Long wave approximation over and beyond the natural time scale

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Abstract

Modulation equations are the central topic of this thesis. In many cases they can be derived by using multiple scaling analysis to get a better understanding for more complicated nonlinear partial differential equations. This approach has been used, for example, in the water wave problem to predict the dynamics of the system. It is essential to justify the validity of the approximation since there are examples where a formally correctly derived modulation equation has not described the real behavior of the solution of the nonlinear partial differential equation over the long time scale.

In the first part of this thesis we justify the Whitham approximation for a coupled system of equations, namely for a Boussinesq-Klein-Gordon system with unstable resonances. The proof is based on an infinite sequence of normal form transformations performed in the space of analytic functions, their convergence and energy estimates.

In contrast to the previously considered model problem, in the second part we analyze the derivation of the Whitham approximation around periodic wave packets for the relevant complex cubic Klein-Gordon equation. Due to two additional eigenvalue curves which lead to additional oscillatory behavior, the analysis here is more complicated than in previously considered situations. Similar to the first part, we work in the space of analytic functions such that our result is valid regardless of the spectral stability of the underlying wave packet. Here, we use Cauchy-Kovalevskaya theory, a sequence of infinitely many normal form transformations and energy estimates.

Finally, in the last part we prove a linear Schrödinger approximation result for a completely integrable system, namely the Korteweg-de Vries equation, over a longer time scale than the natural time scale of the nonlinear Schrödinger approximation. By using inverse scattering theory we show the result due to the fact that the scattering data only satisfies linear equations. This leads to an improved time scale.

Zusammenfassung

Modulationsgleichungen sind in der vorliegenden Doktorarbeit ein zentrales Thema. Sie können in vielen Fällen über Multi-Skalen-Analysis hergeleitet werden, um kompliziertere nichtlineare partielle Differentialgleichungen besser verstehen zu können. Die Herangehensweise ist zum Beispiel bei dem Wasserwellenproblem dazu benutzt worden, um die Dynamik des Systems vorhersagen zu können. Es ist hierbei unerlässlich, die Gültigkeit der Approximation zu rechtfertigen, da es Beispiele gibt, bei welchen eine formal korrekt hergeleitete Modulationsgleichung das tatsächliche Verhalten der Lösung der nichtlinearen partiellen Differentialgleichung nicht über die lange Zeitskala beschreibt.

Im ersten Teil dieser Arbeit rechtfertigen wir die Whitham Approximation für ein gekoppeltes System von Gleichungen, nämlich für das Boussinesq-Klein-Gordon System mit instabilen Resonanzen. Der Beweis beruht auf einer unendlichen Folge von Normalformtransformationen, die im Raum der analytischen Funktionen durchgeführt werden, deren Konvergenz und Energieabschätzungen.

Im Kontrast zu dem vorher betrachteten Modellproblem analysieren wir im zweiten Teil die Herleitung der Whitham Approximation um periodische Wellenpakete für die relevante komplexe kubische Klein-Gordon Gleichung. Durch zwei zusätzliche Eigenwertkurven, die zu zusätzlichem oszillatorischen Verhalten führen, ist die Analysis hier komplizierter als in bisher betrachteten Situationen. Ähnlich wie im ersten Teil arbeiten wir im Raum der analytischen Funktionen, sodass unser Resultat unabhängig von der spektralen Stabilität des zugrunde liegenden Wellenpakets gültig ist. Hierbei benutzen wir Cauchy-Kovalevskayas Theorie, eine Folge unendlich vieler Normalformtransformationen und Energieabschätzungen.

Schließlich beweisen wir im letzten Teil ein lineares Schrödinger Approximationsresultat für ein vollständig integrables System, nämlich die Korteweg-de Vries Gleichung, über einer längeren Zeitskala als der natürlichen Zeitskala der nichtlinearen Schrödinger Approximation. Mit Hilfe von inverser Streutheorie zeigen wir das Resultat, da die Streudaten lediglich lineare Gleichungen erfüllen. Dies führt zu der verbesserten Zeitskala.

Acknowledgments and Declaration

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Last but not least, my thanks go to my family and friends, without their patience and confidence this dissertation would not have been possible.

Ich versichere hiermit, dass die vorliegende Dissertation von mir selbst verfasst wurde und meine eigene Arbeit darstellt, sofern keine anderen Hilfsmittel angegeben sind. Alle Stellen, die dem Wortlaut oder dem Sinn aus anderen Referenzen stammen, sind mit den entsprechenden Informationsquellen ausdrücklich angegeben. Die Arbeit wurde bisher bei keiner anderen akademischen Einrichtung zur Erlangung eines Abschlusses oder einer Qualifikation eingereicht.

I hereby certify that this thesis has been composed by myself and describes my own work unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted and all sources of information have been specifically acknowledged. It has not previously been submitted to any other academic institution to obtain a degree or a qualification.

Stuttgart, 2024 Sarah Hofbauer

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Chapter 1

Introduction

Many physical phenomena, for instance in nonlinear optics, plasma physics, reaction diffusion systems or chemical reactions, can be described by nonlinear partial differential equations for which it is quite difficult to comprehend the dynamics of the system. Therefore, it is essential to approximate these equations by simpler nonlinear evolution equations, called amplitude or modulation equations, in order to gain a deeper understanding for physical problems and to lead to solutions that cannot be derived in the original setting. We want to analyze whether a formal approximation of a solution is a good approach to a true solution of the original dispersive system. Through multiple scaling analysis we can derive modulation equations in the long wave limit which turn out to be universal. We prove the so-called validity of these equations which means that the distance between the exact solution of the original system and the approximation based on the formally derived equation is small on a long time scale.

To illustrate this technique, we give a simple and well-known example. We consider the real cubic Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - u - u^3,$$

with $t, x \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$. The ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t) \mathrm{e}^{\mathrm{i}(k_0 x - \omega t)} + c.c.,$$

with a small perturbation parameter $0 < \varepsilon \ll 1$, group velocity $c_g \in \mathbb{R}$ of the wave packet and *c.c.* standing for the complex conjugated terms, leads to the nonlinear Schrödinger equation (NLS)

$$2\mathrm{i}\omega_0 \partial_T A = (1 - c_g^2)\partial_X^2 A - 3A|A|^2,$$

with the rescaled space variable $X = \varepsilon(x - c_g t)$, the slow time variable $T = \varepsilon^2 t$ and $A(X,T) \in \mathbb{C}$. Here, the amplitude A describes the slowly modulating envelope of a spatially and temporally oscillating wave packet, cf. Figure 1.1 below from [SU17].

It can be shown that the NLS equation makes correct predictions about the behavior of the solutions in the original system. If we replace the cubic nonlinearity by a quadratic nonlinearity in the Klein-Gordon equation, serious difficulties arise due to the fact that

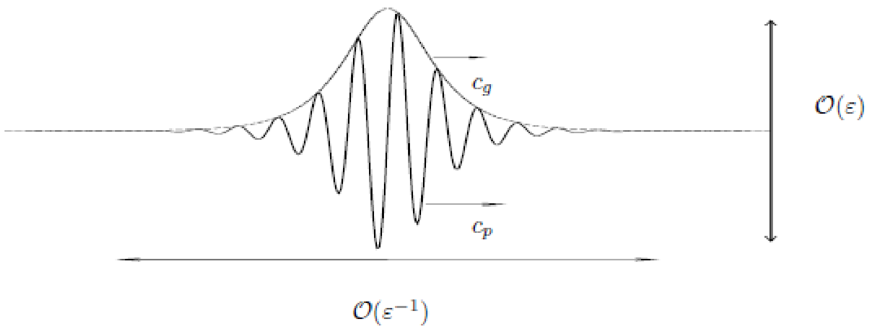

Figure 1.1: A modulating pulse of the Klein-Gordon equation described by the NLS equation. The envelope propagates with the group velocity c_g to the right and modulates the carrier wave $e^{i(k_0 x+\omega_0 t)}$ advancing with the phase velocity c_p. The envelope evolves approximately as a solution of the NLS equation, cf. [SU17].

solutions of order $\mathcal{O}(\varepsilon)$ must be bounded on the long natural time scale of the nonlinear Schrödinger equation $\mathcal{O}(1/\varepsilon^2)$, cf. [SU17, Section 11].

Other famous examples of amplitude equations that can be derived by perturbation analysis include the Korteweg-de Vries equation (KdV)

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0,$$

with $x, t \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$, and the Ginzburg-Landau equation

$$\partial_t A = (1 + i\alpha)\partial_x A + A - (1 + i\beta)A|A|^2,$$

with $\alpha, \beta, x \in \mathbb{R}$, $t \geq 0$ and $A(x,t) \in \mathbb{C}$. The KdV equation appears as an amplitude equation in the description of small spatial and temporal modulations of long waves in various dispersive wave systems, for instance in the water wave problem or in equations from plasma physics. The Ginzburg-Landau equation appears as an amplitude equation in pattern-forming systems.

Among a large class of these modulation equations, in this thesis we consider two different simpler nonlinear evolution equations, namely Whitham's modulation equations (WME) and the nonlinear Schrödinger equation. WME are a universal approximation system for large classes of nonlinear partial differential equations of periodic wave type and describe slow modulations in time and space of a periodic traveling wave in a dispersive wave system.

However, the formal derivation does not guarantee that the solutions of the original system behave as predicted by the approximation equation. Therefore, validity results are important. For instance, if we consider a WME approximation, the solutions of order $\mathcal{O}(1)$ have to be bounded on a long $\mathcal{O}(1/\varepsilon)$-time scale. In general, $\mathcal{O}(1)$-solutions are only bounded on a $\mathcal{O}(1)$-time scale by a simple application of Gronwall's inequality. In comparison, the Korteweg-de Vries approximation describes long waves of amplitudes of order $\mathcal{O}(\varepsilon^2)$ on a $\mathcal{O}(1/\varepsilon^3)$-time scale. This is the reason why it is more complicated to treat a Whitham approximation than a KdV approximation since the smallness of negligible terms has to be ensured only by derivatives, i.e., with a loss of regularity.

First, in this thesis we show the validity of the Whitham approximation for several specific dispersive systems. More specifically, in Chapter 2 we justify the Whitham approximation for a toy problem, namely the Boussinesq-Klein-Gordon system with unstable short-wave resonances. Furthermore, in Chapter 3 we prove the validity of the Whitham approximation for the complex cubic Klein-Gordon equation on the natural time scale. This equation is a relevant example to handle modulations of periodic wave trains in general dispersive systems. In contrast, in Chapter 4 we extend the nonlinear Schrödinger approximation result for the KdV equation, cf. [Sch11], beyond the natural time scale by inverse scattering theory, i.e., from a $\mathcal{O}(1/\varepsilon^2)$-time scale to a $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale with $\delta > 0$ arbitrarily small, but fixed.

Further introductory remarks are explained in the following Sections 1.1, 1.2 and 1.3.

1.1 Whitham approximation for a Boussinesq-Klein-Gordon system

In Chapter 2 we justify the Whitham approximation for a dispersive system with conservation law. More precisely, we consider a Boussinesq-Klein-Gordon (BKG) system with unstable resonances, i.e.,

$$\partial_t^2 u = \alpha^2\partial_x^2 u + \partial_t^2\partial_x^2 u + \alpha^2\partial_x^2(a_{uu}u^2 + 2a_{uv}uv + a_{vv}v^2), \tag{1.1}$$
$$\partial_t^2 v = \partial_x^2 v - v + b_{uu}u^2 + 2b_{uv}uv + b_{vv}v^2, \tag{1.2}$$

where $u = u(x,t), v = v(x,t), x, t \in \mathbb{R}$, with coefficients $a_{uu}, \ldots, b_{vv} \in \mathbb{R}$ and $\alpha > 0$.

The BKG system is a toy problem since the solution of the Klein-Gordon equation represents a quantum scalar field and the Boussinesq equation occurs in the context of the water wave problem. However, the BKG system can be used as a prototype model for a whole class of systems that have similar spectral curves. For more information, see Remark 2.1.4 below.

Inserting the ansatz

$$\psi_u^{\mathrm{wh}}(x,t) = A(\varepsilon x, \varepsilon t) \qquad \text{and} \qquad \psi_v^{\mathrm{wh}}(x,t) = B(\varepsilon x, \varepsilon t),$$

with small perturbation parameter $0 < \varepsilon \ll 1$, into (1.1)-(1.2) yields Whitham's modulation equations (WME) for A, namely

$$\partial_T^2 A = \alpha^2\partial_X^2 A + \alpha^2\partial_X^2(a_{uu}A^2 + 2a_{uv}AB^*(A) + a_{vv}(B^*(A))^2),$$

with the amplitudes $A(X,T), B(X,T) \in \mathbb{R}$ depending on the long temporal variable $T = \varepsilon t$ and on the long spatial variable $X = \varepsilon x$. Here, we use that we can express B as a function of A, i.e., $B = B^*(A)$, for A and B small.

For the BKG system in [DKS16] for $\alpha < 2$, i.e., in case of no additional quadratic resonances, exactly when $\omega_1(k) \neq \omega_2(k)$ is satisfied for all $k \in \mathbb{R}$, cf. left panel of Figure

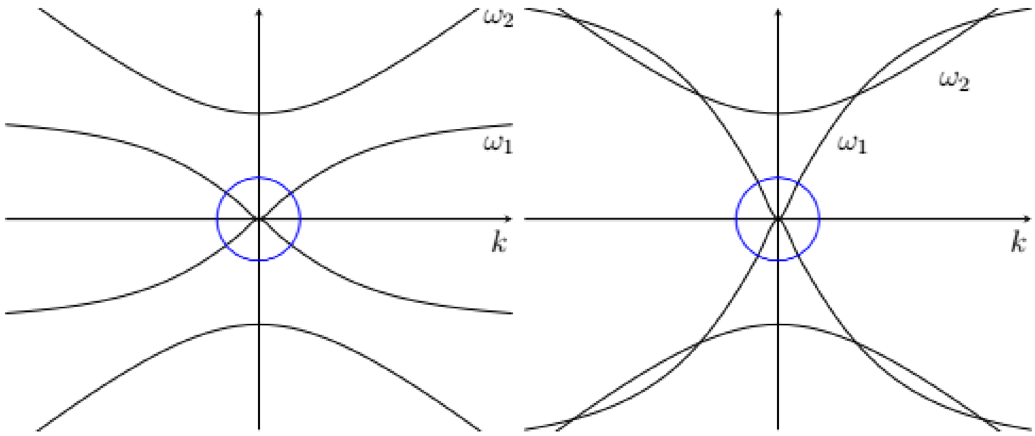

Figure 1.2: The curves of eigenvalues $\pm\omega_1$, $\pm\omega_2$ for the linearized BKG system plotted as a function over the Fourier wave numbers for $\alpha^2 = 1$ (left) and $\alpha^2 = 5$ (right). The modes in the (blue) circles are described by the WME approximation.

1.2, a Whitham approximation result has been established using infinitely many normal form transformations. Hence, it is the purpose of this chapter to cover the Whitham approximation in the case of unstable quadratic resonances, cf. right panel of Figure 1.2, since these unstable resonances were treated for a KdV approximation in [Sch20].

Showing the validity of the Whitham approximation near the trivial solution for the BKG system is a non-trivial task since solutions of order $\mathcal{O}(1)$ have to be bounded over a $\mathcal{O}(1/\varepsilon)$-time scale. It is more complicated to show the justification of a Whitham approximation than the validity of the KdV approximation due to the fact that the smallness of negligible terms in case of Whitham approximation has to be ensured only by derivatives.

In the justification we work in spaces of analytic functions which allows us to control the solutions close to the wave number $k = 0$ by infinitely many normal form transformations and energy estimates. At the other wave numbers where the resonances could lead to a growth the solutions are controlled only by working in spaces of analytic functions in a strip in the complex plane. An artificial damping is obtained by making the strip smaller in time. This procedure leads to a restriction of the time scale.

An advantage of this chapter is to provide a foundation for Chapter 3 such that we can use infinitely many normal form transformations without a detailed discussion.

1.2 Whitham approximation for a complex cubic Klein-Gordon equation

Whitham's modulation equations (WME) can be derived by a multiple scaling perturbation analysis in order to describe slow modulations in time and space of traveling wave solutions. So far there are only few approximation results showing that WME approximations make correct predictions about the dynamics near periodic traveling wave solutions in dispersive and dissipative systems. In [DS09] a validity result is shown for such waves of the nonlinear Schrödinger (NLS) equation

$$\partial_\tau A = \mathrm{i}\nu_1 \partial_\xi^2 A + \mathrm{i}\nu_2 A|A|^2,$$

with $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$, $A(\xi,\tau) \in \mathbb{C}$ and coefficients ν_1, $\nu_2 \in \mathbb{R}$, as original system. For more examples, we refer to Remark 3.1.1. We want to take a next step in the direction of handling modulations of periodic wave trains for general dispersive systems. As a relevant further step, we consider the complex cubic Klein-Gordon (ccKG) equation

$$\partial_t^2 u = \partial_x^2 u - u + \gamma u\,|u|^2\,, \tag{1.3}$$

with $t, x \in \mathbb{R}$, $\gamma \in \{-1, 1\}$ and $u(x,t) \in \mathbb{C}$, which has a family of periodic traveling wave solutions

$$u\,(x,t) = \mathrm{e}^{r_{q,\mu} + \mathrm{i}qx + \mathrm{i}\mu t},$$

where $\mu, q, r_{q,\mu} \in \mathbb{R}$ satisfy

$$-\gamma \mathrm{e}^{2r_{q,\mu}} = \mu^2 - q^2 - 1.$$

The analysis for the ccKG equation is more complicated than the analysis for the NLS equation because there are two additional curves of eigenvalues which lead to an additional oscillatory behavior, see Figure 1.3.

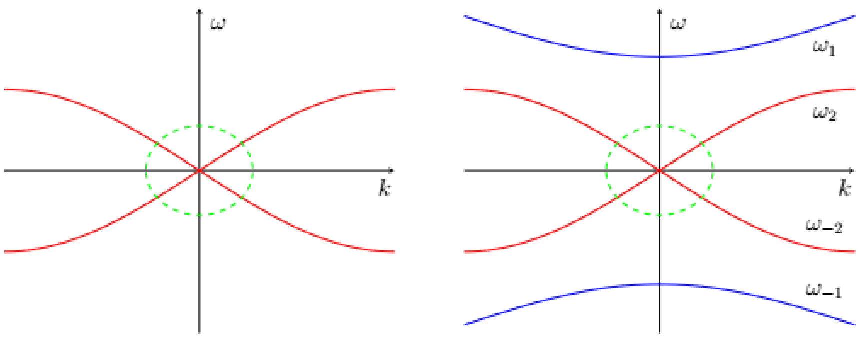

Figure 1.3: The left panel shows the spectral curves (in red) as functions ω over the Fourier wave numbers k for the NLS equation. The right panel shows the spectral curves for the ccKG equation (1.3) with the two additional spectral curves (in blue). WME describe the modes in the small dashed circles (in green).

By introducing polar coordinates

$$u = \mathrm{e}^{r + \mathrm{i}\varphi + r_{0,\mu} + \mathrm{i}\mu t},$$

with $r = r(x,t)$ and $\varphi = \varphi(x,t)$, separating real and imaginary parts and introducing the local temporal and local spatial wave number

$$\vartheta = \partial_t \varphi \qquad \text{and} \qquad \psi = \partial_x \varphi,$$

we can derive WME from (1.3) with the long wave ansatz

$$(r, \psi, \vartheta)(x, t) = (\check{r}, \check{\psi}, \check{\vartheta})(\delta x, \delta t) = (\check{r}, \check{\psi}, \check{\vartheta})(X, T),$$

with $X = \delta x$, $T = \delta t$ and a small perturbation parameter $0 < \delta \ll 1$ instead of ε as in Section 1.1 due to historical reasons. Further details can be found in Remark 3.1.6 below.

In Chapter 3 we prove the justification of the Whitham approximation near a periodic traveling wave solution for the ccKG equation. Such an approximation result is non-trivial since solutions of order $\mathcal{O}(1)$ must be bounded on a long $\mathcal{O}(1/\delta)$-time scale. In general, solutions of order $\mathcal{O}(1)$ are only bounded on a $\mathcal{O}(1)$-time scale. While the spectral curves in the left panel of Figure 1.3 look similar to those of the toy problem in Chapter 2, cf. Figure 1.2, it is more complicated to show the validity of the Whitham approximation for the relevant ccKG equation as in Chapter 2. Some new aspects and problems arise, for instance the diagonalization of the system can only be handled locally which is relevant for the use of normal form transformations. Another difference is that in Chapter 2 we consider a Whitham approximation around the trivial solution. Whereas in Chapter 3 the Whitham approximation is studied to describe slow modulations in time and space of a periodic traveling wave.

The content of this chapter is a joint work with Xian Liao and Guido Schneider and an earlier version of this chapter has already been published as a preprint in [HLS22]. It is the plan of future research to handle modulations of periodic wave trains for the real cubic Klein-Gordon equation, i.e., we want to use a cubic nonlinearity u^3 instead of $u|u|^2$ in (1.3). In this case we have no $\mathbb{S}^1$-symmetry which means that if u is a solution, then $u e^{i\phi}$, with $\phi \in \mathbb{R}$, is also a solution. The $\mathbb{S}^1$-symmetry allows us to focus on a neighborhood of $k = 0$ which would not be possible in general cases.

1.3 A linear Schrödinger approximation for the Korteweg-de Vries equation

There exist various approximation results where the nonlinear Schrödinger (NLS) equation

$$\partial_\tau A = i\nu_1 \partial_\xi^2 A + i\nu_2 A|A|^2,$$

with $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$, $A(\xi, \tau) \in \mathbb{C}$ and coefficients ν_1, $\nu_2 \in \mathbb{R}$, describes slow modulations in time and space of oscillating wave packets in dispersive wave systems for the natural time scale of the NLS approximation, see [Dül21] for a recent overview. We are interested in improving these validity results for the NLS approximation beyond the natural time scale of the NLS approximation for completely integrable systems. We have succeeded in showing a Schrödinger approximation result for the Korteweg-de Vries equation (KdV)

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0,$$

beyond the natural time scale of the Schrödinger approximation using inverse scattering transform (IST) in Chapter 4.

As a first step, we restrict ourselves to initial conditions for which the scattering data contains no eigenvalues. Thus, the solution of the KdV equation is completely determined by the scattering variable b associated to the essential spectrum. We perform an NLS approximation for this b. Due to the fact that b satisfies a linear Airy equation, the approximated equation becomes a linear Schrödinger equation. This is the main reason why we can extend the natural NLS time scale. We only do an error for a linear problem, i.e., the error is small on a time scale of order $\mathcal{O}(1/\varepsilon^{3-\delta})$ with $\delta > 0$ arbitrarily small, but fixed, instead of a $\mathcal{O}(1/\varepsilon^2)$-time scale which corresponds to the natural NLS time scale. Using inverse scattering theory, we can transfer these estimates to the original solution of the KdV equation.

For completely integrable systems one would expect our better approximation. These systems have by definition the property that there is a representation in terms of action and angle variables. The action variables are conserved, while the angle variables satisfy a linear partial differential equation. Hence, we can approximate the frequency of the angle variables up to order $\mathcal{O}(\varepsilon^2)$, i.e., with an error of order $\mathcal{O}(\varepsilon^3)$. This error increases with a rate of order $\mathcal{O}(\varepsilon^3)t$ which means the error is of order $\mathcal{O}(\varepsilon^\delta)$ over a $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale.

It is the plan of future research to transfer these ideas to other completely integrable systems, such as the NLS equation, the derivative NLS equation or the Sine-Gordon equation. We hope that it will be possible to describe the interaction of NLS scaled wave packets for completely integrable systems with this presented theory.

For the convenience of the reader we keep the chapters self-contained and introduce the setting at the beginning of each chapter.

Chapter 2

The Whitham approximation for a Boussinesq-Klein-Gordon system with unstable resonances

Whitham's modulation equations can be derived via multiple scaling analysis for the approximate description of long waves in dispersive systems with a conservation law. In this chapter we show the validity of Whitham's modulation equations for a Boussinesq-Klein-Gordon system with unstable resonances. To handle more complicated systems without a detailed discussion of resonances and without finding a suitable energy, we work in spaces of analytic functions. The proof is based on an infinite series of normal form transformations and energy estimates. The whole chapter is inspired by [Sch20] where the validity of the KdV approximation for the Boussinesq-Klein-Gordon equation with unstable quadratic resonances is shown. In our case it is more complicated due to the fact that the smallness of negligible terms has to be ensured only by derivatives. All sections in this chapter except Section 2.6 are an adaption of the sections in [Sch20]. The main difference is a change in scaling to obtain a Whitham approximation instead of a KdV approximation. This leads to infinitely many normal form transformations which we perform in Section 2.6 in spaces of analytic functions.

2.1 Introduction

We consider the Boussinesq-Klein-Gordon (BKG) system

$$\partial_t^2 u = \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \alpha^2 \partial_x^2 (a_{uu} u^2 + 2a_{uv} uv + a_{vv} v^2), \tag{2.1}$$

$$\partial_t^2 v = \partial_x^2 v - v + b_{uu} u^2 + 2b_{uv} uv + b_{vv} v^2, \tag{2.2}$$

where $u = u(x,t), v = v(x,t), x, t \in \mathbb{R}$, with coefficients $a_{uu}, \ldots, b_{vv} \in \mathbb{R}$ and $\alpha > 0$. For notational simplicity we prepend the constant α^2 to the corresponding terms. We insert the ansatz

$$\psi_u^{\mathrm{wh}}(x,t) = A(\varepsilon x, \varepsilon t) \qquad \text{and} \qquad \psi_v^{\mathrm{wh}}(x,t) = B(\varepsilon x, \varepsilon t), \tag{2.3}$$

with small perturbation parameter $0 < \varepsilon \ll 1$, into (2.1)-(2.2) and get

$$\begin{aligned} \varepsilon^2 \partial_T^2 A &= \alpha^2 \varepsilon^2 \partial_X^2 A + \varepsilon^4 \partial_T^2 \partial_X^2 A + \alpha^2 \varepsilon^2 \partial_X^2 (a_{uu} A^2 + 2a_{uv} AB + a_{vv} B^2), \\ \varepsilon^2 \partial_T^2 B &= \varepsilon^2 \partial_X^2 B - B + b_{uu} A^2 + 2b_{uv} AB + b_{vv} B^2, \end{aligned}$$

respectively in lowest order

$$\partial_T^2 A = \alpha^2 \partial_X^2 A + \alpha^2 \partial_X^2 (a_{uu} A^2 + 2a_{uv} AB + a_{vv} B^2), \tag{2.4}$$

$$0 = -B + b_{uu} A^2 + 2b_{uv} AB + b_{vv} B^2, \tag{2.5}$$

with the long temporal variable $T = \varepsilon t$, the long spatial variable $X = \varepsilon x$ and the amplitudes $A(X, T) \in \mathbb{R}$ and $B(X, T) \in \mathbb{R}$. For small amplitudes A and B the second equation (2.5) can be solved w.r.t. B, i.e., there exists a solution $B = B^*(A)$. Inserting this into the first equation (2.4) we obtain Whitham's modulation equations (WME) for A, precisely

$$\partial_T^2 A = \alpha^2 \partial_X^2 A + \alpha^2 \partial_X^2 (a_{uu} A^2 + 2a_{uv} AB^*(A) + a_{vv} (B^*(A))^2). \tag{2.6}$$

In the following Theorem 2.1.1, our main theorem of this chapter, we state the validity of the WME approximation for the BKG system (2.1)-(2.2) in case of unstable resonances. These resonances occur for $\alpha > 2$, cf. Remark 2.1.3.

Theorem 2.1.1. *For $T_0 > 0$, $C_0 > 0$, $\sigma_A > 0$, $s_A - s \geq 4$, $s \geq 1$ fixed, there exists a C_w such that the following holds. Let A be a solution of WME (2.6) with*

$$\sup_{T \in [0, T_0]} \int_{\mathbb{R}} |\widehat{A}(K, T)| \mathrm{e}^{\sigma_A |K|} (1 + K^2)^{\frac{s_A}{2}} \, \mathrm{d}K \leq C_w, \tag{2.7}$$

and let B be the corresponding solution to the algebraic equation following from (2.5). Then, there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$ and $C_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all initial conditions of (2.1)-(2.2) with

$$\|(u, v)(\cdot, 0) - (\psi_u^{\mathrm{wh}}(\cdot, 0), \psi_v^{\mathrm{wh}}(\cdot, 0))\|_{G^s_{\sigma_A}} \leq C_0 \varepsilon^{3/2},$$

the associated solutions satisfy

$$\sup_{t \in [0, T_1/\varepsilon]} \|(u, v)(\cdot, t) - (\psi_u^{\mathrm{wh}}(\cdot, t), \psi_v^{\mathrm{wh}}(\cdot, t))\|_{H^s} \leq C_1 \varepsilon^{3/2}, \tag{2.8}$$

where the norm $\|\cdot\|_{G^s_{\sigma_A}}$ is defined subsequently in (2.17).

Remark 2.1.2. Solutions of order $\mathcal{O}(1)$ have to be controlled on a $\mathcal{O}(1/\varepsilon)$-time scale of the WME approximation. Therefore, such an approximation result is non-trivial due to the fact that solutions of order $\mathcal{O}(1)$ are in general only bounded on a $\mathcal{O}(1)$-time scale. With the help of estimate (2.8) and Sobolev's embedding theorem we obtain

$$\sup_{t \in [0, T_1/\varepsilon]} \sup_{x \in \mathbb{R}} |(u, v)(x, t) - (\psi_u^{\mathrm{wh}}(x, t), \psi_v^{\mathrm{wh}}(x, t))| \leq C_1 \varepsilon^{3/2}.$$

Remark 2.1.3. The linearization of (2.1)-(2.2) is solved by

$$u(x,t) = \mathrm{e}^{\mathrm{i}kx \pm \mathrm{i}\omega_1(k)t}, \qquad v(x,t) = \mathrm{e}^{\mathrm{i}kx \pm \mathrm{i}\omega_2(k)t},$$

with

$$\omega_1(k) = \frac{\alpha k}{\sqrt{1+k^2}}, \qquad \omega_2(k) = \sqrt{1+k^2}. \tag{2.9}$$

If we switch to the Fourier space, WME describe the behavior of the modes in the u-equation which are strongly concentrated around the wave number $k = 0$, cf. Figure 2.1. For this reason, the expansion $\omega_1(k) = \alpha k + \mathcal{O}(k^3)$ at $k = 0$ is important for the dynamics.

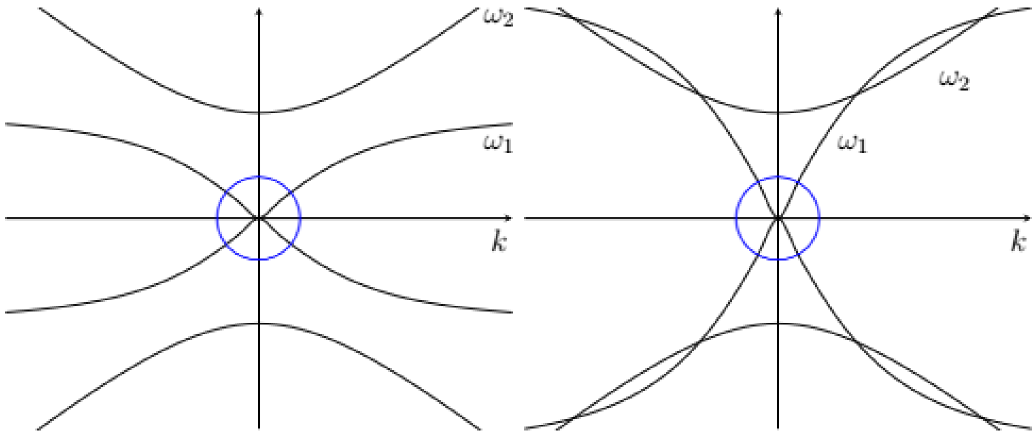

Figure 2.1: The curves of eigenvalues $\pm\omega_1$, $\pm\omega_2$ for the linearized BKG system plotted as a function over the Fourier wave numbers in case $\alpha^2 = 1$ (left) and $\alpha^2 = 5$ (right). The modes in the (blue) circles are described by the WME approximation.

Remark 2.1.4. The BKG system is a toy problem for many systems with similar spectra, such as the poly-atomic FPU problem and the water wave problem over a periodic bottom with $\mathcal{O}(1)$-periodicity and bottom variations of order $\mathcal{O}(1)$. The Boussinesq equation occurs in the context of the water wave problem and the solution of the Klein-Gordon equation can represent a quantum scalar field. The BKG system also looks less complicated than the water wave problem. Nevertheless, for the WME approximation of the BKG system some new difficulties arise which could appear for the water wave problem over a periodic bottom, namely the occurrence of quadratic resonances. The linearized water wave problem over a periodic bottom, which is solved by Bloch modes, has been analyzed in [CGLS18]. It will be a topic of future research to transfer the following analysis to more general systems. The main advantage of the approach in this chapter is to be able to handle more complicated systems without a detailed discussion of resonances and without finding an appropriate energy which would be different for each different system. In Chapter 3 we will use the techniques of this Chapter 2 without going into too much detail in the associated part. A Whitham approximation has so far only been verified for systems with a single pair of curves of eigenvalues $\pm i\omega_1$, with very few exceptions, cf. [CS11, CBCPS12, GMWZ14, BDS19]. In Chapter 3 we also consider two pairs of curves of eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$ for the relevant physical case of the complex cubic Klein-Gordon equation.

Remark 2.1.5. For $\alpha < 2$ there are no additional quadratic resonances in the BKG system which means that $\omega_1(k) \neq \omega_2(k)$ is satisfied for all $k \in \mathbb{R}$. For this case, a validity result was established in [DKS16] for a Whitham approximation using infinitely many normal form transformations in Sobolev spaces. In case of stable quadratic resonances, in [BCS19] it was noted how to introduce an approximation result for all $\alpha \geq 2$, based on [BDS19]. Thus, the purpose of this chapter is to cover the case of unstable quadratic resonances for the Whitham approximation. The KdV approximation of the BKG system is shown in [Sch20]. In our case, the main difficulty is that the smallness of the negligible terms can only be achieved by derivatives. Therefore, an infinite number of normal form transformations must be performed to eliminate the non-resonant nonlinear terms that cannot be included in the energy estimates.

Remark 2.1.6. The plan of proving Theorem 2.1.1 is as follows. We control the solutions close to the wave number $k = 0$ by infinitely many near identity changes of variables and energy estimates. By working in spaces of analytic functions, we can control the solutions outside of a neighborhood of $k = 0$. Functions in spaces of analytic functions correspond to functions which are analytic in a strip around the real axis of width $2\sigma_A$ in the complex plane. In Fourier space, this leads to functions which decay as $e^{-\sigma_A|K|}$ for $|K| \to \infty$, cf. assumption (2.7) in Theorem 2.1.1. For more details see [RS75, Theorem IX.13]. The strip becomes smaller and smaller in time which leads to an artificial damping of the modes $k = \varepsilon K > 0$ under the restriction of time $T_1 \in [0, T_0]$, cf. (2.8).

The plan of the chapter is based on the order of [Sch20] and is as follows. In the next section we ensure that the error resulting from inserting the WME approximation (2.3) into the BKG system (2.1)-(2.2) can be made arbitrarily small. After deriving the error equations in Section 2.3, we introduce the corresponding spaces of analytic functions in Section 2.4. We need these spaces to control the unstable quadratic resonances. In Section 2.5 we estimate the error that we can control in Section 2.2 in the newly introduced spaces. This error can also be made arbitrarily small. To show the validity of the theorem, we need a $\mathcal{O}(1)$-bound of the error on the long $\mathcal{O}(1/\varepsilon)$-time scale. We achieve this bound outside of a neighborhood of the wave number $k = 0$ using the spaces of analytic functions. In this neighborhood of $k = 0$, we use infinitely many normal form transformations in Section 2.6 to eliminate terms that we cannot include in the energy estimates. These near identity changes of variables are performed in spaces of analytic functions. Then, in Section 2.7 we introduce a transformation from the spaces of analytic functions to Sobolev spaces in order to conclude the proof with energy estimates in Sobolev spaces in Section 2.8. In the appendix in Section 2.9 we collect a few estimates that we have used in the proof.

Notation. The Fourier transform of a function u is denoted by $\mathcal{F}u$ or $\widehat{u}$. Possibly different constants which can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$ are denoted by the same symbol C. In the following $\int_{\mathbb{R}}$ can be abbreviated as $\int$.

2.2 Derivation of the WME approximation

In this section we ensure that the error which we make by the WME approximation is sufficiently small. Precisely, we ensure the smallness of the residual terms, i.e., the terms that do not cancel after inserting the WME approximation (2.3) into the BKG system (2.1)-(2.2). By improving the ansatz by adding higher order terms to the previous WME approximation we achieve that the residual can be made smaller.

Inserting the ansatz

$$\psi_u^{\mathrm{wh}}(x,t) = A(\varepsilon x, \varepsilon t) \qquad \text{and} \qquad \psi_v^{\mathrm{wh}}(x,t) = B(\varepsilon x, \varepsilon t), \tag{2.10}$$

into the BKG system (2.1)-(2.2) gives for

$$\begin{aligned} \mathrm{Res}_u(u,v) &= -\partial_t^2 u + \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \alpha^2 \partial_x^2 (a_{uu} u^2 + 2a_{uv} uv + a_{vv} v^2), \\ \mathrm{Res}_v(u,v) &= -\partial_t^2 v + \partial_x^2 v - v + b_{uu} u^2 + 2b_{uv} uv + b_{vv} v^2, \end{aligned}$$

that

$$\begin{aligned} \mathrm{Res}_u(\psi_u^{\mathrm{wh}}, \psi_v^{\mathrm{wh}}) &= \varepsilon^4 \partial_T^2 \partial_X^2 A, \\ \mathrm{Res}_v(\psi_u^{\mathrm{wh}}, \psi_v^{\mathrm{wh}}) &= \varepsilon^2 (-\partial_T^2 B + \partial_X^2 B), \end{aligned}$$

if we choose A and B to satisfy (2.4) and (2.5). By adding higher order terms to the WME approximation we improve the WME ansatz and achieve that the residual terms are sufficiently small, i.e., the residual terms must satisfy $\mathrm{Res}_u(\psi_u^{\mathrm{wh}}, \psi_v^{\mathrm{wh}}) = \mathcal{O}(\varepsilon^4)$ and $\mathrm{Res}_v(\psi_u^{\mathrm{wh}}, \psi_v^{\mathrm{wh}}) = \mathcal{O}(\varepsilon^4)$. To ensure this smallness, we extend our approximation (2.10) in a canonical way to

$$\psi_u(x,t) = A(\varepsilon x, \varepsilon t), \quad \text{and} \quad \psi_v(x,t) = B(\varepsilon x, \varepsilon t) + \varepsilon^2 B_2(\varepsilon x, \varepsilon t). \tag{2.11}$$

This improved approximation provides

$$\begin{aligned} \mathrm{Res}_u(\psi_u, \psi_v) &= \mathcal{O}(\varepsilon^4), \\ \mathrm{Res}_v(\psi_u, \psi_v) &= \varepsilon^2(-\partial_T^2 B + \partial_X^2 B) + \varepsilon^2(-B_2 + 2b_{uv} A B_2 + 2b_{vv} B B_2) + \mathcal{O}(\varepsilon^4) \\ &= \mathcal{O}(\varepsilon^4), \end{aligned}$$

if we choose B_2 to satisfy

$$-\partial_T^2 B + \partial_X^2 B - B_2 + 2b_{uv} A B_2 + 2b_{vv} B B_2 = 0.$$

For A, B sufficiently small there exists a unique solution $B_2 = B_2(A, B)$.

If we estimate the residual terms in L^2-based spaces, we lose a factor $\varepsilon^{-1/2}$ due to the scaling properties of the L^2-norm, i.e.,

$$\left(\int_{\mathbb{R}} |u(\varepsilon x)|^2 \mathrm{d}x\right)^{1/2} = \varepsilon^{-1/2} \left(\int_{\mathbb{R}} |u(X)|^2 \mathrm{d}X\right)^{1/2}.$$

Therefore, we have the following lemma.

Lemma 2.2.1. *For $s_A - s \geq 4$, $s > 1/2$ and $T_0 > 0$ fixed, let $A \in C([0, T_0], H^{s_A})$ be a solution of the WME (2.6) and ψ_u and ψ_v be defined as above. Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{T\in[0,T_0]} \|\mathrm{Res}_u\|_{H^s} \leq C\varepsilon^{7/2} \qquad \textit{and} \qquad \sup_{T\in[0,T_0]} \|\mathrm{Res}_v\|_{H^s} \leq C\varepsilon^{7/2}.$$

Proof. It is straightforward to count the powers of ε . We still need to clarify the assumption $s_A - s \geq 4$. The term $\partial_T^2 B_2$ loses the most regularity since both A and B are sufficiently small. Furthermore, B can be expressed in terms of A and A^2. This term depends on $\partial_T^4 B$ and $\partial_T^2\partial_X^2 B$. So we analyze $\partial_T^2\partial_X^2(A^2)$. We have that $\partial_T^2 A$ depends on $A, \ldots, \partial_X^2 A$ due to the right-hand side of the WME (2.4). By differentiating the WME equation (2.4) twice w.r.t. T, we obtain that $\partial_T^4(A^2)$ can be expressed in terms of $A, \ldots, \partial_X^4 A$. Hence, $\partial_T^2\partial_X^2 B$ can be expressed in terms of $A, \ldots, \partial_X^4 A$. □

If we write the error equations obtained from the BKG system (2.1)-(2.2) as a first order system, the term $\partial_x^{-1}\mathrm{Res}_u$ appears. Therefore, we must also control this expression, which gives us the following lemma.

Lemma 2.2.2. *Under the assumptions of Lemma 2.2.1 we have the estimate*

$$\sup_{T\in[0,T_0]} \left(\|\partial_x^{-1}\mathrm{Res}_u\|_{H^{s+1}}\right) \leq C\varepsilon^{5/2}.$$

Proof. Due to the scaling property of $\partial_x^{-1} = \varepsilon^{-1}\partial_X^{-1}$ we lose a power of ε. We need to show that $\partial_X^{-1}\mathrm{Res}_u$ is again in L^2. This is obvious since all terms have a derivative ∂_X in front of them. Therefore, we are done. □

2.3 The equations for the error

To establish the validity of Theorem 2.1.1, we have to prove a $\mathcal{O}(1)$-bound for error functions $(\varepsilon^{3/2}R_u, \varepsilon^{3/2}R_v)$ which are defined by the difference between a true solution and the improved approximation on a $\mathcal{O}(1/\varepsilon)$-time scale.

Remark 2.3.1. In the following we estimate the difference between a true solution of (2.1)-(2.2) and the improved approximation (2.11). The estimate between the true solution of (2.1)-(2.2) and the original approximation (2.3) follows by the triangle inequality and

$$\sup_{t\in[0,T_0/\varepsilon]} \sup_{x\in\mathbb{R}} |(\psi_u, \psi_v)(x,t) - (\psi_u^{wh}, \psi_v^{wh})(x,t)| \leq C\varepsilon^2.$$

The error functions $(\varepsilon^{3/2}R_u, \varepsilon^{3/2}R_v)$ are defined by the difference between a true solution and the improved approximation, more precisely

$$u = \psi_u + \varepsilon^{3/2}R_u, \qquad v = \psi_v + \varepsilon^{3/2}R_v,$$

and satisfy

$$\begin{aligned}\partial_t^2 R_u &= \alpha^2\partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + 2\alpha^2\partial_x^2 L_u R + \varepsilon^{3/2}\alpha^2\partial_x^2 f_u,\\ \partial_t^2 R_v &= \partial_x^2 R_v - R_v + 2L_v R + \varepsilon^{3/2} f_v,\end{aligned}$$

where

$$\begin{aligned}L_u R &= a_{uu}\psi_u R_u + a_{uv}\psi_u R_v + a_{uv}\psi_v R_u + a_{vv}\psi_v R_v,\\ L_v R &= b_{uu}\psi_u R_u + b_{uv}\psi_u R_v + b_{uv}\psi_v R_u + b_{vv}\psi_v R_v,\\ \varepsilon^{3/2}\partial_x^2 f_u &= \varepsilon^{3/2}\partial_x^2(a_{uu}R_u^2 + 2a_{uv}R_uR_v + a_{vv}R_v^2) + \varepsilon^{-3/2}\mathrm{Res}_u,\\ \varepsilon^{3/2} f_v &= \varepsilon^{3/2}(b_{uu}R_u^2 + 2b_{uv}R_uR_v + b_{vv}R_v^2) + \varepsilon^{-3/2}\mathrm{Res}_v.\end{aligned}$$

We write the system as a first order system

$$\begin{aligned}\partial_t R_u &= \mathrm{i}\omega_1 \widetilde{R}_u,\\ \partial_t \widetilde{R}_u &= \mathrm{i}\omega_1 R_u + 2\mathrm{i}\omega_1 L_u R + \varepsilon^{3/2}\mathrm{i}\omega_1 f_u,\\ \partial_t R_v &= \mathrm{i}\omega_2 \widetilde{R}_v,\\ \partial_t \widetilde{R}_v &= \mathrm{i}\omega_2 R_v + 2(\mathrm{i}\omega_2)^{-1} L_v R + \varepsilon^{3/2}(\mathrm{i}\omega_2)^{-1} f_v,\end{aligned}$$

with ω_1, ω_2 from (2.9). We keep the scaling although $\varepsilon^{-3/2}(\mathrm{i}\omega_2)^{-1}\mathrm{Res}_v$ in $\varepsilon^{3/2}(\mathrm{i}\omega_2)^{-1}f_v$ is of order $\mathcal{O}(\varepsilon^2)$ and $\varepsilon^{-3/2}\mathrm{i}\omega_1\partial_x^{-2}\mathrm{Res}_u$ in $\varepsilon^{3/2}\mathrm{i}\omega_1 f_u$ is of order $\mathcal{O}(\varepsilon)$. Through

$$\mathcal{R}_1 = \frac{1}{\sqrt{2}}(R_u + \widetilde{R}_u), \qquad \mathcal{R}_{-1} = \frac{1}{\sqrt{2}}(R_u - \widetilde{R}_u),$$

and

$$\mathcal{R}_2 = \frac{1}{\sqrt{2}}(R_v + \widetilde{R}_v), \qquad \mathcal{R}_{-2} = \frac{1}{\sqrt{2}}(R_v - \widetilde{R}_v),$$

we diagonalize the linear part and we get

$$\partial_t \mathcal{R}_1 = \mathrm{i}\omega_1\mathcal{R}_1 + \mathrm{i}\omega_1\mathcal{L}_1\mathcal{R} + \frac{1}{\sqrt{2}}\varepsilon^{3/2}\mathrm{i}\omega_1 f_u, \tag{2.12}$$

$$\partial_t \mathcal{R}_2 = \mathrm{i}\omega_2\mathcal{R}_2 + (\mathrm{i}\omega_2)^{-1}\mathcal{L}_2\mathcal{R} + \frac{1}{\sqrt{2}}\varepsilon^{3/2}(\mathrm{i}\omega_2)^{-1} f_v, \tag{2.13}$$

and similarly for $\mathcal{R}_{-1}$ and $\mathcal{R}_{-2}$, with

$$\begin{aligned}\mathcal{L}_1\mathcal{R} &= a_{uu}\psi_u(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{uv}\psi_u(\mathcal{R}_2 + \mathcal{R}_{-2})\\ &\quad + a_{uv}\psi_v(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{vv}\psi_v(\mathcal{R}_2 + \mathcal{R}_{-2}),\\ \mathcal{L}_2\mathcal{R} &= b_{uu}\psi_u(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{uv}\psi_u(\mathcal{R}_2 + \mathcal{R}_{-2})\\ &\quad + b_{uv}\psi_v(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{vv}\psi_v(\mathcal{R}_2 + \mathcal{R}_{-2}).\end{aligned}$$

Since ψ_u and ψ_v are strongly concentrated at $k = 0$, we separate ψ_u and ψ_v into a part concentrated close to $k = 0$ and into the rest. We introduce the mode projection E_δ by $\widehat{E_\delta u} = \widehat{E_\delta}\widehat{u}$, with

$$\widehat{E_\delta}(k) = \begin{cases} 1, & |k| \leq \delta, \\ 0, & |k| > \delta. \end{cases} \tag{2.14}$$

Thus, E_δ^c is defined by $\widehat{E_\delta^c}(k) = 1 - \widehat{E_\delta}(k)$. Since the operator $\widehat{E_\delta^c}(k)$ vanishes in a δ-neighborhood of $k = 0$, we have the estimate $|\widehat{E_\delta^c}(k)| \leq C\varepsilon^\alpha$, for all $\alpha \in \mathbb{R}$, in that δ-neighborhood of $k = 0$. Due to this fact, we have that $E_\delta^c\psi_u$ and $E_\delta^c\psi_v$ is $\mathcal{O}(\varepsilon^{s_A})$-small, for instance w.r.t. the sup-norm, if A is s_A-times continuously differentiable, see Corollary 2.9.5 and Remark 2.9.6 in Section 2.9. We apply this mode projection to the associated terms in (2.12)-(2.13) and obtain the error equations

$$\partial_t \mathcal{R}_1 = \mathrm{i}\omega_1 \mathcal{R}_1 + \mathrm{i}\omega_1 \mathcal{L}_{1,\delta}\mathcal{R} + \frac{1}{\sqrt{2}}\varepsilon^{3/2}\mathrm{i}\omega_1 g_u, \tag{2.15}$$

$$\partial_t \mathcal{R}_2 = \mathrm{i}\omega_2 \mathcal{R}_2 + (\mathrm{i}\omega_2)^{-1} \mathcal{L}_{2,\delta}\mathcal{R} + \frac{1}{\sqrt{2}}\varepsilon^{3/2}(\mathrm{i}\omega_2)^{-1} g_v, \tag{2.16}$$

and similarly for $\mathcal{R}_{-1}$ and $\mathcal{R}_{-2}$, with

$$\begin{aligned} \mathcal{L}_{1,\delta}\mathcal{R} &= a_{uu}(E_\delta\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{uv}(E_\delta\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2}) \\ &\quad + a_{uv}(E_\delta\psi_v)(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{vv}(E_\delta\psi_v)(\mathcal{R}_2 + \mathcal{R}_{-2}), \\ \mathcal{L}_{2,\delta}\mathcal{R} &= b_{uu}(E_\delta\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{uv}(E_\delta\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2}) \\ &\quad + b_{uv}(E_\delta\psi_v)(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{vv}(E_\delta\psi_v)(\mathcal{R}_2 + \mathcal{R}_{-2}), \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{3/2} g_u &= \varepsilon^{3/2} f_u + \sqrt{2}(a_{uu}(E_\delta^c\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{uv}(E_\delta^c\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2}) \\ &\quad + a_{uv}(E_\delta\psi_v^c)(\mathcal{R}_1 + \mathcal{R}_{-1}) + a_{vv}(E_\delta^c\psi_v)(\mathcal{R}_2 + \mathcal{R}_{-2})), \\ \varepsilon^{3/2} g_v &= \varepsilon^{3/2} f_v + \sqrt{2}(b_{uu}(E_\delta^c\psi_u)(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{uv}(E_\delta^c\psi_u)(\mathcal{R}_2 + \mathcal{R}_{-2}) \\ &\quad + b_{uv}(E_\delta\psi_v^c)(\mathcal{R}_1 + \mathcal{R}_{-1}) + b_{vv}(E_\delta^c\psi_v)(\mathcal{R}_2 + \mathcal{R}_{-2})). \end{aligned}$$

2.4 The functional analytic set-up

We introduce the following notation for different function spaces to control the unstable quadratic resonances. The Euclidean inner product is denoted by $(\cdot, \cdot)$ and the associated Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$. We define the Fourier transform by

$$\mathcal{F}(u)(k) = \widehat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}kx} u(x) \mathrm{d}x.$$

For $m \geq 0$ the Sobolev spaces are denoted by

$$H^m = \{u \in L^2(\mathbb{R}) : (1 + |\cdot|^2)^{\frac{m}{2}} \widehat{u}(\cdot) \in L^2(\mathbb{R})\},$$

equipped with the inner product

$$(u,v)_{H^m} = (\widehat{u},\widehat{v})_{L^2_m} = \int_{\mathbb{R}} \left(1+|k|^2\right)^m (\widehat{u}(k),\widehat{v}(k))\,\mathrm{d}k.$$

The induced norm

$$\|u\|_{H^m} = \left(\int_{\mathbb{R}} \left(1+|k|^2\right)^m |\widehat{u}(k)|^2 \mathrm{d}k\right)^{\frac{1}{2}},$$

is equivalent to the usual H^m-norm

$$\|u\|_{H^m} = \left(\sum_{j=0}^{m} \|D^j u\|^2_{L^2}\right)^{\frac{1}{2}},$$

for any $m \in \mathbb{N}$. Finally, we introduce

$$W^m := \left\{ u : u = \mathcal{F}^{-1}(\widehat{u}), \widehat{u} \in L^1(\mathbb{R}), \|u\|_{W^m} = \int_{\mathbb{R}} (1+|k|^m)|\widehat{u}(k)|\,\mathrm{d}k < \infty \right\},$$

for $m \geq 0$. The space $H^{m+\delta}(\mathbb{R})$ is continuously embedded in W^m for any $\delta > 1/2$ by Sobolev's embedding theorem. Moreover, every $u \in W^m$ is $\lfloor m \rfloor$-times continuously differentiable with finite $C_b^{\lfloor m \rfloor}(\mathbb{R})$-norm. The unstable quadratic resonances lead to positive growth rates, cf. Remark 2.1.3, which we have to control. In order to do so, we introduce

$$G^m_\sigma = \{u \in L^2(\mathbb{R}) : \mathrm{e}^{\sigma|\cdot|}(1+|\cdot|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbb{R})\},$$

equipped with the norm

$$\|u\|_{G^m_\sigma} = \left(\int_{\mathbb{R}} |\widehat{u}(k)|^2 \mathrm{e}^{2\sigma|k|}(1+|k|^2)^m \mathrm{d}k\right)^{\frac{1}{2}}, \tag{2.17}$$

where $\sigma \geq 0$ and $m \geq 0$. For $m = 0$, these functions correspond to functions which are analytic in a strip around the real axis in the complex plane $\{z \in \mathbb{C} \colon |\mathrm{Im}\,(z)| < \sigma\}$, cf. [RS75, Theorem IX.13]. Making this strip smaller in time leads to an artificial damping which controls the growth rates and to the restriction on the time scale T_1 instead of T_0. The spaces $\mathcal{W}^m_\sigma$ are defined by

$$\mathcal{W}^m_\sigma = \{u \in C^0_b : \|u\|_{\mathcal{W}^m_\sigma} < \infty\},$$

equipped with the norm

$$\|u\|_{\mathcal{W}^m_\sigma} = \int_{\mathbb{R}} \mathrm{e}^{\sigma|k|}(1+|k|^2)^{m/2}|\widehat{u}(k)|\mathrm{d}k.$$

The spaces G^m_σ are an algebra for every $\sigma \geq 0$ and $m > 1/2$, and the spaces $\mathcal{W}^m_\sigma$ are an algebra for every $\sigma \geq 0$ and $m \geq 0$. For details, see Lemma 2.9.1, Corollary 2.9.2 and Corollary 2.9.3.

2.5 Some first estimates

Since we are working in the new spaces G_σ^m which we introduced above, we have to reformulate Lemma 2.2.1 and Lemma 2.2.2 in G_σ^s-spaces.

Lemma 2.5.1. *For $\sigma_A \geq \sigma \geq 0$, $s_A - s \geq 4$, $s > 1/2$ and $T_0 > 0$ fixed, let $A \in C([0,T_0], \mathcal{W}_{\sigma_A}^{s_A})$ be a solution of WME (2.6) and let ψ_u and ψ_v be defined as above in (2.11). For this approximation, there exists $\varepsilon_0 > 0$ and $C_{\text{res}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{T\in[0,T_0]} (\|\text{Res}_u\|_{G_\sigma^s} + \|\text{Res}_v\|_{G_\sigma^s} + \varepsilon\|\partial_x^{-1}\text{Res}_u\|_{G_\sigma^{s+1}}) \leq C_{\text{res}}\varepsilon^{7/2}.$$

Proof. The proof goes line by line as the proofs of Lemma 2.2.1 and Lemma 2.2.2 by using Lemma 2.9.1 □

From Figure 2.1 and (2.9) we deduce that ω_1 is a bounded operator in G_σ^s and $(\omega_2)^{-1}$ is a bounded operator from G_σ^s to G_σ^{s+1}. Using Lemma 2.5.1 we have that

$$\|\varepsilon^{3/2}\mathrm{i}\omega_1 f_u\|_{G_\sigma^s} + \|\varepsilon^{3/2}(\mathrm{i}\omega_2)^{-1} f_v\|_{G_\sigma^{s+1}} \leq C\varepsilon^{3/2}(\|\mathcal{R}_1\|_{G_\sigma^s}^2 + \|\mathcal{R}_2\|_{G_\sigma^s}^2) + C_{\text{res}}\varepsilon.$$

With these estimates and Corollary 2.9.5 we find that

$$\begin{aligned}
\varepsilon^{3/2}\|\mathrm{i}\omega_1 g_u\|_{G_\sigma^s} &\leq \varepsilon^{3/2}\|\mathrm{i}\omega_1 f_u\|_{G_\sigma^s} + CC_\psi\varepsilon^{s_A-s}(\|\mathcal{R}_1\|_{G_\sigma^s} + \|\mathcal{R}_{-1}\|_{G_\sigma^s}) \\
&\quad + CC_\psi\varepsilon^{s_A-s}(\|\mathcal{R}_2\|_{G_\sigma^s} + \|\mathcal{R}_{-2}\|_{G_\sigma^s}), \\
\varepsilon^{3/2}\|(\mathrm{i}\omega_2)^{-1} g_v\|_{G_\sigma^{s+1}} &\leq \varepsilon^{3/2}\|(\mathrm{i}\omega_2)^{-1} f_v\|_{G_\sigma^{s+1}} + CC_\psi\varepsilon^{s_A-s}(\|\mathcal{R}_1\|_{G_\sigma^s} + \|\mathcal{R}_{-1}\|_{G_\sigma^s}) \\
&\quad + CC_\psi\varepsilon^{s_A-s}(\|\mathcal{R}_2\|_{G_\sigma^s} + \|\mathcal{R}_{-2}\|_{G_\sigma^s}).
\end{aligned}$$

2.6 The normal form transformations

To establish the validity of Theorem 2.1.1, we have to prove a $\mathcal{O}(1)$-bound for the error on the long $\mathcal{O}(1/\varepsilon)$-time scale. The main difficulty comes from the terms of order $\mathcal{O}(1)$ in (2.15)-(2.16) which we have to estimate on the long $\mathcal{O}(1/\varepsilon)$-time scale. The strategy is as follows. Outside of a neighborhood of the wave number $k = 0$, the modes can be controlled by working in the spaces of functions that are analytic in a strip in the complex plane. The terms that do not vanish at the wave number $k = 0$ can be simplified by infinitely many normal form transformations in a δ_0-neighborhood of the origin $k = 0$ with $\delta_0 > 0$ small, but fixed. In this section we perform these near identity changes of variables in the spaces of analytic functions. However, each normal form transformation generates new terms of order $\mathcal{O}(1)$. Some of them are resonant but of long wave form and can be included in the energy estimates. Some of the other terms are non-resonant but not of long wave form. So they can be eliminated again and again by another normal form transformation. This section is an adaption of [DKS16, Section 3]. However, we perform the normal form transformations in Gevrey spaces instead of Sobolev spaces. This means that the underlying spaces for the convergence of infinitely many near identity changes of variables as well as the corresponding estimates have to be adapted, so that we also list them here. In Section 2.8 we handle the transformed limit system by energy estimates in Sobolev spaces since we establish a connection between the spaces of analytic functions and Sobolev spaces in Section 2.7.

2.6.1 The first normal form transformation

To illustrate the procedure, we show how to obtain the first near identity change of variables. Our system has the following form

$$\begin{aligned}\partial_t \mathcal{R}_1 &= \mathrm{i}\omega_1 \mathcal{R}_1 + \mathrm{i}\omega_1 \mathcal{L}_{1,\delta}\mathcal{R} + \frac{1}{\sqrt{2}}\varepsilon^{3/2}\mathrm{i}\omega_1 g_u, \\ \partial_t \mathcal{R}_2 &= \mathrm{i}\omega_2 \mathcal{R}_2 + (\mathrm{i}\omega_2)^{-1}\mathcal{L}_{2,\delta}\mathcal{R} + \frac{1}{\sqrt{2}}\varepsilon^{3/2}(\mathrm{i}\omega_2)^{-1} g_v,\end{aligned}$$

with

$$\begin{aligned}\mathcal{L}_{1,\delta}\mathcal{R} &= a_{uu}(E_\delta\psi_u)(\mathcal{R}_1+\mathcal{R}_{-1}) + a_{uv}(E_\delta\psi_u)(\mathcal{R}_2+\mathcal{R}_{-2}) \\ &\quad + a_{uv}(E_\delta\psi_v)(\mathcal{R}_1+\mathcal{R}_{-1}) + a_{vv}(E_\delta\psi_v)(\mathcal{R}_2+\mathcal{R}_{-2}), \\ \mathcal{L}_{2,\delta}\mathcal{R} &= b_{uu}(E_\delta\psi_u)(\mathcal{R}_1+\mathcal{R}_{-1}) + b_{uv}(E_\delta\psi_u)(\mathcal{R}_2+\mathcal{R}_{-2}) \\ &\quad + b_{uv}(E_\delta\psi_v)(\mathcal{R}_1+\mathcal{R}_{-1}) + b_{vv}(E_\delta\psi_v)(\mathcal{R}_2+\mathcal{R}_{-2}),\end{aligned}$$

and with

$$\begin{aligned}g_u &= g_u(\mathcal{R}_{\pm1},\mathcal{R}_{\pm2}), \\ g_v &= g_v(\mathcal{R}_{\pm1},\mathcal{R}_{\pm2}).\end{aligned}$$

The term

$$\mathrm{i}\omega_1(a_{uu}(E_\delta\psi_u) + a_{uv}(E_\delta\psi_v))(\mathcal{R}_1+\mathcal{R}_{-1}) =: \mathrm{i}\omega_1 S_{1,u}(\psi_1,\mathcal{R}_{\pm1}),$$

with $\psi_1 = a_{uu}\psi_u + a_{uv}\psi_v$, is controlled by energy estimates in the equation of $\mathcal{R}_{\pm1}$. In a δ_0-neighborhood of the wave number $k = 0$, we can eliminate the term

$$\mathrm{i}\omega_1(a_{uv}(E_\delta\psi_u) + a_{vv}(E_\delta\psi_v))(\mathcal{R}_2+\mathcal{R}_{-2}) =: \mathrm{i}\omega_1 S_{2,u}(\psi_2,\mathcal{R}_{\pm2}),$$

with $\psi_2 = a_{uv}\psi_u + a_{vv}\psi_v$, in the $\mathcal{R}_{\pm1}$-equation with infinitely many normal form transformations. Therefore, we introduce a projection E_{δ_0} onto a δ_0-neighborhood of $k = 0$

similar to (2.14). We assume that our system has the following form

$$\begin{aligned}
\partial_t \mathcal{R}_1 &= \mathrm{i}\omega_1 \mathcal{R}_1 + \mathrm{i}\omega_1 (a_{uu}(E_\delta \psi_u) + a_{uv}(E_\delta \psi_v))(\mathcal{R}_1 + \mathcal{R}_{-1}) \\
&\quad + E_{\delta_0} \mathrm{i}\omega_1 (a_{uv}(E_\delta \psi_u) + a_{vv}(E_\delta \psi_v))(\mathcal{R}_2 + \mathcal{R}_{-2}) \\
&\quad + E_{\delta_0}^c \mathrm{i}\omega_1 (a_{uv}(E_\delta \psi_u) + a_{vv}(E_\delta \psi_v))(\mathcal{R}_2 + \mathcal{R}_{-2}) \\
&\quad + \frac{1}{\sqrt{2}} \varepsilon^{3/2} \mathrm{i}\omega_1 g_u, \\
\partial_t \mathcal{R}_2 &= \mathrm{i}\omega_2 \mathcal{R}_2 + (\mathrm{i}\omega_2)^{-1} (b_{uv}(E_\delta \psi_u) + b_{vv}(E_\delta \psi_v))(\mathcal{R}_2) \\
&\quad + E_{\delta_0} (\mathrm{i}\omega_2)^{-1} (b_{uu}(E_\delta \psi_u) + b_{uv}(E_\delta \psi_v))(\mathcal{R}_1 + \mathcal{R}_{-1}) \\
&\quad + E_{\delta_0}^c (\mathrm{i}\omega_2)^{-1} (b_{uu}(E_\delta \psi_u) + b_{uv}(E_\delta \psi_v))(\mathcal{R}_1 + \mathcal{R}_{-1}) \\
&\quad + E_{\delta_0} (\mathrm{i}\omega_2)^{-1} (b_{uv}(E_\delta \psi_u) + b_{vv}(E_\delta \psi_v))(\mathcal{R}_{-2}) \\
&\quad + E_{\delta_0}^c (\mathrm{i}\omega_2)^{-1} (b_{uv}(E_\delta \psi_u) + b_{vv}(E_\delta \psi_v))(\mathcal{R}_{-2}) \\
&\quad + \frac{1}{\sqrt{2}} \varepsilon^{3/2} (\mathrm{i}\omega_2)^{-1} g_v.
\end{aligned}$$

With infinitely many normal form transformations we can eliminate

$$\begin{aligned}
E_{\delta_0} \mathrm{i}\omega_1 (a_{uv}(E_\delta \psi_u) + a_{vv}(E_\delta \psi_v))(\mathcal{R}_2 + \mathcal{R}_{-2}) &=: \mathrm{i}\omega_1 S_{2,u,\delta_0}(\psi_2, \mathcal{R}_{\pm 2}), \\
E_{\delta_0} (\mathrm{i}\omega_2)^{-1} (b_{uu}(E_\delta \psi_u) + b_{uv}(E_\delta \psi_v))(\mathcal{R}_1 + \mathcal{R}_{-1}) &=: (\mathrm{i}\omega_2)^{-1} S_{1,v,\delta_0}(\psi_3, \mathcal{R}_{\pm 1}), \\
E_{\delta_0} (\mathrm{i}\omega_2)^{-1} (b_{uv}(E_\delta \psi_u) + b_{vv}(E_\delta \psi_v))(\mathcal{R}_{-2}) &=: (\mathrm{i}\omega_2)^{-1} S_{2,v,\delta_0}(\psi_4, \mathcal{R}_{-2}),
\end{aligned}$$

with $\psi_2 = a_{uv}\psi_u + a_{vv}\psi_v$, $\psi_3 = b_{uu}\psi_u + b_{uv}\psi_v$ and $\psi_4 = b_{uv}\psi_u + b_{vv}\psi_v$. For the normal form transformations, we switch to the Fourier space. Therefore, we introduce the corresponding Fourier space for functions lying in Gevrey spaces by

$$G_\sigma^{m,*} = \{\widehat{u} : u \in G_\sigma^m\},$$

with

$$\|\widehat{u}\|_{G_\sigma^{m,*}} = \|u\|_{G_\sigma^m}.$$

We take the ansatz

$$\begin{aligned}
\widehat{\mathcal{R}}_{1,2} &= \widehat{\mathcal{R}}_{1,1} + M_1^{(1)}(\widehat{\psi}_2, \widehat{\mathcal{R}}_{\pm 2,1}), \\
\widehat{\mathcal{R}}_{2,2} &= \widehat{\mathcal{R}}_{2,1} + M_{2,1}^{(1)}(\widehat{\psi}_3, \widehat{\mathcal{R}}_{\pm 1,1}) + M_{2,2}^{(1)}(\widehat{\psi}_4, \widehat{\mathcal{R}}_{-2,1}), \\
\widehat{\mathcal{R}}_{-1,2} &= \widehat{\mathcal{R}}_{-1,1} + M_{-1}^{(1)}(\widehat{\psi}_2, \widehat{\mathcal{R}}_{\pm 2,1}), \\
\widehat{\mathcal{R}}_{-2,2} &= \widehat{\mathcal{R}}_{-2,1} + M_{-2,1}^{(1)}(\widehat{\psi}_3, \widehat{\mathcal{R}}_{\pm 1,1}) + M_{-2,2}^{(1)}(\widehat{\psi}_4, \widehat{\mathcal{R}}_{2,1}),
\end{aligned}$$

where $\mathcal{R}_{l,1} = \mathcal{R}_l$ for $l \in \{\pm 1, \pm 2\}$ and where $M_i^{(1)}$ is linear in its second component. The term $\widehat{S}_{2,u,\delta_0}$ has the form

$$\widehat{S}_{2,u,\delta_0}(\widehat{\psi}_2, \widehat{\mathcal{R}}_{\pm 2,1}) = \sum_{l \in \{2,-2\}} \widehat{S}_{2,u,\delta_0,l}(\widehat{\psi}_2, \widehat{\mathcal{R}}_{l,1}),$$

with the following convolution structure

$$\widehat{S}_{2,u,\delta_0,l}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1}) = \int \widehat{s}_{2,u,\delta_0,l}(k,k-m,m)\widehat{\psi}_2(k-m)\widehat{\mathcal{R}}_{l,1}(m)\mathrm{d}m,$$

where

$$\widehat{s}_{2,u,\delta_0,2}(k,k-m,m) = \widehat{E}_{\delta_0}(k)\widehat{E}_{\delta}(k-m),$$

and similarly for $\widehat{s}_{2,u,\delta_0,-2}$.

Therefore, we set

$$M_1^{(1)}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{\pm2,1}) = \sum_{l\in\{-2,2\}} M_{1,l}^{(1)}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1}),$$

with

$$M_{1,l}^{(1)}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1}) = \int \widehat{m}_{1,l}^{(1)}(k,k-m,m)\widehat{\psi}_2(k-m)\widehat{\mathcal{R}}_{l,1}(m)\mathrm{d}m.$$

We differentiate $\widehat{\mathcal{R}}_{1,2}$ with respect to time and obtain

$$\begin{aligned}
\partial_t\widehat{\mathcal{R}}_{1,2} &= \partial_t\widehat{\mathcal{R}}_{1,1} + \sum_{l\in\{-2,2\}} M_{1,l}^{(1)}(\widehat{\psi}_2,\partial_t\widehat{\mathcal{R}}_{l,1}) + \sum_{l\in\{-2,2\}} M_{1,l}^{(1)}(\partial_t\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1})\\
&= \mathrm{i}\widehat{\omega}_1\widehat{\mathcal{R}}_{1,1} + \mathrm{i}\widehat{\omega}_1\widehat{S}_{1,u}(\widehat{\psi}_1,\widehat{\mathcal{R}}_{\pm1,1}) + \mathrm{i}\widehat{\omega}_1\sum_{l\in\{2,-2\}}\widehat{S}_{2,u,\delta_0,l}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1})\\
&\quad +\mathrm{i}\widehat{\omega}_1\sum_{l\in\{2,-2\}}\int \widehat{E}^c_{\delta_0}(k)\widehat{E}_{\delta}(k-m)\widehat{\psi}_2(k-m)\widehat{\mathcal{R}}_{l,1}(m)\mathrm{d}m\\
&\quad +\sum_{l\in\{-2,2\}} M_{1,l}^{(1)}(\widehat{\psi}_2,\mathrm{i}\widehat{\omega}_l\widehat{\mathcal{R}}_{l,1} + (\mathrm{i}\widehat{\omega}_l)^{-1}(\widehat{S}_{1,v}(\widehat{\psi}_3,\widehat{\mathcal{R}}_{\pm1,1}) + \widehat{S}_{2,v}(\widehat{\psi}_4,\widehat{\mathcal{R}}_{\pm2,1}))\\
&\quad +\mathcal{O}(\varepsilon)\\
&= \mathrm{i}\widehat{\omega}_1\widehat{\mathcal{R}}_{1,2} - \mathrm{i}\widehat{\omega}_1\sum_{l\in\{-2,2\}} M_{1,l}^{(1)}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1})\\
&\quad +\mathrm{i}\widehat{\omega}_1\widehat{S}_{1,u}(\widehat{\psi}_1,\widehat{\mathcal{R}}_{\pm1,1}) + \mathrm{i}\widehat{\omega}_1\sum_{l\in\{2,-2\}}\widehat{S}_{2,u,\delta_0,l}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1})\\
&\quad +\mathrm{i}\widehat{\omega}_1\sum_{l\in\{2,-2\}}\int \widehat{E}^c_{\delta_0}(k)\widehat{E}_{\delta}(k-m)\widehat{\psi}_2(k-m)\widehat{\mathcal{R}}_{l,1}(m)\mathrm{d}m\\
&\quad +\sum_{l\in\{-2,2\}} M_{1,l}^{(1)}(\widehat{\psi}_2,\mathrm{i}\widehat{\omega}_l\widehat{\mathcal{R}}_{l,1} + (\mathrm{i}\widehat{\omega}_l)^{-1}(\widehat{S}_{1,v}(\widehat{\psi}_3,\widehat{\mathcal{R}}_{\pm1,1}) + \widehat{S}_{2,v}(\widehat{\psi}_4,\widehat{\mathcal{R}}_{\pm2,1})))\\
&\quad +\mathcal{O}(\varepsilon),
\end{aligned}$$

where we use the fact that $\partial_t\widehat{\psi}_2 = \mathcal{O}(\varepsilon)$. To eliminate $\mathrm{i}\widehat{\omega}_1\widehat{S}_{2,u,\delta_0,l}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1})$, we choose

$$0 \;=\; -\mathrm{i}\widehat{\omega}_1 M_{1,l}^{(1)}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1}) + M_{1,l}^{(1)}(\widehat{\psi}_2,\mathrm{i}\widehat{\omega}_l\widehat{\mathcal{R}}_{l,1}) + \mathrm{i}\widehat{\omega}_1\widehat{S}_{2,u,\delta_0,l}(\widehat{\psi}_2,\widehat{\mathcal{R}}_{l,1}),$$

and get

$$\mathrm{i}(\widehat{\omega}_1(k)-\widehat{\omega}_l(m))\widehat{m}_{1,l}^{(1)}(k,k-m,m) = \mathrm{i}\widehat{\omega}_1(k)\widehat{s}_{2,u,\delta_0,l}(k,k-m,m),$$

respectively

$$\widehat{m}^{(1)}_{1,l}(k,k-m,m) = \frac{\mathrm{i}\widehat{\omega}_1(k)\widehat{s}_{2,u,\delta_0,l}(k,k-m,m)}{\mathrm{i}(\widehat{\omega}_1(k)-\widehat{\omega}_l(m))},$$

for $l \in \{-2,2\}$. Since we have $\widehat{\omega}_1(0) = 0$ and $\widehat{\omega}_2(0) = 1$, the denominator is bounded away from zero for $|k| \le \delta_0$ and $|k-m| \le \delta$ for $\delta_0 > 0$ and $\delta > 0$ sufficiently small. Similarly, we use the method to transform $(\mathrm{i}\widehat{\omega}_2)^{-1}(\widehat{S}_{1,v,\delta_0}(\widehat{\psi}_3, \widehat{\mathcal{R}}_{\pm 1}) + \widehat{S}_{2,v,\delta_0}(\widehat{\psi}_4, \widehat{\mathcal{R}}_{-2}))$. We obtain the non-resonance conditions

$$\widehat{m}^{(1)}_{2,1,l}(k,k-m,m) = \frac{(\mathrm{i}\widehat{\omega}_2(k))^{-1}\widehat{s}_{1,v,\delta_0,l}(k,k-m,m)}{\mathrm{i}(\widehat{\omega}_2(k)-\widehat{\omega}_l(m))},$$

for $l \in \{\pm 1\}$, respectively

$$\widehat{m}^{(1)}_{2,2,-2}(k,k-m,m) = \frac{(\mathrm{i}\widehat{\omega}_2(k))^{-1}\widehat{s}_{2,v,\delta_0,-2}(k,k-m,m)}{\mathrm{i}(\widehat{\omega}_2(k)+\widehat{\omega}_2(m))},$$

where the denominator is bounded away from zero. These non-resonance conditions will be weakened below to make the method applicable to more general systems at the cost of an error of order $\mathcal{O}(\varepsilon)$. The transformation provides

$$\begin{aligned}
\partial_t \widehat{\mathcal{R}}_{1,2} \;=\;& \mathrm{i}\widehat{\omega}_1 \widehat{\mathcal{R}}_{1,2} + \mathrm{i}\widehat{\omega}_1 \widehat{S}_{1,u}(\widehat{\psi}_1, \widehat{\mathcal{R}}_{\pm 1,1}) \\
&+ \mathrm{i}\widehat{\omega}_1 \sum_{l \in \{2,-2\}} \int \widehat{E}^c_{\delta_0}(k)\widehat{E}_\delta(k-m)\widehat{\psi}_2(k-m)\widehat{\mathcal{R}}_{l,1}(m)\mathrm{d}m \\
&+ \sum_{l \in \{-2,2\}} M^{(1)}_{1,l}(\widehat{\psi}_2, (\mathrm{i}\widehat{\omega}_l)^{-1}(\widehat{S}_{1,v}(\widehat{\psi}_3, \widehat{\mathcal{R}}_{\pm 1,1}) + \widehat{S}_{2,v}(\widehat{\psi}_4, \widehat{\mathcal{R}}_{\pm 2,1}))) + \mathcal{O}(\varepsilon),
\end{aligned}$$

and analogously for $\widehat{\mathcal{R}}_{2,2}$, $\widehat{\mathcal{R}}_{-1,2}$ and $\widehat{\mathcal{R}}_{-2,2}$. To eliminate $\widehat{\mathcal{R}}_{j,1}$ on the right hand side of this equation, we invert the normal form transformations via

$$\begin{aligned}
\widehat{\mathcal{R}}_{\pm 1,1} &= \widehat{\mathcal{R}}_{\pm 1,2} + \tilde{M}^{(1)}_{\pm 1}(\widehat{\psi}, \widehat{\mathcal{R}}_{,2}), \\
\widehat{\mathcal{R}}_{\pm 2,1} &= \widehat{\mathcal{R}}_{\pm 2,2} + \tilde{M}^{(1)}_{\pm 2}(\widehat{\psi}, \widehat{\mathcal{R}}_{,2}),
\end{aligned}$$

where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ and $\mathcal{R}_{,j} = (\mathcal{R}_{\pm 1,j}, \mathcal{R}_{\pm 2,j})$. Thus, we have

$$\begin{aligned}
\partial_t \widehat{\mathcal{R}}_{1,2} \;=\;& \mathrm{i}\widehat{\omega}_1 \widehat{\mathcal{R}}_{1,2} + \mathrm{i}\widehat{\omega}_1 \widehat{S}_{1,u}(\widehat{\psi}_1, \widehat{\mathcal{R}}_{\pm 1,2}) + \mathrm{i}\widehat{\omega}_1 \widehat{S}_{1,u}(\widehat{\psi}_1, \tilde{M}^{(1)}_{\pm 1}(\widehat{\psi}, \widehat{\mathcal{R}}_{,2})) \\
&+ \mathrm{i}\widehat{\omega}_1 \sum_{l \in \{2,-2\}} \int \widehat{E}^c_{\delta_0}(k)\widehat{E}_\delta(k-m)\widehat{\psi}_2(k-m)\widehat{\mathcal{R}}_{l,2}(m)\mathrm{d}m \\
&+ \mathrm{i}\widehat{\omega}_1 \sum_{l \in \{2,-2\}} \int \widehat{E}^c_{\delta_0}(k)\widehat{E}_\delta(k-m)\widehat{\psi}_2(k-m)\tilde{M}^{(1)}_{\pm l}(\widehat{\psi}, \widehat{\mathcal{R}}_{,2})(m)\mathrm{d}m \\
&+ \sum_{l \in \{-2,2\}} M^{(1)}_{1,l}(\widehat{\psi}_2, (\mathrm{i}\widehat{\omega}_l)^{-1}(\widehat{S}_{1,v}(\widehat{\psi}_3, \widehat{\mathcal{R}}_{\pm 1,2}) + \widehat{S}_{1,v}(\widehat{\psi}_3, \tilde{M}^{(1)}_{\pm 1}(\widehat{\psi}, \widehat{\mathcal{R}}_{,2})) \\
&\quad + \widehat{S}_{2,v}(\widehat{\psi}_4, \widehat{\mathcal{R}}_{\pm 2,2}) + \widehat{S}_{2,v}(\widehat{\psi}_4, \tilde{M}^{(1)}_{\pm 2}(\widehat{\psi}, \widehat{\mathcal{R}}_{,2})))) \\
&+ \mathcal{O}(\varepsilon).
\end{aligned}$$

As a consequence, new terms of order $\mathcal{O}(1)$ arise. The term $i\widehat{\omega}_1\widehat{S}_{1,u}(\widehat{\psi}_1, \widehat{\mathcal{R}}_{\pm1,2})$ is the only term of order $\mathcal{O}(\|\psi\|)$ in a δ_0-neighborhood of $k = 0$. This term can be controlled with energy estimates, see below Section 2.8. All other terms are of order $\mathcal{O}(\|\psi\|^2)$ or perhaps higher. Some of them are resonant but of long-wave form. These terms can be included in the energy estimates. The other terms are non-resonant. We can eliminate them by another normal form transformation. However, this results in new terms of order $\mathcal{O}(1)$. This procedure can be applied again and again. In the limit, we have to prove the convergence of this procedure. We use the fact that the j-th normal form transformation leads to new terms of order $\mathcal{O}(\|\psi\|^j)$ or possibly higher. This will be discussed in the subsequent sections. The terms outside of the δ_0-neighborhood of the wave number $k = 0$ can be controlled by the artificial damping of the Gevrey spaces.

2.6.2 The recursion formulas

To understand the infinitely many normal form transformations, we analyze the structure of one transformation, especially after performing $j-1$ transformations. The error equations have the following structure

$$\begin{aligned}
\partial_t\widehat{\mathcal{R}}_{1,j}(k,t) &= i\widehat{\omega}_1(k)\widehat{\mathcal{R}}_{1,j}(k,t) + \varepsilon\widehat{p}_{1,j}(k,t)\\
&\quad + i\widehat{\omega}_1(k)\int \widehat{f}^{(j)}_{1,res}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{1,j} + \widehat{\mathcal{R}}_{-1,j})(m,t)\mathrm{d}m\\
&\quad + i\widehat{\omega}_1(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{2,j} + \widehat{\mathcal{R}}_{-2,j})(m,t)\mathrm{d}m\\
&\quad + i\widehat{\omega}_1(k)\int \widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{2,j} + \widehat{\mathcal{R}}_{-2,j})(m,t)\mathrm{d}m,\\
\partial_t\widehat{\mathcal{R}}_{2,j}(k,t) &= i\widehat{\omega}_2(k)\widehat{\mathcal{R}}_{2,j}(k,t) + \varepsilon\widehat{p}_{2,j}(k,t)\\
&\quad + (i\widehat{\omega}_2)^{-1}(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{2,1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{1,j} + \widehat{\mathcal{R}}_{-1,j})(m,t)\mathrm{d}m\\
&\quad + (i\widehat{\omega}_2)^{-1}(k)\int \widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j)}_{2,1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{1,j} + \widehat{\mathcal{R}}_{-1,j})(m,t)\mathrm{d}m\\
&\quad + (i\widehat{\omega}_2)^{-1}(k)\int \widehat{f}^{(j)}_{2,res}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{2,j}(m,t)\mathrm{d}m\\
&\quad + (i\widehat{\omega}_2)^{-1}(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{2,2,non}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{-2,j}(m,t)\mathrm{d}m\\
&\quad + (i\widehat{\omega}_2)^{-1}(k)\int \widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j)}_{2,2,non}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{-2,j}(m,t)\mathrm{d}m.
\end{aligned}$$

For $l \in \{\pm1, \pm2\}$ we define $\widehat{p}_{l,1} = \widehat{p}_l$ and we have

$$\widehat{f}^{(1)}_{l,res}(k,k-m,\varepsilon t) = \widehat{f}^{(1)}_{l,non}(k,k-m,\varepsilon t) = E_\delta(k-m)\widehat{\psi}(k-m,\varepsilon t).$$

Since $\widehat{\mathcal{R}}_{-l,j}$ is complex conjugated to $\widehat{\mathcal{R}}_{l,j}$, we obtain

$$\widehat{f}^{(j)}_{-l,res} = \overline{\widehat{f}^{(j)}_{l,res}}, \quad \widehat{f}^{(j)}_{-l,non} = \overline{\widehat{f}^{(j)}_{l,non}} \quad \text{and} \quad \widehat{p}_{-l,j} = \overline{\widehat{p}_{l,j}},$$

for $j \in \mathbb{N}$ and $l \in \{1,2\}$. Therefore, it suffices to look at $\widehat{\mathcal{R}}_{l,j}$ for $l \in \{1,2\}$. We derive recursion formulas for $\widehat{p}_{l,j}$, $\widehat{f}^{(j)}_{l,res}$ and $\widehat{f}^{(j)}_{l,non}$. We introduce the j-th near identity change of variables by

$$\widehat{\mathcal{R}}_{1,j+1}(k,t) = \widehat{\mathcal{R}}_{1,j}(k,t) + \sum_{l\in\{2,-2\}} \int \widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m, \tag{2.18}$$

$$\begin{aligned} \widehat{\mathcal{R}}_{2,j+1}(k,t) &= \widehat{\mathcal{R}}_{2,j}(k,t) + \sum_{l\in\{1,-1\}} \int \widehat{g}^{(j)}_{2,1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m \\ &\quad + \sum_{l\in\{-2\}} \int \widehat{g}^{(j)}_{2,2,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m. \end{aligned} \tag{2.19}$$

We temporarily assume that the perturbation of the identity (2.18)-(2.19) is invertible and its inverse has the following form

$$\widehat{\mathcal{R}}_{i,j}(k,t) = \widehat{\mathcal{R}}_{i,j+1}(k,t) + \sum_{l\in\{\pm1,\pm2\}} \int \widehat{h}^{(j)}_{i,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j+1}(m,t)\mathrm{d}m,$$

for $i \in \{\pm1,\pm2\}$. We differentiate (2.18) with respect to time and get

$$\begin{aligned} \partial_t\widehat{\mathcal{R}}_{1,j+1}(k,t) &= \partial_t\widehat{\mathcal{R}}_{1,j}(k,t) + \sum_{l\in\{2,-2\}} \int \widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\partial_t\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m \\ &\quad +\varepsilon \sum_{l\in\{2,-2\}} \int \partial_T\widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m. \end{aligned}$$

Inserting the recursion formulas provides

$$\begin{aligned} \partial_t\widehat{\mathcal{R}}_{1,j+1}(k,t) &= \mathrm{i}\widehat{\omega}_1(k)\widehat{\mathcal{R}}_{1,j}(k,t) + \varepsilon\widehat{p}_{1,j}(k,t) \\ &\quad +\mathrm{i}\widehat{\omega}_1(k)\int \widehat{f}^{(j)}_{1,res}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{1,j}+\widehat{\mathcal{R}}_{-1,j})(m,t)\mathrm{d}m \\ &\quad +\mathrm{i}\widehat{\omega}_1(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{2,j}+\widehat{\mathcal{R}}_{-2,j})(m,t)\mathrm{d}m \\ &\quad +\mathrm{i}\widehat{\omega}_1(k)\int \widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{2,j}+\widehat{\mathcal{R}}_{-2,j})(m,t)\mathrm{d}m \\ &\quad +\sum_{l\in\{2,-2\}} \int \widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)[\mathrm{i}\widehat{\omega}_l(m)\widehat{\mathcal{R}}_{l,j}(m,t)+\varepsilon\widehat{p}_{2,j}(m,t) \\ &\quad +(\mathrm{i}\widehat{\omega}_l)^{-1}(m)\int \widehat{E}_{\delta_0}(m)\widehat{f}^{(j)}_{l,1,non}(m,m-m_1,\varepsilon t)(\widehat{\mathcal{R}}_{1,j}+\widehat{\mathcal{R}}_{-1,j})(m_1,t)\mathrm{d}m_1 \\ &\quad +(\mathrm{i}\widehat{\omega}_l)^{-1}(m)\int \widehat{E}^c_{\delta_0}(m)\widehat{f}^{(j)}_{l,1,non}(m,m-m_1,\varepsilon t)(\widehat{\mathcal{R}}_{1,j}+\widehat{\mathcal{R}}_{-1,j})(m_1,t)\mathrm{d}m_1 \\ &\quad +(\mathrm{i}\widehat{\omega}_l)^{-1}(m)\int \widehat{f}^{(j)}_{l,res}(m,m-m_1,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m_1,t)\mathrm{d}m_1 \\ &\quad +(\mathrm{i}\widehat{\omega}_l)^{-1}(m)\int \widehat{E}_{\delta_0}(m)\widehat{f}^{(j)}_{l,2,non}(m,m-m_1,\varepsilon t)\widehat{\mathcal{R}}_{-l,j}(m_1,t)\mathrm{d}m_1 \\ &\quad +(\mathrm{i}\widehat{\omega}_l)^{-1}(m)\int \widehat{E}^c_{\delta_0}(m)\widehat{f}^{(j)}_{l,2,non}(m,m-m_1,\varepsilon t)\widehat{\mathcal{R}}_{-l,j}(m_1,t)\mathrm{d}m_1]\mathrm{d}m \\ &\quad +\mathcal{O}(\varepsilon). \end{aligned}$$

To eliminate the non-resonant terms

$$\mathrm{i}\widehat{\omega}_1(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m,$$

for $l \in \{\pm 2\}$, we proceed as before and choose $\widehat{g}^{(j)}_{1l}$ as follows

$$\begin{aligned} 0 &= -\mathrm{i}\widehat{\omega}_1(k)\int \widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m + \int \mathrm{i}\widehat{\omega}_l(m)\widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m \\ &\quad + \mathrm{i}\widehat{\omega}_1(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m, \end{aligned}$$

respectively

$$\mathrm{i}(\widehat{\omega}_1(k)-\widehat{\omega}_l(m))\widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t) = \mathrm{i}\widehat{\omega}_1(k)\widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t),$$

for $l \in \{\pm 2\}$. Since $\widehat{\psi}$ is strongly concentrated at $k=0$, the difference

$$\varepsilon\widehat{r}_{1,j}(k,t) = \sum_{l\in\{-2,2\}}\int \mathrm{i}(\widehat{\omega}_2(m)-\widehat{\omega}_2(k))\widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m$$

is of order $\mathcal{O}(\varepsilon)$. This can be expected for $j=1$ and for $j \geq 1$ it follows via an induction argument. Hence, we replace the non-resonance condition by

$$\begin{aligned} 0 &= -\mathrm{i}\widehat{\omega}_1(k)\int \widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m + \int \mathrm{i}\widehat{\omega}_l(k)\widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m \\ &\quad + \mathrm{i}\widehat{\omega}_1(k)\int \widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)(\widehat{\mathcal{R}}_{l,j})(m,t)\mathrm{d}m, \end{aligned}$$

for $l \in \{\pm 2\}$, respectively

$$\widehat{g}^{(j)}_{1,l}(k,k-m,\varepsilon t) = \frac{\mathrm{i}\widehat{\omega}_1(k)\widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)}{\mathrm{i}(\widehat{\omega}_1(k)-\widehat{\omega}_l(k))}.$$

This non-resonance condition is weaker and we only make an error of order $\mathcal{O}(\varepsilon)$. For $\mu = \pm 1$ we have by direct calculation

$$
\begin{aligned}
&\widehat{f}^{(j+1)}_{1,res}(k,k-m,\varepsilon t) - \widehat{f}^{(j)}_{1,res}(k,k-m,\varepsilon t)\\
&= \sum_{\kappa\in\{-1,1\}} \int \widehat{f}^{(j)}_{1,res}(k,k-l,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad + \sum_{\lambda\in\{-2,2\}} \int \widehat{E}^{c}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-l,\varepsilon t)\widehat{h}^{(j)}_{\lambda,\mu}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad + \sum_{\lambda\in\{-2,2\}} \int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l,\varepsilon t)(\widehat{\omega}_2)^{-1}(l)\widehat{E}_{\delta_0}(m)\widehat{f}^{(j)}_{\lambda,1,non}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad + \sum_{\lambda\in\{-2,2\},\kappa\in\{-1,1\}} \int\int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\quad \times\widehat{E}_{\delta_0}(l_1)\widehat{f}^{(j)}_{\lambda,1,non}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\\
&\quad + \sum_{\lambda\in\{-2,2\}} \int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l,\varepsilon t)(\widehat{\omega}_2)^{-1}(l)\widehat{E}^{c}_{\delta_0}(m)\widehat{f}^{(j)}_{\lambda,1,non}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad + \sum_{\lambda\in\{-2,2\},\kappa\in\{-1,1\}} \int\int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\quad \times\widehat{E}^{c}_{\delta_0}(l_1)\widehat{f}^{(j)}_{\lambda,1,non}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\\
&\quad + \sum_{\lambda\in\{-2,2\},\kappa\in\{-2,2\}} \int\int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\quad \times\widehat{f}^{(j)}_{\lambda,res}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\\
&\quad + \sum_{\lambda\in\{-2,2\},\kappa\in\{-2,2\}} \int\int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\quad \times\widehat{E}_{\delta_0}(l_1)\widehat{f}^{(j)}_{\lambda,2,non}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\\
&\quad + \sum_{\lambda\in\{-2,2\},\kappa\in\{-2,2\}} \int\int \widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\quad \times\widehat{E}^{c}_{\delta_0}(l_1)\widehat{f}^{(j)}_{\lambda,2,non}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1,
\end{aligned}
$$

where we use the abbreviation

$$
\widehat{\widetilde{g}}^{(j)}_{1,\lambda}(k,k-l,\varepsilon t) = \frac{1}{\widehat{\omega}_1(k)}\widehat{g}^{(j)}_{1,\lambda}(k,k-l,\varepsilon t) = \frac{\mathrm{i}\widehat{E}_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-l,\varepsilon t)}{\mathrm{i}(\widehat{\omega}_1(k)-\widehat{\omega}_\lambda(k))},
$$

for $\lambda \in \{\pm 2\}$. If we summarize some of these terms by using the definition of the projection E_δ with $u = E_\delta u + E_\delta^c u$ for each function u, we simplify this difference to

$$\begin{aligned}
&\widehat{f}^{(j+1)}_{1,res}(k, k-m, \varepsilon t) - \widehat{f}^{(j)}_{1,res}(k, k-m, \varepsilon t) \\
&= \sum_{\kappa \in \{-1,1\}} \int \widehat{f}^{(j)}_{1,res}(k, k-l, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l, l-m) \mathrm{d}l \\
&\quad + \sum_{\lambda \in \{-2,2\}} \int \widehat{E}^c_{\delta_0}(k) \widehat{f}^{(j)}_{1,non}(k, k-l, \varepsilon t) \widehat{h}^{(j)}_{\lambda,\mu}(l, l-m) \mathrm{d}l \\
&\quad + \sum_{\lambda \in \{-2,2\}} \int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l, \varepsilon t) (\widehat{\omega}_2)^{-1}(l) \widehat{f}^{(j)}_{\lambda,1,non}(l, l-m, \varepsilon t) \mathrm{d}l \\
&\quad + \sum_{\lambda \in \{-2,2\}, \kappa \in \{-1,1\}} \int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l_1, \varepsilon t) (\widehat{\omega}_2)^{-1}(l_1) \\
&\qquad \times \widehat{f}^{(j)}_{\lambda,1,non}(l_1, l_1-l_2, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l_2, l_2-m, \varepsilon t) \mathrm{d}l_2 \mathrm{d}l_1 \\
&\quad + \sum_{\lambda \in \{-2,2\}, \kappa \in \{-2,2\}} \int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l_1, \varepsilon t) (\widehat{\omega}_2)^{-1}(l_1) \\
&\qquad \times \widehat{f}^{(j)}_{\lambda,res}(l_1, l_1-l_2, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l_2, l_2-m, \varepsilon t) \mathrm{d}l_2 \mathrm{d}l_1 \\
&\quad + \sum_{\lambda \in \{-2,2\}, \kappa \in \{-2,2\}} \int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l_1, \varepsilon t) (\widehat{\omega}_2)^{-1}(l_1) \\
&\qquad \times \widehat{f}^{(j)}_{\lambda,2,non}(l_1, l_1-l_2, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l_2, l_2-m, \varepsilon t) \mathrm{d}l_2 \mathrm{d}l_1,
\end{aligned}$$

with $\mu \in \{\pm 1\}$. Similarly, for $\mu \in \{\pm 2\}$ we get the following simplified equation

$$\begin{aligned}
&\widehat{E}_{\delta_0}(k) \widehat{f}^{(j+1)}_{1,non}(k, k-m, \varepsilon t) \\
&= \widehat{E}_{\delta_0}(k) \Bigg(\sum_{\kappa \in \{-1,1\}} \int \widehat{f}^{(j)}_{1,res}(k, k-l, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l, l-m, \varepsilon t) \mathrm{d}l \\
&\quad + \sum_{\lambda \in \{-2,2\}, \kappa \in \{-1,1\}} \int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l_1, \varepsilon t) (\widehat{\omega}_2)^{-1}(l_1) \\
&\qquad \times \widehat{f}^{(j)}_{\lambda,1,non}(l_1, l_1-l_2, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l_2, l_2-m, \varepsilon t) \mathrm{d}l_2 \mathrm{d}l_1 \\
&\quad + \sum_{\lambda \in \{-2,2\}} \int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l, \varepsilon t) (\widehat{\omega}_2)^{-1}(l) \widehat{f}^{(j)}_{\lambda,res}(l, l-m, \varepsilon t) \mathrm{d}l \\
&\quad + \sum_{\lambda \in \{-2,2\}, \kappa \in \{-2,2\}} \int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l_1, \varepsilon t) (\widehat{\omega}_2)^{-1}(l_1) \\
&\qquad \times \widehat{f}^{(j)}_{\lambda,res}(l_1, l_1-l_2, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l_2, l_2-m, \varepsilon t) \mathrm{d}l_2 \mathrm{d}l_1 \\
&\quad + \sum_{\lambda \in \{-2,2\}} \int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l, \varepsilon t) (\widehat{\omega}_2)^{-1}(l) \widehat{f}^{(j)}_{\lambda,2,non}(l, l-m, \varepsilon t) \mathrm{d}l \\
&\quad + \sum_{\lambda \in \{-2,2\}, \kappa \in \{-2,2\}} \int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k, k-l_1, \varepsilon t) (\widehat{\omega}_2)^{-1}(l_1) \\
&\qquad \times \widehat{f}^{(j)}_{\lambda,2,non}(l_1, l_1-l_2, \varepsilon t) \widehat{h}^{(j)}_{\kappa,\mu}(l_2, l_2-m, \varepsilon t) \mathrm{d}l_2 \mathrm{d}l_1 \Bigg).
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
&\widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j+1)}_{1,non}(k,k-m,\varepsilon t)-\widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-m,\varepsilon t)\\
&=\sum_{\lambda\in\{-2,2\}}\int \widehat{E}^c_{\delta_0}(k)\widehat{f}^{(j)}_{1,non}(k,k-l,\varepsilon t)\widehat{h}^{(j)}_{\lambda,\mu}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad+\widehat{E}^c_{\delta_0}(k)\Bigg[\sum_{\kappa\in\{-1,1\}}\int \widehat{f}^{(j)}_{1,res}(k,k-l,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad+\sum_{\lambda\in\{-2,2\},\kappa\in\{-1,1\}}\int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\times\widehat{f}^{(j)}_{\lambda,1,non}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\\
&\quad+\sum_{\lambda\in\{-2,2\}}\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k,k-l,\varepsilon t)(\widehat{\omega}_2)^{-1}(l)\widehat{f}^{(j)}_{\lambda,res}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad+\sum_{\lambda\in\{-2,2\},\kappa\in\{-2,2\}}\int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\times\widehat{f}^{(j)}_{\lambda,res}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\\
&\quad+\sum_{\lambda\in\{-2,2\}}\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k,k-l,\varepsilon t)(\widehat{\omega}_2)^{-1}(l)\widehat{f}^{(j)}_{\lambda,2,non}(l,l-m,\varepsilon t)\mathrm{d}l\\
&\quad+\sum_{\lambda\in\{-2,2\},\kappa\in\{-2,2\}}\int\int \widehat{\widehat{g}}^{(j)}_{1,\lambda}(k,k-l_1,\varepsilon t)(\widehat{\omega}_2)^{-1}(l_1)\\
&\quad\times\widehat{f}^{(j)}_{\lambda,2,non}(l_1,l_1-l_2,\varepsilon t)\widehat{h}^{(j)}_{\kappa,\mu}(l_2,l_2-m,\varepsilon t)\mathrm{d}l_2\mathrm{d}l_1\Bigg],
\end{aligned}$$

for $\mu\in\{\pm 2\}$. In contrast to [DKS16], the non-resonant terms outside of the δ_0-neighborhood of $k=0$ will not be eliminated. These terms can be controlled by the artificial damping of the Gevrey spaces if the limit is not infinity. By adding up these terms, we have to ensure that we obtain a convergent series. However, we can choose the rate of decreasing the strip in time large enough to control these terms. Moreover, we have

$$\begin{aligned}
\widehat{p}_{1,j+1}(k,t)-\widehat{p}_{1,j}(k,t) &= \sum_{\lambda\in\{-2,2\}}\int \partial_T\widehat{g}^{(j)}_{1,\lambda}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{\lambda,j}(m,t)\mathrm{d}m\\
&\quad+\sum_{\lambda\in\{-2,2\}}\int \widehat{g}^{(j)}_{1,\lambda}(k,k-m,\varepsilon t)\widehat{p}_{\lambda,j}(m,t)\mathrm{d}m+\widehat{r}_{1,j}(k,t).
\end{aligned}$$

Similarly, we obtain the equations for $\widehat{\mathcal{R}}_2$. Note, that in this case we apply the normal form transformations to the non-resonant terms associated to $\widehat{\mathcal{R}}_{\pm1}$ and $\widehat{\mathcal{R}}_{-2}$.

2.6.3 The functional analytic set-up and the inversion of the normal form transformations

We introduce the norm

$$\|f\|_{X_\sigma^{s,\varepsilon}} := \int \sup_{k\in\mathbb{R}} |\mathrm{e}^{\sigma|k|} f(k,l)| \left(1+\left(\frac{l}{\varepsilon}\right)^2\right)^{s/2} \mathrm{d}l,$$

to control f, g and h which are (infinite) sums of terms of the form $\widehat{\kappa}^{(j)}(k)\varepsilon^{-1}\widehat{\varphi}^{(j)}(\frac{k-m}{\varepsilon})$ where $\widehat{\kappa}^{(j)}$ is Lipschitz continuous and determined by ω_1 and ω_2 and where $\widehat{\varphi}^{(j)}(\cdot,\varepsilon t)$ is determined by $\widehat{\psi}(\cdot,\varepsilon t)$. Young's inequality for convolutions yields

$$\left\| \int \widehat{f}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}(m)\mathrm{d}m \right\|_{G_\sigma^{s,*}} \leq C\|\widehat{f}\|_{X_\sigma^{s,\varepsilon}} \|\widehat{\mathcal{R}}\|_{G_\sigma^{s,*}}.$$

We have the following lemma to control the convolutions of f, g and h in the previous recursion formulas.

Lemma 2.6.1. *For $s>0$ the following estimate holds*

$$\left\| \int f(\cdot_1,\cdot_1-l)g(l,l-\cdot_2)\mathrm{d}l \right\|_{X_\sigma^{s,\varepsilon}} \leq \|f\|_{X_\sigma^{s,\varepsilon}} \|g\|_{X_\sigma^{s,\varepsilon}}.$$

Proof. We follow the steps from the proof of Lemma 3.1 in [DKS16]. With the help of Young's inequality for convolutions in weighted L^1-spaces we have

$$\begin{aligned}
&\int \sup_{k\in\mathbb{R}} \left| \mathrm{e}^{\sigma|k|} \int f(k,k-l)g(l,l-m)\mathrm{d}l \right| (1+((k-m)/\varepsilon)^2)^{s/2}\mathrm{d}(k-m) \\
&\leq \int \sup_{k\in\mathbb{R}} \int \sup_{\tilde{k}\in\mathbb{R}} |\mathrm{e}^{\sigma|\tilde{k}|} f(\tilde{k},k-l)| \sup_{\tilde{k}\in\mathbb{R}} |\mathrm{e}^{\sigma|\tilde{k}|} g(\tilde{k},l-k+m)|\mathrm{d}l(1+(m/\varepsilon)^2)^{s/2}\mathrm{d}m \\
&\leq \int \sup_{k\in\mathbb{R}} \int \sup_{\tilde{k}\in\mathbb{R}} |\mathrm{e}^{\sigma|\tilde{k}|} f(\tilde{k},l)| \sup_{\tilde{k}\in\mathbb{R}} |\mathrm{e}^{\sigma|\tilde{k}|} g(\tilde{k},m-l)|\mathrm{d}l(1+(m/\varepsilon)^2)^{s/2}\mathrm{d}m \\
&\leq C \int \sup_{k\in\mathbb{R}} |\mathrm{e}^{\sigma|k|} f(k,l)|(1+(l/\varepsilon)^2)^{s/2}\mathrm{d}l \int \sup_{k\in\mathbb{R}} |\mathrm{e}^{\sigma|k|} g(k,m)|(1+(m/\varepsilon)^2)^{s/2}\mathrm{d}m \\
&= \|f\|_{X_\sigma^{s,\varepsilon}} \|g\|_{X_\sigma^{s,\varepsilon}}.
\end{aligned}$$

□

For $\|\widehat{g}_{i,l}^{(j)}(\varepsilon t)\|_{X_\sigma^{s,\varepsilon}}$ sufficiently small, independent of $0<\varepsilon\ll 1$, the normal form transformation (2.18)-(2.19) is invertible with the following lemma.

Lemma 2.6.2. *For $\widehat{\mathcal{R}}_{i,j}(t) \in G_\sigma^{s,*}$ with $s \geq 1$ we introduce*

$$\widehat{\mathcal{R}}_{j+1}(k,t) = (I+T^{(j)})(\widehat{\mathcal{R}}_j(k,t)),$$

where

$$T^{(j)}(\widehat{R}_j) = \begin{pmatrix} 0 & 0 & T^{(j)}_{1,2} & T^{(j)}_{1,-2} \\ 0 & 0 & T^{(j)}_{-1,2} & T^{(j)}_{-1,-2} \\ T^{(j)}_{2,1} & T^{(j)}_{2,-1} & 0 & T^{(j)}_{2,-2} \\ T^{(j)}_{-2,1} & T^{(j)}_{-2,-1} & T^{(j)}_{-2,2} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{R}}_{1,j} \\ \widehat{\mathcal{R}}_{-1,j} \\ \widehat{\mathcal{R}}_{2,j} \\ \widehat{\mathcal{R}}_{-2,j} \end{pmatrix},$$

with

$$(T^{(j)}_{i,l}\widehat{\mathcal{R}}_{l,j})(k,t) = \int \widehat{g}^{(j)}_{i,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j}(m,t)\mathrm{d}m.$$

Thus, there exists a $q > 0$ *such that the transformation is bijective and its inverse has the form*

$$\widehat{\mathcal{R}}_{i,j}(k,t) = \widehat{\mathcal{R}}_{i,j+1}(k,t) + \sum_{l\in\{\pm1,\pm2\}} \int \widehat{h}^{(j)}_{i,l}(k,k-m,\varepsilon t)\widehat{\mathcal{R}}_{l,j+1}(m,t)\mathrm{d}m,$$

where

$$\|\widehat{h}^{(j)}(\varepsilon t)\|_{X^{s,\varepsilon}_\sigma} \leq \frac{C\|\widehat{g}^{(j)}(\varepsilon t)\|_{X^{s,\varepsilon}_\sigma}}{1-\|\widehat{g}^{(j)}(\varepsilon t)\|_{X^{s,\varepsilon}_\sigma}},$$

with

$$\|\widehat{g}^{(j)}(\varepsilon t)\|_{X^{s,\varepsilon}_\sigma} = \max_{i,l\in\{\pm1,\pm2\}}\{\|\widehat{g}^{(j)}_{i,l}(\varepsilon t)\|_{X^{s,\varepsilon}_\sigma}\},$$

if

$$\|\widehat{g}^{(j)}_{i,l}(\varepsilon t)\|_{X^{s,\varepsilon}_\sigma} \leq q \tag{2.20}$$

holds for all $i,l \in \{\pm1,\pm2\}$.

Proof. The proof is guided by [DKS16]. Let t be arbitrary, but fixed. The operator $T^{(j)} : (G^{s,*}_\sigma)^4 \to (G^{s,*}_\sigma)^4$ is obviously linear. We introduce

$$T^{(j)}_1 = \begin{pmatrix} T^{(j)}_{1,2} & T^{(j)}_{1,-2} \\ T^{(j)}_{-1,2} & T^{(j)}_{-1,-2} \end{pmatrix}, \quad T^{(j)}_2 = \begin{pmatrix} T^{(j)}_{2,1} & T^{(j)}_{2,-1} \\ T^{(j)}_{-2,1} & T^{(j)}_{-2,-1} \end{pmatrix} \quad \text{and} \quad T^{(j)}_3 = \begin{pmatrix} 0 & T^{(j)}_{2,-2} \\ T^{(j)}_{-2,2} & 0 \end{pmatrix},$$

and set $\|T^{(j)}\| := \max\{\|T^{(j)}_1\|, \|T^{(j)}_2\|, \|T^{(j)}_3\|\}$ with

$$\|T^{(j)}_1\|^2 = \sup_{\|(\widehat{\mathcal{R}}_{2,j},\widehat{\mathcal{R}}_{-2,j})^T\|\leq 1} \left(\sum_{l\in\{\pm2\}} \|T^{(j)}_{1,l}(\widehat{\mathcal{R}}_{l,j})\|^2_{G^{s,*}_\sigma} + \sum_{l\in\{\pm2\}} \|T^{(j)}_{-1,l}(\widehat{\mathcal{R}}_{l,j})\|^2_{G^{s,*}_\sigma} \right).$$

By Lemma 2.6.1 we have

$$\|T^{(j)}_{i,l}\widehat{\mathcal{R}}_{l,j}\|_{G^{s,*}_\sigma} \leq C\|\widehat{g}^{(j)}_{i,l}\|_{X^{s,\varepsilon}_\sigma}\|\widehat{\mathcal{R}}_{l,j}\|_{G^{s,*}_\sigma}.$$

Thus, we have $\|T^{(j)}\| = \mathcal{O}(q)$. For $q > 0$ sufficiently small, we can use Neumann's series to invert

$$(I - (-T^{(j)}))^{-1} = \sum_{\lambda=0}^{\infty} (-T^{(j)})^{\circ\lambda}.$$

Hereby, $(T^{(j)})^{\circ\lambda}$ denotes the λ-times composition of $T^{(j)}$. For each pair $T^{(j)}_{i,l}$ and $T^{(j)}_{s,t}$ we get

$$\begin{aligned}(T^{(j)}_{i,l} \circ T^{(j)}_{s,t})\widehat{\mathcal{R}}_{1,j+1}(k,t) &= \int \widehat{g}^{(j)}_{i,l}(k, k-m, \varepsilon t) \int \widehat{g}^{(j)}_{s,t}(m, m-m_1, \varepsilon t)\widehat{\mathcal{R}}_{1,j+1}(m_1,t)\mathrm{d}m_1\mathrm{d}m \\ &= \int\int \widehat{g}^{(j)}_{i,l}(k, k-m, \varepsilon t)\widehat{g}^{(j)}_{s,t}(m, m-m_1, \varepsilon t)\widehat{\mathcal{R}}_{1,j+1}(m_1,t)\mathrm{d}m_1\mathrm{d}m.\end{aligned}$$

Inductively, we obtain a series of integral kernels. The $X^{s,\varepsilon}_{\sigma}$-norm of $\widehat{h}^{(j)}_{i,k}$ is bounded by

$$\|h^{(j)}\|_{X^{s,\varepsilon}_{\sigma}} \leq C \sum_{l=1}^{\infty} \left(\|g^{(j)}\|_{X^{s,\varepsilon}_{\sigma}}\right)^l = \frac{C\|g^{(j)}\|_{X^{s,\varepsilon}_{\sigma}}}{1 - \|g^{(j)}\|_{X^{s,\varepsilon}_{\sigma}}}.$$

This is exactly the form we have in the Lemma 2.6.2. □

2.6.4 The proof of convergence

In Lemma 2.6.2 we assumed that (2.20) holds. Now, our goal is to show such estimates or even sharper ones for $\widehat{f}^{(j)}_{..}$, $\widehat{g}^{(j)}_{..}$ and $\widehat{h}^{(j)}_{..}$. We adapt Lemma 2.6.1 to the differences and get

$$\begin{aligned}\|\widehat{f}^{(j+1)}_{1,res} - \widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_{\sigma}} \leq\ & \|\widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_{\sigma}} \sum_{\kappa\in\{-1,1\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_{\sigma}} \\ & + \|\widehat{E}^c_{\delta_0}\widehat{f}^{(j)}_{1,non}\|_{X^{s,\varepsilon}_{\sigma}} \sum_{\lambda\in\{-2,2\}} \|\widehat{h}^{(j)}_{\lambda,\mu}\|_{X^{s,\varepsilon}_{\sigma}} \\ & + \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_{\sigma}} \|\widehat{f}^{(j)}_{\lambda,1,non}\|_{X^{s,\varepsilon}_{\sigma}} \qquad (2.21) \\ & + \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_{\sigma}} \|\widehat{f}^{(j)}_{\lambda,1,non}\|_{X^{s,\varepsilon}_{\sigma}} \sum_{\kappa\in\{-1,1\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_{\sigma}} \\ & + \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_{\sigma}} \|\widehat{f}^{(j)}_{\lambda,res}\|_{X^{s,\varepsilon}_{\sigma}} \sum_{\kappa\in\{-2,2\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_{\sigma}} \\ & + \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_{\sigma}} \|\widehat{f}^{(j)}_{\lambda,2,non}\|_{X^{s,\varepsilon}_{\sigma}} \sum_{\kappa\in\{-2,2\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_{\sigma}},\end{aligned}$$

for $\mu \in \{\pm 1\}$. In the δ_0-neighborhood of $k = 0$ we have

$$\begin{aligned}
\|\widehat{E}_{\delta_0}\widehat{f}^{(j+1)}_{1,non}\|_{X^{s,\varepsilon}_\sigma} \leq\ & \|\widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-1,1\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,1,non}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-1,1\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,res}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,res}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-2,2\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,2,non}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,2,non}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-2,2\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma},
\end{aligned}$$

with $\mu \in \{\pm 2\}$. Outside of the δ_0-neighborhood of $k = 0$ we obtain

$$\begin{aligned}
\|\widehat{E}^c_{\delta_0}(\widehat{f}^{(j+1)}_{1,non} - \widehat{f}^{(j)}_{1,non})\|_{X^{s,\varepsilon}_\sigma} \leq\ & \|\widehat{f}^{(j)}_{1,non}\|_{X^{s,\varepsilon}_\sigma} \sum_{\lambda\in\{-2,2\}} \|\widehat{h}^{(j)}_{\lambda,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \|\widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-1,1\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,1,non}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-1,1\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,res}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,res}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-2,2\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,2,non}\|_{X^{s,\varepsilon}_\sigma} \\
&+ \sum_{\lambda\in\{-2,2\}} \|\widehat{\widehat{g}}^{(j)}_{1,\lambda}\|_{X^{s,\varepsilon}_\sigma} \|\widehat{f}^{(j)}_{\lambda,2,non}\|_{X^{s,\varepsilon}_\sigma} \sum_{\kappa\in\{-2,2\}} \|\widehat{h}^{(j)}_{\kappa,\mu}\|_{X^{s,\varepsilon}_\sigma},
\end{aligned}$$

where $\mu \in \{\pm 2\}$. Similarly, we can show inequalities for the corresponding terms of (2.19). In the following lemma, we collect important estimates that imply the main convergence theorem.

Lemma 2.6.3. *There exists a $q > 0$ such that for*

$$\|\widehat{f}^{(1)}_{\nu,res}\|_{X^{s,\varepsilon}_\sigma} + \|\widehat{f}^{(1)}_{\nu,non}\|_{X^{s,\varepsilon}_\sigma} \leq q,$$

with $\nu \in \{\pm 1, \pm 2\}$, we have the following inequalities

a) $\|\widehat{f}^{(j)}_{\kappa,res}\|_{X^{s,\varepsilon}_\sigma} \leq q\frac{1-q^{j/2}}{1-q^{1/2}}$, *b1)* $\|\widehat{E}_{\delta_0}\widehat{f}^{(j)}_{\kappa,non}\|_{X^{s,\varepsilon}_\sigma} \leq q^{(j+1)/2}$,

b2) $\|\widehat{E}^c_{\delta_0}\widehat{f}^{(j)}_{\kappa,non}\|_{X^{s,\varepsilon}_\sigma} \leq q\frac{1-q^{j/2}}{1-q^{1/2}}$,

c) $\|\widehat{g}^{(j)}_{\kappa,\lambda}\|_{X^{s,\varepsilon}_\sigma} \leq C_\omega q^{(j+1)/2}$,

d) $\|\widehat{h}^{(j)}_{\kappa,\lambda}\|_{X^{s,\varepsilon}_\sigma} \leq 2C_\omega q^{(j+1)/2}$,

e) $\|\widehat{f}^{(j+1)}_{\kappa,res} - \widehat{f}^{(j)}_{\kappa,res}\|_{X^{s,\varepsilon}_\sigma} \leq qq^{j/2}$,

f) $\|\widehat{E}^c_{\delta_0}(\widehat{f}^{(j+1)}_{\kappa,non} - \widehat{f}^{(j)}_{\kappa,non})\|_{X^{s,\varepsilon}_\sigma} \leq qq^{j/2}$,

for all $j \in \mathbb{N}$ *and* $\kappa, \lambda \in \{\pm 1, \pm 2\}$ *with*

$$C_\omega = \max_{(\mu,\lambda)\in\substack{\{(\pm1,\pm2),\\ (\pm2,\mp2)\}}} \sup_{k\in\mathbb{R}} |\mathrm{i}\widehat{\omega}_\mu(k) - \mathrm{i}\widehat{\omega}_\lambda(k)|^{-1}.$$

Proof. The proof is based on an induction argument which is also performed in the proof of Lemma 3.3 in [DKS16].

i) For $j = 1$ the estimates *a*), *b*1) and *b*2) follow directly from the assumption of the lemma and the estimate *c*) from the definition and the weaker non-resonance condition. Lemma 2.6.2 implies

$$\|\widehat{h}^{(j)}\|_{X^{s,\varepsilon}_\sigma} \leq \frac{C\|\widehat{g}^{(j)}\|_{X^{s,\varepsilon}_\sigma}}{1 - \|\widehat{g}^{(j)}\|_{X^{s,\varepsilon}_\sigma}}.$$

If q is smaller than $\frac{1}{2C_\omega}$, then we have the estimate *d*) for $j = 1$ based on induction.

ii) In the following we will use the abbreviation

$$\|\widehat{f}^{(j)}_{non}\|_{X^{s,\varepsilon}_\sigma} = \max_{\kappa\in\{\pm1,\pm2\}} \|\widehat{f}^{(j)}_{\kappa,non}\|_{X^{s,\varepsilon}_\sigma}.$$

With (2.21) we have

$$\begin{aligned}
\|\widehat{f}^{(j+1)}_{1,res} - \widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_\sigma} &\leq 4C_\omega\|\widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_\sigma}\|\widehat{f}^{(j)}_{non}\|_{X^{s,\varepsilon}_\sigma} + 4C_\omega\|\widehat{f}^{(j)}_{non}\|^2_{X^{s,\varepsilon}_\sigma} \\
&\quad +2C_\omega\|\widehat{f}^{(j)}_{non}\|^2_{X^{s,\varepsilon}_\sigma} + 8C^2_\omega\|\widehat{f}^{(j)}_{non}\|^3_{X^{s,\varepsilon}_\sigma} \\
&\quad +8C^2_\omega\|\widehat{f}^{(j)}_{non}\|^2_{X^{s,\varepsilon}_\sigma}\|\widehat{f}^{(j)}_{2,res}\|_{X^{s,\varepsilon}_\sigma} + 8C^2_\omega\|\widehat{f}^{(j)}_{non}\|^3_{X^{s,\varepsilon}_\sigma} \\
&\leq 4C_\omega\|\widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_\sigma}\|\widehat{f}^{(j)}_{non}\|_{X^{s,\varepsilon}_\sigma} + 6C_\omega\|\widehat{f}^{(j)}_{non}\|^2_{X^{s,\varepsilon}_\sigma} \\
&\quad +16C^2_\omega\|\widehat{f}^{(j)}_{non}\|^3_{X^{s,\varepsilon}_\sigma} + 8C^2_\omega\|\widehat{f}^{(j)}_{non}\|^2_{X^{s,\varepsilon}_\sigma}\|\widehat{f}^{(j)}_{2,res}\|_{X^{s,\varepsilon}_\sigma}.
\end{aligned}$$

For the induction, we assume

$$\|\widehat{f}^{(j)}_{i,non}\|_{X^{s,\varepsilon}_\sigma} \leq q^{(j+1)/2} \quad \text{and} \quad \|\widehat{f}^{(j)}_{i,res}\|_{X^{s,\varepsilon}_\sigma} \leq q\frac{1-q^{j/2}}{1-q^{1/2}}.$$

This allows us to estimate

$$\begin{aligned}\|\widehat{f}^{(j+1)}_{1,res} - \widehat{f}^{(j)}_{1,res}\|_{X^{s,\varepsilon}_\sigma} &\leq \frac{4C_\omega}{1-q^{1/2}} q^{(j+3)/2} + 6C_\omega q^{j+1} + 16C_\omega^2 q^{(3(j+1))/2} + \frac{8C_\omega^2}{1-q^{1/2}} q^{j+2} \\ &\leq qq^{j/2}.\end{aligned}$$

Therefore, we obtain $e)$ for $q > 0$ sufficiently small. Analogously, outside of the δ_0-neighborhood of $k = 0$ we find

$$\begin{aligned}\|\widehat{E}^c_{\delta_0}(\widehat{f}^{(j+1)}_{1,non} - \widehat{f}^{(j)}_{1,non})\|_{X^{s,\varepsilon}_\sigma} &\leq 4C_\omega q^{j+1} + \frac{6C_\omega}{1-q^{1/2}} q^{(j+3)/2} + 2C_\omega q^{j+1} \\ &\quad + \frac{8C_\omega^2}{1-q^{1/2}} q^{j+2} + 16C_\omega^2 q^{(3(j+1))/2} \\ &\leq q^{(j+2)/2}.\end{aligned}$$

We get $f)$ for $q > 0$ sufficiently small. In addition, we have

$$\widehat{f}^{(j+1)}_{1,res} = \widehat{f}^{(1)}_{1,res} + \sum_{l=1}^{j} (\widehat{f}^{(l+1)}_{1,res} - \widehat{f}^{(l)}_{1,res}).$$

Thus, we obtain

$$\|\widehat{f}^{(j+1)}_{1,res}\|_{X^{s,\varepsilon}_\sigma} \leq q \sum_{l=0}^{j} q^{l/2} = q \frac{1-q^{(j+1)/2}}{1-q^{1/2}}.$$

In a similar way, we proceed for $\|\widehat{f}^{(j)}_{i,non}\|_{X^{s,\varepsilon}_\sigma}$ in the δ_0-neighborhood of $k = 0$ and find

$$\begin{aligned}\|\widehat{E}_{\delta_0}\widehat{f}^{(j+1)}_{i,non}\|_{X^{s,\varepsilon}_\sigma} &\leq \frac{6C_\omega}{1-q^{1/2}} q^{(j+3)/2} + 2C_\omega q^{j+1} + 16C_\omega^2 q^{(3(j+1))/2} + \frac{8C_\omega^2}{1-q^{1/2}} q^{j+2} \\ &\leq q^{(j+2)/2},\end{aligned}$$

for q sufficiently small. Hence, we have shown $b1)$. Outside of the δ_0-neighborhood we get

$$\|\widehat{E}^c_{\delta_0}\widehat{f}^{(j+1)}_{1,non}\|_{X^{s,\varepsilon}_\sigma} \leq q \sum_{l=0}^{j} q^{l/2} = q \frac{1-q^{(j+1)/2}}{1-q^{1/2}},$$

and so we have $b2)$ for $q > 0$ sufficiently small. The estimates $c)$ and $d)$ follow similarly as in part i).

□

Remark 2.6.4. Since $\|\psi\|_{\mathcal{W}^s_\sigma}$ is sufficiently small, we have

$$\|\widehat{f}^{(1)}_{\nu,res}\|_{X^{s,\varepsilon}_\sigma} + \|\widehat{f}^{(1)}_{\nu,non}\|_{X^{s,\varepsilon}_\sigma} \leq q,$$

for a $0 < q \ll 1$ independent of $0 < \varepsilon \ll 1$.

Finally, we need estimates for the higher order terms.

Lemma 2.6.5. *The terms* $\widehat{p}_{l,j+1}$ *can be bounded in the following way*

$$\begin{aligned}\|\widehat{p}_{l,j+1}\|_{G_\sigma^{s,*}} &\leq C(\|\widehat{\mathcal{R}}_{-2,j+1}\|_{G_\sigma^{s,*}} + \|\widehat{\mathcal{R}}_{-1,j+1}\|_{G_\sigma^{s,*}} + \|\widehat{\mathcal{R}}_{1,j+1}\|_{G_\sigma^{s,*}} + \|\widehat{\mathcal{R}}_{2,j+1}\|_{G_\sigma^{s,*}} \\ &\quad + \varepsilon^{1/2}(\|\widehat{\mathcal{R}}_{-2,j+1}\|_{G_\sigma^{s,*}} + \|\widehat{\mathcal{R}}_{-1,j+1}\|_{G_\sigma^{s,*}} + \|\widehat{\mathcal{R}}_{1,j+1}\|_{G_\sigma^{s,*}} + \|\widehat{\mathcal{R}}_{2,j+1}\|_{G_\sigma^{s,*}})^2 + 1),\end{aligned}$$

and

$$\|\widehat{p}_{l,j+1} - \widehat{p}_{l,j}\|_{G_\sigma^{s,*}} \leq Cq^{1/2},$$

with $0 < q \ll 1$ *from Lemma 2.6.1 and a generic constant* C *independent of* j *and* $0 < \varepsilon \ll 1$.

Proof. Using Lemma 2.6.1 we have

$$\begin{aligned}\|\widehat{p}_{l,j+1} - \widehat{p}_{l,j}\|_{G_\sigma^{s,*}} &\leq \sum_{\lambda \in \{-2,2\}} \|\partial_T \widehat{g}_{1,\lambda}^{(j)}\|_{X_\sigma^{s,\varepsilon}} \|\widehat{\mathcal{R}}_{\lambda,j}\|_{G_\sigma^{s,*}} \\ &\quad + \sum_{\lambda \in \{-2,2\}} \|\widehat{g}_{1,\lambda}^{(j)}\|_{X_\sigma^{s,\varepsilon}} \|\widehat{p}_{\lambda,j}\|_{G_\sigma^{s,*}} + \|\widehat{r}_{1,j}\|_{G_\sigma^{s,*}}.\end{aligned}$$

We obtain

$$\|\widehat{r}_{1,j}\|_{G_\sigma^{s,*}} \leq C \sum_{l \in \{2,-2\}} \|\widehat{g}_{1,l}^{(j)}\|_{X_\sigma^{s,\varepsilon}} \|\widehat{\mathcal{R}}_{l,j}\|_{G_\sigma^{s,*}},$$

where C is independent of j similar to Lemma 3.1 of [DS06]. With the help of Lemma 2.6.1 we find that

$$\|\widehat{p}_{,j+1} - \widehat{p}_{,j}\|_{G_\sigma^{s,*}} \leq Cq^{(j+1)/2} \|\widehat{\mathcal{R}}_{,j}\|_{G_\sigma^{s,*}} + Cq^{(j+1)/2} \|\widehat{p}_{,j}\|_{G_\sigma^{s,*}},$$

where $\widehat{p}_{,j} = (\widehat{p}_{-2,j}, \widehat{p}_{-1,j}, \widehat{p}_{1,j}, \widehat{p}_{2,j})$. The inequality shows that $(\widehat{p}_{,j})_{j\in\mathbb{N}}$ converges like a geometric series and that $\widehat{p}_{,j}$ can be estimated by $\widehat{p}_{,1}$. □

2.6.5 Limit system

By Lemma 2.6.1 and Lemma 2.6.5 we have shown for $l \in \{\pm 1, \pm 2\}$ that the non-resonant terms converge to 0 in a δ_0-neighborhood of $k = 0$, i.e.,

$$\|\widehat{E}_{\delta_0} \widehat{f}_{l,non}^{(j)}\|_{X_\sigma^{s,\varepsilon}} \to 0,$$

for $j \to \infty$, the sequences $(\widehat{f}_{l,res}^{(j)})_{j\in\mathbb{N}}$ and $(\widehat{E}_{\delta_0}^c \widehat{f}_{l,non}^{(j)})_{j\in\mathbb{N}}$ are Cauchy sequences in $X_\sigma^{s,\varepsilon}$ and the sequences $(\widehat{p}_{l,j})_{j\in\mathbb{N}}$ and $(\widehat{R}_{l,j})_{j\in\mathbb{N}}$ are Cauchy sequences in $G_\sigma^{s,*}$. Due to the completeness of $X_\sigma^{s,\varepsilon}$ and $G_\sigma^{s,*}$, we conclude that the limits of these sequences exist in $X_\sigma^{s,\varepsilon}$ and $G_\sigma^{s,*}$. After infinitely many near identity changes of variables, we can eliminate the non-resonant terms at leading order ε. We obtain the following limit system in physical space

$$\begin{aligned}\partial_t \mathcal{R}_{1,\infty} &= \mathrm{i}\omega_1 \mathcal{R}_{1,\infty} + \mathrm{i}\omega_1 f_{1,res}(\mathcal{R}_{1,\infty} + \mathcal{R}_{-1,\infty}) + \varepsilon P_1 + \mathrm{i}\omega_1 E_{\delta_0}^c f_{1,non}(\mathcal{R}_{2,\infty} + \mathcal{R}_{-2,\infty}), \\ \partial_t \mathcal{R}_{2,\infty} &= \mathrm{i}\omega_2 \mathcal{R}_{2,\infty} + (\mathrm{i}\omega_2)^{-1} f_{2,res} \mathcal{R}_{2,\infty} + \varepsilon P_2 \\ &\quad + (\mathrm{i}\omega_2)^{-1} E_{\delta_0}^c f_{2,1,non}(\mathcal{R}_{1,\infty} + \mathcal{R}_{-1,\infty}) + (\mathrm{i}\omega_2)^{-1} E_{\delta_0}^c f_{2,2,non} \mathcal{R}_{-2,\infty}.\end{aligned}$$

The system for $\mathcal{R}_{-1,\infty}$ and $\mathcal{R}_{-2,\infty}$ can be found in a similar way. Hereby, we denote $\mathcal{R}_{l,\infty}$, $f_{l,res}$ and P_l as the limits for $j \to \infty$. The higher order terms can be estimated by

$$\begin{aligned} \|P_l\|_{G^s_\sigma} \leq\ & C(\|\mathcal{R}_{-2,\infty}\|_{G^s_\sigma} + \|\mathcal{R}_{-1,\infty}\|_{G^s_\sigma} + \|\mathcal{R}_{1,\infty}\|_{G^s_\sigma} + \|\mathcal{R}_{2,\infty}\|_{G^s_\sigma} \\ & + \varepsilon^{1/2}(\|\mathcal{R}_{-2,\infty}\|_{G^s_\sigma} + \|\mathcal{R}_{-1,\infty}\|_{G^s_\sigma} + \|\mathcal{R}_{1,\infty}\|_{G^s_\sigma} + \|\mathcal{R}_{2,\infty}\|_{G^s_\sigma})^2 + 1). \end{aligned} \tag{2.22}$$

2.7 From analytic to Sobolev functions

Since we want to handle the energy estimates in usual Sobolev spaces similar to [Sch20], we use this part to prepare the system. To control the growth rates at the unstable resonances, we make the strip where the functions are analytic smaller in time. Due to this fact we estimate the error in G^s_σ -spaces with $s \geq 1$ and $\sigma = \sigma(t)$ which is decreasing in time, i.e., we have

$$\sigma\left(t\right) = \sigma_A/\varepsilon - \beta t,$$

for $0 \leq t \leq T_1/\varepsilon$ with $T_1 = \sigma_A/\beta$. In the energy estimates below, $\beta > 0$ has to be chosen sufficiently large which results in a restriction in time. This leads to an artificial damping with respect to this time-dependent norm which controls the growth of the unstable resonances. The solutions of the WME (2.6) satisfy

$$\sup_{T\in[0,T_0]} \|A\left(T\right)\|_{\mathcal{W}^{s_A}_{\sigma_A}} \leq C_\psi.$$

This yields

$$\sup_{T\in[0,T_0/\varepsilon]} \|\psi_{u,v}\left(t\right)\|_{\mathcal{W}^{s_A}_{\sigma_A}} \leq C_\psi.$$

To establish a connection between the spaces of analytic functions and Sobolev spaces, we introduce

$$R_{j,\infty}(t) = S_\omega(t)\mathcal{R}_{j,\infty}(t) \quad \text{and} \quad \psi^*_{u,v}(t) = S_\omega(t)\psi_{u,v}(t),$$

where $S_\omega(t)$ is a multiplication operator defined in Fourier space by

$$\widehat{S}_\omega(k,t) = \mathrm{e}^{(\sigma_A/\varepsilon - \beta t)|k|}.$$

As a direct consequence of these definitions, we obtain the following lemma.

Lemma 2.7.1. *For $t \in [0, \sigma_A/(\beta\varepsilon)]$ the linear mappings $S_\omega(t) : G^s_{\sigma(t)} \to H^s$ and $S_\omega(t) : \mathcal{W}^s_{\sigma(t)} \to W^s$, with $\sigma(t) = (\sigma_A - \beta\varepsilon t)/\varepsilon$, are bijective and bounded with bounded inverse.*

Remark 2.7.2. We introduce the norm

$$\|f\|_{X^{s,\varepsilon}} := \int \sup_{k\in\mathbb{R}} |f(k,l)| \left(1 + \left(\frac{l}{\varepsilon}\right)^2\right)^{s/2} \mathrm{d}l.$$

Furthermore, the linear mapping $S_\omega : X^{s,\varepsilon}_{\sigma(t)} \to X^{s,\varepsilon}$ is bounded with bounded inverse.

The newly defined variables satisfy the limit system

$$\begin{aligned}\partial_t R_{1,\infty} &= -\beta|k|_{op}R_{1,\infty} + \mathrm{i}\omega_1 R_{1,\infty} + \varepsilon P_1^* + S_\omega \mathrm{i}\omega_1 f_{1,res} S_\omega^{-1}(R_{1,\infty} + R_{-1,\infty}) \\ &\quad + S_\omega \mathrm{i}\omega_1 E_{\delta_0}^c f_{1,non} S_\omega^{-1}(R_{2,\infty} + R_{-2,\infty}),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\partial_t R_{2,\infty} &= -\beta|k|_{op}R_{2,\infty} + \mathrm{i}\omega_2 R_{2,\infty} + \varepsilon P_2^* + S_\omega(\mathrm{i}\omega_2)^{-1} f_{2,res} S_\omega^{-1}\mathcal{R}_{2,\infty} \\ &\quad + S_\omega(\mathrm{i}\omega_2)^{-1} E_{\delta_0}^c f_{2,1,non} S_\omega^{-1}(R_{1,\infty} + R_{-1,\infty}) \\ &\quad + S_\omega(\mathrm{i}\omega_2)^{-1} E_{\delta_0}^c f_{2,2,non} S_\omega^{-1} R_{-2,\infty},\end{aligned} \tag{2.24}$$

and similarly for R_{-1} and R_{-2}, where the higher order terms fulfill

$$\begin{aligned}\|P_l^*\|_{H^s} &\leq C(\|R_{-2,\infty}\|_{H^s} + \|R_{-1,\infty}\|_{H^s} + \|R_{1,\infty}\|_{H^s} + \|R_{2,\infty}\|_{H^s} \\ &\quad + \varepsilon^{1/2}(\|R_{-2,\infty}\|_{H^s} + \|R_{-1,\infty}\|_{H^s} + \|R_{1,\infty}\|_{H^s} + \|R_{2,\infty}\|_{H^s})^2 + 1).\end{aligned} \tag{2.25}$$

The operator $|k|_{op}$ is defined by its operation in Fourier space, namely $\widehat{|k|_{op}R}(k) = |k|\widehat{R}(k)$. To finish the proof of Theorem 2.1.1, we need to prove a $\mathcal{O}(1)$-bound for the errors $R_{1,\infty}$ and $R_{2,\infty}$ on the long $\mathcal{O}(1/\varepsilon)$-time scale. The terms of $\mathcal{O}(\varepsilon)$ in (2.23)-(2.24) can easily be controlled with Gronwall's inequality. Since R_{-l} is the complex conjugate of R_l, it is sufficient to use $R_{|l|}$.

Remark 2.7.3. The local existence and uniqueness of solutions of the BKG system (2.1)-(2.2) in G_σ^s-spaces is provided by a simple application of the contraction mapping principle to the variation of constants formula since the BKG system is semilinear. Local existence and uniqueness on the required time interval is guaranteed by the following error estimates which serve as a priori estimates.

2.8 The final energy estimates

In this section we finish the proof by showing that the error R has a $\mathcal{O}(1)$-bound on a $\mathcal{O}(1/\varepsilon)$-time scale. To perform the final energy estimates for the system (2.23)-(2.24), we define an operator Ω by the multiplier $\widehat{\Omega}(k) = \min(\widehat{\omega}_2(k), 4)$ in Fourier space. We will use this operator in the estimates for the term $\mathrm{Re}\,(s_6)$ below. There it leads to a cancelation which shows that $\mathrm{Re}\,(s_6)$ is of order $\mathcal{O}(\varepsilon)$. We define the energy by

$$E_s = \|R_{1,\infty}\|_{H^s}^2 + \left\|\Omega^{1/2}R_{2,\infty}\right\|_{H^s}^2,$$

and compute the time derivative

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_s = \mathrm{Re}\ s_{good} + \sum_{j=1}^{7}\mathrm{Re}\ s_j,$$

where

$$\begin{aligned}
s_{good} &= (R_{1,\infty}, -\beta|k|_{op}R_{1,\infty})_{H^s} + (\Omega^{1/2}R_{2,\infty}, -\beta|k|_{op}\Omega^{1/2}R_{2,\infty})_{H^s}, \\
s_1 &= (R_{1,\infty}, \mathrm{i}\omega_1 R_{1,\infty})_{H^s} + \left(\Omega^{1/2}R_{2,\infty}, \mathrm{i}\omega_2\Omega^{1/2}R_{2,\infty}\right)_{H^s}, \\
s_2 &= (R_{1,\infty}, \varepsilon P_1^*)_{H^s} + (\Omega R_{2,\infty}, \varepsilon P_2^*)_{H^s}, \\
s_3 &= \left(R_{1,\infty}, S_\omega \mathrm{i}\omega_1 f_{1,res} S_\omega^{-1}(R_{1,\infty} + R_{-1,\infty})\right)_{H^s}, \\
s_4 &= \left(R_{1,\infty}, S_\omega \mathrm{i}\omega_1 E^c_{\delta_0} f_{1,non} S_\omega^{-1}(R_{2,\infty} + R_{-2,\infty})\right)_{H^s}, \\
s_5 &= \left(\Omega R_{2,\infty}, S_\omega(\mathrm{i}\omega_2)^{-1} E^c_{\delta_0} f_{2,1,non} S_\omega^{-1}(R_{1,\infty} + R_{-1,\infty})\right)_{H^s}, \\
s_6 &= \left(\Omega R_{2,\infty}, S_\omega(i\omega_2)^{-1} f_{2,res} S_\omega^{-1} R_{2,\infty}\right)_{H^s}, \\
s_7 &= \left(\Omega R_{2,\infty}, S_\omega(\mathrm{i}\omega_2)^{-1} E^c_{\delta_0} f_{2,2,non} S_\omega^{-1} R_{-2,\infty}\right)_{H^s}.
\end{aligned}$$

In the following, we justify that the terms $s_1, \ldots s_7$ are either of order $\mathcal{O}(\varepsilon)$ or can be estimated by the negative terms of order $\mathcal{O}(1)$ collected in s_{good}. For instance, it can be directly concluded that the terms collected in s_1 and s_2 are of order $\mathcal{O}(\varepsilon)$. The terms s_3, s_4, s_5 and s_7 are preceded by ω_1 or E^c_δ or both and thus vanish at the wave number $k = 0$. Therefore, they can be estimated by the "good" terms collected in s_{good}. The most problematic term is s_6. We can show that s_6 must be of order $\mathcal{O}(\varepsilon)$ by using the long wave character of the Whitham approximation. To estimate the terms s_3 and s_5, we use

$$(u, v)_{H^s} = (\widehat{u}, \widehat{v})_{L^2_s},$$

and Parseval's inequality, so we can work with the norm

$$\sum_{j=0}^{s} \int |k|^{2j} |\widehat{u}|^2(k) \mathrm{d}k = \sum_{j=0}^{s} \|\partial_x^j u\|_{L^2}$$

in Fourier space.

2.8.1 Estimates for s_{good}, s_1, and s_2

a) We start with the bound for the linear terms collected in s_{good}. Using the Fourier representation of $|k|_{op}$ we get

$$(R_{1,\infty}, -\beta|k|_{op}R_{1,\infty})_{H^s} = -\beta(\widehat{R}_{1,\infty}, |k|\widehat{R}_{1,\infty})_{L^2_s} = -\beta(|k|^{1/2}\widehat{R}_{1,\infty}, |k|^{1/2}\widehat{R}_{1,\infty})_{L^2_s},$$

and similarly for $(\Omega^{1/2}R_{2,\infty}, -\beta|k|_{op}\Omega^{1/2}R_{2,\infty})_{H^s}$, so that finally

$$s_{good} = -\beta \left\| |k|^{1/2} R_{1,\infty} \right\|^2_{H^s} - \beta \left\| |k|^{1/2} \Omega^{1/2} R_{2,\infty} \right\|^2_{H^s}.$$

b) Using the skew-symmetry of $\mathrm{i}\omega_1$ and $\mathrm{i}\omega_2$ yields

$$s_1 = 0.$$

c) With the help of the Cauchy-Schwarz inequality and (2.25) we obtain

$$|s_2| \leq C\varepsilon E_s + C\varepsilon^{3/2} E_s^{3/2} + C\varepsilon.$$

2.8.2 Estimates for s_3, s_4, s_5 and s_7

The “good” terms collected in s_{good} do not allow us to estimate terms that do not vanish at the wave number $k = 0$. We have to use the fact that the terms s_3, s_4, s_5 and s_7 vanish at the wave number $k = 0$.

a) The terms s_3 and s_4 can be estimated by the “good” terms using the fact that $|\widehat{\omega}_1(k)| \leq C|k|$ for $k \to 0$. The last estimate implies that the symbol of $\vartheta = |k|_{op}^{-1/2}\omega_1$ is bounded at the wave number $k = 0$ and $|\widehat{\vartheta}(k)| \leq C|k|^{1/2}$. We find

$$
\begin{aligned}
|s_3| &= \left|\Big(R_{1,\infty}, S_\omega \mathrm{i}\omega_1 f_{1,res} S_\omega^{-1}(R_{1,\infty} + R_{-1,\infty})\Big)_{H^s}\right| \\
&= \Big|\Big(\widehat{R}_{1,\infty}(k,t), \widehat{S}_\omega(k,t)\mathrm{i}\widehat{\omega}_1(k) \\
&\qquad \times \int \widehat{f}_{1,res}(k,k-m,\varepsilon t)\widehat{S}_\omega^{-1}(m,t)(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})(m,t)\mathrm{d}m\Big)_{L^2_s}\Big| \\
&= \Big|\Big(|k|^{1/2}\widehat{R}_{1,\infty}(k,t), \widehat{S}_\omega(k,t)\mathrm{i}|k|^{-1/2}\widehat{\omega}_1(k) \\
&\qquad \times \int \widehat{f}_{1,res}(k,k-m,\varepsilon t)\widehat{S}_\omega^{-1}(m,t)(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})(m,t)\mathrm{d}m\Big)_{L^2_s}\Big| \\
&\leq C\||k|^{1/2}\widehat{R}_{1,\infty}\|_{L^2_s} \\
&\qquad \left\|\widehat{S}_\omega(k,t)\widehat{\vartheta}(k)\int \widehat{f}_{1,res}(k,k-m,\varepsilon t)\widehat{S}_\omega^{-1}(m,t)(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})(m,t)\mathrm{d}m\right\|_{L^2_s} \\
&\leq C\||k|^{1/2}\widehat{R}_{1,\infty}\|_{L^2_s} \\
&\qquad \left(\left\|\int |k-m|^{1/2}\widehat{S}_\omega(k,t)\widehat{f}_{1,res}(k,k-m,\varepsilon t)(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})(m,t)\mathrm{d}m\right\|_{L^2_s}\right. \\
&\qquad \left. + \left\|\int \widehat{S}_\omega(k,t)\widehat{f}_{1,res}(k,k-m,\varepsilon t)|m|^{1/2}(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})(m,t)\mathrm{d}m\right\|_{L^2_s}\right) \\
&\leq C\||k|_{op}^{1/2}R_{1,\infty}\|_{H^s}\Big(\varepsilon^{1/2}\|\widehat{S}_\omega\widehat{f}_{1,res}\|_{X^{s,\varepsilon}}\|(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})\|_{L^2_s} \\
&\qquad + \|\widehat{S}_\omega\widehat{f}_{1,res}\|_{X^{s,\varepsilon}}\||k|^{1/2}(\widehat{R}_{1,\infty} + \widehat{R}_{-1,\infty})\|_{L^2_s}\Big) \\
&\leq C\left(\varepsilon^{1/2}\||k|_{op}^{1/2}R_{1,\infty}\|_{H^s}\|R_{1,\infty}\|_{H^s} + \||k|_{op}^{1/2}R_{1,\infty}\|_{H^s}^2\right) \\
&\leq C\left(\varepsilon\|R_{1,\infty}\|_{H^s}^2 + \||k|_{op}^{1/2}R_{1,\infty}\|_{H^s}^2\right),
\end{aligned}
$$

where we used Lemma 2.6.1 and that the limit $\widehat{S}_\omega\widehat{f}_{1,res}$ is in $X^{s,\varepsilon}$. Note that $R_{-1,\infty}$ is complex conjugated to $R_{1,\infty}$ and that there exists an isomorphism between H^s and L^2_s. We also used the estimates $S_\omega^{-1}(m,t) < 1$ and $\sqrt{k} \leq \sqrt{k-m} + \sqrt{m}$, with $\mathcal{O}(|k-m|) = \mathcal{O}(\varepsilon)$. The term s_4 can be estimated in exactly the same way as the term s_3, i.e.,

$$\begin{aligned}
|s_4| &= \Big|\Big(R_{1,\infty}, S_\omega \mathrm{i}\omega_1 E^c_{\delta_0} f_{1,non} S_\omega^{-1}(R_{2,\infty}+R_{-2,\infty})\Big)_{H^s}\Big| \\
&= \Big|\Big(\widehat{R}_{1,\infty}(k,t), \widehat{S}_\omega(k,t)\mathrm{i}\widehat{\omega}_1(k)\widehat{E}^c_{\delta_0}(k) \\
&\qquad \times \int \widehat{f}_{1,non}(k,k-m,\varepsilon t)\widehat{S}_\omega^{-1}(m,t)(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})(m)\mathrm{d}m\Big)_{L^2_s}\Big| \\
&= \Big|\Big(|k|^{1/2}\widehat{R}_{1,\infty}(k,t), \widehat{S}_\omega(k,t)\mathrm{i}|k|^{-1/2}\widehat{\omega}_1(k)\widehat{E}^c_{\delta_0}(k) \\
&\qquad \times \int \widehat{f}_{1,non}(k,k-m,\varepsilon t)(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})(m,t)\mathrm{d}m\Big)_{L^2_s}\Big| \\
&\leq C\||k|^{1/2}\widehat{R}_{1,\infty}\|_{L^2_s}\Big\|\widehat{S}_\omega(k)\widehat{\vartheta}(k) \\
&\qquad \times \int \widehat{f}_{1,non}(k,k-m,\varepsilon t)S_\omega^{-1}(m,t)(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})(m,t)\mathrm{d}m\Big\|_{L^2_s} \\
&\leq C\||k|^{1/2}\widehat{R}_{1,\infty}\|_{L^2_s} \\
&\quad \Big(\Big\|\int |k-m|^{1/2}\widehat{S}_\omega(k)\widehat{f}_{1,non}(k,k-m,\varepsilon t)(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})(m,t)\mathrm{d}m\Big\|_{L^2_s} \\
&\quad + \Big\|\int S_\omega(k)\widehat{f}_{1,non}(k,k-m,\varepsilon t)|m|^{1/2}(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})(m,t)\mathrm{d}m\Big\|_{L^2_s}\Big) \\
&\leq C\||k|^{1/2}_{op}R_1\|_{H^s}\Big(\varepsilon^{1/2}\|\widehat{S}_\omega\widehat{f}_{1,non}\|_{X^{s,\varepsilon}}\|(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})\|_{L^2_s} \\
&\quad +\|\widehat{S}_\omega\widehat{f}_{1,non}\|_{X^{s,\varepsilon}}\||k|^{1/2}(\widehat{R}_{2,\infty}+\widehat{R}_{-2,\infty})(m)\|_{L^2_s}\Big) \\
&\leq C\||k|^{1/2}_{op}R_{1,\infty}\|_{H^s}\Big(\varepsilon^{1/2}\|\Omega^{1/2}R_{2,\infty}\|_{H^s}+\||k|^{1/2}_{op}\Omega^{1/2}R_{2,\infty}\|_{H^s}\Big) \\
&\leq C\Big(\||k|^{1/2}_{op}R_{1,\infty}\|^2_{H^s}+\varepsilon\|\Omega^{1/2}R_{2,\infty}\|^2_{H^s}+\||k|^{1/2}_{op}\Omega^{1/2}R_{2,\infty}\|^2_{H^s}\Big).
\end{aligned}$$

b) The remaining terms s_5 and s_7 can be estimated by the "good" terms in exactly the same way as s_3 and s_4. We use the fact that s_5 and s_7 have an E^c_δ in front of them which vanishes at the wave number $k=0$. As a result, s_5 and s_7 vanish at the wave number $k=0$. Finally, we obtain

$$\begin{aligned}
|s_5|+|s_7| &\leq C(\||k|^{1/2}_{op}R_{1,\infty}\|^2_{H^s}+\||k|^{1/2}_{op}\Omega^{1/2}R_{2,\infty}\|^2_{H^s} \\
&\qquad +\varepsilon\|R_{1,\infty}\|^2_{H^s}+\varepsilon\|\Omega^{1/2}R_{2,\infty}\|^2_{H^s}).
\end{aligned}$$

2.8.3 Estimates for s_6

For the Fourier transform of $\mathrm{Re}\,(s_6)$ in case $s=0$ we compute

$$\begin{aligned}
|\widehat{\mathrm{Re}\,(s_6)}| &= \Big|2\mathrm{i}\int\int \widehat{R}_{2,\infty}(k,t)\overline{\widehat{S}_\omega(k,t)\widehat{f}_{2,res}(k,k-m,\varepsilon t)\widehat{S}_\omega^{-1}(m,t)}\widehat{R}_{-2,\infty}(m)\mathrm{d}m\mathrm{d}k \\
&\quad -2\mathrm{i}\int\int \widehat{R}_{-2,\infty}(k,t)\widehat{S}_\omega(k,t)\widehat{f}_{2,res}(k,k-m,\varepsilon t)\widehat{S}_\omega^{-1}(m,t)\widehat{R}_{2,\infty}(m)\mathrm{d}m\mathrm{d}k\Big|,
\end{aligned}$$

where we use $\mathrm{Re}\,(z) = \frac{1}{2}(z + \overline{z})$ for $z \in \mathbb{C}$. Since $\widehat{f}_{2,res}$ is of order $\mathcal{O}(\widehat{\psi})$, we can rewrite $\widehat{f}_{2,res}$ as

$$\begin{aligned}
\widehat{f}_{2,res}(k, k-m, \varepsilon t) &= \widehat{f}_{2,res}(k, k-m, \widehat{\psi}(k-m, \varepsilon t)) \\
&= \widehat{S}_{\omega}^{-1}(k-m, t)\widehat{f}_{2,res}(k, k-m, \widehat{S}_{\omega}(k-m, t)\widehat{\psi}(k-m, \varepsilon t)).
\end{aligned}$$

We estimate

$$\begin{aligned}
|\widehat{\mathrm{Re}\,(s_6)}| \le\ & 2\int\int \Big|\widehat{R}_{2,\infty}(k,t)\widehat{S}_{\omega}(k,t)\widehat{S}_{\omega}^{-1}(k-m,t) \\
&\quad \times \overline{\widehat{f}_{2,res}(k, k-m, \widehat{S}_{\omega}(k-m,t)\widehat{\psi}(k-m,\varepsilon t))}\widehat{S}_{\omega}^{-1}(m,t)\widehat{R}_{-2,\infty}(m) \\
&\quad - \widehat{R}_{-2,\infty}(k,t)\widehat{S}_{\omega}(k,t)\widehat{S}_{\omega}^{-1}(k-m,t) \\
&\quad \times \widehat{f}_{2,res}(k, k-m, \widehat{S}_{\omega}(k-m,t)\widehat{\psi}(k-m,\varepsilon t))\widehat{S}_{\omega}^{-1}(m,t)\widehat{R}_{2,\infty}(m)\Big|\,\mathrm{d}m\mathrm{d}k \\
\le\ & 2\int\int \Big|\widehat{R}_{2,\infty}(k,t)\overline{\widehat{f}_{2,res}(k, k-m, \widehat{\psi}^*(k-m,\varepsilon t))}\widehat{R}_{-2,\infty}(m) \\
&\quad - \widehat{R}_{-2,\infty}(k,t)\widehat{f}_{2,res}(k, k-m, \widehat{\psi}^*(k-m,\varepsilon t))\widehat{R}_{2,\infty}(m)\Big|\,\mathrm{d}m\mathrm{d}k,
\end{aligned}$$

where we use $\widehat{S}_{\omega}(k,t)\widehat{S}_{\omega}^{-1}(k-m,t)\widehat{S}_{\omega}^{-1}(m,t) < 1$ and $\widehat{S}_{\omega}\widehat{\psi} = \widehat{\psi}^*$. In the first term, we switch the role of k and m to obtain

$$\begin{aligned}
|\widehat{\mathrm{Re}\,(s_6)}| \le\ & 2\int\int \Big|\widehat{R}_{2,\infty}(m)\overline{\widehat{f}_{2,res}(m, m-k, \widehat{\psi}^*(m-k,\varepsilon t))}\widehat{R}_{-2,\infty}(k) \\
&\quad - \widehat{R}_{-2,\infty}(k)\widehat{f}_{2,res}(k, k-m, \widehat{\psi}^*(k-m,\varepsilon t))\widehat{R}_{2,\infty}(m)\Big|\,\mathrm{d}m\mathrm{d}k.
\end{aligned}$$

Due to the symmetry of the system (2.23)-(2.24) we have

$$\overline{\widehat{f}_{2,res}(m, m-k, \widehat{\psi}^*(m-k,\varepsilon t))} = \widehat{f}_{2,res}(m, k-m, \widehat{\psi}^*(k-m,\varepsilon t)),$$

and so we get

$$\begin{aligned}
|\widehat{\mathrm{Re}\,(s_6)}| \le\ & 2\int\int \Big|\widehat{R}_{2,\infty}(m)\widehat{f}_{2,res}(m, k-m, \widehat{\psi}^*(k-m,\varepsilon t))\widehat{R}_{-2,\infty}(k) \\
&\quad - \widehat{R}_{-2,\infty}(k)\widehat{f}_{2,res}(k, k-m, \widehat{\psi}^*(k-m,\varepsilon t))\widehat{R}_{2,\infty}(m)\Big|\,\mathrm{d}m\mathrm{d}k.
\end{aligned}$$

Since

$$|\widehat{f}_{2,res}(m, k-m, \widehat{\psi}^*(k-m,\varepsilon t)) - \widehat{f}_{2,res}(k, k-m, \widehat{\psi}^*(k-m,\varepsilon t))| = \mathcal{O}(|k-m|),$$

we estimate

$$|\widehat{\mathrm{Re}\,(s_6)}| \le C\varepsilon E_s,$$

similar to Lemma 4.1 from [DKS16]. For $s > 0$ the estimate for

$$\Big(\partial_x^s \Omega R_{2,\infty}, \partial_x^s S_{\omega}(\mathrm{i}\omega_2)^{-1} f_{2,res} S_{\omega}^{-1}(R_{2,\infty})\Big)_{L^2}$$

is similar to the case $s = 0$. Whenever a derivative falls on $f_{2,res}$ and so on ψ in the second component of the scalar product, we get an additional power of ε. Thus, there is only one term of order $\mathcal{O}(1)$, precisely when all s derivatives after the comma fall on $R_{2,\infty}$. However, this term can be estimated line by line as in the case $s = 0$.

2.8.4 The final estimates

Putting all the previous estimates together yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_s \;\leq\; & (-\beta + C + C\varepsilon^{3/2}E_s^{1/2})\left(\left\| |k|^{1/2} R_{1,\infty}\right\|_{H^s}^2 + \left\| |k|^{1/2}\Omega^{1/2}R_{2,\infty}\right\|_{H^s}^2\right) \\ & + C\varepsilon E_s + C\varepsilon^{3/2}E_s^{3/2} + C\varepsilon,\end{aligned}$$

where C is a constant that is independent of $0 < \varepsilon \ll 1$. We choose $\beta > 0$ to be large enough, but independent of $0 < \varepsilon \ll 1$, that

$$-\beta + C + C\varepsilon^{3/2}E_s^{1/2} < 0 \tag{2.26}$$

is fulfilled. Under this assumption

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_s \;\leq\; C\varepsilon E_s + C\varepsilon^{3/2}E_s^{3/2} + C\varepsilon$$

is satisfied. Then, we choose $\varepsilon > 0$ so small, that

$$\varepsilon^{1/2}E_s^{1/2} \leq 1 \tag{2.27}$$

is fulfilled. Under this assumption we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_s \;\leq\; (C+1)\varepsilon E_s + C\varepsilon,$$

and so Gronwall's inequality implies

$$E_s(t) \leq (E_s(0) + Ct)\mathrm{e}^{(C+1)t} \leq (E_s(0) + CT_0)\mathrm{e}^{(C+1)T_0} =: M = \mathcal{O}(1).$$

The constant M is independent of β, respectively T_1 and $0 < \varepsilon \ll 1$. Therefore, we are done with the proof of Theorem 2.1.1 if we choose $\varepsilon_0 > 0$ to be small enough that $\varepsilon_0^{1/2}M^{1/2} \leq 1$ which guarantees the validity of (2.27). Then, we choose $\beta > 0$ to be large enough that

$$-\beta + C + C\varepsilon_0^{3/2}M^{1/2} < 0$$

which guarantees the validity of (2.26).

2.9 Appendix - Some technical estimates

In this section we collect a number of estimates that we have used previously. They can be found in [Sch20] or as Lemma A.4, Corollary A.5, Corollary A.6, Lemma A.9, and Corollary A.10 in [HdRS23] with their proofs.

Lemma 2.9.1. *The spaces G_σ^s are Banach algebras for $\sigma \geq 0$ and $s > \frac{1}{2}$. Precisely, there exists a σ-independent constant C_s such that*

$$\|uv\|_{G_\sigma^s} \leq C_s\|u\|_{G_\sigma^s}\|v\|_{G_\sigma^s},$$

for all $u, v \in G_\sigma^s$.

In fact, we use the following estimates version of the previous lemma for our error estimates.

Corollary 2.9.2. *For $\delta > 0$, $\sigma \geq 0$ and $s > 1/2$ we have*

$$\|u^2\|_{G^s_\sigma} \leq C_s \|u\|_{G^{1/2+\delta}_\sigma} \|u\|_{G^s_\sigma},$$

for all $u \in G^s_\sigma$.

Corollary 2.9.3. *For $\sigma \geq 0$ and $s \geq 0$ we have*

$$\|uv\|_{G^s_\sigma} \leq C_s \|u\|_{\mathcal{W}^s_\sigma} \|v\|_{G^s_\sigma},$$

for all $u \in \mathcal{W}^s_\sigma$ and $v \in G^s_\sigma$.

The expansion of the kernels in the multilinear maps can be estimated with the following lemma for which we recall the proof, cf. [Sch20, Lemma 7].

Lemma 2.9.4. *Let $\theta_0 \geq 0$, $\theta_\infty \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|g(k)| \leq C \min(|k|^{\theta_0}, (1+|k|)^{\theta_\infty}).$$

Then, for the associated multiplication operator $g_{op} = \mathcal{F}^{-1} g \mathcal{F}$ the following holds. For

i) $\sigma_1 > \sigma_2$ and $m_1, m_2 \geq 0$ or

ii) $\sigma_1 = \sigma_2$ and $m_2 - m_1 \geq \max(\theta_0, \theta_\infty)$,

we have

$$\|g_{op} A(\varepsilon \cdot)\|_{G^{m_1}_{\sigma_1/\varepsilon}} \leq C \varepsilon^{\theta_0 - 1/2} \|A(\cdot)\|_{G^{m_2}_{\sigma_2}},$$

for all $\varepsilon \in (0,1)$.

Proof. We compute

$$\begin{aligned} \|g_{op} A(\varepsilon \cdot)\|_{G^{m_1}_{\sigma_1/\varepsilon}} &\leq \sup_{k \in \mathbb{R}} \left| g(k) \left(1 + \left(\frac{k}{\varepsilon}\right)^2\right)^{\frac{m_1 - m_2}{2}} \mathrm{e}^{(\sigma_1 - \sigma_2)|k|/\varepsilon} \right| \|A(\varepsilon \cdot)\|_{G^{m_2}_{\sigma_2/\varepsilon}} \\ &\leq C \varepsilon^{\theta_0} \, \varepsilon^{-1/2} \|A(\cdot)\|_{G^{m_2}_{\sigma_2}}, \end{aligned}$$

where the scaling properties of the L^2-norm lead to the loss of $\varepsilon^{-1/2}$. □

In contrast, there is no loss of $\varepsilon^{-1/2}$ in $\mathcal{W}^m_\sigma$-spaces due to the scaling invariance of the norm such that we have the following corollary as a direct consequence.

Corollary 2.9.5. *Let $\theta_0 \geq 0$, $\theta_\infty \in \mathbb{R}$, and let $g(k)$ satisfy*

$$|g(k)| \leq C \min(|k|^{\theta_0}, (1+|k|)^{\theta_\infty}).$$

Then, for the associated multiplication operator $g_{op} = \mathcal{F}^{-1} g \mathcal{F}$ the following holds. For *i)* $\sigma_1 > \sigma_2$ *and* $m_1, m_2 \geq 0$ *or*

ii) $\sigma_1 = \sigma_2$ *and* $m_2 - m_1 \geq \max(\theta_0, \theta_\infty)$,

we have

$$\|g_{op}A(\varepsilon\cdot)\|_{\mathcal{W}^{m_1}_{\sigma_1/\varepsilon}} \leq C\varepsilon^{\theta_0}\|A(\cdot)\|_{\mathcal{W}^{m_2}_{\sigma_2}},$$

for all $\varepsilon \in (0,1)$.

Remark 2.9.6. For instance, we use Corollary 2.9.5 to estimate terms with E^c_δ. Due to the definition of the mode projection, $\widehat{E}^c_\delta(k)$ is identical zero in a neighborhood of the origin $k = 0$. Then, we have $|\widehat{E}^c_\delta(k)| \leq C|k|^\alpha$, for every $\alpha \in \mathbb{N}$.

Chapter 3

Validity of the Whitham approximation for a complex cubic Klein-Gordon equation

The complex cubic Klein-Gordon (ccKG) equation possesses a family of periodic traveling wave solutions. Whitham's modulation equations (WME) can be derived by a multiple scaling perturbation analysis in order to describe slow modulations in time and space of these traveling wave solutions. We prove estimates between true solutions of the ccKG equation and their associated WME approximation. The bounds are obtained in Gevrey spaces and hold independently of the spectral stability of the underlying traveling wave solutions. The proof is based on a suitable choice of variables, Cauchy-Kovalevskaya theory, infinitely many near identity changes of variables in Gevrey spaces and energy estimates. The analysis for the ccKG equation is more complicated than the analysis for the nonlinear Schrödinger (NLS) equation which has been handled in the existing literature, cf. [DS09], due to additional curves of eigenvalues leading to an additional oscillatory behavior. The content of this chapter is a joint work with Xian Liao and Guido Schneider and an earlier version of this chapter has already been published as a preprint in [HLS22].

3.1 Introduction

Whitham's modulation equations (WME) can be derived by a multiple scaling perturbation analysis, cf. [Whi74], with a small perturbation parameter $0 < \delta \ll 1$, in order to describe slow modulations in time and space of periodic traveling wave solutions in dispersive and dissipative systems. In this introduction and also in the whole chapter we are going to explain the backgrounds, main ideas as well as some intuitive calculations in a few remarks.

Remark 3.1.1. [WME approximation results for other systems] So far there are only few approximation results showing that WME approximations make correct predictions about the dynamics near periodic traveling wave solutions in dispersive and dissipative systems. In [DS09] such estimates were obtained in Gevrey spaces for such waves of the nonlinear Schrödinger (NLS) equation

$$\partial_\tau A = \mathrm{i}\nu_1 \partial_\xi^2 A + \mathrm{i}\nu_2 A|A|^2,$$

with $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$, $A(\xi,\tau) \in \mathbb{C}$ and coefficients ν_1, $\nu_2 \in \mathbb{R}$, as original system. In [BKS20] such estimates were obtained in Gevrey spaces for such waves of a system of coupled NLS equations as original system. For spectrally stable waves of the NLS equation in [BKZ21] it was shown that WME even make correct predictions for initial data in Sobolev spaces. The only approximation result, we are aware of, for dissipative systems can be found in [HdRS23] where the validity of WME was shown for such waves in the amplitude system which appears at the first instability of the Marangoni problem consisting of a Ginzburg-Landau equation coupled to a diffusive conservation law.

Remark 3.1.2. [Non-triviality of WME approximation results] Such approximation results are non-trivial since solutions of order $\mathcal{O}(1)$ have to be bounded on a long $\mathcal{O}(1/\delta)$-time scale. In general, solutions of order $\mathcal{O}(1)$ are only bounded on a $\mathcal{O}(1)$-time scale. See Remark 3.1.12 below for a more detailed explanation.

Remark 3.1.3. [Periodic traveling waves] As a next step in the direction of handling modulations of periodic wave trains for general dispersive systems, in this chapter, we consider the complex cubic Klein-Gordon (ccKG) equation

$$\partial_t^2 u = \partial_x^2 u - u + \gamma u \left|u\right|^2, \tag{3.1}$$

with $t, x \in \mathbb{R}$, $\gamma \in \{-1, 1\}$ and $u(x,t) \in \mathbb{C}$. It possesses traveling wave solutions

$$u\left(x,t\right) = \mathrm{e}^{r_{q,\mu} + \mathrm{i}qx + \mathrm{i}\mu t}, \tag{3.2}$$

where $\mu, q, r_{q,\mu} \in \mathbb{R}$ satisfy

$$-\gamma \mathrm{e}^{2r_{q,\mu}} = \mu^2 - q^2 - 1.$$

Remark 3.1.4. [$\mathbb{S}^1$-symmetry] All these equations have in common that the nonlinearities possess an $\mathbb{S}^1$-symmetry, i.e., with u also $u\mathrm{e}^{\mathrm{i}\phi}$, with $\phi \in \mathbb{R}$, is a solution. As a consequence, the underlying periodic traveling wave solutions are harmonic which easily allows us to extract a local wave number variable which is necessary for the derivation of WME.

Remark 3.1.5. [Two additional spectral curves for ccKG] What makes the analysis for the ccKG equation more complicated than the analysis for the NLS equation are two additional curves of eigenvalues which lead to an additional oscillatory behavior, see Figure 3.1 below.

Remark 3.1.6. [Derivation of WME for $|\mu| > 1/\sqrt{3}$] For notational simplicity, here in the introduction, we derive WME for the ccKG equation (3.1) for the wave train (3.2)

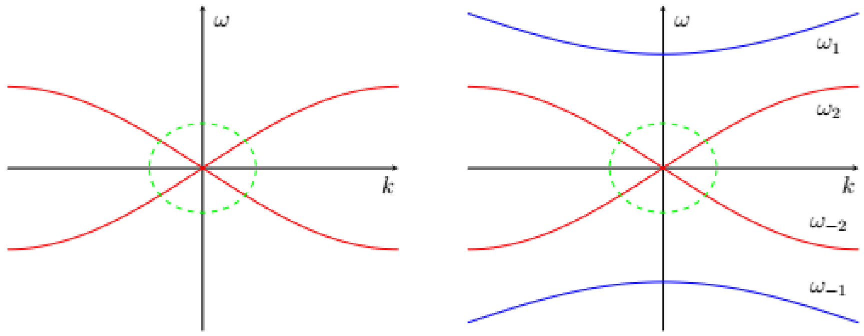

Figure 3.1: The left panel shows the spectral curves (in red) as functions ω over the Fourier wave numbers k for the NLS equation. The right panel shows the spectral curves for the ccKG equation (3.1) with the two additional spectral curves (in blue). WME describe the modes in the small dashed circles (in green).

associated to $q = 0$, where we have $\mu^2 = 1 - \gamma e^{2r_{0,\mu}}$.
Step 1: We introduce polar coordinates

$$u = e^{r+i\varphi+r_{0,\mu}+i\mu t}, \tag{3.3}$$

with $r = r(x,t)$ and $\varphi = \varphi(x,t)$. Using

$$\begin{aligned} \frac{\partial_t u}{u} &= \partial_t r + i\partial_t\varphi + i\mu, \\ \frac{\partial_t^2 u}{u} &= (\partial_t r + i\partial_t\varphi + i\mu)^2 + (\partial_t^2 r + i\partial_t^2\varphi), \end{aligned}$$

and similar expressions for $(\frac{\partial_x u}{u})$ and $(\frac{\partial_x^2 u}{u})$, we find by separating real and imaginary parts that

$$\begin{aligned} \partial_t^2 r - (\partial_t\varphi + \mu)^2 + (\partial_t r)^2 &= \partial_x^2 r - (\partial_x\varphi)^2 + (\partial_x r)^2 - 1 + \gamma e^{2r+2r_{0,\mu}}, \\ 2(\partial_t r)(\partial_t\varphi + \mu) + \partial_t^2\varphi &= 2(\partial_x r)(\partial_x\varphi) + \partial_x^2\varphi. \end{aligned}$$

Step 2: We introduce the local temporal wave number $\vartheta = \partial_t\varphi$ and the local spatial wave number $\psi = \partial_x\varphi$ for which we obtain

$$\partial_t^2 r = \partial_x^2 r + \vartheta^2 + 2\mu\vartheta - (\partial_t r)^2 - \psi^2 + (\partial_x r)^2 + \gamma e^{2r_{0,\mu}}(e^{2r} - 1), \tag{3.4}$$

$$\partial_t\vartheta = 2(\partial_x r)\psi + \partial_x\psi - 2(\partial_t r)(\vartheta + \mu), \tag{3.5}$$

$$\partial_t\psi = \partial_x\vartheta, \tag{3.6}$$

by using

$$(\partial_t\varphi + \mu)^2 = (\vartheta + \mu)^2 = \vartheta^2 + 2\mu\vartheta + \mu^2 = \vartheta^2 + 2\mu\vartheta + 1 - \gamma e^{2r_{0,\mu}}.$$

Step 3: For the derivation of WME we make the long wave ansatz

$$(r, \psi, \vartheta)(x, t) = (\check{r}, \check{\psi}, \check{\vartheta})(\delta x, \delta t) = (\check{r}, \check{\psi}, \check{\vartheta})(X, T),$$

with $X = \delta x$, $T = \delta t$ and a small perturbation parameter $0 < \delta \ll 1$. Ignoring higher order terms yields the system

$$\begin{aligned} 0 &= \check{\vartheta}^2 + 2\mu\check{\vartheta} - \check{\psi}^2 + \gamma e^{2r_{0,\mu}}(e^{2\check{r}} - 1), && (3.7) \\ \partial_T \check{\vartheta} &= 2(\partial_X \check{r})\check{\psi} + \partial_X \check{\psi} - 2(\partial_T \check{r})(\check{\vartheta} + \mu), && (3.8) \\ \partial_T \check{\psi} &= \partial_X \check{\vartheta}. && (3.9) \end{aligned}$$

Step 4: Since the second equation (3.8) contains derivatives of $\check{\vartheta}$ and $\check{r}$ w.r.t. T, it turns out to be advantageous to work with the variables

$$\begin{pmatrix} \check{v} \\ \check{w} \end{pmatrix} = \begin{pmatrix} b & 2\mu \\ 2\mu & 1 \end{pmatrix} \begin{pmatrix} \check{r} \\ \check{\vartheta} \end{pmatrix}, \tag{3.10}$$

which are linear combinations of $(\check{r}, \check{\vartheta})$ with $b = 2\gamma e^{2r_{0,\mu}} = 2 - 2\mu^2$. Equivalently, $(\check{r}, \check{\vartheta})$ are linear combinations of $(\check{v}, \check{w})$, if $|\mu| > 1/\sqrt{3}$, precisely

$$\begin{pmatrix} \check{r} \\ \check{\vartheta} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} 1 & -2\mu \\ -2\mu & b \end{pmatrix} \begin{pmatrix} \check{v} \\ \check{w} \end{pmatrix} =: \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \check{v} \\ \check{w} \end{pmatrix}, \tag{3.11}$$

with $D = -4\mu^2 + b = -(6\mu^2 - 2) = 2 - 6\mu^2 < 0$. Equation (3.7) is then of the form

$$\check{v} = f_v(\check{v}, \check{w}, \check{\psi}),$$

with f_v at least quadratic in its arguments. For $\check{w}$ and $\check{\psi}$ sufficiently small this equation can be solved with respect to $\check{v}$, i.e., there exists a nonlinear function g_v such that

$$\check{v} = g_v(\check{w}, \check{\psi}), \tag{3.12}$$

with g_v at least quadratic in its arguments.
Step 5: Using the second equation (3.8) in the form

$$\partial_T \check{\vartheta} + 2\mu \partial_T \check{r} = 2(\partial_X \check{r})\check{\psi} + \partial_X \check{\psi} - 2(\partial_T \check{r})\check{\vartheta},$$

we rewrite the $\check{\vartheta}$-equation into

$$\begin{aligned} \partial_T \check{w} &= 2(\partial_X(a_1\check{v} + a_2\check{w}))\check{\psi} + \partial_X \check{\psi} - 2(\partial_T(a_1\check{v} + a_2\check{w}))(a_3\check{v} + a_4\check{w}) \\ &= 2\check{\psi}(\partial_X(a_1\check{v} + a_2\check{w})) + \partial_X \check{\psi} - 2(a_3\check{v} + a_4\check{w})a_2\partial_T \check{w} \\ &\quad -2(a_3\check{v} + a_4\check{w})a_1(\ell_1(\check{w}, \check{\psi})\partial_T \check{w} + \ell_2(\check{w}, \check{\psi})\partial_T \check{\psi}), \end{aligned}$$

with ℓ_1 and ℓ_2 at least linear in its arguments. We can replace $\partial_T \check{\psi}$ by the right hand side of the third equation (3.9) which is of the form

$$\partial_T \check{\psi} = \partial_X(a_3\check{v} + a_4\check{w}).$$

Hence, for $\check{w}$ and $\check{\psi}$ sufficiently small the second equation (3.8) can be solved with respect to $\partial_T \check{w}$, i.e., there exists a nonlinear function g_w such that

$$\partial_T \check{w} = g_w(\check{v}, \check{w}, \check{\psi}, \partial_X \check{v}, \partial_X \check{w}, \partial_X \check{\psi}),$$

where g_w is of the form

$$g_w = g_{w,1}(\check{v}, \check{w}, \check{\psi})\partial_X \check{v} + g_{w,2}(\check{v}, \check{w}, \check{\psi})\partial_X \check{w} + g_{w,3}(\check{v}, \check{w}, \check{\psi})\partial_X \check{\psi}.$$

Substituting $\check{v}$ by $g_v(\check{w}, \check{\psi})$ finally yields WME for $(\check{w}, \check{\psi})$ given by

$$\partial_T \check{w} = g_w(g_v(\check{w}, \check{\psi}), \check{w}, \check{\psi}, \partial_X g_v(\check{w}, \check{\psi}), \partial_X \check{w}, \partial_X \check{\psi}), \tag{3.13}$$

$$\partial_T \check{\psi} = \partial_X (a_3 g_v(\check{w}, \check{\psi}) + a_4 \check{w}), \tag{3.14}$$

describing the modes in the (green dashed) circle in the right panel of Figure 3.1, where the right-hand side of the $\check{w}$-equation (3.13) can be written as

$$g_{w,4}(\check{w}, \check{\psi})\partial_X \check{w} + g_{w,5}(\check{w}, \check{\psi})\partial_X \check{\psi},$$

where $g_{w,4}$ and $g_{w,5}$ are nonlinear smooth functions in their arguments.

Remark 3.1.7. [Benjamin-Feir (in-)stability and well-posedness of WME in Sobolev spaces] Depending on the values of μ and γ WME (3.13)-(3.14) can be well- or ill-posed in Sobolev spaces. It turns out that, equivalently, the periodic wave train is spectrally stable in the first case and spectrally unstable in the second case. The first situation is called the Benjamin-Feir stable and the second situation the Benjamin-Feir unstable case, cf. Remark 3.1.11 and Section 3.2.1.

In the following we prove estimates between true solutions of the ccKG equation (3.4)-(3.6) and their associated WME approximation (3.13)-(3.14). The bounds are obtained in Gevrey spaces and hold independently of the spectral stability of the underlying traveling wave solutions.

Definition 3.1.8. *The Gevrey spaces*

$$G_\sigma^m = \{u \in L^2 : \|u\|_{G_\sigma^m} < \infty\}$$

are Hilbert spaces equipped with the inner product

$$(u, v)_{G_\sigma^m} = \int e^{2\sigma|k|} \left(1 + |k|^2\right)^m \widehat{u}(k)\overline{\widehat{v}(k)} dk,$$

for $\sigma \geq 0$ and $m \geq 0$.

Notation: Here and in the following $\int_{\mathbb{R}}$ is abbreviated as $\int$.

Remark 3.1.9. Since the right hand sides of WME (3.13)-(3.14) only contain first order derivatives, local-in-time existence and uniqueness of solutions in Gevrey spaces for WME is well known by the Cauchy-Kovalevskaya theory, see Section 3.4 below.

Our approximation result in the case $q = 0$, i.e., for $u = e^{r_{0,\mu}+i\mu t}$, with $\mu^2 = 1 - \gamma e^{2r_{0,\mu}}$, is as follows.

Theorem 3.1.10. *Let $|\mu| > 1/\sqrt{3}$, $\sigma_0 > 0$ and $m \geq 3$. Then, for all $T_0 > 0$ there exist C_1, C_2, T_1, $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the following holds. Let $(\check{w}_{app}, \check{\psi}_{app}) \in C([0,T_0], G^{m+1}_{\sigma_0}) \cap C^1((0,T_0], G^m_{\sigma_0})$ be a solution of WME (3.13)-(3.14) satisfying*

$$\sup_{T\in[0,T_0]} \|(\check{w}_{app}, \check{\psi}_{app})\|_{G^{m+1}_{\sigma_0}} \leq C_1,$$

let $\check{v}_{app}$ be the corresponding solution to the algebraic equation (3.12) and $(\check{r}_{app}, \check{\vartheta}_{app}, \check{\psi}_{app})$ is the approximation constructed from $(\check{v}_{app}, \check{w}_{app}, \check{\psi}_{app})$ by (3.11). Then, there exist solutions (r, ϑ, ψ) of (3.4)-(3.6) with

$$\sup_{t\in[0,T_1/\delta]} \sup_{x\in\mathbb{R}} |(r, \vartheta, \psi)(x,t) - (\check{r}_{app}, \check{\vartheta}_{app}, \check{\psi}_{app})(\delta x, \delta t)| \leq C_2 \delta^{3/2}.$$

Remark 3.1.11. [Benjamin-Feir (in-)stability and Range of μ] The approximation result covers the Benjamin-Feir stable case, $|\mu| \geq 1$, and the Benjamin-Feir unstable case, $|\mu| \in (1/\sqrt{3}, 1)$, cf. Figure 3.2. For $|\mu| \leq 1/\sqrt{3}$ it cannot be expected that WME make correct predictions, cf. Figure 3.3 and Remark 3.10.2.

Remark 3.1.12. As already said in Remark 3.1.2, the above validity result is a non-trivial task. The WME approximation and the associated solution are of order $\mathcal{O}(1)$ for $\delta \to 0$. Therefore, a simple application of Gronwall's inequality would only provide the boundedness of the solutions on a $\mathcal{O}(1)$-time scale, but not on the natural $\mathcal{O}(1/\delta)$-time scale of the WME approximation.

Remark 3.1.13. Although Theorem 3.1.10 is not optimal in the sense that the possible approximation time T_1/δ is possibly smaller than T_0/δ, we do establish an approximation result on the natural $\mathcal{O}(1/\delta)$-time scale of the WME approximation.

Remark 3.1.14. [Counter-examples] There is a number of counter-examples where formally derived amplitude equations make wrong predictions about the dynamics of original systems on the natural time scale of the amplitude equations, cf. [Sch95, SSZ15, HS20, BSSZ20, FS22].

The plan of the chapter is as follows. In the next section we go on with some further remarks. We start with the Benjamin-Feir instability in Section 3.2.1 where we explain that there are stable and unstable wave trains. Modulations of small amplitude wave trains of the ccKG equation (3.1) can be described by an NLS approximation. Therefore, in Section 3.2.2 we relate our approximation result to the associated approximation results for the NLS equation. The WME approximation is a long wave limit approximation like the Korteweg-de Vries (KdV) approximation or the inviscid Burgers approximation. Hence, in Section 3.2.3 we relate our result to other long wave approximation results and formulate the associated approximation results for modulations of wave trains in the ccKG equation (3.1). Finally, in Section 3.2.4 we explain the ideas of the proof of Theorem 3.1.10. In Section 3.3 we redo some calculations for $q \neq 0$ and plot a few spectral curves which look qualitatively different from the spectral curves for $q = 0$. The rest of the chapter is devoted to the proof of the main theorem 3.1.10. We use Cauchy-Kovalevskaya theory in Section 3.4 to obtain local-in-time existence and uniqueness of solutions to WME (3.13)-(3.14). After a partial diagonalization of the error equations in Section 3.5, we use infinitely many normal form transformations in Section 3.6 to get rid of the new oscillatory terms appearing in the right panel of Figure 3.1, similar as in [DKS16]. After some preparations in Section 3.7 and Section 3.8, we obtain a system for which in Section 3.9 we can use energy estimates, similar to the ones used for WME in Section 3.4, to control the solutions close to the wave number $k = 0$. At wave numbers bounded away from $k = 0$ we use the artificial damping coming from the time-dependent scale of Gevrey spaces to control the solutions. This finally yields the validity of the main theorem 3.1.10. We close this chapter with Section 3.10 where we discuss related questions such as difficulties and strategies to obtain estimates in Sobolev spaces. In an appendix we collect some calculations about the spectral curves plotted subsequently in Figure 3.5 for the case $q \neq 0$.

3.2 Some further remarks

As in the introduction, all further explanations in this section are still made for the wave trains (3.2) with the wave number $q = 0$.

3.2.1 The Benjamin-Feir instability

In this section we explain that there are stable and unstable wave trains. The so-called Benjamin-Feir instability is a long wave instability.

Remark 3.2.1. The linearization of (3.4)-(3.6) is given by

$$\begin{aligned} \partial_t^2 r &= \partial_x^2 r + 2\mu\vartheta + 2\gamma e^{2r_{0,\mu}} r, \\ \partial_t \vartheta &= \partial_x \psi - 2\mu(\partial_t r), \\ \partial_t \psi &= \partial_x \vartheta, \end{aligned}$$

where for $q = 0$ we have

$$-\gamma e^{2r_{0,\mu}} = \mu^2 - 1.$$

The Fourier ansatz

$$(r, \vartheta, \psi) = e^{ikx + i\omega t}(\widehat{r}, \widehat{\vartheta}, \widehat{\psi})$$

yields the dispersion relations

$$\begin{aligned} -\omega^2 \widehat{r} &= -k^2 \widehat{r} + 2\mu\widehat{\vartheta} - 2(\mu^2 - 1)\widehat{r}, \\ i\omega\widehat{\vartheta} &= ik\widehat{\psi} - 2\mu(i\omega\widehat{r}), \\ i\omega\widehat{\psi} &= ik\widehat{\vartheta}. \end{aligned}$$

For $k = 0$ we find

$$-\omega^2(-\omega^2 + 6\mu^2 - 2) = 0.$$

Hence, we have two eigenvalues equal to zero and the rest of the eigenvalues bounded away from zero for $|\mu| > 1/\sqrt{3}$, which is exactly the spectral situation necessary for the derivation of WME, cf. [BDS19].

Remark 3.2.2. [Spectral curves $\pm i\omega_{1,2}$] For the calculation of the dispersion relation we have to solve

$$\det \begin{pmatrix} \omega^2 - k^2 - 2(\mu^2 - 1) & 2\mu & 0 \\ -2\mu i\omega & -i\omega & ik \\ 0 & ik & -i\omega \end{pmatrix} = 0.$$

We find

$$(\omega^2 - k^2 - 2(\mu^2 - 1))(-\omega^2 + k^2) + 4\mu^2\omega^2 = 0,$$

respectively

$$\omega^4 - \omega^2(2k^2 + 6\mu^2 - 2) + k^4 + 2k^2(\mu^2 - 1) = 0,$$

and so

$$2\omega_{1,2}^2 = (2k^2 + 6\mu^2 - 2) \pm \sqrt{(2k^2 + 6\mu^2 - 2)^2 - 4(k^4 + 2k^2(\mu^2 - 1))},$$

which yields

$$\omega_{1,2}^2 = (k^2 + 3\mu^2 - 1) \pm \sqrt{4k^2\mu^2 + (3\mu^2 - 1)^2}.$$

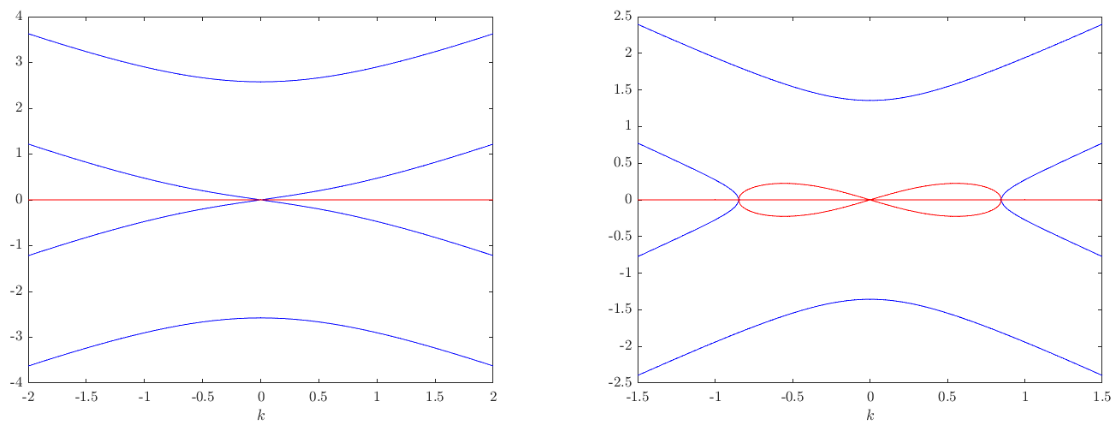

Figure 3.2: The left panel shows the spectral curves of (3.4)-(3.6) for $\mu = 1.2 \geq 1$ as functions over the Fourier wave numbers k. They are purely imaginary (in blue) since the real part (in red) vanishes identically. The right panel shows the imaginary part of the spectral curves of (3.4)-(3.6) for $\mu = 0.8 \in (1/\sqrt{3}, 1)$ (in blue). The eigenvalues with vanishing imaginary part have non-zero real part (in red), i.e., there is a so-called Benjamin-Feir instability.

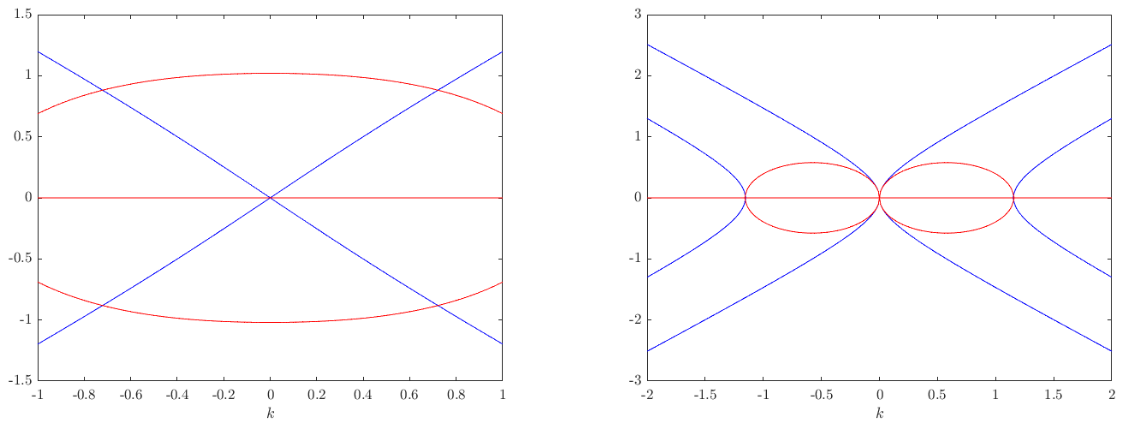

Figure 3.3: The left panel shows the real (in red) and imaginary part (in blue) of the spectral curves of (3.4)-(3.6) for $\mu = 0.4 < 1/\sqrt{3}$ as functions over the Fourier wave numbers k. The right panel shows the same at the threshold $\mu = 1/\sqrt{3}$, cf. Remark 3.1.11.

Figure 3.2 and Figure 3.3 show that the traveling wave solutions are only spectrally stable for $|\mu| \geq 1$. In this region an approximation result with initial conditions in Sobolev spaces would be desirable. For $|\mu| \in (1/\sqrt{3}, 1)$ we have a Benjamin-Feir instability and so only an approximation result with initial conditions in Gevrey spaces can be expected. For $|\mu| < 1/\sqrt{3}$ the modes associated to $\omega = 0$ are imaginary again. However, for $k = 0$ there are now modes with positive growth rates, cf. Figure 3.3, and so it cannot be expected that the WME approximation makes correct predictions, cf. [HS20] for an example of a non-approximation result.

3.2.2 The NLS limit

In the following remarks we explain how the previous WME approximation results from [DS09,BKZ21] for the NLS equation are related to our result for the ccKG equation stated in Theorem 3.1.10.

Remark 3.2.3. [NLS as modulation equation for ccKG] Inserting the multiple scaling ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x-ct), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)},$$

with $\xi = \varepsilon(x - ct)$ and $\tau = \varepsilon^2 t$, into the ccKG equation (3.1) and then equating the coefficients in front of $\varepsilon^n e^{i(k_0 x - \omega_0 t)}$ to zero for $n = 1, 2, 3$ gives the dispersion relation

$$\omega_0^2 = k_0^2 + 1,$$

the group velocity

$$c = k_0/\omega_0,$$

and shows that in lowest order A has to satisfy the NLS equation

$$-2\mathrm{i}\omega_0\partial_\tau A = (1-c^2)\partial_\xi^2 A + \gamma A|A|^2.$$

Remark 3.2.4. [WME for NLS] The normalized NLS equation

$$\partial_\tau U = \mathrm{i}\partial_\xi^2 U + \gamma \mathrm{i} U|U|^2,$$

with $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$, $U(\xi,\tau) \in \mathbb{C}$ and $\gamma = \pm 1$, possesses traveling wave solutions

$$U_{\varrho,q}(\xi,\tau) = \mathrm{e}^{\varrho + \mathrm{i}\widetilde{\omega}\tau + \mathrm{i}\phi_0 + \mathrm{i}q\xi},$$

where $\varrho, \widetilde{\omega}, \phi_0, q \in \mathbb{R}$ satisfy

$$\widetilde{\omega} = -q^2 + \gamma \mathrm{e}^{2\varrho}. \tag{3.15}$$

WME describe slow modulations in time and in space of these waves. For notational simplicity, we restrict ourselves in the rest of this remark to modulations of the wave train to the case $q = \varrho = \phi_0 = 0$. Hence, (3.15) becomes $\widetilde{\omega} = \gamma$.
Step 1: For the derivation we introduce polar coordinates, with radius also in exponential form, in a uniformly rotating frame. The NLS equation in such polar coordinates

$$U(\xi,\tau) = \mathrm{e}^{r(\xi,\tau) + \mathrm{i}\phi(\xi,\tau) + \mathrm{i}\gamma\tau}$$

is then given by

$$\partial_\tau r = -\partial_\xi^2\phi - 2(\partial_\xi r)(\partial_\xi \phi), \tag{3.16}$$
$$\partial_\tau \phi = \partial_\xi^2 r - (\partial_\xi\phi)^2 + (\partial_\xi r)^2 + \gamma(\mathrm{e}^{2r} - 1). \tag{3.17}$$

Step 2: Introducing the local spatial wave number $\partial_\xi \phi = \psi$ yields

$$\begin{aligned} \partial_\tau r &= -\partial_\xi \psi - 2(\partial_\xi r)\psi, \\ \partial_\tau \psi &= \partial_\xi^3 r - \partial_\xi(\psi^2) + \partial_\xi(\partial_\xi r)^2 + 2\gamma \mathrm{e}^{2r}\partial_\xi r. \end{aligned}$$

Step 3: The long wave ansatz

$$r(\xi,\tau) = \check{r}(\delta\xi,\delta\tau), \quad \psi(\xi,\tau) = \check{\psi}(\delta\xi,\delta\tau),$$

with $0 < \delta \ll 1$, leads to

$$\begin{aligned} \partial_T \check{r} &= -\partial_X \check{\psi} - 2(\partial_X \check{r})\check{\psi}, \\ \partial_T \check{\psi} &= \delta^2\partial_X^3 \check{r} - \partial_X(\check{\psi}^2) + \delta^2\partial_X(\partial_X \check{r})^2 + 2\gamma \mathrm{e}^{2\check{r}}\partial_X \check{r}, \end{aligned}$$

where $T = \delta\tau$ and $X = \delta\xi$. Ignoring the higher order terms gives WME

$$\begin{aligned} \partial_T \check{r} &= -\partial_X \check{\psi} - 2(\partial_X \check{r})\check{\psi}, \\ \partial_T \check{\psi} &= -\partial_X(\check{\psi}^2) + 2\gamma \mathrm{e}^{2\check{r}}\partial_X \check{r}. \end{aligned}$$

In case $\gamma = 1$ we recover the Benjamin-Feir instability, cf. [BM95], i.e., the linearization

$$\partial_T \check{r} = -\partial_X \check{\psi}, \qquad \partial_T \check{\psi} = 2\gamma \partial_X \check{r},$$

is an elliptic system and so ill-posed in Sobolev spaces for $\gamma = 1$. However, even for $\gamma = 1$ WME still possess local-in-time solutions in the space of functions which are analytic in a strip around the real axis in the complex plane, i.e., in Gevrey spaces (see Definition 3.1.8) by Cauchy-Kovalevskaya theory.

Remark 3.2.5. [WME for ccKG via NLS] We derive WME for the ccKG equation (3.1) in the NLS limit introduced in the two previous remarks. In this limit we consider modulations of the traveling wave solution

$$u(x,t) = \varepsilon \mathrm{e}^{\mathrm{i}\omega t},$$

of the ccKG equation (3.1) with

$$\omega = \sqrt{1-\gamma\varepsilon^2} = 1 - \frac{\gamma}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^4).$$

Step 1: For the derivation of WME we introduce polar coordinates

$$u(x,t) = \varepsilon \mathrm{e}^{r(\varepsilon x, \varepsilon^2 t) + \mathrm{i}\varphi(\varepsilon x, \varepsilon^2 t) + \mathrm{i}\omega t}.$$

Using

$$\begin{aligned} \frac{\partial_t u}{u} &= \varepsilon^2 \partial_\tau r + \mathrm{i}\varepsilon^2 \partial_\tau \varphi + \mathrm{i} - \mathrm{i}\varepsilon^2\gamma/2 + \mathcal{O}(\varepsilon^4), \\ \frac{\partial_t^2 u}{u} &= (\varepsilon^2 \partial_\tau r + \mathrm{i}\varepsilon^2 \partial_\tau \varphi + \mathrm{i} - \mathrm{i}\varepsilon^2\gamma/2 + \mathcal{O}(\varepsilon^4))^2 + (\varepsilon^4 \partial_\tau^2 r + \mathrm{i}\varepsilon^4 \partial_\tau^2 \varphi), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial_x u}{u} &= \varepsilon \partial_\xi r + \mathrm{i}\varepsilon \partial_\xi \varphi, \\ \frac{\partial_x^2 u}{u} &= (\varepsilon \partial_\xi r + \mathrm{i}\varepsilon \partial_\xi \varphi)^2 + (\varepsilon^2 \partial_\xi^2 r + \mathrm{i}\varepsilon^2 \partial_\xi^2 \varphi), \end{aligned}$$

where $\xi = \varepsilon x$ and $\tau = \varepsilon^2 t$, and separating real and imaginary parts we find

$$\begin{aligned} &\varepsilon^4 \partial_\tau^2 r - \left(\varepsilon^2 \partial_\tau \varphi + 1 - \gamma\varepsilon^2/2 + \mathcal{O}(\varepsilon^4)\right)^2 + \left(\varepsilon^2 \partial_\tau r\right)^2 \\ &\qquad = \varepsilon^2 \partial_\xi^2 r - (\varepsilon \partial_\xi \varphi)^2 + (\varepsilon \partial_\xi r)^2 - 1 + \varepsilon^2 \gamma \mathrm{e}^{2r}, \\ &2\varepsilon^2 (\partial_\tau r) \left(\varepsilon^2 \partial_\tau \varphi + 1 - \gamma\varepsilon^2/2 + \mathcal{O}(\varepsilon^4)\right) + \varepsilon^4 \partial_\tau^2 \varphi \\ &\qquad = 2 (\varepsilon \partial_\xi r)(\varepsilon \partial_\xi \varphi) + \varepsilon^2 \partial_\xi^2 \varphi. \end{aligned}$$

In lowest order we obtain

$$\begin{aligned} -2\partial_\tau \varphi &= \partial_\xi^2 r - (\partial_\xi \varphi)^2 + (\partial_\xi r)^2 + \gamma(\mathrm{e}^{2r} - 1), \\ 2\partial_\tau r &= 2 (\partial_\xi r)(\partial_\xi \varphi) + \partial_\xi^2 \varphi, \end{aligned}$$

which is (3.16)-(3.17) up to rescaling $\tau \to -2\tau$.
Step 2: In order to relate the equations (3.16)-(3.17) to WME (3.7)-(3.9) for the ccKG equation, we use the variables from above, namely $\psi = \partial_\xi \phi$ and $\vartheta = \partial_\tau \phi$. We find

$$\begin{aligned} \partial_\tau r &= -\partial_\xi \psi - 2(\partial_\xi r)\psi, \\ \vartheta &= \partial_\xi^2 r - \psi^2 + (\partial_\xi r)^2 + \gamma(\mathrm{e}^{2r} - 1), \\ \partial_\tau \psi &= \partial_\xi \vartheta. \end{aligned}$$

Differentiating the ϑ-equation w.r.t. τ and replacing then $\partial_\tau r$ and $\partial_\tau \psi$ on the new right-hand side by the right-hand sides of the $\partial_\tau r$- and $\partial_\tau \psi$-equations gives the system (3.7)-(3.9) from above.

Remark 3.2.6. Finally, we show how the Benjamin-Feir instability criterion for the ccKG equation and for the associated NLS equation fit together. With the notations of Remark 3.1.6 we have $\mu^2 = 1 - \gamma \mathrm{e}^{2r_{0,\mu}}$ and get from (3.7) that

$$\check{r}^*(\check{\vartheta}, \check{\psi}) = -\frac{\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \check{\vartheta} + h.o.t.$$

using the implicit function theorem. We find

$$\begin{aligned} \partial_T \check{r}^* &= -\frac{\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \partial_T \check{\vartheta} + h.o.t., \\ \partial_X \check{r}^* &= -\frac{\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \partial_X \check{\vartheta} + h.o.t.. \end{aligned}$$

Inserting this in the above equation (3.8) yields

$$\begin{aligned} \partial_T \check{\vartheta} &= 2\left(-\frac{\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \partial_X \check{\vartheta}\right) \check{\psi} + \partial_X \check{\psi} - 2\left(-\frac{\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \partial_T \check{\vartheta}\right)\left(\check{\vartheta} + \mu\right) \\ &= -\frac{2\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \left(\partial_X \check{\vartheta}\right) \check{\psi} + \partial_X \check{\psi} + \frac{2\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \left(\partial_T \check{\vartheta}\right) \check{\vartheta} + \frac{2\mu}{\gamma \mathrm{e}^{2r_{0,\mu}}} \left(\partial_T \check{\vartheta}\right) \mu. \end{aligned}$$

The linearization of this equation is given by

$$\partial_T \check{\vartheta} = \partial_X \check{\psi} + \frac{2\mu^2}{\gamma \mathrm{e}^{2r_{0,\mu}}} \partial_T \check{\vartheta} = \partial_X \check{\psi} + \frac{2\left(1 - \gamma \mathrm{e}^{2r_{0,\mu}}\right)}{\gamma \mathrm{e}^{2r_{0,\mu}}} \partial_T \check{\vartheta}.$$

Hence, we find

$$\partial_T \check{\vartheta} = \frac{1}{\left(3 - \frac{2}{\gamma \mathrm{e}^{2r_{0,\mu}}}\right)} \partial_X \check{\psi} = \frac{1}{(3 - 2\gamma^{-1} \mathrm{e}^{-2r_{0,\mu}})} \partial_X \check{\psi}.$$

Since additionally the equation (3.9), namely $\partial_T \check{\psi} = \partial_X \check{\vartheta}$, holds, the Benjamin-Feir instability occurs for $3 - 2\gamma^{-1} \mathrm{e}^{-2r_{0,\mu}} < 0$. In this case WME are ill-posed in Sobolev spaces. In the NLS limit we have $r_{0,\mu} \to -\infty$ and so $3 \ll \mathrm{e}^{-2r_{0,\mu}}$. Therefore, the sign of γ decides about the stability and instability in the NLS limit. The unstable case is possible for $\gamma > 0$.

3.2.3 Other long wave limit approximations

WME appear as a long wave approximation with wave length $\mathcal{O}(1/\delta)$. Other long wave approximations are the KdV approximation or the inviscid Burgers equation. The KdV approximation describes long waves of amplitude $\mathcal{O}(\delta^2)$ on a $\mathcal{O}(1/\delta^3)$-time scale whereas, as we have seen, the WME approximation describes long waves of amplitude $\mathcal{O}(1)$ on a $\mathcal{O}(1/\delta)$-time scale.

Remark 3.2.7. [KdV approximation] In order to obtain a KdV equation

$$\partial_T A = \nu_1 \partial_X^3 A + \nu_2 A \partial_X A, \tag{3.18}$$

with coefficients ν_1, $\nu_2 \in \mathbb{R}$, for (3.4)-(3.6) with $|\mu| > 1$, we make the ansatz

$$\begin{pmatrix} r_{kdv} \\ \vartheta_{kdv} \\ \psi_{kdv} \end{pmatrix} (x,t) = \delta^2 A(\delta(x-ct), \delta^3 t) V,$$

where $c \in \mathbb{R}$ is the group velocity and $V \in \mathbb{C}^3$ an eigenvector to the eigenvalue 0 associated to one of the two curves $\omega_{\pm 2}$ plotted in the (green dashed) circle of the right panel of Figure 3.1. Then, similar to Theorem 3.1.10 the following approximation result can be established.

Theorem 3.2.8. *Let $|\mu| > 1$, $\sigma_0 > 0$ and $m \geq 5$. Then, for all T_0 and C_1 there exist C_2, T_1, $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the following holds. Let $A \in C([0,T_0], G_{\sigma_0}^{m+3}) \cap C^1((0,T_0], G_{\sigma_0}^m)$ be a solution of the KdV equation* (3.18) *satisfying*

$$\sup_{T\in[0,T_0]} \|A\|_{G_{\sigma_0}^{m+3}} \leq C_1.$$

Then, there exist solutions (r, ϑ, ψ) of (3.4)-(3.6) *with*

$$\sup_{t\in[0,T_1/\delta^3]} \sup_{x\in\mathbb{R}} |(r,\vartheta,\psi)(x,t) - (r_{kdv}, \vartheta_{kdv}, \psi_{kdv})(x,t)| \leq C_2 \delta^{7/2}.$$

Remark 3.2.9. We explain in the subsequent Remark 3.2.14 how the proof of Theorem 3.1.10 has to be modified for proving Theorem 3.2.8. There, we also explain why Theorem 3.1.10 and Theorem 3.2.8 are formulated differently. We refrain from formulating a similar result for the approximation by an inviscid Burgers equation, cf. [BDS19].

Remark 3.2.10. The spectral situation, plotted in the left panel of Figure 3.1, appears for various systems with a spatially homogeneous background state and so for this spectral situation various KdV approximation results exist, for instance for the water wave problem, cf. [Cra85, SW00b, Dül12], or the FPU-system, cf. [SW00a]. Less results are available for the spectral situation plotted in the right panel of Figure 3.1. Only recently methods have been developed for the description of the long wave limit of spatially homogeneous systems by KdV approximations, cf. [CS11,Sch20], and WME approximations for such systems. In the justification analysis of the KdV approximation the new oscillatory modes are eliminated by some normal form transformations. In the justification analysis of the WME approximation a new serious difficulty occurs, namely the fact that due to the scaling of the WME ansatz infinitely many normal form transformations have to be performed, cf. [DKS16] and Chapter 2. In [BDS19] for a Boussinesq equation with spatially periodic coefficients the validity of the WME approximation was established with a suitable chosen energy.

Remark 3.2.11. The spectral situation, plotted in the left panel of Figure 3.1, also appears in the situation described in Remark 3.1.1, namely in the description of modulations of periodic wave trains, and so beside the already mentioned WME approximation results, cf. [DS09, BKZ21], also KdV approximation results, cf. [BGSS09, BGSS10, CR10, CDS14], do exist for the NLS equation. However, slow modulations in time and space of periodic traveling wave solutions with a spectral situation as plotted in the right panel of Figure 3.1 have not been considered before.

3.2.4 Idea of the proof

Remark 3.2.12. [Ideas in the Proof of Theorem 3.1.10] The strategy of the proof is as follows. By using Cauchy-Kovalevskaya theory in Gevrey spaces we obtain local-in-time existence and uniqueness of solutions to WME (3.13)-(3.14). Another application of the Cauchy-Kovalevskaya theory yields the local-in-time existence of higher order approximations in Gevrey spaces. These higher order approximations are necessary for the proof of the main result, Theorem 3.1.10. The solutions of the error equations are controlled with methods from [DKS16], described at the end of Remark 3.2.10. Since we have to perform infinitely many normal form transformations, we must show the convergence of this procedure. Energy estimates for the limit system provide the final argument to finish the proof of Theorem 3.1.10.

Remark 3.2.13. [Differences to [DKS16]] Although our proof is based on the overall idea of [DKS16], there are a number of differences between the analysis from [DKS16] and the analysis of the present chapter. Here, the normal form transformations are only made in a neighborhood of wave numbers at $k = 0$. Therefore, in the present chapter the validity of non-resonance conditions is only necessary in this neighborhood but not on the whole real line like in [DKS16]. However, by this restriction due to some incompatibility of some Fourier modes supports infinitely many new terms are created. For their infinite sum absolute convergence must also be shown. Moreover, since the Benjamin-Feir unstable situation is included, the estimates from [DKS16] have to be transferred from Sobolev spaces to Gevrey spaces, cf. Chapter 2.

Remark 3.2.14. [Ideas of the proof of Theorem 3.2.8] The KdV approximation result in Theorem 3.2.8 can similarly be proven as the Whitham approximation result in Theorem 3.1.10. Thanks to the smaller size of the solutions one normal form transformation is sufficient for the elimination of the oscillatory terms. This is the reason why Theorem 3.2.8 and Theorem 3.1.10 are formulated differently. The smallness of the solutions of WME is needed for the convergence of the infinitely many normal form transformations. Finally, we can also use the same energy estimates as for the WME approximation for the validity of the KdV approximation.

3.3 The case $q \neq 0$

In the previous Remark 3.1.6 we restricted ourselves to the case $q = 0$. In this section we derive the evolution equations in the case $q \neq 0$ and investigate the linear stability of the associated wave trains. Calculations for determining the stability regions in the (μ, q)-parameter plane can be found in Appendix 3.11.

3.3.1 The evolution equations

Here we redo the calculations to derive (3.4)-(3.6) at the beginning of Remark 3.1.6, for the case $q \neq 0$.

Remark 3.3.1. [(r, ϑ, ψ)**-system for** $q \neq 0$] We introduce polar coordinates

$$u = \mathrm{e}^{r+\mathrm{i}\varphi+r_{q,\mu}+\mathrm{i}\mu t+\mathrm{i}qx},$$

with $r = r(x,t)$, $\varphi = \varphi(x,t)$ and $-\gamma \mathrm{e}^{2r_{q,\mu}} = \mu^2 - q^2 - 1$. Inserting this into the ccKG equation (3.1) and separating real and imaginary parts gives

$$\begin{aligned}
\partial_t^2 r &= -(\partial_t r)^2 + (\partial_t \varphi)^2 + 2\mu\partial_t\varphi + (\partial_x r)^2 - (\partial_x\varphi)^2 - 2q\partial_x\varphi + \partial_x^2 r \\
&\quad +\gamma \mathrm{e}^{2r_{q,\mu}}\left(\mathrm{e}^{2r}-1\right), \\
\partial_t^2\varphi &= -2\partial_t r\partial_t\varphi - 2\mu\partial_t r + 2\partial_x r\partial_x\varphi + 2q\partial_x r + \partial_x^2\varphi.
\end{aligned}$$

As in Remark 3.1.6, we introduce the local spatial wave number $\psi = \partial_x\varphi$ and the local temporal wave number $\vartheta = \partial_t\varphi$ for which we obtain the evolutionary system

$$\begin{aligned}
\partial_t^2 r &= \partial_x^2 r + \vartheta^2 + 2\mu\vartheta - (\partial_t r)^2 - \psi^2 + (\partial_x r)^2 - 2q\psi \\
&\quad +\gamma \mathrm{e}^{2r_{q,\mu}}\left(\mathrm{e}^{2r}-1\right), && (3.19)\\
\partial_t\vartheta &= 2(\partial_x r)(\psi+q) + \partial_x\psi - 2(\partial_t r)(\vartheta+\mu), && (3.20)\\
\partial_t\psi &= \partial_x\vartheta. && (3.21)
\end{aligned}$$

3.3.2 Linear stability analysis

In this section we redo the linear stability analysis from Section 3.2.1 for the periodic wave trains in the case $q \neq 0$, i.e., we consider the linear stability of $(r, \vartheta, \psi) = (0,0,0)$ of (3.19)-(3.21).

Remark 3.3.2. The linearization of (3.19)-(3.21) at the origin is given by

$$\begin{aligned}
\partial_t^2 r &= \partial_x^2 r + 2\mu\vartheta - 2q\psi + 2\gamma e^{2r_{q,\mu}} r, \\
\partial_t \vartheta &= \partial_x \psi - 2\mu(\partial_t r) + 2q(\partial_x r), \\
\partial_t \psi &= \partial_x \vartheta,
\end{aligned}$$

which yields the spectral problem

$$\begin{aligned}
-\omega^2 \widehat{r} &= -k^2 \widehat{r} + 2\mu\widehat{\vartheta} - 2q\widehat{\psi} - 2(\mu^2 - 1 - q^2)\widehat{r}, \\
\mathrm{i}\omega\widehat{\vartheta} &= \mathrm{i}k\widehat{\psi} - 2\mu(\mathrm{i}\omega\widehat{r}) + 2q\mathrm{i}k\widehat{r}, \\
\mathrm{i}\omega\widehat{\psi} &= \mathrm{i}k\widehat{\vartheta},
\end{aligned}$$

where we used $\mu^2 = 1 + q^2 - \gamma e^{2r_{q,\mu}}$.

Remark 3.3.3. For the calculation of the eigenvalues $\omega = \omega(k)$ we have to solve

$$\det \begin{pmatrix} \omega^2 - k^2 - 2(\mu^2 - 1 - q^2) & 2\mu & -2q \\ -2\mu\mathrm{i}\omega + 2q\mathrm{i}k & -\mathrm{i}\omega & \mathrm{i}k \\ 0 & \mathrm{i}k & -\mathrm{i}\omega \end{pmatrix} = 0.$$

We find

$$(\omega^2 - k^2 - 2(\mu^2 - 1 - q^2))(-\omega^2 + k^2) - (-2\mu\mathrm{i}\omega + 2q\mathrm{i}k)^2 = 0,$$

respectively

$$\omega^4 - \omega^2(2k^2 + 6\mu^2 - 2 - 2q^2) + \omega(8\mu qk) + k^4 + 2k^2(\mu^2 - 1 - 3q^2) = 0,$$

which no longer can be solved explicitly w.r.t. ω. For $k = 0$ we obtain

$$\omega^2(\omega^2 - 6\mu^2 + 2 + 2q^2) = 0.$$

Hence, we have two eigenvalues equal to zero and two eigenvalues with

$$\omega^2 = 6\mu^2 - 2 - 2q^2.$$

If $\mu^2 > \frac{1}{3} + \frac{1}{3}q^2$, then $\omega_{\pm 1} = \pm\sqrt{6\mu^2 - 2 - 2q^2} \in \mathbb{R}$, cf. Figure 3.5 below. Otherwise $\mu^2 < \frac{1}{3} + \frac{1}{3}q^2$, then $\omega_{\pm 1} = \pm\mathrm{i}\sqrt{2 + 2q^2 - 6\mu^2} \in \mathrm{i}\mathbb{R}$.

Remark 3.3.4. Figure 3.4 shows the different stability/instability regions in the (μ, q)-parameter plane. In the (yellow) area $\mathcal{P}_{\text{stab}}$ the spectral curves show a similar behavior as the ones in the left panel of Figure 3.2. In that case WME approximations can be derived with the same techniques like for $q = 0$. In the (white) area $\mathcal{P}_{\text{rest}}$ there are eigenvalues $\mathrm{i}\omega$ with positive real part at the wave number $k = 0$, see end of Remark 3.3.3. In this region it cannot be expected that the WME approximation makes correct predictions. Typical spectral curves for the parameter region $\mathcal{P}_{\text{benj}}$ are shown in the left panel of Figure 3.5. It shows a Benjamin-Feir instability for $q \neq 0$. Since the derivation of WME needs a spectral situation as shown in Figure 3.2, we concentrate in the following on parameters outside the parameter region $\mathcal{P}_{\text{rest}}$, cf. Remark 3.10.2.

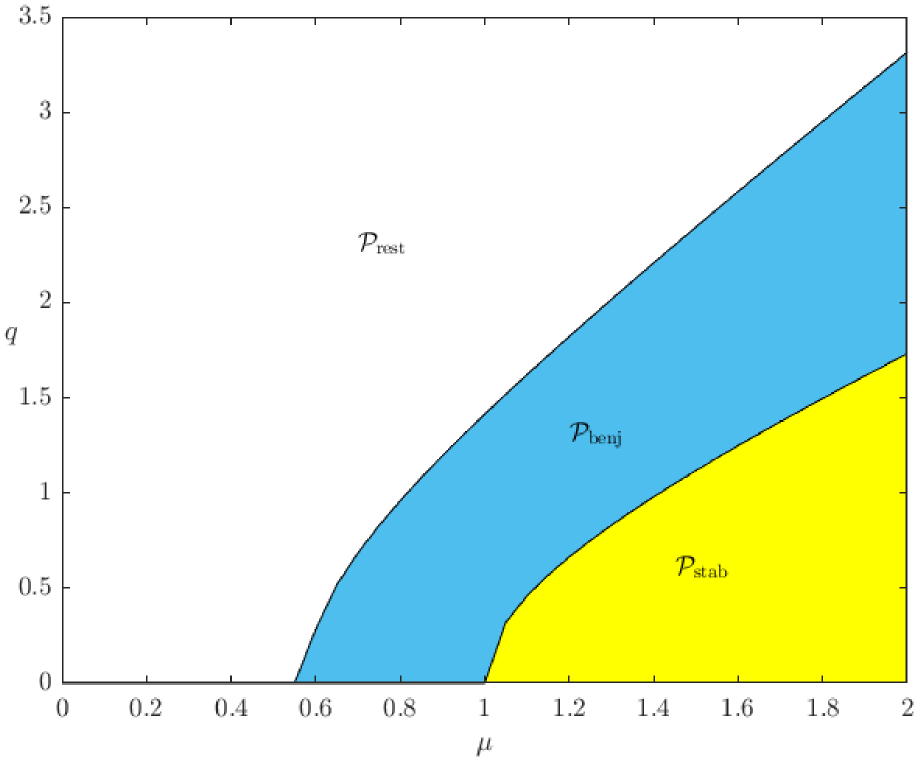

Figure 3.4: In the (μ, q)-parameter plane we identify regions where the spectral curves look qualitatively different. The boundary between the parameter region $\mathcal{P}_{\text{rest}}$ and $\mathcal{P}_{\text{benj}}$ is determined by the equality $1 + q^2 - 3\mu^2 = 0$ and the boundary between the parameter region $\mathcal{P}_{\text{benj}}$ and $\mathcal{P}_{\text{stab}}$ is determined by the equality $1 + 3q^2 - \mu^2 = 0$. For some details, see Appendix 3.11. In $\mathcal{P}_{\text{stab}}$ the wave trains are spectrally stable, in $\mathcal{P}_{\text{benj}}$ they exhibit a Benjamin-Feir instability and in $\mathcal{P}_{\text{rest}}$ WME are not valid.

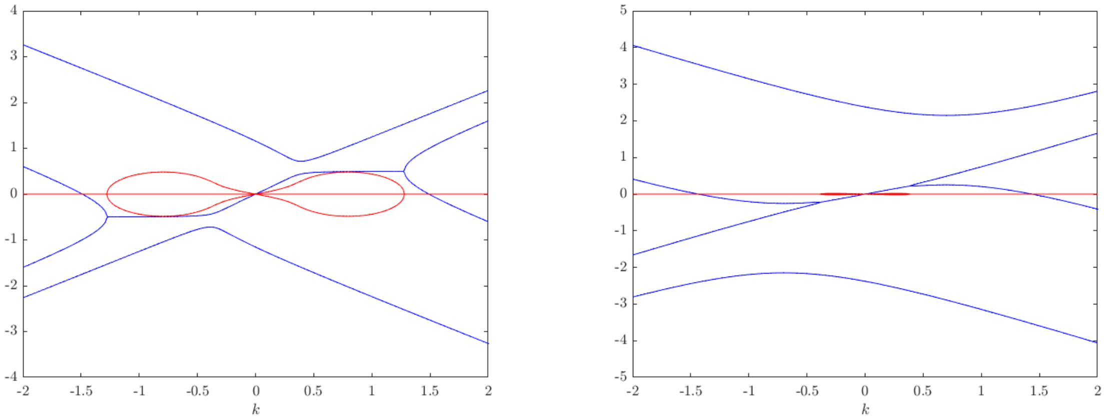

Figure 3.5: The left panel shows the real (in red) and imaginary part (in blue) of the spectral curves as functions over the Fourier wave numbers k of (3.19)-(3.21) for $\mu = 0.8$ and $q = 0.5$ which is located in $\mathcal{P}_{\text{benj}}$. The right panel shows the same for $\mu = 1.2$ and $q = 0.7$ which is close to the boundary of $\mathcal{P}_{\text{benj}}$ and $\mathcal{P}_{\text{stab}}$.

3.4 The improved WME approximation

For estimating the error made by the WME approximation we need that the residual terms, i.e., the terms which do not cancel after inserting the WME approximation into the ccKG equation (3.1) are sufficiently small. The residual can be made smaller by adding higher order terms to the previous WME approximation. This section contains the construction of such an improved WME approximation. The subsequent analysis is an adaption of [HdRS23, Section 2]. The local-in-time existence and uniqueness of solutions of the approximation equations is guaranteed by an application of the Cauchy-Kovalevskaya theory in Gevrey spaces.

3.4.1 Some preparations

In the next Remark 3.4.1 we collect some inequalities which we will use in the following.

Remark 3.4.1. [Estimates in Gevrey spaces] a) We use that G_σ^m is an algebra for $m > 1/2$ and $\sigma \geq 0$. Then in addition, $u, v \in G_\sigma^m$ implies $uv \in G_\sigma^m$ and

$$\|uv\|_{G_\sigma^m} \leq C_m \|u\|_{G_\sigma^m} \|v\|_{G_\sigma^m}, \tag{3.22}$$

where the constant $C_m > 0$ is independent of $\sigma \geq 0$. In case that u and v are vector-valued, the product is replaced by an inner product on $\mathbb{R}^d$. Formula (3.22) can be improved to

$$\|uv\|_{G_\sigma^{m_1}} \leq C_{m_1,m_2}(\|u\|_{G_\sigma^{m_1}} \|v\|_{G_\sigma^{m_2}} + \|u\|_{G_\sigma^{m_2}} \|v\|_{G_\sigma^{m_1}}),$$

which holds for all $\sigma \geq 0$ and $m_j > 1/2$ for $j = 1, 2$, where the constant C_{m_1,m_2} is independent of $\sigma \geq 0$.

b) Let ϕ be any entire function with $\phi(0) = 0$. Then, for any $m > 1/2$ there exists an entire function $\phi_m(z)$ which is monotonically increasing on $\mathbb{R}_+$ and satisfies $\phi_m(0) = 0$ such that we have

$$\|\phi(u)\|_{G_\sigma^m} \leq \phi_m(\|u\|_{G_\sigma^m}), \tag{3.23}$$

for all $u \in G_\sigma^m$.

c) Functions $u \in G_\sigma^m$ can be extended to functions that are analytic on the strip $\{z \in \mathbb{C}\colon |\mathrm{Im}(z)| < \sigma\}$ by the Paley-Wiener Theorem, cf. [RS75, Theorem IX.13]. It is easy to see that for any $\sigma_1 > \sigma_2 \geq 0$ and any $m \geq 0$ we have the continuous embedding $G_{\sigma_1}^0 \hookrightarrow G_{\sigma_2}^m$.

d) Since $\|u(\delta x)\|_{L^2(\mathrm{d}x)} = \mathcal{O}(\delta^{-1/2})$, the WME approximation will be of order $\mathcal{O}(\delta^{-1/2})$ in the L^2-based spaces G_σ^m. In order to estimate the WME approximation without this loss of powers of δ we introduce the following spaces.

Definition 3.4.2. [Spaces $\mathcal{W}_\sigma^m$] *The spaces*

$$\mathcal{W}_\sigma^m = \{u \in C_b^0 : \|u\|_{\mathcal{W}_\sigma^m} < \infty\}$$

are equipped with the norm

$$\|u\|_{\mathcal{W}_\sigma^m} = \int \mathrm{e}^{\sigma|k|} \left(1 + |k|^2\right)^{m/2} |\widehat{u}(k)| \mathrm{d}k,$$

for $\sigma \geq 0$ and $m \geq 0$.

In the following we use

$$\|uv\|_{G_\sigma^m} \leq C_m \|u\|_{\mathcal{W}_\sigma^m} \|v\|_{G_\sigma^m} \quad \text{and} \quad \|uv\|_{\mathcal{W}_\sigma^m} \leq C_m \|u\|_{\mathcal{W}_\sigma^m} \|v\|_{\mathcal{W}_\sigma^m},$$

with $m, \sigma \geq 0$, where the constant $C_m > 0$ is independent of $\sigma \geq 0$.

3.4.2 The structure of the problem

For the derivation of the higher order approximations it turns out that the notational efforts can be reduced tremendously if we exploit the structure of the underlying system (3.4)-(3.6) for (r, ϑ, ψ) after applying the transformation $(\check{r}, \check{\vartheta}) \to (\check{v}, \check{w})$ as in (3.10). The equations for (v, w, ψ) are obtained by substituting (r, ϑ) in terms of (v, w) in (3.4)-(3.6) as explained in the above Remark 3.1.6, i.e., we find

$$\begin{aligned} \partial_t^2(a_1 v + a_2 w) &= \partial_x^2(a_1 v + a_2 w) + (a_3 v + a_4 w)^2 + v \\ &\quad - (\partial_t(a_1 v + a_2 w))^2 - \psi^2 + (\partial_x(a_1 v + a_2 w))^2 \\ &\quad + \gamma \mathrm{e}^{2r_{0,\mu}}(\mathrm{e}^{2(a_1 v + a_2 w)} - 1 - 2(a_1 v + a_2 w)), \end{aligned} \tag{3.24}$$

$$\begin{aligned} \partial_t w &= 2\psi(\partial_x(a_1 v + a_2 w)) + \partial_x \psi \\ &\quad - 2(\partial_t(a_1 v + a_2 w))(a_3 v + a_4 w), \end{aligned} \tag{3.25}$$

$$\partial_t \psi = \partial_x(a_3 v + a_4 w). \tag{3.26}$$

We make the long wave ansatz in (3.24)-(3.26) and set

$$(v,w,\psi)(x,t) = (\check{v},\check{w},\check{\psi})(\delta x,\delta t) = (\check{v},\check{w},\check{\psi})(X,T).$$

We obtain

$$\begin{aligned}
\delta^2\partial_T^2(a_1\check{v}+a_2\check{w}) &= \delta^2\partial_X^2(a_1\check{v}+a_2\check{w}) + (a_3\check{v}+a_4\check{w})^2 + \check{v} \\
&\quad -\delta^2\left(\partial_T(a_1\check{v}+a_2\check{w})\right)^2 - \check{\psi}^2 + \delta^2\left(\partial_X(a_1\check{v}+a_2\check{w})\right)^2 \\
&\quad +\gamma e^{2r_{0,\mu}}(e^{2(a_1\check{v}+a_2\check{w})} - 1 - 2(a_1\check{v}+a_2\check{w})), && (3.27)\\
\partial_T\check{w} &= 2\check{\psi}(\partial_X(a_1\check{v}+a_2\check{w})) + \partial_X\check{\psi} \\
&\quad -2(\partial_T(a_1\check{v}+a_2\check{w}))(a_3\check{v}+a_4\check{w}), && (3.28)\\
\partial_T\check{\psi} &= \partial_X(a_3\check{v}+a_4\check{w}). && (3.29)
\end{aligned}$$

The resulting system (3.27)-(3.29) for $(\check{v},\check{w},\check{\psi})$ is then of the form

$$\begin{aligned}
0 &= \mathbf{M}_v(\mathbf{v},\mathbf{u}) + \delta^2\mathbf{F}_v(D_X^2\mathbf{v}, D_X^2\mathbf{u}, D_T^2\mathbf{v}, D_T^2\mathbf{u}), && (3.30)\\
\partial_T\mathbf{u} &= \mathbf{M}_u(\mathbf{v},\mathbf{u})\partial_X(\mathbf{v},\mathbf{u}) + \mathbf{M}_{T,u}(\mathbf{v},\mathbf{u})\partial_T(\mathbf{v},\mathbf{u}), && (3.31)
\end{aligned}$$

with $\mathbf{v} = \check{v}$ and $\mathbf{u} = (\check{w},\check{\psi})$, where $\mathbf{M}_v(\mathbf{v},\mathbf{u})$, $\mathbf{M}_u(\mathbf{v},\mathbf{u})$ and $\mathbf{M}_{T,u}(\mathbf{v},\mathbf{u})$ are entire (matrix-valued) functions of their arguments. The function $\mathbf{F}_v$ is polynomial in $D_X^2\mathbf{v} = (\mathbf{v},\partial_X\mathbf{v},\partial_X^2\mathbf{v})$, $D_X^2\mathbf{u} = (\mathbf{u},\partial_X\mathbf{u},\partial_X^2\mathbf{u})$, $D_T^2\mathbf{v} = (\mathbf{v},\partial_T\mathbf{v},\partial_T^2\mathbf{v})$ and $D_T^2\mathbf{u} = (\mathbf{u},\partial_T\mathbf{u},\partial_T^2\mathbf{u})$, with the additional property that $\mathbf{F}_v$ is linear in $\partial_T^2\mathbf{v}$ and $\partial_T^2\mathbf{u}$. In the following we use the fact that the only linear term in $\mathbf{M}_v(\mathbf{v},\mathbf{u})$ is $\mathbf{v} = \check{v}$ and that $\mathbf{M}_{T,u}(\mathbf{v},\mathbf{u})$ is linear in its arguments.

3.4.3 Derivation of the amplitude equations

The residual contains the terms which do not cancel after inserting the approximation into the original system. Adding higher order terms to the WME approximation (3.13)-(3.14) allows us to make the residual sufficiently small for our purposes. Like in [BKS20,HdRS23] we consider an improved approximation $(\mathbf{v},\mathbf{u})$ of the form

$$\begin{aligned}
\mathbf{v}(X,T,\delta) &= \mathbf{v}^0(X,T) + \delta^2\mathbf{v}^1(X,T) + \delta^4\mathbf{v}^2(X,T) + h.o.t.,\\
\mathbf{u}(X,T,\delta) &= \mathbf{u}^0(X,T) + \delta^2\mathbf{u}^1(X,T) + \delta^4\mathbf{u}^2(X,T) + h.o.t..
\end{aligned}$$

I. In lowest order: We insert the ansatz into (3.30)-(3.31) and equate the coefficients in front of the δ^s to zero. In lowest order, i.e., here at δ^0 we get

$$\begin{aligned}
0 &= \mathbf{M}_v(\mathbf{v}^0,\mathbf{u}^0),\\
\partial_T\mathbf{u}^0 &= \mathbf{M}_u(\mathbf{v}^0,\mathbf{u}^0)\partial_X(\mathbf{v}^0,\mathbf{u}^0) + \mathbf{M}_{T,u}(\mathbf{v}^0,\mathbf{u}^0)\partial_T(\mathbf{v}^0,\mathbf{u}^0).
\end{aligned}$$

In the following lines we use the properties, listed at the end of Section 3.4.2. By the implicit function theorem the first equation can be solved w.r.t. $\mathbf{v}^0$ for $\mathbf{u}^0$ sufficiently small. Inserting the solution $\mathbf{v}^0 = \mathbf{v}^0(\mathbf{u}^0)$ in the second equation and solving then the second equation w.r.t. $\partial_T \mathbf{u}^0$ for $\mathbf{u^0}$ sufficiently small by Neumann's series yields

$$\partial_T \mathbf{u}^0 = \mathbf{M}(\mathbf{u}^0)\partial_X(\mathbf{u}^0), \tag{3.32}$$

with $\mathbf{u}^0|_{T=0} = \mathbf{u}_0$, which coincides with WME (3.12)-(3.14).

II. Higher order: The governing equations for $(\mathbf{v}^n, \mathbf{u}^n), n \in \mathbb{N}$, arise at δ^{2n}. We obtain linear inhomogeneous equations of the form

$$\begin{aligned} 0 &= \widetilde{\mathbf{M}}_v(\mathbf{v}^n, \mathbf{v}^0, \mathbf{u}^n, \mathbf{u}^0) \\ &\quad + \mathbf{F}_{v,n}\Big(D_X^2\mathbf{v}^0, D_X^2\mathbf{u}^0, D_T^2\mathbf{v}^0, D_T^2\mathbf{u}^0, ... \\ &\qquad ..., D_X^2\mathbf{v}^{n-1}, D_X^2\mathbf{u}^{n-1}, D_T^2\mathbf{v}^{n-1}, D_T^2\mathbf{u}^{n-1}\Big), \\ \partial_T\mathbf{u}^n &= \mathbf{M}_u(\mathbf{v}^0,\mathbf{u}^0)\partial_X(\mathbf{v}^n,\mathbf{u}^n) + D\mathbf{M}_u(\mathbf{v}^0,\mathbf{u}^0)[(\mathbf{v}^n,\mathbf{u}^n)]\partial_X(\mathbf{v}^0,\mathbf{u}^0) \\ &\quad + \mathbf{M}_{T,u}(\mathbf{v}^0,\mathbf{u}^0)\partial_T(\mathbf{v}^n,\mathbf{u}^n) + D\mathbf{M}_{T,u}(\mathbf{v}^0,\mathbf{u}^0)[(\mathbf{v}^n,\mathbf{u}^n)]\partial_T(\mathbf{v}^0,\mathbf{u}^0) \\ &\quad + \mathbf{F}_{u,n}(D_X\mathbf{v}^0, D_X\mathbf{u}^0, D_T\mathbf{v}^0, D_T\mathbf{u}^0, ... \\ &\qquad ..., D_X\mathbf{v}^{n-1}, D_X\mathbf{u}^{n-1}, D_T\mathbf{v}^{n-1}, D_T\mathbf{u}^{n-1}). \end{aligned}$$

Herein, $D\mathbf{M}_u(\mathbf{v}^0,\mathbf{u}^0)[(\mathbf{v}^n,\mathbf{u}^n)]$ denotes the linearization of the map $(v,u) \mapsto \mathbf{M}_u(v,u)$ at the point $(\mathbf{v}^0,\mathbf{u}^0)$ applied to $(\mathbf{v}^n,\mathbf{u}^n)$. $D\mathbf{M}_{T,u}(\mathbf{v}^0,\mathbf{u}^0)[(\mathbf{v}^n,\mathbf{u}^n)]$ is analogously defined. As above we can use the implicit function theorem to solve the first equation with respect to

$$\mathbf{v}^n = \mathbf{v}^n\Big(D_X^2\mathbf{v}^0, D_X^2\mathbf{u}^0, D_T^2\mathbf{v}^0, D_T^2\mathbf{u}^0, ..., D_X^2\mathbf{v}^{n-1}, D_X^2\mathbf{u}^{n-1}, D_T^2\mathbf{v}^{n-1}, D_T^2\mathbf{u}^{n-1}, \mathbf{u}^n\Big),$$

for sufficiently small and sufficiently smooth data, where $\mathbf{v}^n$ is analytic in a neighborhood of 0 and where $\mathbf{v}^n = 0$ for $(D_X^2\mathbf{v}^0, \ldots, D_T^2\mathbf{u}^{n-1}) = 0$ with the help of the properties, listed at the end of Section 3.4.2. By induction, $\mathbf{v}^{n-1}$ and therefore $D_T^2\mathbf{v}^{n-1}$ and $D_X^2\mathbf{v}^{n-1}$ are completely described in terms of $(\mathbf{v}^0, \mathbf{u}^0, ..., \mathbf{v}^{n-2}, \mathbf{u}^{n-2}, \mathbf{u}^{n-1})$ and its temporal and spatial derivatives. Iteratively, $(\mathbf{v}^0, ..., \mathbf{v}^{n-1}, \mathbf{v}^n)$ is determined in terms of $(\mathbf{u}^0, ..., \mathbf{u}^{n-1}, \mathbf{u}^n)$. Thus, we get the equation

$$\begin{aligned} \partial_T\mathbf{u}^n &= \widetilde{\mathbf{M}}_u(\mathbf{u}^0)\partial_X\mathbf{u}^n + \widetilde{D\mathbf{M}}_u(\mathbf{u}^0)[\mathbf{u}^n]\partial_X\mathbf{u}^0 \\ &\quad + \widetilde{\mathbf{M}}_{T,u}(\mathbf{u}^0)\partial_T\mathbf{u}^n + \widetilde{D\mathbf{M}}_{T,u}(\mathbf{u}^0)[(\mathbf{u}^n)]\partial_T(\mathbf{u}^0) \\ &\quad + \mathbf{F}_n(D_X\mathbf{v}^0, D_X\mathbf{u}^0, D_T\mathbf{v}^0, D_T\mathbf{u}^0, ... \\ &\qquad ..., D_X\mathbf{v}^{n-1}, D_X\mathbf{u}^{n-1}, D_T\mathbf{v}^{n-1}, D_T\mathbf{u}^{n-1}), \end{aligned} \tag{3.33}$$

with zero initial data for $n \geq 1$. Here, $\widetilde{\mathbf{M}}_u$, $\widetilde{\mathbf{M}}_{T,u}$, $\mathbf{F}_n$ and the linearizations $\widetilde{D\mathbf{M}}_u(\mathbf{u}^0)$ and $\widetilde{D\mathbf{M}}_{T,u}(\mathbf{u}^0)$ are entire (matrix-valued) functions and $\mathbf{F}_n(0) = 0$. Similar to above we apply Neumann's series to solve (3.33) w.r.t. $\partial_T\mathbf{u}^n$ for sufficiently small and smooth data. Finally, this yields

$$\begin{aligned} \partial_T\mathbf{u}^n &= \widetilde{\mathbf{M}}_u^*(\mathbf{u}^0)\partial_X\mathbf{u}^n + \widetilde{D\mathbf{M}}_u^*(\mathbf{u}^0)[\mathbf{u}^n]\partial_X\mathbf{u}^0 \\ &\quad + \widetilde{D\mathbf{M}}_{T,u}^*(\mathbf{u}^0)[(\mathbf{u}^n)]\partial_T(\mathbf{u}^0) \\ &\quad + \mathbf{F}_n^*(D_X\mathbf{v}^0, D_X\mathbf{u}^0, D_T\mathbf{v}^0, D_T\mathbf{u}^0, ... \\ &\qquad ..., D_X\mathbf{v}^{n-1}, D_X\mathbf{u}^{n-1}, D_T\mathbf{v}^{n-1}, D_T\mathbf{u}^{n-1}), \end{aligned} \tag{3.34}$$

for $n \geq 1$. In the following we explain how to solve these equations and how to obtain estimates for the residual.

3.4.4 Cauchy-Kovalevskaya theory in Gevrey spaces

As preparation for the subsequent error estimates, we would like to show for the simple example of WME

$$\partial_T \mathbf{u} = \mathbf{M}(\mathbf{u})\partial_X \mathbf{u}, \quad \mathbf{u}\Big|_{T=0} = \mathbf{u}_0 , \qquad X \in \mathbb{R},\, T \geq 0, \tag{3.35}$$

where $\mathbf{u} = \mathbf{u}(X,T)$ is an unknown function taking values in $\mathbb{R}^d$, how to obtain estimates in a time-dependent scale of Gevrey spaces. The following Cauchy-Kovalevskaya theorem provides local-in-time existence and uniqueness of solutions in Gevrey spaces for (3.35), cf. [BKS20, HdRS23].

Theorem 3.4.3. *Let $m > 1$ and $R, \sigma_0 > 0$. Then, for every $\mathbf{u}_0 \in G^m_{\sigma_0}$ with $2\|\mathbf{u}_0\|_{G^m_{\sigma_0}} < R$ and $\sigma_1 \in (0,\sigma_0)$ there exists an $\eta = \eta(R, m, \sigma_0, \sigma_1) > 0$ such that for $T_0 = (\sigma_0 - \sigma_1)/\eta$ there exists a local solution $\mathbf{u} \in C^1\Big((0, T_0], G^{m-1}_{\sigma_1}\Big) \cap C\Big([0, T_0], G^m_{\sigma_1}\Big)$ to (3.35) satisfying*

$$\sup_{T\in[0,T_0]} \|\mathbf{u}(T)\|_{G^m_{\sigma_1}} \leq R. \tag{3.36}$$

Proof. For a complete proof we refer to the existing literature, cf. [Saf95, Theorem 1.1]. Here, we restrict ourselves to the question how to obtain the bound (3.36). We define $|k|_{op} := \sqrt{-\partial_x^2}$ as a Fourier multiplier operator. Multiplication of (3.35) by

$$\mathrm{e}^{2\sigma(T)|k|_{op}} \left(1 + |k|^2_{op}\right)^m \mathbf{u},$$

where $\sigma(T) = \sigma_0 - \eta T$, and integration w.r.t. $X \in \mathbb{R}$ leads to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\|\mathbf{u}\|^2_{G^m_{\sigma(T)}} &+ \eta \||k|^{1/2}_{op}\mathbf{u}\|^2_{G^m_{\sigma(T)}} \\ &= \mathrm{Re}\left(((\mathbf{M}(\mathbf{u}) - \mathbf{M}(0))\partial_X \mathbf{u}, \mathbf{u})_{G^m_{\sigma(T)}} + (\mathbf{M}(0)\partial_X \mathbf{u}, \mathbf{u})_{G^m_{\sigma(T)}}\right).\end{aligned}$$

By the Cauchy-Schwarz like inequality

$$\mathrm{Re}(\mathbf{u}, \mathbf{v})_{G^m_\sigma} \leq \|\mathbf{u}\|_{G^{m-1/2}_\sigma} \|\mathbf{v}\|_{G^{m+1/2}_\sigma} ,$$

Remark 3.4.1 with (3.22) and (3.23), and the assumption $m - \frac{1}{2} > \frac{1}{2}$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\|\mathbf{u}\|^2_{G^m_{\sigma(T)}} + \eta\||k|^{1/2}_{op}\mathbf{u}\|^2_{G^m_{\sigma(T)}} \leq \|\mathbf{M}(0)\|\|\mathbf{u}\|^2_{G^{m+1/2}_{\sigma(T)}} + \phi_m(\|\mathbf{u}\|_{G^{m-1/2}_{\sigma(T)}})\|\mathbf{u}\|^2_{G^{m+1/2}_{\sigma(T)}} ,$$

where ϕ_m is an entire function which is monotonically increasing on $\mathbb{R}_+$ and satisfies $\phi_m(0) = 0$. Finally, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\|\mathbf{u}\|^2_{G^m_{\sigma(T)}} + \left(\eta - \|\mathbf{M}(0)\| - \phi_m(\|\mathbf{u}\|_{G^m_{\sigma(T)}})\right)\|\mathbf{u}\|^2_{G^{m+1/2}_{\sigma(T)}} \leq \eta\||k|^{1/2}_{op}\mathbf{u}\|^2_{G^0_{\sigma(T)}}.$$

Choosing η so large that

$$\eta > \|\mathbf{M}(0)\| + \phi_m(R)$$

yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\|\mathbf{u}\|^2_{G^m_{\sigma(T)}} \leq \eta\|\mathbf{u}\|^2_{G^m_{\sigma(T)}}, \quad \text{resp.} \quad \|\mathbf{u}\|^2_{G^m_{\sigma(T)}} \leq \|\mathbf{u}\|^2_{G^m_{\sigma(0)}} \mathrm{e}^{\eta T}.$$

Setting $R = \|\mathbf{u}\|_{G^m_{\sigma(0)}} \mathrm{e}^{\eta T_0/2}$ finally yields (3.36) by continuity. □

Remark 3.4.4. [Artificial damping thanks to decrease of σ in Gevrey spaces] This approach leads to some artificial damping of the solutions at Fourier wave numbers $k \neq 0$ which we will explicitly use later.

3.4.5 Approximate solutions for the perturbed problem

Line by line as in [HdRS23] we obtain

Theorem 3.4.5. *Let $m > 1$ and $\sigma_0 > 0$. Suppose there exists a local-in-time solution*

$$\mathbf{u}^0 \in C^1\Big((0, T_0], G^{m-1}_{\sigma_0}\Big) \cap C\Big([0, T_0], G^m_{\sigma_0}\Big)$$

to (3.32). *Then, for every $\sigma_1 \in (0, \sigma_0)$, $n \in \mathbb{N}$ and for all $0 < k \leq n$ there exist $T_1 = T_1(\sigma_1, k+1) \leq T_1(\sigma_1, k) \leq T_0$ and solutions*

$$\mathbf{u}^k \in C^1\Big((0, T_1], G^{m-1}_{\sigma_1}\Big) \cap C\Big([0, T_1], G^m_{\sigma_1}\Big)$$

to (3.34).

Then, the n-th order approximations are given by

$$\begin{aligned}\tilde{\mathbf{v}}^n(T) &= \mathbf{v}^0(T) + \delta^2\mathbf{v}^1(T) + \cdots + \delta^{2n}\mathbf{v}^n(T)\,,\\ \tilde{\mathbf{u}}^n(T) &= \mathbf{u}^0(T) + \delta^2\mathbf{u}^1(T) + \cdots + \delta^{2n}\mathbf{u}^n(T)\,,\end{aligned}$$

with corresponding residuals of (3.30)-(3.31)

$$\begin{aligned}\mathsf{Res}^n_v(T) &= \mathbf{M}_v(\tilde{\mathbf{v}}^n, \tilde{\mathbf{u}}^n) + \delta^2\mathbf{F}_v(D^2_X\tilde{\mathbf{v}}^n, D^2_X\tilde{\mathbf{u}}^n, D^2_T\tilde{\mathbf{v}}^n, D^2_T\tilde{\mathbf{u}}^n),\\ \mathsf{Res}^n_u(T) &= -\partial_T\tilde{\mathbf{u}}^n + \mathbf{M}_u(\tilde{\mathbf{v}}^n, \tilde{\mathbf{u}}^n)\partial_X(\tilde{\mathbf{v}}^n, \tilde{\mathbf{u}}^n) + \mathbf{M}_{T,u}(\tilde{\mathbf{v}}^n, \tilde{\mathbf{u}}^n)\partial_T(\tilde{\mathbf{v}}^n, \tilde{\mathbf{u}}^n).\end{aligned}$$

By the above construction and Theorem 3.4.5 we directly obtain

Corollary 3.4.6. *Assume that the hypotheses of Theorem 3.4.5 are met. Then, for every $n \in \mathbb{N}_0$ the approximate solutions $(\tilde{\mathbf{v}}^n, \tilde{\mathbf{u}}^n)$ and residuals $(\mathsf{Res}^n_v, \mathsf{Res}^n_u)$ are in $C\Big([0, T_1], G^m_{\tilde{\sigma}_1}\Big)$ for all $\tilde{\sigma}_1 \in [0, \sigma_1)$. Furthermore, there exists a constant $C > 0$ such that we have*

$$\begin{aligned}\sup_{T\in[0,T_1]} \|(\tilde{\mathbf{v}}^n(T), \tilde{\mathbf{u}}^n(T)) - (\mathbf{v}^0(T), \mathbf{u}^0(T))\|_{G^m_{\tilde{\sigma}_1}} &\leq C\delta^2,\\ \sup_{T\in[0,T_1]} \|(\mathsf{Res}^n_v, \mathsf{Res}^n_u)(T)\|_{G^m_{\tilde{\sigma}_1}} &\leq C\delta^{2n+2}.\end{aligned}$$

3.5 The equations for the error

For notational simplicity the following analysis is carried out for $q = 0$. It will be obvious that the proof will also work for $q \neq 0$ in the parameter regimes $\mathcal{P}_{\text{benj}}$ and $\mathcal{P}_{\text{stab}}$, cf. Figure 3.4. For estimating the difference between the WME approximation $(\widetilde{\mathbf{v}}^n, \widetilde{\mathbf{u}}^n)$ and true solutions $(r, \vartheta, \psi)(x, t)$ of (3.4)-(3.6) on the long $\mathcal{O}(1/\delta)$-time scale, we separate the modes in a neighborhood of the wave number $k = 0$ from the modes bounded away from the wave number $k = 0$. In the neighborhood of $k = 0$ we use normal form transformations and energy estimates similar to the ones in Section 3.4 to get rid of the terms of order $\mathcal{O}(1)$ in the equations for the error. Outside of this neighborhood we will use that the time-dependent scale of Gevrey spaces additionally leads to some artificial damping of the solutions at Fourier wave numbers $k \neq 0$.

Step 1: Our starting point is system (3.4)-(3.6) which we write as first order system

$$\partial_t V = LV + N(V), \tag{3.37}$$

where

$$V = \begin{pmatrix} r \\ \widetilde{r} \\ \vartheta \\ \psi \end{pmatrix}, \qquad LV = \begin{pmatrix} \widetilde{r} \\ \partial_x^2 r + 2\mu\vartheta + 2(1-\mu^2)r \\ \partial_x \psi - 2\mu\widetilde{r} \\ \partial_x \vartheta \end{pmatrix},$$

and

$$N(V) = \begin{pmatrix} 0 \\ \vartheta^2 - \widetilde{r}^2 - \psi^2 + (\partial_x r)^2 + (1-\mu^2)(\mathrm{e}^{2r} - 1 - 2r) \\ 2(\partial_x r)\psi - 2\widetilde{r}\vartheta \\ 0 \end{pmatrix}.$$

We introduce the error function $\mathcal{R}$ made by the associated WME approximation Ψ through $V = \Psi + \delta^{3/2}\mathcal{R}$, where $\Psi = \Psi^n = (\check{r}^n, \partial_t \check{r}^n, \check{\vartheta}^n, \check{\psi}^n)^T$ which is a function of $(\check{v}^n, \check{w}^n, \check{\psi}^n) = (\mathbf{v}^n, \mathbf{u}^n)$ by the transformation (3.10) where $(\mathbf{v}^n, \mathbf{u}^n)$ is the WME approximation constructed in Section 3.4.

The error function $\mathcal{R}$ satisfies

$$\partial_t \mathcal{R} = L\mathcal{R} + DN(\Psi)\mathcal{R} + \delta^{3/2} G(\mathcal{R}) + \delta^{-3/2}\mathrm{Res}_{\mathcal{R}}, \tag{3.38}$$

with $\mathcal{R}(0) = 0$, where $DN(\Psi)\mathcal{R}$ stands for the Ψ-dependent terms which are linear in $\mathcal{R}$, and $G(\mathcal{R})$ for the terms which are non-linear in $\mathcal{R}$, i.e.,

$$\delta^{3/2} G(\mathcal{R}) = \delta^{-3/2}(N(\Psi + \delta^{3/2}\mathcal{R}) - N(\Psi)) - DN(\Psi)\mathcal{R}.$$

In our notation, we suppress the fact that G depends on Ψ, too. In contrast to $L\mathcal{R}$ and $DN(\Psi)\mathcal{R}$, the term G is of order $\mathcal{O}(\delta^{3/2})$, and therefore, it makes no problems for obtaining a $\mathcal{O}(1)$-bound for $\mathcal{R}$ on the long $\mathcal{O}(1/\delta)$-time scale. The residual $\mathrm{Res}_{\mathcal{R}}$ contains all terms which do not cancel after inserting the WME approximation Ψ into (3.37). Since the residual $\mathrm{Res}_{\mathcal{R}}$ is obtained by a number of simple transformations from the residuals $(\mathsf{Res}_v^n, \mathsf{Res}_u^n)$, the residual $\mathrm{Res}_{\mathcal{R}}$ obeys the following estimates.

Corollary 3.5.1. *Assume that the hypotheses of Theorem 3.4.5 are met. Then, for every $n \in \mathbb{N}_0$ the approximate solution Ψ^n and residual $\mathrm{Res}_{\mathcal{R}}$ are in $C\Big([0,T_1], G^m_{\tilde{\sigma}_1}\Big)$ for all $\tilde{\sigma}_1 \in [0,\sigma_1)$. Furthermore, there exists a constant $C>0$ such that we have*

$$\begin{aligned} \sup_{T\in[0,T_1]} \|\Psi^n(T) - \Psi^0(T)\|_{G^m_{\tilde{\sigma}_1}} &\leq C\delta^2, \\ \sup_{T\in[0,T_1]} \|\mathrm{Res}_{\mathcal{R}}(T)\|_{G^m_{\tilde{\sigma}_1}} &\leq C\delta^{2n+2}. \end{aligned}$$

So the residual $\mathrm{Res}_{\mathcal{R}}$ is sufficiently small for n large enough and also makes no problems for obtaining a $\mathcal{O}(1)$-bound for $\mathcal{R}$ on the long $\mathcal{O}(1/\delta)$-time scale.

Step 2: As mentioned above, the error will be handled differently for different Fourier wave numbers. So for the separation of the modes we introduce the mode projections $E = E(\frac{1}{\mathrm{i}}\partial_x)$, such that $\widehat{Ef}(k) = \widehat{E}(k)\widehat{f}(k)$ with

$$\widehat{E}(k) = \begin{cases} 1, & |k| \leq \delta_c, \\ 0, & |k| > \delta_c, \end{cases}$$

and $E^c = \mathrm{Id} - E$, such that $\widehat{E^c f}(k) = \widehat{E}^c(k)\widehat{f}(k) = (1-\widehat{E}(k))\widehat{f}(k)$ on the complementary part for a $\delta_c > 0$, independent of $0 < \delta \ll 1$. Since $\widehat{E}^c(k)$ is equal to zero in a δ_c-neighborhood of $k=0$, we have in this neighborhood the following estimate

$$|\widehat{E}^c(k)| \leq C|k|^\alpha,$$

for every $\alpha \in \mathbb{N}$. Due to this fact we have that $E^c\Psi$ is of the order $\mathcal{O}(\delta^s)$ if Ψ is chosen s times more differentiable than the error. This is a consequence of the following lemma, cf. [Sch20, Lemma 7].

Lemma 3.5.2. *Let $\theta_0 \geq 0$, $\theta_\infty \in \mathbb{R}$ and $\widehat{E}^c : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|\widehat{E}^c(k)| \leq C \min(|k|^{\theta_0}, (1+|k|)^{\theta_\infty}).$$

Then, for the associated operator $E^c = \mathcal{F}^{-1}\widehat{E}^c\mathcal{F}$ the following holds. For

a) $\sigma_1 > \sigma_2$ and $m_1, m_2 \geq 0$ or

b) $\sigma_1 = \sigma_2$ and $m_2 - m_1 \geq \max(\theta_0, \theta_\infty)$,

there exists a $C>0$ such that

$$\|E^c\Psi(\delta\cdot)\|_{G^{m_1}_{\sigma_1/\delta}} \leq C\delta^{\theta_0 - \frac{1}{2}}\|\Psi(\cdot)\|_{G^{m_2}_{\sigma_2}},$$

for all $\delta \in (0,1)$.

Hereby, the loss of $\delta^{-1/2}$ is due to the scaling properties of the L^2-norm. We introduce the new error functions $R = E\mathcal{R}$ and $R^c = E^c\mathcal{R}$ which satisfy

$$\partial_t R = LR + EDN(\Psi)R + EDN(\Psi)R^c + \delta^{3/2}EG(R+R^c) + \delta^{-3/2}E\mathrm{Res}_{\mathcal{R}}, \quad (3.39)$$

$$\begin{aligned} \partial_t R^c = {} & LR^c + E^cDN(\Psi)R + E^cDN(\Psi)R^c + \delta^{3/2}E^cG(R+R^c) \quad (3.40) \\ & + \delta^{-3/2}E^c\mathrm{Res}_{\mathcal{R}}, \end{aligned}$$

by applying E and E^c to (3.38). For controlling the error functions R and R^c on the long $\mathcal{O}(1/\delta)$-time scale we have to get rid of the $\mathcal{O}(1)$-terms on the right-hand sides. It turns out that the terms

$$E^c DN(\Psi)R + E^c DN(\Psi)R^c$$

in (3.40) can be controlled by the artificial damping obtained from the time-dependent scale of Gevrey spaces, see Section 3.9 below. We use normal form transformations and energy estimates to get rid of the terms $EDN(\Psi)R + EDN(\Psi)R^c$ in (3.39) with a Fourier support in a neighborhood of the Fourier wave number $k = 0$.

Step 3: For the normal form transformations in the neighborhood around $k = 0$ it is advantageous to diagonalize the linearized system. A diagonalization is possible since L is of the form

$$\widehat{L}(k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2(1-\mu^2) + \mathcal{O}(k^2) & 0 & 2\mu & 0 \\ 0 & -2\mu & 0 & \mathrm{i}k \\ 0 & 0 & \mathrm{i}k & 0 \end{pmatrix},$$

for $k \to 0$. There are two eigenvalues $\mathrm{i}\omega_{\pm 1}$ of order $\mathcal{O}(1)$ and two eigenvalues $\mathrm{i}\omega_{\pm 2}$ of order $\mathcal{O}(k)$. The associated eigenvectors are denoted by $\widehat{\varphi}_{\pm 1}(k)$ and $\widehat{\varphi}_{\pm 2}(k)$. For $R^* = S^{-1}R$, with the matrix $\widehat{S}(k) = (\widehat{\varphi}_1(k), \widehat{\varphi}_{-1}(k), \widehat{\varphi}_2(k), \widehat{\varphi}_{-2}(k))$ for $|k| \leq \delta_c$, we find

$$\begin{aligned} \partial_t R^* &= \Lambda R^* + ES^{-1}(DN(\Psi)SR^*) + ES^{-1}(DN(\Psi)R^c) \\ &\quad + \delta^{3/2} ES^{-1} G(SR^* + R^c) + \delta^{-3/2} ES^{-1} \mathrm{Res}_{\mathcal{R}}, \end{aligned} \tag{3.41}$$

where

$$\widehat{\Lambda}(k) = S^{-1}(k)\widehat{L}(k)S(k) = \begin{pmatrix} \mathrm{i}\omega_1(k) & 0 & 0 & \\ 0 & \mathrm{i}\omega_{-1}(k) & 0 & 0 \\ 0 & 0 & \mathrm{i}\omega_2(k) & 0 \\ 0 & 0 & 0 & \mathrm{i}\omega_{-2}(k) \end{pmatrix}.$$

Step 4: Since Ψ is strongly concentrated at the wave number $k = 0$, the part $E^c\Psi$ is $\mathcal{O}(\delta^s)$ in the spaces used subsequently if Ψ is chosen s times more differentiable than the error. The approximation Ψ appears in the equations for the error not only linearly but also nonlinearly. Due to product estimates (3.22) the application of E^c on the nonlinear terms in Ψ is of the order $\mathcal{O}(\delta^s)$ if Ψ is chosen s times more differentiable than the error. Hence, we separate Ψ into

$$\Psi = \Psi_0 + \Psi_r,$$

with

$$\Psi_0 = E\Psi \quad \text{and} \quad \Psi_r = E^c\Psi,$$

and write

$$\begin{aligned} ES^{-1}(DN(\Psi)SR^*) &= ES^{-1}(DN(\Psi_0)SR^*) \\ &\quad + (ES^{-1}(DN(\Psi_0 + \Psi_r)SR^*) - ES^{-1}(DN(\Psi_0)SR^*)), \end{aligned}$$

where by the above argument

$$ES^{-1}(DN(\Psi_0 + \Psi_r)SR^*) - ES^{-1}(DN(\Psi_0)SR^*) = \mathcal{O}(\delta^s).$$

Hence, we rewrite (3.41) into

$$\partial_t R^* = \Lambda R^* + ES^{-1}(DN(\Psi_0)(SR^* + R^c)) + H_0^*, \tag{3.42}$$

where

$$\begin{aligned} H_0^* &= ES^{-1}(DN(\Psi_0 + \Psi_r)(SR^* + R^c)) - ES^{-1}(DN(\Psi_0)(SR^* + R^c)) \\ &\quad + \delta^{3/2} ES^{-1} G(SR^* + R^c) + \delta^{-3/2} ES^{-1} \mathrm{Res}_{\mathcal{R}}. \end{aligned}$$

Step 5: The Fourier support can be restricted further. In fact, the term

$$ES^{-1}(DN(\Psi_0)(SR^* + R^c))$$

can be written as

$$\begin{aligned} ES^{-1}(DN(\Psi_0)(SR^* + R^c)) &= E\left(\left(\sum_{n=1}^{\infty} L_n(\Psi_0)\right)(SR^* + R^c)\right) \\ &= \sum_{n=1}^{\infty} E((L_n(\Psi_0))(SR^* + R^c)), \end{aligned}$$

where the operators L_n are linear operators with $L_n(\Psi_0) = \mathcal{O}(\|\Psi_0\|^n)$. This term is split further into

$$\sum_{n=1}^{\infty} E((EL_n(\Psi_0))(SR^* + R^c)) + \sum_{n=1}^{\infty} E((E^c L_n(\Psi_0))(SR^* + R^c)),$$

where by the above argument

$$\sum_{n=1}^{\infty} E((E^c L_n(\Psi_0))(SR^* + R^c)) = \mathcal{O}(\delta^s).$$

With the previous remarks we write (3.42) as

$$\partial_t R^* = \Lambda R^* + \sum_{n=1}^{\infty} E((EL_n(\Psi_0))(SR^* + R^c)) + H^*, \tag{3.43}$$

where

$$H^* = H_0^* + \sum_{n=1}^{\infty} E((E^c L_n(\Psi_0))(SR^* + R^c)).$$

We use normal form transformations to simplify $\sum_{n=1}^{\infty} E((EL_n(\Psi_0))(SR^* + R^c))$ as far as possible for subsequently applying energy estimates to get rid of the remaining terms. The term H^* does not make problems to prove bounds for R on the long $\mathcal{O}(1/\delta)$-time scale since all terms in H^* are at least of order $\mathcal{O}(\delta)$.

Step 6: We separate (3.43) in its components. With R^*, respectively R^c, written as $(R_1, R_{-1}, R_2, R_{-2})^T$, with some slight abuse of notation, we have

$$\partial_t R_1 = \mathrm{i}\omega_1 R_1 + \sum_{n=1}^{\infty} \sum_{j=\pm1,\pm2} B_{n,1,j}(\Psi_0, R_j) + H_1^*, \tag{3.44}$$

$$\partial_t R_2 = \mathrm{i}\omega_2 R_2 + \sum_{n=1}^{\infty} \sum_{j=\pm1,\pm2} B_{n,2,j}(\Psi_0, R_j) + H_2^*, \tag{3.45}$$

and similarly for R_{-1} and R_{-2}, where $\omega_j R_j$ is understood as

$$\widehat{\omega_j R_j}(k) = \widehat{\omega_j}(k)\widehat{R_j}(k),$$

for $j \in \{\pm1, \pm2\}$, where

$$B_{n,j_1,j_2}(\Psi_0, R_{j_2}) = \varphi_{j_1}^* \cdot E((EL_n(\Psi_0))(\varphi_{j_2}^* \cdot (SR^* + R^c)\varphi_{j_2})),$$

and where $\widehat{\varphi}_j^*(k)$ are the adjoint eigenvectors of $\widehat{L}(k)$, with the property

$$(\widehat{\varphi}_i^*(k), \widehat{\varphi}_j(k))_{\mathbb{C}^4} = \delta_{ij}.$$

Obviously, the remaining terms H_j^* satisfy the estimates

$$\|H_j^*\|_{G_\sigma^m} \leq C_1\delta\|\mathcal{R}\|_{G_\sigma^m} + C_2\delta^{3/2}\|\mathcal{R}\|_{G_\sigma^m}^2 + C_3\delta,$$

for all $\sigma \geq 0$ and $m > 1/2$, where the constants C_j can be chosen independently of σ.

Remark 3.5.3. [Separation of the equations (3.44) **-** (3.45)] Although (3.44) - (3.45) can be written as one equation, we will not do so, since in the following the nonlinear terms are handled differently and have different properties, cf. Section 3.9.

Remark 3.5.4. [Terms $\mathbf{B_{n,j_1,j_2}(\Psi_0, R_{j_2})}$] The terms $B_{n,j_1,j_2}(\Psi_0, R_{j_2})$ consist of a mode projection E applied on a product of a term $EL_n(\Psi_0)$ with Fourier support in $[-\delta_c, \delta_c]$ and an error term $SR^* + R^c$. In Fourier space the term is mainly a convolution with some kernel $\widehat{b}_{n,j_1,j_2}$, i.e.,

$$B_{n,j_1,\widehat{j_2(\Psi_0, R_{j_2})}}(k) = \int \widehat{b}_{n,j_1,j_2}(k, k-k', k')\widehat{\Psi}_0^{*n}(k-k')\widehat{R}_{j_2}(k')\mathrm{d}k',$$

where $\widehat{\Psi}_0^{*n}$ corresponds to a n-times convolution of $\widehat{\Psi}_0$. More details about this form can be found in Section 3.1 in [DKS16]. This term is non-zero if $|k| \leq \delta_c$ and if $|k-k'| \leq \delta_c$. As a consequence, only $|k'| \leq 2\delta_c$ has to be considered. In general, the kernels are more complicated and depend on more variables.

3.6 The series of normal form transformations

In the following we use the notation

$$\rho = \mathcal{O}(\|\Psi_0\|^n \|\mathcal{R}\|),$$

if

$$\|\rho\|_{G_\sigma^m} \le C\|\Psi_0\|^n_{G_\sigma^m} \|\mathcal{R}\|_{G_\sigma^m},$$

for all $\sigma \ge 0$ and $m > 1/2$, where the constant C can be chosen independently of σ.

Our goal is to prove a $\mathcal{O}(1)$-bound for $R_{\pm 1}$, $R_{\pm 2}$, and R^c on a $\mathcal{O}(1/\delta)$-time scale. As already said, as a next step on this path, we simplify (3.44)-(3.45) by eliminating all non-resonant terms of order $\mathcal{O}(1)$ by near-identity changes of variables. System (3.44)-(3.45) has a similar structure as [DKS16, system (21)-(24)] and so it can be expected that the non-resonant terms $B_{n,1,\pm 2}$, $B_{n,1,-1}$ and $B_{n,2,\pm 1}$ and similarly $B_{n,-1,\pm 2}$, $B_{n,-1,1}$ and $B_{n,-2,\pm 1}$ can be eliminated with convergent infinite series of normal form transformations, cf. Chapter 2. With the first normal form transformation we eliminate the terms $B_{1,1,\pm 2}$, $B_{1,1,-1}$ and $B_{1,2,\pm 1}$, with the second normal form transformation we eliminate updated versions of the terms $B_{2,1,\pm 2}$, $B_{2,1,-1}$ and $B_{2,2,\pm 1}$, etc.

Step 1: [The first normal form transformation] To illustrate the procedure, we show how to obtain the first near identity change of variables. We set

$$\begin{aligned} R_{1,1} &= R_1 + \sum_{j_2 \in \{-1,\pm 2\}} M_{1,1,j_2}(\Psi_0, R_{j_2}), \\ R_{2,1} &= R_2 + \sum_{j_2 = \pm 1} M_{1,2,j_2}(\Psi_0, R_{j_2}). \end{aligned}$$

The operators M_{1,j_1,j_2} and B_{1,j_1,j_2} are linear in the error functions R_{j_2} and possess a convolution structure in Fourier space, i.e.,

$$\widehat{M_{1,j_1,j_2}(\Psi_0, R_{j_2})}(k,t) = \int \widehat{m}_{1,j_1,j_2}(k, k-k', k')\widehat{\Psi}_0(k-k', \delta t)\widehat{R_{j_2}}(k',t)\mathrm{d}k',$$

with kernel $\widehat{m}_{1,j_1,j_2}$ and similarly for B_{1,j_1,j_2} with kernel $\widehat{b}_{1,j_1,j_2}$.

We differentiate $R_{1,1}$ w.r.t. time and obtain

$$\begin{aligned} \partial_t R_{1,1} &= \partial_t R_1 + \sum_{j_2 \in \{-1,\pm 2\}} (M_{1,1,j_2}(\partial_t \Psi_0, R_{j_2}) + M_{1,1,j_2}(\Psi_0, \partial_t R_{j_2})) \\ &= \mathrm{i}\omega_1 R_1 + \sum_{j_2 \in \{\pm 1, \pm 2\}} B_{1,1,j_2}(\Psi_0, R_{j_2}) + \mathcal{O}(\|\Psi_0\|^2\|\mathcal{R}\|) + \mathcal{O}(\delta) \\ &\quad + \sum_{j_2 \in \{-1,\pm 2\}} M_{1,1,j_2}\left(\Psi_0, \mathrm{i}\omega_{j_2} R_{j_2} + \sum_{j_2' \in \{\pm 1,\pm 2\}} B_{1,j_2,j_2'}(\Psi_0, R_{j_2'})\right) + \mathcal{O}(\delta) \\ &= \mathrm{i}\omega_1 R_{1,1} - \mathrm{i}\omega_1 \sum_{j_2 \in \{-1,\pm 2\}} M_{1,1,j_2}(\Psi_0, R_{j_2}) + \sum_{j_2 \in \{\pm 1,\pm 2\}} B_{1,1,j_2}(\Psi_0, R_{j_2}) \\ &\quad + \sum_{j_2 \in \{-1,\pm 2\}} M_{1,1,j_2}\left(\Psi_0, \mathrm{i}\omega_{j_2} R_{j_2} + \sum_{j_2' \in \{\pm 1,\pm 2\}} B_{1,j_2,j_2'}(\Psi_0, R_{j_2'})\right) \\ &\quad + \mathcal{O}(\|\Psi_0\|^2\|\mathcal{R}\|) + \mathcal{O}(\delta), \end{aligned}$$

where we used that $\partial_t \Psi_0 = \mathcal{O}(\delta)$ due to the long wave character of Ψ_0. In order to eliminate the terms $B_{1,1,j}$ for $j \in \{-1, \pm 2\}$, we choose $M_{1,1,j}$ to satisfy

$$\begin{aligned} -\mathrm{i}\omega_1 M_{1,1,2}(\Psi_0, R_2) + M_{1,1,2}(\Psi_0, \mathrm{i}\omega_2 R_2) + B_{1,1,2}(\Psi_0, R_2) &= 0, \\ -\mathrm{i}\omega_1 M_{1,1,-2}(\Psi_0, R_{-2}) + M_{1,1,-2}(\Psi_0, \mathrm{i}\omega_{-2} R_{-2}) + B_{1,1,-2}(\Psi_0, R_{-2}) &= 0, \\ -\mathrm{i}\omega_1 M_{1,1,-1}(\Psi_0, R_{-1}) + M_{1,1,-1}(\Psi_0, \mathrm{i}\omega_{-1} R_{-1}) + B_{1,1,-1}(\Psi_0, R_{-1}) &= 0, \end{aligned}$$

i.e., we set

$$\begin{aligned} \widehat{m}_{1,1,2}(k, k-k', k') &= \frac{\widehat{b}_{1,1,2}(k, k-k', k')}{\mathrm{i}\widehat{\omega}_1(k) - \mathrm{i}\widehat{\omega}_2(k')}, \\ \widehat{m}_{1,1,-2}(k, k-k', k') &= \frac{\widehat{b}_{1,1,-2}(k, k-k', k')}{\mathrm{i}\widehat{\omega}_1(k) - \mathrm{i}\widehat{\omega}_{-2}(k')}, \\ \widehat{m}_{1,1,-1}(k, k-k', k') &= \frac{\widehat{b}_{1,1,-1}(k, k-k', k')}{\mathrm{i}\widehat{\omega}_1(k) - \mathrm{i}\widehat{\omega}_{-1}(k')}. \end{aligned}$$

Since $|k| \le \delta_c$ and $|k'| \le 2\delta_c$, the denominator is non-zero for $\delta_c > 0$ sufficiently small, and so the $\widehat{m}_{1,1,\pm 2}$ and $\widehat{m}_{1,1,-1}$ are well-defined and bounded. As a consequence, the $M_{1,1,\pm 2}$ are bounded mappings in all G^m_σ-spaces. After this transformation we have

$$\begin{aligned} \partial_t R_{1,1} &= \mathrm{i}\omega_1 R_{1,1} + B_{1,1,1}(\Psi_0, R_1) + \mathcal{O}(\|\Psi_0\|^2 \|\mathcal{R}\|) + \mathcal{O}(\delta) \\ &\quad + \sum_{j \in \{-1, \pm 2\}} M_{1,1,j}\left(\Psi_0, \sum_{j_1 \in \{\pm 1, \pm 2\}} B_{1,j,j_1}(\Psi_0, R_{j_1})\right) + \mathcal{O}(\delta) \\ &= \mathrm{i}\omega_1 R_{1,1} + B_{1,1,1}(\Psi_0, R_1) + \mathcal{O}(\|\Psi_0\|^2 \|\mathcal{R}\|) + \mathcal{O}(\delta). \end{aligned}$$

We do exactly the same with $R_{2,1}$ and obtain

$$\begin{aligned} \widehat{m}_{1,2,1}(k, k-k', k') = \frac{\widehat{b}_{1,2,1}(k, k-k', k')}{\mathrm{i}\widehat{\omega}_2(k) - \mathrm{i}\widehat{\omega}_1(k')}, \\ \widehat{m}_{1,2,-1}(k, k-k', k') = \frac{\widehat{b}_{1,2,-1}(k, k-k', k')}{\mathrm{i}\widehat{\omega}_2(k) - \mathrm{i}\widehat{\omega}_{-1}(k')}, \end{aligned}$$

such that finally

$$\begin{aligned} \partial_t R_{2,1} &= \mathrm{i}\omega_2 R_{2,1} + \sum_{j=\pm 2} B_{1,2,j}(\Psi_0, R_j) + \mathcal{O}(\|\Psi_0\|^2 \|\mathcal{R}\|) + \mathcal{O}(\delta) \\ &\quad + \sum_{j=\pm 1} M_{1,2,j}\left(\Psi_0, \sum_{j_1 = \pm 1, \pm 2} B_{1,j,j_1}(\Psi_0, R_{j_1})\right) + \mathcal{O}(\delta) \\ &= \mathrm{i}\omega_2 R_{2,1} + \sum_{j=\pm 2} B_{1,2,j}(\Psi_0, R_j) + \mathcal{O}(\|\Psi_0\|^2 \|\mathcal{R}\|) + \mathcal{O}(\delta). \end{aligned}$$

Hence, new terms of order $\mathcal{O}(1)$ are created by this strategy and so this procedure must be performed again and again. The newly created terms by the second normal form transformation are at most of order $\mathcal{O}(\|\Psi_0\|^2)$, then $\mathcal{O}(\|\Psi_0\|^3)$ by the third transformation, etc., such that a geometric series in $\|\Psi_0\|$ can be used as convergent majorant for $\|\Psi_0\| = \mathcal{O}(1)$, but sufficiently small.

Since the bilinear functions B are of order $\mathcal{O}(1)$, the norm of the normal form transformations M is $\mathcal{O}(1)$-bounded by the norm of Ψ_0 and R. This yields the invertibility of the near identity changes of variables with the help of Neumann's series for $\|\Psi_0\| = \mathcal{O}(1)$, but sufficiently small.

Remark 3.6.1. [Preparation of the first system] Before we eliminate the non-resonant terms of order $\mathcal{O}(\|\Psi_0\|^2\|\mathcal{R}\|)$ with a transformation

$$
\begin{aligned}
R_{1,2} &= R_{1,1} + \sum_{j\in\{-1,\pm 2\}} M_{2,1,j}(\Psi_0, R_j),\\
R_{2,2} &= R_{2,1} + \sum_{j=\pm 1} M_{2,2,j}(\Psi_0, R_j),
\end{aligned}
$$

we again prepare the system as in (3.43). Since the non-resonance condition for the elimination of the non-resonant terms of order $\mathcal{O}(\|\Psi_0\|^2\|\mathcal{R}\|)$ is again $\mathrm{i}\widehat{\omega}_2(k) - \mathrm{i}\widehat{\omega}_1(k') \neq 0$, for $|k| \leq \delta_c$ and $|k'| \leq 2\delta_c$, these terms can be eliminated as in the first step.

Step 2: [Induction step] The normal form transformations

$$
\begin{aligned}
R_{1,m+1} &= R_{1,m} + \sum_{j\in\{-1,\pm 2\}} M_{m+1,1,j}(\Psi_0, R_j),\\
R_{2,m+1} &= R_{2,m} + \sum_{j=\pm 1} M_{m+1,2,j}(\Psi_0, R_j),
\end{aligned}
$$

lead to the same non-resonance condition. Hence, the non-resonant terms of order $\mathcal{O}(\|\Psi_0\|^{m+1}\|\mathcal{R}\|)$ can be eliminated. Before each step we prepare the system as in the first step in (3.43), cf. Remark 3.6.1.

By this approach we have a sequence of problems

$$
\begin{aligned}
\partial_t R_{1,\nu} &= \mathrm{i}\omega_1 R_{1,\nu} + \widetilde{B_{1,1,\nu}}(\Psi_0, R_{1,\nu}) + \sum_{j\in\{-1,\pm 2\}} \widetilde{B_{1,j,\nu}}(\Psi_0, R_{j,\nu}) + H^*_{1,\nu},\\
\partial_t R_{2,\nu} &= \mathrm{i}\omega_2 R_{2,\nu} + \sum_{j=\pm 2} \widetilde{B_{2,j,\nu}}(\Psi_0, R_{j,\nu}) + \sum_{j=\pm 1} \widetilde{B_{2,j,\nu}}(\Psi_0, R_{j,\nu}) + H^*_{2,\nu},
\end{aligned}
$$

with

$$
\begin{aligned}
\widetilde{B_{1,1,\nu}}(\Psi_0, R_{1,\nu}) &= \mathcal{O}(\|\Psi_0\|\|\mathcal{R}\|),\\
\sum_{j\in\{-1,\pm 2\}} \widetilde{B_{1,j,\nu}}(\Psi_0, R_{j,\nu}) &= \mathcal{O}(\|\Psi_0\|^{\nu}\|\mathcal{R}\|),\\
\sum_{j=\pm 2} \widetilde{B_{2,j,\nu}}(\Psi_0, R_{j,\nu}) &= \mathcal{O}(\|\Psi_0\|\|\mathcal{R}\|),\\
\sum_{j=\pm 1} \widetilde{B_{2,j,\nu}}(\Psi_0, R_{j,\nu}) &= \mathcal{O}(\|\Psi_0\|^{\nu}\|\mathcal{R}\|),
\end{aligned}
$$

$H^*_{1,\nu} = \mathcal{O}(\delta)$, $H^*_{2,\nu} = \mathcal{O}(\delta)$, and a sequence of normal form transformations

$$R_{1,\nu+1} = R_{1,\nu} + \sum_{j\in\{-1,\pm2\}} \widetilde{M_{1,j,\nu}}(\Psi_0, R_{j,\nu}),$$
$$R_{2,\nu+1} = R_{2,\nu} + \sum_{j=\pm1} \widetilde{M_{2,j,\nu}}(\Psi_0, R_{j,\nu}),$$

with

$$\sum_{j\in\{-1,\pm2\}} \widetilde{M_{1,j,\nu}}(\Psi_0, R_{j,\nu}) = \mathcal{O}(\|\Psi_0\|^\nu \|\mathcal{R}\|),$$
$$\sum_{j=\pm1} \widetilde{M_{2,j,\nu}}(\Psi_0, R_{j,\nu}) = \mathcal{O}(\|\Psi_0\|^\nu \|\mathcal{R}\|).$$

Step 3: [Limit system] After these infinitely many normal form transformations the limit system for $\nu \to \infty$ has the following structure

$$\partial_t R_{1,\infty} = \mathrm{i}\omega_1 R_{1,\infty} + B_{1,1,\infty}(\Psi_0, R_{1,\infty}) + H^*_{1,\infty}, \tag{3.46}$$
$$\partial_t R_{2,\infty} = \mathrm{i}\omega_2 R_{2,\infty} + \sum_{j=\pm2} B_{2,j,\infty}(\Psi_0, R_{j,\infty}) + H^*_{2,\infty}, \tag{3.47}$$

and similarly for $R_{-1,\infty}$ and $R_{-2,\infty}$. For $j \in \{\pm1, \pm2\}$ the nonlinear terms obey the estimates

$$\|H^*_{j,\infty}\|_{G^m_\sigma} \leq C\delta(\|(R_{1,\infty}, R_{2,\infty})\|_{G^m_\sigma} + \delta^{1/2}(\|(R_{1,\infty}, R_{2,\infty})\|^2_{G^m_\sigma} + \|R^c\|^2_{G^m_\sigma}) + 1), \tag{3.48}$$

since we eliminated the linear terms in $R_{\pm1}$ which have no δ in front in the $R_{\pm2}$-equation, and vice versa interchanging the role of $R_{\pm2}$ and $R_{\pm1}$. Since $R_{-j,\infty}$ is complex conjugated to $R_{j,\infty}$ for $j \in \{\pm1, \pm2\}$, it suffices to include $R_{1,\infty}$ and $R_{2,\infty}$ in the estimates. As already said, convergence holds for all appearing terms since they can be bounded by a geometric series in $\|\Psi_0\|$ as convergent majorant, for $\|\Psi_0\| = \mathcal{O}(1)$, but sufficiently small.

3.7 Some further preparations

Before performing the energy estimates for obtaining a $\mathcal{O}(1)$-bound for $R_{\pm1,\infty}$, $R_{\pm2,\infty}$ and R^c on a $\mathcal{O}(1/\delta)$-time scale, we need some additional preparations.

Remark 3.7.1. For notational simplicity it turns out to be advantageous if all components of $R_{\cdot,\infty}$ and R^c have the same regularity. This is automatically fulfilled for $R_{\pm1,\infty}$ and $R_{\pm2,\infty}$ since they all have a compact support in Fourier space. However, for R^c this is not the case and so we introduce the multiplication operator $\mathcal{M}$ defined by its symbol $\widehat{\mathcal{M}}(k) = (1 + k^2)^{1/2}$ in Fourier space. Since in Equation (3.37) we have $\widetilde{r} \in G^m_\sigma$, $\vartheta \in G^m_\sigma$ and $\psi \in G^m_\sigma$, but $r \in G^{m+1}_\sigma$, we introduce $\mathbf{r} = \mathcal{M}r \in G^m_\sigma$ and from (3.37) we find

$$\partial_t \mathbf{V} = \mathbf{L}\mathbf{V} + \mathbf{N}(\mathbf{V}),$$

where

$$\mathbf{V}=\begin{pmatrix}\mathbf{r}\\ \widetilde{r}\\ \vartheta\\ \psi\end{pmatrix},\qquad \mathbf{LV}=\begin{pmatrix}\mathcal{M}\widetilde{r}\\ \partial_x^2(\mathcal{M}^{-1}\mathbf{r})+2\mu\vartheta+2(1-\mu^2)(\mathcal{M}^{-1}\mathbf{r})\\ \partial_x\psi-2\mu\widetilde{r}\\ \partial_x\vartheta\end{pmatrix},$$

and

$$\mathbf{N}(\mathbf{V})=\begin{pmatrix}0\\ \vartheta^2-\widetilde{r}^2-\psi^2+(\partial_x(\mathcal{M}^{-1}\mathbf{r}))^2+(1-\mu^2)(\mathrm{e}^{2(\mathcal{M}^{-1}\mathbf{r})}-1-2(\mathcal{M}^{-1}\mathbf{r}))\\ 2(\partial_x(\mathcal{M}^{-1}\mathbf{r}))\psi-2\widetilde{r}\vartheta\\ 0\end{pmatrix}.$$

We introduce the new error function $\widetilde{\mathcal{R}}$ made by the WME approximation Ψ through $\mathbf{V}=\Psi+\delta^{3/2}\widetilde{\mathcal{R}}$ similar to Step 1 in Section 3.5 and set $\mathbf{R}^c=E^c\widetilde{\mathcal{R}}$. Since $R_{\pm1,\infty}$ and $R_{\pm2,\infty}$ have a compact support in Fourier space, it suffices to handle the energy estimates below with respect to $R_{\pm1,\infty}$ and $R_{\pm2,\infty}$ and $\mathbf{R}^c$. For the transformed error part $\mathbf{R}^c$ we obtain a system of the form

$$\partial_t\mathbf{R}^c = \mathbf{LR}^c+H_c^*,$$

with

$$\begin{aligned}\|H_c^*\|_{G_\sigma^m} \leq\ & C_1\|\mathbf{R}^c\|_{G_\sigma^m}+C_2\delta(\|(R_{1,\infty},R_{2,\infty})\|_{G_\sigma^m}\\ &+C_3\delta^{3/2}(\|(R_{1,\infty},R_{2,\infty})\|_{G_\sigma^m}+\|\mathbf{R}^c\|_{G_\sigma^m})^2+C_4\delta.\end{aligned}\tag{3.49}$$

In this estimate the terms linear in $R_{\pm1,\infty}$ and $R_{\pm2,\infty}$ are at least of order $\mathcal{O}(\delta)$ since these terms come from the procedure described in Section 3.5 to achieve system (3.43), similar to the end of Section 3.6.

Remark 3.7.2. [Structure of the nonlinear terms in $R_{2,\infty}$-equation] In order to apply the ideas from Section 3.4 in the energy estimates of $R_{j,\infty}$, we need an additional structure in the limit system (3.46)-(3.47), namely that the terms $\sum_{j=\pm2}B_{2,j,\infty}(\Psi_0,R_{j,\infty})$ can be written in Fourier space as

$$\sum_{j=\pm2}\int\widehat{b}^*_{2,j,\infty}(k,k-k',k')\widehat{\Psi}_0(k-k',\delta t)\mathrm{i}k'\widehat{R}_{j,\infty}(k',t)\mathrm{d}k'+\mathcal{O}(\delta).$$

This follows directly from the representation (3.30)-(3.31) because the terms where a derivative falls on Ψ_0 give an additional $\mathcal{O}(\delta)$ and can be included in $H^*_{2,\infty}$.

Remark 3.7.3. [Structure of the nonlinear terms in $R_{\pm1,\infty}$-equation] Through the diagonalization of the linear system $R_{1,\infty}$ and $R_{-1,\infty}$ have the same regularity. Due to this fact and in comparison of the original system (3.37) and the diagonalized system (3.41), the term $B_{1,1,\infty}$ has the form

$$B_{1,1,\infty}(\Psi_0,R_{1,\infty})=\frac{1}{\mathrm{i}\omega_1}B^*_{1,1,\infty}(\Psi_0,R_{1,\infty}).$$

3.8 From analytic to Sobolev functions

We want to perform the energy estimates in usual Sobolev spaces, defined through

$$H^m = \left\{ u \in L^2(\mathbb{R}) : (1 + |\cdot|^2)^{m/2} \widehat{u}(\cdot) \in L^2(\mathbb{R}) \right\},$$

with the inner product

$$(u, v)_{H^m} = (\widehat{u}, \widehat{v})_{L^2_m} = \int (1 + |k|^2)^m (\widehat{u}(k), \widehat{v}(k)) \mathrm{d}k,$$

whereby $(\cdot, \cdot)$ denotes the Euclidean inner product, similar to [Sch20]. The induced norm is equivalent to the usual H^m-norm for any $m \in \mathbb{N}$. In addition, we introduce

$$W^m = \left\{ u : u = \mathcal{F}^{-1}(\widehat{u}), \widehat{u} \in L^1(\mathbb{R}), \|u\|_{W^m} = \int (1 + |k|^m) |\widehat{u}(k)| \mathrm{d}k < \infty \right\},$$

for $m > 0$.

We establish a connection between the spaces of analytic functions and Sobolev spaces through

$$\widehat{\mathcal{R}}_j(k, t) = \widehat{S}_\omega(k, t) \widehat{R}_j(k, t),$$

where $\widehat{S}_\omega$ is a multiplication operator which is defined by

$$\widehat{S}_\omega(k, t) = \mathrm{e}^{(\sigma_0/\delta - \eta t)|k|}.$$

As a direct consequence of these definitions, we obtain the following lemma.

Lemma 3.8.1. *For $t \in [0, \sigma_0/(\eta\delta)]$ the linear mappings $S_\omega(t) : G^m_{\sigma(t)} \to H^m$ and $S_\omega(t) : \mathcal{W}^m_{\sigma(t)} \to W^m$ with $\sigma(t) = \sigma_0/\delta - \eta t$ are bijective and bounded with bounded inverse.*

The new defined variables satisfy the system

$$\begin{aligned}
\partial_t \mathcal{R}_{1,\infty} &= -\eta |k|_{op} \mathcal{R}_{1,\infty} + \mathrm{i}\omega_1 \mathcal{R}_{1,\infty} + S_\omega B_{1,1,\infty}(\Psi_0, S_\omega^{-1} \mathcal{R}_{1,\infty}) + \tilde{H}^*_{1,\infty}, \\
\partial_t \mathcal{R}_{2,\infty} &= -\eta |k|_{op} \mathcal{R}_{2,\infty} + \mathrm{i}\omega_2 \mathcal{R}_{2,\infty} + S_\omega \sum_{j=\pm 2} B_{2,j,\infty}(\Psi_0, S_\omega^{-1} \mathcal{R}_{j,\infty}) + \tilde{H}^*_{2,\infty}, \\
\partial_t \mathcal{R}^c &= -\eta |k|_{op} \mathcal{R}^c + \mathbf{L} \mathcal{R}^c + \tilde{H}^*_c,
\end{aligned}$$

and similarly for $\mathcal{R}_{-1,\infty}$ and $\mathcal{R}_{-2,\infty}$, where as before $|k|_{op}$ is defined as the Fourier multiplier operator through $(\widehat{|k|_{op} u})(k) = |k| \widehat{u}(k)$. Using Lemma 3.8.1 and (3.48) and (3.49) we obtain

$$\begin{aligned}
\|\tilde{H}^*_{j,\infty}\|_{H^m} &\leq C\delta(\|(\mathcal{R}_{1,\infty}, \mathcal{R}_{2,\infty})\|_{H^m} \\
&\qquad + \delta^{1/2}(\|(\mathcal{R}_{1,\infty}, \mathcal{R}_{2,\infty})\|^2_{H^m} + \|\mathcal{R}^c\|^2_{H^m}) + 1), \\
\|\tilde{H}^*_c\|_{H^m} &\leq C\|\mathcal{R}^c\|_{H^m} + C\delta(\|(\mathcal{R}_{1,\infty}, \mathcal{R}_{2,\infty})\|_{H^m} \\
&\qquad + C\delta^{3/2}(\|(\mathcal{R}_{1,\infty}, \mathcal{R}_{2,\infty})\|_{H^m} + \|\mathcal{R}^c\|_{H^m})^2 + C\delta.
\end{aligned}$$

3.9 Error estimates in Gevrey spaces

Now we have all ingredients to perform the final energy estimates. For the energy

$$E(t) = \|\omega_1^{1/2}\mathcal{R}_{1,\infty}\|_{H^m}^2 + \|\mathcal{R}_{2,\infty}\|_{H^m}^2 + \|\mathcal{R}^c\|_{H^m}^2,$$

we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E = \mathrm{Re}\sum_{j=1}^{11} s_j,$$

where

$$\begin{aligned}
s_1 &= -\eta\| |k|_{op}^{1/2}\omega_1^{1/2}\mathcal{R}_{1,\infty}\|_{H^m}^2,\\
s_2 &= (\omega_1^{1/2}\mathcal{R}_{1,\infty}, \mathrm{i}\omega_1^{3/2}\mathcal{R}_{1,\infty})_{H^m},\\
s_3 &= \Big(\omega_1\mathcal{R}_{1,\infty}, S_\omega B_{1,1,\infty}(\Psi_0, S_\omega^{-1}\mathcal{R}_{1,\infty})\Big)_{H^m},\\
s_4 &= (\omega_1\mathcal{R}_{1,\infty}, \tilde{H}_{1,\infty}^*)_{H^m},\\
s_5 &= -\eta\| |k|_{op}^{1/2}\mathcal{R}_{2,\infty}\|_{H^m}^2,\\
s_6 &= (\mathcal{R}_{2,\infty}, \mathrm{i}\omega_2\mathcal{R}_{2,\infty})_{H^m},\\
s_7 &= \left(\mathcal{R}_{2,\infty}, S_\omega \sum_{j=\pm2} B_{2,j,\infty}(\Psi_0, S_\omega^{-1}\mathcal{R}_{j,\infty})\right)_{H^m},\\
s_8 &= (\mathcal{R}_{2,\infty}, \tilde{H}_{2,\infty}^*)_{H^m},\\
s_9 &= -\eta\| |k|_{op}^{1/2}\mathcal{R}^c\|_{H^m}^2,\\
s_{10} &= (\mathcal{R}^c, \mathbf{L}\mathcal{R}^c)_{H^m},\\
s_{11} &= (\mathcal{R}^c, \tilde{H}_c^*)_{H^m}.
\end{aligned}$$

Since $\mathcal{R}_{-j,\infty}$ is the complex conjugate of $\mathcal{R}_{j,\infty}$ for $j \in \{1, 2\}$, it suffices to include $\mathcal{R}_{1,\infty}$ and $\mathcal{R}_{2,\infty}$ in the energy.

In the following we estimate the "bad" terms s_2, s_3, s_4, s_6, s_7, s_8, s_{10} and s_{11} by the "good" artificial damping terms s_1, s_5 and s_9.

$\mathbf{s_2}$: Since $\mathrm{i}\omega_1$ is a skew-symmetric operator in the parameter regimes under consideration, we have

$$s_2 = 0.$$

$\mathbf{s_6}$: In the Benjamin-Feir stable situation, cf. left panel of Figure 3.2, $\mathrm{i}\omega_2$ is a skew-symmetric operator which yields

$$s_6 = 0.$$

In the Benjamin-Feir unstable situation, cf. right panel of Figure 3.2, $\mathrm{i}\omega_2$ grows at most as $C|k|$ such that

$$|s_6| \leq C_6\| |k|_{op}^{1/2}\mathcal{R}_{2,\infty}\|_{H^m}^2.$$

Next, we go on with the higher order terms.

$\mathbf{s_4}$, $\mathbf{s_8}$, $\mathbf{s_{11}}$: We find

$$\begin{aligned}|s_4| &\leq C\|\omega_1\mathcal{R}_{1,\infty}\|_{H^m}\|\tilde{H}^*_{1,\infty}\|_{H^m}\\ &\leq C_4\delta\left(E+\delta^{1/2}E^{3/2}+1\right),\end{aligned}$$

where we used $a \leq 1+a^2$. Similarly, we obtain

$$\begin{aligned}|s_8| &\leq C\|\mathcal{R}_{2,\infty}\|_{H^m}\|\tilde{H}^*_{2,\infty}\|_{H^m}\\ &\leq C_8\delta\left(E+\delta^{1/2}E^{3/2}+1\right).\end{aligned}$$

Finally, we estimate

$$\begin{aligned}|s_{11}| &\leq C\|\mathcal{R}^c\|_{H^m}\|\tilde{H}^*_c\|_{H^m}\\ &\leq C_{11}(\|\mathcal{R}^c\|^2_{H^m}+\delta(E+\delta^{1/2}E^{3/2}+1).\end{aligned}$$

It remains to estimate s_3, s_7, and s_{10}.

$\mathbf{s_{10}}$: We start with the energy estimates for the linear term $\mathbf{L}\mathcal{R}^c$. The parameter regions we are working in include the possibility of Benjamin-Feir unstable wave trains. Hence, the eigenvalues of $\mathbf{L}$ can be bounded from above by $C|k|$, cf. the right panel of Figure 3.2, and so we obtain the rather rough estimate

$$|s_{10}| \leq C_{10}\||k|^{1/2}_{op}\mathcal{R}^c\|^2_{H^m}.$$

$\mathbf{s_7}$: With Remark 3.7.2 the term s_7 can be rewritten as

$$\begin{aligned}s_7 &= \Bigg(\widehat{\mathcal{R}}_{2,\infty}(\cdot),\widehat{S}_\omega(\cdot,t)\sum_{j=\pm2}\int\widehat{b}^*_{2,j,\infty}(\cdot,\cdot-k',k')\\ &\qquad\times\widehat{S}^{-1}_\omega(\cdot-k',t)\widehat{\Psi}_*(\cdot-k',\delta t)\widehat{S}^{-1}_\omega(k',t)\mathrm{i}k'\widehat{\mathcal{R}}_{j,\infty}(k')\mathrm{d}k'+\mathcal{O}(\delta)\Bigg)_{L^2_m},\end{aligned}$$

where $\widehat{S}_\omega\widehat{\Psi}_0 = \widehat{\Psi}_*$. Using $\widehat{S}_\omega(k,t)\widehat{S}^{-1}_\omega(k-k',t)\widehat{S}^{-1}_\omega(k',t) \leq 1$ and integration by parts with $|k|^{1/2}_{op}$ yields

$$|s_7| \leq C(\|\Psi_*\|_{W^m}\||k|^{1/2}_{op}\mathcal{R}_{\pm2,\infty}\|^2_{H^m}+\||k|^{1/2}_{op}\Psi_*\|_{W^m}\|\mathcal{R}_{\pm2,\infty}\|_{H^m}\||k|^{1/2}_{op}\mathcal{R}_{\pm2,\infty}\|_{H^m}+\mathcal{O}(\delta)).$$

Next, we use that $\|\Psi_*\|_{W^m}=\mathcal{O}(1)$, $\||k|^{1/2}_{op}\Psi_*\|_{W^m}=\mathcal{O}(\delta^{1/2})$ and $\delta^{1/2}ab \leq a^2+\delta b^2$ such that

$$\begin{aligned}|s_7| &\leq C(\||k|^{1/2}_{op}\mathcal{R}_{\pm2,\infty}\|^2_{H^m}+\delta^{1/2}\|\mathcal{R}_{\pm2,\infty}\|_{H^m}\||k|^{1/2}_{op}\mathcal{R}_{\pm2,\infty}\|_{H^m}+\mathcal{O}(\delta))\\ &\leq C_7(\||k|^{1/2}_{op}\mathcal{R}_{\pm2,\infty}\|^2_{H^m}+\delta\|\mathcal{R}_{\pm2,\infty}\|^2_{H^m}).\end{aligned}$$

$\mathbf{s_3}$: For the Fourier transform of $\mathrm{Re}(s_3)$ in case $m=0$ we compute

$$\begin{aligned}|\mathrm{Re}(s_3)| &= \Bigg|\frac{1}{2}\int\int\widehat{\omega}_1(k)\widehat{\mathcal{R}}_{1,\infty}(k)\overline{\widehat{S}_\omega(k,t)\frac{1}{\mathrm{i}\widehat{\omega}_1(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')}\\ &\qquad\overline{\times\widehat{S}^{-1}_\omega(k-k',t)\widehat{\Psi}_*(k-k',\delta t)\widehat{S}^{-1}_\omega(k',t)\widehat{\mathcal{R}}_{1,\infty}}\\ &\qquad+\overline{\widehat{\omega}_1(k)\widehat{\mathcal{R}}_{1,\infty}(k)}\widehat{S}_\omega(k,t)\frac{1}{\mathrm{i}\widehat{\omega}_1(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\\ &\qquad\times\widehat{S}^{-1}_\omega(k-k',t)\widehat{\Psi}_*(k-k',\delta t)\widehat{S}^{-1}_\omega(k',t)\widehat{\mathcal{R}}_{1,\infty}\mathrm{d}k'\mathrm{d}k\Bigg|,\end{aligned}$$

where we used $\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z})$ for $z \in \mathbb{C}$. Using $\widehat{S}_\omega(k,t)\widehat{S}_\omega^{-1}(k-k',t)\widehat{S}_\omega^{-1}(k',t) \leq 1$ again yields the estimate

$$\begin{aligned}|\mathrm{Re}(s_3)| \;\leq\; & \frac{1}{2}\int\int\Bigg|\Bigg(\widehat{\omega}_1(k)\widehat{\mathcal{R}}_{1,\infty}(k)\overline{\frac{1}{\mathrm{i}\widehat{\omega}_1(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')} \\ & +\overline{\widehat{\omega}_1(k)\widehat{\mathcal{R}}_{1,\infty}(k)}\frac{1}{\mathrm{i}\widehat{\omega}_1(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')\Bigg)\Bigg|\,\mathrm{d}k'\mathrm{d}k.\end{aligned}$$

Since $\overline{\mathrm{i}\omega_1} = -\mathrm{i}\omega_1$, we obtain

$$\begin{aligned}|\mathrm{Re}(s_3)| \;\leq\; & \frac{1}{2}\int\int\Big|\widehat{\mathcal{R}}_{1,\infty}(k)\overline{\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')} \\ & -\overline{\widehat{\mathcal{R}}_{1,\infty}(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')\Big|\,\mathrm{d}k'\mathrm{d}k.\end{aligned}$$

In the first line we switch the role of k and k' to get

$$\begin{aligned}|\mathrm{Re}(s_3)| \;\leq\; & \frac{1}{2}\int\int\Big|\widehat{\mathcal{R}}_{1,\infty}(k')\overline{\widehat{b}^*_{1,1,\infty}(k',k'-k,k)\widehat{\Psi}_*(k'-k,\delta t)\widehat{\mathcal{R}}_{1,\infty}(k)} \\ & -\overline{\widehat{\mathcal{R}}_{1,\infty}(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')\Big|\,\mathrm{d}k'\mathrm{d}k.\end{aligned}$$

Since $\overline{\widehat{\Psi}_*(k'-k,\delta t)} = \widehat{\Psi}_*(k-k',\delta t)$, we have

$$\begin{aligned}|\mathrm{Re}(s_3)| \;\leq\; & \frac{1}{2}\int\int\Big|\widehat{\mathcal{R}}_{1,\infty}(k')\overline{\widehat{b}^*_{1,1,\infty}(k',k'-k,k)}\widehat{\Psi}_*(k-k',\delta t)\overline{\widehat{\mathcal{R}}_{1,\infty}(k)} \\ & -\overline{\widehat{\mathcal{R}}_{1,\infty}(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')\Big|\,\mathrm{d}k'\mathrm{d}k.\end{aligned}$$

Due to the symmetry of (3.44)-(3.45) and of the corresponding complex conjugate equations, we obtain

$$\overline{\widehat{b}^*_{1,1,\infty}(k',k'-k,k)} = \widehat{b}^*_{1,1,\infty}(k,k-k',k'),$$

and so we find

$$\begin{aligned}|\mathrm{Re}(s_3)| \;\leq\; & \frac{1}{2}\int\int\Big|\widehat{\mathcal{R}}_{1,\infty}(k')\widehat{b}^*_{1,1,\infty}(k',k-k',k)\widehat{\Psi}_*(k-k',\delta t)\overline{\widehat{\mathcal{R}}_{1,\infty}(k)} \\ & -\overline{\widehat{\mathcal{R}}_{1,\infty}(k)}\widehat{b}^*_{1,1,\infty}(k,k-k',k')\widehat{\Psi}_*(k-k',\delta t)\widehat{\mathcal{R}}_{1,\infty}(k')\Big|\,\mathrm{d}k'\mathrm{d}k \\ = \; & \frac{1}{2}\int\int\Big|\widehat{\mathcal{R}}_{1,\infty}(k')(\widehat{b}^*_{1,1,\infty}(k',k-k',k)-\widehat{b}^*_{1,1,\infty}(k,k-k',k')) \\ & \times\widehat{\Psi}_*(k-k',\delta t)\overline{\widehat{\mathcal{R}}_{1,\infty}(k)}\Big|\,\mathrm{d}k'\mathrm{d}k.\end{aligned}$$

Since

$$|\widehat{b}^*_{1,1,\infty}(k',k-k',k)-\widehat{b}^*_{1,1,\infty}(k,k-k',k')| = \mathcal{O}(|k-k'|),$$

we have

$$|\mathrm{Re}(s_3)| \leq C_3\delta E,$$

due to the concentration of $\widehat{\Psi}_*(k-k', \delta t)$ at $k-k'=0$, similar to Lemma 4.1 from [DKS16]. For the estimates in H^m for $m \geq 1$ we can use the fact that whenever a derivative falls on Ψ_* we gain an additional power of δ. The term where all m derivatives fall on $\mathcal{R}_{\pm 1,\infty}$ in the second component of the scalar product can be estimated line by line as for $m=0$ above.

The final estimates: The previous estimates yield

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E &\leq s_1 + s_5 + s_9 + |\mathrm{Re}(s_3)| + |s_4| + |s_6| + |s_7| + |s_8| + |s_{10}| + |s_{11}| \\
&\leq -\eta \| |k|_{op}^{1/2} \omega_1^{1/2} \mathcal{R}_{1,\infty}\|_{H^m}^2 - \eta \| |k|_{op}^{1/2} \mathcal{R}_{2,\infty}\|_{H^m}^2 - \eta \| |k|_{op}^{1/2} \mathcal{R}^c\|_{H^m}^2 \\
&\quad + C_6 \| |k|_{op}^{1/2} \mathcal{R}_{2,\infty}\|_{H^m}^2 \\
&\quad + (C_4 + C_8)\delta \left(E + \delta^{1/2} E^{3/2} + 1\right) \\
&\quad + C_{11}(\|\mathcal{R}^c\|_{H^m}^2 + \delta(E + \delta^{1/2}E^{3/2} + 1)) \\
&\quad + C_{10} \| |k|_{op}^{1/2} \mathcal{R}^c\|_{H^m}^2 + C_7(\| |k|_{op}^{1/2} \mathcal{R}_{\pm 2,\infty}\|_{H^m}^2 + \delta \|\mathcal{R}_{\pm 2,\infty}\|_{H^m}^2) \\
&\quad + C_3 \delta E \\
&\leq (-\eta) \| |k|_{op}^{1/2} \omega_1^{1/2} \mathcal{R}_{1,\infty}\|_{H^m}^2 \\
&\quad + (-\eta + C_6 + C_7) \| |k|_{op}^{1/2} \mathcal{R}_{2,\infty}\|_{H^m}^2 \\
&\quad + (-\eta + C_{10} + C_{11}) \| |k|_{op}^{1/2} \mathcal{R}^c\|_{H^m}^2 \\
&\quad + (C_3 + C_4 + C_7 + C_8 + C_{11})\delta \left(E + \delta^{1/2}E^{3/2} + 1\right) \\
&\leq (C_3 + C_4 + C_7 + C_8 + C_{11})\delta \left(E + \delta^{1/2}E^{3/2} + 1\right),
\end{aligned}$$

if $\eta > 0$ is chosen so large that

$$-\eta + C_6 + C_7 < 0, \qquad -\eta + C_{10} + C_{11} < 0.$$

Then, we choose $\delta > 0$ so small that

$$\delta^{1/2} E^{1/2} \leq 1 \tag{3.50}$$

is fulfilled. Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \leq (C+1)\delta E + C\delta.$$

With the help of Gronwall's inequality we obtain

$$E(t) \leq (E(0) + C\delta t)\mathrm{e}^{(C+1)\delta t} \leq (E(0) + CT_0)\mathrm{e}^{(C+1)T_0} = M = \mathcal{O}(1).$$

The constant M is independent of η, T_1 and $0 < \delta \ll 1$. We choose $\delta_0 > 0$ sufficiently small such that $\delta_0^{1/2} M^{1/2} < 1$ is satisfied. This guarantees the validity of (3.50). As a consequence, this proves Theorem 3.1.10.

3.10 Discussion

Remark 3.10.1. Since the Benjamin-Feir instability occurs for $\gamma = 1$ respectively $|\mu| < 1$, it is not possible to replace the Gevrey spaces by classical Sobolev spaces, like the plot of the spectral curves in the right panel of Figure 3.2 shows. However, it is a natural question whether it might be possible to work in Sobolev spaces for $\gamma = -1$ respectively $|\mu| > 1$. In this case the wave train is spectrally stable and such a WME approximation result can be found in [BKZ21] for the NLS equation. It will be the topic of future research to prove a similar result for the ccKG equation (3.1).

Remark 3.10.2. Although in the parameter region $\mathcal{P}_{\text{rest}}$, cf. the left panel of Figure 3.3, WME can be derived, it cannot be expected that the associated WME approximation makes correct predictions on the long $\mathcal{O}(1/\delta)$-time scale. In the left panel of Figure 3.3 we have a smooth curve of eigenvalues with positive real part of order $\mathcal{O}(1)$ at the wave number $k = 0$. This leads to growth rates of order $\mathcal{O}(\exp(1/\delta))$ on the long $\mathcal{O}(1/\delta)$-time scale. Therefore, to come to the long $\mathcal{O}(1/\delta)$-time scale, by nonlinear interaction of the other modes and initially only terms of order $\mathcal{O}(\exp(-1/\delta))$ can be allowed. However, this is not the case and so the WME approximation fails to make correct predictions in $\mathcal{P}_{\text{rest}}$, cf. [HS20] for an example of a non-approximation result.

Remark 3.10.3. Finally, we remark that the reconstruction of the solution in physical variables (3.3) requires the spatial integration of the local wave number $\psi = \partial_x \varphi$ to reconstruct the phase φ. As a consequence, in the original u-variable only a local in space approximation result can be obtained. The size of the spatial domain where the WME approximation makes correct predictions is proportional to the inverse order of the higher order WME approximation constructed in Section 3.4.5. For details about that see for instance [DS09, BKS20].

3.11 Appendix - Stability regions for $q \neq 0$

In this section we consider the case $q \neq 0$ and explain where the parameter regions plotted in Figure 3.4 come from.

Remark 3.11.1. To analyze the case $q \neq 0$, we look at the system (3.19)-(3.21), derive WME and redo the calculations from Remark 3.2.6. We have

$$\gamma e^{2r_{q,\mu}} = 1 + q^2 - \mu^2,$$

and we make the long wave ansatz

$$(r, \psi, \vartheta)(x, t) = (\check{r}, \check{\psi}, \check{\vartheta})(\delta x, \delta t) = (\check{r}, \check{\psi}, \check{\vartheta})(X, T),$$

with $X = \delta x$, $T = \delta t$ and a small perturbation parameter $0 < \delta \ll 1$. Ignoring higher order terms yields the system

$$\begin{aligned}
0 &= \check{\vartheta}^2 + 2\mu\check{\vartheta} - \check{\psi}^2 - 2q\check{\psi} + \gamma \mathrm{e}^{2r_{q,\mu}}(\mathrm{e}^{2\check{r}} - 1), \\
\partial_T \check{\vartheta} &= 2(\partial_X \check{r})(\check{\psi} + q) + \partial_X \check{\psi} - 2(\partial_T \check{r})(\check{\vartheta} + \mu), \\
\partial_T \check{\psi} &= \partial_X \check{\vartheta}.
\end{aligned}$$

For $\check{\vartheta}$ and $\check{\psi}$ small we can solve the first equation w.r.t. $\check{r}$ and get

$$\check{r}^*(\check{\vartheta}, \check{\psi}) = -\frac{\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\check{\vartheta} + \frac{q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\check{\psi} + h.o.t.,$$

with the partial temporal and spatial derivatives

$$\begin{aligned}
\partial_T \check{r}^* &= -\frac{\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_T\check{\vartheta} + \frac{q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_T\check{\psi} + h.o.t., \\
\partial_X \check{r}^* &= -\frac{\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\vartheta} + \frac{q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\psi} + h.o.t..
\end{aligned}$$

Inserting this in the equations for $\check{\vartheta}$ and $\check{\psi}$ yields

$$\begin{aligned}
\partial_T \check{\vartheta} =& 2\left(-\frac{\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\vartheta} + \frac{q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\psi}\right)(\check{\psi} + q) + \partial_X\check{\psi} \\
&- 2\left(-\frac{\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_T\check{\vartheta} + \frac{q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_T\check{\psi}\right)\left(\check{\vartheta} + \mu\right) \\
=& -\frac{2\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\left(\partial_X\check{\vartheta}\right)(\check{\psi} + q) + \frac{2q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\psi}(\check{\psi} + q) + \partial_X\check{\psi} \\
&+ \frac{2\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\left(\partial_T\check{\vartheta}\right)(\check{\vartheta} + \mu) - \frac{2q}{\gamma \mathrm{e}^{2r_{q,\mu}}}\left(\partial_T\check{\psi}\right)(\check{\vartheta} + \mu),
\end{aligned}$$

where we used $\partial_T\check{\psi} = \partial_X\check{\vartheta}$. The linearization of this equation is given by

$$\partial_T\check{\vartheta} = \partial_X\check{\psi} + \frac{2\mu^2}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_T\check{\vartheta} - \frac{4q\mu}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\vartheta} + \frac{2q^2}{\gamma \mathrm{e}^{2r_{q,\mu}}}\partial_X\check{\psi}.$$

Hence, we find

$$\begin{aligned}
\left(1 - \frac{2\mu^2}{\gamma \mathrm{e}^{2r_{q,\mu}}}\right)\partial_T\check{\vartheta} = \left(\frac{1 + q^2 - \mu^2 - 2\mu^2}{1 + q^2 - \mu^2}\right)\partial_T\check{\vartheta} &= \left(\frac{1 + q^2 - 3\mu^2}{1 + q^2 - \mu^2}\right)\partial_T\check{\vartheta} \\
&= -\frac{4q\mu}{1 + q^2 - \mu^2}\partial_X\check{\vartheta} + \frac{1 + q^2 - \mu^2 + 2q^2}{1 + q^2 - \mu^2}\partial_X\check{\psi} \\
&= -\frac{4q\mu}{1 + q^2 - \mu^2}\partial_X\check{\vartheta} + \frac{1 + 3q^2 - \mu^2}{1 + q^2 - \mu^2}\partial_X\check{\psi},
\end{aligned}$$

and so finally in the long wave limit

$$\begin{aligned}
\partial_T\check{\vartheta} &= -\frac{4q\mu}{1 + q^2 - 3\mu^2}\partial_X\check{\vartheta} + \frac{1 + 3q^2 - \mu^2}{1 + q^2 - 3\mu^2}\partial_X\check{\psi}, \\
\partial_T\check{\psi} &= \partial_X\check{\vartheta}.
\end{aligned}$$

Thus, the sign of $\frac{1+3q^2-\mu^2}{1+q^2-3\mu^2}$ determines the stability or instability of the wave train w.r.t. long wave perturbations. If $1+3q^2-\mu^2$ and $1+q^2-3\mu^2$ are both smaller than zero, the spectral curves look similar as the ones in the left panel of Figure 3.2. Spectral curves for other parameter values are plotted in Figure 3.5.

Chapter 4

A linear Schrödinger approximation for the KdV equation via inverse scattering transform beyond the natural NLS time scale

We are interested in improving validity results for the nonlinear Schrödinger approximation beyond the natural time scale for completely integrable systems. As a first step, we consider this approximation for the Korteweg-de Vries equation with initial conditions for which the scattering data contains no eigenvalues. By performing a linear Schrödinger approximation for the scattering data, the error made by this approximation has only to be estimated for a purely linear problem which gives estimates beyond the natural nonlinear Schrödinger time scale. The inverse scattering transform allows us to transfer these estimates to the original variables.

4.1 Introduction

The nonlinear Schrödinger (NLS) equation describes slow modulations in time and space of oscillating wave packets in dispersive wave systems. It was derived through a multiple scaling perturbation ansatz in [Zak68] first. Various approximation results have been established in the mean-time, cf. [Kal88, KSM92, Sch05, TW12], see [Dül21] for a recent overview. We are interested in improving these validity results for the NLS approximation beyond the natural time scale of the NLS approximation for completely integrable systems. As a first step in this direction, in this chapter, we consider this question for the KdV equation as an example of such a completely integrable system.

Using the Miura transformation, cf. [DJ89], and Gronwall's inequality, in [Sch11] a simple proof was given that the NLS approximation

$$\varepsilon\psi_u = \varepsilon A(\varepsilon(x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c. + \mathcal{O}(\varepsilon^2)$$

makes correct predictions about the dynamics of the KdV equation

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \tag{4.1}$$

if A is chosen to be a solution of the NLS equation

$$\partial_2 A = -3\mathrm{i}k\partial_1^2 A - 6\mathrm{i}kA|A|^2. \tag{4.2}$$

In detail, it was shown

Theorem 4.1.1. *Fix $s \geq 1$ and let $A \in C([0,T_0], H^{s+4})$ be a solution of the NLS equation* (4.2)*. Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions of the KdV equation* (4.1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\cdot,t) - \varepsilon\psi_u(\cdot,t)\|_{H^s} \leq C\varepsilon^{3/2}.$$

As already said, we are interested in improving such validity results for the NLS approximation beyond the natural $\mathcal{O}(1/\varepsilon^2)$-NLS time scale for completely integrable systems, here the KdV equation. We do so by restricting ourselves to initial conditions of the KdV equation for which the scattering data contains no eigenvalues and by performing an NLS approximation for the scattering variable b associated to the essential spectrum. Since the equation for b is linear, the NLS equation degenerates into a linear Schrödinger equation. The error made by this approximation has to be estimated for a linear problem which gives estimates beyond the natural NLS time scale, cf. Section 4.3. Hence, our approach allows us to extend the approximation time from $\mathcal{O}(1/\varepsilon^2)$ to $\mathcal{O}(1/\varepsilon^{3-\delta})$ with $\delta > 0$ arbitrarily small, but fixed. The inverse scattering transform finally allows us to transfer these results to the original variables, cf. Section 4.4 and Section 4.5. The Chapter is closed with some discussions in Section 4.6.

Notation. Throughout this chapter many possible different constants are denoted with the same symbol C if they can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$. The Sobolev space H^s of s times weakly differentiable functions is equipped with the norm

$$\|u\|_{H^s} = \left(\sum_{j=0}^{s} \int |\partial_x^j u(x)|^2 \mathrm{d}x\right)^{1/2}.$$

The weighted Lebesgue space L_s^2 is equipped with the norm

$$\|\widehat{u}\|_{L_s^2} = \left(\int |\widehat{u}(k)|^2 (1+k^2)^s \mathrm{d}k\right)^{1/2}.$$

4.2 IST for the KdV equation

It is well known that the KdV equation

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u$$

can be solved with the help of the inverse scattering transform (IST). Since this theory plays a fundamental role in the following, we recall its basics for completeness. For more details we refer to [DJ89].

For a solution $u = u(x,t)$ of the KdV equation we consider the associated quantum mechanical scattering problem, namely

$$L\psi = -\partial_x^2\psi - u\psi = \lambda\psi. \tag{4.3}$$

i) The scattering problem is to find the eigenvalues/spectral values $\lambda_k(t)$ and the associated eigenfunctions $\psi_k(\cdot,t)$ for a given $u = u(\cdot,t)$ where k is in some index set I.

ii) The inverse scattering problem is to reconstruct $u = u(\cdot,t)$ from the scattering data $\lambda_k(t)$ and $\psi_k(\cdot,t)$ for $k \in I$.

4.2.1 The scattering problem

The KdV equation is a completely integrable Hamiltonian system for which there exists a Lax pair formulation

$$\partial_t L = ML - LM,$$

with L defined in (4.3) and $M\psi = -4\partial_x^3\psi - 3(u\partial_x\psi + (\partial_x u)\psi)$. The Lax pair representation implies that the eigenvalues/spectral values $\lambda_k(t)$ of the operator L are independent of time. The eigenfunctions $\psi_k(\cdot,t)$ satisfy

$$\partial_t\psi_k(\cdot,t) = M\psi_k(\cdot,t).$$

For spatially localized u the operator L possesses essential spectrum $[0,\infty)$ and a finite number, say N, of negative eigenvalues $\lambda_n \in \mathbb{R}$, with $n = 1,\ldots,N$.

i) The eigenfunctions to the negative eigenvalues decay with some exponential rate for $|x| \to \infty$, in particular we have

$$\psi_n(x,t) \sim c_n(t)\mathrm{e}^{-\kappa_n x},$$

for $x \to \infty$, where $\kappa_n^2 = -\lambda_n$ and $\kappa_n > 0$. It turns out that the coefficient $c_n(t)$ satisfies the simple evolution equation

$$\partial_t c_n = 4\kappa_n^3 c_n,$$

which is solved by $c_n(t) = c_n(0)\mathrm{e}^{4\kappa_n^3 t}$.

ii) The eigenfunctions to the spectral values $\lambda_k = k^2$ for $k \in \mathbb{R}$ are of the form

$$\psi_k(x,t) \sim \mathrm{e}^{-\mathrm{i}kx} + \widehat{b}(k,t)\mathrm{e}^{\mathrm{i}kx}, \qquad \text{for } x \to \infty,$$

and

$$\psi_k(x,t) \sim \widehat{a}(k,t)\mathrm{e}^{-\mathrm{i}kx}, \qquad \text{for } x \to -\infty.$$

It turns out that because of

$$|\widehat{a}(k,t)|^2 + |\widehat{b}(k,t)|^2 = 1,$$

it is sufficient to control the coefficients $\widehat{b}(k,t)$ which satisfy the simple evolution equations

$$\partial_t \widehat{b}(k,t) = 8\mathrm{i}k^3 \widehat{b}(k,t). \tag{4.4}$$

4.2.2 The inverse scattering problem

The solution $u = u(x,t)$ can be reconstructed from the scattering data by solving the Gelfand-Levitan-Marchenko equation

$$K(x,y,t) + F(x+y,t) + \int_x^\infty K(x,z,t)F(y+z,t)\mathrm{d}z = 0 \tag{4.5}$$

for $K(x,y,t)$ with $y \geq x$ where

$$F(x,t) = \sum_{j=1}^{n} c_j^2(t)\mathrm{e}^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\widehat{b}(k,t)\mathrm{d}k.$$

The solution is then given by

$$u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x^+,t),$$

where x^+ indicates that the derivative is computed as right-hand limit in the second variable. The time t appears in these calculations only as a parameter. In the integral equation (4.5) also the variable x is a parameter.

4.3 The approximation for the scattering data

In this section we construct a Schrödinger approximation for the scattering variables $b(k,t)$, i.e., in the following we consider the case of no eigenvalues, i.e., we assume $N = 0$ and comment on this assumption later on in Section 4.6.

The evolution equation (4.4) for the scattering variables $\widehat{b}(k,t)$ is solved by

$$\widehat{b}(k,t) = \mathrm{e}^{8\mathrm{i}k^3 t}\widehat{b}(k,0).$$

If k is interpreted as Fourier wave number and $\widehat{b}$ as Fourier transform of a function b, then b satisfies the so-called Airy equation

$$\partial_t b(x,t) = -8\partial_x^3 b(x,t). \tag{4.6}$$

For this equation we make the ansatz

$$b(x,t) = \varepsilon A(\varepsilon(x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + c.c..$$

Plugging this ansatz into (4.6) and equating the coefficients of $\varepsilon^n \mathrm{e}^{\mathrm{i}(k_0 x-\omega_0 t)}$ to zero gives the linear dispersion relation $\omega_0 = -8k_0^3$ at $\mathcal{O}(\varepsilon)$, the group velocity $c = -24k_0^2$ at $\mathcal{O}(\varepsilon^2)$, and the linear Schrödinger equation

$$\partial_T A = -24\mathrm{i}k_0 \partial_X^2 A \tag{4.7}$$

at $\mathcal{O}(\varepsilon^3)$. For this equation we have the global existence of solutions in every H^s for each $s \geq 0$ with the bound

$$\begin{aligned} \|A(\cdot,T)\|_{H^s} &= \|\mathrm{e}^{24\mathrm{i}k_0 k^2 T}\widehat{A}(K,0)(1+K^2)^{s/2}\|_{L^2_s(\mathrm{d}K)} \\ &= \|\widehat{A}(K,0)(1+K^2)^{s/2}\|_{L^2_s(\mathrm{d}K)} = \|A(\cdot,0)\|_{H^s}. \end{aligned} \tag{4.8}$$

We have the following approximation result:

Theorem 4.3.1. *For each $s \geq 0$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A \in C([0,\infty), H^{s+3})$ be a solution of the linear Schrödinger equation* (4.7)*. Then, for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions b of the Airy equation* (4.6) *such that for all $t_0 \geq 0$ we have*

$$\sup_{t\in[0,t_0]} \|b(x,t) - (\varepsilon A(\varepsilon(x-ct),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)} + c.c.)\|_{H^s(\mathrm{d}x)} \leq C\varepsilon^{7/2} t_0 \|A(0)\|_{H^{s+3}}.$$

Proof. Let

$$R(x,t) = b(x,t) - (\varepsilon A(\varepsilon(x-ct),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)} + c.c.),$$

with $R|_{t=0} = 0$. This error function satisfies

$$\partial_t R = -8\partial_x^3 R - \varepsilon^4(8\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)}\partial_X^3 A + c.c.).$$

Applying the variation of constants formula yields

$$R(t) = -\int_0^t \mathrm{e}^{-8\partial_x^3(t-\tau)}\varepsilon^4(8\mathrm{e}^{\mathrm{i}(k_0x-\omega_0\tau)}\partial_X^3 A + c.c.)(\tau)\mathrm{d}\tau.$$

Taking care of the fact that we lose a factor $\varepsilon^{-1/2}$ due to the scaling properties of the L^2-norm under $x \mapsto \varepsilon x$, we immediately find the estimate

$$\|R(\cdot,t)\|_{H^s} \leq C\varepsilon^4 t\varepsilon^{-1/2} \sup_{\tau\in[0,t]} \|A(\cdot,\tau)\|_{H^{s+3}} \leq C\varepsilon^{7/2} t\|A(\cdot,0)\|_{H^{s+3}},$$

due to (4.8). □

Corollary 4.3.2. *For each $s \geq 0$ and $\delta \in (0,1]$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A \in C([0,\infty), H^{s+3})$ be a solution of the linear Schrödinger equation* (4.7)*. Then, for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions b of the Airy equation* (4.6) *such that*

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|b(x,t) - (\varepsilon A(\varepsilon(x-ct),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)} + c.c.)\|_{H^s(\mathrm{d}x)} \leq C\varepsilon^{1/2+\delta}.$$

Remark 4.3.3. The error of order $\mathcal{O}(\varepsilon^{1/2+\delta})$ is still smaller than the solution and the approximation which both are of order $\mathcal{O}(\varepsilon^{1/2})$ in H^s. Thus, we improved the approximation time from $\mathcal{O}(1/\varepsilon^2)$ to $\mathcal{O}(1/\varepsilon^{3-\delta})$ with $\delta > 0$ arbitrarily small, but fixed.

Remark 4.3.4. The Schrödinger equation shows a decay rate like $T^{-1/2}$ for $T \to \infty$, whereas the Airy equation shows a decay rate like $t^{-1/3}$ for $t \to \infty$. Due to the strong concentration of the Fourier modes of the Schrödinger approximation at $k = k_0$, for $A \in H^s$ the part around $k = 0$, showing the slower decay rate $t^{-1/3}$, is ε^s initially. This part and the Schrödinger part at $k = k_0$ are of the same order if $\varepsilon^s t^{-1/3} = T^{-1/2} = (\varepsilon^2 t)^{-1/2}$, i.e., for $t = 1/\varepsilon^{6(s+1)} \gg 1/\varepsilon^3$.

Higher order approximations can be computed, too.

Remark 4.3.5. The ansatz for the computation of higher order approximations is given by

$$b(x,t) = \sum_{n=1}^{N} \varepsilon^n A_n(\varepsilon(x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + c.c.,$$

leading to the approximation equations

$$\partial_T A_1 = -24\mathrm{i}k_0 \partial_X^2 A_1, \qquad \partial_T A_n = -24\mathrm{i}k_0 \partial_X^2 A_n - 8\partial_X^3 A_{n-1},$$

for $n \in \{2, \dots, N\}$. These approximation equations for $n \geq 2$ can be solved with the variation of constants formula

$$A_n(T) = -\int_0^T \mathrm{e}^{-24\mathrm{i}k_0\partial_X^2(T-\tau)} 8\partial_X^3 A_{n-1}(\tau)\mathrm{d}\tau,$$

where we have chosen vanishing initial conditions $A_n(\cdot, 0) = 0$ for $n \in \{2, \dots, N\}$. This immediately gives the estimate

$$\sup_{0\leq\tau\leq T} \|A_n(\cdot,\tau)\|_{H^s} \leq CT \sup_{0\leq\tau\leq T} \|A_{n-1}(\cdot,\tau)\|_{H^{s+3}}.$$

Therefore, we need

$$A_1 \in H^{s+3N}, \quad A_2 \in H^{s+3N-3}, \quad A_3 \in H^{s+3N-6}, \dots, A_N \in H^{s+3}.$$

The error function then satisfies

$$\partial_t R = -8\partial_x^3 R - \varepsilon^{N+3}(8\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}\partial_X^3 A_N + c.c.).$$

Remark 4.3.6. For obtaining estimates for the higher order approximation on the long $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale with $\delta > 0$ arbitrarily small, but fixed, we modify the ansatz into

$$b(x,t) = \sum_{n=1}^{N} \varepsilon^{1+(n-1)\delta} A_n(\varepsilon(x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + c.c.,$$

leading to the approximation equations

$$\partial_T A_1 = -24\mathrm{i}k_0\partial_X^2 A_1, \qquad \partial_T A_n = -24\mathrm{i}k_0\partial_X^2 A_n - 8\varepsilon^{1-\delta}\partial_X^3 A_{n-1}.$$

Since

$$\sup_{0\le\tau\le T}\|A_n(\cdot,\tau)\|_{H^s} \le C\varepsilon^{1-\delta}T \sup_{0\le\tau\le T}\|A_{n-1}(\cdot,\tau)\|_{H^{s+3}},$$

all A_n remain $\mathcal{O}(1)$-bounded for $t\in[0,1/\varepsilon^{3-\delta}]$. The error function then satisfies

$$\partial_t R = -8\partial_x^3 R - \varepsilon^{1+(N-1)\delta+3}(8\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}\partial_X^3 A_N + c.c.),$$

and so

$$\|R(\cdot,t)\|_{H^s} \le C\varepsilon^{1+(N-1)\delta+3}t\varepsilon^{-1/2}\sup_{\tau\in[0,t]}\|A_N(\cdot,\tau)\|_{H^{s+3}}.$$

Thus, we have proved

Theorem 4.3.7. *For each $N\in\mathbb{N}$, $s\ge 0$ and $\delta\in(0,1]$ there exist $C>0$ and $\varepsilon_0>0$ such that the following holds. Let $A_1\in C([0,\infty),H^{s+3N})$ be a solution of the linear Schrödinger equation* (4.7) *and let the A_n be solutions of*

$$\partial_T A_n = -24\mathrm{i}k_0\partial_X^2 A_n - 8\varepsilon^{1-\delta}\partial_X^3 A_{n-1}, \qquad A_n|_{T=0}=0,$$

for $n=2,\dots,N$. Then, for all $\varepsilon\in(0,\varepsilon_0)$ there are solutions b of the Airy equation (4.6) *such that*

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]}\|b(x,t)-\varepsilon\Psi_N(x,t)\|_{H^s(dx)} \le C\varepsilon^{1/2+N\delta},$$

where

$$\varepsilon\Psi_N(x,t) = \sum_{n=1}^N \varepsilon^{1+(n-1)\delta}A_n(\varepsilon(x-ct),\varepsilon^2t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c..$$

Remark 4.3.8. Sobolev's embedding theorem immediately yields

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]}\sup_{x\in\mathbb{R}}|b(x,t)-\varepsilon\Psi_N(x,t)| \le C\varepsilon^{1/2+N\delta},$$

which is a non-void estimate if $1/2+N\delta>1$. Then, the error of order $\mathcal{O}(\varepsilon^{1/2+N\delta})$ is smaller than the solution and the approximation which both are of order $\mathcal{O}(\varepsilon)$ in C_b^0.

Remark 4.3.9. Since we are handling linear inhomogeneous equations, the above analysis holds in various other function spaces. For the subsequent analysis, we need a L^∞-bound in Fourier space. We rewrite Remark 4.3.6 to obtain estimates for the higher order approximation on the long $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale with $\delta>0$ arbitrarily small. The modified ansatz in Fourier space is given by

$$\widehat{b}(k,t) = \sum_{n=1}^N \varepsilon^{(n-1)\delta}\widehat{A}_n(\varepsilon^{-1}(k-k_0),\varepsilon^2t)\mathrm{e}^{\mathrm{i}(-\omega_0t)-\mathrm{i}c(k-k_0)t} + c.c.f.,$$

leading to the approximation equations

$$\partial_T \widehat{A}_1 = 24\mathrm{i}k_0 K^2 \widehat{A}_1, \qquad \partial_T \widehat{A}_n = 24\mathrm{i}k_0 K^2 \widehat{A}_n + 8\mathrm{i}\varepsilon^{1-\delta} K^3 \widehat{A}_{n-1},$$

where *c.c.f.* corresponds to the complex conjugate in Fourier space. Since

$$\sup_{0\le\tau\le T} \|\widehat{A}_n(\cdot,\tau)\|_{L^\infty_s} \le C\varepsilon^{1-\delta} T \sup_{0\le\tau\le T} \|\widehat{A}_{n-1}(\cdot,\tau)\|_{L^\infty_{s+3}},$$

where

$$\|\widehat{A}(\cdot,\tau)\|_{L^\infty_s} = \sup_{k\in\mathbb{R}} |\widehat{A}(k)(1+k^2)^{s/2}|,$$

all $\widehat{A}_n$ remain $\mathcal{O}(1)$-bounded for $t \in [0, 1/\varepsilon^{3-\delta}]$. Then, the error function satisfies

$$\partial_t \widehat{R} = 8\mathrm{i}k^3 \widehat{R} - \varepsilon^{(N-1)\delta+3}(8\mathrm{e}^{\mathrm{i}(-\omega_0 t)-\mathrm{i}c(k-k_0)t}(\mathrm{i}K)^3 \widehat{A}_N + c.c.f.),$$

and so

$$\|\widehat{R}(\cdot,t)\|_{L^\infty_s} \le C\varepsilon^{(N-1)\delta+3} t \sup_{\tau\in[0,t]} \|\widehat{A}_N(\cdot,\tau)\|_{L^\infty_{s+3}}$$

for all $t \in [0, 1/\varepsilon^{3-\delta}]$.

4.4 The approximation of the KdV solutions via IST

In this section we use the Gelfand-Levitan-Marchenko equation to construct the approximation $\varepsilon\Psi_u$ for the KdV equation (4.1) associated to the linear Schrödinger approximation $\varepsilon\Psi_b = \varepsilon\Psi_N$ and $\beta = 1/2 + N\delta$ from Theorem 4.3.7. We compute

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\widehat{b}(k,t)\mathrm{d}k = b(x,t),$$

for the solutions constructed in Section 4.3, i.e., for $b = \varepsilon\Psi_b$. Then, we set

$$\varepsilon\Psi_u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}(\varepsilon\Psi_K)(x,x^+,t),$$

where $\varepsilon\Psi_K$ is an approximate solution of

$$\varepsilon\Psi_K(x,y,t) + \varepsilon\Psi_b(x+y,t) + \varepsilon^2\int_x^\infty \Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z = 0, \tag{4.9}$$

with $y \ge x$. In the following we explain how to compute $\varepsilon\Psi_K$ iteratively. We have

$$\varepsilon\Psi_b(x,t) = \varepsilon A(\varepsilon(x-ct),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)} + c.c..$$

i) For approximately solving (4.9) we use perturbation theory. We make the ansatz

$$\varepsilon\Psi_K(x,y,t) = \varepsilon K_1(\varepsilon x,\varepsilon y,t)\mathrm{e}^{\mathrm{i}(k_0x+k_0y-\omega_0 t)} + c.c. + h.o.t.,$$

and compute

$$
\begin{aligned}
&\varepsilon^2 \int_x^\infty \Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z \\
= \; & \int_x^\infty \varepsilon K_1(\varepsilon x, \varepsilon z, t)\varepsilon A(\varepsilon(y+z-ct),\varepsilon^2 t)\mathrm{e}^{2\mathrm{i}k_0 z}\mathrm{d}z \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
= \; & \varepsilon^2 K_1(\varepsilon x, \varepsilon z, t) A(\varepsilon(y+z-ct),\varepsilon^2 t) \left.\frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{2\mathrm{i}k_0}\right|_{z=x}^{\infty} \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
& - \int_x^\infty \varepsilon^3 \partial_Z (K_1(\varepsilon x, \varepsilon z, t) A(\varepsilon(y+z-ct),\varepsilon^2 t)) \frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{2\mathrm{i}k_0}\mathrm{d}z \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
= \; & -\varepsilon^2 K_1(\varepsilon x, \varepsilon x, t) A(\varepsilon(y+x-ct),\varepsilon^2 t) \frac{\mathrm{e}^{2\mathrm{i}k_0 x}}{2\mathrm{i}k_0} \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
& + \varepsilon^3 \partial_Z (K_1(\varepsilon x, \varepsilon x, t) A(\varepsilon(y+x-ct),\varepsilon^2 t)) \frac{\mathrm{e}^{2\mathrm{i}k_0 x}}{(2\mathrm{i}k_0)^2} \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
& - \int_x^\infty \varepsilon^4 \partial_Z^2 (K_1(\varepsilon x, \varepsilon z, t) A(\varepsilon(y+z-ct),\varepsilon^2 t)) \frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{(2\mathrm{i}k_0)^2}\mathrm{d}z \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
= \; & \ldots = \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

such that equating the coefficient of $\varepsilon \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-\mathrm{i}\omega_0 t}$ in (4.9) to zero yields

$$K_1(\varepsilon x, \varepsilon y, t) = -A(\varepsilon(x+y-ct),\varepsilon^2 t).$$

The solution of the KdV equation is then given by

$$u(x,t) = \varepsilon u_1(x,t) + h.o.t.,$$

where

$$
\begin{aligned}
u_1(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_1(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(k_0 x + k_0 x - \omega_0 t)} + c.c.) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}(A(\varepsilon(x+x-ct),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0 x + k_0 x - \omega_0 t)} + c.c.) \\
&= 4\mathrm{i}k_0 A(\varepsilon(2x-ct),\varepsilon^2 t)\mathrm{e}^{2\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} + c.c. \\
&\quad + 4\varepsilon(\partial_X A)(\varepsilon(2x-ct),\varepsilon^2 t)\mathrm{e}^{2\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} + c.c..
\end{aligned}
$$

ii) For getting rid of the terms of order $\mathcal{O}(\varepsilon^2)$ at $\mathrm{e}^{\mathrm{i}(3k_0 x + k_0 y - 2\omega_0 t)}$, we extend our ansatz to

$$
\begin{aligned}
\varepsilon \Psi_K(x,y,t) &= \varepsilon K_1(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(k_0 x + k_0 y - \omega_0 t)} + c.c. \\
&\quad + \varepsilon^2 K_2(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0 x + k_0 y - 2\omega_0 t)} + c.c. + h.o.t..
\end{aligned}
$$

Equating the coefficient of $\varepsilon^2 \mathrm{e}^{3\mathrm{i}k_0 x + \mathrm{i}k_0 y}\mathrm{e}^{-2\mathrm{i}\omega_0 t}$ in (4.9) to zero yields

$$
\begin{aligned}
K_2(\varepsilon x, \varepsilon y, t) &= \frac{1}{2\mathrm{i}k_0} K_1(\varepsilon x, \varepsilon x, t) A(\varepsilon(y+x-ct),\varepsilon^2 t) \\
&= -\frac{1}{2\mathrm{i}k_0} A(\varepsilon(2x-ct),\varepsilon^2 t) A(\varepsilon(y+x-ct),\varepsilon^2 t).
\end{aligned}
$$

The next order solution of the KdV equation is then given by

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + h.o.t.,$$

where

$$\begin{aligned} u_2(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_2(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0x-2\omega_0 t)} + c.c.) \\ &= \frac{1}{\mathrm{i}k_0}\frac{\mathrm{d}}{\mathrm{d}x}(A^2(\varepsilon(2x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)} + c.c.) \\ &= 4A^2(\varepsilon(2x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)} + c.c. \\ &\quad + \frac{2}{\mathrm{i}k_0}\varepsilon(\partial_X(A^2))(\varepsilon(2x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)} + c.c.. \end{aligned}$$

iii) We use the same idea to get rid of the terms of order $\mathcal{O}(\varepsilon^3)$ at $\mathrm{e}^{\mathrm{i}(5k_0x+k_0y-3\omega_0 t)}$. Again we extend our ansatz to

$$\begin{aligned} \varepsilon\Psi_K(x,y,t) &= \varepsilon K_1(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(k_0x+k_0y-\omega_0 t)} + c.c. \\ &\quad + \varepsilon^2 K_2(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)} + c.c. \\ &\quad + \varepsilon^3 K_3(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(5k_0x+k_0y-3\omega_0 t)} + c.c. + h.o.t.. \end{aligned}$$

Equating the coefficient of $\varepsilon^3\mathrm{e}^{5\mathrm{i}k_0x+\mathrm{i}k_0y}\mathrm{e}^{-3\mathrm{i}\omega_0 t}$ in (4.9) to zero yields

$$\begin{aligned} K_3(\varepsilon x, \varepsilon y, t) &= \frac{1}{2\mathrm{i}k_0}K_2(\varepsilon x, \varepsilon x, t)A(\varepsilon(y+x-ct), \varepsilon^2 t) \\ &= \frac{1}{4k_0^2}A^2(\varepsilon(2x-ct), \varepsilon^2 t)A(\varepsilon(y+x-ct), \varepsilon^2 t). \end{aligned}$$

The next order solution of the KdV equation is then given by

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + h.o.t.,$$

where

$$\begin{aligned} u_3(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_3(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(5k_0x+k_0x-3\omega_0 t)} + c.c.) \\ &= -\frac{1}{2k_0^2}\frac{\mathrm{d}}{\mathrm{d}x}(A^3(\varepsilon(2x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(6k_0x-3\omega_0 t)} + c.c. \\ &= -\frac{3\mathrm{i}}{k_0}A^3(\varepsilon(2x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(6k_0x-3\omega_0 t)} + c.c. \\ &\quad - \frac{1}{k_0^2}\varepsilon(\partial_X(A^3))(\varepsilon(2x-ct), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(6k_0x-3\omega_0 t)} + c.c.. \end{aligned}$$

iv) As a last example we explain how to eliminate the terms of order $\mathcal{O}(\varepsilon^3)$ at $\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)}$. We extend our ansatz to

$$\begin{aligned} \varepsilon\Psi_K(x,y,t) &= \varepsilon K_1(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(k_0x+k_0y-\omega_0 t)} + c.c. \\ &\quad + \varepsilon^2 K_2(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)} + c.c. \\ &\quad + \varepsilon^3 K_3(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(5k_0x+k_0y-3\omega_0 t)} + c.c. \\ &\quad + \varepsilon^3 K_{2,1}(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)} + c.c. + h.o.t.. \end{aligned}$$

Equating the coefficient of $\varepsilon^3 \mathrm{e}^{3\mathrm{i}k_0x+\mathrm{i}k_0y}\mathrm{e}^{-2\mathrm{i}\omega_0 t}$ in (4.9) to zero yields

$$\begin{aligned} K_{2,1}(\varepsilon x,\varepsilon y,t) &= -\frac{1}{(2\mathrm{i}k_0)^2}\partial_X\left(K_1(\varepsilon x,\varepsilon x,t)A(\varepsilon(y+x-ct),\varepsilon^2 t)\right) \\ &= -\frac{1}{4k_0^2}\partial_X\left(A(\varepsilon(2x-ct),\varepsilon^2 t)A(\varepsilon(y+x-ct),\varepsilon^2 t)\right). \end{aligned}$$

The next order solution of the KdV equation is then given by

$$u(x,t)=\varepsilon u_1(x,t)+\varepsilon^2 u_2(x,t)+\varepsilon^3 u_3(x,t)+\varepsilon^3 u_{2,1}(x,t)+h.o.t.,$$

where

$$\begin{aligned} u_{2,1}(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_{2,1}(\varepsilon x,\varepsilon x^+,t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0x-2\omega_0 t)}+c.c.) \\ &= -\frac{1}{2k_0^2}\frac{\mathrm{d}}{\mathrm{d}x}(\partial_X\left(A^2(\varepsilon(2x-ct),\varepsilon^2 t)\right)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)}+c.c. \\ &= -\frac{2\mathrm{i}}{k_0}(\partial_X(A^2))\left(\varepsilon(2x-ct),\varepsilon^2 t\right)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)}+c.c. \\ &\quad -\frac{1}{k_0^2}\varepsilon(\partial_X^2(A^2))\left(\varepsilon(2x-ct),\varepsilon^2 t\right)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)}+c.c.. \end{aligned}$$

Remark 4.4.1. These calculations can be performed up to an arbitrary order. For solving (4.9) we make the ansatz

$$\varepsilon\Psi_K(x,y,t)=\sum_{n\in I_N}\sum_{m=0}^{M_{N,n}}\varepsilon^{\beta(n)+m}K_{n,m}(\varepsilon x,\varepsilon y,t)\mathrm{e}^{\mathrm{i}((2n-1)k_0x+k_0y-n\omega_0 t)},$$

with $\beta(n)=1+||n|-1|$, $I_N=\{-N,-N+1,\ldots,N-1,N\}$ and sufficiently large numbers $M_{N,n}\in\mathbb{N}_0$. To derive equations for the $K_{n,m}$ and therefore $u_{n,m}$, we do analogous steps as before with $K_j=K_{j,0}$ and $u_j=u_{j,0}$ for $j\in\mathbb{N}$. Hence, we can conclude

$$\varepsilon\Psi_u(x,t)=-2\frac{\mathrm{d}}{\mathrm{d}x}(\varepsilon\Psi_K(x,x^+,t)). \tag{4.10}$$

4.5 Error estimates via IST

The kernel K is a sum of the approximation kernel $\varepsilon\Psi_K$ constructed in Section 4.4 and an error $\varepsilon^\beta R_K$. Plugging

$$K(x,y,t)=\varepsilon\Psi_K(x,y,t)+\varepsilon^\beta R_K(x,y,t)$$

into the Gelfand-Levitan-Marchenko equation (4.5) yields

$$\begin{aligned} &\varepsilon\Psi_K(x,y,t)+\varepsilon^\beta R_K(x,y,t)+\varepsilon\Psi_b(x+y,t)+\varepsilon^\beta R_b(x+y,t) \\ &\quad+\int_x^\infty(\varepsilon\Psi_K(x,z,t)+\varepsilon^\beta R_K(x,z,t))(\varepsilon\Psi_b(y+z,t)+\varepsilon^\beta R_b(y+z,t))\mathrm{d}z=0, \end{aligned}$$

and so

$$R_K + s_{inh} + s_{lin} + s_{non} + s_{res} = 0, \tag{4.11}$$

with

$$\begin{aligned}
s_{inh}(x,y,t) &= R_b(x+y,t) + \varepsilon \int_x^\infty \Psi_K(x,z,t) R_b(y+z,t) \mathrm{d}z, \\
s_{lin}(x,y,t) &= \varepsilon \int_x^\infty R_K(x,z,t) \Psi_b(y+z,t) \mathrm{d}z, \\
s_{non}(x,y,t) &= \varepsilon^\beta \int_x^\infty R_K(x,z,t) R_b(y+z,t) \mathrm{d}z, \\
s_{res}(x,y,t) &= \varepsilon^{-\beta}(\varepsilon \Psi_K(x,y,t) + \varepsilon \Psi_b(x+y,t) \\
&\qquad + \varepsilon^2 \int_x^\infty \Psi_K(x,z,t) \Psi_b(y+z,t) \mathrm{d}z).
\end{aligned}$$

The function $R_K(x,y,t)$ vanishes identically for $y < x$ since the Gelfand-Levitan-Marchenko equation (4.5) is only valid for $y \geq x$.

The structure of (4.11) is as follows:

- The term s_{inh} is independent of R_K and does not contain residual terms.
- The term s_{lin} is linear in R_K. This term can be estimated with the help of energy estimates.
- The term s_{non} is nonlinear in R_K and will be of higher order due to the ε^β in front.
- The term s_{res} is the residual, i.e., it contains the terms which do not cancel after inserting the formal approximations Ψ_K and Ψ_b into the Gelfand-Levitan-Marchenko equation (4.9). The remaining terms can be written as

$$\begin{aligned}
s_{res} &= \varepsilon^{-\beta}(\varepsilon \Psi_K(x,y,t) + \varepsilon \Psi_b(x+y,t) \\
&\qquad + \varepsilon^2 \int_x^\infty \Psi_K(x,z,t) \Psi_b(y+z,t) \mathrm{d}z) \\
&= \varepsilon^{-\beta} \left(\int_x^\infty \varepsilon^{N+1} K_{rest}(x,z,t) \Psi_b(y+z,t) \mathrm{d}z \right),
\end{aligned}$$

  where K_{rest} is a function of $A, \ldots, \partial_X^{s_A} A$, with s_A a number depending on N, since we balanced in the Chapter 4.4 all smaller powers of ε.

4.5.1 Outline

Equation (4.11) will be solved for every fixed t. Since (4.11) is formally of the form R_K plus some small perturbation in R_K plus some inhomogeneity, we will use Neumann's series to solve (4.11) w.r.t. R_K.

Multiplication of (4.11) with $R_K(x,y,t)$ and integration w.r.t. y yields

$$\int_{-\infty}^\infty |R_K(x,y,t)|^2 \mathrm{d}y + r_{inh}(x,t) + r_{lin}(x,t) + r_{non}(x,t) + r_{res}(x,t) = 0, \tag{4.12}$$

with

$$\begin{aligned} r_{inh}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{inh}(x,y,t)\mathrm{d}y, \\ r_{lin}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{lin}(x,y,t)\mathrm{d}y, \\ r_{non}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{non}(x,y,t)\mathrm{d}y, \\ r_{res}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{res}(x,y,t)\mathrm{d}y. \end{aligned}$$

Remark 4.5.1. In the following we use the fundamental identity, cf. [Seg73, p. 727],

$$\begin{aligned} & \int_{-\infty}^{\infty}\int_{x}^{\infty} \varepsilon R_1(x,z,t)\Psi_b(y+z,t)R_2(x,y,t)\mathrm{d}z\mathrm{d}y \\ = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon R_1(x,z,t)\Psi_b(y+z,t)R_2(x,y,t)\mathrm{d}z\mathrm{d}y \\ = & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon R_1(x,z,t)\int_{-\infty}^{\infty}\widehat{\Psi}_b(k,t)\mathrm{e}^{\mathrm{i}k(y+z)}\mathrm{d}kR_2(x,y,t)\mathrm{d}z\mathrm{d}y \\ = & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon R_1(x,z,t)\mathrm{e}^{\mathrm{i}kz}\widehat{\Psi}_b(k,t)R_2(x,y,t)\mathrm{e}^{\mathrm{i}ky}\mathrm{d}z\mathrm{d}y\mathrm{d}k \\ = & \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon\widehat{\Psi}_b(k,t)\widehat{R}_1(x,-k,t)\widehat{R}_2(x,-k,t)\mathrm{d}k \\ = & \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon\widehat{\Psi}_b(k,t)\overline{\widehat{R}_1(x,k,t)\widehat{R}_2(x,k,t)}\mathrm{d}k, \end{aligned}$$

where we used the definition of the Fourier transform and $R_1(x,z,t)=0$ or $R_2(x,z,t)=0$ for $x>z$ since $R_K(x,z,t)=0$ for $x>z$.

4.5.2 Estimates for the inhomogeneous term r_{inh}

In this subsection we are going to estimate

$$\begin{aligned} r_{inh}(x,t) = & \int_{-\infty}^{\infty} R_K(x,y,t)R_b(x+y,t)\mathrm{d}y \\ & +\varepsilon\int_{-\infty}^{\infty} R_K(x,y,t)\int_{x}^{\infty}\Psi_K(x,z,t)R_b(y+z,t)\mathrm{d}z\mathrm{d}y. \end{aligned}$$

With the Cauchy-Schwarz inequality we find

$$\left|\int_{-\infty}^{\infty} R_K(x,y,t)R_b(x+y,t)\mathrm{d}y\right| \leq \|R_K(x,\cdot,t)\|_{L^2}\|R_b(\cdot,t)\|_{L^2},$$

and with Remark 4.5.1, Plancherel's identity and Young's inequality that

$$\begin{aligned} & \varepsilon\left|\int_{-\infty}^{\infty} R_K(x,y,t)\int_{x}^{\infty}\Psi_K(x,z,t)R_b(y+z,t)\mathrm{d}z\mathrm{d}y\right| \\ & \leq C\varepsilon\sup_{k\in\mathbb{R}}|\widehat{R}_b(\cdot,t)|\|R_K(x,\cdot,t)\|_{L^2}\|\Psi_K(x,\cdot,t)\|_{L^2}. \end{aligned}$$

Hence, we obtain the estimate

$$|r_{inh}(x,t)| \leq (C_{0,1}+C_{0,2}\varepsilon)\|R_K(x,\cdot,t)\|_{L^2} \leq 2C_{0,1}\|R_K(x,\cdot,t)\|_{L^2},$$

with constants

$$\begin{aligned} C_{0,1} &= \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|R_b(\cdot,t)\|_{L^2}, \\ C_{0,2}\varepsilon &= C\varepsilon \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{k\in\mathbb{R}} |\widehat{R}_b(\cdot,t)| \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{x\in\mathbb{R}} \|\Psi_K(x,\cdot,t)\|_{L^2}, \end{aligned}$$

if $\varepsilon > 0$ is chosen so small that $C_{0,2}\varepsilon \leq C_{0,1}$.

4.5.3 Estimates for the linear term r_{lin}

With Remark 4.5.1 and Plancherel's identity we find

$$\varepsilon\left|\int_{-\infty}^{\infty} R_K(x,y,t)\int_x^{\infty} R_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z\mathrm{d}y\right| \leq \varepsilon\|\widehat{\Psi}_b\|_{L^\infty}\|R_K(x,\cdot,t)\|_{L^2}^2.$$

We assume

$$\sup_{K\in\mathbb{R}} |\widehat{A}(K,T)| \leq \sup_{K\in\mathbb{R}} |\widehat{A}(K,0)| \leq 1-\delta' < 1, \tag{4.13}$$

for $T \geq 0$ and a $\delta' \in (0,1)$ which is a natural restriction due to our underlying problem which leads to (4.13). Then, the triangle inequality yields

$$\|\varepsilon\widehat{\Psi}_b\|_{L^\infty} \leq \|\widehat{A}_b\|_{L^\infty} + \mathcal{O}(\varepsilon) \leq 1-\delta'/2, \tag{4.14}$$

for $\varepsilon > 0$ sufficiently small. Therefore, we achieve

$$|r_{lin}(x,t)| \leq C_1\|R_K(x,\cdot,t)\|_{L^2}^2,$$

with a $C_1 < 1$.

4.5.4 Estimates for the nonlinear term r_{non}

In this subsection we are going to estimate

$$r_{non}(x,t) = \varepsilon^\beta \int_{-\infty}^{\infty} R_K(x,y,t)\int_x^{\infty} R_K(x,z,t)R_b(y+z,t)\mathrm{d}z\mathrm{d}y.$$

Again with Remark 4.5.1 and Plancherel's identity this can be estimated by

$$|r_{non}(x,t)| \leq C\varepsilon^\beta\|\widehat{R}_b(\cdot,t)\|_{L^\infty}\|R_K(x,\cdot,t)\|_{L^2}^2.$$

4.5.5 Estimates for the residual term r_{res}

In this subsection we are going to estimate

$$\begin{aligned} r_{res}(x,t) &= \varepsilon^{-\beta}\int_{-\infty}^{\infty} R_K(x,y,t)\Big(\varepsilon\Psi_K(x,y,t)+\varepsilon\Psi_b(x+y,t) \\ &\qquad +\varepsilon^2\int_x^{\infty}\Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z\Big)\mathrm{d}y \\ &= \varepsilon^{-\beta}\int_{-\infty}^{\infty} R_K(x,y,t)\left(\int_x^{\infty}\varepsilon^{N+1}K_{rest}(x,z,t)\Psi_b(y+z,t)\mathrm{d}z\right)\mathrm{d}y. \end{aligned}$$

With Remark 4.5.1 we find

$$\begin{aligned} r_{res}(x,t) &= \varepsilon^{-\beta}\int_{-\infty}^{\infty} R_K(x,y,t)\left(\varepsilon^{N+1}\int_x^{\infty}K_{rest}(x,z,t)\Psi_b(y+z,t)\mathrm{d}z\right)\mathrm{d}y \\ &= \varepsilon^{N+1-\beta}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\Psi}_b(k,t)\overline{\widehat{R}_K(x,k,t)\widehat{K}_{rest}(x,k,t)}\mathrm{d}k. \end{aligned}$$

The function K_{rest} can be expressed in terms of the functions $A, ..., \partial_X^{s_A}A$ with s_A a number depending on N. It contains terms which are at least quadratic in A and its derivatives. Moreover, there are no terms in K_{rest} w.r.t. A and its derivatives of power bigger than $N+1$. Since multiplication becomes convolution under Fourier transform, we have to estimate for instance

$$\|\widehat{A}^{*(N+1)}\|_{L^2}.$$

By Young's inequality for convolutions and the embedding $L^2_s\subset L^1$ for $s>1/2$ we obtain for instance

$$\|\widehat{A}^{*(N+1)}\|_{L^2}\le C\|\widehat{A}^{*(N)}\|_{L^1}\|\widehat{A}\|_{L^2}\le C\|\widehat{A}\|_{L^1}^N\|\widehat{A}\|_{L^2}\le C\|\widehat{A}\|_{L^2_s}^{N+1}\le C\|A\|_{H^s}^{N+1},$$

and analogously for the terms containing derivatives of A such that all terms in K_{rest} can be estimated in terms of $C\|A\|^j_{H^{s+s_A}}$ for $j=2,\dots,N+1$.

By the Cauchy-Schwarz inequality, Plancherel's identity and estimate (4.14) we obtain

$$\begin{aligned} |r_{res}(x,t)| &= \left|\varepsilon^{N+1-\beta}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\Psi}_b(k,t)\overline{\widehat{R}_K(x,k,t)\widehat{K}_{rest}(x,k,t)}\mathrm{d}k\right| \\ &\le \varepsilon^{N-\beta}\varepsilon\|\widehat{\Psi}_b(\cdot,t)\|_{L^\infty}\|R_K(x,\cdot,t)\|_{L^2}\|K_{rest}(x,\cdot,t)\|_{L^2} \\ &\le C\varepsilon^{N-\beta}\varepsilon\|\widehat{\Psi}_b(\cdot,t)\|_{L^\infty}\|R_K(x,\cdot,t)\|_{L^2}\sum_{j=2}^{N+1}\|A(x,\cdot,t)\|^j_{H^{s+s_A}} \\ &\le C\varepsilon^{N-\beta}C_1\|R_K(x,\cdot,t)\|_{L^2}\sum_{j=2}^{N+1}\|A(x,\cdot,t)\|^j_{H^{s+s_A}} \\ &\le C_2\varepsilon^{N-\beta}\|R_K(x,\cdot,t)\|_{L^2}, \end{aligned}$$

with the constant

$$C_2=\sup_{t\in[0,1/\varepsilon^{3-\delta}]}\sup_{x\in\mathbb{R}}\left(CC_1\sum_{j=2}^{N+1}\|A(x,\cdot,t)\|^j_{H^{s+s_A}}\right).$$

4.5.6 Final estimates

In the following we choose N so large that $N-\beta \geq 1$. From (4.12) we immediately find

$$\int_{-\infty}^{\infty} |R_K(x,y,t)|^2 \mathrm{d}y \quad \leq \quad |r_{inh}(x,t)| + |r_{lin}(x,t)| + |r_{non}(x,t)| + |r_{res}(x,t)|.$$

First, Young's inequality gives

$$|r_{inh}(x,t)| \leq 2C_{0,1}\|R_K(x,\cdot,t)\|_{L^2} \leq \delta_1^2 \|R_K(x,\cdot,t)\|_{L^2}^2 + \frac{C_{0,1}}{\delta_1^2},$$

and

$$|r_{res}(x,t)| \leq C_2 \varepsilon^{N-\beta}\|R_K(x,\cdot,t)\|_{L^2} \leq \frac{C_2^2}{4} + \varepsilon^{2(N-\beta)}\|R_K(x,\cdot,t)\|_{L^2}^2,$$

such that

$$\begin{aligned}\|R_K(x,\cdot,t)\|_{L^2}^2 \quad \leq \quad & \delta_1^2 \|R_K(x,\cdot,t)\|_{L^2}^2 + \frac{C_{0,1}}{\delta_1^2} \\ & + C_1 \|R_K(x,\cdot,t)\|_{L^2}^2 \\ & + C\varepsilon^\beta C_B \|R_K(x,\cdot,t)\|_{L^2}^2 \\ & + \frac{C_2^2}{4} + \varepsilon^{2(N-\beta)}\|R_K(x,\cdot,t)\|_{L^2}^2,\end{aligned}$$

with $C_1 < 1$ and

$$C_B = \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|\widehat{R}_b(\cdot,t)\|_{L^\infty}.$$

Rearranging the terms we obtain

$$(1-\delta_1^2 - C_1 - C\varepsilon^\beta C_B - \varepsilon^{2(N-\beta)})\|R_K(x,\cdot,t)\|_{L^2}^2 \quad \leq \quad \frac{C_{0,1}}{\delta_1^2} + \frac{C_2^2}{4}.$$

Choosing $\delta_1 > 0$ and $\varepsilon > 0$ so small that

$$\delta_1^2 + C\varepsilon^\beta C_B + \varepsilon^{2(N-\beta)} \leq \frac{1-C_1}{2}$$

gives

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{x\in\mathbb{R}} \|R_K(x,\cdot,t)\|_{L^2} \leq 2(1-C_1)^{-1}\left(\frac{C_{0,1}}{\delta_1^2} + \frac{C_2^2}{4}\right) =: C_R = \mathcal{O}(1),$$

and hence

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{x\in\mathbb{R}} \|K(x,\cdot,t) - \varepsilon\Psi_K(x,\cdot,t)\|_{L^2} \leq C_R \varepsilon^\beta.$$

In exactly the same way, we prove

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{x\in\mathbb{R}} \|\partial_x^{s_x}\partial_y^{s_y}(K(x,\cdot,t) - \varepsilon\Psi_K(x,\cdot,t))\|_{L^2} \leq C_R \varepsilon^\beta,$$

for $0 \leq s_x + s_y \leq s$. Therefore, we find

$$u(x,t) - \varepsilon\Psi_u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}(K(x,x^+,t) - \varepsilon\Psi_K(x,x^+,t)),$$

and so

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|u(x,t) - \varepsilon\Psi_u(x,t)\|_{H^{s-1}} \leq C\varepsilon^{\beta}.$$

Hence, we have proven

Theorem 4.5.2. *For each $N \in \mathbb{N}$, $s \geq 0$ and $\delta \in (0,1]$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A_1 \in C([0,\infty),(\mathcal{F}^{-1}L^\infty_{s+1+3N}) \cap H^{s+1+3N})$ be a solution of the linear Schrödinger equation* (4.7) *and let the A_n be solutions of*

$$\partial_T A_n = -24\mathrm{i}k_0\partial_X^2 A_n - 8\varepsilon^{1-\delta}\partial_X^3 A_{n-1}, \qquad A_n|_{T=0} = 0,$$

for $n = 2,\dots,N$. Then, for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions u of the KdV equation (4.1) *such that*

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|u(x,t) - \varepsilon\Psi_u(x,t)\|_{H^s(\mathrm{d}x)} \leq C\varepsilon^{1/2+N\delta},$$

with $\varepsilon\Psi_u$ as constructed in (4.10).

4.6 Discussion

In the previous sections we used the Gelfand-Levitan-Marchenko equation and the evolution of the scattering data to construct a linear Schrödinger approximation for the KdV equation. Although at a first view this detour only seems to be of theoretical use. The transfer of a nonlinear PDE problem into a pure integration problem allowed us to extend the approximation time beyond the natural NLS time scale.

Remark 4.6.1. Our result is what you expect for completely integrable systems for which a representation in action and angle variables does exist. The action variables are conserved. The frequency of the angle variables are approximated up to order $\mathcal{O}(\varepsilon^2)$, i.e. with an error of order $\mathcal{O}(\varepsilon^3)$. Then, the error for these variables grows like $\mathcal{O}(\varepsilon^3)t$ which is of order $\mathcal{O}(\varepsilon^\delta)$ for a $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale.

Remark 4.6.2. The inverse scattering approach for the KdV equation and the NLS equation have been related in [ZK86]. Note that in parts our expansion is different from the expansion used in [ZK86]. Such correspondences have been analyzed in a number of other papers, cf. [BCP02, KP03].

Remark 4.6.3. It is the goal of future research to describe the interaction of NLS scaled wave packets for completely integrable systems. Is it possible to extend the separation of internal and interaction dynamics of NLS scaled wave packets with different carrier waves for completely integrable systems even further than in the existing literature [PW95, CSU07, CCSU08, CS12, SC15]?

Remark 4.6.4. Due to the scaling properties of the slow spatial variable $X \sim \varepsilon x$ in the NLS ansatz and the scaling properties of the KdV solitons $\sim \varepsilon^2 A_{soliton}(\varepsilon(x - ct))$, we expect that the discrete eigenvalues in the scattering data are of order $\mathcal{O}(\varepsilon^2)$ for a general NLS ansatz in the original KdV equation. Rigorous estimates for the number and size of the eigenvalues can be found with the help of Lieb-Thirring inequalities, cf. [FLW23]. However, a detailed analysis with respect to this question is beyond the scope of this thesis.

Remark 4.6.5. For the KdV equation only the defocusing NLS equation can be derived. However, given the IST approach for the KdV equation the sign in front of the cubic term of the NLS equation did not play any role in the analysis.

Remark 4.6.6. The Schrödinger equation shows a dispersive decay $\sim 1/\sqrt{T}$. Hence for $t = \mathcal{O}(1/\varepsilon^{3-\delta})$, respectively $T = \mathcal{O}(1/\varepsilon^{1-\delta})$, we have that the solutions are of order $\mathcal{O}(\varepsilon^{1/2-\delta/2})$ if they are initially of order $\mathcal{O}(1)$ or of order $\mathcal{O}(\varepsilon^{3/2-\delta/2})$ if they are initially of order $\mathcal{O}(\varepsilon)$ like for the NLS approximation. For solutions of order $\mathcal{O}(\varepsilon^{3/2-\delta/2})$ for the mKdV equation Gronwall's inequality easily gives estimates on a time scale of order $\mathcal{O}(1/\varepsilon^{3-\delta})$ w.r.t. t. However, in our situation the solutions are much bigger than $\mathcal{O}(\varepsilon^{3/2-\delta/2})$ and only of that order at the end of the time interval. Therefore, our result is non-trivial.

Remark 4.6.7. Due to the decay of the solutions, cf Remark 4.6.6, we have a global in time approximation result with an error $\mathcal{O}(1/\varepsilon^{3-\delta})$.

Remark 4.6.8. It is the purpose of future research to transfer the presented analysis to other dispersive completely integrable systems, such as the sine-Gordon equation, the NLS equation or the Toda-lattice.

Remark 4.6.9. It remains to compute the initial scattering data for b for a given initial NLS ansatz

$$u(x) = \varepsilon A_1(\varepsilon x)\mathrm{e}^{\mathrm{i}k_0 x} + \varepsilon A_{-1}(\varepsilon x)\mathrm{e}^{-\mathrm{i}k_0 x} + h.o.t..$$

The scattering problem

$$-\partial_x^2\psi - u(x)\psi = -k^2\psi$$

in lowest order leads to $k = k_0/2$ and

$$\mathrm{i}k\partial_X B_1 - A_1 B_{-1} = 0, \qquad \mathrm{i}k\partial_X B_{-1} - A_{-1} B_1 = 0,$$

where

$$\psi(x) = \varepsilon B_1(\varepsilon x)\mathrm{e}^{\mathrm{i}kx} + \varepsilon B_{-1}(\varepsilon x)\mathrm{e}^{-\mathrm{i}kx} + h.o.t..$$

This shows where the relation between the spatial scale of the scattering data and the original spatial scale comes from. It is the purpose of future research to replace the special initial conditions for the scattering data b by scattering data which allows us to handle all NLS approximations.

Remark 4.6.10. The previous analysis will work for all other members of the KdV-hierarchy, too.

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