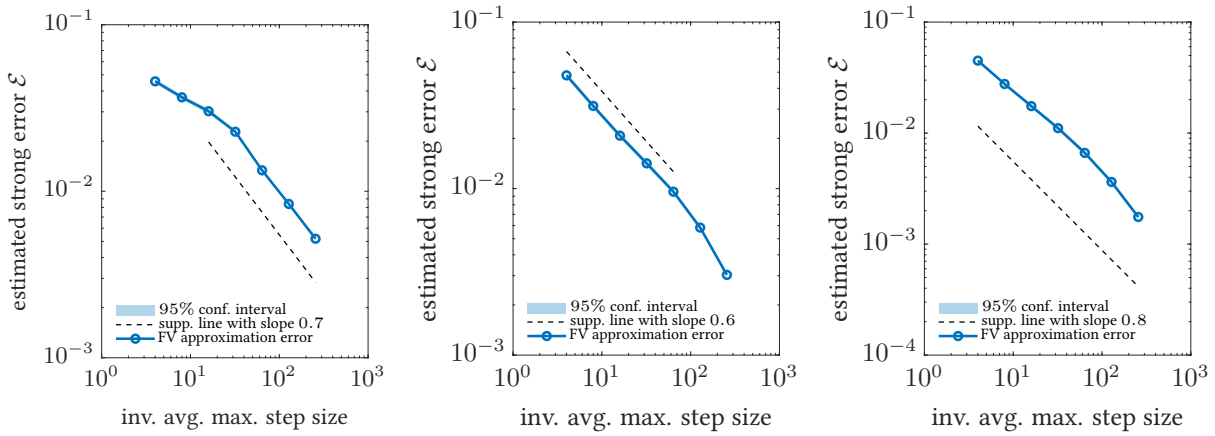
(b) Correlation length $\rho = 1/10$.

Figure 8.9: Solution to the randomized Burgers' equation at the final time $T = 1$ with an underlying log-Gaussian random field (GRF), which is characterized by a Matérn covariance operator with varying correlation length ρ.

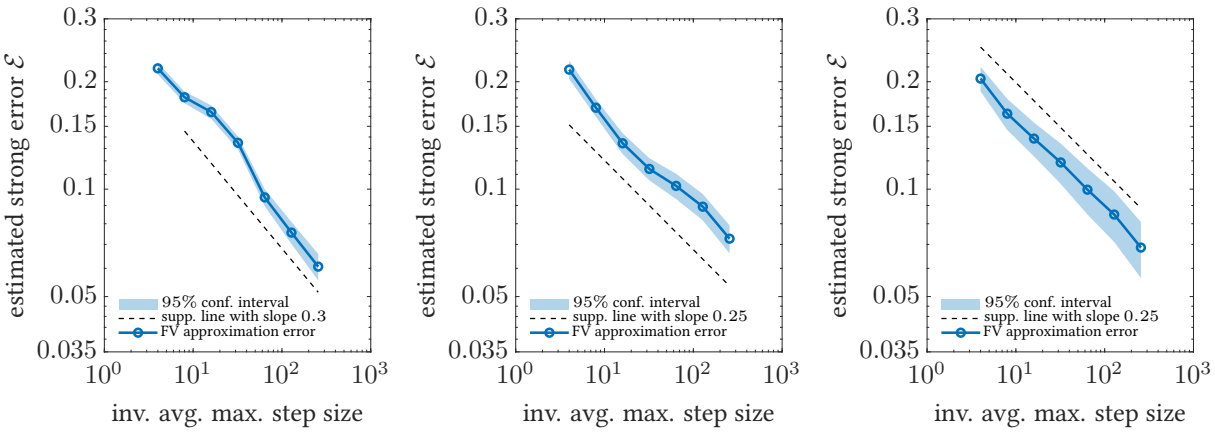
As we can observe in Figure 8.9, roughly speaking, a smaller correlation length leads to a higher spatial variation of the coefficient. Here, Figure 8.9b shows the solution corresponidng to a log-Gaussian random field with correlation length $\rho = 1/10$. The rather small spatial variation leads to the impression that the solution also does not change significantly due to changes in the coefficient. Contrary to this, Figure 8.9a depicts the solution corresponding to a correlation length $\rho = 1/100$, which has much more spatial variation. In this case, the solution inherits the varying behavior of the coefficient.

As the effect of the correlation length ρ has a significant influence on the behavior of the solution, it appears most natural that this parameter also affects the pathwise convergence of approximations. Therefore, the strong error $\mathcal{E}$ of approximated solutions is investigated, which is defined by Equation (8.5). To estimate the random convergence rates of the finite volume method, we again employ a Monte Carlo estimator with 20 samples. The results given in Figure 8.10 demonstrate that the effect of the correlation length ρ on the convergence rates depends on the considered error measure.

Considering the $\mathcal{L}^1$-error in Figures 8.10a – 8.10c we observe that a decreasing correlation length leads to lower convergence rates. This behavior is similar to the one discovered by investigating the Matérn smoothness parameter. In contrast to this, the convergence rate for the $\mathcal{L}^\infty$-error seems not to be



(a) $\mathcal{L}^1$-error for $\rho = 1/100$. (b) $\mathcal{L}^1$-error for $\rho = 1/20$. (c) $\mathcal{L}^1$-error for $\rho = 1/10$.



(d) $\mathcal{L}^\infty$-error for $\rho = 1/100$. (e) $\mathcal{L}^\infty$-error for $\rho = 1/20$. (f) $\mathcal{L}^\infty$-error for $\rho = 1/10$.

Figure 8.10: Convergence of (equidistant) finite volume (FV) approximations to the entopy solution u with an underlying log-Gaussian random field, which is characterized by the Matérn covariance operator for varying correlation lengths ρ. Each error estimation is based on a Monte Carlo estimator with 20 samples.

affected by the correlation length of the correlation length of the Matérn covariance operator. However, while 20 samples were sufficient for the $\mathcal{L}^1$-error to yield a narrow confidence interval, this is broader for the $\mathcal{L}^\infty$-error. Moreover, with increasing correlation length, the confidence interval seems to become broader for this error measure.

8.4 Pathwise convergence study

In this section, we investigate the pathwise convergence of the samplewise jump-adapted meshing and samplewise jump-adapted wave-cell meshing compared to a standard equidistant discretization. Therefore, we simulate the stochastic discontinuous-flux Burgers' Equation (8.1) equipped with various Lévy-type random fields, which have different qualitative properties. In all of the presented experiments, we consider the *strong error*, which is given by

$$\mathcal{E}(\omega) := \left\| u_\Delta^{\mathrm{ref}}(\omega, \cdot, T) - u_\Delta(\omega, \cdot, T) \right\|_{\mathcal{L}^\star(\mathbb{X};\mathbb{R})},$$

where $\mathcal{L}^\star(\mathbb{X};\mathbb{R})$ is either $\mathcal{L}^1(\mathbb{X};\mathbb{R})$, $\mathcal{L}^2(\mathbb{X};\mathbb{R})$ or $\mathcal{L}^\infty(\mathbb{X};\mathbb{R})$. Furthermore, u_Δ denotes the approximated solution computed with the finite volume method and u_Δ^{ref} is a reference solution. As in the previous sections, let us stress that the $\mathcal{L}^\infty$-error should not be confused with the error in the Chebyshev norm $\mathcal{L}^\infty\big(\mathbb{T};\mathcal{L}^1(\mathbb{X})\big)$. Instead, we consider the (spatial) $\mathcal{L}^\infty(\mathbb{X})$-error. Throughout all experiments in this section, we set a random initial condition u_0 to Equation (8.1), which is given by $u_0(x) = 0.3\sin(\pi x)$. Additionally, for all experiments in this section, the reference solution u_Δ^{ref} is simulated via the jump-adapted wave-cell finite volume method with a maximum step size bound $\Delta_{x,\mathrm{bound}}$ set to one fourth of the smallest step size bound considered in the numerical experiment at hand. Since the pathwise error $\mathcal{E}$ depends on the stochastic parameter $\omega \in \Omega$, it is approximated via a Monte Carlo estimation. Additionally, the relevant quantity for the error computation is the maximum spatial step size $\Delta_{x,\max} := \max_i \Delta_{\boldsymbol{x}}^i$. Its average over the computed samples is denoted by $\overline{\Delta_{x,\max}}$.

For the pathwise convergence study, we consider three different Lévy-type random fields in the subsequent sections: First, in Section 8.4.1, we consider a Poisson jump field with a squared-exponential Gaussian random field. Here, the Gaussian part is smooth and the jump heights are not bounded a-priori. Afterwards, in Section 8.4.2, we investigate a rough Gaussian part with jump heights that satisfy a-priori bounds. We conclude this pathwise convergence study by discussing a random field that has small inclusions in Section 8.4.3.

8.4.1 Poisson field with squared-exponential Gaussian random field

We start the pathwise convergence study by considering a *Poisson jump field with squared-exponential Gaussian random field*, which is a particular type of pathwise bounded Lévy-type random field given by Equation (8.3). The deterministic mean function $\bar{\mathfrak{a}}$ is set to $\bar{\mathfrak{a}} \equiv 0$ and the functional Φ is chosen as $\Phi(\xi) = \exp(\xi)$ for some scalar value $\xi \in \mathbb{R}$. Thus, according to the Definition 8.1 of Gauss-type random fields, the Gauss-type random field $\mathcal{G}$ has the form $\Phi\big(\mathcal{G}(\omega, x)\big)$ for some truncated Gaussian random field $\mathcal{G}$. Since on the spatial domain $\mathbb{X}$, no truncation is necessary, we consider $\mathcal{G}$ to be a Gaussian random field, which is characterized by the *squared-exponential covariance operator* Q_{SE}. This covariance operator is defined as a mapping $Q_{\mathrm{SE}} : \mathcal{L}^2(\mathbb{X}_\mathcal{G};\mathbb{R}) \to \mathcal{L}^2(\mathbb{X}_\mathcal{G};\mathbb{R})$ given by

$$[Q_{\mathrm{SE}}\,\psi](y) := \int_{\mathbb{X}} \sigma^2 \exp\left(\frac{|x-y|}{2\rho^2}\right)\psi(x)\,\mathrm{d}x \quad \text{for all } \psi \in \mathcal{L}^2(\mathbb{X}_\mathcal{G};\mathbb{R})\,.$$

Here, the variance σ^2 was set to $\sigma^2 = 0.1$ and the correlation length ρ was chosen as $\rho = 0.1$. The Gaussian random field $\mathcal{G}$ is approximated via a truncated Karhunen-Loéve expansion (compare Theorem 2.30) based on the spectral basis of the covariance operator Q_{SE}. This spectral basis is itself approximated via Nyström's method [286].

It remains to specify the Poisson jump field of the random field $\mathfrak{a}$. Therefore, let the number of jumps τ be a Poisson-distributed random variable with $\tau \sim \mathrm{Poi}(5)+1$, which guarantees that $\mathfrak{a}$ has at least one jump. The corresponding jump positions are uniformly distributed across the domain $\mathbb{X}$, i.e., $\mathfrak{d}_j \sim \mathcal{U}(\mathbb{X})$. As the name of the random field already suggests, the jump heights $\mathfrak{p}_i$ are Poisson-distributed. In particular, we set $\mathfrak{p}_i \sim \mathrm{Poi}(5) + 1$, which ensures the positivity of the jump heights. Note, the constructed jump field does not admit any global bound that is satisfied for all samples, since the Poisson distribution is unbounded. We can only find a samplewise bound by evaluating the maximum jump height.

In Figure 8.11, we illustrate the constructed jump-advection coefficient $\mathfrak{a}$ by visualizing two samples. As one can see, the squared-exponential covariance operator leads to a very smooth Gaussian random field. Also, the absolute values of the jump coefficient $\mathfrak{a}$ vary strongly across the spatial domain.

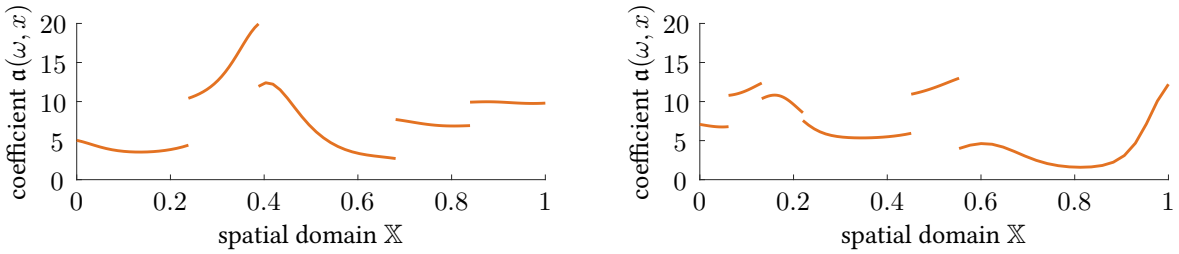
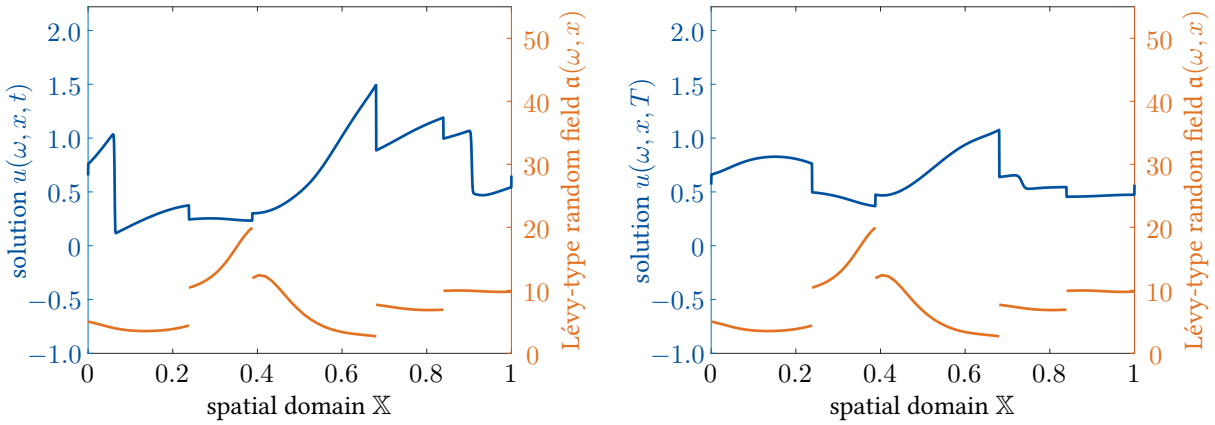


Figure 8.11: Two samples of a Lévy-type random field with a Poisson jump field and a squared-exponential Gaussian random field.

Let us now demonstrate, how a solution to the stochastic Burgers' Equation (8.1) is influenced by the underlying discontinuous random field. Therefore, the first examplary jump-advection coefficient in Figure 8.11 is employed and the corresponding solution is depicted in Figure 8.12 at two different times.



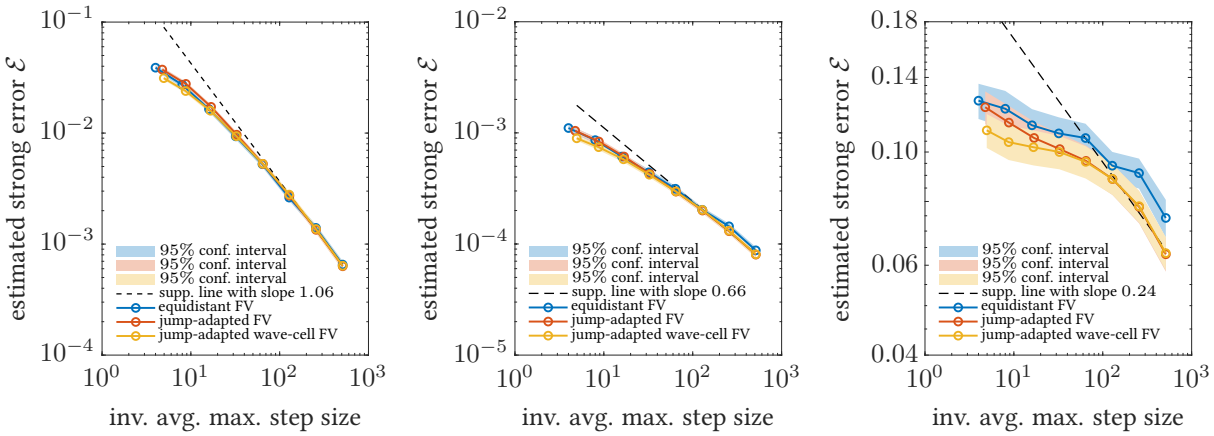
(a) Solution $u(\omega, x, t)$ at time $t \approx 0.09$ with the underlying random field.

(b) Solution $u(\omega, x, T)$ at the final time $T = 1.0$ with the underlying random field.

Figure 8.12: Solution to the stochastic discontinuous-flux Burgers' equation for a Lévy-type random field with a Poisson jump field and a squared-exponential Gaussian random field.

Due to the spatial jumps in the flux function, multiple discontinuities appear in the solution, which can be seen at both times $t \approx 0.09$ and $t = T = 1$. Here, let us stress that there exist two different types of discontinuities in the solution: Standing contact discontinuities at the jump discontinuities and moving shock-waves. In Figure 8.12a, these moving shock-waves are located at $x \approx 0.1$ and $x \approx 0.9$. At the later time $t = T = 1$, there exists only one such shock-wave, which is located at $x \approx 0.75$. This is shown in Figure 8.12b. Also, note that the smoothness of the Gaussian random field seems to be passed over to the solution.

With this impression on the qualitative behavior of the solution, we can now turn to the main investigation of this section, which discusses the pathise convergence of approximations for the various discretization techniques discussed in Section 8.2. For this purpose, Figure 8.13 shows the pathwise $\mathcal{L}^1$- and $\mathcal{L}^2$-error averaged for 50 samples of the random entropy solution to Equation (8.1). One immediately



(a) Pathwise $\mathcal{L}^1$-error. (b) Pathwise $\mathcal{L}^2$-error. (c) Pathwise $\mathcal{L}^\infty$-error.

Figure 8.13: Pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error of finite volume (FV) approximations for a Poisson jump field with a squared-exponential Gaussian random field. Each error is estimated via a Monte Carlo estimator using 50 samples.

notices that the chosen discretization scheme barely influences the order of convergence. Even the absolute error values do not seem to differ significantly. This behavior is by far not surprising: For the chosen random field, the jump discontinuities are widely scattered across the domain. Therefore, even coarse discretizations are able to resolve these discontinuities. The only exception to this behavior is the convergence of the $\mathcal{L}^\infty$-error of finite volume approximations. Here, the jump-adapted discretization techniques lead to a (slightly) improved error constant. Additionally, for coarse discretizations, this constant is lower for the jump-adapted wave-cell meshing than for the normal discretization with jump-adaptivity. Finally, let us note that 50 Monte Carlo samples lead to a narrow confidence interval for the $\mathcal{L}^1$- and $\mathcal{L}^2$-error. However, for the $\mathcal{L}^\infty$-error, the confidence interval for the approximation error is quiet large.

8.4.2 Alternating jump field with exponential Gaussian random field

We continue our pathwise convergence study by considering a different Lévy-type random field, an *alternating jump field with exponential Gaussian random field*. As in the previous case, we set the deterministic mean function $\overline{\mathfrak{a}}$ to $\overline{\mathfrak{a}} \equiv 0$ and the functional Φ is chosen as $\Phi(\xi) = \exp(\xi)$ for some scalar value $\xi \in \mathbb{R}$. Therefore, the Gauss-type random field $\mathcal{G}$ is of the form $\mathcal{G}(\omega, x) = \Phi\big(\mathcal{G}(\omega, x)\big)$. Here, $\mathcal{G}$ is a (truncated) Gaussian random field characterized by the *exponential covariance operator* Q_{E}, which corresponds to the Matérn covariance operator Q_{M} with smoothness parameter $\nu = 1/2$. Thus, the resulting Gaussian random field has rather rough paths. The variance σ^2 is set to $\sigma^2 = 0.1$ and the correlation length ρ is chosen as $\rho = 0.1$. The resulting Gaussian random field $\mathcal{G}$ is approximated via the truncated Karhunen-Loéve expansion (Theorem 2.30). For this, the spectral basis of the exponential covariance operator Q_{E} is approximated via Nyström's method [286].

We continue by specifying the alternating (random) jump field of the jump-advection coefficient $\mathfrak{a}$. The structure of this jump field is very similar to the one of the previous section: The number of jumps τ is a Poisson-distributed random variable with $\tau \sim \mathrm{Poi}(5) + 1$, such that we have at least one discontinuity. The corresponding jump positions are uniformly distributed across the domain $\mathbb{X}$, i.e., $\mathfrak{d}_j \sim \mathcal{U}(\mathbb{X})$. The

main difference to the Poisson-distributed jump field of the previous section are the jump heights $\mathbb{p}_i$. These are defined as

$$\mathbb{p}_i = \begin{cases} \mathcal{U}([\frac{1}{4}, \frac{3}{4}]) & \text{for } i \text{ odd} , \\ \mathcal{U}([\frac{5}{4}, \frac{7}{4}]) & \text{for } i \text{ even} . \end{cases}$$

By the above construction, the alternating jump field admits global bounds on the coefficient that hold for all samples (instead of only samplewise bounds for the Poisson jump field). To illustrate the constructed jump-advection coefficient, we visualize two samples of the random field in Figure 8.14.

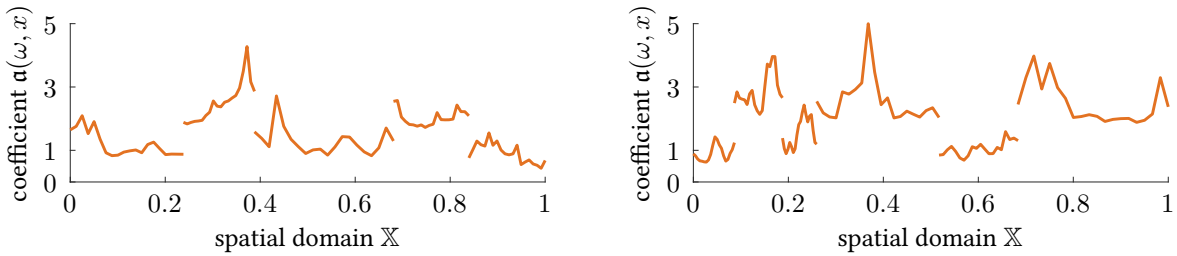
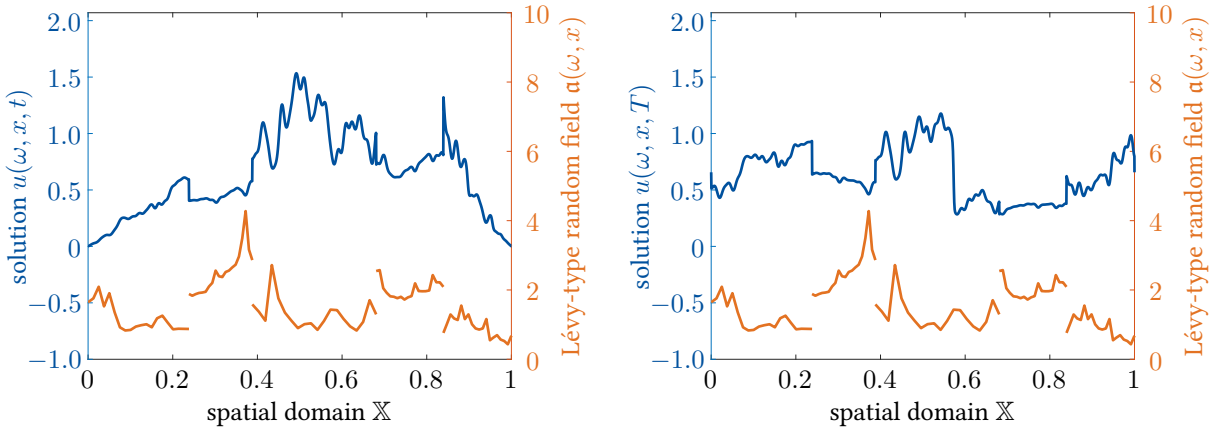


Figure 8.14: Two samples of a Lévy-type random field consisting of an alternating jump field and an exponential Gaussian random field.

One immediately observes the roughness of the Gaussian random field due to the exponential covariance operator. This is just what one expects, due to the results of Section 8.3.3, where the influence of the Matérn smoothness parameter was investigated. Also, due to the structure of the coefficient, the Gaussian part of the random field varies more and jump discontinuities are not necessarily the main cause for spatial variation. In Figure 8.15, we demonstrate how the entropy solution to the stochastic Burgers' Equation (8.1) is affected by the underlying jump-advection coefficient. Here, the first exemplary random field sample of Figure 8.14 is employed and the corresponding solution is shown at two different time steps.



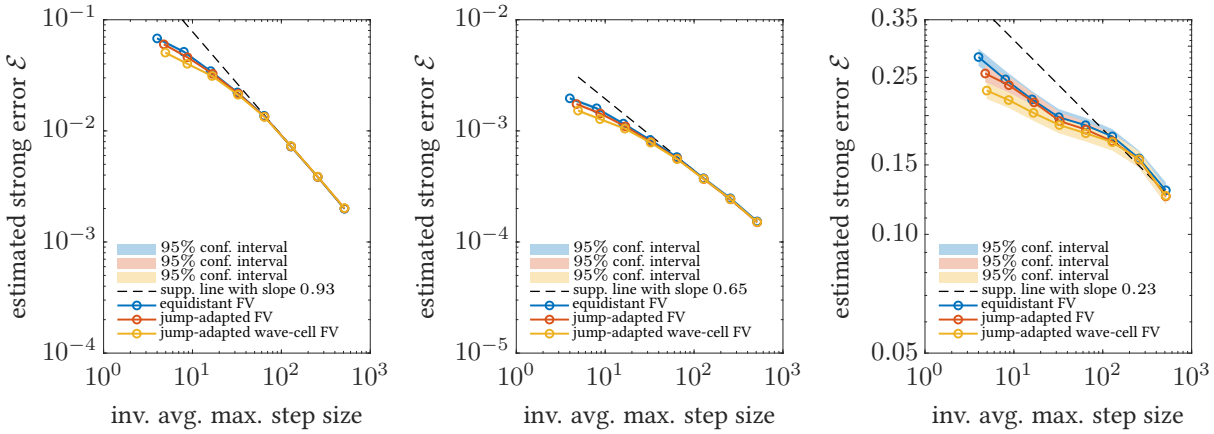
(a) Solution $u(\omega, x, t)$ at time $t \approx 0.09$ with the underlying random field.

(b) Solution $u(\omega, x, T)$ at the final time $T = 1.0$ with the underlying random field.

Figure 8.15: Pathwise solution to the random discontinuous-flux Burgers' equation for a Lévy-type random field with an alternating jump field and an exponential Gaussian random field.

As in the previous setting, the spatial jumps in the flux function lead to multiple discontinuities in the solution. This behavior can be observed at both time steps $t \approx 0.09$ (Figure 8.15a) and $t = T = 1$ (Figure 8.15b). While the contact discontinuities that appear at the jumps of the flux function are easily detected, the occurring moving shock-waves are less apparent due to the low regularity of the solution. However, in the solution at time $t = 1$ depicted in Figure 8.15b such a shock-wave appears at approximately $x \approx 0.6$.

To conclude the investigation of this kind of Lévy-type random field, we now discuss the pathwise convergence of approximations of solutions. Therefore, finite volume simulations with underlying equidistant, jump-adapted and jump-adapted wave-cell discretizations are evaluated. Since the approximation error is random, Figure 8.16 depicts the pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error averaged over 50 Monte Carlo samples of the random entroy solution.



(a) Pathwise $\mathcal{L}^1$-error. (b) Pathwise $\mathcal{L}^2$-error. (c) Pathwise $\mathcal{L}^\infty$-error.

Figure 8.16: Pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error of finite volume (FV) approximations of the random entropy solution u for an alternating jump field with an exponential Gaussian random field. Each error is estimated via a Monte Carlo estimator using 50 samples.

Even though we have changed some of the characteristics of the random field, such as the smoothness of the log-Gaussian random field, the convergence results are very similar to those of the previous section. All three discretization methods lead to similar convergence rates. While it is not surprising, let us note that the order of convergence is highest for the $\mathcal{L}^1$-error and lowest for the $\mathcal{L}^\infty$-error. Again, 50 Monte Carlo samples of the random approximation error are sufficient for a small confidence interval in the $\mathcal{L}^1$- and $\mathcal{L}^2$-convergence. For the $\mathcal{L}^\infty$-error, the confidence of the error estimation is much lower. As a last comment, let us mention that the jump-adaptivity is able to slightly improve the error for coarse grids and the jump-adapted wave-cell discretization marginally improves this effect. This behavior can best be seen for the $\mathcal{L}^\infty$-error of the finite volume approximations.

8.4.3 Jump field with random inclusions

For the last coefficient in this pathwise convergence study, we consider a *jump field with random inclusions*. For this particular random field, let the number of inclusion τ be given as $\tau \sim \mathrm{Poi}(10) + 1$, which guarantees that at least one inclusion is contained in the coefficient. The positions of the inclusions

are uniformly distributed over the spatial domain $\mathbb{X}$, i.e., $\mathbb{x}_i \sim \mathcal{U}(\mathbb{X})$. Each inclusion has a random size, which is given by $\mathbb{l}_i \sim \mathcal{U}([10^{-5}, 10^{-3}])$. Let now $\mathfrak{X} \subset \mathbb{X}$ denote the union of all inclusions. With this set,the jump-advection coefficient is defined via

$$\mathfrak{a}(\omega, x) \coloneqq \begin{cases} \mathbb{h}_k & \text{for } x \in \mathfrak{X}\,, \\ 1 & \text{otherwise}\,, \end{cases} \quad \text{with} \quad \mathbb{h}_k \sim \begin{cases} \mathrm{Poi}(30) & \text{for } k = 0\,, \\ \frac{1}{\xi}, \xi \sim \mathrm{Poi}(30) & \text{for } k = 1\,. \end{cases}$$

Here, $k \sim \mathrm{Ber}(1, 1/2)$ is a Bernoulli-$1/2$-distributed random variable, defining whether the random inclusion height is distributed as $\mathbb{h}_0$ or as $\mathbb{h}_1$. As for the previous coefficients, we provide two samples of the constructed jump field with random inclusions to illustrate the behavior of this jump-advection coefficient in Figure 8.17.

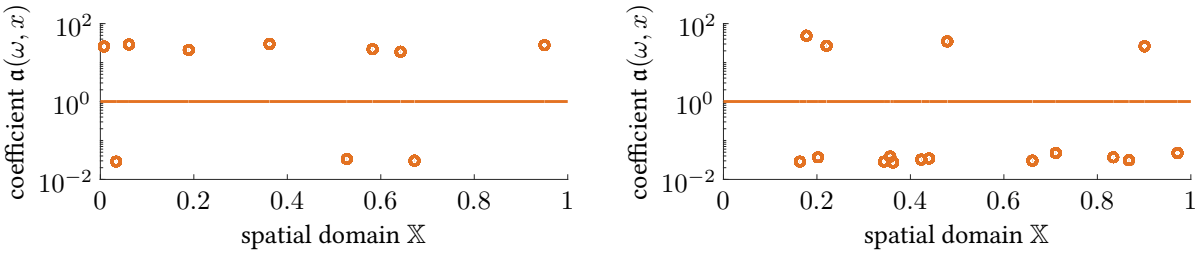
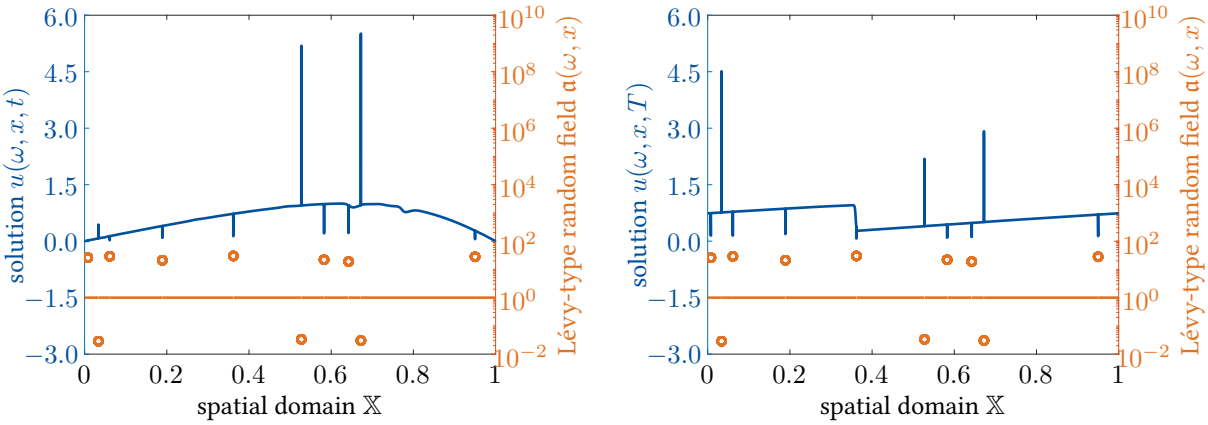


Figure 8.17: Two samples of a jump-advection coefficient with random Poisson-distributed inclusions.

To improve visibility of the random inclusions, they are highlighted by a surrounding circle. By construction, the inclusions have a very small size, but one also observes that the number of inclusions varies quiet strongly and that two inclusions can appear very close to each other. Note that the character of the random field is very different to the two coefficients of the preceeding discussion.

In Figure 8.18, we demonstrate the impact of the jump-advection coefficient on the random entropy solution. Therefore, Figure 8.18a shows the solution at time $t \approx 0.09$, while Figure 8.18b depicts the solution at the final computation time $t = T = 1$.



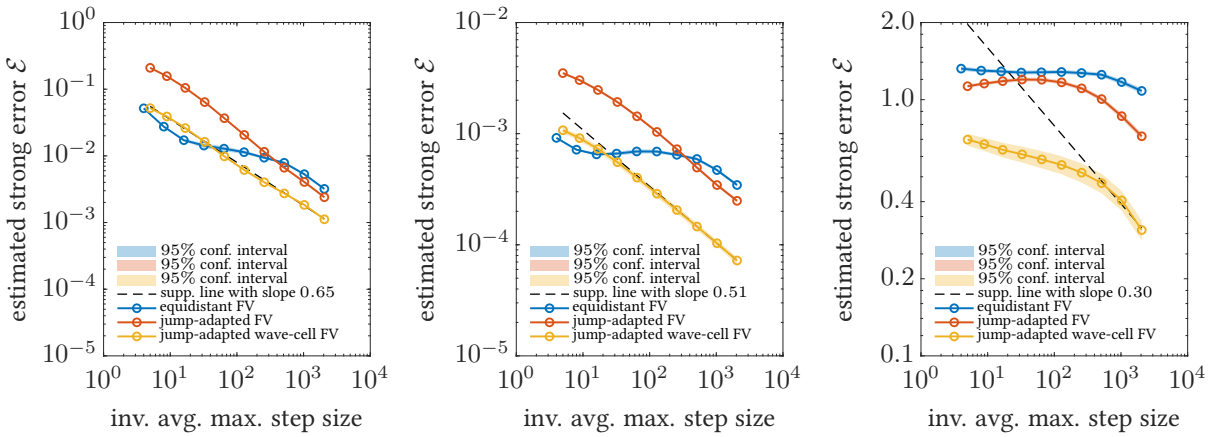
(a) Solution $u(\omega, x, t)$ at time $t \approx 0.09$ with the underlying random field.

(b) Solution $u(\omega, x, T)$ at the final time $T = 1.0$ with the underlying random field.

Figure 8.18: Solution to the stochastic discontinuous-flux Burgers' equation for a constant jump field with random Poisson-distributed inclusions.

Here, the inclusions are again marked with a surrounding circle to improve visibility. As one immediately observes, the small inclusions lead to significant peaks, which point up- or downwards depending on the inclusion height. Additionally, the random inclusions lead to multiple moving shock-waves, which can be seen in Figure 8.18a at approximately $x \approx 0.65$ and $x \approx 0.8$. At the final time $t = 1$, there is only one moving shock-wave, which is located at approximately $x \approx 0.4$. Another interesting observation is the temporal development of the peaks: While the peaks on the left side of the domain ($x \approx 0.05$) are very small for the earlier time $t \approx 0.09$, they are significantly higher at the final time $t \approx 1$. On the other hand, the eruption of the peaks around $x \approx 0.6$ are very tall in the beginning of the simulation and their height is decreasing over time. Also, discontinuities leading to an inclusion with lower value of the advection coefficient have a much stronger impact on peaks in the solution than inclusions on which the coefficient value increases.

Due to the small inclusions and the resulting peaks, this random field is very promising to demonstrate the qualitative differences of the proposed discretization techniques. Therefore, in Figure 8.19, the estimated pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error are visualized.



(a) Pathwise $\mathcal{L}^1$-error. (b) Pathwise $\mathcal{L}^2$-error. (c) Pathwise $\mathcal{L}^\infty$-error.

Figure 8.19: Pathwise $\mathcal{L}^1$-, $\mathcal{L}^2$- and $\mathcal{L}^\infty$-error of finite volume (FV) approximations of the random entropy solution u for a constant jump field with random Poisson-distributed inclusions. Each error is estimated via a Monte Carlo estimator using 100 samples.

Here, the pathwise error estimation is based on a Monte Carlo estimator with 100 samples. Independent of the considered norm, the jump-adapted discretizations are converging at a similar rate. However, the standard equidistant finite volume approximation does not seem to converge at all or at a very low rate. This is by far not surprising: The equidistant discretization is not able to resolve the small inclusions. Thus, the standard finite volume method needs a much finer resolution of the grid to resolve the flux discontinuities than the jump-adapted discretization techniques require. Nevertheless, asymptotically, we expect all three discretizations to converge at the same convergence rate.

Furthermore, let us stress that the jump-adapted wave-cell discretization leads to a much better error constant compared to the jump-adapted finite volume method. This behavior can be explained by the better approximation of the standing-wave profiles resulting from the discontinuities of the flux function. However, in all three error norms, the jump-adapted wave-cell discretization appears to result in a lower confidence of the error estimation.

8.5 Estimation of stochastic moments

To conclude the chapter on numerical simulations of one-dimensional random scalar conservation laws, this section investigates the estimation of stochastic moments of the adapted entropy solution u. In particular, the convergence behavior of the multilevel Monte Carlo (MLMC) estimator introduced in Definition 2.35 is studied. For the investigation, we simulate the stochastic discontinuous-flux Burgers' Equation (8.1), where the jump-advection coefficient is given as a Lévy-type random field. As the convergence behavior of the MLMC method is influenced by the pathwise convergence behavior, the same Lévy-type random fields as in Section 8.4 are employed. However, since the pathwise convergence in Section 8.4 was similar for the Poisson jump field with squared-exponential Gaussian random field and for the alternating jump field with exponential Gaussian random field, we restrict ourselves to the latter one. The corresponding convergence behavior of the multilevel Monte Carlo estimator is compared to the estimation for a jump field with random inclusions. Throughout this section, we consider the *weak error* of the moment estimation, which is given by

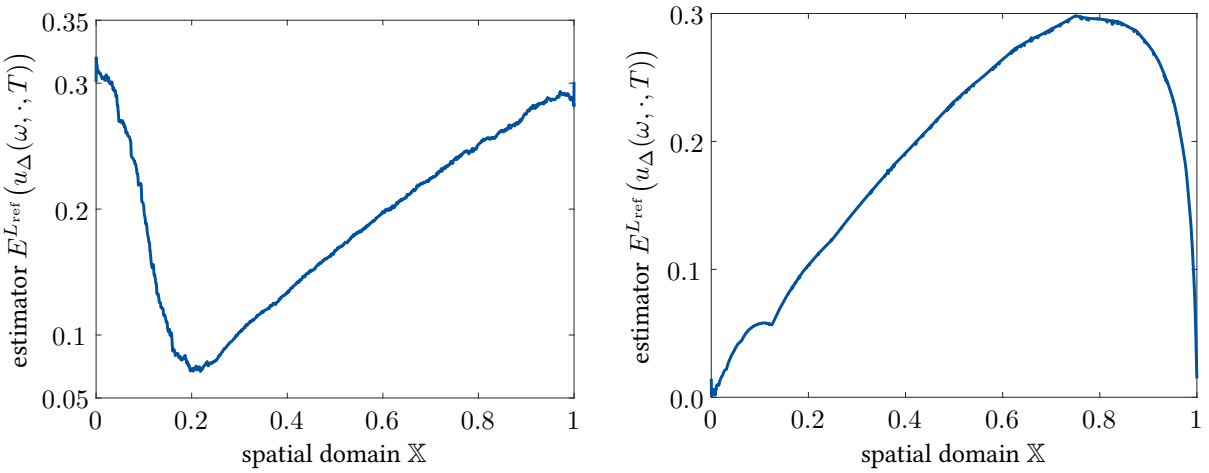
$$\mathcal{E}_w(\omega) \coloneqq \mathbb{E}\left(\left\| E^{L_{\mathrm{ref}}}\big(u_\Delta(\omega,\cdot,T)\big) - E^{L}\big(u_\Delta(\omega,\cdot,T)\big)\right\|_{\mathcal{L}^\star(\mathbb{X};\mathbb{R})}\right) ,$$

where $\mathcal{L}^\star(\mathbb{X};\mathbb{R})$ is either $\mathcal{L}^1(\mathbb{X};\mathbb{R})$, $\mathcal{L}^2(\mathbb{X};\mathbb{R})$ or $\mathcal{L}^\infty(\mathbb{X};\mathbb{R})$. Furthermore, u_Δ denotes the finite volume approximation of the adapted entropy solution u and E^L denotes the MLMC estimator based on $L \in \mathbb{N}$ levels. For all experiments, the reference estimation $E^{L_{\mathrm{ref}}}\big(u_\Delta(\omega,\cdot,T)\big)$ is computed by an MLMC estimator using $L_{\mathrm{ref}} = 7$ levels. The finite volume approximation u_Δ for this reference estimation was computed with an underlying jump-adapted wave-cell meshing. The spatial step size for the coarsest level was set to $\Delta^0_{x,\max} \coloneqq 1/2$ and the level-dependent spatial step sizes are given by $\Delta^l_{x,\max} = 2^{-l}\Delta^0_{x,\max}$ for $l = 1,\dots,L$.

Before we turn to the convergence study, the impact of the random jump-advection coefficients on the multilevel Monte Carlo estimators is demonstrated. Therefore, Figure 8.20 depicts the reference estimator $E^{L_{\mathrm{ref}}}\big(u_\Delta(\omega,\cdot,T)\big)$ for both types of underlying Lévy-type random fields.

One immediately notices that the estimated solutions appear to be steadier than the underlying pathwise solutions. While the roughness of the exponential Gaussian random field is still visible in Figure 8.20a, one barely notices the peaks contained in pathwise solutions of the jump field with random inclusions in Figure 8.20b. This is clear, since in both cases the (multilevel) Monte Carlo estimator approximates the expectation as an average of computed samples. Consequently, the estimators appear rather smooth in the sense that they seem related to pathwise solutions based on continuous flux functions. In this case, a traveling shock-wave appears, which can be seen in Figure 8.20a at approximately $x \approx 0.1$ and close to the right boundary of the spatial domain $\mathbb{X}$ in Figure 8.20b.

With these qualitative characterizations of the multilevel Monte Carlo estimators in mind, the weak approximation error and its convergence behavior can be investigated. At this point, let us stress that the multilevel Monte Carlo scheme considered in the subsequent experiments is not optimized with respect to the underlying meshing method. In particular, the number of samples computed on each discretization level is not optimal, since they are not fitted towards the decay of the variance of the correction term in the multilevel Monte Carlo estimation (2.6). As a result, we cannot expect the approximation to converge out-of-the-box. The presented experiments should rather be seen as an investigation on

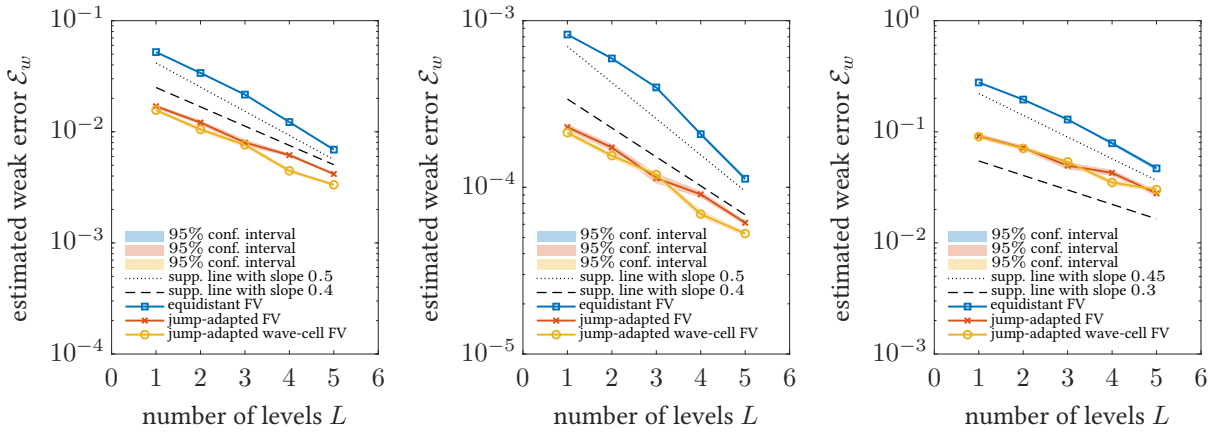


(a) Alternating jump field with exponential Gaussian random field.

(b) Jump field with random inclusions.

Figure 8.20: Multilevel Monte Carlo (MLMC) estimates of the adapted entropy solution u to the randomized discontinuous-flux Burgers' equation with different underlying Lévy-type random fields $\mathfrak{a}$.

which meshing scheme is suitable to enable the multilevel Monte Carlo method without the necessity of changing the algorithm to obtain convergence. To approximate the expectation in the weak error, a Monte Carlo estimation is employed, for which the number of simulations was set to $M = 20$. The error estimation for the multilevel Monte Carlo estimates with underlying alternating jump field and exponential Gaussian random field is visualized in Figure 8.21.



(a) Weak $\mathcal{L}^1$-error.

(b) Weak $\mathcal{L}^2$-error.

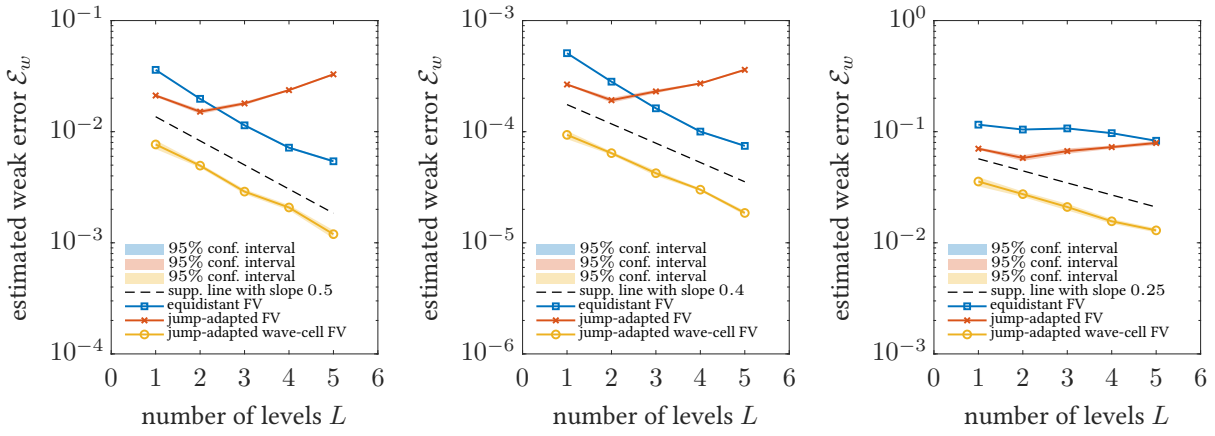
(c) Weak $\mathcal{L}^\infty$-error.

Figure 8.21: Convergence of multilevel Monte Carlo estimates of finite volume (FV) approximations of the random adapted entropy solution u to the randomized Burgers' equation with underlying alternating jump field with exponential Gaussian random field for different number of levels. Each error is estimated via a Monte Carlo estimator using 20 samples.

Figure 8.21 clearly shows that the jump-adapted meshing algorithms lead to better finite volume approximations than the classical equidistant discretization. This is not suprising, since the discontinuities in the coefficient are better resolved in the jump-adapted meshing. On coarse discretizations, this leads to

a finer initial grid, which explains the better error constant. As for the pathwise convergence study, the jump-adapted wave-cell meshing does not seem to significantly improve the error approximation. Nevertheless, in the shown numerical experiment, the equidistant meshing converges at a faster rate. This is caused by the same argumentation: As coarse level have a finer resolution, increasing the number of levels (and thus the spatial refinement) has a reduced effect on the quality of the approximation. If we would further increase the number of levels L, we expect the convergence rates of the three methods to become similar. One should also note the decreased convergence rates for the weak $\mathcal{L}^\infty$-error, which is analogous to the pathwise convergence results.

As a last investigation of this section, the weak convergence behavior of the multilevel Monte Carlo estimator is analyzed for the case of an underlying jump field with random inclusions. The expected value in the weak error is estimated by a Monte Carlo estimation using $M = 10$ samples. This reduced number of Monte Carlo samples is caused by higher run times of the simulations for the jump field with random inclusions. The corresponding results are depicted in Figure 8.22.



(a) Weak $\mathcal{L}^1$-error. (b) Weak $\mathcal{L}^2$-error. (c) Weak $\mathcal{L}^\infty$-error.

Figure 8.22: Convergence of multilevel Monte Carlo estimators of finite volume (FV) approximations of the random adapted entropy solution u to the randomized Burgers' equation with an underlying jump field having random inclusions for different number of levels. Each weak error is estimated via a Monte Carlo estimator based on 10 samples.

At first glance the findings may seem partly unintuitive, since the multilevel Monte Carlo estimation based on jump-adapted finite volume grids does not seem to converge at all. However, with an in-depth look, this behavior appears completely reasonable: For MLMC estimators with $L = 1$ or $L = 2$ levels, the expectation of the solution is better approximated than with equidistant discretizations, since the flux discontinuities are better resolved. Unfortunately, the extremly high-amplitude peaks in the pathwise solutions caused by the jump field with random inclusions badly affect the computation of the correction term in the multilevel Monte Carlo estimator given by Equation (2.6). Computing the difference of two approximations with differing discretizations is classically done by interpolating the coarse approximation onto the finer grid and evaluating the difference on this fine discretization. Now, for the jump-adapted finite volume method, the cells next to the peaks in the solution can be very broad, which induces a significant interpolation error into the computation. Since this effect is most significant for coarse step size bounds, the MLMC estimation based on jump-adapted finite volume approximations does not converge for the jump field with random inclusions. Since this problem of computing the

correction terms with an induced interpolation error affects the overall approximation, this behavior can be observed in all three considered error norms. Recall, the multilevel Monte Carlo method was not optimized regarding the underlying discretization scheme. Thus, to obtain convergence of the estimation, adapting the approximation algorithm would be necessary. This could include a different definition of the considered discretization levels as well as changing the number of samples computed at each level. However, such an adaptation would reduce the efficiency of the multilevel Monte Carlo method compared to employing a different discretization scheme.

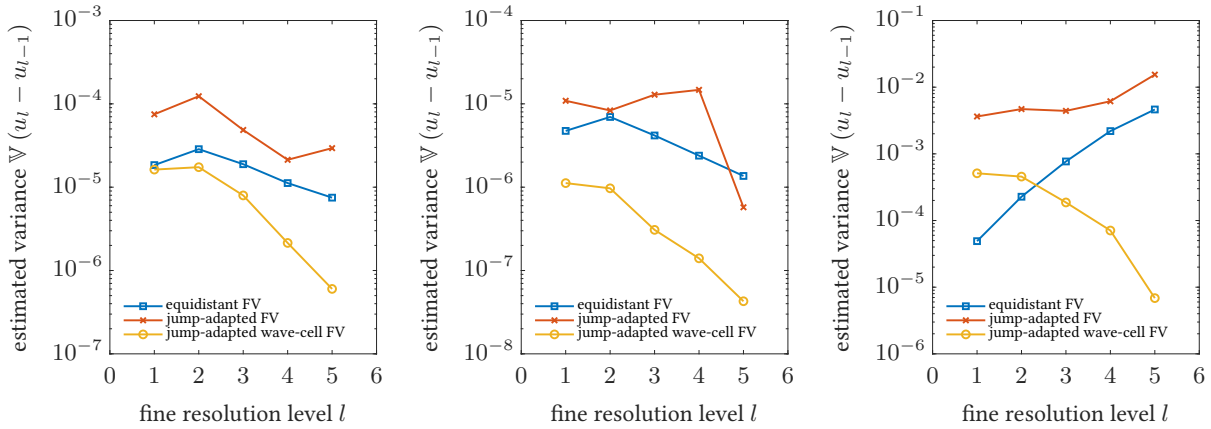
Due to the construction of jump-adapted wave-cell discretizations, the (interpolation) problems described above vanish and the MLMC estimation converges as one expects. Clearly, introducing small cells next to the standing-wave peaks in the pathwise solution overcomes the problems for computing the correction terms. As a result, the multilevel Monte Carlo estimations based on jump-adapted wave-cell discretizations converge as expected. Analogously to previous experiments, the convergence rate is highest for the weak $\mathcal{L}^1$-error and decreases to the lowest convergence rate for the weak $\mathcal{L}^\infty$-error. This effect is similar for the convergence of MLMC estimators based on an equidistant finite volume approximation. Since the peaks in the solution are not resolved with these equidistant grids, the correction term can be computed easily without the problems caused by jump-adaptivity. Therefore, the equidistant multilevel Monte Carlo estimators converge at a similar rate as those based on jump-adapted wave-cell discretizations. Here, one can also notice that the jump-adaptation of the grid significantly improves the error constant of the approximations.

Summarizing the above argumentation, the MLMC estimators based on jump-adapted finite volume grids do not converge, since the variance of the difference between two resolutions, i.e., $\mathbb{V}(u_l - u_{l-1})$, does not decrease. A similar argument holds for the equidistant discretization, whose approximation error is measured in the $\mathcal{L}^\infty$-norm. To verify this argument, we investigate the decay of the term $\mathbb{V}(u_l - u_{l-1})$ in the multilevel Monte Carlo estimator (2.6). Since we cannot compute the variance $\mathbb{V}(U)$ of a random variable U directly, it is estimated via the *sample variance* of a set $\mathbb{U} = \left\{U^{(i)}\right\}_{i=1}^{M}$ consisting of M i.i.d. samples of the random variable U. Based on such a set, the sample variance is defined es

$$\mathbb{V}(\mathbb{U}) = \frac{\sum_{i=1}^{M}(U^{(i)} - \bar{\mathbb{U}})^2}{M-1},$$

where $\bar{\mathbb{U}} = E_M(\mathbb{U}) = {}^1\!/_M \sum_{i=1}^{M} U^{(i)}$ is the *sample mean* of the set $\mathbb{U}$. For the subsequent numerical experiment, the variance $\mathbb{V}(u_l - u_{l-1})$ is estimated via the empirical variance of a set of 1000 samples of the term $(u_l - u_{l-1})$, with $l = 1, \ldots, 5$. The estimated variance $\mathbb{V}(u_l - u_{l-1})$ of these samples is depicted in Figure 8.23.

One immediately notices that the variance based on jump-adapted wave-cell finite volume approximations decreases in all error norms. Contrary to this, the variance based on standard jump-adapted discretizations does not decrease in any of the considered norms. In fact, it even increases for the $\mathcal{L}^\infty$-norm. For the equidistant approximations, the variance decreases in the $\mathcal{L}^1$- and $\mathcal{L}^2$-error norm. However, the variance increases in the $\mathcal{L}^\infty$-norm. Consequently, the investigation of the variance completely supports our findings on the convergence behavior of the multilevel Monte Carlo estimators in Figure 8.22: The estimators based on jump-adapted wave-cell discretizations converge in all three considered error norms, since the variance decreases. For the equidistant grids, the multilevel Monte Carlo estimators converge in the $\mathcal{L}^1$- and $\mathcal{L}^2$-norm due to the variance decay. However, in the $\mathcal{L}^\infty$-norm,



(a) Weak $\mathcal{L}^1$-error. (b) Weak $\mathcal{L}^2$-error. (c) Weak $\mathcal{L}^\infty$-error.

Figure 8.23: Variance of the correction terms in the multilevel Monte Carlo estimators of finite volume (FV) approximations of the random adapted entropy solution u to the randomized Burgers' equation. The underlying jump field has random inclusions and the variance of the correction terms is shown for various discretization levels. Each variance is estimated using 1000 samples of the correction term.

the MLMC estimator does not converge as the variance increases. This is also the reason, why the estimators based on standard jump-adapted meshes do not converge in any error norm. Overall, these results demonstrate the ability of the jump-adapted wave-cell meshing for accurately computing random solutions and approximating their stochastic moments.

Conclusion and future research

9

This dissertation has adopted the approach of uncertainty quantification with Lévy-type random fields of [262] to the modeling, analysis and numerical simulation of hyperbolic conservation laws. As a main contribution, the presented research extends the theoretical frameworks of scalar conservation laws with discontinuous flux functions to allow for random spatial jumps in the flux functions. In particular, the novel theoretical frameworks are pioneering works by not requiring *strong measurability* of the flux function while ensuring well-posedness of random entropy solutions.

From a modeling perspective, the main contribution of this thesis is the development of random (families of) admissibility germs. As describing the behavior of an entropy solution across flux discontinuities is part of the modeling process, extending the unifying framework of admissibility germs enables such modeling in the presence of uncertainties. While this framework requires the existence of strong traces, the presented investigation of random adapted entropy solutions in the Audusse-Perthame setting and for the case of Panov-type flux functions appears complementary to the theory of admissibility germs. These types of admissibility conditions are particularly relevant for applications in which a confinement assumption on the entropy solution appears to be unnatural. In all the presented theoretical well-posedness frameworks, strong measurability of the random entropy solution is established without leveraging standard results, as these are not available for general discontinuous flux functions. To overcome these challenges, an innovative (proof) strategy utilizing set-valued mappings and the novel notion of entropy functionals is employed. At this point, let us stress that the presented research also extends the presentation of admissibility germs for the case of compound flux discontinuities. While this extension is straightforward in the deterministic case, in the randomized setting various technical issues appear, which were addressed and resolved in this thesis.

The numerical simulation of random scalar conservation laws is addressed through the invention of a meshing strategy, which accounts for the standing-wave profiles that result from stationary flux discontinuities in the adapted entropy framework of Audusse-Perthame. This new approach for creating (finite volume) meshes demonstrates its potential to outperform standard techniques, especially in the case of flux functions, whose advection coefficient contains small inclusions or has jumps concentrated in clusters. In particular, the novel meshing strategy can lead to a significant reduction in the samplewise variance of the computed solution.

Future research directions

Coming back to the holistic idea of addressing the modeling, analysis and the numerical simulation of random scalar conservation laws with discontinuous flux functions, open questions and various topics for future research directions arise quiet naturally. From a modeling perspective, integrating (experimental) data into the Lévy-type random field poses an interesting question. While there are standard methods such as Gaussian processes [286] to achieve this in continuous random fields, we are not aware of approaches for including data into discontinuous coefficients. Additionally, extending the presented frameworks to temporal discontinuities or allowing for a discontinuous dependence of the flux function of the solution variable appears natural.

Turning to the theoretical analysis, the required *joint measurability* for random families of admissibility germs can be difficult to verify. Reducing this assumption to *separate measurability* is an interesting open question. However, such an extension is not straightforward and many technical challenges arise. Analogously, a further generalization of the presented frameworks to flux functions of bounded variation as well as to merely $\mathcal{L}^\infty$-valued solutions appears as an interesting problem. Unfortunately, this generalization poses delicate technical issues of measurability, which require novel proof strategies. For multi-dimensional conservation laws, we required rather unnatural assumptions such as additional regularity of the flux discontinuity interface. Avoiding these assumptions to allow for generalized geometries, such as merging lines, would enable more realistic applications. A promising approach in this direction was introduced in the deterministic setting of [8] by the development of singular Kružkov admissibility conditions.
While this dissertation focused on scalar conservation laws, future investigations could address hyperbolic systems of conservation laws with flux discontinuities. In this setting, a particularly interesting question is given by the case, in which more than one term can have discontinuities.

The consideration of systems of conservation laws with discontinuous flux functions also raise interesting challenges for the numerical simulation of solutions. Restricting ourselves to multi-dimensional scalar conservation laws again, an interesting question is given by possible extensions of the jump-adapted wave-cell meshing strategy to the multi-dimensional setting. To account for standing wave-fronts an extension may be rather straightforward. However, as also travelling wave-fronts may appear at flux discontinuities, tracking these fronts may require a pathwise discretization with moving meshes [144, 258]. Comparing the performance of the presented finite volume methods with Discontinuous Galerkin or Front tracking approaches may lead to new insights and improved pathwise approximation schemes. Conducting numerical analysis of the presented methods is an important, yet challenging task. Especially the random convergence rates of finite volume approximations are fundamental for a variety of research questions. Adapting multilevel Monte Carlo methods to such random convergence rates is only one of them. In particular, leveraging knowledge about the structure of the solution in combination with convergence rates estimates may enable (more) efficient methods to quantify the uncertainty of random scalar conservation laws with discontinous flux functions.

Finally, we want to address potential future research questions that are related to the recent advances and successes of *Machine Learning*, in particular with a focus on deep neural networks (DNN). Sampling Lévy-type random fields may be computationally expensive, especially if Fourier inversion needs to be applied. Here, the approach of [56] seems promising, where DNNs have been used for the reconstruction of heterogeneous materials, such as porous rocks. In the sampling process of Lévy-type random fields

such neural networks may be able to omit the approximation of the eigenbasis in a Karhunen-Loéve expansion or other bottlenecks of current techniques. Also, approximating the random entropy solution of nonlinear scalar conservation laws via deep neural networks is an interesting question. However, the lack of regularity of solutions poses many challenges as its characteristic curves are not necessarily smooth (see, e.g., the discussion of [175]).

III

Appendices

Examples of (random) remainder functions

A

In this appendix, various examples of (random) remainder functions are presented and discussed. Let us stress that all of the occurring remainder terms are inspired by the choices introduced by Andreianov et al. in [13]. For each presented choice of the remainder function, the two imposed assumptions of joint measurability as well as integrability are verified (compare Assumptions 3.27 and 3.46 for the sole discontinuity setting of Assumptions 4.23 and 4.35 for the case of a compound flux discontinuity). For the ease of presentation, we restrict ourselves to the sole flux discontinuity setting. However, extending the subsequent results to the compound flux discontinuity case is straightforward up to heavy notational technicalities.

A.1 Remainder function via flux oscillation

While the remainder function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{dist}}$ is a simple choice, which is easy to handle in proofs, it requires global Lipschitz continuity of the flux functions. A more sophisticated choice, that does only requires local instead of global Lipschitz continuity, is the remainder function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{Osc}}$ based on the oscillation of the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$. Subsequently, we properly define this remainder function.

Definition A.1 (Remainder function via flux oscillation)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole flux discontinuity. Additionally, let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be locally Lipschitz continuous. Furthermore, let $\mathrm{Osc} : \mathcal{C}(\mathbb{R}; \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined via

$$\mathrm{Osc}(g; k, \mathfrak{u}) := \max\Big\{ |g(r) - g(s)| \,\Big|\, \min\{k, \mathfrak{u}\} \leq r \leq s \leq \max\{k, \mathfrak{u}\} \Big\},$$

be the oscillation of a continuous function $g \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ between the values $k, \mathfrak{u} \in \mathbb{R}$. Then, the function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{Osc}} : \Omega \times \mathbb{X}_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$\mathfrak{R}_{\mathfrak{G}}^{\mathrm{Osc}}(\omega, \mathfrak{x}; (k^l, k^r)) := 2 \inf_{(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathfrak{G}(\omega; \mathfrak{x})} \Big(\mathrm{Osc}\left(\mathfrak{f}^l(\omega, \cdot); k^l, \mathfrak{u}^l\right) + \mathrm{Osc}\left(\mathfrak{f}^r(\omega, \cdot); k^r, \mathfrak{u}^r\right) \Big),$$

is called the remainder function via the flux oscillation. ◆

As in the previous section, we start our discussion by showing that the remainder function based on the flux oscillation is jointly measurable as soon as the underlying admissibility germs $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ are jointly measurable.

Proposition A.2 (Joint measurability of remainder function via flux oscillation)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity hypersurface and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be locally Lipschitz continuous. Furthermore, let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs that satisfies the joint measurability Assumption 3.16. Then, the remainder function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}} : \Omega \times \mathbb{X}_\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ associated to $\mathfrak{G}$ is jointly measurable. ◆

Proof. Recall that by the stochastic measurability Assumption 3.41, the flux functions $\mathfrak{f}^{l,r}$ are measurable with respect to the stochastic parameter $\omega \in \Omega$. This implies that the oscillation functions of $\mathfrak{f}^{l,r}$ are measurable. Based on the discussion of [246, Chapter 4], set convergence does not distinguish open and closed sets. Therefore, we can identify the germ $\mathfrak{G}$ with its closure $\mathrm{cl}\big(\mathfrak{G}\big)$. However, by a result analogous to Lemma 2.15, the infimum over a closed set A is continuous with respect to this closed set $A \in \mathcal{CS}(\mathbb{R}^2)$.

Consequently, the remainder function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_\mathbb{T}$ and continuous with respect to the couple of entropy values $(k^l, k^r) \in \mathbb{R}^2$. Thus, the remainder function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ is Carathéodory and by [5, Lemma 4.51] jointly measurable. ■

The investigation of the remainder function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ based on the flux oscillation is concluded with the next proposition, which shows that the function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ admits a locally Lebesgue integrable majorant and, therefore, is also locally Lebesgue integrable.

Proposition A.3 (Integrability of remainder function via flux oscillation)**:**
Let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs such that for every stochastic parameter $\omega \in \Omega$ there exists a solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Then, the remainder function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ is locally integrable in the sense of Assumption 3.46, meaning that for each compact set $K \subset \mathbb{R}^2$ and fixed stochastic parameter $\omega \in \Omega$, the function

$$\mathcal{M}^{\mathfrak{R}}_K(\omega, \mathfrak{x}) \coloneqq \sup_{\boldsymbol{k} \in K} \big| \mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}(\omega, \mathfrak{x}, \boldsymbol{k}) \big|$$

is locally Lebesgue integrable, i.e., $\mathcal{M}^{\mathfrak{R}}_K(\omega, \cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_\mathbb{T}; \mathbb{R})$. ◆

Proof. First, note that for every stochastic parameter $\omega \in \Omega$, there exists a $\mathfrak{G}$-entropy solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Therefore, there also exists a (possibly random) interval $\mathbb{U}(\omega) \subset \mathbb{R}$ such that the solution u satisfies $u(\omega, \cdot, \cdot) \in \mathbb{U}(\omega)$ for almost every $(\boldsymbol{x}, t) \in \mathbb{X} \times \mathbb{T}$. However, this implies that $\mathfrak{G}(\omega, \mathfrak{x}) \cap \mathbb{U}(\omega)^2 \neq \emptyset$ for almost every spatio-temporal coordinate $\mathfrak{x} \in \mathbb{X}_\mathbb{T}$. Otherwise, this would be a contradiction to the existence of a $\mathfrak{G}$-entropy solution.

Since the flux functions $\mathfrak{f}^{l,r}$ are locally Lipschitz continuous by Assumption (F-2), for a fixed continuous function $g \in \mathcal{C}(\mathbb{R}; \mathbb{R})$, the oscillation function $\mathrm{Osc}(g; \cdot, \cdot)$ is continuous in both arguments. Now, let $K \subset \mathbb{R}^2$ be a compact set. Then, by exploiting the boundedness of the oscillation function over this compact set, the remainder function $\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ can be bounded by

$$\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}\big(\omega, \mathfrak{x}; (k^l, k^r)\big) \leq 4 C_{\mathfrak{f}} \,,$$

where the constant $C_{\mathfrak{f}} > 0$ depends on the compact set K, the flux functions $\mathfrak{f}^{l,r}$ and the admissibility germ $\mathfrak{G}$. However, this upper bound implies that $\mathcal{M}_K^{\mathfrak{R}}(\omega, \cdot) \in \mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R}_{\geq 0})$, which concludes the proof due to the inclusion $\mathcal{L}^\infty(\mathbb{X}_\mathbb{T}; \mathbb{R}_{\geq 0}) \subset \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_\mathbb{T}; \mathbb{R})$. ■

A.2 Remainder function via modulus of continuity

In this section, we discuss the remainder function $\mathfrak{R}^m_{\mathfrak{G}}$, which is based on the modulus of continuity of the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$. We start by providing the following definition.

Definition A.4 (Remainder function via modulus of continuity):
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole discontinuity hypersurface and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be locally Lipschitz continuous. Additionally, let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs. Furthermore, for any continuous function $g \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ let $m(g; \cdot) : [0, \infty] \to [0, \infty]$ denote the associated modulus of continuity defined as

$$m(g; \delta) := \max\Big\{ |g(r) - g(s)| \;\Big|\; \min\{k, \mathfrak{u}\} \leq r \leq s \leq \max\{k, \mathfrak{u}\}, |r - s| \leq \delta \Big\}.$$

Then, the function $\mathfrak{R}^m_{\mathfrak{G}} : \Omega \times \mathbb{X}_\mathbb{T} \to \mathbb{R}_{\geq 0}$ given by

$$\mathfrak{R}^m_{\mathfrak{G}}(\omega, \mathfrak{x}; (k^l, k^r)) := 2 \inf_{(\mathfrak{u}^l, \mathfrak{u}^r) \in \mathfrak{G}(\omega; \mathfrak{x})} \Big(m\big(\mathfrak{f}^l(\omega, \cdot); \big|k^l - \mathfrak{u}^l\big|\big) + m\big(\mathfrak{f}^r(\omega, \cdot); |k^r - \mathfrak{u}^r|\big)\Big),$$

is called the remainder function via the modulus of continuity. ◆

With this definition available, we can start by verifying the joint measurability Assumption 3.27 and the integrability Assumption 3.46. To do this, the subsequent proposition shows that the remainder function $\mathfrak{R}^m_{\mathfrak{G}}$ based on the modulus of continuity of the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ is jointly measurable as soon as the underlying family of admissibility germs $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ is jointly measurable.

Proposition A.5 (Joint measurability of remainder function via modulus of continuity):
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_\mathbb{T}$ be a random sole flux discontinuity and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be locally Lipschitz continuous. Additionally, let $\mathfrak{G} : \Omega \times \mathbb{X}_\mathbb{T} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs that satisfies the joint measurability Assumption 3.16. Then, the remainder function $\mathfrak{R}^m_{\mathfrak{G}}$ associated to $\mathfrak{G}$ is jointly measurable. ◆

Proof. By the stochastic measurability Assumption 3.41, the flux functions $\mathfrak{f}^{l,r}$ are measurable with respect to the stochastic parameter $\omega \in \Omega$. Therefore, by its construction, the modulus of continuity m of the functions $\mathfrak{f}^{l,r}$ is also measurable.

Since set convergence (and the metric on the hyperspace of closed subsets) does not distinguish open and closed sets, we can identify each admissibility germ $\mathfrak{G}(\omega, \mathfrak{x})$ by its closure $\mathrm{cl}\big(\mathfrak{G}(\omega, \mathfrak{x})\big)$. Thereby, a result analogous to Lemma 2.15 yields that the infimum over a closed set $A \in \mathcal{CS}(\mathbb{R}^2)$ is continuous with respect to the set A.

As a result, the remainder function $\mathfrak{R}^m_{\mathfrak{G}}$ based on the modulus of continuity is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_\mathbb{T}$ due to the joint measurability of the admissibility germ $\mathfrak{G}$. Additionally, the function $\mathfrak{R}^m_{\mathfrak{G}}$ is continuous with respect to the pair of entropy values $(k^l, k^r) \in \mathbb{R}^2$. Consequently, [5, Lemma 4.51] implies that the remainder term $\mathfrak{R}^m_{\mathfrak{G}}$ is jointly measurable as a Carathéodory function. ■

As a last property of the remainder function $\mathfrak{R}_{\mathfrak{G}}^{m}$ based on the modulus of continuity, we show that the function $\mathfrak{R}_{\mathfrak{G}}^{m}$ satisfies the local Lebesgue integrability Assumption 3.46. The corresponding result is argued in the following Proposition, which concludes the discussion of the remainder function $\mathfrak{R}_{\mathfrak{G}}^{m}$.

Proposition A.6 (Integrability of remainder function via modulus of continuity)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole flux discontinuity and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be locally Lipschitz continuous. Furthermore, let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs such that for every stochastic parameter $\omega \in \Omega$ there exists a solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^{\infty}(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Then, the remainder function $\mathfrak{R}_{\mathfrak{G}}^{m}$ is locally integrable in the sense of Assumption 3.46, meaning that for each compact set $K \subset \mathbb{R}^2$ and fixed stochastic parameter $\omega \in \Omega$, the function

$$\mathcal{M}_K^{\mathfrak{R}}(\omega, \mathfrak{x}) \coloneqq \sup_{\boldsymbol{k} \in K} |\mathfrak{R}_{\mathfrak{G}}^{m}(\omega, \mathfrak{x}, \boldsymbol{k})|$$

is locally Lebesgue integrable, i.e., $\mathcal{M}_K^{\mathfrak{R}}(\omega, \cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$. ◆

Proof. By hypothesis, for any stochastic parameter $\omega \in \Omega$, there exists a $\mathfrak{G}$-entropy solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^{\infty}(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Thus, there exists a (possibly random) interval $\mathbb{U}(\omega) \subset \mathbb{R}$ such that for almost every $(\boldsymbol{x}, t) \in \mathbb{X} \times \mathbb{T}$ the solution u is contained in $\mathbb{U}(\omega)$. Consequently, for almost every spatio-temporal point $\mathfrak{x} \in \mathbb{X}_{\mathbb{T}}$, the condition $\mathfrak{G}(\omega, \mathfrak{x}) \cap \mathbb{U}(\omega)^2 \neq \emptyset$ is fulfilled. Otherwise, this would be a contradiction to the existence hypothesis of a $\mathfrak{G}$-entropy solution u.

Since the flux functions $\mathfrak{f}^{l,r}$ are locally Lipschitz continuous by Assumption (F-2), there exists a constant $C_{\mathfrak{f}} \in \mathbb{R}_{>0}$ such that $m\Big(\mathfrak{f}^{l,r}(\omega, \cdot); \mathbb{d}_{\infty}\big(K, \mathbb{U}(\omega)^2\big)\Big) \leq C_{\mathfrak{f}}$ is satisfied. Here, $K \subset \mathbb{R}^2$ is some arbitrary compact set of $\mathbb{R}^2$. Note that the constant $C_{\mathfrak{f}}$ may depend on the stochastic parameter $\omega \in \Omega$. However, this estimation allows us to bound the remainder function via

$$\mathfrak{R}_{\mathfrak{G}}^{m}\big(\omega, \mathfrak{x}; (k^l, k^r)\big) \leq 4C_{\mathfrak{f}} \, .$$

Thereby, we also obtain that the function $\mathcal{M}_K^{\mathfrak{R}}$ satisfies $\mathcal{M}_K^{\mathfrak{R}}(\omega, \cdot) \in \mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}}; \mathbb{R}_{\geq 0})$. Since every essentially bounded function is also locally integrable, this concludes the proof. ■

A.3 Remainder function via flux variation

As a last remainder function in this appendix, we discuss the remainder function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{Var}}$, which is defined via the variation of the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$. Similarly to the first remainder function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{dist}}$, which required global Lipschitz continuity of the flux functions, we need to impose an additional requirement on the fluxes $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ to be able to employ this remainder function. In particular, the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ need to be of bounded variation, i.e., $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$. The precise definition is given subsequently.

Definition A.7 (Remainder function via flux variation)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole flux discontinuity and let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs. Then, if the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ are of bounded variation, i.e., $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$, we can define the remainder function via the flux variation *as the function*

$\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}} : \Omega \times \mathbb{X}_{\mathbb{T}} \to \mathbb{R}_{\geq 0}$ *given by*

$$\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}(\omega, \mathfrak{x}; (k^l, k^r)) \coloneqq 2 \inf_{(\mathrm{u}^l, \mathrm{u}^r) \in \mathfrak{G}(\omega; \mathfrak{x})} \Big(\mathcal{V}^{\mathrm{u}^l}_{k^l}\big(\mathfrak{f}^l(\omega, \cdot)\big) + \mathcal{V}^{\mathrm{u}^r}_{k^r}\big(\mathfrak{f}^r(\omega, \cdot)\big) \Big).$$

Here, for any function g that is of locally bounded variation, the function $\mathcal{V}^s_r(g)$ defined as

$$\mathcal{V}^s_r(g) \coloneqq \left| \int_r^s \big| g'(\tau) \big| \, \mathrm{d}\tau \right|$$

denotes the variation of g between r and s. ◆

As soon as the random family of admissibility germs $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ is jointly measurable, we can show that the remainder function $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ based on the flux variation is jointly measurable. The corresponding result is proven in the subsequent proposition.

Proposition A.8 (Joint measurability of remainder function via flux variation)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole flux discontinuity and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be of locally bounded variation. Furthermore, let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs that satisfies the joint measurability Assumption 3.16. Then, the remainder function $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ associated to $\mathfrak{G}$ is jointly measurable. ◆

Proof. By Assumption 3.41, the flux functions $\mathfrak{f}^{l,r}$ are measurable with respect to the stochastic parameter $\omega \in \Omega$. Additionally, since the functions $\mathfrak{f}^{l,r}$ have locally bounded variation, the Fubini and Tonelli Theorems [106, Theorems 2.37 and 2.39] imply that the variation $\mathcal{V}^s_r\big(\mathfrak{f}^{l,r}(\omega, \cdot)\big)$ is also measurable. Here, $r, s \in \mathbb{R}$ are two arbitrary values defining the compact interval for evaluating the variation.

By the discussion in [246, Chapter 4], set convergence does not distinguish between open and closed sets. Therefore, we can identify each admissibility germ $\mathfrak{G}(\omega, \mathfrak{x})$ by its closure $\mathrm{cl}\big(\mathfrak{G}(\omega, \mathfrak{x})\big)$. An analogous result to Lemma 2.15 yields that the infimum over a closed set $A \in \mathcal{CS}(\mathbb{R}^2)$ is continuous with respect to the closed set A. This implies that the remainder function $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ based on the flux variation is jointly measurable in $(\omega, \mathfrak{x}) \in \Omega \times \mathbb{X}_{\mathbb{T}}$ due to the joint measurability of the underlying admissibility germ $\mathfrak{G}$. Additionally, by its construction, the remainder function $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ is continuous in the pair of entropy values $(k^l, k^r) \in \mathbb{R}^2$. This concludes the proof, since the remainder term $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ is Carathéodory and thereby jointly measurable due to [5, Lemma 4.51]. ■

To conclude this appendix on (random) remainder functions, it remains to argue that the function $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ satisfies the local Lebesgue integrability Assumption 3.46. This statement is proven in the following proposition, which also concludes the discussion of $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$.

Proposition A.9 (Integrability of remainder function via flux variation)**:**
Let $\mathfrak{D} : \Omega \rightrightarrows \mathbb{X}_{\mathbb{T}}$ be a random sole discontinuity hypersurface and let the flux functions $\mathfrak{f}^{l,r}(\omega, \mathfrak{x}, \cdot)$ be of locally bounded variation. Additionally, let $\mathfrak{G} : \Omega \times \mathbb{X}_{\mathbb{T}} \rightrightarrows \mathbb{R}^2$ be a random family of $\mathcal{L}^1$-dissipative admissibility germs such that for every $\omega \in \Omega$ there exists a solution $u(\omega, \cdot, \cdot) \in \mathcal{L}^\infty(\mathbb{X} \times \mathbb{T}; \mathbb{R})$. Then, the remainder function $\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ is locally integrable in the sense of Assumption 3.46, meaning that for each compact set $K \subset \mathbb{R}^2$ and fixed stochastic parameter $\omega \in \Omega$, the function

$$\mathcal{M}^{\mathfrak{R}}_K(\omega, \mathfrak{x}) \coloneqq \sup_{\boldsymbol{k} \in K} \big| \mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}(\omega, \mathfrak{x}, \boldsymbol{k}) \big|$$

is locally Lebesgue integrable, i.e., $\mathcal{M}^{\mathfrak{R}}_K(\omega, \cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}}; \mathbb{R})$. ◆

Proof. By construction, choosing the remainder function $\mathfrak{R}_{\mathfrak{G}}^{\mathrm{Var}}$ based on the variation of the flux functions $\mathfrak{f}^{l,r}$ is only possible if these satisfy $\mathfrak{f}^{l,r}(\omega,\cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$. Furthermore, by the hypothesis on the existence of a $\mathfrak{G}$-entropy solution $u(\omega,\cdot,\cdot) \in \mathcal{L}^{\infty}(\mathbb{X}\times\mathbb{T};\mathbb{R})$, there exists a (possibly random) interval $\mathbb{U}(\omega) \subset \mathbb{R}$ that contains the solution $u(\omega,\cdot,\cdot)$ for almost every $(\boldsymbol{x},t) \in \mathbb{X}\times\mathbb{T}$. Additionally, let $K \subset \mathbb{R}^2$ be a compact set. Then, there exists another (possibly random) closed and bounded set $M(\omega) \coloneqq M_l \times M_r \subset \mathbb{R}^2$ that contains $K \cap \mathbb{U}^2(\omega)$. Here, the two sets $M_{l,r} \coloneqq [\underline{M}_{l,r}, \overline{M}_{l,r}] \subset \mathbb{R}$ are compact intervals.

Based on this construction of the compact set M and the locally bounded variation hypothesis on the flux functions $\mathfrak{f}^{l,r}$, there exist constants $0 < V_{l,r} < \infty$, which might depend on the stochastic parameter $\omega \in \Omega$, such that $\mathcal{V}_{\underline{M}_{l,r}}^{\overline{M}_{l,r}}\big(\mathfrak{f}^{l,r}(\omega,\cdot)\big) \leq V_{l,r} < \infty$. However, based on this estimation, we can bound the remainder function via the expression

$$\mathfrak{R}_{\mathfrak{G}}^{\mathrm{Var}}(\omega,\mathfrak{x};(k^l,k^r)) \leq 2(V_l + V_r)\,.$$

This implies that the function $\mathcal{M}_K^{\mathfrak{R}}$ is essentially bounded, i.e., $\mathcal{M}_K^{\mathfrak{R}}(\omega,\cdot) \in \mathcal{L}^{\infty}(\mathbb{X};\mathbb{R}_{\geq 0})$. Consequently, we have shown the assertion, since every essentially bounded function is also locally integrable due to the inclusion $\mathcal{L}^{\infty}(\mathbb{X}_{\mathbb{T}};\mathbb{R}) \subset \mathcal{L}^1_{\mathrm{loc}}(\mathbb{X}_{\mathbb{T}};\mathbb{R})$. ■

$\mathcal{L}^\infty$-valued flux functions are not strongly measurable

B

In this appendix, we provide a counterexample that shows the lack of strong measurability for general flux functions having random spatial discontinuities. For simplicity, we restrict ourselves to the one-dimensional case and let the flux function $\mathfrak{f}$ have a multiplicative form. Consequently, we can write the flux function $\mathfrak{f}$ as

$$\mathfrak{f}(\omega, x, v) \coloneqq \mathfrak{a}(\omega, x) f(v) \ ,$$

where $\mathfrak{a}$ is a random jump coefficient and f is a continuous function[32] that depends solely on the value $v \in \mathbb{R}$. Due to this continuity, the strong measurability of $\mathfrak{f}$ only depends on the strong measurability of the jump coefficient $\mathfrak{a}$. Since all of the admissibility criteria considered in this thesis require the spatial dependency of the flux function to be bounded, the most general form for this coefficient is as a function $\mathfrak{a} : \Omega \to \mathcal{L}^\infty(\mathbb{X}; \mathbb{R})$. Here, the spatial domain $\mathbb{X}$ is defined as $\mathbb{X} = \mathbb{R}$.

The main idea of the counterexample is to construct a random jump coefficient $\mathfrak{a}$, such that the coefficient satisfies the conditions to be employed in the scalar discontinuous-flux conservation law, but $\mathfrak{a}$ is not strongly measurable. Therefore, define a partition $\mathfrak{T}$ of $\mathbb{X} = \mathbb{R}$ as $\mathbb{R} \coloneqq \mathrm{cl}\big(\mathfrak{T}_0(\omega)\big) \cup \mathrm{cl}\big(\mathfrak{T}_1(\omega)\big)$, where the partition parts $\mathfrak{T}_{0,1}(\omega)$ are given as $\mathfrak{T}_0(\omega) \coloneqq \big(-\infty, \mathfrak{x}(\omega)\big)$ and $\mathfrak{T}_1(\omega) \coloneqq \big(\mathfrak{x}(\omega), \infty\big)$ with $\mathfrak{x} \sim \mathcal{U}\big((0,1)\big)$. Here, $\mathrm{cl}\big(E\big)$ denotes the closure of a set E and $\mathcal{U}\big((0,1)\big)$ is the uniform distribution over the interval $(0,1)$. Based on this partition, we can define the random jump coefficient

$$\mathfrak{a} : \Omega \times \mathbb{X} \to \{0,1\} \qquad (\omega, x) \mapsto \mathbb{1}_{\mathfrak{T}_0(\omega)}(x) = \mathbb{1}_{\left(-\infty, \mathfrak{x}(\omega)\right)}(x) \ .$$

Obviously, this jump coefficient is bounded and can be written as a function-valued random variable $\mathfrak{a} : \Omega \to \mathcal{L}^\infty(\mathbb{R}; \mathbb{R})$. Considering the image of this random variable, we obtain

$$\mathfrak{a}\{\Omega\} \coloneqq \big\{\mathfrak{a}(\omega, \cdot) \mid \omega \in \Omega\big\} = \big\{\mathbb{1}_{\left(-\infty, \mathfrak{x}(\omega)\right)}(\cdot) \mid \mathfrak{x} \in (0,1)\big\} \subset \mathcal{L}^\infty(\mathbb{R}; \mathbb{R}) \ .$$

This immediately implies that $\mathfrak{a}\{\Omega\}$ is an uncountable set. Additionally, for two functions $\alpha, \beta \in \mathfrak{a}\{\Omega\}$ we have that $\|\alpha - \beta\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} = 1$ as soon as $\alpha \neq \beta$.

[32] The particular form and/or additional assumptions on the function f depend on the underlying admissibility criterion. Since this counterexample intends to be general, we do not specify such a condition in this section.

With this set $\mathfrak{a}\{\Omega\}$ at hand, we can now conclude via a contradiction argument that $\mathfrak{a}$ is not strongly measurable. Therefore, assume that the set $\mathfrak{a}\{\Omega\}$ is separable. This implies that there exists a countable set $\mathbb{C} \subset \mathcal{L}^\infty(\mathbb{R};\mathbb{R})$ such that for any function $\alpha \in \mathfrak{a}\{\Omega\}$ there is a sequence $(\alpha_n, n \in \mathbb{N}) \subset \mathbb{C}$ satisfying

$$\lim_{n\to\infty} \|\alpha - \alpha_n\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} = 0 \,.$$

Let now two points $\nu, o \in (0,1)$ be given that satisfy $\nu \neq o$. For these two distinct points, the relation $\left\|\mathbb{1}_{(-\infty,\nu)} - \mathbb{1}_{(-\infty,o)}\right\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} = 1$ holds. Furthermore, for any scalar value $\varepsilon > 0$, there exist two functions $\beta_\nu, \beta_o \in \mathbb{C}$ that satisfy

$$\left\|\mathbb{1}_{(-\infty,\nu)} - \beta_\nu\right\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} < \varepsilon \qquad \text{and} \qquad \left\|\mathbb{1}_{(-\infty,o)} - \beta_o\right\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} < \varepsilon \,,$$

respectively. However, for a fixed value $\varepsilon < 1/2$, the backward triangle inequality yields

$$\begin{aligned}\left\|\mathbb{1}_{(-\infty,\nu)} - \beta_o\right\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} &\geq \left\|\mathbb{1}_{(-\infty,\nu)} - \mathbb{1}_{(-\infty,o)}\right\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} - \left\|\mathbb{1}_{(-\infty,o)} - \beta_o\right\|_{\mathcal{L}^\infty(\mathbb{R};\mathbb{R})} \\ &> 1 - \varepsilon > \varepsilon \,.\end{aligned}$$

On the other hand, this means that for any pair of distinct points $(\nu, o) \in (0,1)^2$ with $\nu \neq o$ and for fixed value $\varepsilon \in (0, 1/2)$, the approximating functions $\beta_\nu, \beta_o \in \mathbb{C}$ have to be distinct, i.e., $\beta_\nu \neq \beta_o$. However, this implies that we can define a set

$$\mathfrak{U} \coloneqq \bigcup_{\nu\in(0,1)} \beta_\nu \,,$$

which is uncountable. But this is a contradiction to $\mathbb{C}$ being countable, because by construction it holds that $\mathfrak{U} \subset \mathbb{C}$. Consequently, the function-valued random variable $\mathfrak{a} : \Omega \to \mathcal{L}^\infty(\mathbb{R};\mathbb{R})$ cannot be strongly measurable.

Bibliography

[1] A. Abdulle et al. *Multilevel Monte Carlo methods for stochastic elliptic multiscale PDEs.* Multiscale Modeling & Simulation, Vol. 11: 1033–1070, 2013.

[2] R. Abgrall and S. Mishra. *Uncertainty quantification for hyperbolic systems of conservation laws.* Vol. 18: 507–544, 2017.

[3] A. Adimurthi et al. *Optimal entropy solutions for conservation laws with discontinuous flux-functions.* Journal of Hyperbolic Differential Equations, Vol. 2: 783–837, 2005.

[4] Adimurthi et al. *Godunov-Type Methods for Conservation Laws with a Flux Function Discontinuous in Space.* SIAM Journal on Numerical Analysis, Vol. 42: 179–208, 2005.

[5] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis.* Springer–Verlag, Berlin, Heidelberg, 2006.

[6] M. Alkämper et al. *The DUNE-ALUGrid Module.* Archive of Numerical Software, Vol. 4: 1–28, 2016.

[7] L. Ambrosio et al. *Functions of bounded variation and free discontinuity problems.* Vol. 254: Clarendon Press Oxford, 2000.

[8] B. Andreianov. *New approaches to describing admissibility of solutions of scalar conservation laws with discontinuous flux.* ESAIM: Proceedings and Surveys, Vol. 50: 40–65, 2015.

[9] B. Andreianov and C. Cancès. *Vanishing capillarity solutions of Buckley–Leverett equation with gravity in two-rocks' medium.* Computational Geosciences, Vol. 17: 551–572, 2013.

[10] B. Andreianov and D. Mitrović. *Entropy conditions for scalar conservation laws with discontinuous flux revisited.* in: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis.* Vol. 32: 6 Elsevier 2015. 1307–1335

[11] B. Andreianov et al. *Finite volume schemes for locally constrained conservation laws.* Numerische Mathematik, Vol. 115: 609–645, 2010.

[12] B. Andreianov et al. *On vanishing viscosity approximation of conservation laws with discontinuous flux.* Networks & Heterogeneous Media, Vol. 5: 617–633, 2010.

[13] B. Andreianov et al. *A theory of L^1–dissipative solvers for scalar conservation laws with discontinuous flux.* Archive for Rational Mechanics and Analysis, Vol. 201: 27–86, 2011.

[14] B. Andreianov et al. *Approximating the vanishing capillarity limit of two-phase flow in multi-dimensional heterogeneous porous medium.* ZAMM – Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 94: 655–667, 2014.

[15] S. S. Antman. *Problems in nonlinear elasticity.* Nonlinear Problems of Elasticity, 513–584, 2005.

[16] D. Applebaum. *Lévy processes and stochastic calculus.* 2009.

[17] J.-P. Aubin and H. Frankowska. *Set-valued analysis.* Springer Science & Business Media, 2009.

[18] E. Audusse and B. Perthame. *Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies.* Proceedings of the Royal Society of Edinburgh Section A: Mathematics, Vol. 135: 253–265, 2005.

[19] I. Babuška et al. *A Stochastic Collocation Method for Elliptic Partial Differential Equations with Random Input Data.* SIAM Journal on Numerical Analysis, Vol. 45: 1005–1034, 2007.

[20] F. Bachmann and J. Vovelle. *Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients.* Communications in Partial Differential Equations, Vol. 31: 371–395, 2006.

[21] J. Badwaik and A. M. Ruf. *Convergence rates of monotone schemes for conservation laws with discontinuous flux.* SIAM Journal on Numerical Analysis, Vol. 58: 607–629, 2020.

[22] J. Badwaik et al. *Multilevel Monte Carlo finite volume methods for random conservation laws with discontinuous flux.* ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 55: 1039–1065, 2021.

[23] P. Baiti and H. K. Jenssen. *Well-Posedness for a Class of 2×2 Conservation Laws with L^∞ Data.* Journal of Differential Equations, Vol. 140: 161–185, 1997.

[24] C. T. H. Baker. *The numerical treatment of integral equations.* Oxford University Press, 1977.

[25] A. Barth and I. Kröker. *Finite Volume Methods for Hyperbolic Partial Differential Equations with Spatial Noise.* in: *Theory, Numerics and Applications of Hyperbolic Problems I.* Cham: Springer International Publishing, 2018. 125–135

[26] A. Barth and A. Lang. *Multilevel Monte Carlo method with applications to stochastic partial differential equations.* International Journal of Computer Mathematics, Vol. 89: 2479–2498, 2012.

[27] A. Barth and A. Stein. *A study of elliptic partial differential equations with jump diffusion coefficients.* Journal on Uncertainty Quantification, Vol. 6: 1707–1743, 2018.

[28] A. Barth and A. Stein. *Approximation and simulation of infinite-dimensional Lévy processes.* Stochastics and Partial Differential Equations: Analysis and Computations, Vol. 6: 286–334, 2018.

[29] A. Barth and A. Stein *A stochastic transport problem with Lévy noise: Fully discrete numerical approximation.* Preprint arXiv:1910.14657, 2019

[30] A. Barth and A. Stein. *Numerical analysis for time-dependent advection-diffusion problems with random discontinuous coefficients.* ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 56: 1545–1578, 2022.

[31] A. Barth et al. *Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients.* Numerische Mathematik, Vol. 119: 123–161, 2011.

[32] A. Barth et al. *Multilevel Monte Carlo method for parabolic stochastic partial differential equations.* BIT Numerical Mathematics, Vol. 53: 3–27, 2013.

[33] A. Barth et al. *Computational uncertainty quantification for a clarifier-thickener model with several random perturbations: A hybrid stochastic Galerkin approach.* Computers & Chemical Engineering, Vol. 89: 11–26, 2016.

[34] A. Barth et al. *Multilevel Monte Carlo simulation of statistical solutions to the Navier–Stokes equations.* 209–227, 2016.

[35] C. Bauzet et al. *The Cauchy problem for conservation laws with a multiplicative stochastic perturbation.* Journal of Hyperbolic Differential Equations, Vol. 9: 661–709, 2012.

[36] C. Bauzet et al. *A degenerate parabolic–hyperbolic Cauchy problem with a stochastic force.* Journal of Hyperbolic Differential Equations, Vol. 12: 501–533, 2015.

[37] J. Bear. *Dynamics of fluids in porous media.* Courier Corporation, 2013.

[38] S. Bianchini and A. Bressan. *Vanishing viscosity solutions of nonlinear hyperbolic systems.* Annals of mathematics, 223–342, 2005.

[39] M. Blatt et al. *The distributed and unified numerics environment, version 2.4.* Archive of Numerical Software, Vol. 4: 13–29, 2016.

[40] M. P. Bonkile et al. *A systematic literature review of Burgers' equation with recent advances.* Pramana, Vol. 90: 1–21, 2018.

[41] L. Brencher and A. Barth. *Hyperbolic Conservation Laws with Stochastic Discontinuous Flux Functions.* in: *International Conference on Finite Volumes for Complex Applications.* Springer, 2020. 265–273

[42] L. Brencher and A. Barth *Scalar conservation laws with random discontinuous flux functions.* Submitted to: ESAIM: Mathematical Modelling and Numerical Analysis, arXiv preprint arXiv:2107.00549, 2022

[43] A. Bressan. *Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem.* Vol. 20: Oxford University Press on Demand, 2000.

[44] S. E. Buckley and M. C. Leverett. *Mechanism of Fluid Displacement in Sands.* Transactions of the AIME, Vol. 146: 107–116, 1942.

[45] M. Bulíček et al. *On scalar hyperbolic conservation laws with a discontinuous flux.* Mathematical Models and Methods in Applied Sciences, Vol. 21: 89–113, 2011.

[46] M. Bulíček et al. *Multi-dimensional scalar conservation laws with fluxes discontinuous in the unknown and the spatial variable.* Mathematical Models and Methods in Applied Sciences, Vol. 23: 407–439, 2013.

[47] M. Bulíček et al. *On unified theory for scalar conservation laws with fluxes and sources discontinuous with respect to the unknown.* Journal of Differential Equations, Vol. 262: 313–364, 2017.

[48] R. Bürger et al. *A multiclass Lighthill-Whitham-Richards traffic model with a discontinuous velocity function.* Networks & Heterogeneous Media, Vol. 16: 187, 2021.

[49] R. Bürger et al. *A front tracking approach to a model of continuous sedimentation in ideal clarifier–thickener units.* Nonlinear Analysis: Real World Applications, Vol. 4: 457–481, 2003.

[50] R. Bürger et al. *Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units.* Numerische Mathematik, Vol. 97: 25–65, 2004.

[51] R. Bürger et al. *A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units.* SIAM Journal on Applied Mathematics, Vol. 65: 882–940, 2005.

[52] R. Bürger et al. *Difference schemes, entropy solutions, and speedup impulse for an inhomogeneous kinematic traffic flow model.* Networks & Heterogeneous Media, Vol. 3: 1, 2008.

[53] R. Bürger et al. *A conservation law with discontinuous flux modelling traffic flow with abruptly changing road surface conditions.* Hyperbolic Problems: Theory, Numerics and Applications, Vol. 67: 455–464, 2009.

[54] R. Bürger et al. *An Engquist–Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections.* SIAM Journal on Numerical Analysis, Vol. 47: 1684–1712, 2009.

[55] R. Bürger et al. *A hybrid stochastic Galerkin method for uncertainty quantification applied to a conservation law modelling a clarifier-thickener unit.* ZAMM – Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 94: 793–817, 2014.

[56] R. Cang et al. *Microstructure representation and reconstruction of heterogeneous materials via deep belief network for computational material design.* Journal of Mechanical Design, Vol. 139: 071404, 2017.

[57] J. Carrillo. *Conservation laws with discontinuous flux functions and boundary condition.* Journal of evolution equations, Vol. 3: 283–301, 2003.

[58] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions.* Vol. 580: Springer, 2006.

[59] J. Charrier. *Strong and weak error estimates for elliptic partial differential equations with random coefficients.* SIAM Journal on numerical analysis, Vol. 50: 216–246, 2012.

[60] J. Charrier et al. *Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel Monte Carlo methods.* SIAM Journal on Numerical Analysis, Vol. 51: 322–352, 2013.

[61] G.-Q. Chen et al. *Hyperbolic conservation laws with discontinuous fluxes and hydrodynamic limit for particle systems.* Journal of Differential Equations, Vol. 245: 3095–3126, 2008.

[62] G.-Q. Chen et al. *On nonlinear stochastic balance laws.* Archive for Rational Mechanics and Analysis, Vol. 204: 707–743, 2012.

[63] S.-K. Choi et al. *Polynomial Chaos Expansion with Latin Hypercube Sampling for Estimating Response Variability.* AIAA Journal, Vol. 42: 1191–1198, 2004.

[64] K.-l. Chung. *Note on some strong laws of large numbers.* American Journal of Mathematics, Vol. 69: 189–192, 1947.

[65] P. G. Ciarlet. *Mathematical elasticity: Three-dimensional elasticity.* SIAM, 2021.

[66] K. A. Cliffe et al. *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients.* Computing and Visualization in Science, Vol. 14: 3–15, 2011.

[67] B. Cockburn et al. *An error estimate for finite volume methods for multidimensional conservation laws.* mathematics of computation, Vol. 63: 77–103, 1994.

[68] G. M. Coclite and N. H. Risebro. *Conservation laws with time dependent discontinuous coefficients.* SIAM Journal on Mathematical Analysis, Vol. 36: 1293–1309, 2005.

[69] G. M. Coclite et al. *Traffic flow on a road network.* SIAM Journal on Mathematical Analysis, Vol. 36: 1862–1886, 2005.

[70] G. M. Coclite et al. *On a discontinuous nonlocal conservation law modeling material flow on conveyor belts.* 2020.

[71] A. Cohen et al. *Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's.* Analysis and Applications, Vol. 9: 11–47, 2011.

[72] D. L. Cohn. *Measure theory.* Springer, 2013.

[73] P. Constantin and C. Foias. *Navier-stokes equations.* University of Chicago Press, 1988.

[74] G. Crasta and V. De Cicco. *On the chain rule formulas for divergences and applications to conservation laws.* Nonlinear Analysis: Theory, Methods & Applications, Vol. 153: 275–293, 2017.

[75] G. Crasta et al. *Kinetic Formulation and Uniqueness for Scalar Conservation Laws with Discontinuous Flux.* Communications in Partial Differential Equations, Vol. 40: 694–726, 2015.

[76] G. Crasta et al. *Structure of solutions of multidimensional conservation laws with discontinuous flux and applications to uniqueness.* Archive for Rational Mechanics and Analysis, Vol. 221: 961–985, 2016.

[77] T. Crestaux et al. *Polynomial chaos expansion for sensitivity analysis.* Reliability Engineering & System Safety, Vol. 94: Special Issue on Sensitivity Analysis, 1161–1172, 2009.

[78] M. Croci et al. *Efficient white noise sampling and coupling for multilevel Monte Carlo with nonnested meshes.* SIAM/ASA Journal on Uncertainty Quantification, Vol. 6: 1630–1655, 2018.

[79] C. M. Dafermos. *Polygonal approximations of solutions of the initial value problem for a conservation law.* Journal of mathematical analysis and applications, Vol. 38: 33–41, 1972.

[80] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics.* Springer, 2000.

[81] C. F. Daganzo. *Fundamentals of transportation and traffic operations.* Pergamon Oxford, 1997.

[82] L. P. Dake. *Fundamentals of reservoir engineering.* Elsevier, 1983.

[83] A. Debussche and J. Vovelle. *Scalar conservation laws with stochastic forcing, revised version.* Journal of Functional Analysis, 2014.

[84] A. Debussche and J. Vovelle. *Scalar conservation laws with stochastic forcing.* Journal of Functional Analysis, Vol. 259: 1014–1042, 2010.

[85] A. Debussche and J. Vovelle. *Invariant measure of scalar first-order conservation laws with stochastic forcing.* Probability Theory and Related Fields, Vol. 163: 575–611, 2015.

[86] A. Debussche et al. *Degenerate parabolic stochastic partial differential equations: Quasilinear case.* The Annals of Probability, Vol. 44: 1916–1955, 2016.

[87] J.-P. Dias and M. Figueira. *On the Riemann problem for some discontinuous systems of conservation laws describing phase transitions.* Communications on Pure & Applied Analysis, Vol. 3: 53, 2004.

[88] J.-P. Dias and M. Figueira. *On the approximation of the solutions of the Riemann problem for a discontinuous conservation law.* Bulletin of the Brazilian Mathematical Society, Vol. 36: 115–125, 2005.

[89] J.-P. Dias et al. *Solutions to a scalar discontinuous conservation law in a limit case of phase transitions.* Journal of Mathematical Fluid Mechanics, Vol. 7: 153–163, 2005.

[90] S. Diehl. *On scalar conservation laws with point source and discontinuous flux function.* SIAM Journal on Mathematical Analysis, Vol. 26: 1425–1451, 1995.

[91] S. Diehl. *A conservation law with point source and discontinuous flux function modelling continuous sedimentation.* SIAM Journal on Applied Mathematics, Vol. 56: 388–419, 1996.

[92] S. Diehl. *Scalar conservation laws with discontinuous flux function: I. The viscous profile condition.* Communications in mathematical physics, Vol. 176: 23–44, 1996.

[93] S. Diehl. *A uniqueness condition for nonlinear convection-diffusion equations with discontinuous coefficients.* Journal of Hyperbolic Differential Equations, Vol. 6: 127–159, 2009.

[94] S. Diehl and N.-O. Wallin. *Scalar conservation laws with discontinuous flux function: II. On the stability of the viscous profiles.* Communications in mathematical physics, Vol. 176: 45–71, 1996.

[95] R. J. DiPerna. *Measure-valued solutions to conservation laws.* Archive for Rational Mechanics and Analysis, Vol. 88: 223–270, 1985.

[96] M. Eigel et al. *Adaptive stochastic galerkin fem.* Computer Methods in Applied Mechanics and Engineering, Vol. 270: 247–269, 2014.

[97] A. Einstein. In: *Sidelights on relativity.* chap. Geometry and experience (translated by G.B. Jeffery & W. Perrett). Methuen & Co. London, 1922.

[98] J. Elstrodt. *Maß-und Integrationstheorie*. Vol. 7: Springer, 1996.

[99] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Chapman and Hall/CRC, 2015.

[100] N. Even. *On Hydrodynamic Limits and Conservation Laws*. PhD thesis. Universität Würzburg, 2009.

[101] R. Eymard et al. *Finite volume methods*. Handbook of numerical analysis, Vol. 7: 713–1018, 2000.

[102] H. Federer. *Geometric measure theory*. Springer, 2014.

[103] J. Feng and D. Nualart. *Stochastic scalar conservation laws*. Journal of Functional Analysis, Vol. 255: 313–373, 2008.

[104] E. Fermi and R. D. Richtmyer. *Note on census-taking in Monte-Carlo calculations*. 1948.

[105] G. Fishman. *Monte Carlo: concepts, algorithms, and applications*. Springer Science & Business Media, 2013.

[106] G. B. Folland. *Real analysis: modern techniques and their applications*. Vol. 40: John Wiley & Sons, 1999.

[107] P. K. Friz and B. Gess. *Stochastic scalar conservation laws driven by rough paths*. Vol. 33: 933–963, 2016.

[108] T. W. Gamelin and R. E. Greene. *Introduction to Topology*. Dover books on mathematics Dover Publications, 1999.

[109] I. M. Gel'fand. *Some problems in the theory of quasi-linear equations*. Uspekhi Matematicheskikh Nauk, Vol. 14: 87–158, 1959.

[110] A. Geletu. *Introduction to topological spaces and set-valued maps (Lecture notes)*. Ilmenau University of Technology: Institute of Mathematics. Department of Operations Research and Stochastics, 2006.

[111] B. Gess and M. Hofmanová. *Well-posedness and regularity for quasilinear degenerate parabolic-hyperbolic SPDE*. The Annals of Probability, Vol. 46: 2495–2544, 2018.

[112] B. Gess and P. E. Souganidis. *Long-Time Behavior, Invariant Measures, and Regularizing Effects for Stochastic Scalar Conservation Laws*. Communications on Pure and Applied Mathematics, Vol. 70: 1562–1597, 2017.

[113] S. S. Ghoshal and A. Jana. *Optimal jump set in hyperbolic conservation laws*. Journal of Hyperbolic Differential Equations, Vol. 17: 765–784, 2020.

[114] S. S. Ghoshal et al. *Convergence of a Godunov scheme to an Audusse–Perthame adapted entropy solution for conservation laws with BV spatial flux*. Numerische Mathematik, Vol. 146: 629–659, 2020.

[115] S. S. Ghoshal et al. *Well-posedness for conservation laws with spatial heterogeneities and a study of BV regularity*. arXiv preprint arXiv:2010.13695, 2020.

[116] S. S. Ghoshal et al. *A Godunov type scheme and error estimates for scalar conservation laws with Panov-type discontinuous flux*. Numerische Mathematik, 1–25, 2022.

[117] J. Gil-Pelaez. *Note on the inversion theorem*. Biometrika, Vol. 38: 481–482, 1951.

[118] M. B. Giles. *Multilevel monte carlo path simulation*. Operations research, Vol. 56: 607–617, 2008.

[119] M. B. Giles. *Multilevel monte carlo methods*. Acta Numerica, Vol. 24: 259–328, 2015.

[120] T. Gimse. *Conservation laws with discontinuous flux functions.* SIAM Journal on Mathematical Analysis, Vol. 24: 279–289, 1993.

[121] T. Gimse and N. H. Risebro. *Riemann problems with a discontinuous flux function.* in: *Proceedings of Third International Conference on Hyperbolic Problems.* Vol. 1: 1991. 488–502

[122] T. Gimse and N. H. Risebro. *Solution of the Cauchy problem for a conservation law with a discontinuous flux function.* SIAM Journal on Mathematical Analysis, Vol. 23: 635–648, 1992.

[123] T. Gimse and N. H. Risebro. *A note on reservoir simulation for heterogeneous porous media.* Transport in porous media, Vol. 10: 257–270, 1993.

[124] P. Glasserman. *Monte Carlo methods in financial engineering.* Vol. 53: Springer Science & Business Media, 2013.

[125] Y. Guo and W. Strauss. *Stability of semiconductor states with insulating and contact boundary conditions.* Archive for rational mechanics and analysis, Vol. 179: 1–30, 2006.

[126] P. Gwiazda et al. *Multi-dimensional scalar balance laws with discontinuous flux.* Journal of Functional Analysis, Vol. 267: 2846–2883, 2014.

[127] P. R. Halmos. *Measure theory.* Springer, 2013.

[128] J. Hammersley. *Monte carlo methods.* Springer Science & Business Media, 2013.

[129] A. Harten et al. *On finite-difference approximations and entropy conditions for shocks.* Communications on pure and applied mathematics, Vol. 29: 297–322, 1976.

[130] S. Heinrich. *Multilevel monte carlo methods.* in: *International Conference on Large-Scale Scientific Computing.* Springer 2001. 58–67

[131] J. S. Hesthaven. *Numerical methods for conservation laws: From analysis to algorithms.* Society for Industrial and Applied Mathematics, 2017.

[132] H. Heuser. *Lehrbuch der analysis.* Springer-Verlag, 2013.

[133] C. Hirsch. *Numerical computation of internal and external flows: The fundamentals of computational fluid dynamics.* Elsevier, 2007.

[134] H. Hoel et al. *Path-dependent convex conservation laws.* Journal of Differential Equations, Vol. 265: 2708–2744, 2018.

[135] J. Hoffmann-Jørgensen and G. Pisier. *The law of large numbers and the central limit theorem in Banach spaces.* The Annals of Probability, 587–599, 1976.

[136] M. Hofmanova. *Scalar conservation laws with rough flux and stochastic forcing.* Stochastics and Partial Differential Equations: Analysis and Computations, Vol. 4: 635–690, 2016.

[137] M. Hofmanová. *Degenerate parabolic stochastic partial differential equations.* Stochastic Processes and their Applications, Vol. 123: 4294–4336, 2013.

[138] M. Hofmanová. *A Bhatnagar–Gross–Krook approximation to stochastic scalar conservation laws.* in: *Annales de l'IHP Probabilités et statistiques.* Vol. 51: 4 2015. 1500–1528

[139] H. Holden and N. H. Risebro. *Conservation laws with a random source.* Applied Mathematics and Optimization, Vol. 36: 229–241, 1997.

[140] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws.* Vol. 152: Springer, 2015.

[141] R. Hooke. *Lectures de potentia restitutiva, or of spring explaining the power of springing bodies, 1678.* John Martyn, 2016.

[142] E. Hopf. *The partial differential equation* $u_t + uu_x = \mu_{xx}$. Communications on Pure and Applied mathematics, Vol. 3: 201–230, 1950.

[143] S. Hu and N. S. Papageorgiou. *Handbook of multivalued analysis: Volume II: Applications*. Vol. 500: Springer Science & Business Media, 2013.

[144] W. Huang and R. D. Russell. *Adaptive moving mesh methods*. Vol. 174: Springer Science & Business Media, 2010.

[145] P. Hughett. *Error bounds for numerical inversion of a probability characteristic function*. SIAM Journal on Numerical Analysis, Vol. 35: 1368–1392, 1998.

[146] P.-H. Hugoniot. *Sur la propagation du mouvement dans les corps et spécialement dans les gaz parfaits (première partie)*. Journal de l'École Polytechnique, Vol. 57: 3–97, 1887.

[147] P.-H. Hugoniot. *Sur la propagation du mouvement dans les corps et spécialement dans les gaz parfaits (deuxième partie)*. Journal de l'École Polytechnique, Vol. 58: 1–125, 1889.

[148] R. Iturriaga and K. Khanin. *Burgers turbulence and random Lagrangian systems*. Communications in Mathematical Physics, Vol. 232: 377–428, 2003.

[149] J. Jaffré. *Numerical calculation of the flux across an interface between two rock types of a porous medium for a two-phase flow*. Hyperbolic problems: theory, numerics, applications (Stony Brook, NY, 1994), 165–177, 1996.

[150] J. Jimenez. *Scalar conservation law with discontinuous flux in a bounded domain*. in: *Conference Publications*. Vol. 2007: Special American Institute of Mathematical Sciences 2007. 520

[151] J. Jimenez. *Mathematical analysis of a scalar multidimensional conservation law with discontinuous flux*. Journal of Evolution Equations, Vol. 11: 553–576, 2011.

[152] J. Jimenez and L. Lévi. *Entropy formulations for a class of scalar conservations laws with space-discontinuous flux functions in a bounded domain*. Journal of Engineering Mathematics, Vol. 60: 319–335, 2008.

[153] S. Jin and L. Pareschi. *Uncertainty Quantification for Hyperbolic and Kinetic Equations*. Springer, 2017.

[154] E. F. Kaasschieter. *Solving the Buckley–Leverett equation with gravity in a heterogeneous porous medium*. Computational Geosciences, Vol. 3: 23–48, 1999.

[155] K. H. Karlsen and J. D. Towers. *Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux*. Chinese Annals of Mathematics, Vol. 25: 287–318, 2004.

[156] K. H. Karlsen and J. D. Towers. *Convergence of a Godunov scheme for conservation laws with a discontinuous flux lacking the crossing condition*. Journal of Hyperbolic Differential Equations, Vol. 14: 671–701, 2017.

[157] K. H. Karlsen et al. *Upwind difference approximations for degenerate parabolic convection–diffusion equations with a discontinuous coefficient*. IMA Journal of Numerical Analysis, Vol. 22: 623–664, 2002.

[158] K. H. Karlsen et al. L^1 *Stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*. Preprint series. Pure Mathematics, 2003.

[159] K. H. Karlsen and E. B. Storrøsten. *On stochastic conservation laws and Malliavin calculus*. Journal of Functional Analysis, Vol. 272: 421–497, 2017.

[160] A. Kechris. *Classical Descriptive Set Theory*. Graduate Texts in Mathematics Springer New York, 2012.

[161] S. Ken-Iti. *Lévy processes and infinitely divisible distributions.* Cambridge university press, 1999.

[162] A. Khintchine. *Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze.* Mathematics of the US Sbornik, Vol. 2: 79–120, 1937.

[163] J. U. Kim. *On a stochastic scalar conservation law.* Indiana University Mathematics Journal, 227–256, 2003.

[164] K. Kirchner and C. Schwab. *Monte Carlo convergence rates for k th moments in Banach spaces.* arXiv preprint arXiv:2212.03797, 2022.

[165] R. A. Klausen and N. H. Risebro. *Stability of conservation laws with discontinuous coefficients.* Journal of Differential Equations, Vol. 157: 41–60, 1999.

[166] C. Klingenberg and N. H. Risebro. *Convex conservation laws with discontinuous coefficients. Existence, uniqueness and asymptotic behavior.* Communications in Partial Differential Equations, Vol. 20: 1959–1990, 1995.

[167] C. Klingenberg and N. H. Risebro. *Stability of a resonant system of conservation laws modeling polymer flow with gravitation.* Journal of Differential Equations, Vol. 170: 344–380, 2001.

[168] U. Koley et al. *A multilevel Monte Carlo finite difference method for random scalar degenerate convection–diffusion equations.* Journal of Hyperbolic Differential Equations, Vol. 14: 415–454, 2017.

[169] D. P. Kroese et al. *Handbook of monte carlo methods.* Vol. 706: John Wiley & Sons, 2013.

[170] D. Kröner. *Numerical schemes for conservation laws.* Vol. 22: Wiley Chichester, 1997.

[171] S. N. Kružkov and E. Y. Panov. *Conservative quasilinear first-order laws with an infinite domain of dependence on the initial data.* in: *Soviet Mathematics Doklady.* Vol. 42: 1991. 316–321

[172] S. N. Kružkov and E. Y. Panov. *Osgood's type conditions for uniqueness of entropy solutions to Cauchy problem for quasilinear conservation laws of the first order.* Annali dell'Università di Ferrara, Vol. 40: 31–54, 1994.

[173] S. N. Kružkov. *First order quasilinear equations in several independent variables.* Mathematics of the USSR-Sbornik, Vol. 10: 217, 1970.

[174] N. Kuznetsov. *Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation.* USSR Computational Mathematics and Mathematical Physics, Vol. 16: 105–119, 1976.

[175] F. Laakmann and P. Petersen. *Efficient approximation of solutions of parametric linear transport equations by ReLU DNNs.* Advances in Computational Mathematics, Vol. 47: 1–32, 2021.

[176] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow.* Vol. 2: Gordon and Breach New York, 1969.

[177] O. Le Maître and O. M. Knio. *Spectral methods for uncertainty quantification: with applications to computational fluid dynamics.* Springer Science & Business Media, 2010.

[178] O. P. Le Maître et al. *A stochastic projection method for fluid flow: II. Random process.* Journal of computational Physics, Vol. 181: 9–44, 2002.

[179] M. Ledoux and M. Talagrand *Probability in Banach spaces. Classics in Mathematics.* 2011

[180] J. M. Lee. *Smooth manifolds.* Springer, 2013.

[181] P. G. LeFloch and J. Novotny. *Hyperbolic Systems of Conservation Laws: The theory of classical and nonclassical shock waves.* Applied Mechanics Reviews, Vol. 56: B53–B54, 2003.

[182] R. J. LeVeque. *Numerical methods for conservation laws*. Vol. 132: Springer, 1992.

[183] R. J. LeVeque. *Finite volume methods for hyperbolic problems*. Vol. 31: Cambridge university press, 2002.

[184] P. Lévy. *Sur les intégrales dont les éléments sont des variables aléatoires indépendantes*. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, Vol. 3: 337–366, 1934.

[185] P. Lévy. *L'addition des variables aléatoires définies sur une circonférence*. Bulletin de la Société mathématique de France, Vol. 67: 1–41, 1939.

[186] P. Lévy. *Théorie de l'addition des variables aléatoires*. Gauthier-Villars, 1954.

[187] M. J. Lighthill and G. B. Whitham. *On kinematic waves II. A theory of traffic flow on long crowded roads*. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, Vol. 229: 317–345, 1955.

[188] P.-L. Lions et al. *A kinetic formulation of multidimensional scalar conservation laws and related equations*. Journal of the American Mathematical Society, Vol. 7: 169–191, 1994.

[189] P.-L. Lions et al. *Scalar conservation laws with rough (stochastic) fluxes*. Stochastic Partial Differential Equations: Analysis and Computations, Vol. 1: 664–686, 2013.

[190] P.-L. Lions et al. *Scalar conservation laws with rough (stochastic) fluxes: the spatially dependent case*. Stochastic Partial Differential Equations: Analysis and Computations, Vol. 2: 517–538, 2014.

[191] L. H. Loomis and S. Sternberg. *Advanced calculus*. World Scientific, 1968.

[192] Y. Lu et al. *The entropy solutions for the Lighthill-Whitham-Richards traffic flow model with a discontinuous flow-density relationship*. Transportation Science, Vol. 43: 511–530, 2009.

[193] B. J. Lucier. *A moving mesh numerical method for hyperbolic conservation laws*. Mathematics of computation, Vol. 46: 59–69, 1986.

[194] F. Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*. 135 Cambridge University Press, 2012.

[195] P. A. Markowich et al. *Semiconductor equations*. Springer Science & Business Media, 2012.

[196] S. Martin and J. Vovelle. *Convergence of implicit finite volume methods for scalar conservation laws with discontinuous flux function*. ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 42: 699–727, 2008.

[197] R. Merkle and A. Barth. *Multilevel Monte Carlo estimators for elliptic PDEs with Lévy-type diffusion coefficient*. BIT Numerical Mathematics, 1–39, 2022.

[198] R. Merkle and A. Barth. *On Some Distributional Properties of Subordinated Gaussian Random Fields*. Methodology and Computing in Applied Probability, 1–28, 2022.

[199] R. Merkle and A. Barth. *Subordinated Gaussian random fields in elliptic partial differential equations*. Stochastics and Partial Differential Equations: Analysis and Computations, 1–49, 2022.

[200] B. Merlet and J. Vovelle. *Error estimate for finite volume scheme*. Numerische Mathematik, Vol. 106: 129–155, 2007.

[201] N. Metropolis et al. *The beginning of the Monte Carlo method*. Los Alamos Science, Vol. 15: 125–130, 1987.

[202] N. Metropolis and S. Ulam. *The monte carlo method*. Journal of the American statistical association, Vol. 44: 335–341, 1949.

[203] F. Meyer et al. *A posteriori error analysis for random scalar conservation laws using the stochastic Galerkin method*. IMA Journal of Numerical Analysis, Vol. 40: 1094–1121, 2020.

[204] S. Mishra. *Convergence of upwind finite difference schemes for a scalar conservation law with indefinite discontinuities in the flux function.* SIAM Journal on Numerical Analysis, Vol. 43: 559–577, 2005.

[205] S. Mishra. *Numerical methods for conservation laws with discontinuous coefficients.* in: *Handbook of Numerical Analysis.* Vol. 18: Elsevier, 2017. 479–506

[206] S. Mishra, G. Gowda, et al. *Convergence of Godunov type methods for a conservation law with a spatially varying discontinuous flux function.* Mathematics of Computation, Vol. 76: 1219–1242, 2007.

[207] S. Mishra and G. D. V. Gowda. *Conservation law with the flux function discontinuous in the space variable–II: Convex–concave type fluxes and generalized entropy solutions.* Journal of Computational and Applied Mathematics, Vol. 203: 310–344, 2007.

[208] S. Mishra and C. Schwab. *Sparse tensor multi-level Monte Carlo finite volume methods for hyperbolic conservation laws with random initial data.* Mathematics of Computation, Vol. 81: 1979–2018, 2012.

[209] S. Mishra et al. *Multi-level Monte Carlo finite volume methods for nonlinear systems of conservation laws in multi-dimensions.* Journal of Computational Physics, Vol. 231: 3365–3388, 2012.

[210] S. Mishra et al. *Multilevel Monte Carlo finite volume methods for shallow water equations with uncertain topography in multi-dimensions.* SIAM Journal on Scientific Computing, Vol. 34: B761–B784, 2012.

[211] S. Mishra et al. *Numerical solution of scalar conservation laws with random flux functions.* SIAM/ASA Journal on Uncertainty Quantification, Vol. 4: 552–591, 2016.

[212] D. Mitrovic. *Proper entropy conditions for scalar conservation laws with discontinuous flux.* arXiv preprint arXiv:1202.5853, 2012.

[213] A. E. Moyer. *Robert Hooke's Ambiguous Presentation of "Hooke's Law".* Isis, Vol. 68: 266–275, 1977.

[214] F. Müller et al. *Multilevel Monte Carlo for two phase flow and Buckley–Leverett transport in random heterogeneous porous media.* Journal of Computational Physics, Vol. 250: 685–702, 2013.

[215] N. Müller and W. Bock. *Stochastic perturbation of the Lighthill–Whitham–Richards model via the method of stochastic characteristics.* Journal of Mathematics in Industry, Vol. 11: 1–13, 2021.

[216] T. Müller-Gronbach et al. *Monte Carlo-Algorithmen.* Springer-Verlag, 2012.

[217] F. Murat. *L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$.* Journal de mathématiques pures et appliquées, Vol. 60: 309–322, 1981.

[218] T. Musha and H. Higuchi. *Traffic current fluctuation and the Burgers equation.* Japanese journal of applied physics, Vol. 17: 811, 1978.

[219] T. Nagatani. *Density waves in traffic flow.* Physical Review E, Vol. 61: 3564–3570, 2000.

[220] C. L. M. H. Navier. *Mémoire sur les lois du mouvement des fluides.* Mémoires de l'Académie Royale des Sciences de l'Institut de France, Vol. 6: 389–440, 1823.

[221] F. Nobile et al. *A Sparse Grid Stochastic Collocation Method for Partial Differential Equations with Random Input Data.* SIAM Journal on Numerical Analysis, Vol. 46: 2309–2345, 2008.

[222] F. Nobile and R. Tempone. *Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients.* International journal for numerical methods in engineering, Vol. 80: 979–1006, 2009.

[223] S. Oladyshkin and W. Nowak. *Data-driven uncertainty quantification using the arbitrary polynomial chaos expansion.* Reliability Engineering & System Safety, Vol. 106: 179–190, 2012.

[224] O. A. Oleinik. *Discontinuous solutions of non-linear differential equations.* Uspekhi Matematicheskikh Nauk, Vol. 12: 3–73, 1957.

[225] O. A. Oleinik. *Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation.* Uspekhi Matematicheskikh Nauk, Vol. 14: 165–170, 1959.

[226] P. Orlandi. *The burgers equation.* in: *Fluid Flow Phenomena.* Springer, 2000. 40–50

[227] E. Y. Panov. *Generalized solutions of the Cauchy problem for quasilinear conservation laws.* PhD thesis. 1991.

[228] E. Y. Panov. *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws.* Journal of Hyperbolic Differential Equations, Vol. 2: 885–908, 2005.

[229] E. Y. Panov. *Existence of strong traces for quasi-solutions of multidimensional conservation laws.* Journal of Hyperbolic Differential Equations, Vol. 4: 729–770, 2007.

[230] E. Y. Panov. *On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux.* Journal of Hyperbolic Differential Equations, Vol. 6: 525–548, 2009.

[231] E. Y. Panov. *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux.* Archive for Rational Mechanics and Analysis, Vol. 195: 643–673, 2010.

[232] B. Perthame et al. *A simple derivation of BV bounds for inhomogeneous relaxation systems.* Communications in Mathematical Sciences, Vol. 13: 577–586, 2015.

[233] S. Peszat and J. Zabczyk. *Stochastic partial differential equations with Lévy noise: An evolution equation approach.* Cambridge University Press, 2007.

[234] B. Piccoli and M. Tournus. *A general BV existence result for conservation laws with spatial heterogeneities.* SIAM Journal on Mathematical Analysis, Vol. 50: 2901–2927, 2018.

[235] G. Poëtte et al. *Uncertainty quantification for systems of conservation laws.* Journal of Computational Physics, Vol. 228: 2443–2467, 2009.

[236] J. Pöschel. *Etwas mehr Analysis: Eine Einführung in die mehrdimensionale Analysis.* Springer-Verlag, 2014.

[237] W. H. Press et al. *Numerical recipes 3rd edition: The art of scientific computing.* Cambridge university press, 2007.

[238] K. R. Rajagopal. *On implicit constitutive theories.* Applications of Mathematics, Vol. 48: 279–319, 2003.

[239] K. R. Rajagopal. *On implicit constitutive theories for fluids.* Journal of Fluid Mechanics, Vol. 550: 243–249, 2006.

[240] K. R. Rajagopal and A. R. Srinivasa. *On the thermodynamics of fluids defined by implicit constitutive relations.* Zeitschrift für angewandte Mathematik und Physik, Vol. 59: 715–729, 2008.

[241] W. J. M. Rankine. *On the thermodynamic theory of waves of finite longitudinal disturbance.* Philosophical Transactions of the Royal Society of London, 277–288, 1870.

[242] M. Reed. *Methods of modern mathematical physics: Functional analysis.* Elsevier, 2012.

[243] P. I. Richards. *Shock waves on the highway.* Operations research, Vol. 4: 42–51, 1956.

[244] N. H. Risebro et al. *Multilevel Monte Carlo front-tracking for random scalar conservation laws.* BIT Numerical Mathematics, Vol. 56: 263–292, 2016.

[245] S. M. Robinson. *Regularity and stability for convex multivalued functions.* Mathematics of Operations Research, Vol. 1: 130–143, 1976.

[246] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis.* Vol. 317: Springer Science & Business Media, 2009.

[247] R. Y. Rubinstein and D. P. Kroese. *Simulation and the Monte Carlo method.* Vol. 10: John Wiley & Sons, 2016.

[248] W. Rudin. *Real and complex analysis. 1987.* 1987.

[249] A. M. Ruf. *Flux-stability for conservation laws with discontinuous flux and convergence rates of the front tracking method.* IMA Journal of Numerical Analysis, Vol. 42: 1116–1142, 2022.

[250] J. Rychlewski. *On Hooke's law.* Journal of Applied Mathematics and Mechanics, Vol. 48: 303–314, 1984.

[251] F. Sabac. *The optimal convergence rate of monotone finite difference methods for hyperbolic conservation laws.* SIAM journal on numerical analysis, Vol. 34: 2306–2318, 1997.

[252] O. Sander. *DUNE — The Distributed and Unified Numerics Environment.* Vol. 140: Springer Nature, 2020.

[253] B. Saussereau and I. L. Stoica. *Scalar conservation laws with fractional stochastic forcing: Existence, uniqueness and invariant measure.* Stochastic Processes and their Applications, Vol. 122: 1456–1486, 2012.

[254] C. Schwab and C. J. Gittelson. *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs.* Acta Numerica, Vol. 20: 291–467, 2011.

[255] N. Seguin and J. Vovelle. *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients.* Mathematical Models and Methods in Applied Sciences, Vol. 13: 221–257, 2003.

[256] K. Sepahvand et al. *Uncertainty Quantification in stochastic systems using polynomial chaos expansion.* International Journal of Applied Mechanics, Vol. 02: 305–353, 2010.

[257] D. Serre. *Systems of Conservation Laws 1: Hyperbolicity, entropies, shock waves.* Cambridge University Press, 1999.

[258] R. Sevilla. *Mesh Generation and Adaptation: Cutting-Edge Techniques.* Springer Nature, 2022.

[259] J. Shen et al. *Spectral methods: algorithms, analysis and applications.* Vol. 41: Springer Science & Business Media, 2011.

[260] R. C. Smith. *Uncertainty quantification: theory, implementation, and applications.* Vol. 12: Society for Industrial and Applied Mathematics, 2013.

[261] S. Solem. *Convergence rates of the front tracking method for conservation laws in the Wasserstein distances.* SIAM Journal on Numerical Analysis, Vol. 56: 3648–3666, 2018.

[262] A. Stein. *Uncertainty quantification with Lévy-type random fields.* PhD thesis. 2020.

[263] A. Stein and A. Barth. *A Multilevel Monte Carlo Algorithm for Parabolic Advection-Diffusion Problems with Discontinuous Coefficients.* Monte Carlo and Quasi-Monte Carlo Methods: MCQMC 2018, Rennes, France, July 1–6, Vol. 324: 445, 2020.

[264] G. G. Stokes. *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids.* Transactions of the Cambridge Philosophical Society, Vol. 8: 1880.

[265] A. Szepessy. *An existence result for scalar conservation laws using measure valued solutions.* Communications in Partial Differential Equations, Vol. 14: 1329–1350, 1989.

[266] A. L. Teckentrup et al. *Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients.* Numerische Mathematik, Vol. 125: 569–600, 2013.

[267] R. Temam. *Navier-Stokes equations: theory and numerical analysis.* Vol. 343: American Mathematical Soc., 2001.

[268] J. D. Towers. *Convergence of a difference scheme for conservation laws with a discontinuous flux.* SIAM Journal on Numerical Analysis, Vol. 38: 681–698, 2000.

[269] J. D. Towers. *A difference scheme for conservation laws with a discontinuous flux: the nonconvex case.* SIAM Journal on Numerical Analysis, Vol. 39: 1197–1218, 2001.

[270] J. D. Towers. *Convergence of the Godunov scheme for a scalar conservation law with time and space discontinuities.* Journal of Hyperbolic Differential Equations, Vol. 15: 175–190, 2018.

[271] J. D. Towers. *A splitting algorithm for LWR traffic models with flux discontinuous in the unknown.* Journal of Computational Physics, Vol. 421: 109722, 2020.

[272] J. D. Towers. *An existence result for conservation laws having BV spatial flux heterogeneities - without concavity.* Journal of Differential Equations, Vol. 269: 5754–5764, 2020.

[273] J. Tryoen et al. *Adaptive anisotropic spectral stochastic methods for uncertain scalar conservation laws.* SIAM Journal on Scientific Computing, Vol. 34: A2459–A2481, 2012.

[274] S. Ulam et al. *Statistical methods in neutron diffusion.* LAMS-551, Los Alamos National Laboratory, 1–22, 1947.

[275] G. Vallet and P. Wittbold. *On a stochastic first-order hyperbolic equation in a bounded domain.* Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 12: 613–651, 2009.

[276] A. Vasseur. *Strong traces for solutions of multidimensional scalar conservation laws.* Archive for rational mechanics and analysis, Vol. 160: 181–193, 2001.

[277] J.-P. Vila. *Convergence and error estimates in finite volume schemes for general multidimensional scalar conservation laws. I. Explicite monotone schemes.* ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 28: 267–295, 1994.

[278] A. I. Vol'pert. *The spaces and quasilinear equations.* Mathematics of the USSR-Sbornik, Vol. 2: 225, 1967.

[279] J. Von Neumann and S. Ulam. *Monte carlo method.* National Bureau of Standards Applied Mathematics Series, Vol. 12: 36, 1951.

[280] F. W. Warner. *Foundations of differentiable manifolds and Lie groups.* Vol. 94: Springer Science & Business Media, 1983.

[281] J. Wehr and J. Xin. *Front speed in the Burgers equation with a random flux.* Journal of statistical physics, Vol. 88: 843–871, 1997.

[282] E. Weinan et al. *Invariant measures for Burgers equation with stochastic forcing.* Annals of Mathematics, 877–960, 2000.

[283] X. Wen and S. Jin. *Convergence of an immersed interface upwind scheme for linear advection equations with piecewise constant coefficients I: L^1-error estimates.* Journal of Computational Mathematics, 1–22, 2008.

[284] N. Wiener. *The homogeneous chaos.* American Journal of Mathematics, Vol. 60: 897–936, 1938.

[285] J. K. Wiens et al. *Riemann solver for a kinematic wave traffic model with discontinuous flux.* Journal of Computational Physics, Vol. 242: 1–23, 2013.

[286] C. K. I. Williams and C. E. Rasmussen. *Gaussian processes for machine learning*. Vol. 2: MIT press Cambridge, MA, 2006.

[287] D. Xiu. *Numerical Methods for Stochastic Computations*. Princeton University Press, 2010.

[288] D. Xiu and J. S. Hesthaven. *High-Order Collocation Methods for Differential Equations with Random Inputs*. SIAM Journal on Scientific Computing, Vol. 27: 1118–1139, 2005.

[289] D. Xiu and G. E. Karniadakis. *The Wiener–Askey Polynomial Chaos for Stochastic Differential Equations*. SIAM Journal on Scientific Computing, Vol. 24: 619–644, 2002.

[290] V. A. Zorich and O. Paniagua. *Mathematical analysis II*. Vol. 220: Springer, 2016.

Nomenclature

Random scalar conservation laws

Ω	Stochastic sample space
Σ	σ-algebra over the sample space Ω
$\mathbb{P}$	Probability measure
ω	Stochastic variable
$\mathbb{X}$	Spatial domain
$\mathbb{T}$	Time interval
$\mathbb{X}_\mathbb{T}$	Space-time domain
T	End time of the problem
d	Dimension of the domain $\mathbb{X} = \mathbb{R}^d$
t	Temporal variable
$\boldsymbol{x}$	Spatial variable
$\mathfrak{x}$	Space-time variable
$\boldsymbol{x}_{k:n}$	Vector containing the k-th to n-th entry of the vector $\boldsymbol{x}$
u	Solution / unknown of the (random) scalar conservation law
u_0	Initial condition of the random scalar conservation law
q	Stochastic regularity of the random initial condition
p	Spatial regularity of the random initial condition
$\mathfrak{f}$	Discontinuous flux function of the random scalar conservation law
f	Separated (random) flux function only depending on the solution value
$L_\mathfrak{f}$	(Local) Lipschitz constant for flux function (with respect to the solution variable)
$\boldsymbol{g}$	Continuous function of Panov-type flux
$\mathfrak{Z}$	Carathéodory function of Panov-type flux

(Random) Sole flux discontinuities

$\mathfrak{D}$	Sole discontinuity hypersurface of the flux function $\mathfrak{f}$
$\mathfrak{d}$	Space-time point of the discontinuity hypersurface $\mathfrak{D}$
$\Phi^\mathfrak{D}$	Function, whose graph describes the discontinuity hypersurface
$\Psi^\mathfrak{D}$	Function whose zero-level set describes the discontinuity manifold
$\mathfrak{n}_\mathfrak{D}$	Normal unit vector of the discontinuity hypersurface $\mathfrak{D}$
$\widehat{\mathfrak{n}_\mathfrak{D}}$	Extension of the normal vector field to whole space-time domain
$\boldsymbol{y}$	Spatial Variable of lower-dimensional space, used in/for parametrizations
$\mathfrak{y}$	Space-time variable of lower-dimensional space, used in/for parametrizations
$\boldsymbol{P}_\mathfrak{D}$	Parametrization of the flux discontinuity manifold
$\bullet^{l,r}$	Specification for $\bullet$ being restricted to the left/right part of the sole discontinuity

Compound flux discontinuity description

$\mathfrak{D}$	Compound flux discontinuity
$\mathfrak{D}_\mathfrak{i}$	i-th sole discontinuity contained in compund discontinuity $\mathfrak{D}$
$\mathfrak{N}_\mathfrak{D}$	Number of sole discontinuities in the compound flux discontinuity $\mathfrak{D}$
$\mathfrak{N}_{\mathbb{X}_\mathbb{T}}$	Number of space-time domain parts resulting from a compound flux discontinuity $\mathfrak{D}$

$\mathcal{I}_{\mathfrak{D}}$ (Possibly random) index set for sole discontinuities contained in a compound flux discontinuity

$\mathfrak{I}_{\mathfrak{D}}$ Set of intersection points in the compound flux discontinuity $\mathfrak{D}$

$\mathfrak{I}^{o}_{\mathfrak{D}}$ Subset having measure zero of the set $\mathfrak{I}_{\mathfrak{D}}$ of intersection points in the compound flux discontinuity

$\mathfrak{C}_{\mathfrak{D}}$ (Random) Partition up to a null set of the compound flux discontinuity

$\mathfrak{C}^{\kappa}_{\mathfrak{D}}$ κ-th part of the partition $\mathfrak{C}_{\mathfrak{D}}$

$\boldsymbol{P}_{\mathfrak{C}^{\kappa}_{\mathfrak{D}}}$ Parametrization of κ-th partition part $\mathfrak{C}^{\kappa}_{\mathfrak{D}}$

$\mathcal{S}_{\omega}$ Selection of sole discontinuity containing a given partition part

$\mathbf{n}_{\mathfrak{C}^{\kappa}_{\mathfrak{D}}}$ Normal unit vector field of the partition part $\mathfrak{C}^{\kappa}_{\mathfrak{D}}$

$\widehat{\mathbf{n}}_{\mathfrak{C}^{\kappa}_{\mathfrak{D}}}$ Extension of the normal vector field $\mathbf{n}_{\mathfrak{C}^{\kappa}_{\mathfrak{D}}}$ to the whole space-time domain

$\mathbf{n}_{\mathfrak{D}_{s_\omega}}$ Normal vector of the sole discontinuity containing a given partition part

$\widehat{\mathbf{n}}_{\mathfrak{D}_{s_\omega}}$ Extension of the normal vector field $\mathbf{n}_{\mathfrak{D}_{s_\omega}}$ to the whole space-time domain

$\mathbb{C}^{l,r}_{\mathfrak{i}}$ Left- and right space-time domain parts of the i-th sole discontinuity of the compound flux discontinuity $\mathfrak{D}$

$\mathbb{F}$ Random function for showing measurability of domain parts resulting from compound flux discontinuity

$\mathbb{G}$ Random function for showing measurability of domain parts resulting from compound flux discontinuity

(Random) Admissibility germs and remainder functions

$\mathfrak{G}$ Admissibility germ

$\mathfrak{G}^{*}$ Dual germ of $\mathfrak{G}$

$\widehat{\mathfrak{G}}$ extension of germ $\mathfrak{G}$

$\mathfrak{G}_{\mathrm{RH}}$ Rankine-Hugoniot germ

$\mathfrak{G}_{\mathrm{VV}}$ Vanishing viscosity germ

$\mathfrak{R}_{\mathfrak{G}}$ Remainder function corresponding to admissibility germ $\mathfrak{G}$

$\mathfrak{R}^{\mathrm{dist}}_{\mathfrak{G}}$ Remainder function associated to germ $\mathfrak{G}$ based on Euclidean distance

$\mathfrak{R}^{\mathrm{Osc}}_{\mathfrak{G}}$ Remainder function based on oscillation

$\mathfrak{R}^{\mathrm{Var}}_{\mathfrak{G}}$ Remainder function associated to germ $\mathfrak{G}$ based on locally bounded variation

$\mathfrak{R}^{m}_{\mathfrak{G}}$ Remainder function associated to germ $\mathfrak{G}$ based on modulus of continuity

$C_{\mathfrak{f}}$ Constant for remainder function based on Euclidean distance

Π Function for describing condition of germ as set-valued mapping

$\mathcal{S}^{l,r}_{\mathfrak{f}}$ Selection function for the left and right fluxes for some part $\mathfrak{C}^{\kappa}_{\mathfrak{D}}$ of the compound flux discontinuity's partition $\mathfrak{C}_{\mathfrak{D}}$

$(\mathfrak{u}^{l}, \mathfrak{u}^{r})$ Scalar tuple of admissibility germ

$(\hat{\mathfrak{u}}^{l}, \hat{\mathfrak{u}}^{r})$ Tuple of scalar values of dual germ

Π_{RH} Function for describing condition of Rankine-Hugoniot germ

Θ_{VV} Correspondence describing conditions of vanishing viscosity germ

ξ Space-time ratio $\xi := {}^{x}/_{t}$ for defining self-similar solutions

λ Function for describing correspondence overlapping

$\mathbb{l}$, $\mathbb{r}$ Correspondences describing flux conditions of vanishing viscosity germ $\mathfrak{G}_{\mathrm{VV}}$

$\Lambda_{\cdot}$ Correspondence for describing flux conditions of vanishing viscosity germ as a level-set

$H^{\cdot}$ Function for describing condition on flux in vanishing viscosity germ

$\mathbb{O}$ Function for describing overlapping sets

Random (adapted) entropy inequalities

k Constant in Kružkov entropy condition

k Adapted Kružkov entropy for sole/compound flux discontinuities

$\mathfrak{q}$ (Random) entropy flux

$k^{l,r}$ Left/right value of entropy pair $\boldsymbol{k}$

$\boldsymbol{k}$ Pair of entropy constants (k^{l}, k^{r}) in the sole flux discontinuity setting

$\boldsymbol{k}$ Sequence of Kružkov entropy values for compound flux discontinuity setting

$k^{\mathbb{i}}$ $\mathbb{i}$-th value of the Kružkov entropy sequence $\boldsymbol{k}$

$\boldsymbol{k}_{\mathbb{i}:\infty}$	Subsequence of Kružkov entropy sequence $\boldsymbol{k}$ starting at the $\mathbb{i}$-th element
$\mathbb{K}_m$	Set of fixed entropy sequence having an arbitrary value $k^m \in \mathbb{Q}$ at index $m \in \mathbb{N}$
$\mathcal{s}_{\boldsymbol{k}}$	Selection function for selecting the entropy value corresponding to a domain part resulting from a compound flux discontinuity
$\mathbb{I}_{\mathfrak{R}}$	Abbreviation symbol for integrand of remainder function in integral over (sole) discontinuity hypersurface

Audusse-Perthame and Panov frameworks

α	Steady-state parameter for adapted entropy frameworks
k_α	Function describing the adapted entropy in the sense of Audusse-Perthame for steady-state parameter α
$k_\alpha^\pm$	Stochastic steady-state solution(s) for parameter α in Audusse-Perthame setting
k_α	Solution of steady-state equation for value α with Panov-type flux function
k	Function that describes the steady-state solution k_α by taking the steady-state parameter α as an additional argument
$\mathfrak{p}_u$	Borel function for singular Kružkov entropy condition
$\mathfrak{D}$	Set of measure zero containing discontinuities in Audusse-Perthame formulation
$\mathfrak{d}$	Jump interface of flux in one-dimensional problems
$u_{\mathfrak{m}}$	Function describing the modality point of an Audusse-Perthame flux function
$\mathfrak{m}_0$	Point splitting a convex or concave Audusse-Perthame flux function into monotone parts
$\mathfrak{f}_-$	Lower (pathwise) bound of flux function in Audusse-Perthame framework
$\mathfrak{f}_+$	Upper (pathwise) bound of flux function in Audusse-Perthame framework
$\mathfrak{f}_\pm^{-1}$	inverse function of u-dependency of one-dimensional flux function in multiplicative Audusse-Perthame flux
$\mathbb{w}$	Bound on inverse flux function $\mathfrak{f}_\pm^{-1}$ of Audusse-Perthame fluxes
I	Real-valued interval for local Lipschitz continuity of Audusse-Perthame flux functions
$\mathcal{M}_{\mathfrak{f}}$	Pathwise bound on Audusse-Perthame flux function
R_k	Constant for describing the majorant of steady-state solutions with Panov-type flux functions
$\mathcal{M}_{R_k}$	Majorant of the steady-state function k for all steady-state parameters in the interval $[-R_k, R_k]$

Pathwise existence and uniqueness

$\mathbb{U}$	Bounded interval containing the values of the entropy solution u
$\underline{\mathbb{w}}$	Lower bound of confinement interval
$\overline{\mathbb{w}}$	Upper bound of confinement interval
$\tilde{u}$	Second solution variable for Kato inequality
$\tilde{u_0}$	Second initial condition for Kato inequality
ψ	Test function for variational formulations
h	Truncation variable for test function truncation (for showing equivalency of definitions)
ψ_h	Truncated test function for showing equivalency of definitions via admissibility germs and entropy inequalities as well as proving Kato inequality
η	Viscosity parameter for vanishing viscosity approach
u^η	Solution of vanishing viscosity problem
u_0^η	Initial condition of the vanishing viscosity problem
$\mathbb{s}$	Shock speed
$\mathcal{W}$	Interval, which boundaries correspond to the positive and negative $\mathcal{L}^\infty$-norm of the solution, respectively
$\Omega_{\omega,\boldsymbol{x}}^{k_\alpha^\pm}$	Abbreviation for the difference of two entropy fluxes in the Audusse-Perthame setting
a	Lower bound of interval of $\mathcal{L}^1$ contraction in Audusse-Perthame setting
b	Upper bound of interval of $\mathcal{L}^1$ contraction in Audusse-Perthame setting

Strong measurability of entropy solutions

$\mathbb{J}_{\psi}^{\boldsymbol{k}}$	Entropy functional for entropy pair $\boldsymbol{k}$ with ψ as the test function
$\mathbb{J}_{i,N}^{\boldsymbol{k}}$	Modified entropy functional for entropy pair $\boldsymbol{k}$ and, i-th basis function of $\mathcal{D}_N$
Ξ	Multifunction mapping onto the set of the entropy solutions in measurability proof
$\Xi^{\boldsymbol{k}}$	Correspondence containing all entropy solutions for a fixed entropy $\boldsymbol{k}$
$\Xi_{i,N}^{\boldsymbol{k}}$	Set-valued mapping onto set of functions satisfying the modified entropy condition
$\mathfrak{Q}_{\omega,\mathfrak{x}}^{\boldsymbol{k}}$	Abbreviation for the difference of two entropy fluxes in the sole discontinuity setting
${}^{s}\mathfrak{Q}_{\omega,\mathfrak{x}}^{\nu}$	Function describing the difference of two entropy flux functions having compound discontinuities
$\boldsymbol{S}_{\mathfrak{D}}$	Support of the test function overlapping with discontinuity hypersurface
$\boldsymbol{J}(\cdot)$	Determinant of transformation
$\mathbb{D}^{(\cdot)}$	(Random) integral domain depending on the sign value of the entropy flux $\mathfrak{q}$
${}^{s}\mathbb{D}_{(\cdot)}^{(\cdot)}$	Integral domain depending on sign value of entropy flux $\mathfrak{q}$ with compound discontinuity
ψ^i	i-th basis function of the truncated test function space $\mathcal{D}_N$
$\mathcal{M}_K^{\mathfrak{R}}$	Majorant of the remainder function corresponding to the compact set K
$\mathbb{A}$	Set containing the steady-state parameter depending on specific assumption
$C_{\mathbb{J}}$	Constant for continuous dependence results
C_{ψ}	Constant resulting from boundedness of test function
R_{ψ}	Constant for specific test function, such that function is contained in truncated test function space (see proof of measurability)
N	Value describing the interval $[-N, N]$ which contains the truncated test function support in measurability proofs

Lévy-type random fields

$\mathfrak{a}$	Stochastic jump coefficient of random flux function
$\mathfrak{a}_{-}$	Lower bound of random jump coefficient
$\mathfrak{a}_{+}$	Upper bound of random jump coefficient
$\overline{\mathfrak{a}}$	Mean field of Lévy-type random field
Φ	Functional applied to Gaussian random field for Lévy-type random fields
$\mathcal{G}$	Gaussian random field for construction of Lévy-type random fields
$\mathcal{G}$	Bounded Gaussian term in the Lévy-type random field
$\mathcal{G}_{+}$	Upper bound on the Gauss-type random field $\mathcal{G}$
$\mathfrak{P}$	Jump field in construction of Lévy-type random field
$\mathfrak{P}_j$	j-th value of jump field $\mathfrak{P}$
$\mathfrak{P}_{+}$	Upper bound on the jump field $\mathfrak{P}$
$\mathfrak{P}^{\mathrm{up,down}}$	Jump field for investigation of distance between jumps for jumping up/down
$\mathfrak{T}$	Partition of the domain for construction the jump field $\mathfrak{P}$
τ	Number of jumps in jump field $\mathfrak{P}$ of Lévy-type random field
Λ	Jump measure corresponding to the partition $\mathfrak{T}$ of the jump field $\mathfrak{P}$
$\mathfrak{p}_i$	i-th entry of jump height sequence for jump field
$\mathfrak{X}$	Set of all inclusions of a random inclusion field
$\mathfrak{x}_i$	Position of the i-th inclusion of a random inclusion field
$\mathfrak{l}_i$	Size of the i-th inclusion of a random inclusion field
$\mathfrak{h}_i$	Random variable describing the i-th jump height in a jump-advection coefficient $\mathfrak{a}$ having random inclusions
Q_{M}	Matérn covariance operator
Q_{SE}	Squared-exponential covariance operator
Q_{E}	Exponential covariance operator
σ^2	Variance for Matérn covariance kernel definition
ρ	Correlation length for Matérn covariance kernel definition
ν	Smoothness parameter of Matérn covariance kernel definition
K_{ν}	Modified Bessel function of second kind with ν degrees of freedem

$\mathbb{X}_{\mathcal{G}}$	Domain of interest for defining a Gauss-type random field
$\mathfrak{a}$	Slope angle of the fracture in the piecewise homogeneous medium
$\mathfrak{a}_i$	Angle of the i-th fracture in the heterogeneous medium for two-phase flow saturation problem
$\mathfrak{y}$	Interception point for fracture in piecewise homogeneous medium
N_{KL}	Truncation index of Karhunen-Loéve expansion
$\mathrm{Ber}(\cdot)$	Bernoulli distribution
$\mathcal{U}(\cdot)$	Uniform distribution
$\mathrm{Poi}(\cdot)$	Poisson distribution

Numerical methods and experiments

$\mathcal{O}(\cdot)$	Order of convergence
$\mathcal{E}$	Strong error
$\mathcal{E}_{\mathrm{RMS}}$	Root-mean-squared-error
$\mathcal{E}_w$	Weak error
$u_{\Delta}^{\mathrm{ref}}$	Numerical reference solution based on finer discretization
u_{Δ}	Numerical approximation of the solution
$U_{\mathcal{X}}^m$	Numerical approximation of the solution at time step m on cell $\mathcal{X}$
$\mathbb{T}_{\Delta}$	Time discretization of $\mathbb{T}$
t^m	m-th time point of the time discretization
Δ_t^m	Time step size of m-th time step
C_{CFL}^m	Constant for CFL condition in time step m
$\mathbb{X}_{\Delta}$	Spatial mesh of the domain $\mathbb{X}$
$\mathcal{X}_i$	i-th cell of the spatial mesh $\mathbb{X}_{\Delta}$
$\mathcal{Y}$	Alternative cell for finite volume description
$\Gamma_{\mathcal{X}}$	Boundary of the cell $\mathcal{X}$
$\mathcal{X}_{\mathrm{c}}$	Midpoint of cell $\mathcal{X}$
$\Delta\mathcal{X}$	Diameter of cell $\mathcal{X}$
$\lvert\mathcal{X}\rvert$	Volume of cell $\mathcal{X}$
$\boldsymbol{n}_{\mathcal{X}}$	Normal vector of the cell $\mathcal{X}$
$N(\mathcal{X})$	Set of cells sharing an edge with cell $\mathcal{X}$
$G_{\mathcal{X},\mathcal{Y}}^m$	Numerical flux from cell $\mathcal{X}$ to cell $\mathcal{Y}$ at time t^m
$G_{\mathcal{X},\mathcal{Y}}^{\mathrm{God}}$	Godunov flux from cell $\mathcal{X}$ to $\mathcal{Y}$
θ	Parameter for minimization and maximization in Godunov flux
$\Delta_{\boldsymbol{x}}^i$	Spatial step size of the i-th cell
$\Delta_{x,\mathrm{bound}}$	Upper bound on spatial step size for discretization
$\Delta_{x,\mathrm{max}}$	Maximum space step size of the discretization
$\overline{\Delta_{x,\mathrm{max}}}$	Maximum space step size of averaged over multiple discretizations
$\Delta_{x,\mathrm{max}}^l$	Maximum step size restriction of discretization on level l of a discretization hierarchy
$\mathfrak{d}_j$	j-th jump interface in the flux of a one-dimensional scalar conservation law
$\mathfrak{w}_{l,r}^i$	Left and right wave-cells of the i-th grid cell of one-dimensional problem
$\Delta\mathfrak{w}$	Size of the wave-cells
L_{ref}	Number of levels in the reference MLMC estimation
$E^{L_{\mathrm{ref}}}$	Reference MLMC estimation
$\mathbb{k}_o$	Permeability of medium, e.g., oil for two-phase flow saturation problem
$\mathbb{k}_w$	Alternative permeability of medium, e.g., water for two-phase flow saturation problem
$\boldsymbol{F}_{\mathrm{in}}$	Inflow flux for two-phase flow saturation problem
$\boldsymbol{F}_{\mathrm{out}}$	Outflow flux for two-phase flow saturation problem
$\mathfrak{c}$	Scaling of the initial condition for investigation of explicit/implicit time integration
$\mathfrak{h}$	Size of jump area $\mathfrak{H}^{\mathfrak{h}}$ for jump field parameter dependency investigation
$\mathfrak{H}^{\mathfrak{h}}$	Area, whose boundaries define jumps for parameter sudy on distance between jumps
$\mathfrak{H}_{\mathrm{c}}^{\mathfrak{h}}$	Midpoint of the jump area $\mathfrak{H}^{\mathfrak{h}}$

Stochastics and (multilevel) Monte Carlo methods

X	Random variable for random variable definition
$\mathbb{E}(X)$	Expectation of a random variable X
$\mathbb{V}(X)$	Variance of a random variable X
$\mathbb{C}\mathrm{ov}(X)$	Covariance of the random variable X
$\mathbb{M}_k(X)$	k-th moment of X
$\mathbb{L}$	Set of strongly measurable functions with finite moments
$\mathcal{N}$	Null set for defining Lebesgue-Bochner spaces
x'	Element of the dual space of X
x	Expected value for a weakly measurable random variable X
x''	Dunford integral of a weakly measurable random variable X
$\mathbb{E}_P(X)$	Pettis expectation of random variable X
Q	Covariance operator
λ	Eigenvalues of the covariance operator in KL expansion
$\boldsymbol{v}$	Eigenfunctions of the covariance operator
Z	Standard normally distributed random variable in KL expansion
M	Number of samples in MC estimator
$E_M(X)$	Monte Carlo estimator of X
L	Number of levels in the MLMC estimation
M_l	Number of MLMC samples on level l
$E^L(X)$	MLMC estimator with L levels
$\Delta^l_{x,\max}$	Step size for MLMC estimation on level l
C_{MLMC}	Constant of RSME estimation of MLMC

Preliminaries and appendices

Ξ	Correspondence for discussion preliminaries on set-valued mappings
ρ	Parameter for pseudo-distances of sets
s_g	Minimal selector
$\mathbb{1}$	Set-dependent indicator function
$\mathfrak{Z}_{\boldsymbol{x}}$	Closed nonempty sets containing $\boldsymbol{x}$
u_{L}, u_{R}	Left / right value for defining Riemann problems
$\mathcal{H}^d$	d-dimensional Hausdorff measure
s	Hausdorff dimension for defining Hausdorff measure
δ	Parameter for defining Hausdorff measure
$\mathcal{H}^s_\delta$	Covering for defining Hausdorff measure
U_j	j-th set of covering for definition of Hausdorff measure
$C^{\mathcal{H}}_d$	Normalization constant for equality of d-dimensional Hausdorff and Lebesgue measure
$\mathcal{L}^d$	d-dimensional Lebesgue measure
$\mathcal{I}$	Set consisting of half-open intervals
$\mathfrak{x}$	Position of jump field in appendix
$\mathbb{C}$	Countable subset of $\mathcal{L}^\infty$ for counterexample of strong measurability of essentially bounded flux functions
$\mathfrak{U}$	Uncountable subset of $\mathcal{L}^\infty$ for counterexample of strong measurability of essentially bounded flux functions

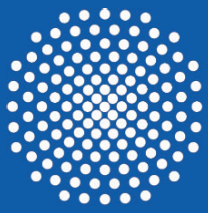
Function spaces

$\mathbb{N}$	Natural numbers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathcal{C}$	Space of continuous functions
$\mathcal{C}^\infty_{\mathrm{c}}$	Space of smooth functions with compact support
$\mathcal{L}^p$	Space of all Lebesgue p-integrable functions
$\mathcal{L}^p_{\mathrm{loc}}$	Space of all locally Lebesgue p-integrable functions
$\mathcal{L}^p_w$	Space of weakly measurable functions, such that the dual pairing is Lebesgue-p-integrable for all dual elements
$\mathcal{S}$	Separable subspace of $\mathcal{L}^\infty(\mathbb{X}_{\mathbb{T}})$
$\mathcal{D}$	Space of all test functions
$\mathcal{D}_N$	Subspace of $\mathcal{D}$ containing functions whose support is contained in $[0,N]\times\mathbb{B}_N$
$\mathcal{BV}$	Space of functions having bounded variation
$\mathcal{BV}_{\mathrm{loc}}$	Space of functions having locally bounded variation

$\mathcal{SBV}$	Space of special functions of bounded variation
$\mathcal{CS}(A)$	Space of nonempty closed subsets of A
$\mathbb{R}^{\leq\mathbb{N}}$	Space of all finite and infinite real-valued sequences
$\ell^\infty(\mathbb{R})$	Space of all bounded real-valued sequences

Math operators and symbols

$\partial_t f$	Time derivative of f
$\partial_x f$	One-dimensional spatial derivative of function f
$\operatorname{div}_{\boldsymbol{x}} f$	Divergence of f
$\nabla_{\boldsymbol{x}} f$	Spatial gradient of function f
$\nabla_{\mathfrak{x}} f$	Space-time gradient of function f
Δf	Laplace of function f
$\boldsymbol{D}$	Differential of (linear) map
$\operatorname{Gr} f$	Graph of a function f
$\gamma^{l,r} g$	Strong one-sided traces of function g
$\gamma^0 g$	Strong initial trace of function g
$\operatorname{dom} f$	domain of the function f
$\operatorname{supp} f$	Support of function f
$\operatorname{Osc}(f)$	Oscillation of function f
$\mathcal{V}_a^b(f)$	Variation of function f on interval (a,b)
$m(f)$	Modulus of continuity of a function f
$f\big\vert_A$	Restriction of f to the set A
$\lim_{x \searrow a} f$	Right-sided limit of function f at point a
$\lim_{x \nearrow a} f$	Left-sided limit of function f at point a
$\Vert f \Vert$	Norm of function f
$\operatorname{sign}(\cdot)$	Sign function
$\mathbb{1}_A$	Indicator function of the set A
dist	Euclidean distance function
$[\bullet]$	Equivalence class of $\bullet$
Tr	Trace operator
$\otimes$	Direct product
$\fint$	Integral symbol for mean integral
$\langle \cdot, \cdot \rangle$	Dual Pairing
$\langle \cdot, \cdot \rangle$	Inner product
$L(H)$	Set of linear operators over H
$L_N^+(H)$	Set of all nonnegative, symmetric, nuclear operators on H
$\{\cdot\}$	Set
$\operatorname{cl}(A)$	Closure of a set A
ch	Convex hull
diam	diameter of a set
$\mathbb{d}$	Metric on hyperspace of closed sets
A^{c}	Complement of set A
$\mathcal{B}$	Borel sigma algebra
$\mathcal{P}$	Power set
$(\mu)^{ac}$	Absolutely continuous part of a Radon measure μ
$(\mu)^s$	Singular part of a Radon measure μ
$\vert\xi\vert$	Absolute value of scalar value $\xi \in \mathbb{R}$
B'	Dual space of the Banach space B
B''	Double dual space of the Banach space B
$\mathbf{0}_{\mathbb{R}^d}$	Zero vector in $\mathbb{R}^d$
$\mathbf{1}$	Constant one vector in $\mathbb{R}^d$
const.	Constant value
A^{T}	Transpose of matrix A
$\boldsymbol{e}_i$	t-th canonical unit vector
$\mathbb{B}_r(\boldsymbol{x})$	Ball with radius r around $\boldsymbol{x}$



University of Stuttgart
Germany

SimTech