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The KdV approximation for a system with unstable resonances

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The KdV equation can be derived via multiple scaling analysis for the approximate description of long waves in dispersive systems with a conservation law. In this paper, we justify this approximation for a system with unstable resonances by proving estimates between the KdV approximation and true solutions of the original system. By working in spaces of analytic functions, the approach will allow us to handle more complicated systems without a detailed discussion of the resonances and without finding a suitableenergy.

KEYWORDS
long-wave approximation, unstable resonances

MSC CLASSIFICATION
35C20; 35Q35; 35Q53

1 | INTRODUCTION

We consider the Boussinesq-Klein-Gordon (BKG) system

$$\partial_t^2 u = \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \alpha^2 \partial_x^2 (a_{uu} u^2 + 2a_{uv} uv + a_{vv} v^2), \tag{1}$$

$$\partial_t^2 v = \partial_x^2 v - v + b_{uu} u^2 + 2b_{uv} uv + b_{vv} v^2, \tag{2}$$

where $u = u(x,t), v = v(x,t), x, t \in \mathbb{R}$, with coefficients $\alpha > 0, a_{uu}, \dots, b_{vv} \in \mathbb{R}$. Inserting the ansatz

$$\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t) = \varepsilon^2 A(\varepsilon(x - \alpha t), \varepsilon^3 t) \quad \text{and} \quad \varepsilon^2 \psi_v^{\mathrm{KdV}}(x,t) = 0, \tag{3}$$

with small perturbation parameter $0 < \varepsilon^2 \ll 1$, into (1) and (2) yields the KdV equation

$$\partial_T A = v_1 \partial_X^3 A + v_2 \partial_X (A^2), \tag{4}$$

with coefficients $v_1, v_2 \in \mathbb{R}$. The amplitude $A(X,T) \in \mathbb{R}$ depends on the long temporal variable $T = \varepsilon^3 t$ and on the long spatial variable $X = \varepsilon(x - \alpha t)$.

We are interested in the validity of the KdV approximation for the BKG system in case of unstable resonances; ie, in case $\alpha > 2$; cf Remark 5. For notational simplicity in the subsequent formulae, we put α^2 in front of the nonlinear terms in (1). We prove the following:

Theorem 1. *Fix $T_0 > 0$, $C_0 > 0$, $\mu_A > 0$, $s_A - s \geq 8$, $s \geq 1$, and let A be a solution of the KdV Equation 4 with*

$$\sup_{T \in [0,T_0]} \int_{\mathbb{R}} |\hat{A}(K,T)| e^{\mu_A |K|} (1 + K^2)^{\frac{s_A}{2}} dK \leq C_0. \tag{5}$$

Then, there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$, and $C_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all initial conditions of (1) and (2) with

$$\|(u,v)(\cdot,0) - (\varepsilon^2 \psi_u^{KdV}(\cdot,0),0)\|_{H^\infty_{\mu_A,s}} \leq C_0 \varepsilon^{7/2}.$$

The associated solutions satisfy

$$\sup_{t\in[0,T_1/\varepsilon^3]} \|(u,v)(\cdot,t) - (\varepsilon^2 \psi_u^{KdV}(\cdot,t),0)\|_{H^s} \leq C_1 \varepsilon^{7/2}, \tag{6}$$

where the norm $\|\cdot\|_{H^\infty_{\mu_A,s}}$ is defined subsequently in (12).

Remark 1. Such an approximation result is nontrivial since solutions of order $\mathcal{O}(\varepsilon^2)$ have to be controlled on an $\mathcal{O}(1/\varepsilon^3)$ timescale. The estimate (6) and Sobolev's embedding theorem imply

$$\sup_{t\in[0,T_1/\varepsilon^3]} \sup_{x\in\mathbb{R}} |(u,v)(x,t) - (\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t),0)| \leq C_1 \varepsilon^{7/2}.$$

Remark 2. The linearized problem is solved by

$$u(x,t) = e^{ikx \pm i\omega_u(k)t}, \qquad v(x,t) = e^{ikx \pm i\omega_v(k)t},$$

with

$$\omega_u(k) = \frac{\alpha k}{\sqrt{1+k^2}}, \qquad \omega_v(k) = \sqrt{1+k^2}. \tag{7}$$

In Fourier space, the KdV equation describes the modes in the u equation, which are strongly concentrated around the wave number $k = 0$; cf Figure 1. Therefore, the expansion $\omega_u(k) = \alpha k - \frac{1}{2}\alpha k^3 + \mathcal{O}(k^5)$ at $k = 0$ plays an important role for the dynamics. We have $\nu_1 = -\frac{1}{2}\alpha$ in (4).

Remark 3. Historically, the KdV equation has been derived for the so-called water wave problem first. Approximation results have been established in a number of papers. They are either based on energy estimates (cf Craig[1] and Schneider and Wayne,[2] Schneider and Wayne,[3] and Duell[4]) or on the use of analytic functions (cf Kano and Nishida[5] and Schneider[6]).

Remark 4. Although the BKG system looks less complicated than the water wave problem, for the KdV approximation of the BKG system, some features occur, which are not present for the water wave problem over a flat bottom, namely, the occurrence of quadratic resonances, like they occur for the water wave problem over a periodic bottom. The linearized water wave problem over a periodic bottom, which is solved by Bloch modes, has been analyzed in Craig et al.[7]

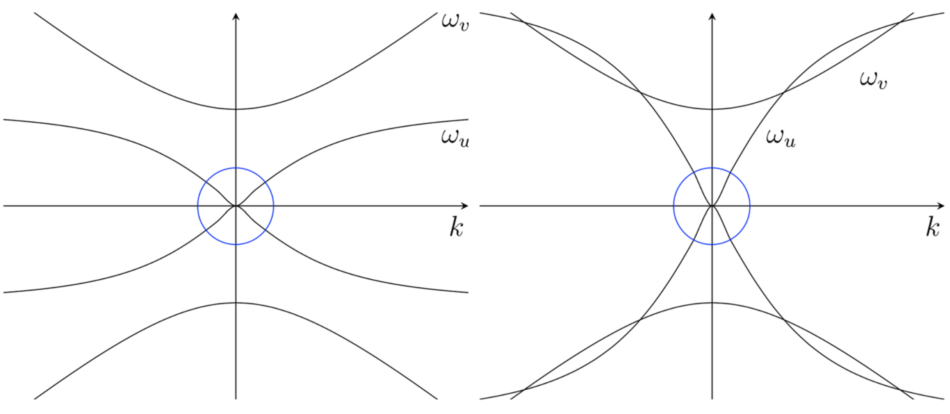

FIGURE 1 The curves of eigenvalues $\pm\omega_u$, $\pm\omega_v$ for the linearized BKG system plotted as a function over the Fourier wave numbers in case $\alpha^2 = 1$ (left) and $\alpha^2 = 5$ (right). The modes in the circles are described by the KdV approximation [Colour figure can be viewed at wileyonlinelibrary.com]

Remark 5. For $\alpha > 2$, the curves ω_u and ω_v intersect at two wave numbers k_1 and k_2; cf the right panel of Figure 1. In Bauer et al,[8] it has been explained that there are $\frac{2\pi}{k_1}$ spatially periodic solutions of the form

$$\begin{aligned} u &= \varepsilon^2 A(\varepsilon^2 t) + \varepsilon^n A_1(\varepsilon^2 t) e^{i\omega_u(k_1)t} e^{ik_1 x} + \varepsilon^n A_{-1}(\varepsilon^2 t) e^{-i\omega_u(-k_1)t} e^{-ik_1 x}, \\ v &= \varepsilon^n B_1(\varepsilon^2 t) e^{i\omega_v(k_1)t} e^{ik_1 x} + \varepsilon^n B_{-1}(\varepsilon^2 t) e^{i\omega_v(-k_1)t} e^{-ik_1 x}, \end{aligned}$$

which satisfy $\partial_\tau^2 A = 0$ and

$$\partial_\tau \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad \text{with} \quad M = \frac{1}{i\omega_u(k_1)} \begin{pmatrix} -\alpha^2 a_{uu} k_1^2 A & -\alpha^2 a_{uv} k_1^2 A \\ b_{uu} A & b_{uv} A \end{pmatrix},$$

where $\tau = \varepsilon^2 t$. By suitably choosing the coefficients a_{uu}, a_{uv}, b_{uu}, and b_{uv}, the matrix M has eigenvalues with nonvanishing real part. Hence, growth rates $e^{\beta\tau} = e^{\beta\varepsilon^2 t} = e^{\beta T/\varepsilon}$ with a $\beta > 0$ occur. These allow us to bring $\varepsilon^n A_1$ and $\varepsilon^n B_1$, which are both initially of order $\mathcal{O}(\varepsilon^n)$, to an order $\mathcal{O}(\varepsilon^2)$ at a time $T = \mathcal{O}((n-2)\varepsilon|\ln(\varepsilon)|) \ll 1$. Therefore, we have that $v = \mathcal{O}(\varepsilon^2)$ on a timescale much smaller than the natural timescale of the KdV equation. See Figure 2. Hence, in this situation, the KdV approximation makes wrong predictions. It can only make correct predictions if initially, $\varepsilon^n A_1$ and $\varepsilon^n B_1$ are chosen exponentially small w.r.t. ε; cf Assumption (5) in Theorem 1. Without excluding the possibility of unstable resonances, the restriction to analytic solutions and to $T_1 \in (0, T_0]$ cannot be avoided.

Remark 6. For the BKG system in Chong and Schneider[9] for $\alpha < 2$, ie, in case of no additional quadratic resonances, ie, in case $\omega_u(k) \neq \omega_v(k)$ for all $k \in \mathbb{R}$, a KdV approximation result has been established using normal form transformations. Based on Bauer et al,[10] it has been explained in Bauer et al[8] how to establish an approximation result for all $\alpha \geq 2$ in case of stable quadratic resonances, ie, in case that all eigenvalues of M are purely imaginary or have negative real part. Hence, it is the purpose of this paper to cover the case of unstable quadratic resonances, ie, when M has at least one eigenvalue with positive real part.

Remark 7. The proof of Theorem 1 is based on a control of the solutions close to the wave number $k = 0$ by energy estimates and normal transformations. At the other wave numbers, the solutions are solely controlled by working in spaces of analytic functions. Functions that are analytic in a strip in the complex plane around the real axis of width $2\mu_A$ correspond in Fourier space to functions, which decay as $e^{-\mu_A|K|}$ for $|K| \to \infty$; cf Assumption (5) in Theorem 1. See Reed and Simon.[11], Theorem IX.13 By making the strip smaller in time, an artificial damping of the modes with $k = \varepsilon K > 0$ is obtained. This procedure leads to the restriction on $T_1 \in [0, T_0]$; cf (17).

Remark 8. The BKG system is a prototype model for a whole class of systems. Elements of this class are the poly-atomic FPU problem and the water wave problem over a periodic bottom with $\mathcal{O}(1)$ periodicity and bottom variations of $\mathcal{O}(1)$. The transfer of the following analysis to these systems will be the subject of future research. The main strength of the approach of the present paper is that it will allow us to handle such more complicated systems without a detailed discussion of the resonances and without finding a suitable energy, which will be different for every system. Except of very few exceptions (cf Chong and Schneider,[9] Bauer et al,[10] Chirilus-Bruckner et al,[12] and Gaison et al[13]), the KdV approximation so far has only been justified for systems with a single pair of curves of eigenvalues $\pm i\omega_u$.

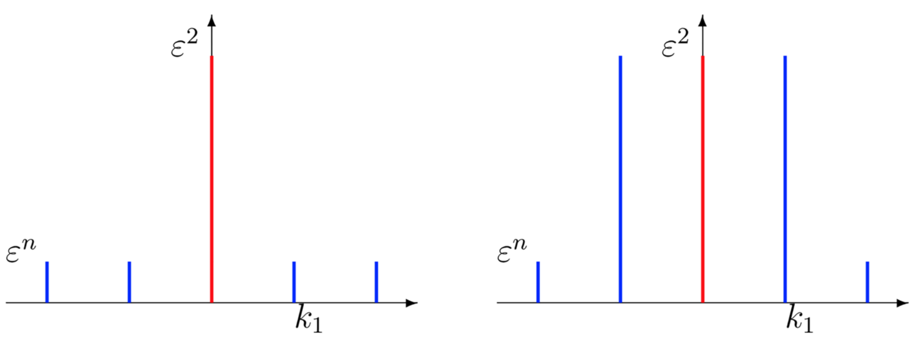

FIGURE 2 Left panel: the mode distribution of a valid KdV approximation for $t = 0$. Right panel: the mode distribution for $t = \mathcal{O}(|\ln\varepsilon|/\varepsilon^2) \ll \mathcal{O}(1/\varepsilon^3)$. The KdV approximation is no longer valid in the right picture, since the modes at $\pm k_1$ are of the same order as the KdV modes at $k = 0$ [Colour figure can be viewed at wileyonlinelibrary.com]

Notation. The Fourier transform of a function u is denoted by $\mathcal{F}u$ or $\hat{u}$. Possibly different constants that can be chosen independently of the small perturbation parameter $0 < \varepsilon^2 \ll 1$ are denoted with the same symbol C.

2 | DERIVATION OF THE KDV APPROXIMATION

Inserting the ansatz

$$\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t) = \varepsilon^2 A(\varepsilon(x-\alpha t), \varepsilon^3 t) \qquad \text{and} \qquad \varepsilon^2 \psi_v^{\mathrm{KdV}} = 0. \tag{8}$$

into the BKG system gives

$$\begin{aligned} \mathrm{Res}_u &= -\partial_t^2 u + \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \alpha^2 \partial_x^2 (a_{uu} u^2 + 2a_{uv} uv + a_{vv} v^2) \\ &= \varepsilon^8 \partial_T^2 A, \\ \mathrm{Res}_v &= -\partial_t^2 v + \partial_x^2 v - v + b_{uu} u^2 + 2b_{uv} uv + b_{vv} v^2 \\ &= \varepsilon^4 b_{uu} A^2. \end{aligned}$$

If we choose

$$-2\alpha \partial_T \partial_X A = \alpha^2 \partial_X^4 A + \alpha^2 \partial_X^2 (a_{uu} A^2),$$

respectively,

$$\partial_T A = -\frac{\alpha}{2} \partial_X^3 A - \frac{\alpha a_{uu}}{2} \partial_X (A^2). \tag{9}$$

The residuals Res_u and Res_v contain the terms, which do not cancel after inserting the approximation into the BKG system. For our subsequent error estimates, we need $\mathrm{Res}_u = \mathcal{O}(\varepsilon^8)$ and $\mathrm{Res}_v = \mathcal{O}(\varepsilon^8)$. In order to achieve this goal, we have to extend our approximation of v by higher order terms. Therefore, our final approximation is given by

$$\begin{aligned} \varepsilon^2 \psi_u(x,t) &= \varepsilon^2 A(\varepsilon(x-\alpha t), \varepsilon^3 t), \\ \varepsilon^4 \psi_v(x,t) &= \varepsilon^4 B_1(\varepsilon(x-\alpha t), \varepsilon^3 t) + \varepsilon^6 B_2(\varepsilon(x-\alpha t), \varepsilon^3 t). \end{aligned}$$

For this improved approximation, we find

$$\begin{aligned} \mathrm{Res}_u &= \mathcal{O}(\varepsilon^8), \\ \mathrm{Res}_v &= \varepsilon^4(-B_1 + b_{uu} A^2) + \varepsilon^6(-B_2 + (1-\alpha^2)\partial_X^2 B_1 + 2b_{uv} A B_1) + \mathcal{O}(\varepsilon^8) \\ &= \mathcal{O}(\varepsilon^8), \end{aligned}$$

if we choose B_1 and B_2 to satisfy

$$-B_1 + b_{uu} A^2 = 0 \quad \text{and} - B_2 + (1-\alpha^2)\partial_X^2 B_1 + 2b_{uv} A B_1 = 0.$$

Due to $(\int_{\mathbb{R}} |u(\varepsilon x)|^2 dx)^{1/2} = \varepsilon^{-1/2} (\int_{\mathbb{R}} |u(X)|^2 dX)^{1/2}$, we lose a factor $\varepsilon^{-1/2}$ if we compute the magnitude of the residual w.r.t. powers of ε in L^2-based spaces. Therefore, we have the following.

Lemma 1. *Fix $s_A - s \geq 8$, $s > 1/2$, and $T_0 > 0$. Let $A \in C([0,T_0], H^{s_A})$ be a solution of the KdV Equation (9) and $\varepsilon^2 \psi_u$ and $\varepsilon^4 \psi_v$ be defined as above. Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\sup_{T \in [0,T_0]} \|Res_u\|_{H^s} \leq C\varepsilon^{15/2} \quad \text{and} \quad \sup_{T \in [0,T_0]} \|Res_v\|_{H^s} \leq C\varepsilon^{15/2}.$$

Proof. Counting the powers of ε is straightforward. Hence, it remains to discuss the assumption $s_A - s \geq 8$. The term that loses most regularity is $\partial_T^2 B_2$, which can be expressed in terms of $\partial_T^2 (AB_1)$ and $\partial_T^2 \partial_X^2 B_1$. Since B_1 can be expressed in terms of A^2, it is sufficient to control $\partial_T^2 \partial_X^2 (A^2)$. We have that $\partial_T A$ can be expressed via the right-hand side of the

KdV equation in terms of $A, \ldots, \partial_X^3 A$. Differentiating the KdV equation w.r.t. T shows that $\partial_T^2(A^2)$ can be expressed in terms of $A, \ldots, \partial_X^6 A$. Therefore, $\partial_T^2 \partial_X^2 B$ can be expressed in terms of $A, \ldots, \partial_X^8 A$. □

Then, writing the equations for the error, obtained from the BKG systems (1) and (2), as a first order system, a term $\partial_x^{-1}\mathrm{Res}_u$ occurs.

Lemma 2. *Under the assumption of Lemma 1, we have the estimate*

$$\sup_{T\in[0,T_0]} \|\partial_x^{-1} Res_u\|_{H^{s+1}} \leq C\varepsilon^{13/2}.$$

Proof. The loss of ε^{-1} comes from $\partial_x^{-1} = \varepsilon^{-1}\partial_X^{-1}$. We need to show that $\partial_x^{-1}\mathrm{Res}_u$ is again in L^2. This is obvious for all terms, which have a derivative ∂_X in front, ie, all terms except $\partial_T^2 A$. We have

$$\partial_T^2 A = \partial_T(\partial_T A) = \partial_T(\nu_1 \partial_X^3 A + \nu_2 \partial_X(A^2)) = \partial_X \partial_T(\nu_1 \partial_X^2 A + \nu_2(A^2)).$$

Therefore, we are done. □

3 | THE EQUATIONS FOR THE ERROR

The error functions $(\varepsilon^{7/2}R_u, \varepsilon^{7/2}R_v)$, defined by

$$u = \varepsilon^2\psi_u + \varepsilon^{7/2}R_u, \qquad v = \varepsilon^4\psi_v + \varepsilon^{7/2}R_v,$$

satisfy

$$\begin{aligned}\partial_t^2 R_u &= \alpha^2\partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + 2\varepsilon^2\alpha^2\partial_x^2(a_{uu}\psi_u R_u + a_{uv}\psi_u R_v) \\ &\quad +\varepsilon^3\alpha^2\partial_x^2 f_u, \end{aligned} \tag{10}$$

$$\partial_t^2 R_v = \partial_x^2 R_v - R_v + 2\varepsilon^2(b_{uu}\psi_u R_u + b_{uv}\psi_u R_v) + \varepsilon^3 f_v, \tag{11}$$

where

$$\begin{aligned}\varepsilon^3\partial_x^2 f_u &= \varepsilon^{7/2}\partial_x^2(a_{uu}R_u^2 + 2a_{uv}R_uR_v + a_{vv}R_v^2) \\ &\quad +2\varepsilon^4\partial_x^2(a_{uv}\psi_v R_u + a_{vv}\psi_v R_v) + \varepsilon^{-7/2}\mathrm{Res}_u, \\ \varepsilon^3 f_v &= \varepsilon^{7/2}(b_{uu}R_u^2 + 2b_{uv}R_uR_v + b_{vv}R_v^2) \\ &\quad +2\varepsilon^4(b_{uv}\psi_v R_u + b_{vv}\psi_v R_v) + \varepsilon^{-7/2}\mathrm{Res}_v.\end{aligned}$$

This system is written as first-order system, with ω_u, ω_v from (7),

$$\begin{aligned}\partial_t R_u &= i\omega_u \tilde{R}_u, \\ \partial_t \tilde{R}_u &= i\omega_u R_u + 2\varepsilon^2 i\omega_u(a_{uu}\psi_u R_u + a_{uv}\psi_u R_v) + \varepsilon^3 i\omega_u f_u, \\ \partial_t R_v &= i\omega_v \tilde{R}_v, \\ \partial_t \tilde{R}_v &= i\omega_v R_v + 2\varepsilon^2(i\omega_v)^{-1}(b_{uu}\psi_u R_u + b_{uv}\psi_u R_v) + \varepsilon^3(i\omega_v)^{-1} f_v.\end{aligned}$$

Although $\varepsilon^{-7/2}(i\omega_v)^{-1}\mathrm{Res}_v$ in $\varepsilon^3(i\omega_v)^{-1}f_v$ is of order $\mathcal{O}(\varepsilon^4)$, we keep the scaling since $\varepsilon^{-7/2}i\omega_u\partial_x^{-2}\mathrm{Res}_u$ in $\varepsilon^3 i\omega_u f_u$ is of order $\mathcal{O}(\varepsilon^3)$ due to Lemma 2. After diagonalization,

$$\mathcal{R}_1 = \frac{1}{\sqrt{2}}(R_u + \tilde{R}_u), \qquad \mathcal{R}_{-1} = \frac{1}{\sqrt{2}}(R_u - \tilde{R}_u),$$

and

$$\mathcal{R}_2 = \frac{1}{\sqrt{2}}(R_v + \tilde{R}_v), \qquad \mathcal{R}_{-2} = \frac{1}{\sqrt{2}}(R_v - \tilde{R}_v),$$

of the linear part, we obtain

$$\begin{aligned}
\partial_t \mathcal{R}_1 &= i\omega_u \mathcal{R}_1 + \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u f_u \\
&\quad + \varepsilon^2 i\omega_u (a_{uu}\psi_u(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{uv}\psi_u(\mathcal{R}_2 + \mathcal{R}_{-2})), \\
\partial_t \mathcal{R}_2 &= i\omega_v \mathcal{R}_2 + \frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} f_v \\
&\quad + \varepsilon^2 (i\omega_v)^{-1}(b_{uu}\psi_u(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{uv}\psi_u(\mathcal{R}_2 + \mathcal{R}_{-2})),
\end{aligned}$$

and similar for $\mathcal{R}_{-1}$ and $\mathcal{R}_{-2}$.

Since $\varepsilon^2\psi_u$ is strongly concentrated at $k = 0$, we separate $\varepsilon^2\psi_u$ into a part concentrated close to $k = 0$ and into the rest. For $\delta > 0$, we define the mode projection E_δ via $\widehat{E_\delta u} = \hat{E}_\delta \hat{u}$, where $\hat{E}_\delta(k) = 1$ for $|k| \leq \delta$ and $\hat{E}_\delta(k) = 0$ elsewhere. Moreover, we define E_δ^c via $\hat{E}_\delta^c(k) = 1 - \hat{E}_\delta(k)$. Since $E_\delta^c \psi_u$ is $\mathcal{O}(\varepsilon^{s_A})$-small, for instance, w.r.t. the sup-norm, if A is s_A times continuously differentiable (see Corollary 3 and Remark 11 in Appendix A), we write the equations for the error as

$$\begin{aligned}
\partial_t \mathcal{R}_1 &= i\omega_u \mathcal{R}_1 + \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u g_u \\
&\quad + \varepsilon^2 i\omega_u (a_{uu}(E_\delta\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{uv}(E_\delta\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2})), \\
\partial_t \mathcal{R}_2 &= i\omega_v \mathcal{R}_2 + \frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} g_v \\
&\quad + \varepsilon^2 (i\omega_v)^{-1}(b_{uu}(E_\delta\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{uv}(E_\delta\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2})),
\end{aligned}$$

and similar for $\mathcal{R}_{-1}$ and $\mathcal{R}_{-2}$, where

$$g_u = g_u(\mathcal{R}_1, \mathcal{R}_2), \qquad g_v = g_v(\mathcal{R}_1, \mathcal{R}_2),$$

with

$$\begin{aligned}
\varepsilon^3 g_u &= \varepsilon^3 f_u + \sqrt{2}\varepsilon^2 (a_{uu}(E_\delta^c\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{uv}(E_\delta^c\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2})), \\
\varepsilon^3 g_v &= \varepsilon^3 f_v + \sqrt{2}\varepsilon^2 (b_{uu}(E_\delta^c\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{uv}(E_\delta^c\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2})).
\end{aligned}$$

4 | THE FUNCTIONAL ANALYTIC SET-UP

In order to control the unstable resonances, we introduce a number of function spaces. By $(\cdot,\cdot)$, we denote the Euclidean inner product, and by $|\cdot|$, the associated Euclidean norm in $\mathbb{R}^d$. The Fourier transform is denoted by

$$\mathcal{F}(u)(k) = \hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x) dx.$$

For $m \geq 0$, we define the Sobolev spaces

$$H^m = \{u \in L^2(\mathbb{R}) : (1 + |\cdot|^2)^{\frac{m}{2}} \hat{u}(\cdot) \in L^2(\mathbb{R})\},$$

equipped with the inner product

$$(u, v)_{H^m} = (\hat{u}, \hat{v})_{L^2_m} = \int_{\mathbb{R}} \left(1 + |k|^2\right)^m (\hat{u}(k), \hat{v}(k))\, dk.$$

For any $m \in \mathbb{N}$, the induced norm is equivalent to the usual H^m-norm. Finally, for $m \geq 0$, we introduce

$$W^m := \left\{ u : u = \mathcal{F}^{-1}(\hat{u}), \hat{u} \in L^1(\mathbb{R}), \|u\|_{W_m} = \int_{\mathbb{R}} (1 + |k|^m)|\hat{u}(k)| \, dk < \infty \right\}.$$

By Sobolev's embedding theorem, the space $H^{m+\delta}(\mathbb{R})$ is continuously embedded into W^m for each $\delta > 1/2$. Moreover, every $u \in W^m$ is $\lfloor m \rfloor$ times continuously differentiable with finite $C_b^{\lfloor m \rfloor}(\mathbb{R})$-norm.

In order to control the positive growth rates, possibly occurring at the resonances (cf Remark 5), we work in the space

$$H^\infty_{\mu,m} = \{u \in L^2(\mathbb{R}) : e^{\mu|\cdot|}(1 + |\cdot|^2)^{\frac{m}{2}} \hat{u} \in L^2(\mathbb{R})\},$$

equipped with the norm

$$\|u\|_{H^\infty_{\mu,m}} = \left(\int_{\mathbb{R}} |\hat{u}(k)|^2 e^{2\mu|k|} (1 + |k|^2)^m dk \right)^{\frac{1}{2}}, \tag{12}$$

where $\mu \geq 0$ and $m \geq 0$. Functions $u \in H^\infty_{\mu,0}$ can be extended to functions that are analytic on the strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \mu\}$; cf Reed and Simon.[11, Theorem IX.13] Similarly, we define the spaces $W^\infty_{\mu,m}$.

In our notations of the spaces and norms, we do not distinguish between scalar and vector-valued functions. The spaces $H^\infty_{\mu,m}$ are closed under point-wise multiplication for every $\mu \geq 0$ and $m > 1/2$, and the spaces $W^\infty_{\mu,m}$ for every $\mu \geq 0$ and $m \geq 0$. For details, see Lemma 6, Corollary 1, and Corollary 2.

5 | SOME FIRST ESTIMATES

In this section, we collect various estimates, which are necessary for the proof of Theorem 1. We start by rewriting Lemma 1 and Lemma 2 into $H^\infty_{\mu,s}$ spaces.

Lemma 3. *Fix $\mu_A \geq \mu \geq 0$, $s_A - s \geq 8$, $s > 1/2$, and $T_0 > 0$. Let $A \in C([0, T_0], W^\infty_{\mu_A, s_A})$ be a solution of the KdV Equation (9), and let $\varepsilon^2\psi_u$ and $\varepsilon^4\psi_v$ be defined as above. For this approximation, then there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\sup_{T \in [0,T_0]} (\|Res_u\|_{H^\infty_{\mu,s}} + \|Res_v\|_{H^\infty_{\mu,s}} + \varepsilon\|\partial_x^{-1} Res_u\|_{H^\infty_{\mu,s+1}}) \leq C_{res}\varepsilon^{15/2}.$$

Proof. Using Lemma 6 from the appendix the proof goes line for line as the proofs of Lemma 1 and Lemma 2. □

Since ω_u is a bounded operator in $H^\infty_{\mu,s}$ and since $(\omega_v)^{-1}$ is a bounded operator from $H^\infty_{\mu,s}$ to $H^\infty_{\mu,s+1}$ we have, using Lemma 3, that

$$\|\varepsilon^3 i\omega_u f_u\|_{H^\infty_{\mu,s}} + \|\varepsilon^3 (i\omega_v)^{-1} f_v\|_{H^\infty_{\mu,s+1}} \leq C\varepsilon^{7/2}(\|R_u\|^2_{H^\infty_{\mu,s}} + \|R_v\|^2_{H^\infty_{\mu,s}}) + C\varepsilon^4(\|\psi_v\|_{W^\infty_{\mu,s}}\|R_u\|_{H^\infty_{\mu,s}} + \|\psi_v\|_{W^\infty_{\mu,s}}\|R_v\|_{H^\infty_{\mu,s}}) + C_{\mathrm{res}}\varepsilon^3.$$

With these estimates and Corollary 3, we find

$$\begin{aligned}
\varepsilon^3\|i\omega_u g_u\|_{H^\infty_{\mu,s}} &\leq \varepsilon^3\|f_u\|_{H^\infty_{\mu,s}} + C\varepsilon^2 C_\psi \varepsilon^{s_A - s}(\|\mathcal{R}_1\|_{H^\infty_{\mu,s}} + \|\mathcal{R}_{-1}\|_{H^\infty_{\mu,s}}) \\
&\quad + C\varepsilon^2 C_\psi \varepsilon^{s_A - s}(\|\mathcal{R}_2\|_{H^\infty_{\mu,s}} + \|\mathcal{R}_{-2}\|_{H^\infty_{\mu,s}}), \\
\varepsilon^3\|(i\omega_v)^{-1} g_v\|_{H^\infty_{\mu,s+1}} &\leq \varepsilon^3\|f_v\|_{H^\infty_{\mu,s+1}} + C\varepsilon^2 C_\psi \varepsilon^{s_A - s}(\|\mathcal{R}_1\|_{H^\infty_{\mu,s}} + \|\mathcal{R}_{-1}\|_{H^\infty_{\mu,s}}) \\
&\quad + C\varepsilon^2 C_\psi \varepsilon^{s_A - s}(\|\mathcal{R}_2\|_{H^\infty_{\mu,s}} + \|\mathcal{R}_{-2}\|_{H^\infty_{\mu,s}}).
\end{aligned}$$

6 | FROM ANALYTIC TO SOBOLEV FUNCTIONS

In order to control the unstable resonances, we solve the equations for the error in $H^\infty_{\mu,s}$ spaces with $s \geq 1$ and $\mu = \mu(t)$ decreasing in time. In detail, we choose

$$\mu(t) = \mu_A/\varepsilon - \beta\varepsilon^2 t,$$

for $0 \leq t \leq T_1/\varepsilon^3$ with $T_1 = \mu_A/\beta$. In order to satisfy the subsequent condition (17), we have to choose $\beta > 0$ sufficiently large, which gives the restriction on the approximation time. The growing unstable resonant modes are damped w.r.t. this time-dependent norm for $\beta > 0$ sufficiently big. Since the solutions of the KdV equation satisfy

$$\sup_{T\in[0,T_0]} \|A(T)\|_{W^\infty_{\mu_A,s_A}} \leq C_\psi,$$

we have

$$\sup_{T\in[0,T_0/\varepsilon^3]} \left\|\varepsilon^2\psi_u(t)\right\|_{W^\infty_{\mu_A/\varepsilon,s_A}} \leq C_\psi\varepsilon^2.$$

In order to work in usual Sobolev, we introduce

$$R_j(t) = S_\omega(t)\mathcal{R}_j(t),$$

with $S_\omega(t)$ a multiplication operator defined in Fourier space by

$$\hat{S}_\omega(k,t) = e^{(\mu_A/\varepsilon-\beta\varepsilon^2 t)|k|}.$$

As a direct consequence of the definitions, we have the following.

Lemma 4. *For $t \in [0, \mu_A/(\beta\varepsilon^3)]$, the linear mappings $S_\omega(t) : H^\infty_{\mu(t),s} \to H^s$ and $S_\omega(t) : W^\infty_{\mu(t),s} \to W^s$, with $\mu(t) = (\mu_A - \eta\varepsilon^3 t)/\varepsilon$, are bijective and bounded with bounded inverse.*

The new variables satisfy

$$\begin{aligned}
\partial_t R_1 = & -\beta\varepsilon^2|k|_{op}R_1 + i\omega_u R_1 + \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u g_1 \\
& +\varepsilon^2 i\omega_u S_\omega(t)(a_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_1+R_{-1}) \\
& +a_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_2+R_{-2})),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\partial_t R_2 = & -\beta\varepsilon^2|k|_{op}R_2 + i\omega_v R_2 + \frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} g_2 \\
& +\varepsilon^2 (i\omega_v)^{-1} S_\omega(t)(b_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_1+R_{-1}) \\
& +b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_2+R_{-2})),
\end{aligned} \tag{14}$$

and similar for R_{-1} and R_{-2}, where

$$\begin{aligned}
g_1 &= g_1(t,R_1,R_2) = S_\omega g_u(S_\omega^{-1}R_1, S_\omega^{-1}R_2),\\
g_2 &= g_2(t,R_1,R_2) = S_\omega g_v(S_\omega^{-1}R_1, S_\omega^{-1}R_2).
\end{aligned}$$

The operator $|k|_{op}$ is defined via its operation in Fourier space $\widehat{|k|_{op}R}(k) = |k|\hat{R}(k)$.

In order to estimate R_1 and R_2 on the long $\mathcal{O}(1/\varepsilon^3)$ timescale, the terms of order $\mathcal{O}(\varepsilon^3)$ in (13) and (6) are no problem. In the end, they can be controlled easily with Gronwall's inequality. Using Lemma 4, Lemma 6, and the previous estimates, we find

$$\begin{aligned}
& \varepsilon^3\|i\omega_u g_1\|_{H^s} + \varepsilon^3\|(i\omega_v)^{-1}g_2\|_{H^{s+1}} \\
& \leq C\varepsilon^{7/2}(\|R_1\|_{H^s}^2 + \|R_2\|_{H^s}^2) \\
& \quad +C\varepsilon^4(\|\psi_v\|_{W^\infty_{\mu(t),s}}\|R_1\|_{H^s} + \|\psi_v\|_{W^\infty_{\mu(t),s}}\|R_2\|_{H^s}) + C_{\mathrm{res}}\varepsilon^3 \\
& \quad +C\varepsilon^2 C_\psi\varepsilon^{s_A-s}(\|R_1\|_{H^s} + \|R_2\|_{H^s}).
\end{aligned}$$

Hence, the major difficulty to come to the long $\mathcal{O}(1/\varepsilon^3)$ timescale is the control of the terms of order $\mathcal{O}(\varepsilon^2)$ in (13) and (6). Our strategy to handle the terms of order $\mathcal{O}(\varepsilon^2)$ is as follows. The modes to wave numbers outside a neighborhood of zero are controlled with the sectorial operator $-\beta\varepsilon^2|k|_{op}$. For terms not vanishing at $k = 0$, this operator is of no use. For wave numbers in a δ_0 neighborhood of the origin with $\delta_0 > 0$ small, but fixed, the error equations are simplified by a number of normal form transformations, ie, by a number of near identity change of variables. These normal form transformations are provided in Section 7. In Section 8, the solutions of the transformed system are then estimated by energy estimates.

Remark 9. Since the BKG systems (1) and (2) are a semilinear system local existence and uniqueness in $H^{\infty}_{\mu,s}$ spaces follow from a simple application of the contraction mapping principle to the variation of constant formula. The subsequent error estimates serve as a priori estimates to guarantee existence and uniqueness on the required time interval.

7 | THE NORMAL FORM TRANSFORMATION

In this section, we provide a number of near identity change of variables to eliminate terms close to the wave number $k = 0$ of formal order $\mathcal{O}(\varepsilon^2)$, which cannot be controlled by the sectorial operator $-\beta\varepsilon^2|k|_{op}$ or which finally turn out to be of order $\mathcal{O}(\varepsilon^3)$. Before we do so, we recall the basics of normal form transformations.

Remark 10. For the abstract evolutionary system

$$\partial_t u = Au + \varepsilon^2 N_Q(u) + \varepsilon^3 N_c(u),$$

we seek a near identity change of coordinates $v = u - \varepsilon^2 K(u)$ to eliminate the terms $\varepsilon^2 N_Q(u)$ and to transfer them into higher order terms, like $\varepsilon^3 N_c(u)$. We find

$$\begin{aligned}\partial_t v &= \partial_t u - \varepsilon^2 K'(u)\partial_t u \\ &= Au + \varepsilon^2 N_Q(u) + \varepsilon^3 N_c(u) - \varepsilon^2 K'(u)Au - \varepsilon^4 K'(u)N_Q(u) - \varepsilon^5 K'(u)N_c(u) \\ &= Av + \varepsilon^2 AK(u) - \varepsilon^2 K'(u)Au + \varepsilon^2 N_Q(u) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

In order to eliminate $\varepsilon^2 N_Q$, we choose $\varepsilon^2 K$ to satisfy

$$AK(u) - K'(u)Au + N_Q(u) = 0,$$

such that after the transformation

$$\partial_t v = Av + \mathcal{O}(\varepsilon^3).$$

Hence, after the transform, the terms of order $\mathcal{O}(\varepsilon^2)$ are eliminated, and only terms of order $\mathcal{O}(\varepsilon^3)$ remain.

In the following, we provide such normal form transformations with the goal, which have explained above.

- The term $\varepsilon^2 E_{\delta_0} i\omega_u S_\omega(t)(a_{uv}(E_\delta \psi_u)S_\omega^{-1}(t)(R_2 + R_{-2}))$ in the R_1 equation can be written in Fourier space as

$$\varepsilon^2 \int q_{1,2}(k, k-l, l)\hat{\psi}_u(k-l)\hat{R}_2(l)dl + \varepsilon^2 \int q_{1,-2}(k, k-l, l)\hat{\psi}_u(k-l)\hat{R}_{-2}(l)dl,$$

with

$$q_{1,2}(k, k-l, l) = \hat{E}_{\delta_0}(k) i\hat{\omega}_u(k)\hat{S}_\omega(k,t)a_{uv}\hat{E}_\delta(k-l)S_\omega^{-1}(l,t)$$

and similarly for $q_{1,-2}(k, k-l, l)$. Following the existing literature (cf Sanders et al[14] and Schneider and Uecker[15]), it is obvious that this term can be eliminated by a near identity change of variables

$$\hat{R}_3 = \hat{R}_1 + \varepsilon^2 \int b_{1,2}(k, k-l, l)\hat{\psi}_u(k-l)\hat{R}_2(l)dl + \varepsilon^2 \int b_{1,-2}(k, k-l, l)\hat{\psi}_u(k-l)\hat{R}_{-2}(l)dl,$$

where

$$b_{1,\pm 2}(k, k-l, l) = \frac{q_{1,\pm 2}(k, k-l, l)}{i\omega_u(k) - i\omega_u(k-l) \mp i\omega_v(l) - \beta\varepsilon^2(|k| - |k-l| - |l|)}.$$

Since $\omega_u(0) = 0$ and $\omega_v(0) = 1$, the denominator is bounded away from 0 for $|k| \leq \delta_0$ and $|k-l| \leq \delta$ if $\delta_0 > 0$ and $\delta > 0$ are sufficiently small.

• The term $\varepsilon^2 E_{\delta_0}(i\omega_v)^{-1}S_\omega(t)(b_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_1+R_{-1}))$ and the term $\varepsilon^2 E_{\delta_0}(i\omega_v)^{-1}S_\omega(t)(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_{-2}))$ in the R_2 equation can be written in Fourier space as

$$\begin{aligned}&\varepsilon^2\int q_{2,1}(k,k-l,l)\hat{\psi}_u(k-l)\hat{R}_1(l)dl+\varepsilon^2\int q_{2,-1}(k,k-l,l)\hat{\psi}_u(k-l)\hat{R_{-2}}(l)dl\\&+\varepsilon^2\int q_{2,-2}(k,k-l,l)\hat{\psi}_u(k-l)\hat{R_{-2}}(l)dl,\end{aligned}$$

with

$$q_{2,1}(k,k-l,l)=\hat{E_{\delta_0}}(k)(i\hat{\omega}_v(k))^{-1}\hat{S}(k,t)b_{uu}\hat{E_\delta}(k-l)S^{-1}(l,t),$$

and similarly for $q_{2,-1}(k,k-l,l)$ and $q_{2,-2}(k,k-l,l)$. Again, this term can be eliminated by a near identity change of variables

$$\begin{aligned}\hat{R}_4&=\hat{R}_2+\varepsilon^2\int b_{2,1}(k,k-l,l)\hat{\psi}_u(k-l)\hat{R}_1(l)dl\\&\quad+\varepsilon^2\int b_{2,-1}(k,k-l,l)\hat{\psi}_u(k-l)\hat{R_{-2}}(l)dl\\&\quad+\varepsilon^2\int b_{2,-2}(k,k-l,l)\hat{\psi}_u(k-l)\hat{R_{-2}}(l)dl,\end{aligned}$$

where

$$b_{2,\pm1}(k,k-l,l)=\frac{q_{2,\pm1}(k,k-l,l)}{i\omega_v(k)-i\omega_u(k-l)\mp i\omega_u(l)-\beta\varepsilon^2(|k|-|k-l|-|l|)},$$

and

$$b_{2,-2}(k,k-l,l)=\frac{q_{2,-2}(k,k-l,l)}{i\omega_v(k)-i\omega_u(k-l)+i\omega_v(l)-\beta\varepsilon^2(|k|-|k-l|-|l|)}.$$

Again, since $\omega_u(0)=0$ and $\omega_v(0)=1$, the denominator is bounded away from 0 for $|k|\le\delta_0$ and $|k-l|\le\delta$ if $\delta_0>0$ and $\delta>0$ are sufficiently small. Estimates such as

$$\|\varepsilon^2\int b(k,k-l,l)\psi_u(k-l)R(l)dl\|_{L^2_s(dk)}\le\varepsilon^2(\sup_{k,l}|b(k,k-l,l)|)\|\psi_u\|_{W^s}\|R\|_{H^s},$$

then imply the following.

Lemma 5. *There exist $\varepsilon_0>0$ and $C>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$, the transformation $(R_1,R_2)\mapsto(R_3,R_4)$ is bijective with*

$$\|R_1-R_3\|_{H_s}+\|R_2-R_4\|_{H_s}\le C\varepsilon^2(\|R_1\|_{H^s}+(\|R_2\|_{H^s}).$$

After the transformation, our system is of the form

$$\begin{aligned}\partial_t R_3&=-\beta\varepsilon^2|k|_{op}R_3+i\omega_u R_3+\frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u g_3\\&\quad+\varepsilon^2 i\omega_u S(t)(a_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3+R_{-3}))\\&\quad+\varepsilon^2 i\omega_u S_\omega(t)E_\delta^c(a_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_4+R_{-4})),\\\partial_t R_4&=-\beta\varepsilon^2|k|_{op}R_4+i\omega_v R_4+\frac{1}{\sqrt{2}}\varepsilon^3(i\omega_v)^{-1}g_4\\&\quad+\varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3+R_{-3}))\\&\quad+\varepsilon^2(i\omega_v)^{-1}S_\omega(t)(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_4))\\&\quad+\varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_{-4})),\end{aligned}$$

and similar for R_{-3} and R_{-4}. The terms with g_3 and g_4 come from g_1 and g_2 and from higher order terms obtained via the normal form transformation. Therefore, using $s_A - s \geq 2$, the terms with g_3 and g_4 obey the estimates

$$\begin{aligned}&\varepsilon^3\|i\omega_u S_\omega(t)g_3\|_{H^s} + \varepsilon^3\|(i\omega_v)^{-1}S_\omega(t)g_4\|_{H^{s+1}}\\ &\leq C\varepsilon^{7/2}(\|R_3\|_{H^s}^2 + \|R_4\|_{H^s}^2)\\ &\quad + C_1\varepsilon^4(\|R_3\|_{H^s} + \|R_4\|_{H^s}) + C_{\text{res}}\varepsilon^3,\end{aligned} \tag{15}$$

where C_1 is a constant independent of $0 < \varepsilon \ll 1$, solely depending on $\|A\|_{W^\infty_{\mu_A,s_A}}$.

8 | THE FINAL ENERGY ESTIMATES

We have now all ingredients to perform the final energy estimates. We define an operator Ω via the multiplier $\hat{\Omega}(k) = \min(\omega_v(k), 4)$ in Fourier space. This operator is used in the estimates for the subsequent term Res_6. It leads there to a cancelation, which shows that Res_6 is of order $\mathcal{O}(\varepsilon^3)$.

We start now to estimate the time derivative of

$$E_s = \|R_3\|_{H^s}^2 + \left\|\Omega^{1/2}R_4\right\|_{H^s}^2.$$

We compute

$$\frac{1}{2}\frac{d}{dt}E_s = \text{Re } s_{good} + \sum_{j=1}^{8}\text{Re } s_j,$$

where

$$\begin{aligned}
s_{good} &= (R_3, -\beta\varepsilon^2|k|_{op}R_3)_{H^s} + (\Omega^{1/2}R_4, -\beta\varepsilon^2|k|_{op}\Omega^{1/2}R_4)_{H^s},\\
s_1 &= (R_3, i\omega_u R_3)_{H^s} + \left(\Omega^{1/2}R_4, i\omega_v\Omega^{1/2}R_4\right)_{H^s},\\
s_2 &= \left(R_3, \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u S_\omega(t)g_3\right)_{H^s} + \left(\Omega R_4, \frac{1}{\sqrt{2}}\varepsilon^3(i\omega_v)^{-1}S(t)g_4\right)_{H^s},\\
s_3 &= \left(R_3, \varepsilon^2 i\omega_u S_\omega(t)(a_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3}))\right)_{H^s},\\
s_4 &= \left(R_3, \varepsilon^2 i\omega_u S_\omega(t)E_\delta^c(a_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_4 + R_{-4}))\right)_{H^s},\\
s_5 &= \left(\Omega R_4, \varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3}))\right)_{H^s},\\
s_6 &= \left(\Omega R_4, \varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_4))\right)_{H^s},\\
s_7 &= \left(\Omega R_4, \varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_4))\right)_{H^s},\\
s_8 &= \left(\Omega R_4, \varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_{-4}))\right)_{H^s}.
\end{aligned}$$

The terms $s_1, \ldots s_8$ are either of order $\mathcal{O}(\varepsilon^3)$ or can be estimated by the negative terms of order $\mathcal{O}(\varepsilon^2)$ collected in s_{good}. The terms collected in s_1 and s_2 can easily be estimated to be $\mathcal{O}(\varepsilon^3)$. The terms s_3, s_4, s_5, s_7, and s_8 have either ω_u or E_δ^c in front and thus vanish at the wave number $k = 0$. They can be estimated by the good terms collected in s_{good}. The term s_6 can be shown to be of order $\mathcal{O}(\varepsilon^3)$ using the long wave character of the KdV approximation.

8.1 | Estimates for s_{good}, s_1, and s_2

(a) We start with the bound on the linear terms collected in s_{good}. Using the Fourier representation of $|k|_{op}$ gives

$$(R_3, -\beta\varepsilon^2|k|_{op}R_3)_{H^s} = -\beta\varepsilon^2(\hat{R}_3, |k|\hat{R}_3)_{L_s^2} = -\beta\varepsilon^2(|k|^{1/2}\hat{R}_3, |k|^{1/2}\hat{R}_3)_{L_s^2},$$

and similar for $(\Omega^{1/2}R_4, -\beta\varepsilon^2|k|_{op}\Omega^{1/2}R_4)_{H^s}$ such that finally,

$$s_{good} = -\beta\varepsilon^2\left\||k|^{1/2}R_3\right\|_{H^s} - \beta\varepsilon^2\left\||k|^{1/2}\Omega^{1/2}R_4\right\|_{H^s}. \tag{16}$$

(b) Using the skew symmetry of $i\omega_u$ and $i\omega_v$ yields

$$s_1 = 0.$$

(c) Using the Cauchy-Schwarz inequality and (7) yields

$$|s_2| \le C\varepsilon^3 E_s + C\varepsilon^{7/2} E_s^{3/2} + C\varepsilon^3.$$

8.2 | Estimates for s_3, s_4, s_5, s_7, and s_8

The good terms collected in s_{good} do not allow us to estimate terms at the wave number $k = 0$. We have to use the fact, that the terms s_3, s_4, s_5, s_7, and s_8 vanish at the wave number $k = 0$, too.

(a) The terms s_3 and s_4 can be estimated by the "good" terms using the fact that $|\hat{\omega}_u(k)| \le C|k|$ for $k \to 0$. The last estimate implies that the symbol of $\vartheta = |k|_{op}^{-1/2}\omega_u$ is bounded at the wave number $k = 0$. We find

$$\begin{aligned}
|s_3| &= |(R_3, \varepsilon^2 i\omega_u S_\omega(t)(a_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3})))_{H^s}| \\
&= \left|\left(|k|_{op}^{1/2}R_3, \varepsilon^2 i|k|_{op}^{-1/2}\omega_u S_\omega(t)(a_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3}))\right)_{H^s}\right| \\
&\le C\varepsilon^2 \||k|_{op}^{1/2}R_3\|_{H^s}\|\vartheta S_\omega(t)(a_{uu}(E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3}))\|_{H^s} \\
&\le C\varepsilon^2 \||k|_{op}^{1/2}R_3\|_{H^s} \times (\|S_\omega^{-1}(t)\vartheta^{1/2}\psi_u\|_{W_s}\|R_3\|_{H^s} + \|S_\omega^{-1}(t)\psi_u\|_{W_s}\|\vartheta^{1/2}R_3\|_{H^s}) \\
&\le C\varepsilon^{5/2}\||k|_{op}^{1/2}R_3\|_{H^s}\|R_3\|_{H^s} + C\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}\||k|_{op}^{1/2}R_3\|_{H^s} \\
&\le C(\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}^2 + \varepsilon^3\|R_3\|_{H^s}^2),
\end{aligned}$$

where we used that $\|S_\omega^{-1}(t)\vartheta^{1/2}\psi_u\|_{W_s} = \mathcal{O}(\varepsilon^{1/2})$ due to Corollary 3 applied to $|\hat{\vartheta}^{1/2}(k)| \le C|k|^{1/2}$. In the last line, we used $\varepsilon^{5/2}ab \le \varepsilon^2 a^2 + \varepsilon^3 b^2$.

The term s_4 can be estimated in exactly the same as the term s_3. The last lines have to be modified into

$$\begin{aligned}
|s_4| &\le C\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s} \times (\|S_\omega^{-1}(t)\vartheta^{1/2}\psi_u\|_{W_s}\|R_4\|_{H^s} + \|S_\omega^{-1}(t)\psi_u\|_{W_s}\|\vartheta^{1/2}R_4\|_{H^s}) \\
&\le C\varepsilon^{5/2}\||k|_{op}^{1/2}R_3\|_{H^s}\|R_4\|_{H^s} + C\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}\||k|_{op}^{1/2}R_4\|_{H^s} \\
&\le C(\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}^2 + \varepsilon^2\||k|_{op}^{1/2}\Omega^{1/2}R_4\|_{H^s}^2 + \varepsilon^3\|\Omega^{1/2}R_4\|_{H^s}^2).
\end{aligned}$$

(b) The remaining terms s_5, s_7, and s_8 can be estimated by the "good" terms in exactly the same way as s_3 and s_4 using the fact that s_5, s_7, and s_8 have an E_δ^c in front which vanishes at the wave number $k = 0$, too. We finally obtain

$$|s_5| + |s_7| + |s_8| \le C(\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}^2 + \varepsilon^2\||k|_{op}^{1/2}\Omega^{1/2}R_4\|_{H^s}^2 + \varepsilon^3\|R_3\|_{H^s}^2 + \varepsilon^3\|\Omega^{1/2}R_4\|_{H^s}^2).$$

8.3 | Estimates for s_6

For the Fourier transform of $\mathrm{Re}s_6$ in case $s = 0$, we obtain

$$\begin{aligned}
\hat{\mathrm{Re}}\, s_6 &= 2i\varepsilon^2 \int\int \hat{R_{-4}}(k)\hat{S_\omega}(k,t)\hat{E}_\delta(k)b_{uv}\hat{E}_\delta(k-l)\hat{\psi_u}(k-l)\hat{S_\omega^{-1}}(l,t)\hat{R}_4(l)dldk \\
&\quad -2i\int \varepsilon^2 \int \hat{R}_4(k)\hat{S}_\omega(k,t)\hat{E}_\delta(k)\overline{b_{uv}}\hat{E}_\delta(k-l)\,\hat{}\,(\bar{k}-\psi_u l)\hat{S_\omega^{-1}}(l,t)\hat{R_{-4}}(l)dldk \\
&= 2i\varepsilon^2 \int\int \hat{R_{-4}}(k)\hat{S}_\omega(k,t)\hat{E}_\delta(k)b_{uv}\hat{E}_\delta(k-l)\hat{\psi_u}(k-l)\hat{S_\omega^{-1}}(l,t)\hat{R}_4(l)dldk \\
&\quad -2i\varepsilon^2 \int\int \hat{R}_4(l)\hat{S_\omega}(l,t)\hat{E}_\delta(l)\overline{b_{uv}}\hat{E}_\delta(l-k)\hat{\psi_u}(l-k)\hat{S_\omega^{-1}}(k,t)\hat{R_{-4}}(k)dldk \\
&= 2i\varepsilon^2 \int\int q_0(k, k-l, l)\hat{R_{-4}}(k)\hat{\psi_u}(k-l)\hat{R}_4(l)dldk,
\end{aligned}$$

with

$$q_0(k,k-l,l) = 2i\hat{S}_\omega(k,t)\hat{E}_\delta(k)b_{uv}\hat{E}_\delta(k-l)\widehat{S_\omega^{-1}}(l,t) - 2i\hat{S}_\omega(l,t)\hat{E}_\delta(l)\overline{b_{uv}}\hat{E}_\delta(l-k)\widehat{S_\omega^{-1}}(k,t),$$

where we used $\widehat{\psi_u}(l-k) = \widehat{\psi_u}(k-l)$ which holds due to the fact that ψ_u is real-valued. By definition we have

$$b_{uv}\hat{E}_\delta(k-l) = \overline{b_{uv}}\hat{E}_\delta(l-k) \in \mathbb{R},$$

and so $q_0(k,0,k) = 0$ for all $k \in \mathbb{R}$. Since we have a compact set of wave numbers involved here, this implies $|q_0(k,k-l,l)| \leq C|k-l|$. As a consequence, we can apply Corollary 3 and obtain

$$\int\int q_0(k,k-l,l)\widehat{R_{-4}}(k)\hat{\psi}_u(k-l)\hat{R}_4(l)dldk = \mathcal{O}(\varepsilon),$$

respectively, $|s_6| \leq C\varepsilon^3 E_s$. The case $s > 0$ works very similarly. In order to estimate

$$\left(\partial_x^s \Omega R_4, \partial_x^s \varepsilon^2 (i\omega_v)^{-1} S_\omega(t) E_\delta(b_{uv}(E_\delta\psi_u)S_\omega^{-1}(t)(R_4))\right)_{L^2},$$

we use the fact that after the comma, whenever a derivative falls on ψ_u, we gain an additional power of ε. Hence, there is only one term of order $\mathcal{O}(\varepsilon^2)$; namely, then, all s derivatives after the comma fall on R_4. But this term can be estimated line for line as the case $s = 0$.

8.4 | Putting all estimates together

Using the previous estimates yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_s \leq\ & (-\beta\varepsilon^2 + C\varepsilon^2 + C\varepsilon^{7/2}E_s^{1/2})(\left\||k|^{1/2}R_3\right\|_{H^s} + \left\||k|^{1/2}\Omega^{1/2}R_4\right\|_{H^s})\\ & + C\varepsilon^3 E_s + C\varepsilon^{7/2}E_s^{3/2} + C\varepsilon^3,\end{aligned}$$

with C a constant, which is independent of $0 < \varepsilon \ll 1$. Subsequently, we choose $\beta > 0$ so large but independently of $0 < \varepsilon \ll 1$ that

$$-\beta + C + C\varepsilon^{3/2}E_s^{1/2} < 0, \tag{17}$$

will be satisfied. Under this assumption, we have

$$\frac{1}{2}\frac{d}{dt}E_s \leq C\varepsilon^3 E_s + C\varepsilon^{7/2}E_s^{3/2} + C\varepsilon^3.$$

In the following, we choose $\varepsilon > 0$ so small that

$$\varepsilon^{1/2}E_s^{1/2} \leq 1, \tag{18}$$

will be satisfied. Under this assumption, we then have

$$\frac{d}{dt}E_s \leq (C+1)\varepsilon^3 E_s + C\varepsilon^3,$$

and so Gronwall's inequality implies

$$E_s(t) \leq (E_s(0) + Ct)e^{(C+1)t} \leq (E_s(0) + CT_0)e^{(C+1)T_0} =: M = \mathcal{O}(1).$$

The constant M is independent of β, respectively, T_1 and $0 < \varepsilon \ll 1$. We are done, if we choose $\varepsilon_0 > 0$ so small that $\varepsilon_0^{1/2}M^{1/2} \leq 1$, which guarantees the validity of (18), and then $\beta > 0$ so large that

$$-\beta + C + C\varepsilon_0^{3/2}M^{1/2} < 0,$$

which guarantees the validity of (17).

9 | SOME TECHNICAL ESTIMATES

In this section, we collect a number of estimates, which we used in previous sections. Together with their proofs, they can be found as Lemma A.4, Corollary A.5, Corollary A.6, Lemma A.9, and Corollary A.10 in Haas et al.[16] We start with some estimates for the nonlinear terms.

Lemma 6. *The spaces $H^\infty_{\mu,s}$ are Banach algebras for $\mu \geq 0$ and $s > \frac{1}{2}$. In detail, there exists a μ-independent constant C_s such that*

$$\|uv\|_{H^\infty_{\mu,s}} \leq C_s \|u\|_{H^\infty_{\mu,s}} \|v\|_{H^\infty_{\mu,s}},$$

for all $u, v \in H^\infty_{\mu,s}$.

For our error estimates, we need the following tame estimates version of this lemma.

Corollary 1. *For $\delta > 0$, $\mu \geq 0$ and $s > 1/2$, we have*

$$\|u^2\|_{H^\infty_{\mu,s}} \leq C_s \|u\|_{H^\infty_{\mu,1/2+\delta}} \|u\|_{H^\infty_{\mu,s}},$$

for all $u \in H^\infty_{\mu,s}$.

Corollary 2. *For $\mu \geq 0$ and $s \geq 0$, we have*

$$\|uv\|_{H^\infty_{\mu,s}} \leq C_s \|u\|_{W_{\mu,s}} \|v\|_{H^\infty_{\mu,s}},$$

for all $u \in W^\infty_{\mu,s}$ and $v \in H^\infty_{\mu,s}$.

The expansion of the kernels in the multilinear maps can be estimated with the following lemma, for which we recall the proof.

Lemma 7. *Let $\vartheta_0 \geq 0$, $\vartheta_\infty \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|g(k)| \leq C \min(|k|^{\vartheta_0}, (1+|k|)^{\vartheta_\infty}).$$

Then, for the associated multiplication operator $g_{op} = \mathcal{F}^{-1} g \mathcal{F}$, the following holds. For (a) $\mu_1 > \mu_2$ and $m_1, m_2 \geq 0$ or (b) $\mu_1 = \mu_2$ and $m_2 - m_1 \geq \max(\vartheta_0, \vartheta_\infty)$, we have

$$\|g_{op} A(\varepsilon \cdot)\|_{H^\infty_{\mu_1/\varepsilon, m_1}} \leq C \varepsilon^{\vartheta_0 - 1/2} \|A(\cdot)\|_{H^\infty_{\mu_2, m_2}},$$

for all $\varepsilon \in (0, 1)$.

Proof. This follows immediately from the fact that the left-hand side of this inequality can be estimated by

$$\begin{aligned} &\leq \sup_{k \in \mathbb{R}} \left| g(k) \left(1 + \left(\frac{k}{\varepsilon}\right)^2\right)^{\frac{m_1 - m_2}{2}} e^{(\mu_1 - \mu_2)|k|/\varepsilon} \right| \quad \|A(\varepsilon \cdot)\|_{H^\infty_{\mu_2/\varepsilon, m_2}}, \\ &\leq C \varepsilon^{\vartheta_0} \quad \varepsilon^{-1/2} \|A(\cdot)\|_{H^\infty_{\mu_2, m_2}}, \end{aligned}$$

where the loss of $\varepsilon^{-1/2}$ is due to the scaling properties of the L^2-norm. □

In $W^\infty_{\mu,m}$ spaces, there is no $\varepsilon^{-1/2}$ loss due to the scaling invariance of the norm, and so, we have the following as a direct consequence.

Corollary 3. *Let $\vartheta_0 \geq 0$, $\vartheta_\infty \in \mathbb{R}$, and let $g(k)$ satisfy*

$$|g(k)| \leq C \min(|k|^{\vartheta_0}, (1+|k|)^{\vartheta_\infty}).$$

Then, for the associated multiplication operator $g_{op} = \mathcal{F}^{-1} g \mathcal{F}$*, the following holds. For (a)* $\mu_1 > \mu_2$ *and* $m_1, m_2 \geq 0$ *or (b)* $\mu_1 = \mu_2$ *and* $m_2 - m_1 \geq \max(\vartheta_0, \vartheta_\infty)$*, we have*

$$\|g_{op} A(\varepsilon \cdot)\|_{W_{\mu_1/\varepsilon, m_1}} \leq C \varepsilon^{\vartheta_0} \|A(\cdot)\|_{W_{\mu_2, m_2}},$$

for all $\varepsilon \in (0, 1)$.

Remark 11. This corollary is used, for instance, to estimate E_δ^c. Since $\hat{E}_\delta^c(k)$ is identical zero in a neighborhood of the origin, we have $|\hat{E}_\delta^c(k)| \leq C|k|^r$ for every $r \in \mathbb{N}$.

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