On the uniqueness of the Calderón Problem and its application in Electrical Impedance Tomography

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Abstract

This thesis addresses several questions about the uniqueness and reconstruction of the conductivity γ from knowledge of the boundary information encapsulated in the Dirichlet-to-Neumann map Λ_γ. This problem is well-known in the literature as *Calderón problem.*

In two dimensions, we extend the uniqueness of Calderón problem in two dimensions for complex conductivities with curves of discontinuity based on the stationary phase method and the introduction of new exponentially growing solutions.

In three dimensions, we extend the result established by Nachman for real conductivities with two derivatives, by noting that most of the proof holds with the need of extending some of the results to encapsulate the complex case. Moreover, we establish a methodology to recover the complex conductivity from small complex frequencies, but some open questions are left about this reconstruction process.

Furthermore, we reduce the differentiability condition for uniqueness to hold. We have shown that the Dirichlet-to-Neumann map uniquely determines complex conductivities with one derivative. Our approach is completely novel and introduces a quaternionic analysis approach to deal with the problem in three dimensions.

With the quaternionic framework we also introduce a possible path to show uniqueness for real conductivities in L^∞. This is a step in the direction of a complete answer to Calderón's question in three dimensions.

This problem is also relevant for practical applications, in particular medical imaging where it is used in Electrical Impedance Tomography (EIT). For practical implementations, a reconstruction algorithm is required to transform the boundary measurements into a conductivity profile. We use iterative methods to obtain a reconstruction method and our goal is to provide a simple and effective way to compute the required Jacobian matrix This approach is based in *automatic differentiation (AD) tools* .

We show that AD can be used to efficiently and effectively compute the Jacobian matrix of a numerical method that simulates the voltages measurements. Further, we show that this computation is as effective as analytical closed-forms applied in general iterative method in order to reconstruct the conductivity profile.

Zusammenfassung

Diese Dissertation beschäftigt sich mit mehreren Fragen zur eindeutigen Rekonstruierbarkeit der Leitfähigkeit γ aus in der Dirichlet-zu-Neumann-Abbildung Λ_γ enthaltenen Randinformationen. Dieses Problem ist in der Literatur als *Calderón-Problem* bekannt.

In zwei Dimensionen erweitern wir die Eindeutigkeit des Calderón-Problems auf für komplexe Leitfähigkeiten mit Unstetigkeiten entlang von Kurven. Das Resultat basiert auf der Methode der stationären Phase und der Einführung eines neuen Typs exponentiell wachsender Lösungen.

In drei Dimensionen erweitern wir das Resultat von Nachman für reelle Leitfähigkeiten der Klasse C^2 unter Beibehaltung wesentlicher Beweisideen auf komplexe γ. Darüber hinaus etablieren wir eine Methode, die Leitfähigkeit unter Nutzung niedriger Frequenzen zu rekonstruieren. Allerdings bleiben einige offene Fragen bei diesem Rekonstruktionsprozess.

Witerhin reduzieren wir die Differenzierbarkeitsbedingungen an γ für eindeutige Rekonstruierbarkeit. Dabei haben wir gezeigt, dass die Dirichlet-zu-Neumann-Abbildung einfach differenzierbare komplexe Leitfähigkeiten eindeutig bestimmt. Unser Ansatz ist neu und basiert in drei Raumdimensionen auf quaternionischer Analysis.

In diesem quaternionischen Rahmen zeigen wir einen möglichen Weg zum Beweis der Eindeutigkeit für reelle Leitfähigkeiten $\gamma \in L^\infty$ auf. Dies ist ein wichtiger Schritt in Richtung einer vollständigen Beantwortung des Calderón-Problems in drei Dimensionen.

Dieses Problem ist auch für praktische Anwendungen von Bedeutung, insbesondere in der medizinischen Bildgebung, wo es in der *Elektrischen Impedanztomographie (EIT)* verwendet wird. Für praktische Umsetzungen ist ein Rekonstruktionsalgorithmus erforderlich, um die Randwertmessungen in ein Leitfähigkeitsprofil zu transformieren.

Wir verwenden iterative Methoden, um ein Rekonstruktionsverfahren zu erhalten. Dabei ist es unser Ziel, eine einfache und effektive Möglichkeit zur Berechnung der erforderlichen Jacobimatrix des zugrundeliegenden Simulationsverfahrens bereitzustellen. Dieser Ansatz basiert auf *automatischen Differenzierungs-Tools* (AD).

Wir zeigen, dass AD verwendet werden kann, um die Jacobimatrix der numerischen Methode, die die Spannungsmessungen simuliert, effizient und effektiv zu berechnen. Des Weiteren zeigen wir, dass diese Berechnung ebenso effektiv ist wie analytische geschlossene Formen, die in allgemeinen iterativen Methoden zur Rekonstruktion des Leitfähigkeitsprofils angewendet werden.

Statutory Declaration

I hereby formally declare that I have written the submitted thesis independently.

I did not use any outside support except for the quoted literature and other sources mentioned in the text. I clearly marked and separately listed all of the literature and all of the other sources, both directly and indirectly employed when producing this thesis.

The thesis itself in the same or similar form has not been submitted to any examination body and has not been published. The electronic copy of this thesis provided for examination purposes is identical in content and format to the submitted hard copies.

Stuttgart, January 28, 2023

Ivan Pombo

Contents

Preprints and publications

This thesis is based and extends the following publications:

- ∇ *Automatic differentiation as an effective tool in Electrical Impedance Tomography*
  Ivan Pombo, Luis Sarmento
  submitted on 26 Jan 2023
  arXiv:
  Reference: [78]

- ∇ *CGO-Faddeev approach for complex conductivities with regular jumps in two dimensions*
  Ivan Pombo
  accepted by *Inverse Problems*
  arXiv: 1903.03485 [math-ph] (2019)
  Reference: [75]

- ∇ *Reconstructions from boundary measurements: complex conductivities*
  Ivan Pombo
  accepted by *Harmonic Analysis and Partial Differential Equations*
  arXiv: 2112.09894 [math.AP] (2021)
  Reference: [76]

- ∇ *Uniqueness of the inverse conductivity problem once-differentiable complex conductivities in three dimensions*
  Ivan Pombo
  submitted on 20 Jan 2023
  arXiv: 2301.08663 [math.AP] (2023)
  Reference: [77]

Chapter 1

Introduction

1.1 Motivation

While working in a research project for an Argentinian geophysical prospecting company in the 1950s, Alberto Calderón dealt with the possibility of determining the electrical conductivity inside a domain by making electrical measurements at the boundary. Only in 1980 he was convinced by colleagues to publish the problem and a partial answer for it [18]. This work pioneered a new area of mathematical research in inverse boundary value problems.

Let us start by defining the Calderón problem as initially posed. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$. Let $\sigma : \Omega \to \mathbb{C}$ a measurable function with positive lower bound, i.e., $\sigma \geq c > 0$. Then given $f \in H^{1/2}(\partial\Omega)$ let $u \in H^1(\Omega)$ be the unique solution to

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f. \end{cases} \tag{1.1}$$

Physically, this equation describes the behavior of an electrical potential u inside a body with *isotropic conductivity* σ, when a voltage f is applied at its surface. We call it *conductivity equation* and it describes the *direct problem.* Uniqueness of solutions is guaranteed for $\sigma \in L^\infty(\Omega)$, $\sigma \geq c > 0$ by Theorem 4.10 in chapter 4 of McLean's book [62] through the study of strongly elliptic and coercive differential operators.

These solutions allow the determination of the so-called Dirichlet-to-Neumann map, which is formally defined as:

$$\begin{aligned} \Lambda_\sigma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f &\mapsto \sigma \frac{\partial u}{\partial n}, \end{aligned} \tag{1.2}$$

where n is the unit outward normal vector to $\partial\Omega$. Given a voltage applied to the boundary we have that $\Lambda_\sigma f$ represents the electrical current passing $\partial\Omega$. Here, we said formally because this strong definition only holds when we have more regular boundary values. In any case, the proper weak formulation is given as

$$\langle g, \Lambda_\sigma f\rangle_{H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega)} = \int_\Omega \sigma\nabla u \cdot \nabla w \, dV(x), \tag{1.3}$$

for $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = g$.

The Dirichlet-to-Neumann map (DtN) describes the relation between voltage and electrical current on the boundary of Ω. In this sense, Calderón was interested in knowing if we can determine the internal conductivity σ from the outside information. Mathematically, the Calderón problem is succinctly stated as:

> "*Can we uniquely determine the conductivity* $\sigma \in L^\infty(\Omega)$, $\sigma \geq c > 0$ *from its Dirichlet-to-Neumann map* Λ_σ*? If so, how can we reconstruct it?*"

Initially, Calderón was able to show that the linearized problem at constant conductivities has a unique solution, which sets an initial framework for all further developments.

We remark that Calderón problem is posed under the most general conductivities. The only requirement is that σ is bounded below, in order to have uniqueness of the direct problem we mentioned above. Otherwise, we assume conductivities that are bounded.

However, this has proven to be a difficult task. The initial results obtained by researchers required more regularity assumptions. Iteratively, conditions have been relaxed to achieve $L^\infty(\Omega)$ conductivities.

Kohn and Vogelius were one of the firsts dealing with the Calderón problem. In [56] they established that all derivatives of conductivities at the boundary are uniquely determined by the Dirichlet-to-Neumann map. Hence, by analytical continuation this shows that if σ is real-analytic in Ω then it is uniquely determined by Λ_σ. As a first step, this work established an essential foundation for later ones, since the determination of conductivities at the boundary was essential for many of them.

In 1986, Sylvester and Uhlmann [87] extended the work of Calderón by showing uniqueness for $\sigma \in W^{3,\infty}(\Omega)$ close to 1 in two dimensions. This work established the foundations for all uniqueness proofs that came afterwards by introducing the concept of *exponentially growing solutions* depending on a complex parameter.

In 1988, Novikov [72] solved the more general inverse scattering problem for bounded potentials and, hence, for twice-differentiable conductivities. One key idea that persisted to later proofs and reconstruction formulas is the concept of a scattering transform. This transform relates the complex frequency defining the exponential growing solutions to the boundary information, in particular the Dirichlet-to-Neumann map, and the potential inside.

In 1987, Sylvester and Uhlmann [88] used these solutions to extract information about the Fourier transform of σ at every frequency from asymptotics for large complex frequencies. With it, they have proven that smooth conductivities are uniquely determined by DtN map in $\mathbb{R}^n$, $n \geq 3$.

In the year afterwards, Nachman in [68] extended these results for conductivities with only two derivatives over domains in $\mathbb{R}^n$, $n \geq 3$. Not only he was able to provide a uniqueness result, it is the first work explicitly working in the Calderón problem that introduces the concept of *scattering transform* and how one can reconstruct the conductivity from it. In this work, Nachman transforms the conductivity equation into a Schrödinger equation and reconstructs the potential from boundary data.

Up to 1988, global uniqueness for smooth conductivities was still an open question in two dimensions. The difficulty arises from the fact that the inverse problem is no longer overdetermined and all of the information about Λ_σ needs to be used, while for $n \geq 3$ the large complex frequency information has been sufficient for global uniqueness. All proofs mentioned above relied on this over determinacy.

Afterwards, Sun and Uhlmann [86] were the first to extend these results in order to obtain uniqueness for generic conductivities with three-derivatives. In 1996, Nachman [69] improved them by loosening the condition for two derivatives and at the same time providing a reconstruction method based on $\bar{\partial}$-equation.

Based on this result and similar ideas, Brown and Uhlmann [15] decreased the regularity in order to only require $\sigma \in W^{1,p}(\Omega)$. They use the Dirac scattering equation to transform the problem and establish the uniqueness result before setting it for the conductivities. Based on this proof a reconstruction method was obtained in [55].

Only in 2006 a full answer to Calderón problem was given by Astala and Päivärinta [8] for two-dimensions. They used the theory of quasi-conformal mappings to show the existence and uniqueness of exponentially growing solutions to a Beltrami equation obtained from the conductivity equation without requiring derivatives of σ.

Unfortunately, since most of the tools used to prove uniqueness in two-dimensions are based on complex analysis they cannot be immediately used for higher dimensions.

The best known result for higher dimensions was obtained by Caro and Rodgers [19] for $\sigma \in W^{1,\infty}(\Omega)$ for Lipschitz domain and around the same time by Haberman for conductivities with unbounded gradient [36].

In the meantime an extension of Calderón problem to admittivities $\gamma = \sigma + i\omega\epsilon$ appeared. The admittivity considers both the conductivity and permittivity of the body Ω which may change with respect to the current frequency ω. This problem is more physically natural to study since it incorporates information about the frequency of the electrical current. This was avoided previously when deducing from Maxwell equations the conductivity equation (1.1) by considering it was negligible.

This generalization is important for application purposes, since it helps distinguish more inner domains from the extra information provided by the permittivity and the admittivity relation with the current frequency. For simplicity, we also call γ *complex conductivity* through out the text.

With respect to the direct problem nothing changes and we can still consider equation (1.1) with σ extended to γ. Here, the condition $\operatorname{Re}\gamma = \sigma \geq c > 0$ is still assumed. In fact, it is necessary to ensure existence and uniqueness of the electrical potential $u \in H^1(\Omega)$ solution to the extended conductivity equation for a given voltage potential $f \in H^{1/2}(\Omega)$. For reference purposes, the *complex conductivity equation* is thus given as follows:

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f. \end{cases} \tag{1.4}$$

We remark that for $\gamma \in L^\infty(\Omega)$ with the above condition on σ the uniqueness and existence of a solution is guaranteed by Theorem 4.10 of McLean's book [62]. Furthermore, the respective Dirichlet-to-Neumann map still holds analogously by:

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f &\mapsto \gamma \frac{\partial u}{\partial n}. \end{aligned} \tag{1.5}$$

The proper weak formulation is similarly given for $w \in H^1(\Omega),\ w|_{\partial\Omega} = g$ through

$$\langle g, \Lambda_\gamma f \rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = \int_\Omega \gamma \nabla u \cdot \nabla w \, dV(x). \tag{1.6}$$

The Calderón problem is formulated almost as before with the slight consideration of the frequencies:

> "*Can we uniquely determine the conductivity* $\gamma \in L^\infty(\Omega)$, $\operatorname{Re}\gamma = \sigma \geq c > 0$ *from its Dirichlet-to-Neumann map* Λ_γ *at a single frequency* ω_0*? If so, how can we reconstruct it?*"

While for some researchers there was evidence that some results could be easily extended for complex conductivities, the first uniqueness result was published in 2000 by Francini [30] for $\gamma \in W^{2,\infty}(\Omega)$ with small frequencies in two dimensions. This work based itself on the results obtained by Brown and Uhlmann for real conductivities with the Dirac system [15]. The smallness assumption pends on a perturbation argument to show uniqueness of exponentially growing solutions. Furthermore, these solutions still exist when γ is in $W^{1,\infty}(\Omega)$ but the uniqueness proof presented requires higher regularity.

In 2008, Bukhgeim [16] presented a novel approach that drops the smallness requirement for $\gamma \in W^{2,\infty}(\Omega)$ in two dimensions. His work is not specifically focused on the conductivity equation, instead it is centered around the Schrödinger equation and Dirac system. Through the stationary phase method a uniqueness result for the potential is obtained from large complex frequency asymptotics. Joining this result with Theorem 5.1 of Francini the uniqueness proof follows for the complex conductivity.

The Bukhgeim scheme was extended by other works [6, 28, 89]. Another work to keep in mind [58] devised a reconstruction method based on $\bar{\partial}$-equation. However, none of these were able to decrease the regularity to once-differentiable complex conductivities.

The first work to decrease the regularity assumption to $\gamma \in W^{1,\infty}(\Omega)$ was obtained in [57] by Lakshtanov, Tejero and Vainberg. The authors combined the works of Francini and Bukhgeim to establish a uniqueness proof based on scattering data for large complex frequencies.

The only extension of this result was obtained in [75] for complex conductivities with discontinuity over curves. The author introduced a new set of exponentially growing solutions that allows the determination of the complex conductivity on a set of special points, designated by *admissible points*. As far as we are aware this is the best known result in two dimensions present in the literature.

Furthermore, notice that all of the above works were for the two-dimensional case. In fact, there is no explicit reference for the three-dimensional Calderón problem with complex conductivities. However, based on results for the Schrödinger inverse problem present in [70] a uniqueness proof for $\gamma \in W^{2,\infty}(\Omega)$ can be obtained by establishing a relation between the Dirichlet-to-Neumann map for the potential and the complex conductivity.

Notice that there is a common line for all uniqueness and reconstruction methods. Here we highlight the main steps:

1. Transform the conductivity equation (1.4) into another boundary value problem for which the coefficient of interest is not affected by derivatives. Examples in the literature involved the Schrödinger equation, Dirac system and Beltrami equation.

2. Extend this problem into the whole space and convert it into an integral equation.

3. Study a set of exponentially growing solutions depending on a complex parameter for the integral equation.

4. Either establish a scattering transform that is related with the boundary measurements and with the exponential growing solutions, or establish an identity that makes use of the exponential growing solutions to generate a Fourier transform of the coefficient. An example of the latter is Alessandrini's identity [5].

5. Devise a relation between the scattering data or the identity to obtain the coefficient of interest;

6. Finally, relate the coefficient with the conductivity by connecting the Dirichlet-to-Neumann map to the scattering data or identity.

The last steps vary slightly according to the equation we are dealing with and the respective exponentially growing solutions. Moreover, the scattering transform has been mostly used for possible reconstruction methods, while the identity are uniquely used for the uniqueness proof.

Besides the mathematical interest, Calderón's problem also has a role in practical applications. As initially mentioned, Calderón had an application in mind to introduce it, namely geophysical prospecting. As of today, one of its largest applications is in medical imaging, where it is called *Electrical impedance tomography (EIT).*

Electrical Impedance Tomography (EIT) is a non-invasive imaging method that produces images by first determining electrical conductivity inside a subject using only electrical measurements obtained at its surface. More specifically, sinusoidal currents are applied to the subject through electrodes placed in certain locations at the surface of the object, and the resulting voltages are then measured, making it possible to infer certain internal properties of the objects. EIT is a low-cost method and since it only applies low amplitude currents is non-harmful for living beings. Further, it allows for real-time monitoring of various subjects in harsh conditions. There are applications of this technology for medical purposes, in scenarios such as ventilation monitoring, detecting brain hemorrhages and breast cancer. Besides that, it is also used in geophysical imaging, flow analysis and other industrial purposes. For further insight into the applications, see [3, 26, 92].

In order to reconstruct the electrical conductivity from the measurements one can use either *direct* or *iterative* reconstruction methods.

Direct methods are scarce, but yet they are powerful. The idea behind these methods is to obtain the conductivity through the Dirichlet-to-Neumann map by solving a set of steps in once. The most general is based on the $\bar{\partial}$-equation that was initially proposed by Nachman in [69] and successfully implemented in a stable manner by [48]. Further, theoretical extensions have been done in [54], application to simulated data in [53, 66, 67] and for experimental setup and *in vivo* data see [49, 50, 65]. Another $\bar{\partial}$-method was based on the uniqueness proof in [15] and the reconstruction method established in [55]. This method was immediately tested for complex conductivities since the work in [30] is heavily based on the real conductivities work [37] and tested in experimental data in [38] and [41].

In the last couple of years, there has been an attempt to use the direct methods Nachman introduced in [68] with implementations proposed on the following articles [23–25]. More testing with simulated electrode data has been shown in [42], with even an unreferenced extension to complex conductivities. All of these algorithms rely on the asymptotic behavior of exponentially growing solutions with respect to large complex frequencies. As such, since it is computationally impossible to obtain a limiting behavior these methodologies are very unstable when noise corrupts the electrical measurements.

Recall that the physical nature of the problem is inherently three-dimensional. While the direct methods for two-dimensions have been successfully tested and implemented to obtain results in practice, there is still a long way to go for direct algorithms in three-dimensions. In particular, when complex conductivities are involved.

Therefore, in three-dimensions EIT has been mostly solved by iterative methods. These methods can be further split into two categories: linear and non-linear. On the first case, a linearization of EIT problem is applied. These methods only allow for the detection of small changes with respect to the initial approximation. While they are fast to be applied in practice, they lack the resolution required for clinical practice. The latter methods are based on iteratively improving an approximation of the conductivity by solving on each step a PDE to simulate new measurements and compared them with the real measured ones. Furthermore, the computation of the Jacobian matrix, with respect to a numerical model, is required to appropriately update the conductivity. Both steps require careful consideration since they are heavy computational tasks.

Some of the relevant work in this direction are more focused on the appropriate choice of optimization algorithm, see [13,74,82,90] for a nice review of the literature. The software package *EIDORS* is available in MATLAB and was implemented based on this work [4].

From this confined literature review we verify the existence of several open problems in the literature, in particular, in terms of complex conductivities and the computational hardship of iterative methods. To this end, we have solved some of them and give a brief summary of the results obtained in the following section.

1.2 Main results

The aim of the present work is to prove uniqueness of the Calderón problem under various assumptions and to simplify the reconstruction procedure under an optimization framework.

The following theorem gives uniqueness of Calderón problem for complex conductivities in two dimensions with a discontinuity curve. We refer to Chapter 3 for the theorem and proof.

Theorem 1.2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and $\Gamma_j, j = 1, ..., m$ be a set of closed Lipschitz curves with interior domains denoted by $\mathcal{D}_j$ and their union denoted as $\mathcal{D}$.*

Let $\gamma = \sigma + i\omega\epsilon \in \cup_{k=1}^{n} W^{2,\infty}(\mathcal{D}_k) \cup W^{2,\infty}(\Omega \setminus \mathcal{D})$ with $\operatorname{Re}\gamma(x) \geq c > 0$ almost everywhere. Further, we denote by γ^- the traces from the inside of the curves and γ^+ from the outside.

If the jumps $\sqrt{\frac{\gamma^-}{\gamma^+}} - 1$ are small enough in $L^\infty(\Gamma_j)$ over all curves $j = 1, ..., n$, then the Dirichlet-to-Neumann map Λ_γ uniquely determines the conductivity γ in a set of proper admissible points.

While this theorem does not show uniqueness over the full domain, it is the best result known so far for complex conductivities, since all previous results still require continuity of γ.

In three dimensions, the following theorem combines the results obtained in chapter 4 and stated separately in Theorems 4.1.1 and 4.1.2. The nature of the proofs is based on similar ideas, but the tools used in the proof is completely different.

Theorem 1.2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let γ_i for $i = 1, 2$ be two complex-valued conductivities with $\operatorname{Re}\gamma_i \geq c > 0$ and Λ_{γ_i} be their respective Dirichlet-to-Neumann maps.*

(i) If Ω is a $C^{1,1}$-domain and $\gamma_i \in C^{1,1}(\Omega)$, we have that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$ in Ω.

(ii) If $\gamma_i \in W^{1,\inf ty}(\Omega)$ then it holds that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$ in Ω.

Notice that the second part is an improvement over the first one. This was possible by applying quaternionic analysis to extend of the Dirac system of Brown, Uhlmann [15] and other authors (see [57]) to three-dimensions. Further, this last theorem still requires continuity of the conductivity, but it matches the best yet known result, even for the case when γ is purely real.

Calderón problem with discontinuous conductivities still remains an open question in three dimensions, even for real conductivities. The proof in two dimensions is built on complex analysis and quasi-holomorphic mappings which are unavailable for higher dimensions with the same properties. In Chapter 5, we establish a framework with some open questions remaining to obtain the uniqueness proof for real conductivities. Thus, the goal is to prove the following conjecture.

Conjecture 1.2.3. *Let Ω be a bounded Lipschitz domain. Let $\sigma_i \in L^\infty$ for $i = 1, 2$ and denote the Λ_{σ_i} as their respective Dirichlet-to-Neumann maps.*

Then, it holds:

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \Rightarrow \sigma_1 = \sigma_2 \text{ in } \Omega$$

Recall that the Calderón problem also encapsulates the reconstruction process. With this in mind we also established some ground work in the reconstruction process through iterative methods. In Chapter 6, we have shown that automatic differentiation (AD) method is effective at computing the required Jacobian matrix used in most iterative methods and, therefore, in solving the inverse problem. The core of this work reveals that this method can be used to iterate faster on the optimization solvers, rather than on the efficiently implementation of the analytical form of the Jacobian matrix. Hence, we have shown the effectiveness of AD to solve the inverse problem.

Chapter 2

Preliminaries

2.1 Some functional analysis

In this section we introduce the basic definitions of functional analysis that are required for the following chapters. A complete introduction can be found in many textbooks, *e.g.*, [2, 32, 62]. The framework we will introduced is based on the first two.

We start by introducing bounded domains and their boundaries, then we define Lebesgue spaces and Sobolev spaces on both. Thereafter, we introduce the space embeddings and when they are compact.

2.1.1 Domains and properties

Let Ω be a bounded and open set of $\mathbb{R}^n$ and define $\partial\Omega = \bar{\Omega} \cap (\mathbb{R}^n \setminus \Omega)$ to be its boundary. For most results on this thesis concerning Sobolev spaces we need our domain Ω to satisfy some properties described below, as well as a certain regularity on its boundary. All the definitions below are present in [2].

Definition 2.1.1 (Lipschitz Domain). *We say that Ω is a Lipschitz domain if its boundary can be locally represented by Lipschitz continuous function. Namely, if for any $x \in \partial\Omega$ there exists a neighborhood of x, $V_x \subset \mathbb{R}^n$, such that $V_x \cap \partial\Omega$ is the graph of a Lipschitz continuous function under a proper local coordinate system.*

In this sense, for some $N \geq 1$, there exists $\Omega^1, \dots, \Omega^N$ connected components of the Lipschitz domain, such that $\Omega = \bigcup_{j=1}^{N} \Omega^j$ and for each $j \in \{1, ..., N\}$ the boundary $\partial\Omega^j$ is given by the graph of a Lipschitz function $\phi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ through $\partial\Omega^j = \{x \in \mathbb{R}^n : x_n = \phi(x') \,\forall\, x' \in \mathbb{R}^{n-1}\}$.

Moreover, for any Lipschitz domain Ω there exists a surface measure $d\sigma$ and we can define the integration over the boundary also through:

$$\int_{\partial\Omega} f(x)\, d\sigma(x) = \sum_{j=1}^{N} \int_{\partial\Omega^j} f(x_1, x_2, ..., x_{n-1}, \phi_j(x_1, x_2, ..., x_{n-1})) \sqrt{1 + |\nabla\phi_j(x_1, ..., x_{n-1})|^2}\, dx_1 ... dx_{n-1};$$

By Rademacher theorem and the fact that is Lipschitz, ϕ_j is differentiable almost everywhere and bounded for any $j \in \{1, ..., N\}$.

Definition 2.1.2 (Higher regularity domains). *We say that Ω is of class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one-to-one mapping ψ of B onto $D \subset \mathbb{R}^n$ such that:*

(i) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$;

(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;

(iii) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

Definition 2.1.3 (Cone Condition). *Let v be a nonzero vector in $\mathbb{R}^n$ and let $\angle(x, v)$ be the angle between the position vector $x \neq 0$ and v. For such given v, and $\rho > 0$, and κ satisfying to $0 < \kappa \leq \pi$, the set*

$$C = \{x \in \mathbb{R}^n : x = 0 \; or \; 0 < |x| \leq \rho, \angle(x, v) \leq \pi/2\}$$

is called a finite cone of height ρ, axis direction v, and aperture angle κ with vertex at the origin.

We say that a domain Ω satisfies the ***cone condition*** *if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. Note that C_x need not be obtained from C by a parallel translation, but simply by rigid motions (rotation, reflection and/or translation).*

2.1.2 L^p and Sobolev Spaces

Definition 2.1.4 (Lebesgue spaces). *Let Ω be a domain in $\mathbb{R}^n$ and let $p \in [1, \infty]$. We denote by $L^p(\Omega)$ the class of all measurable functions u defined on Ω for which*

$$\int_\Omega |u(x)|^p \, dx < \infty, \quad if 1 \leq p < \infty \tag{2.1}$$

$$\sup_{x \in \Omega} |u(x)| < \infty \tag{2.2}$$

Two functions are said equivalent if they are equal almost everywhere in Ω. In this sense, an element of $L^p(\Omega)$ is just a representative of an equivalence class of measurable functions satisfying (2.1 or 2.2) , and we will work with these equivalence classes of functions without making further comments.

The space is a Banach Space with the norm

$$\|u\|_{L^p(\Omega)} = \left[\int_\Omega |u(x)|^p \, dx\right]^{1/p} \tag{2.3}$$

$$\|u\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |u(x)| \tag{2.4}$$

Moreover, for $p = 2$, the space is in fact a Hilbert Space, with the inner product:

$$\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)} \, dx \tag{2.5}$$

In our thesis, the measure dx will always represent the Lebesgue measure in $\mathbb{R}^n$, although, the above statement work for any measure.

The concept of Sobolev Spaces comes from the definition of *weak derivative*, which we will not present here, and from the above definition ofL^p space.

Definition 2.1.5 (Sobolev spaces). *Let Ω be a domain in $\mathbb{R}^n$, $p \in [1, \infty]$ and k be a positive integer. We denote by $W^{k,p}(\Omega)$ the space of functions such that all the weak (or distributional) partial derivatives D^α, for $0 \leq |\alpha| \leq k$, are in $L^p(\Omega)$, that is:*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \, for \, 0 \leq |\alpha| \leq k\} \tag{2.6}$$

This space is a Banach space under the norm:

$$\|u\|_{W^{k,p}(\Omega)} = \left[\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right]^{1/p}, \quad if 1 \leq p < \infty, \tag{2.7}$$

$$\|u\|_{W^{k,\infty}} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)} \tag{2.8}$$

Similarly to above, when $p = 2$ it is an Hilbert space, under the sum of the inner products in L^2 of all the derivatives. Hence, we change notations in this case: $W^{k,2}(\Omega) = H^k(\Omega)$.

Theorem 2.1.6 (L^p **embeddings over bounded domains**)**.** *Suppose that Ω is a domain such that $\mathrm{vol}(\Omega) = \int_\Omega 1 \, dx < \infty$. Let $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$ then $u \in L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \leq (\mathrm{vol}(\Omega))^{1/p - 1/q} \|u\|_{L^q(\Omega)} \tag{2.9}$$

We denote this embedding as $L^q(\Omega) \hookrightarrow L^p(\Omega)$.

Theorem 2.1.7 (**Sobolev Embedding Theorem**)**.** *Let Ω be a domain in $\mathbb{R}^n$. Let $j \geq 0$ and $m \geq 1$ be integers and let $1 \leq p < \infty$.*

PART I *Suppose that Ω satisfies the cone condition.*

Case A *If either $mp > n$ or $m = n$ and $p = 1$, then*

$$W^{j+m,p}(\Omega) \to C^j_B(\Omega). \tag{2.10}$$

For $p \leq q \leq \infty$, we have

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{2.11}$$

Case B *If $mp < n$ then, for $p \leq q \leq p^* = np/(n - mp)$, we have*

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{2.12}$$

PART II *Suppose that Ω is a Lipschitz domain. Then, we have the following refinement holds: if $mp > n > (m - 1)p$ then*

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\Omega) \quad for \, 0 < \lambda \leq m - (n/p). \tag{2.13}$$

Theorem 2.1.8 (**Sobolev Inequality**). *When $mp < n$ then there exists a finite constant K such that:*

$$\|\phi\|_{L^{\tilde{p}}(\mathbb{R}^n)} \leq K \left(\sum_{|\alpha|=m} \|D^\alpha \phi\|^p_{L^p(\mathbb{R}^n)} \right)^{1/p} \quad \text{for } \tilde{p} = np/(n-mp) \text{ and } \phi \in C_c^\infty(\mathbb{R}^n). \tag{2.14}$$

In particular, for $n \geq 2$ and $1 \leq p < 2$ we get

$$\|\phi\|_{L^{\tilde{p}}(\mathbb{R}^n)} \leq K \|\nabla \phi\|_{L^p(\mathbb{R}^n)} \tag{2.15}$$

Theorem 2.1.9 (**Rellich-Kondrachov Theorem**). *Let Ω be a domain in $\mathbb{R}^n$ and Ω_0 a bounded sub-domain of Ω. Let $j \geq 0$ and $m \geq 1$ be integers and let $p \in [1, \infty)$.*

PART I *If Ω satisfies the cone condition and $mp \leq n$, then the following embedding is compact:*

$$W^{m,p}(\Omega) \to L^q(\Omega_0), \ \text{if } 1 \leq q \leq np/(n-mp). \tag{2.16}$$

PART II *If Ω satisfies the cone condition and $mp > n$, then the following embedding is compact:*

$$W^{m,p}(\Omega) \to C_B(\Omega_0). \tag{2.17}$$

Theorem 2.1.10 (**Trace operator**). *Let Ω be a domain in $\mathbb{R}^n$. We define the trace operator in this domain by:*

$$\begin{aligned} \mathrm{tr} : \mathcal{D}(\bar{\Omega}) &\to \mathcal{D}(\Gamma) \\ u &\mapsto tr(u) = u|_\Gamma. \end{aligned}$$

(i) If Ω is a $C^{k-1,1}$ and $\frac{1}{2} < s \leq k$, then tr *has a unique extension to a bounded linear operator*

$$\mathrm{tr} : H^s(\Omega) \to H^{s-1/2}(\Gamma).$$

(ii) If Ω be a Lipschitz domain and $\frac{1}{2} < s < \frac{3}{2}$, then tr *has a unique extension to a bounded linear operator*

$$\mathrm{tr} : H^s(\Omega) \to H^{s-1/2}(\Gamma).$$

Both extensions have a continuous right inverse.

2.1.3 Other related results

Here we introduce some of the results that are required throughout the chapters but due not fit any specific section.

Theorem 2.1.11 (**Maximum and minimum principles for elliptic operators**, [32])**.** *We consider the differential operator L defined as:*

$$Lu = \sum_{i,j=1}^{n} D_i(a^{ij}(x)D_j u(x)),$$

whose coefficients a^{ij} $(i,\, j \;=\; 1, ..., n)$ are assumed to be measurable functions on a domain $\Omega \in \mathbb{R}^n$ (we consider this operator in a weak sense).

Let the operator L satisfy the conditions:

1. *The operator L is strictly elliptic: $\forall x \in \Omega, \xi \in \mathbb{R}^n \exists \lambda > 0 :\ \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$,*
2. *L has bounded coefficients: $\sum_{i,j=1}^{n} |a^{ij}(x)|^2 \leq \Lambda^2$*

Then, if $u \in H^1(\Omega)$ is a solution of $Lu = 0$, with $u|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega)$ we have

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} f^{+} \quad \inf_{\Omega} u \leq \sup_{\partial\Omega} f^{-},$$

where $f = f^{+} + f^{-}$, $f^{+} = \max\{f(x), 0\}$, $f^{-} = \min\{f(x), 0\}$.

Theorem 2.1.12 (**Hardy-Littlewood-Sobolev inequality**, [84])**.** *Let $f \in L^p(\mathbb{R}^n)$. We defined the Riesz potential of f by:*

$$\left(I_\alpha f\right)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad \textit{for } 0 < \alpha < n \tag{2.18}$$

with $\gamma(\alpha) = \pi^{n/2}2^{\alpha}\Gamma(\alpha/2) \Big/ \Gamma\left((n-\alpha)/2\right)$. Moreover, for $1 < p$, $q < \infty$, such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, it holds

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq A_{p,q}\|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, for $1 < p < 2$, we have:

$$\|I_1(f)\|_{L^{\tilde{p}}(\mathbb{R}^2)} \leq A_{\tilde{p}}\|f\|_{L^p(\mathbb{R}^2)},$$

where $\tilde{p}$ is the Sobolev conjugate $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}$.

Theorem 2.1.13 (**Riemann-Lebesgue lemma**, [12])**.** *Let f be a function in $L^1(\mathbb{R}^n)$. Then its Fourier transform is defined as:*

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx,$$

and fulfills

$$\mathcal{F}f(\xi) \to 0, \;\; as\, |\xi| \to +\infty.$$

The result holds also by extension for L^2 functions.

For any $f \in L^2(\partial\Omega)$ where Ω is a Lipschitz domain, we have by the above integration over the surface that the Riemann-Lebesgue lemma also holds for this functions, since

$$\sqrt{1 + |\nabla\phi_j(x')|^2} f(x') \in L^2(\mathbb{R}^{n-1}).$$

Moreover, the Fourier transform is invertible in $L^2(\mathbb{R}^n)$ and the inverse is densely defined as

$$\mathcal{F}^{-1}f(\xi) := \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\, dx.$$

2.2 Quaternionic analysis

We introduce here the quaternionic framework necessary for the work presented in the Chapter 4 and 5. This introduction is heavily based on the books of Gürlebeck and Sprössig [34, 35].

Real quaternions

Let e_0, e_1, e_2, e_3 be the basic elements in $\mathbb{R}^4$ where each e_k identifies with the 4-tuple for which the $k+1$-th component has the number one and is zero otherwise. An arbitrary element $x \in \mathbb{R}^4$ now has the representation $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$.

The idea of quaternionic analysis is to introduce a product in $\mathbb{R}^4$. To be an algebra the basic elements must fulfill the following conditions:

(i) $e_1^2 = e_2^2 = e_3^2 = -1$,

(ii) $e_1e_2 = e_3, \; e_2e_3 = e_1 \; e_3e_1 = e_2$,

(iii) $e_ie_j + e_je_i = 0, \; (i, j = 1, 2, 3; \; i \neq j)$.

The tuple $(\mathbb{R}^4, \cdot)$ is called *algebra of real quaternions*. The quaternions were initial described by W.R. Hamilton in 1843 and thus we denote them by $\mathbb{H}$ in his honor. By simplicity we can assume that $e_0 = 1$ and thus in general a element $x \in \mathbb{H}$ can be written as

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3, \quad \text{with } x_j \in \mathbb{R}, \; j = 0, 1, 2, 3. \tag{2.19}$$

For $x, y \in \mathbb{H}$ we denote xy for the resulting quaternion product. We note that this product is not commutative and a simple example is given by $e_1e_2 = -e_2e_1$. Now we denote the *scalar part* by $\mathrm{Sc}\, x = x_0$ and the *vector part* by $\mathrm{Vec}\, x = x_1e_1 + x_2e_2 + x_3e_3$. The quaternion

$$\bar{x} = x_0 - (x_1e_1 + x_2e_2 + x_3e_3), \tag{2.20}$$

is called the *conjugate* of $x \in \mathbb{H}$. For two quaternions $x, y \in \mathbb{H}$ it holds that $\overline{xy} = \bar{y}\bar{x}$. It also holds that $\mathrm{Sc}\, x = \frac{1}{2}(x + \bar{x})$ and $\mathrm{Vec}\, x = \frac{1}{2}(x - \bar{x})$.

The inner product in $\mathbb{R}^4$ that is given as $\langle x, y\rangle_{\mathbb{R}^4} = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ can also be written through quaternions as

$$\langle x, y\rangle_{\mathbb{R}^4} = \mathrm{Sc}(\bar{x}y) = \mathrm{Sc}(x\bar{y}) = \frac{1}{2}(x\bar{y} + y\bar{x}).$$

We note that a quaternion-valued inner product is also possible and given as $\langle x, y\rangle_{\mathbb{H}} = \bar{x}y$. With this inner product we have that $\mathbb{H}$ is an Hilbert space and the resulting norm is the usual Euclidean norm, $\|x\|_{\mathbb{H}} := \sqrt{\langle x, x\rangle_{\mathbb{H}}} = \sqrt{\bar{x}x} = \sqrt{|x|^2}$.

Complex quaternions

An extension of the quaternions to consider, is assuming x_j are complex-valued. This allows us to later on to introduce the exponentially growing solutions.

Let us consider $x \in \mathbb{C}^4$. Then we can define a complex quaternion in the same way by $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3$. We denote this space by $\mathbb{C} \otimes \mathbb{H}$. Here, we use the same generators (e_0, e_1, e_2, e_3) as before, with the same multiplication rules, however, the coefficients of the quaternion can be complex-valued. A complex quaternion can also be denoted as $x = x_R + ix_I$ with x_R, $x_C \in \mathbb{H}$

Here, we want to clarify the conjugation one can apply, since both the complex numbers and quaternions can be conjugated.

First and foremost, we keep the notation for when we only apply the quaternion conjugation, that is, for $x \in \mathbb{C} \otimes \mathbb{H}$ we denote the quaternion conjugation as

$$\bar{x} = \bar{x}_R + i\bar{x}_I = x_0 - (x_1e_1 + x_2e_2 + x_3e_3), \quad x_j \in \mathbb{C}. \tag{2.21}$$

In second, we can define the complex conjugation of a complex quaternion as

$$\bar{x}^c = x_R - ix_I = \bar{x}_0^{\mathbb{C}} + \bar{x}_1^{\mathbb{C}}e_1 + \bar{x}_2^{\mathbb{C}}e_2 + \bar{x}_3^{\mathbb{C}}e_3, \tag{2.22}$$

where for $z \in \mathbb{C}$ we define $\bar{z}^{\mathbb{C}}$ as the usual complex conjugation.

Finally, the quaternion and complex conjugation together is denoted as

$$\bar{x}^{\dagger} = \bar{x}_R - i\bar{x}_I. \tag{2.23}$$

Similarly to the real case, we introduce an associated inner product and norm in $\mathbb{C} \otimes \mathbb{H}$ by means of this conjugation as follows:

$$\langle x, y\rangle_{\mathbb{C}^4} := \mathrm{Sc}\left(\bar{x}^{\dagger}y\right) \text{ and } \|x\|_{\mathbb{C}\otimes\mathbb{H}} := \sqrt{\mathrm{Sc}\,(\bar{x}^{\dagger}x)}. \tag{2.24}$$

2.2.1 Quaternion-valued functions

In the interest of the work established later, we define the following quaternionic-valued functions

$$\begin{aligned} &f:\mathbb{R}^3\to\mathbb{C}\otimes\mathbb{H}\\ &f(x)=f_0(x)+f_1(x)e_1+f_2(x)e_2+f_3(x)e_3 \end{aligned}$$

where $f_j:\mathbb{R}^3\to\mathbb{C}$.

Furthermore, from now one we relate the vector $x\in\mathbb{R}^3,\ x=(x_0,x_1,x_2)$ with a quaternion through the same notation as $x=x_0+x_1e_1+x_2e_2$. It will be clear from the context if we are using one or another.

The Banach spaces L^p, $W^{n,p}$ of $\mathbb{C}\otimes\mathbb{H}$-valued functions are defines by requiring that each component is in such space. On $L^2(\Omega)$ we introduce the $\mathbb{C}\otimes\mathbb{H}$-valued inner product through

$$\langle f,g\rangle=\int_\Omega \bar{f}^\dagger(x)g(x)\,dx. \tag{2.25}$$

Analogously to the Wirtinger derivatives in complex analysis, we establish the Cauchy-Riemann operators through the coordinates $(x_0,\,x_1,\,x_2)$ of $\mathbb{R}^3$ by

$$D=\partial_0+e_1\partial_1+e_2\partial_2, \tag{2.26}$$

where ∂_j is the derivative with respect to the x_j, $j=0,1,2$ variable, and

$$\bar{D}=\partial_0-e_1\partial_1-e_2\partial_2. \tag{2.27}$$

The vector part of the Cauchy-Riemann operator is designated as Dirac operator. It holds that $D\bar{D}=\Delta$ where Δ is the Laplace operator.

Analogous to holomorphic functions we introduce the following definition.

Definition 2.2.1. *Let Ω be a domain in $\mathbb{R}^3$. Any function $f:\Omega\to\mathbb{C}\otimes\mathbb{H}$ is called a monogenic function if $Df=0$ over Ω.*

Moreover, due to the non-commutative nature of quaternions the derivative rule for products of functions does not hold in the same way as in complex or real analysis. In this sense, we introduce a Leibniz rule for quaternionic functions.

Lemma 2.2.2. *Let $f:\mathbb{R}^3\to\mathbb{C}\times\mathbb{H}$ and $g:\mathbb{R}^3\to\mathbb{C}\times\mathbb{H}$ be two quaternionic-valued differentiable functions.*

Then, the following rule for the derivative of their product holds:

$$D\,(fg)=(Df)\,g-\bar{f}\left(\bar{D}g\right)+2\mathrm{Sc}\;(fD)g, \tag{2.28}$$

with $\mathrm{Sc}\;(fD)=f_0\partial_0-\sum_{k=1}^{2}(f_k\partial_k)$.

We remark that if f is a scalar-valued function one recovers the classical Leibniz's rule

$$D(fg)=(Df)g+f(Dg).$$

A bit of Operator Theory

Let Ω be a bounded domain and $f : \Omega \to \mathbb{C} \otimes \mathbb{H}$. We introduce now the Teoderescu transform in the quaternion, also just called the T-operator, by

$$(Tf)(x) = -\frac{1}{\omega_3} \int_\Omega \frac{\overline{y-x}}{|y-x|^3} f(y)\, dy \quad \text{for } x \in \Omega, \tag{2.29}$$

where $E(x, y) = -\frac{1}{\omega_3} \frac{\overline{y-x}}{|y-x|^3}$ is the generalized Cauchy kernel and $\omega_3 = 4\pi$ stands for the surface area of the unit sphere in $\mathbb{R}^3$. This operator is the right-inverse for the Cauchy-Riemann operator, *i.e.*, for $x \in \Omega$ and $f \in L^p(\Omega)$ it holds that $DTf = f$.

The operator T has the following mapping properties.

Theorem 2.2.3. *Let Ω be a bounded domain in $\mathbb{R}^3$.*

(i) For $1 < p < \infty$ the operator $T : W^{k,p}(\Omega) \to W^{k+1,p}(\Omega)$ is bounded for $k \in \mathbb{N}_0$.

(ii) For $1 < p < 3$ the operator $T : L^p(\Omega) \to L^{p'}(\Omega)$ with $p' < \frac{3p}{3-p}$

Theorem 2.2.4. *Let Ω be a domain in $\mathbb{R}^3$ and let $1 \leq p < 3$. In particular, Ω can be $\mathbb{R}^3$. Then, the operator $T : L^p(\Omega) \to L^{\tilde{p}}(\Omega)$ is a bounded operator for $\tilde{p} = 3p/(3-p)$.*

Proof. The proof of this theorem is immediate from Hardy-Littlewood-Sobolev inequality introduced above in Theorem 2.1.12. □

Furthermore, we introduce the boundary integral operator for $x \notin \partial\Omega$

$$(F_{\partial\Omega} f)(x) = \frac{1}{\omega_3} \int_{\partial\Omega} \frac{\overline{y-x}}{|y-x|^3} \alpha(y) f(y)\, dS(y), \tag{2.30}$$

where $\alpha(y)$ is the outward pointing normal unit vector to $\partial\Omega$ at y and dS is the measure at the boundary.

We get the well-known Borel-Pompeiu formula

$$(F_{\partial\Omega} f)(x) + (TDf)(x) = f(x) \text{ for } x \in \Omega.$$

Obviously, $DF_{\partial\Omega} = 0$ holds through this formula and $F_{\partial\Omega}$ acts from $W^{k-\frac{1}{p},p}(\partial\Omega)$ into $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$ and $1 < p < \infty$.

One of the other well-known results we will need for our work is the Plemelj-Sokhotzki formula is obtained by taking the trace of the boundary integral operator.

To present it, we introduce another operator over the boundary of Ω.

Proposition 2.2.5. *Let $1 < p < \infty$, $k \in \mathbb{N}$ and $f \in W^{k,p}(\partial\Omega)$. Then the integral exists*

$$(S_{\partial\Omega} f) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\overline{y-x}}{|y-x|^3} \alpha(y) f(y)\, dS(y) \tag{2.31}$$

for all points $x \in \Omega$ in the sense of Cauchy principal value. Furthermore, the operator $S_{\partial\Omega}$ is bounded in $W^{k,p}(\partial\Omega)$.

The Plemelj-Sokhotzki formula is now given as follows.

Theorem 2.2.6. *Let $1 < p < \infty$, $k \in \mathbb{N}$ and $f \in W^{k,p}(\partial\Omega)$. By taking by taking the non-tangential limit of $F_{\partial\Omega}$ it holds*

$$\lim_{\substack{x \to x_0,\\ x\in\Omega,\, x_0\in\partial\Omega}} \left(F_{\partial\Omega} f\right)(x) = \frac{1}{2}\left(f(x_0) + \left(S_{\partial\Omega} f\right)(x_0)\right).$$

One of the corollaries concerns the limit to the boundary acting as a projector. That is,

Corollary 2.2.7. *Let $P_{\partial\Omega}$ denote the projection onto the space of all $\mathbb{H}-$valued functions which may be extended to a monogenic function over the domain Ω.*

Then this projection may be represented as

$$P_{\partial\Omega} = \frac{1}{2}\left(I + S_{\partial\Omega}\right).$$

Chapter 3

Complex Conductivities in 2D with jumps

This chapter focus on the Calderón problem for complex conductivities in two dimensions.

One of the core ideas in this line of work is to use stationary phase method, which uses high oscillatory behavior of exponentials to cancel out in the integral except at the stationary points of the exponential phase. From this, we can reconstruct the conductivity through oscillatory integrals. The method was introduced by Bukhgeim in [16] and is the best known approach to establish uniqueness for complex conductivities in two dimensions.

Our work deepens this method by the introduction of a new family of exponentially growing solutions and the study of respective integral equations. The first version was published in [75]. Here, we present a more in depth version with simplified proofs. These new family of solutions leads us to uniqueness of complex-conductivities which can be smooth enough except for certain curves, where they possess a discontinuity.

3.1 The Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and define $\gamma \in L^\infty(\Omega)$ to be an isotropic complex conductivity $\gamma = \sigma + i\omega\epsilon$, as defined in the Introduction. It is also called admittivity in literature.

The *direct problem* we focus here is given in Eq. (1.4) and we recall it here for simplicity.

Let $f \in H^{1/2}(\partial\Omega)$ be a voltage established at the boundary $\partial\Omega$. Then, we want to find the unique electrical potential $u \in H^1(\Omega)$ that fulfills the *conductivity equation*:

$$\begin{cases} \nabla \cdot (\gamma \nabla u) & = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} & = f. \end{cases} \tag{3.1}$$

Further, recall that uniqueness is guaranteed in $H^1(\Omega)$, as long as $\mathrm{Re}\ \gamma \geq c > 0$. As such, we can define the Dirichlet-to-Neumann map from this solution. Formally, it is defined as:

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f &\mapsto \gamma\frac{\partial u}{\partial n}, \end{aligned} \tag{3.2}$$

where n is the unit outward normal vector to $\partial\Omega$. However, in the strong sense it only holds in cases when we have more regularity of the boundary values.

In any case, the proper weak formulation is given as

$$\langle g, \Lambda_\gamma f\rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = \int_\Omega \gamma\nabla u \cdot \nabla v \, dx, \quad \text{for } v \in H^1(\Omega),\ v|_{\partial\Omega} = g. \tag{3.3}$$

As mentioned in the introduction, in most cases we require stronger assumptions on γ to prove uniqueness of Calderón problem.

Here, we initial consider the that the complex conductivity has a discontinuity along a single closed curve Γ, see Figure 3.1 for an example, but are smooth otherwise. We denote by $\mathcal{D}$ the interior region inside Γ. In the last section of this chapter, we consider the general case with more than one discontinuity curve, which proves the general Theorem 1.2.1.

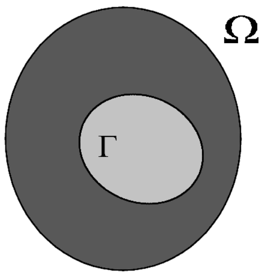


Figure 3.1: Complex conductivity γ which is smooth inside and outside the domains divided by the curve Γ.

Due to this discontinuity the solution u of (3.1) must fulfill the following transmission condition:

$$\begin{cases} u^-(x) - u^+(x) & = 0 \\ \gamma^- \frac{\partial u^-}{\partial n}(x) - \gamma^+ \frac{\partial u^+}{\partial n}(x) & = 0 \end{cases} \qquad \text{for } x \in \Gamma, \tag{3.4}$$

where u^-, u^+ and γ^-, γ^+ represent the traces of these functions at Γ from inside and outside, respectively.

For the sake of presenting the main result of this chapter, we introduce the concept of an *admissible point*. These points are where uniqueness of the conductivity is guaranteed from the Dirichlet-to-Neumann map. Their definition arises from the results required to show uniqueness.

Definition 3.1.1. *We say that a point $w \in \Omega$ is an* ***admissible point*** *if there is a constant $\lambda_\Omega \in \mathbb{C}$ such that*

$$A_w := \sup_{x\in\bar{\Omega}} \operatorname{Re}\left[\lambda_\Omega (x-w)^2\right] < 1/2$$
$$B_w := \sup_{x\in\bar{\mathcal{D}}} \operatorname{Re}\left[\lambda_\Omega (x-w)^2\right] < -1/2$$

Furthermore, if w is an admissible point and $A_w + B_w < 0$ then we say that w is a ***proper admissible point****.*

These points are essential to build the full uniqueness proof, as they allow to introduce a new set of exponentially growing solutions and to define a reconstruction formula based on them. With this in mind, we present one of the main results in this chapter which holds for one discontinuity curve. Later on, we explain how to generalize the proof so that the conductivities can possess more than one discontinuity curve.

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\Gamma \subset \Omega$ be a closed Lipschitz curve, with interior denoted by $\mathcal{D}$. Further, let $\gamma \in W^{2,\infty}(\mathcal{D}) \cup W^{2,\infty}(\Omega\setminus\bar{\mathcal{D}})$ be a complex-valued conductivity such that $\operatorname{Re}\gamma \geq c > 0$.*

Then, if $\sqrt{\frac{\gamma^-}{\gamma^+}} - 1$ is small enough on $L^\infty(\Gamma)$ we have that the Dirichlet-to-Neumann map Λ_γ uniquely determines the conductivity γ in any proper admissible point.

We have to remark that this only gives uniqueness in the proper admissible points, but this approach allows to overcome the current limitation of Lipschitz conductivities in the current literature. Further, while the condition $\sqrt{\frac{\gamma^-}{\gamma^+}} - 1$ being small means that the jump of γ should be small, it is a technical restriction.

3.2 The relation with Dirac equation

One of the essential steps in solving the Calderón problem has been the conversion into another similar inverse problem. In what follows we use complex analysis to pass our problem into an **Inverse Dirac problem**. In this new setting it is easier to prove uniqueness since the important parameter is not affected by any derivatives. Thereafter, we can use this result to prove uniqueness for the Calderón problem.

Hereby, we identify any point $x \in \mathbb{R}^2$ by its respective complex number $z := x_1 + ix_2 \in \mathbb{C}$. Assume, we know the complex conductivity γ and let $u \in H^1(\Omega)$ be the unique solution to (3.1) for a given voltage at the boundary. We define the vector function $\phi := (\phi_1, \phi_2) = \gamma^{1/2}\left(\partial u, \bar{\partial} u\right)^t$

and compute $\left(\bar\partial\phi_1, \partial\phi_2\right)$ to obtain the following system of equations

$$\begin{cases} \bar\partial\phi_1 & = \bar\partial\left(\gamma^{1/2}\partial u\right) \\ \partial\phi_2 & = \partial\left(\gamma^{1/2}\bar\partial u\right) \end{cases} \Leftrightarrow \begin{cases} \bar\partial\phi_1 & = \left(\bar\partial\gamma^{1/2}\right)\partial u + \gamma^{1/2}\left(\bar\partial\partial u\right) \\ \partial\phi_2 & = \left(\partial\gamma^{1/2}\right)\bar\partial u + \gamma^{1/2}\left(\partial\bar\partial u\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} \bar\partial\phi_1 & = \frac{1}{2}\gamma^{-1/2}\bar\partial\gamma\partial u + \frac{1}{4}\gamma^{1/2}\Delta u \\ \partial\phi_2 & = \frac{1}{2}\gamma^{-1/2}\partial\gamma\bar\partial u + \frac{1}{4}\gamma^{1/2}\Delta u \end{cases} \Leftrightarrow \begin{cases} \bar\partial\phi_1 & = \frac{1}{2}\gamma^{-1/2}\bar\partial\gamma\partial u - \frac{1}{4}\gamma^{-1/2}\nabla\gamma\cdot\nabla u \\ \partial\phi_2 & = \frac{1}{2}\gamma^{-1/2}\partial\gamma\bar\partial u - \frac{1}{4}\gamma^{-1/2}\nabla\gamma\cdot\nabla u. \end{cases}$$

The last step follows due to u being a solution of the conductivity equation $\nabla\cdot(\gamma\nabla u) = 0$, which by expansion leads to $\gamma^{1/2}\Delta u = -\gamma^{-1/2}\nabla\gamma\cdot\nabla u$.

Further notice that $\nabla\gamma\cdot\nabla u = 2\left(\bar\partial\gamma\partial u + \partial\gamma\bar\partial u\right)$. Substituting and making the adequate subtraction of the other terms leaves us with

$$\begin{cases} \bar\partial\phi_1 & = -\frac{1}{2}\gamma^{-1/2}\partial\gamma\bar\partial u \\ \partial\phi_2 & = -\frac{1}{2}\gamma^{-1/2}\bar\partial\gamma\partial u \end{cases} \Leftrightarrow \begin{cases} \bar\partial\phi_1 & = -\frac{1}{2}\frac{\partial\gamma}{\gamma}\phi_2 \\ \partial\phi_2 & = -\frac{1}{2}\frac{\bar\partial\gamma}{\gamma}\phi_1 \end{cases}$$

Recall that any solution u of (3.1) will be defined on $\Omega\setminus\Gamma$ with a transmission condition coupling the function inside and outside the curve.

Therefore from u we can determine a solution $\phi := \gamma^{1/2}\left(\partial u, \bar\partial u\right)^t$ to the Dirac system

$$\begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix}\phi(z) = q(z)\phi(z), \quad z\in\Omega\setminus\Gamma. \tag{3.5}$$

where q is a **potential** and defined also on $\Omega\setminus\Gamma$ by

$$q(z) = \begin{pmatrix} 0 & q_{12}(z) \\ q_{21}(z) & 0 \end{pmatrix}, \quad q_{12} = -\frac{1}{2}\frac{\partial\gamma}{\gamma}, \quad q_{21} = -\frac{1}{2}\frac{\bar\partial\gamma}{\gamma}. \tag{3.6}$$

The usual treatment of Calderón problems is to study the transformed equation over the whole space $\mathbb{C}\setminus\Gamma$, which allows us to attribute non-physical asymptotic behavior to our solutions. First, we assume that $\gamma\equiv 1$ near $\partial\Omega$ and further extend it by $\gamma = 1$ outside Ω, that is, over $\mathbb{C}\setminus\Omega$. The fact that we are assuming $\gamma = 1$ near the boundary is not restrictive, since we are able to enlarge the domain with this condition and the problems will be related, see section 2 of [8] for a well described explanation of this construction.

Our study focuses on solutions ϕ to the system of equations (3.5) in $\mathbb{C}\setminus\Gamma$ and on the unique determination of the potential q from boundary data.

Additionally to the system of equations, we still need a missing piece that ϕ must fulfill to complete the connection. Since u satisfies a transmission condition in Γ we must also define one for ϕ.

Let $n(z) = (n_x(z), n_y(z))$ denote the unit outer normal vector in Γ, given as $\nu(z) = n_x(z) + i\,n_y(z)$ in $\mathbb{C}$. Then, we relate the transmission condition (3.4) to solutions of Dirac system through next lemma.

Lemma 3.2.1. *The transmission condition (3.4) takes the form*

$$\begin{pmatrix} \phi_1^+ - \phi_1^- \\ \phi_2^+ - \phi_2^+ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha + \frac{1}{\alpha} - 2 & (\alpha - \frac{1}{\alpha})\bar{\nu}^2 \\ (\alpha - \frac{1}{\alpha})\nu^2 & \alpha + \frac{1}{\alpha} - 2 \end{pmatrix} \begin{pmatrix} \phi_1^- \\ \phi_2^- \end{pmatrix} \tag{3.7}$$

where $\alpha = \sqrt{\frac{\gamma^-}{\gamma^+}}$.

Proof. Let $\tau(z) = (-n_y(z), n_x(z))$ be a unit tangent vector to Γ. Applying the tangential derivative to the first condition $u^+ - u^- = 0$ of (3.4) we obtain

$$\frac{\partial}{\partial \tau}(u^+(z) - u^-(z)) = 0.$$

Multiplying this equation by $\sqrt{\gamma^+}$ leads to $\sqrt{\gamma^+}u_\tau^+ = \sqrt{\gamma^+}u_\tau^-$, where $u_\tau := \frac{\partial u}{\partial \tau}$. Subtracting on both sides $\sqrt{\gamma^-}u_\tau^-$ we get

$$\begin{aligned} \sqrt{\gamma^+}u_\tau^+ - \sqrt{\gamma^-}u_\tau^- &= \left(\sqrt{\gamma^+} - \sqrt{\gamma^-}\right) u_\tau^- \\ &= \sqrt{\gamma^-}\left(\sqrt{\frac{\gamma^+}{\gamma^-}} - 1\right) u_\tau^- = \sqrt{\gamma^-}\left(\frac{1}{\alpha} - 1\right) u_\tau^-. \end{aligned}$$

Let us denote as well the normal derivative by $u_n := \frac{\partial u}{\partial n}$ for ease of notation. Immediately, we get from the second condition of (3.4) that $u_n^+ = \frac{\gamma^-}{\gamma^+}u_n^-$. Hence, we find

$$\begin{aligned} \sqrt{\gamma^+}u_n^+ - \sqrt{\gamma^-}u_n^- &= \sqrt{\gamma^+}\left(\frac{\gamma^-}{\gamma^+}\right) u_n^- - \sqrt{\gamma^-}u_n^- \\ &= \left(\frac{\gamma^-}{\sqrt{\gamma^+}} - \sqrt{\gamma^-}\right) u_n^- = \sqrt{\gamma^-}(\alpha - 1)u_n^-. \end{aligned}$$

Combining the relations between complex derivatives, normal and tangential derivatives given by

$$\partial u = \frac{1}{2}\left(\bar{\nu}u_n - i\bar{\nu}u_\tau\right) \tag{3.8}$$

$$\bar{\partial} u = \frac{1}{2}\left(\nu u_n + i\nu u_\tau\right) \tag{3.9}$$

we can establish a relation for $\phi_i^+ - \phi_i^-$, $i = 1, 2$ together with conditions derived above. We obtain

$$\begin{aligned} \phi_1^+ - \phi_1^- &= \sqrt{\gamma^+}\partial u^+ - \sqrt{\gamma^-}\partial u^- \\ &= \frac{\bar{\nu}}{2}\left[\sqrt{\gamma^+}u_n^+ - \sqrt{\gamma^-}u_n^-\right] - \frac{i\bar{\nu}}{2}\left[\sqrt{\gamma^+}u_\tau^+ - \sqrt{\gamma^-}u_\tau^-\right] \\ &= \frac{\bar{\nu}}{2}\sqrt{\gamma^-}(\alpha - 1)u_n^- - \frac{i\bar{\nu}}{2}\sqrt{\gamma^-}\left(\frac{1}{\alpha} - 1\right) u_\tau^- \end{aligned}$$

and

$$\begin{aligned}\phi_2^+ - \phi_2^- &= \sqrt{\gamma^+}\bar\partial u^+ - \sqrt{\gamma^-}\bar\partial u^- \\ &= \frac{\nu}{2}\left[\sqrt{\gamma^+}u_n^+ - \sqrt{\gamma^-}u_n^-\right] + \frac{i\nu}{2}\left[\sqrt{\gamma^+}u_\tau^+ - \sqrt{\gamma^-}u_\tau^-\right] \\ &= \frac{\nu}{2}\sqrt{\gamma^-}(\alpha-1)u_n^- + \frac{i\nu}{2}\sqrt{\gamma^-}\left(\frac{1}{\alpha}-1\right)u_\tau^-\end{aligned}$$

These relations take the matrix form

$$\begin{pmatrix}\phi_1^+ - \phi_1^- \\ \phi_2^+ - \phi_2^-\end{pmatrix} = \frac{1}{2}\begin{pmatrix}(\alpha-1)\,\bar\nu & \left(\frac{1}{\alpha}-1\right)(-i\bar\nu) \\ (\alpha-1)\,\nu & \left(\frac{1}{\alpha}-1\right)(i\nu)\end{pmatrix}\begin{pmatrix}\sqrt{\gamma^-}u_n^- \\ \sqrt{\gamma^-}u_\tau^-\end{pmatrix}. \tag{3.10}$$

To finalize, we need to convert the right-hand side vector. This easily follows by conditions (3.8) and (3.9). Multiplying by $\sqrt{\gamma}$ both equations, writing them in matrix form and taking the trace from inside, gives:

$$\begin{pmatrix}\sqrt{\gamma^-}\partial u^- \\ \sqrt{\gamma^-}\bar\partial u^-\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\bar\nu & -i\bar\nu \\ \nu & i\nu\end{pmatrix}\begin{pmatrix}\sqrt{\gamma^-}u_n^- \\ \sqrt{\gamma^-}u_\tau^-\end{pmatrix}.$$

By definition of ϕ through u, the left-hand side is exactly ϕ. Inverting the two-by-two matrix with unit outer normal ν, we obtain

$$\begin{pmatrix}\sqrt{\gamma^-}u_n^- \\ \sqrt{\gamma^-}u_\tau^-\end{pmatrix} = \begin{pmatrix}\nu & \bar\nu \\ i\nu & -i\bar\nu\end{pmatrix}\begin{pmatrix}\phi_1^- \\ \phi_2^-\end{pmatrix}.$$

Joining this expression with (3.10) leads to the desired result. □

We are now ready to explore solutions of Dirac system that satisfy the equation (3.5) and transmission condition (3.7). For such, we construct a new set of exponential growing solutions that allows reconstruction of the potential later on.

3.3 Exponentially growing solutions

The second piece of the puzzle in Calderón problem uniqueness proofs has been exponential growing solutions. In our case, we start from typical Bukhgeim exponential functions, also used in [6, 8, 47, 57, 71], to name a few.

For this purpose, we consider solutions ϕ have the following asymptotic behavior:

$$\begin{cases}\phi_1(z,w,\lambda) & \to e^{i\lambda(z-w)^2/2}\left[U(z,w,\lambda)+o(1)\right] \\ \phi_2(x,w,\lambda) & \to e^{-i\overline{\lambda(z-w)^2}/2}o(1)\end{cases} \quad \text{as } |z|\to\infty, \tag{3.11}$$

where $z, w \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and U is an entire function in z, that we define later on.

Since these asymptotics complicate the specific study of the equation, we consider the following related functions:

$$\mu_1(z,w,\lambda) = \phi_1(z,w,\lambda)e^{-i\frac{\lambda}{2}(z-w)^2}, \tag{3.12}$$

$$\mu_2(z,w,\lambda) = \phi_2(z,w,\lambda)e^{i\overline{\lambda(z-w)^2}/2}. \tag{3.13}$$

Given that ϕ fulfills the Dirac system (3.5) in $\mathbb{C}\setminus\Gamma$ and the exponential functions $e^{-i\frac{\lambda}{2}(z-w)^2}$, $e^{i\overline{\lambda(z-w)^2}/2}$ are holomorphic and anti-holomorphic, respectively, then by direct substitution we see that μ is a solution to the following system

$$\begin{aligned}\begin{pmatrix}\bar\partial & 0\\ 0 & \partial\end{pmatrix}\mu(z) &= \begin{pmatrix}0 & q_{12}(z)e^{-i\mathrm{Re}(\lambda(z-w)^2)}\\ q_{21}(z)e^{i\mathrm{Re}(\lambda(z-w)^2)} & 0\end{pmatrix}\mu(z)\\ &=: Q_\lambda(z)\mu(z), \quad z\in\Omega\setminus\Gamma. \end{aligned}\tag{3.14}$$

Analogously, we obtain the transmission condition in μ:

$$\begin{aligned}\begin{pmatrix}\mu_1^+ - \mu_1^-\\ \mu_2^+ - \mu_2^-\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}\alpha+\frac{1}{\alpha}-2 & (\alpha-\frac{1}{\alpha})\bar\nu^2 e^{-i\mathrm{Re}(\lambda(z-w)^2)}\\ (\alpha-\frac{1}{\alpha})\nu^2 e^{i\mathrm{Re}(\lambda(z-w)^2)} & \alpha+\frac{1}{\alpha}-2\end{pmatrix}\begin{pmatrix}\mu_1^-\\ \mu_2^-\end{pmatrix}\\ &=: A_\lambda(z)\mu^-, \quad z\in\Gamma.\end{aligned}\tag{3.15}$$

Our interest now resides in solving (3.14) with (3.15) and fulfilling the respective asymptotics

$$(\mu_1,\mu_2)^t \to (U,0)^t \text{ as } |x|\to\infty.$$

To start our study, we determine the corresponding integral equation and define appropriate spaces to establish existence and uniqueness of a solution.

Proposition 3.3.1. *Let μ be a solution of (3.14) satisfying (3.15) and has asymptotics (3.11). Then μ is a solution of the following integral equation:*

$$[I + PA_\lambda - DQ_\lambda]\,\mu = \begin{pmatrix}U\\ 0\end{pmatrix}, \tag{3.16}$$

where I is the identity operator, $D = \begin{pmatrix}\bar\partial^{-1} & 0\\ 0 & \partial^{-1}\end{pmatrix}$ with

$$\bar\partial^{-1}f(z) = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(\xi)}{\xi - z}\,d\sigma_\xi$$

and ∂^{-1} is given through the integral kernel $\overline{(\xi-z)}^{-1}$ and $d\sigma_\xi = d\xi_1 d\xi_2$.

Furthermore, $P = \begin{pmatrix}P^+ & 0\\ 0 & P^-\end{pmatrix}$ is a matrix operator defined by Cauchy projector and its complex adjoint, respectively:

$$P^+f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz, \quad P^-f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{\overline{z-w}}\,d\bar z, \quad w\in\mathbb{C}. \tag{3.17}$$

Proof. The approach we follow for this proof is based on [58]. Let $\mathcal{G}$ be an arbitrary bounded domain with smooth boundary and $f \in C^1(\bar{\mathcal{G}})$. Then Cauchy-Green formulas hold as follows

$$
\begin{aligned}
f(z) &= -\frac{1}{\pi}\int_{\mathcal{G}} \frac{\partial f(\xi)}{\partial \bar{\xi}} \frac{1}{\xi - z}\, d\sigma_\xi + \frac{1}{2\pi i}\int_{\partial\mathcal{G}} \frac{f(\xi)}{\xi - z}\, d\xi, \quad z \in \mathcal{G},\\
0 &= -\frac{1}{\pi}\int_{\mathcal{G}} \frac{\partial f(\xi)}{\partial \bar{\xi}} \frac{1}{\xi - z}\, d\sigma_\xi + \frac{1}{2\pi i}\int_{\partial\mathcal{G}} \frac{f(\xi)}{\xi - z}\, d\xi, \quad z \notin \bar{\mathcal{G}}.
\end{aligned}
$$

Now, for each $z \in \mathcal{G}$ we denote by D_R a disk of radius R and centered at z, and take $D_R^- = D_R \setminus \bar{\mathcal{D}}$. We recall that $\mathcal{D}$ is the interior part of Γ.

Without loss of generality, we assume that $z \in \mathcal{D}$ and $f = \mu_1$ in both formulae. Further, taking $\mathcal{G} = \mathcal{D}$ in the first formula, $\mathcal{G} = D_R^-$ in the second and adding them up together we obtain

$$
\begin{aligned}
\mu_1(z,\lambda) &= \frac{1}{2\pi i}\int_{D_R\setminus\Gamma} \frac{\partial \mu_1(\xi)}{\partial \bar{\xi}} \frac{1}{\xi - z}\, d\sigma_\xi + \frac{1}{2\pi i}\int_{\Gamma} \frac{\mu_1^-(\xi)}{\xi - z}\, d\xi + \frac{1}{2\pi i}\int_{\partial D_R\setminus\mathcal{D}} \frac{\mu_1^+(\xi)}{\xi - z}\, d\xi\\
&= \frac{1}{2\pi i}\int_{D_R\setminus\Gamma} \left(Q_\lambda \mu\right)_1(\xi) \frac{1}{\xi - z}\, d\sigma_\xi + \frac{1}{2\pi i}\int_{\Gamma} \frac{[\mu_1](\xi)}{\xi - z}\, d\xi + \frac{1}{2\pi i}\int_{\partial D_R} \frac{\mu_1^+(\xi)}{\xi - z}\, d\xi
\end{aligned}
$$

since μ_1 is a solution to (3.14) and $[\mu_1] = \mu_1^- - \mu_1^+$.

Noticing, that μ_1 converges at infinity to an entire function U and using Cauchy integral formula $U(z) = \frac{1}{2\pi i}\int_{\partial D_R} \frac{U(\xi)}{\xi - z}\, d\xi$, we obtain by taking the limit $R \to \infty$

$$
\mu_1(z,\lambda) - \frac{1}{2\pi i}\int_{\mathbb{C}} \left(Q_\lambda \mu\right)_1(\xi) \frac{1}{\xi - z}\, d\sigma_\xi + \frac{1}{2\pi i}\int_{\Gamma} \frac{\left(A_\lambda \mu\right)_1(\xi)}{\xi - z}\, d\xi = U(z).
$$

The equation also holds for $z \in D_R^-$, with analogous computations. The case for μ_2 follows analogously through adjoint Cauchy-Green formulas and 0 being a trivial entire function. □

3.4 Function spaces

To study our resulting integral equations we need to establish appropriate function spaces. In other works with an approach based on the Dirac system, like Brown and Uhlmann [15] and Lakshtanov, Tejero and Vainberg [57], the used spaces are mixed in z, λ of the form:

$$
L_z^\infty(L_\lambda^q).
$$

These spaces are essential to show a contraction argument for the corresponding integral operators. Actually, in the above work, the integral equation is based on $I - DQ_\lambda$. Therefore, this is our starting point to define the appropriate space, which allows us to reuse some arguments presented in those papers.

Since the transmission condition adds a projector to our integral equation, it is essential to have a space that deals with it. Given we are dealing with a Cauchy integral and the respective projections, there is a very well understood interplay between them and Hardy spaces, see for example [27]. In this sense, we combine the above mixed space with another one coming from Hardy spaces.

Let us introduce our desired spaces and show some of the interesting properties!

Let $L_z^\infty(B)$ be the space of bounded functions in $z \in \mathbb{C}$ with values in a Banach space B. Thus, picking $B = L_\lambda^p(|\lambda| > R)$ we introduce the first space as

$$\mathcal{H}_1^p := \left\{ f = (f_1, f_2)^t : f_j \in L_z^\infty \left(L_\lambda^p(|\lambda| > R) \right) \text{ and continuous in } z \right\} \tag{3.18}$$

To simplify notation ahead, we introduce the following function space:

$$S := \left\{ g : \Gamma \times \{\lambda \in \mathbb{R} : \lambda > R\} \to \mathbb{C} \text{ s.t. } \sum_{i=1}^{2} \int_{|\lambda|>R} \int_\Gamma |g_i(z, \lambda)|^p \, d|z| d\sigma_\lambda \right\}.$$

Now, we define our second space through Cauchy projector P by

$$\mathcal{H}_2^p := \left\{ F \in \mathcal{R}(P) : \sum_{i=1}^{2} \int_{|\lambda|>R} \int_\Gamma |F_i^-(z, \lambda)|^p \, d|z| d\sigma_\lambda \right\},$$

where $d\sigma_\lambda = d\lambda_1 d\lambda_2$ and $\mathcal{R}(P)$ is the range of the matrix operator P with domain S. To be clear, this means that if $F \in \mathcal{H}_2^p$ there exists a function $f \in S$ such that $F = Pf$ and it fulfills $F^- = f$. Moreover, we endow this space with the norm

$$\|F\|_{\mathcal{H}_2^p}^p := \sum_{i=1}^{2} \int_{|\lambda|>R} \int_\Gamma |F_i^-(z, \lambda)|^p \, d|z| d\sigma_\lambda = \sum_{i=1}^{2} \int_{|\lambda|>R} \int_\Gamma |f_i(z, \lambda)|^p \, d|z| d\sigma_\lambda.$$

Finally, our space is given as

$$\mathcal{H}^p = \mathcal{H}_1^p + \mathcal{H}_2^p$$

and endowed with the norm

$$\|t\|_{\mathcal{H}^p} = \inf_{\substack{u+v=t \\ u \in \mathcal{H}_1^p, v \in \mathcal{H}_2^p}} \max \left(\|u\|_{\mathcal{H}_1^p}, \|v\|_{\mathcal{H}_2^p} \right). \tag{3.19}$$

We remark that intersection and union of two Banach spaces are correctly defined if all terms can be continuously embedded into a common locally convex space, see [17] and [60]. In our situation this common locally convex space is endowed with semi-norms

$$\int_{|\lambda|>R} \int_{\mathcal{G}} \frac{1}{|\lambda|^2} |f(z, \lambda)| \, d\sigma_z d\sigma_\lambda,$$

where $\mathcal{G}$ is an arbitrary domain and $d\sigma_z = dxdy$.

If $f \in \mathcal{H}_1^p$ the embedding is evident. For $f \in \mathcal{H}_2^p$ we have

$$\|Pf\|_{L^p(\mathcal{G})} \le C \|f\|_{L^p(\Gamma)},$$

so that $\left[\|Pf\|_{L^p(\mathcal{G})} \right]^p \le \left[\|f\|_{L^p(\Gamma)} \right]^p$ and

$$\int \left(\int_{\mathcal{G}} |Pf(z)|^p \, d\sigma_z \right) d\sigma_\lambda \le \int [\|f\|_{L^p(\Gamma)}]^p \, d\sigma_\lambda = \int_{|\lambda|>R} \int_\Gamma |f(z, \lambda)|^p \, d|z| d\sigma_\lambda.$$

The boundedness of each semi-norm follows from continuity of embedding $L^p(\mathcal{G})$ into $L^1(\mathcal{G})$.

Lemma 3.4.1. *The operators $\widehat{P}_{\pm} : f \to (Pf)|_{\Gamma^{\pm}}$ are bounded in the space with norm*

$$\left[\int_R^\infty \int_\Gamma |f(z,\lambda)|^p \, d|z| d\sigma_\lambda\right]^{1/p}.$$

Proof. During the proof, we omit the sign $\pm$ in the projectors. From continuity of Cauchy projectors in $L^p(\Gamma)$ it follows

$$\|\widehat{P}f\|_{L^p(\Gamma)} \leq C\|f\|_{L^p(\Gamma)}.$$

and therefore

$$\left(\|\widehat{P}f\|_{L^p(\Gamma)}\right)^p \leq C^p \left(\|f\|_{L^p(\Gamma)}\right)^p.$$

Finally

$$\begin{aligned}
\|P\widehat{P}f\|_{\mathcal{H}_2^p} &= \int_{|\lambda|>R} \left(\|\widehat{P}f\|_{L^p(\Gamma)}\right)^p \, d\sigma_\lambda \\
&\leq C^p \int_{|\lambda|>R} \left(\|f\|_{L^p(\Gamma)}\right)^p \, d\sigma_\lambda = C^p \int_{|\lambda|>R} \int_\Gamma |f(z,\lambda)|^p \, d|z| d\sigma_\lambda.
\end{aligned}$$

□

Lemma 3.4.2. *Let $u \in \mathcal{H}_1^p$. Then $P(u|_\Gamma) \in \mathcal{H}_2^p$.*

Proof. From the definition of $\mathcal{H}_1^p$, combined with u being a continuous function, we get

$$\|u\|_{L^p_\lambda} \in L^\infty_z(\Gamma).$$

Since Γ is a bounded set, the L^p norm is bounded by the L^∞ norm and, therefore

$$\|\|u\|_{L^p_\lambda}\|_{L^p_z(\Gamma)} \leq C\|u\|_{\mathcal{H}_1^p}.$$

To finish, note that the left-hand side is $\mathcal{H}_2^p$ norm. □

Using the space $\mathcal{H}^p$ we now study existence and uniqueness of solutions to the integral equation (3.16).

3.5 Study of the integral equation

In its present form, we are not able to tackle the integral equation (3.16). Brown and Uhlmann used a simple but efficient trick in [15] to transform it into another integral equation. This trick was later adapted in [57] for the case of exponential functions with quadratic phase. In our work, we take it a step further, since extra terms appear due to the projection operators.

This simple trick is based on multiplying our integral equation by $(I + DQ_\lambda)$, which leads to:

$$(I + DQ_\lambda)(I - DQ_\lambda + PA_\lambda)\mu = (I + DQ_\lambda)\begin{pmatrix} U \\ 0 \end{pmatrix}$$

$$\Leftrightarrow [I - DQ_\lambda DQ_\lambda + DQ_\lambda PA_\lambda + PA_\lambda]\mu = (I + DQ_\lambda)\begin{pmatrix} U \\ 0 \end{pmatrix}.$$

Basically it cancels out the single DQ_λ term of previous equation, since even though bounded it is harder to verify the decay in terms of λ parameter.

By setting $M = PA_\lambda - DQ_\lambda DQ_\lambda + DQ_\lambda PA_\lambda$, we can re-writhe the integral equation as:

$$(I + M)\,\mu = (I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix}. \tag{3.20}$$

The existence and uniqueness of solutions to this integral equation follows in two steps: first we show that M is a contraction in $\mathcal{H}^p$ for $R > 0$ large enough and small jumps; thereafter, we show $(I + DQ_\lambda)$ is in $\mathcal{H}^p$, in some sense. The latter means we need to add an extra term to control the behavior at infinity, which is part of the reason why we introduce a new set of **admissible points**.

Lemma 3.5.1. *Let* $\operatorname{Re}\gamma \geq c > 0$, $p > 1$ *and* $R > 0$. *Further, let the matrix operator* A_λ *be in* $L^\infty(\Gamma)$. *Then,*

1. $DQ_\lambda PA_\lambda$ *and* $DQ_\lambda DQ_\lambda$ *are bounded operators in* $\mathcal{H}^p$*;*
2. *There, exists an* $R > 0$ *large enough, such that for all* $|\lambda| > R$ *the operators* $DQ_\lambda PA_\lambda$ *and* $DQ_\lambda DQ_\lambda$ *are contractions.*
3. *If the jump* $\alpha = \sqrt{\frac{\gamma^-}{\gamma^+}}$ *is close to* 1 *in* $L^\infty(\Gamma)$*, then* PA_λ *is a contraction in* $\mathcal{H}^p$*.*

Proof. In order to estimate $\|(DQ_\lambda PA_\lambda)t\|_{\mathcal{H}^p}$ and $\|(DQ_\lambda DQ_\lambda)t\|_{\mathcal{H}^p}$ (recall Definition 3.19) we consider the representation $t = u + v$ where infimum is (almost) achieved. It is easy to see that the desired estimate follows from these operators being a contraction in each of the spaces, $\mathcal{H}_1^p$ and $\mathcal{H}_2^p$. This fact can be shown as follows.

In Lemma 2.1 of [57] it was proved that $DQ_\lambda DQ_\lambda$ is bounded in $\mathcal{H}_1^p$. The proof that it is also a contraction in $\mathcal{H}_2^p$ and the statement for $DQ_\lambda PA_\lambda$ follows in a similar manner.

Here, we show it for $DQ_\lambda PA_\lambda$. By definition we have:

$$DQ_\lambda PA_\lambda u(z) = \begin{cases} \int_\Gamma [A_\lambda u]_2(z_2) G_1(z, z_2, \lambda, w)\, dz_2 \\ \int_\Gamma [A_\lambda u]_1(z_2) G_2(z, z_2, \lambda, w)\, dz_2 \end{cases}$$

where

$$G(z, z_2, \lambda, w) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{cases} (2\pi i)^{-2} \int_\Omega \frac{e^{-i\operatorname{Re}(\lambda(z_1 - w)^2)/2}}{z_1 - z} \frac{Q_{12}(z_1)}{\bar{z}_2 - \bar{z}_1}\, d\sigma_{z_1} \\ (2\pi i)^{-2} \int_\Omega \frac{e^{i\operatorname{Re}(\lambda(z_1 - w)^2)/2}}{\bar{z}_1 - \bar{z}} \frac{Q_{21}(z_1)}{z_2 - z_1}\, d\sigma_{z_1} \end{cases}. \tag{3.21}$$

By following a similar estimation to proof of Lemma 2.1 as in [57], we obtain by the stationary phase approximation:

$$\sup_{\substack{z \\ |\lambda| > R}} \|G_i(z, \cdot, \lambda, w)\|_{L^q_{z_2}(\Gamma)} \leq \frac{1}{R}, \qquad 1/p + 1/q = 1 \text{ and } i = 1, 2.$$

Thus

$$|DQ_\lambda PA_\lambda u|(z) \leq \|G(z, \cdot, \lambda, w)\|_{L^q_{z_2}(\Gamma)} \|A_\lambda u\|_{L^p_{z_2}(\Gamma)}.$$

Then, we have for

$$\|DQ_\lambda PA_\lambda u(z)\|_{L^p_\lambda} \leq \|G(z,\cdot,\lambda,w)\|_{L^q_{z_2}(\Gamma)}\|A_\lambda\|_{L^\infty(\Gamma)}\|u\|_{\mathcal{H}^p_2}$$

where we use that $u \in \mathcal{H}^p_2$ is equal to $\|u\|_{L^p_{z_2}(\Gamma)} \in L^p_\lambda$. The final estimate follows from definitions of both spaces and the above uniform bound on G_i.

If we take $R > 0$ large enough then it follows that $DQ_\lambda DQ_\lambda$ and $DQ_\lambda PA_\lambda$ are contractions in $\mathcal{H}^p$ as long as $\|A_\lambda\|_{L^\infty(\Gamma)}$ is finite.

By definition of $\mathcal{H}^p$, boundedness of PA_λ follows from usual L^p boundedness. Since this operator will not have the same dependence on λ as the others we need the jump to be close enough to 1 so that the supremum norm of A_λ in z on Γ is small enough and allows the operator norm to be less than 1.

A rough estimate for this norm is given in terms of the jump by:

$$\|A_\lambda\|_{L^\infty(\Gamma)} \leq 2\,|\alpha - 1| \left(1 + \left|\frac{1}{\alpha}\right|\right) \leq 4\epsilon,$$

where $\epsilon > 0$ is an upper bound for $|\alpha - 1|$. Hence for PA_λ to be a contraction on $\mathcal{H}^p$ we need that

$$|\alpha - 1| \leq \frac{1}{4\|P\|_{\mathcal{H}^p}}.$$

□

3.5.1 Hausdorff-Young type inequality

Before studying the integral equation we introduce the so-called Hausdorff-Young type inequality. This inequality will be used from now on in many proofs. It is a basic estimate initially obtained in [57], that we extend here in order to consider vaster exponentially growing solutions.

We show an auxiliary result to start.

Lemma 3.5.2. *For $1 \leq q < 2$. Then the following estimate is valid for an arbitrary $a \in \mathbb{C}$ and constant $C = C(q, M)$:*

$$\left\|\frac{1}{u(\sqrt{u} - a)}\right\|_{L^q(u:|u|<M)} \leq C|a|^{1-\delta}$$

Proof. To start we apply a change of variables $u = |a|^2 v\, d\sigma_u = |a|^4 d\sigma_v$ to find

$$\begin{aligned}
\left\|\frac{1}{u(\sqrt{u} - a)}\right\|_{L^q(u:|u|<M)} &= \left[\int_{|v|<M/|a|^2} \left|\frac{1}{|a|^2 v\,(|a|\sqrt{v} - |a|\dot{a})}\right|^q |a|^4\, d\sigma_v\right]^{1/q}, \text{ where } \dot{a} = \frac{a}{|a|} \\
&= \left[\int_{|v|<M/|a|^2} |a|^{4-3q} \left|\frac{1}{v\,(\sqrt{v} - a)}\right|^q d\sigma_v\right]^{1/q} \\
&= |a|^{\frac{4}{q}-3} \left[\int_{|v|<M/|a|^2} \left|\frac{1}{|v|^q\,|\sqrt{v} - \dot{a}|^q}\right|^q d\sigma_v\right]^{1/q}.
\end{aligned}$$

Without loss of generality we assume that $M > 2$ and split the proof into two cases. In the case $|a| \geq 1$, then $\frac{M}{|a|^2} \leq M$ and the above computation reduces to

$$\left\| \frac{1}{u(\sqrt{u}-a)} \right\|_{L^q(u:|u|<M} = |a|^{\frac{4}{q}-3} \left[\int_{|v|<M} \frac{1}{|v|^q \, |\sqrt{v}-\dot{a}|^q} \, d\sigma_v \right]^{1/q}.$$

Since $\dot{a}$ has norm 1, we can split the integral into two regions where we have a singularity of order $|u|^q$ on one and $|u|^{q/2}$ in the other. Both parts are easily bounded by a constant, since $q, q/2 < 2$ and we are in two dimensions. Therefore, the norm above is bounded by $C|a|^{4/q-3}$.

In the other case, $|a| < 1$ it is almost the same idea, but we just need to understand how the domain with $R/|a|^2$ works. First we split the integral into two regions:

$$\int_{|v|<M/|a|^2} \frac{1}{|v|^q |\sqrt{v}-\dot{a}|^q} \, d\sigma_v = \int_{|v|<M} \frac{1}{|v|^q |\sqrt{v}-\dot{a}|^q} \, d\sigma_v + \int_{M\leq|v|\leq M/|a|^2} \frac{1}{|v|^q |\sqrt{v}-\dot{a}|^q} \, d\sigma_v.$$

The first integral on the right can be bounded by a constant exactly like the previous scenario with $|a| \geq 1$. As such, we only provide an estimate for the second one. With the reverse triangle inequality help we get

$$\begin{aligned}
\int_{M\leq|v|\leq M/|a|^2} \frac{1}{|v|^q |\sqrt{v}-\dot{a}|^q} \, d\sigma_v &\leq \int_{M\leq|v|\leq M/|a|^2} \frac{1}{|v|^q \left||v|^{1/2}-1\right|^q} \, d\sigma_v \\
&\leq \int_{M\leq|v|\leq M/|a|^2} \frac{1}{\left(|v|^{1/2}-1\right)^{2q} \left||v|^{1/2}-1\right|^q} \, d\sigma_v,
\end{aligned}$$

which follows since $2 \leq M \leq |v| \Rightarrow \left||v|^{1/2}-1\right| = |v|^{1/2}-1$ and $\left(|v|^{1/2}\right)^2 \geq \left(|v|^{1/2}-1\right)^2$. Then we only need to estimate

$$\begin{aligned}
\int_{M\leq|v|\leq M/|a|^2} \frac{1}{|v|^q |\sqrt{v}-\dot{a}|^q} \, d\sigma_v &\leq \int_{M\leq|v|\leq M/|a|^2} \frac{1}{\left(|v|^{1/2}-1\right)^{3q}} d\sigma_v \\
&\leq C_\theta \int_M^{M/|a|^2} \frac{r}{\left(r^{1/2}-1\right)^{3q}} \, dr.
\end{aligned}$$

By applying polar change of variables and thereafter setting $t = r^{1/2} - 1$, $dr = 2(t+1)dt$ we have

$$\begin{aligned}
\int_{M\leq|v|\leq M/|a|^2} \frac{1}{|v|^q |\sqrt{v}-\dot{a}|^q} \, d\sigma_v &\leq C_\theta \int_M^{M/|a|^2} \frac{r}{\left(r^{1/2}-1\right)^3 q} \, dr \\
&= C_\theta \int_{\sqrt{M}-1}^{\sqrt{M}/|a|-1} \frac{2(t+1)^3}{t^{3q}} \, dt \\
&\leq C_\theta \int_{\sqrt{M}-1}^{\sqrt{M}/|a|-1} t^{3-3q} \, dt = Ct^{2-3q}\Big|_{\sqrt{M}-1}^{\sqrt{M}/|a|-1} \\
&= \left[\tilde{C} + \tilde{C}(M/|a|-1)^{2-3q} \right] \leq C.
\end{aligned}$$

Due to $1 \leq q < 2$ it holds $2 - 3q < 0$. As such $M - 1 < \frac{M}{|a|} - 1 \Rightarrow \frac{1}{M/|a|-1} < 1/(M-1)$ implies the bound by a constant.

Therefore, we obtain for both cases

$$\left\| \frac{1}{u\,(\sqrt{u}-a)} \right\|_{L^q(|u|<M)} \leq C|a|^{4/q-3} \leq C|a|^{-1+\delta},$$

where $\delta = \frac{4}{q} - 2 > 0$. $\square$

We are ready to introduce an Hausdorff-Young type inequality that takes into account the quadratic phase and the sub-exponential term given by U. A first approach in this sense was obtained in [57] with just the quadratic phase. To control the sub-exponential growth term we divide by $|\lambda|^{A_w}$, and therefore the behavior is analogous to the initial estimate provided by Lakshtanov, Vainberg and Tejero.

Lemma 3.5.3. *Let Ω be a bounded Lipschitz domain and $\phi \in L^\infty(\Omega)$ a function with compact support in Ω. Let $z \in \mathbb{C}$ and $w \in \operatorname{supp}\phi$. Further, let $p > 2$.*

Then, for $\lambda_\Omega \in \mathbb{C} \setminus \{0\}$ with $A_w = \sup_{z\in\bar{\Omega}} \operatorname{Re}(\lambda_\Omega (z-w)^2)$ it holds:

$$\left\| \frac{1}{|\lambda|^{A_w}} \int_{\mathbb{C}} \phi(z) \frac{e^{i\operatorname{Re}(\lambda(z-w)^2)} e^{\ln|\lambda| \lambda_\Omega (z-w)^2}}{z-\zeta}\, d\sigma_z \right\|_{L^p_\lambda(|\lambda|>R)} \leq C_{\Omega,\delta} \frac{\|\phi\|_\infty}{|\zeta - w|^{1-\delta}}, \tag{3.22}$$

with $\delta(p) > 0$ and $C_{\Omega,\delta} > 0$ a constant only depending on Ω, δ.

Proof. We start by defining the internal integral as

$$F(w,\zeta;\lambda) = \int_{\mathbb{C}} \phi(z) \frac{e^{i\operatorname{Re}(\lambda(z-w)^2)} e^{\ln|\lambda|\lambda_\Omega(z-w)^2}}{z-\zeta}\, d\sigma_z.$$

The objective is to first understand the behavior in terms of λ of $F(w,\zeta;\lambda)$ and only later we show the inequality in terms of λ. First, lets apply the change of variables

$$u = (z-w)^2 \quad \text{and} \quad d\sigma_u = 4|z-w|^2\, d\sigma_z, \quad \text{which leads to}$$

$$F(w,\zeta;\lambda) = \sum_{\pm} \frac{1}{4} \int_{\mathbb{C}} \phi(w \pm \sqrt{u}) \frac{e^{i\operatorname{Re}(\lambda u)} e^{\ln|\lambda|\lambda_\Omega u}}{|u|\,[\pm\sqrt{u} - (\zeta - w)]}\, d\sigma_u,$$

where now $\operatorname{Re}(\lambda_\Omega u) \leq A_w$ in $w \pm \sqrt{u} \in \operatorname{supp}\phi$.

Further, we simplify the function we are evaluating like

$$\psi_w^{\pm}(u) = \frac{\phi(w \pm \sqrt{u})}{|u|\,[\pm\sqrt{u} - (\zeta - w)]}$$

with support equal to $\operatorname{supp}\phi$.

With these in mind, we apply another change of variables defined by $\hat{u} = u - u_0$ where u_0 is given as

$$u_0 = \operatorname{argmax}_{w\pm\sqrt{u}\in\operatorname{supp}\psi_w^{\pm}} \operatorname{Re}(\lambda_\Omega u), \quad \text{which implies } \operatorname{Re}\,(\lambda_\Omega \hat{u}) < 0.$$

This allows us to take out of integration the possible growing exponential. As such, we obtain:

$$F(w,\zeta;\lambda) = \sum_{\pm} \frac{1}{4} e^{i\mathrm{Re}(\lambda u_0)} e^{\ln|\lambda|\lambda_\Omega u_0} \int_{\mathbb{C}} \psi_w^{\pm}(\hat{u}+u_0) e^{i\mathrm{Re}(\lambda\hat{u})} e^{\ln|\lambda|\lambda_\Omega \hat{u}}\, d\sigma_{\hat{u}}.$$

To complete, we do yet another change of variables as follows $\tilde{u} = \lambda_\Omega \hat{u} \Rightarrow \mathrm{Re}(\tilde{u}) < 0$ in $\mathrm{supp}\psi_w^{\pm}$ and $\hat{u} = \frac{\bar{\lambda}_\Omega}{|\lambda_\Omega|^2} \Rightarrow d\sigma_{\hat{u}} = \frac{1}{|\lambda|^2}\, d\sigma_{\tilde{u}}$. Applying it we obtain

$$F(w,\zeta;\lambda) = \sum_{\pm} \frac{1}{4|\lambda_\Omega|^2} e^{i\mathrm{Re}(\lambda u_0)} e^{\ln|\lambda|\lambda_\Omega u_0} \int_{\mathbb{C}} \psi_w^{\pm}(\tilde{u}/\lambda_\Omega + u_0) e^{i\mathrm{Re}\left(\tilde{u}\frac{\lambda}{\lambda_\Omega}\right)} e^{\ln|\lambda|\tilde{u}}\, d\sigma_{\hat{u}}.$$

Remark that all changes of variables applied are meant to take out growing terms from the non-complex exponential, which may imply growth.

Notice, that if we set $\tilde{u} = x+iy$ then support of $\psi_w^{\pm}(\tilde{u}/\lambda_\Omega + u_0)$ is a subset of $\{z = x+iy \in \mathbb{C} : x < 0\}$. With this, we now obtain:

$$F(w,\zeta;\lambda) = \sum_{\pm} \frac{1}{4|\lambda_\Omega|^2} e^{i\mathrm{Re}(\lambda u_0)} e^{\ln|\lambda|\lambda_\Omega u_0} \int_{\mathbb{R}} \int_0^\infty \psi_w^{\pm}\left(\frac{x+iy}{\lambda_\Omega} + u_0\right) e^{-z_{\lambda_\Omega}(\lambda)x}\, dx\, e^{i\tilde{z}_{\lambda_\Omega}(\lambda)y}\, dy,$$

where we have established the following definitions:

$$z_{\lambda_\Omega}(\lambda) = \ln|\lambda| + i\frac{\lambda_x}{\lambda_\Omega},$$

that is a monotone curve for $|\lambda| > R$ and is in $\mathbb{C}_+$, in the sense that both real and imaginary terms are monotone functions. Further, we define

$$\tilde{z}_{\lambda_\Omega}(\lambda) = \ln|\lambda| - \frac{\lambda_y}{\lambda_\Omega}.$$

We are ready now to bring back the desired estimate. Given the last expression for $F(w,\zeta;\lambda)$ we get:

$$\begin{aligned}
&\left\| \frac{1}{|\lambda|^{A_w}} \int_{\mathbb{C}} \phi(z) \frac{e^{i\mathrm{Re}(\lambda(z-w)^2)} e^{\ln|\lambda|\lambda_\Omega(z-w)^2}}{z-\zeta}\, d\sigma_z \right\|_{L^p_\lambda(|\lambda|>R)} = \left\| \frac{1}{|\lambda|^{A_w}} F(w,\zeta;\lambda) \right\|_{L^p_\lambda(|\lambda|>R)} \\
&\le \sum_{\pm} \frac{1}{4|\lambda_\Omega|^2} \left[\int_{|\theta|=1} \int_R^\infty \left| \int_{\mathbb{R}} \int_0^\infty \psi_w^{\pm}\left(\frac{x+iy}{\lambda_\Omega} + u_0\right) e^{-z(\tau\theta)x}\, dx\, e^{i\tilde{z}(\tau\theta)y}\, dy \right|^p d\tau d\theta \right]^{1/p} \\
&\le \sum_{\pm} \frac{1}{4|\lambda_\Omega|^2} \int_{|\theta|=1} \int_{\mathbb{R}} \left[\int_R^\infty \left| \int_0^\infty \psi_w^{\pm}\left(\frac{x+iy}{\lambda_\Omega} + u_0\right) e^{-z(\tau\theta)x}\, dx \right|^p d\lambda \right]^{1/p} dy d\theta,
\end{aligned}$$

where we have changed to polar coordinates $\lambda = \tau\theta$ and applied Minkowski integral inequality to switch integrals in θ, τ and y.

The idea is to use a result of Sadov and Merzon that proves an Hausdorff-Young inequality over curves in $\mathbb{C}_+$ that are either convex or monotone, which is exactly our case, see [63] and [80].

As such, given $p>2$ and its Hölder conjugate $\frac{1}{q}=1-\frac{1}{p}$ it follows

$$\begin{aligned}\left\|\frac{1}{|\lambda|^{A_w}}F(w,\zeta;\lambda)\right\|_{L^p_\lambda(|\lambda|>R)} \leq &\sum_{\pm}\frac{1}{4|\lambda_\Omega|^2}\int_{|\theta|=1}\int_{\mathbb{R}}\left[\int_0^\infty\left|\psi\pm_w\left(\frac{x+iy}{\lambda_\Omega}+u_0\right)\right|^q dx\right]^{1/q}dy\\ \leq &C_s\|\psi^\pm_w\|_{L^q(|u|<M)},\end{aligned}$$

since the Hausdorff-Young inequality constant does not depend on the curve, and therefore not on θ. Further, the last inequality arises for M large enough so that $|u|<M$ includes the support of $\psi^\pm_w$ and due to

$$\int_{\mathbb{R}}\left[\int_0^\infty\left|\psi\pm_w\left(\frac{x+iy}{\lambda_\Omega}+u_0\right)\right|^q dx\right]^{1/q}dy\leq\|\psi^\pm_w\|_{L^q(|u|<M)},$$

by applying the Hölder inequality in L^1 with the characteristic function of the support of $\psi^\pm_w$.

Given the previous Lemma 3.5.2, we then obtain our desired inequality:

$$\left\|\frac{1}{|\lambda|^{A_w}}\int_{\mathbb{C}}\phi(z)\frac{e^{i\mathrm{Re}(\lambda(z-w)^2)}e^{\ln|\lambda|\lambda_\Omega(z-w)^2}}{z-\zeta}d\sigma_z\right\|_{L^p_\lambda(|\lambda|>R)}\leq C_{\Omega,\delta}\frac{\|\phi\|_\infty}{|z-w|^{1-\delta}}.$$

□

3.5.2 Enrichment of the set of exponentially growing solutions

As mentioned above the second step to study (3.20) is to show that the right-hand side in (3.20) is also in $\mathcal{H}^p$. However, we have not yet specified which entire function U we choose. The purpose of it is to show how far we can go until we need to define it. In fact, many works can follow from this by taking variants of U and obtain further results.

Given that we already have exponential growth, our approach is to define a term U that is sub-exponentially growing. For this purpose, our choice is given by

$$U(z,w;\lambda)=e^{\ln|\lambda|\cdot\lambda_\Omega(z-w)^2}=|\lambda|^{\lambda_\Omega(z-w)^2},\tag{3.23}$$

where $\lambda_\Omega\in\mathbb{C}$ is fixed with respect to w. In essence, we will only work with functions U such that w is an admissible point with respect to λ_Ω, as established in Definition 3.1.1.

We start by not making any requirements on $\lambda_\Omega\in\mathbb{C}$ and denote for $w\in\Omega$:

$$\begin{aligned}A_w&:=\sup_{z\in\bar\Omega}\ \mathrm{Re}\left(\lambda_\Omega(z-w)^2\right)<\infty\\ B_w&:=\sup_{z\in\bar{\mathcal{D}}}\ \mathrm{Re}\left(\lambda_\Omega(z-w)^2\right)<\infty.\end{aligned}$$

In order for a unique solution μ in $\mathcal{H}^p$ to (3.20) to exist we show the following lemma.

Lemma 3.5.4. *Let $p>2$ and $R>0$ large enough. If $w\in\Omega\setminus\mathbb{C}$ such that there exists $\lambda_\Omega\in\mathbb{C}$ for which $(A_w-B_w)>\frac{2}{p}$, then*

$$\frac{1}{|\lambda|^{A_w}}\left(I+DQ_\lambda\right)\begin{pmatrix}U\\0\end{pmatrix}\in\mathcal{H}^p.$$

Proof. Since the norm in $\mathcal{H}^p$ is given as

$$\|f\|_{\mathcal{H}_p} = \inf_{\substack{u+v=t \\ u\in\mathcal{H}_1^p, v\in\mathcal{H}_2^p}} \max\left[\|u\|_{\mathcal{H}_1^p}, \|v\|_{\mathcal{H}_2^p}\right],$$

one possible partition (u, v) is given by

$$u = 0, \text{ and } v = \frac{1}{|\lambda|^{A_w}}\left(I + DQ_\lambda\right)\begin{pmatrix} U \\ 0 \end{pmatrix}.$$

Therefore, the norm in $\mathcal{H}^p$ is less than the norm in $\mathcal{H}_2^p$ with this split. Thus, we prove that $\frac{1}{|\lambda|^{A_w}}(U, 0)^t \in \mathcal{H}_2^p$ and $\frac{1}{|\lambda|^{A_w}}DQ_\lambda(U, 0)^t \in \mathcal{H}_2^p$. We start with the first.

$$\begin{aligned}
\left\|\frac{1}{|\lambda|^{A_w}}\begin{pmatrix} U \\ 0 \end{pmatrix}\right\|_{\mathcal{H}_2^p} &= \left[\int_{|\lambda|>R}\int_\Gamma \left|\frac{1}{|\lambda|^{A_w}}U(z, w, \lambda)\right|^p d|z| d\sigma_\lambda\right]^{1/p} \\
&= \left[\int_{|\lambda|>R}\int_\Gamma \left|\frac{1}{|\lambda|^{A_w}}e^{\ln|\lambda|\lambda_\Omega(z-w)^2}\right|^p d|z| d\sigma_\lambda\right]^{1/p} \\
&\le \left[\int_{|\lambda|>R}\int_\Gamma \left(\frac{1}{|\lambda|^{A_w}}e^{\ln|\lambda|\mathrm{Re}\left(\lambda_\Omega(z-w)^2\right)}\right)^p d|z| d\sigma_\lambda\right]^{1/p}, \text{ since } |e^{a+bi}| = e^a, \\
&\le \left[\int_{|\lambda|>R}\int_\Gamma \left(\frac{1}{|\lambda|^{A_w}}|\lambda|^{B_w}\right)^p d|z| d\sigma_\lambda\right]^{1/p} \le |\Gamma|\left[\int_{|\lambda|>R}|\lambda|^{(B_w-A_w)p}\, d\sigma_\lambda\right]^{1/p} \\
&\le |\Gamma|\left[\int_R^\infty \tau^{(B_w-A_w)p+1}\, d\tau\right] \le |\Gamma|\left[\frac{\tau^{(B_w-A_w)p+2}}{(B_w-A_w)p+2}\right]\Bigg|_R^\infty \\
&< \infty, \text{ if } (B_w - A_w)p + 2 < 0
\end{aligned}$$

By hypothesis $(A_w - B_w) > 2/p$ the last inequality holds.

Let us look into the second term. Due to matrices D and Q_λ being diagonal and anti-diagonal, respectively, the norm simplifies as

$$\begin{aligned}
\left\|\frac{1}{|\lambda|^{A_w}}DQ_\lambda\begin{pmatrix} U \\ 0 \end{pmatrix}\right\|_{\mathcal{H}_2^p} &= \left\|\frac{1}{|\lambda|^{A_w}}\begin{pmatrix} 0 \\ \partial^{-1}(Q_{21}U) \end{pmatrix}\right\|_{\mathcal{H}_2^p} \\
&\le \left[\int_{|\lambda|>R}\int_\Gamma \left|\frac{1}{|\lambda|^{A_w}}\partial^{-1}(Q_{21}U)(z)\right|^p d|z| d\sigma_\lambda\right]^{1/p},
\end{aligned}$$

where Q_{21} is the respective entry of Q_λ. Therefore, by Fubini theorem it holds

$$\begin{aligned}
&\left\|\frac{1}{|\lambda|^{A_w}}DQ_\lambda\begin{pmatrix} U \\ 0 \end{pmatrix}\right\|_{\mathcal{H}_2^p} \le \left[\int_\Gamma\left(\int_{|\lambda|>R}\left|\frac{1}{|\lambda|^{A_w}}\partial^{-1}(Q_{21}U)(z)\right|^p d\sigma_\lambda\right)^{p/p} d|z|\right]^{1/p} \\
&\le \left[\int_\Gamma\left(\int_{|\lambda|>R}\left|\frac{1}{2\pi i|\lambda|^{A_w}}\int_{\mathbb{C}} e^{\ln|\lambda|\lambda_\Omega(\xi-w)^2}e^{i\mathrm{Re}(\lambda(\xi-w)^2)}\frac{q_{21}(\xi)}{\xi - z}\, d\sigma_\xi\right|^p d\sigma_\lambda\right) d|z|\right]^{1/p} \\
&\le \int_\Gamma C\frac{\|q_{21}\|_\infty}{|z-w|^{1-\delta}}\, d|z| < \infty.
\end{aligned}$$

The last inequality follows by an Hausdorff-Young type inequality on Lemma 3.5.3. $\square$

This result requires

$$A_w - B_w > \frac{2}{p}.$$

We are yet to restrict our conditions to the ones in Definition 3.1.1. The results are established with conditions on A_w, B_w depending on p and only by the end we specify the definition of an admissible point based on all of the conditions.

Together with Lemma 3.5.1 we now show existence and uniqueness of solutions to (3.20).

Proposition 3.5.5. *Let $p > 2$ and $w \in \mathbb{C} \setminus \Gamma$ such that there exists $\lambda_\Omega \in \mathbb{C}$ for which $A_w - B_w > 2/p$. Moreover, let $R > 0$ be large enough and $\alpha - 1 \in L^\infty(\Gamma)$ for Γ a known Lipschitz curve be small enough.*

Then, there exists a unique solution $\tilde{\mu}$ in $\mathcal{H}^p$ to the integral equation:

$$(I + M)\,\tilde{\mu} = \frac{1}{|\lambda|^{A_w}}\,(I + DQ_\lambda)\begin{pmatrix} U \\ 0 \end{pmatrix},$$

where $\tilde{\mu} = \frac{1}{|\lambda|^{A_w}}\mu$ and μ solves (3.20).

Existence and uniqueness allows us to define a key concept for reconstruction methods of the Calderón problem. This key idea is the **scattering transform** and we introduce it for our case.

3.6 Reconstruction from the scattering transform

The importance of admissible points becomes clear in this section. Since the work of Bukhgeim the quadratic phase points w allow application of the stationary phase method to reconstruct potentials over those points.

Our work is not much different and is heavily based on this idea. However, due to the discontinuity a new term is introduced in our scattering transform, when compared with the works [16, 57].

Our scattering transform is defined through an integral over $\partial\Omega$ with the boundary data.

Definition 3.6.1. *Let $\Omega \subset \mathbb{C}$ be a bounded Lipschitz domain and Γ a connected Lipschitz curve inside Ω.*

For $q \in W^{2,\infty}(\mathcal{D}) \cup W^{2,\infty}(\Omega \setminus \bar{\mathcal{D}})$ and μ the corresponding solution to the integral equation (3.20), we define at each admissible point $w \in \mathbb{C} \setminus \Gamma$ the ***scattering transform*** *through*

$$h(\lambda, w) = \int_{\partial\Omega} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}}\mu_2(z)\, d\bar{z}. \tag{3.24}$$

The idea behind any scattering transform is to make a connection between boundary data, in particular the Dirichlet-to-Neumann map, and the parameter inside, now with an emphasis in the potential q. Later on, we describe how to uniquely determine this scattering transform from Λ_γ.

To transform boundary information to inside information, we use Green identities, *i.e.*, for $f \in C^1(\bar{\Omega})$ by

$$\int_{\partial\Omega} f\, d\bar{z} = -2i \int_{\Omega} \partial f\, d\sigma_z.$$

Since our function is defined over two domains with a transmission condition over Γ, we need to take that into account when connecting it with the potential. Initially let us assume $f \in C^1(\bar{\Omega})$ then we get

$$\begin{aligned}
\int_{\partial\Omega} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} f(z)\, d\bar{z} &= -2i \int_{\Omega} \partial\left(e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} f(z)\right) d\sigma_z \\
&= -2i \int_{\mathcal{D}} \partial\left(e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} f(z)\right) d\sigma_z - 2i \int_{\Omega\setminus\overline{\mathcal{D}}} \partial\left(e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} f(z)\right) d\sigma_z \\
&= \int_{\Gamma} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} f(z)\, d\sigma_z - 2i \int_{\Omega\setminus\overline{\mathcal{D}}} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} \partial\left(f(z)\right) d\sigma_z,
\end{aligned}$$

where we have used that $\bar{U}$ is an anti-holomorphic function in $\mathbb{C}$. By a density argument it also holds for $\mu \in \mathcal{H}^p$ and applying this to our scattering data we obtain

$$\begin{aligned}
h(\lambda, w) &= \int_{\Gamma} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} \mu_2(z)\, d\sigma_z + 2i \int_{\Omega\setminus\overline{\mathcal{D}}} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} \partial\left(\mu_2(z)\right) d\sigma_z, \\
&= \int_{\Gamma} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} \mu_2(z)\, d\sigma_z + 2i \int_{\Omega\setminus\overline{\mathcal{D}}} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} q_{21}(z) e^{i\mathrm{Re}(\lambda(z-w)^2)} \mu_1(z)\, d\sigma_z. \quad (3.25)
\end{aligned}$$

The last equation is fundamental to reconstruct the potential. Informally, our plan is to first kill the boundary integral and afterwards obtain the potential q at the admissible points w. Let us proceed in this direction.

To facilitate our presentation, we define the following operator:

$$T_\lambda(G)(w) = 2i \int_{\Omega\setminus\overline{\mathcal{D}}} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}} q_{21}(z) e^{i\mathrm{Re}(\lambda(z-w)^2)} G(z)\, d\sigma_z. \quad (3.26)$$

Furthermore, due to D and Q_λ being diagonal and anti-diagonal matrices, respectively, remark that

$$\left[(I + DQ_\lambda)\begin{pmatrix} U \\ 0 \end{pmatrix}\right]_1 = U.$$

With all this in mind, we are ready to present a reconstruction theorem.

Theorem 3.6.2. *Let Ω be a bounded Lipschitz domain and $\Gamma \subset \Omega$ a closed Lipschitz curve. Let q be a potential function obtained from a complex conductivity $\gamma \in W^{2,\infty}(\mathcal{D}) \cup W^{2,\infty}(\Omega \setminus \bar{\mathcal{D}})$ satisfying* $\mathrm{Re}\,\gamma \geq c > 0$.

If the jump $\alpha - 1$ is small enough in $L^\infty(\Gamma)$ and w is an admissible point in $\Omega \setminus \bar{\mathcal{D}}$, then we can reconstruct the potential in any proper admissible point w through the formula

$$q_{21}(w) = \lim_{R\to\infty} \frac{1}{4\pi \ln 2} \int_{R<|\lambda|<2R} \frac{h(\lambda, w)}{|\lambda|}\, d\sigma_\lambda. \quad (3.27)$$

Observe that this theorem is not equivalent to our main Theorem 3.1.2 in this chapter. The missing link is to obtain the scattering data uniquely from Λ_γ.

To prove this result and obtain a reconstruction formula, we are required to piece various results together which we are going to establish next. Thereafter finalize the proof of the above theorem in a clean manner.

From expression (3.25) we have two integrals. As already mentioned, our approach starts by showing the first integral divided by $|\lambda|$ is in $L^1_\lambda(|\lambda| > R)$ and thereafter we split μ_1 into further terms. Set f as

$$f = \mu - (I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \tag{3.28}$$

where μ solves (3.20). Applying the operator $(I + M)$ to both sides leads to

$$\begin{aligned} (I+M)f &= (I+M)\mu - (I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} - M\,(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \\ &= -M\,(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix}. \end{aligned}$$

Since for $w \in \mathbb{C}$ satisfying $A_w - B_w > 2/p$, we have

$$\frac{1}{|\lambda|^{A_w}}\,(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \in \mathcal{H}^p$$

and M is a bounded operator and even a contraction if $R > 0$ is large enough, the above equation has a unique solution

$$\frac{1}{|\lambda|^{A_w}} f \in \mathcal{H}^p.$$

Therefore we can decompose μ_1 in the term $T_\lambda[\mu_1]$, in order to understand its behaviour as $R \to \infty$ through f, M and U in $\mathcal{H}^p$. Hereby, we get

$$\begin{aligned} \mu_1 &= f_1 + \left[(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \right]_1 = f_1 + U \qquad (3.29) \\ &= -\,[Mf]_1 - \left[M(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \right]_1 + U, \qquad (3.30) \end{aligned}$$

which substituted in $T_\lambda[\mu_1]$ gives

$$T_\lambda\,(\mu_1) = T_\lambda\,([Mf]_1) + T_\lambda \left(\left[M(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \right]_1 \right) + T_\lambda\,(U)\,. \tag{3.31}$$

The proof of Theorem 3.6.2 follows by showing the first two elements are in $L^q_\lambda(|\lambda| > R)$ with $1 \le q < 2$ and the last one reconstructs the potential at the desired admissible points.

3.6.1 Asymptotic behavior of T_λ

Let us start by studying the asymptotic behaviour of T_λ. Hereby, we rely heavily on the Hausdorff-Young type inequality . For the first term in (3.25) we can state the following lemma.

Lemma 3.6.3. *Let $p > 2$ and Γ a closed Lipschitz curve in Ω. For $R > 0$ be large enough and $w \in \mathbb{C} \setminus \Gamma$ such that there exists $\lambda_\Omega \in \mathbb{C}$ fulfilling the condition*

$$A_w + B_w < \frac{2}{p} - 1,$$

we have

$$\frac{1}{|\lambda|} \int_\Gamma e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} \in L^1_\lambda(\lambda : |\lambda| > R). \tag{3.32}$$

Proof. Let μ be a unique solution to (3.20) given by Proposition 3.5.5 for $R > 0$ large enough. Further, we define f through (3.28) such that

$$\begin{aligned} \mu_2 &= f_2 + \left[(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \right]_2 \\ &= f_2 + [DQ_\lambda]_2 = f_2 + \partial^{-1}\left(Q_{21} U\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{|\lambda|>R} \frac{1}{|\lambda|} \left| \int_\Gamma e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} \right| d\sigma_\lambda = \int_{|\lambda|>R} \frac{1}{|\lambda|} \left| \int_\Gamma |\lambda|^{A_w} e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \frac{\mu_2(z)}{|\lambda|^{A_w}}\, d\bar{z} \right| d\sigma_\lambda \\ &\leq \int_{|\lambda|>R} \int_\Gamma \frac{1}{|\lambda|} \left| |\lambda|^{A_w} e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \right| \left| \frac{1}{|\lambda|^{A_w}} \mu_2(z) \right| d|\bar{z}| d\sigma_\lambda \\ &\leq \left[\int_{|\lambda|>R} \int_\Gamma \left| \frac{1}{|\lambda|^{1-(A_w+B_w)}} \right|^q d|\bar{z}| d\sigma_\lambda \right]^{1/q} \left[\int_{|\lambda|>R} \int_\Gamma \left| \frac{1}{|\lambda|^{A_w}} \mu_2(z) \right|^p d|\bar{z}| d\sigma_\lambda \right]^{1/p}, \end{aligned}$$

which follows by Hölder inequality with $p > 2$. Given that $\frac{1}{|\lambda|^{A_w}} \mu_2 \in \mathcal{H}^p$ we obtain

$$\begin{aligned} &\int_{|\lambda|>R} \frac{1}{|\lambda|} \left| \int_\Gamma e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} \right| d\sigma_\lambda \leq |\Gamma| \left\| \frac{1}{|\lambda|^{A_w}} \mu_2 \right\|_{\mathcal{H}^p} \left[\int_{|\lambda|>R} |\lambda|^{(A_w+B_w-1)q}\, d\sigma_\lambda \right]^{1/q} \\ &\leq |\Gamma| \left\| \frac{1}{|\lambda|^{A_w}} \mu_2 \right\|_{\mathcal{H}^p} \left[\int_R^\infty \tau^{1+(A_w+B_w-1)q}\, d\tau \right]^{1/q} \\ &\leq |\Gamma| \left\| \frac{1}{|\lambda|^{A_w}} \mu_2 \right\|_{\mathcal{H}^p} \left[\frac{\tau^{2+q(A_w+B_w-1)}}{2 + q(A_w + B_w - 1)} \right]^{1/q} \Bigg|_R^\infty < \infty, \end{aligned}$$

if $2 + q(A_w + B_w - 1) < 0 \Leftrightarrow A_w + B_w < 2 - 1/q = 2/p - 1$ which follows by hypothesis. $\square$

We remark that the conditions in this Lemma and Lemma 3.5.4 for a point $w \in \mathbb{C} \setminus \Gamma$, given for $p > 2$ as

$$\begin{cases} A_w - B_w > \frac{2}{p} \\ A_w + B_w < \frac{2}{p} - 1 \end{cases} \tag{3.33}$$

define the admissible points.

Further, let us look into the terms containing f in the scattering transform.

Lemma 3.6.4. *Let $q < 2$ and $R > 0$ be large enough. Further, let f be defined as in (3.28).*

Then, for $U(z, w, \lambda)$ defined through $w, \lambda_\Omega \in \mathbb{C}$ with $A_w < 1/2$, it holds

$$\frac{1}{|\lambda|} T_\lambda \left(M(I + DQ_\lambda) \begin{pmatrix} U \\ 0 \end{pmatrix} \right) \in L^q_\lambda(\lambda : |\lambda| > R), \tag{3.34}$$

and

$$\frac{1}{|\lambda|} T_\lambda \left(Mf \right) \in L^q_\lambda(\lambda : |\lambda| > R). \tag{3.35}$$

Proof. Given the structure of $M = P\tilde{A}_\lambda + D\tilde{Q}_\lambda - D\tilde{Q}_\lambda D\tilde{Q}_\lambda$ and that $\frac{1}{|\lambda|}T_\lambda$ is a linear operator, it is enough to show that each term applied to both, $(U, 0)^t$ and $DQ_\lambda(U, 0)^t$, belongs to $L^p_\lambda(\lambda : |\lambda| > R)$.

We look directly into the computations of each term. By using Fubini's Theorem, Minkowski integral inequality, Hölder inequality, and Lemma 3.5.3 we can show that all of these terms are in fact in $L^q_\lambda(\lambda : |\lambda| > R)$. Since the computations for each term follow roughly the same lines, for convenience of the reader, we present just the computation in one of these cases. We look at the term

$$\frac{1}{|\lambda|} T_\lambda \left(DQ_\lambda DQ_\lambda \begin{pmatrix} U \\ 0 \end{pmatrix} \right) \in L^q_\lambda(\lambda : |\lambda| > R).$$

Let us denote $\rho(z) = i\mathrm{Re}[\lambda(z-w)^2] + \ln|\lambda|\lambda_\Omega(z-w)^2$ and $A_w = \sup_{z\in\overline{\Omega}} \mathrm{Re}[\lambda_\Omega(z-w)^2] < 1/2$.

$$\begin{aligned}
&\left\| \frac{1}{|\lambda|} T_\lambda \left(DQ_\lambda DQ_\lambda \begin{pmatrix} U \\ 0 \end{pmatrix} \right) \right\|_{L^q(\lambda:|\lambda|>R)} = \\
&= \Bigg[\int_{|\lambda|>R} \Bigg| \frac{1}{4\pi^2|\lambda|} \int_{\Omega\setminus\overline{\mathcal{D}}} e^{\overline{\rho(z)}} q_{21}(z) \\
&\qquad \cdot \int_\Omega \frac{e^{-i\,\mathrm{Re}[\lambda(z_1-w)^2]}}{z_1 - z} q_{12}(z_1) \int_\Omega \frac{e^{\rho(z_2)}}{z_2 - z_1} q_{21}(z_2)\, d\sigma_{z_2} d\sigma_{z_1} d\sigma_z \Bigg|^q d\sigma_\lambda \Bigg]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\int_{|\lambda|>R}\left|\frac{1}{4\pi^2|\lambda|}\int_\Omega\left(\int_{\Omega\setminus\overline{\mathcal{D}}}\frac{e^{\overline{\rho(z)}}}{z_1-z}q_{21}(z)\,d\sigma_z\right)\left(\int_\Omega\frac{e^{\rho(z_2)}}{\overline{\overline{z_2-z_1}}}q_{21}(z_2)\,d\sigma_{z_2}\right)\right.\right.\\
&\qquad\qquad\qquad\left.\left.\cdot\, q_{12}(z_1)e^{-i\operatorname{Re}[\lambda(z_1-w)^2]}\,d\sigma_{z_1}\right|^q\,d\sigma_\lambda\right]^{1/q}\\
&\le \int_\Omega\left[\int_{|\lambda|>R}\left|\frac{|\lambda|^{2A_w}}{|\lambda|}\left(\frac{1}{|\lambda|^{A_w}}\int_{\Omega\setminus D}\frac{e^{\overline{\rho(z)}}}{z_1-z}q_{21}(z)\,d\sigma_z\right)\right.\right.\\
&\qquad\qquad\left.\left.\cdot\left(\frac{1}{|\lambda|^{A_w}}\int_\Omega\frac{e^{\rho(z_2)}}{\overline{\overline{z_2-z_1}}}q_{21}(z_2)\,d\sigma_{z_2}\right)\right|^q\,d\sigma_\lambda\right]^{1/q}|q_{12}(z_1)|\,d\sigma_{z_1}\\
&\le \|Q\|_{L^\infty}\int_\Omega\left\|\frac{1}{|\lambda|^{A_w}}\int_{\Omega\setminus\overline{\mathcal{D}}}\frac{e^{\overline{\rho(z)}}}{z_1-z}q_{21}(z)\,d\sigma_z\right\|_{L^{2q}(\lambda:|\lambda|>R)}\\
&\qquad\qquad\cdot\left\|\frac{1}{|\lambda|^{A_w}}\int_\Omega\frac{e^{\rho(z_2)}}{\overline{\overline{z_2-z_1}}}q_{21}(z_2)\,d\sigma_{z_2}\right\|_{L^{2q}(\lambda:|\lambda|>R)}d\sigma_{z_1}\\
&\le C\|Q\|_{L^\infty}\int_\Omega\frac{1}{|z_1-w|^{1-\delta}}\frac{1}{|z_1-w|^{1-\delta}}\,d\sigma_{z_1}<\infty.
\end{aligned}
$$

Notice that $2q>2$ and we can use the Hausdorff-Young inequality in Lemma 3.5.3. Thus, these calculations we obtain (3.34).

To show (3.35) we have shown previously that $\frac{1}{|\lambda|^{A_w}}f\in\mathcal{H}^p$ for any $p>2$. We consider T applied to each term of M. Again, we present only the computations for the case $\frac{1}{|\lambda|}T_\lambda\left(D\tilde{Q}_\lambda D\tilde{Q}_\lambda f\right)$, since the other computations follow analogously, with special attention to the behavior of $\frac{1}{|\lambda|^{A_w}}f$. Furthermore, we only present the calculation for the first term of the vector.

$$
\begin{aligned}
&\left[\int_{|\lambda|>R}\left|\frac{1}{|\lambda|}\int_{\Omega\setminus\overline{\mathcal{D}}}e^{\overline{\rho(z)}}q_{21}(z)\int_\Omega\frac{e^{-i\operatorname{Re}(\lambda(z_1-w)^2)}}{z_1-z}q_{12}(z_1)\right.\right.\\
&\qquad\qquad\left.\left.\cdot\int_\Omega\frac{e^{i\operatorname{Re}(\lambda(z-w)^2)}}{\overline{\overline{z_2-z_1}}}q_{21}(z_2)f_1(z_2)\,d\sigma_{z_2}\,d\sigma_{z_1}\,d\sigma_z\right|^q\,d\sigma_\lambda\right]^{1/q}\\
&\le C\|Q\|_\infty^3\int_\Omega\int_\Omega\frac{1}{|z_2-z_1|}\frac{1}{|z_1-w|^{1-\delta}}\left\|\frac{1}{|\lambda|^{A_w}}f_1(z_2)\right\|_{L^{2q}_\lambda}d\sigma_{z_2}\,d\sigma_{z_1}\\
&\le C\|Q\|_\infty^3\left\|\frac{1}{|\lambda|^{A_w}}f_1\right\|_{\mathcal{H}^{2q}}\int_\Omega\int_\Omega\frac{1}{|z_2-z_1|}\frac{1}{|z_1-w|^{1-\delta}}\,d\sigma_{z_2}\,d\sigma_{z_1}<\infty
\end{aligned}
$$

Since $2q>2$ and $\frac{1}{|\lambda|^{A_w}}f\in\mathcal{H}^{2q}$ the norm is finite and boundedness of integrals easily follows. □

3.6.2 Stationary phase method

As we have seen the reconstruction formula is obtained from the asymptotic behavior of $T_\lambda U$. The decaying properties of this integral in terms of λ are a variant of the stationary phase method. From Hörmander's and Stein's books (see [46,85]) it is well-known that the stationary phase method works as long as the exponential phase has negative real part. However, on our case we cannot guarantee such hypothesis.

Starting from unpublished ideas by Hoop, Holman and Uhlmann [45], that we obtained from Chapter drafts available online of a new book we derive the required asymptotics. In essence, due to the presence of an exponential term like $e^{\ln|\lambda|\phi(z,w)}$ we lose some decay, but it still provides enough decay for our desired reconstruction formula.

The original result in [75] was proven directly and without a link to the stationary phase method. As such, decaying properties were only roughly proven but our goal is to obtain some clear understanding of their behaviour.

Therefore our objective is to understand the asymptotic behaviour for integrals of the form

$$I(w,\lambda) = \int e^{i\mathrm{Re}\left(\lambda(z-w)^2\right)} a_{\lambda,w}(z)\, d\sigma_z \tag{3.36}$$

as $|\lambda| \to \infty$.

Intuitively, exponentials with phase $\mathrm{Re}\left(\lambda(z-w)^2\right)$ oscillate rapidly with respect to z for large $|\lambda|$. These oscillations cancel out in the integral except at the stationary points of $\mathrm{Re}\left(\lambda(z-w)^2\right)$, that is, points where the gradient is zero. In our particular case they are $z = w$ for each fixed $w \in \mathbb{C}$.

Some of the proofs can work for rather general $a_{\lambda,w}$ but require more smoothness than we desire. As such, we denote by

$$a_{\lambda,w}(z) = e^{2\ln|\lambda|\mathrm{Re}\left(\lambda_\Omega(z-w)^2\right)}\phi(z), \tag{3.37}$$

and only require $\phi \in W^{1,\infty}(\Omega)$ to be compactly supported in Ω and $w \in \Omega$ to be an admissible point, that is, there exists $\lambda_\Omega \in \mathbb{C}$ such that $A_w = \sup_{z\in\bar{\Omega}} \mathrm{Re}\left(\lambda_\Omega(z-w)^2\right) < \frac{1}{2}$.

Some of the estimates are variants of Hausdorff-Young type inequality. In fact we combine its proof with integration by parts to bring further asymptotically decay. This works particularly well for our $a_{\lambda,w}$ of interest in (3.37).

First and foremost, the following asymptotic behaviour of $a_{\lambda,w}$ holds.

Lemma 3.6.5. *Let $\phi \in W^{1,\infty}(\Omega)$ for Ω a bounded Lipschitz domain with $\operatorname{supp}\phi \subset \Omega$. Further, let $w \in \Omega$ and define for $\lambda_\Omega \in \mathbb{C}$*

$$A_w := \sup_{z\in\bar{\Omega}} \mathrm{Re}\left(\lambda_\Omega(z-w)^2\right). \tag{3.38}$$

Then, for $a_{\lambda,w} = e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)}\phi(z)$ we have

$$\|a_{\lambda,w}\|_{W^{1,\infty}(\Omega)} = O\left(|\lambda|^{2A_w+\delta}\right)$$

with $\delta > 0$ arbitrarily small.

Proof. By applying derivatives the term $\ln|\lambda|$ drops from the exponential. At infinity has behavior $O(|\lambda|^{\delta})$ where $\delta > 0$ can be extremely small.

Furthermore, $e^{2\ln|\lambda|\mathrm{Re}\left(\lambda_\Omega(z-w)^2\right)} = O(|\lambda|^{2A_w})$ by hypothesis. Applying derivatives to $a_{\lambda,w}$ and joining all of these with a compactly supported ϕ in $W^{1,\infty}(\Omega)$ we obtain our desired estimate. □

The first result establishes asymptotically decay on $|\lambda|$ when w is not in the support of ϕ.

Lemma 3.6.6. *Let $a_{\lambda,w} = e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)}\phi(z)$ with $\phi \in W^{1,\infty}(\Omega)$ compactly supported on a bounded Lipschitz domain Ω such that $A_w < 1/2$.*

Suppose that $w \in \Omega$ is not in the support of ϕ. Then, the oscillatory integral

$$I(w,\lambda) := \int e^{i\mathrm{Re}(\lambda(z-w)^2)} a_{\lambda,w}(z)\, d\sigma_z, \tag{3.39}$$

has the following behavior $I(w,\lambda)/|\lambda| \in L^1(\lambda : |\lambda| > R)$.

Proof. Since by hypothesis $w \in \Omega$ is not on the support of $a_{\lambda,w}$, then there exists an $\epsilon > 0$ such that $\operatorname{supp} a_{\lambda,w} \subset \{z \in \Omega |\, |z-w| > \epsilon\}$.

Hence, we can apply integration by parts over $|z-w| > \epsilon$ and since the functions have compact support the boundary integral disappears. As such, we obtain with an interplay of $\lambda = \tau\theta$ and the change of variables $u = (z-w)^2$ the following:

$$\begin{aligned}
I_w(\lambda) &:= \int_{|z-w|>\epsilon} e^{i\tau\mathrm{Re}(\theta(z-w)^2)} e^{\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)}\phi(z)\, d\sigma_z \\
&= \sum_{\pm} \int_{|u|>\sqrt{\epsilon}} e^{i\tau\mathrm{Re}(\theta u)} e^{\ln|\lambda|\mathrm{Re}(\lambda_\Omega u)} \frac{\phi(w \pm \sqrt{u})}{|u|}\, d\sigma_u \\
&= \sum_{\pm} \frac{1}{i\lambda/2 + \ln|\lambda|\lambda_\Omega} \int_{|u|>\sqrt{\epsilon}} e^{i\tau\mathrm{Re}(\theta u)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega u)} \partial_u\left(\frac{\phi(w \pm \sqrt{u})}{|u|}\right) d\sigma_u
\end{aligned}$$

Now, the last derivative is given as follows:

$$\begin{aligned}
\partial_u\left(\frac{\phi(w \pm \sqrt{u})}{|u|}\right) &= \frac{\pm\partial_u\sqrt{u}\,(\partial_z\phi(w \pm \sqrt{u}))\,|u| - \phi(w \pm \sqrt{u})\partial_u|u|}{|u|^2} \\
&= \frac{\pm\frac{1}{2\sqrt{u}}\,(\partial_z\phi(w \pm \sqrt{u}))\,|u| - \phi(w \pm \sqrt{u})\frac{\bar{u}}{2|u|}}{|u|^2} \\
\Rightarrow \left|\partial_u\left(\frac{\phi(w \pm \sqrt{u})}{|u|}\right)\right| &\le \frac{\|\phi\|_{W^{1,\infty}}\chi_{\operatorname{supp}\phi}(u)}{|u|^{3/2}},
\end{aligned}$$

where χ is the indicator function of the support of ϕ. Notice that, we have used the fact that ϕ is a Lipschitz function and w not in its support.

Now, the idea is to use exactly the approach provided in the proof of Lemma 3.5.3 to find our estimate. Define

$$\psi(u) = \partial_u\left(\frac{\phi(w \pm \sqrt{u})}{|u|}\right)\chi_{|u|>\sqrt{\epsilon}}(u).$$

It follows by Hölder inequality for $p > 2$:

$$\int_{|\lambda|>R} \left| \frac{I_w(\lambda)}{|\lambda|} \right| d\sigma_\lambda \leq 2 \int_{|\lambda|>R} \left| \frac{1}{|\lambda|} \frac{1}{|\lambda|} \int_{|u|>\sqrt{\epsilon}} e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \psi(u)\, d\sigma_u \right| d\sigma_\lambda$$
$$\leq 2 \left[\int_{|\lambda|>R} \frac{1}{|\lambda|^{(2-2A_w)q}} d\sigma_\lambda \right]^{1/q} \left\| \frac{1}{|\lambda|^{2A_w}} \int e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \psi(u)\, d\sigma_u \right\|_{L^p(|\lambda|>R)}.$$

Following the proof of the Hausdorff-Young type inequality, we can make a change of variables to cut-off the growth of the exponential term with $|\lambda|^{2A_w}$ in the outside. Thereafter, we can apply Sadov's Hausdorff-Young inequality for convex curves [63], as before, and obtain:

$$\left\| \frac{1}{|\lambda|^{2A_w}} \int e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \psi(u)\, d\sigma_u \right\|_{L^p(|\lambda|>R)} \leq C \|\psi\|_{L^q(\sqrt{\epsilon}<|u|<M)}$$

where M is chosen large enough so that $\{u : |u| < M\}$ includes the support of ψ.

Hence, we obtain by a change of variables through polar coordinates

$$\int_{|\lambda|>R} \left| \frac{I_w(\lambda)}{|\lambda|} \right| d\sigma_\lambda \leq C \left(\left[\tau^{2-(2-2A_w)q} \right]_R^\infty \right)^{1/q} \|\psi\|_{L^q(|u|<M)} < C' \|\phi\|_{W^{1,\infty}},$$

where the last inequality follows by applying Hölder generalized inequality to control the singularity $|u|^{-3/2}$. The result holds as long as $A_w < 1/p$ and thus follows with $A_w < 1/2$. $\square$

Theorem 3.6.7. *Let $a_{\lambda,w} \in C^\infty(\mathbb{C})$ be arbitrary with compact support and $\|a_{\lambda,w}\|_{W^{5,\infty}} = O(|\lambda|^{m+\delta})$ with $\delta > 0$.*

If $w \in \Omega$ is in the support of $a_{\lambda,w}$, then the oscillatory integral has the following asymptotic behaviour:

$$\left| I(w,\lambda) - \frac{\pi}{|\lambda|} a_{\lambda,w}(w) \right| = O(|\lambda|^{m+\delta-2}), \tag{3.40}$$

where $\delta > 0$.

Proof. For simplicity, we consider $\lambda = \tau\theta$, where $|\lambda| = \tau$ and $\theta \in S^1$.

$$I(w,\lambda) = \int_{\mathbb{C}} e^{i\mathrm{Re}(\lambda(z-w)^2)} a_{\lambda,w}(z) d\sigma_z = \int_{\mathbb{C}} e^{i\mathrm{Re}(\lambda z^2)} a_{\lambda,w}(z+w)\, d\sigma_z$$
$$= -2i \int_{\mathbb{R}^2} e^{-i\tau\langle Ax,x\rangle/2} a_\lambda(x+w)\, dx_1 dx_2$$

where we make a change of variables and used $z = x_1 + ix_2$ and $w = w_1 + iw_2$ or $w = (w_1, w_2)$ and $A = \begin{pmatrix} \theta_1 & -\theta_2 \\ -\theta_2 & -\theta_1 \end{pmatrix}$.

Applying $\mathcal{F}^{-1}$ and $\mathcal{F}$, by using the definition of $\mathcal{F}^{-1}$ on $\mathcal{S}'(\mathbb{R}^2)$, which is possible since a_λ has compact support, we obtain

$$\begin{aligned}I(w,\lambda) &= -2i\int \mathcal{F}_x\left[e^{-i\tau\langle Ax,x\rangle/2}\right](\xi)\mathcal{F}_x^{-1}[a_{\lambda,w}(x+w)](\xi)\,d\xi\\ &= -2i\int\left(\frac{\pi}{\tau}e^{-\langle A^{-1}\xi,\xi\rangle/2}\right)\mathcal{F}_x^{-1}[a_{\lambda,w}(x+w)](\xi)\,d\xi\\ &= -2i\frac{\pi}{\tau}\int e^{-\frac{i}{4\tau}\mathrm{Re}(\theta(\xi_1+i\xi_2)^2)}\mathcal{F}_x^{-1}[a_{\lambda,w}(x+w)](\xi)\,d\xi.\end{aligned}$$

Next, we split the exponential function e^b like:

$$e^b = 1 + b\left(\frac{e^b-1}{b}\right),$$

where $g(b)\in C^\infty(\mathbb{C})$ satisfies $|g(b)|\le|e^b|$ and setting $b=-\frac{i}{4\tau}\mathrm{Re}\left(\theta(\xi_1+i\xi_2)^2\right)$,

$$\begin{aligned}I(w,\lambda) = \frac{\pi}{\tau}&\int\mathcal{F}_x^{-1}[a_{\lambda,w}(x+w)](\xi)\,d\sigma_\xi\\ &-2i\frac{\pi}{\tau}\int\frac{-i}{4\tau}\mathrm{Re}\left(\theta(\xi_1+i\xi_2)^2\right)\mathcal{F}_x^{-1}[a_{\lambda,w}(x+w)](\xi)\,g(b)\,d\xi\\ = \frac{\pi}{\tau}&\int\mathcal{F}_x^{-1}[a_{\lambda,w}(x+w)](\xi)\,d\sigma_\xi\\ &-\frac{2\pi}{\tau^2}\int\mathcal{F}_x^{-1}\left[\mathrm{Re}\left(\theta(\partial_{x_1}+i\partial_{x_2})^2\right)a_{\lambda,w}(x+w)\right](\xi)g(b)\,d\xi.\end{aligned}$$

Finally, by $\int\mathcal{F}^{-1}(f)(\xi)\,d\xi = f(0)$ it holds:

$$\begin{aligned}\left|I_{a_\lambda,\phi}-\frac{\pi}{\tau}a_\lambda(w)\right| &= \left|\frac{2\pi}{\tau^2}\int\mathcal{F}_x^{-1}\left[\mathrm{Re}\left(\theta(\partial_{x_1}+i\partial_{x_2})^2\right)a_\lambda(x+w)\right](\xi)g(b)\,d\xi\right|\\ &\le\frac{2\pi}{\tau^2}\sup|g(b)|\left\|\mathcal{F}_x^{-1}\left[\mathrm{Re}\left(\theta\partial_z^2\right)a_{\lambda,w}\right]\right\|_{L^1(\Omega)} < \|\mathrm{Re}(\theta\partial_z^2)a_{\lambda,w}\|_{W^{3,\infty}}\\ &= O(|\lambda|^{m+\delta-2}).\end{aligned}$$

□

Theorem 3.6.8. *Let $\phi\in W^{1,\infty}(\Omega)$ for Ω a bounded Lipschitz domain with* $\operatorname{supp}\phi\subset\mathcal{G}$. *Further, let $w\in\Omega$ and define for $\lambda_\Omega\in\mathbb{C}$*

$$A_w := \sup_{z\in\bar{\Omega}}\mathrm{Re}\left(\lambda_\Omega(z-w)^2\right) < 1/2. \tag{3.41}$$

Then, for $R>0$ large enough the following asymptotic formula holds

$$\int_{\mathcal{G}} e^{i\mathrm{Re}\left(\lambda(z-w)^2\right)}e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)}\phi(z)\,d\sigma_z = 2\pi\frac{\phi(w)}{|\lambda|}+O(|\lambda|^{2A_w+\delta-2})+R_w(\lambda), \tag{3.42}$$

with $R_w(\lambda)/|\lambda|\in L^1(|\lambda|>R)$.

Proof. The proof follows from joining the previous results with an estimate based on ϕ being Lipschitz.

Let $\epsilon > 0$ and split Ω as follows

$$\Omega_1 = \{z \in \Omega : |z-w| < 3\epsilon\},\ \Omega_2 = \{z \in \Omega : |z-w| > \epsilon\}.$$

We define a partition of unity with $\alpha_1, \alpha_2 \in C^\infty(\Omega)$ such that $\alpha_1 + \alpha_2 = 1$ and α_j compactly supported in Ω_j.

Accordingly, we split our integral of interest:

$$\begin{aligned}\int_\Omega e^{i\mathrm{Re}\left(\lambda(z-w)^2\right)} e^{2\ln|\lambda| \mathrm{Re}\left(\lambda_\Omega (z-w)^2\right)} \phi(z)\, d\sigma_z = &\int_{\Omega_1} e^{i\mathrm{Re}\left(\lambda(z-w)^2\right)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)} \alpha_1(z)\phi(z)\, d\sigma_z \\ &+ \int_{\Omega_2} e^{i\mathrm{Re}\left(\lambda(z-w)^2\right)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)} \alpha_2(z)\phi(z)\, d\sigma_z.\end{aligned}$$

Remark that w is not inside Ω_2 and setting $a_{\lambda,w}(z) = e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)}\alpha_2(z)\phi(z)$ it holds for arbitrary $\delta > 0$ that $\|a_{\lambda,w}\|_{W^{1,\infty}(\Omega_2)} = O(|\lambda|^{2A_w+\delta})$. Therefore, $a_{\lambda,w}$ fits the hypothesis of lemma 3.6.6. As such, setting

$${}^1R_w(\lambda) := \int_{\Omega_2} e^{i\tau\mathrm{Re}(\theta(z-w)^2)} a_{\lambda,w}(z)\, d\sigma_z$$

it holds that ${}^1R_w(\lambda)\,/|\lambda| \in L^1(|\lambda| > R)$.

To integrate over Ω_1 we start by decomposing ϕ in two terms:

$$\phi(z) = \phi(w) + [\phi(z) - \phi(w)]\,.$$

Due to independence of the integration variable we bring $\phi(w)$ outside of integration and obtain:

$$\begin{aligned}&\int_{\Omega_1} e^{i\tau\mathrm{Re}(\theta(z-w)^2)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)} \alpha_1(z)\phi(z)\, d\sigma_z = \\ &\qquad \phi(w)\int_{\Omega_1} e^{i\tau\mathrm{Re}(\theta(z-w)^2)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)} \alpha_1(z)\, d\sigma_z \\ &\qquad + \int_{\Omega_1} e^{i\tau\mathrm{Re}(\theta(z-w)^2)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)} \alpha_1(z)\left[\phi(z) - \phi(w)\right]\, d\sigma_z.\end{aligned}$$

For the first integral, we define $b_{\lambda,w} = e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)}\alpha_1(z)$ and verify that b is under the conditions of Theorem 3.6.7. Therefore, it holds

$$\begin{aligned}\phi(w)\int_{\Omega_1} e^{i\tau\mathrm{Re}(\theta(z-w)^2)} e^{2\ln|\lambda|\mathrm{Re}(\lambda_\Omega(z-w)^2)} \alpha_1(z)\, d\sigma_z &= \phi(w)\left[\frac{\pi}{|\lambda|} b_{\lambda,w}(w) + O(|\lambda|^{2A_w+\delta-2})\right] \\ &= \frac{\pi}{|\lambda|}\phi(w) + O\left(|\lambda|^{2A_w+\delta-2}\right),\end{aligned}$$

since $b_{\lambda,w}(w) = 1$ by definition of α_1 and cancellation of the exponential. This results in the reconstruction part.

We cast our focus on second term of integration, that we define as ${}^2R_w(\lambda)$. To finish we show ${}^2R_w(\lambda)\,/|\lambda| \in L^1(\lambda : |\lambda| > R)$.

The idea is analogous to Lemma 3.5.3. Without loss of generality we dismiss α_1 as it fulfilled its purpose. We start by applying a change of variables $u = (z-w)^2$ and obtain

$$\begin{aligned}
&\int_{\Omega_1} e^{i\tau \mathrm{Re}(\theta(z-w)^2)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega (z-w)^2)} \left(\phi(z) - \phi(w)\right)\, d\sigma_z \\
&\quad = \sum_{\pm} \int e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \frac{\phi(w \pm \sqrt{u} - \phi(w))}{|u|}\, d\sigma_u \\
&\quad = \sum_{\pm} \int \frac{1}{i\tau\theta + 2\ln\tau\lambda_\Omega} e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \partial_u \left(\frac{\phi(w \pm \sqrt{u}) - \phi(w)}{|u|} \right) d\sigma_u.
\end{aligned}$$

Due to $\phi \in W^{1,\infty}(\Omega)$ and thus being a Lipschitz continuous function it holds

$$\left| \partial_u \left(\frac{\phi(w \pm \sqrt{u}) - \phi(w)}{|u|} \right) \right| = \left| \frac{\partial_z \phi(w \pm \sqrt{u})}{2\sqrt{u}|u|} - \frac{\left(\phi(w \pm \sqrt{u}) - \phi(w)\right) \bar{u}}{2|u|^3} \right| \le \frac{C}{|u|^{3/2}} + \frac{L}{|u|^{3/2}}.$$

We define $\psi(u) := \partial_u \left(\frac{\phi(w \pm \sqrt{u}) - \phi(w)}{|u|} \right)$ and choose $M > 0$ large enough so that $\{u |\, |u| < M\}$ includes the support of ψ.

To complete the estimate we apply Hölder inequality to integration in terms of λ with $p = 2$ as follows:

$$\begin{aligned}
\int_{|\lambda|>R} \left| \frac{1}{|\lambda|} R_w(\lambda) \right| d\sigma_\lambda &\le \int_{|\lambda|>R} \left| \frac{1}{|\lambda|^{2-2A_w}} \left(\sum_{\pm} \frac{1}{|\lambda|^{2A_w}} \int e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \psi(u)\, d\sigma_u \right) \right| d\sigma_\lambda \\
&\le 2 \left[\int_{|\lambda|>R} \frac{1}{|\lambda|^{(2-2A_w)2}}\, d\sigma_\lambda \right]^{1/2} \left\| \frac{1}{|\lambda|^{2A_w}} \int e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \psi(u)\, d\sigma_u \right\|_{L^2(|\lambda|>R)} \\
&\le C \left[\tau^{2-(2-2A_w)2} \right]_R^\infty \|\psi\|_{L^2(|u|<M)} < \infty,
\end{aligned}$$

where the boundedness follows since $2 - (2-2A_w)2 < 0 \Leftrightarrow A_w < 1/2$ and the inequality follows as above.

Further, the estimate

$$\left\| \frac{1}{|\lambda|^{2A_w}} \int e^{i\tau \mathrm{Re}(\theta u)} e^{2\ln|\lambda| \mathrm{Re}(\lambda_\Omega u)} \psi(u)\, d\sigma_u \right\|_{L^2(|\lambda|>R)} \le C_M \|\psi\|_{L^2(|u|<M)},$$

follows exactly as in the proof of Lemma 3.5.3 by cutting exponential growth $2\ln|\lambda|$ with $|\lambda|^{2A_w}$ and apply Hausdorff-Young inequality for $p = 2$. Notice that the Lemma 3.5.3 does not hold for $p = 2$, but this is because of the kernel $\frac{1}{z-w}$ being present under the integral. □

We can relax the condition on admissible point much more, that is, A_w can be a bit larger than 1/2. But that requires extra smoothness conditions on ϕ in order to apply further integration by parts, which leads to more decay. In fact, with more derivatives we do not strictly require the Hausdorff-Young type inequality.

These results allows us now to bring back the scattering transform and provide a simple proof to the reconstruction formula.

3.6.3 Proof of the reconstruction formula

As we have already indicated before, the reconstruction formula follows by decomposing the solution μ_1 of (3.20) into further terms like (3.28). Hence, we can re-write the scattering transform over the domain (3.25) with this expansion and to finish we just need to know the asymptotic behavior of all terms as $R \to \infty$.

Since these asymptotics were proved in the previous subsection, we collect now all results together to provide the reconstruction proof.

For this purpose, we note that the reconstruction is only possible on the points $w \in \Omega$ that fulfill the following conditions for $p > 2$:

$$A_w < 1/2, \quad A_w - B_w > \frac{2}{p}, \quad A_w + B_w < \frac{2}{p} - 1.$$

From these conditions, we now deduce a loose condition for B_w. The first condition implies that $-A_w > -1/2$, which together with the second condition leads to

$$\begin{aligned} A_w - B_w > \frac{2}{p} &\Leftrightarrow -B_w > \frac{2}{p} - A_w > \frac{2}{p} - 1/2 \\ &\Leftrightarrow B_w < 1/2 - \frac{2}{p}. \end{aligned}$$

Further, since $p > 2$ we have that $-1/2 < 1/2 - 2/p < 1/2$. Thus, if $B_w < -1/2$ the condition deduced above is met.

As such in a loose manner we obtain the Definition 3.1.1 of admissible points. We recall that $w \in \Omega$ is an admissible point if there exists a $\lambda_\Omega \in \mathbb{C}$ such that:

$$\begin{aligned} A_w &:= \sup_{z \in \overline{\Omega}} \mathrm{Re}(\lambda_\Omega (z-w)^2) < 1/2, \\ B_w &:= \sup_{z \in \bar{\mathcal{D}}} \mathrm{Re}(\lambda_\Omega (z-w)^2) < -1/2. \end{aligned}$$

Note that this definition originated with independence of the chosen $p > 2$. The last condition can be simplified to $A_w + B_w < 0$ by choosing an appropriate p during the proof of Lemma 3.6.3.

With this definition we are ready to prove the main theorem of this chapter on proper admissible points.

Proof of Theorem 3.6.2

Let us re-write the scattering transform over the whole domain.

$$\begin{aligned} \frac{h(\lambda, w)}{|\lambda|} &= \frac{1}{|\lambda|} \int_\Gamma e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\sigma_z \\ &+ \frac{2i}{|\lambda|} \int_{\Omega \setminus \bar{\mathcal{D}}} e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} q_{21}(z) e^{i\mathrm{Re}(\lambda (z-w)^2)} \mu_1(z)\, d\sigma_z \\ &= \frac{1}{|\lambda|} \int_\Gamma e^{\ln|\lambda| \overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} + \frac{1}{|\lambda|} T_\lambda(\mu_1). \end{aligned}$$

Applying the splitting (3.29) to μ it holds that

$$\mu_1 = -\left[Mf\right]_1 - \left[M\left(I + DQ_\lambda\right)\begin{pmatrix} U \\ 0 \end{pmatrix}\right]_1 + U.$$

Therefore, the scattering data follows as

$$\begin{aligned}\frac{h(\lambda,w)}{|\lambda|} &= \frac{1}{|\lambda|}\int_\Gamma e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}}\mu_2(z)\,d\sigma_z - \frac{1}{|\lambda|}T_\lambda\left(\left[Mf\right]_1\right) \\ &\quad - \frac{1}{|\lambda|}T_\lambda\left(\left[M\left(I+Q_\lambda\right)\begin{pmatrix} U \\ 0 \end{pmatrix}\right]_1\right) + \frac{1}{|\lambda|}T_\lambda U\end{aligned}$$

Due to Lemmas 3.6.3 and 3.6.4 we have that:

$$\begin{aligned}&\lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{1}{|\lambda|}\int_\Gamma e^{\ln\lambda\overline{\lambda_\Omega(z-w)^2}}\mu_2(z)\,d\bar{z}d\sigma_\lambda = 0, \\ &\lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{1}{|\lambda|}T_\lambda\left(\left[Mf\right]_1\right)\,d\sigma_\lambda = 0, \\ &\lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{1}{|\lambda|}T_\lambda\left(\left[M\left(I+Q_\lambda\right)\begin{pmatrix} U \\ 0 \end{pmatrix}\right]_1\right)\,d\sigma_\lambda = 0.\end{aligned}$$

As such, it follows from Lemma 3.6.8 that:

$$\begin{aligned}\lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{h(\lambda,w)}{|\lambda|}\,d\sigma_\lambda &= \lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{2\pi}{|\lambda|^2}q_{21}(w)\,d\sigma_\lambda \\ &\quad + \lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{C}{|\lambda|^{3-2A_w+\delta}}\,d\sigma_\lambda + \lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{1}{|\lambda|}\left|R_w(\lambda)\right|\,d\sigma_\lambda.\end{aligned}$$

The last term goes to zero, since we have shown that $\frac{1}{|\lambda|}R_w(\lambda) \in L^1(|\lambda| > R)$. The second term can be treated quickly with polar coordinates as

$$\lim_{R\to\infty}\int_{R<|\lambda|<2R}\frac{1}{|\lambda|^{3-2A_w-\delta}}\,d\sigma_\lambda = \lim_{R\to\infty}2\pi\int_{R<\tau<2R}\tau^{2A_w+\delta-2}\,d\tau = \lim_{R\to\infty}\left[\tau^{2A_w+\delta-1}\right]\Big|_R^{2R} = 0,$$

holds for $2A_w + \delta - 1 < 0$, which is true since $\delta > 0$ can be arbitrarily small and $A_w < 1/2$.

As such, the limit as $R \to \infty$ of the scattering transform only depends on the first term, which is trivially computed at the admissible points $w \in \Omega \setminus \bar{\mathcal{D}}$:

$$\int_{R<|\lambda|<2R}\frac{2\pi}{|\lambda|^2}q_{21}(w)\,d\lambda = 4\pi q_{21}(w)\,\left[\ln\tau\right]\Big|_R^{2R} = 4\pi\ln 2\,q_{21}(w).$$

□

The conditions of the theorem are mostly concerned with existence and uniqueness of the solutions μ, since the scattering transform is defined through them and they need to have the appropriate asymptotics for everything to hold together.

The nature of admissible points is very visible through the proofs, but we want to highlight that is also an essential step for application of the stationary phase method in the last step of reconstruction formula.

Remark, that the work we have achieved so far only allows us to reconstruct the potential from data $\{\mu|_{\partial\Omega}\}$. In order to uniquely determine the complex conductivity we show how to establish a one-to-one connection between this boundary data and the Dirichlet-to-Neumann map. With this connection Theorem 3.6.2 proves the Calderón problem for complex conductivities with a discontinuity curve.

3.7 Scattering data for the Dirac equation via the Dirichlet-to-Neumann map

Our next objective is to establish a relation between the Dirichlet-to-Neumann map for equation (3.1) and the traces of the solutions (3.5) on $\partial\Omega$. For this purpose, we define the Cauchy data set for a potential $q \in L^\infty(\Omega)$ by

$$\mathcal{T}_q := \left\{ \phi|_{\partial\Omega} : \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ is a solution of (3.5) and (3.4)}, \quad \phi_1, \phi_2 \in H^1(\Omega) \right\}. \tag{3.43}$$

Let $u \in H^2(\Omega \setminus \bar{\mathcal{D}}) \cap H^2(\mathcal{D})$ be a solution of (3.1) with $u|_{\partial\Omega} = f \in H^{3/2}(\partial\Omega)$. Consider $\phi = \gamma^{1/2}(\partial u, \bar{\partial} u) \in H^1(\Omega \setminus \overline{\mathcal{D}}) \cap H^1(\mathcal{D})$. Then, formally

$$\phi|_{\partial\Omega} = \frac{1}{2} \begin{pmatrix} \bar{\nu} & -i\bar{\nu} \\ \nu & i\nu \end{pmatrix} \begin{pmatrix} \Lambda_\gamma f \\ \partial_s f \end{pmatrix}, \tag{3.44}$$

where Λ_γ is the co-normal D-t-N map and ∂_s is the operator of the tangential derivative. Inverting we get

$$\begin{pmatrix} \Lambda_\gamma f \\ \partial_s f \end{pmatrix} = \begin{pmatrix} \nu & \bar{\nu} \\ i\nu & -i\bar{\nu} \end{pmatrix} \phi|_{\partial\Omega}. \tag{3.45}$$

We normalize ∂_s^{-1} in such a way that

$$\int_{\partial\Omega} \partial_s^{-1} f ds = 0.$$

Then (3.45) can be written as *a boundary relation*

$$(I - i\Lambda_\gamma \partial_s^{-1})(\nu\phi_1|_{\partial\Omega}) = (I + i\Lambda_\gamma \partial_s^{-1})(\bar{\nu}\phi_2|_{\partial\Omega}) \tag{3.46}$$

Let us show the generalization of [55, Thm 3.2] where $\gamma \in C^{1+\epsilon}(\mathbb{R}^2)$ to the case of non-continuous γ.

Theorem 3.7.1. *Let $\gamma \in W^{1,\infty}(\Omega \setminus \bar{\mathcal{D}}) \cap W^{1,\infty}(\mathcal{D})$ with* $\operatorname{Re} \gamma \geq c > 0$. *Further, let q be the complex potential defined through γ by (3.6). Then, the respective Cauchy data satisfies the following equality:*

$$\mathcal{T}_q = \left\{ (h_1, h_2)^t \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) : (I - i\Lambda_\gamma \partial_s^{-1})(\nu h_1) = (I + i\Lambda_\gamma \partial_s^{-1})(\bar{\nu} h_2) \right\} \tag{3.47}$$

Proof. First we show that any pair $(h_1, h_2)^t \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ that satisfies the boundary relation above is in $\mathcal{T}_q$. Consider a solution $u \in H^2(\Omega \setminus \bar{\mathcal{D}}) \cap H^2(\mathcal{D})$ of (3.1) with the boundary condition

$$u|_{\partial\Omega} = i\partial_s^{-1}(\nu h_1 - \bar{\nu} h_2) \in H^{3/2}(\partial\Omega).$$

Since $\gamma \in W^{1,\infty}(\Omega \setminus \bar{\mathcal{D}}) \cap W^{1,\infty}(\mathcal{D})$ and γ is separated from zero, it follows that $\gamma^{1/2} \in W^{1,\infty}(\Omega \setminus \bar{\mathcal{D}}) \cap W^{1,\infty}(\mathcal{D})$. Then, both components of the vector $\phi = \gamma^{1/2}(\partial u, \bar{\partial} u)^t$ belong to $H^1(\Omega \setminus \bar{\mathcal{D}}) \cap H^1(\mathcal{D})$ and ϕ satisfies (3.5). The fact $\phi|_{\partial\Omega} = (h_1, h_2)^t$ follows from (3.44) and (3.46).

Conversely, we start with a solution $\phi \in H^1(\Omega \setminus \bar{\mathcal{D}}) \cap H^1(\mathcal{D})$ of (3.5) satisfying (3.7) on Γ. From (3.5) and (3.6) the following compatibility condition holds on Γ

$$\bar{\partial}(\gamma^{-1/2}\phi_1) = \partial(\gamma^{-1/2}\phi_2).$$

Poincaré lemma ensures the existence of a function u such that

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial u \\ \bar{\partial} u \end{pmatrix} \quad \text{on} \quad \Omega \setminus \Gamma.$$

It is easy to check that u is a solution to (3.1) on $\Omega \setminus \Gamma$ and belongs to $H^2(\Omega \setminus \overline{\mathcal{D}}) \cap H^2(\mathcal{D})$. Moreover, through the Poincaré Lemma and (3.7) it satisfies the transmission condition (3.4). Then, (3.44)-(3.46) proves that $h = \phi|_{\partial\Omega}$ satisfies the boundary relation stated in the theorem. □

We want to highlight that the uniqueness proof and the boundary determination require different regularities of the conductivity γ. At least, this shades light in possible improvements on this methodology, if we are able to surpass the necessity of two-derivatives of γ on the reconstruction formula obtained by the stationary phase method.

A future idea to explore concerns the actual determination of the trace of ϕ at $\partial\Omega$ from the Dirichlet-to-Neumann map Λ_γ. This is an essential step to obtain a reconstruction method based on the uniqueness proof given in Theorem 3.6.2. A hint to this is given in [55, Th. III.3.] for $\gamma \in C^{1+\varepsilon}(\mathbb{R}^2)$, although in here the exponential growing solutions are of the type e^{ikz}.

For such, we denote $S_{\lambda,w} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$

$$S_{\lambda,w} f(z) = \frac{1}{i\pi} \int_{\partial\Omega} f(\varsigma) \frac{e^{-\lambda(z-w)^2 + \lambda(\varsigma - w)^2}}{\varsigma - z} d\varsigma$$

The integral is to be understood in the sense of principal value. This allows to state the following conjecture for our method.

Conjecture 3.7.2. *The only pair $(h_1, h_2) \in H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ which satisfies*

$$(I - S_{\lambda,w})h_1 = 2e^{\lambda(z-w)^2}, \tag{3.48}$$

$$(I - \overline{S_{\lambda,w}})h_2 = 0, \tag{3.49}$$

$$(I - i\Lambda_\gamma \partial_s^{-1})(\nu h_1) = (I + i\Lambda_\gamma \partial_s^{-1})(\bar{\nu} h_2) \tag{3.50}$$

is $(\phi_1|_{\partial\Omega}\,,\ \phi_2|_{\partial\Omega})$*, where* ϕ_1*,* ϕ_2 *are the solutions of Dirac equation (3.5) satisfying the asymptotics (3.11).*

3.8 Complex conductivities with more discontinuity curves

Assuming that the conductivity only has one discontinuity curve is a rather restrictive assumption when one thinks of applications. The existence of one discontinuity curve may be representative of a tumour inside a breast, however it does not represent clinical scenarios of lung ventilation or brain haemorrhages.

To consider these new scenarios one needs to extend the previous work for more than discontinuity curve. Without loss of generality we show the procedure for the extension with two discontinuity curves, see Figure 3.2 for the two possibilities.

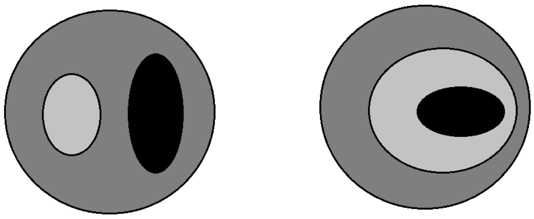


Figure 3.2: Complex conductivities with more than one discontinuity curve. The left-hand side represents complex conductivities for which there are two separate discontinuity curves and on the right-hand domain represents scenarios where we have a discontinuity curve inside the interior domain of the other.

There are a few key steps essential for the generalization. By their respective order they are:

1. Devising a transmission condition for each discontinuity curve;
2. Determining the integral equation for the more than one discontinuity curve;
3. Generalizing the function spaces;
4. Determining the scattering transform and apply the stationary phase method to obtain an analogous reconstruction formula.

Since most of the results are analogous we briefly describe how each step follows without being extensive about them.

First and foremost, the transmission condition can be obtained exactly the same as in Lemma 3.4. Instead of one transmission condition we will have two, one for each discontinuity curve. This is a clear procedure since we will need also to consider two transmission conditions for the conductivity equation. Furthermore, we also will take into account two jumps α_1, α_2 to construct these conditions over the two discontinuity curves Γ_1, Γ_2.

Remark that the exponentially growing solutions can be exactly the same and no further consideration needs to be taken. As such, we can obtain the μ solutions to (4.71) in $\Omega\backslash(\Gamma_1\cup\Gamma_2)$. Transforming the above transmission conditions for μ we obtain two matrix operators $A_{\lambda,1}, A_{\lambda,2}$ with respect to the condition over each curve.

The second step follows now swiftly and using the ideas in Proposition 3.3.1 we obtain an analogous integral equation:

$$[I + PA_{\lambda,1} + PA_{\lambda,2} - DQ_\lambda]\,\mu = \begin{pmatrix} U \\ 0 \end{pmatrix}. \tag{3.51}$$

To study this equation the function spaces need to be carefully adapted. Recall that the space $\mathcal{H}^p_2$ was introduced to deal with the behavior over the discontinuity curve. As such, we split this space in order to deal with all curves. We can define $\mathcal{H}^p_2 = \mathcal{H}^p_{2,1} + \mathcal{H}^p_{2,2}$ where each $\mathcal{H}^p_{2,j}$ space treats its respective curve Γ_j like in the single scenario. In the space $\mathcal{H}^p_2$ we consider the norm

$$\|f\|_{\mathcal{H}^p_2} := \min_{\substack{f=f_1+f_2,\\ f_j\in\mathcal{H}^p_{2,j}}} \left[\|f_1\|_{\mathcal{H}^p_{2,1}} + \|f_2\|_{\mathcal{H}^p_{2,2}}\right].$$

Finally, we multiply the new integral equation by $(I + DQ_\lambda)$ and obtain an operator M defined as:

$$M = PA_{\lambda,1} + PA_{\lambda,2} + DQ_\lambda\left(PA_{\lambda,1} + PA_{\lambda,2}\right) - DQ_\lambda DQ_\lambda.$$

This operator is a contraction in $\mathcal{H}^p$ for a large $R > 0$ as long as the jumps α_j are close to 1 in $L^\infty(\Gamma_j), j = 1, 2$. This allows us to ensure the existence and uniqueness of a solution μ to the integral equation $(I + M)\mu = (I + DQ_\lambda)(U, 0)^t$ in $\mathcal{H}^p$ as achieved in Lemma 3.5.4 with the multiplication by the factor $1/|\lambda|^{A_w}$. The proof already relies on the notion of an admissible point. Here, the only different aspect is the definition of B_w, which we set for $\mathcal{D}_j = \operatorname{int}\Gamma_j$ to be

$$B_w = \sup_{z\in\mathcal{D}_1\cup\mathcal{D}_2} \operatorname{Re}\left[\lambda_\Omega(z-w)^2\right] < -1/2.$$

Lastly, one needs to understand how the scattering data relates with the potential. Here, there is a splitting between both scenarios.

If the inner domains of the curves are disjoint, like on the left case presented in Figure 3.2, then the scattering data is analogous. In this case, we have two terms that consider the curves given as

$$\int_{\Gamma_j} e^{\ln|\lambda|\overline{\lambda_\Omega(z-w)^2}}\mu_2(z)\,d\bar{z},$$

where μ_2 is the trace from the inner domain of Γ_j. Thereafter, the asymptotics are proven as in the extension of Lemma 3.6.3.

For the other scenario, care needs to be taken, because the integral over the inner curve has to account for the behavior from the inside and outside. Let us see how to obtain it. For such, consider that Γ_1 is the inner curve and Γ_2 the outer one. Further, let $\mathcal{D}_j = \mathrm{int}\Gamma_j$

$$
\begin{aligned}
h(\lambda, w) &= \int_{\partial\Omega} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} = -2i \int_{\Omega} \partial \left(e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z) \right) d\sigma(z) \\
&= -2i \left[\int_{\mathcal{D}_1} + \int_{\mathcal{D}_2 \backslash \mathcal{D}_1} + \int_{\Omega \backslash \mathcal{D}_2} \partial \left(e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z) \right) d\sigma(z) \right] \\
&= \int_{\Gamma_1} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} + \int_{\partial\mathcal{D}_2 \backslash \mathcal{D}_1} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} \\
&\quad - 2i \int_{\Omega \backslash \mathcal{D}_2} \partial \left(e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z) \right) d\sigma(z) \\
&= \int_{\Gamma_1} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \left(\mu_2^-(z) - \mu_2^+(z) \right) d\bar{z} + \int_{\Gamma_2} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z)\, d\bar{z} \\
&\quad - 2i \int_{\Omega \backslash \mathcal{D}_2} \partial \left(e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \mu_2(z) \right) d\sigma(z).
\end{aligned}
$$

Due to the transmission condition over the Γ_1 the first integral is given as:

$$
\int_{\Gamma_1} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \left(\mu_2^-(z) - \mu_2^+(z) \right) d\bar{z} = - \int_{\Gamma_1} e^{\ln|\lambda|\overline{\lambda_\Omega (z-w)^2}} \left[A_{\lambda,1} \mu \right]_2 d\bar{z}.
$$

The decay this last integral at infinity holds as in Lemma 3.6.3 by bringing the term $A_{\lambda,1}$ since it is in $L^\infty(\Gamma_1)$.

Essentially, for both scenarios these type of terms disappear when taking the limit of $R \to \infty$ on the reconstruction formula. Furthermore, the stationary phase method allows to obtain the potential q at the admissible points.

Hence, we are still able to reconstruct the potential uniquely when it has more than one discontinuity curve which shows the full picture of Theorem 1.2.1.

Chapter 4

Complex conductivities in three dimensions

In this chapter we consider the Calderón problem in the case of complex conductivities in three dimensions.

Our focus lies in the uniqueness question posed by Calderón for L^∞ conductivities. The history of the Calderón problem starts with stronger assumptions, like C^∞ conductivities [88], and proceeds to improve the results with new techniques but a similar proof structure. Here we present two lines of work based in different techniques that successively decreases the conductivity smoothness requirements.

We introduce an approach for uniqueness of complex conductivities in $C^{1,1}$ and an approach when they belong to $W^{1,\infty}$. The first focusses heavily on the work of Nachman [68] while the latter focusses on novel work with quaternionic analysis.

We start by introducing the general problem and theorems we want to prove. Thereafter, we split our presentation into two sections. One focusing on $C^{1,1}$ conductivities and making an extension of the works [22] and [68] for complex conductivities. The second focuses on $W^{1,\infty}$ and is based on quaternionic analysis to introduce a Dirac inverse problem and new exponentially growing solutions.

4.1 The problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and define $\gamma \in L^\infty(\Omega)$ to be an isotropic complex conductivity $\gamma = \sigma + i\omega\epsilon$, as defined in the Introduction.

The *direct problem* we focus here is given in Eq. (1.4) and we recall it here for simplicity. Let $f \in H^{1/2}(\partial\Omega)$ be a voltage established at the boundary $\partial\Omega$. Then, we want to find the unique electrical potential $u \in H^1(\Omega)$ that fulfills the *conductivity equation*:

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, \text{ in } \Omega \\ u|_{\partial\Omega} = f. \end{cases} \tag{4.1}$$

Further, recall that uniqueness is guaranteed in $H^1(\Omega)$ for $\operatorname{Re}\gamma \geq c > 0$ and as such, we can define the Dirichlet-to-Neumann map from this solution. Formally, it is defined as:

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$f \mapsto \gamma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}.$$

This provides all the boundary information we can acquire in practice, and therefore we can formulate the *inverse Calderón problem* through it.

In this chapter, we work on two different assumption settings. Our main objective will be the proof of the following two theorems.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded $C^{1,1}$-domain. Let $\gamma_i \in C^{1,1}(\Omega)$ with $\operatorname{Re}\gamma_i \geq c > 0$ for $i = 1, 2$ and denote Λ_{γ_i} as their respective Dirichlet-to-Neumann maps.*

$$\textit{If } \Lambda_{\gamma_1} = \Lambda_{\gamma_2} \textit{ then } \gamma_1 = \gamma_2 \textit{ in } \Omega.$$

Theorem 4.1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $\gamma_i \in W^{1,\infty}(\Omega)$ with $\operatorname{Re}\gamma_i \geq c > 0$ for $i = 1, 2$ and denote Λ_{γ_i} as their respective Dirichlet-to-Neumann maps.*

$$\textit{If } \Lambda_{\gamma_1} = \Lambda_{\gamma_2} \textit{ then } \gamma_1 = \gamma_2 \textit{ in } \Omega.$$

4.2 Complex conductivities in $C^{1,1}(\Omega)$

Nachman's work [68] starts by connecting the Calderón problem with an analogous Schrödinger inverse problem. The focus is to obtain a reconstruction method to determine a real conductivity from its Dirichlet-to-Neumann map. Uniqueness was established slightly earlier in [70] by Nachman, Sylvester and Uhlmann. Furthermore, it is clearly extended to complex conductivities with positive real-part, since there was no restrictive assumption for this in the proof.

In this section, we give a brief overview of the uniqueness proof and thereafter proceed to present the extension of Nachman's reconstruction proof to complex conductivities in $C^{1,1}(\Omega)$ in $C^{1,1}$ domains. To be explicit, we state clearly where new results are required to fit the case of complex conductivities.

One problem of this approach is related with the asymptotic behaviour of the reconstruction formula, which is hard to implement in a stable manner. To tackle this, we also extend the work in [22] for complex conductivities in order to obtain a more stable reconstruction procedure.

However, this is based on a smallness condition of the complex conductivity and therefore, it is just a small step towards a general reconstruction method.

As always, we start by transforming our initial equation (4.1) into another one, this method in particular changes it into a Schrödinger equation.

4.2.1 The relation with Schrödinger equation

The initial relation was established in [87] and allows a connection with scattering theory and with the many tools already established for that. This transformation was one of the starting points for uniqueness proofs of Calderón problems.

Let $u \in H^1(\Omega)$ be the unique solution of (4.1) with boundary values $f \in H^{1/2}(\partial\Omega)$. Then, the substitution $u = \gamma^{-1/2} w$ yields with the potential $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$ the equation

$$\begin{cases} -\Delta w + qw = 0, \text{ in } \Omega \\ w|_{\partial\Omega} = \gamma^{1/2} f. \end{cases} \tag{4.2}$$

Remark since $\operatorname{Re}\gamma > 0$ and $\gamma \in C^{1,1}(\Omega)$ then $\gamma^{1/2}$ is well-defined and twice-weakly differentiable. Therefore, the potential q is also well-defined and in $L^\infty(\Omega)$.

Now, from these assumptions on γ we have that 0 is not a Dirichlet eigenvalue of $\nabla\cdot(\gamma\nabla u)$. As such, and due to the bijection between solutions of both problems (4.1) and (4.2) for $\gamma \in C^{1,1}$, we have that 0 is also not a Dirichlet eigenvalue of the Schrödinger operator. In the proofs Nachman always made an emphasis of this to provide mathematical generality. Since our interest resides in complex conductivities we use it as an assumption in order to avoid repetition of the results. To this end we define a special type of potential:

Definition 4.2.1. *Let $\gamma \in C^{1,1}(\Omega)$ be a complex conductivity with $\operatorname{Re}\gamma \geq c > 0$, thus 0 is not a Dirichlet eigenvalue for the operator $\nabla \cdot (\gamma\nabla u)$.*

*We call the following potential a **complex conductivity potential***

$$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}, \tag{4.3}$$

and therefore 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$.

In general, if 0 is not a Dirichlet eigenvalue of the Schrödinger operator in Ω, then the respective Dirichlet-to-Neumann map Λ_q is well-defined from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ and formally is given by

$$\Lambda_q g = \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega},$$

where w is the unique solution of $(-\Delta + q)w = 0$, $w|_{\partial\Omega} = g$.

The inverse problem is to determine the conductivity potential q from the Dirichlet-to-Neumann map Λ_q uniquely. In this manner, we are focusing on the Schrödinger equation.

However, for the proof of Theorem 4.1.1 we need to establish a connection between Λ_γ and Λ_q for complex conductivities and its respective potential.

In fact, let γ be a complex conductivity, $f \in H^{1/2}(\partial\Omega)$ boundary value, and $u \in H^1(\Omega)$ the unique solution of $\nabla\cdot(\gamma\nabla u) = 0$ in Ω, with $u|_{\partial\Omega} = f$. Since $\gamma \in C^{1,1}(\Omega)$ it holds that $w = \gamma^{1/2}u$ is the unique solution of $(-\Delta + q)w = 0$ in Ω with $w|_{\partial\Omega} = \gamma^{1/2}f$.

Applying the normal derivative at the boundary to w we get:

$$\begin{aligned}\Lambda_q(\gamma^{1/2}f) = \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} &= \left.\left[\frac{1}{2}\gamma^{-1/2}\frac{\partial\gamma}{\partial n}u + \gamma^{1/2}\frac{\partial u}{\partial n}\right]\right|_{\partial\Omega}\\ &= \frac{1}{2}\gamma^{-1/2}\frac{\partial\gamma}{\partial n}f + \gamma^{-1/2}\Lambda_\gamma f\end{aligned}$$

which assuming $g = \gamma^{1/2}f$ leads to

$$\Lambda_q = \gamma^{-1/2}\left(\frac{1}{2}\frac{\partial\gamma}{\partial n} + \Lambda_\gamma\right)\gamma^{-1/2}.$$

Therefore, for us to determine Λ_q we also require γ and $\frac{\partial\gamma}{\partial n}$ at $\partial\Omega$. We will show that these boundary values are uniquely determined by Λ_γ, which implies that Λ_q is uniquely determined from Λ_γ.

4.2.2 Exponentially growing solutions

We start by extending our equation to $\mathbb{R}^n$. Since, q is in $L^\infty(\Omega)$ we can extend it by 0 outside Ω and study solutions of

$$-\Delta\psi + q\psi = 0 \text{ in } \mathbb{R}^n. \tag{4.4}$$

Thanks to Sylvester and Uhlmann [88] we can now transfer the problem to one at infinity, through the, by now famous, exponentially growing solutions given as:

$$\psi = e^{ix\cdot\zeta}\left(1 + \mu(x,\zeta)\right) \text{ for } \zeta \in \mathbb{C}^n \text{ such that } \zeta\cdot\zeta = 0. \tag{4.5}$$

Substituting into Schrödinger equation (4.4) we get that μ must fulfill

$$-\Delta\mu - 2i\zeta\cdot\nabla\mu + q\mu = -q \text{ in } \mathbb{R}^n, \tag{4.6}$$

which is obtained through harmonic nature of the exponential function, given that we assume $\zeta \in \mathbb{C}^n$ satisfies $\zeta\cdot\zeta = 0$. The choice of exponential functions is indeed done with this purpose in mind.

From scattering theory we inherited Faddeev-Green's function (see [29]) which takes a principal role in the study of the above equation (4.6). It is defined as $G_\zeta(x) = e^{ix\cdot\zeta}g_\zeta(x)$ where g_ζ is the fundamental solution of operator $(-\Delta - 2i\zeta\cdot\nabla)$ given as:

$$g_\zeta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi}\,d\xi. \tag{4.7}$$

By definition of g_ζ it holds that G_ζ is a fundamental solution of the Laplacian, and therefore it differs from the classical one G_0 by an harmonic function H_ζ, that is,

$$G_\zeta(x) = G_0(x) + H_\zeta(x).$$

This decomposition is important to provide estimates about the solutions w.

Given a fundamental solution G_ζ of the Laplace operator, we obtain an integral equation associated with (4.4)

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int G_\zeta(x-y)q(y)\psi(y,\zeta)\,dy \tag{4.8}$$

and with the assumption (4.5), we obtain the following integral equation for μ:

$$\mu + g_\zeta * (q\mu) = -g_\zeta * q. \tag{4.9}$$

These integral equations are essential to study exponentially growing solutions. We show that there exists a unique solution to (4.9) for large values of $|\zeta|$. This result is based on a g_ζ estimate on a weighted L^2 space, initially introduced in [88] in the case of real conductivities. However, this last assumption is not necessary and the proof works even if q is a complex potential.

Let $\langle x\rangle = (1+|x|^2)^{1/2}$. We define a weighted L^2-space for $\delta \in \mathbb{R}$ as

$$L^2_\delta(\mathbb{R}^n) := \left\{ f : \|f\|_\delta := \|\langle x\rangle^\delta f\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

Then the convolution operators with g_ζ and G_ζ satisfy the following estimates.

Proposition 4.2.2. *For all $\zeta \in \mathbb{C}^n$ with $\zeta\cdot\zeta = 0$ and $|\zeta| \geq a$ the convolution operator with g_ζ satisfies*

$$\|g_\zeta * f\|_{\delta-1} \leq \frac{c(\delta,a)}{|\zeta|}\|f\|_\delta, \qquad \textit{for } 0<\delta<1. \tag{4.10}$$

Moreover, let $H^2_\delta(\Omega) := \left\{ f : D^\alpha f \in L^2_{-\delta}(\mathbb{R}^n), 0 \leq |\alpha| \leq 2 \right\}$ be the weighted Sobolev space with norm

$$\|f\|_{2,\delta} = \left(\sum_{|\alpha|\leq 2} \|D^\alpha f\|^2_\delta \right)^{1/2}.$$

Then, for any $\zeta \in \mathbb{C}^n$ with $\zeta\cdot\zeta = 0$ it holds for $\delta \in (\frac{1}{2}, 1)$ that

$$\|g_\zeta * w\|_{2,-\delta} \leq c(\delta,\zeta)\|w\|_{2,\delta}.$$

Furthermore, under the definition

$$\boldsymbol{G}_\zeta w(x) = \int_\Omega G_\zeta(x-y)w(y)\,dy$$

it holds that

$$\|\boldsymbol{G}_\zeta w\|_{H^2(\Omega)} \leq c(\zeta,\Omega)\|w\|_{L^2(\Omega)}.$$

Proof. The first estimate can be found in [88, Corollary 2.2] while the rest is in [68, Lemma 2.11] □

With these estimates, uniqueness of solutions to (4.4) with the desired asymptotics follows quickly.

Corollary 4.2.3. *Let $0 < \delta < 1$ and $q \in L^\infty(\Omega)$ be complex-valued potential and extended to zero outside Ω.*

Then there exists an $R > 0$ such that for all $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$ and $|\zeta| > R$ the integral equation (4.8) is uniquely solvable with $e^{-ix\cdot\zeta}\psi(x,\zeta) - 1 \in L^2_{\delta-1}(\mathbb{R}^n)$. Furthermore, it holds

$$\|e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\|_{\delta-1} \leq \frac{\tilde{c}(R,\delta)}{|\zeta|}\|q\|_\delta. \tag{4.11}$$

Proof. The proof can be found in [70] and [68], but we present it here for clarity.

Let $M_q\phi = q\phi$ be the multiplication operator with q. We start by showing that for $q \in L^\infty(\mathbb{R}^n)$ with compact support $M_q : L^2_{\delta-1}(\mathbb{R}^n) \to L^2_\delta(\mathbb{R}^n)$ is a bounded operator.

Indeed, we have for $f \in L^2_{\delta-1}(\mathbb{R}^n)$

$$\begin{aligned}
\|M_q f\|_\delta &= \left[\int_{\mathbb{R}^n} (1+|x|^2)^\delta |q(x)f(x)|^2\,dx\right]^{1/2} \\
&= \left[\int_{\mathbb{R}^n} \left(1+|x|^2\right)|q(x)|^2\left(1+|x|^2\right)^{\delta-1}|f(x)|^2\,dx\right]^{1/2} \\
&\leq \|\langle x\rangle q\|_\infty \|f\|_{\delta-1} < c\|f\|_{\delta-1},
\end{aligned}$$

where $c = \|\langle x\rangle q\|_\infty$ is finite due to the compact support of q.

Now, we define the operator $A_\zeta = C_\zeta M_q$, where C_ζ is the convolution with g_ζ and is given by

$$A_\zeta f(x) = \int_{\mathbb{R}^n} g_\zeta(x-y)q(y)f(y)\,dy = C_\zeta M_q f \tag{4.12}$$

By proposition 4.2.2 for $|\zeta| \geq R$ we obtain

$$\|A_\zeta f\|_{\delta-1} = \|C_\zeta M_q f\|_{\delta-1} \leq \frac{c(\delta,R)}{|\zeta|}\|M_q f\|_\delta \leq \frac{c(\delta,R)}{|\zeta|}\,\|\langle x\rangle q\|_\infty\,\|f\|_{\delta-1}$$

Therefore, A_ζ is bounded in $L^2_{\delta-1}(\mathbb{R}^n)$ and if

$$|\zeta| > R := c(\delta,R)\,\|\langle x\rangle q\|_\infty$$

then A_ζ is a contraction, which implies that $I + A_\zeta$ is invertible by Neumann series.

Since $q \in L^\infty$ and has compact support then it is in L^2_δ and therefore the right-hand side of (4.9) is in $L^2_{\delta-1}$. Hence, the unique solution to (4.6) is given by

$$\mu(x,\zeta) = -\left[I + A_\zeta\right]^{-1}(g_\zeta * q).$$

From here, we have

$$\psi = e^{ix\cdot\zeta}\left(1 - [I + A_\zeta]^{-1}(g_\zeta * q)\right)$$

solves the integral equation (4.8). Furthermore, the estimate (4.11) easily follows from $[I + A_\zeta]^{-1}$ being bounded in $L^2_{\delta-1}$ and the estimate (4.10) in Proposition 4.2.2.

To establish uniqueness, suppose that there exists two solutions ψ_1, ψ_2 of (4.8) such that

$$\mu_j = e^{-ix\cdot\zeta}\psi_j - 1 \in L^2_{\delta-1}.$$

Then, the difference $\mu_1 - \mu_2 = e^{-ix\cdot\zeta}(\psi_1 - \psi_2)$ is also in $L^2_{\delta-1}$ and both fulfil the equation

$$[I + A_\zeta]\mu_j = -g_\zeta * q.$$

This implies $[I + A_\zeta]\left(e^{-ix\cdot\zeta}(\psi_1 - \psi_2)\right) = 0$ and thus $\psi_1 \equiv \psi_2$ by the invertibility of $I + A_\zeta$ in $L^2_{\delta-1}$. Hence, uniqueness of the integral equation (4.8) is established for exponentially growing solutions. $\square$

Remark, that during the proof there is no requirement of q being real or complex, since it follows by a contraction argument.

When a solution does not exist or is not unique, we define ζ as an *exceptional point.* The above result guarantees that there are no large exceptional points. To be clear, we define them as follows.

Definition 4.2.4. *Let $q \in L^\infty(\Omega)$ complex-valued and extended to zero outside Ω.*

*Let $\zeta \in \mathcal{V} := \{\zeta \in \mathbb{C}^n \setminus \{0\} | \, \zeta \cdot \zeta = 0\}$. Then we call $\zeta \in \mathcal{V}$ an **exceptional point** for q if there is no unique exponential growing solution of $(-\Delta + q)\psi = 0$ in $\mathbb{R}^n$, that is, there is no unique solution of the type*

$$\psi(x, \zeta) := e^{ix\cdot\zeta}\left(1 + \mu(x, \zeta)\right), \text{ with } \mu \in L^2_{\delta-1}(\mathbb{R}^n),\ 0 < \delta < 1. \tag{4.13}$$

4.2.3 Uniqueness of complex conductivities

The uniqueness proof given in [70] for potentials $q \in L^\infty$ only requires large ζ. There is no explicit mention of potentials being real-valued, and from an overview of the used arguments and results, it is clear that this is also not required. For completeness, we follow this proof to establish uniqueness for complex conductivity potentials q from their Dirichlet-to-Neumann map Λ_q. Thereafter, we connect the dots with further results from the literature on differential equations for complex conductivities to provide a proof of Theorem 4.1.1.

A first step is to show that the exponentially growing solutions outside Ω are uniquely identified by the Dirichlet-to-Neumann map. Recall, for complex conductivity potentials q, 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$ in Ω.

Lemma 4.2.5. *Let q_1, $q_2 \in L^\infty(\Omega)$ be complex conductivity potentials, extended to zero outside Ω. Further, let $\zeta \in \mathcal{V}$ a non-exceptional point for q_1, q_2. Suppose that $\Lambda_{q_1} = \Lambda_{q_2}$ and ψ_1, ψ_2 are the unique solutions of $(-\Delta + q_j)\psi_j = 0$ in $\mathbb{R}^n$ of the form $e^{ix\cdot\zeta}(1+\psi_j)$. Then*

$$\psi_1 = \psi_2, \qquad \text{in } \mathbb{R}^n \setminus \Omega.$$

Proof. Since 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$, let $v \in H^1(\Omega)$ be the unique solution of

$$\begin{aligned} -\Delta v + q_2\, v = 0, &\qquad \text{in } \Omega \\ v|_{\partial\Omega} = \psi_1|_{\partial\Omega}\,. & \end{aligned}$$

Then we define

$$h = \begin{cases} v, & \text{in } \Omega \\ \psi_1, & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}.$$

Since $\Lambda_{q_1} = \Lambda_{q_2}$ it holds that $\Lambda_{q_1}\, \psi_1|_{\partial\Omega} = \Lambda_{q_2}\, \psi_1|_{\partial\Omega}$ and thus $\frac{\partial\psi_1}{\partial n} = \frac{\partial v}{\partial n}$. This implies that h as well as $\frac{\partial h}{\partial n}$ is continuous over $\partial\Omega$. Therefore, h solves $-\Delta h + q_2\, h = 0$ in $\mathbb{R}^n$ and has the appropriate asymptotics since ψ_1 has them. By the uniqueness theorem it follows that $h = \psi_2$ and thus $\psi_1 = \psi_2$ in $\mathbb{R}^n \setminus \Omega$. □

With this equality outside, we can now show uniqueness of potential inside like [70]. Thereafter, we can use this result to prove Theorem 4.1.1.

Theorem 4.2.6. *Let q_1, $q_2 \in L^\infty(\Omega)$ be complex conductivity potentials extended to zero outside Ω.*

$$\textit{If } \Lambda_{q_1} = \Lambda_{q_2}, \textit{ then } q_1 = q_2.$$

Proof. Let $k \in \mathbb{R}^n$ be fixed and for $m, s \in \mathbb{R}^n$ we set

$$\zeta = \frac{1}{2}\left((k+s) + im\right) \text{ and } \tilde{\zeta} = \frac{1}{2}\left((k-s) - im\right)$$

with $k \cdot s = k \cdot m = s \cdot m = 0$ and $|k|^2 + |s|^2 = |m|^2$. The ζ, $\tilde{\zeta}$ are in $\mathbb{C}^n$ and fulfil the condition $\zeta \cdot \zeta = 0$. Hence, taking s, m large enough we obtain solutions ψ_j of the integral equation (4.8) for their respective potentials with respect to $\tilde{\zeta}$ by Corollary 4.2.3.

Since $\Delta e^{ix\cdot\zeta} = 0$, by applying the second Green identity we obtain the following relation for each $j = 1, 2$:

$$\begin{aligned} \int_\Omega e^{ix\cdot\zeta} q_j(x)\psi_j(x)\, dx &= \int_\Omega e^{ix\cdot\zeta}\Delta\psi_j(x) - \psi_j \Delta e^{ix\cdot\zeta}\, dx \\ &= \int_{\partial\Omega} e^{ix\cdot\zeta}\frac{\partial\psi_j}{\partial n} - \psi_j\,(n\cdot i\zeta)\, e^{ix\cdot\zeta}\, dS(x), \end{aligned}$$

where dS is the surface element.

By hypothesis $\Lambda_{q_1} = \Lambda_{q_2}$ it holds from Lemma 4.2.5 that $\psi_1|_{\partial\Omega} = \psi_2|_{\partial\Omega}$.

As such, it further holds

$$\left.\frac{\partial \psi_1}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \psi_2}{\partial n}\right|_{\partial\Omega}$$

as ψ_j solve the interior problem $(-\Delta + q_j)\psi_j = 0$ with boundary value $\psi_j|_{\partial\Omega}$.

Hence, the fact that the right-hand side of the integral above is equal for both q_j implies the relation:

$$\int_\Omega e^{ix\cdot\zeta}\,(q_1\mu_1 - q_2\mu_2)\,dx = 0.$$

Due to the asymptotic behaviour of ψ_j with respect to $\tilde{\zeta}$ we get

$$\int_\Omega e^{ix\cdot(\zeta+\tilde{\zeta})}\,(q_1 - q_2)\,dx = \int_\Omega e^{ix\cdot(\zeta+\tilde{\zeta})}\,(q_1\mu_1 - q_2\mu_2)\,dx.$$

Using $\zeta + \tilde{\zeta} = k$ and taking modulus we obtain by Cauchy-Schwarz inequality and Corollary 4.2.3 the inequality

$$\begin{aligned}\left|\int_\Omega e^{ix\cdot k}\,(q_1 - q_2)\,dx\right| &\leq \sum_{j=1}^{2}\int_\Omega |q_j\mu_j| \leq \sum_{j=1}^{2}\|q_j\|_{1-\delta}\|\mu_j\|_{\delta-1}\\ &\leq \sum_{j=1}^{2}\frac{C}{|\tilde{\zeta}|}\|q_j\|_{1-\delta}\|q_j\|_{\delta}.\end{aligned}$$

Since $\tilde{\zeta}$ was arbitrarily depending on s, we can take the limit as $|s| \to \infty$. This implies that the left-hand side equals to zero for each fixed $k \in \mathbb{R}^n$. As such, for all $k \in \mathbb{R}^n$ it holds

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2)\,dx = \int_{\mathbb{R}^3} e^{ix\cdot k}(q_1 - q_2)\,dx = 0.$$

Therefore, by Fourier inversion theorem we obtain $q_1 = q_2$ in Ω. □

Again, recall that there is no requirement or reference to q being complex-valued during the proofs. All presented work proves uniqueness for complex conductivity potentials q from their respective Dirichlet-to-Neumann map Λ_q. This was implicitly present in the literature, in particular, see [70].

However, it is not yet clear or immediate how this proof also implies uniqueness of the respective complex conductivity γ from its Dirichlet-to-Neumann map Λ_γ.

Our contribution is to establish this connection by making clear what results are required for this to happen. With it, we prove Theorem 4.1.1.

As we have seen a first step is to show that Λ_q is uniquely determined by Λ_γ, $\gamma|_{\partial\Omega}$ and $\left.\frac{\partial\gamma}{\partial n}\right|_{\partial\Omega}$. Furthermore, these boundary values are uniquely determined by Λ_γ. We prove it later on by also extending results in [69] to complex conductivities.

To finish, we just need to determine uniquely γ from q. The idea arises from [68] where the used result is strictly for real-valued potentials. However for complex conductivities it follows by a result in the book of Gilbarg and Trudinger [32] together with a perturbation argument.

The restriction may have just been a causality due disinterest in complex conductivities at that time.

The result to fix the gap is related with uniqueness of the interior problem with a complex potential.

Proposition 4.2.7. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Suppose that $q \in L^\infty(\bar{\Omega})$ is complex conductivity potential.*

Then for every $f \in H^{3/2}(\partial\Omega)$ there is a unique $u \in H^2(\Omega)$ such that

$$\begin{cases} (-\Delta + q)\, u = 0 \ \text{in}\ \Omega \\ u|_{\partial\Omega} = f. \end{cases} \tag{4.14}$$

The solution operator is defined by $P_q f := w$ and has the mapping property

$$P_q : H^{3/2}(\partial\Omega) \to H^2(\Omega).$$

Moreover, the Dirichlet-to-Neumann map operator has the mapping property

$$\Lambda_q : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega).$$

Proof. The proof follows by studying first a Laplace interior problem and showing that multiplication by q is a compact operator from $H^2(\Omega)$ to $L^2(\Omega)$.

Thus, let

$$P_0 : H^2(\Omega) \to L^2(\Omega) \times H^{3/2}(\partial\Omega), \quad u \mapsto (\,-\Delta u,\, \mathrm{tr}\ u\,)\,.$$

By definition of $H^2(\Omega)$ and trace properties for $C^{1,1}$-domains the operator P_0 is linear and bounded. By Theorem 9.15. of [32], there always exists a unique solution in $H^2(\Omega)$ of

$$\begin{cases} -\Delta u = f, \\ u|_{\partial\Omega} = g. \end{cases}$$

Therefore, the operator P_0 is bijective and invertible. In particular, it is Fredholm of index zero.

Analogously, we define the operator

$$P_q : H^2(\Omega) \to L^2(\partial\Omega) \times H^{3/2}(\partial\Omega), \quad u \mapsto (\,[-\Delta + q]\, u,\ \mathrm{tr}\ u)\,.$$

Then, the difference of operators $P_q - P_0$ maps u to $(qu,\ 0)$ over the same spaces. Since, the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it immediately follows that multiplication by $q \in L^\infty(\Omega)$ is a compact operator. Hence, by definition $P_q - P_0$ is a compact operator. Since $P_q = P_0 + (P_q - P_0)$ is the sum of a Fredholm operator of index zero and a compact operator, it is a Fredholm of index zero.

Thus, to show invertibility we prove that ker $P_q = \{0\}$. Let $u \in$ ker P_q. By definition this implies u is a solution in $H^2(\Omega)$ of

$$\begin{cases} (-\Delta\psi + q)u = 0, \\ u|_{\partial\Omega} = 0, \end{cases}$$

but due to the assumption of 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$, given that q is a complex conductivity potential in Ω it follows that $u \equiv 0$. □

Since our main assumption is $\gamma \in C^{1,1}(\bar{\Omega})$, it is in $H^2(\Omega)$. Therefore, for potentials q given by a complex conductivity the following statement holds.

Corollary 4.2.8. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $\gamma \in C^{1,1}(\bar{\Omega})$ such that* Re $\gamma \geq c > 0$.

Further, let $q \in L^\infty(\Omega)$ be given as $q = \Delta\gamma^{1/2}/\gamma^{1/2}$ for which 0 *is not a Dirichlet eigenvalue of* $(-\Delta + q)$. *Then the unique solution $\psi \in H^2(\Omega)$ of*

$$\begin{cases} -\Delta\psi + q\psi = 0 \\ \psi|_{\partial\Omega} = \gamma^{1/2} \end{cases} \tag{4.15}$$

is $\psi \equiv \gamma^{1/2}$.

The proof immediately follows by Proposition 4.2.7, and it establishes a unique correspondence between complex conductivity potentials and complex conductivities. Therefore, we have that Λ_γ uniquely establishes γ proving Theorem 4.1.1.

4.2.4 Preliminaries for reconstruction method

The first part of Calderón problem is answered for complex conductivities $\gamma \in C^{1,1}(\Omega)$, that is, γ is uniquely determined by Λ_γ. However, in practical terms the interest resides more on the second part, the reconstruction of γ from the measurements encapsulated in the Dirichlet-to-Neumann map.

The novel nature of Nachman's work in [68] resides precisely on the reconstruction method for real conductivities. One of the initial results is based on the scattering transform and the fact we can reconstruct the potential q from its asymptotics. Due to instability of the limiting procedure, Nachman was not satisfied, so he was able to obtain a procedure to reconstruct the potential from asymptotic behaviour of the solutions ψ we described before, but by taking integration. This becomes a more stable process of reconstruction. However, it is still to be computationally implemented and studied.

Recently, in [39] and [40] there has been an implementation of Nachman's work based on the asymptotic behaviour of the scattering transform, that is, the first unstable method we described in the previous paragraph. They introduce a regularization strategy, like the one initially done by Siltanen, Mueller and Isaacson for the 2D case [48]. While their focus is in real conductivities, they indeed apply the method for Dirichlet-to-Neumann maps given from complex-conductivities and obtain very reasonable results. However, there was no mention of Nachman's work being also feasible for this case.

Therefore, the following subsections are dedicated to this purpose. We extend Nachman's reconstruction methods to complex conductivities in $C^{1,1}(\Omega)$. Furthermore, we explain how further results from [22] can also be obtained for complex conductivities. This may be feasible for future numerical implementations.

In this sense, our work fills the gap in the literature for complex conductivities. Even though minimalistic most experts assume it to be true. Due to this, we restrict ourselves to stating some of the results and pointing to the original proofs.

To start, we introduce some preliminaries results concerned with boundary value operators that are independent of the nature of the conductivity and are connected with Faddeev-Green function G_ζ.

Analogously to the classical single and double layer potentials we define the respective operators for G_ζ. The single layer operator is defined as

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x-y) f(y)\, ds(y)$$

and the double layer as

$$D_\zeta f(x) = \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial n}(x-y) f(y)\, ds(y).$$

Moreover, taking the trace of double layer potential it holds

$$B_\zeta f(x) := \text{ p.v. } \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial n}(x-y) f(y)\, ds(y), \text{ for } x \in \partial\Omega.$$

Since the singularity of G_ζ for x near y is the same as G_0, it is locally integrable on $\partial\Omega$ and the trace of S_ζ is continuous on the $\partial\Omega$.

We state here the properties that Nachman established in its Appendix and that are essential to establish the proof of the reconstruction method.

Proposition 4.2.9. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$.*

(i) For $0 \leq s \leq 1$

$$\|S_\zeta f\|_{H^{s+1}(\partial\Omega)} \leq c(\zeta, s) \|f\|_{H^s(\partial\Omega)}. \tag{4.16}$$

(ii) For $0 \leq s \leq \frac{3}{2}$ we have that B_ζ is bounded in $H^s(\partial\Omega)$.

Let ρ_0 be a number large enough so that $\bar{\Omega} \subset \{x : |x| < \rho_0\}$. For any $\rho > \rho_0$ we define $\Omega'_\rho = \{x : x \notin \bar{\Omega},\, |x| < \rho\}$.

Lemma 4.2.10. *If $f \in H^{1/2}(\partial\Omega)$, the function $\phi = S_\zeta f$ has the following properties*

(i) $\Delta\phi = 0$ *in* $\mathbb{R}^n \setminus \partial\Omega$.

(ii) $\phi \in H^2(\Omega)$ *and* $\phi \in H^2(\Omega'_\rho)$ *for any* $\rho > \rho_0$.

(iii) ϕ *satisfies an analogue to the Sommerfeld radiation condition. For almost every x it holds*

$$\lim_{\rho\to\infty} \int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial\phi}{\partial n(y)} - \phi(y)\frac{\partial G_\zeta}{\partial n(y)}(x-y) \right] ds(y) = 0. \tag{4.17}$$

In fact, for $\rho > \rho_0$ the above identity holds for $|x| < \rho$ even without taking the limit.

(iv) Let $B^\dagger_\zeta$ denote the operator on the boundary

$$B^\dagger_\zeta f(x) = \text{ p.v. } \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial n(x)}(x-y) f(y)\, ds(y). \tag{4.18}$$

It follows that the (non-tangential) limits $\partial\phi/\partial n_+$, $\partial\phi/\partial n_-$ of the normal derivative of ϕ as the boundary is approached from the outside and inside Ω, respectively, are given by

$$\frac{\partial\phi}{\partial n_\pm} = \mp\frac{1}{2} f(x) + B^\dagger_\zeta f(x), \qquad \text{for almost every } x \in \partial\Omega. \tag{4.19}$$

(v) The boundary values ϕ_+, ϕ_- of ϕ from outside and inside of Ω, respectively, are identical as elements of $H^{3/2}(\partial\Omega)$ and agree with the trace of the single layer potential $S_\zeta f$.

Lemma 4.2.11. *If $f \in H^{3/2}(\partial\Omega)$ the function $\psi = D_\zeta f$ defined in $\mathbb{R}^n \setminus \partial\Omega$ has the properties (i), (ii) and (iii) of Lemma 4.2.10.*

Moreover, the non-tangential limits ψ_+, ψ_- of ψ as we approach the boundary from outside and inside of Ω, respectively, exist and satisfy

$$\psi_\pm(x) = \pm\frac{1}{2} f(x) + B_\zeta f(x), \text{ for almost every } x \in \partial\Omega. \tag{4.20}$$

Finally, to establish the existence of solutions for small values of ζ we also require the following estimate on the single layer potential, obtained in [22]. This is based on the difference $H_\zeta = G_\zeta - G_0$ being an harmonic function and on the properties of this operator being related with G_ζ. We have:

Lemma 4.2.12. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. The Faddeev fundamental solution G_ζ can be given through the decomposition*

$$G_\zeta(x) = G_0(x) + H_\zeta(x),$$

where G_0 is the classical fundamental solution and H_ζ is an harmonic function.

Moreover, the single and double layer operators have a similar decomposition and, for our own convenience, we present here the case for the single layer. For $f \in H^{1/2}(\partial\Omega)$ we have

$$S_\zeta f(x) = S_0 f(x) + \int_{\partial\Omega} H_\zeta(x-y) f(y)\, ds(y) =: S_0 f(x) + \mathcal{H}_\zeta f(x).$$

Further, it holds

$$\|\mathcal{H}_\zeta\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{3/2}(\partial\Omega))} \leq C|\zeta|^{n-2},$$

where the constant C only depends on the domain.

The idea now is to establish a relation between solutions behavior outside and on the boundary $\partial\Omega$.

4.2.5 From the outside to the boundary $\partial\Omega$

The properties of the boundary operator above allow us to establish a one-to-one correspondence between the solution of a boundary integral equation and of the following exterior problem

$$\begin{cases} (i) & \Delta\psi = 0, & \text{in } \Omega' := \mathbb{R}^n \setminus \bar{\Omega}, \\ (ii) & \psi \in H^2(\Omega'_\rho), & \text{for any } \rho > \rho_0, \\ (iii) & \psi(x,\zeta) - e^{ix\cdot\zeta} & \text{satisfies (4.17)}, \\ (iv) & \frac{\partial\psi}{\partial n_+} = \Lambda_q \psi & \text{on } \partial\Omega. \end{cases} \tag{4.21}$$

In this section, we assume that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$ and $q \in L^\infty(\Omega)$ is a complex conductivity potential, i.e., 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$.

Moving forward, we provide all proofs necessary to establish the reconstruction method. Remark, that most of them follow directly from Nachman's work [68]. We highlight, again, the new pieces needed to put the puzzle together.

Our first step is to establish a relation between a boundary integral equation and the above exterior problem (4.21). In essence, we are only demonstrating that if a solution to the boundary integral equation exists than we can define a solution for the exterior problem from it, and vice-versa.

Lemma 4.2.13. *Let $\zeta \in \mathcal{V}$.*

(a) Suppose ψ solves the exterior problem (4.21). Then its trace $f_\zeta = \psi_+ = \psi|_{\partial\Omega}$ solves the boundary integral equation

$$f_\zeta = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta. \tag{4.22}$$

(b) Conversely, suppose $f_\zeta \in H^{3/2}(\partial\Omega)$ solves (4.22). Then the function $\psi(x,\zeta)$ defined for $x \in \Omega'$ by

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - (S_\zeta \Lambda_q - D_\zeta)\, f_\zeta(x) \tag{4.23}$$

solves the above exterior problem under all conditions. Furthermore, $\psi|_{\partial\Omega} = f_\zeta$.

Proof. a) Assume ψ solves (4.21). We apply Green's identity to G_ζ and ψ in Ω'_ρ, $\rho > \rho_0$. It holds

$$\begin{aligned}\left(\int_{|y|=\rho} - \int_{\partial\Omega}\right) & \left[G_\zeta(x-y)\frac{\partial\psi}{\partial n_+} - \psi_+(y,\zeta)\frac{\partial G_\zeta}{\partial n_+(y)}(x-y)\right] ds(y) \\ &= \int_{\Omega'_\rho} \left[G_\zeta(x-y)\Delta\psi(y,\zeta) - \psi(y,\zeta)\Delta_y G_\zeta(x-y)\right] dy.\end{aligned} \tag{4.24}$$

Since ψ is harmonic on Ω'_ρ and G_ζ is the fundamental solution of $-\Delta$ we obtain for arbitrary $x \in \Omega'_\rho$

$$\begin{aligned}\psi(x,\zeta) = & \int_{|y|=\rho}\left[G_\zeta(x-y)\frac{\partial\left(\psi - e^{iy\cdot\zeta}\right)}{\partial n} - \left(\psi - e^{iy\cdot\zeta}\right)\frac{\partial G_\zeta}{\partial n_+(y)}(x-y)\right] ds(y) \\ & + \int_{|y|=\rho}\left[G_\zeta(x-y)\frac{\partial e^{iy\cdot\zeta}}{\partial n} - e^{iy\cdot\zeta}\frac{\partial G_\zeta}{\partial n(y)}(x-y)\right] ds(y) \\ & - \int_{\partial\Omega} G_\zeta(x-y)\frac{\partial\psi}{\partial n_+}\, ds(y) - \int_{\partial\Omega}\psi_+(y,\zeta)\frac{\partial G_\zeta}{\partial n(y)}(x-y)\, ds(y)\end{aligned} \tag{4.25}$$

By hypothesis (4.21-iii) the first integral vanishes. The function $e^{iy\cdot\zeta}$ is harmonic and a re-application of Green's identity to the second integral on $|y| < \rho$ equals $e^{ix\cdot\zeta}$. Finally, due to (4.21- iv) the last integral is $[S_\zeta\Lambda_q - D_\zeta]\,\psi$. Thus, for $x \in \Omega'$ the function ψ fulfills the identity

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - [S_\zeta\Lambda_q - D_\zeta]\, f_\zeta.$$

Taking the non-tangential limit to the boundary from the outside by Lemmas 4.2.10, 4.2.11 we obtain

$$f_\zeta(x) = e^{ix\cdot\zeta} - \left[S_\zeta\Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta(x).$$

b) Conversely, suppose $f_\zeta \in H^{3/2}(\partial\Omega)$ solves the boundary integral equation (4.22). Define a function ψ in Ω' by

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - [S_\zeta\Lambda_q - D_\zeta]\, f_\zeta(x). \tag{4.26}$$

We show that this ψ solves the exterior problem (4.21) from properties of the single and double layer (Lemma 4.2.10 and 4.2.11).

It is immediate to see that ψ fulfills the condition i) of (4.21), since for $\zeta\cdot\zeta = 0$ the exponential $e^{ix\cdot\zeta}$ is harmonic, and $S_\zeta\Lambda_q f_\zeta$, $D_\zeta f_\zeta$ are harmonic in Ω' by the above mentioned lemmas. Moreover, it holds that $S_\zeta \Lambda_q f_\zeta$, $D_\zeta f_\zeta \in H^2(\Omega'_\rho)$, $\rho > \rho_0$ and further the identity (4.17) also holds. Hence, the conditions ii) and iii) of the exterior problem follow.

To show the last condition, we approach the boundary $\partial\Omega$ non-tangentially from the outside and we obtain, as in part a),

$$\psi|_{\partial\Omega} = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta.$$

By virtue of f_ζ fulfilling the boundary integral equation the right-hand side equals f_ζ and therefore $\psi|_{\partial\Omega} = f_\zeta$. From this and the first three properties of (4.21), that we already showed ψ fulfills, we can obtain analogously to part a)

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta\left(\frac{\partial\psi}{\partial n_+}\right) + D_\zeta f_\zeta, \qquad \text{for } x \in \Omega'. \tag{4.27}$$

Subtracting both formulae of ψ, (4.27) and (4.26), the following equality holds throughout Ω'

$$S_\zeta\left[\Lambda_q f_\zeta - \frac{\partial\psi}{\partial n_+}\right] = 0. \tag{4.28}$$

By taking traces from the outside, it actually holds on the boundary $\partial\Omega$. We are reminded that $S_\zeta\left[\Lambda_q f_\zeta - \frac{\partial\psi}{\partial n_+}\right]$ is harmonic in $\mathbb{R}^n \setminus \partial\Omega$ and since the trace is 0 on $\partial\Omega$ uniqueness of the interior problem for $q \equiv 0$ implies that the equality (4.28) holds everywhere. Then, its normal derivatives will be zero and subtracting them on $\partial\Omega$ with the help of (4.19) we obtain

$$[\Lambda_q - \partial\psi/\partial n_+] = \frac{\partial S_\zeta\left[\Lambda_q - \partial\psi/\partial n_+\right]}{\partial n_-} - \frac{\partial S_\zeta\left[\Lambda_q - \partial\psi/\partial n_+\right]}{\partial n_+} = 0. \tag{4.29}$$

Thus the last condition of the exterior problem follows. □

The end goal is to connect the boundary integral equation to exponential growing solutions of Faddeev integral equation (4.8). For such, we relate uniquely the exterior problem to these solutions, and due to the previous result, a connection immediately follows with the boundary integral equation.

Lemma 4.2.14. *Let $\zeta \in \mathcal{V}$.*

(a) Suppose $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is a solution of

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\psi(y,\zeta).$$

Then the restriction of ψ to Ω' solves the exterior problem (4.21) and fulfils the respective conditions i)-iv).

(b) Conversely, if ψ solves the exterior problem (4.21), there is a unique solution $\tilde{\psi} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ of the integral equation (4.8), such that $\tilde{\psi} = \psi$ in Ω'.

Proof. a) From the proposition 4.2.2 it follows $\psi \in H^2_{\text{loc}}(\mathbb{R}^n)$, which immediately implies condition ii) of the exterior problem. Moreover, in $\mathbb{R}^n$ it holds $(-\Delta + q)\psi = 0$, thus due to $q \equiv 0$ on Ω' the condition i) holds, i.e., $-\Delta\psi = 0$ in Ω'.

Applying Green identity on $|y| < \rho$ yields

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \psi}{\partial n(y)} - \psi(y,\zeta)\frac{\partial G_\zeta}{\partial n(y)}(x-y) \right] ds(y)$$
$$= \int_{|y|<\rho} G_\zeta(x-y)q(y)\psi(y,\zeta)\,dy + \psi(x,\zeta), \text{ for a.e. } x \text{ with } |x| < \rho.$$

Now, we can choose ρ large in order to contain the support of q. Since ψ solves the integral equation this means that the right-hand side equals $e^{ix\cdot\zeta}$. Moreover, we already showed that

$$e^{ix\cdot\zeta} = \int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial e^{iy\cdot\zeta}}{\partial n(y)} - e^{iy\cdot\zeta}\frac{\partial G_\zeta}{\partial n(y)}(x-y) \right] ds(y).$$

Then passing the exponential to the right-hand side, we obtain

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \left(\psi - e^{iy\cdot\zeta}\right)}{\partial n(y)} - \left(\psi(y,\zeta) - e^{iy\cdot\zeta}\right)\frac{\partial G_\zeta}{\partial n(y)}(x-y) \right] ds(y) = 0$$

for all $\rho > \rho_0$. Thus condition iii) follows by taking the limit as $\rho \to \infty$.

Immediately, we can see that $\Lambda_q\psi_- = \frac{\partial\psi}{\partial n_-}$ and since $\psi \in H^2$ in a two-sided neighbourhood of $\partial\Omega$ it holds that $\psi_- = \psi_+$ and $\frac{\partial\psi}{\partial n_-} = \frac{\partial\psi}{\partial n_+}$. This leads to ψ fulfilling the condition iv). Therefore, the restriction of ψ to Ω' solves the exterior problem (4.21).

b) Suppose ψ defined in Ω' solves the exterior problem (4.21). Set $\tilde{\psi}$ by $\tilde{\psi} = P_q\psi_+$ in Ω and $\tilde{\psi} = \psi$ in Ω'. Then on $\partial\Omega$,

$$\tilde{\psi}_- = (P_q\psi_+) = \psi_+ = \tilde{\psi}_+$$

and

$$\frac{\partial\tilde{\psi}}{\partial n_-} = \Lambda_q\psi_+ = \frac{\partial\psi}{\partial n_+} = \frac{\partial\tilde{\psi}}{\partial n_+}$$

due to iv). Thus $\tilde{\psi}$ solves $(-\Delta + q)\tilde{\psi} = 0$ on $\mathbb{R}^n$. Applying Green's formula in $|y| < \rho$ yields

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \psi}{\partial n(y)} - \psi(y,\zeta)\frac{\partial G_\zeta}{\partial n(y)}(x-y) \right] ds(y)$$
$$= \int_{|y|<\rho} G_\zeta(x-y)q(y)\tilde{\psi}(y,\zeta)\,dy + \tilde{\psi}(x,\zeta)$$

for almost every x with $|x| < \rho$. Thus by letting $\rho \to \infty$ the radiation condition iii) implies that the left-hand side is $e^{ix\cdot\zeta}$. Hence $\tilde{\psi}$ verifies the desired integral equation in $\mathbb{R}^n$.

To finalize we prove that this extension is unique. Suppose that we have two extensions $\tilde{\psi}^1, \tilde{\psi}^2 \in L^2_{\text{loc}}(\mathbb{R}^n)$ of ψ which agree in Ω' and solve the integral equation everywhere. As in part a), we see that $\tilde{\psi}^1, \tilde{\psi}^2 \in H^2_{\text{loc}}(\mathbb{R}^n)$ and $(-\Delta + q)\tilde{\psi}^j = 0$ in $\mathbb{R}^n$ for $j = 1$, 2. Hence, they are in H^2 on a two-sided neighbourhood of $\partial\Omega$. This implies that $\tilde{\psi}^j_+ = \tilde{\psi}^j_-$, for $j = 1, 2$, which promptly leads to $\tilde{\psi}^1_- = \tilde{\psi}^2_-$ since they agree on Ω'. Now, from the uniqueness of the interior problem it follows that $\tilde{\psi}^1 = \tilde{\psi}^2$. $\square$

With Lemma 4.2.13 and 4.2.14 we have therefore established a relation between the boundary integral equation and exponential growing solutions of the type (4.5) for the Schrödinger equation in $\mathbb{R}^n$.

One interesting remark, there is no explicit requirement of ζ being large. Hence, by showing that the boundary integral equation is uniquely solvable for small values of ζ we guarantee the existence of exponential growing solutions for these ζ.

In the other direction, due to Corollary 4.2.3 we guarantee the existence of a unique solution for the boundary integral equation for large values of ζ.

Moreover, in all proofs above there is no explicit difference for q being real or complex.

The importance of this relation lies on the boundary integral equation only being dependent on information we can obtain in practice, i.e., the Dirichlet-to-Neumann map.

Keeping this in mind, we focus on the solvability of the boundary integral equation. The following proposition glues together the papers [22,68] and applies them to the complex potential. It uses the uniqueness of the interior problem that we established in Proposition 4.2.7 for this case.

Proposition 4.2.15. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let q be a complex conductivity potential in $L^\infty(\Omega)$.*

Define

$$K_\zeta = S_\zeta \Lambda_q - B_\zeta - \frac{1}{2} I \tag{4.30}$$

and for any $\zeta \in \mathcal{V}$ it holds

(a) The operators K_0, K_ζ are compact on $H^{3/2}(\partial\Omega)$.

(b) If $\operatorname{Re} q \geq 0$, then $I + K_0$ is invertible in $H^{3/2}(\partial\Omega)$.

(c) If $\operatorname{Re} q \geq 0$ there exists an $\epsilon > 0$ with $|\zeta| < \epsilon$ for which the operator $I + K_\zeta$ is invertible in $H^{3/2}(\Omega)$.

(d) There exists an $R > 0$ such that for all $|\zeta| > R$ the operator $I + K_\zeta$ is invertible in $H^{3/2}(\partial\Omega)$.

Proof. Part a) follows by a compact embedding. Let $f \in H^{3/2}(\partial\Omega)$ and set $w = P_q f$ as the solution of interior Dirichlet problem (4.14). For $x \in \Omega$ we use the Green's formula to obtain

$$\int_\Omega G_\zeta(x-y)\Delta w(y)\, dy + w(x) = [S_\zeta \Lambda_q - D_\zeta]\, f(x)$$

which is equivalent to

$$\int_\Omega G_\zeta(x-y) q(y) P_q f(y)\, dy + w(x) = [S_\zeta \Lambda_q - D_\zeta]\, f(x)$$

By letting x approach the boundary non-tangentially from the inside we thus obtain

$$\mathrm{tr}\ (G_\zeta * (qP_q f)) + f(x) = S_\zeta \Lambda_q f(x) - \left[-\frac{1}{2} f(x) + B_\zeta f(x)\right]$$

and therefore

$$\left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2} I\right] f = \mathrm{tr}\ (G_\zeta * (qP_q f))\,.$$

Hence, our desired operator satisfies this factorization, where the following mapping properties hold

$P_q : H^{3/2}(\partial\Omega) \to H^2(\Omega)$;

$\imath : H^2(\Omega) \to L^2(\Omega)$ is a compact embedding;

$M_q : L^2(\Omega) \to L^2(\Omega)$;

$\mathbf{G}_\zeta : L^2(\Omega) \to H^2(\Omega)$ convolution with G_ζ, which we prove up next;

$\mathrm{tr} : H^2(\Omega) \to H^{3/2}(\partial\Omega)$.

The operator defined before as $\mathrm{tr}\ (G_\zeta * (qP_q f))$ is now given as $\mathrm{tr}\ (G_\zeta M_q \imath P_q)\, f$.

The compactness of the embedding implies compactness of the desired operator.

b) Let $\zeta = 0$. In this case G_0 is the classical fundamental solution and the corresponding operators are the classical ones. By part a), we already know that $S_0\Lambda_q - B_0 - \frac{1}{2}I$ is compact on $H^{3/2}(\partial\Omega)$. Then $I + K_0 = \left[\frac{1}{2}I + S_0\Lambda_q - B_0\right]$ is Fredholm of index zero on $H^{3/2}(\partial\Omega)$. Therefore, it is enough to show injectivity.

Let $h \in H^{3/2}(\partial\Omega)$ such that $\left[\frac{1}{2}I + S_0\Lambda_q - B_0\right] h = 0$. Define $w = -S_0\Lambda_q h + D_0 h$. Then w is harmonic in $\mathbb{R}^n$, $w \in H^2(\Omega)$ and $w \in H^2(\Omega'_\rho)$ by Lemma 4.2.10 and 4.2.11. Moreover, approaching the boundary non-tangentially by the inside we obtain

$$w_- = -S_0\Lambda_q h + \left(-\frac{1}{2} h + B_0 h\right) = -\left[\frac{1}{2} I + S_0\Lambda_q - B_0\right] h = 0.$$

Since the problem $-\Delta w = 0,\ w|_{\partial\Omega} = 0$ is uniquely solvable in $H^2(\Omega)$ it follows that $w \equiv 0$ in Ω and thus $\frac{\partial w}{\partial n_-} = 0$ on $\partial\Omega$.

By using the jump relations for the single and double layer operator (see [62]), we can deduce that

$$[w] = w_+ - w_- = w_+ = [D_0 h] = h$$

and

$$\left[\frac{\partial w}{\partial n}\right] = \frac{\partial w}{\partial n_+} = -\left[\frac{\partial}{\partial n} S_0 \Lambda_q h\right] = \Lambda_q h.$$

Now, by Proposition 4.2.7 there is a unique solution $u \in H^2(\Omega)$ of

$$\begin{cases} (-\Delta + q)\, u = 0, \\ u|_{\partial\Omega} = h, \end{cases}$$

such that $\Lambda_q h = \frac{\partial u}{\partial n_-}\Big|_{\partial\Omega}$. We set

$$v = \begin{cases} u, \text{ in } \Omega \\ w, \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$

and see that $u_- = w_+ = h$ and $\frac{\partial u}{\partial n_-} = \frac{\partial w}{\partial n_+} = \Lambda_q$, thus it holds that v and $\frac{\partial v}{\partial n}$ are continuous over the boundary $\partial\Omega$. Therefore $v \in H^2(B_\rho(0))$, $\rho > 0$ and it solves $-\Delta v + qv = 0$ in $\mathbb{R}^n$, since $q \equiv 0$, in $\mathbb{R}^n \setminus \Omega$.

Let $\chi_\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ in $B_{\rho-\epsilon}(0)$ and $\chi \equiv 0$ in $\rho - \epsilon < |x| < \rho$, for $\epsilon > 0$ small enough.

Then for $\phi \in H^1(\mathbb{R}^n)$ it follows by Green's identity

$$\int_{|x|<\rho} (-\Delta v + qv)\,(\chi\phi)\; dx = 0,$$

which is equivalent to

$$\int_{|x|<\rho} \nabla v \cdot \nabla(\chi\phi) + qv(\chi\phi)\, dx = 0,$$

as well as

$$\int_\Omega \nabla v \cdot \nabla\phi + qv\phi\, dx + \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla(\chi\phi)\, dx = 0.$$

In particular we can take $\phi = \bar{v}$ and since w is given through the classical single and double layer it follows that $\nabla w \in L^2(B_\rho(0) \setminus \bar{\Omega})$. Thus taking the limit as $\rho \to \infty$

$$\begin{aligned} &\int_\Omega |\nabla v|^2\phi + q|v|^2\, dx + \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla(\chi\bar{w})\, dx = 0, \\ &\int_\Omega |\nabla v|^2\phi + q|v|^2\, dx + \int_{B_\rho(0)\setminus\Omega} |\nabla w|^2\, dx = \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla((1-\chi)\bar{w})\, dx, \end{aligned}$$

which yields

$$\int_{\mathbb{R}^n} |\nabla v|^2 + q|v|^2\, dx = 0$$

and therefore

$$\int_{\mathbb{R}^n} |\nabla v|^2 + (\operatorname{Re} q)|v|^2\, dx = 0.$$

Now, we can apply Hardy's inequality for $H^1(\mathbb{R}^n)$

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^n} |x|^{-2}|v|^2\, dx \leq \int_{\mathbb{R}^n} |\nabla v|^2\, dx$$

to finally obtain the condition

$$\int_{\mathbb{R}^n} \left[\frac{(d-2)^2}{4|x|^2} + (\mathrm{Re}\ q(x)) \right] |v|^2 \, dx \leq 0.$$

Hence, for Re $q \geq 0$ this implies that $v \equiv 0$ in $\mathbb{R}^n$. Thus $h \equiv 0$ in $\partial\Omega$. Hence we obtain invertibility in the case $\zeta = 0$. Notice that we have been loose on the requirement for q, since this is enough for the purpose of complex conductivity, but this proof works for potentials that satisfy the estimate Re $q(x) \geq -\frac{(d-2)^2}{4|x|^2}$.

Part c) follows quite easily by the fact that the set of invertible operators is open. However, we present the result with the help of some estimates and Neumann series.

For $h \in H^{3/2}(\partial\Omega)$, it holds that K_ζ as defined in (4.30) also has the form $K_\zeta h = S_\zeta(\Lambda_q - \Lambda_0)h$, since due to Green's formula we have $B_\zeta = -\frac{1}{2}I + S_\zeta\Lambda_0$.

Moreover, by lemma 4.2.12 and for $h \in H^{3/2}(\partial\Omega)$ we have the decomposition $S_\zeta(\Lambda_q - \Lambda_0)h = S_0(\Lambda_q - \Lambda_0)h + \mathcal{H}_\zeta(\Lambda_q - \Lambda_0)h$. Moreover, we also have by the lemma the estimate

$$\|\mathcal{H}_\zeta(\Lambda_q - \Lambda_0)h\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta|^{n-2}\|(\Lambda_q - \Lambda_0)h\|_{H^{1/2}(\partial\Omega)} \leq C|\zeta|^{n-2}\|h\|_{H^{3/2}(\partial\Omega)}.$$

From the invertibility of $I + K_0$ we obtain the decomposition

$$[I + K_\zeta] = I + K_0 + \mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right) = (I + K_0)\left(I + (I + K_0)^{-1}\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\right)$$

and if

$$\|\left(I + K_0\right)^{-1}\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} < 1$$

we obtain invertibility for $I + K_\zeta$ in $H^{3/2}(\partial\Omega)$. This norm can be translated to an estimate for ζ by the above on $\mathcal{H}_\zeta$. We have

$$\begin{aligned}
&\|\left(I + K_0\right)^{-1}\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} \\
&\qquad\qquad \leq C|\zeta|^{n-2}\left\|(I + K_0)^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}\|\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} < 1.
\end{aligned}$$

Hence, for

$$|\zeta| < \left[\frac{1}{\left\|(I + K_0)^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}\|\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}}\right]^{1/(n-2)} =: \epsilon,$$

invertibility is obtained by Neumann series.

Part d) uses the existence of exponentially growing solutions for large values of $|\zeta|$. Let $R > 0$ be large enough such that for $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$, $|\zeta| > R$ we have unique exponential growing solutions of (4.8), corollary 4.2.3 . Under these conditions, we have shown that $K_\zeta := S_\zeta\Lambda_q - B_\zeta - \frac{1}{2}I$ is compact in $H^{3/2}(\partial\Omega)$. Therefore, $I + K_\zeta$ is a Fredholm operator of index zero in $H^{3/2}(\partial\Omega)$. We need to show that the kernel is empty to prove that it is invertible.

Let $g \in H^{3/2}(\partial\Omega)$ be in $\ker K$. Then $h = [-S_\zeta \Lambda_q + D_\zeta]g$ solves the exterior problem i), ii), iv) and fulfils the radiation condition (4.17) (the proof is analogous to Lemma 4.2.13).

Moreover, we can extend h to a solution $\tilde{h}$ of $\tilde{h} = -\int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\tilde{h}(y)\,dy$ in all of $\mathbb{R}^n$ (analogous to the previous lemma). By the estimates on G_ζ we note that $e^{-ix\cdot\zeta}\tilde{h} \in L^2_{\delta-1}(\mathbb{R}^n)$, $0 < \delta < 1$ and

$$e^{-ix\cdot\zeta}\tilde{h} = -A_\zeta(e^{-ix\cdot\zeta}\tilde{h})$$

with A_ζ defined as in (4.12). Since, we took $R > 0$ large enough then A_ζ is a contraction in $L^2_{\delta-1}(\mathbb{R}^n)$ and this forces $\tilde{h} \equiv 0$. Therefore,

$$g \equiv 0 \text{ and } I + K_\zeta \text{ is invertible in } H^{3/2}(\partial\Omega)$$

and the statement is proven. □

We would like to remark that the proof of b) in Proposition 4.2.15 needed some tweaks to accommodate the complex potential and the spaces in focus. The core idea is the same: prove invertibility by showing injectivity due to the compactness of K_0. In [22] the proof is established over $H^{1/2}(\partial\Omega)$, but due to our interest in connecting with Nachman's work we showed it for $H^{3/2}(\partial\Omega)$.

The beauty of this proposition is that now we can solve the boundary integral equation for small and large values of $|\zeta|$ and obtain ψ on $\partial\Omega$ by

$$\psi(x,\zeta) = \left[\frac{1}{2}I + S_\zeta\Lambda_q - B_\zeta\right]^{-1}\left(e^{ix\cdot\zeta}\right).$$

As we remarked before, this immediately brings that there exists exponentially growing solutions of (4.8) for large, but most importantly, small values of ζ and we have a method to obtained their boundary values.

The importance of these boundary values lies in the definition of the scattering transform. The scattering transform is an essential tool in the reconstruction and uniqueness proofs of Calderón problem, since it connects the boundary data with inside information about the potential and devises a method to recover it.

As such, all of this work allows us to obtain the following theorem:

Theorem 4.2.16. *Suppose that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in L^\infty(\Omega)$ be complex-valued and suppose that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω.*

We define the scattering transform for non-exceptional points $\zeta \in \mathcal{V}$ by

$$\mathbf{t}(\xi,\zeta) = \int_{\mathbb{R}^3} e^{-ix\cdot(\zeta+\xi)} q(x)\psi(x,\zeta)\,dx,\ \xi \in \mathbb{R}^n. \tag{4.31}$$

Then, for each $\xi \in \mathbb{R}^n$ we can compute the scattering transform for the non-exceptional points $\zeta \in \mathcal{V}_\xi := \left\{\zeta \in \mathbb{C}^n \setminus \{0\} : \zeta\cdot\zeta = 0,\ |\xi|^2 + 2\zeta\cdot\xi = 0\right\}$ from the solutions of the boundary integral equation by

$$\mathbf{t}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\zeta+\xi)}\left[\Lambda_q + i(\xi+\zeta)\cdot n\right]\psi(x,\zeta)\,ds(x),\ \xi \in \mathbb{R}^n. \tag{4.32}$$

Proof. From the Lemma 4.2.13 and 4.2.14 we obtain unique exponentially growing solutions of (4.8) by the one-to-one relation with the boundary integral (4.22). Therefore, by Green identity it holds

$$\begin{aligned}\mathbf{t}(\xi,\zeta) &= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)\,dx \\ &= \int_\Omega e^{-ix\cdot(\xi+\zeta)} \Delta\psi(x,\zeta) - \left(\Delta e^{-ix\cdot(\zeta+\xi)}\right)\psi(x,\zeta)\,dx \\ &= \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)} \left[\Lambda_q + i(\xi+\zeta)\cdot n\right]\psi(x,\zeta)\,ds(x)\end{aligned}$$

for $\xi \in \mathbb{R}^n$ and $\zeta \in \mathcal{V}_\xi$ such that the boundary integral equation has a unique solution. □

4.2.6 From t to γ

Finally, we show how to obtain the desired potential from the scattering transform. This subsection is completely focused on this aspect and brings forward two possible methods. One of them arises from joining results in [22] and [68].

The first method uses large asymptotics of the scattering transform to obtain the potential Fourier transform. Unfortunately, this requires solving the boundary integral equation for large values ζ, which is inherently very unstable. In [39] they avoid the boundary integral equation by using the approximation $\psi(x,\zeta) \approx e^{ix\cdot\zeta}$ to compute the scattering transform. As we have already mentioned, this even allowed them to reconstruct complex potentials. However, it is only an approximation and still contains some reconstruction flaws.

The second method still requires some extra results for practical implementation to be feasible. It is based on the reconstruction of the complex conductivity from exponentially growing solutions for small values of ζ in Ω. However, there is yet to be an appropriate study of the method to determine these solutions in Ω. In two dimensions, this was established in [69] through a $\bar{\partial}$-method. However, it is still an open question how to solve similar equations introduced by Nachman in [68] for small values of ζ.

The first method is based on the following asymptotic:

Theorem 4.2.17. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in L^\infty(\Omega)$ be a complex conductivity potential extended to zero outside Ω. Then for $|\zeta| > R$ and $0 < \delta < 1$ it holds:*

$$|\mathbf{t}(\xi,\zeta) - \hat{q}(\xi)| \leq \frac{\tilde{c}(\delta,R)}{|\zeta|}\|q\|_\delta^2 \quad \textit{for all } \xi \in \mathbb{R}^n. \tag{4.33}$$

Proof. The proof follows trivially by the corollary 4.2.3. If $q \in L^\infty(\Omega)$ is a complex-valued and compactly supported potential it follows that $\hat{q}$ is well-defined and holds true

$$\begin{aligned}|\mathbf{t}(\xi,\zeta) - \hat{q}(\xi)| &= \left|\int e^{-ix\cdot\xi} q(x)\left[e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\right]\,dx\right| \\ &\leq \|q\|_{1-\delta}\|e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\|_{\delta-1} \leq \frac{\tilde{c}(\delta,R)}{|\zeta|}\|q\|_\infty^2.\end{aligned}$$

□

This is enough to reconstruct the Fourier transform of the potential q and thereafter one can obtain γ by solving the Schrödinger equation with boundary value $\gamma^{1/2}$ as in Corollary 4.2.8. Again, we remark that the reconstruction from large complex frequencies is a very unstable method and therefore there is still interest in obtaining a more stable direct reconstruction method.

The purpose of the second method is to fix this instability. It is based on the so-called $\bar{\partial}$ equation and on compatibility equations satisfied by $\mathbf{t}$ known since ([1,9,44],), Nachman was able to derive a equation in three dimensions, which allows to obtain solutions μ that eventually permits the computation of $\hat{q}$ from $\mathbf{t}(\xi,\zeta)$ for $\xi \in \mathbb{R}^n$, $|\zeta| \geq M$, $(\xi+\zeta)^2 = 0$ and its derivative in ζ. Although more elaborate than in two dimensions, this method does not require taking the limit of $|\zeta| \to \infty$. In essence, the proof follows through for the complex-potential with some slight adaptations.

For such, let $\psi(x,\zeta)$ be the solution of (4.8) with $e^{-ix\cdot\zeta}\psi(x,\zeta) - 1 \in L^2_{\delta-1}(\mathbb{R}^n)$, that is, ζ is not an exceptional point. Define,

$$\mu(x,\zeta) := |q(x)|e^{-ix\cdot\zeta}\psi(x,\zeta) \tag{4.34}$$

then μ solves the following integral equation

$$\mu(x,\zeta) = |q(x)| - \tilde{A}_\zeta \mu(\cdot,\zeta) \tag{4.35}$$

where

$$\tilde{A}_\zeta f(x) := |q(x)| \int_{\mathbb{R}^n} g_\zeta(x-y)\tilde{q}(y)f(y)\,dy$$

with $\tilde{q}(x) = q(x)/|q(x)|$ in the support of q and 0 otherwise. Moreover, the scattering transform is given through

$$\mathbf{t}(\xi,\zeta) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\tilde{q}(y)\mu(x,\zeta)\,dx. \tag{4.36}$$

Lemma 4.2.18. *Suppose $q \in L^\infty(\mathbb{R}^n)$ is complex-valued potential and as compact support. Let $R > Mc(\delta,a)\|q(x)\langle x\rangle^{1-\delta}\|_{L^\infty}$ with $\delta \in (0,1)$, $c(\delta,a)$ as in Proposition 4.2.2 and M a constant depending on the support of q.*

(i) If $|\zeta| > R$, $\zeta\cdot\zeta = 0$, then (4.35) has a unique solution $\mu(\cdot,\zeta)$ in $L^2(\mathbb{R}^n)$ with compact support.

(ii) For $|\zeta| > R$, $\zeta\cdot\zeta = 0$ and all $w \in \mathbb{C}^n$ with $w\cdot\bar{\zeta} = 0$,

$$w \cdot \frac{\partial\mu}{\partial\bar{\zeta}}(x,\zeta) = \frac{-1}{(2\pi)^{n-1}} \int e^{ix\cdot\xi} w\cdot\xi\,\delta(|\xi|^2 + 2\zeta\cdot\xi)t(\xi,\zeta)\mu(x,\zeta+\xi)\,d\xi. \tag{4.37}$$

Proof. (i) By Proposition 4.2.2 we show that $\tilde{A}_\zeta$ is bounded in $L^2(\mathbb{R}^n)$. The idea follows from Corollary 4.2.3.

Let $f \in L^2(\mathbb{R}^n)$ and $\delta \in (0,1)$ then

$$\begin{aligned}\|\tilde{A}_\zeta f\|_{L^2} &= \||q(x)|g_\zeta * (\tilde{q}f)\|_{L^2} \le \|\langle x\rangle^{1-\delta}|q|\|_\infty \|g_\zeta * (\tilde{q}f)\|_{\delta-1} \\ &\le \frac{c(\delta,a)}{|\zeta|}\left\|\langle x\rangle^{1-\delta}|q|\right\|_\infty \|\tilde{q}f\|_\delta \le \frac{c(\delta,a)}{|\zeta|}\left\|\langle x\rangle^{1-\delta}|q|\right\|_\infty \left\|\tilde{q}\langle x\rangle^\delta\right\|_\infty \|f\|_{L^2} \\ &\frac{Mc(\delta,a)}{|\zeta|}\left\|\langle x\rangle^{1-\delta}|q|\right\|_\infty \|f\|_{L^2},\end{aligned}$$

where $M > 0$ such that $\|\langle x\rangle^\delta\|_\infty \le M$. The norm of A_ζ is less than

$$Mc(\delta,a)|\zeta|^{-1}\|q\langle x\rangle^{1-\delta}\|_\infty,$$

so if $|\zeta| \ge R$ we have that $\tilde{A}_\zeta$ is a contraction and we obtain the unique solution as

$$\mu(\cdot,\zeta) = (I+\tilde{A}_\zeta)^{-1}|q|^{1/2}. \tag{4.38}$$

(ii) We start by differentiating the distribution g_ζ defined in (4.7):

$$w \cdot \frac{\partial g}{\partial \bar{\zeta}} = \frac{1}{(2\pi)^{n-1}} \int e^{ix\cdot\xi} w \cdot \xi \delta(\xi^2 + 2\zeta\cdot\xi)\, d\xi. \tag{4.39}$$

We claim that the operator

$$w \cdot \frac{\partial}{\partial \bar{\zeta}} \tilde{A}_\zeta f(x) = \frac{|q(x)|}{(2\pi)^{n-1}} \int e^{ix\cdot\xi} w \cdot \xi \delta(\xi^2 + 2\zeta\cdot\xi)(\tilde{q}f)^{(}\xi)\, d\xi \tag{4.40}$$

is bounded on $L^2(\mathbb{R}^n)$. To see this, we regard the right-hand side of the formulation as the composition of the following operators:

(a) the operator $M_{\tilde{q}} : L^2\mathbb{R}^n \to L^2_{\delta+1}(\mathbb{R}^n)$ given by multiplication with $\tilde{q}$,

(b) the operator $\mathcal{F}_{\mathcal{M}_\zeta} : L^2_{\delta+1}(\mathbb{R}^n) \to L^2(\mathcal{M}_\zeta)$ which takes f to $\hat{f}\Big|_{\mathcal{M}_\zeta}$ that is the restriction of the Fourier transform to $\mathcal{M}_\zeta = \{\xi \in \mathbb{R}^n : \xi^2 + 2\zeta\cdot\xi = 0\}$,

(c) the operator $M_{w\cdot\xi} : L^2(\mathcal{M}_\zeta) \to L^2(\mathcal{M}_\zeta)$ defined as multiplication by $w\cdot\xi$,

(d) the operator $S : L^2(\mathcal{M}_\zeta) \to L^2_{-(\delta+1)}(\mathbb{R}^n)$ given by

$$h \to \frac{1}{(2\pi)^{n-1}} \int e^{ix\cdot\xi} \delta(\xi^2 + 2\zeta\cdot\xi) h(\xi)\, d\xi, \tag{4.41}$$

(e) the operator $M_{|q|} : L^2_{-(\delta+1)} \to L^2(\mathbb{R}^n)$ of multiplication by $|q|$.

The operators $\mathcal{F}_{\mathcal{M}_\zeta}$ and S are bounded by Theorem IX.39 in [79] by noting that $\mathcal{M}_\zeta$ is a compact sub-manifold of $\mathbb{R}^n$ of co-dimension 2. The multiplier $M_{w\cdot\xi}$ is bounded by $2|w||\zeta|$ on $\mathcal{M}_\zeta$. Further, both $M_{\tilde{q}}$ and $M_{|q|}$ both have norms less than $\|q(x)\langle x\rangle^{1+\delta}\|_\infty^{1/2}$. Differentiating (4.38) we get for $|\zeta| > R$ the following equation

$$w \cdot \frac{\partial \mu}{\partial \bar{\zeta}} = -\left(I + \tilde{A}_\zeta\right)^{-1}\left(w \cdot \frac{\partial}{\partial \bar{\zeta}}\tilde{A}_\zeta\right)\left(I + \tilde{A}_\zeta\right)^{-1}|q|. \tag{4.42}$$

Due to the boundedness and continuity as a function in ξ for $\tilde{q}\mu \in L^1$ we obtain by the Fourier Transform

$$\mathcal{F}\left[\tilde{q}\left(I+\tilde{A}_\zeta\right)^{-1}|q|\right](\xi) = \int e^{-ix\cdot\xi}\tilde{q}(x)\mu(x,\zeta)\,dx = t(\xi,\zeta). \tag{4.43}$$

Due to (4.40) we now get

$$w\cdot\frac{\partial\mu}{\partial\bar{\zeta}} = -\frac{1}{(2\pi)^{n-1}}\left(I+\tilde{A}_\zeta\right)^{-1}\int |q(x)|e^{ix\cdot\xi}w\cdot\xi\delta(\xi^2+2\zeta\cdot\xi)t(\xi,\zeta)\,d\xi \tag{4.44}$$

the integral over $\mathcal{M}_\zeta$ being absolutely convergent. Now observe that

$$e^{-ix\cdot\xi}g(x\cdot\zeta) = g(x,\zeta+\xi),\ \text{if}\ \xi^2+2\zeta\cdot\xi = 0,$$

hence, for such ξ and $f \in L^2(\mathbb{R}^n)$, we get

$$\tilde{A}_\zeta(e^{i\langle\cdot,\xi\rangle}f) = e^{i\langle\cdot,\xi\rangle}\tilde{A}_{\zeta+\xi}f. \tag{4.45}$$

If $\xi^2+2\zeta\cdot\xi = 0$, then $|\zeta+\xi| = |\zeta| \geq R$; thus $I+\tilde{A}_{\zeta+\xi}$ is invertible and

$$\left(I+\tilde{A}_\zeta\right)^{-1}\left(e^{i\langle\cdot,\xi\rangle}|q|\right) = e^{i\langle\cdot,\xi\angle}\left(I+\tilde{A}_{\zeta+\xi}\right)^{-1}|q| \tag{4.46}$$

$$= e^{i\langle\cdot,\xi\rangle}\mu\left(\mu,\zeta+\xi\right). \tag{4.47}$$

Substituting the last expression in (4.44) yields the desired equation.

□

The above Lemma shows that the $\bar{\partial}$-equation for μ can be solved for large complex frequency values. However, it is yet to be shown if it is possible to uniquely solve it for small complex frequency values. While a unique solution μ in $L^2(\mathbb{R}^n)$ to (4.35) can be proven analogously through the boundary integral equation, the $\bar{\partial}$-equation as of now requires the invertibility of $(I+\tilde{A}_\zeta)$. This has only been obtained for large complex frequency values of ζ.

Due to the invertibility of boundary integral operator we have shown that there are no exceptional points near 0 for complex-valued potentials. The only restriction we need to impose for this purpose is that γ is equal to 1 near the boundary. This is not a strict restriction since we can change slightly the domain to obtain it and the new Dirichlet-to-Neumann map follows from the previous one accordingly.

Therefore, analogously to [22] we are able to obtain an estimate for non-exceptional points close to zero.

Lemma 4.2.19. *Let $\gamma \in C^{1,1}(\Omega)$ be the complex-conductivity with $\sigma \geq c > 0$, $\epsilon \geq 0$, $\omega \in \mathbb{R}^+$ and suppose $\gamma \equiv 1$ near $\partial\Omega$. Set $q = (\Delta\gamma^{1/2})/\gamma^{1/2} \in L^\infty(\Omega)$.*

For $\zeta \in \mathcal{V}$ sufficiently small and $\psi \in H^{3/2}(\partial\Omega)$ the corresponding boundary integral solution of (4.22), it holds

$$\|\psi(\cdot,\zeta)-1\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta|. \tag{4.48}$$

Proof. Recall that for $h \in H^{3/2}(\partial\Omega)$ it holds that K_ζ as defined in (4.30) can also be obtained through $K_\zeta h = S_\zeta(\Lambda_q - \Lambda_0)h$, since due to Green's formula we have $B_\zeta = -\frac{1}{2}I + S_\zeta\Lambda_0$.

As such, boundary integral equation solutions fulfill

$$\psi(x,\zeta) - 1 = \left(e^{ix\cdot\zeta} - 1\right) - K_\zeta\left(\psi(x,\zeta) - 1\right).$$

The introduction of 1 to K_ζ follows from $\Lambda_q 1 = 0$, since by Proposition 4.2.7 the unique H^2-solution of $(-\Delta + q)u = 0,\ u|_{\partial\Omega} = 1$ is $\gamma^{1/2}$, which is constant near $\partial\Omega$. Furthermore, $\Lambda_0 1 = 0$ due to $w = 1$ being the unique harmonic function in $H^2(\Omega)$ with boundary value 1.

Under the conditions on γ it holds that Re $q > 0$ and hence by proposition 4.2.15 it holds that $[I + K_\zeta]$ is invertible in $H^{3/2}(\partial\Omega)$ for small ζ and hence,

$$\psi - 1 = [I + K_\zeta]^{-1}\left(e^{ix\cdot\zeta} - 1\right).$$

By Taylor series we have the estimate $\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial\Omega)} \leq C_1|\zeta|$ for small values of ζ, and $\left\|[I + K_\zeta]^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}$ is uniformly bounded for small $|\zeta|$ due to Neumann series inversion. Thus,

$$\|\psi - 1\|_{H^{3/2}(\partial\Omega)} \leq C_2\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial\Omega)} \leq C_3|\zeta|$$

and the statement follows. $\square$

Theorem 4.2.20. *Let $\gamma \in C^{1,1}(\Omega)$ be the complex-conductivity with $\sigma \geq c > 0$, $\epsilon \geq 0, \omega \in \mathbb{R}^+$ and suppose $\gamma \equiv 1$ near $\partial\Omega$. Set $q = (\Delta\gamma^{1/2})/\gamma^{1/2} \in L^\infty(\Omega)$.*

For $\zeta \in \mathcal{V}$ small enough such that (4.8) has unique exponentially growing solutions $\psi(x,\zeta)$, it holds

$$\|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{L^2(\Omega)} \leq C|\zeta|. \tag{4.49}$$

Proof. Since $\gamma = 1$ near the boundary $\partial\Omega$ we have that $\gamma^{1/2}$ is the unique $H^2(\Omega)$ solution of

$$\begin{cases} -\Delta u + qu = 0, \text{ in } \Omega \\ u|_{\partial\Omega} = 1. \end{cases}$$

By the elliptic estimates, we obtain that

$$\begin{aligned} \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{L^2(\Omega)} &\leq \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{H^2(\Omega)} \\ &\leq \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta| \end{aligned}$$

and the statement follows. $\square$

This theorem states that we can reconstruct the complex-conductivity from the exponentially growing solutions by

$$\gamma(x)^{1/2} = \lim_{|\zeta|\to 0} \psi(x,\zeta), \quad \text{for a.e. } x \in \Omega. \tag{4.50}$$

However, recall that for small ζ we only know how to obtain the boundary values of the exponential growing solutions from the boundary measurements. To provide a reconstruction of γ in Ω it is necessary to compute these solutions for all small enough ζ inside Ω from the scattering data or the Dirichlet-to-Neumann map. This might be possible by the $\bar{\partial}$-equation, but it is still an open question.

The required layout to obtain a $\bar{\partial}$ reconstruction method for complex conductivities is as follows:

1. **$\bar{\partial}$ equation for ζ non-exceptional and considerably small:** To establish the $\bar{\partial}$-equation (4.37) we required the invertibility of the operator $(I + \tilde{A}_\zeta)$ in $L^2(\mathbb{R}^n)$, however this is only available for $|\zeta| > R$ and R large enough to guarantee a contraction argument. In the case of small values a different proof for the $\bar{\partial}-$equation is required, since the above argument is not available. Recall, that the existence of exponentially growing solutions was obtained through the boundary integral equation, but this does not guarantee the invertibility of the operator for small-values.

2. **Solvability of $\bar{\partial}$ equation:** In Lemma 4.2.18 we have deduce the $\bar{\partial}$ equation that connects the scattering transform to the exponentially growing solutions. However, the unique solvability of this equation is still an open question. In this sense, we need to study the equation in the space $\mathcal{V} \setminus \{\zeta \in \mathbb{C}^n : \epsilon \leq |\zeta| < R\}$ and thereafter we can obtain the conductivity through the limiting procedure provided in (4.50). Similar work was established in two-dimensions in [58].

In order to conclude the reconstruction and uniqueness proofs of $\gamma \in C^{1,1}(\Omega)$ from its Dirichlet-to-Neumann map Λ_γ, we have to establish a relation between Λ_q and Λ_γ, where q is the complex conductivity potential. This is our task for the next subsection.

4.2.7 Reconstruction of Λ_q from the boundary measurements Λ_γ

The Dirichlet-to-Neumann map Λ_γ is bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. Moreover, it is properly defined through

$$\langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \tag{4.51}$$

where u is the unique $H^1(\Omega)$ solution of the interior problem $\nabla \cdot (\gamma \nabla u) = 0$ in Ω and $u|_{\partial\Omega} = f$ and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$.

We can also define the Dirichlet-to-Neumann map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ for the Schrödinger operator through the weak-formulation as

$$\langle \Lambda_q \tilde{f}, \tilde{g} \rangle = \int_\Omega \nabla \tilde{u} \cdot \nabla \tilde{v} + q \tilde{u} \tilde{v} \, dx, \quad \text{for all } \tilde{v} \in H^1(\Omega), \text{ s.t. } \tilde{v}|_{\partial\Omega} = \tilde{g},$$

and $\tilde{u} \in H^1(\Omega)$ is the unique solution to $(-\Delta + q)\tilde{u} = 0, \text{ in } \Omega, \ \tilde{u}|_{\partial\Omega} = \tilde{f}$.

As in the real case, since both problems are interconnected we can obtain Λ_q from Λ_γ by

$$\Lambda_q = \gamma^{-1/2} \left[\Lambda_\gamma + \frac{1}{2} \frac{\partial \gamma}{\partial n} \right] \gamma^{-1/2}. \tag{4.52}$$

This brings to light that we can determine Λ_q from Λ_γ and the boundary values $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial n}\Big|_{\partial\Omega}$. Moreover, if $\gamma \equiv 1$ near $\partial\Omega$ then for $\gamma \in W^{2,\infty}(\Omega)$ it holds that $\Lambda_q = \Lambda_\gamma$. Otherwise, we need to obtain a method to reconstruct these boundary values.

There are many results to compute these boundary values. However, most of them need further smoothness. Nachman has proven the best result for the conditions in play. In [69] he showed that the boundary values can be obtained without further smoothness assumptions. Following his proof we see that there is no explicit requirement of γ being real, besides the fact that $\operatorname{Re}\gamma \geq c > 0$ and uniqueness of the Dirichlet problem in $H^1(\Omega)$ holds. Hence, we can quickly extend the result for complex-conductivities in $W^{2,\infty}(\Omega)$.

In this section, we will work with a domain $\Omega \in \mathbb{R}^n$, for $n \geq 2$, which is of Lipschitz type. Hence the domain can be partitioned into $N \geq 1$ connected components $\Omega^1, ..., \Omega^N$. Further, we defined the Neumann-to-Dirichlet map R (NtD) on the space

$$\mathring{H}^{-1/2}(\partial\Omega) = \{h \in H^{-1/2}(\partial\Omega) : \langle h, 1\rangle_{\partial\Omega^j} = 0,\ j = 1, ..., N\} \tag{4.53}$$

by $Rh = w|_{\partial\Omega}$, with $w \in H^1(\Omega)$ the weak solution (unique modulo functions constant on each Ω^j) of $\Delta w = 0$, $\frac{\partial w}{\partial n} = h$.

For the proofs we will need to define a new function. Let $x_0 \in \partial\Omega^{j_0}$, let $U = B \times I \subset \mathbb{R}^n$ be a cylindrical neighborhood with coordinates chosen so as to have $\Omega \cap U = \Omega^{j_0} \cap U = \{(x', x_n) \in U : x_n < \phi(x')\}$ with ϕ a Lipschitz function. Let $h \in L^2(\partial\Omega)$ with support in $\partial\Omega^{j_0} \cap U$. For any $\eta \in \mathbb{R}^{n-1} \times \{0\}$, we define the function h_η to be identically zero on $\partial\Omega^{j_0}$ for $j \neq j_0$, and

$$h_\eta(x) = h(x) e^{-ix\cdot\eta} - \frac{1}{|\partial\Omega^{j_0}|} \int_{\partial\Omega\cap U} h(y) e^{-iy\cdot\eta}\, d\sigma(y) \text{ for } x \in \partial\Omega^{j_0}. \tag{4.54}$$

First, we establish an auxiliary lemma that connects the conductivity to the Dirichlet-to-Neumann map and Neumann-to-Dirichlet map at once. Notice, that these results hold for general dimensions.

Lemma 4.2.21. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ with connect components $\Omega^1, ..., \Omega^N$. Assume $\gamma \in W^{1,r}(\Omega)$ for $r > n$ and* $\operatorname{Re}\gamma \geq c > 0$.

Then for any $f \in H^{1/2}(\partial\Omega)$ and $h \in \mathring{H}^{-1/2}(\partial\Omega)$ the identity holds

$$\langle h, (\gamma - \mathcal{R}\Lambda_\gamma) f\rangle = \int_\Omega u \nabla w \cdot \nabla\gamma, \tag{4.55}$$

where $u \in H^1(\Omega)$ is the solution of $\nabla\cdot(\gamma\nabla u) = 0$, $u|_{\partial\Omega} = f$, and $w \in H^1(\Omega)$ is a weak solution of $\Delta w = 0$ in Ω with $\frac{\partial w}{\partial n} = h$ and $\mathcal{R}$ denotes the Neumann-to-Dirichlet map.

Proof. Since w is a weak solution of the Neumann problem it follows by the Green identity that for any $v \in H^1(\Omega)$ the following identity holds:

$$\begin{aligned} \langle h, v|_{\partial\Omega}\rangle &= \int_{\partial\Omega} hv|_{\partial\Omega}\, dS = \int_{\partial\Omega} \frac{\partial w}{\partial n}\bigg|_{\partial\Omega} v|_{\partial\Omega}\, dS(x) = \int_{\Omega} v\Delta w + \nabla v\cdot\nabla w\, dV(x) \\ &= \int_{\Omega} \nabla v\cdot\nabla w\, dV(x). \end{aligned} \tag{4.56}$$

Moreover, by definition we have $Rh = w|_{\partial\Omega}$ (modulo constant functions on each $\partial\Omega^j$), so by the weak definition of the Dirichlet-to-Neumann map (1.6), presented above, it follows

$$\langle Rh, \Lambda_\gamma f\rangle = \int_{\Omega} \gamma\nabla w\cdot\nabla u\, dV(x), \tag{4.57}$$

where the constant mentioned above does not affect the computations because by the same weak formulation we have $\langle 1, \Lambda_\gamma f\rangle_{\partial\Omega^j} = 0$.

Multiplication by γ is a bounded operator on $H^1(\Omega)$ due to the Sobolev embedding of $W^{1,r}(\Omega) \subset C_B(\Omega)$. Therefore, we can apply (4.56) to $v = \gamma u$

$$\langle h, \gamma f\rangle = \int_{\Omega} \nabla w\cdot\nabla(\gamma u) = \int_{\Omega} \gamma\nabla w\cdot\nabla u + u\nabla w\cdot\nabla\gamma\, dV(x). \tag{4.58}$$

Now, we subtract the equation (4.57) from the above one (4.58). Hereby, we use the R symmetry property, which is obtained through application of the Green identities. Thus, we have

$$\langle h, \gamma f\rangle - \langle h, R\Lambda_\gamma f\rangle = \langle h, \gamma f\rangle - \langle Rh, \Lambda_\gamma f\rangle = \int_{\Omega} u\nabla w\cdot\nabla\gamma\, dV(x).$$

□

In second, we establish a relation for the normal derivative of the conductivity at $\partial\Omega$.

Lemma 4.2.22. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 2$. Assume γ is in $W^{2,p}(\Omega)$, $p > n/2$ and* $\operatorname{Re}\gamma \geq c > 0$*. For any f, $g \in H^{1/2}(\partial\Omega)$ the identity holds*

$$\langle g, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial n}\right) f\rangle = \int_{\Omega} 2v\nabla(u-u_0)\cdot\nabla\gamma + v(2u-u_0)\Delta\gamma\, dV(x),$$

where u, u_0, v are respectively the $H^1(\Omega)$ solutions of $\nabla\cdot(\gamma\nabla u) = 0$, $\Delta u_0 = 0$ and $\Delta v = 0$, in Ω, with $u|_{\partial\Omega} = f$, $u_0|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$.

Proof. First by Sobolev embedding theorem it follows that $\gamma v \in H^1(\Omega)$, and again by the DtN definition (1.6) we have

$$\langle g, \gamma\Lambda_1 f\rangle = \langle \gamma g, \Lambda_1 f\rangle = \int_{\Omega} \nabla(\gamma v)\cdot\nabla u\, dV(x). \tag{4.59}$$

Moreover, by noting that $\gamma(2u - u^0)|_{\partial\Omega} = \gamma f$ and $\gamma(2u - u^0) \in H^1(\Omega)$ and that Λ_1 is the Dirichlet-to-Neumann map of the Laplacian we obtain

$$\langle g, \Lambda_1\gamma f\rangle = \int_{\Omega} \nabla v\cdot\nabla(\gamma(2u-u^0))\, dV(x). \tag{4.60}$$

By hypothesis we have $\gamma \in W^{2,p}(\Omega), p > \frac{n}{2}$, thus Sobolev embedding theorem implies that $\frac{\partial\gamma}{\partial n}$ defines a bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ defined via through:

$$\left\langle g, \frac{\partial\gamma}{\partial n} f \right\rangle = \int_{\partial\Omega} fg\frac{\partial\gamma}{\partial n}\, dS(x);$$

which by applying Green identities also satisfies

$$\left\langle g, \frac{\partial\gamma}{\partial n} f \right\rangle = \int_{\Omega} \nabla(uv)\cdot\nabla\gamma + uv\Delta\gamma\, dV(x) \tag{4.61}$$

for any $u, v \in H^1(\Omega)$ with traces f, g on $\partial\Omega$, respectively.

Now, if we use the above equation with $2u - u^0$ instead of just u, we obtain

$$\begin{aligned}\left\langle g, \frac{\partial\gamma}{\partial n} f \right\rangle &= \int_{\Omega} \nabla((2u-u^0)v)\cdot\nabla\gamma + (2u-u^0)v\Delta\gamma\, dV(x)\\ &= \int_{\Omega}(2u-u^0)\nabla v\cdot\nabla\gamma + v\nabla(2u-u^0)\cdot\nabla\gamma + (2u-u_0)v\Delta\gamma\, dV(x).\end{aligned}$$

Hence, by using this formula, the (1.6) and the above weak formulations (4.59), (4.60) it holds

$$\begin{aligned}\left\langle g, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial n}\right) f \right\rangle &= 2\left\langle g, \Lambda_\gamma f\right\rangle - \left\langle g, \Lambda_1\gamma f\right\rangle - \left\langle g, \gamma\Lambda_1 f\right\rangle + \left\langle g, \frac{\partial\gamma}{\partial n} f\right\rangle\\ &= \int_{\Omega} 2\gamma\nabla v\cdot\nabla u - \int_{\Omega}\nabla v\cdot\nabla(\gamma(2u-u^0))\, dV(x)\\ &- \int_{\Omega}\gamma\nabla v\cdot\nabla u^0 + v\nabla\gamma\cdot\nabla u^0 + (2u-u^0)\nabla v\cdot\nabla\gamma\, dV(x)\\ &+ \int_{\Omega} v\nabla(2u-u^0)\cdot\nabla\gamma + (2u-u^0)v\Delta\gamma\, dV(x)\\ &= \int_{\Omega} 2v\nabla(u-u^0)\cdot\nabla\gamma + v(2u-u^0)\Delta\gamma\, dV(x).\end{aligned}$$

The last equality follows by expansion of all terms and canceling out terms. Consequently, the result follows. □

From both auxiliary lemmas we now obtain the boundary reconstruction formulas.

Theorem 4.2.23. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 2$. Suppose $\gamma \in W^{1,r}(\Omega)$, $r > n$ and $\operatorname{Re}\gamma \geq c > 0$. Let $x_0 \in \partial\Omega$ and let U be a cylindrical neighborhood of x_0 as described above. Then:*

(i) $\gamma|_{\partial\Omega\cap U}$ can be recovered from Λ_γ by

$$\langle h, \gamma f\rangle = \lim_{\substack{|\eta|\to\infty\\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, \mathcal{R}\Lambda_\gamma e^{-i\langle\cdot,\eta\rangle} f\rangle, \tag{4.62}$$

with $f \in H^{1/2}(\partial\Omega)\cap C(\partial\Omega)$ and $h \in L^2(\Omega)$ are assumed supported in $U\cap\partial\Omega$ and h_η is defined as in (4.54).

(ii) If, moreover, $\gamma \in W^{2,r}$, $r > n/2$, then for any continuous function f, g in $H^{1/2}(\partial\Omega)$ with support in $\partial\Omega \cap \partial\Omega$ holds

$$\left\langle g, \frac{\partial \gamma}{\partial n} f \right\rangle = \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle g, e^{-i\langle \cdot,\eta\rangle} \left(\gamma\Lambda_1 + \Lambda_1\gamma - 2\Lambda_\gamma\right) e^{i\langle \cdot,\eta\rangle} f\rangle. \tag{4.63}$$

Proof. 1. We start by defining $f_\eta = e^{i\langle \cdot,\eta\rangle} f$ and apply Lemma 4.2.21 to this function and h_η. The condition $h_\eta \in \overset{\circ}{H}{}^{-1/2}(\partial\Omega)$ follows by duality $L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$ and due to computation in $\partial\Omega \cap U$ and the supported being in this space for the other parts of the boundary.

Hence, it follows

$$\langle h_\eta, (\gamma - R\Lambda_\gamma) f_\eta\rangle = \int_\Omega u_\eta \nabla w_\eta \cdot \nabla\gamma \, dV(x),$$

where $\nabla\cdot(\gamma\nabla u_\eta) = 0$, $u_\eta|_{\partial\Omega} = f_\eta$ and $\Delta w_\eta = 0$, $\frac{\partial w_\eta}{\partial n} = h_\eta$.

From this we obtain the inequality

$$|\langle h_\eta, (\gamma - R\Lambda_\gamma) f_\eta\rangle| \le \|u_\eta\|_{L^\infty(\Omega)} \|\nabla w_\eta\|_{L^2(\Omega)} \|\nabla\gamma\|_{L^2(\Omega)}. \tag{4.64}$$

The above estimate is finite because the weak solutions u_η satisfy $\|u_\eta\|_{L^\infty(\Omega)} \le c\|f\|_{L^\infty(\partial\Omega)}$ due to Theorem 2.1.11 of [32] and the operator taking h_η into the solution of the Neumann problem w_η is bounded from $L^2(\partial\Omega)$ to $H^{3/2}(\Omega)$.

The functions h_η converge weakly in $L^2(\partial\Omega)$ to zero by Riemann-Lebesgue lemma. Moreover, compact operators map weakly convergent sequences to strongly convergent sequences, hence, due to the embedding $H^{3/2}(\Omega)$ in $H^1(\Omega)$ being compact and the composition with a compact operator still being compact, follows that w_η converges to 0 in the H^1 norm.

With this convergence in mind and with (4.64) it follows:

$$\begin{aligned}
\lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, R\Lambda_\gamma f_\eta\rangle &= \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, (\gamma - R\Lambda_\gamma) f_\eta\rangle + \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, \gamma f_\eta\rangle \\
&= \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, \gamma f_\eta\rangle \\
&= \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \int_{\partial\Omega} h_\eta(x)\gamma(x) e^{ix\cdot\eta} f(x)\, dS(x) \\
&= \int_{\partial\Omega} h(x)\gamma(x) f(x)\, dS(x) \\
&\quad - \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \frac{1}{|\partial\Omega\cap U|} \int_{\partial\Omega\cap U} h(y) e^{-iy\cdot\eta}\, d\sigma(y) \int_{\partial\Omega} f_\eta(x)\, dS(x) \\
&= \langle h, \gamma f\rangle .
\end{aligned}$$

2. Now, consider g_η and let u^0_η, $v_\eta \in H^1(\Omega)$ be the solution to $\Delta u^0_\eta = 0$, $u^0_\eta|_{\partial\Omega} = 0$ and $\Delta v_\eta = 0$, $v_\eta|_{\partial\Omega} = g_\eta$, respectively. As in part (i) we have $\|u_\eta\|_{L^\infty(\Omega)}$, $\|u^0_\eta\|_{L^\infty(\Omega)}$ and $\|v_\eta\|_{L^\infty(\Omega)}$ are bounded uniformly in η due to Theorem 2.1.11 of [32].

The function $u_\eta - u^0_\eta$ satisfies

$$\Delta(u_\eta - u^0_\eta) = -(\nabla\gamma/\gamma)\cdot\nabla u_\eta, \text{ and } (u_\eta - u^0_\eta)|_{\partial\Omega} = 0. \tag{4.65}$$

Moreover, the Laplacian Δ has a bounded inverse as an operator from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ (Lax-Milgram). By hypothesis $\gamma \in W^{2,p}(\Omega)$, $p > n/2$, therefore, both γ and it's first derivative are bounded on the closure of Ω. Then it follows by embeddings that

$$\begin{aligned}
\left\|u_\eta - u^0_\eta\right\|_{H^1(\Omega)} \le c\left\|\frac{1}{\gamma}\nabla\gamma\cdot\nabla u_\eta\right\|_{H^{-1}(\Omega)} &\le \frac{c\|\gamma\|_{L^\infty(\Omega)}\|\gamma'\|_{L^\infty(\Omega)}}{c_0}\|\nabla u_\eta\|_{H^{-1}(\Omega)} \\
&\le c'\|u_\eta\|_{L^\infty(\Omega)} \le c''\|f\|_{L^\infty(\Omega)}.
\end{aligned} \tag{4.66}$$

Now Lemma 4.2.22, together with the Hölder inequality for $1/2 = 1/p_n + (1/p - 1/n)$ and the Sobolev embedding $W^{1,p}(\Omega) \to L^{np/(n-p)}(\Omega)$ gives

$$\begin{aligned}
\left|\left\langle g_\eta, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial n}\right)\right\rangle\right| &\le 2\int_\Omega \left|v_\eta\nabla(u_\eta - u^0_\eta)\cdot\nabla\gamma\right| + \left|v_\eta(2u_\eta - u^0_\eta)\Delta\gamma\right| dV(x) \\
&\le 2\left\|\nabla(u_\eta - u^0_\eta)\right\|_{L^2(\Omega)}\left\|v_\eta\nabla\gamma\right\|_{L^2(\Omega)} \\
&\qquad + c\|f\|_{L^\infty(\Omega)}\|\gamma\|_{W^{2,p}(\Omega)}\|v_\eta\|_{L^{p'}(\Omega)} \\
&\le 2\left\|u_\eta - u^0_\eta\right\|_{H^1(\Omega)}\left\|v_\eta\right\|_{L^{p_n}(\Omega)}\left\|\nabla\gamma\right\|_{L^{np/(p-n)}} \\
&\qquad + c\|f\|_{L^\infty(\Omega)}\|\gamma\|_{W^{2,p}(\Omega)}\|v_\eta\|_{L^{p'}(\Omega)} \\
&\le c''\|f\|_{L^\infty(\Omega)}\|v_\eta\|_{L^{p_n}(\Omega)}\|\nabla\gamma\|_{W^{1,p}(\Omega)} \\
&\qquad + c\|f\|_{L^\infty(\Omega)}\|\gamma\|_{W^{2,p}(\Omega)}\|v_\eta\|_{L^{p'}(\Omega)} \\
&\le C\|f\|_{L^\infty(\Omega)}\|\gamma\|_{W^{2,p}(\Omega)}\left(\|v_\eta\|_{L^{p_n}(\Omega)} + \|v_\eta\|_{L^{p'}(\Omega)}\right).
\end{aligned}$$

The solution operator for the Dirichlet problem is bounded from $L^2(\partial\Omega)$ to $H^{1/2}(\Omega)$. Similarly as above, by Riemann-Lebesgue lemma g_η converges weakly in $L^2(\partial\Omega)$ so by compactness of the embedding $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ it follows that v_η converges to zero in the L^2-norm. Now convergence in the $L^{p_n}(\Omega)$ immediately follows by L^p inclusions if $p_n \le 2$, and by interpolation if $2 < p_n < \infty$, since in this case

$$\|v_\eta\|_{L^{p_n}(\Omega)} \le \|v_\eta\|^{1-2/p_n}_{L^\infty(\Omega)}\|v_\eta\|^{2/p_n}_{L^2(\Omega)} \le c\|g\|^{1-2/p_n}_{L^\infty(\partial\Omega)}\|v_\eta\|^{2/p_n}_{L^2(\Omega)}.$$

Due to L^p inclusions, it follows that $L^{p_n}(\Omega) \subset L^{p'}(\Omega)$, hence, we get convergence to zero in the $L^{p'}$-norm.

The desired formulation follows immediately by re-arranging the terms:

$$\left\langle g, \frac{\partial\gamma}{\partial n}f\right\rangle \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \left\langle e^{-i\langle\cdot,\eta\rangle}g, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial n}\right)f\right\rangle.$$

$\square$

4.3 Complex conductivities in $W^{1,\infty}(\Omega)$

The above work is a considerable restriction on the set of possible complex-conductivities in three-dimensions. The goal would be to show uniqueness for $L^{\infty}(\Omega)$ conductivities. Even in the real case this is yet to be achieved, with the best result know so far to be for $W^{1,n}(\Omega) \cap L^{\infty}(\Omega)$ real conductivities. Hence, our focus is to lower the condition on complex conductivities to $W^{1,\infty}(\Omega)$.

In this section we will explicitly prove Theorem 4.1.2 for complex-conductivities in $W^{1,\infty}(\Omega)$. The work follows again by transforming the equation into another one that is more easy to deal with.

The basis of our work starts from Brown and Uhlmann approach with complex analysis [15] for real-conductivities in two-dimensions. Similarly, we transform our equation into a Dirac system of equations and study it under new exponential growing solutions. To study this system we apply ideas from [15, 57, 58] to ensure the existence of solutions and, thereafter, to obtain a reconstruction formula for complex potentials obtained from conductivities in $W^{1,\infty}(\Omega)$.

The only missing step is the connection between Dirichlet-to-Neumann map Λ_γ and exponential growing solutions outside of Ω. In a completely novel approach due to the restrictions of quaternionic analysis, we establish the necessary relation to avoid the use of Poincaré Lemma, which was used by the works cited above. Essentially, the use of Cauchy-Riemann operators D and $\bar{D}$, that extend the Wirtinger derivatives of 2D, ends up being a slight restriction, since through them we cannot obtain all partial derivatives. Such endeavor would require introduction of more than two derivative operators, which would complicate the study of exponentially growing solutions. This work is based on the paper [77].

Let's start by explaining the transformation into a Dirac system of equations!

4.3.1 The relation with Dirac system of equations

The transformation of conductivity equation (4.1) into a Dirac system arises exactly as in [15], where Brown and Uhlmann initially introduced it. However, in a quaternionic framework we need to be careful with the non-commutative nature of quaternions. The only constraint from this is a non-compact formulation of the system.

Afterwards, we obtain a system of equations based on the Cauchy-Riemann operator D and the focus is to first solve the inverse Dirac scattering problem and subsequently we prove uniqueness for Calderón problem from it.

Recall from preliminaries that any point $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ can be connected with a quaternion through $x := x_0 + x_1e_1 + x_2e_2 \in \mathbb{H}$. Further, the Cauchy-Riemann operator and its conjugate are defined as

$$D = \partial_0 + \partial_1 e_1 + \partial_2 e_2$$
$$\bar{D} = \partial_0 - \partial_1 e_1 - \partial_2 e_2.$$

There is a slight abuse of notation here when compared with chapter 3, since there we used D to define the matrix Cauchy operator which inverts the complex Wirtinger derivatives and the inverse operator of D is denoted by T.

Assume we know the complex conductivity $\gamma \in W^{1,\infty}(\Omega)$ and let u be a solution to (4.1) in $H^1(\Omega)$ for some boundary function. Let us define

$$\phi = \gamma^{1/2} \left(\bar{D}u, Du \right)^t$$

and remark that $\gamma^{1/2}$ is well-defined since it is contained in $\mathbb{C}^+$. Then, ϕ solves the system

$$\begin{cases} D\phi_1 & = \phi_2 q_1, \\ \bar{D}\phi_2 & = \phi_1 q_2, \end{cases} \quad \text{in } \Omega. \tag{4.67}$$

where $q_1 = -\frac{1}{2}\frac{\bar{D}\gamma}{\gamma}$ and $q_2 = -\frac{1}{2}\frac{D\gamma}{\gamma}$. This transformation arises as follows:

$$\begin{aligned} D\phi_1 = D\left(\gamma^{1/2}\bar{D}u\right) &= D\gamma^{1/2}\bar{D}u + \gamma^{1/2}\Delta u \\ &= D\gamma^{1/2}\bar{D}u - \gamma^{-1/2}\nabla\gamma \cdot \nabla u \\ &= D\gamma^{1/2}\bar{D}u - \frac{1}{2}\gamma^{-1/2}\left(D\gamma\bar{D}u + \mathrm{D}u\bar{D}\gamma\right) \\ &= -\frac{1}{2}\left(\gamma^{1/2}Du\right)\frac{\bar{D}\gamma}{\gamma} = \phi_2 q_1 \end{aligned}$$

The other equation follows analogously. Again, since $q = (q_1, q_2)$ is in $L^\infty(\Omega)$ then we can extend it by 0 outside Ω, with the goal to study solutions over $\mathbb{R}^3$ of:

$$\begin{cases} D\phi_1 & = \phi_2 q_1, \\ \bar{D}\phi_2 & = \phi_1 q_2, \end{cases} \quad \text{in } \mathbb{R}^3. \tag{4.68}$$

4.3.2 Exponentially growing solutions

Starting from the previous section exponential functions, initially introduced by Sylvester and Uhlmann [88], we derive an exponential behavior that fits our needs. These being the requirement that our functions are monogenic, i.e., $Df = 0$. The idea is to facilitate the study of the system of equations, since as usual, we focus on the non-exponential part to thereafter obtain a feasible reconstruction formula.

As such, we start from the exponential behavior defined by $e^{x\cdot\zeta}$, with $\zeta := (\zeta_0, \zeta_1, \zeta_2) \in \mathbb{C}^3$ fulfilling $\zeta \cdot \zeta := \zeta_0^2 + \zeta_1^2 + \zeta_2^2 = 0$. Our first objective is to obtain a exponential function $\mathcal{E}$ that is growing in some directions and $D\mathcal{E} = 0$. Due to $\zeta\cdot\zeta = 0$, the function $e^{x\cdot\zeta}$ is harmonic. Further, the Laplace operator is factorized by Cauchy-Riemann operator through:

$$\Delta = \partial_{x_0}^2 + \partial_{x_1}^2 + \partial_{x_2}^2 = D\bar{D}.$$

With all of this in mind, we obtain a monogenic exponential function through:

$$\begin{aligned}\Delta e^{x\cdot\zeta} = 0 \Leftrightarrow D\left(\bar{D}e^{x\cdot\zeta}\right) = 0 \\ \Leftrightarrow D\left(\left(\zeta_0 - \zeta_1 e_1 - \zeta_2 e_2\right) e^{x\cdot\zeta}\,\bar{\zeta}\right) = 0 \Leftrightarrow D\left(\bar{\zeta}e^{x\cdot\zeta}\right) = 0\end{aligned}$$

where now ζ is also defined as a quaternion by $\zeta = \zeta_0 + e_1\zeta_1 + e_2\zeta_2 \in \mathbb{C}\otimes\mathbb{H}$.

In this way, we obtain a monogenic exponential function defined as

$$\mathcal{E}_1(x,\zeta) = \bar{\zeta}e^{x\cdot\zeta}.$$

Tweaking this idea for our purposes, we can also establish another exponential function through

$$\mathcal{E}_2(x,\zeta) = \bar{\zeta}^c e^{x\cdot\bar{\zeta}^{\mathbb{C}}}$$

which is an anti-monogenic function, *i.e.*, $\bar{D}\mathcal{E}_2 = 0$. Further, in quaternion form the required condition is transformed to

$$\zeta\cdot\zeta = \zeta\,\bar{\zeta} = \zeta_0^2 + \zeta_1^2 + \zeta_2^2 = 0.$$

We make a clear statement of when ζ is a complex-quaternion or a complex-vector, but in most cases it will be clear from context: it is a vector if it is in the exponent and a quaternion otherwise.

With this exponential behavior we now look for solutions ϕ of (4.68) with the following asymptotics:

$$\begin{cases}\phi_1 &= e^{x\cdot\zeta}\bar{\zeta}\mu_1,\\ \phi_2 &= e^{x\cdot\bar{\zeta}^{\mathbb{C}}}\bar{\zeta}^c\mu_2.\end{cases} \tag{4.69}$$

Plugging them in on (4.68), we have $\mu = (\mu_1,\mu_2)$ solving:

$$\begin{cases}D\left(\bar{\zeta}\mu_1\right) &= e^{-x\cdot(\zeta-\bar{\zeta}^{\mathbb{C}})}\bar{\zeta}^c\mu_2 q_1\\ \bar{D}\left(\bar{\zeta}^c\mu_2\right) &= e^{x\cdot(\zeta-\bar{\zeta}^{\mathbb{C}})}\bar{\zeta}\mu_1 q_2,\end{cases} \tag{4.70}$$

which follows immediately by Leibniz rule of multiplication and the fact that our exponentials are monogenic and anti-monogenic ($D\mathcal{E}_1 = 0$, $\bar{D}\mathcal{E}_2 = 0$).

Setting $\tilde{\mu}_1 = \bar{\zeta}\mu_1$, $\tilde{\mu}_2 = \bar{\zeta}^c\mu_2$ and substituting we have the equations:

$$\begin{cases}D\tilde{\mu}_1 &= e^{-x\cdot\left(\zeta-\bar{\zeta}^{\mathbb{C}}\right)}\tilde{\mu}_2 q_1\\ \bar{D}\tilde{\mu}_2 &= e^{x\cdot\left(\zeta-\bar{\zeta}^{\mathbb{C}}\right)}\tilde{\mu}_1 q_2.\end{cases} \tag{4.71}$$

Further, we assume

$$\tilde{\mu} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as } |x| \to \infty.$$

The system of equations leads us to an integral equation from which we can extract interesting behavior for $\zeta \to \infty$.

In order to ensure existence and uniqueness of solutions to (4.71) we transform it into a system of integral equations based on the T operator and $e^{\pm x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})}$.

4.3.3 Integral equation and function spaces

In order to obtain a integral equation, we approach our system (4.71) similarly to [57], but again we need to be careful due to the non-commutative nature of quaternions.

Recall, that $DT = \bar{D}\bar{T} = I$ (in appropriate spaces). Since, $D1 = 0$ we can subtract on the first equation to obtain $\tilde{\mu}_1 - 1$ and thereafter, we apply the $T, \bar{T}$ operator from the right to achieve:

$$\begin{cases} \tilde{\mu}_1 = 1 + T\left[e^{-x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})}\tilde{\mu}_2 q_1\right] \\ \\ \tilde{\mu}_2 = \bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})}\tilde{\mu}_1 q_2\right]. \end{cases}$$

Substituting $\tilde{\mu}_1, \tilde{\mu}_2$ as given in the other equation we get:

$$\begin{cases} \tilde{\mu}_1 = 1 + T\left[e^{-x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})}\bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})}\tilde{\mu}_1 q_2\right] q_1\right] \\ \\ \tilde{\mu}_2 = \bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})} q_2\right] + \bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})} T\left[e^{-x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})}\tilde{\mu}_2 q_1\right] q_2\right]. \end{cases}$$

Denoting two new operators M_1, M_2 by

$$M_1 f = T\left[e^{-x\cdot(\zeta - \bar{\zeta}^{c})}\bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{c})} f q_2\right] q_1\right] \tag{4.72}$$

$$M_2 f = \bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{c})} T\left[e^{-x\cdot(\zeta - \bar{\zeta}^{c})} f q_1\right] q_2\right], \tag{4.73}$$

the integral system of equations simplifies to:

$$\begin{cases} \tilde{\mu}_1 = 1 + M^1\tilde{\mu}_1 \\ \tilde{\mu}_2 = \bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})} q_2\right] + M^2\tilde{\mu}_2 \end{cases} \Leftrightarrow \begin{cases} [I - M^1](\tilde{\mu}_1 - 1) = M^1 1 \\ [I - M^2](\tilde{\mu}_2) = \bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})} q_2\right]. \end{cases} \tag{4.74}$$

The goal now is to study uniqueness and existence of these equations. We approach this task by proving that M is a contraction in certain space and the right-hand side is in it.

Ahead of time, we remark there is no need to work with all $\zeta \in \mathbb{C} \otimes \mathbb{H}$ that fulfill $\zeta\bar{\zeta} = 0$. In fact, we make an appropriate choice through $k = k_0 + k_1 e_1 + k_2 e_2 \in \mathbb{H}$ by

$$\zeta = k^{\perp} + i\frac{k}{2}, \quad k^{\perp} \cdot k = 0$$

and $k^\perp$ can be algorithmically found, i.e., it can be viewed as a function of k.

This already allows us to simplify the definition of our function space, which we define as

$$S = L_x^\infty(L_k^p(|k| > R)), \tag{4.75}$$

where the norm in x is taken over $\mathbb{R}^3$ and $R > 0$ is a constant.

In this space we prove that the operator M is a contraction in S:

Lemma 4.3.1. *Let $1 \leq p \leq \infty$. Then*

$$\lim_{R\to\infty} \|M^j\|_S = 0.$$

Proof. Let us assume, without loss of generality, that f is a scalar function, which implies that the result also holds for quaternionic-valued functions by $M(f+g) = Mf + Mg$. Further, we present the proof for M^1. For M^2 is analogous.

Recall, that we choose $\zeta \in \mathbb{C}_{(2)}$ with respect to $k \in \mathbb{R}_{(2)}$ as $\zeta = k^\perp + i\frac{k}{2}, \quad k^\perp \cdot k = 0$.

In vector form it implies that $\zeta - \zeta^c = ik$ which simplifies our computations to:

$$\begin{aligned} M^1 f(x) &= \int_{\mathbb{R}^3} e^{-w\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})} \frac{\overline{x-w}}{|x-w|^3} \int_{\mathbb{R}^3} e^{y\cdot(\zeta-\bar{\zeta}^{\mathbb{C}})} \frac{w-y}{|w-y|^3} f(y) q_2(y)\, dy\, q_1(w)\, dw \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iw\cdot k} \frac{\overline{x-w}}{|x-w|^3} e^{iy\cdot k} \frac{w-y}{|w-y|^3} f(y) q_2(y) q_1(w)\, dwdy \\ &= \int_{\mathbb{R}^3} A(x,y;k) f(y)\, dy, \end{aligned}$$

where

$$A(x,y;k) = \int_{\mathbb{R}^3} e^{-i(w-y)\cdot k} \frac{\overline{x-y}}{|x-y|^3} \frac{w-y}{|w-y|^3} q_2(y) q_1(w)\, dw.$$

Due to the compact support of the potential q_2, it holds that A has compact support on the second variable.

Let us now apply the norm in terms of k to it and Minkowski integral inequality:

$$\begin{aligned} \|Mf(x,\cdot)\|_{L^p(|k|>R)} &= \left[\int_{|k|>R} |Mf(x,\zeta)|^p \, d\sigma_\zeta\right]^{1/p} \\ &= \left[\int_{|k|>R} \left|\int_\Omega A(x,y;k) f(y)\, dy\right|^p d\sigma_k\right]^{1/p} \\ &\leq \int_\Omega \left[\int_{|k|>R} |A(x,y;k) f(y)|^p \, d\sigma_k\right]^{1/p} dy \\ &\leq \int_\Omega \sup_k |A(x,y;k)| \, dy \, \|f\|_S. \end{aligned}$$

In order to complete the proof we show that the first integral goes to zero as $R \to \infty$.

Let A^s be given with the extra factor $\alpha(s|x-w|)\alpha(s|w-y|)$ in the integrand, where $\alpha \in C^\infty$ is 1 outside a neighborhood of the origin and 0 inside a smaller neighborhood of it.

Since,

$$\int_{B_1(0)} \int_{B_1(0)} \frac{1}{|w|^2} \frac{1}{|w-y|^2} \, dw \, dy,$$

it holds that for any $\epsilon > 0$ there exists an $s > 0$ such that:

$$\int_{\Omega} |A - A^s| \, dy < \epsilon.$$

Further, we denote $A^{s_0,n}$ the function A^{s_0} with potentials q_1, q_2 replaced by their L^1 smooth approximation Q_1^n, $Q_2^n \in C^\infty$. Since the other factors are bounded it holds

$$\int_{\Omega} |A^{s_0} - A^{s_0,n}| \, dy < \epsilon.$$

Now it is enough to show that $A^{s_0,n} \to 0$ as $|k| \to 0$ uniformly.

Since all integrands functions are in C^∞ and uniformly bounded, the result immediately follows by Riemann-Lebesgue. $\square$

The proof of this lemma is much more clear with the restriction of ζ through $k \in \mathbb{R}^3$. This lemma points to a clear statement that M is a contraction in S.

Corollary 4.3.2. *Let $1 \leq p \leq \infty$. There exists an $R > 0$ big enough such that M is a contraction operator in S and $[I - M]$ is invertible in S, where I is the identity matrix.*

Proof. By Lemma 4.3.1, we have that $\lim_{R\to\infty} \|M\|_S = 0$, therefore by definition of limit, there exists $R > 0$ big enough such that $\|M\|_S < 1$. As such, M is a contraction in S. Neumann series immediately gives that $I - M$ is invertible in S. $\square$

Existence and uniqueness of (4.74) is now ensured by showing for $R > 0$ the right-hand side is in S.

Lemma 4.3.3. *Let $p > 2$. Then there exists $R > 0$ such that*

$$M^1 1 \in S, \tag{4.76}$$

$$\bar{T}\left[e^{x\cdot(\zeta - \bar{\zeta}^{\mathbb{C}})} q_2\right] \in S. \tag{4.77}$$

Proof. Once again recall that $\zeta = \left(k^\perp + i\frac{k}{2}\right)$ for $k \in \mathbb{R}^3$. First we show that $M^1 1 \in S$. We have

$$M^1 1 = \int_{\Omega} \int_{\Omega} e^{-iw\cdot k} \frac{\overline{x-w}}{|x-w|^3} \frac{w-y}{|w-y|^3} e^{iy\cdot k} q_2(y) q_1(w) \, dy \, dw,$$

and applying the L^p norm in k followed with Minkowski integral inequality we obtain

$$\left[\int_{|k|>R} |M^1 1|^p dk\right]^{1/p} \leq \int_{\Omega} \frac{|q_1(w)|}{|x-w|^2} \left[\int_{|k|>R} \left| \int_{\Omega} e^{iy\cdot k} \frac{w-y}{|w-y|^3} q_2(y) dy \right|^p dk\right]^{1/p} dw.$$

The inner most integral resembles a Fourier transform, hence, applying the Hausdorff-Young inequality for $p > 2$ we have

$$\left[\int_{|k|>R}\left|\int_{\Omega} e^{iy\cdot k}\frac{w-y}{|w-y|^3}q_2(y)\,dy\right|^p dk\right]^{1/p} \leq \left[\int_{\Omega}\frac{|q_2(y)|^{p'}}{|w-y|^{2p'}}\,dy\right]^{1/p'} < C\|q_2\|_\infty,$$

where the last inequality follows quickly by Young's convolution inequality and Riesz type estimate of the kernel.

Therefore, by the same Riesz type estimate it holds:

$$\left[\int_{|k|>R}|M^1 1|^p\,dk\right]^{1/p} \leq C\|q_2\|_\infty\int_\Omega\frac{|q_1(w)|}{|x-w|^2}\,dw \leq C'\|q_2\|_\infty\|q_1\|_\infty.$$

To complete the proof we need to show statement (4.77). Similarly, to the above proof, we have by Hausdorff-Young Inequality, Young's convolution inequality and a Riesz type estimate the following:

$$\begin{aligned}\left[\int_{|k|>R}\left|\int_{\mathbb{R}^3} e^{iy\cdot k}\frac{x-y}{|x-y|^3}q_2(y)\,d\sigma_y\right|^p d\sigma_k\right]^{1/p} &\leq \left[\int_{\mathbb{R}^3}\left|\frac{x-y}{|x-y|^3}q_2(y)\right|^{p'} d\sigma_y\right]^{1/p'}\\ &\leq C\|q_2\|_\infty.\end{aligned}$$

□

All of the lemmas above lead us to conclude existence and uniqueness of solutions $(\tilde{\mu}_1, \tilde{\mu}_2)$ solving (4.74), which we resume in the following result:

Corollary 4.3.4. *Let $q \in L^\infty(\Omega)$ extended by zero outside Ω. Further, let $p > 2$ and $R > 0$ be big enough. Then there exists a unique solution*

$$\left(\phi_1 e^{-x\cdot\zeta(k)} - 1,\, \phi_2 e^{-x\cdot\bar{\zeta}^{\mathbb{C}}(k)}\right) \in L_x^\infty(L_k^p(|k| > R)) \tag{4.78}$$

to the system (4.74).

4.3.4 Reconstruction from scattering data

These solutions are key to define the scattering data, which transforms information on the domain boundary into knowledge about the potential inside Ω. To accomplish it, we mix ideas from [57] and [68] with quaternionic theory to obtain a well-defined scattering data and to get the potential from its asymptotic behaviour.

Starting from Clifford-Green theorem

$$\int_\Omega \left[g(x)\left(\bar{D}f(x)\right) + \left(g(x)\bar{D}\right)f(x)\right]\,dx = \int_{\partial\Omega} g(x)\overline{\eta(x)}f(x)\,dS_x$$

and using $g(x; i\xi + \zeta) = (i\xi + \zeta)e^{-x\cdot(i\xi+\zeta)}$ for $\xi \in \mathbb{R}^3 \cong \mathbb{R}_{(2)}$ such that $(i\xi + \zeta)\cdot(i\xi + \zeta) = 0$. This implies that $g\bar{D} = 0$.

We define the scattering data as:

$$h(\xi,\zeta) = (i\xi+\zeta)\int_{\partial\Omega} e^{-x\cdot(i\xi+\zeta)}\overline{\eta(x)}\phi_2(x,\zeta)\,dx. \tag{4.79}$$

Applying now Clifford-Green theorem, we obtain another form for the scattering data:

$$\begin{aligned}
h(\xi,\zeta) &= (i\xi+\zeta)\int_{\Omega} e^{-x\cdot(i\xi+\zeta)}\bar{D}\phi_2(x,\zeta)\,dx \\
&= (i\xi+\zeta)\int_{\Omega} e^{-ix\cdot\xi}\left(e^{-x\cdot\zeta}\phi_1(x,\zeta)\right)q_2(x)\,dx, \quad \text{by } \bar{D}\phi_2 = \phi_1 q_2 \\
&= (i\xi+\zeta)\int_{\Omega} e^{-ix\cdot\xi}\left(\bar{\zeta}\mu_1(x,\zeta)\right)q_2(x)\,dx \\
&= i\xi\int_{\Omega} e^{-ix\cdot\xi}\tilde{\mu}_1(x,\zeta)q_2(x)\,dx, \quad \text{since } \bar{\zeta}\zeta = 0 \\
&= i\xi\hat{q}_2(\xi) + i\xi\int_{\Omega} e^{-ix\cdot\xi}\left[\tilde{\mu}_1(x,\zeta)-1\right]q_2(x)\,dx
\end{aligned}$$

where we use the definition of the Fourier transform for the potential q_2.

Thus, we have:

$$\hat{q}_2(\xi) = \frac{h(\xi,\zeta)}{i\xi} - \int_{\Omega} e^{-ix\cdot\xi}\left[\tilde{\mu}_1(x,\zeta)-1\right]q_2(x)\,dx \tag{4.80}$$

This is yet not enough to reconstruct the potential, since the integral acts as a residual in the reconstruction and requires data that we technically do not have. Therefore, we integrate everything over an annulus in k as

$$\begin{aligned}
\int_{R<|k|<2R}\frac{\hat{q}_2(\xi)}{|k|^3}\,dk = \frac{1}{i\xi}&\int_{R<|k|<2R}\frac{h(\xi,\zeta(k))}{|k|^3}\,dk \\
&- \int_{R<|k|<2R}\frac{1}{|k|^3}\int_{\Omega} e^{-ix\cdot\xi}\left[\hat{\mu}_1(x,\zeta(k))-1\right]q_2(x)\,dx.
\end{aligned}$$

On the left-side integral, we can take out the Fourier transform of potential, since it does not depend on k. Taking the limit as $R\to\infty$ leads the second integral on the right to decay to zero, and therefore, we obtain a reconstruction formula.

Theorem 4.3.5. *Let $\Omega\subset\mathbb{R}^3$ a bounded Lipschitz domain, $q\in L^\infty(\Omega)$ be a complex-valued potential obtained through a conductivity $\gamma\in W^{1,\infty}(\Omega)$, $\operatorname{Re}\gamma\geq c>0$. Then, we can reconstruct the potential from*

$$\hat{q}_2(\xi) = \frac{1}{4\pi\ln(2)}\lim_{R\to\infty}\frac{1}{i\xi}\int_{R<|k|<2R}\frac{h(\xi,\zeta(k))}{|k|^3}\,dk \tag{4.81}$$

Proof. The scattering data is defined from solutions of the Dirac system (4.74) and therefore it holds that $\tilde{\mu}_1 - 1\in S$. Starting from (4.81) we obtain by integrating the right-hand side for any $\xi\in\mathbb{R}^3$:

$$\begin{aligned}
4\pi\ln 2\hat{q}_2(\xi) = \frac{1}{i\xi}&\int_{R<|k|<2R}\frac{h(\xi,\zeta(k))}{|k|^3}\,dk \\
&- \int_{R<|k|<2R}\frac{1}{|k|^3}\int_{\Omega} e^{-ix\cdot\xi}\left[\tilde{\mu}_1(x,\zeta(k))-1\right]q_2(x)\,dx.
\end{aligned}$$

Let $p > 2$ and $1/p + 1/q = 1$. We estimate the last integral:

$$\begin{aligned}
&\left| \int_{R<|k|<2R} \frac{1}{|k|^3} \int_\Omega e^{-ix\cdot\xi} \left[\tilde{\mu}_1(x,\zeta(k)) - 1\right] q_2(x)\, dx \right| \leq \\
&\leq \int_{R<|k|<2R} \frac{1}{|k|^3} \int_\Omega \left| e^{-ix\cdot\xi} \left[\tilde{\mu}_1(x,\zeta(k)) - 1\right] q_2(x) \right| dx \\
&\leq C_\Omega \|q\|_\infty \int_{R<|k|<2R} \frac{1}{|k|^3} \sup_x |\tilde{\mu}_1(x,\zeta(k)) - 1|\; dk \\
&\leq C_\Omega \|q\|_\infty \left[\int_{R<|k|<2R} \frac{1}{|k|^{3q}}\, dk \right]^{1/q} \left[\int_{R<|k|<2R} \sup_x |\tilde{\mu}_1(x,\zeta(k)) - 1|^p\; dk \right]^{1/p} \\
&\leq C_\Omega \|q\|_\infty \|\tilde{\mu}_1 - 1\|_S \left[\int_{R<|k|<2R} \frac{1}{|k|^{3q}}\, dk \right]^{1/q} .
\end{aligned}$$

Taking the limit as $R \to 0$, the integral that is left goes to zero for $q > 1$, which implies the desired decay to zero and leaves us with our reconstruction formula. □

Since, scattering data is uniquely defined from the boundary values of ϕ, then the potential q, obtained from a complex-conductivity, is uniquely determined by them, as well.

To show that complex-conductivities $\gamma \in W^{1,\infty}(\Omega)$ are uniquely determined from their Dirichlet-to-Neumann map, we need to establish a unique relation between Λ_γ and the boundary values of ϕ.

4.3.5 From Λ_γ to the scattering data

To establish a relation between Λ_γ for $\gamma \in W^{1,\infty}(\Omega)$ and the boundary values of respective solutions to (4.74), we need to introduce some new results that establish the existence of a solution u from ϕ. Recall, that if we have a solution u to (4.1), then we may obtain solutions ϕ to the integral equation. Our objective is to achieve the reverse.

This result is more clearly understood in two dimensions since the derivative operators are intrinsically connected with Poincaré lemma for complex forms. However, in quaternions we would require the use of four derivative operators to obtain an analogous result. Even though possible, this would end up creating difficulties on the existence of exponential growing solutions!

Lemma 4.3.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. If h is a scalar-valued and harmonic function then*

$$\mathrm{Vec}(S_{\partial\Omega} h) = 0 \Rightarrow h|_{\partial\Omega} \text{ is constant.}$$

Proof. First, note that $I + S_{\partial\Omega} = P_{\partial\Omega}$ is a projector and by Proposition 2.5.12 and Corollary 2.5.15 of [34] it holds that $P_{\partial\Omega} h$ is the boundary value of a monogenic function in Ω.

Since h is a scalar-valued function it holds that

$$\begin{aligned}
P_{\partial\Omega} h &= \mathrm{Sc}(P_{\partial\Omega} h) + \mathrm{Vec}(P_{\partial\Omega} h) \\
&= (h + \mathrm{Sc}(S_{\partial\Omega} h)) + \mathrm{Vec}(S_{\partial\Omega} h).
\end{aligned}$$

Let $w = (h + \mathrm{Sc}(S_{\partial\Omega}h))$ and $v = \mathrm{Vec}(S_{\partial\Omega}h)$. Now, we denote f as the monogenic extension of $P_{\partial\Omega}h$ in Ω, as such, the boundary values of f fulfill tr $f = w + v$. Note that by hypothesis we have that $v|_{\partial\Omega} = 0$.

Hence, f is also an harmonic function, which implies that the scalar and vector components are harmonic:

$$\begin{cases} \Delta(\mathrm{Vec}\ f) = 0 \\ \mathrm{Vec}\ f|_{\partial\Omega} = 0. \end{cases}$$

By a mean value theorem or a maximum principle it holds that Vec $f = 0$. Due to this and f being monogenic we obtain that $Df = 0 \Leftrightarrow D(\mathrm{Sc}\ f) = 0$. Thus, Sc $f = c$ since D is a quaternionic operator.

Consequently, the boundary values are also constant, which means that $w = c$ in $\partial\Omega$. Since, $\mathrm{Sc}(S_{\partial\Omega}h)$ is an averaging operator it holds that $h = c$. □

In order to connect them we introduce the following result:

Proposition 4.3.7. *Let Ω be a bounded domain in $\mathbb{R}^3$. Let $\phi = (\phi_1, \phi_2)$ be a solution of the Dirac system (4.67) for a potential $q \in L^\infty(\Omega)$ associated with a complex-conductivity $\gamma \in W^{1,\infty}(\Omega)$.*

If $\phi_1 = \bar{\phi}_2$ then there exists a unique solution u of:

$$\begin{cases} \bar{D}u = \gamma^{-1/2}\phi_1, \\ Du = \gamma^{-1/2}\phi_2. \end{cases} \tag{4.82}$$

Further, this function fulfills the conductivity equation

$$\nabla \cdot (\gamma \nabla u) = 0 \ \textit{in}\ \Omega.$$

Proof. Suppose that (u, v) are solutions to the following equations:

$$\begin{cases} \bar{D}u = \gamma^{-1/2}\phi_1 \\ Dv = \gamma^{-1/2}\phi_2. \end{cases}$$

From applying the operator D and $\bar{D}$ to the first and second equation respectively, we obtain from $\phi_2 = \bar{\phi}_1$ and $q_2 = \bar{q}_1$ the following:

$$\begin{aligned}
\Delta u &= D(\gamma^{-1/2}\phi_1) = D(\gamma^{-1/2})\phi_1 + \gamma^{-1/2} D\phi_1 \\
&= -\frac{1}{2}\gamma^{-3/2}(D\gamma\phi_1) + \gamma^{-1/2}\phi_2 q_1 \\
&= \gamma^{-1/2}\left[q_2\phi_1 + \phi_2 q_1\right] = \gamma^{-1/2}\left[\bar{q}_1\phi_1 + \bar{\phi}_1 q_1\right] \\
&= \gamma^{-1/2}\mathrm{Sc}\ (\bar{\phi}_1 q_1), \quad \text{ and} \\
\Delta v &= \bar{D}(\gamma^{-1/2}\phi_2) = \bar{D}(\gamma^{-1/2})\phi_2 + \gamma^{-1/2}\bar{D}\phi_2 \\
&= -\frac{1}{2}\gamma^{-3/2}(\bar{D}\gamma)\phi_2 + \gamma^{-1/2}\phi_1 q_2 \\
&= \gamma^{-1/2}\left[q_1\phi_2 + \phi_1 q_2\right] = \gamma^{-1/2}\left[q_1\bar{\phi}_1 + \phi_1\bar{q}_1\right] \\
&= \gamma^{-1/2}\mathrm{Sc}\ (\phi_1\bar{q}_1).
\end{aligned}$$

The first thing to notice is that both equations imply that u and v must be scalar-valued functions. Further, it holds that

$$\begin{aligned}
\Delta(u - v) = \gamma^{-1/2}&\left[\mathrm{Sc}\ (\bar{\phi}_1 q_1) - \mathrm{Sc}\ (\phi_1\bar{q}_1)\right] \\
&= \gamma^{-1/2}\left[\mathrm{Sc}\ \left(\bar{\phi}_1 q_1\right) - \mathrm{Sc}\ \overline{(q_1\bar{\phi}_1)}\right] = 0.
\end{aligned}$$

Therefore, $h = u - v$ is a scalar-valued harmonic function. Our objective is to show that $h \equiv 0$, thus showing that $u = v$.

For such, let us consider the theory of integral transforms in quaternionic analysis. We have

$$\begin{aligned}
u &= \bar{T}(\gamma^{-1/2}\phi_1) + \bar{F}_{\partial\Omega}(\gamma^{-1/2}\phi_1), \text{ and} \\
u &= \bar{T}(\gamma^{-1/2}\phi_1) + \bar{F}_{\partial\Omega}(u),
\end{aligned}$$

which implies that

$$\bar{F}_{\partial\Omega}(\gamma^{-1/2}\phi_1) = \bar{F}_{\partial\Omega} u.$$

Analogously, we obtain

$$F_{\partial\Omega}(\gamma^{-1/2}\phi_2) = F_{\partial\Omega} v.$$

Here, we can extrapolate from the first equation and from u being scalar-valued that

$$\begin{aligned}
\overline{\gamma^{-1/2}\phi_1} F_{\partial\Omega} &= F_{\partial\Omega} u \\
\Leftrightarrow \gamma^{-1/2}\phi_2 F_{\partial\Omega} &= F_{\partial\Omega} u.
\end{aligned}$$

Applying the operator $F_{\partial\Omega}$ on the other side, we obtain:

$$\begin{aligned}
&F_{\partial\Omega}F_{\partial\Omega} u = F_{\partial\Omega}F_{\partial\Omega}(\gamma^{-1/2}\phi_2)F_{\partial\Omega} \ \text{ and } \ F_{\partial\Omega}F_{\partial\Omega} v = F_{\partial\Omega}(\gamma^{-1/2}\phi_2)F_{\partial\Omega} \\
&\Rightarrow F_{\partial\Omega}F_{\partial\Omega} h = F_{\partial\Omega}F_{\partial\Omega}(u - v) = 0.
\end{aligned}$$

If we take the trace on both sides, the operator becomes a projector thus we obtain that tr $F_{\partial\Omega}h = 0$. Now, through the Sokhotski-Plemelj formula we obtain:

$$\text{tr } F_{\partial\Omega}h = h|_{\partial\Omega} + S_{\partial\Omega}h = 0, \text{ at } \partial\Omega.$$

Since h is a scalar-valued function that we decompose this formulation with the scalar and vector part to obtain two conditions:

$$\begin{cases} h + \text{Sc}(S_{\partial\Omega}h) = 0, \\ \text{Vec}(S_{\partial\Omega}h) = 0. \end{cases}$$

Through the second condition and Lemma 4.3.6 we have that h is constant over $\partial\Omega$.

Now, given that h is a scalar constant, the first condition reduces to:

$$h(1 + \text{Sc}(S_{\partial\Omega}1)) = 0.$$

By [34] we obtain that $1 + \text{Sc}(S_{\partial\Omega}1) = 1/2$ in $\partial\Omega$. Therefore, we conclude that $h \equiv 0$ in $\partial\Omega$. Given that h is harmonic, this immediately implies that $h = 0$ in Ω.

Therefore, we obtain $u = v$, and therefore there exists a unique solution to the initial system through the T and $F_{\partial\Omega}$ operators in Ω.

To finalize, we only need to show that u fulfills the conductivity equation in Ω.

Bringing the first equation to light

$$\bar{D}u = \gamma^{-1/2}\phi_1,$$

changing the side of the conductivity we get $\gamma^{1/2}\bar{D}u = \phi_1$ and applying the D operator to both sides now brings

$$\begin{aligned} & D\left(\gamma^{1/2}\bar{D}u\right) = D\phi_1 \\ \Leftrightarrow\quad & D\left(\gamma^{1/2}\right)\bar{D}u + \gamma^{1/2}\Delta u = \phi_2 q_1 \\ \Leftrightarrow\quad & D\left(\gamma^{1/2}\right)\bar{D}u + \gamma^{1/2}\Delta u = \gamma^{-1/2}Du\frac{1}{2}\frac{\bar{D}\gamma}{\gamma} \\ \Leftrightarrow\quad & \frac{1}{2}\gamma^{1/2}D\gamma\bar{D}u + \gamma^{1/2}\Delta u + \frac{1}{2}Du\frac{\bar{D}\gamma}{\gamma^{1/2}} = 0 \\ \Leftrightarrow\quad & \nabla\gamma \cdot \nabla u + \gamma\Delta u = 0 \Leftrightarrow \nabla \cdot (\gamma\nabla u) = 0. \end{aligned}$$

□

Let us recall the main theorem, that we are now able to prove with all these pieces we assembled.

Theorem 4.1.2 *Let $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain, $\gamma_i \in W^{1,\infty}(\Omega)$, $i = 1, 2$ be two complex-valued conductivities with* $\text{Re } \gamma_i \geq c > 0$.

$$\textit{If } \Lambda_{\gamma_1} = \Lambda_{\gamma_2}, \textit{ then } \gamma_1 = \gamma_2.$$

Proof. Due to Theorem 4.3.5, we only need to show the scattering data h for $|k| >> 1$ is uniquely determined by the Dirichlet-to-Neumann map Λ_γ.

For such, let us start with two conductivities γ_1, γ_2 in $W^{1,\infty}(\Omega)$ for Ω a bounded domain. By hypothesis $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and thus by [76] we have $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$.

Further, we can extend γ_j, $j = 1, 2$ outside Ω in such a way that in $\mathbb{R}^3 \setminus \Omega$ and $\gamma_j - 1 \in W^{1,\infty}_{\text{comp}}(\mathbb{R}^3)$. Let q_j, ϕ^j, μ^j, h_j, $j = 1, 2$ be the potential and the solution in (4.67), the function in (5.9), and the scattering data in (4.79) all associated with the conductivity γ_j.

Due to the scattering formulation at the boundary $\partial\Omega$, then we just want to know if $\phi^1 = \phi^2$ on $\partial\Omega$ when $|k| >> 1$.

First, by Proposition 3.4, we know that there exists an u_1 such that

$$\phi^1 = \gamma_1^{1/2}(\bar{D}u_1, Du_1)^t,$$

which is a solution to

$$\nabla \cdot (\gamma_1 \nabla u_1) = 0 \text{ in } \mathbb{R}^3.$$

Now, let us define u_2 by

$$u_2 = \begin{cases} u_1 & \text{in } \mathbb{R}^3 \setminus \Omega, \\ \hat{u} & \text{in } \Omega. \end{cases}$$

where $\hat{u}$ is the solution to the Dirichlet problem

$$\begin{cases} \nabla \cdot (\gamma_2 \nabla \hat{u}) = 0 & \text{in } \Omega, \\ \hat{u} = u_1 & \text{on } \partial\Omega. \end{cases}$$

Let $g \in C_c^\infty(\mathbb{R}^3)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^3} \gamma_2 \nabla u_2 \nabla g \, dx &= \int_{\mathbb{R}^3 \setminus \Omega} \gamma_1 \nabla u_1 \nabla g \, dx + \int_\Omega \gamma_2 \nabla \hat{u} \nabla g \, dx \\ &= -\int_{\partial\Omega} \Lambda_{\gamma_1} \left[u_1|_{\partial\Omega}\right] g \, ds_x + \int_{\partial\Omega} \Lambda_{\gamma_2} \left[\hat{u}|_{\partial\Omega}\right] g \, ds_x \\ &= 0. \end{aligned}$$

Hence, u_2 is the solution of $\nabla \cdot (\gamma_2 \nabla u_2) = 0$ in $\mathbb{R}^3$. Further, the following function

$$\psi^2 = \gamma_2^{1/2} \left(\bar{D}u_2, Du_2\right)^t$$

is the solution of (4.67) where the potential is given by γ_2.

Furthermore, ψ^2 has the asymptotics of ϕ^1 in $\mathbb{R}^3 \setminus \Omega$, thus by Lemma 3.1 and 3.2 it will be the unique solution of the respective integral equation of (4.67). Thus, ψ^2 will be equal ϕ^2 when $|k| > R$. Since, on the outside $\psi^2 \equiv \phi^1$. Then we obtain:

$$\phi^1 = \phi^2 \text{ in } \mathbb{R}^3 \setminus \Omega.$$

In particular, we have equality at the boundary $\partial\Omega$. So, this implies that if the Dirichlet-to-Neumann maps are equal the respective scattering data will also be the same. Thus, the Dirichlet-to-Neumann map uniquely determines the potential q.

From the definition of q, we can uniquely determine the conductivity γ up to a constant, which in the end is defined by $\gamma|_{\partial\Omega}$ which is uniquely determined by the Dirichlet-to-Neumann map Λ_γ.

$\square$

As such, we conclude our proof of uniqueness for complex-conductivities in $W^{1,\infty}(\Omega)$ from the Dirichlet-to-Neumann map Λ_γ. Notice that (4.3.5) even provides a reconstruction formula, but as mentioned in the previous section it is very unstable for computational purposes.

There is no yet known result in the literature for conductivities just in $L^\infty(\Omega)$, even for real ones. The next chapter is focused on this, but it only provides a possible path to achieve it.

Chapter 5

Real Conductivities in 3D

The chapter objective is meant as an exploratory work and to establish a possible framework to solve the full Calderón problem in three-dimensions for real-conductivities $\sigma \in L^\infty(\Omega)$.

The main ideas arose from the work of Santacesaria [81], where he uses quaternionic analysis to establish a link with theoretical results presented for two-dimensions on [8]. It starts by making a connection with a Beltrami equation of quaternionic nature, which allows an immediate relation between the real conductivity and a real-valued coefficient μ without the need for derivatives. This builds the foundations for a possible solution to Calderón problem with $\sigma \in L^\infty$.

In this seminal paper, Santacesaria provides a uniqueness proof analogous to Calderón in the original paper [18] for the linearized problem at constant conductivities, which we will not present here. However, this is the furthest he is able to achieve.

We provide a bit further insight into the quaternionic problem. We obtain an integral equation for exponential growing solutions and present the essential pieces that are needed to conclude the proof, and that we weren't able to achieve for now.

5.1 The problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and define $\sigma \in L^\infty(\Omega)$ to be an isotropic real conductivity which is positively lower bounded, *i.e.*, $\sigma(x) \geq c > 0$, $\forall x \in \Omega$. We remark that this is the first time we are restricting our assumptions just for real conductivities.

The **Direct Problem** concerns the determination of an electrical potential $u \in H^1(\Omega)$, for a given voltage $f \in H^{1/2}(\partial\Omega)$ set at the boundary, satisfying the *conductivity equation*:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, \text{ in } \Omega, \\ u|_{\partial\Omega} = f \end{cases} \tag{5.1}$$

Once again, this problem is uniquely solvable in $H^1(\Omega)$ due to the positive lower bound of

σ. From the corresponding solutions u we define the Dirichlet-to-Neumann map as before:

$$\Lambda_\sigma : \phi \in H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$f \mapsto \sigma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}.$$

Recall, that Calderón problem is mathematically stated as follows:

"Given Λ_σ find if $\sigma \in L^\infty(\Omega)$ is uniquely determined by it and if so reconstruct it."

This has been an open question for a long time and the best known result in three-dimensions is for the case of Lipschitz conductivities. Here, we are providing a framework for the following conjecture:

Conjecture 5.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $\sigma_i \in L^\infty(\Omega)$ for $i = 1, 2$ and denote Λ_{σ_i} as their respective Dirichlet-to-Neumann maps.*

$$\text{If } \Lambda_{\sigma_1} = \Lambda_{\sigma_2} \text{ then } \sigma_1 = \sigma_2 \text{ in } \Omega.$$

We introduce now the Quaternionic-Beltrami equation and an approach based on it that may provide a bridge to obtain the uniqueness result.

5.2 The relation with Quaternionic-Beltrami equation

In this section, we convert our initial problem into a simpler but analogous one. The key is that in the latter problem we do not require any derivatives on σ. However, this approach is not as easy as others for σ with derivatives. Even in two-dimensions, Astala and Päivärinta [8] require the introduction of quasi-conformal theory to obtain a uniqueness proof.

In three dimensions we do not yet know the path, but the theory of quasi-conformal mappings breaks down for quaternions.

Now, without loss of generality, lets assume that $\Omega = \mathcal{B}$ and $\sigma \equiv 1$ outside of it (see [81]). Our equation in (5.1) can be rewritten through

$$d \star (\sigma du) = 0,$$

where d is the exterior derivative and $\star$ the Hodge star. From the properties of both operators we obtain the following Lemma.

Lemma 5.2.1. *Let $\Omega \subset \mathbb{R}^3, n \geq 3$ be the unit ball, $\sigma \in L^\infty(\Omega)$ bounded from below and $u \in H^1(\Omega)$ a solution to the conductivity equation (5.1). Then, there exists a $(n-2)$ form ω, unique up to $d\nu$ for a $(n-3)$ form ν, such that*

$$\begin{cases} d\omega = \star\sigma du = 0, \\ d \star \left(\frac{1}{\sigma} d\omega\right) = 0. \end{cases} \tag{5.2}$$

Proof. Poincaré lemma states that in $\mathcal{B}$ every closed form ($d\omega = 0$) is exact.

In this sense, let $\alpha = \star\sigma du$, then $d\alpha = d\star(\sigma du) = 0$, for $u \in H^1(\Omega)$ unique solution of (5.1). Thus, α is a closed form in $\mathcal{B}$. By Poincaré lemma, there is a $(n-2)$ form ω such that $d\omega = \alpha$, i.e., ω is exact, and unique up to $d\chi$ with χ a $(n-3)$ form.

□

In the particular case of three dimensions, ($n = 3$), we have

$$\omega = u_0 dx + u_1 dy + u_2 dz.$$

From (5.2) we can extract that $\nabla \times (u_0, u_1, u_2) = \sigma\nabla u$. Recall, that ω is unique only up to $d\omega$. However, for our purpose we can establish a one-to-one correspondence by choosing ω in order to have $\nabla \cdot (u_0, u_1, u_2) = 0$.

Our objective now is to establish an analogue of the Beltrami equation in the realm of quaternions. In order to this, we define the following function

$$\phi = u - e_3(u_0 + u_1 e_1 + u_2 e_2). \tag{5.3}$$

Now by applying the Cauchy-Riemann operator D given as $D = \partial_0 + \partial_1 e_1 + \partial_2 e_2$ to ϕ leads to:

$$D\phi = Du - D(u_0 e_3 + u_1 e_1 - u_2 e_2) = Du - (\partial_0 + \partial_1 e_1 + \partial_2 e_2)(u_0 e_3 + u_1 e_2 - u_2 e_1) \tag{5.4}$$

$$= Du - [(\partial_1 u_2 - \partial_1 u_2) + e_1(\partial_2 u_0 - \partial_0 u_2) + e_2(\partial_0 u_1 - \partial_1 u_0) + e_3(\partial_0 u_0 + \partial_1 u_1 + \partial_2 u_2)] \tag{5.5}$$

Before proceeding we formulate the previous expression $\nabla \times (u_0, u_1, u_2) = \sigma\nabla u$ in terms of quaternions. We define the cross-product as:

$$\nabla \times (u_0 + u_1 e_1 + u_2 e_2) := \begin{vmatrix} e_0 & e_1 & e_2 \\ \partial_0 & \partial_1 & \partial_2 \\ u_0 & u_1 & u_2 \end{vmatrix} = e_0(\partial_1 u_2 - \partial_2 u_1) + e_1(\partial_2 u_0 - \partial_0 u_2) + e_2(\partial_0 u_1 - \partial_1 u_0)$$

Further, σDu is the quaternionic form of the expression $\sigma\nabla u$, since u is a real-valued function solving (5.1). Therefore, the previous relation is now given in quaternions as

$$\nabla \times (u_0 + u_1 e_1 + u_2 e_2) = \sigma Du.$$

This leads now to

$$\begin{aligned} D\phi &= Du - \nabla \times (u_0 + u_1 e_1 + u_2 e_2) + e_3 \nabla \cdot (u_0, u_1, u_2) \\ &= Du - \sigma Du, \end{aligned}$$

since we choose ν such that $\nabla \cdot (u_0, u_1, u_2) = 0$.

Analogously, applying D to $\bar{\phi} = u + (u_0 - u_1e_1 - u_2e_2)e_3$ leads to:

$$\begin{aligned} D\bar{\phi} &= Du + (\partial_0 + e_1\partial_1 + e_2\partial_2)(u_0e_3 + u_1e_2 - u_2e_1) \\ &= Du + \nabla \times (u_0 + u_1e_1 + u_2e_2) + e_3\nabla \cdot (u_0, u_1, u_2) = Du + \sigma Du \end{aligned}$$

The above expressions imply that

$$Du = \frac{1}{1-\sigma} D\phi \quad \text{and} \quad Du = \frac{1}{(1+\sigma)} D\bar{\phi}.$$

Joining them together we obtain the Quaternionic-Beltrami equation:

$$D\phi = \mu D\bar{\phi}, \text{ where } \mu = \frac{1-\sigma}{1+\sigma}. \tag{5.6}$$

With this definition and the fact that $\sigma \geq c > 0$ it is easily seen that

$$|\mu(x)| < 1, \text{ in } \mathbb{R}^3. \tag{5.7}$$

Furthermore, the support of μ is also restricted to $\mathcal{B}$.

To study the problem, as typical, we introduce the exponential growing solutions that seem to be appropriate for the exploration and that were initially described in [81].

5.3 Exponentially growing solutions

Due to the nature of exponential functions with quaternionic values we cannot make a clear connection to Astala and Päivärinta work [8]. The first thing to notice is that these exponentials would not be monogenic, which has been essential in every piece since it simplifies a lot of equations for the non-exponential term of the solutions. Furthermore, with quaternions we also lose the simple derivative-rule for multiplication.

These are the two-main aspects to keep in mind for the following presentation. We start from the usual exponential complex functions initially presented in the work of Sylvester and Uhlmann, [88].

These exponential functions are defined for $\zeta = (\zeta_0, \zeta_1, \zeta_2) \in \mathbb{C}^3 \setminus \{0\}$ by

$$e^{ix\cdot\zeta} \quad \text{with } x \in \mathbb{R}^3. \tag{5.8}$$

The essence of these functions is that if $\zeta \cdot \zeta := \zeta_0^2 + \zeta_1^2 + \zeta_2^2 = 0$ then they are harmonic functions, which is only possible if ζ is complex-valued.

Since, the Laplace operator can be decomposed in terms of the D and $\bar{D}$ we can find exponential functions related to these ones and that are monogenic.

$$\Delta e^{x\cdot\zeta} = 0 \Leftrightarrow D(\bar{D}e^{x\cdot\zeta}) = 0.$$

In this sense, the functions we are looking for can be defined as

$$\mathcal{E}(x,\zeta) = \bar{D}e^{x\cdot\zeta} = (\zeta_0 - \zeta_1 e_1 - \zeta_2 e_2)e^{x\cdot\zeta} =: \bar{\zeta}e^{x\cdot\zeta}.$$

Here, we denote by $\bar{\cdot}^{\mathbb{H}}$ the quaternionic conjugation of a complex-valued quaternion. Throughout the chapter we simply denote it by $\bar{\cdot}$ since we do not apply complex conjugation on this work.

Here, we are mixing two notations for ζ but they are related by the connection of $\mathbb{C}^3$ and $\mathbb{C}\otimes\mathbb{H}$ through:

$$\zeta = (\zeta_0, \zeta_1, \zeta_2) \in \mathbb{C}^3 \Leftrightarrow \zeta = \zeta_0 + \zeta_1 e_1 + \zeta_2 e_2 \in \mathbb{C}\otimes\mathbb{H}.$$

With this connection the have the equivalence between conditions:

$$\zeta\cdot\zeta = 0 \Leftrightarrow \bar{\zeta}\zeta = \zeta\bar{\zeta} = 0.$$

Now, we define our exponential growing solutions to the quaternionic-Beltrami equation as:

$$\phi(x,\zeta) = \mathcal{E}(x,\zeta)m(x,\zeta), \tag{5.9}$$

with a quaternionic complex-valued function.

Since, we fixed the exponential behavior we formalize now how the function m should behave in terms of the differential equation. For such, we substitute (5.9) in (5.6) as follows

$$D(\mathcal{E}m) = \mu D(\bar{m}\bar{\mathcal{E}}).$$

With the Leibniz rule in the particular case a function is scalar valued (see the Preliminaries), the above derivatives can be expanded as

$$\begin{aligned}
D\left(e^{x\cdot\zeta}\bar{\zeta}m(x,\zeta)\right) &= \left(De^{x\cdot\zeta}\right)\bar{\zeta}m(x,\zeta) + e^{x\cdot\zeta}\left(D\bar{\zeta}m(x,\zeta)\right) \\
&= \zeta\bar{\zeta}m(x\zeta) + e^{x,\cdot\zeta}\left(D\bar{\zeta}m(x,\zeta)\right) = e^{x\cdot\zeta}\left(D\bar{\zeta}m(x,\zeta)\right) \\
D\left(e^{x\cdot\zeta}\overline{m(x,\zeta)}\zeta\right) &= \left(De^{x\cdot\zeta}\right)\overline{m(x,\zeta)}\zeta + e^{x\cdot\zeta}\left(D\overline{m(x,\zeta)}\zeta\right) \\
&= e^{x\cdot\zeta}\zeta\overline{m(x,\zeta)}\zeta + e^{x\cdot\zeta}\left(D\overline{\bar{\zeta}m(x,\zeta)}\right)
\end{aligned}$$

By joining both expressions in the quaternionic-Beltrami equation, we obtain:

$$D(\bar{\zeta}m(x,\zeta)) = \mu(x)\left[\zeta\overline{\bar{\zeta}m(x,\zeta)} + D\overline{\bar{\zeta}m(x,\zeta)}\right]. \tag{5.10}$$

Since $\bar{\zeta}m(x,\zeta)$ is a common term through out the equation, we can define

$$\psi(x,\zeta) = \bar{\zeta}m(x,\zeta),$$

in order to obtain the simpler equation

$$D\psi = \mu\left[\zeta\bar{\psi} + D\bar{\psi}\right]. \tag{5.11}$$

Notice that due to the definition of ψ the initial function ϕ solving (5.6) can be defined through it as:

$$\phi(x,\zeta) = e^{x\cdot\zeta}\psi(x,\zeta).$$

This hides the term that makes the exponential function monogenic inside ψ and allows a simpler connection with the same decomposition made in two-dimensions. It also simplifies the analysis that follows.

In the next section, we introduce a bit of the operator theory in quaternionic analysis and introduce a integral equation corresponding to (5.11).

5.4 Study of the integral equation

The study of exponential growing solutions to the quaternionic-Beltrami equation is always coupled with the study of integral equations for the equation with non-exponential growth, in our case (5.11). This is the focus of this section, where we use the inverse operator of D to derive it.

We now introduce an extension of the T-operator to $\mathbb{R}^3$ by

$$Th(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\overline{x-y}}{|x-y|^3} h(y)\, dy, \tag{5.12}$$

which acts as a left and right-inverse operator like $DTg = g$, $TDg = g$, for $g \in W^{1,p}(\mathbb{R}^3)$.

Further, we set $L^p(\Omega)$ over a bounded domain through $L^p(\Omega) := \left\{ g \in L^p(\mathbb{R}^3) : g|_{\mathbb{R}^3\setminus\Omega \equiv 0} \right\}$.

Proposition 5.4.1. *Let Ω be a bounded Lipschitz domain and $h \in L^p(\Omega)$ for $1 < p < \infty$. Then, $Th(x)$ exists for all $x \in \mathbb{R}^3 \setminus \bar{\Omega}$ and it fulfills the inequality*

$$|Th(x)| \leq \frac{1}{4\pi}\mathrm{dist}(x,\Omega)^{-2}\|h\|_{L^p(\Omega)}.$$

Proof. For $h \in L^p(\Omega)$ we get that

$$\begin{aligned} |Th(x)| &\leq \frac{1}{4\pi}\left|\int_\Omega \frac{\overline{x-y}}{|x-y|^3}h(y)\,dy\right| \leq \frac{1}{4\pi}\int_\Omega \frac{1}{|x-y|^2}|h(y)|\,dy \\ &\leq \frac{1}{4\pi}\mathrm{dist}(x,\Omega)^{-2}\|h\|_{L^1(\Omega)} \leq \frac{C_\Omega}{4\pi}\mathrm{dist}(x,\Omega)^{-2}\|h\|_{L^p(\Omega)}. \end{aligned}$$

□

Proposition 5.4.2. *Let $\Omega \subset \mathbb{R}^3$ a bounded domain and let $p > 3/2$. Then,*

(i) $T : L^p(\Omega) \to W^{1,p}(\mathbb{R}^3)$ is a bounded operator,

(ii) If $3/2 < p <$ $T : L^p(\Omega) \to L^p(\mathbb{R}^3)$ is a compact operator.

Proof. (i) The proof follows by the boundedness of the operator over bounded domains, *i.e.*, $T : L^p(\Omega) \to W^{1,p}(\Omega)$ and the decay at infinity presented in Proposition 5.4.1.

(ii) This proof can be treated similarly as in the Lemma 4.2 of the preprint of [69].

The operator $T : L^p(\Omega) \to L^p(\mathbb{R}^3)$ can be consider in $L^p(\mathbb{R}^3)$ as the operator T defined in (5.12) multiplied with the characteristic function of the domain, like $T\chi_\Omega$.

Under this definition, this operator is compact if and only if dual operator is compact. Thus, by duality we study the compactness of $\chi_\Omega \bar{T}$, where here $\bar{T}$ is equivalent to the operator T but with the kernel without the conjugation. Accordingly, the dual operator is now defined over the dual space $L^r(\mathbb{R}^3)$ of $L^p(\mathbb{R}^3)$ with $1 < r < 3$. Thus, we have

$$\|\chi_\Omega \bar{T} f\|_{L^r(\mathbb{R}^3)} \leq \|\chi_\Omega\|_{L^3(\mathbb{R}^3)} \|\bar{T} f\|_{L^{\tilde{r}}(\mathbb{R}^3)}, \tag{5.13}$$

where $\tilde{r}$ is the Hölder conjugate of r defined as $\frac{1}{\tilde{r}} = \frac{1}{r} - \frac{1}{3}$.

Now suppose that χ is a function in C^1 with compact support in Ω. Then, it holds that

$$\|\nabla\left(\chi_\Omega T f\right)\|_{L^r(\mathbb{R}^3)} \leq \|\nabla\chi\|_{L^3(\mathbb{R}^3)} \|\partial^{-1} f\|_{L^{\tilde{r}}(\mathbb{R}^3)} + \|\chi\|_{L^\infty(\mathbb{R}^3)} \|\nabla T f\|_{L^r(\mathbb{R}^3)} \leq c\|f\|_{L^r(\mathbb{R}^3)}.$$

Therefore, the image under χT of the unit ball in $L^r(\mathbb{R}^3)$ lies in $\{u \in L^r(\Omega) : \|u\|_{L^r(\Omega)} \leq c, \|\nabla u\|_{L^r(\Omega)} \leq c\}$ which is compactly embedding into $L^r(\Omega)$ by the Rellich-Kondrachov theorem. Thus χT is a compact operator in $L^r(\mathbb{R}^3)$.

Now, let χ_Ω be the characteristic function of the domain Ω and let $\{\chi_k\}$ be a sequence of C^1 functions of compact support converging to χ_Ω in $L^3(\Omega)$. As we have shown, the corresponding operators $\chi_k T$ are compact and norm convergent, similarly to the above estimate (5.13), hence their limit, too, is a compact operator.

□

The other operator of interest is defined as

$$Sg = D\overline{Tg},$$

which is analogous to the Π-operator introduced in [83], further studied in [33] and extended in [11].

Since, we already known the Fourier symbol of the integral operator T (see [33]) we can establish the symbol of S. However, due to the non-commutative nature of quaternions it holds that

$$\mathcal{F}\left(D\overline{Th}(x)\right)(\xi) = \xi\, \overline{\hat{h}(\xi)}\, \frac{\bar{\xi}}{|\xi|^2} \tag{5.14}$$

From this we can visualize that S in $L^2(\mathbb{R}^3)$ is an isometry and the norm is equal to 1. The norm

$$D\psi = \mu\left(D\bar{\psi} + \zeta\bar{\psi}\right) \text{ for } x \in \mathbb{R}^3.$$

Furthermore, for the general $L^p(\mathbb{R}^3)$ spaces Theorem 16 in [11] provides the following estimate for the norm:

$$\|S\|_{L^p(\mathbb{R}^3)} \leq 6 \max \left\{ p-1, \frac{1}{p-1} \right\}.$$

One of the most interesting conjectures on the $\Pi-$ operator, that could be extended for our case, was introduced by Iwaniec in [51] and states that:

$$\|\Pi\|_{L^p} = \begin{cases} \frac{1}{p-1}, & 1 < p \leq 2, \\ p-1, & 2 < p < \infty. \end{cases} \tag{5.15}$$

Let us proceed to determine the integral equation of interest. We assume further that our desired solution has the following asymptotic behavior:

$$\psi = 1 + \eta \quad \text{ and } \eta \to 0, |x| \to \infty. \tag{5.16}$$

With this in mind, we proceed as in Proposition 4.1 of [8], where $TD = I$ in $W^{1,p}(\mathbb{R}^3)$

$$\begin{aligned} D\eta - \mu D\bar{\eta} &= \mu\zeta\,(1+\bar{\eta}) \\ \Leftrightarrow D\eta - \mu D\overline{TD\eta} &= \mu\zeta(1+\bar{\eta}) \\ \Leftrightarrow [I - \mu S]\,(D\eta) &= \mu\zeta(1+\bar{\eta}). \end{aligned}$$

At least for $p = 2$ we know that μS is a contraction in $L^2(\mathbb{R}^3)$ due to $\|\mu\|_\infty < 1$. In case the conjecture 5.15 is true, it holds that:

$$\|\mu S\|_{L^p} < \|\mu\|_\infty (p-1), \text{ for } p > 2,$$

which implies that we have a contraction when

$$p < 1 + \frac{1}{\|\mu\|_\infty}.$$

This leads to:

$$\begin{aligned} D\eta &= [I - \mu S]^{-1}\,(\mu\zeta(1+\bar{\eta})) \\ \Leftrightarrow \eta &= T\,[I - \mu S]^{-1}\,(\mu\zeta(1+\bar{\eta})) \\ \Leftrightarrow [I - K_\zeta]\eta &= K_\zeta 1, \text{ where } K_\zeta = T\,[I - \mu S]^{-1}\,(\mu\zeta\,\bar{\cdot}\,). \end{aligned}$$

Due to the compactness properties of T we want to show that K_ζ will be a compact operator, which implies $I - K_\zeta$ is Fredholm of index zero. If this holds, then uniqueness of the integral equation only depends on $\ker[I - K_\zeta]$ being trivial.

Lemma 5.4.3. *Let $p > 3/2$, $\zeta \in \mathbb{C} \otimes \mathbb{H}$ and $\mu \in L^\infty(\mathbb{R}^3)$ with compact support in $\mathcal{B}$ such that μS is a contraction in $L^p(\mathbb{R}^3)$. Then, in $L^p(\mathbb{R}^3)$ it holds:*

- supp $[I-\mu S]^{-1}(\mu\zeta\bar{g}) \subset \mathcal{B}$;
- $K_\zeta := T[I-\mu S]^{-1}(\mu\zeta\,\bar{\cdot})$ *is compact;*
- $I-K_\zeta$ *is Fredholm of index zero.*

To study the above equation, it follows by Fredholm theory that we just need to show $I-K_\zeta$ is injective.

Conjecture 5.4.4. *Let $p>3/2$, $\zeta \in \mathbb{C}\otimes\mathbb{H}$ and $\mu \in L^\infty(\mathbb{R}^3)$ with compact support in $\mathcal{B}$ such that μS is a contraction in $L^p(\mathbb{R}^3)$. We denote $|\zeta|^2 := |\zeta_0|^2+|\zeta_1|^2+|\zeta_2|^2$. Then there exists an $R>0$ such that for $|\zeta|>R$ the operator $I-K_\zeta$ is invertible in $L^p(\mathbb{R}^3)$.*

Therefore, there exists a unique solution $\phi(x,\zeta)$ of the form:

$$\phi(x,\zeta) = e^{x\cdot\zeta}(1+\eta), \tag{5.17}$$

where η is the solution of

$$[I-K_\zeta]\,\eta = K_\zeta 1. \tag{5.18}$$

In the following section we base ourselves on the exponentially growing solutions to prove we can uniquely determine σ from the Dirichlet-to-Neumann map Λ_σ.

5.5 From Λ_σ to σ

The approach we establish up next was initially derived in [81] where he used this result to show an alternative proof for uniqueness of the linearized problem at constant conductivities, initially shown by Calderón in its seminal paper [18].

As a follow up and with the exponentially growing solutions in mind, we apply those ideas to obtain a formula that leads to the unique determination of the conductivity $\sigma \in L^\infty(\Omega)$ from the Dirichlet-to-Neumann map Λ_σ, that depends only on the existence of the solutions ψ for large complex frequencies with asymptotically decay in ζ.

To start we introduce the generalization of Alessandrini's identity, obtained by Santacesaria in [81].

Proposition 5.5.1. *Let σ_1, $\sigma_2 \in L^\infty(\Omega)$ be two positively lower bounded conductivities and let Λ_1, Λ_2 be their corresponding Dirichlet-to-Neumann maps. Then, for every g_1, $g_2 \in H^{1/2}(\partial\Omega)$ we have the identity:*

$$\langle g_1, (\Lambda_2-\Lambda_1)\,g_2\rangle_{H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega)} = \frac{1}{2}\int_\Omega (\mu_1-\mu_2)\left\langle D\bar{\phi}_1,\, D\bar{\phi}_2\right\rangle_{\mathbb{R}^4} dx, \tag{5.19}$$

where $u_j = \mathrm{Sc}\,\phi_j$ solves

$$\nabla\cdot(\sigma_j\nabla u_j) = 0, \text{ in } \Omega,\ u_j = g_j \text{ on } \partial\Omega,$$

and ϕ_j satisfies $D\phi_j = \mu_j D\bar{\phi}_j$ in Ω with $\mu_j = \frac{1-\sigma_j}{1+\sigma_j}$.

Proof. First, notice that for a quaternionic-valued functions f it holds

$$\bar{f} + f = 2\mathrm{Sc}f \quad \text{and} \quad \bar{f} - f = 2\mathrm{Vec}f \tag{5.20}$$

by the conjugation properties. Our solutions are connected with the conductivity equation solution u_j by $\phi_j = u_j - e_3(u_{j,0} + u_{j,1}e_1 + u_{j,2}e_2)$, $j = 1, 2$ with $\nabla \times (u_{j,0}, u_{j,1}, u_{j,2}) = \sigma_j \nabla u_j$.

Hence, by the above relations and the computation (5.4) we obtain:

$$\frac{D\bar{\phi}_j + D\phi_j}{2} = \partial_0 u_j + \partial_1 u_j e_1 + \partial_2 u_j e_2, \tag{5.21}$$

$$\frac{D\bar{\phi}_j - D\phi_j}{2} = \mathrm{curl}_0 + \mathrm{curl}_1 e_1 + \mathrm{curl}_2 e_2 + \nabla \cdot (u_{j,0}, u_{j,1}, u_{j,2}) e_3, \tag{5.22}$$

where we have denoted $(\mathrm{curl}_0, \mathrm{curl}_1, \mathrm{curl}_2 e_2) = \nabla \times (u_{j,0}, u_{j,1}, u_{j,2})$.

By Green's formulas one readily obtains from the Dirichlet-to-Neumann maps the following Alessandrini's identity for the Calderón problem [5]:

$$\langle g_1, (\Lambda_2 - \Lambda_1) g_2 \rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = \int_\Omega (\sigma_2 - \sigma_1) \nabla u_1 \cdot \nabla u_2 \, dx.$$

Here $\cdot$ denotes the Euclidean inner product. For simplicity purposes we denote U_j be the vector field $U_j := (u_{j,0}, u_{j,1}, u_{j,2})$ that fulfills $\nabla \times (U_j) = \sigma_j \nabla u_j$. Then

$$\langle f_1, (\Lambda_2 - \Lambda_1) f_2 \rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = \int_\Omega (\nabla \times U_2) \cdot \nabla u_1 - (\nabla \times U_1) \cdot \nabla u_2 \, dx. \tag{5.23}$$

With the above identities (5.20) it holds that the inner products under the integral are given as:

$$\begin{aligned}(\nabla \times U_2) \cdot \nabla u_1 - (\nabla \times U_1) \cdot \nabla u_2 &= \frac{1}{4} \langle D\bar{\phi}_2 - D\phi_2, D\bar{\phi}_1 + D\phi_1 \rangle_{\mathbb{H}} \\ &\quad - \frac{1}{4} \langle D\bar{\phi}_2 + D\phi_2, D\bar{\phi}_1 - D\phi_1 \rangle_{\mathbb{H}}\end{aligned}$$

Given that ϕ_j satisfy the respective Beltrami equation $D\phi_j = \mu D\bar{\phi}_j$, we can substitute in the above formula to obtain:

$$\begin{aligned}(\nabla \times U_2) \cdot \nabla u_1 - (\nabla \times U_1) \cdot \nabla u_2 &= \frac{1}{4} \langle (1 - \mu_2) D\phi_2, (1 + \mu_1) D\bar{\phi}_1 \rangle_{\mathbb{R}^4} \\ &\quad - \frac{1}{4} \langle (1 + \mu_2) D\bar{\phi}_2, (1 - \mu_1) D\bar{\phi}_1 \rangle_{\mathbb{R}^4} \\ &= \frac{1}{4} \left((1 - \mu_2)(1 + \mu_1) - (1 + \mu_2)(1 - \mu_1) \right) \langle D\bar{\phi}_2, D\bar{\phi}_1 \rangle_{\mathbb{R}^4} \\ &= \frac{\mu_1 - \mu_2}{2} \langle D\bar{\phi}_2, D\bar{\phi}_1 \rangle_{\mathbb{R}^4}.\end{aligned}$$

Substituting in the variation of Alessandrini identity (5.23) we obtain our desired identity (5.19). $\square$

By applying this identity when the Dirichlet-to-Neumann maps Λ_1, Λ_2 are equal to each other, we obtain the unique determination of the coefficients μ_j. That is, if for $\sigma_j \in L^\infty(\Omega), \sigma_j \geq c > 0, j = 1, 2$ we have $\Lambda_1 = \Lambda_2$ then Alessandrini identity implies that:

$$\int_\Omega (\mu_1 - \mu_2) \left\langle D\bar{\phi}_1,\, D\bar{\phi}_2 \right\rangle_{\mathbb{R}^4} dx = 0. \tag{5.24}$$

where ϕ_j are exponentially growing solutions associated with the spectral parameter ζ^j. We already know, that at least for some $\zeta^j \in \mathbb{C}^3$ with $\zeta^j \cdot \zeta^j = 0$ the following types of solutions exist in $W^{1,p}_{\mathrm{loc}}(\Omega)$

$$\begin{aligned} \phi_j &= \left(e^{x\cdot\zeta^j}\bar{\zeta}^j\right) m_j, \qquad \text{and recall that for } \psi_j = \bar{\zeta}^j m_j \\ D\bar{\phi}_j &= e^{x\cdot\zeta^j}\left(\left(D\bar{m}_j\right)\zeta^j + \zeta^j\bar{m}_j\zeta^j\right) = e^{x\cdot\zeta^j}\left(D\bar{\psi}_j + \zeta^j\bar{\psi}_j\right). \end{aligned}$$

Substituting in (5.24) we obtain:

$$\int_\Omega (\mu_1 - \mu_2)e^{x\cdot(\zeta^1+\zeta^2)} \left\langle D\bar{\psi}_1 + \zeta^1\bar{\psi}_1, D\bar{\psi}_2 + \zeta^2\bar{\psi}_2 \right\rangle_{\mathbb{R}^4} dx = 0. \tag{5.25}$$

Since $\zeta^j \in \mathbb{C}^3$ it is possible to pick them such that $\zeta^1 + \zeta^2 = ik$, $k \in \mathbb{R}^3$, but ζ^j are not purely imaginary.

Moreover, from our previous assumptions on exponentially growing solutions we further take the asymptotics $\psi_j = 1 + \eta_j$. This leads to:

$$\begin{aligned} \left\langle \zeta^1, \zeta^2 \right\rangle_{\mathbb{R}^4} \int_\Omega (\mu_1 - \mu_2)e^{ix\cdot k}\, dx = \int_\Omega (\mu_1 - \mu_2)e^{ix\cdot k}\Big(&\left\langle D\bar{\eta}_1 + \zeta^1\bar{\eta}_1, D\bar{\eta}_2 + \zeta^2\bar{\eta}_2 \right\rangle_{\mathbb{R}^4} \\ &+ \left\langle D\bar{\eta}_1 + \zeta^1\bar{\eta}_1, \zeta^2 \right\rangle_{\mathbb{H}} + \left\langle \zeta^1, D\bar{\eta}_2 + \zeta^2\bar{\eta}_2 \right\rangle_{\mathbb{R}^4} \Big)\, dx. \end{aligned} \tag{5.26}$$

Now, the objective is to show that as $|\zeta^j| \to \infty$ the scalar products on the right-side converge to zero. With this we obtain:

$$\int_\Omega (\mu_1 - \mu_2)e^{ix\cdot k}\, dx = 0 \Rightarrow \mu_1 = \mu_2.$$

This implication is easily obtained by Fourier inversion theorem. The following deductions are the basis to prove Conjecture 5.1.1, that is, Λ_σ uniquely determines $\sigma \in L^\infty(\Omega)$.

5.5.1 Open questions

In order to use the above uniqueness theorem there are still two missing pieces one needs to show before finalizing the proof. Both are related with the exponentially growing solutions ϕ that solve the Quaternionic-Beltrami equation.

First and foremost, we need to prove Conjecture 5.4.4 that guarantees existence and uniqueness of exponentially growing solutions to the Beltrami equation for large complex frequencies. These are the solutions plugged in the Alessandrini identity.

Recall that the Beltrami equation (5.6) for our type of solutions is equivalent to the integral equation (5.18), that we make explicit here again

$$[I - K_\zeta]\,\eta = K_\zeta 1.$$

So far, we have shown that K_ζ is a compact operator in $L^p(\mathbb{R}^3)$ and thus the left-hand side operator is Fredholm of index zero. Thus, to prove invertibility one needs to show that $\ker(I - K_\zeta)$ is empty.

In the related work of Astala and Päivärinta in two dimensions [8] this was obtained through the study of pseudo-analytic functions in [10], [91]. This study ends up connecting with quasi-conformal homeomorphism and the study of the Beltrami equation. The key idea is provided in their paper and more extensively in Theorem 8.5.3 of [7] and is based on obtaining an analogue of Liouville theorem through a distortion inequality.

In higher dimensions, suppose $f \in W^{1,n}_{loc}(\mathbb{R}^n)$ satisfies the distortion inequality

$$|Df|^n \leq KJ(x,f) + \gamma(x)|f|^n$$

with $\gamma \in L^{n\pm}(\mathbb{R}^n)$, D is the Jacobian and $J(x,f)$ given by its determinant. As far as we are aware it is still an open question to show that f is continuous and, additionally, if $f \to 0$ as $x \to \infty$ then is $f \equiv 0$.

In two dimensions this inequality simplifies to $|\bar{\partial} f(z)| \leq k|\partial f(z)| + \gamma(z)|f(z)$ due to the relation between D and the Wirtinger derivatives $\bar{\partial}, \partial$. Astala and Päivärinta used the above result that was established through the theory of pseudo-analytic functions to show the desired uniqueness of the Beltrami equation.

This would be a step to show uniqueness of solutions for all $\zeta \in \mathbb{C}^3$, but a generalization of pseudo-analytic functions and their study is necessary for quaternions and Clifford algebras. At this time, it is an infeasible methodology to prove uniqueness.

Another approach which seems more viable, as of now, is to obtain an estimate of K_ζ in terms of $|\zeta|$, that shows its decay as $|\zeta| \to \infty$. This would bring a contraction principle and we could obtain the inverse operator to $[I - K_\zeta]$ by Neumann series. This path follows the uniqueness proofs provided in [88], [68] and others.

Finally, to show the decaying properties on the right-hand side of (5.26) one needs to show how the exponentially growing solutions behave in terms of $|\zeta|$.

Conjecture 5.5.2. *Let $\eta_j \in L^p(\mathbb{R}^3)$, $p > 3$ be the solutions in (5.4.4) with respect to $\sigma_j \in L^\infty(\Omega), j = 1, 2$. Then, in some topology to be defined, they fulfill the following asymptotics as $\zeta_j \to \infty$*

$$D\bar{\eta} + \zeta\bar{\eta} \to 0 \qquad \textit{as } |\zeta| \to \infty,$$

or

$$\eta \to 0 \qquad \textit{as } |\zeta| \to \infty.$$

An aspect to notice is that in two-dimensions Nachman was able to prove a similar decay to the last one based on the decay properties $(\partial + ik)^{-1}$, see Lemma 1.2 in [69], which is related to the operator above $D + \zeta$. However, for the latter one a different methodology needs to be used to prove this decay, since we are not able to obtain the decomposition of $(\partial + ik) = e^{-i(kz+\bar{k}\bar{z})}\partial e^{i(kz+\bar{k}\bar{z})}$ due to the quaternionic exponential not commuting with the differential operators.

This chapter provides a novel framework that extends the one proposed by Santacesaria [81] to solve Calderón problem for $\sigma \in L^\infty(\Omega)$. It gives insight into the quaternionic framework and how one can use it to extend some of the results provided in [8] for three-dimensions.

Chapter 6

Automatic Differentiation in EIT

In medical practice, the Calderón problem is designated by *Electrical Impedance Tomography (EIT)* and the goal is to reconstruct the conductivity from boundary measurements through a finite number of electrodes.

This chapter focuses on the reconstruction of conductivities through iterative methods in an optimization framework. These methods require the computation of derivatives of a complex numerical method. Thus, in particular, we focus on the problem of effectively computing these derivatives in a simple manner.

For this purpose we introduce the Automatic Differentiation (AD) method to compute derivatives of complex programs and apply it to solve EIT through iterative methods. This method can be applied to other inverse problems, where the computation of derivatives are required. For this purpose, we show that AD is an effective method to compute derivatives of a finite element method (FEM) with respect to a parameterization of the conductivity and use it to solve the EIT inverse problem. This work is an extension of [78].

6.1 The problem

An inverse problem can be always posed as a minimization problem, where the goal is to find a parameter approximation that matches the measurements acquired in practice. For this purpose, we need to numerically compute those measurements, verify if they are close to the ones measured in practice and, if not, update the parameter accordingly.

Iterative methods are essentially important for updating the approximation of the parameter. Most of them require the computation of derivatives either of the loss function or of the numerical method in terms of the parameter variables. For non-linear inverse problems, this implies the computation of the derivative of complex programs, for which it can be hard or impossible to deduce a closed analytical formula for the derivative.

Hence, our goal is to study the effectiveness of the automatic differentiation method when applied to inverse problems. This study is done on the inverse problem of EIT. This choice is not only to keep the thesis contained in one main theme, but also because we can obtain a closed analytical formula for the derivative under particular assumptions.

Introduction to EIT

A particularly relevant application of EIT is in the early determination of breast cancer, specifically for young women where the risks of the ionizing X-rays of mammographies outweigh the benefits of regular check-ups. Fig. 6.1 describes one simplified EIT scenario where the blue region represents cancer inside the breast Ω. The assumption is that healthy and cancerous tissue have different conductivity values σ_1, σ_2, respectively. The goal is to locate a potential region affected by cancer from measurements on the surface $\partial\Omega$.

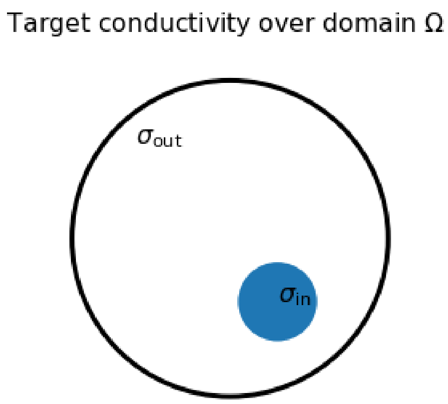


Figure 6.1: Example of a target conductivity over the domain Ω that represents a simple model of breast cancer where tumors have higher conductivity than the background. The domain Ω is represented by the black circumference which has a conductivity of σ_{out}. In a blue circle it is represented a region with different conductivity σ_{in} from the background one σ_{out}.

The measurements are obtained by injecting into the domain Ω a fixed set of different *electrical current patterns* I_j. Each I_j is defined by injecting electrical current through all electrodes in a particular manner, *i.e.*, for L electrodes we have $I_j = (I_{j,1}, ..., I_{j,L})$. Simultaneously, we measure the resulting voltages V_j for each current pattern, hence obtaining a voltage measurement at each electrode, thus $V_j = (V_{j,1}, ..., V_{j,L})$. This leads to a set of *true measurements* denoted by $m_j = (I_j, V_j)$. Then, the corresponding *inverse problem* is to determine the electrical conductivity over Ω that leads to these measurements. In the particular case of Fig. 6.1 we want to determine σ_{out} and σ_{in} and the location of the anomaly (in blue).

This is a hard problem because in general there is no analytical expression that maps a set of electrical measurements back to respective conductivity values.

To solve this inverse problem we first need to understand how to solve the *direct problem*, that is, computing electrical measurements V_j for a given set of currents I_j and conductivity.

The direct problem has an easier solution, since the propagation of electrical current through the domain obeys well-known physical laws.

Many methods for solving the direct problem are described in the literature, *e.g.*, the finite element method [82], the boundary element method [31], and, more recently Deep Learning methods [52].

Independently of the numerical method used to solve the direct problem, such a procedure is commonly designated as *simulation*. Hence, for a given conductivity profile we can obtain through a simulation method the electrical measurements denoted as $m_j^{\text{Sim}} = (I_j, V_j^{\text{Sim}})$, for each different current pattern with $j = 1, ..., N$. We can, thus, define an operator that maps conductivity into voltage measurements, designated by *direct operator*, given as:

$$\mathbf{Sim} : \sigma \mapsto V^{\mathbf{Sim}} = (V_{1,1}^{\text{Sim}}, .., V_{j,l}^{\text{Sim}}, ..., V_{N,L}^{\text{Sim}}) \in \mathbb{R}^{L \cdot N}, \tag{6.1}$$

where $V_{j,l}^{\text{Sim}}$ represent voltages measured at the l-th electrode for the j-th current pattern.

Our goal is to find a conductivity profile that matches measurements $m = (m_1, ..., m_N)$. Thus, we can formulate EIT as the following minimization problem by making use of the direct operator **Sim**:

$$\min_{\sigma} \frac{1}{2} \left\| \mathbf{Sim}(\sigma) - m^{\text{true}} \right\|_2^2 . \tag{6.2}$$

We use the L^2-norm here for simplicity, but, in general, we could use any other norm as long as it is differentiable.

Most classical methods for solving this minimization problem are based on iteratively improving the solution. The update requires computing the derivative of both the loss function and the **Sim** operator.

To solve the inverse problem under an optimization framework we introduce in section 6.7 the *Levenberg-Marquardt algorithm* [59], [61]. It is a simple quasi-Newton method that only requires the Jacobian of the **Sim** operator.

In essence, the main challenges to solve inverse problems in an optimization framework with classic iterative methods are:

- ensure that the simulator is once differentiable with respect to a conductivity parameterization;
- devise a method to compute the respective derivatives of the simulator.

Our study explores a simulation operator obtained through FEM, which is already well established for EIT, see [64].

When the **Sim** operator is given by FEM we can deduce an analytical formula for the derivative with respect to the conductivity variation. It is simply obtained with respect to a conductivity discretization over the FEM mesh, see Fig. 6.2. As such, it requires derivative

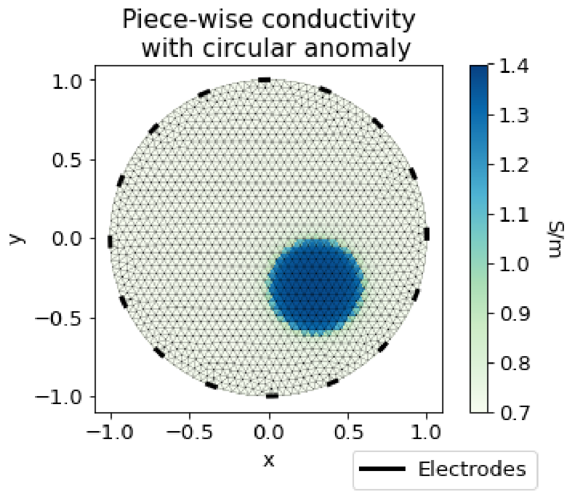


Figure 6.2: Circular anomaly defined over a triangular FEM mesh. Electrodes are attached to the boundary, black lines.

computations with respect to conductivity values over *all elements of the mesh*. If the conductivity is defined through a different parameterization we can obtain the respective derivatives by the chain rule of differentiation. For such endeavor, the analytical formulation needs to be adapted and derived for each particular parameterization of the conductivity. Accordingly, this method is hard to derive and implement, see [43] and [90].

Automatic differentiation (AD) is a method that automatically evaluates exact derivatives for complex programs. It exploits the simple mathematical operations the programs are built on, to automatically compute the derivative through the chain rule. While the initial concept was developed in the sixties [93], only lately with advancements in hardware and efficient implementations, like JAX [14], it has gained traction for application in general problems.

In this work, we explore automatic differentiation as an alternative to manual methods for computing the Jacobian of differentiable simulators. In particular, the goal is to validate its effectiveness in solving the EIT inverse problem. By doing so we show its versatility compared with analytic formulation and moreover verify its viability for high resolution images.

The validation is done by comparing the absolute error between solutions obtained by solving the minimization problem with both methods to compute the derivatives and the absolute error compared with the true solution. We evaluate the maximum difference between both Jacobian computations to check if they are evaluating to the same result. Then as a second set of checks, we explore the memory consumption of AD and show that it is still in reasonable terms for high resolution meshes.

Our end goal is to show feasibility and practicality of this approach as a tool for lowering the entry barrier for other inverse problems in partial differential equations, where AD can also be applied.

6.2 Automatic differentiation method

AD is a set of techniques to evaluate the derivative of a function specified by a computer program. No matter how complicated they are, any computer program is based on a simple set of arithmetic operations and functions, like addition, multiplication, trigonometric functions, exponentials, etc. We can encode the derivative rule for all of these simple operations and build up the full derivative of our complex program through the chain-rule. AD evaluates derivatives with exact precision.

There are two modes for AD implementation: forward-mode and reverse-mode. In any case, they are not hard to implement through operator overloading techniques. The difficult part is to provide an efficient and optimal computation of these modes. However, at the present moment there are libraries that provide efficient implementations of AD for both modes, like JAX for Python.

The first step in AD is the creation of a computational graph of our program, that explains the decomposition into simpler operations for which we know the derivative. Let us exemplify for the following function $f(x_1, x_2) = \sin(x_1 \cdot x_2) + e^{x_1}$. The first step is to break things apart into simpler operations:

$$
\begin{aligned}
&w_1 = x_1, w_2 = x_2 \\
&w_3 = w_1 \cdot w_2 \\
&w_4 = \sin(w_3) \\
&w_5 = e^{w_1} \\
&w_6 = w_4 + w_5 =: f(w_1, w_2)
\end{aligned}
$$

This decomposition is more easily visualized through the computational graph in Fig. 6.3.

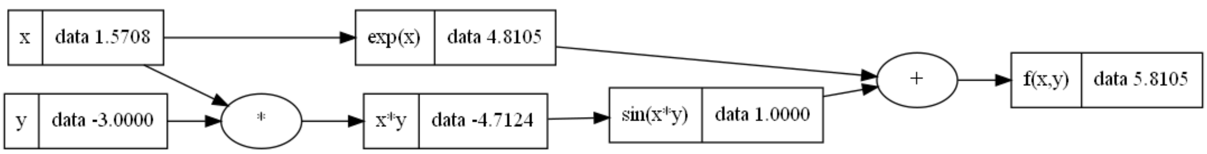


Figure 6.3: Computational Graph of $f(x_1, x_2) = \sin(x_1 \cdot x_2) + e^{x_1}$ evaluated at $(\pi/2, -3)$.

With the computational graph in mind, forward-mode computes derivatives from bottom-to-top, that is from the variables to output. As such, it allows the derivative computation of all outputs with respect to a single variable. It can evaluate the derivative simultaneously with the function, and thus it is proportional to the original code complexity. In this terms, it is more efficient for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m >> n$.

Reverse-mode of AD works the other way around, that it is, top-to-bottom. First, it requires a forward evaluation of all the variables, and thereafter it starts computing the derivatives from output values for the variables involved immediately, doing that successively until the input variables. Therefore, it allows evaluation of the gradient of an single output function. As such, it is way more efficient for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m << n$.

A familiar example is given by neural networks that are described by much more weights than output variables, in this particular example the reverse-mode is known as back-propagation.

One possible limitation to take into account in AD arises from the computational graph we described. Due to the computer program complexity this computational graph can be very expensive to establish and keep in memory. In such scenarios, where the Jacobian is obtained from a very complex graph, instead of a compact formula like analytic formulation, it can take a long time to be evaluated. As such, AD is not a tool to be inserted into play whenever needed and considerations must be made when implementing the **Sim** operator, to avoid some of these flaws.

To bypass this problem, JAX can encode loops and conditionals in primitive operations that are inherent from the domain-specific compilers for linear algebra (XLA). Otherwise, the loops are unrolled into a set of operations (may be smaller than the general loop, but) that increases the computational graph size. With the primitives in mind, this will be encoded on the graph with a single operation, for which we already know the derivative.

With AD the focus is completely in an optimal implementation of the **Sim** operator, which is essential to obtain a very efficient inverse problem solver (even with analytical computation of derivatives). Thereafter, thinking about both modes, we can apply forward-mode to compute efficiently the derivatives of **Sim** with respect to the parameterization $(r, c_x, c_y, \sigma_{in}, \sigma_{out})$.

Being aware of the inherent problems with both methods is essential for a proper implementation of the inverse solver.

6.3 Establishing a case study

To make a clear comparison between both methods for computing derivatives, in this section we introduce a clear case where we establish our study.

This case is based on certain choices for the voltage measurement setup to provide a transparent framework for all the experiments.

6.3.1 EIT scenario for comparison

To demonstrate our claims it is enough to assume that our subjects are two-dimensional. Since electrical current propagates in three dimensions, this is not physically accurate, but it simplifies the construction of our case study.

EIT is an *ill-posed* inverse problem [64] and thus we need to take into account the possible instability of the problem, *i.e.*, small variations in the measurements may imply large variation on parameters solution. In practice, this makes it hard to solve the inverse problem since true measurements, captured with real-world measuring devices, always contain noise. Therefore, solutions for noisy input data can be distinct from the true solution.

Due to this, it becomes hard to accurately determine a very large number of parameters from a small number of measurements. An example would be a conductivity defined over a fine mesh which has a value at each mesh element, see Fig. 6.2.

To mitigate this problem we want to make as many measurements as possible. However, the possible number of distinct measurements is constrained by the quantity of electrodes. This occurs since for L electrodes there are only $L-1$ linearly independent current patterns for which the voltage measurements bring independent information of the conductivity, see [64].

The best way to mitigate this issue is to work on simpler cases. By doing so we can reduce the possible parameter space and have less variability on the solutions, like in Fig. 6.1. This does not fix completely potential instability issues, but now the measurements represent a smaller space of possible solutions.

For the sake of comparison we wish to make, it is enough to focuses on conductivities with a circular region of distinct conductivity value from the background, see Fig. 6.1 and 6.2. These conductivities are parameterized by their center (c_x, c_y) inside the domain Ω, radius r and conductivity value inside and outside σ_{in}, σ_{out}, respectively.

We work with this simplification for two particular reasons:

- it is easier to obtain a solution to the inverse problem due to the parameterization of such region being given by only a few parameters;

- it is one of the most complex cases for which we are still able to deduce the analytic formula of the derivative.

The second reason arises from the need to compute derivatives of **Sim** with respect to the parameterization variables. As such, we need to make sure that the parameterization is differentiable. Our choice of circular regions is based on this, since it is easy to define a smooth parameterization. For regions with corners two smoothing procedures would be required, one to smooth the corners and another to smooth the parameterization.

For the reasons above, in our experiments we assume the existence of a *single* circular anomaly with conductivity value different from the background, like in Fig. 6.1. We introduce now the EIT model, the conductivity parameterization definition and the measurement setup we use to proceed with out comparison.

6.3.2 Voltage measuring setup

We introduce here the measuring setup that is applied for the direct problem.

We have defined the **Sim** operator in (6.1) and now we simplify its definition accordingly to the case study and the measurement setup.

As mentioned before, by attaching L electrodes at the surface $\partial\Omega$, we can at most apply $L-1$ linearly independent current patterns $I_j \in \mathbb{R}^L$ with $j = 1, ..., L-1$. The **Sim** operator is obtained by solving the direct problem for each I_j and determine the respective voltages $V_j \in \mathbb{R}^L$ over the electrodes.

The more measurements we can perform the better we are able to potentially reconstruct the conductivity. Therefore, we need to choose $L-1$ linearly independent current patterns. This choice is non-trivial. One possibility presented in the literature [64] is obtained by injecting currents in a wave pattern through the electrodes according to

$$I_{j,l} = \begin{cases} A\cos(j\theta_l), & j = 1, ..., \frac{L}{2}, \\ A\sin\left((j - \frac{L}{2})\theta_l\right), & j = \frac{L}{2} + 1, ..., L-1 \end{cases} \tag{6.3}$$

with $\theta_l = \frac{2\pi}{L} l$ and A the constant current amplitude. These patterns have been shown to obtain the best result on the detection of conductivities profiles with small anomalies in the regions furthest from the boundary [64].

The experiments are performed in the following setting:

- Ω is a circular domain with radius 10cm;
- Current amplitude of $A = 3$mA, which is a reasonable value for human subjects, and the voltages are measured in (mV);
- Attach $L = 16$ electrodes equally spaced at the boundary with each having fixed length $\pi/64$.

We refer to Figure 6.2 for a visual representation of the setting.

With respect to the fixed case study with circular anomalies and the voltage measurement setup the simulator is now given as

$$\begin{aligned} \mathbf{Sim} : \mathbb{R}^5 &\to \mathbb{R}^{L(L-1)} \\ (r, c_x, c_y, \sigma_{in}, \sigma_{out}) &\mapsto (V_1^{\mathbf{Sim}}, ..., V_{j,l}^{\mathbf{Sim}}, ..., V_{L-1,L}^{\mathbf{Sim}}) \end{aligned} \tag{6.4}$$

with $V_{j,l}^{\mathbf{Sim}} \in \mathbb{R}$ being the voltage measurement on the l-th electrode obtained by the direct problem solution for the trigonometric current pattern I_j.

6.4 Modeling EIT

6.4.1 Direct problem

For human subjects, current propagation inside the domain is described by the complete electrode model (CEM) [21]. It accounts for the finite nature of electrodes, for the current injection through them and for the electro-chemical effects happening between skin and electrode surface.

Let Ω describe the subject region we are evaluating. To establish a measurement setup, we attach L electrodes at the subject boundary $\partial\Omega$. Through them we apply an electrical *current pattern* $I = (I_1, ..., I_L)$ into Ω. The objective is to find the electrical potential u inside and the voltages at electrodes $V = (V_1, ..., V_L)$ that fulfill the system of equations describing the CEM:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & \text{in } \Omega, \\ \int_{E_l} \sigma \frac{\partial u}{\partial \nu}\, dS = I_l, & l = 1, 2, ..., L \\ \sigma \frac{\partial u}{\partial \nu} = 0, & \text{in } \partial\Omega \setminus \cup_{l=1}^{L} E_l \\ u + z_l \sigma \frac{\partial u}{\partial \nu}\big|_{E_l} = V_l, & l = 1, 2, ..., L \end{cases} \tag{6.5}$$

where σ is the conductivity distribution.

The first equation represents electrical *current diffusion.* The second and third define the insertion of current through electrodes, meaning current spreads through the whole electrode before being inserted into the domain and in regions without electrodes there isn't current flowing. Finally, the last equations model the electrochemical effects at interface of skin-electrode, with z_l designated as *contact impedance* representing the resistance at that interface.

To ensure the existence and uniqueness of a solution, the current pattern must satisfy Kirchoff's law and we fix a *reference voltage condition*:

$$\sum_{l=1}^{L} I_l = 0 \quad \text{and} \quad \sum_{l=1}^{L} V_l = 0. \tag{6.6}$$

6.4.2 FEM formulation of the direct problem

In order to simulate the voltage measurements with respect to a certain conductivity we apply the finite element method to the complete electrode model.

In order to apply the FEM, we introduce the variational equation that describes fully (6.5). In [20] it has been derived and shown that (u, V) is a weak-solution of (6.5) if for all $(w, W) \in H^1(\Omega) \times \mathbb{R}^L$ we have:

$$\int_{\Omega} \sigma \nabla u \cdot \nabla v dx + \sum_{l=1}^{L} \frac{1}{z_l} \int_{E_l} (u - V_l)(w - W_l)\, dS = \sum_{l=1}^{L} I_l W_l \tag{6.7}$$

This formulation joins every condition of (6.5) together into one equation. The first integral describes the propagation of current throughout the domain, while the second represents skin-electrode interface condition and the right-hand side explains the insertion of current.

FEM allows the transformation of the continuous problem, described by the variational equation (6.7) into a discrete system of equations that can be handled by linear algebra methods.

A detailed explanation is provided in any FEM book, and specifically for EIT in [64]. Hereby, we specify the assumptions that we are using to implement FEM.

The first step in any implementation is the discretization of the subject domain Ω into smaller elements. For this purpose we choose the *DistMesh algorithm*, developed by Per-Olof Persson and Gilbert Strang in [73]. This algorithm is simple to implement in practice, and we have extended it to consider L equidistant electrodes at the surface $\partial\Omega$ with a pre-defined size. We denote by N and K the number of nodes and elements of the mesh, respectively.

In a second step, we approximate our solutions u, U through a finite number of basis functions. In particular, we approximate them as

$$u^h(x,y) = \sum_{i=1}^{N} \alpha_i \phi_i(x,y) \tag{6.8}$$

$$V^h = \sum_{k=1}^{L-1} \beta_k \eta_k, \tag{6.9}$$

where ϕ_i, η_k are basis functions.

The basis functions ϕ_i is defined for each node (x_i, y_i) as a linear function over each element, thus it is piece-wise linear and can over each element behaves as

$$\phi_i(x,y) = a_i x_i + b_i y_i + c_i. \tag{6.10}$$

The coefficients a_i, b_i, c_i are defined implicitly by the condition

$$\phi_i(x_j, y_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{6.11}$$

We remark that due to this last condition and the piecewise linear nature of the basis functions ϕ_i it holds that they are zero over the elements that do not contain (x_i, y_i) as a node.

Moreover, the basis functions over the boundary $\eta_k \in \mathbb{R}^L$ are chosen to ensure that the reference voltage condition (6.6) is fulfilled and thus they can be defined as

$$\begin{aligned} \eta_1 &= (1, -1, 0, ..., 0)^T, \\ \eta_2 &= (1, 0, -1, 0, ..., 0)^T, \\ &\vdots \\ \eta_{L-1} &= (1, 0, ..., 0, -1)^T. \end{aligned}$$

The approximate solutions u^h and V^h for the direct problem are now determined by the coefficients

$$\alpha = [\alpha_1, ..., \alpha_N] \in \mathbb{R}^N, \quad \beta = [\beta_1, ..., \beta_{L-1}] \in \mathbb{R}^{L-1}. \tag{6.12}$$

FEM allows us to obtain a system of linear equations characterizing them. This is achieved by inserting (u^h, V^h) into the variational equation (6.7), together with different choices of $(w, W) = (\phi_i, \eta_j)$. Gathering all possibilities leads to a linear system of equations:

$$A\theta = \tilde{I}, \tag{6.13}$$

where $\theta = [\alpha, \beta] \in \mathbb{R}^{N+L-1}$ and $\tilde{I}$ is described through the current pattern I applied at the electrodes as follows:

$$\tilde{I} = \left[\vec{0}, I_1 - I_2, I_1 - I_3, ..., I_1 - I_L\right] \in \mathbb{R}^{N+(L-1)}. \tag{6.14}$$

The stiffness matrix A can be computed in terms of four blocks

$$A = \begin{pmatrix} B^1 + B^2 & C \\ C^T & D \end{pmatrix}. \tag{6.15}$$

Each term is defined through integration over the domain and over the electrodes like:

$$B^1_{ij} = \int_\Omega \sigma \nabla\phi_i \cdot \nabla\phi_j \, dx, \quad i, j = 1, 2, ..., N \tag{6.16}$$

$$B^2_{ij} = \sum_{l=1}^{L} \frac{1}{z_l} \int_{E_l} \phi_i \phi_j \, dS, \quad i, j = 1, 2, ..., N \tag{6.17}$$

$$C_{ij} = -\left[\frac{1}{z_1} \int_{E_1} \phi_i \, dS - \frac{1}{z_{j+1}} \int_{E_{j+1}} \phi_i \, dS\right], \; i = 1, 2, ..., N, \; j = 1, 2, ..., L-1 \tag{6.18}$$

$$D_{ij} = \begin{cases} \frac{|E_1|}{z_1}, & i \neq j \\ \frac{|E_1|}{z_1} + \frac{|E_{j+1}|}{z_{j+1}}, & i = j \end{cases}, \quad i, j = 1, ..., L-1, \tag{6.19}$$

with $|E_j|$ being the electrode area.

The derivation of each block arises from application of two different basis functions on the weak formulation. A full description was done in [64].

After solving the system for θ, the voltages V^h are obtained by multiplication with the basis functions matrix M defined as:

$$M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix} \tag{6.20}$$

through

$$V^h = M\beta.$$

One detail we want to point out regarding FEM implementation concerns the conductivity parameterization. For computational purposes, we assume that σ is piece-wise constant, meaning that it is constant at each element. Mathematically, is defined as

$$\sigma(x, y) = \sum_{k=1}^{K} \sigma_k \chi_k(x, y), \tag{6.21}$$

where K is the total number of elements and χ_k is the indicator function of the k-th element.

From this we can already simplify the computation of the matrix B^1. Further, recall that the basis functions ϕ_i are only non-zero over the elements for which the node (x_i, y_i) takes part. Hence, we have that the inner product of two basis function is only non-zero when the nodes that defined them are part of at least one element. Therefore, the computation of the matrix B^1 is simplified to

$$B^1_{ij} = \sum_{\{k:\, i,j \in T_k\}} \sigma_k \int_{T_k} \nabla\phi_i \cdot \nabla\phi_j \, dx. \tag{6.22}$$

In turn, the matrix B^1 is sparse and analogously we have that the other blocks, and the stiffness matrix A, also are sparse.

The parameterization of σ is essential to compute the voltages variation V^h with respect to a conductivity variation, i.e., the derivative. If a parameterization was not applied to σ, then it would be described as a function from Ω to $\mathbb{R}$. For the latter case a derivative still exists, but in a theoretically sense, which was described in [43].

Finally, we need to choose a method to solve the linear system of equations. The equation (6.5) being elliptic implies that the stiffness matrix A is positive definite. This together with the matrix A being sparse and large implies that the best solver is the *conjugate gradient method (CG)*.

6.5 Modeling the circular anomaly

In this section, we define the conductivity parameterization formally introduced in Section 6.3.

The parameterization is done through a *level-set*, *i.e.*, a function that has positive sign inside the region it describes, negative on the outside and equal to zero on the region boundary. In particular, a *circle level-set* $\mathrm{LS}(x,y)$ can be defined through a center $c = (c_x, c_y)$ and a radius r as follows

$$\mathrm{LS}(x,y) = r^2 - \left[(x-c_x)^2 + (y-c_y)^2\right]. \tag{6.23}$$

As expected, the circle is fully defined by its center and radius. With this level-set, the conductivity of interest in this paper is given through

$$\sigma(x,y) = \sigma_{in} H(\mathrm{LS}(x,y)) + \sigma_{out}\left(1 - H(\mathrm{LS}(x,y))\right) \tag{6.24}$$

where $H(z)$ is the *Heaviside function* that equals 1 if $z > 0$ and 0 otherwise.

Under this formulation σ is not differentiable due to the discontinuity of H at $z = 0$. In order to attain differentiability, we use a smooth approximation of the Heaviside function given as

$$H^\epsilon(z) = \frac{1}{\pi}\arctan\left(\frac{z}{\epsilon}\right) + \frac{1}{2}.$$

The conductivity σ is instead established in terms of H^ϵ, where $\epsilon > 0$ works as a smoothing parameter. The smaller it is the closer H^ϵ is to H.

This smoothing procedure is necessary both for the analytical computation as well as AD. In fact, we need to take into account the mathematical differentiability for a proper implementation of derivatives through AD. For example, JAX AD applies the derivative to H by following the conditional operations `if else`, which implies a derivative of 0 everywhere, which is not true for $z = 0$.

6.6 Derivation of Levenberg-Marquardt method

A simple method to solve inverse problems under such an optimization framework is *Levenberg-Marquardt* method.

It is a general method since it is independent of the simulator and the method used for differentiating it. As such, it allows us to demonstrate the effectiveness of various methods to compute the derivatives, in particular, of automatic differentiation.

Let us recall the minimization problem introduced in (6.2) and given as

$$\min_{\sigma} \frac{1}{2} \left\| \mathbf{Sim}(\sigma) - m^{\text{true}} \right\|_2^2, \tag{6.25}$$

where m^{true} is a set of true measured voltages with respect to N currents applied, as already introduced.

The goal is to iteratively improve an approximate solution of the minimization problem (6.2) through

$$\sigma^{k+1} = \sigma^k + \delta\sigma^k \tag{6.26}$$

where $\delta\sigma^k$ is an *update step* and σ^k is the current approximate solution. This process is done until a satisfactory solution is found. Each method to solve the minimization problem is defined by the computation of the update step $\delta\sigma^k$.

The Levenberg-Marquardt method is a particular type of quasi-Newton methods. We start by deducing the general form of quasi-Newton methods and there after funnel on our chosen method.

We hereby assume that σ is discretely given by a parameterization, i.e., $\sigma \in \mathbb{R}^p$. This simplifies simulation and, more importantly, the derivatives computation process which is now done with respect to each variable σ_i, $i = 1, ..., p$. An example is seen in Figure 6.1 where $\sigma = (\sigma_{\text{in}}, \sigma_{\text{out}})$.

Denote by $\mathcal{L}(\sigma)$ the loss function in (6.25). Then, assuming that we have an initial guess σ_0, we can re-write it as

$$\mathcal{L}(\sigma + \delta\sigma) = \frac{1}{2} \left\| \mathbf{Sim}(\sigma_0 + \delta\sigma) - m^{\text{true}} \right\|_2^2 \tag{6.27}$$

with an intent to minimize with respect to the parameter variation $\delta\sigma$, which denotes the *update step*.

Now, we apply the Taylor expansion to the loss function defined in (6.27), which up to the quadratic term is given by

$$\mathcal{L}(\sigma + \delta\sigma) = \mathcal{L}(\sigma) + \mathcal{L}'(\sigma)\delta\sigma + \frac{1}{2}\mathcal{L}''(\sigma)(\delta\sigma)^2 + O(\delta\sigma^3), \tag{6.28}$$

where $\mathcal{L}'(\sigma)$ and $\mathcal{L}''(\sigma)$ denotes the gradient and Hessian of the objective function $\mathcal{L}$, with respect to parameters defining σ.

A minimum with respect to $\delta\sigma$ has gradient zero. Thus, we apply the gradient to the Taylor expansion in (6.28), like,

$$\frac{\partial \mathcal{L}}{\partial \delta\sigma}(\sigma + \delta\sigma) = \mathcal{L}'(\sigma) + \mathcal{L}''(\sigma)\delta\sigma.$$

Setting the gradient equal to zero yields

$$0 = \mathcal{L}'(\sigma) + \mathcal{L}''(\sigma)\delta\sigma \Leftrightarrow \delta\sigma = -\left[\mathcal{L}''(\sigma)\right]^{-1}\mathcal{L}'(\sigma).$$

Since only **Sim** depends on the conductivity parameterization we can compute the gradient and Hessian through:

$$\mathcal{L}'(\sigma) = J(\sigma)^T\left(\mathbf{Sim}(\sigma) - m^{\text{true}}\right) \tag{6.29}$$

$$\mathcal{L}''(\sigma) = J(\sigma)^T J(\sigma) + \sum_i \left[\mathbf{Sim}_i(\sigma)\right]''\left(\mathbf{Sim}_i(\sigma) - m_i^{\text{true}}\right), \tag{6.30}$$

where J is the Jacobian of simulated voltages $\mathbf{Sim}(\sigma)$ with respect to the parameterization of σ.

Up until here, the derivation is general for quasi-Newton methods.

The Levenberg-Marquardt method distinguishes itself from other quasi-Newton methods by avoiding the computation of second order derivatives. It substitutes this computation by a scaled identity matrix $\lambda_{LM}I$, $\lambda \in \mathbb{R}^+$, which acts as a regularizer by improving the condition number of the Hessian matrix to be inverted. Now, the update can be computed through:

$$\delta\sigma_{LM} = -\left[J(\sigma)^T J(\sigma) + \lambda_{LM}I\right]^{-1} J(\sigma)^T\left(\mathbf{Sim}(\sigma) - m^{\text{true}}\right). \tag{6.31}$$

As described in equation (6.26), we apply this update rule iteratively, in order to improve the approximate solution until a satisfactory solution is found.

6.7 Derivatives computation

In order to solve the inverse problem in a minimization framework, we need to compute derivatives of the **Sim** operator. In this section, we deduce the analytical formula and explain how to apply AD to **Sim**, in order to obtain the derivatives with respect to the parameters of interest.

We recall that the direct solver and **Sim** is independent of the derivative computation method, but the derivative will depend on it.

6.7.1 Analytical computation

We recall that (6.4) we have that the FEM simulator operator is given by

$$\begin{aligned} \mathbf{Sim} : \mathbb{R}^5 &\to \mathbb{R}^{L(L-1)} \\ (c_x, c_y, r, \sigma_{in}, \sigma_{out}) &\mapsto V^{\mathbf{Sim}} = \left(V_1^{\mathbf{Sim}}, ..., V_{j,l}^{\mathbf{Sim}}, ..., V_{L-1,L}^{\mathbf{Sim}}\right) \end{aligned} \tag{6.32}$$

From now forward we denote $V_n \in \mathbb{R}^L$ for the voltages measured j-th current pattern.

The Jacobian matrix $J \in \mathbb{R}^{L(L-1)\times 5}$ is given by

$$J = \begin{pmatrix} \frac{\partial V^{\mathbf{Sim}}}{\partial c_x} & \frac{\partial V^{\mathbf{Sim}}}{\partial c_y} & \frac{\partial V^{\mathbf{Sim}}}{\partial r} & \frac{\partial V^{\mathbf{Sim}}}{\partial \sigma_{in}} & \frac{\partial V^{\mathbf{Sim}}}{\partial \sigma_{out}} \end{pmatrix} \tag{6.33}$$

In order to provide an analytical formulation, we specifically focus on the computation of derivatives for each V_n with respect to a single parameter, which if done for all $n = 1, ..., L-1$ determines one column of the Jacobian.

Our method of choice to simulate the measurements is the FEM that we described for the CEM in Section 6.4.

The FEM solution is based on the coefficients $\theta = (\alpha, \beta) \in \mathbb{R}^{N+L-1}$, where α and β describe the electrical potential in Ω and the voltages at the electrodes through (6.8).

Accordingly, we denote for each current pattern I_j the FEM solution by $\theta_j = [\alpha_j, \beta_j] \in \mathbb{R}^{N+L-1}$ with respect to $\tilde{I}_j$ on the right-hand side of the FEM system of equations defined in (6.14).

With this in mind, the voltages are computed by $V_j = M\beta_j$ where M is the matrix defined in (6.20) with the basis functions at the electrodes.

Now, if we define $\tilde{M} = [\hat{0}\ M] \in \mathbb{R}^{L\times(N+L-1)}$ then we have

$$V_n = \tilde{M}\theta_n = \tilde{M}A^{-1}\tilde{I}_n. \tag{6.34}$$

As such, it holds for any parameter w of $\{c_x, c_y, r, \sigma_{in}, \sigma_{out}\}$ that:

$$\frac{\partial V_n}{\partial w} = \frac{\partial\left(\tilde{M}A^{-1}\tilde{I}_n\right)}{\partial w}.$$

Since neither $\tilde{M}$ and $\tilde{I}_n$ depend on the conductivity and, therefore, on any of the parameters, it holds that

$$\frac{\partial V_n}{\partial w} = \tilde{M}\frac{\partial A^{-1}}{\partial w}\tilde{I}_n = -\tilde{M}A^{-1}\frac{\partial A}{\partial w}A^{-1}\tilde{I}_n \tag{6.35}$$

with the last equality following from matrix calculus properties.

Thus, in essence, the computation resumes to the stiffness matrix derivative and noticing that $A^{-1}\tilde{I}_n = \theta_n$. Setting $\gamma = \tilde{M}A^{-1}$ the computation of the derivative in (6.35) simplifies to

$$\frac{\partial V_n}{\partial w} = -\gamma^T\frac{\partial A}{\partial w}\theta_n. \tag{6.36}$$

As such, the focus is on the computation of $\frac{\partial A}{\partial w}$. The stiffness matrix A is composed of four blocks, like,

$$\begin{bmatrix} B^1 + B^2 & C \\ C^T & D \end{bmatrix}.$$

The block B^1 is the only one depending on the conductivity. Due to its definition there is a clear way of computing the derivatives of B^1 with respect to each σ_k (see the Appendix for further details on its definition):

$$\frac{\partial B^1_{ij}}{\partial \sigma_k} = \begin{cases} \int_{T_k} \nabla\phi_i \cdot \nabla\phi_j \, dx, & \text{if } i, j \in T_k \\ 0, & \text{otherwise.} \end{cases} \tag{6.37}$$

Furthermore, the resulting matrix is independent of σ therefore it can be precomputed at the start and re-used.

Through the chain rule we have that

$$\frac{\partial B^1_{ij}}{\partial w} = \sum_{k=0}^{K} \frac{\partial B^1_{ij}}{\partial \sigma_k} \frac{\partial \sigma_k}{\partial w}. \tag{6.38}$$

We note that due to sparsity of the matrix defined in Eq. (6.37) it can be assembled very efficiently. However, this optimal performance is an extra layer of complexity that needs to be solved manually and AD takes care of that automatically.

Note that due to Eq. (6.36) to compute the Jacobian we need to determine γ. Since, A is a very large sparse matrix the best way to do determine it is by solving the adjoint system equivalent to $\gamma = \tilde{M} A^{-1}$ given as

$$A^T \gamma = \tilde{M}^T \text{ with } \gamma \in \mathbb{R}^{N+(L-1)\times L}. \tag{6.39}$$

Due to A being dependent on the conductivity, this system needs to be solve once at each iteration of the inverse solver.

A final formula for the derivatives in Eq. (6.33) is obtained after we solve the adjoint system (6.39) and compute the derivative of B^1 as in (6.38) and is given as:

$$\frac{\partial V_n}{\partial w} = -\gamma^T \begin{bmatrix} \frac{\partial B^1}{\partial w} & 0 \\ 0 & 0 \end{bmatrix} \theta_n \tag{6.40}$$

Through this demonstration, we have seen that it can be very tedious to deduce and implement the analytical derivatives for complex problems. For simple functions, an analytical derivative in compact form takes the lead in efficiency, however we want to experiment with the case of more complex functions.

6.7.2 Automatic differentiation application in EIT

In this section, we use JAX automatic differentiation toolbox [14] to obtain the Jacobian. Further details about the inner workings of AD and JAX are explained in Section 6.2. Here, we describe the implementation in practice.

Since our direct operator **Sim** has more output variables than input variables we note that the most-efficient AD mode is the forward-mode.

To use JAX AD our focus is the implementation of a differentiable simulator **Sim**. Our preparation of the case study as ensured that we obtain a fully differentiable direct operator. Hence, our **Sim** operator is differentiable with respect to the parameterization variables $(r, c_x, c_y, \sigma_{in}, \sigma_{out})$ that define the anomaly, as introduced in Section 6.5.

This preparation are a requirement for both derivative methods, but now the derivative computation with AD is simple to be implemented through JAX.

To do so, we implement a routine that defines the direct operator **Sim** given in (6.4). The implementation is established through the solution of the direct problem through FEM, that we here hide as the *simulator* method. Listing 1 provides the routine with all of these in mind.

In order to compute the Jacobian defined in Eq. (6.33) with JAX one only needs to call `jax.jacfwd(direct_operator)` for our direct operator as in Listing 2.

To establish the inverse solver these function definitions are redundant and we can immediately call `simulator` and `jax.jacfwd(direct_operator)` in the inverse solver routine. This definition is just for visualization purposes in this section.

6.8 Experimental setup

To compare both analytic and automatic differentiation methods, we explore their evaluation at different conductivities, and how they fit in to solve the inverse problem. For the latter, we consider two particular cases for the inverse problem. The first case, that we label as the case of **fixed conductivities** is simpler. We want to determine only the location parameters (r, c_x, c_y) and we assume the conductivity values inside σ_{in} and outside σ_{out} are fixed. This scenario can represent breast cancer, for example, where we know a priori conductivity values of different tissues, and we are only concerned in determining the anomaly location.

The second case, that we label as the case of **general conductivities**, we want to determine all parameters $(r, c_x, c_y, \sigma_{in}, \sigma_{out})$. This is a more general scenario where we only know there is a circular anomaly and want to characterize it in terms of location, radius and conductivity.

Recall that we fix a voltage measurement setup to simplify the comparison. Our only interest is to show that AD is as good as analytical methods in terms of solution accuracy. Further, we show that the memory requirements for AD scale reasonably well with the mesh resolution, to show that AD can be effectively implemented in more realistic cases involving more complex

```python
import jax

def direct_operator(anomaly_parameters):
        """Simulate measurements for given
        input function with JAX.

        Args:
        anomaly_parameters: Array of shape
        (5,) with parametrization variables
        of circular anomalies.

        Returns:
        measurements: Array of shape
        (nmb_electrodes(nmb_electrodes-1),) that
        contains the voltage measurements for all
        current patterns.
        """

        # Compute measurements
        measurements = simulator(anomaly_parameters)

        return measurements
```

Listing 1: Definition of the direct operator through a general simulator method.

scenarios and 3D meshes.

All of the experiments have been run in a machine with the following hardware specifications:

- CPU *Intel Core i5-12400F* (released in Q1 2022, 12th gen., 4.4 GHz, 6 cores, 12 threads, 64 GB RAM);

- GPU *NVIDIA GeForce RTX 3070* (released in Q4 2020, 6144 CUDA cores, 8 GB memory).

We chose this machine because it has typical med-range specs and can be considered as a good example of an affordable solution for the numerical computation, compatible with the lower cost of EIT. We remark that besides automatic differentiation, JAX excels in optimizing the performance for a given hardware. Therefore, we have not performed any specific optimization, but appropriate care as been taken in implementation.

```python
def jacobian(anomaly_parameters):
    """Compute Jacobian with JAX AD

    Args:
    anomaly_parameters: 1d array of shape
    (5,) with parametrization variables
    of circular anomalies.

    Returns:
    Jacobian matrix of shape
    (nmb_electrodes(nmb_electrodes-1), 5).
    """

    # Define the jacobian through forward-mode
    jacobian = jax.jacfwd(direct_operator)

    return jacobian(anomaly_parameters)
```

Listing 2: Computation of the Jacobian matrix through JAX automatic differentiation toolbox.

6.8.1 Establish a ground truth

In order to have a "lab" setup, *i.e.*, one we can control from start to finish, we define a voltage measurements dataset through simulation. For such, we randomly initialize our conductivity parameterization under a certain range of parameters and determine their respective voltage measurements m.

To test new inverse solvers we need to generate measurements with the highest resolution possible to avoid the so-called *inverse crimes*. Such crimes occur by using the same resolution to obtain m and **Sim** operator computationally. By doing it, we do not account for errors arising from the approximate nature of the direct solver, which occurs when using true measurements obtained by a real-world measuring device, which adequately we can think as having infinite resolution. As such, we need to choose a higher mesh resolution for m than for **Sim** operator, since they are obtained both through FEM.

With this in mind we generate our ground truth dataset of voltage measurements with the highest possible resolution for our hardware specifications. In our work, it was established with a FEM mesh of 5815 elements that is set accordingly to have each element with a edge length of $h = 0.035$ relative to the domain size.

Furthermore, we generate the dataset through the following random initialization of the anomaly parameters:

- Uniformly generate conductivity centers anywhere inside the domain $\Omega = B_1(0)$. For such, uniformly generate an angle between $[0, 2\pi]$. Then, we uniformly generate a value in $[0, 1]$ to obtain a radius sample by taking square root of it. Joining both through polar coordinates gives an almost uniformly sampled set of 2D points inside Ω;
- Uniformly generate anomaly radius, taking into consideration the center position generated on the previous point, so that anomalies are strictly in Ω. As such, for each center we select the anomaly radius uniformly from $[0.1, 1 - |c|]$, where $|c|$ is the distance from center to origin;
- Uniformly generate conductivity values inside σ_{in} from $[1, 1.6]$ S/m and outside σ_{out} from $[0.6, 1.]$ S/m. Such values do not encapsulate any particular medical or industrial scenario.

Our model assumes that contact impedances on each electrode are fixed and have value $z = 5 \times 10^{-6}\,\Omega{\cdot}\text{m}$ and recall that Ω is a disk of radius 1.

6.9 Results

To study the effectiveness of AD in EIT we split our study in three sections.

In the first section, we provide a sanity check on the Jacobian evaluation with both methods, that is, we compute the Jacobian matrix through both methods and evaluate if they are evaluating equally.

In the second and third section, we solve the inverse problem for different anomalies but provide a distinct analysis. On the second section we visually present the reconstructions with both methods and compare them with the true anomaly. In the latter one, we present a large analysis for two separate datasets each with 1000 cases that is based on error analysis.

The two datasets differ in the amount of variables we try to reconstruction. One is for the case of **fixed conductivities** where we randomly generate 1000 anomalies and compute the respective measurements with fixed conductivity value inside of $\sigma_{\text{in}} = 1.4$ S/m and outside of $\sigma_{\text{out}} = 0.7$ S/m. Another is for the case of **general conductivities** where we randomly generate 1000 anomalies and compute their measurements as described above.

6.9.1 Sanity check

The sanity check is to verify if the Jacobian computed through automatic differentiation and the analytic formulation match. This is what we already expect since AD applies the chain-rule of differentiation to FEM, which is exactly what we have done by hand to determine the analytic formulation. The Frobenius norm of the Jacobian difference is given as:

$$\left\| J^{AD} - J^{analytic} \right\|_{Fro} \quad \text{where } \|A\|_{Fro} = \left[\sum_{i,j}^{n,m} |a_{ij}|^2 \right]^{1/2}.$$

Further, we computed the Jacobian with both methods for 100 randomly generated general conductivities described in section 6.8. Thereafter, we compute their difference and applied the Frobenius norm in order to obtain an array with dimension 100.

To verify the assumption that both should evaluate to almost the same values we make an histogram of the losses and provide some statistics, namely, mean, variance, maximum and minimum. This results are provided in Fig. 6.4 and Table 6.1.

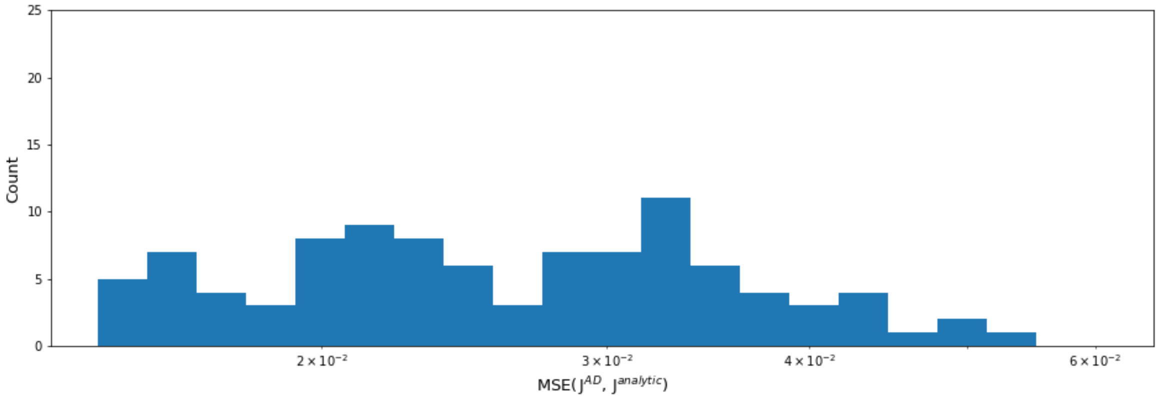


Figure 6.4: Histogram of Jacobian error with both derivative methods evaluated with Frobenius norm.

	Mean	S^2	Max. Error	Min. Error
$\left\Vert J^{AD} - J^{analytic} \right\Vert_{Fro}$	0.0271	7.94e-05	0.0552	0.0146

Table 6.1: Statistic analysis of the error between Jacobian matrices obtained through the Frobenius norm.

Statistically we can infer that the Jacobian match closely together with maximum error of 0.0552 and an average of 0.0271. Indeed, the histogram confirms that most evaluations are close together, with only some outliers compared with the overall picture. Further, these outliers might just be rounding off errors and are not worrisome since the error is still considerably small.

6.9.2 Inverse problem reconstructions

To check the convergence of the inverse solver with both methods we randomly generate 4 different conductivities: two where the conductivity values inside and outside are fixed to be $(1.4, 0.7)$ and one where the conductivity values are to be determined.

The four scenarios were randomly generated and used to generate the respective voltage measurements. Thereafter, we solve the inverse problem with a 5210 elements mesh, that is established with $h = 0.037$.

The results for the conductivity profiles with fixed σ_{in} and σ_{out} can be visualized in Figures 6.5 and 6.6.

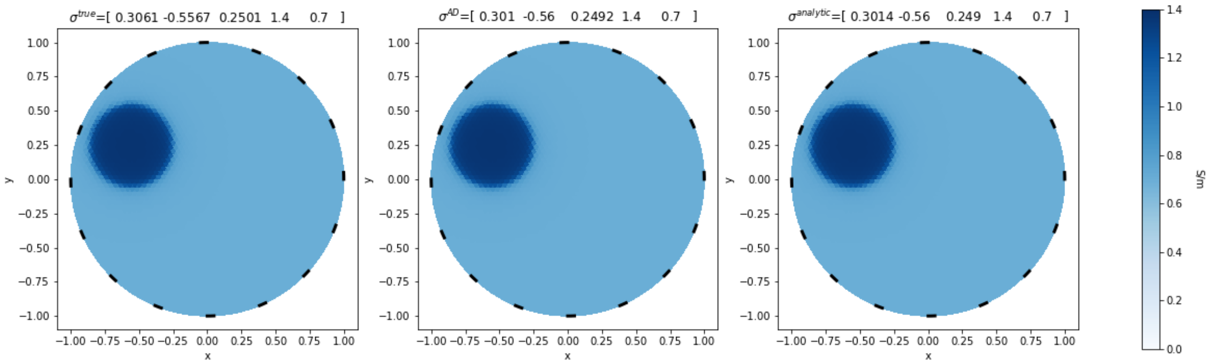


Figure 6.5: Reconstruction of the anomaly $\sigma^{\text{true}} = (0.3061, -0.5567, 0.2501, 1.4, 0.7)$ with fixed conductivity values. The left most plot presents the true conductivity profile. The middle and right plots present the reconstructions with the AD and analytical method, respectively.

We notice that both reconstructions with fixed conductivities are practically the same as the true conductivity profile. Further, the reconstructions are more similar to each other than to the true profile.

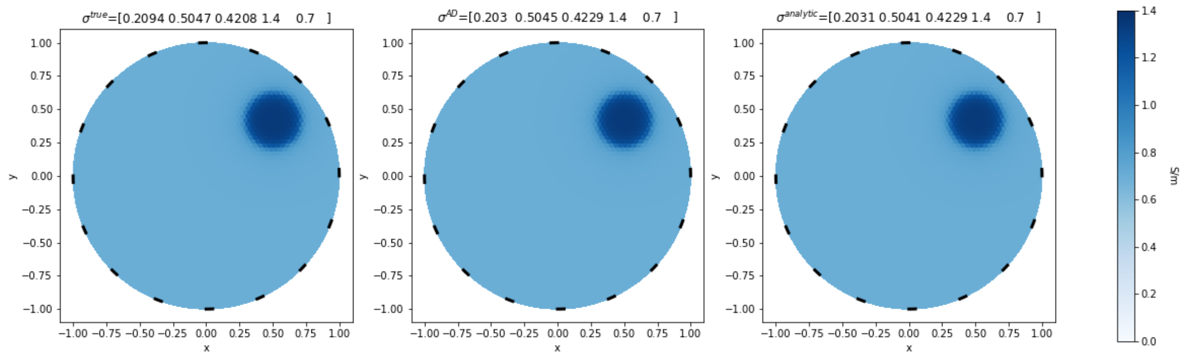


Figure 6.6: Reconstruction of the anomaly $\sigma^{\text{true}} = (0.2094, 0.5047, 0.4208, 1.4, 0.7)$ with fixed conductivity values. The left most plot presents the true conductivity profile. The middle and right plots present the reconstructions with the AD and analytical method, respectively.

In Figures 6.7 and 6.8 present the conductivity profiles comparing the reconstructed anomalies in the general setting.

Notice that both reconstructions for general conductivities underestimate and overestimate some variable parameters. Namely, in Fig. 6.7 we see that for both reconstructions method the radius of the anomaly is underestimated and the conductivity value inside the anomaly, σ_{in}, is overestimated. In Fig. 6.8 the reverse happens, the radius of the anomaly is overestimated and the conductivity inside, σ_{in}, is underestimated.

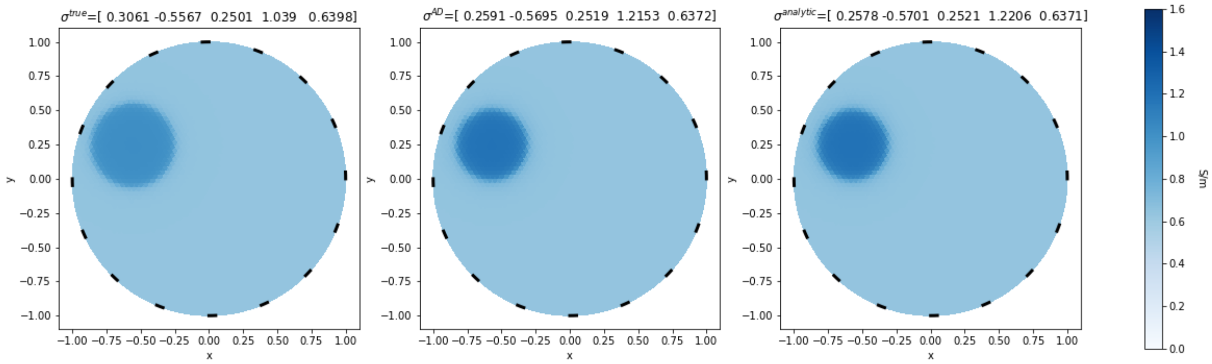


Figure 6.7: Reconstruction of the anomaly $\sigma^{\text{true}} = (0.3061, -0.5567, 0.2501, 1.039, 0.6398)$ with general conductivity values. The left most plot presents the true conductivity profile. The middle and right plots present the reconstructions with the AD and analytical method, respectively.

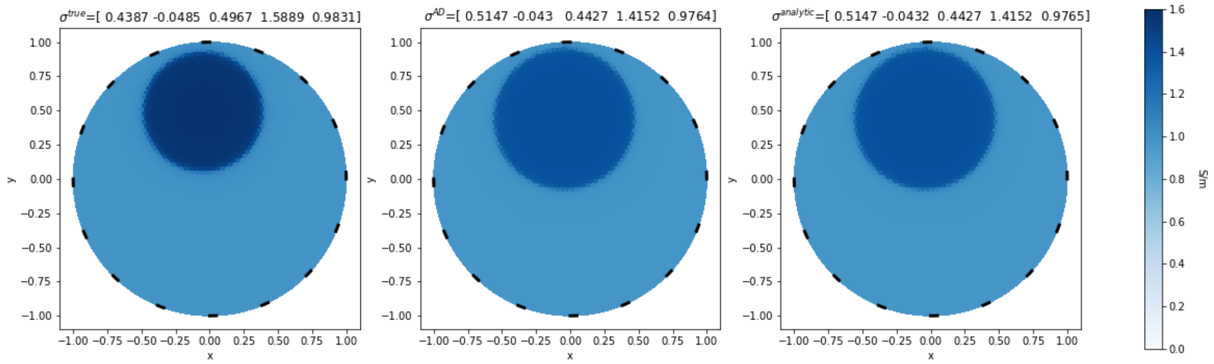


Figure 6.8: Reconstruction of the anomaly $\sigma^{\text{true}} = (0.4387, -0.0485, 0.4967, 1.5889, 0.9831)$ with general conductivity values. The left most plot presents the true conductivity profile. The middle and right plots present the reconstructions with the AD and analytical method, respectively.

Independently of the error in the reconstructions, the figures for the general conductivities point out that the reconstructions obtained through the AD and analytical method are identical. Hence, this is a first hint for the effectiveness of AD to solve inverse problems and shows that it can be as good as the analytical method.

6.9.3 Inverse problem analysis

In order to solve the inverse problem for the two datasets described above, we use a FEM mesh with 5210 elements that is set by $h = 0.037$ to define the **Sim** operator, in order to avoid inverse crimes. Our chosen inverse solver is the Levenberg-Marquardt method with a line search algorithm on each iteration. Further, we establish two stopping criteria based on a maximum number of iterations equal to 20 and a relative mean squared loss

$$\frac{1}{2}\frac{\left\|\mathbf{Sim}(\sigma) - m^{\text{true}}\right\|_2^2}{\|m^{\text{true}}\|_2^2} < \xi \tag{6.41}$$

with a feasible threshold of $\xi = 0.001$. This choice was established empirically, since after that it becomes hard to improve the anomaly reconstruction.

Let σ^{AD} and σ^{analytic} be the solutions obtained through the inverse solver with the different methods to compute the derivative. In order to verify the effectiveness of AD in solving the EIT inverse problem we evaluate how σ^{AD} and σ^{analytic} compare with the true solution σ^{true} and how they compare with each other. This evaluation is based on the mean squared error between the anomalies, *i.e.*, for two different anomaly parameterizations σ_1, σ_2 we evaluate

$$\text{MSE}(\sigma_1,\, \sigma_2) := \|\sigma_1 - \sigma_2\|_2.$$

In essence, we compute

$$\text{MSE}(\sigma^{\text{true}},\, \sigma^{\text{AD}}),\ \text{MSE}(\sigma^{\text{true}},\, \sigma^{\text{analytic}}),\ \text{MSE}(\sigma^{\text{AD}},\, \sigma^{\text{analytic}}).$$

Then, we perform an analysis of the mean squared errors by computing simple statistics of the mean, variance, maximum and minimum error, and by plotting the histogram with a logarithmic scale in the x-axis.

We remark that the following analysis is focused on a general analysis on the reconstructions obtained through the different methods and does not verifies the nature of the errors obtained, *i.e.*, we do not check if the errors are occurring for one specific parameter or for small/large values of those same parameters.

Case 1: Fixed Conductivities

In this case our goal is to determine the anomaly parameterized by $\sigma^{\text{true}} = (r, c_x, c_y)$, since we know a priori that the conductivity inside and outside are $\sigma_{in} = 1.4$ S/m and $\sigma_{out} = 0.7$ S/m, respectively. Here, we denote σ^{true} as the conductivity we aim to discover and m^{true} for the respective measurements.

We start from our measurements dataset for the **fixed conductivities** with a set of 1000 voltage measurements corresponding to different anomalies. This number of experiments was constrained by time and hardware capabilities.

The statistic analysis for this case is given in Table 6.2 and the histogram for the different mean squared errors is in Fig. 6.9.

The histogram presented in the Fig. 6.9 shows that the distribution of the mean squared errors $\text{MSE}(\sigma^{\text{true}}, \sigma_{\text{AD}})$ and $\text{MSE}(\sigma^{\text{true}}, \sigma_{\text{analytic}})$ is similar. Notice that the mean squared errors in both cases are concentrated around 10^{-2} with a set of outliers with error higher than 0.1.

	Mean	S^2	Max.	Min.
$\mathrm{MSE}(\sigma^{\mathrm{true}}, \sigma^{\mathrm{AD}})$	0.0456	0.0059	0.4177	0.0020
$\mathrm{MSE}(\sigma^{\mathrm{true}}, \sigma^{\mathrm{analytic}})$	0.0455	0.0057	0.4007	0.0020
$\mathrm{MSE}(\sigma^{\mathrm{AD}}, \sigma^{\mathrm{analytic}})$	0.002	2.64e-4	0.2702	1.51e-5

Table 6.2: Statistics of mean squared errors of fixed conductivities, case 1, that compares the reconstructed conductivities obtained through the different derivative methods with the true anomalies.

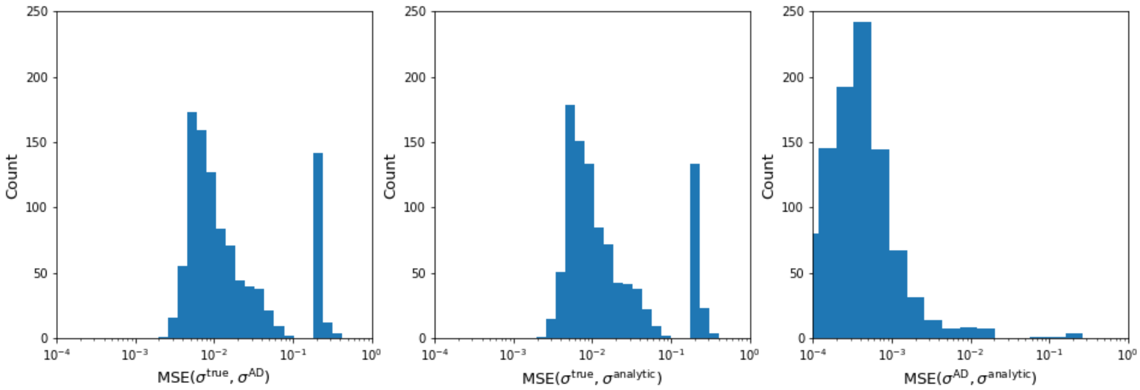


Figure 6.9: Histogram of the mean squared errors of fixed conductivities, case 1, comparing the reconstructed anomalies obtained through the different derivative methods with the true anomalies.

However, this outliers occur in the same proportion for both methods. In analysis, this shows that the inverse solver with automatic differentiation matches that with the analytic derivative.

Furthermore, in Fig. 6.9 the histogram on the right shows that the distribution of mean squared errors between reconstructions $\mathrm{MSE}(\sigma^{\mathrm{AD}}, \sigma^{\mathrm{analytic}})$ is highly concentrated around 10^{-3}. There are some reconstructions that are diverging between both methods, but their error is in the order of 0.1. Again, this highlights again the effectiveness of AD compared with the analytic method. However, there are some outliers that shows divergence in the reconstructions between both methods. These errors seem to be related with round-off errors when we combine this analysis with the sanity check for the Jacobian.

To complete the discussion of this case, we allude to the statistics Table 6.2. We point to the mean and variance of the different mean squared errors. This shows that on average the reconstruction obtained with AD is much closer with the analytic one than with the true anomalies. Furthermore, the variance between these reconstructions is very small. Once again it shows the effectiveness of AD to match the analytic derivative method and that other inverse solver methods need to be improved in order to obtain better reconstruction results.

Case 2: General Conductivities

For this case the objective is to determine the general anomaly parameterization given by $\sigma^{\text{true}} = (r, c_x, c_y, \sigma_{\text{in}}, \sigma_{\text{out}})$. Again, we denote σ^{true} as the conductivity we aim to discover and m^{true} for the respective measurements.

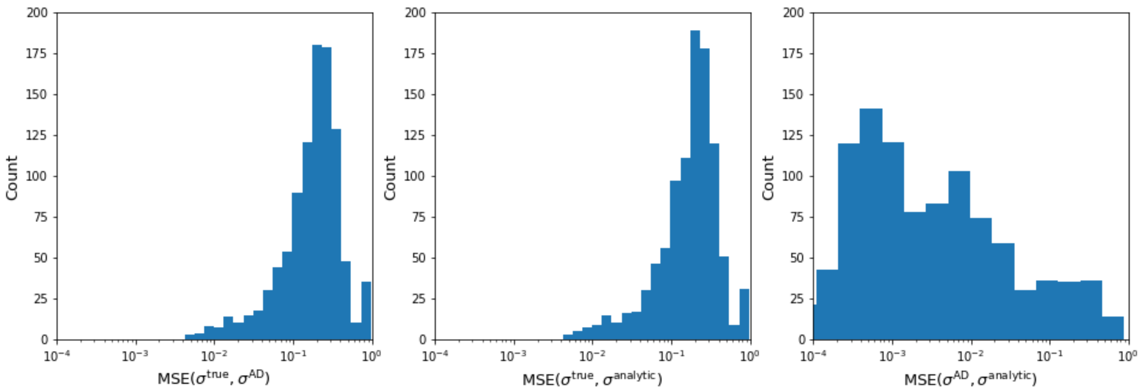
We start from the measurements dataset for the **general conductivities** with the set of 1000 voltage measurements corresponding to the different anomalies. Recall, that in this generation we have assumed that σ_{in} is always greater than σ_{out},

The statistic analysis for this case is given in Table 6.3 and the histogram for the different mean squared errors is in Fig. 6.10.

Table 6.3: Statistics of mean squared errors of general conductivities that compares the reconstructed conductivities obtained through the different derivative methods with the true anomalies.

	Mean	S^2	Max. Error	Min. Error
MSE($\sigma^{\text{true}}, \sigma^{\text{AD}}$)	0.2264	0.0292	0.9698	0.0042
MSE($\sigma^{\text{true}}, \sigma^{\text{analytic}}$)	0.2215	0.0273	0.9706	0.0042
MSE($\sigma^{\text{AD}}, \sigma^{\text{analytic}}$)	0.039	0.0134	0.8838	4.4e-6

Figure 6.10: Histogram of the mean squared errors of general conductivities that compares the reconstructed anomalies obtained through the different derivative methods with the true anomalies.



The histogram presented in the Fig. 6.10 shows that the distribution of the mean squared errors MSE($\sigma^{\text{true}}, \sigma^{\text{AD}}$) and MSE($\sigma^{\text{true}}, \sigma^{\text{analytic}}$) is similar. In analysis, this shows that the inverse solver with automatic differentiation matches that with the analytic derivative. Further, notice that the mean squared errors in both cases are concentrated around 10^{-1}. In fact by setting a threshold, we verified that there are at most 50 reconstructions for both methods where the mean squared error with the true anomaly is higher than 0.5, which together with the histograms shows that the vast majority of reconstructions is successful.

Furthermore, the histogram on the right-hand side of Figure 6.10 presents the histogram of the mean squared errors between reconstructions MSE($\sigma^{\text{AD}}, \sigma^{\text{analytic}}$) shows that the errors are more concentrated around the interval $[10^{-4}, 10^{-2}]$. Again, this highlights again the equivalence of AD compared with the analytic method. However, there are some outliers that shows divergence in the reconstructions between both methods. These errors seem to be related with round-off errors when we combine this analysis with the sanity check for the Jacobian.

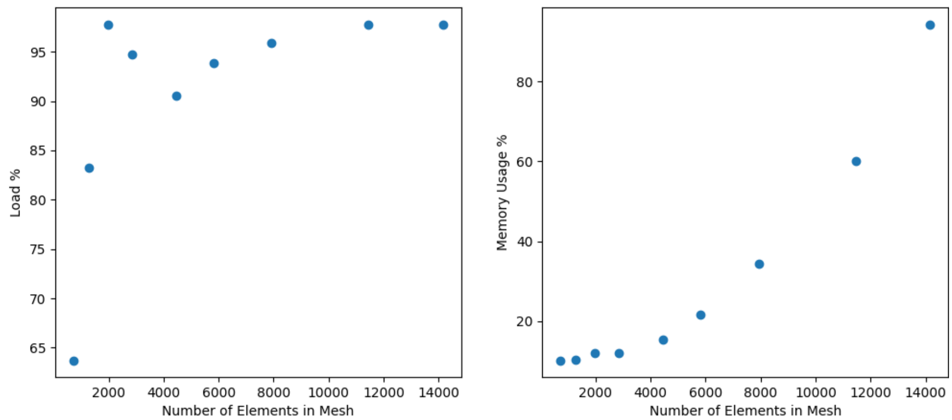
To complete the discussion of this case, we allude to the statistics Table 6.3. The only aspect we would like to point out here is the mean of the different mean squared errors. This shows that on average the reconstruction obtained with AD is much closer with the analytic one than with the true anomalies. Once again it shows the effectiveness of AD to match the analytic derivative method and that other inverse solver methods need to be improved in order to obtain better reconstruction results.

6.9.4 Computational performance of AD

The viability of AD also depends of its scaling capabilities. Namely, we want to understand if increasing the number of mesh elements, and therefore the resolution and accuracy of the FEM turns AD unfeasible. This is relevant because AD requires the construction of a computational graph for the direct problem and then applies the chain-rule throughout the nodes of the graph to compute the derivatives. As the number of mesh elements increases the computational graph becomes larger and can be unfeasible to use it for derivative computation.

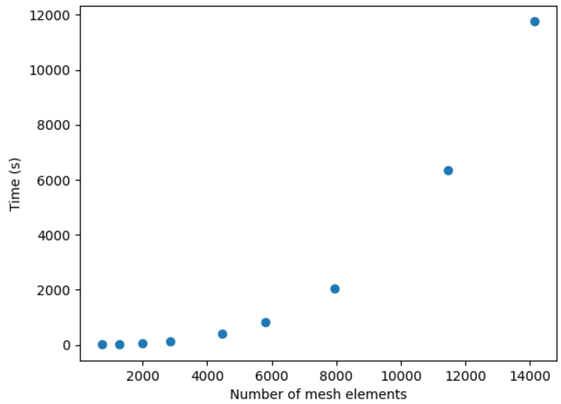
In order to understand this behavior, we compute for ten different mesh sizes the Jacobian matrix of 100 distinct general anomalies, randomly generated as described before. For each mesh size we measure the average GPU memory and load usage through the python package `GPUtil`.

Figure 6.11: Percentage of GPU load and memory usage with respect to the number of mesh elements.



In Fig. 6.11 we plot the average of GPU load and memory usage percent for each of the

Figure 6.12: Time elapsed to compute Jacobian matrices for 100 random anomalies with respect to the number of mesh elements.



different mesh resolutions and in Fig. 6.12 we plot the time that took to compute the Jacobian matrices with respect to each mesh resolution.

It is clear from both figures the growth in GPU memory usage and time to execute this experiment. Moreover, for meshes with more than 15000 elements we require more than 8Gb of GPU memory. As of now, we cannot understand the order of growth and further experiments with finer resolution are needed.

6.10 Conclusion

In this chapter we have compared the effectiveness of AD to solve inverse problems against classical methods with analytical formulations of the derivative. We have shown how to adequately construct a FEM differentiable simulator in the context of inverse problems. We successfully introduced automatic-differentiation for solving inverse problems in an optimization framework, in particular, electrical impedance tomography. We have shown that AD provides a simple way of computing derivatives of complex operators, for example, arising from solutions of partial differential equations, with respect to a set of parameters.

We have shown that AD is indeed effective to compute the derivatives, since it matches the analytical computation up to minimal error. Further, it was used to solve the electrical impedance tomography inverse problem and we shown that it is even superior to analytical methods, in terms of time and resources.

The analytical formulation is nothing more than an application of differentiation rules to the FEM formulation of the direct operator. By construction AD essentially executes the same process, but automatically. As such, AD and the analytical formulation can be even performing the same operations, but the fact that AD is a plug-and-play tool makes it advantageous to use for complex operators.

Moreover, it has proven more efficient since it takes less time on average to solve any particular EIT problem in our case study and scales well with the mesh resolution. This indicates that with the right hardware AD can be efficiently executed for large-scale problems.

With this tool, we can cast our focus into an efficient implementation of the direct problem solvers, which is way more understood in literature, and on the methods to solve the inverse problem. It allows freedom to experiment and deal with difficult equations, without much thought, bringing focus to the practical application at hand.

Further, we expect that AD extends nicely to higher dimensions, while the analytic formulation will require some re-implementation to accommodate the three dimensional shapes of anomalies.

Future studies are interested in testing how AD easily handles different shapes of anomalies, as well as three-dimensions.

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