Singular limits in KGZ systems and the DNLS approximation in case of quadratic nonlinearities

Von der Fakultät für Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

Vorgelegt von

Raphael Taraca

aus Pfullendorf

Hauptberichter: Prof. Dr. Guido Schneider

Mitberichter: Prof. Dr. Wolf-Patrick Düll
Prof. Dr. C. Eugene Wayne

Tag der mündlichen Prüfung: 26. April 2024

Institut für Analysis, Dynamik und Modellierung

2024

Contents

Zusammenfassung

Im ersten Teil dieser Arbeit betrachten wir das Klein-Gordon-Zakharov-System (ein Modell in der Plasmaphysik) sowie verwandte Systeme in Abhängigkeit von einem kleinen Störungsparameter $0 < \varepsilon \ll 1$. Im singulären Grenzfall $\varepsilon \to 0$ leiten wir weitere Systeme her. Mit Hilfe von Energieabschätzungen und Normalformtheorie beweisen wir erstmals Approximationsresultate zwischen Lösungen der ursprünglichen Systeme und Lösungen der Grenzwertsysteme im Falle räumlich periodischer Randbedingungen. Die wesentlichen Schwierigkeiten ergeben sich aus den Abschätzungen für das Residuum, der Konstruktion von Approximationen höherer Ordnung und der Tatsache, dass die Nichtresonanzbedingungen zum Anwenden der Normalformtheorie nicht immer erfüllt sind. Für den Fall, dass die Normalformtransformation an Regularität verliert, verwenden wir eine modifizierte Energie wie sie bei der Approximationstheorie für quasilineare Systeme ihre Anwendung hat.
Im zweiten Teil rechtfertigen wir die Derivative NLS-Approximation im Falle quadratischer Nichtlinearitäten. Die DNLS-Gleichung taucht im Wasserwellenproblem als Modulationsgleichung auf. Die Rechtfertigung ist eine nichttriviale Aufgabe, da Lösungen der Ordnung $\mathcal{O}(\varepsilon^{1/2})$ auf einer $\mathcal{O}(\varepsilon^{-2})$-Zeitskala für $0 < \varepsilon \ll 1$ kontrolliert werden müssen. Wir leiten die DNLS-Gleichung mittels Multiskalenanalysis her und zeigen ein Approximationsresultat, indem wir Energieabschätzungen beweisen und mehrere Normalformtransformationen verwenden. Hierbei treten komplexere Resonanzstrukturen wie totale Resonanzen, Resonanzen zweiter Ordnung und zusätzliche Resonanzen erster Ordnung auf. Bei quadratischer Nichtlinearität können diese Resonanzen stabil oder instabil werden. Für den stabilen Fall beweisen wir ein Approximationsresultat, für den instabilen Fall ein Nicht-Approximationsresultat.

Abstract

In the first part of this thesis, we consider the Klein-Gordon-Zakharov system (a model in plasma physics) and related systems depending on a small perturbation parameter $0 < \varepsilon \ll 1$. In the singular limit $\varepsilon \to 0$, we derive further systems. By using energy estimates and normal form theory, for the first time we prove approximation results between solutions of the original systems and solutions of the limit systems in the case of spatially periodic boundary conditions. The main difficulties arise in the estimates for the residual, the construction of higher order approximations and the fact that the non-resonance conditions for applying the normal form theory are not always satisfied. In case that the normal form transform loses regularity, we use a modified energy that is reminiscent of applications in approximation theory for quasilinear systems.
In the second part, we justify the Derivative NLS approximation in the case of quadratic nonlinearities. The DNLS equation appears in the water wave problem as a modulation equation. The justification is a non-trivial task since solutions of order $\mathcal{O}(\varepsilon^{1/2})$ have to be controlled on an $\mathcal{O}(\varepsilon^{-2})$ time scale for $0 < \varepsilon \ll 1$. We derive the DNLS equation via multiple scaling analysis and show an approximation result by using energy estimates and applying several normal form transformations. In doing so, more complex resonance structures such as total resonances, second order resonances, and additional first order resonances appear. For a quadratic nonlinearity, these resonances can become stable or unstable. For the stable case, we prove an approximation result; for the unstable case, we prove a non-approximation result.

Danksagung

An dieser Stelle möchte ich mich bei all den Personen bedanken, die mich bei der Erstellung dieser Doktorarbeit unterstützt haben.

Mein größter Dank gilt meinem Betreuer Prof. Dr. Guido Schneider, der mir diese Doktorarbeit erst ermöglicht hat. Er war mir nicht nur während meiner Promotion, sondern auch während meines Masterstudiums immer eine große Hilfe, stand jederzeit für Fragen zur Verfügung und hatte stets konstruktive Ideen, wenn ich mal nicht weiterwusste. Die zahlreichen Gespräche auf fachlicher und persönlicher Ebene waren mir immer eine Freude.

Ebenso möchte ich mich bei den Mitberichtern dieser Arbeit bedanken. Dabei gilt mein Dank Prof. Dr. Wolf-Patrick Düll, der auch meine Bachelorarbeit betreut hat und während meines Studiums häufig für Fragen verfügbar war, sowie Prof. Dr. Eugene Wayne, dem ich für das konstruktive Gespräch über meine Forschung danke. Ein großes Dankeschön geht auch an meine (teils ehemaligen) Kolleginnen und Kollegen vom IADM, die jederzeit für eine positive und lebendige Arbeitsatmosphäre gesorgt haben. Insbesondere danke ich Katja Stefanie Engstler, die mich zu Beginn meiner Promotion herzlich empfangen hat und mir bei administrativen Angelegenheiten stets weiterhelfen konnte.

Mein ganz besonderer Dank gilt meiner Familie, meinen Freunden sowie meiner Freundin Gabriela für die Kraft, Motivation und jahrelange Unterstützung, die mich durch mein Studium und meine Promotion getragen hat. Sie alle waren mir immer eine verlässliche Stütze und dafür kann ich mich nicht genug bei ihnen bedanken.

Selbstständigkeitserklärung

Hiermit versichere ich, Raphael Taraca, diese Dissertation selbstständig verfasst, alle benutzten Hilfsmittel vollständig angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer entnommen wurde. Die Arbeit wurde bisher noch nicht einer akademischen Institution vorgelegt.

Stuttgart, 2024 Raphael Taraca

Chapter 1

Introduction

Modulation theory refers to a concept that has applications in various fields of science and technology, cf. [SU17, §10-12]. The underlying mathematical idea is to use knowledge of the behaviour of solutions of so-called modulation equations to get a better understanding of the dynamics of solutions of more complex physical systems, such as pattern forming systems, the water wave problem, systems from nonlinear optics, etc. For this purpose, we need to provide approximation results for these systems. This thesis deals with modulation theory and is divided into two parts.
The aim of the first part in Chapter 2 is to improve and to extend the existing literature on the approximation theory of the Klein-Gordon-Zakharov (KGZ) system and related systems describing so-called Langmuir waves in plasma. These systems have many applications in both astrophysical situations and laboratory experiments, cf. [SS99] for more details. The KGZ system

$$\partial_t^2 u = \partial_x^2 u - u - uv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2),$$

with $x, t \in \mathbb{R}$, $u(x,t) \in \mathbb{C}$, and $v(x,t) \in \mathbb{R}$, describes the interaction between Langmuir waves and ion sound waves in plasma. Here, $v(x,t)$ is proportional to the ion density fluctuation from a constant equilibrium density and $u(x,t)$ is proportional to the electric field. The Zakharov system

$$2i\partial_t u = \partial_x^2 u - uv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2),$$

with $x, t \in \mathbb{R}$, $u(x,t) \in \mathbb{C}$, and $v(x,t) \in \mathbb{R}$, describes the propagation of Langmuir waves in an ionized plasma via the envelope $u(x,t)$ of the electric field and the deviation $v(x,t)$ of the ion density from the equilibrium density. By rescaling the terms in these systems by a small perturbation parameter $0 < \varepsilon \ll 1$ and by considering the limit case $\varepsilon \to 0$, effective systems for the slow dynamics are obtained. For small values of the parameter $0 < \varepsilon \ll 1$, our goal is to estimate the

distance between solutions of the regular limit system and true solutions of the original singular system by means of energy estimates and normal form theory. We improve the existing literature by showing convergence rates and by considering these systems for the first time on the torus where the methods from the literature, such as dispersive decay estimates, no longer apply. Moreover, our studies include estimates for the residual and the construction of higher order approximations. At the beginning of Chapter 2, we present all limits we consider, the problems we encounter, and the methods we use to solve these.
In the second part of this thesis, in Chapter 3, we extend the approximation theory of the Derivative Nonlinear Schrödinger (DNLS) equation

$$i\partial_T A = \nu_1 \partial_X^2 A + \nu_2 A|A|^2 + i\nu_3 |A|^2 \partial_X A + i\nu_4 A^2 \partial_X \overline{A} + \nu_5 A|A|^4,$$

with $T \geq 0$, $X \in \mathbb{R}$, $A(X,T) \in \mathbb{C}$, and coefficients $\nu_j \in \mathbb{R}$ for $j = 1, \ldots, 5$. Via multiple scaling perturbation analysis, the DNLS equation can be derived from dispersive wave equations, such as nonlinear Klein-Gordon equations of the form

$$\partial_t^2 u = \partial_x^2 u - u + f(\partial_x, u).$$

This is done to describe slow modulations in time and space of the envelope of a spatially and temporarily oscillating wave packet of the form

$$u(x,t) \approx \varepsilon^{1/2} \psi(x,t) = \varepsilon^{1/2} A(\varepsilon(x - c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)},$$

see Figure 1.1. Hereby, $c_g \in \mathbb{R}$ is the linear group velocity, $k_0 \in \mathbb{R}$ the basic spatial wave number, $\omega_0 \in \mathbb{R}$ the basic temporal wave number, and $0 < \varepsilon \ll 1$ a small perturbation parameter.

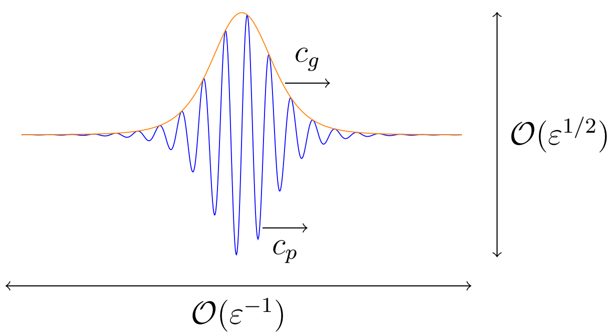


Figure 1.1: The envelope (in orange) of a wave packet that advances with group velocity c_g and modulates the underlying oscillatory wave (in blue) advancing with phase velocity c_p.

The DNLS equation occurs when the cubic coefficient for the associated NLS equation vanishes for the basic spatial wave number of the underlying slowly modulated wave packet. In [HS22a, HS22b], the DNLS approximation was already justified for a cubic Klein-Gordon equation. In Chapter 3, we extend the theory of the DNLS approximation by giving a first proof of the DNLS approximation for a quadratic Klein-Gordon equation. This is a highly non-trivial problem since the approximation is of order $\mathcal{O}(\varepsilon^{1/2})$ and solutions have to be controlled on an $\mathcal{O}(\varepsilon^{-2})$ time scale. We show the approximation result by using energy estimates and normal form theory where additional resonance structures occur in comparison to the cubic case. Further, we give a first proof of the failure of the DNLS approximation for a particular Klein-Gordon equation in the case of spatially periodic boundary conditions.

Notation. Throughout the thesis, we use the following notation which will be supplemented in the individual sections. The Fourier transform of a function $u \in L^2(\mathbb{R}, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is defined by

$$\mathcal{F}(u)(k) = \widehat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ikx} \, \mathrm{d}x.$$

The inverse Fourier transform of a function $\widehat{u} : \mathbb{R} \to \mathbb{K}$ is given by

$$\mathcal{F}^{-1}(\widehat{u})(x) = u(x) = \int_{\mathbb{R}} \widehat{u}(k) e^{ikx} \, \mathrm{d}k.$$

The multiplication $(uv)(x) = u(x)v(x)$ in physical space corresponds, in Fourier space, to the convolution

$$(\widehat{u} * \widehat{v})(k) = \int_{\mathbb{R}} \widehat{u}(k-l)\widehat{v}(l) \, \mathrm{d}l.$$

The Sobolev space $H^s(\mathbb{R}, \mathbb{K})$, $s \geq 0$, is the space of functions from $\mathbb{R}$ into $\mathbb{K}$, for which the norm

$$\|u\|_{H^s(\mathbb{R},\mathbb{K})} = \left(\int_{\mathbb{R}} |\widehat{u}(k)|^2 (1 + |k|^2)^s \, \mathrm{d}k \right)^{1/2}$$

is finite. Many possibly different constants are denoted by the same symbol C, if they can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$. With *c.c.*, we denote the complex conjugate of an expression.

Chapter 2

KGZ and related systems

We consider singular limits of the Klein-Gordon-Zakharov (KGZ) and related systems which depend on a small perturbation parameter $0 < \varepsilon \ll 1$. We are interested in these systems due to their relevance in plasma physics. Moreover, the resonance structures for these systems in the singular limit also appear for other relevant systems. The aim of this chapter is to prove approximation results for the KGZ and related systems by estimating the distance between their solutions and solutions of the regular systems obtained by taking the singular limit $\varepsilon \to 0$. Consider, for example, the KGZ system

$$\partial_t^2 u = \partial_x^2 u - u - uv, \qquad \varepsilon^2 \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2).$$

In the singular limit $\varepsilon \to 0$, we first obtain $v = -|u|^2$ and finally the Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - u + u|u|^2.$$

This limit has already been studied in [DSS16] for higher dimensions. In [DSS16], an approximation result was shown for $\mathbb{R}^d$ with $d \geq 3$. It turned out that, on a n-dimensional torus, the approximation result only holds for a modified Klein-Gordon equation. In this chapter, we consider additional limits of related systems:

(i) In Section 2.1, we consider the Zakharov system

$$2i\partial_t u = \partial_x^2 u - uv, \qquad \varepsilon^2 \partial_t^2 v = \partial_x^2 v + \partial_x^2 |u|^2.$$

By letting $\varepsilon \to 0$, we first obtain $v = -|u|^2$ and finally the NLS equation

$$2i\partial_t u = \partial_x^2 u + |u|^2 u.$$

This limit is the most straightforward one and has already been studied in [AA88, SW86]. However, these studies focus on convergence and not on error bounds. In Section 2.1, for the first time, we show an approximation result for the Zakharov system in case of periodic boundary conditions by means of energy estimates. It turns out that we need a smallness condition for solutions of the NLS equation, which was not considered in the corresponding literature. Moreover, we provide estimates for the residual and construct a higher order approximation in order to make the residual arbitrarily small. We remark that the estimates for the residual on the real line are different from the estimates in case of periodic boundary conditions.

(ii) In Section 2.2, we consider the KGZ system

$$\varepsilon^2\partial_t^2 u = \partial_x^2 u - \varepsilon^{-2}u - uv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2).$$

In the singular limit $\varepsilon \to 0$ with the ansatz $u(x,t) = w(x,t)e^{i\varepsilon^{-2}t}$, we obtain the Zakharov system

$$2i\partial_t w = \partial_x^2 w - wv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|w|^2).$$

In [BBC96, CEGT04], the same ansatz has been considered but with a focus on convergence and not on error bounds. In [Sch19], the Zakharov approximation of the Klein-Gordon-Zakharov system was already justified under the assumption that the solutions of the Zakharov system are analytic within a strip in the complex plane. In Section 2.2, we show an approximation result on the real line by using energy estimates. Before that, we have to perform a normal form transformation to eliminate problematic terms. Here, no resonances occur. In addition to that, we provide estimates for the residual and construct a higher order approximation.

(iii) In Section 2.3, we consider the KGZ system in the form

$$\varepsilon^2\partial_t^2 u = \partial_x^2 u - \varepsilon^{-2}u - uv, \qquad \gamma^2\varepsilon^2\partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2),$$

with a parameter $\gamma \in \mathbb{R}$. In the singular limit $\varepsilon \to 0$ with the ansatz $u(x,t) = w(x,t)e^{i\varepsilon^{-2}t}$, we obtain the NLS equation

$$2i\partial_t w = \partial_x^2 w + w|w|^2.$$

We show an approximation result for $|\gamma| \geq 1$ in case of periodic boundary conditions by means of energy estimates and normal form transformations. Here, we have to distinguish between the following two cases. For $|\gamma| > 1$, the normal form transformations are bounded, while, for $|\gamma| = 1$, they lose regularity. We can solve this problem by including the normal form transformations into the energy as it has been done in the justification of the

NLS approximation for quasilinear dispersive systems, cf. [Due17, HITW15]. The case $|\gamma| > 1$ has already been considered in [MN05][1] but with a completely different method of proof. In [MN05], dispersive decay estimates were used which are not applicable in our setting due to the periodic boundary conditions. In contrast to [MN05], we provide estimates for the residual and construct a higher order approximation. To the author's knowledge, the case $|\gamma| = 1$ has not been considered in any of the existing literature so far.

We emphasize that the limits, which are considered in (i), (ii) and (iii), build on each other. To be more precise, in Section 2.1, we use energy estimates to prove the main result. In Section 2.2, we use similar energy estimates with the additional difficulty that problematic terms have to be eliminated by a normal form transformation. In Section 2.3, we proceed analogously, whereby the normal form transformation is more complex due to the scaling of the original system.

(iv) In Section 2.4, we consider the KGZ system

$$\varepsilon^2\partial_t^2 u = \partial_x^2 u - \varepsilon^{-2}u - uv, \qquad \frac{\gamma^2}{4}\varepsilon^4\partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2),$$

with a parameter $\gamma \in \mathbb{R}$. In the singular limit $\varepsilon \to 0$ with the ansatz

$$\begin{aligned} u(x,t) &= w(x,t)e^{i\varepsilon^{-2}t} + c.c, \\ v(x,t) &= v_0(x,t) + v_{0,+}(x,t)e^{2i\varepsilon^{-2}t} + v_{0,-}(x,t)e^{-2i\varepsilon^{-2}t}, \end{aligned}$$

we obtain the singular NLS equation

$$2i\partial_t w = \partial_x^2 w + 2w|w|^2 - \overline{w}\mathcal{A}_\gamma(w^2),$$

where $\mathcal{A}_\gamma = -\partial_x^2(\gamma^2 + \partial_x^2)^{-1}$. This limit has already been studied in [MN10]. In Section 2.4, we provide estimates for the residual under spatially periodic boundary conditions. In this context, we choose the period so that the operator $\mathcal{A}_\gamma$ is well-defined in Fourier space. Further, we construct a higher order approximation in order to make the residual arbitrarily small. However, the estimates for the error using our method of proof remain open since in the normal form transformation we lose too much powers of ε.

[1] Note that the paramater γ here corresponds to the parameter $1/\gamma$ in [MN05].

2.1 From the Zakharov system to the NLS equation on the torus

2.1.1 Introduction

In this section, we consider the singular limit of the Zakharov system in which the NLS equation is obtained as regular limit system. Our goal is to estimate the distance between the solutions obtained through the regular limit system and the true solutions of the Zakharov system for small values of the perturbation parameter $0 < \varepsilon \ll 1$. In detail, we consider the Zakharov system in the form

$$2i\partial_t u = \partial_x^2 u - uv, \tag{2.1}$$

$$\varepsilon^2 \partial_t^2 v = \partial_x^2 v + \partial_x^2 |u|^2, \tag{2.2}$$

for $u = u(x,t) \in \mathbb{C}$, $v = v(x,t), x, t \in \mathbb{R}$ with spatially 2π-periodic boundary conditions, where $0 < \varepsilon \ll 1$ is a small perturbation parameter. In the singular limit $\varepsilon \to 0$, we first obtain $v = -|u|^2$ and finally the NLS equation

$$2i\partial_t u = \partial_x^2 u + |u|^2 u \tag{2.3}$$

with spatially 2π-periodic boundary conditions. This corresponds to the spectral situation in Figure 2.1. It is a goal of this section to give a proof of

Theorem 2.1.1. *There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0,T_0], H^6)$ be a solution of the NLS equation* (2.3) *with spatially 2π-periodic boundary conditions and*

$$\sup_{t\in[0,T_0]} \|u_0(\cdot,t)\|_{H^6} = C_u < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of the Zakharov system (2.1)–(2.2) *with spatially 2π-periodic boundary conditions satisfying*

$$\sup_{t\in[0,T_0]} \|(u,v)(\cdot,t) - (u_0, -|u_0|^2)(\cdot,t)\|_{H^1\times L^2} \leq C\varepsilon^2.$$

Remark 2.1.2. Such estimates have been shown in [AA88, SW86] for $x \in \mathbb{R}^d$ with $d \in \{1,2,3\}$ with the help of energy estimates in order to study the asymptotic behaviour of the solutions of the Zakharov system (2.1)-(2.2) when ε goes to zero. The approximation only holds, if the nonlinear part on the right-hand side of (2.1) has a negative sign. For a positive sign, there exists a counterexample which shows that the NLS approximation fails to make correct predictions about the dynamics

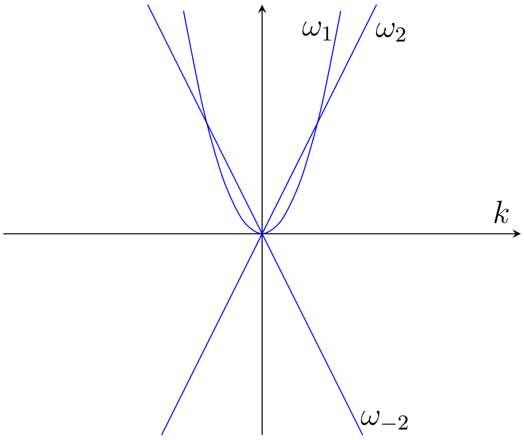


Figure 2.1: The linearized Zakharov system is solved by $u(x,t) = e^{ikx+i\omega_1(k)t}$ and $v(x,t) = e^{ikx+i\omega_{\pm 2}(k)t}$ where $\omega_1(k) = \frac{1}{2}k^2$ and $\omega_{\pm 2}(k) = \pm\varepsilon^{-1}k$ for $k \in \mathbb{R}$. The figure shows the curves of eigenvalues ω_1 and $\omega_{\pm 2}$. The intersection point is at $k = \mathcal{O}(\varepsilon^{-1})$.

of the Zakharov system, cf. [BSSZ20]. In this section, we consider the Zakharov system (2.1)-(2.2) on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and improve the approximation rate by constructing a higher order approximation with vanishing mean value, cf. Section 2.1.3. Although our energy estimates are very similar to the ones of [AA88], the higher order approximation allows us to reduce the number of necessary energy estimates slightly.

Remark 2.1.3. The Zakharov system was introduced by Zakharov ([Zak72]) to describe the propagation of Langmuir waves in an ionized plasma via the electric field u and the deviation v of the ions' equilibrium density. It can be derived directly from and justified for Maxwell's equation coupled with Euler's equation, cf. [Tex07].

Remark 2.1.4. The Zakharov system can be rewritten as a semilinear evolutionary system for which local existence and uniqueness of solutions in Sobolev spaces can be established using semigroup theory, cf. [OT92]. Going back to the original variables for the Zakharov system (2.1)-(2.2), there is local existence and uniqueness for $(u, v, \partial_t v) \in H^{s+2} \times H^{s+1} \times H^s$, $s \geq 0$.

Remark 2.1.5. The local existence and uniqueness of solutions $u \in H^s$, $s \geq 1$, of the NLS equation (2.3) is well known. It follows by using semigroup theory and a standard fixed point argument applied to the variation of constants formula.

The approximation theorem is proved by using energy estimates and Gronwall's inequality. In the next section, we bound the residual terms appearing for the Zakharov system.

Notation. We use the notation from Chapter 1. Further, we write $\int$ for $\int_{\mathbb{T}}$ and H^s for $H^s(\mathbb{T},\mathbb{K})$, unless otherwise specified.

2.1.2 Estimates for the residual

The residual of (2.1)–(2.2) is given by

$$\begin{aligned}\mathrm{Res}_u(u,v) &= -2i\partial_t u + \partial_x^2 u - uv,\\ \mathrm{Res}_v(u,v) &= -\varepsilon^2\partial_t^2 v + \partial_x^2 v + \partial_x^2|u|^2\end{aligned}$$

and contains all terms which do not cancel after inserting the approximation into the Zakharov system. If we directly choose $v = -|u|^2$ and u to satisfy the NLS equation, the residual will be of order $\mathcal{O}(\varepsilon^2)$, which is not sufficient for our proof of the approximation theorem, Theorem 2.1.1. Therefore, we introduce an improved approximation which brings the residual Res_v from $\mathcal{O}(\varepsilon^2)$ to $\mathcal{O}(\varepsilon^4)$. Inserting the extended ansatz

$$\psi_u(x,t) = u_0(x,t), \qquad \psi_v(x,t) = v_0(x,t) + \varepsilon^2 v_2(x,t) \tag{2.4}$$

into the Zakharov system gives at ε^0 that

$$2i\partial_t u_0 = \partial_x^2 u_0 - u_0 v_0, \qquad 0 = \partial_x^2 v_0 + \partial_x^2(|u_0|^2),$$

and at ε^2 that

$$\partial_t^2 v_0 = \partial_x^2 v_2.$$

We choose $v_0 = -|u_0|^2$ and then u_0 to satisfy the NLS equation

$$2i\partial_t u_0 = \partial_x^2 u_0 + u_0|u_0|^2. \tag{2.5}$$

Next, we set

$$\widehat{v}_2(k,t) = -k^{-2}\partial_t^2\widehat{v}_0(k,0) \tag{2.6}$$

for $k \in \mathbb{Z}\setminus\{0\}$. In order to have v_2 well-defined, due to the periodic boundary conditions, it is sufficient to show that the mean value of v_0 is conserved. This holds due to

$$\partial_t \int v_0\,\mathrm{d}x = -\partial_t \int |u_0|^2\,\mathrm{d}x = 0,$$

which is the conservation of the L^2-norm for the solutions of the NLS equation. Therefore, we define $\widehat{v}_2(0,t) = \widehat{v}_2(0,0)$. Since then, by construction all $\partial_x^{-n}\partial_t^m v_2$ for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ are well-defined and have a vanishing mean value.

Remark 2.1.6. The function v_2 can also be well-defined, if $x \in \mathbb{T}$ is replaced with $x \in \mathbb{R}$ since

$$\partial_t^2 v_0 = \frac{1}{4}\left(\partial_x^4(|u_0|^2) - 4\partial_x^2(\partial_x u_0 \partial_x \overline{u_0}) + \partial_x^2(|u_0|^4)\right),$$

as well as

$$v_2 = \frac{1}{4}\left(\partial_x^2(|u_0|^2) - 4|\partial_x u_0|^2 + |u_0|^4\right) = v_2^*(u_0).$$

If ψ_u and ψ_v are defined as in (2.4), we find for the residual that

$$\mathrm{Res}_u(\psi_u, \psi_v) = -\varepsilon^2 u_0 v_2, \qquad \mathrm{Res}_v(\psi_u, \psi_v) = -\varepsilon^4 \partial_t^2 v_2.$$

Thus, we directly obtain the following lemma.

Lemma 2.1.7. *Let $s \geq 0$ and let $u_0 \in C([0, T_0], H^{s+6})$ be a solution of the NLS equation* (2.3) *with spatially 2π-periodic boundary conditions. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{H^{s+4}} \leq C_{res}\varepsilon^2, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\psi_u, \psi_v)\|_{H^s} \leq C_{res}\varepsilon^4.$$

Proof. In order to estimate $\partial_t^2 v_2$ in H^s, we can use the representation of v_2 in terms of u_0, which can be found in Remark 2.1.6, and the NLS equation to express time derivatives of v_0 by space derivatives of v_0. Thus, the function has to be in H^{s+6}. The rest of the proof is straightforward. □

In the equations for the error, not only the residual appears but also $\partial_x^{-1}\mathrm{Res}_v$. Hence, we have to estimate the term $\varepsilon^4 \partial_x^{-1}\partial_t^2 v_2$, too. As above, due to the periodic boundary conditions, it would be sufficient to prove that the mean value of v_2 is conserved in order to have the term $\partial_x^{-1}\partial_t^2 v_2$ bounded in some function space on the torus $\mathbb{T}$, but we already proved this above. Therefore, we have

Lemma 2.1.8. *Let $u_0 \in C([0, T_0], H^6)$ be a solution of the NLS equation* (2.3) *with spatially 2π-periodic boundary conditions. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\psi_u, \psi_v)\|_{L^2} = \sup_{t\in[0,T_0]} \varepsilon^4 \|\partial_x^{-1}\partial_t^2 v_2\|_{L^2} \leq C_{res}\varepsilon^4.$$

Remark 2.1.9. For $x \in \mathbb{R}$ a serious difficulty occurs at that point. In this case, we have to choose $v_2 = v_2^*(u_0)$ from Remark 2.1.6 for which however we find

$$\partial_t \int_{\mathbb{R}} v_2 \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \partial_x v_0 \,\mathrm{Im}(u_0 \partial_x \overline{u_0}) \,\mathrm{d}x \neq 0$$

after a straightforward calculation.

2.1.3 Higher order approximation

For computing higher order approximations, in case $x \in \mathbb{T}$, we make the ansatz

$$\psi_{u,n}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k} u_{2k}(x,t), \qquad \psi_{v,n}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k} v_{2k}(x,t) \tag{2.7}$$

with the goal to make the residual even smaller. Then, as before $v_0 = -|u_0|^2$ and u_0 solves the NLS equation (2.5). For $k \in \{1, \dots, n\}$, the functions u_{2k} solve inhomogeneous linear Schrödinger equations of the form

$$2i\partial_t u_{2k} = \partial_x^2 u_{2k} - u_0 v_{2k} - u_{2k} v_0 - F_{2k}(u_0, \dots, u_{2(k-1)}),$$

and the functions v_{2k} satisfy

$$\partial_t^2 v_{2(k-1)} = \partial_x^2 v_{2k} + \partial_x^2 G_{2k}(u_0, \dots, u_{2k}), \tag{2.8}$$

where F_{2k} and G_{2k} are quadratic mappings. Suppose that $v_{2(k-1)}$ has a vanishing mean value. We look for v_{2k} having a vanishing mean value. Since G_{2k}, in general, will not have a vanishing mean value, we add a constant $\beta_{2k} \in \mathbb{C}$ to v_{2k} to get rid of the non-vanishing mean value of G_{2k}. We can do this since the constant will cancel in (2.8). Then, we set

$$v_{2k} = \partial_x^{-2}\partial_t^2 v_{2(k-1)} - G_{2k}(u_0, \dots, u_{2k}) + \frac{1}{2\pi}\int G_{2k}(u_0, \dots, u_{2k})(x)\,\mathrm{d}x \tag{2.9}$$

and

$$\beta_{2k} = \frac{1}{2\pi}\int G_{2k}(u_0, \dots, u_{2k})(x)\,\mathrm{d}x.$$

Remark 2.1.10. It is unclear how to solve this problem for $x \in \mathbb{R}$. In space dimensions $d \geq 3$, one may use that $\Delta^{-1} : L^2 \cap L^1 \to L^2$ is a bounded operator, and that nonlinear terms will be in L^1 due to Cauchy-Schwarz inequality, if the u_{2k} and v_{2k} are in some Sobolev space.

We formulate the following lemma.

Lemma 2.1.11. *Let $n \in \mathbb{N}$ and $s \geq 0$. Further, let $u_0 \in C([0,T_0], H^{s+2n+5})$ be a solution of the NLS equation (2.3) with spatially 2π-periodic boundary conditions. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an approximation $(\psi_{u,n}, \psi_{v,n})$ of the form (2.7) with*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\psi_{u,n}, \psi_{v,n})\|_{H^{s+1}} \leq C_{res}\varepsilon^{2n+2}, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\psi_{u,n}, \psi_{v,n})\|_{H^{s}} \leq C_{res}\varepsilon^{2n+2},$$

and

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\psi_{u,n}, \psi_{v,n})\|_{H^{s}} \leq C_{res}\varepsilon^{2n+2}.$$

Proof. The term which contains the most derivatives in the residual is $\partial_t^2 v_{2n}$. After repeatedly replacing v_{2n} with the right-hand side of (2.9), inductively the term $\partial_x^{-2n}\partial_t^{2n+2} v_0$ appears. We remark that $v_0 = -|u_0|^2$ and that u_0 solves the NLS equation (2.3), i.e., each time derivative of v_0 generates two spatial derivatives of u_0. Therefore, in order to estimate Res_u in H^{s+1}, we have to assume that $u_0 \in H^{s+2n+5}$. The estimates for Res_v in H^s are straightforward. We note that each term in Res_v has either a spatial derivative in front or has a vanishing mean value by construction. □

2.1.4 Estimates for the error

Proof of Theorem 2.1.1. We introduce the error $\varepsilon^2(R_u, R_v)$ made by the improved approximation (ψ_u, ψ_v) by

$$(u, v)(x, t) = (\psi_u, \psi_v)(x, t) + \varepsilon^2(R_u, R_v)(x, t).$$

The error functions R_u and R_v satisfy

$$2i\partial_t R_u = \partial_x^2 R_u - \psi_u R_v - \psi_v R_u - \varepsilon^2 R_u R_v + \varepsilon^{-2}\mathrm{Res}_u, \qquad (2.10)$$

$$\varepsilon^2 \partial_t^2 R_v = \partial_x^2 R_v + \partial_x^2(\overline{\psi_u} R_u) + \partial_x^2(\psi_u \overline{R_u}) + \varepsilon^2 \partial_x^2 |R_u|^2 + \varepsilon^{-2}\mathrm{Res}_v. \qquad (2.11)$$

Next, we follow [AA88] and multiply the first equation with $-i\overline{R_u}$ and integrate this equation w.r.t. x. Since ψ_v and R_v are real-valued, we have

$$\mathrm{Re}\int i\overline{R_u}\psi_v R_u \,\mathrm{d}x = 0, \qquad \mathrm{Re}\int i\overline{R_u} R_u R_v \,\mathrm{d}x = 0.$$

Therefore, adding the complex conjugate yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|R_u\|_{L^2}^2 = \mathrm{Re}\int i\overline{R_u}\psi_u R_v \,\mathrm{d}x - \mathrm{Re}\int i\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u \,\mathrm{d}x.$$

Further, we multiply the first equation with $\partial_t \overline{R_u}$ and integrate this equation w.r.t. x. By adding its complex conjugate, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x R_u\|_{L^2}^2 = & -\int (\psi_u R_v \partial_t \overline{R_u} + \overline{\psi_u} R_v \partial_t R_u)\,\mathrm{d}x \\ & -\int (\psi_v R_u \partial_t \overline{R_u} + \psi_v \overline{R_u}\partial_t R_u)\,\mathrm{d}x \\ & -\varepsilon^2 \int (R_v R_u \partial_t \overline{R_u} + R_v \overline{R_u}\partial_t R_u)\,\mathrm{d}x \\ & +2\mathrm{Re}\int \partial_t \overline{R_u}\varepsilon^{-2}\mathrm{Res}_u \,\mathrm{d}x.\end{aligned}$$

Multiplying the second equation with $\partial_x^{-2}\partial_t R_v$ and integrating w.r.t. x yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2 \\
&= -\int(\partial_t R_v)\overline{\psi_u}R_u\,\mathrm{d}x - \int(\partial_t R_v)\psi_u\overline{R_u}\,\mathrm{d}x - \varepsilon^2\int|R_u|^2\partial_t R_v\,\mathrm{d}x \\
&\quad + \int(\partial_x^{-1}\partial_t R_v)\partial_x^{-1}(\varepsilon^{-2}\mathrm{Res}_v)\,\mathrm{d}x.
\end{aligned}$$

Adding these resulting equations gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|R_u\|_{L^2}^2 + \|\partial_x R_u\|_{L^2}^2 + \frac{1}{2}\|R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2\right) \\
&= \mathrm{Re}\int i\overline{R_u}\psi_u R_v\,\mathrm{d}x - \mathrm{Re}\int i\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u\,\mathrm{d}x + 2\mathrm{Re}\int\partial_t\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u\,\mathrm{d}x \\
&\quad -\int(\psi_u R_v\partial_t\overline{R_u} + \overline{\psi_u}R_v\partial_t R_u)\,\mathrm{d}x - \int(\psi_v R_u\partial_t\overline{R_u} + \psi_v\overline{R_u}\partial_t R_u)\,\mathrm{d}x \\
&\quad -\varepsilon^2\int(R_v R_u\partial_t\overline{R_u} + R_v\overline{R_u}\partial_t R_u)\,\mathrm{d}x - \int(\partial_t R_v)\overline{\psi_u}R_u\,\mathrm{d}x - \int(\partial_t R_v)\psi_u\overline{R_u}\,\mathrm{d}x \\
&\quad -\varepsilon^2\int|R_u|^2\partial_t R_v\,\mathrm{d}x + \int(\partial_x^{-1}\partial_t R_v)(\varepsilon^{-2}\mathrm{Res}_v)\,\mathrm{d}x \\
&= \mathrm{Re}\int i\overline{R_u}\psi_u R_v\,\mathrm{d}x - \mathrm{Re}\int i\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u\,\mathrm{d}x + 2\mathrm{Re}\int\partial_t\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u\,\mathrm{d}x \\
&\quad +\int(\partial_x^{-1}\partial_t R_v)\partial_x^{-1}(\varepsilon^{-2}\mathrm{Res}_v)\,\mathrm{d}x - \varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int R_v|R_u|^2\,\mathrm{d}x \\
&\quad -\frac{\mathrm{d}}{\mathrm{d}t}\int\psi_v|R_u|^2\,\mathrm{d}x + \int(\partial_t\psi_v)|R_u|^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int\psi_u R_v\overline{R_u}\,\mathrm{d}x + \int(\partial_t\psi_u)R_v\overline{R_u}\,\mathrm{d}x \\
&\quad -\frac{\mathrm{d}}{\mathrm{d}t}\int\overline{\psi_u}R_v R_u\,\mathrm{d}x + \int(\partial_t\overline{\psi_u})R_v R_u\,\mathrm{d}x.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
E &= \|R_u\|_{L^2}^2 + \|\partial_x R_u\|_{L^2}^2 + \|R_v\|_{L^2}^2 + \varepsilon^2\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2 + \int\psi_v|R_u|^2\,\mathrm{d}x \\
&\quad + \int\psi_u R_v\overline{R_u}\,\mathrm{d}x + \int\overline{\psi_u}R_v R_u\,\mathrm{d}x + \varepsilon^2\int R_v|R_u|^2\,\mathrm{d}x
\end{aligned}$$

satisfies the estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E &\leq \|\psi_u\|_{L^\infty}\|R_u\|_{L^2}\|R_v\|_{L^2} + \varepsilon^{-2}\|R_u\|_{L^2}\|\mathrm{Res}_u\|_{L^2} \\
&\quad +2\mathrm{Re}\left|\int\partial_t R_u\varepsilon^{-2}\overline{\mathrm{Res}_u}\,\mathrm{d}x\right| + \|\partial_x^{-1}\partial_t R_v\|_{L^2}\|\partial_x^{-1}(\varepsilon^{-2}\mathrm{Res}_v)\|_{L^2} \\
&\quad +2\|\partial_t\psi_u\|_{L^\infty}\|R_v\|_{L^2}\|R_u\|_{L^2} + \|\partial_t\psi_v\|_{L^\infty}\|R_u\|_{L^2}^2.
\end{aligned}$$

In order to estimate the expression $\mathrm{Re}\left|\int \partial_t R_u \varepsilon^{-2}\overline{\mathrm{Res}_u}\,\mathrm{d}x\right|$, we replace $\partial_t R_u$ with the right-hand side of (2.10). Integration by parts finally gives

$$\begin{aligned}
&\mathrm{Re}\left|\int \partial_t R_u \varepsilon^{-2}\overline{\mathrm{Res}_u}\,\mathrm{d}x\right| \\
&\quad = \mathrm{Re}\left|\int (\partial_x^2 R_u - \psi_u R_v - \psi_v R_u - \varepsilon^2 R_u R_v + \varepsilon^{-2}\mathrm{Res}_u)\varepsilon^{-2}\overline{\mathrm{Res}_u}\,\mathrm{d}x\right| \\
&\quad \le \varepsilon^{-2}\|R_u\|_{H^1}\|\mathrm{Res}_u\|_{H^1} + \varepsilon^{-2}\|\psi_u\|_{L^\infty}\|R_v\|_{L^2}\|\mathrm{Res}_u\|_{L^2} \\
&\qquad + \varepsilon^{-2}\|\psi_v\|_{L^\infty}\|R_u\|_{L^2}\|\mathrm{Res}_u\|_{L^2} + \|R_u\|_{H^1}\|R_v\|_{L^2}\|\mathrm{Res}_u\|_{L^2} + \varepsilon^{-4}\|\mathrm{Res}_u\|_{L^2}^2 \\
&\quad \le C_0 E^{1/2} + C_{res}\varepsilon^2 E + C_{res}^2.
\end{aligned}$$

Hence, by using $E^{1/2} \le 1 + E$ and Lemma 2.1.8, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E &\le C_0 + C_1 E + \varepsilon^2\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2 + \varepsilon^{-2}\|\partial_x^{-1}(\varepsilon^{-2}\mathrm{Res}_v)\|_{L^2}^2 \\
&\le C_2 E + C_3.
\end{aligned}$$

Consequently, with Gronwall's inequality, we have $E(t) \le M$ for all $t \in [0, T_0]$ for a constant $M = \mathcal{O}(1)$. For u_0 sufficiently small but $\mathcal{O}(1)$, the square root of the energy on the left-hand side is equivalent to the $H^1 \times L^2$-norm of (R_u, R_v) . □

2.1.5 Higher regularity

The aim of this section is to give a proof of the following approximation result, which gives estimates for the error in Sobolev spaces with higher regularity.

Theorem 2.1.12. *Let $s \in \mathbb{N}_0$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0, T_0], H^{s+6})$ be a solution of the NLS equation* (2.3) *with spatially 2π-periodic boundary conditions and*

$$\sup_{t\in[0,T_0]} \|u_0(\cdot,t)\|_{H^{s+6}} = C_u < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of the Zakharov system (2.1)–(2.2) *with spatially 2π-periodic boundary conditions satisfying*

$$\sup_{t\in[0,T_0]} \|(u,v)(\cdot,t) - (u_0, -|u_0|^2)(\cdot,t)\|_{H^{s+1}\times H^s} \le C\varepsilon^2.$$

Proof. First, we apply the operator ∂_x^s to the system (2.10)–(2.11). Then, we multiply the first equation with $\partial_t\partial_x^s\overline{R_u}$, integrate w.r.t. x, and add its complex conjugate. Further, we multiply the second equation with $\partial_x^{s-2}\partial_t R_v$ and integrate w.r.t. x. Adding the resulting equations together yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{s+1}R_u\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^s R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{s-1}\partial_t R_v\|_{L^2}^2 \\
&= \; -\int (\partial_x^s(\psi_u R_v)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(\psi_u\overline{R_u})\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&\quad\; -\int (\partial_x^s(\overline{\psi_u}R_v)\partial_t\partial_x^s R_u + \partial_x^s(\overline{\psi_u}R_u)\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&\quad\; -\int (\partial_x^s(\psi_v R_u)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(\psi_v\overline{R_u})\partial_t\partial_x^s R_u)\,\mathrm{d}x \\
&\quad\; -\varepsilon^2\int (\partial_x^s(R_v R_u)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(R_v\overline{R_u})\partial_t\partial_x^s R_u + \partial_x^s(|R_u|^2)\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&\quad\; +2\mathrm{Re}\int \partial_t\partial_x^s\overline{R_u}\varepsilon^{-2}\partial_x^s\mathrm{Res}_u\,\mathrm{d}x + \int (\partial_x^{s-1}\partial_t R_v)\partial_x^{s-1}(\varepsilon^{-2}\mathrm{Res}_v)\,\mathrm{d}x.
\end{aligned}$$

In contrast to the proof of Theorem 2.1.1, the problematic terms on the right-hand side cannot directly be written as a time derivative. Therefore, we have to rewrite some terms on the right-hand side. For the first two integrals, we use the Leibniz rule in order to get

$$\begin{aligned}
&\int (\partial_x^s(\psi_u R_v)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(\psi_u\overline{R_u})\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&= \int (\psi_u\partial_x^s R_v\partial_t\partial_x^s\overline{R_u} + \psi_u\partial_x^s\overline{R_u}\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u}\,\mathrm{d}x + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}\overline{R_u}\partial_t\partial_x^s R_v\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \psi_u\partial_x^s R_v\partial_x^s\overline{R_u}\,\mathrm{d}x - \int \partial_t\psi_u\partial_x^s R_v\partial_x^s\overline{R_u}\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u}\,\mathrm{d}x \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{s}\binom{s}{k}\int (\partial_x^k\psi_u\partial_x^{s-k}\overline{R_u}\partial_x^s R_v)\,\mathrm{d}x - \sum_{k=1}^{s}\binom{s}{k}\int \partial_t(\partial_x^k\psi_u\partial_x^{s-k}\overline{R_u})\partial_x^s R_v\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \partial_x^s(\psi_u\overline{R_u})\partial_x^s R_v\,\mathrm{d}x - \int \partial_t\psi_u\partial_x^s R_v\partial_x^s\overline{R_u}\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u}\,\mathrm{d}x - \sum_{k=1}^{s}\binom{s}{k}\int \partial_t(\partial_x^k\psi_u\partial_x^{s-k}\overline{R_u})\partial_x^s R_v\,\mathrm{d}x.
\end{aligned}$$

The third integral will be estimated subsequently. For the fourth integral on the right-hand side, we again apply the Leibniz rule and obtain

$$\begin{aligned}
&\int (\partial_x^s(R_vR_u)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(R_v\overline{R_u})\partial_t\partial_x^s R_u + \partial_x^s(|R_u|^2)\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&\quad = \int (R_u\partial_x^s R_v\partial_t\partial_x^s\overline{R_u} + \partial_x^s R_v\overline{R_u}\partial_t\partial_x^s R_u + \partial_x^s R_u\overline{R_u}\partial_t\partial_x^s R_v + \partial_x^s\overline{R_u}R_u\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
&\qquad + \sum_{k=1}^{s}\binom{s}{k}\int (\partial_x^k R_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u} + \partial_x^k\overline{R_u}\partial_x^{s-k}R_v\partial_t\partial_x^s R_u)\,\mathrm{d}x \\
&\qquad + \sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x^k R_u\partial_x^{s-k}\overline{R_u}\partial_t\partial_x^s R_v\,\mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int (R_u\partial_x^s R_v\partial_x^s\overline{R_u} + \partial_x^s R_v\overline{R_u}\partial_x^s R_u)\,\mathrm{d}x - \int (\partial_t R_u\partial_x^s R_v\partial_x^s\overline{R_u} + \partial_x^s R_v\partial_t\overline{R_u}\partial_x^s R_u)\,\mathrm{d}x \\
&\qquad + 2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k R_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u}\,\mathrm{d}x + \sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x^k R_u\partial_x^{s-k}\overline{R_u}\partial_t\partial_x^s R_v\,\mathrm{d}x.
\end{aligned}$$

Besides, we define the energy $\mathcal{E} = E + \widetilde{E}$, where

$$\begin{aligned}
\widetilde{E} &= \|\partial_x^{s+1}R_u\|_{L^2}^2 + \frac{1}{2}\|\partial_x^s R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\|\partial_x^{s-1}\partial_t R_v\|_{L^2}^2 \\
&\quad + \int (\partial_x^s(\psi_u\overline{R_u})\partial_x^s R_v + \partial_x^s(\overline{\psi_u}R_u)\partial_x^s R_v)\,\mathrm{d}x \\
&\quad + \varepsilon^2\int (R_u\partial_x^s R_v\partial_x^s\overline{R_u} + \partial_x^s R_v\overline{R_u}\partial_x^s R_u)\,\mathrm{d}x.
\end{aligned}$$

Then, for u_0 sufficiently small but $\mathcal{O}(1)$, $\mathcal{E}$ is equivalent to the $H^{s+1}\times H^s$-norm of (R_u, R_v). We remark that the term $\partial_t\partial_x^s R_u$ cannot be estimated directly by the energy $\mathcal{E}$ since the term $\partial_x^{s+2}R_u$ occurs after replacing the time derivative by the right-hand side of (2.10). However, via integration by parts, we can shift the derivatives away from these terms. Thus, the energy $\widetilde{E}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E} = \sum_{j=1}^{9} I_j,$$

where

$$|I_1| = 2\left|\mathrm{Re}\int \partial_t\psi_u\partial_x^s R_v\partial_x^s\overline{R_u}\,\mathrm{d}x\right| \le C\|\partial_t\psi_u\|_{L^\infty}\|R_v\|_{H^s}\|R_u\|_{H^s},$$

$$|I_2| = 2\left|\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x(\partial_x^k\psi_u\partial_x^{s-k}R_v)\partial_t\partial_x^{s-1}\overline{R_u}\,\mathrm{d}x\right| \le C\|\psi_u\|_{H^{s+1}}\|R_v\|_{H^s}\|\partial_t R_u\|_{H^{s-1}},$$

$$|I_3| = 2\left|\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_t(\partial_x^k\psi_u\partial_x^{s-k}\overline{R_u})\partial_x^s R_v\,\mathrm{d}x\right|$$
$$\le C(\|\partial_t\psi_u\|_{H^s}\|R_u\|_{H^{s-1}} + \|\psi_u\|_{H^s}\|\partial_t R_u\|_{H^{s-1}})\|R_v\|_{H^s},$$

$$|I_4| = 2\left|\mathrm{Re}\int \partial_x^{s+1}(\psi_v R_u)\partial_t\partial_x^{s-1}\overline{R_u}\,\mathrm{d}x\right| \le C\|\psi_u\|_{H^{s+1}}\|R_u\|_{H^{s+1}}\|\partial_t R_u\|_{H^{s-1}},$$

$$|I_5| = 2\varepsilon^2\left|\mathrm{Re}\int \partial_t R_u\partial_x^s R_v\partial_x^s\overline{R_u}\,\mathrm{d}x\right| \le C\varepsilon^2\|\partial_t R_u\|_{L^2}\|R_v\|_{H^s}\|R_u\|_{H^{s+1}},$$

$$|I_6| = 2\varepsilon^2\left|\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x(\partial_x^k R_u\partial_x^{s-k}R_v)\partial_t\partial_x^{s-1}\overline{R_u}\,\mathrm{d}x\right| \le C\varepsilon^2\|R_u\|_{H^{s+1}}\|R_v\|_{H^s}\|\partial_t R_u\|_{H^{s-1}},$$

$$|I_7| = \varepsilon^2\left|\sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x(\partial_x^k R_u\partial_x^{s-k}\overline{R_u})\partial_t\partial_x^{s-1}R_v\,\mathrm{d}x\right| \le C\varepsilon^2\|R_u\|_{H^{s+1}}\|R_u\|_{H^{s+1}}\|\partial_t\partial_x^{s-1}R_v\|_{L^2},$$

$$|I_8| = 2\left|\mathrm{Re}\int \partial_t\partial_x^{s-1}\overline{R_u}\varepsilon^{-2}\partial_x^{s+1}\mathrm{Res}_u\,\mathrm{d}x\right| \le C\varepsilon^{-2}\|\partial_t R_u\|_{H^{s-1}}\|\mathrm{Res}_u\|_{H^{s+1}},$$

$$|I_9| = \left|\int (\partial_x^{s-1}\partial_t R_v)\partial_x^{s-1}(\varepsilon^{-2}\mathrm{Res}_v)\,\mathrm{d}x\right| \le C\varepsilon^{-2}\|\partial_t\partial_x^{s-1}R_v\|_{L^2}\|\mathrm{Res}_v\|_{H^{s-1}}.$$

We note that $\mathcal{E}^{1/2} \le 1 + \mathcal{E}$. Since R_u satisfies (2.10), it follows that

$$\|\partial_t R_u\|_{H^{s-1}} \le C + C\mathcal{E}^{1/2} + C\varepsilon\mathcal{E}.$$

Using Lemma 2.1.7 and the calculations from the previous section, we can conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \le C\mathcal{E} + C$$

for $\varepsilon\mathcal{E}^{3/2} \le 1$. With Gronwall's inequality, we have $\mathcal{E}(t) \le M$ for all $t \in [0, T_0]$ for a constant $M = \mathcal{O}(1)$. Finally, the result follows from choosing $\varepsilon_0 > 0$ sufficiently small such that $\varepsilon_0 M^{3/2} \le 1$. □

Remark 2.1.13. For u_0 in a Sobolev space with sufficiently high regularity, the approximation rate can be significantly increased in both Theorem 2.1.1 and Theorem 2.1.12. In the following, we outline how to achieve this. Unlike in Section 2.1.4, for the error we make the ansatz

$$(u, v)(x, t) = (\psi_{u,n}, \psi_{v,n})(x, t) + \varepsilon^\beta(R_u, R_v)(x, t)$$

where $\beta \geq 2$ and $(\psi_{u,n}, \psi_{v,n})$ is the higher order approximation (2.7). Then, the error functions R_u and R_v satisfy

$$\begin{aligned} 2i\partial_t R_u &= \partial_x^2 R_u - \psi_u R_v - \psi_v R_u - \varepsilon^\beta R_u R_v + \varepsilon^{-\beta}\mathrm{Res}_u(\psi_{u,n}, \psi_{v,n}), \\ \varepsilon^2\partial_t^2 R_v &= \partial_x^2 R_v + \partial_x^2(\overline{\psi_u} R_u) + \partial_x^2(\psi_u \overline{R_u}) + \varepsilon^\beta \partial_x^2 |R_u|^2 + \varepsilon^{-\beta}\mathrm{Res}_v(\psi_{u,n}, \psi_{v,n}). \end{aligned}$$

The energy estimates are analogous to those in Section 2.1.4 and Section 2.1.5. In the time derivative of the energy $\mathcal{E}$, the term which loses most powers of ε is given by

$$\int (\partial_x^{s-1}\partial_t R_v)\partial_x^{s-1}(\varepsilon^{-\beta}\mathrm{Res}_v)\,\mathrm{d}x.$$

With Lemma 2.1.11, we can choose $\beta = 2n+1$ in order to obtain

$$\left|\int (\partial_x^{s-1}\partial_t R_v)\partial_x^{s-1}(\varepsilon^{-\beta}\mathrm{Res}_v)\,\mathrm{d}x\right| \leq C\varepsilon^{-\beta-1}\|\mathrm{Res}_v\|_{H^{s-1}}\mathcal{E}^{1/2} = C\mathcal{E}^{1/2}.$$

In total, we have the following theorem.

Theorem 2.1.14. *Let $n \in \mathbb{N}$ and $s \in \mathbb{N}_0$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0,T_0], H^{s+2n+5})$ be a solution of the NLS equation* (2.3) *with spatially 2π-periodic boundary conditions and*

$$\sup_{t\in[0,T_0]} \|u_0(\cdot,t)\|_{H^{s+2n+5}} = C_u < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u,v) of the Zakharov system (2.1)–(2.2) *with spatially 2π-periodic boundary conditions satisfying*

$$\sup_{t\in[0,T_0]} \|(u,v)(\cdot,t) - (\psi_{u,n}, \psi_{v,n})(\cdot,t)\|_{H^{s+1}\times H^s} \leq C\varepsilon^{2n+1}.$$

2.2 From KGZ to Zakharov

2.2.1 Introduction

In this section, we consider a KGZ system on the real line with a small parameter $\varepsilon > 0$ such that we obtain a Zakharov system in the limit $\varepsilon \to 0$. The proof of the corresponding approximation result is similar to the one in Section 2.1, where we have considered the limit from the Zakharov system to the NLS equation. However, in the limit from the KGZ to the Zakharov system, an additional difficulty is the elimination of oscillating terms by applying a normal form transform.

Remark 2.2.1. The KGZ system is a model from plasma physics which is used to describe the interaction between so-called Langmuir waves and ion sound waves in plasma. Here, $v(x,t)$ is proportional to the ion density fluctuation from a constant equilibrium density and $u(x,t)$ is proportional to the electric field. It is derived from a coupled system that consists of the Euler equation for the electrons and ions and the Maxwell equation for the electric field. For details, we refer to [Tex07].

We consider the KGZ system in the form

$$\varepsilon^2 \partial_t^2 u = \partial_x^2 u - \varepsilon^{-2} u - uv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2) \tag{2.12}$$

with $u(x,t) \in \mathbb{C}$, $v(x,t), x, t \in \mathbb{R}$, and $0 < \varepsilon \ll 1$. This corresponds to the spectral situation in Figure 2.2.

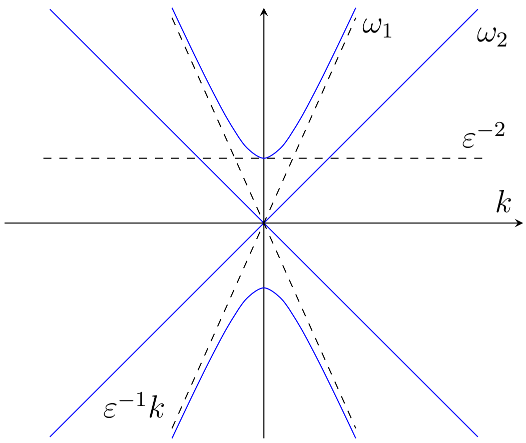


Figure 2.2: The spectral situation corresponding to the linearized KGZ system with $k \in \mathbb{R}$. It is solved by $u(x,t) = e^{ikx+i\omega_{\pm1}(k)t}$ and $v(x,t) = e^{ikx+i\omega_{\pm2}(k)t}$ where $\omega_{\pm1}(k) = \pm\varepsilon^{-2}\sqrt{1+(\varepsilon k)^2}$ and $\omega_{\pm2}(k) = \pm k$. We emphasize that ω_1 asymptotically scales like $\varepsilon^{-1}|k|$.

In the singular limit $\varepsilon \to 0$ with the ansatz

$$u(x,t) = \Psi_u(x,t) = \psi_u(x,t)e^{i\varepsilon^{-2}t}, \qquad v(x,t) = \psi_v(x,t), \tag{2.13}$$

the Zakharov system

$$2i\partial_t\psi_u = \partial_x^2\psi_u - \psi_u\psi_v, \qquad \partial_t^2\psi_v = \partial_x^2\psi_v + \partial_x^2(|\psi_u|^2) \tag{2.14}$$

can be derived from the KGZ system (2.12). Our goal is to prove that the Zakharov system (2.14) correctly predicts the dynamics of the KGZ system (2.12) for small values of $\varepsilon > 0$. Specifically, we have the following approximation result.

Theorem 2.2.2. *Let $s \in \mathbb{N}$. There is a $C_{max} > 0$ such that for all $C_u, C_v \in [0, C_{max})$ the following holds. Let $(\psi_u, \psi_v) \in C([0,T_0], H^{s+5} \times H^{s+4})$ be a solution of the Zakharov system (2.14) with*

$$\sup_{t\in[0,T_0]} \|\psi_u\|_{H^{s+5}} =: C_u < \infty, \qquad \sup_{t\in[0,T_0]} \|\psi_v\|_{H^{s+4}} =: C_v < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of the KGZ system (2.12) satisfying

$$\sup_{t\in[0,T_0]} \|(u,v) - (\psi_u e^{i\varepsilon^{-2}t}, \psi_v)\|_{H^{s+1}\times H^s} \leq C\varepsilon^2.$$

Remark 2.2.3. The KGZ system (2.12) can be written as a semilinear evolutionary system. The local existence and uniqueness of solutions $(u, v) \in H^{s+1} \times H^s$, $s \geq 1$, of the KGZ system follows with a standard fixed point argument applied to the variation of constants formula, cf. [DSS16].

Remark 2.2.4. The Zakharov system can be written as a semilinear evolutionary system [Sch19] for which we have local existence and uniqueness in $H^{s-1} \times H^s \times H^s$, $s \geq 1$, using semigroup theory [Paz83]. Reverting to the original variables of the Zakharov system (2.14), we then have local existence and uniqueness for $(\psi_u, \psi_v, \partial_t\psi_v) \in H^{s+1} \times H^s \times H^{s-1}$, cf. [OT92].

Remark 2.2.5. The Zakharov system can be used for a robust numerical description of the KGZ system (2.12) for small values of ε. Resolving the highly oscillatory behavior of the solutions in this regime is numerically very delicate. Severe time step restrictions have to be imposed, which results in high computational costs. However, this can be avoided by passing to the regular limit system, cf. [BSS20]. This also applies to all other limits that are considered in Section 2.

Remark 2.2.6. This approximation question has been addressed in a number of papers, cf. [BBC96, CEGT04, Sch19]. In [Tex07], the Zakharov approximation has been justified for the original Euler-Maxwell system.

Remark 2.2.7. For the KGZ system, many different singular limits have been considered and a number of approximation results have been established in the literature, cf. [BBC96, MN02, MN05, MN08, MN10]. In particular, the same ansatz (2.13) has been considered in [BBC96]. However, the focus of [BBC96] is on convergence and not on error bounds.

Remark 2.2.8. The proof of the approximation theorem given in this section only holds, if the nonlinear part on the right-hand side of the u-equation of (2.12) has a negative sign. In [Sch19], the Zakharov approximation of the KGZ system is justified under the assumption that the solutions of the Zakharov system are analytic in a strip in the complex plane. There, the proof holds in both cases.

Notation. We use the notation from Chapter 1. We write $\int$ for $\int_{\mathbb{R}}$ and H^s for $H^s(\mathbb{R}, \mathbb{K})$, unless otherwise specified.

2.2.2 Estimates for the residual

First, we want to estimate the residual that contains all terms which do not cancel after inserting the approximation into the original system (2.12). Inserting the ansatz (2.13) into the u- and v-equation yields

$$\varepsilon^2 \partial_t^2 \psi_u + 2i\partial_t \psi_u = \partial_x^2 \psi_u - \psi_u \psi_v, \qquad \partial_t^2 \psi_v = \partial_x^2 \psi_v + \partial_x^2(|\psi_u|^2).$$

We choose (ψ_u, ψ_v) to satisfy the Zakharov system (2.14). The remaining terms are collected in the residual, namely,

$$\mathrm{Res}_u(\Psi_u, \psi_v) = -\varepsilon^2 \partial_t^2 \psi_u, \quad \mathrm{Res}_v(\Psi_u, \psi_v) = 0.$$

In order to estimate the residual, we substitute $\varepsilon^2 \partial_t^2 \psi_u$ by the right-hand side of the first equation of (2.14). More precisely, we have

$$\begin{aligned}
\|\partial_t^2 \psi_u\|_{H^{s+1}} &= C\|\partial_t \left(\partial_x^2 \psi_u - \psi_u \psi_v\right)\|_{H^{s+1}} \\
&= C\|\partial_x^2(\partial_t \psi_u) - \partial_t \psi_u \psi_v - \psi_u \partial_t \psi_v\|_{H^{s+1}} \\
&\le C\|\partial_x^2(\partial_x^2 \psi_u - \psi_u \psi_v)\|_{H^{s+1}} + C\|(\partial_x^2 \psi_u - \psi_u \psi_v)\psi_v\|_{H^{s+1}} \\
&\quad + C\|\psi_u \partial_t \psi_v\|_{H^{s+1}} \\
&\le C\|\psi_u\|_{H^{s+5}} + C\|\psi_u\|_{H^{s+3}}\|\psi_v\|_{H^{s+3}} \\
&\qquad + C\|\psi_v\|_{H^{s+1}}(\|\psi_u\|_{H^{s+3}} + \|\psi_u\|_{H^{s+1}}\|\psi_v\|_{H^{s+1}}) \\
&\qquad + C\|\psi_u\|_{H^{s+1}}\|\partial_t \psi_v\|_{H^{s+1}}.
\end{aligned}$$

Here, we can bound $\partial_t \psi_v$ in H^{s+1} since $\partial_t \psi_v \in H^{s+3}$ as a consequence of Remark 2.2.4 and the assumptions on (ψ_u, ψ_v) given in Theorem 2.2.2. Thus, we have shown the following lemma.

Lemma 2.2.9. *Let $s \geq 0$ and let $(\psi_u, \psi_v) \in C([0,T_0], H^{s+5} \times H^{s+4})$ be a solution of the Zakharov system* (2.14)*. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\Psi_u, \psi_v)\|_{H^{s+1}} \leq C_{res}\varepsilon^2, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\Psi_u, \psi_v)\|_{H^s} = 0.$$

2.2.3 Higher order approximation

We can make the residual arbitrarily small by constructing a higher order approximation. Consider the system

$$\varepsilon^2\partial_t^2 w + 2i\partial_t w = \partial_x^2 w - wv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|w|^2).$$

For $n \in \mathbb{N}$, we make the improved ansatz

$$w = \psi_{u,n} = \sum_{k=0}^{n} \varepsilon^{2k} w_{2k}, \quad v = \psi_{v,n} = \sum_{k=0}^{n} \varepsilon^{2k} v_{2k}. \tag{2.15}$$

Then, $(w_0, v_0) = (\psi_u, \psi_v)$ solves the Zakharov system (2.14) and (w_{2k}, v_{2k}), $k \in \{1, \dots, n\}$, solve linear inhomogeneous Zakharov systems of the form

$$\begin{aligned} 2i\partial_t w_{2k} &= \partial_x^2 w_{2k} - w_{2k}v_0 - w_0 v_{2k} - \partial_t^2 w_{2(k-1)} \\ &\qquad - F_{2k}(w_0, \dots, w_{2(k-1)}, v_0, \dots, v_{2(k-1)}), \\ \partial_t^2 v_{2k} &= \partial_x^2 v_{2k} + \partial_x^2(w_{2k}\overline{w_0} + \overline{w_{2k}}w_0) + \partial_x^2(G_{2k}(w_0, \dots, w_{2(k-1)})), \end{aligned} \tag{2.16}$$

where F_{2k}, G_{2k} are quadratic mappings. Hence, all terms up to order $\mathcal{O}(\varepsilon^{2n})$ cancel and only terms of at least order $\mathcal{O}(\varepsilon^{2n+2})$ remain. The term which contains the most derivatives in both Res_u and Res_v is $\partial_t^2 w_{2n}$. We replace the time derivative with the right-hand side of the first equation of (2.16). Then, the term $\partial_t^3 w_{2(k-1)}$ appears. After repeating this process n times, the term $\partial_t^{n+2} w_0$ appears. Further, we repeatedly replace the time derivatives of w_0 with the right-hand side of the first equation of the Zakharov system (2.14). Then, the terms $\partial_x^{2n+4} w_0$ and $\partial_x^{2n+2} v_0$ appear. Therefore, in order to estimate Res_u in H^{s+1}, we have to assume that $w_0 \in H^{s+2n+5}$ and $v_0 \in H^{s+2n+3}$. The estimates for Res_v in H^s are straightforward. We note that, by the choice (2.16), each term in Res_v has a spatial derivative in front. Thus, we can conclude the following lemma.

Lemma 2.2.10. *Let $n \in \mathbb{N}$ and $s \geq 0$. Further, let $(\psi_u, \psi_v) \in C([0,T_0], H^{s+2n+5} \times H^{s+2n+3})$ be a solution of the Zakharov system* (2.14)*. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an approximation $(\Psi_{u,n}, \psi_{v,n})$, where $\Psi_{u,n} = \psi_{u,n}e^{i\varepsilon^{-2}t}$ and where $(\psi_{u,n}, \psi_{v,n})$ is of the form* (2.15)*, with*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\Psi_{u,n}, \psi_{v,n})\|_{H^{s+1}} \leq C_{res}\varepsilon^{2n+2}, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\Psi_{u,n}, \psi_{v,n})\|_{H^s} \leq C_{res}\varepsilon^{2n+2},$$

and

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\Psi_{u,n},\psi_{v,n})\|_{H^s} \leq C_{res}\varepsilon^{2n+2}.$$

2.2.4 Error equation and normal form transform

We write the KGZ system (2.12) as

$$\partial_t^2 u = -\omega_1^2 u - \varepsilon^{-2}uv, \qquad \partial_t^2 v = -\omega_2^2 v - \omega_2^2(|u|^2),$$

where, in Fourier space,

$$\omega_1^2(k) = \varepsilon^{-2}(k^2+\varepsilon^{-2}) = \varepsilon^{-4}(1+(\varepsilon k)^2), \qquad \omega_2^2(k) = k^2.$$

We define the error $\varepsilon^2(R_u,R_v)$ of the approximation (Ψ_u,ψ_v) by

$$(u,v)(x,t) = (\Psi_u,\psi_v)(x,t) + \varepsilon^2(R_u,R_v)(x,t). \tag{2.17}$$

The error functions R_u and R_v satisfy

$$\begin{aligned}
\partial_t^2 R_u &= -\omega_1^2 R_u - \varepsilon^{-2}(\Psi_u R_v + \psi_v R_u + \varepsilon^2 R_u R_v) + \varepsilon^{-4}\mathrm{Res}_u,\\
\partial_t^2 R_v &= -\omega_2^2 R_v - \omega_2^2(\overline{\Psi}_u R_u + \Psi_u \overline{R}_u + \varepsilon^2|R_u|^2) + \varepsilon^{-2}\mathrm{Res}_v.
\end{aligned}$$

We rewrite this system as the first order system

$$\begin{aligned}
\partial_t R_u &= i\omega_1 \widetilde{R}_u,\\
\partial_t \widetilde{R}_u &= i\omega_1 R_u - \varepsilon^{-2}(i\omega_1)^{-1}(\Psi_u R_v + \psi_v R_u + \varepsilon^2 R_u R_v) + \varepsilon^{-4}(i\omega_1)^{-1}\mathrm{Res}_u,\\
\partial_t R_v &= i\omega_2 \widetilde{R}_v,\\
\partial_t \widetilde{R}_v &= i\omega_2 R_v + i\omega_2(\overline{\Psi}_u R_u + \Psi_u \overline{R}_u + \varepsilon^2|R_u|^2) + \varepsilon^{-2}(i\omega_2)^{-1}\mathrm{Res}_v.
\end{aligned}$$

By introducing

$$\begin{aligned}
&R_u = R_1 + R_{-1},\ \widetilde{R}_u = R_1 - R_{-1} \quad \text{resp.} \quad 2R_1 = R_u + \widetilde{R}_u,\ 2R_{-1} = R_u - \widetilde{R}_u,\\
&R_v = R_2 + R_{-2},\ \widetilde{R}_v = R_2 - R_{-2} \quad \text{resp.} \quad 2R_2 = R_v + \widetilde{R}_v,\ 2R_{-2} = R_v - \widetilde{R}_v,
\end{aligned}$$

we diagonalize this system and find for $R_{\pm 1}$ and $R_{\pm 2}$

$$\begin{aligned}
\partial_t R_{\pm 1} &= \pm i\omega_1 R_{\pm 1} \mp \varepsilon^{-2}(2i\omega_1)^{-1}\big(\Psi_u(R_2+R_{-2}) + \psi_v(R_1+R_{-1})\\
&\qquad + \varepsilon^2(R_1+R_{-1})(R_2+R_{-2})\big) \pm \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u,\\
\partial_t R_{\pm 2} &= \pm i\omega_2 R_{\pm 2} \pm \frac{1}{2} i\omega_2\big(\overline{\Psi}_u(R_1+R_{-1}) + \Psi_u(\overline{R}_1+\overline{R}_{-1})\\
&\qquad + \varepsilon^2|R_1+R_{-1}|^2\big) \pm \varepsilon^{-2}(2i\omega_2)^{-1}\mathrm{Res}_v.
\end{aligned} \tag{2.18}$$

In order to estimate the solutions of this system, we split the right-hand side into terms which can be handled by energy estimates, collected in $\mathcal{B}_1$, terms which can be handled by normal form transformations, collected in $\mathcal{B}_2$, and terms with sufficiently high order in ε, collected in $\mathcal{G}$. Thus, we can write the system as

$$\partial_t \mathcal{R} = \Lambda\mathcal{R} + \mathcal{B}_1(\Psi,\mathcal{R}) + \mathcal{B}_2(\Psi,\mathcal{R}) + \mathcal{G}(\Psi,\mathcal{R}) + \varepsilon^{-4}\mathrm{RES}(\Psi),$$

where

$$\mathcal{R} = (R_1, R_{-1}, R_2, R_{-2})^T, \quad \Lambda = \mathrm{diag}(i\omega_1, -i\omega_1, i\omega_2, -i\omega_2),$$
$$\Psi = (\psi_1, \psi_{-1}, \psi_2, \psi_{-2})^T = (\Psi_u, \overline{\Psi}_u, \psi_v, -\psi_v)^T,$$

and

$$\mathcal{B}_1(\Psi,\mathcal{R}) = \begin{pmatrix} -\varepsilon^{-2}(2i\omega_1)^{-1}(\Psi_u(R_2+R_{-2}) + \psi_v R_1 + \varepsilon^2 R_1(R_2+R_{-2})) \\ \varepsilon^{-2}(2i\omega_1)^{-1}(\psi_v R_{-1} + \varepsilon^2 R_{-1}(R_2+R_{-2})) \\ \frac{1}{2}i\omega_2(\overline{\Psi}_u R_1 + \Psi_u \overline{R}_1 + \varepsilon^2(|R_1|^2 + |R_{-1}|^2)) \\ -\frac{1}{2}i\omega_2(\overline{\Psi}_u R_1 + \Psi_u \overline{R}_1 + \varepsilon^2(|R_1|^2 + |R_{-1}|^2)) \end{pmatrix},$$

$$\mathcal{B}_2(\Psi,\mathcal{R}) = \begin{pmatrix} -\varepsilon^{-2}(2i\omega_1)^{-1}(\psi_v R_{-1}) \\ \varepsilon^{-2}(2i\omega_1)^{-1}(\Psi_u(R_2+R_{-2}) + \psi_v R_1) \\ \frac{1}{2}i\omega_2(\overline{\Psi}_u R_{-1} + \Psi_u \overline{R}_{-1}) \\ -\frac{1}{2}i\omega_2(\overline{\Psi}_u R_{-1} + \Psi_u \overline{R}_{-1}) \end{pmatrix},$$

$$\begin{aligned}\mathcal{G}(\Psi,\mathcal{R}) &= (g_1(\Psi,\mathcal{R}), g_{-1}(\Psi,\mathcal{R}), g_2(\Psi,\mathcal{R}), g_{-2}(\Psi,\mathcal{R}))^T \\ &= \begin{pmatrix} -(2i\omega_1)^{-1}(R_{-1}(R_2+R_{-2})) \\ (2i\omega_1)^{-1}(R_1(R_2+R_{-2})) \\ \frac{1}{2}\varepsilon^2 i\omega_2(R_1\overline{R_{-1}} + \overline{R_1}R_{-1}) \\ -\frac{1}{2}\varepsilon^2 i\omega_2(R_1\overline{R_{-1}} + \overline{R_1}R_{-1}) \end{pmatrix},\end{aligned}$$

$$\mathrm{RES}(\Psi) = \begin{pmatrix} (2i\omega_1)^{-1}\mathrm{Res}_u(\Psi_u,\psi_v) \\ -(2i\omega_1)^{-1}\mathrm{Res}_u(\Psi_u,\psi_v) \\ \varepsilon^2(2i\omega_2)^{-1}\mathrm{Res}_v(\Psi_u,\psi_v) \\ -\varepsilon^2(2i\omega_2)^{-1}\mathrm{Res}_v(\Psi_u,\psi_v) \end{pmatrix}.$$

(i) $\mathcal{G}(\Psi,\mathcal{R})$ contains all terms which provide high enough orders w.r.t. ε and cause no difficulties in arriving at the $\mathcal{O}(1)$ time scale. More precisely, by using

$$\varepsilon^2\omega_1(k) = \sqrt{1+(\varepsilon k)^2} \geq 1 \qquad \text{for all } k \in \mathbb{R},$$

and assuming $(\psi_u, \psi_v) \in H^{s+5} \times H^{s+4}$, we have

$$\begin{aligned} \|g_1\|_{H^s} &\leq \|\varepsilon^2\omega_1 g_1\|_{H^s} \leq C\varepsilon^2 \|R_{-1}\|_{H^{s+1}} \|R_2 + R_{-2}\|_{H^s}, \\ \|g_{-1}\|_{H^s} &\leq \|\varepsilon^2\omega_1 g_{-1}\|_{H^s} \leq C\varepsilon^2 \|R_1\|_{H^{s+1}} \|R_2 + R_{-2}\|_{H^s}, \\ \|g_2\|_{H^s} &\leq C\varepsilon^2 \|R_1\|_{H^{s+1}} \|R_{-1}\|_{H^{s+1}}, \\ \|g_{-2}\|_{H^s} &\leq C\varepsilon^2 \|R_1\|_{H^{s+1}} \|R_{-1}\|_{H^{s+1}}. \end{aligned} \tag{2.19}$$

(ii) In order to get rid of the $\mathcal{O}(\varepsilon^{-2})$ terms in $\mathcal{B}_2$, we use normal form transformations. We aim to eliminate $\mathcal{B}_2$ with the near identity change of coordinates

$$\widetilde{\mathcal{R}} = \mathcal{R} + \mathcal{Q}(\Psi, \mathcal{R}),$$

where $\widetilde{\mathcal{R}} = (\widetilde{R}_1, \widetilde{R}_{-1}, \widetilde{R}_2, \widetilde{R}_{-2})^T$. Here, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_{-1}, \mathcal{Q}_2, \mathcal{Q}_{-2})^T$ consists of bilinear mappings $\mathcal{Q}_j$, $j \in \{\pm 1, \pm 2\}$, which, in Fourier space, have the form

$$\widehat{\mathcal{Q}}_j(\widehat{\Psi}, \widehat{\mathcal{R}}) = \sum_{j_1, j_2 \in \{\pm 1, \pm 2\}} \widehat{\mathcal{Q}}_{j,j_1,j_2}(\widehat{\psi}_{j_1}, \widehat{R}_{j_2})$$

with

$$\widehat{\mathcal{Q}}_{j,j_1,j_2}(\widehat{\psi}_{j_1}, \widehat{R}_{j_2}) = \int q_{j,j_1,j_2}(k) \widehat{\psi}_{j_1}(k-l) \widehat{R}_{j_2}(l) \, \mathrm{d}l, \quad j \in \{\pm 1\},$$

and

$$\begin{aligned} \widehat{\mathcal{Q}}_{j,j_1,j_2}(\widehat{\psi}_{j_1}, \widehat{R}_{j_2}) = & \int q_{j,j_1,j_2}(k) \overline{\widehat{\psi}_{j_1}}(k-l) \widehat{R}_{j_2}(l) \, \mathrm{d}l \\ & + \int q'_{j,j_1,j_2}(k) \widehat{\psi}_{j_1}(k-l) \overline{\widehat{R}_{j_2}}(l) \, \mathrm{d}l, \quad j \in \{\pm 2\}. \end{aligned}$$

We also write

$$\mathcal{Q}(\Psi, \mathcal{R}) = \sum_{j_1, j_2 \in \{\pm 1, \pm 2\}} \mathcal{Q}_{j_1,j_2}(\psi_{j_1}, R_{j_2}),$$

where $\mathcal{Q}_{j,j_1,j_2}$ are the components of $\mathcal{Q}_{j_1,j_2}$. Using the bilinearity of $\mathcal{Q}$, we obtain

$$\begin{aligned} \partial_t \widetilde{\mathcal{R}} &= \partial_t \mathcal{R} + \mathcal{Q}(\partial_t \Psi, \mathcal{R}) + \mathcal{Q}(\Psi, \partial_t \mathcal{R}) \\ &= \Lambda \widetilde{\mathcal{R}} - \underline{\Lambda \mathcal{Q}(\Psi, \mathcal{R})} + \underline{\mathcal{Q}(\Lambda(0)\Psi, \mathcal{R})} + \underline{\sum_{j_1,j_2} i\omega_{j_2} \mathcal{Q}_{j_1,j_2}(\psi_{j_1}, R_{j_2})} \\ &\quad + \mathcal{B}_1(\Psi, \mathcal{R}) + \underline{\mathcal{B}_2(\Psi, \mathcal{R})} + \mathcal{G}(\Psi, \mathcal{R}) + \varepsilon^{-4} \mathrm{RES}(\Psi) \\ &\quad + \mathcal{Q}(\partial_t \Psi - \Lambda(0)\Psi, \mathcal{R}) + \mathcal{Q}(\Psi, \partial_t \mathcal{R} - \Lambda \mathcal{R}) \\ &\quad + \mathcal{Q}(\Psi, \Lambda \mathcal{R}) - \sum_{j_1,j_2} i\omega_{j_2} \mathcal{Q}_{j_1,j_2}(\psi_{j_1}, R_{j_2}). \end{aligned} \tag{2.20}$$

In order to eliminate the problematic $\mathcal{B}_2$-terms, we want to choose $\mathcal{Q}$ in such a way that the underlined terms cancel. Hence, $\mathcal{Q}$ has to satisfy

$$\begin{aligned}\mathcal{B}_2(\Psi,\mathcal{R}) &= \Lambda\mathcal{Q}(\Psi,\mathcal{R}) - \mathcal{Q}(\Lambda(0)\Psi,\mathcal{R}) - \sum_{j_1,j_2} i\omega_{j_2}\mathcal{Q}_{j_1,j_2}(\psi_{j_1},R_{j_2})\\ &= \sum_{j_1,j_2}\Lambda\mathcal{Q}_{j_1,j_2}(\psi_{j_1},R_{j_2}) - \sum_{j_1,j_2} i\omega_{j_1}(0)\mathcal{Q}_{j_1,j_2}(\psi_{j_1},R_{j_2})\\ &\quad - \sum_{j_1,j_2} i\omega_{j_2}\mathcal{Q}_{j_1,j_2}(\psi_{j_1},R_{j_2})\\ &= \sum_{j_1,j_2}(\Lambda - i\omega_{j_1}(0)I - i\omega_{j_2}I)\mathcal{Q}_{j_1,j_2}(\psi_{j_1},R_{j_2}),\end{aligned}$$

where I denotes the unit matrix. In Fourier space, this leads to

$$\begin{aligned} q_{\pm1,2,\mp1}(k) &= \varepsilon^{-2}\frac{1}{2\omega_1(k)}(\omega_1(k) \mp \omega_2(0) + \omega_1(k))^{-1},\\ q_{-1,1,\pm2}(k) &= \varepsilon^{-2}\frac{1}{2\omega_1(k)}(\omega_1(k) + \omega_1(0) \pm \omega_2(k))^{-1},\\ q_{2,1,-1}(k) = q'_{-2,1,-1}(k) &= \frac{1}{2}\omega_2(k)(\omega_2(k) + \omega_1(0) + \omega_1(k))^{-1},\\ q_{-2,1,-1}(k) = q'_{2,1,-1}(k) &= \frac{1}{2}\omega_2(k)(\omega_2(k) - \omega_1(0) - \omega_1(k))^{-1}.\end{aligned}$$

The remaining coefficients are set to 0. In order to show the boundedness of the mapping $\mathcal{Q}$, we need the following lemma.

Lemma 2.2.11. *Let $s \geq 0$ and let $h \in (H^s(\mathbb{R},\mathbb{C}))^4$. The mappings $R \mapsto \mathcal{Q}_{\pm1}(h,R)$ are continuous from $(H^s(\mathbb{R},\mathbb{C}))^4$ into $H^{s+1}(\mathbb{R},\mathbb{C})$ and the mappings $R \mapsto \mathcal{Q}_{\pm2}(h,R)$ are continuous from $(H^s(\mathbb{R},\mathbb{C}))^4$ into $H^s(\mathbb{R},\mathbb{C})$. In particular, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\begin{aligned}\|\mathcal{Q}_{\pm1}(h,\mathcal{R})\|_{H^s} &\leq C\varepsilon^2\|h\|_{(H^s)^4}\|\mathcal{R}\|_{(H^s)^4},\\ \|\mathcal{Q}_{\pm1}(h,\mathcal{R})\|_{H^{s+1}} &\leq C\varepsilon\|h\|_{(H^s)^4}\|\mathcal{R}\|_{(H^s)^4},\\ \|\mathcal{Q}_{\pm2}(h,\mathcal{R})\|_{H^s} &\leq C\varepsilon\|h\|_{(H^s)^4}\|\mathcal{R}\|_{(H^s)^4}.\end{aligned} \tag{2.21}$$

Proof. Let $k \in \mathbb{R}$. We recall that $\omega_1(k) = \varepsilon^{-2}\sqrt{1+(\varepsilon k)^2}$ and $\omega_2(k) = k$. In order to prove (2.21), we have to bound the functions $(1+|\cdot|)q_{\pm1,j_1,j_2}$ and $q_{\pm2,j_1,j_2}$ for

$j_1, j_2 \in \{\pm1, \pm2\}$. By using $\sqrt{1+(\varepsilon k)^2} \geq \varepsilon|k| \geq \pm\varepsilon^2 k$, we obtain

$$
\begin{aligned}
|q_{\pm1,2,\mp1}(k)| &= \frac{1}{2\sqrt{1+(\varepsilon k)^2}} \frac{\varepsilon^2}{\sqrt{1+(\varepsilon k)^2} \mp 0 + \sqrt{1+(\varepsilon k)^2}} = \frac{1}{4}\frac{\varepsilon^2}{1+(\varepsilon k)^2} \leq \frac{1}{4}\varepsilon^2, \\
|k||q_{\pm1,2,\mp1}(k)| &= \frac{1}{4}\varepsilon^2 \frac{|k|}{1+(\varepsilon k)^2} = \frac{1}{4}\varepsilon \frac{\varepsilon|k|}{1+(\varepsilon k)^2} \leq \frac{1}{4}\varepsilon, \\
|q_{-1,1,\pm2}(k)| &= \frac{1}{2\sqrt{1+(\varepsilon k)^2}} \frac{\varepsilon^2}{\sqrt{1+(\varepsilon k)^2} + 1 \pm \varepsilon^2 k} \leq \frac{1}{2}\varepsilon^2, \\
|k||q_{-1,1,\pm2}(k)| &= \frac{1}{2}\varepsilon \frac{\varepsilon|k|}{\sqrt{1+(\varepsilon k)^2}} \frac{1}{\sqrt{1+(\varepsilon k)^2} + 1 \pm \varepsilon^2 k} \leq \frac{1}{2}\varepsilon.
\end{aligned}
$$

Thus, it directly follows that

$$
\sup_{j_1,j_2\in\{\pm1,\pm2\},\, k\in\mathbb{R}} |q_{\pm1,j_1,j_2}(k)| = \mathcal{O}(\varepsilon^2), \qquad \sup_{j_1,j_2\in\{\pm1,\pm2\},\, k\in\mathbb{R}} |(1+|k|)q_{\pm1,j_1,j_2}(k)| = \mathcal{O}(\varepsilon).
$$

Further, the function

$$
q_{2,1,-1}(k) = \frac{1}{2}\frac{k}{k+\varepsilon^{-2}+\varepsilon^{-2}\sqrt{1+(\varepsilon k)^2}} = \frac{1}{2}\varepsilon^2 \frac{k}{\sqrt{1+(\varepsilon k)^2}+1+\varepsilon^2 k}
$$

is strictly increasing as

$$
\frac{\mathrm{d}}{\mathrm{d}k} q_{2,1,-1}(k) = \frac{1}{2}\varepsilon^2 \frac{1}{\sqrt{1+(\varepsilon k)^2}} \frac{1+\sqrt{1+(\varepsilon k)^2}}{(\sqrt{1+(\varepsilon k)^2}+1+\varepsilon^2 k)^2} > 0.
$$

Thus, $q_{2,1,-1}$ is $\mathcal{O}(\varepsilon)$ bounded since

$$
\begin{aligned}
\lim_{k\to\infty} q_{2,1,-1}(k) &= \frac{1}{2}\varepsilon^2 \lim_{k\to\infty} \frac{1}{\sqrt{k^{-2}+\varepsilon^2}+k^{-1}+\varepsilon^2} = \frac{\varepsilon}{2(1+\varepsilon)} = \mathcal{O}(\varepsilon), \\
\lim_{k\to-\infty} q_{2,1,-1}(k) &= \frac{1}{2}\varepsilon^2 \lim_{k\to\infty} \frac{-1}{\sqrt{k^{-2}+\varepsilon^2}+k^{-1}-\varepsilon^2} = \frac{\varepsilon}{2(\varepsilon-1)} = \mathcal{O}(\varepsilon).
\end{aligned}
$$

By analogous calculations, the remaining function $q_{-2,1,-1}$ is strictly decreasing and $\mathcal{O}(\varepsilon)$ bounded. Thus, we get

$$
\sup_{j_1,j_2\in\{\pm1,\pm2\},\, k\in\mathbb{R}} |q_{\pm2,j_1,j_2}(k)| = \mathcal{O}(\varepsilon).
$$

The assertion now follows from the multiplicativity of the H^s-norm. □

With Lemma 2.2.11, we conclude that $\widetilde{\mathcal{R}}$ is a small perturbation of $\mathcal{R}$. In particular, we have that

$$
\begin{aligned}
\|\widetilde{R}_{\pm2} - R_{\pm2}\|_{H^s} &\leq C\varepsilon\|\mathcal{R}\|_{(H^s)^4}, \\
\|\widetilde{R}_{\pm1} - R_{\pm1}\|_{H^{s+1}} &\leq C\varepsilon\|\mathcal{R}\|_{(H^s)^4}.
\end{aligned}
\tag{2.22}
$$

Thus, the normal form transformation is invertible for $\varepsilon > 0$ sufficiently small. After the elimination, according to the calculations in (2.20), we are left with

$$\partial_t \widetilde{\mathcal{R}} = \Lambda \widetilde{\mathcal{R}} + \mathcal{B}_1(\Psi, \widetilde{\mathcal{R}}) + \mathcal{H}(\Psi, \mathcal{R}) + \varepsilon^{-4}\mathrm{RES}(\Psi),$$

where

$$\begin{aligned} \mathcal{H}(\Psi, \mathcal{R}) &= (h_1(\Psi, \mathcal{R}), h_{-1}(\Psi, \mathcal{R}), h_2(\Psi, \mathcal{R}), h_{-2}(\Psi, \mathcal{R}))^T \\ &= s_1 + s_2 + s_3 + s_4 + s_5 \end{aligned} \tag{2.23}$$

with

$$\begin{aligned} s_1 &= \mathcal{Q}(\partial_t \Psi - \Lambda(0)\Psi, \mathcal{R}), \\ s_2 &= \mathcal{Q}(\Psi, \partial_t \mathcal{R} - \Lambda \mathcal{R}), \\ s_3 &= \mathcal{B}_1(\Psi, \mathcal{R}) - \mathcal{B}_1(\Psi, \widetilde{\mathcal{R}}), \\ s_4 &= \mathcal{G}(\Psi, \mathcal{R}), \\ s_5 &= \mathcal{Q}(\Psi, \Lambda \mathcal{R}) - \sum_{j_1, j_2} i\omega_{j_2} \mathcal{Q}_{j_1, j_2}(\psi_{j_1}, R_{j_2}). \end{aligned}$$

With the help of Lemma 2.2.11, we can bound $\mathcal{H}(\Psi, \mathcal{R})$ in $(H^s)^4$ given the assumption that $(\psi_u, \psi_v) \in H^{s+5} \times H^{s+4}$ and $(R_1, R_{-1}, R_2, R_{-2}) \in H^{s+1} \times H^{s+1} \times H^s \times H^s$.

Bound for s_1: Via (2.21), we get

$$\|s_1\|_{(H^s)^4} = \|\mathcal{Q}(\partial_t \Psi - \Lambda(0)\Psi, \mathcal{R})\|_{(H^s)^4} \leq C\varepsilon \|\partial_t \Psi - \Lambda(0)\Psi\|_{(H^s)^4} \|\mathcal{R}\|_{(H^s)^4}.$$

It is to note that $\omega_1(0) = \varepsilon^{-2}$ and $\omega_2(0) = 0$. Therefore, we estimate

$$\begin{aligned} \|\partial_t \Psi - \Lambda(0)\Psi\|_{(H^s)^4} &\leq C\|\partial_t(\psi_u e^{i\varepsilon^{-2}t}) - i\omega_1(0)\psi_u e^{i\varepsilon^{-2}t}\|_{H^s} + C\|\partial_t \psi_v - i\omega_2(0)\psi_v\|_{H^s} \\ &= C\|\partial_t \psi_u\|_{H^s} + C\|\partial_t \psi_v\|_{H^s}. \end{aligned}$$

By exploiting that (ψ_u, ψ_v) solves the Zakharov system (2.14), we obtain

$$\|\partial_t \psi_u\|_{H^s} \leq C\|\partial_x^2 \psi_u\|_{H^s} + C\|\psi_u \psi_v\|_{H^s} \leq C\|\psi_u\|_{H^{s+2}} + C\|\psi_v\|_{H^s}\|\psi_u\|_{H^s}.$$

With Remark 2.2.4, we have $\|\partial_t \psi_v\|_{H^s} \leq C$. Thus, we conclude

$$\|s_1\|_{(H^s)^4} \leq C\varepsilon \|\mathcal{R}\|_{(H^s)^4}.$$

Bound for s_2: With (2.21), we first get

$$\|s_2\|_{(H^s)^4} \leq C\varepsilon \|\partial_t \mathcal{R} - \Lambda \mathcal{R}\|_{(H^s)^4}.$$

Further, we estimate

$$\|\partial_t \mathcal{R} - \Lambda \mathcal{R}\|_{(H^s)^4} \leq C(\|\partial_t R_1 - i\omega_1 R_1\|_{H^s} + \|\partial_t R_2 - i\omega_2 R_2\|_{H^s})$$

by replacing the time derivatives $\partial_t R_1$ and $\partial_t R_2$ with the right-hand side of the error equation (2.18). With Lemma 2.2.9 and Sobolev's embedding theorem, we directly obtain

$$\begin{aligned}\|\partial_t \mathcal{R} - \Lambda \mathcal{R}\|_{(H^s)^4} \leq C(1 &+ \|\mathcal{R}\|_{(H^s)^4} + \|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} \\ &+ \varepsilon^2(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}})\|\mathcal{R}\|_{(H^s)^4}),\end{aligned}$$

which gives us a bound for s_2.

Bound for s_3: We divide the term s_3 into a linear and a nonlinear part, i.e., $s_3 = s_3^{lin} + s_3^{nonlin}$. By linearity, we can express s_3^{lin} as a function of the difference $\widetilde{\mathcal{R}} - \mathcal{R}$, which we can replace with $\mathcal{Q}(\Psi, \mathcal{R})$. Thus, with Lemma 2.2.11, we can estimate the linear part of s_3 in $(H^s)^4$ by

$$\left\|s_3^{lin}\right\|_{(H^s)^4} \leq C\varepsilon \|\mathcal{R}\|_{(H^s)^4}.$$

Using (2.22), the nonlinear part in s_3 can be estimated by

$$\begin{aligned}\left\|s_3^{nonlin}\right\|_{(H^s)^4} \leq\ & C\varepsilon^2(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}})\|R_2 + R_{-2}\|_{H^s} \\ &+ C\varepsilon^2(\|R_1\|^2_{H^{s+1}} + \|R_{-1}\|^2_{H^{s+1}}) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

Bounds for s_4: With (2.19), we have

$$\|s_4\|_{(H^s)^4} \leq C\varepsilon^2(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}})(\|\mathcal{R}\|_{(H^s)^4} + \|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}}).$$

Bounds for s_5: Subsequently, w.l.o.g., let $j \in \{\pm 1\}$. With the triangle inequality, we obtain

$$\begin{aligned}&\left|\big(\widehat{\mathcal{Q}}_j(\widehat{\Psi}, \widehat{\Lambda \mathcal{R}}) - \sum_{j_1, j_2} i\omega_{j_2} \widehat{\mathcal{Q}}_{j,j_1,j_2}(\widehat{\psi}_{j_1}, \widehat{R}_{j_2})\big)(k)\right| \\ &= \left|\sum_{j_1,j_2} \int (i\omega_{j_2}(k) - i\omega_{j_2}(l)) q_{j,j_1,j_2}(k) \widehat{\psi}_{j_1}(k-l) \widehat{R}_{j_2}(l)\, \mathrm{d}l\right| \\ &\leq \sup_{j \in \{\pm 1\}, j_1, j_2 \in \{\pm 1, \pm 2\}, k \in \mathbb{R}} |q_{j,j_1,j_2}(k)| \cdot \sum_{j_1,j_2} \int |\omega_{j_2}(k) - \omega_{j_2}(l)| |\widehat{\psi}_{j_1}(k-l)| |\widehat{R}_{j_2}(l)|\, \mathrm{d}l.\end{aligned}$$

Next, we use the relation

$$|\omega_1(k) - \omega_1(l)| \leq \frac{1}{2}|k-l||k+l| \leq \frac{1}{2}\left(|k-l|^2 + 2|l||k-l|\right),$$

which follows from a Taylor expansion, in order to obtain

$$\begin{aligned}
&\left\| \int |\omega_1(\cdot) - \omega_1(l)| \widehat{\psi}_{j_1}(\cdot - l) \widehat{R}_{\pm 1}(l) \, \mathrm{d}l \right\|_{H^s} \\
&\quad \leq \left\| \int \frac{1}{2} |\cdot - l|^2 \widehat{\psi}_{j_1}(\cdot - l) \widehat{R}_{\pm 1}(l) \, \mathrm{d}l \right\|_{H^s} + \left\| \int |\cdot - l| \widehat{\psi}_{j_1}(\cdot - l) |l| \widehat{R}_{\pm 1}(l) \, \mathrm{d}l \right\|_{H^s} \\
&\quad \leq \frac{1}{2} \|\partial_x^2 \psi_{j_1}\|_{H^s} \|R_{\pm 1}\|_{H^s} + \|\partial_x \psi_{j_1}\|_{H^s} \|\partial_x R_{\pm 1}\|_{H^s} \\
&\quad \leq C \|\psi_{j_1}\|_{H^{s+2}} \|R_{\pm 1}\|_{H^{s+1}}.
\end{aligned}$$

By using $|\omega_2(k) - \omega_2(l)| = |k-l|$, we find

$$\begin{aligned}
\left\| \int |\omega_2(\cdot) - \omega_2(l)| \widehat{\psi}_{j_1}(\cdot - l) \widehat{R}_{\pm 2}(l) \, \mathrm{d}l \right\|_{H^s} &= \left\| \int |\cdot - l| \widehat{\psi}_{j_1}(\cdot - l) \widehat{R}_{\pm 2}(l) \, \mathrm{d}l \right\|_{H^s} \\
&\leq \|\psi_{j_1}\|_{H^{s+1}} \|R_{\pm 2}\|_{H^s}
\end{aligned}$$

Thus, we only generate terms of order $\mathcal{O}(1)$ when replacing $\omega_{j_2}(l)$ with $\omega_{j_2}(k)$. Moreover, since

$$\sup_{j,j_1,j_2 \in \{\pm 1, \pm 2\}, k \in \mathbb{R}} |q_{j,j_1,j_2}(k)| = \mathcal{O}(\varepsilon),$$

we conclude

$$\left\| \mathcal{Q}(\Psi, \Lambda \mathcal{R}) - \sum_{j_1, j_2} i\omega_{j_2} \mathcal{Q}_{j_1,j_2}(\psi_{j_1}, R_{j_2}) \right\|_{(H^s)^4} \leq C\varepsilon (\|\mathcal{R}\|_{(H^s)^4} + \|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}}).$$

2.2.5 Estimates for the error

In this section, we want to prove Theorem 2.2.2. The proof is a non-trivial task since we have $\partial_t^2 u = \mathcal{O}(\varepsilon^{-2})$, while solutions have to be bounded on an $\mathcal{O}(1)$ time scale. The idea is to derive an energy $\mathcal{E}$ with $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathcal{O}(1)$ in order to control the $\mathcal{O}(\varepsilon^{-2})$ terms. We consider the system resulting from the elimination in the previous section

$$\partial_t \widetilde{\mathcal{R}} = \Lambda \widetilde{\mathcal{R}} + \mathcal{B}_1(\Psi, \widetilde{\mathcal{R}}) + \mathcal{H}(\Psi, \mathcal{R}) + \varepsilon^{-4}\mathrm{RES}(\Psi).$$

In the following, we make the ansatz $\widetilde{R}_{\pm 1} = W_{\pm 1} e^{\pm i\varepsilon^{-2}t}$. The idea behind this ansatz is to shift the linear operator ω_1 of the KGZ system (2.12) by $-\varepsilon^{-2}$ such that, in Fourier space, the resulting operator $\widetilde{\omega}_1 = \omega_1 - \varepsilon^{-2}$ touches the origin.

Then, the spectral situation corresponding to the linearized KGZ system for the new variables $W_{\pm 1}$ is similar to the one of the Zakharov system from Section 2.1, cf. Figure 2.1 and Figure 2.2. The reason for this is that, in Fourier space, the linear operator from the Zakharov system is the second order Taylor expansion of $\widetilde{\omega}_1$ at the wave number $k = 0$. Hence, we can adapt the proof of the approximation result in Section 2.1 to the situation here. The ansatz yields

$$\begin{aligned}
\partial_t W_1 &= i\widetilde{\omega}_1 W_1 - \varepsilon^{-2}(2i\omega_1)^{-1}\left(\psi_u(\widetilde{R}_2 + \widetilde{R}_{-2}) + \psi_v W_1 + \varepsilon^2 W_1(\widetilde{R}_2 + \widetilde{R}_{-2})\right) \\
&\quad + h_1(\Psi, \mathcal{R}) + \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u, \\
\partial_t W_{-1} &= -i\widetilde{\omega}_1 W_{-1} + \varepsilon^{-2}(2i\omega_1)^{-1}\left(\psi_v W_{-1} + \varepsilon^2 W_{-1}(\widetilde{R}_2 + \widetilde{R}_{-2})\right) \\
&\quad + h_{-1}(\Psi, \mathcal{R}) - \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u, \\
\partial_t \widetilde{R}_{\pm 2} &= \pm i\omega_2 \widetilde{R}_{\pm 2} \pm \frac{1}{2} i\omega_2\left(\psi_u \overline{W_1} + \overline{\psi_u} W_1 + \varepsilon^2 |W_1|^2 + \varepsilon^2 |W_{-1}|^2\right) \\
&\quad + h_{\pm 2}(\Psi, \mathcal{R}) \pm \varepsilon^{-2}(2i\omega_2)^{-1}\mathrm{Res}_v.
\end{aligned}$$

Defining the variables

$$W_v = \widetilde{R}_2 + \widetilde{R}_{-2}, \quad W_q = \widetilde{R}_2 - \widetilde{R}_{-2}$$

allows us to write the error equation as

$$\begin{aligned}
\partial_t W_1 &= i\widetilde{\omega}_1 W_1 - \varepsilon^{-2}(2i\omega_1)^{-1}\left(\psi_u W_v + \psi_v W_1 + \varepsilon^2 W_1 W_v\right) \\
&\quad + h_1(\Psi, \mathcal{R}) + \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u, \\
\partial_t W_{-1} &= -i\widetilde{\omega}_1 W_{-1} + \varepsilon^{-2}(2i\omega_1)^{-1}\left(\psi_v W_{-1} + \varepsilon^2 W_{-1} W_v\right) \\
&\quad + h_{-1}(\Psi, \mathcal{R}) - \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u, \\
\partial_t W_v &= \partial_x W_q + h_2(\Psi, \mathcal{R}) + h_{-2}(\Psi, \mathcal{R}), \\
\partial_t W_q &= \partial_x W_v + \partial_x\left(\psi_u \overline{W_1} + \overline{\psi_u} W_1 + \varepsilon^2 |W_1|^2 + \varepsilon^2 |W_{-1}|^2\right) \\
&\quad + h_2(\Psi, \mathcal{R}) - h_{-2}(\Psi, \mathcal{R}) + \varepsilon^{-2}\partial_x^{-1}\mathrm{Res}_v.
\end{aligned} \tag{2.24}$$

We observe that there are no more terms with a factor $e^{\pm i\varepsilon^{-2}t}$ as all oscillatory terms have already been eliminated by a normal form transformation. This is necessary for the following energy estimates since we will rewrite some problematic terms as time derivatives. The next step is now to derive an energy. For this, we first apply the operator ∂_x^l, with $l \in \{0, s\}$, to the error system (2.24). Multiplying the resulting $W_{\pm 1}$-equation with $\partial_x^l \overline{W_{\pm 1}}$, integrating this equation w.r.t. x, and adding the complex conjugate yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^l W_1\|_{L^2}^2 &= \mathrm{Im}\int (\varepsilon^2\omega_1)^{-1}\partial_x^l(\overline{\psi_u} W_v + \psi_v \overline{W_1} + \varepsilon^2 \overline{W_1} W_v)\partial_x^l W_1 \,\mathrm{d}x \\
&\quad + 2\mathrm{Re}\int \partial_x^l h_1(\Psi, \mathcal{R})\partial_x^l \overline{W_1}\,\mathrm{d}x + \mathrm{Im}\int \varepsilon^{-4}(\omega_1)^{-1}\partial_x^l \mathrm{Res}_u \partial_x^l \overline{W_1}\,\mathrm{d}x
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^l W_{-1}\|_{L^2}^2 = -\mathrm{Im}\int (\varepsilon^2\omega_1)^{-1}\partial_x^l(\psi_v\overline{W_{-1}} + \varepsilon^2\overline{W_{-1}}W_v)\partial_x^l W_{-1}\,\mathrm{d}x$$
$$+ 2\mathrm{Re}\int \partial_x^l h_{-1}(\Psi,\mathcal{R})\partial_x^l\overline{W_{-1}}\,\mathrm{d}x - \mathrm{Im}\int \varepsilon^{-4}(\omega_1)^{-1}\partial_x^l \mathrm{Res}_u\partial_x^l\overline{W_{-1}}\,\mathrm{d}x.$$

Multiplying the $W_{\pm1}$-equation with $2i\varepsilon^2\omega_1\partial_x^l\partial_t\overline{W_{\pm1}}$, integrating this equation w.r.t. x, and adding the complex conjugate yields

$$\begin{aligned}
&2\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int \widetilde{\omega}_1\partial_x^l W_1\omega_1\partial_x^l\overline{W_1}\,\mathrm{d}x \\
&\quad= -\int (\partial_x^l(\psi_v W_1)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(\psi_u W_v)\partial_t\partial_x^l\overline{W_1} + \varepsilon^2\partial_x^l(W_vW_1)\partial_t\partial_x^l\overline{W_1})\,\mathrm{d}x \\
&\qquad - \int (\partial_x^l(\psi_v\overline{W_1})\partial_t\partial_x^l W_1 + \partial_x^l(\overline{\psi_u}W_v)\partial_t\partial_x^l W_1 + \varepsilon^2\partial_x^l(W_v\overline{W_1})\partial_t\partial_x^l W_1)\,\mathrm{d}x \\
&\qquad - 4\mathrm{Im}\int \partial_x^l h_1(\Psi,\mathcal{R})\varepsilon^2\omega_1\partial_x^l\partial_t\overline{W_1}\,\mathrm{d}x \\
&\qquad + 2\varepsilon^{-2}\mathrm{Re}\int \partial_x^l\overline{\mathrm{Res}_u}\partial_x^l\partial_t W_1\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&2\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int \widetilde{\omega}_1\partial_x^l W_{-1}\omega_1\partial_x^l\overline{W_{-1}}\,\mathrm{d}x \\
&\quad= -\int (\partial_x^l(\psi_v W_{-1})\partial_t\partial_x^l\overline{W_{-1}} + \varepsilon^2\partial_x^l(W_vW_{-1})\partial_t\partial_x^l\overline{W_{-1}})\,\mathrm{d}x \\
&\qquad - \int (\partial_x^l(\psi_v\overline{W_{-1}})\partial_t\partial_x^l W_{-1} + \varepsilon^2\partial_x^l(W_v\overline{W_{-1}})\partial_t\partial_x^l W_{-1})\,\mathrm{d}x \\
&\qquad + 4\mathrm{Im}\int \partial_x^l h_{-1}(\Psi,\mathcal{R})\varepsilon^2\omega_1\partial_x^l\partial_t\overline{W_{-1}}\,\mathrm{d}x \\
&\qquad + 2\varepsilon^{-2}\mathrm{Re}\int \partial_x^l\overline{\mathrm{Res}_u}\partial_x^l\partial_t W_{-1}\,\mathrm{d}x.
\end{aligned}$$

Here, we can write

$$2\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int \widetilde{\omega}_1\partial_x^l W_{\pm1}\omega_1\partial_x^l\overline{W_{\pm1}}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varepsilon\widetilde{\omega}_1\partial_x^l W_{\pm1}\|_{L^2}^2 + \|\partial_x^{l+1}W_{\pm1}\|_{L^2}^2\right)$$

due to the relation

$$2\varepsilon^2\omega_1(\cdot)\widetilde{\omega}_1(\cdot) = (\varepsilon\widetilde{\omega}_1(\cdot))^2 + (\cdot)^2.$$

We multiply the W_v-equation with $\partial_t\partial_x^{l-1}W_q$ and the W_q-equation with $\partial_t\partial_x^{l-1}W_v$ and integrate the equations w.r.t. x. Summing the resulting equations yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x^l W_q\|_{L^2}^2+\|\partial_x^l W_v\|_{L^2}^2\right) = &-\int (\partial_x^l(\overline{\psi_u}W_1)\partial_t\partial_x^l W_v + \partial_x^l(\psi_u\overline{W_1})\partial_t\partial_x^l W_v)\,\mathrm{d}x\\ &-\int (\varepsilon^2\partial_x^l(|W_1|^2)\partial_t\partial_x^l W_v + \varepsilon^2\partial_x^l(|W_{-1}|^2)\partial_t\partial_x^l W_v)\,\mathrm{d}x\\ &+\int \partial_x^l(h_2(\Psi,\mathcal{R})+h_{-2}(\Psi,\mathcal{R}))\partial_t\partial_x^{l-1}W_q\,\mathrm{d}x\\ &+\int \partial_x^l(h_2(\Psi,\mathcal{R})-h_{-2}(\Psi,\mathcal{R}))\partial_t\partial_x^{l-1}W_v\,\mathrm{d}x\\ &+\varepsilon^{-2}\int \partial_x^{l-1}\mathrm{Res}_v\partial_t\partial_x^{l-1}W_v\,\mathrm{d}x.\end{aligned}$$

Then, we define the energy $\mathcal{E}$ by $\mathcal{E} = E_0 + E_s + E_*$ with

$$\begin{aligned}E_l = &\|\partial_x^l W_1\|_{L^2}^2 + \|\partial_x^l W_{-1}\|_{L^2}^2 + \frac{1}{2}\|\partial_x^l W_q\|_{L^2}^2 + \frac{1}{2}\|\partial_x^l W_v\|_{L^2}^2\\ &+\|\varepsilon\widetilde{\omega}_1\partial_x^l W_1\|_{L^2}^2 + \|\varepsilon\widetilde{\omega}_1\partial_x^l W_{-1}\|_{L^2}^2 + \|\partial_x^{l+1}W_1\|_{L^2}^2 + \|\partial_x^{l+1}W_{-1}\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}E_* = &\int \left(\overline{\psi_u}W_1W_v + \psi_u\overline{W_1}W_v + (|W_1|^2+|W_{-1}|^2)(\psi_v+\varepsilon^2W_v)\right)\mathrm{d}x\\ &+\int (\partial_x^s(\psi_u\overline{W_1})\partial_x^s W_v + \partial_x^s(\overline{\psi_u}W_1)\partial_x^s W_v)\,\mathrm{d}x\\ &+\varepsilon^2\int (W_1\partial_x^s W_v\partial_x^s\overline{W_1} + \overline{W_1}\partial_x^s W_v\partial_x^s W_1)\,\mathrm{d}x\\ &+\varepsilon^2\int (W_{-1}\partial_x^s W_v\partial_x^s\overline{W_{-1}} + \overline{W_{-1}}\partial_x^s W_v\partial_x^s W_{-1})\,\mathrm{d}x.\end{aligned}$$

In Section 2.2.6, we will explain in detail where the part E_* of the energy $\mathcal{E}$ comes from. There, we also find some auxiliary calculations in order to understand the following representation of the time derivative of the energy $\mathcal{E}$. With the results from Section 2.2.6, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \frac{\mathrm{d}}{\mathrm{d}t}E_* - \sum_{l\in\{0,s\}} \Bigg(& \int (\partial_x^l(\psi_v W_1)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(\psi_v\overline{W_1})\partial_t\partial_x^l W_1)\,\mathrm{d}x \\
& + \int (\partial_x^l(\psi_v W_{-1})\partial_t\partial_x^l\overline{W_{-1}} + \partial_x^l(\psi_v\overline{W_{-1}})\partial_t\partial_x^l W_{-1})\,\mathrm{d}x \\
& + \int (\partial_x^l(\psi_u W_v)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(\psi_u\overline{W_1})\partial_t\partial_x^l W_v)\,\mathrm{d}x \\
& + \int (\partial_x^l(\overline{\psi_u} W_v)\partial_t\partial_x^l W_1 + \partial_x^l(\overline{\psi_u} W_1)\partial_t\partial_x^l W_v)\,\mathrm{d}x \\
& + \varepsilon^2 \int (\partial_x^l(W_v W_1)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(W_v\overline{W_1})\partial_t\partial_x^l W_1 + \partial_x^l(|W_1|^2)\partial_t\partial_x^l W_v)\,\mathrm{d}x \\
& + \varepsilon^2 \int (\partial_x^l(W_v W_{-1})\partial_t\partial_x^l\overline{W_{-1}} + \partial_x^l(W_v\overline{W_{-1}})\partial_t\partial_x^l W_{-1} + \partial_x^l(|W_{-1}|^2)\partial_t\partial_x^l W_v)\,\mathrm{d}x \Bigg) \\
& + \sum_{l\in\{0,s\}}\sum_{i=1}^{11} r_{l,i} \\
= \sum_{i=1}^{8} t_i + \sum_{l\in\{0,s\}}\sum_{i=1}^{11} r_{l,i}.
\end{aligned}$$

Here, $t_1, \dots, t_8$ are given by (2.26) and

$$
\begin{aligned}
r_{l,1} &= \mathrm{Im} \int (\varepsilon^2\omega_1)^{-1}\partial_x^l(\overline{\psi_u}W_v + \psi_v\overline{W_1} + \varepsilon^2\overline{W_1}W_v)\partial_x^l W_1 \,\mathrm{d}x, \\
r_{l,2} &= -\mathrm{Im} \int (\varepsilon^2\omega_1)^{-1}\partial_x^l(\psi_v\overline{W_{-1}} + \varepsilon^2\overline{W_{-1}}W_v)\partial_x^l W_{-1} \,\mathrm{d}x, \\
r_{l,3} &= -4\mathrm{Im} \int \partial_x^l h_1(\Psi, \mathcal{R})\varepsilon^2\omega_1\partial_x^l\partial_t\overline{W_1} \,\mathrm{d}x, \\
r_{l,4} &= 4\mathrm{Im} \int \partial_x^l h_{-1}(\Psi, \mathcal{R})\varepsilon^2\omega_1\partial_x^l\partial_t\overline{W_{-1}} \,\mathrm{d}x, \\
r_{l,5} &= 2\varepsilon^{-2}\mathrm{Re} \int \partial_x^l\overline{\mathrm{Res}_u}\partial_x^l\partial_t W_1 \,\mathrm{d}x, \\
r_{l,6} &= 2\varepsilon^{-2}\mathrm{Re} \int \partial_x^l\overline{\mathrm{Res}_u}\partial_x^l\partial_t W_{-1} \,\mathrm{d}x, \\
r_{l,7} &= 2\mathrm{Re} \int \partial_x^l h_1(\Psi, \mathcal{R})\partial_x^l\overline{W_1} \,\mathrm{d}x, \\
r_{l,8} &= 2\mathrm{Re} \int \partial_x^l h_{-1}(\Psi, \mathcal{R})\partial_x^l\overline{W_{-1}} \,\mathrm{d}x, \\
r_{l,9} &= \mathrm{Im} \int \varepsilon^{-4}(\omega_1)^{-1}\partial_x^l\mathrm{Res}_u\partial_x^l\overline{W_1} \,\mathrm{d}x, \\
r_{l,10} &= -\mathrm{Im} \int \varepsilon^{-4}(\omega_1)^{-1}\partial_x^l\mathrm{Res}_u\partial_x^l\overline{W_{-1}} \,\mathrm{d}x, \\
r_{l,11} &= \int \partial_x^l(h_2(\Psi, \mathcal{R}) + h_{-2}(\Psi, \mathcal{R}))\partial_t\partial_x^{l-1}W_q \,\mathrm{d}x \\
&\quad + \int \partial_x^l(h_2(\Psi, \mathcal{R}) - h_{-2}(\Psi, \mathcal{R}))\partial_t\partial_x^{l-1}W_v \,\mathrm{d}x \\
&\quad + \varepsilon^{-2} \int \partial_x^{l-1}\mathrm{Res}_v\partial_t\partial_x^{l-1}W_v \,\mathrm{d}x.
\end{aligned}
$$

Energy equivalence: The following lemma shows that the square root of the energy $\mathcal{E}$ is equivalent to the $H^{s+1} \times H^{s+1} \times H^s \times H^s$-norm of the error functions $(R_1, R_{-1}, R_2, R_{-2})$ for sufficiently small $\varepsilon > 0$ and under additional assumptions on the functions ψ_u and ψ_v.

Lemma 2.2.12. *Let $s \geq 0$. There is a $C_{max} > 0$ such that for all $C_u, C_v \in [0, C_{max})$ the following holds. Let $(\psi_u, \psi_v) \in C([0, T_0], H^{s+5} \times H^{s+4})$ be a solution of the Zakharov system* (2.14) *with*

$$
\sup_{t\in[0,T_0]} \|\psi_u\|_{H^{s+5}} =: C_u < \infty, \qquad \sup_{t\in[0,T_0]} \|\psi_v\|_{H^{s+4}} =: C_v < \infty.
$$

Then, there exist $\varepsilon_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned}\big(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} + \|R_2\|_{H^s} + \|R_{-2}\|_{H^s}\big)^2 \\ \leq C_1\mathcal{E} \leq C_2\big(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} + \|R_2\|_{H^s} + \|R_{-2}\|_{H^s}\big)^2.\end{aligned}$$

Proof. We note that $\widetilde{\mathcal{R}}$ is a small perturbation of $\mathcal{R}$ according to Lemma 2.2.11, see also (2.22). Further, we have

$$\|\widetilde{R}_{\pm 1}\|_{H^{s+1}} = \|W_{\pm 1}\|_{H^{s+1}} \sim \|W_{\pm 1}\|_{H^s} + \|\partial_x W_{\pm 1}\|_{H^s}.$$

Making use of the fact that W_v and W_q are linear combinations of $\widetilde{R}_2$ and $\widetilde{R}_{-2}$, the square root of the energy $E_0 + E_s$ allows to estimate the $H^{s+1} \times H^{s+1} \times H^s \times H^s$-norm of $(R_1, R_{-1}, R_2, R_{-2})$, if $\varepsilon > 0$ is sufficiently small. It remains to show that E_* is a small perturbation of $E_0 + E_s$. Obviously, it is

$$|\mathcal{E}_*| \leq 6C_{max}(E_0 + E_s) + 4\varepsilon^2(E_0 + E_s)^{3/2}.$$

Thus, the result follows, if we choose $C_{max} > 0$ and $\varepsilon_0 > 0$ sufficiently small. □

According to Lemma 2.2.12, it is sufficient to find an $\mathcal{O}(1)$ bound for $\mathcal{E}$ in order to prove Theorem 2.2.2. This is achieved by estimating $r_{s,1}, \ldots, r_{s,11}$ in terms of $\mathcal{E}$ and applying Gronwall's inequality. The terms $r_{0,1}, \ldots, r_{0,11}$ can be estimated in the same way. Estimates for the terms $t_1, \ldots, t_8$ can be found in Section 2.2.6. We will use the inequality $\mathcal{E}^{1/2} \leq 1 + \mathcal{E}$ in several places.

Bound for $\mathcal{H}$: For the subsequent estimates, we need a precise bound for $\mathcal{H}$, which is defined in (2.23). We note that each term in the first two components of $\mathcal{Q}$, $\mathcal{B}_1$ and $\mathcal{G}$ contains a factor $(\varepsilon^2\omega_1)^{-1}$. Thus, with the estimates for $s_1, \ldots, s_5$, we have

$$\begin{aligned}\|h_1\|_{H^s} &\leq \|\varepsilon^2\omega_1 h_1\|_{H^s} \leq C\varepsilon + C\varepsilon\mathcal{E}, \\ \|h_{-1}\|_{H^s} &\leq \|\varepsilon^2\omega_1 h_{-1}\|_{H^s} \leq C\varepsilon + C\varepsilon\mathcal{E}, \\ \|h_2\|_{H^s} &\leq C\varepsilon + C\varepsilon\mathcal{E}, \\ \|h_{-2}\|_{H^s} &\leq C\varepsilon + C\varepsilon\mathcal{E}.\end{aligned} \tag{2.25}$$

Trivial bounds: We use the Cauchy-Schwarz inequality, Sobolev's embedding theorem, Lemma 2.2.9, (2.25), and the relation

$$\|(\varepsilon^2\omega_1)^{-1}\|_{L^\infty} = \sup_{k\in\mathbb{R}} \frac{1}{\sqrt{1 + (\varepsilon k)^2}} \leq 1.$$

Then, a pure counting of ε-powers directly yields

$$\begin{aligned}
|r_{s,1}| &\le C\mathcal{E}^{1/2} + C\varepsilon^2\mathcal{E}^{3/2},\\
|r_{s,2}| &\le C\mathcal{E}^{1/2} + C\varepsilon^2\mathcal{E}^{3/2},\\
|r_{s,7}| &\le C\varepsilon\mathcal{E}^{1/2} + C\varepsilon\mathcal{E}^{3/2},\\
|r_{s,8}| &\le C\varepsilon\mathcal{E}^{1/2} + C\varepsilon\mathcal{E}^{3/2},\\
|r_{s,9}| &\le C\mathcal{E}^{1/2},\\
|r_{s,10}| &\le C\mathcal{E}^{1/2}.
\end{aligned}$$

Bounds for $r_{s,3}$ and $r_{s,4}$: In the definition of $r_{s,3}$, we replace $\partial_t W_1$ with the right-hand side of the error equation (2.24). Then, we obtain

$$\begin{aligned}
\int \partial_x^s\overline{h_1(\Psi,\mathcal{R})}\varepsilon^2\omega_1\partial_x^s\partial_t W_1\,\mathrm{d}x &= \int \partial_x^s\overline{h_1(\Psi,\mathcal{R})}\varepsilon^2\omega_1 i\widetilde{\omega}_1\partial_x^s W_1\,\mathrm{d}x\\
&\quad - \int \partial_x^s\overline{h_1(\Psi,\mathcal{R})}\frac{1}{2i}\partial_x^s\left(\psi_u W_v + \psi_v W_1 + \varepsilon^2 W_1 W_v\right)\mathrm{d}x\\
&\quad + \int \partial_x^s\overline{h_1(\Psi,\mathcal{R})}\varepsilon^2\omega_1\partial_x^s h_1(\Psi,\mathcal{R})\,\mathrm{d}x\\
&\quad + \int \partial_x^s\overline{h_1(\Psi,\mathcal{R})}\varepsilon^{-2}\frac{1}{2i}\partial_x^s\mathrm{Res}_u\,\mathrm{d}x\\
&= I_{s,1} + I_{s,2} + I_{s,3} + I_{s,4}.
\end{aligned}$$

With the Cauchy-Schwarz inequality and (2.25), we find

$$\begin{aligned}
|I_{s,1}| &\le C\varepsilon^{-1}\|\varepsilon^2\omega_1 h_1(\Psi,\mathcal{R})\|_{H^s}\|\varepsilon\widetilde{\omega}_1 W_1\|_{H^s} \le C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^2,\\
|I_{s,2}| &\le \|h_1(\Psi,\mathcal{R})\|_{H^s}\left(\|\psi_u\|_{H^s}\|W_v\|_{H^s} + \|\psi_v\|_{H^s}\|W_1\|_{H^s} + \varepsilon^2\|W_1\|_{H^s}\|W_v\|_{H^s}\right)\\
&\le C\varepsilon + C\varepsilon\mathcal{E} + C\varepsilon^2\mathcal{E}^2,\\
|I_{s,3}| &\le C\|h_1(\Psi,\mathcal{R})\|_{H^s}\|\varepsilon^2\omega_1 h_1(\Psi,\mathcal{R})\|_{H^s} \le C\varepsilon + C\varepsilon\mathcal{E} + C\varepsilon^2\mathcal{E}^2,\\
|I_{s,4}| &\le \varepsilon^{-2}\|h_1(\Psi,\mathcal{R})\|_{H^s}\|\mathrm{Res}_u\|_{H^s} \le C\varepsilon + C\varepsilon\mathcal{E}.
\end{aligned}$$

This yields

$$|r_{s,3}| \le C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^2.$$

Analogously, we obtain

$$|r_{s,4}| \le C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^2.$$

Bounds for $r_{s,5}$ and $r_{s,6}$: In the definition of $r_{s,5}$, we replace $\partial_t W_1$ with the right-hand side of the error equation (2.24). This yields

$$\begin{aligned}
&\varepsilon^{-2}\int \partial_x^s\overline{\mathrm{Res}_u}\partial_t\partial_x^s W_1\,\mathrm{d}x \\
&\quad= \varepsilon^{-2}\int i\widetilde{\omega}_1\partial_x^s W_1\partial_x^s\overline{\mathrm{Res}_u}\,\mathrm{d}x \\
&\qquad - \varepsilon^{-4}\int (2i\omega_1)^{-1}\partial_x^s(\psi_u W_v + \psi_v W_1 + \varepsilon^2 W_1 W_v)\partial_x^s\overline{\mathrm{Res}_u}\,\mathrm{d}x \\
&\qquad + \varepsilon^{-2}\int \partial_x^s h_1(\Psi,\mathcal{R})\partial_x^s\overline{\mathrm{Res}_u}\,\mathrm{d}x + \varepsilon^{-6}\int (2i\omega_1)^{-1}\partial_x^s\mathrm{Res}_u\partial_x^s\overline{\mathrm{Res}_u}\,\mathrm{d}x \\
&\quad= I_{s,5} + I_{s,6} + I_{s,7} + I_{s,8}.
\end{aligned}$$

Since $|\widetilde{\omega}_1(k)| = |\omega_1(k) - \varepsilon^{-2}| \le \frac{1}{2}k^2$, we can use integration by parts to obtain

$$|I_{s,5}| \le C\varepsilon^{-2}\|W_1\|_{H^{s+1}}\|\mathrm{Res}_u\|_{H^{s+1}} \le C\mathcal{E}^{1/2}.$$

With Sobolev's embedding theorem, Lemma 2.2.9, and (2.25), we have

$$\begin{aligned}
|I_{s,6}| &\le C\varepsilon^{-2}\|\mathrm{Res}_u\|_{H^s}(\|W_1\|_{H^s} + \|W_v\|_{H^s} + \|W_1\|_{H^{s+1}}\|W_v\|_{H^s}) \le C + C\mathcal{E}, \\
|I_{s,7}| &\le \varepsilon^{-2}\|h_1(\Psi,\mathcal{R})\|_{H^s}\|\mathrm{Res}_u\|_{H^s} \le C\varepsilon + C\varepsilon\mathcal{E}, \\
|I_{s,8}| &\le \varepsilon^{-4}\|\mathrm{Res}_u\|_{H^s}^2 \le C.
\end{aligned}$$

This yields

$$|r_{s,5}| \le C + C\mathcal{E}$$

and, analogously,

$$|r_{s,6}| \le C + C\mathcal{E}.$$

Bound for $r_{s,11}$: In $r_{s,11}$, we replace the time derivatives with the right-hand side of the last two equations of the error equation (2.24). Then, we obtain

$$\begin{aligned}
r_{s,11} = &\int \partial_x^s(h_2(\Psi,\mathcal{R}) + h_{-2}(\Psi,\mathcal{R}))\partial_x^s(W_v + \psi_u\overline{W_1} + \overline{\psi_u}W_1 + \varepsilon^2|W_1|^2 + \varepsilon^2|W_{-1}|^2)\,\mathrm{d}x \\
&+ \int \partial_x^s(h_2(\Psi,\mathcal{R}) - h_{-2}(\Psi,\mathcal{R}))\partial_x^s W_q\,\mathrm{d}x + \varepsilon^{-2}\int \partial_x^{s-1}\mathrm{Res}_v\partial_x^s W_q\,\mathrm{d}x \\
&+ \int \partial_x^{s-1}(h_2(\Psi,\mathcal{R}) + h_{-2}(\Psi,\mathcal{R}))\varepsilon^{-2}\partial_x^{s-1}\mathrm{Res}_v(\Psi)\,\mathrm{d}x \\
&+ \int \partial_x^s(h_2(\Psi,\mathcal{R}) + h_{-2}(\Psi,\mathcal{R}))\varepsilon^{-2}\partial_x^{s-2}\mathrm{Res}_v(\Psi)\,\mathrm{d}x \\
&+ \int \partial_x^{s-1}(h_2(\Psi,\mathcal{R}) + h_{-2}(\Psi,\mathcal{R}))\partial_x^s(h_2(\Psi,\mathcal{R}) - h_{-2}(\Psi,\mathcal{R}))\,\mathrm{d}x \\
&+ \int \partial_x^{s-1}(h_2(\Psi,\mathcal{R}) - h_{-2}(\Psi,\mathcal{R}))\partial_x^s(h_2(\Psi,\mathcal{R}) + h_{-2}(\Psi,\mathcal{R}))\,\mathrm{d}x.
\end{aligned}$$

The last four lines vanish after integration by parts. Thus, with (2.25), the term $r_{s,11}$ can be estimated by

$$|r_{s,11}| \le C\varepsilon + C\varepsilon\mathcal{E} + C\varepsilon^2\mathcal{E}^2.$$

Final estimates: Finally, after using $\mathcal{E}^{1/2} \le 1 + \mathcal{E}$ and $\varepsilon\mathcal{E}^2 \le 1$, we are left with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \le C + C\mathcal{E}.$$

With Gronwall's inequality, we have $\mathcal{E}(t) \le M$ for all $t \in [0, T_0]$ for a constant $M = \mathcal{O}(1)$. We choose $\varepsilon_0 > 0$ such that $\varepsilon_0 M^2 \le 1$. This concludes the proof of Theorem 2.2.2. □

Remark 2.2.13. Analogous to Remark 2.1.13, the approximation rate can be increased in Theorem 2.2.2. Instead of making the ansatz (2.17), we can introduce the error by

$$(u, v)(x, t) = (\psi_{u,n} e^{i\varepsilon^{-2}t}, \psi_{v,n})(x, t) + \varepsilon^\beta (R_u, R_v)(x, t),$$

where $(\psi_{u,n}, \psi_{v,n})$ is the higher order ansatz from Section 2.2.3. Then, the error estimates are analogous to those that have just been made. In order to find an $\mathcal{O}(1)$ bound of the $H^{s+1} \times H^s$-norm of (R_u, R_v), we can choose $\beta = 2n + 2$. Therefore, we have the following theorem.

Theorem 2.2.14. *Let $n \in \mathbb{N}$ and $s \in \mathbb{N}$. There is a $C_{max} > 0$ such that for all $C_u, C_v \in [0, C_{max})$ the following holds. Let $(\psi_u, \psi_v) \in C([0, T_0], H^{s+2n+5} \times H^{s+2n+4})$ be a solution of the Zakharov system* (2.14) *with*

$$\sup_{t\in[0,T_0]} \|\psi_u\|_{H^{s+2n+5}} =: C_u < \infty, \qquad \sup_{t\in[0,T_0]} \|\psi_v\|_{H^{s+2n+4}} =: C_v < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of the KGZ system (2.12) *satisfying*

$$\sup_{t\in[0,T_0]} \|(u, v) - (\psi_{u,n} e^{i\varepsilon^{-2}t}, \psi_{v,n})\|_{H^{s+1}\times H^s} \le C\varepsilon^{2n+2}.$$

2.2.6 Auxiliary calculations and estimates

In the following, we show some calculations and estimates which are necessary to understand the energy estimates from Section 2.2.5. For this purpose, we consider the terms

$$\begin{aligned}
t_{l,1} &= \int (\partial_x^l(\psi_v W_1)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(\psi_v\overline{W_1})\partial_t\partial_x^l W_1)\,\mathrm{d}x, \\
t_{l,2} &= \int (\partial_x^l(\psi_v W_{-1})\partial_t\partial_x^l\overline{W_{-1}} + \partial_x^l(\psi_v\overline{W_{-1}})\partial_t\partial_x^l W_{-1})\,\mathrm{d}x, \\
t_{l,3} &= \int (\partial_x^l(\psi_u W_v)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(\psi_u\overline{W_1})\partial_t\partial_x^l W_v)\,\mathrm{d}x, \\
t_{l,4} &= \int (\partial_x^l(\overline{\psi_u} W_v)\partial_t\partial_x^l W_1 + \partial_x^l(\overline{\psi_u} W_1)\partial_t\partial_x^l W_v)\,\mathrm{d}x, \\
t_{l,5} &= \varepsilon^2 \int (\partial_x^l(W_v W_1)\partial_t\partial_x^l\overline{W_1} + \partial_x^l(W_v\overline{W_1})\partial_t\partial_x^l W_1 + \partial_x^l(|W_1|^2)\partial_t\partial_x^l W_v)\,\mathrm{d}x, \\
t_{l,6} &= \varepsilon^2 \int (\partial_x^l(W_v W_{-1})\partial_t\partial_x^l\overline{W_{-1}} + \partial_x^l(W_v\overline{W_{-1}})\partial_t\partial_x^l W_{-1} + \partial_x^l(|W_{-1}|^2)\partial_t\partial_x^l W_v)\,\mathrm{d}x,
\end{aligned}$$

which appear in the time derivative of E_l. The problem with these terms is the occurence of the term $i\widetilde{\omega}_1\partial_x^s W_{\pm1}$ in the time derivate of $\partial_x^s W_{\pm1}$. Since $|\widetilde{\omega}_1(k)| \leq \frac{1}{2}k^2$, the H^s-norm of the term $i\widetilde{\omega}_1 W_{\pm1}$ can only be estimated by the H^{s+2}-norm of $W_{\pm1}$, which is not contained in the energy $\mathcal{E}$. Analogously, the H^s-norm of the time derivate of W_v can only be estimated by the H^{s+1}-norm of W_q, which is also not contained in the energy $\mathcal{E}$. However, we can rewrite the terms $t_{l,i}$ such that the terms containing the highest possible derivative of $W_{\pm1}$ and W_v will be written as a time derivative and all other terms can be estimated in terms of the energy $\mathcal{E}$. In the following, we proceed as in Section 2.1.5. For $l = 0$, we obtain

$$\begin{aligned}
\sum_{i=1}^{6} t_{0,i} = \frac{\mathrm{d}}{\mathrm{d}t}\int (\overline{\psi_u} W_1 W_v + \psi_u\overline{W_1}W_v + (|W_1|^2 + |W_{-1}|^2)(\psi_v + \varepsilon^2 W_v))\,\mathrm{d}x \\
- \int (\partial_t\psi_v|W_1|^2 + \partial_t\psi_v|W_{-1}|^2 + \partial_t\psi_u W_v\overline{W_1} + \partial_t\overline{\psi_u}W_v W_1)\,\mathrm{d}x.
\end{aligned}$$

For $l = s \in \mathbb{N}$, the terms cannot be written as a time derivative directly. Thus, we have to proceed separately. The terms $t_{s,1}$ and $t_{s,2}$ do not have to be written as a time derivative since, subsequently, they will be estimated directly. For the term

$t_{s,3}$ and, analogously, for the term $t_{s,4}$, we use the Leibniz rule and obtain

$$
\begin{aligned}
&\int (\partial_x^s(\psi_u W_v)\partial_t\partial_x^s\overline{W_1} + \partial_x^s(\psi_u\overline{W_1})\partial_t\partial_x^s W_v)\,\mathrm{d}x \\
&= \int (\psi_u\partial_x^s W_v\partial_t\partial_x^s\overline{W_1} + \psi_u\partial_x^s\overline{W_1}\partial_t\partial_x^s W_v)\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_1}\,\mathrm{d}x + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}\overline{W_1}\partial_t\partial_x^s W_v\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \psi_u\partial_x^s W_v\partial_x^s\overline{W_1}\,\mathrm{d}x - \int \partial_t\psi_u\partial_x^s W_v\partial_x^s\overline{W_1}\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_1}\,\mathrm{d}x \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}\overline{W_1}\partial_x^s W_v\,\mathrm{d}x - \sum_{k=1}^{s}\binom{s}{k}\int \partial_t(\partial_x^k\psi_u\partial_x^{s-k}\overline{W_1})\partial_x^s W_v\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \partial_x^s(\psi_u\overline{W_1})\partial_x^s W_v\,\mathrm{d}x - \int \partial_t\psi_u\partial_x^s W_v\partial_x^s\overline{W_1}\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_1}\,\mathrm{d}x - \sum_{k=1}^{s}\binom{s}{k}\int \partial_t(\partial_x^k\psi_u\partial_x^{s-k}\overline{W_1})\partial_x^s W_v\,\mathrm{d}x.
\end{aligned}
$$

For $t_{s,5}$ and, analogously, for the term $t_{s,6}$, we again use the Leibniz rule and find

$$
\begin{aligned}
&\int (\partial_x^s(W_v W_1)\partial_t\partial_x^s\overline{W_1} + \partial_x^s(W_v\overline{W_1})\partial_t\partial_x^s W_1 + \partial_x^s(|W_1|^2)\partial_t\partial_x^s W_v)\,\mathrm{d}x \\
&\ = \int (W_1\partial_x^s W_v\partial_t\partial_x^s\overline{W_1} + \partial_x^s W_v\overline{W_1}\partial_t\partial_x^s W_1 + \partial_x^s W_1\overline{W_1}\partial_t\partial_x^s W_v + \partial_x^s\overline{W_1}W_1\partial_t\partial_x^s W_v)\,\mathrm{d}x \\
&\quad\ + \sum_{k=1}^{s}\binom{s}{k}\int (\partial_x^k W_1\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_1} + \partial_x^k\overline{W_1}\partial_x^{s-k}W_v\partial_t\partial_x^s W_1)\,\mathrm{d}x \\
&\quad\ + \sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x^k W_1\partial_x^{s-k}\overline{W_1}\partial_t\partial_x^s W_v\,\mathrm{d}x \\
&\ = \frac{\mathrm{d}}{\mathrm{d}t}\int (W_1\partial_x^s W_v\partial_x^s\overline{W_1} + \partial_x^s W_v\overline{W_1}\partial_x^s W_1)\,\mathrm{d}x - \int (\partial_t W_1\partial_x^s W_v\partial_x^s\overline{W_1} + \partial_x^s W_v\partial_t\overline{W_1}\partial_x^s W_1)\,\mathrm{d}x \\
&\quad\ + 2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k W_1\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_1}\,\mathrm{d}x + \sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x^k W_1\partial_x^{s-k}\overline{W_1}\partial_t\partial_x^s W_v\,\mathrm{d}x.
\end{aligned}
$$

The terms, that we have written as a time derivative in these calculations, will be collected in the energy E_* which is a part of the full energy $\mathcal{E}$. The remaining terms t_i will be estimated subsequently. They are given by

$$\begin{aligned}
t_1 &= \int (\partial_t\psi_v|W_1|^2 + \partial_t\psi_v|W_{-1}|^2 + \partial_t\psi_u W_v\overline{W_1} + \partial_t\overline{\psi_u}W_vW_1)\,\mathrm{d}x,\\
t_2 &= -\sum_{j\in\{\pm1\}} \int (\partial_x^s(\psi_v W_j)\partial_t\partial_x^s\overline{W_j} + \partial_x^s(\psi_v\overline{W_j})\partial_t\partial_x^s W_j)\,\mathrm{d}x,\\
t_3 &= 2\mathrm{Re}\int \partial_t\psi_u\partial_x^s W_v\partial_x^s\overline{W_1}\,\mathrm{d}x,\\
t_4 &= -2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k\psi_u\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_1}\,\mathrm{d}x,\\
t_5 &= 2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_t(\partial_x^k\psi_u\partial_x^{s-k}\overline{W_1})\partial_x^s W_v\,\mathrm{d}x,\\
t_6 &= \varepsilon^2\sum_{j\in\{\pm1\}}\int (\partial_t W_j\partial_x^s W_v\partial_x^s\overline{W_j} + \partial_x^s W_v\partial_t\overline{W_j}\partial_x^s W_j)\,\mathrm{d}x,\\
t_7 &= -\varepsilon^2\sum_{j\in\{\pm1\}} 2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x^k W_j\partial_x^{s-k}W_v\partial_t\partial_x^s\overline{W_j}\,\mathrm{d}x,\\
t_8 &= -\varepsilon^2\sum_{j\in\{\pm1\}}\sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x^k W_j\partial_x^{s-k}\overline{W_j}\partial_t\partial_x^s W_v\,\mathrm{d}x.
\end{aligned} \tag{2.26}$$

Bounds for t_1, t_3, t_5 and t_6: We use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|t_1| &\le C\|W_1\|_{L^2}^2 + C\|W_{-1}\|_{L^2}^2 + C\|W_v\|_{L^2}\|W_1\|_{L^2} \le C\mathcal{E},\\
|t_3| &\le C\|W_v\|_{H^s}\|W_1\|_{H^s} \le C\mathcal{E},\\
|t_5| &\le C(\|\partial_t\psi_u\|_{H^s}\|W_1\|_{H^{s-1}} + \|\psi_u\|_{H^s}\|\partial_t W_1\|_{H^{s-1}})\|W_v\|_{H^s} \le C + C\mathcal{E} + C\varepsilon\mathcal{E}^{3/2},\\
|t_6| &\le \varepsilon^2\sum_{j\in\{\pm1\}}\|\partial_t W_j\|_{L^2}\|W_v\|_{H^s}\|W_j\|_{H^{s+1}} \le C\varepsilon^2\mathcal{E}^{3/2}.
\end{aligned}$$

Bounds for t_2, t_4, t_7 and t_8: As described above, the H^s-norm of $\partial_t W_{\pm1}$ and $\partial_t W_v$ cannot be estimated by the energy $\mathcal{E}$. However, in the H^{s-1}-norm, we have

$$\begin{aligned}
\|\partial_t W_j\|_{H^{s-1}} &\le C + C\mathcal{E}^{1/2} + C\varepsilon\mathcal{E},\\
\|\partial_t W_v\|_{H^{s-1}} &\le C + C\mathcal{E}^{1/2} + C\varepsilon\mathcal{E},
\end{aligned}$$

where we have used the estimate $|\widetilde{\omega}_1(k)| \le \frac{1}{2}k^2$. Thus, we use integration by parts in order to shift one derivative away from the problematic terms. This yields

$$\begin{aligned}
t_2 &= \sum_{j\in\{\pm1\}} \int (\partial_x^{s+1}(\psi_v W_j)\partial_t\partial_x^{s-1}\overline{W_j} + \partial_x^{s+1}(\psi_v \overline{W_j})\partial_t\partial_x^{s-1}W_j)\,\mathrm{d}x,\\
t_4 &= 2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x(\partial_x^k\psi_u\partial_x^{s-k}W_v)\partial_t\partial_x^{s-1}\overline{W_1}\,\mathrm{d}x,\\
t_7 &= \varepsilon^2\sum_{j\in\{\pm1\}} 2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int \partial_x(\partial_x^k W_j\partial_x^{s-k}W_v)\partial_t\partial_x^{s-1}\overline{W_j}\,\mathrm{d}x,\\
t_8 &= \varepsilon^2\sum_{j\in\{\pm1\}}\sum_{k=1}^{s-1}\binom{s}{k}\int \partial_x(\partial_x^k W_j\partial_x^{s-k}\overline{W_j})\partial_t\partial_x^{s-1}W_v\,\mathrm{d}x.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
|t_2| &\le C\sum_{j\in\{\pm1\}}\|\psi_v\|_{H^{s+1}}\|W_j\|_{H^{s+1}}\|\partial_t\partial_x^{s-1}W_j\|_{H^{s-1}} \le C + C\mathcal{E} + C\varepsilon\mathcal{E}^{3/2},\\
|t_4| &\le C\|\psi_u\|_{H^{s+1}}\|W_v\|_{H^s}\|\partial_t\partial_x^{s-1}W_1\|_{H^{s-1}} \le C + C\mathcal{E} + C\varepsilon\mathcal{E}^{3/2},\\
|t_7| &\le C\varepsilon^2\sum_{j\in\{\pm1\}}\|W_j\|_{H^{s+1}}\|W_v\|_{H^s}\|\partial_t W_j\|_{H^{s-1}} \le C\varepsilon^2\mathcal{E} + C\varepsilon^3\mathcal{E}^2,\\
|t_8| &\le C\varepsilon^2\sum_{j\in\{\pm1\}}\|W_j\|_{H^s}^2\|\partial_t W_v\|_{H^{s-1}} \le C\varepsilon^2\mathcal{E} + C\varepsilon^3\mathcal{E}^2.
\end{aligned}$$

2.3 From KGZ to NLS on the torus

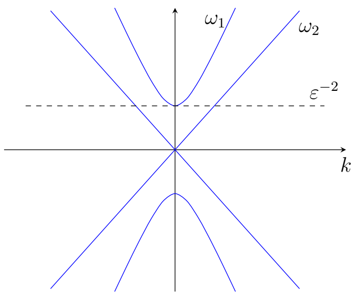
We consider the singular limit from the KGZ system to the NLS equation, where the v-equation of the KGZ system depends on a parameter $\gamma \in \mathbb{R}$, $|\gamma| \geq 1$. In order to prove an approximation result, we distinguish the cases $|\gamma| = 1$ and $|\gamma| > 1$ as for each case different difficulties occur.

2.3.1 Introduction

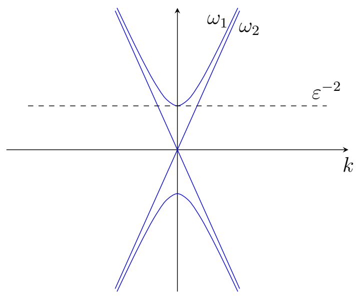
We consider the Klein-Gordon-Zakharov system

$$\varepsilon^2 \partial_t^2 u = \partial_x^2 u - \varepsilon^{-2} u - uv, \qquad \gamma^2 \varepsilon^2 \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2) \tag{2.27}$$

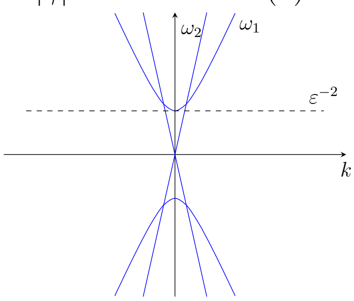
on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with a parameter $\gamma \in \mathbb{R} \setminus (-1, 1)$, where $u(x,t) \in \mathbb{C}$, $v(x,t), x, t \in \mathbb{R}$, and $0 < \varepsilon \ll 1$. This corresponds to the spectral situation in Figure 2.3. The spectral situation depends on the choice of the parameter γ.



(a) The case $|\gamma| > 1$.



(b) The case $|\gamma| = 1$.



(c) The case $|\gamma| < 1$.

Figure 2.3: Spectral situation corresponding to the linearized KGZ system with $k \in \mathbb{R}$. The system is solved by $u(x,t) = e^{ikx + i\omega_{\pm 1}(k)t}$ and $v(x,t) = e^{ikx + i\omega_{\pm 2}(k)t}$ where $\omega_{\pm 1}(k) = \pm \varepsilon^{-2}\sqrt{1 + (\varepsilon k)^2}$ and $\omega_{\pm 2}(k) = \pm(\gamma\varepsilon)^{-1}k$. We note that $\omega_{\pm 1}$ asymptotically behaves like $\pm|\omega_2|$ for $|\gamma| = 1$. The case (c) is not considered here, but we show it anyway for illustration purposes.

In the singular limit $\varepsilon \to 0$, the NLS equation

$$2i\partial_t \psi_u = \partial_x^2 \psi_u + \psi_u |\psi_u|^2 \tag{2.28}$$

with spatially 2π-periodic boundary conditions can be derived with the ansatz

$$u(x,t) = \psi_u(x,t) e^{i\varepsilon^{-2}t}, \qquad v(x,t) = \psi_v(x,t), \tag{2.29}$$

where ψ_u and ψ_v are spatially 2π-periodic. Our goal is to prove that the NLS equation (2.28) with spatially 2π-periodic boundary conditions makes correct predictions about the dynamics of the KGZ system (2.27), with $|\gamma| \geq 1$, on the one-dimensional torus $\mathbb{T}$ for small values of $\varepsilon > 0$. We have the following approximation result.

Theorem 2.3.1. *Let $s \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ with $|\gamma| \geq 1$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $\psi_u \in C([0,T_0], H^{s+5})$ be a solution of the NLS equation* (2.28) *with spatially 2π-periodic boundary conditions and*

$$\sup_{t \in [0,T_0]} \|\psi_u\|_{H^{s+5}} =: C_u < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions (u,v) of the KGZ system (2.27) *with spatially 2π-periodic boundary conditions satisfying*

$$\sup_{t \in [0,T_0]} \|(u,v) - (\psi_u e^{i\varepsilon^{-2}t}, -|\psi_u|^2)\|_{H^{s+1} \times H^s} \leq C\varepsilon^2.$$

In Section 2.3.5, we prove Theorem 2.3.1 by using energy estimates and Gronwall's inequality. Ahead of that, in Section 2.3.2 and Section 2.3.3, we use periodic boundary conditions in order to estimate the residual and construct a higher order approximation. In Section 2.3.4, we use normal form transformations to eliminate problematic terms. It turns out that the normal form transformations are bounded for $|\gamma| > 1$, while they lose regularity for $|\gamma| = 1$. Hence, instead of carrying out the normal form transformations directly, we only use them to define an energy, cf. [Due17, HITW15].

Remark 2.3.2. We have local existence and uniqueness of solutions $(u,v) \in H^{s+1} \times H^s$, $s \geq 1$, of the KGZ system (2.27), cf. Remark 2.2.3, and local existence and uniqueness of solutions $\psi_u \in H^s$, $s \geq 1$, of the NLS equation (2.28), cf. Remark 2.1.5.

Remark 2.3.3. If we consider spatially L-periodic boundary conditions for $L > 0$, with our methods it is also possible to prove an approximation result for $|\gamma| < 1$. In this case, resonances occur which can be handled by using the periodic boundary conditions such that each $k \in (2\pi/L)\mathbb{Z}$ is bounded away from the resonances. However, since the resonances are of order $\mathcal{O}(\varepsilon^{-1})$, the period L should be of order $\mathcal{O}(\varepsilon)$. Since this corresponds to spatially constant functions, the statement of such a theorem would be empty.

Remark 2.3.4. The same ansatz as in equation (2.29) has been considered in [MN05] with a parameter $\gamma \in \mathbb{R}$, $|\gamma| > 1$. However, our proof differs significantly from that in [MN05] since dispersive decay estimates are used there, which are not applicable in the case of periodic boundary conditions. Moreover, our work involves residual estimates and the construction of a higher order approximation. The case $|\gamma| = 1$ is not discussed in the literature.

Remark 2.3.5. The proof of the approximation theorem given in Section 2.3.5 only holds, if the nonlinear part on the right-hand side of the u-equation of (2.27) has a negative sign.

Notation. We use the notation from Chapter 1. We write $\int$ for $\int_{\mathbb{T}}$ and H^s for $H^s(\mathbb{T}, \mathbb{K})$, unless otherwise specified.

2.3.2 Estimates for the residual

The residual of the original system (2.27) contains all terms which do not cancel after inserting the approximation into this system. In this section, we want to estimate the residual for $x \in \mathbb{T}$ and construct an approximation such that the residual becomes sufficiently small. Inserting the ansatz (2.29) into the u- and v-equation yields

$$\varepsilon^2 \partial_t^2 \psi_u + 2i\partial_t \psi_u = \partial_x^2 \psi_u - \psi_u \psi_v, \qquad \gamma^2 \varepsilon^2 \partial_t^2 \psi_v = \partial_x^2 \psi_v + \partial_x^2(|\psi_u|^2). \tag{2.30}$$

We can choose ψ_u to satisfy the NLS equation (2.28) and $\psi_v = -|\psi_u|^2$. However, this would result in a residual of order $\mathcal{O}(\varepsilon^2)$, which is not sufficient to prove Theorem 2.3.1. In order to make the residual smaller, we make the improved ansatz

$$u(x,t) = \Psi_u(x,t) = \psi_u(x,t)e^{i\varepsilon^{-2}t}, \qquad v(x,t) = \Psi_v(x,t) = \psi_v(x,t) + \varepsilon^2 \psi_{v,2}(x,t).$$

After inserting this ansatz into the KGZ system (2.27), a comparison of coefficients yields

$$2i\partial_t \psi_u = \partial_x^2 \psi_u - \psi_u \psi_v, \qquad 0 = \partial_x^2 \psi_v + \partial_x^2(|\psi_u|^2)$$

at order ε^0 and

$$\gamma^2 \partial_t^2 \psi_v = \partial_x^2 \psi_{v,2}$$

at order ε^2. We choose $\psi_v = -|\psi_u|^2$ and ψ_u to satisfy the NLS equation (2.28). Further, we set

$$\widehat{\psi}_{v,2}(k,t) = \gamma^2 k^{-2} \partial_t^2 \widehat{\psi}_v(k,0) \tag{2.31}$$

for $k \in \mathbb{Z} \setminus \{0\}$. We remark that, up to some constants, the functions ψ_u, ψ_v and $\psi_{v,2}$ are defined in the same way as the functions u_0, v_0 and v_2 in Section 2.1.2, where the limit from the Zakharov system to the NLS equation is considered. In fact, we can proceed analogously here. As the mean value of v_0 is conserved for spatially 2π-periodic boundary conditions, we can set $\widehat{\psi}_{v,2}(0,t) = \widehat{\psi}_{v,2}(0,0)$. Thus, the functions $\partial_x^{-n}\partial_t^n \psi_{v,2}$ for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ are well-defined and have a vanishing mean value. With the choices (2.28) and (2.31), the residual of (2.27) is given by

$$\mathrm{Res}_u(\Psi_u, \Psi_v) = -\varepsilon^2(\partial_t^2 \psi_u + \psi_u \psi_{v,2}), \qquad \mathrm{Res}_v(\Psi_u, \Psi_v) = -\gamma^2 \varepsilon^4 \partial_t^2 \psi_{v,2}.$$

Thus, we can formulate the following lemma.

Lemma 2.3.6. *Let $s \geq 0$ and $\gamma \in \mathbb{R} \setminus \{0\}$. Let $\psi_u \in C([0,T_0], H^{s+6})$ be a solution of the NLS equation* (2.28) *with spatially 2π-periodic boundary conditions. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\Psi_u, \Psi_v)\|_{H^{s+2}} \leq C_{res}\varepsilon^2, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\Psi_u, \Psi_v)\|_{H^s} \leq C_{res}\varepsilon^4.$$

Proof. We can estimate $\|\partial_t^2 \psi_{v,2}\|_{H^s}$ in terms of $\|\psi_u\|_{H^{s+6}}$ due to

$$\psi_{v,2} = \frac{1}{4}\gamma^2 \left(\partial_x^2(|u_0|^2) - 4|\partial_x u_0|^2 + |u_0|^4\right)$$

and the fact that ψ_u solves the NLS equation. For the same reason, we can estimate $\|\partial_t^2 \psi_u\|_{H^s}$ in terms of $\|\psi_u\|_{H^{s+4}}$. Thus, we directly get an estimate for $\|\mathrm{Res}_u(\Psi_u, \Psi_v)\|_{H^{s+2}}$ when assuming $\psi_u \in C([0,T_0], H^{s+6})$. □

For the subsequent error estimates, it is sufficient to find a bound for $\partial_x^{-1}\mathrm{Res}_v$. We note that Res_v has no derivative in front but it has a vanishing mean value since it contains the term $\psi_{v,2}$. Therefore, we can directly conclude the following lemma.

Lemma 2.3.7. *Let $\gamma \in \mathbb{R} \setminus \{0\}$. Let $\psi_u \in C([0,T_0], H^6)$ be a solution of the NLS equation* (2.28) *with spatially 2π-periodic boundary conditions. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\Psi_u, \Psi_v)\|_{L^2} = \sup_{t\in[0,T_0]} \gamma^2\varepsilon^4 \|\partial_x^{-1}\partial_t^2 v_2\|_{L^2} \leq C_{res}\varepsilon^4.$$

2.3.3 Higher order approximation

In order to make the residual arbitrarily small, we insert the ansatz

$$\psi_{u,n} = \sum_{k=0}^{n} \varepsilon^{2k} u_{2k}, \qquad \psi_{v,n} = \sum_{k=0}^{n} \varepsilon^{2k} v_{2k} \tag{2.32}$$

into (2.30). Then, $v_0 = -|u_0|^2$ and $u_0 = \psi_u$ solves the NLS equation (2.28). For $k \geq 1$, the functions u_{2k} solve inhomogeneous linear Schrödinger equations of the form

$$2i\partial_t u_{2k} = \partial_x^2 u_{2k} - u_{2k}v_0 - u_0 v_{2k} - \partial_t^2 u_{2(k-1)} - F_{2k}(u_0, \dots, u_{2(k-1)})$$

and the functions v_{2k} satisfy

$$\gamma^2 \partial_t^2 v_{2(k-1)} = \partial_x^2 v_{2k} + \partial_x^2 G_{2k}(u_0, \dots, u_{2k}), \tag{2.33}$$

where F_{2k} and G_{2k} are quadratic mappings. Analogous to Section 2.1.3, we set

$$v_{2k} = \gamma^2 \partial_x^{-2} \partial_t^2 v_{2(k-1)} - G_{2k}(u_0, \dots, u_{2k}) + \frac{1}{2\pi} \int G_{2k}(u_0, \dots, u_{2k})(x) \, \mathrm{d}x$$

in order to achieve that, on the one hand, (2.33) is satisfied, and, on the other hand, the v_{2k} have a vanishing mean value. The following lemma is a direct consequence of Lemma 2.1.11 and Lemma 2.2.10.

Lemma 2.3.8. *Let $n \in \mathbb{N}$ and $s \geq 0$. Further, let $\psi_u \in C([0, T_0], H^{s+2n+5})$ be a solution of the NLS equation (2.28) with spatially 2π-periodic boundary conditions. Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an approximation $(\Psi_{u,n}, \psi_{v,n})$, where $\Psi_{u,n} = \psi_{u,n} e^{i\varepsilon^{-2}t}$ and where $(\psi_{u,n}, \psi_{v,n})$ is of the form (2.32), with*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\Psi_{u,n}, \psi_{v,n})\|_{H^{s+1}} \leq C_{res}\varepsilon^{2n+2}, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\Psi_{u,n}, \psi_{v,n})\|_{H^s} \leq C_{res}\varepsilon^{2n+2},$$

and

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\Psi_{u,n}, \psi_{v,n})\|_{H^s} \leq C_{res}\varepsilon^{2n+2}.$$

2.3.4 Error equation and normal form transform

In this section, we derive an equation for the error which is made by the approximation (Ψ_u, Ψ_v). Since there are some oscillating terms left in the error equation, we use normal form transformations to eliminate them. First, we write the KGZ system (2.27) as

$$\partial_t^2 u = -\omega_1^2 u - \varepsilon^{-2} uv, \qquad \partial_t^2 v = -\omega_2^2 v - \omega_2^2(|u|^2),$$

where, in Fourier space,

$$\omega_1^2(k) = \varepsilon^{-2}(k^2 + \varepsilon^{-2}) = \varepsilon^{-4}(1 + (\varepsilon k)^2), \qquad \omega_2^2(k) = \gamma^{-2}\varepsilon^{-2}k^2.$$

We introduce the error $\varepsilon^2(R_u, R_v)$ made by the improved approximation (Ψ_u, Ψ_v) by

$$(u, v)(x, t) = (\Psi_u, \Psi_v)(x, t) + \varepsilon^2(R_u, R_v)(x, t).$$

The error functions R_u and R_v satisfy

$$\begin{aligned} \partial_t^2 R_u &= -\omega_1^2 R_u - \varepsilon^{-2}(\Psi_u R_v + \Psi_v R_u + \varepsilon^2 R_u R_v) + \varepsilon^{-4}\mathrm{Res}_u, \\ \partial_t^2 R_v &= -\omega_2^2 R_v - \omega_2^2(\overline{\Psi}_u R_u + \Psi_u \overline{R}_u + \varepsilon^2 |R_u|^2) + \varepsilon^{-4}\mathrm{Res}_v. \end{aligned}$$

We rewrite this system as the first order system

$$\begin{aligned} \partial_t R_u &= i\omega_1 \widetilde{R}_u, \\ \partial_t \widetilde{R}_u &= i\omega_1 R_u - \varepsilon^{-2}(i\omega_1)^{-1}(\Psi_u R_v + \Psi_v R_u + \varepsilon^2 R_u R_v) + \varepsilon^{-4}(i\omega_1)^{-1}\mathrm{Res}_u, \\ \partial_t R_v &= i\omega_2 R_q, \\ \partial_t R_q &= i\omega_2 R_v + i\omega_2(\overline{\Psi}_u R_u + \Psi_u \overline{R}_u + \varepsilon^2 |R_u|^2) + \varepsilon^{-4}(i\omega_2)^{-1}\mathrm{Res}_v. \end{aligned}$$

By introducing

$$\begin{aligned} &R_u = R_1 + R_{-1},\ \widetilde{R}_u = R_1 - R_{-1} \quad \text{resp.} \quad 2R_1 = R_u + \widetilde{R}_u,\ 2R_{-1} = R_u - \widetilde{R}_u, \\ &R_v = R_2 + R_{-2},\ R_q = R_2 - R_{-2} \quad \text{resp.} \quad 2R_2 = R_v + R_q,\ 2R_{-2} = R_v - R_q, \end{aligned}$$

we diagonalize this system and find for $R_{\pm 1}$ and $R_{\pm 2}$

$$\begin{aligned} \partial_t R_{\pm 1} = &\pm i\omega_1 R_{\pm 1} \mp \varepsilon^{-2}(2i\omega_1)^{-1}\big(\Psi_u(R_2 + R_{-2}) + \Psi_v(R_1 + R_{-1}) \\ &+ \varepsilon^2(R_1 + R_{-1})(R_2 + R_{-2})\big) \pm \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u, \\ \partial_t R_{\pm 2} = &\pm i\omega_2 R_{\pm 2} \pm \frac{1}{2} i\omega_2\big(\overline{\Psi}_u(R_1 + R_{-1}) + \Psi_u(\overline{R}_1 + \overline{R}_{-1}) \\ &+ \varepsilon^2|R_1 + R_{-1}|^2\big) \pm \varepsilon^{-4}(2i\omega_2)^{-1}\mathrm{Res}_v. \end{aligned} \tag{2.34}$$

In the energy estimates, cf. Section 2.3.5, we will work with the system (2.34). However, in order to understand the subsequent normal form transformation, we divide the right-hand side of (2.34) into non-oscillating terms that will be collected in $\mathcal{B}_1$ and oscillating terms that will be collected in $\mathcal{B}_2$ and $\mathcal{B}_3$. Therefore, we write

$$\partial_t \mathcal{R} = \Lambda\mathcal{R} + \mathcal{B}_1(\Psi_u, \mathcal{R}) + \mathcal{B}_2(\Psi_u, \mathcal{R}) + \mathcal{B}_3(\psi_v, \mathcal{R}) + \mathcal{G}(\Psi_u, \mathcal{R}) + \varepsilon^{-4}\mathrm{RES}(\Psi),$$

where

$$\mathcal{R} = (R_1, R_{-1}, R_2, R_{-2})^T, \quad \Lambda = \mathrm{diag}(i\omega_1, -i\omega_1, i\omega_2, -i\omega_2), \quad \Psi = (\Psi_u, \Psi_v),$$

and

$$\mathcal{B}_1(\Psi_u, \mathcal{R}) = \begin{pmatrix} -\varepsilon^{-2}(2i\omega_1)^{-1}(\Psi_u(R_2 + R_{-2}) + \psi_v R_1 + \varepsilon^2 R_1(R_2 + R_{-2})) \\ \varepsilon^{-2}(2i\omega_1)^{-1}(\psi_v R_{-1} + \varepsilon^2 R_{-1}(R_2 + R_{-2})) \\ \frac{1}{2} i\omega_2(\overline{\Psi}_u R_1 + \Psi_u \overline{R}_1 + \varepsilon^2(|R_1|^2 + |R_{-1}|^2)) \\ -\frac{1}{2} i\omega_2(\overline{\Psi}_u R_1 + \Psi_u \overline{R}_1 + \varepsilon^2(|R_1|^2 + |R_{-1}|^2)) \end{pmatrix},$$

$$\mathcal{B}_2(\Psi_u, \mathcal{R}) = \begin{pmatrix} 0 \\ \varepsilon^{-2}(2i\omega_1)^{-1}(\Psi_u(R_2 + R_{-2})) \\ \frac{1}{2} i\omega_2(\overline{\Psi}_u R_{-1} + \Psi_u \overline{R}_{-1}) \\ -\frac{1}{2} i\omega_2(\overline{\Psi}_u R_{-1} + \Psi_u \overline{R}_{-1}) \end{pmatrix},$$

$$\mathcal{B}_3(\psi_v, \mathcal{R}) = \begin{pmatrix} -\varepsilon^{-2}(2i\omega_1)^{-1}(\psi_v R_{-1}) \\ \varepsilon^{-2}(2i\omega_1)^{-1}(\psi_v R_1) \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{G}(\Psi_u, \mathcal{R}) = \begin{pmatrix} -(2i\omega_1)^{-1}(R_{-1}(R_2 + R_{-2}) + \psi_{v,2}(R_1 + R_{-1})) \\ (2i\omega_1)^{-1}(R_1(R_2 + R_{-2}) + \psi_{v,2}(R_1 + R_{-1})) \\ \frac{1}{2}\varepsilon^2 i\omega_2(R_1\overline{R_{-1}} + \overline{R_1}R_{-1}) \\ -\frac{1}{2}\varepsilon^2 i\omega_2(R_1\overline{R_{-1}} + \overline{R_1}R_{-1}) \end{pmatrix},$$

$$\mathrm{RES}(\Psi) = \begin{pmatrix} (2i\omega_1)^{-1}\mathrm{Res}_u(\Psi) \\ -(2i\omega_1)^{-1}\mathrm{Res}_u(\Psi) \\ (2i\omega_2)^{-1}\mathrm{Res}_v(\Psi) \\ -(2i\omega_2)^{-1}\mathrm{Res}_v(\Psi) \end{pmatrix}.$$

$\mathcal{G}(\Psi_u, \mathcal{R})$ contains all terms with sufficiently high order in ε such that they cause no difficulties when passing to the $\mathcal{O}(1)$ time scale. In order to get rid of the $\mathcal{O}(\varepsilon^{-2})$ terms, we use energy estimates to control $\mathcal{B}_1$ and normal form transformations to control $\mathcal{B}_2$ and $\mathcal{B}_3$. Specifically, we try to eliminate $\mathcal{B}_2$ and $\mathcal{B}_3$ with the near identity change of coordinates

$$\widetilde{\mathcal{R}} = \mathcal{R} + \mathcal{Q}(\Psi_u, \mathcal{R}) + \mathcal{P}(\psi_v, \mathcal{R}).$$

We have $\widetilde{\mathcal{R}} = (\widetilde{R}_1, \widetilde{R}_{-1}, \widetilde{R}_2, \widetilde{R}_{-2})^T$. The mapping $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_{-1}, \mathcal{Q}_2, \mathcal{Q}_{-2})^T$ consists of bilinear mappings $\mathcal{Q}_j$ of the form

$$\widehat{\mathcal{Q}}_j(\widehat{\Psi}_u, \widehat{\mathcal{R}}) = \sum_{j_1 \in \{\pm 1, \pm 2\}} \widehat{\mathcal{Q}}_{j,j_1}(\widehat{\Psi}_u, \widehat{R}_{j_1}),$$

in Fourier space, with

$$\widehat{\mathcal{Q}}_{j,j_1}(\widehat{\Psi}_u, \widehat{R}_{j_1}) = \int q_{j,j_1}(k)\widehat{\Psi}_u(k-l)\widehat{R}_{j_1}(l)\,\mathrm{d}l, \quad j \in \{\pm 1\},$$

and

$$\widehat{\mathcal{Q}}_{j,j_1}(\widehat{\Psi}_u, \widehat{R}_{j_1}) = \int q_{j,j_1}(k)\overline{\widehat{\Psi}_u}(k-l)\widehat{R}_{j_1}(l)\,\mathrm{d}l \\ + \int q'_{j,j_1}(k)\widehat{\Psi}_u(k-l)\overline{\widehat{R}_{j_1}}(l)\,\mathrm{d}l, \quad j \in \{\pm 2\}.$$

Further, $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_{-1}, 0, 0)^T$ consists of bilinear mappings $\mathcal{P}_j$ which have the form

$$\widehat{\mathcal{P}}_j(\widehat{\psi}_v, \widehat{\mathcal{R}}) = \sum_{j_1 \in \{\pm 1, \pm 2\}} \widehat{\mathcal{P}}_{j,j_1}(\widehat{\psi}_v, \widehat{R}_{j_1}),$$

in Fourier space, with

$$\widehat{\mathcal{P}}_{j,j_1}(\widehat{\psi}_v, \widehat{R}_{j_1}) = \int p_{j,j_1}(k)\widehat{\psi}_v(k-l)\widehat{R}_{j_1}(l)\,\mathrm{d}l, \quad j \in \{\pm 1\}.$$

In the following, we also write

$$\widehat{\mathcal{Q}}(\widehat{\Psi}_u, \widehat{\mathcal{R}}) = \sum_{j_1 \in \{\pm 1, \pm 2\}} \widehat{\mathbf{Q}}_{j_1}(\widehat{\Psi}_u, \widehat{R}_{j_1}), \qquad \widehat{\mathcal{P}}(\widehat{\psi}_v, \widehat{\mathcal{R}}) = \sum_{j_1 \in \{\pm 1, \pm 2\}} \widehat{\mathbf{P}}_{j_1}(\widehat{\psi}_v, \widehat{R}_{j_1}).$$

We understand $\mathcal{Q}_j(\Psi_u, \mathcal{R})$ as the components of $\mathcal{Q}(\Psi_u, \mathcal{R})$ and $\mathbf{Q}_{j_1}(\Psi_u, R_{j_1})$ as the vector with components $\mathcal{Q}_{j,j_1}(\Psi_u, R_{j_1})$. The same holds for $\mathcal{P}$. By the linearity of the mappings $\mathcal{Q}$ and $\mathcal{P}$, we obtain

$$\begin{aligned} \partial_t \widetilde{\mathcal{R}} &= \partial_t \mathcal{R} + \mathcal{Q}(\partial_t \Psi_u, \mathcal{R}) + \mathcal{Q}(\Psi_u, \partial_t \mathcal{R}) + \mathcal{P}(\partial_t \psi_v, \mathcal{R}) + \mathcal{P}(\psi_v, \partial_t \mathcal{R}) \\ &= \Lambda \widetilde{\mathcal{R}} - \underline{\Lambda \mathcal{Q}(\Psi_u, \mathcal{R})} - \underline{\underline{\Lambda \mathcal{P}(\psi_v, \mathcal{R})}} \\ &\quad + \mathcal{B}_1(\Psi_u, \mathcal{R}) + \underline{\mathcal{B}_2(\Psi_u, \mathcal{R})} + \underline{\underline{\mathcal{B}_3(\psi_v, \mathcal{R})}} + \mathcal{G}(\Psi_u, \mathcal{R}) + \varepsilon^{-4}\mathrm{RES}(\Psi) \\ &\quad + \underline{\mathcal{Q}(i\omega_1(0)\Psi_u, \mathcal{R})} + \mathcal{Q}(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R}) \\ &\quad + \underline{\sum_{j_1 \in \{\pm 1, \pm 2\}} i\omega_{j_1} \mathbf{Q}_{j_1}(\Psi_u, R_{j_1})} + \mathcal{Q}(\Psi_u, \partial_t \mathcal{R} - \Lambda \mathcal{R}) \\ &\quad + \mathcal{Q}(\Psi_u, \Lambda \mathcal{R}) - \sum_{j_1 \in \{\pm 1, \pm 2\}} i\omega_{j_1} \mathbf{Q}_{j_1}(\Psi_u, R_{j_1}) \\ &\quad + \mathcal{P}(\partial_t \psi_v, \mathcal{R}) + \underline{\underline{\sum_{j_1 \in \{\pm 1, \pm 2\}} i\omega_{j_1} \mathbf{P}_{j_1}(\psi_v, R_{j_1})}} + \mathcal{P}(\psi_v, \partial_t \mathcal{R} - \Lambda \mathcal{R}) \\ &\quad + \mathcal{P}(\psi_v, \Lambda \mathcal{R}) - \sum_{j_1 \in \{\pm 1, \pm 2\}} i\omega_{j_1} \mathbf{P}_{j_1}(\psi_v, R_{j_1}). \end{aligned}$$

In order to eliminate the problematic $\mathcal{B}_2$- and $\mathcal{B}_3$-terms, we set the equally underlined terms to zero. Thus, $\mathcal{Q}$ and $\mathcal{P}$ have to satisfy

$$\begin{aligned} \mathcal{B}_2(\Psi_u, \mathcal{R}) &= \Lambda \mathcal{Q}(\Psi_u, \mathcal{R}) - \mathcal{Q}(i\omega_1(0)\Psi_u, \mathcal{R}) - \sum_{j_1 \in \{\pm 1, \pm 2\}} i\omega_{j_1} \mathbf{Q}_{j_1}(\Psi_u, R_{j_1}), \\ \mathcal{B}_3(\psi_v, \mathcal{R}) &= \Lambda \mathcal{P}(\psi_v, \mathcal{R}) - \sum_{j_1 \in \{\pm 1, \pm 2\}} i\omega_{j_1} \mathbf{P}_{j_1}(\psi_v, R_{j_1}). \end{aligned} \tag{2.35}$$

In Fourier space, this leads to

$$\begin{aligned} p_{\pm 1, \mp 1}(k) &= \varepsilon^{-2} \frac{1}{2\omega_1(k)} (\omega_1(k) + \omega_1(k))^{-1}, \\ q_{-1, \pm 2}(k) &= \varepsilon^{-2} \frac{1}{2\omega_1(k)} (\omega_1(k) + \omega_1(0) \pm \omega_2(k))^{-1}, \\ q_{2,-1}(k) = q'_{-2,-1}(k) &= \frac{1}{2}\omega_2(k)(\omega_2(k) + \omega_1(0) + \omega_1(k))^{-1}, \\ q_{-2,-1}(k) = q'_{2,-1}(k) &= \frac{1}{2}\omega_2(k)(\omega_2(k) - \omega_1(0) - \omega_1(k))^{-1}. \end{aligned}$$

Since no further terms have to be eliminated, the remaining kernels are set equal to 0. In the following two lemmas, depending on the value of the parameter γ, we will analyze the properties of the mappings $\mathcal{R} \mapsto \mathcal{Q}_j(\Psi_u, \mathcal{R})$ and $\mathcal{R} \mapsto \mathcal{P}_j(\psi_v, \mathcal{R})$ for $j \in \{\pm 1, \pm 2\}$.

Lemma 2.3.9. *Let $s \geq 0$ and $\gamma \in \mathbb{R}$ with $|\gamma| > 1$. Fix $h \in H^s(\mathbb{R}, \mathbb{C})$. Then, $\mathcal{R} \mapsto \mathcal{Q}_{\pm 1}(h, \mathcal{R})$ and $\mathcal{R} \mapsto \mathcal{P}_{\pm 1}(h, \mathcal{R})$ define continuous mappings from $(H^s(\mathbb{R}, \mathbb{C}))^4$ into $H^{s+1}(\mathbb{R}, \mathbb{C})$ and $\mathcal{R} \mapsto \mathcal{Q}_{\pm 2}(h, \mathcal{R})$ define continuous mappings from $(H^s(\mathbb{R}, \mathbb{C}))^4$ into $H^s(\mathbb{R}, \mathbb{C})$. More precisely, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\begin{aligned} \|\mathcal{Q}_{\pm 1}(h, \mathcal{R})\|_{H^s} &\leq C\varepsilon^2 \|h\|_{H^s} (\|R_2\|_{H^s} + \|R_{-2}\|_{H^s}), \\ \|\mathcal{Q}_{\pm 1}(h, \mathcal{R})\|_{H^{s+1}} &\leq C\varepsilon \|h\|_{H^s} (\|R_2\|_{H^s} + \|R_{-2}\|_{H^s}), \\ \|\mathcal{Q}_{\pm 2}(h, \mathcal{R})\|_{H^s} &\leq C \|h\|_{H^s} (\|R_1\|_{H^s} + \|R_{-1}\|_{H^s}), \\ \|\mathcal{P}_{\pm 1}(h, \mathcal{R})\|_{H^s} &\leq C\varepsilon^2 \|h\|_{H^s} (\|R_1\|_{H^s} + \|R_{-1}\|_{H^s}), \\ \|\mathcal{P}_{\pm 1}(h, \mathcal{R})\|_{H^{s+1}} &\leq C\varepsilon \|h\|_{H^s} (\|R_1\|_{H^s} + \|R_{-1}\|_{H^s}). \end{aligned} \tag{2.36}$$

Additionally, for $h \in H^{s+1}(\mathbb{R}, \mathbb{C})$, it holds

$$\|\mathcal{Q}_{\pm 2}(h, \mathcal{R})\|_{H^s} \leq C\varepsilon \|h\|_{H^{s+1}} (\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}}). \tag{2.37}$$

Proof. Let $k \in \mathbb{R}$. We recall that $\omega_1(k) = \varepsilon^{-2}\sqrt{1 + (\varepsilon k)^2}$ and $\omega_2(k) = \gamma^{-1}\varepsilon^{-1}k$. We show that the functions $(1 + |\cdot|)q_{-1,\pm 2}$, $(1 + |\cdot|)p_{\pm 1, \mp 1}$ and $q_{\pm 2, -1}$ are bounded for $|\gamma| > 1$. By using $\sqrt{1+(\varepsilon k)^2} > \varepsilon|k| \geq \pm\varepsilon k \geq \pm\gamma^{-1}\varepsilon k$, we obtain

$$\begin{aligned}
\sup_{k\in\mathbb{R}} |p_{\pm1,\mp1}(k)| &= \varepsilon^2 \cdot \sup_{k\in\mathbb{R}} \left| \frac{1}{2\sqrt{1+(\varepsilon k)^2}} \frac{1}{\sqrt{1+(\varepsilon k)^2}+\sqrt{1+(\varepsilon k)^2}} \right| \\
&= \frac{1}{4}\varepsilon^2 \cdot \sup_{k\in\mathbb{R}} \left| \frac{1}{1+(\varepsilon k)^2} \right| \leq \frac{1}{4}\varepsilon^2, \\
\sup_{k\in\mathbb{R}} |k \cdot p_{\pm1,\mp1}(k)| &= \frac{1}{4}\varepsilon \cdot \sup_{k\in\mathbb{R}} \left| \frac{\varepsilon k}{1+(\varepsilon k)^2} \right| \leq \frac{1}{4}\varepsilon, \\
\sup_{k\in\mathbb{R}} |q_{-1,\pm2}(k)| &= \frac{1}{2}\varepsilon^2 \cdot \sup_{k\in\mathbb{R}} \left| \frac{1}{\sqrt{1+(\varepsilon k)^2}} \frac{1}{\sqrt{1+(\varepsilon k)^2}+1\pm\gamma^{-1}\varepsilon k} \right| \leq \frac{1}{2}\varepsilon^2, \\
\sup_{k\in\mathbb{R}} |k \cdot q_{-1,\pm2}(k)| &= \frac{1}{2}\varepsilon \cdot \sup_{k\in\mathbb{R}} \left| \frac{\varepsilon k}{\sqrt{1+(\varepsilon k)^2}} \frac{1}{\sqrt{1+(\varepsilon k)^2}+1\pm\gamma^{-1}\varepsilon k} \right| \leq \frac{1}{2}\varepsilon.
\end{aligned}$$

The function $q_{2,-1}$ is strictly increasing due to the relation

$$\frac{\mathrm{d}}{\mathrm{d}k} q_{2,-1}(k) = \frac{1}{2}\varepsilon\gamma \frac{1+\sqrt{1+(\varepsilon k)^2}}{\sqrt{1+(\varepsilon k)^2}(\gamma\sqrt{1+(\varepsilon k)^2}+\gamma+\varepsilon k)^2} > 0.$$

Thus, $q_{2,-1}$ is $\mathcal{O}(1)$ bounded as we have

$$\begin{aligned}
\lim_{k\to\infty} q_{2,-1}(k) &= \lim_{k\to\infty} \frac{1}{2} \frac{\gamma^{-1}\varepsilon k}{\gamma^{-1}\varepsilon k + 1 + \sqrt{1+(\varepsilon k)^2}} = \frac{1}{2(1+\gamma)}, \\
\lim_{k\to-\infty} q_{2,-1}(k) &= \lim_{k\to\infty} \frac{1}{2} \frac{-\gamma^{-1}\varepsilon k}{-\gamma^{-1}\varepsilon k + 1 + \sqrt{1+(\varepsilon k)^2}} = \frac{1}{2(1-\gamma)}.
\end{aligned}$$

Analogously, one can show that $q_{-2,-1}$ is strictly decreasing and $\mathcal{O}(1)$ bounded. Thus, we get

$$\sup_{k\in\mathbb{R}} |q_{\pm2,-1}(k)| \leq C.$$

With Sobolev's embedding theorem, we now directly obtain (2.36). The estimate (2.37) follows from Sobolev's embedding theorem and

$$\begin{aligned}
|q_{2,-1}(k)| &\leq \frac{1}{2}|k| \cdot \sup_{k\in\mathbb{R}} \left| \frac{\gamma^{-1}\varepsilon}{\gamma^{-1}\varepsilon k + 1 + \sqrt{1+(\varepsilon k)^2}} \right| \leq C\varepsilon|k|, \\
|q_{-2,-1}(k)| &= \frac{1}{2}|k| \cdot \sup_{k\in\mathbb{R}} \left| \frac{\gamma^{-1}\varepsilon}{\gamma^{-1}\varepsilon k - 1 - \sqrt{1+(\varepsilon k)^2}} \right| \leq C\varepsilon|k|.
\end{aligned}$$

□

Lemma 2.3.10. *Let $s \geq 0$ and $\gamma \in \mathbb{R}$ with $|\gamma| = 1$. Fix $h \in H^{s+1}(\mathbb{R}, \mathbb{C})$. Then, $\mathcal{R} \mapsto \mathcal{Q}_{\pm 1}(h, \mathcal{R})$ and $\mathcal{R} \mapsto \mathcal{P}_{\pm 1}(h, \mathcal{R})$ define continuous mappings from $(H^s(\mathbb{R}, \mathbb{C}))^4$ into $H^{s+1}(\mathbb{R}, \mathbb{C})$ and $\mathcal{R} \mapsto \mathcal{Q}_{\pm 2}(h, \mathcal{R})$ define continuous mappings from $(H^{s+1}(\mathbb{R}, \mathbb{C}))^2 \times (H^s(\mathbb{R}, \mathbb{C}))^2$ into $H^s(\mathbb{R}, \mathbb{C})$. More precisely, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\begin{aligned}
\|\mathcal{Q}_{\pm 1}(h, \mathcal{R})\|_{H^s} &\leq C\varepsilon^2 \|h\|_{H^s}(\|R_2\|_{H^s} + \|R_{-2}\|_{H^s}), \\
\|\mathcal{Q}_{\pm 1}(h, \mathcal{R})\|_{H^{s+1}} &\leq C\varepsilon \|h\|_{H^s}(\|R_2\|_{H^s} + \|R_{-2}\|_{H^s}), \\
\|\mathcal{Q}_{\pm 2}(h, \mathcal{R})\|_{H^s} &\leq C\varepsilon \|h\|_{H^{s+1}}(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}}), \\
\|\mathcal{P}_{\pm 1}(h, \mathcal{R})\|_{H^s} &\leq C\varepsilon^2 \|h\|_{H^s}(\|R_1\|_{H^s} + \|R_{-1}\|_{H^s}), \\
\|\mathcal{P}_{\pm 1}(h, \mathcal{R})\|_{H^{s+1}} &\leq C\varepsilon \|h\|_{H^s}(\|R_1\|_{H^s} + \|R_{-1}\|_{H^s}).
\end{aligned} \tag{2.38}$$

Proof. We want to show that the functions $(1 + |\cdot|)q_{-1,\pm 2}$ and $(1 + |\cdot|)p_{\pm 1,\mp 1}$ are bounded and that the functions $q_{\pm 2,-1}$ are asymptotically linear. By using $\sqrt{1 + (\varepsilon k)^2} > \varepsilon|k| \geq \pm\varepsilon k$, analogous to the proof of Lemma 2.3.9, we can conclude

$$\begin{aligned}
\sup_{k \in \mathbb{R}} |p_{\pm 1,\mp 1}(k)| &= \frac{1}{4}\varepsilon^2 \cdot \sup_{k \in \mathbb{R}} \left| \frac{1}{1 + (\varepsilon k)^2} \right| \leq \frac{1}{4}\varepsilon^2, \\
\sup_{k \in \mathbb{R}} |k \cdot p_{\pm 1,\mp 1}(k)| &= \frac{1}{4}\varepsilon \cdot \sup_{k \in \mathbb{R}} \left| \frac{\varepsilon k}{1 + (\varepsilon k)^2} \right| \leq \frac{1}{4}\varepsilon, \\
\sup_{k \in \mathbb{R}} |q_{-1,\pm 2}(k)| &= \frac{1}{2}\varepsilon^2 \cdot \sup_{k \in \mathbb{R}} \left| \frac{1}{\sqrt{1 + (\varepsilon k)^2}} \frac{1}{\sqrt{1 + (\varepsilon k)^2} + 1 \pm \varepsilon k} \right| \leq \frac{1}{2}\varepsilon^2, \\
\sup_{k \in \mathbb{R}} |k \cdot q_{-1,\pm 2}(k)| &= \frac{1}{2}\varepsilon \cdot \sup_{k \in \mathbb{R}} \left| \frac{\varepsilon k}{\sqrt{1 + (\varepsilon k)^2}} \frac{1}{\sqrt{1 + (\varepsilon k)^2} + 1 \pm \varepsilon k} \right| \leq \frac{1}{2}\varepsilon.
\end{aligned}$$

The remaining kernels $q_{\pm 2,-1}$ are not bounded. Instead, we have that

$$q_{\pm 2,-1}(k) = \pm\frac{1}{2}\varepsilon k + \frac{1}{2} + \mathcal{O}(k^{-1}) \quad \text{for } k \to \mp\infty.$$

Thus, with

$$\begin{aligned}
|q_{2,-1}(k)| &\leq \frac{1}{2}\varepsilon|k| \cdot \sup_{k \in \mathbb{R}} \left| \frac{1}{\varepsilon k + 1 + \sqrt{1 + (\varepsilon k)^2}} \right| \leq C\varepsilon|k|, \\
|q_{-2,-1}(k)| &\leq \frac{1}{2}\varepsilon|k| \cdot \sup_{k \in \mathbb{R}} \left| \frac{1}{\varepsilon k - 1 - \sqrt{1 + (\varepsilon k)^2}} \right| \leq C\varepsilon|k|.
\end{aligned}$$

and Sobolev's embedding theorem we directly obtain (2.38). □

Remark 2.3.11. Lemma 2.3.9 and Lemma 2.3.10 were formulated and proved for $x \in \mathbb{R}$. However, they are also valid for $x \in \mathbb{T}$.

2.3.5 Estimates for the error

In this section, we want to prove Theorem 2.3.1. The proof is a non-trivial task since $\partial_t^2 u, \partial_t^2 v = \mathcal{O}(\varepsilon^{-2})$ but solutions have to be bounded on an $\mathcal{O}(1)$ time scale. The idea is to define an energy $\mathcal{E}$, which is equivalent to the $H^{s+1}\times H^{s+1}\times H^s\times H^s$ norm of $\mathcal{R}$ and satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathcal{O}(1)$. In order to eliminate the problematic $\mathcal{B}_2$- and $\mathcal{B}_3$-terms from Section 2.3.4, the energy $\mathcal{E}$ has to contain the mappings $\mathcal{Q}$ and $\mathcal{P}$ in such a way that these problematic terms cancel in the time derivative of $\mathcal{E}$. However, there are some terms left of order $\mathcal{O}(\varepsilon^{-1})$. By setting $R_{\pm 1} = W_{\pm 1} e^{\pm i\varepsilon^{-2}t}$, we can write the sum of all remaining $\mathcal{O}(\varepsilon^{-1})$ terms as a time derivative and, thus, include them in the energy $\mathcal{E}$. Analogous to Section 2.2.5, the idea of the ansatz is to shift the linear part of the R_1-equation by $-\varepsilon^{-2}$ such that the resulting spectral situation of the linearized W_1-equation is similar to the situation in Section 2.1, cf. Figure 2.1 and Figure 2.3. It turns out that the time derivative of $\mathcal{E}$ then only contains terms of order $\mathcal{O}(1)$ which can be estimated by the energy $\mathcal{E}$. An application of Gronwall's inequality delivers an $\mathcal{O}(1)$ bound of the error $\mathcal{R}$. In the following, we consider the system

$$\begin{aligned}
\partial_t R_{\pm 1} &= \pm i\omega_1 R_{\pm 1} \mp \varepsilon^{-2}(2i\omega_1)^{-1}\left(\Psi_u R_v + \Psi_v(R_1 + R_{-1}) + \varepsilon^2 (R_1 + R_{-1})R_v\right) \\
&\quad \pm \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u, \\
\partial_t R_v &= i\omega_2 R_q, \\
\partial_t R_q &= i\omega_2 R_v + i\omega_2(\overline{\Psi_u}(R_1 + R_{-1}) + \Psi_u(\overline{R_1} + \overline{R_{-1}}) + \varepsilon^2 |R_1 + R_{-1}|^2) \\
&\quad + \varepsilon^{-4}(i\omega_2)^{-1}\mathrm{Res}_v,
\end{aligned}$$

which was derived in Section 2.3.4, and the energy

$$\begin{aligned}
E_l &= \|\partial_x^l R_1\|_{L^2}^2 + \|\partial_x^l R_{-1}\|_{L^2}^2 + \frac{1}{2}\|\partial_x^l R_q\|_{L^2}^2 + \frac{1}{2}\|\partial_x^l R_v\|_{L^2}^2 \\
&\quad + \|\varepsilon\widetilde{\omega}_1 \partial_x^l R_1\|_{L^2}^2 + \|\varepsilon\widetilde{\omega}_1 \partial_x^l R_{-1}\|_{L^2}^2 + \|\partial_x^{l+1} R_1\|_{L^2}^2 + \|\partial_x^{l+1} R_{-1}\|_{L^2}^2,
\end{aligned}$$

which was derived in Section 2.2.5. Subsequently, we calculate the time derivative of E_l step by step. We will underline all terms that need to be eliminated. First, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x^l R_1\|_{L^2}^2 + \|\partial_x^l R_{-1}\|_{L^2}^2\right) = I_{l,0} + I_{l,1} + I_{l,2},$$

where

$$\begin{aligned}
I_{l,0} &= \int (i\omega_1 \partial_x^l R_1 \partial_x^l \overline{R_1} + c.c.)\,\mathrm{d}x, \\
I_{l,1} &= \int (-i\omega_1 \partial_x^l R_{-1} \partial_x^l \overline{R_{-1}} + c.c.)\,\mathrm{d}x, \\
I_{l,2} &= \mathrm{Im}\int (\varepsilon^2\omega_1)^{-1}\partial_x^l(\overline{\Psi_u} R_v + (\Psi_v + \varepsilon^2 R_v)(\overline{R_1} + \overline{R_{-1}}) - \varepsilon^{-2}\overline{\mathrm{Res}_u})\partial_x^l (R_1 - R_{-1})\,\mathrm{d}x.
\end{aligned}$$

Next, we calculate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\partial_x^l R_q\|_{L^2}^2+\frac{1}{2}\|\partial_x^l R_v\|_{L^2}^2\right)\\
&\quad=\int i\omega_2\partial_x^l\left(\overline{\Psi_u}(R_1+\underline{R_{-1}})+\Psi_u(\overline{R_1}+\underline{\overline{R_{-1}}})+\varepsilon^2(|R_1|^2+|R_{-1}|^2)\right)\partial_x^l R_q\,\mathrm{d}x\\
&\qquad+I_{l,3}+I_{l,4}+I_{l,5},
\end{aligned}$$

where

$$\begin{aligned}
I_{l,3}&=\int(\partial_x^l R_q i\omega_2\partial_x^l R_v+\partial_x^l R_v i\omega_2\partial_x^l R_q)\,\mathrm{d}x,\\
I_{l,4}&=\int\varepsilon^2 i\omega_2\partial_x^l(R_1\overline{R_{-1}}+\overline{R_1}R_{-1})\partial_x^l R_q\,\mathrm{d}x,\\
I_{l,5}&=\int\varepsilon^{-4}(i\omega_2)^{-1}\partial_x^l\mathrm{Res}_v\partial_x^l R_q\,\mathrm{d}x.
\end{aligned}$$

We continue with the time derivative of the second line in the energy E_l, where we use that

$$2\varepsilon^2\widetilde{\omega}_1(\cdot)\omega_1(\cdot)=(\varepsilon\widetilde{\omega}_1(\cdot))^2+(\cdot)^2.$$

We obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varepsilon\widetilde{\omega}_1\partial_x^l R_1\|_{L^2}^2+\|\varepsilon\widetilde{\omega}_1\partial_x^l R_{-1}\|_{L^2}^2+\|\partial_x^{l+1}R_1\|_{L^2}^2+\|\partial_x^{l+1}R_{-1}\|_{L^2}^2\right)\\
&\quad=\int\left(i\partial_x^l\left(\Psi_u R_v+\psi_v(R_1+\underline{R_{-1}})+\varepsilon^2 R_1 R_v\right)\widetilde{\omega}_1\partial_x^l\overline{R_1}+c.c.\right)\mathrm{d}x\\
&\qquad-\int\left(i\partial_x^l\left(\underline{\Psi_u R_v}+\psi_v(\underline{R_1}+R_{-1})+\varepsilon^2 R_{-1}R_v\right)\widetilde{\omega}_1\partial_x^l\overline{R_{-1}}+c.c.\right)\mathrm{d}x\\
&\qquad+I_{l,6}+I_{l,7}+I_{l,8}+I_{l,9},
\end{aligned}$$

where

$$\begin{aligned}
I_{l,6}&=\int\left((i\omega_1\varepsilon\widetilde{\omega}_1\partial_x^l R_1\varepsilon\widetilde{\omega}_1\partial_x^l\overline{R_1}+i\omega_1\partial_x^{l+1}R_1\partial_x^{l+1}\overline{R_1})+c.c.\right)\mathrm{d}x,\\
I_{l,7}&=\int\left(-(i\omega_1\varepsilon\widetilde{\omega}_1\partial_x^l R_{-1}\varepsilon\widetilde{\omega}_1\partial_x^l\overline{R_{-1}}+i\omega_1\partial_x^{l+1}R_{-1}\partial_x^{l+1}\overline{R_{-1}})+c.c.\right)\mathrm{d}x,\\
I_{l,8}&=\varepsilon^2\int\Bigg(\Big(i\partial_x^l(R_{-1}R_v)\widetilde{\omega}_1\partial_x^l\overline{R_1}-i\partial_x^l(R_1R_v)\widetilde{\omega}_1\partial_x^l\overline{R_{-1}}\\
&\qquad+i\partial_x^l(\psi_{v,2}(R_1+R_{-1}))\widetilde{\omega}_1\partial_x^l(\overline{R_1}-\overline{R_{-1}})\Big)+c.c.\Bigg)\mathrm{d}x,\\
I_{l,9}&=2\mathrm{Im}\int\varepsilon^{-2}\widetilde{\omega}_1\partial_x^l\mathrm{Res}_u\partial_x^l(\overline{R_1}-\overline{R_{-1}})\,\mathrm{d}x.
\end{aligned}$$

The terms $I_{l,j}$ will be estimated subsequently. The remaining terms in the time derivative of E_l, including the underlined terms, cannot be estimated directly since they are at least of order $\mathcal{O}(\varepsilon^{-1})$. In order to eliminate the underlined terms, we add an additional energy $\widetilde{E}_l$ to the energy E_l. According to Section 2.3.4, the energy $\widetilde{E}_l$ has to depend on the mappings $\mathcal{Q}$ and $\mathcal{P}$. Thus, we define

$$\begin{aligned}\widetilde{E}_l = &\int \left(2\partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\Psi_u, \mathcal{R}) + 2\partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\Psi_u, \mathcal{R})\right) \mathrm{d}x \\ &+ \int \left(\partial_x^l \overline{R_1} 2\varepsilon^2 \widetilde{\omega}_1 \omega_1 \partial_x^l (\mathcal{Q}_1(\Psi_u, \mathcal{R}) + \mathcal{P}_1(\psi_v, \mathcal{R})) + c.c.\right) \mathrm{d}x \\ &+ \int \left(\partial_x^l \overline{R_{-1}} 2\varepsilon^2 \widetilde{\omega}_1 \omega_1 \partial_x^l (\mathcal{Q}_{-1}(\Psi_u, \mathcal{R}) + \mathcal{P}_{-1}(\psi_v, \mathcal{R})) + c.c.\right) \mathrm{d}x.\end{aligned}$$

In the following, we calculate the time derivative of $\widetilde{E}_l$. We use that R_j solves the error equation (2.34). Using integration by parts yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \int \left(2\partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\Psi_u, \mathcal{R}) + 2\partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\Psi_u, \mathcal{R})\right) \mathrm{d}x \\ &= 2\int \big(\partial_t \partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\Psi_u, \mathcal{R}) + \partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\partial_t \Psi_u, \mathcal{R}) + \partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\Psi_u, \partial_t \mathcal{R}) \\ &\qquad + \partial_t \partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\Psi_u, \mathcal{R}) + \partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\partial_t \Psi_u, \mathcal{R}) + \partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\Psi_u, \partial_t \mathcal{R})\big) \mathrm{d}x \\ &= 2\int \Bigg(\partial_x^l R_2 \partial_x^l \Bigg(-i\omega_2 \mathcal{Q}_2(\Psi_u, \mathcal{R}) + \mathcal{Q}_2(i\omega_1(0)\Psi_u, \mathcal{R}) + \sum_{j_1} i\omega_{j_1} \mathcal{Q}_{2,j_1}(\Psi_u, R_{j_1})\Bigg) \\ &\qquad + \partial_x^l R_{-2} \partial_x^l \Bigg(i\omega_2 \mathcal{Q}_{-2}(\Psi_u, \mathcal{R}) + \mathcal{Q}_{-2}(i\omega_1(0)\Psi_u, \mathcal{R}) + \sum_{j_1} i\omega_{j_1} \mathcal{Q}_{-2,j_1}(\Psi_u, R_{j_1})\Bigg)\Bigg) \mathrm{d}x \\ &\quad + I_{l,10} + I_{l,11} + I_{l,12} + I_{l,13},\end{aligned}$$

where

$$\begin{aligned}I_{l,10} &= 2\int \left(\partial_x^l (\partial_t R_2 - i\omega_2 R_2) \partial_x^l \mathcal{Q}_2 + \partial_x^l (\partial_t R_{-2} + i\omega_2 R_{-2}) \partial_x^l \mathcal{Q}_{-2}\right) \mathrm{d}x, \\ I_{l,11} &= 2\int \left(\partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R}) + \partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R})\right) \mathrm{d}x, \\ I_{l,12} &= 2\int \left(\partial_x^l R_2 \partial_x^l \mathcal{Q}_2(\Psi_u, \partial_t \mathcal{R} - \Lambda\mathcal{R}) + \partial_x^l R_{-2} \partial_x^l \mathcal{Q}_{-2}(\Psi_u, \partial_t \mathcal{R} - \Lambda\mathcal{R})\right) \mathrm{d}x, \\ I_{l,13} &= 2\int \Bigg(\partial_x^l R_2 \partial_x^l \Bigg(\mathcal{Q}_2(\Psi_u, \Lambda\mathcal{R}) - \sum_{j_1} i\omega_{j_1} \mathcal{Q}_{2,j_1}(\Psi_u, R_{j_1})\Bigg) \\ &\qquad + \partial_x^l R_{-2} \partial_x^l \Bigg(\mathcal{Q}_{-2}(\Psi_u, \Lambda\mathcal{R}) - \sum_{j_1} i\omega_{j_1} \mathcal{Q}_{-2,j_1}(\Psi_u, R_{j_1})\Bigg)\Bigg) \mathrm{d}x.\end{aligned}$$

Further, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \big(\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\Psi_u,\mathcal{R})+\mathcal{P}_1(\psi_v,\mathcal{R}))+c.c.\big)\,\mathrm{d}x\\
&=\int \big(\partial_t\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\Psi_u,\mathcal{R})+\mathcal{P}_1(\psi_v,\mathcal{R}))+c.c.\\
&\qquad+\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\partial_t\Psi_u,\mathcal{R})+\mathcal{P}_1(\partial_t\psi_v,\mathcal{R}))+c.c.\\
&\qquad+\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\Psi_u,\partial_t\mathcal{R})+\mathcal{P}_1(\psi_v,\partial_t\mathcal{R}))+c.c.\big)\,\mathrm{d}x\\
&=\int\bigg(\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\\
&\qquad\times\partial_x^l\bigg(\bigg(-i\omega_1\mathcal{Q}_1(\Psi_u,\mathcal{R})+\mathcal{Q}_1(i\omega_1(0)\Psi,\mathcal{R})+\sum_{j_1}i\omega_{j_1}\mathcal{Q}_{1,j_1}(\Psi_u,R_{j_1})\bigg)\\
&\qquad\quad+\bigg(-i\omega_1\mathcal{P}_1(\psi_v,\mathcal{R})+\sum_{j_1}i\omega_{j_1}\mathcal{P}_{1,j_1}(\psi_v,R_{j_1})\bigg)\bigg)+c.c.\bigg)\,\mathrm{d}x\\
&\qquad+I_{l,14}+I_{l,15}+I_{l,16}+I_{l,17},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{l,14}&=\int\big(\partial_x^l(\overline{\partial_tR_1-i\omega_1R_1})2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\Psi_u,\mathcal{R})+\mathcal{P}_1(\psi_v,\mathcal{R}))+c.c.\big)\,\mathrm{d}x,\\
I_{l,15}&=\int\big(\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\partial_t\Psi_u-i\omega_1(0)\Psi_u,\mathcal{R})+\mathcal{P}_1(\partial_t\psi_v,\mathcal{R}))+c.c.\big)\,\mathrm{d}x,\\
I_{l,16}&=\int\big(\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_1(\Psi_u,\partial_t\mathcal{R}-\Lambda\mathcal{R})+\mathcal{P}_1(\psi_v,\partial_t\mathcal{R}-\Lambda\mathcal{R}))+c.c.\big)\,\mathrm{d}x,\\
I_{l,17}&=\int\bigg(\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l\bigg(\mathcal{Q}_1(\Psi_u,\Lambda\mathcal{R})-\sum_{j_1}i\omega_{j_1}\mathcal{Q}_{1,j_1}(\Psi_u,R_{j_1})\bigg)+c.c.\\
&\qquad+\partial_x^l\overline{R_1}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l\bigg(\mathcal{P}_1(\psi_v,\Lambda\mathcal{R})-\sum_{j_1}i\omega_{j_1}\mathcal{P}_{1,j_1}(\psi_v,R_{j_1})\bigg)+c.c.\bigg)\,\mathrm{d}x.
\end{aligned}
$$

Moreover, it is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \big(\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\Psi_u,\mathcal{R})+\mathcal{P}_{-1}(\psi_v,\mathcal{R}))+c.c.\big)\,\mathrm{d}x\\
&=\int \big(\partial_t\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\Psi_u,\mathcal{R})+\mathcal{P}_{-1}(\psi_v,\mathcal{R}))+c.c.\\
&\qquad+\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\partial_t\Psi_u,\mathcal{R})+\mathcal{P}_{-1}(\partial_t\psi_v,\mathcal{R}))+c.c.\\
&\qquad+\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\Psi_u,\partial_t\mathcal{R})+\mathcal{P}_{-1}(\psi_v,\partial_t\mathcal{R}))+c.c.\big)\,\mathrm{d}x\\
&=\int \Bigg(\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\\
&\qquad\times\partial_x^l\Bigg(\Bigg(i\omega_1\mathcal{Q}_{-1}(\Psi_u,\mathcal{R})+\mathcal{Q}_{-1}(i\omega_1(0)\Psi,\mathcal{R})+\sum_{j_1}i\omega_{j_1}\mathcal{Q}_{-1,j_1}(\Psi_u,R_{j_1})\Bigg)\\
&\qquad\qquad+\Bigg(i\omega_1\mathcal{P}_{-1}(\psi_v,\mathcal{R})+\sum_{j_1}i\omega_{j_1}\mathcal{P}_{-1,j_1}(\psi_v,R_{j_1})\Bigg)\Bigg)+c.c.\Bigg)\,\mathrm{d}x\\
&\qquad+I_{l,18}+I_{l,19}+I_{l,20}+I_{l,21},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{l,18}&=\int \big(\partial_x^l(\overline{\partial_tR_{-1}+i\omega_1R_{-1}})2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\Psi_u,\mathcal{R})+\mathcal{P}_{-1}(\psi_v,\mathcal{R}))+c.c.\big)\,\mathrm{d}x,\\
I_{l,19}&=\int \big(\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\partial_t\Psi_u-i\omega_1(0)\Psi_u,\mathcal{R})+\mathcal{P}_{-1}(\partial_t\psi_v,\mathcal{R}))+c.c.\big)\,\mathrm{d}x,\\
I_{l,20}&=\int \big(\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l(\mathcal{Q}_{-1}(\Psi_u,\partial_t\mathcal{R}-\Lambda\mathcal{R})+\mathcal{P}_{-1}(\psi_v,\partial_t\mathcal{R}-\Lambda\mathcal{R}))+c.c.\big)\,\mathrm{d}x,\\
I_{l,21}&=\int \Bigg(\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l\Bigg(\mathcal{Q}_{-1}(\Psi_u,\Lambda\mathcal{R})-\sum_{j_1}i\omega_{j_1}\mathcal{Q}_{-1,j_1}(\Psi_u,R_{j_1})\Bigg)+c.c.\\
&\qquad+\partial_x^l\overline{R_{-1}}2\varepsilon^2\widetilde{\omega}_1\omega_1\partial_x^l\Bigg(\mathcal{P}_{-1}(\psi_v,\Lambda\mathcal{R})-\sum_{j_1}i\omega_{j_1}\mathcal{P}_{-1,j_1}(\psi_v,R_{j_1})\Bigg)+c.c.\Bigg)\,\mathrm{d}x.
\end{aligned}
$$

We consider the time derivate of $E_l+\widetilde{E}_l$. By using the diagonalized variables

$$
R_v=R_2+R_{-2},\qquad R_q=R_2-R_{-2}
$$

and the construction of the mappings $\mathcal{Q}_j$, cf. (2.35), the remaining terms in the time derivative of $\widetilde{E}_l$, except for the terms $I_{l,j}$, cancel with the underlined terms in the time derivate of E_l. Hence, we can set $R_{\pm 1} = W_{\pm 1} e^{\pm i\varepsilon^{-2}t}$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E_l + \widetilde{E}_l) = &\int i\omega_2 \partial_x^l \left(\overline{\psi_u} W_1 + \psi_u \overline{W_1} + \varepsilon^2(|W_1|^2 + |W_{-1}|^2)\right) \partial_x^l R_q \,\mathrm{d}x \\ &+ \int \left(i\partial_x^l \left(\psi_u R_v + \psi_v W_1 + \varepsilon^2 W_1 R_v\right) \widetilde{\omega}_1 \partial_x^l \overline{W_1} + c.c.\right) \mathrm{d}x \\ &- \int \left(i\partial_x^l \left(\psi_v W_{-1} + \varepsilon^2 W_{-1} R_v\right) \widetilde{\omega}_1 \partial_x^l \overline{W_{-1}} + c.c.\right) \mathrm{d}x \\ &+ \sum_{j=0}^{21} I_{l,j}.\end{aligned}$$

We proceed to rewrite the right-hand side of this equation. For the first integral, we use integration by parts and the fact that $i\omega_2 R_q = \partial_t R_v$. For the second and third integral, we use that W_1 and W_{-1} solve

$$\begin{aligned}\partial_t W_1 &= i\widetilde{\omega}_1 W_1 - \varepsilon^{-2}(2i\omega_1)^{-1}\left(\psi_u R_v + (\Psi_v + \varepsilon^2 R_v)(W_1 + W_{-1}e^{-2i\varepsilon^{-2}t})\right) \\ &\quad + \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u e^{-i\varepsilon^{-2}t}, \\ \partial_t W_{-1} &= -i\widetilde{\omega}_1 W_{-1} + \varepsilon^{-2}(2i\omega_1)^{-1}\left(\psi_u R_v e^{2i\varepsilon^{-2}t} + (\Psi_v + \varepsilon^2 R_v)(W_1 e^{2i\varepsilon^{-2}t} + W_{-1})\right) \\ &\quad - \varepsilon^{-4}(2i\omega_1)^{-1}\mathrm{Res}_u e^{i\varepsilon^{-2}t}.\end{aligned} \tag{2.39}$$

Therefore, we can combine the integrals in the time derivative of $E_l + \widetilde{E}_l$ to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(E_l + \widetilde{E}_l) \\ &= -\Bigg(\int (\partial_x^l(\psi_v W_1)\partial_t\partial_x^l \overline{W_1} + \partial_x^l(\psi_v \overline{W_1})\partial_t\partial_x^l W_1)\,\mathrm{d}x \\ &\quad + \int (\partial_x^l(\psi_v W_{-1})\partial_t\partial_x^l \overline{W_{-1}} + \partial_x^l(\psi_v \overline{W_{-1}})\partial_t\partial_x^l W_{-1})\,\mathrm{d}x \\ &\quad + \int (\partial_x^l(\psi_u R_v)\partial_t\partial_x^l \overline{W_1} + \partial_x^l(\psi_u \overline{W_1})\partial_t\partial_x^l R_v)\,\mathrm{d}x \\ &\quad + \int (\partial_x^l(\overline{\psi_u} R_v)\partial_t\partial_x^l W_1 + \partial_x^l(\overline{\psi_u} W_1)\partial_t\partial_x^l R_v)\,\mathrm{d}x \\ &\quad + \varepsilon^2 \int (\partial_x^l(R_v W_1)\partial_t\partial_x^l \overline{W_1} + \partial_x^l(R_v \overline{W_1})\partial_t\partial_x^l W_1 + \partial_x^l(|W_1|^2)\partial_t\partial_x^l R_v)\,\mathrm{d}x \\ &\quad + \varepsilon^2 \int (\partial_x^l(R_v W_{-1})\partial_t\partial_x^l \overline{W_{-1}} + \partial_x^l(R_v \overline{W_{-1}})\partial_t\partial_x^l W_{-1} + \partial_x^l(|W_{-1}|^2)\partial_t\partial_x^l R_v)\,\mathrm{d}x\Bigg) \\ &\quad + \sum_{j=0}^{23} I_{l,j},\end{aligned}$$

where

$$I_{l,22} = \int \left(\partial_x^l(\psi_u R_v + \psi_v W_1 + \varepsilon^2 W_1 R_v)\partial_x^l(\overline{\partial_t W_1 - i\widetilde{\omega}_1 W_1}) + c.c.\right) \mathrm{d}x,$$

$$I_{l,23} = \int \left(\partial_x^l(\psi_v W_{-1} + \varepsilon^2 W_{-1} R_v)\partial_x^l(\overline{\partial_t W_{-1} + i\widetilde{\omega}_1 W_{-1}}) + c.c.\right) \mathrm{d}x.$$

With the results from Section 2.2.6, we can conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \sum_{i=1}^{8} t_i + \sum_{l\in\{0,s\}} \sum_{j=0}^{23} I_{l,j},$$

where the terms t_i are given by (2.26), with W_v replaced by R_v, and the full energy $\mathcal{E}$ is defined by

$$\mathcal{E} = E_0 + E_s + \widetilde{E}_0 + \widetilde{E}_s + E_*,$$

with

$$\begin{aligned} E_* = &\int \left(\overline{\psi_u} W_1 R_v + \psi_u \overline{W_1} R_v + (|W_1|^2 + |W_{-1}|^2)(\psi_v + \varepsilon^2 R_v)\right) \mathrm{d}x \\ &+ \int (\partial_x^s(\psi_u \overline{W_1})\partial_x^s R_v + \partial_x^s(\overline{\psi_u} W_1)\partial_x^s R_v) \, \mathrm{d}x \\ &+ \varepsilon^2 \int (W_1 \partial_x^s R_v \partial_x^s \overline{W_1} + \overline{W_1}\partial_x^s R_v \partial_x^s W_1) \, \mathrm{d}x \\ &+ \varepsilon^2 \int (W_{-1} \partial_x^s R_v \partial_x^s \overline{W_{-1}} + \overline{W_{-1}}\partial_x^s R_v \partial_x^s W_{-1}) \, \mathrm{d}x. \end{aligned}$$

Energy equivalence: In the following lemma, we show that the square root of the energy $\mathcal{E}$ is equivalent to the $H^{s+1} \times H^{s+1} \times H^s \times H^s$-norm of the error function $(R_1, R_{-1}, R_2, R_{-2})$. We note that, with (2.36) and (2.37), we can infer (2.38).

Lemma 2.3.12. *Let $s \geq 0$ and $\gamma \in \mathbb{R}$ with $|\gamma| \geq 1$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $\psi_u \in C([0,T_0], H^{s+5})$ be a solution of the NLS equation* (2.28) *with spatially 2π-periodic boundary conditions and*

$$\sup_{t\in[0,T_0]} \|\psi_u\|_{H^{s+5}} =: C_u < \infty.$$

Then, there exist $\varepsilon_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned} &(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} + \|R_2\|_{H^s} + \|R_{-2}\|_{H^s})^2 \\ &\qquad \leq C_1 \mathcal{E} \leq C_2(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} + \|R_2\|_{H^s} + \|R_{-2}\|_{H^s})^2. \end{aligned}$$

Proof. Since we have that

$$\|R_{\pm 1}\|_{H^{s+1}} \sim \|R_{\pm 1}\|_{H^s} + \|\partial_x R_{\pm 1}\|_{H^s},$$

the square root of the energy E_0+E_s allows to estimate the $H^{s+1}\times H^{s+1}\times H^s\times H^s$-norm of $(R_1, R_{-1}, R_2, R_{-2})$. Therefore, it suffices to estimate $\widetilde{E}_0 + \widetilde{E}_s$ and E_* by $E_0 + E_s$. In the following, we use the inequality $\varepsilon\omega_1(k) \leq \varepsilon^{-1} + |k|$ for all $k \in \mathbb{R}$ and, consequently,

$$\|\varepsilon\omega_1 f\|_{H^s} \leq \varepsilon^{-1}\|f\|_{H^s} + \|f\|_{H^{s+1}} \quad \text{for } f \in H^{s+1}. \tag{2.40}$$

With (2.38), we can estimate $\widetilde{E}_0 + \widetilde{E}_s$ by $E_0 + E_s$. Omitting the arguments for the sake of brevity, we obtain

$$\begin{aligned}
&|\widetilde{E}_0 + \widetilde{E}_s| \\
&\leq C\|R_v\|_{H^s}(\|\mathcal{Q}_2\|_{H^s} + \|\mathcal{Q}_{-2}\|_{H^s}) + C\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}(\|\varepsilon\omega_1\mathcal{Q}_1\|_{H^s} + \|\varepsilon\omega_1\mathcal{P}_1\|_{H^s}) \\
&\qquad + C\|\varepsilon\widetilde{\omega}_1 R_{-1}\|_{H^s}(\|\varepsilon\omega_1\mathcal{Q}_{-1}\|_{H^s} + \|\varepsilon\omega_1\mathcal{P}_{-1}\|_{H^s}) \\
&\leq C(E_0+E_s)^{1/2}(\|\mathcal{Q}_2\|_{H^s} + \|\mathcal{Q}_{-2}\|_{H^s} + \varepsilon^{-1}\|\mathcal{Q}_1\|_{H^s} + \varepsilon^{-1}\|\mathcal{Q}_{-1}\|_{H^s} \\
&\qquad + \|\mathcal{Q}_1\|_{H^{s+1}} + \|\mathcal{Q}_{-1}\|_{H^{s+1}} + \varepsilon^{-1}\|\mathcal{P}_1\|_{H^s} + \varepsilon^{-1}\|\mathcal{P}_{-1}\|_{H^s} + \|\mathcal{P}_1\|_{H^{s+1}} + \|\mathcal{P}_{-1}\|_{H^{s+1}}) \\
&\leq C\varepsilon(E_0+E_s).
\end{aligned}$$

Further, with Sobolev's embedding theorem, we obtain

$$\begin{aligned}
|E_*| &\leq C\|\psi_v\|_{L^\infty}(\|W_1\|_{H^1}^2 + \|W_{-1}\|_{H^1}^2) + C\varepsilon^2\|R_v\|_{H^s}(\|W_1\|_{H^{s+1}}^2 + \|W_{-1}\|_{H^{s+1}}^2) \\
&\qquad + C\|\psi_u\|_{H^s}\|R_v\|_{H^s}\|W_1\|_{H^{s+1}} \\
&\leq C_{max}^2(E_0+E_s) + C_{max}(E_0+E_s) + C\varepsilon^2(E_0+E_s)^{3/2}.
\end{aligned}$$

The result now follows by choosing $\varepsilon_0 > 0$ and $C_{max} > 0$ sufficiently small. □

In Section 2.2.6, we already have estimates for the terms $t_1, \ldots, t_8$. Thus, in order to use Gronwall's inequality, it remains to estimate $I_{s,0}, \ldots, I_{s,23}$ by the energy $\mathcal{E}$. The estimates for $I_{0,0}, \ldots, I_{0,23}$ are obtained in the same way.

Bounds for $I_{s,0}$, $I_{s,1}$, $I_{s,3}$, $I_{s,6}$ and $I_{s,7}$: Due to the skew symmetry of $i\omega_j$, we obtain

$$I_{s,0} = I_{s,1} = I_{s,3} = I_{s,6} = I_{s,7} = 0.$$

Bounds for $I_{s,2}$, $I_{s,4}$, $I_{s,5}$ and $I_{s,8}$: We use the Cauchy-Schwarz inequality, Sobolev's embedding theorem, Lemma 2.3.6, Lemma 2.3.7, and $\|(\varepsilon^2\omega_1)^{-1}\|_{L^\infty} \leq C$.

Then, we obtain

$$\begin{aligned}|I_{s,2}| &\leq C(\|R_v\|_{H^s} + (1+\varepsilon^2\|R_v\|_{H^s})(\|R_{-1}\|_{H^s} + \|R_1\|_{H^s}) + \varepsilon^{-2}\|\mathrm{Res}_u\|_{H^s}) \\ &\qquad \times (\|R_{-1}\|_{H^{s+1}} + \|R_1\|_{H^{s+1}}) \\ &\leq C\mathcal{E}^{1/2} + C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2}, \\ |I_{s,4}| &\leq C\varepsilon\|R_1\|_{H^{s+1}}\|R_{-1}\|_{H^{s+1}}\|R_q\|_{H^s} \leq C\varepsilon\mathcal{E}^{3/2}, \\ |I_{s,5}| &\leq C\varepsilon^{-3}\|\partial_x^{-1}\mathrm{Res}_v\|_{H^s}\|R_q\|_{H^s} \leq C\varepsilon\mathcal{E}^{1/2}, \\ |I_{s,8}| &\leq C\varepsilon\|R_v\|_{H^s}(\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}\|R_{-1}\|_{H^{s+1}} + \|\varepsilon\widetilde{\omega}_1 R_{-1}\|_{H^s}\|R_1\|_{H^{s+1}}) \\ &\qquad + C\varepsilon(\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s} + \|\varepsilon\widetilde{\omega}_1 R_{-1}\|_{H^s})(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}}) \\ &\leq C\varepsilon\mathcal{E} + C\varepsilon\mathcal{E}^{3/2}.\end{aligned}$$

Bounds for $I_{s,9}$: By using a Taylor expansion, we find

$$|\widetilde{\omega}_1(k)| = |\omega_1(k) - \varepsilon^{-2}| \leq \frac{1}{2}k^2.$$

Thus, we use integration by parts and Lemma 2.3.6 to obtain

$$|I_{s,9}| \leq C\varepsilon^{-2}\|\mathrm{Res}_u\|_{H^{s+1}}(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}}) \leq C\mathcal{E}^{1/2}.$$

Bounds for $I_{s,10}$, $I_{s,14}$ and $I_{s,18}$: In the definition of $I_{s,10}$, we replace $\partial_t R_2$ with the right-hand side of the error equation (2.34). Then, we obtain

$$\begin{aligned}|I_{s,10}| &\leq C\|\partial_t R_2 - i\omega_2 R_2\|_{H^s}(\|\mathcal{Q}_2(\Psi_u,\mathcal{R})\|_{H^s} + \|\mathcal{Q}_{-2}(\Psi_u,\mathcal{R})\|_{H^s}) \\ &\leq C\varepsilon^{-1}(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} + \varepsilon^2(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}})^2 + \varepsilon^{-3}\|\partial_x^{-1}\mathrm{Res}_v\|_{H^s}) \\ &\qquad \times (\|\mathcal{Q}_2(\Psi_u,\mathcal{R})\|_{H^s} + \|\mathcal{Q}_{-2}(\Psi_u,\mathcal{R})\|_{H^s})\end{aligned}$$

Further, in the definition of $I_{s,14}$, we replace $\partial_t R_1$ with the right-hand side of the error equation (2.34). Since for all $k \in \mathbb{R}$ it holds $\widetilde{\omega}_1(k) \leq \varepsilon^{-1}|k|$, we can use

$$\|\widetilde{\omega}_1 f\|_{H^s} \leq \varepsilon^{-1}\|f\|_{H^{s+1}} \quad \text{for } f \in H^{s+1}$$

to obtain

$$\begin{aligned}|I_{s,14}| &\leq C\|\varepsilon^2\omega_1(\partial_t R_1 - i\omega_1 R_1)\|_{H^s}(\|\widetilde{\omega}_1\mathcal{Q}_1(\Psi_u,\mathcal{R})\|_{H^s} + \|\widetilde{\omega}_1\mathcal{P}_1(\psi_v,\mathcal{R})\|_{H^s}) \\ &\leq C\|\Psi_u(R_2 + R_{-2}) + (\Psi_v + \varepsilon^2(R_2 + R_{-2}))(R_1 + R_{-1}) - \varepsilon^{-2}\mathrm{Res}_u\|_{H^s} \\ &\qquad \times \varepsilon^{-1}(\|\mathcal{Q}_1(\Psi_u,\mathcal{R})\|_{H^{s+1}} + \|\mathcal{P}_1(\psi_v,\mathcal{R})\|_{H^{s+1}}).\end{aligned}$$

With Lemma 2.3.6, Lemma 2.3.7, and (2.38), we conclude

$$|I_{s,10}|, |I_{s,14}| \leq C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2}$$

and, analogously,

$$|I_{s,18}| \leq C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2}.$$

Bounds for $I_{s,11}$, $I_{s,15}$ and $I_{s,19}$: First, we write

$$|\partial_t \Psi_u - i\omega_1(0)\Psi_u| = |\partial_t(\psi_u e^{i\varepsilon^{-2}t}) - i\varepsilon^{-2}\psi_u e^{i\varepsilon^{-2}t}| = |\partial_t \psi_u|.$$

Since $\psi_v = -|\psi_u|^2$ and ψ_u solves the NLS equation (2.28), we obtain

$$\begin{aligned}\|\partial_t \Psi_u - i\omega_1(0)\Psi_u\|_{H^s} &\leq C\|\partial_x^2 \psi_u\|_{H^s} + C\|\psi_u|\psi_u|^2\|_{H^s} \leq C\|\psi_u\|_{H^{s+2}} + C\|\psi_u\|_{H^s}^3,\\ \|\partial_t \psi_v\|_{H^s} &\leq C\|\partial_t \psi_u\|_{H^s}\|\psi_u\|_{H^s} \leq C(\|\psi_u\|_{H^{s+2}} + C\|\psi_u\|_{H^s}^3)\|\psi_u\|_{H^s}.\end{aligned}$$

Thus, with (2.38), we estimate

$$\begin{aligned}|I_{s,11}| &\leq C\|R_2\|_{H^s}\|\mathcal{Q}_2(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R})\|_{H^s}\\ &\quad + C\|R_{-2}\|_{H^s}\|\mathcal{Q}_{-2}(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R})\|_{H^s}\\ &\leq C\varepsilon\|\partial_t \Psi_u - i\omega_1(0)\Psi_u\|_{H^s}(\|R_2\|_{H^s} + \|R_{-2}\|_{H^s})(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}})\\ &\leq C\varepsilon\mathcal{E}.\end{aligned}$$

With (2.38) and (2.40), we conclude

$$\begin{aligned}|I_{s,15}| &\leq C\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}(\varepsilon^{-1}\|\mathcal{Q}_1(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R})\|_{H^s}\\ &\qquad + \|\mathcal{Q}_1(\partial_t \Psi_u - i\omega_1(0)\Psi_u, \mathcal{R})\|_{H^{s+1}}\\ &\qquad + \varepsilon^{-1}\|\mathcal{P}_1(\partial_t \psi, \mathcal{R})\|_{H^s} + \|\mathcal{P}_1(\partial_t \psi_v, \mathcal{R})\|_{H^{s+1}})\\ &\leq C\varepsilon\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}\|\partial_t \Psi_u - i\omega_1(0)\Psi_u\|_{H^{s+1}}(\|R_2\|_{H^s} + \|R_{-2}\|_{H^s})\\ &\qquad + C\varepsilon\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}\|\partial_t \psi_v\|_{H^{s+1}}(\|R_1\|_{H^s} + \|R_{-1}\|_{H^s})\\ &\leq C\varepsilon\mathcal{E}\end{aligned}$$

and, analogously,

$$|I_{s,19}| \leq C\varepsilon\mathcal{E}.$$

Bounds for $I_{s,12}$, $I_{s,16}$ and $I_{s,20}$: We replace $\partial_t R_j$ with the right-hand side of the error equation (2.34). With Lemma 2.3.6 and Lemma 2.3.7, we obtain

$$\begin{aligned}\|\varepsilon^2\omega_1(\partial_t R_{\pm 1} \mp i\omega_1 R_{\pm 1})\|_{H^s} &\leq C(\|\mathcal{R}\|_{(H^s)^4} + \varepsilon^2\|R_1 + R_{-1}\|_{H^{s+1}}\|R_2 + R_{-2}\|_{H^s} + 1),\\ \|\partial_t R_{\pm 2} \mp i\omega_2 R_{\pm 2}\|_{H^s} &\leq C\varepsilon^{-1}(\|R_1\|_{H^{s+1}} + \|R_{-1}\|_{H^{s+1}} + \varepsilon^2\|R_1 + R_{-1}\|_{H^{s+1}}^2 + \varepsilon^2).\end{aligned}$$

Further, since for all $k \in \mathbb{R}$ we have $(1 + |k|^2)^{1/2} \leq \varepsilon\omega_1(k)$, we can use

$$\|f\|_{H^{s+1}} \leq C\varepsilon^{-1}\|\varepsilon^2\omega_1 f\|_{H^s} \quad \text{for } f \in H^{s+1}$$

to obtain

$$\|\partial_t R_{\pm 1} \mp i\omega_1 R_{\pm 1}\|_{H^{s+1}} \leq C\varepsilon^{-1}\|\varepsilon^2\omega_1(\partial_t R_{\pm 1} \mp i\omega_1 R_{\pm 1})\|_{H^s}.$$

Therefore, with (2.38), we get

$$\begin{aligned}|I_{s,12}| &\le C\|R_2\|_{H^s}\|\mathcal{Q}_2(\Psi_u, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^s} + C\|R_{-2}\|_{H^s}\|\mathcal{Q}_{-2}(\Psi_u, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^s} \\ &\le C\varepsilon\mathcal{E}^{1/2}(\|\partial_t R_1 - i\omega_1 R_1\|_{H^{s+1}} + \|\partial_t R_{-1} + i\omega_1 R_{-1}\|_{H^{s+1}}) \\ &\le C\mathcal{E}^{1/2}(\|\varepsilon^2\omega_1(\partial_t R_1 - i\omega_1 R_1)\|_{H^s} + \|\varepsilon^2\omega_1(\partial_t R_{-1} + i\omega_1 R_{-1})\|_{H^s}) \\ &\le C\mathcal{E}^{1/2} + C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2}.\end{aligned}$$

Furthermore, with (2.38) and (2.40), we find

$$\begin{aligned}|I_{s,16}| &\le C\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}(\|\varepsilon\omega_1\mathcal{Q}_1(\Psi_u, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^s} + \|\varepsilon\omega_1\mathcal{P}_1(\psi_v, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^s}) \\ &\le C\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}(\|\varepsilon^{-1}\mathcal{Q}_1(\Psi_u, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^s} + \|\mathcal{Q}_1(\Psi_u, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^{s+1}}) \\ &\quad + C\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}(\|\varepsilon^{-1}\mathcal{P}_1(\psi_v, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^s} + \|\mathcal{P}_1(\psi_v, \partial_t\mathcal{R}-\Lambda\mathcal{R})\|_{H^{s+1}}) \\ &\le C\varepsilon\|\varepsilon\widetilde{\omega}_1 R_1\|_{H^s}\|\partial_t\mathcal{R}-\Lambda\mathcal{R}\|_{(H^s)^4} \\ &\le C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2}\end{aligned}$$

and, analogously,

$$|I_{s,20}| \le C + C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2}.$$

Bounds for $I_{s,13}$, $I_{s,17}$ and $I_{s,21}$: We use

$$|\omega_2(k) - \omega_2(l)| = \varepsilon^{-1}|\gamma|^{-1}|k-l|$$

and the Lipschitz continuity of ω_1, namely,

$$|\omega_1(k) - \omega_1(l)| \le \varepsilon^{-1}|k-l|,$$

in order to obtain

$$\begin{aligned}\left\|\int |\omega_{j_1}(\cdot) - \omega_{j_1}(l)|\widehat{\Psi}_u(\cdot - l)\widehat{R}_{j_1}(l)\,\mathrm{d}l\right\|_{H^s} &\le C\left\|\int \varepsilon^{-1}|\cdot - l|\widehat{\Psi}_u(\cdot - l)\widehat{R}_{j_1}(l)\,\mathrm{d}l\right\|_{H^s} \\ &\le C\varepsilon^{-1}\|\partial_x\Psi_u\|_{H^s}\|R_{j_1}\|_{H^s}.\end{aligned}$$

We remark that the loss of ε-powers in this estimate causes no problems since, according to the proofs of Lemma 2.3.9 and Lemma 2.3.10, for all $k \in \mathbb{R}$ we have

$$\begin{aligned}&|q_{-1,\pm 2}(k)| \le C\varepsilon^2, \qquad |k \cdot q_{-1,\pm 2}(k)| \le C\varepsilon, \qquad |q_{\pm 2,-1}(k)| \le C\varepsilon|k|, \\ &|p_{\pm 1,\mp 1}(k)| \le C\varepsilon^2, \qquad |k \cdot p_{\pm 1,\mp 1}(k)| \le C\varepsilon.\end{aligned}$$

Therefore, by using (2.38), we can conclude

$$\begin{aligned}
&\Big\|\mathcal{Q}_{\pm1}(\Psi_u,\Lambda\mathcal{R}) - \sum_{j_1\in\{\pm1,\pm2\}} i\omega_{j_1}\mathcal{Q}_{\pm1,j_1}(\Psi_u,R_{j_1})\Big\|_{H^s}\\
&\qquad\qquad\leq C\varepsilon\|\partial_x\Psi_u\|_{H^s}(\|R_2\|_{H^s}+\|R_{-2}\|_{H^s}),\\
&\Big\|\mathcal{Q}_{\pm1}(\Psi_u,\Lambda\mathcal{R}) - \sum_{j_1\in\{\pm1,\pm2\}} i\omega_{j_1}\mathcal{Q}_{\pm1,j_1}(\Psi_u,R_{j_1})\Big\|_{H^{s+1}}\\
&\qquad\qquad\leq C\|\partial_x\Psi_u\|_{H^s}(\|R_2\|_{H^s}+\|R_{-2}\|_{H^s}),\\
&\Big\|\mathcal{Q}_{\pm2}(\Psi_u,\Lambda\mathcal{R}) - \sum_{j_1\in\{\pm1,\pm2\}} i\omega_{j_1}\mathcal{Q}_{\pm2,j_1}(\Psi_u,R_{j_1})\Big\|_{H^s}\\
&\qquad\qquad\leq C\|\partial_x\Psi_u\|_{H^{s+1}}(\|R_1\|_{H^{s+1}}+\|R_{-1}\|_{H^{s+1}}),\\
&\Big\|\mathcal{P}_{\pm1}(\psi_v,\Lambda\mathcal{R}) - \sum_{j_1\in\{\pm1,\pm2\}} i\omega_{j_1}\mathcal{P}_{\pm1,j_1}(\psi_v,R_{j_1})\Big\|_{H^s}\\
&\qquad\qquad\leq C\varepsilon\|\partial_x\psi_v\|_{H^s}(\|R_1\|_{H^s}+\|R_{-1}\|_{H^s}),\\
&\Big\|\mathcal{P}_{\pm1}(\psi_v,\Lambda\mathcal{R}) - \sum_{j_1\in\{\pm1,\pm2\}} i\omega_{j_1}\mathcal{P}_{\pm1,j_1}(\psi_v,R_{j_1})\Big\|_{H^{s+1}}\\
&\qquad\qquad\leq C\|\partial_x\psi_v\|_{H^s}(\|R_1\|_{H^s}+\|R_{-1}\|_{H^s}).
\end{aligned}$$

With the Cauchy-Schwarz inequality and (2.40), we obtain

$$\begin{aligned}
|I_{s,13}| &\leq C\mathcal{E},\\
|I_{s,17}| &\leq C\mathcal{E},\\
|I_{s,21}| &\leq C\mathcal{E}.
\end{aligned}$$

Bounds for $I_{s,22}$ and $I_{s,23}$: We replace $\partial_t W_1$ with the right-hand side of (2.39). Sobolev's embedding theorem and Lemma 2.3.6 yield

$$\begin{aligned}
|I_{s,22}| &\leq C(\|R_v\|_{H^s}+\|W_1\|_{H^s}+\varepsilon^2\|R_v\|_{H^s}\|W_1\|_{H^{s+1}})\|\partial_t W_1 - i\widetilde{\omega}_1 W_1\|_{H^s}\\
&\leq C(\|R_v\|_{H^s}+\|W_1\|_{H^s}+\varepsilon^2\|R_v\|_{H^s}\|W_1\|_{H^{s+1}})\\
&\qquad\times(\|R_v\|_{H^s}+(C+\varepsilon^2\|R_v\|_{H^s})(\|W_1\|_{H^s}+\|W_{-1}\|_{H^s})+\varepsilon^{-2}\|\mathrm{Res}_u\|_{H^s})\\
&\leq C\mathcal{E}^{1/2}+C\mathcal{E}+C\varepsilon^2\mathcal{E}^{3/2}+C\varepsilon^4\mathcal{E}^2
\end{aligned}$$

and, analogously,

$$|I_{s,23}| \leq C\mathcal{E}^{1/2}+C\mathcal{E}+C\varepsilon^2\mathcal{E}^{3/2}+C\varepsilon^4\mathcal{E}^2.$$

Final estimates: Finally, after using $\mathcal{E}^{1/2}\leq 1+\mathcal{E}$ and $\varepsilon\mathcal{E}\leq 1$ we are left with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq C + C\mathcal{E}.$$

With Gronwall's inequality, we have $\mathcal{E}(t) \leq M$ for all $t \in [0, T_0]$ for a constant $M = \mathcal{O}(1)$. We choose $\varepsilon_0 > 0$ sufficiently small such that $\varepsilon_0 M \leq 1$. This completes the proof of Theorem 2.3.1.

Remark 2.3.13. In fact, we can increase the approximation rate in Theorem 2.3.1. The following outlines how this is done. As in Remark 2.1.13, we introduce the error by

$$(u, v)(x, t) = (\psi_{u,n} e^{i\varepsilon^{-2} t}, \psi_{v,n})(x, t) + \varepsilon^{\beta}(R_u, R_v)(x, t),$$

where $(\psi_{u,n}, \psi_{v,n})$ is the higher order approximation from Section 2.3.3. In order to find an $\mathcal{O}(1)$ bound of the $H^{s+1} \times H^s$-norm of (R_u, R_v), we can apply the above energy estimates and choose $\beta = 2n+1$. Therefore, we have the following theorem.

Theorem 2.3.14. *Let $n \in \mathbb{N}$, $s \in \mathbb{N}$, and $\gamma \in \mathbb{R}$ with $|\gamma| \geq 1$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $\psi_u \in C([0, T_0], H^{s+2n+5})$ be a solution of the NLS equation* (2.28) *with spatially 2π-periodic boundary conditions and*

$$\sup_{t \in [0, T_0]} \|\psi_u\|_{H^{s+2n+5}} =: C_u < \infty.$$

Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of the KGZ system (2.27) *with spatially 2π-periodic boundary conditions satisfying*

$$\sup_{t \in [0, T_0]} \|(u, v) - (\psi_{u,n} e^{i\varepsilon^{-2} t}, \psi_{v,n})\|_{H^{s+1} \times H^s} \leq C\varepsilon^{2n+1}.$$

2.4 From KGZ to singular NLS on the torus

In this section, we consider a KGZ system with a parameter $\varepsilon > 0$ such that we obtain a singular NLS equation in the limit $\varepsilon \to 0$. We consider spatially periodic boundary conditions in order to have this NLS equation well-defined in Fourier space. Besides, we give estimates for the residual and construct a higher order approximation. However, error estimates are not possible since we lose too much powers of ε in the normal form transformation.

2.4.1 Introduction

We consider the KGZ system

$$\varepsilon^2 \partial_t^2 u = \partial_x^2 u - \varepsilon^{-2} u - uv, \qquad \frac{\gamma^2}{4}\varepsilon^4 \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2) \tag{2.41}$$

posed on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(L\mathbb{Z})$ for $L > 0$ with $u(x,t), v(x,t), x, t \in \mathbb{R}, \gamma \in \mathbb{R} \setminus \{0\}$, and $0 < \varepsilon \ll 1$. This corresponds to the spectral situation in Figure 2.4.

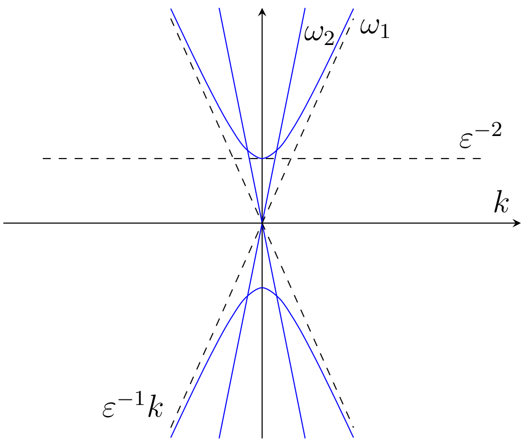


Figure 2.4: The spectral situation corresponding to the linearized KGZ system with $k \in \mathbb{R}$ and $\gamma = 2$. It is solved by $u(x,t) = e^{ikx+i\omega_{\pm 1}(k)t}$ and $v(x,t) = e^{ikx+i\omega_{\pm 2}(k)t}$ where $\omega_{\pm 1}(k) = \pm\varepsilon^{-2}\sqrt{1+(\varepsilon k)^2}$ and $\omega_{\pm 2}(k) = \pm\frac{2}{\gamma}\varepsilon^{-2}k$. We note that $\omega_1(k)$ asymptotically behaves like $\varepsilon^{-1}|k|$. Since $|\omega_1(k)| = \mathcal{O}(\varepsilon^{-1}|k|)$ and $|\omega_2(k)| = \mathcal{O}(\varepsilon^{-2}|k|)$, the spectral situation is similar for every choice of the parameter γ.

We make the ansatz

$$\begin{aligned} u(x,t) &= \Psi_u(x,t) = \psi_u(x,t)e^{i\varepsilon^{-2}t} + c.c., \\ v(x,t) &= \Psi_v(x,t) = \psi_v(x,t) + \psi_{v,+}(x,t)e^{2i\varepsilon^{-2}t} + \psi_{v,-}(x,t)e^{-2i\varepsilon^{-2}t}, \end{aligned} \tag{2.42}$$

where ψ_u, ψ_v and $\psi_{v,\pm}$ are spatially L-periodic. In the singular limit $\varepsilon \to 0$, this yields

$$\psi_v = -2|\psi_u|^2, \qquad \psi_{v,+} = \mathcal{A}_\gamma \psi_u^2, \qquad \psi_{v,-} = \mathcal{A}_\gamma \overline{\psi_u}^2, \tag{2.43}$$

where the operator $\mathcal{A}_\gamma$ is defined by $\mathcal{A}_\gamma = -\partial_x^2(\gamma^2 + \partial_x^2)^{-1}$. Then, one can derive the singular NLS equation

$$2i\partial_t \psi_u = \partial_x^2 \psi_u + 2\psi_u |\psi_u|^2 - \overline{\psi_u} \mathcal{A}_\gamma(\psi_u^2), \tag{2.44}$$

with spatially L-periodic boundary conditions in the limit $\varepsilon \to 0$.

Remark 2.4.1. In Fourier space, the operator $\widehat{\mathcal{A}}_\gamma(k) = k^2/(\gamma^2 - k^2)$ is not well-defined for wave numbers $k \in \mathbb{R}$ as singularities arise for $|k| = |\gamma|$. Since we consider periodic boundary conditions, we can choose the period $L > 0$ in such a way that we are bounded away from the singularities for $k \in (2\pi/L)\mathbb{Z}$. Therefore, the operator is well-defined on the torus $\mathbb{T}$ for a reasonably chosen period L. Furthermore, $\mathcal{A}_\gamma$ is self-adjoint w.r.t. to the L^2-scalar product since $-\partial_x^2$ and $(\gamma^2 + \partial_x^2)^{-1}$ are self-adjoint differential operators w.r.t. to the L^2-scalar product.

Remark 2.4.2. According to Remark 2.2.3, we have local existence and uniqueness of solutions $(u, v) \in H^{s+1} \times H^s$, $s \geq 1$, of the KGZ system (2.41).

Remark 2.4.3. Due to Remark 2.4.1, the singular operator $\mathcal{A}_\gamma$ is well-defined on the torus $\mathbb{T}$. Therefore, analogous to Remark 2.1.5, we have local existence and uniqueness of solutions $u \in H^s$, $s \geq 1$, of the singular NLS equation (2.44) on the torus $\mathbb{T}$.

Remark 2.4.4. The same limit has been considered in [MN10], where formal convergence results can be found. Furthermore, [MN10] involves neither residual estimates nor error bounds and the method of proof is completely different from our methods.

Our goal would be to prove that the KGZ system (2.41) makes correct predictions about the dynamics of the singular NLS equation (2.44) for small values of $\varepsilon > 0$. But with our methods of proof we lose too many powers of ε in the normal form transformations such that error estimates are not possible at that point. In the subsequent section, we provide estimates for the residual on the torus $\mathbb{T}$ and construct a higher order approximation in order to make the residual arbitrarily small.

Notation. We use the notation from Chapter 1. We write $\int$ for $\int_{\mathbb{T}}$ and H^s for $H^s(\mathbb{T}, \mathbb{K})$, unless otherwise specified.

2.4.2 Estimates for the residual and higher order approximation

The residual of (2.41) is given by

$$\begin{aligned}\mathrm{Res}_u(u,v) &= -\varepsilon^2\partial_t^2 u + \partial_x^2 u - \varepsilon^{-2}u - uv,\\ \mathrm{Res}_v(u,v) &= -\frac{\gamma^2}{4}\varepsilon^4\partial_t^2 v + \partial_x^2 v + \partial_x^2(|u|^2).\end{aligned}$$

It contains all the terms which do not cancel after inserting the approximation into the KGZ system. If we insert the approximation (2.42) into the residual and choose $\psi_v = -2|\psi_u|^2$, $\psi_{v,+} = \mathcal{A}_\gamma(\psi_u^2)$, and ψ_u to satisfy the singular NLS equation (2.44), the residual Res_u will be of order $\mathcal{O}(1)$ and the residual Res_v will be of order $\mathcal{O}(\varepsilon^2)$. However, in this section, we are interested in making the residual arbitrarily small. Thus, we introduce the following higher order approximation

$$\begin{aligned}\Psi_{u,n} &= \sum_{k=0}^{n}\sum_{j=0}^{k}\varepsilon^{2k}(\psi_{u,2k,2j+1}e^{(2j+1)i\varepsilon^{-2}t} + c.c.),\\ \Psi_{v,n} &= \sum_{k=0}^{n}\varepsilon^{2k}(\psi_{v,2k,0} + \sum_{j=1}^{k+1}(\psi_{v,2k,2j}e^{2ji\varepsilon^{-2}t} + c.c.)).\end{aligned} \tag{2.45}$$

Then, the residual of the KGZ system (2.41) is given by

$$\begin{aligned}&\mathrm{Res}_u(\Psi_{u,n},\Psi_{v,n})\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}\Big(-\varepsilon^{2(k+1)}\partial_t^2\psi_{u,2k,2j+1} - 2(2j+1)i\varepsilon^{2k}\partial_t\psi_{u,2k,2j+1}\\ &\quad + \varepsilon^{2k}\partial_x^2\psi_{u,2k,2j+1} + ((2j+1)^2-1)\varepsilon^{2(k-1)}\psi_{u,2k,2j+1}\Big)e^{(2j+1)i\varepsilon^{-2}t} + c.c.\\ &\quad - \Psi_{u,n}\Psi_{v,n}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_v(\Psi_{u,n},\Psi_{v,n}) &= \sum_{k=0}^{n}\Big(-\frac{\gamma^2}{4}\varepsilon^{2(k+2)}\partial_t^2\psi_{v,2k,0} + \varepsilon^{2k}\partial_x^2\psi_{v,2k,0}\Big)\\ &\quad + \sum_{k=0}^{n}\sum_{j=1}^{k+1}\Big(-\frac{\gamma^2}{4}\varepsilon^{2(k+2)}\partial_t^2\psi_{v,2k,2j} - ji\gamma^2\varepsilon^{2(k+1)}\partial_t\psi_{v,2k,2j}\\ &\qquad + \varepsilon^{2k}(j^2\gamma^2 + \partial_x^2)\psi_{v,2k,2j}\Big)e^{2ji\varepsilon^{-2}t} + c.c.\\ &\quad + \partial_x^2(|\Psi_{u,n}|^2).\end{aligned}$$

We remark that, in the definition of $\Psi_{u,n}$ and $\Psi_{v,n}$, we have $\psi_{u,0,1} = \psi_u$, $\psi_{v,0,0} = \psi_v$, and $\psi_{v,0,2} = \psi_{v,+}$. In the following, we consider several ε-balances. First, we consider the terms contained in $\mathrm{Res}_u(\Psi_{u,n}, \Psi_{v,n})$:

- We consider the $\mathcal{O}(\varepsilon^{2k} e^{i\varepsilon^{-2}t})$ terms for $0 \leq k \leq n$. At $\varepsilon^0 e^{i\varepsilon^{-2}t}$, we find the singular NLS equation (2.44) for $\psi_{u,0,1}$. At $\varepsilon^{2k} e^{i\varepsilon^{-2}t}$, $k \geq 1$, since for $j = 0$ we have $(2j+1)^2 = 1$, we find that the functions $\psi_{u,2k,1}$ solve linear inhomogeneous Schrödinger equations of the form

$$2i\partial_t \psi_{u,2k,1} = \partial_x^2 \psi_{u,2k,1} - \partial_t^2 \psi_{u,2(k-1),1} + G_{2k,1}, \tag{2.46}$$

  where $G_{2k,1}$ is a quadratic mapping which does only depend linearly on $\psi_{u,2k,1}$.

- At $\varepsilon^{2(k-1)} e^{(2j+1)i\varepsilon^{-2}t}$, with $1 \leq k \leq n$ and $1 \leq j \leq k$, we find that the functions $\psi_{u,2k,2j+1}$ solve algebraic equations of the form

$$\begin{aligned} \psi_{u,2k,2j+1} = ((2j+1)^2 - 1)^{-1} \big(& \partial_t^2 \psi_{u,2(k-2),2j+1} + 2(2j+1) i \partial_t \psi_{u,2(k-1),2j+1} \\ & - \partial_x^2 \psi_{u,2(k-1),2j+1} + G_{2k,2j+1} \big), \end{aligned}$$

  where $G_{2k,2j+1}$ is a quadratic mapping which does not depend on $\psi_{u,2k,2j+1}$. We note that we set $\psi_{u,m,2j+1} = 0$ for $m < 0$.

Next, we consider the terms contained in $\mathrm{Res}_v(\Psi_{u,n}, \Psi_{v,n})$:

- We consider the $\mathcal{O}(\varepsilon^{2k})$ terms for $0 \leq k \leq n$. At ε^0, we find that

$$\partial_x^2 \psi_{v,0,0} + 2\partial_x^2 (|\psi_{u,0,1}|^2) = 0, \tag{2.47}$$

  and at ε^2, we find that

$$\partial_x^2 \psi_{v,2,0} + 2\partial_x^2 (\psi_{u,0,1} \overline{\psi_{u,2,1}} + \overline{\psi_{u,0,1}} \psi_{u,2,1}) = 0. \tag{2.48}$$

  At ε^{2k}, $k \geq 2$, we find that the functions $\psi_{v,2k,0}$ solve algebraic equations of the form

$$\frac{\gamma^2}{4} \partial_t^2 \psi_{v,2(k-2),0} - \partial_x^2 \psi_{v,2k,0} = \partial_x^2 H_{2k,0}, \tag{2.49}$$

  where $H_{2k,0}$ is a quadratic mapping which does not depend on $\psi_{v,2k,0}$. Suppose now that $\psi_{v,2(k-2),0}$ has a vanishing mean value. We look for $\psi_{v,2k,0}$ satisfying (2.49) and having a vanishing mean value. According to Section 2.1.2, we can achieve this by setting

$$\psi_{v,2k,0} = \frac{\gamma^2}{4} \partial_x^{-2} \partial_t^2 \psi_{v,2(k-2),0} - H_{2k,0} + \frac{1}{L} \int H_{2k,0}(x) \, \mathrm{d}x.$$

Therefore, it remains to show that $\psi_{v,0,0}$ resp. $\psi_{v,2,0}$ satisfy (2.47) resp. (2.48) and have a vanishing mean value, because then this also applies inductively to $\psi_{v,4k,0}$ resp. $\psi_{v,4k+2,0}$, with $k \geq 1$. Using integration by parts and the self-adjointness of the operator $\mathcal{A}_\gamma$, see Remark 2.4.1, we obtain

$$\partial_t \int_{\mathbb{T}} |\psi_{u,0,1}|^2 \,\mathrm{d}x = 0,$$

which is the conservation of the L^2-norm for the solutions of the NLS equation. Thus, $\psi_{v,0,0} = -2|\psi_{u,0,1}|^2$ satisfies (2.47) and has a vanishing mean value. Moreover, if we set

$$\psi_{v,2,0} = -2(\psi_{u,0,1}\overline{\psi_{u,2,1}} + \overline{\psi_{u,0,1}}\psi_{u,2,1}),$$

then (2.48) is satisfied, but $\psi_{v,2,0}$ has a non-vanishing mean value. To fix this problem, we set

$$\psi_{v,2,0} = \widetilde{\psi}_{v,2,0} - \frac{1}{L}\int \widetilde{\psi}_{v,2,0}(x)\,\mathrm{d}x,$$

where

$$\widetilde{\psi}_{v,2,0} = -2(\psi_{u,0,1}\overline{\psi_{u,2,1}} + \overline{\psi_{u,0,1}}\psi_{u,2,1}).$$

Then, (2.48) is still satisfied and the mean value of $\psi_{v,2,0}$ vanishes.

- At $\varepsilon^{2k}e^{2ji\varepsilon^{-2}t}$, with $0 \leq k \leq n$ and $1 \leq j \leq k+1$, we find that the functions $\psi_{v,2k,2j}$ solve algebraic equations of the form

$$-\frac{\gamma^2}{4}\partial_t^2\psi_{v,2(k-2),2j} - ji\gamma^2\partial_t\psi_{v,2(k-1),2j} + (j^2\gamma^2 + \partial_x^2)\psi_{v,2k,2j} + \partial_x^2 H_{2k,2j} = 0,$$

where $H_{2k,2j}$ is a quadratic mapping which does not depend on $\psi_{v,2k,2j}$. We note that we set $\psi_{v,m,2j} = 0$ for $m < 0$. In order to solve this equation for $\psi_{v,2k,2j}$, we have to choose the period L in such a way that, in Fourier space, each wave number in $(2\pi/L)\mathbb{Z}$ is $\mathcal{O}(1)$ bounded away from the wave numbers $k_{\pm j,*} = \pm j\gamma$. Then, we set

$$\psi_{v,2k,2j} = (j^2\gamma^2 + \partial_x^2)^{-1}\big(\frac{\gamma^2}{4}\partial_t^2\psi_{v,2(k-2),2j} + ji\gamma^2\partial_t\psi_{v,2(k-1),2j}\big) + \mathcal{A}_{j\gamma}H_{2k,2j},$$

and $\psi_{v,2k,2j}$ is well-defined due to the choice of the period L.

In total, all terms up to order $\mathcal{O}(\varepsilon^{2n-2})$ cancel in Res_u and all terms up to order $\mathcal{O}(\varepsilon^{2n})$ cancel in Res_v. The term which needs the most derivatives in both Res_u and Res_v is $\partial_t^2\psi_{u,2n,1}$. As in Section 2.2.3, we can repeatedly replace the time derivatives of $\psi_{u,2n,1}$ with the right-hand side of (2.46). Then, the term $\partial_t^{n+2}\psi_{u,0,1}$

appears. Further, we can repeatedly replace the time derivatives of $\psi_{u,0,1}$ with the right-hand side of (2.44). Then, the term $\partial_x^{2n+4}\psi_{u,0,1}$ appears. Therefore, in order to estimate Res_u in H^{s+1}, we have to assume that $\psi_{u,0,1} \in H^{s+2n+5}$. The estimates for Res_v in H^s are straightforward. We note that each term in Res_v has either a spatial derivative in front or has a vanishing mean value by construction. Thus, we can formulate the following theorem.

Theorem 2.4.5. *Let $n \in \mathbb{N}$, $s \geq 0$, and $\gamma \in \mathbb{R} \setminus \{0\}$. Consider spatially L-periodic boundary conditions with $L > 0$ satisfying that each $k \in (2\pi/L)\mathbb{Z}$ is $\mathcal{O}(1)$ bounded away from the wave numbers $k_{\pm j,*} = \pm j\gamma$ for all $1 \leq j \leq n+1$. Further, let $\psi_u \in C([0,T_0], H^{s+2n+5})$ be a solution of the singular NLS equation (2.44). Then, there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there is an approximation $(\Psi_{u,n}, \Psi_{v,n})$ of the form (2.45) with*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\Psi_{u,n}, \Psi_{v,n})\|_{H^{s+1}} \leq C_{res}\varepsilon^{2n}, \quad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\Psi_{u,n}, \Psi_{v,n})\|_{H^s} \leq C_{res}\varepsilon^{2n+2},$$

and

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\Psi_{u,n}, \Psi_{v,n})\|_{H^s} \leq C_{res}\varepsilon^{2n+2}.$$

Chapter 3

The validity of the Derivative NLS approximation for systems with quadratic nonlinearities

3.1 Introduction

Modulation equations or amplitude equations are relatively simple PDEs, which can be derived by perturbation analysis. They are used for modeling more complicated PDEs in the sense that the modulation equation makes correct predictions about the dynamics of the original systems. There are many PDEs which serve as modulation equations for various dissipative and dispersive systems. For instance, the Ginzburg-Landau equation, which has been justified for pattern forming systems [SH77, CE90, Sch94a, Sch94b], or the Korteweg-De Vries (KdV) equation, which has been justified for some dispersive equations in plasma physics, the Fermi–Pasta–Ulam system and the water wave equation, cf. [Cra85, SW00, SW02, Dül12]. Another example of such a modulation equation is the Nonlinear Schrödinger (NLS) equation, which can be derived as an amplitude equation describing slow modulations in time and space of oscillatory wave packets in dispersive wave systems, for instance the quadratic and cubic Klein-Gordon equation, the water wave problem [Zak68, Osb10], waves in DNA [Pel11], Bose-Einstein condensates [SH94], and, most importantly, systems from nonlinear optics, e.g., [Agr01]. For more details, we refer to [SU17, §10-12]. In this chapter, we consider the Derivative Nonlinear Schrödinger (DNLS) equation

$$i\partial_T A = \nu_1 \partial_X^2 A + \nu_2 A|A|^2 + i\nu_3 |A|^2 \partial_X A + i\nu_4 A^2 \partial_X \overline{A} + \nu_5 A|A|^4, \tag{3.1}$$

with $T \geq 0$, $X \in \mathbb{R}$, $A(X,T) \in \mathbb{C}$, and coefficients $\nu_j \in \mathbb{R}$ for $j = 1,\ldots,5$. The DNLS equation has been derived, for instance, as a long wave limit equation from the one-dimensional compressible magnetohydrodynamical equations in the presence of the Hall effect [Mjø76, CLPS99, JLPS19], and the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas [MOMT76]. In this chapter, we are interested in its appearance as an envelope equation describing slow modulations in time and space of an oscillating wave packet $e^{i(k_0x-\omega_0t)}$. The ansatz for the derivation of the DNLS equation is given by

$$u(x,t)=\varepsilon^{1/2}\psi_{\text{DNLS}}=\varepsilon^{1/2}\left(A(\varepsilon(x-c_gt),\varepsilon^2t)e^{i(k_0x-\omega_0t)}+c.c.\right)+\mathcal{O}(\varepsilon).$$

Hereby, c_g is the linear group velocity, $k_0=1$ the basic spatial wave number, ω_0 the basic temporal wave number, and $0<\varepsilon\ll 1$ a small perturbation parameter. For original dispersive wave systems with a quadratic resp. cubic nonlinearity, the justification of the DNLS equation is more complicated than the justification of the NLS approximation [Kal88, BSTU06, TW12, DSW16]. The reason for this is that in the error equation, terms of order $\mathcal{O}(\varepsilon^{1/2})$ resp. $\mathcal{O}(\varepsilon)$ have to be controlled on an $\mathcal{O}(\varepsilon^{-2})$ time scale. There already exist proofs of the DNLS approximation for a special cubic Klein-Gordon equation in case of analytic solutions as well as solutions in Sobolev spaces [HS22a, HS22b]. In this chapter, we consider a Klein-Gordon equation with a quadratic nonlinearity. This one is actually a general Klein-Gordon equation for which the DNLS approximation can be justified, since the justification includes all problems which occur in case of higher order nonlinearities. For the sake of simplicity, and with the intention of handling upcoming resonances, we also add a fourth order term to the nonlinearity, cf. Remark 3.1.4. Thus, we consider the most simple toy problem, namely, the following Klein-Gordon equation with a special nonlinearity

$$\partial_t^2u=\partial_x^2u-u+\varrho(\partial_x)u^2+\varrho_1(\partial_x)u^4, \tag{3.2}$$

with $x\in\mathbb{R}$, $t\in\mathbb{R}$, $u(x,t)\in\mathbb{R}$, and

$$\varrho(\partial_x)=-(1-\partial_x^2)^{-1}(1+\partial_x^2), \qquad \text{resp.} \qquad \varrho(ik)=\frac{k^2-1}{k^2+1}.$$

For the choice of the operator $\varrho_1(\partial_x)$, we refer to Theorem 3.1.1. By inserting the ansatz

$$u(x,t)=\varepsilon^{1/2}\psi_{\text{DNLS}}=\varepsilon^{1/2}(a_1+a_{-1})+\varepsilon(a_2+a_{-2})+\varepsilon a_0, \tag{3.3}$$

where $a_{-j}=\overline{a_j}$ and

$$a_j(x,t)=A_j(\varepsilon(x-c_gt),\varepsilon^2t)e^{ji(k_0x-\omega_0t)}, \qquad j=0,1,2,$$

with $k_0=1$, into the equation (3.2) and equating the coefficients in front of $\mathbf{E}:=e^{i(k_0x-\omega_0t)}$ with zero, we obtain the linear dispersion relation $\omega_0^2=k_0^2+1$ at $\mathcal{O}(\varepsilon^{1/2})$, and the linear group velocity $c_g=k_0/\omega_0$ at $\mathcal{O}(\varepsilon^{3/2})$. Using the expansion

$$\varrho(i+i\varepsilon K)=1+\varepsilon K+\mathcal{O}(\varepsilon^2),$$

gives at $\mathcal{O}(\varepsilon^{5/2})$

$$\begin{aligned}-2i\omega_0\partial_T A_1 &= (1-c_g^2)\partial_X^2 A_1 - 2i\partial_X(\overline{A_1}A_2 + A_0A_1)\\ &\qquad + \varrho_1(k_0)(4A_1^3\overline{A_2} + 12\overline{A_1}|A_1|^2A_2 + 12A_1|A_1|^2A_0).\end{aligned}$$

At $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon\mathbf{E}^2)$, we find the relations

$$\begin{aligned}\mathcal{O}(\varepsilon): &\quad A_0 = 2\varrho(0)|A_1|^2,\\ \mathcal{O}(\varepsilon\mathbf{E}^2): &\quad A_2 = \frac{\varrho(2k_0)}{-4\omega_0^2+4k_0^2+1}A_1^2,\end{aligned}$$

which are well-defined since the denominator does not vanish as a result of

$$-m^2\omega_0^2 + m^2k_0^2 + 1 = -(m\omega(k_0))^2 + \omega(mk_0)^2 \neq 0 \qquad \text{for all } m \geq 2. \tag{3.4}$$

Inserting the equations for A_0 and A_2 into the equation for A_1 finally yields the DNLS equation

$$-2i\omega_0\partial_T A_1 = (1-c_g^2)\partial_X^2 A_1 - 2i\gamma_1\partial_X(A_1|A_1|^2) + \gamma_2(A_1|A_1|^4), \tag{3.5}$$

where

$$\gamma_1(k_0) = 2\varrho(0) + \frac{\varrho(2k_0)}{-4\omega_0^2+4k_0^2+1}, \quad \gamma_2(k_0) = \varrho_1(k_0)\left(24\varrho(0) + \frac{16\varrho(2k_0)}{-4\omega_0^2+4k_0^2+1}\right).$$

The goal of this chapter is to prove that the DNLS equation (3.5) makes correct predictions about the dynamics of the Klein-Gordon equation (3.2), i.e., that the following approximation property holds.

Theorem 3.1.1. *Let the operator ϱ_1 in* (3.2) *be chosen such that the subsequent condition* (3.28) *is satisfied. Let $s_A \geq 6$ and $A_1 \in C([0,T_0], H^{s_A})$ be a solution of the DNLS equation* (3.5). *Then, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions u of the Klein-Gordon model* (3.2) *such that*

$$\sup_{t\in[0,T_0/\varepsilon^2]}\sup_{x\in\mathbb{R}}\left|u(x,t) - \varepsilon^{1/2}\psi_{DNLS}(x,t)\right| \leq C\varepsilon^{3/2},$$

where $\varepsilon^{1/2}\psi_{DNLS}$ is given by (3.3).

The condition (3.28) is necessary to handle resonances appearing at the fourth order terms. We can always choose the operator ϱ_1 to satisfy this condition. Only then, the approximation result holds. Otherwise, if the condition is not satisfied, we will be able to prove a non-approximation result, cf. Section 3.6.

Remark 3.1.2. The proof of Theorem 3.1.1 is a non-trivial task as solutions of order $\mathcal{O}(\varepsilon^{1/2})$ have to be controlled on an $\mathcal{O}(\varepsilon^{-2})$ time scale. Since we have a quadratic nonlinearity, a simple application of Gronwall's inequality would only give control on an $\mathcal{O}(\varepsilon^{-1/2})$ time scale. The proof is based on energy estimates and normal form transformations. In contrast to the DNLS approximation, for the cubic Klein-Gordon equation, cf. [HS22a, HS22b], we have to perform multiple normal form transformations. Thereby, the same difficulties occur as in the cubic case due to total resonances and second order resonances. In addition to that, new difficulties occur due to further resonant terms arising from the fourth order terms and due to more problematic terms produced by the elimination of the second order resonant terms.

Remark 3.1.3. In contrast to the DNLS approximation (3.3), for the quadratic Klein-Gordon equation, the ansatz for the derivation of the NLS equation

$$i\partial_T A = \nu_1 \partial_X^2 A + \widetilde{\nu}_2 A|A|^2, \tag{3.6}$$

with coefficients $\nu_1, \widetilde{\nu}_2 \in \mathbb{R}$, is given by

$$u(x,t) = \varepsilon\psi_{\text{NLS}} = \varepsilon(a_1 + a_{-1}) + \varepsilon^2(a_2 + a_{-2}) + \varepsilon^2 a_0.$$

The DNLS approximation appears in the degenerated situation when the cubic coefficient $\widetilde{\nu}_2 = \widetilde{\nu}_2(k_0)$ in the NLS equation (3.6) vanishes for the chosen basic wave number k_0. In this case, the DNLS equation takes the role of the NLS equation. This situation appears, for instance, in the water wave problem for certain values of surface tension and basic spatial wave number k_0, cf. [AS81].

Remark 3.1.4. For the derivation of the DNLS equation, in the nonlinearity of the original system only the terms of order two, three, four and five are relevant. For the approximation result, only the terms of order two, three and four are problematic. The fifth order terms do not cause any problems since in the equation for the error, see Section 3.3, they are of order $\mathcal{O}(\varepsilon^2)$. Thus, they can be easily controlled on the natural time scale $\mathcal{O}(\varepsilon^{-2})$ of the DNLS approximation. Hence, in order to avoid longer calculations, we discard any fifth order terms in our model problem. We also discard any third order terms since, in the equation for the error, the cubic terms will be produced by the quadratic terms anyway. Nevertheless, we include fourth order terms because we need them to handle upcoming resonances. In total, for completeness we could also consider third and fifth order terms but for notational simplicity we discard them. Our model problem (3.2) is sufficient enough in the sense that all terms, which are necessary to derive the DNLS equation, are contained and all possible difficulties for the proof of the approximation result are covered.

Remark 3.1.5. In case that $\gamma_2 = 0$, the DNLS equation (3.5) is a completely integrable PDE solvable through the inverse scattering method [KN78, JLPS19]. It was shown in [TF80] that smooth solutions exist uniquely in Sobolev spaces H^s with $s > 3/2$. Further papers [TF80, HO92, Tak99, CKS$^+$02, Wu15, GW16, JLPS19, BP20] extend these results to solutions of lower regularity and investigate global existence. However, for $u_0 \in H^s$ with $s < 1/2$, the Cauchy problem is ill-posed and uniform continuity with respect to initial conditions fails [Tak99].

This chapter is structured as follows. In the subsequent section, we construct a higher order DNLS approximation such that the so-called residual of the Klein-Gordon equation (3.2) becomes sufficiently small. The residual of (3.2) contains all terms, which do not cancel after inserting an approximation into the original system (3.2), and measures how much the approximation fails to solve the original system. Further, we sketch how to estimate the residual.
We continue by preparing the proof of the justification of the DNLS approximation considering solutions in Sobolev spaces. In Section 3.3, we derive an equation for the error which is an ε^β-scaled difference of the real solution of the original system and the higher order DNLS approximation. Further, we write this error equation as a first order equation and diagonalize it. We note that a goal of this chapter is to find an $\mathcal{O}(1)$ bound for the error over the natural $\mathcal{O}(\varepsilon^{-2})$ time scale of the DNLS approximation. However, since the DNLS approximation is of order $\mathcal{O}(\varepsilon^{1/2})$, the problem occurs that quadratic, cubic and quartic terms of order $\mathcal{O}(\varepsilon^{1/2})$, $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^{3/2})$ show up in the error equation. As these are obviously not $\mathcal{O}(1)$ bounded on the $\mathcal{O}(\varepsilon^{-2})$ time scale, they have to be eliminated by so-called normal form transformations.
In order to eliminate the problematic terms which are oscillatory in time, in Section 3.4, we perform these three normal form transformations and an additional one. The use of normal form transformations goes back to [Kal88]. When using normal form transformations, a number of non-resonance conditions has to be satisfied. For the first normal form transformation, see Subsection 3.4.1, the non-resonance conditions are satisfied such that the $\mathcal{O}(\varepsilon^{1/2})$ terms can be eliminated without further problems. For the second normal form transformation, see Subsection 3.4.2, the non-resonance conditions for the elimination of terms of the form $ib^j_{j_1,j_2,j_3}(\widehat{a}_{j_1} * \widehat{a}_{j_2} * \widehat{R}_{j_3})$, with $j, j_1, j_2, j_3 \in \{\pm 1\}$, in the j-th component of the error equation reads

$$j\omega(k) + j_1\omega(k_0) + j_2\omega(k_0) - j_3\omega(k - j_1k_0 - j_2k_0) \neq 0.$$

It turns out that some terms violate the non-resonance conditions, i.e., resonances occur which prevent the elimination of these resonant $\mathcal{O}(\varepsilon)$ terms. In detail, we have so-called totally resonant terms and second order resonant terms:

- The totally resonant terms correspond to the indices with $j_3 = j$ and $j_2 =$

$-j_1$ such that the left-hand side of the non-resonance condition completely vanishes for all $k \in \mathbb{R}$. This seems to be a huge problem, but it turns out that the totally resonant terms can be controlled by energy estimates.

- The second order resonant terms correspond to the indices with $j_3 = j = -1$ and $j_2 = j_1$ such that the left-hand side of the non-resonance condition vanishes quadratically for the wave number $k = j_1$. In order to solve this problem, by adding and subtracting irrelevant terms of order $\mathcal{O}(\varepsilon^2)$ to and from the error equation, one can achieve to shift the quadratic singularity $\mathcal{O}(\varepsilon^2)$ away from zero. Thus, the non-resonance condition can be satisfied but we lose powers of ε. However, in the energy estimates most of these terms gain an ε power. The reason for this is the choice of the quadratic nonlinearity which also vanishes for the same wave number $k = j_1$.

The problems that stem from these resonances were already solved in [HS22b] for the cubic Klein-Gordon equation and can be handled the same way in our case. However, in the third normal form transformation, see Subsection 3.4.3, further resonances occur. To be more precise, for the resonant quartic terms the left-hand side of the corresponding non-resonance conditions vanishes linearly for the resonant wave numbers. By a certain choice of the operator ϱ_1, in the original system, one can make the resonances stable in the sense that the resonant terms can be controlled by energy estimates, cf. [Sch05]. Furthermore, we need to perform an additional normal form transformation, see Subsection 3.4.4, in order to eliminate certain sixth order terms generated by the second order resonant terms. This is due to the fact that the operator ϱ_1, in general, does not vanish for the basic wave number k_0. Here, additional resonances arise. However, as these are bounded away from the second order resonances at $k = j_1 k_0$, we do the elimination by cutting off the normal form transformation around the wave numbers $k = j_1 k_0$.
In Section 3.5, we close the proof of Theorem 3.1.1 by giving energy estimates. In order to get the $\mathcal{O}(1)$ boundedness of the error, we include the four normal form transformations to the energy and use Gronwall's inequality.
In fact, approximation results should not be taken for granted. There are a number of counterexamples which show that the approximation fails, cf. [Sch95, SSZ15, HS18, BSSZ20]. In Section 3.6, we consider the case where we choose the operator ϱ_1 in a way so that the resonances appearing at the quartic terms become unstable. By modifying the original system, we can achieve that these unstable resonant wave numbers lie on an integer multiple of the basic wave number k_0. Then, in case of spatially $2\pi/k_0$-periodic boundary conditions, we give a rigorous proof that the DNLS approximation fails to predict the behaviour of solutions of the modified original system. This proof resembles the one that a spectrally unstable fixed point is unstable, cf. [SU17, §2.3].

Finally, in Section 3.7, we discuss whether the DNLS approximation is still valid when we consider solutions that are analytic in a strip of the complex plane instead of solutions in Sobolev spaces. It turns out that the alternative method for proving the approximation property, cf. [Sch98, DHSZ16, HS22a], can be applied without further problems. According to [HS22a], in our approximation theorem, there is a restriction regarding the time due to the resonant cubic terms. However, despite of the resonances appearing at the fourth order terms, we show that there is no additional time restriction as it should normally be the case.

Notation. We use the notation from Chapter 1. Further, we define the space $L^p_s(\mathbb{R},\mathbb{K})$ by

$$u \in L^p_s(\mathbb{R},\mathbb{K}) \Leftrightarrow u(\cdot)(1+(\cdot)^2)^{s/2} \in L^p(\mathbb{R},\mathbb{K}).$$

We write $\int$ for $\int_{\mathbb{R}}$, H^s for $H^s(\mathbb{R},\mathbb{K})$, and L^p_s for $L^p_s(\mathbb{R},\mathbb{K})$, unless otherwise specified.

3.2 The higher order DNLS approximation

The residual of (3.2) is defined by

$$\mathrm{Res}(u) = -\partial_t^2 u + \partial_x^2 u - u + \varrho(\partial_x)u^2 + \varrho_1(\partial_x)u^4 \tag{3.7}$$

and contains all terms which do not cancel after inserting an approximation into the original system (3.2). To be more precise, the residual measures how much an approximation fails to solve the Klein-Gordon equation. When inserting the DNLS approximation (3.3) into the original equation (3.2), the residual is of order $\mathcal{O}(\varepsilon^2)$ which is not sufficient enough to prove Theorem 3.1.1. For the further course of this chapter, we need the residual to be at least of order $\mathcal{O}(\varepsilon^4)$. We can achieve this by adding higher order terms to our DNLS approximation (3.3) in order to eliminate all terms up to order $\mathcal{O}(\varepsilon^{7/2})$.
First, we modify the original approximation $\varepsilon^{1/2}\psi_{\mathrm{DNLS}}$ by replacing A_1 in the definition of $\varepsilon^{1/2}\psi_{\mathrm{DNLS}}$ by

$$A_1^c\left(\varepsilon(\cdot - c_g t), \varepsilon^2 t\right) = \mathcal{F}^{-1}\left(\chi_{[-\delta,\delta]}(\cdot)\mathcal{F}\left(A_1\left(\varepsilon(\cdot - c_g t), \varepsilon^2 t\right)\right)(\cdot)\right),$$

where $\chi_{[-\delta,\delta]}$ is the characteristic function on the interval $[-\delta,\delta]$ and $\delta \in (0, k_0/20)$ is a fixed chosen constant that is independent of the parameter ε. The error made by replacing A_1 by A_1^c is of order $\mathcal{O}(\varepsilon^5)$ due to the estimate

$$\|\chi_{[-\delta,\delta]}\varepsilon^{-1}\widehat{f}(\varepsilon^{-1}\cdot) - \varepsilon^{-1}\widehat{f}(\varepsilon^{-1}\cdot)\|_{L^2_m} \leq C(\delta)\varepsilon^{m+M-1/2}\|f\|_{H^{m+M}}$$

and the fact that $A_1 \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ solves the DNLS equation (3.5), cf. [SU17, §11.5] for more details. Since A_1^c has a bounded support in Fourier space,

the use of A_1^c instead of A_1 subsequently allows us to control the terms in the normal form transformations more efficiently.
We continue by constructing an improved DNLS approximation for (3.2) such that the residual is of formal order $\mathcal{O}(\varepsilon^4)$ which is necessary to prove Theorem 3.1.1. The higher order DNLS approximation $\varepsilon^{1/2}\Psi$ is given by

$$\varepsilon^{1/2}\Psi(x,t) = \varepsilon^{1/2}\Psi_{\mathrm{DNLS}}(x,t) + \varepsilon^{3/2}\Psi_q(x,t), \tag{3.8}$$

where

$$\begin{aligned}\varepsilon^{3/2}\Psi_q(x,t) = {} & \big(\varepsilon^{3/2}a_{1,1}(x,t) + c.c.\big) + \sum_{n=0,1,2}\big(\varepsilon^{3/2+n}a_{3,n}(x,t) + c.c.\big) \\ & + \sum_{n=0,1}\big(\varepsilon^{5/2+n}a_{5,n}(x,t) + c.c.\big) + \big(\varepsilon^{7/2}a_{7,0}(x,t) + c.c.\big) \\ & + \sum_{n=1,2}\varepsilon^{1+n}a_{0,n}(x,t) + \sum_{n=1,2}\big(\varepsilon^{1+n}a_{2,n}(x,t) + c.c.\big) \\ & + \sum_{n=0,1}\big(\varepsilon^{2+n}a_{4,n}(x,t) + c.c.\big) + \big(\varepsilon^{3}a_{6,0}(x,t) + c.c.\big)\end{aligned}$$

and

$$a_{j,n}(x,t) = A_{j,n}(\varepsilon(x-c_g t), \varepsilon^2 t)e^{ji(k_0 x - \omega_0 t)}.$$

Remark 3.2.1. In contrast to [HS22b], the ansatz (3.8) for the higher order DNLS approximation not only contains terms of order $\mathcal{O}(\varepsilon^{1/2+n})$ but also terms of order $\mathcal{O}(\varepsilon^n)$. Therefore, the ansatz (3.8) can be also used for a more general Klein-Gordon model including third and fifth order terms.

Let $\mathbf{E} := e^{i(k_0 x - \omega_0 t)}$. Since, in Fourier space, $A_1^c\mathbf{E}$ has a small support near the wave number k_0 and since we have a polynomial nonlinearity, also the $A_{j,n}$ can be chosen so that, in Fourier space, the support of $A_{j,n}\mathbf{E}^j$ lies in a small neighborhood of the wave number jk_0. If we insert the improved approximation (3.8) into the residual (3.7), equating the coefficients at $\mathcal{O}(\varepsilon^{1/2}\mathbf{E})$ and $\mathcal{O}(\varepsilon^{3/2}\mathbf{E})$ to zero yields $\omega_0^2 = k_0^2+1$ and $c_g = k_0/\omega_0$. Equating the coefficients at $\mathcal{O}(\varepsilon^{5/2}\mathbf{E})$ to zero yields the DNLS equation (3.5). Equating the coefficients at $\mathcal{O}(\varepsilon^{7/2}\mathbf{E})$ to zero, we obtain that $A_{1,1}$ is determined by solving a linear, but inhomogeneous, Schrödinger equation. For this equation, the inhomogeneous term only depends on A_1^c. Equating the coefficients at $\mathcal{O}(\varepsilon^{n+1}\mathbf{E}^j)$ and $\mathcal{O}(\varepsilon^{3/2+n}\mathbf{E}^j)$ with $|j| \neq 1$ to zero results in linear algebraic equations for $A_{j,n}$ which can be solved with respect to $A_{j,n}$ since their coefficients do not vanish due to (3.4). For more details, we refer to [SU17, §11.2]. Finally, all terms up to formal order $\mathcal{O}(\varepsilon^{7/2})$ vanish such that we are left with terms of order $\mathcal{O}(\varepsilon^4)$ depending on A_1^c. Since $A_1 \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ and due to the fact that A_1 solves the DNLS equation (3.5), these remaining terms can be easily estimated in H^s. Hence, we can conclude the following lemma.

Lemma 3.2.2. *Let $s_A \geq 6$ and $A_1 \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ be a solution of the DNLS equation* (3.5) *with*

$$\sup_{T\in[0,T_0]} \|A_1\|_{H^{s_A}} \leq C_A.$$

Then, for all $s \geq 0$ there exist C_{Res}, C_Ψ, and $\varepsilon_0 > 0$ depending on C_A such that for all $\varepsilon \in (0,\varepsilon_0)$ the approximation $\varepsilon^{1/2}\Psi$, defined in (3.8), *satisfies*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{H^s} \leq C_{Res}\varepsilon^4, \tag{3.9}$$

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon^{1/2}\Psi - \varepsilon^{1/2}\Psi_{DNLS}\|_{C_b^0} \leq C_\Psi \varepsilon^{3/2}, \tag{3.10}$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \left(\|\widehat{a}_1\|_{L^1_{s+1}} + \|\widehat{a}_{-1}\|_{L^1_{s+1}} + \|\widehat{\Psi}_q\|_{L^1_{s+1}} \right) \leq C_\Psi. \tag{3.11}$$

Remark 3.2.3. We state a number of remarks concerning Lemma 3.2.2.

a) We refrain from recalling the proof since such estimates are carried out in the existing literature of the NLS approximation. For further details, we refer to [SU17, §11]. We note that the residual can be made arbitrarily small by adding even higher order terms to the DNLS approximation.

b) We note that, due to the scaling property of the L^2-norm with respect to the scaling $X = \varepsilon x$, we lose half an ε-power when estimating A_1 in H^s. Instead, we use the estimate

$$\|a_1 f\|_{H^s} \leq C\|a_1\|_{C_b^s} \|f\|_{H^s} \leq C\|\widehat{a}_1\|_{L^1_s} \|f\|_{H^s}$$

which prevents the loss of powers in ε due to (3.11).

c) The reason for the order of regularity $s_A \geq 6$ is as follows. In the inhomogeneity of the equation for $A_{1,1}$, the term $2c_g\partial_X\partial_T A_1$ occurs. After replacing the time derivative with the right-hand side of the DNLS equation (3.5), the term $\partial_X^3 A_1$ appears, i.e., we need $s_A \geq 3$ for the well-posedness of the equation for $A_{1,1}$. Finally, since in the residual the term $\partial_T^2 A_{1,1}E$ occurs and since there is no higher spatial derivative of A_1, we need $s_A \geq 6$.

3.3 The error equation

Our model problem is of the form

$$\partial_t^2 u = -\omega_{op}^2 u - \omega_{op}\rho_{op}u^2 + \omega_{op}\rho_{1,op}u^4, \tag{3.12}$$

with pseudo differential operators ω_{op}, ρ_{op} and $\rho_{1,op}$ defined by

$$\omega(k) = \operatorname{sign}(k)\sqrt{1+k^2}, \qquad \rho(k) = -\frac{\varrho(k)}{\omega(k)}, \qquad \rho_1(k) = \frac{\varrho_1(k)}{\omega(k)}.$$

In order to estimate the error $\varepsilon^\beta R$ that is made by the improved DNLS approximation $\varepsilon^{1/2}\Psi$, we make the following ansatz

$$u = \varepsilon^{1/2}\Psi + \varepsilon^\beta R \tag{3.13}$$

for $\beta = 2$. The error function R satisfies

$$\begin{aligned}\partial_t^2 R = &-\omega_{op}^2 R - \omega_{op}\rho_{op}(2\varepsilon^{1/2}\Psi R + \varepsilon^2 R^2) \\ &+ \omega_{op}\rho_{1,op}(4\varepsilon^{3/2}\Psi^3 R + 6\varepsilon^3\Psi^2 R^2 + 4\varepsilon^{9/2}\Psi R^3 + \varepsilon^6 R^4) + \varepsilon^{-2}\mathrm{Res}(\varepsilon^{1/2}\Psi).\end{aligned}$$

We write this equation as a first order system

$$\begin{aligned}\partial_t R &= i\omega_{op}\widetilde{R}, \\ \partial_t \widetilde{R} &= i\omega_{op}R_u + i\rho_{op}(2\varepsilon^{1/2}\Psi R + \varepsilon^2 R^2) \\ &\quad - i\rho_{1,op}(4\varepsilon^{3/2}\Psi^3 R + 6\varepsilon^3\Psi^2 R^2 + 4\varepsilon^{9/2}\Psi R^3 + \varepsilon^6 R^4) \\ &\quad + (i\omega_{op})^{-1}\varepsilon^{-2}\mathrm{Res}(\varepsilon^{1/2}\Psi).\end{aligned}$$

Via the transformation

$$\mathcal{R} = \begin{pmatrix} R_1 \\ R_{-1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} R \\ \widetilde{R} \end{pmatrix},$$

we obtain a diagonalized first order system. This system is written as

$$\begin{aligned}\partial_t \mathcal{R} = &\Lambda\mathcal{R} + \varepsilon^{1/2}\mathcal{B}_1(\Upsilon, \mathcal{R}) + \varepsilon\mathcal{B}_2(\Upsilon, \Upsilon, \mathcal{R}) + \varepsilon^{3/2}\mathcal{B}_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R}) \\ &+ \varepsilon^{3/2}\mathcal{B}_4(\Upsilon, \mathcal{R}) + \varepsilon^2\mathcal{B}_5(\Upsilon, \mathcal{R}) + \varepsilon^{-2}\mathrm{RES}(\varepsilon^{1/2}\Psi),\end{aligned} \tag{3.14}$$

where $\Upsilon = (a_1, a_{-1})^T$ and where, in Fourier space, $\widehat{\Lambda}(k) = \operatorname{diag}(i\omega(k), -i\omega(k))$ is a skew symmetric operator. Further, we have

$$\widehat{\mathcal{B}}_1(\widehat{\Upsilon}, \widehat{\mathcal{R}})(k,t) = i\rho(k) \begin{pmatrix} (\widehat{a}_1 + \widehat{a}_{-1}) * (\widehat{R}_1 + \widehat{R}_{-1}) \\ -(\widehat{a}_1 + \widehat{a}_{-1}) * (\widehat{R}_1 + \widehat{R}_{-1}) \end{pmatrix} (k,t),$$

$$\widehat{\mathcal{B}}_2(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}})(k,t) = i\rho(k) \begin{pmatrix} (\widehat{a}_0 + \widehat{a}_2 + \widehat{a}_{-2}) * (\widehat{R}_1 + \widehat{R}_{-1}) \\ -(\widehat{a}_0 + \widehat{a}_2 + \widehat{a}_{-2}) * (\widehat{R}_1 + \widehat{R}_{-1}) \end{pmatrix} (k,t),$$

$$\widehat{\mathcal{B}}_3(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}})(k,t) = i\rho(k) \begin{pmatrix} (\widehat{a}_3 + \widehat{a}_{-3}) * (\widehat{R}_1 + \widehat{R}_{-1}) \\ -(\widehat{a}_3 + \widehat{a}_{-3}) * (\widehat{R}_1 + \widehat{R}_{-1}) \end{pmatrix} (k,t),$$

$$- 2i\frac{\varrho_1(k)}{\omega(k)} \begin{pmatrix} (\widehat{a}_1 + \widehat{a}_{-1})^{*3} * (\widehat{R}_1 + \widehat{R}_{-1}) \\ -(\widehat{a}_1 + \widehat{a}_{-1})^{*3} * (\widehat{R}_1 + \widehat{R}_{-1}) \end{pmatrix} (k,t),$$

$$\widehat{\mathcal{B}}_4(\widehat{\Upsilon}, \widehat{\mathcal{R}})(k,t) = i\rho(k) \begin{pmatrix} (\widehat{a}_{1,1} + \widehat{a}_{-1,1}) * (\widehat{R}_1 + \widehat{R}_{-1}) \\ -(\widehat{a}_{1,1} + \widehat{a}_{-1,1}) * (\widehat{R}_1 + \widehat{R}_{-1}) \end{pmatrix} (k,t)$$

and

$$\widehat{\mathrm{RES}}(\varepsilon^{1/2}\widehat{\Psi})(k,t) = \frac{1}{2i\omega(k)} \begin{pmatrix} \widehat{\mathrm{Res}}(\varepsilon^{1/2}\widehat{\Psi}) \\ -\widehat{\mathrm{Res}}(\varepsilon^{1/2}\widehat{\Psi}) \end{pmatrix} (k,t).$$

The mapping $\varepsilon^2\mathcal{B}_5(\Upsilon, \mathcal{R})$ contains all terms of at least order $\mathcal{O}(\varepsilon^2)$ such that we have

$$\left\|\varepsilon^2\mathcal{B}_5(\Upsilon, \mathcal{R})\right\|_{(H^s)^2} \leq C\varepsilon^2 \left(\|R\|_{H^s} + \|R\|_{H^s}^2 + \varepsilon^{5/2}\|R\|_{H^s}^3 + \varepsilon^4\|R\|_{H^s}^4\right).$$

3.4 The normal form transformations

We note that, due to Lemma 3.2.2, the last two terms on the right-hand side of (3.14) are at least of order $\mathcal{O}(\varepsilon^2)$. These terms cause no problems in uniformly bounding the error $\mathcal{R}$ on the long time scale $\mathcal{O}(\varepsilon^{-2})$. Since Λ is a skew symmetric operator, the first term on the right-hand side does not cause any problems, too. However, all other terms are at most of order $\mathcal{O}(\varepsilon^{3/2})$ which is a serious problem in bounding the error independently of the parameter ε. Hence, we try to get rid of the problematic terms by making a near identity change of variables. By these so-called normal form transformations one can gain half an ε-power, if a number of non-resonance conditions is satisfied. Due to the fact that for every normal form transformation we normally gain half an ε-power and since we also have to eliminate terms of order $\mathcal{O}(\varepsilon^{1/2})$, we have to perform three normal form transformations and an additional one since we have to distinguish between fourth and sixth order terms of order $\mathcal{O}(\varepsilon^{3/2})$. However, the non-resonance condition is not always satisfied which prevents the elimination of some terms.

3.4.1 The first normal form transformation

We consider the diagonalized first order system (3.14). By the first normal form transformation, we want to get rid of the term $\varepsilon^{1/2}\mathcal{B}_1(\Upsilon,\mathcal{R})$. This term is of the form

$$\widehat{\mathcal{B}}_1^j(\widehat{\Upsilon},\widehat{\mathcal{R}}) = \sum_{j_1,j_2\in\{\pm1\}} ib^j_{j_1,j_2}(k)(\widehat{a}_{j_1} * \widehat{R}_{j_2})(k), \qquad j\in\{\pm1\},$$

where $\mathcal{B}_1^{\pm1}$ denote the components of $\mathcal{B}_1$. Obviously, we have that

$$\sup_{k\in\mathbb{R}} \left|b^j_{j_1,j_2}(k)\right| \leq C < \infty.$$

In order to eliminate the terms above, we make the near identity change of variables

$$\widetilde{\mathcal{R}} = \mathcal{R} + \varepsilon^{1/2}\mathcal{Q}(\Upsilon,\mathcal{R}), \tag{3.15}$$

where $\mathcal{Q} = (\mathcal{Q}^1, \mathcal{Q}^{-1})^T$ is a bilinear mapping which, in Fourier space, has the form

$$\widehat{\mathcal{Q}}^j(\widehat{\Upsilon},\widehat{\mathcal{R}}) = \sum_{j_1,j_2\in\{\pm1\}} q^j_{j_1,j_2}(k)(\widehat{a}_{j_1} * \widehat{R}_{j_2})(k), \qquad j\in\{\pm1\}.$$

We also write

$$\widehat{\mathcal{Q}}(\widehat{\Upsilon},\widehat{\mathcal{R}}) = \sum_{j_1,j_2} \widehat{\mathcal{Q}}_{j_1,j_2}(\widehat{a}_{j_1},\widehat{R}_{j_2}),$$

where

$$\widehat{\mathcal{Q}}^j_{j_1,j_2}(\widehat{a}_{j_1},\widehat{R}_{j_2}) = q^j_{j_1,j_2}(k)(\widehat{a}_{j_1} * \widehat{R}_{j_2})(k)$$

denote the components of $\mathcal{Q}_{j_1,j_2}$. It is well-known that the reduced non-resonance condition for the elimination of a term of the form $ib^j_{j_1,j_2}(\widehat{a}_{j_1} * \widehat{R}_{j_2})$ reads

$$\inf_{k\in\mathbb{R}} \left|L^j_{j_1,j_2}(k)\right| := \inf_{k\in\mathbb{R}} \left|j\omega(k) + j_1\omega(k_0) - j_2\omega(k-j_1k_0)\right| \geq C > 0$$

and is satisfied, cf. [SU17]. Thus, we can set

$$q^j_{j_1,j_2}(k) = \frac{b^j_{j_1,j_2}(k)}{j\omega(k) + j_1\omega(k_0) - j_2\omega(k-j_1k_0)},$$

where

$$\sup_{k\in\mathbb{R}} \left|q^j_{j_1,j_2}(k)\right| \leq C < \infty. \tag{3.16}$$

Therefore, the first normal form transformation can be inverted for $\varepsilon > 0$ sufficiently small. Finally, by construction of the mapping $\mathcal{Q}$, the problematic $\mathcal{O}(\varepsilon^{1/2})$

terms cancel and we are left with

$$\begin{aligned}\partial_t \widetilde{\mathcal{R}} = \Lambda \widetilde{\mathcal{R}} &+ \varepsilon \left(\mathcal{B}_2(\Upsilon, \Upsilon, \mathcal{R}) + \mathcal{Q}(\Upsilon, \mathcal{B}_1(\Upsilon, \mathcal{R}))\right) \\ &+ \varepsilon^{3/2}\mathcal{B}_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R}) + \varepsilon^{3/2}\mathcal{Q}(\Upsilon, \mathcal{B}_2(\Upsilon, \Upsilon, \mathcal{R})) \\ &+ \varepsilon^{1/2}\mathcal{Q}(\Upsilon, \Lambda\mathcal{R}) - \varepsilon^{1/2}\sum_{j_1,j_2} j_2 i\omega(-i\partial_x - j_1 k_0)\mathcal{Q}_{j_1,j_2}(a_{j_1}, R_{j_2}) \\ &+ \varepsilon^{1/2}\mathcal{Q}(\partial_t \Upsilon + \Lambda(k_0)\Upsilon, \mathcal{R}) + \widetilde{\mathcal{G}}(\Upsilon, \mathcal{R}),\end{aligned} \tag{3.17}$$

where $\omega(-i\partial_x)$ is defined via its Fourier transformation $\mathcal{F}(\omega(-i\partial_x))(k) = \omega(k)$ and where the mapping $\widetilde{\mathcal{G}}(\Upsilon, \mathcal{R})$ contains all terms, for which one can show that

$$\|\widetilde{\mathcal{G}}_j(\Upsilon, \mathcal{R})\|_{H^s} \leq C\varepsilon^2 \left(\|R\|_{H^s} + \|R\|_{H^s}^2 + \varepsilon^{5/2}\|R\|_{H^s}^3 + \varepsilon^4\|R\|_{H^s}^4\right). \tag{3.18}$$

We note that the equation (3.17) does not contain any terms of order $\mathcal{O}(\varepsilon^{1/2})$ anymore. The three terms with prefactor $\varepsilon^{1/2}$ are actually of order $\mathcal{O}(\varepsilon^{3/2})$ since for the last term we have

$$\partial_t \Upsilon + \Lambda(k_0)\Upsilon = \begin{pmatrix} -c_g \varepsilon \partial_X A_1 e^{i(k_0 x - \omega_0 t)} + \varepsilon^2 \partial_T A_1 e^{i(k_0 x - \omega_0 t)} \\ -c_g \varepsilon \partial_X A_{-1} e^{-i(k_0 x - \omega_0 t)} + \varepsilon^2 \partial_T A_{-1} e^{-i(k_0 x - \omega_0 t)} \end{pmatrix} \tag{3.19}$$

and for the first two terms we use the following Lemma 3.4.1, cf. [SU17], in combination with the Lipschitz-continuity of ω.

Lemma 3.4.1. *Let $n, m \in \mathbb{N}$ and let $g(k)$ satisfy $|g(k)| \leq C|k - k_0|^n$. Then,*

$$\left\| g(\cdot)\varepsilon^{-1}\widehat{A_1}\left(\frac{\cdot - k_0}{\varepsilon}\right)\right\|_{L^1_m} \leq C\varepsilon^n \|\widehat{A_1}\|_{L^1_{m+n}}.$$

With this lemma, we could reduce the non-resonance condition to a one-dimensional problem, cf. [SU17, §11.5] for more details.

Remark 3.4.2. From the above, we can conclude that the terms with prefactor $\varepsilon^{1/2}$ are actually of order $\mathcal{O}(\varepsilon^{3/2})$, which is still not sufficient enough to prove Theorem 3.1.1. Therefore, we have to eliminate them by another normal form transformation. However, these mentioned terms are of the form $\varepsilon^{3/2}\widetilde{\mathcal{B}}(\Upsilon, \mathcal{R})$, where $\widetilde{\mathcal{B}}$ is a bilinear mapping. Thus, the elimination goes exactly like the one just carried out, i.e., no further problems, such as resonances, occur. Since the same holds for the term $\varepsilon^{3/2}\mathcal{B}_4(\Upsilon, \mathcal{R})$, we refrain from carrying out the normal form transformations in the remainder of Section 3.4.

3.4.2 The second normal form transformation

In this subsection, similar problems occur which were already handled in the Klein-Gordon equation with a cubic nonlinearity [HS22b]. Thus, we only summarize the procedure briefly.
We consider the equation (3.17) after the first normal form transformation. By the second normal form transformation, we want to eliminate the terms of order $\mathcal{O}(\varepsilon)$, namely,

$$\begin{aligned}&\widehat{\mathcal{B}}_2^j(\widehat{\Upsilon},\widehat{\Upsilon},\widehat{\mathcal{R}})+\widehat{\mathcal{Q}}^j(\widehat{\Upsilon},\widehat{\mathcal{B}}_1(\widehat{\Upsilon},\widehat{\mathcal{R}}))\\&=\sum_{j_1,j_2,j_3\in\{\pm1\}} ib^j_{j_1,j_2,j_3}(k)(\widehat{a}_{j_1}*\widehat{a}_{j_2}*\widehat{R}_{j_3})(k),\quad j\in\{\pm1\},\end{aligned}$$

where with (3.16) we have

$$\sup_{k\in\mathbb{R}}\left|b^j_{j_1,j_2,j_3}(k)\right|\leq C<\infty.$$

In order to do so, we make the near identity change of variables

$$\breve{\mathcal{R}}=\widetilde{\mathcal{R}}+\varepsilon\mathcal{P}(\Upsilon,\Upsilon,\mathcal{R}), \tag{3.20}$$

where $\mathcal{P}=(\mathcal{P}^1,\mathcal{P}^{-1})^T$ is a trilinear mapping which, in Fourier space, has the form

$$\widehat{\mathcal{P}}^j(\widehat{\Upsilon},\widehat{\Upsilon},\widehat{\mathcal{R}})=\sum_{j_1,j_2,j_3\in\{\pm1\}} p^j_{j_1,j_2,j_3}(k)(\widehat{a}_{j_1}*\widehat{a}_{j_2}*\widehat{R}_{j_3})(k),\qquad j\in\{\pm1\}.$$

For the elimination of the term $ib^j_{j_1,j_2,j_3}(\widehat{a}_{j_1}*\widehat{a}_{j_2}*\widehat{R}_{j_3})$, the following reduced non-resonance condition has to be satisfied

$$\inf_{k\in\mathbb{R}}\left|L^j_{j_1,j_2,j_3}(k)\right|\geq C>0, \tag{3.21}$$

where

$$L^j_{j_1,j_2,j_3}(k):=j\omega(k)+j_1\omega(k_0)+j_2\omega(k_0)-j_3\omega(k-j_1k_0-j_2k_0).$$

It turns out that there are terms that violate the non-resonance condition (3.21), including totally resonant terms as well as second order resonant terms.

The totally resonant terms: For the indices $(j,j_1,j_2,j_3)=(j,j_1,-j_1,j)$, there is a total resonance since obviously for all $k\in\mathbb{R}$ it holds

$$j\omega(k)+j_1\omega(k_0)-j_1\omega(k_0)-j\omega(k)=0.$$

Therefore, a normal form transformation is not possible. However, an elimination is not even necessary since it turns out that the totally resonant terms can be controlled by energy estimates. To be more precise, when calculating the evolution of the L^2-norm of $\breve{R}_j$, the totally resonant terms arise in the form

$$C\varepsilon \int \big(\overline{\breve{R}_j}\; ib^j_{j_1,-j_1,j}(a_{j_1}a_{-j_1}\breve{R}_j) + c.c.\big)\,\mathrm{d}x.$$

By using the skew symmetry of $ib^j_{j_1,-j_1,j}$ and $\partial_x(a_1 a_{-1}) = \mathcal{O}(\varepsilon)$ in combination with Plancherel's identity and Lemma 3.4.1, one can show that ([HS22b, Lemma 4.2])

$$\varepsilon \left| \int \big(\overline{\breve{R}_j}\; ib^j_{j_1,-j_1,j}(a_{j_1}a_{-j_1}\breve{R}_j) + c.c.\big)\,\mathrm{d}x \right| \leq C\varepsilon^2 \|\breve{R}_j\|^2_{L^2}. \tag{3.22}$$

Hence, an elimination is really not necessary and we set $p^j_{j_1,-j_1,j} = 0$.

The second order resonant terms: For $(j, j_1, j_2, j_3) = (-1, j_1, j_1, -1)$, the non-resonance condition is also not satisfied. To be more precise, we have a second order resonance since the term

$$\omega(k) - 2j_1\omega(k_0) - \omega(k - 2j_1k_0) = j_1\omega''(j_1)(k - j_1)^2 + \mathcal{O}(|k - j_1|^3)$$

vanishes quadratically at $k = j_1$. Thus, an elimination of these terms by a normal form transformation seems to be not possible. However, the quadratic singularity can be shifted $\mathcal{O}(\varepsilon^2)$ away from zero by adding and subtracting terms of the form

$$j_1 i\kappa\varepsilon^3 p^{-1}_{j_1,j_1,-1} a_{j_1} a_{j_1} R_{-1}$$

to and from the equation for R_{-1} with $\kappa = \mathcal{O}(1)$ chosen sufficiently large. Subsequently, the added term will be included in the definition of the normal form transformation, while the subtracted counterpart is of order $\mathcal{O}(\varepsilon^2)$ and can be easily estimated by Gronwall's inequality. Due to the added terms in the error equation (3.17), the second order resonance is shifted $\mathcal{O}(\varepsilon^2)$ away from the k-axis. More precisely, the non-resonance condition (3.21) is transformed into

$$\inf_{k\in\mathbb{R}} \left| L^{-1}_{j_1,j_1,-1}(k) - j_1\kappa\varepsilon^2 \right| \geq C > 0,$$

which is now satisfied. Therefore, we can set

$$p^{-1}_{j_1,j_1,-1}(k) = \frac{b^{-1}_{j_1,j_1,-1}(k)}{L^{-1}_{j_1,j_1,-1}(k) - j_1\kappa\varepsilon^2}.$$

We would expect that $p^{-1}_{j_1,j_1,-1}$ is of order $\mathcal{O}(\varepsilon^{-2})$. However, we note that the numerator $b^{-1}_{j_1,j_1,-1}$ also vanishes at $k = j_1 k_0$ since $\mathcal{B}^j_2$ and $\mathcal{Q}^j$ both contain ϱ as an factor. Thus, we have that

$$\left|\varepsilon p^{-1}_{j_1,j_1,-1}(k)\right| \approx \varepsilon \left| \frac{k - j_1 k_0}{(k - j_1 k_0)^2 + \kappa\varepsilon^2} \right|$$

is of order $\mathcal{O}(1)$ for $\varepsilon \to 0$ and of order $\mathcal{O}(\varepsilon + \kappa^{-1})$ for $\kappa \to \infty$. Further, we remark that we gain another power of ε when multiplying $p^{-1}_{j_1,j_1,-1}$ by a function which vanishes at $k = j_1 k_0$.

The non-resonant terms: For the remaining indices, the non-resonance condition (3.21) is satisfied. Thus, we can easily eliminate these non-resonant terms by setting

$$p^j_{j_1,j_2,j_3}(k) = \frac{b^j_{j_1,j_2,j_3}(k)}{j\omega(k) + j_1\omega(k_0) + j_2\omega(k_0) - j_3\omega(k - j_1 k_0 - j_2 k_0)}$$

for $(j_2, j_3) \neq (-j_1, j)$ and $(j, j_2, j_3) \neq (-1, j_1, -1)$. Finally, for all $(j, j_2, j_3) \neq (-1, j_1, -1)$ and a neighborhood $U_{\widetilde{\delta}}(j_1)$ around $k = j_1$ with a radius $\widetilde{\delta} > 0$ sufficiently small, we have that

$$\begin{aligned} \sup_{k\in\mathbb{R}} \left|p^j_{j_1,j_2,j_3}(k)\right| &\leq C < \infty, \\ \sup_{k\notin U_{\widetilde{\delta}}(j_1)} \left|p^{-1}_{j_1,j_1,-1}(k)\right| &\leq C < \infty, \\ \sup_{k\in U_{\widetilde{\delta}}(j_1)} \left|\varepsilon p^{-1}_{j_1,j_1,-1}(k)\right| &\leq C < \infty. \end{aligned} \tag{3.23}$$

Consequently, the second normal form transformation is invertible for $\varepsilon > 0$ sufficiently small and $\kappa > 0$ sufficiently large. In total, after the second normal form transformation, we are left with the following system

$$\begin{aligned} \partial_t \breve{R}_j = \Lambda \breve{R}_j &+ \varepsilon^{3/2}\mathcal{B}^j_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R}) + \varepsilon^{3/2}\mathcal{Q}^j(\Upsilon, \mathcal{B}_2(\Upsilon, \Upsilon, \mathcal{R})) \\ &+ \varepsilon^{3/2}\mathcal{P}^j(\Upsilon, \Upsilon, \mathcal{B}_1(\Upsilon, \mathcal{R})) - \varepsilon^{3/2}\sum_{j_1} ib^j_{j_1,-j_1,j}(a_{j_1}a_{-j_1}\mathcal{Q}^j(\Upsilon, \mathcal{R})) \\ &+ \varepsilon^{5/2}\mathcal{P}^j(\Upsilon, \Upsilon, \mathcal{B}_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R})) + \breve{\mathcal{G}}^j(\Upsilon, \mathcal{R}), \end{aligned} \tag{3.24}$$

with $\breve{\mathcal{G}}(\Upsilon, \mathcal{R})$ obeying the same property (3.18) as $\widetilde{\mathcal{G}}(\Upsilon, \mathcal{R})$. The second term in the second line comes from the totally resonant terms when replacing R_j by $\breve{R}_j$.

Remark 3.4.3. We write $\mathcal{P} = \mathcal{P}_{\text{SOR}} + \mathcal{P}_{\text{NON}}$, where $\mathcal{P}_{\text{SOR}}$ contains all the second order resonant terms in $\mathcal{P}$, and $\mathcal{P}_{\text{NON}}$ contains the non-resonant ones. We remark that $\varepsilon^{5/2}\mathcal{P}_{\text{SOR}}(\Upsilon, \Upsilon, \mathcal{B}_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R}))$ is actually of order $\mathcal{O}(\varepsilon^{3/2})$ since the mapping $\mathcal{B}_3$ from (3.14) contains terms with a prefactor $\varrho_1(k)$ which, in general, does not vanish for $k = \pm 1$. Thus, near the wave numbers $\pm k_0$ the corresponding kernels are of order $\mathcal{O}(\varepsilon^{-1})$. Therefore, we have to perform an additional normal form transformation, see Section 3.4.4.

3.4.3 The third normal form transformation

By the third normal form transformation, we want to get rid of the quartic terms of order $\mathcal{O}(\varepsilon^{3/2})$, namely,

$$\begin{aligned} &\widehat{\mathcal{B}}_3^j(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}}) + \widehat{\mathcal{Q}}^j(\widehat{\Upsilon}, \widehat{\mathcal{B}}_2(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}})) \\ &\qquad + \widehat{\mathcal{P}}^j(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{B}}_1(\widehat{\Upsilon}, \widehat{\mathcal{R}})) - \sum_{j_1} i b^j_{j_1,-j_1,j}(\widehat{a}_{j_1} * \widehat{a}_{-j_1} * \widehat{\mathcal{Q}}^j(\widehat{\Upsilon}, \widehat{\mathcal{R}})) \\ &= \sum_{j_1,j_2,j_3,j_4 \in \{\pm 1\}} i b^j_{j_1,j_2,j_3,j_4}(k)(\widehat{a}_{j_1} * \widehat{a}_{j_2} * \widehat{a}_{j_3} * \widehat{R}_{j_4})(k), \quad j \in \{\pm 1\}, \end{aligned}$$

where by (3.16) and (3.23) we have that

$$\sup_{k \in \mathbb{R}} \left| b^j_{j_1,j_2,j_3,j_4}(k) \right| \leq C < \infty.$$

In order to carry out the elimination, we make the near identity change of variables

$$\check{\mathcal{R}} = \breve{\mathcal{R}} + \varepsilon^{3/2}\mathcal{S}(\Upsilon, \Upsilon, \Upsilon, \mathcal{R}), \tag{3.25}$$

where $\mathcal{S} = (\mathcal{S}^1, \mathcal{S}^{-1})^T$ is a multilinear mapping which, in Fourier space, has the form

$$\widehat{\mathcal{S}}^j(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}}) = \sum_{j_1,j_2,j_3,j_4 \in \{\pm 1\}} s^j_{j_1,j_2,j_3,s_4}(k)(\widehat{a}_{j_1} * \widehat{a}_{j_2} * \widehat{a}_{j_3} * \widehat{R}_{j_4})(k), \qquad j \in \{\pm 1\}.$$

For the elimination of the term $i b^j_{j_1,j_2,j_3,j_4}(\widehat{a}_{j_1} * \widehat{a}_{j_2} * \widehat{a}_{j_3} * \widehat{R}_{j_4})$, the following reduced non-resonance condition has to be satisfied

$$\inf_{k \in \mathbb{R}} \left| L^j_{j_1,j_2,j_3,j_4}(k) \right| \geq C > 0, \tag{3.26}$$

where

$$L^j_{j_1,j_2,j_3,j_4}(k) := j\omega(k) + (j_1 + j_2 + j_3)\omega(k_0) - j_4\omega(k - (j_1 + j_2 + j_3)k_0).$$

We notice that there are further resonant terms. In contrast to the previous normal form transformation, the left-hand side of (3.26) does not touch the k-axis but intersects it. Thus, we have to handle them in a different way. Let us consider the system (3.24), which we write as

$$\partial_t \breve{R}_j = ji\omega \breve{R}_j + \varepsilon^{3/2} \sum_{j_1,j_2,j_3,j_4 \in \{\pm 1\}} ib^j_{j_1,j_2,j_3,j_4} a_{j_1} a_{j_2} a_{j_3} R_{j_4} + \breve{\mathcal{G}}^j(\Upsilon, \mathcal{R}), \tag{3.27}$$

and the energy

$$E = \sum_{j\in\{\pm 1\}} \int \sigma_j |\breve{R}_j|^2 \,\mathrm{d}x,$$

with some operators $\sigma_{\pm 1} = \sigma_{\pm 1}(-i\partial_x)$. We ignore the first term in the last line of (3.24) since this term gets eliminated in the subsequent section anyway. By taking the time derivative of the energy E, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} E &= \sum_{j\in\{\pm 1\}} \int \big(\sigma_j \partial_t \breve{R}_j \overline{\breve{R}_j} + c.c.\big) \,\mathrm{d}x \\ &= \sum_{j\in\{\pm 1\}} \int \Bigg(\varepsilon^{3/2} \sigma_j \sum_{j_1,j_2,j_3,j_4 \in \{\pm 1\}} \big(ib^j_{j_1,j_2,j_3,j_4} a_{j_1} a_{j_2} a_{j_3} R_{j_4} \overline{R_j} + c.c.\big)\Bigg) \,\mathrm{d}x + \mathcal{O}(\varepsilon^2),\end{aligned}$$

where we used the skew symmetry of $i\omega$, the estimate (3.18), and $\breve{R}_j = R_j + \mathcal{O}(\varepsilon^{1/2})$. In order to get rid of the $\mathcal{O}(\varepsilon^{3/2})$ terms, we add

$$\widetilde{E} = \sum_{j\in\{\pm 1\}} \int \big(\varepsilon^{3/2} \mathcal{S}^j(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}) \overline{\breve{R}_j} + c.c.\big) \,\mathrm{d}x$$

to the energy E. Taking the time derivative yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(E + \widetilde{E}) \\ &= \sum_{j\in\{\pm 1\}} \int \Bigg(\varepsilon^{3/2} \sigma_j \sum_{j_1,j_2,j_3,j_4 \in \{\pm 1\}} \big(ib^j_{j_1,j_2,j_3,j_4} a_{j_1} a_{j_2} a_{j_3} R_{j_4} \overline{R_j} + c.c.\big)\Bigg) \,\mathrm{d}x \\ &\qquad - \int \Bigg(\varepsilon^{3/2} \sum_{j_1,j_2,j_3,j_4 \in \{\pm 1\}} \big(iL^j_{j_1,j_2,j_3,j_4} s^j_{j_1,j_2,j_3,j_4} a_{j_1} a_{j_2} a_{j_3} R_{j_4} \overline{R_j} + c.c.\big)\Bigg) \,\mathrm{d}x + \mathcal{O}(\varepsilon^2).\end{aligned}$$

The non-resonant terms: All terms with indices $(j_2, j_3, j_4) \neq (j_1, j_1, -j)$ are non-resonant and, thus, can be easily eliminated by setting

$$s^j_{j_1,j_2,j_3,j_4}(k) = \sigma_j(k) b^j_{j_1,j_2,j_3,j_4}(k) (L^j_{j_1,j_2,j_3,j_4}(k))^{-1}.$$

The first order resonant terms: For $(j, j_1, j_2, j_3, j_4) = (j, j_1, j_1, j_1, -j)$ resonances occur. To be more precise, $L^j_{j_1,j_1,j_1,-j}$ vanishes linearly at $k = k^j_{j_1} := j_1\left(\frac{3}{2} - j\sqrt{\frac{5}{2}}\right)$, i.e., $L^j_{j_1,j_1,j_1,-j}(k^j_{j_1}) = 0$ and $\partial_k L^j_{j_1,j_1,j_1,-j}(k^j_{j_1}) \neq 0$. Thus, a direct elimination is not possible and we have to study the resonant terms in a more detailed way.

Remark 3.4.4. To understand the behaviour at the resonant wave numbers k_j, where $(\cdot)_j := (\cdot)^j_{j,j,j,-j}$ with $j = \pm 1$, in Fourier space, we consider a subsystem of (3.27), namely,

$$\partial_t \widehat{\breve{R}}_j(k_j, t) = ji\omega(k_j)\widehat{\breve{R}}_j(k_j, t) + \varepsilon^{3/2} ib_j(k_j)\widehat{a}^3_j(3k_0)\widehat{\breve{R}}_{-j}(k_{-j}, t) + \mathcal{O}(\varepsilon^2).$$

We make the ansatz $\widehat{v}_j(k_j, t) = e^{ji\omega(k_j)t}\widehat{\breve{R}}_j(k_j, t)$ to obtain

$$\partial_t \widehat{v}_j(k_j, t) = \varepsilon^{3/2} ib_j(k_j)\widehat{a}^3_j(3k_0)\widehat{v}_{-j}(k_{-j}, t) + \mathcal{O}(\varepsilon^2)$$

resp.

$$\partial_t^2 \widehat{v}_1(k_1, t) = -\varepsilon^3 b_1(k_1)b_{-1}(k_{-1})|\widehat{a}_1|^6\widehat{v}_1(k_1, t) + \mathcal{O}(\varepsilon^2).$$

Thus, for

$$b_1(k_1)b_{-1}(k_{-1}) > 0, \tag{3.28}$$

we expect that the resonances can be stable in the sense that the resonant modes grow as $\mathcal{O}(\exp(i\varepsilon^{3/2}t))$ and stay bounded on the natural time scale $\mathcal{O}(\varepsilon^{-2})$ of the DNLS approximation. To be more specific, we consider the energy

$$\mathbb{E} = \sum_j \sigma_j(k_j)|\widehat{v}_j(k_j)|^2.$$

By taking the time derivative of $\mathbb{E}$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E} &= \sum_j \sigma_1(k_j)\left(\partial_t\widehat{v}_j(k_j)\overline{\widehat{v}_j(k_j)} + \widehat{v}_j(k_j)\partial_t\overline{\widehat{v}_j(k_j)}\right) \\
&= i\varepsilon^{3/2}\left(\sigma_1(k_1)b_1(k_1) - \sigma_{-1}(k_{-1})b_{-1}(k_{-1})\right) \\
&\quad \times\left(\widehat{A}^3_1(3k_0)\widehat{v}_{-1}(k_{-1})\overline{\widehat{v}_1(k_1)} - \widehat{A}^3_{-1}(3k_0)\overline{\widehat{v}_{-1}(k_{-1})}\widehat{v}_1(k_1)\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

At this point, we note that the operator ϱ_1 in the original system (3.2) was not chosen yet. The operator appears in the error equation (3.14) within the definition of the term $\mathcal{B}_3$, but not in the definition of $\mathcal{Q}$ and $\mathcal{P}$ such that, in Fourier space, the $b_j(k)$ are of the form

$$b_j(k) = f_j(k) - 2j\omega(k)^{-1}\varrho_1(k)$$

for certain functions f_j. We note that $\omega(k)^{-1} > 0$, for $k > 0$, and $\omega(k)^{-1} < 0$, for $k < 0$. Since the k_j's have the same sign and are bounded away from k_0, we can choose ϱ_1 appropriately, but independent of the parameter ε, such that (3.28) is satisfied. A possible choice would be, for instance, $\varrho_1(k) = |\alpha|$, where the parameter $\alpha \in \mathbb{R}$ is sufficiently large. Consequently, we can choose σ_1, σ_{-1} the same sign so that the energy $\mathbb{E}$ is positive definite and

$$\sigma_1(k_1)b_1(k_1) - \sigma_{-1}(k_{-1})b_{-1}(k_{-1}) = 0. \tag{3.29}$$

Finally, we can conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E} = \mathcal{O}(\varepsilon^2).$$

We call the resonances stable, if (3.28) holds, and unstable, if not.

We proceed to make the considerations of Remark 3.4.4 rigorously. Since the non-resonant terms cancel, with Plancherel's identity, we are left with

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(E + \widetilde{E}) \\
&= 2\pi \sum_{j\in\{\pm1\}} \Bigg(\iint \Big(\varepsilon^{3/2}\sigma_j(k) \sum_{j_1\in\{\pm1\}} \big(i b^j_{j_1}(k) \hat{a}^{*3}_{j_1}(k - l - 3j_1k_0)\hat{R}_{-j}(l)\overline{\hat{R}_j(k)} + c.c.\big) \Big) \mathrm{d}l\,\mathrm{d}k \\
&\quad - \iint \Big(\varepsilon^{3/2} \sum_{j_1\in\{\pm1\}} \big(i L^j_{j_1}(k) s^j_{j_1}(k) \hat{a}^{*3}_{j_1}(k - l - 3j_1k_0)\hat{R}_{-j}(l)\overline{\hat{R}_j(k)} + c.c.\big) \Big) \mathrm{d}l\,\mathrm{d}k \Bigg) \\
&\quad + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

in Fourier space, where $(\cdot)^j_{j_1} := (\cdot)^j_{j_1,j_1,j_1,-j}$. In order to achieve $\frac{\mathrm{d}}{\mathrm{d}t}(E + \widetilde{E}) = \mathcal{O}(\varepsilon^2)$, we have to find $s^j_{j_1}$'s and σ_j's such that

$$\begin{aligned}
&\sum_{j,j_1\in\{\pm1\}} \iint \big(i(\sigma_j(k)b^j_{j_1}(k) - L^j_{j_1}(k)s^j_{j_1}(k))\hat{a}^{*3}_{j_1}(k - l - 3j_1k_0)\hat{R}_{-j}(l)\overline{\hat{R}_j(k)} + c.c.\big) \mathrm{d}l\,\mathrm{d}k \\
&= \sum_{j_1\in\{\pm1\}} \iint \Big(\big(i(\sigma_1(k)b^1_{j_1}(k) - L^1_{j_1}(k)s^1_{j_1}(k))\hat{a}^{*3}_{j_1}(k - l - 3j_1k_0)\hat{R}_{-1}(l)\overline{\hat{R}_1(k)} + c.c.\big) \\
&\qquad + \big(i(\sigma_{-1}(k)b^{-1}_{j_1}(k) - L^{-1}_{j_1}(k)s^{-1}_{j_1}(k))\hat{a}^{*3}_{j_1}(k - l - 3j_1k_0)\hat{R}_{1}(l)\overline{\hat{R}_{-1}(k)} + c.c.\big) \Big) \mathrm{d}l\,\mathrm{d}k \\
&= \sum_{j_1\in\{\pm1\}} \iint \Big(\big(i(\sigma_1(k)b^1_{j_1}(k) - L^1_{j_1}(k)s^1_{j_1}(k))\hat{a}^{*3}_{j_1}(k - l - 3j_1k_0)\hat{R}_{-1}(l)\overline{\hat{R}_1(k)} + c.c.\big) \\
&\qquad - \big(i(\sigma_{-1}(k)b^{-1}_{j_1}(k) - L^{-1}_{j_1}(k)s^{-1}_{j_1}(k))\hat{a}^{*3}_{-j_1}(l - k + 3j_1k_0)\overline{\hat{R}_{1}(l)}\hat{R}_{-1}(k) + c.c.\big) \Big) \mathrm{d}l\,\mathrm{d}k \\
&= 0.
\end{aligned}$$

Swapping the integration variables in the last line yields

$$\sum_{j_1\in\{\pm1\}} \iint \Big(\Big[i(\sigma_1(k)b^1_{j_1}(k) - L^1_{j_1}(k)s^1_{j_1}(k))\hat{a}^{*3}_{j_1}(k-l-3j_1k_0) \\ - i(\sigma_{-1}(l)b^{-1}_{j_1}(l) - L^{-1}_{j_1}(l)s^{-1}_{j_1}(l))\hat{a}^{*3}_{-j_1}(k-l+3j_1k_0)\Big] \hat{R}_{-1}(l)\overline{\hat{R}_1(k)} + c.c.\Big) \, \mathrm{d}l\, \mathrm{d}k \\ = 0.$$

Therefore, we have to find $s^j_{j_1}$'s and σ_j's such that

$$i(\sigma_1(k)b^1_1(k) - L^1_1(k)s^1_1(k)) - i(\sigma_{-1}(l)b^{-1}_{-1}(l) - L^{-1}_{-1}(l)s^{-1}_{-1}(l)) = 0$$

and

$$i(\sigma_1(k)b^1_{-1}(k) - L^1_{-1}(k)s^1_{-1}(k)) - i(\sigma_{-1}(l)b^{-1}_1(l) - L^{-1}_1(l)s^{-1}_1(l)) = 0,$$

where we restrict ourselves to the indices $(j, j_1) = (j, j)$. For the remaining indices $(j, j_1) = (j, -j)$, the procedure can be applied analogously. In the following, we use the indices $(\cdot)_j := (\cdot)^j_{j,j,j,-j}$, if $(\cdot)^j_{j,j,j,-j}$ is defined. Outside a sufficiently small neighborhood of the resonant wave numbers k_j, we choose $\sigma_j = 1$ and

$$s_j(k) = b_j(k)(L_j(k))^{-1}.$$

Inside the neighborhood of the resonant wave numbers, the problem then consists in finding s_j's and σ_j's such that

$$0 = ig_1(k) - iL_1(k)s_1(k) + i\sigma_1(k_1)b_1(k_1) \\ - (ig_{-1}(l) - iL_{-1}(l)s_{-1}(l)) - i\sigma_{-1}(k_{-1})b_{-1}(k_{-1}),$$

where

$$g_j(\cdot) = (\sigma_j(\cdot) - \sigma_j(k_j))b_j(\cdot) + \sigma_j(k_j)(b_j(\cdot) - b_j(k_j)).$$

According to Remark 3.4.4, we can choose ϱ_1 in (3.2) in such a way that

$$\sigma_1(k_1)b_1(k_1) - \sigma_{-1}(k_{-1})b_{-1}(k_{-1})$$

vanishes. Finally, inside the neighborhood of the resonant wave numbers, we can set

$$s_j(\cdot) = (L_j(\cdot))^{-1}g_j(\cdot). \tag{3.30}$$

We note that we have

$$|g_j(\cdot)| \le C|\cdot - k_j| \quad \text{and} \quad |L_j(\cdot)| \le C|\cdot - k_j|,$$

due to the relation $\partial_k L_j(k_j) \neq 0$, such that the resonance becomes trivial in the sense that the numerator and denominator both vanish for the same wave number. Accordingly, by analogous approach for the s^j_{-j}'s, we have the boundedness of $s^j_{j_1}$ near $k^j_{j_1}$. Finally, by a suitable choice of the operator ϱ_1, for all possible indices we have that

$$\sup_{k\in\mathbb{R}} \left|s^j_{j_1,j_2,j_3,j_4}(k)\right| \leq C < \infty. \tag{3.31}$$

Consequently, the third normal form transformation is invertible for $\varepsilon > 0$ sufficiently small. After the elimination, we are left with the system

$$\partial_t \check{\mathcal{R}} = \Lambda \check{\mathcal{R}} + \varepsilon^{5/2} \mathcal{P}_{\mathrm{SOR}}(\Upsilon, \Upsilon, \mathcal{B}_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R})) + \check{\mathcal{G}}(\Upsilon, \mathcal{R}), \tag{3.32}$$

with $\check{\mathcal{G}}(\Upsilon, \mathcal{R})$ obeying the same property (3.18) as $\widetilde{\mathcal{G}}(\Upsilon, \mathcal{R})$.

3.4.4 The fourth normal form transformation

According to Remark 3.4.3, we have to get rid of the last remaining term in (3.32). We note that this step is not necessary, if the operator ϱ_1 in (3.2) vanishes at $k = \pm k_0$. Since the second order resonant terms correspond to the indices $(j, j_1, j_2, j_3) = (-1, j_1, j_1, -1)$, the term to be eliminated is of the form

$$\begin{aligned} &\varepsilon^{5/2} \widehat{\mathcal{P}}^{-1}_{\mathrm{SOR}}(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{B}}_3(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}})) \\ &= \varepsilon^{5/2} \sum_{j_1,\ldots,j_5 \in \{\pm 1\}} i b^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k) (\widehat{a}^{*2}_{j_1} * \widehat{a}_{j_2} * \widehat{a}_{j_3} * \widehat{a}_{j_4} * \widehat{R}_{j_5})(k), \end{aligned}$$

where, with (3.16) and (3.23), there exists a neighborhood $U_{\widetilde{\delta}}(j_1 k_0)$ of $j_1 k_0$ with radius $\widetilde{\delta} > 0$ sufficiently small such that

$$\sup_{k\in\mathbb{R}} \left|\varepsilon b^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k)\right| \leq C < \infty \quad \text{for all } k \in U_{\widetilde{\delta}}(j_1 k_0)$$

and

$$\sup_{k\in\mathbb{R}} \left|b^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k)\right| \leq C < \infty \quad \text{for all } k \notin U_{\widetilde{\delta}}(j_1 k_0).$$

Hence, all terms restricted to the complement of this neighborhood are of order $\mathcal{O}(\varepsilon^{5/2})$ and do not have to be eliminated at all. Inside of this neighborhood, they are of order $\mathcal{O}(\varepsilon^{3/2})$ and have to be eliminated. The corresponding reduced non-resonance condition reads

$$\inf_{k\in\mathbb{R}} \left|L^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k)\right| \geq C > 0, \tag{3.33}$$

where

$$L^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k) := -\omega(k) + (2j_1 + j_2 + j_3 + j_4)\omega(k_0) - j_5\omega(k - (2j_1 + j_2 + j_3 + j_4)k_0).$$

For the indices

$$j = -1, \quad j_1 = j_2 + j_3 + j_4, \quad j_5 = 1,$$

we get the exact same resonances as in the third normal form transformation. Beside these, for the indices

$$j = -1, \quad j_1 = j_2 = j_3 = j_4, \quad j_5 = 1,$$

we get further first order resonances, namely, at $k = j_1\left(\frac{5}{2} + \sqrt{\frac{21}{2}}\right)$. We remark that all these resonant wave numbers are sufficiently bounded away from any integer multiple of the wave numbers $\pm k_0$. Therefore, we can choose $\widetilde{\delta} > 0$ sufficiently small such that all upcoming resonances lie outside of $U_{\widetilde{\delta}}(j_1 k_0)$. As a consequence, the part which has to be eliminated is non-resonant. In order to make this rigorously, we make the near identity change of variables

$$\acute{\mathcal{R}} = \check{\mathcal{R}} + \varepsilon^{3/2}\mathcal{Z}(\Upsilon, \Upsilon, \Upsilon, \Upsilon, \Upsilon, \mathcal{R}), \tag{3.34}$$

where $\mathcal{Z} = (\mathcal{Z}^1, \mathcal{Z}^{-1})^T$ is a multilinear mapping which, in Fourier space, has the form

$$\widehat{\mathcal{Z}}^j(\widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\Upsilon}, \widehat{\mathcal{R}}) = \sum_{j_1,\ldots,j_6 \in \{\pm 1\}} z^j_{j_1,j_2,j_3,j_4,j_5,j_6}(k)(\widehat{a}_{j_1} * \widehat{a}_{j_2} * \widehat{a}_{j_3} * \widehat{a}_{j_4} * \widehat{a}_{j_5} * \widehat{R}_{j_6})(k), \quad j \in \{\pm 1\}.$$

Further, we define the cut-off functions $E^n_{j_1}$ and $E^r_{j_1}$ by

$$E^r_{j_1}(k) = \begin{cases} 1, & k \in U_{\widetilde{\delta}}(j_1 k_0) \\ 0, & \text{else} \end{cases}, \qquad E^n_{j_1} = 1 - E^r_{j_1}.$$

Then, we set

$$z^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k) = \varepsilon E^r_{j_1}(k) b^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k)(L^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}(k))^{-1}$$

and all remaining kernels to zero. Hence, the fourth normal form transformation is invertible for $\varepsilon > 0$ sufficiently small.

3.5 Estimates for the error

In this section, we complete the proof of Theorem 3.1.1. We use energy estimates to control the error R_j. In order to eliminate problematic terms, we have to include the normal form transformations from Section 3.4 in our energy. The estimates for the non-resonant and totally resonant terms will be straightforward. The most interesting aspect of this section is the handling of the second order resonant terms and the first order resonant terms.

3.5.1 Equivalence of the energy and the Sobolev norm

Let the energy $\mathcal{E}$ be defined by $\mathcal{E} = \mathcal{E}^0 + \mathcal{E}^l$, where $\mathcal{E}^l = E_0^l + E_1^l + E_2^l$ and

$$\begin{aligned} E_0^l &= \sum_{j\in\{\pm1\}} \int \sigma_j |\partial_x^l \breve{R}_j|^2 \,\mathrm{d}x, \\ E_1^l &= \sum_{j\in\{\pm1\}} \int \left(\varepsilon^{3/2} \partial_x^l \mathcal{S}^j(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}) \partial_x^l \overline{\breve{R}_j} + c.c.\right) \mathrm{d}x, \\ E_2^l &= \sum_{j\in\{\pm1\}} \int \left(\varepsilon^{3/2} \sigma_j \partial_x^l \mathcal{Z}^j(\Upsilon,\Upsilon,\Upsilon,\Upsilon,\Upsilon,\mathcal{R}) \partial_x^l \overline{\breve{R}_j} + c.c.\right) \mathrm{d}x. \end{aligned}$$

We note that due to the relation

$$\breve{R}_j = R_j + \varepsilon^{1/2} \mathcal{Q}^j(\Upsilon,\mathcal{R}) + \varepsilon \mathcal{P}^j(\Upsilon,\Upsilon,\mathcal{R}),$$

the energy $\mathcal{E}$ contains all terms which are necessary to eliminate the problematic terms. Since the mappings $\mathcal{Q}$, $\varepsilon\mathcal{P}$, $\mathcal{S}$ and $\mathcal{Z}$ are all $\mathcal{O}(1)$ bounded, one can conclude the equivalence of the energy $\mathcal{E}$ and the H^l-norm of the error terms R_j.

Lemma 3.5.1. *There exist $\varepsilon_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$(\|R_1\|_{H^l} + \|R_{-1}\|_{H^l})^2 \leq C_1 \mathcal{E} \leq C_2 (\|R_1\|_{H^l} + \|R_{-1}\|_{H^l})^2.$$

Proof. We note that $\varepsilon p^{-1}_{j_1,j_1,-1}$ is $\mathcal{O}(1)$ but can be made small by choosing $\kappa > 0$ sufficiently large and independent of ε. Hence, all terms with an ε-factor in front are a small perturbation of the H^l-norm of R_j. □

3.5.2 Energy estimates

In the following, we calculate the time derivative of the energy E_0^l. By combining the results from Section 3.4.1 and Section 3.4.2, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_0^l = \sum_{j\in\{\pm 1\}} \int \Big(\big(& \varepsilon^{3/2}\sigma_j \partial_x^l (\mathcal{B}_3^j(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}) + \mathcal{Q}^j(\Upsilon, \mathcal{B}_2(\Upsilon,\Upsilon,\mathcal{R})) \\
& + \mathcal{P}^j(\Upsilon,\Upsilon,\mathcal{B}_1(\Upsilon,\mathcal{R})) - \sum_{j_1} i b^j_{j_1,-j_1,j}(a_{j_1}a_{-j_1}\mathcal{Q}^j(\Upsilon,\mathcal{R})))\overline{\partial_x^l \check{R}_j} \\
& + \varepsilon^{5/2}\sigma_j \partial_x^l \mathcal{P}^j(\Upsilon,\Upsilon,B_3(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}))\overline{\partial_x^l \check{R}_j}\big) + c.c.\Big)\,\mathrm{d}x \\
& + \sum_{j\in\{\pm 1\}} \sum_{i=0}^{8} I_{i,j},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{0,j} &= \int \big(j\sigma_j i\omega |\partial_x^l \breve{R}_j|^2 + c.c.\big)\,\mathrm{d}x, \\
I_{1,j} &= \sum_{j_1,j_2} \int \big(\varepsilon^{1/2}\sigma_j \partial_x^l ((ib^j_{j_1,j_2} - iL^j_{j_1,j_2} q^j_{j_1,j_2}) a_{j_1} R_{j_2}) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{2,j} &= \int \big(\varepsilon^{1/2}\sigma_j \partial_x^l (\mathcal{Q}^j((\partial_t + \Lambda(k_0))\Upsilon, \mathcal{R}) \\
&\qquad + \mathcal{Q}^j(\Upsilon, \Lambda\mathcal{R}) - \sum_{j_1,j_2} j_2 i\omega(-i\partial_x - j_1 k_0) \mathcal{Q}^j_{j_1,j_2}(a_{j_1}, R_{j_2})) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{3,j} &= \int \big(\varepsilon^{3/2}\sigma_j \partial_x^l \mathcal{B}_4^j(\Upsilon, \mathcal{R}) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{4,j} &= \int \big(\sigma_j \partial_x^l (\varepsilon^2 B_5^j(\Upsilon, \mathcal{R}) + j\varepsilon^{-2}(2i\omega)^{-1}\mathrm{Res}(\varepsilon^{1/2}\Psi)) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{5,j} &= \int \big(\varepsilon^{1/2}\sigma_j \partial_x^l \mathcal{Q}^j(\Upsilon, \partial_t\mathcal{R} - \Lambda\mathcal{R} - \varepsilon^{1/2}\mathcal{B}_1(\Upsilon, \mathcal{R}) - \varepsilon\mathcal{B}_2(\Upsilon,\Upsilon,\mathcal{R})) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{6,j} &= \sum_{j_1,j_2,j_3} \int \big(\varepsilon\sigma_j \partial_x^l ((ib^j_{j_1,j_2,j_3} - iL^j_{j_1,j_2,j_3} p^j_{j_1,j_2,j_3}) a_{j_1} a_{j_2} R_{j_3} \\
&\qquad + \varepsilon^{1/2} \sum_{j_1} ib^j_{j_1,-j_1,j}(a_{j_1} a_{-j_1} \mathcal{Q}^j(\Upsilon, \mathcal{R}))) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{7,j} &= \int \big(\varepsilon\sigma_j \partial_x^l (\mathcal{P}^j((\partial_t + \Lambda(k_0))\Upsilon, \Upsilon, \mathcal{R}) + \mathcal{P}^j(\Upsilon, (\partial_t + \Lambda(k_0))\Upsilon, \mathcal{R}) \\
&\qquad + \mathcal{P}^j(\Upsilon, \Upsilon, \Lambda\mathcal{R}) \\
&\qquad - \sum_{j_1,j_2,j_3} j_3 i\omega(-i\partial_x - (j_1 + j_2)k_0) \mathcal{P}^j_{j_1,j_2,j_3}(a_{j_1}, a_{j_2}, R_{j_3})) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\
I_{8,j} &= \int \big(\varepsilon\sigma_j \partial_x^l \mathcal{P}^j(\Upsilon, \Upsilon, \partial_t\mathcal{R} - \Lambda\mathcal{R} - \varepsilon^{1/2}\mathcal{B}_1(\Upsilon, \mathcal{R}) - \varepsilon^{3/2} B_3(\Upsilon,\Upsilon,\Upsilon,\mathcal{R})) \\
&\qquad \times \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x.
\end{aligned}
$$

According to Section 3.4.3, adding E_1^l to E_0^l yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(E_0^l + E_1^l) = &\sum_{j\in\{\pm 1\}} \int \big(\varepsilon^{5/2}\sigma_j \partial_x^l \mathcal{P}^j(\Upsilon, \Upsilon, B_3(\Upsilon,\Upsilon,\Upsilon,\mathcal{R})) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x \\
&+ \sum_{j\in\{\pm 1\}} \sum_{i=0}^{12} I_{i,j},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{9,j} = & \sum_{j_1,j_2,j_3,j_4} \int \big(\varepsilon^{3/2}\partial_x^l((i\sigma_j b^j_{j_1,j_2,j_3,j_4} - iL^j_{j_1,j_2,j_3,j_4} s^j_{j_1,j_2,j_3,j_4}) a_{j_1} a_{j_2} a_{j_3} R_{j_4}) \\
& \times \overline{\partial_x^l \breve{R}_j} + c.c.\big)\, \mathrm{d}x, \\
I_{10,j} = & \int \big(\varepsilon^{3/2}\partial_x^l(\mathcal{S}^j((\partial_t + \Lambda(k_0))\Upsilon, \Upsilon, \Upsilon, \mathcal{R}) \\
& + \mathcal{S}^j(\Upsilon, (\partial_t + \Lambda(k_0))\Upsilon, \Upsilon, \mathcal{R}) + \mathcal{S}^j(\Upsilon, \Upsilon, (\partial_t + \Lambda(k_0))\Upsilon, \mathcal{R}) \\
& + \mathcal{S}^j(\Upsilon, \Upsilon, \Upsilon, \Lambda\mathcal{R}) \\
& - \sum_{j_1,j_2,j_3,j_4} j_4 i\omega(-i\partial_x - (j_1+j_2+j_3)k_0)\mathcal{S}^j_{j_1,j_2,j_3,j_4}(a_{j_1}, a_{j_2}, a_{j_3}, R_{j_4})) \\
& \times \overline{\partial_x^l \breve{R}_j} + c.c.\big)\, \mathrm{d}x, \\
I_{11,j} = & \int \big(\varepsilon^{3/2}\partial_x^l \mathcal{S}^j(\Upsilon, \Upsilon, \Upsilon, \partial_t\mathcal{R} - \Lambda\mathcal{R})\overline{\partial_x^l \breve{R}_j} + c.c.\big)\, \mathrm{d}x, \\
I_{12,j} = & \int \big(\varepsilon^{3/2}\partial_x^l \mathcal{S}^j(\Upsilon, \Upsilon, \Upsilon, \mathcal{R})\overline{\partial_x^l(\partial_t \breve{R}_j - ji\omega \breve{R}_j)} + c.c.\big)\, \mathrm{d}x.
\end{aligned}
$$

Finally, with Section 3.4.4, we are left with

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^l = \sum_{j\in\{\pm 1\}} \sum_{i=0}^{17} I_{i,j}, \tag{3.35}
$$

where

$$
\begin{aligned}
I_{13,1} &= \int \big(\varepsilon^{5/2}\sigma_1 \partial_x^l \mathcal{P}^1(\Upsilon,\Upsilon,B_3(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}))\overline{\partial_x^l \breve{R}_1} + c.c.\big)\,\mathrm{d}x,\\
I_{13,-1} &= \int \big(\varepsilon^{5/2}\sigma_{-1}\partial_x^l(\mathcal{P}^{-1}_{\mathrm{NON}}(\Upsilon,\Upsilon,B_3(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}))\\
&\qquad + \sum_{j_1} E^n_{j_1}\mathcal{P}^{-1}_{\mathrm{SOR}}(a_{j_1},a_{j_1},B_3(\Upsilon,\Upsilon,\Upsilon,\mathcal{R})))\overline{\partial_x^l \breve{R}_{-1}} + c.c.\big)\,\mathrm{d}x,\\
I_{14,1} &= 0,\\
I_{14,-1} &= \sum_{j_1,j_2,j_3,j_4,j_5} \int \big(\varepsilon^{3/2}\sigma_{-1}\partial_x^l((i\varepsilon E^r_{j_1} b^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5}\\
&\qquad - iL^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5} z^{-1}_{j_1,j_1,j_2,j_3,j_4,j_5})a^2_{j_1}a_{j_2}a_{j_3}a_{j_4}R_{j_5})\overline{\partial_x^l \breve{R}_{-1}} + c.c.\big)\,\mathrm{d}x,\\
I_{15,j} &= \int \big(\varepsilon^{3/2}\sigma_j\partial_x^l(\mathcal{Z}^j((\partial_t+\Lambda(k_0))\Upsilon,\Upsilon,\Upsilon,\Upsilon,\Upsilon,\mathcal{R})\\
&\qquad + \ldots + \mathcal{Z}^j(\Upsilon,\Upsilon,\Upsilon,\Upsilon,(\partial_t+\Lambda(k_0))\Upsilon,\mathcal{R})\\
&\qquad + \mathcal{Z}^j(\Upsilon,\ldots,\Upsilon,\Lambda\mathcal{R})\\
&\qquad - \sum_{j_1,\ldots,j_6} j_6 i\omega(-i\partial_x - (j_1+\ldots+j_5)k_0)\mathcal{Z}^j_{j_1,\ldots,j_6}(a_{j_1},\ldots,a_{j_5},R_{j_6}))\\
&\qquad \times \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x,\\
I_{16,j} &= \int \big(\varepsilon^{3/2}\sigma_j\partial_x^l \mathcal{Z}^j(\Upsilon,\Upsilon,\Upsilon,\Upsilon,\Upsilon,\partial_t\mathcal{R}-\Lambda\mathcal{R})\overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x,\\
I_{17,j} &= \int \big(\varepsilon^{3/2}\sigma_j\partial_x^l \mathcal{Z}^j(\Upsilon,\Upsilon,\Upsilon,\Upsilon,\Upsilon,\mathcal{R})\overline{\partial_x^l(\partial_t\breve{R}_j - ji\omega\breve{R}_j)} + c.c.\big)\,\mathrm{d}x.
\end{aligned}
$$

In order to find an $\mathcal{O}(1)$ bound of the error R_j on the $\mathcal{O}(\varepsilon^{-2})$ time scale, we want to apply Gronwall's inequality. Thus, the right-hand side of (3.35) should be subsequently estimated by $\mathcal{O}(\varepsilon^2)(1+\mathcal{E})$.

3.5.3 Bounds on $I_{0,j},\ldots,I_{17,j}$

We want to find $\mathcal{O}(\varepsilon^2)$ bounds for the terms $I_{i,j}$, $i=0,\ldots,17$. We note that $\mathcal{P}^j$ is bounded but can be of order $\mathcal{O}(\varepsilon^{-1})$ such that the estimation is more than a pure counting of powers of ε. We use $\mathcal{E}^{1/2}\leq 1+\mathcal{E}$ several times and choose $\varepsilon_0>0$ sufficiently small such that $\varepsilon^{1/2}\mathcal{E}\leq 1$.

Trivial bounds: First, we show some trivial estimates.

(i) Using the skew symmetry of $i\omega$, we directly obtain

$$I_{0,j} = 0.$$

(ii) Since in the first and fourth normal form transform no resonances occur, cf. Section 3.4.1 and Section 3.4.4, we can conclude that

$$I_{1,j} = I_{14,j} = 0.$$

(iii) The following terms can be directly estimated by a pure counting of ε powers using Lemma 3.2.2 and the $\mathcal{O}(1)$ boundedness of the mappings $\mathcal{Q}^j$, $\mathcal{S}^j$ and $\mathcal{Z}^j$

$$\begin{aligned}
|I_{4,j}| &\leq C\varepsilon^2(1+\mathcal{E}),\\
|I_{5,j}| &\leq C\varepsilon^2(1+\mathcal{E}),\\
|I_{11,j}| &\leq C\varepsilon^2(1+\mathcal{E}),\\
|I_{12,j}| &\leq C\varepsilon^2(1+\mathcal{E}),\\
|I_{16,j}| &\leq C\varepsilon^2(1+\mathcal{E}),\\
|I_{17,j}| &\leq C\varepsilon^2(1+\mathcal{E}).
\end{aligned}$$

(iv) With Lemma 3.4.1, the identity (3.19), and the $\mathcal{O}(1)$ boundedness of the mappings $\mathcal{Q}^j$, $\mathcal{S}^j$ and $\mathcal{Z}^j$, we can conclude

$$\begin{aligned}
|I_{10,j}| &\leq C\varepsilon^2(1+\mathcal{E}),\\
|I_{15,j}| &\leq C\varepsilon^2(1+\mathcal{E}).
\end{aligned}$$

Remark on $I_{2,j}$ and $I_{3,j}$: As we conclude from Section 3.4.1, the terms $I_{2,j}$ and $I_{3,j}$ are actually of order $\mathcal{O}(\varepsilon^{3/2})(1+\mathcal{E})$, which is obviously not sufficient enough to prove Theorem 3.1.1. As a consequence of Remark 3.4.2, we can eliminate these terms by another normal form transformation without resonances occuring. Thus, we can handle $I_{2,j}$ and $I_{3,j}$ by adding another term containing this normal form transformation to the energy $\mathcal{E}$. However, here no further problems arise and, for the sake of brevity, we refrain from carrying this out.

Auxiliary remark: One problem is that $\varepsilon p^{-1}_{j_1,j_1,-1}(k) = \mathcal{O}(1)$ holds due to the second order resonance. Thus, for some terms we first do not obtain an $\mathcal{O}(\varepsilon^2)$ bound. However, we note that $\varrho(k)$ vanishes at $k = jk_0$. Hence, for terms which contain a factor $\varrho(k)$, we have

$$\varrho(k)p^{-1}_{j_1,j_1,-1}(k) \approx C\frac{(k-j_1k_0)^2}{(k-j_1k_0)^2+\kappa\varepsilon^2} = \mathcal{O}(1). \tag{3.36}$$

This also applies to terms that contain any factor which vanishes at $k = jk_0$.

Bounds on $I_{6,j}$: We write

$$\begin{aligned} I_{6,j} = &\sum_{j_1,j_2,j_3} \int \big(\varepsilon\sigma_j \partial_x^l ((ib^j_{j_1,j_2,j_3} - iL^j_{j_1,j_2,j_3} p^j_{j_1,j_2,j_3}) a_{j_1} a_{j_2} R_{j_3}) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x \\ &+ \sum_{j_1} \int \big(\varepsilon^{3/2}\sigma_j \partial_x^l (ib^j_{j_1,-j_1,j}(a_{j_1} a_{-j_1} \mathcal{Q}^j(\Upsilon,\mathcal{R}))) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x \\ =:&\; r_{1,j} + r_{2,j}. \end{aligned}$$

In the following, we use our results from Section 3.4.2. Since the non-resonant terms cancel in $r_{1,j}$, we are left with the totally resonant terms (TOT) and the second order resonant (SOR) terms, i.e.,

$$r_{1,j} = r_{1,j;TOT} + r_{1,j;SOR},$$

where

$$\begin{aligned} r_{1,j;TOT} &= \sum_{j_1\in\{\pm 1\}} \int \big(\varepsilon\sigma_j \partial_x^l (ib^j_{j_1,-j_1,j}(a_{j_1} a_{-j_1} R_j)) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x, \\ r_{1,-1;SOR} &= \sum_{j_1\in\{\pm 1\}} \int \big(\varepsilon\sigma_{-1} \partial_x^l ((ib^{-1}_{j_1,j_1,-1} - iL^{-1}_{j_1,j_1,-1} p^{-1}_{j_1,j_1,-1}) a_{j_1} a_{j_1} R_{-1}) \overline{\partial_x^l \breve{R}_{-1}} + c.c.\big)\,\mathrm{d}x. \end{aligned}$$

We add $r_{2,j}$ to $r_{1,j;TOT}$ and obtain

$$\begin{aligned} r_{1,j;TOT} + r_{2,j} = &\sum_{j_1\in\{\pm 1\}} \int \big(\varepsilon\sigma_j \partial_x^l (ib^j_{j_1,-j_1,j}(a_{j_1} a_{-j_1} \breve{R}_j)) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x \\ &- \sum_{j_1\in\{\pm 1\}} \int \big(\varepsilon^2\sigma_j \partial_x^l (ib^j_{j_1,-j_1,j}(a_{j_1} a_{-j_1} \mathcal{P}^j(\Upsilon,\Upsilon,\Upsilon,\mathcal{R}))) \overline{\partial_x^l \breve{R}_j} + c.c.\big)\,\mathrm{d}x \\ =:&\; s_{1,j} + s_{2,j}. \end{aligned}$$

With (3.22), we already have

$$|s_{1,j}| \leq C\varepsilon^2\mathcal{E}.$$

For $s_{2,j}$, we use that $b^j_{j_1,-j_1,j}(k)$ contains a factor $\varrho(k)$. Then, by using the relation (3.36), we find

$$|s_{2,j}| \leq C\varepsilon^2\mathcal{E}.$$

For the second order resonant terms, using the definition of $p^{-1}_{j_1,j_1,-1}$, we conclude

$$r_{1,-1;SOR} = -\sum_{j_1\in\{\pm 1\}} \int \big(j_1\varepsilon^3\sigma_{-1} \partial_x^l (i\kappa p^{-1}_{j_1,j_1,-1} a_{j_1} a_{j_1} R_{-1}) \overline{\partial_x^l \breve{R}_{-1}} + c.c.\big)\,\mathrm{d}x,$$

which can be easily estimated by $|r_{1,-1;SOR}| \leq C\varepsilon^2\mathcal{E}$. In total, we have

$$|I_{6,j}| \leq C\varepsilon^2\mathcal{E}.$$

Bounds on $I_{7,j}$: In the non-resonant part of $I_{7,j}$, we gain a power of ε by using (3.19) for the first line and Lemma 3.4.1 for the second and third line. Thus, $I_{7,j;NON}$ can directly be estimated by $\mathcal{O}(\varepsilon^2)\mathcal{E}$ and it remains to estimate the second order resonant part, which is a non-trivial task since $\varepsilon p^{-1}_{j_1,j_1,-1} = \mathcal{O}(1)$. By using a Taylor expansion, we find

$$i(\omega(n) - \omega(k-2j_1)) = i\omega'(k-2j_1)(n-k+2j_1) + \mathcal{O}((n-k+2j_1)^2).$$

Then, with Plancherel's identity and (3.19), for the second order resonant part of $I_{7,j}$, we obtain

$$\begin{aligned} I_{7,j;SOR} = 2\pi \sum_{j_1} \int \big(& 2\varepsilon^2 \sigma_{-1}(k)(ik)^l(\omega'(k-2j_1) - c_g)p^{-1}_{j_1,j_1,-1}(k) \\ & \times \widehat{\partial_X a}_{j_1}(k-m)\widehat{a}_{j_1}(m-n)\widehat{R}_{-1}(n)\overline{(ik)^l \widehat{R}_{-1}(k)} + c.c.\big)\, \mathrm{d}n\, \mathrm{d}m\, \mathrm{d}k \\ & + \mathcal{O}(\varepsilon^2)\mathcal{E}. \end{aligned}$$

Due to the relation

$$\omega'(k-2j_1) - c_g = \omega'(k-2j_1) - \omega'(-j_1) = \mathcal{O}(k-j_1),$$

we have

$$(\omega'(k-2j_1) - c_g)p^{-1}_{j_1,j_1,-1}(k) = \mathcal{O}(1).$$

Thus, we can conclude

$$|I_{7,j;SOR}| \leq C\varepsilon^2\mathcal{E}.$$

Bounds on $I_{8,j}$: All terms contained in the difference

$$\partial_t \mathcal{R} - \Lambda\mathcal{R} - \varepsilon^{1/2}\mathcal{B}_1(\Upsilon, \mathcal{R}) - \varepsilon^{3/2}\mathcal{B}_3(\Upsilon, \Upsilon, \Upsilon, \mathcal{R})$$

have a ϱ as a factor, except for the residual. For the terms with prefactor ϱ, we can use (3.36). For the residual, with Lemma 3.2.2 and a pure counting of ε powers, we conclude

$$|I_{8,j}| \leq C\varepsilon^2\mathcal{E}.$$

Bounds on $I_{9,j}$: This term was already analyzed in Section 3.4.3. The non-resonant terms cancel and, outside of a neighborhood of the resonant wave numbers, also the resonant terms cancel. Then, inside of the neighborhood of the resonant wave numbers, with Plancherel's identity, we are left with

$$\sum_j I_{9,j}$$
$$= 2\pi \sum_{j_1 \in \{\pm 1\}} \iint \Big(\big[i(\sigma_1(k)b^1_{j_1}(k) - L^1_{j_1}(k)s^1_{j_1}(k)) - i(\sigma_{-1}(n)b^{-1}_{-j_1}(n) - L^{-1}_{-j_1}(n)s^{-1}_{-j_1}(n))\big]$$
$$\times (ik)^l \hat{a}^{*3}_{j_1}(k-n-3j_1k_0)\hat{R}_{-1}(n)\overline{(ik)^l\hat{R}_1(k)} + c.c.\Big)\,\mathrm{d}n\,\mathrm{d}k$$

By the choice (3.30), we obtain

$$\sum_j I_{9,j} = 2\pi \sum_{j_1 \in \{\pm 1\}} \iint \big(i(\sigma_1(k^1_{j_1})b^1_{j_1}(k^1_{j_1}) - \sigma_{-1}(k^{-1}_{-j_1})b^{-1}_{-j_1}(k^{-1}_{-j_1}))$$
$$\times (ik)^l \hat{a}^{*3}_{j_1}(k-n-3j_1k_0)\hat{R}_{-1}(n)\overline{(ik)^l\hat{R}_1(k)} + c.c.\big)\,\mathrm{d}n\,\mathrm{d}k.$$

According to Section 3.4.3, this expression vanishes for an appropriate choice of the operator ϱ_1.

Bounds on $I_{13,j}$: Since the mapping $\mathcal{P}^1$ does not contain any second order resonant terms, the mapping is $\mathcal{O}(1)$ bounded. The same holds for $\mathcal{P}^{-1}_{\mathrm{NON}}$. Further, the cut-off function $E^n_{j_1}$ is chosen in such a way that we are sufficiently bounded away from the wave numbers $\pm k_0$ in Fourier space. In total, we find

$$|I_{13,j}| \leq C\varepsilon^2\mathcal{E}.$$

3.5.4 Gronwall's inequality

By the bounds on $I_{i,j}$, we finally achieved to show

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq C\varepsilon^2(1+\mathcal{E}).$$

Using Gronwall's inequality, one obtains the $\mathcal{O}(1)$ boundedness of $\mathcal{E}$ for all $t \in [0, T_0/\varepsilon^2]$ as long as $\varepsilon_0 > 0$ is chosen sufficiently small. Consequently, with Lemma 3.5.1, we have the $\mathcal{O}(1)$ boundedness of $\mathcal{R}$ in $H^l \times H^l$ for all $t \in [0, T_0/\varepsilon^2]$ and for $\varepsilon_0 > 0$ chosen sufficiently small. Finally, Theorem 3.1.1 follows from Sobolev's embedding theorem $H^1 \subset C^0_b$ combined with the triangle inequality, and (3.10).

3.6 A non-approximation result

According to Remark 3.4.4, the proof of Theorem 3.1.1 is based on the choice of the operator ϱ_1 to make the resonances stable. In this section, we prove that non-trivial resonances are able to destroy solutions far before the end of the natural time scale of the DNLS approximation. In detail, for spatially $2\pi/k_0$-periodic solutions, we prove that the DNLS approximation breaks down after a time scale $\mathcal{O}(\varepsilon^{-3/2}|\ln(\varepsilon)|)$ which is much smaller than the natural time scale $\mathcal{O}(\varepsilon^{-2})$ of the DNLS approximation. In order to do so, we want to investigate the situation where the operator ϱ_1 is chosen in such a way that the resonances are unstable. We construct a counterexample such that Theorem 3.1.1 does not necessarily hold, and so the solutions of the original system behave differently than predicted by the DNLS equation (3.5).
As already pointed out, we consider spatially $2\pi/k_0$-periodic boundary conditions, i.e., we have $k \in k_0\mathbb{Z}$ in Fourier space. Thus, in order to prove that the DNLS approximation makes wrong predictions, we need the resonances to be an integer multiple of the basic wave number $k_0 = 1$. In the Klein-Gordon equation (3.2), this is not the case. However, this can be achieved by replacing the linear operator $\partial_x^2 - 1$ with $\partial_x^2 - 4$, for example. Furthermore, according to the subsequent Remark 3.6.2, we discard the quadratic terms and, instead, we consider the reduced original system

$$\partial_t^2 u = \partial_x^2 u - 4u + \varrho_1(\partial_x)u^4, \tag{3.37}$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, $u(x,t) \in \mathbb{R}$. The operator $\varrho_1(\partial_x)$ is chosen such that (3.28) is not satisfied. A possible choice of $\varrho_1(\partial_x)$ would be, for instance, $\varrho_1(k) = -|\alpha|$, where the parameter $\alpha \in \mathbb{R}$ is sufficiently large. We insert the DNLS approximation

$$u(x,t) = \varepsilon^{1/2}\psi_{\mathrm{DNLS}} = \varepsilon^{1/2} A_1(\varepsilon(x - c_g t), \varepsilon^2 t)e^{i(k_0 x - \omega_0 t)} + c.c.$$

into (3.37). Then, the dispersion relation is given by $\omega_0^2 = 4 + k_0^2$ and the DNLS equation changes into

$$-2i\omega_0 \partial_T A_1 = (1 - c_g^2)\partial_X^2 A_1. \tag{3.38}$$

We remark that spatially periodic boundary conditions on the original system correspond to X-independent solutions of (3.38). Thus, the ansatz of the approximation is of the form

$$u_{\mathrm{per}}(x,t) = \varepsilon^{1/2}\psi_{\mathrm{DNLS}}^{\mathrm{per}}(x,t) = \varepsilon^{1/2} A_1(\varepsilon^2 t)e^{i(k_0 x - \omega_0 t)} + c.c. \tag{3.39}$$

and the DNLS equation (3.38) degenerates into the ODE

$$-2i\omega_0 \partial_T A_1 = 0 \qquad \text{resp.} \qquad A_1 \equiv C \in \mathbb{C}. \tag{3.40}$$

With the modified equation (3.37), we skip the first two normal form transformations in the proof of Theorem 3.1.1 presented in the previous sections. The difference now is that the resonant wave numbers, which arise in the fourth order terms, are given by $k_{j_1}^j = \frac{1}{2}j_1(5-3j)$, which are actually an integer multiple of k_0. According to Remark 3.4.4, with the choice of the operator ϱ_1, we expect that the resonances can be unstable in the sense that the resonant modes grow as $\mathcal{O}(\exp(\varepsilon^{3/2}t))$ which is not $\mathcal{O}(\varepsilon^{-2})$ bounded on the natural time scale $\mathcal{O}(\varepsilon^{-2})$ of the DNLS approximation. Hence, the DNLS equation (3.38) can make wrong predictions about the dynamics of the Klein-Gordon equation (3.37). The purpose of this section is to give a rigorous proof of the failure of the DNLS approximation in case of periodic boundary conditions, i.e., to give a proof of the following theorem.

Theorem 3.6.1. *Let the operator ϱ_1 be chosen such that* (3.28) *is not satisfied. Consider the Klein-Gordon equation* (3.37) *with spatially $2\pi/k_0$-periodic boundary conditions. Further, let A_1 be a solution of* (3.40)*. Then, for all $n \geq 2$ there exist $\varepsilon_0 > 0$, $C_1 > 0$, $C_2 > 0$ and $\eta \in (2/3, 2)$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are spatially $2\pi/k_0$-periodic solutions of* (3.37) *satisfying initially*

$$\|u_{per}(\cdot,0) - \varepsilon^{1/2}\psi_{DNLS}^{per}(\cdot,0)\|_{H^1_{per}} + \|\partial_t u_{per}(\cdot,0) - \varepsilon^{1/2}\partial_t\psi_{DNLS}^{per}(\cdot,0)\|_{H^1_{per}} \leq C_1\varepsilon^n$$

for which the associated solutions satisfy

$$\sup_{t\in[0,T_0/\varepsilon^\eta]} \sup_{x\in\mathbb{R}} \left|u_{per}(x,t) - \varepsilon^{1/2}\psi_{DNLS}^{per}(x,t)\right| \geq C_2\varepsilon^{1/2},$$

where ψ_{DNLS}^{per} is given by (3.39)*.*

In simple terms, Theorem 3.6.1 states that under $2\pi/k_0$-spatially perdiodic boundary conditions, the error made by the DNLS approximation is of the same order as both the solution of the original system and the DNLS approximation, although the initial error is sufficiently small. This happens far before the end of the natural time scale of the DNLS approximation. Therefore, the DNLS approximation fails to predict the behaviour of solutions of the original system. The proof of the theorem resembles the proof that a spectrally unstable fixed point is unstable, cf. [SU17, §2.3].

Remark 3.6.2. When considering the full model (3.2) containing also a quadratic nonlinearity, we have to face even more complicated problems since, in the subsequent error equation, terms of the form $\mathcal{O}(1)R^2$ occur. Since these terms should be of order $\mathcal{O}(\varepsilon)$, we would have to eliminate them which is only possible under loss of regularity. We note that the non-resonance condition cannot be reduced this time since, in Fourier space, no spatially localized approximation is contained in the problematic terms. However, this problem is automatically fixed since in the diagonalized first order error equation we divide by ω, which gains one derivative. Nevertheless, after the elimination, terms of the form $\mathcal{O}(1)R^3$ occur. Since these should be of order $\mathcal{O}(\varepsilon^{1/2})$, we would have to eliminate them by another normal form transformation. Hereby, the non-resonance condition is not satisfied for an uncountable set of wave numbers. Thus, an elimination is not possible. However, since the focus of this section lies on the proof of the failure of the DNLS approximation we discard the quadratic terms in the original system and only consider quartic terms.

3.6.1 Some preparations

Analogous to Section 3.3, we derive the error equation for the system (3.37). But instead of making the ansatz (3.13) with $\beta = 2$ as before, we take $\beta = 0$. Consequently, the error satisfies

$$\partial_t^2 R = -\omega_{op}^2 R + \omega_{op}\rho_{1,op}(4\varepsilon^{3/2}\Psi^3 R + 6\varepsilon\Psi^2 R^2 + 4\varepsilon^{1/2}\Psi R^3 + R^4) + \mathrm{Res}(\varepsilon^{1/2}\Psi).$$

When writing it as a diagonalized first order system, for the j-th component of the error we obtain

$$\partial_t \check{R}_j = ji\omega\check{R}_j + \varepsilon^{3/2} \sum_{j_1,j_2,j_3,j_4\in\{\pm 1\}} ib^j_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}a_{j_3}\check{R}_{j_4} + \check{\mathcal{G}}^j(\Upsilon, \check{\mathcal{R}}),$$

with $\check{\mathcal{G}}^j(\Upsilon, \check{\mathcal{R}})$ satisfying

$$\|\check{\mathcal{G}}^j(\Upsilon, \check{\mathcal{R}})\|_{H^s} \leq C\left(\varepsilon\|\check{R}\|_{H^s}^2 + \varepsilon^{1/2}\|\check{R}\|_{H^s}^3 + \|\check{R}\|_{H^s}^4 + \|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{H^s}\right).$$

According to Section 3.4.3, we perform the normal form transformation (3.25) to first eliminate only the non-resonant quartic terms. Then, we obtain

$$\partial_t \check{R}_j = ji\omega\check{R}_j + \varepsilon^{3/2}(ib^j_1 a_1^3 + ib^j_{-1}a_{-1}^3)\check{R}_{-j} + \check{g}_j(\check{R}),$$

with $\check{g}_j$ obeying

$$\|\check{g}_j(\check{R})\|_{H^s} \leq C\left(\varepsilon^2\|\check{R}\|_{H^s} + \varepsilon\|\check{R}\|_{H^s}^2 + \varepsilon^{1/2}\|\check{R}\|_{H^s}^3 + \|\check{R}\|_{H^s}^4 + \|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{H^s}\right).$$

By slightly reshaping the equation, we can restrict ourselves to the system

$$\partial_t \check{R}_j = ji\omega\check{R}_j + \varepsilon^{3/2} ib^j_j a_j^3 \check{R}_{-j} + \check{g}_j(\check{R}).$$

Since we consider spatially $2\pi/k_0$-periodic solutions, we make the Fourier ansatz

$$\check{R}_{\mathrm{per}}(x,t) = \sum_{k\in\mathbb{Z}} r_k(t)e^{kx}.$$

Ahead of that, we eliminate all remaining quartic terms except for the resonant terms around the resonant wave numbers $k_1 := k_1^1$ and $k_2 := k_{-1}^1$. Then, for the Fourier coefficients we obtain

$$\begin{aligned}\partial_t r_{1;k_1} &= i\omega_{k_1} r_{1;k_1} + \varepsilon^{3/2} i b_{1;k_1} a_1^3 r_{-1;k_2} + g_1(r_{\pm 1;k_1}),\\ \partial_t r_{-1;k_2} &= -i\omega_{k_2} r_{-1;k_2} + \varepsilon^{3/2} i b_{-1;k_2} a_{-1}^3 r_{1;k_1} + g_{-1}(r_{\pm 1;k_2}),\\ \partial_t r_{j;k} &= j i\omega_k r_{j;k} + g_j(r_{\pm 1;k})\end{aligned}$$

for $k \neq k_{\pm 1}, k_{\pm 2}$. Accordingly, for g_j we have

$$\|\hat{g}_j(\hat{r}_{\pm 1})\|_{\ell_s^2} \leq C\left(\varepsilon^2 \|\hat{r}\|_{\ell_s^2} + \varepsilon \|\hat{r}\|_{\ell_s^2}^2 + \varepsilon^{1/2} \|\hat{r}\|_{\ell_s^2}^3 + \|\hat{r}\|_{\ell_s^2}^4 + \|\widehat{\mathrm{Res}}(\varepsilon^{1/2}\hat{\Psi})\|_{\ell_s^2}\right).$$

Remark 3.6.3. *The Fourier transformation $u \mapsto \hat{u} = (u_k)_{k\in\mathbb{Z}}$ is an isomorphism between H_{per}^s and*

$$\ell_s^2 = \{\hat{u} : \mathbb{Z} \to \mathbb{C} : \ \|\hat{u}\|_{\ell_s^2}^2 = \sum_{k\in\mathbb{Z}} |u_k|^2 (1+k^2)^s < \infty\}.$$

Since H_{per}^s is closed under multiplication for $s > 1/2$, the same holds for ℓ_s^2, too.

By setting

$$r_{j;k} = e^{j i\omega_k t} v_{j;k},$$

we obtain

$$\begin{aligned}\partial_t v_{1;k_1} &= \varepsilon^{3/2} i b_{1;k_1} a_1^3 v_{-1;k_2} + h_1(r_{\pm 1;k_1}),\\ \partial_t v_{-1;k_2} &= \varepsilon^{3/2} i b_{-1;k_2} a_{-1}^3 v_{1;k_1} + h_{-1}(r_{\pm 1;k_2}),\\ \partial_t v_{j;k} &= h_j(r_{\pm 1;k}),\end{aligned}$$

where the nonlinear terms h_j again satisfy

$$\|\hat{h}_j(\hat{r}_{\pm 1})\|_{\ell_s^2} \leq C\left(\varepsilon^2 \|\hat{r}\|_{\ell_s^2} + \varepsilon \|\hat{r}\|_{\ell_s^2}^2 + \varepsilon^{1/2} \|\hat{r}\|_{\ell_s^2}^3 + \|\hat{r}\|_{\ell_s^2}^4 + \|\widehat{\mathrm{Res}}(\varepsilon^{1/2}\hat{\Psi})\|_{\ell_s^2}\right).$$

We linearize the first two equations, differentiate the first linearized equation and insert the second one to obtain a second order ODE for $v_{1;k_1}$

$$\partial_t^2 v_{1;k_1} = -\varepsilon^3 b_{1;k_1} b_{-1;k_2} |a_1|^6 v_{1;k_1}.$$

Thus, the eigenvalues $j\varepsilon^{3/2}\mu$ of the $v_{j;k_{1/2}}$-part are given by

$$\mu = (-b_{1;k_1} b_{-1;k_2})^{1/2} |a_1|^3.$$

We note that, for the following section, we need these eigenvalues to be real-valued. In order to achieve that, we choose the operator ϱ_1 in such a way that

$$b_{1;k_1} b_{-1;k_2} < 0. \tag{3.41}$$

Since (3.29) cannot be fulfilled, the resonances become unstable and the eigenvalues are real-valued. Consequently, we can diagonalize the equations for $v_{j;k_{1/2}}$ and finally obtain

$$\begin{aligned} \partial_t R_u &= \varepsilon^{3/2}\mu R_u + h_u, \\ \partial_t R_s &= -\varepsilon^{3/2}\mu R_s + h_s, \end{aligned}$$

where h_u and h_s obey the same properties as the h_j. These equations obviously show exponential growth since growth rates of order $\mathcal{O}(\exp(\varepsilon^{3/2}t))$ occur which are not $\mathcal{O}(\varepsilon^{-2})$ bounded on the natural time scale $\mathcal{O}(\varepsilon^{-2})$ of the DNLS approximation.

3.6.2 Estimates for the unstable sector

We define the quantity E by $E = E_u - E_s$, where

$$E_u = |R_u|^2, \qquad E_s = |R_s|^2 + \sum_{j\in\{\pm 1\},\ k\in\mathbb{Z}\setminus\{\pm k_1, \pm k_2\}} |v_{j;k}|^2(1+k^2).$$

In the following, we estimate the time derivative of E. Keeping the estimates as simple as possible, we assume that $\mu < 1$. For $\varepsilon_0 > 0$ sufficiently small, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E &= 2\varepsilon^{3/2}\mu E_u + 2\varepsilon^{3/2}\mu E_s \\ &\qquad + 2\mathrm{Re}(\overline{R_u} h_u - \overline{R_s} h_s) - 2 \sum_{j\in\{\pm 1\},\ k\in\mathbb{Z}\setminus\{\pm k_1, \pm k_2\}} \mathrm{Re}(\overline{v_{j;k}} h_{j;k})(1+k^2) \\ &\geq 2\varepsilon^{3/2}\mu E_u + 2\varepsilon^{3/2}\mu E_s \\ &\qquad - 2\left|\overline{R_u} h_u\right| - 2\left|\overline{R_s} h_s\right| - 2\left| \sum_{j\in\{\pm 1\},\ k\in\mathbb{Z}\setminus\{\pm k_1, \pm k_2\}} \mathrm{Re}(\overline{v_{j;k}} h_{j;k})(1+k^2)\right| \\ &\geq 2\varepsilon^{3/2}\mu E_u - C_1\varepsilon^2 E_u - C_1\varepsilon^2 E_s - C_2\varepsilon E_u^{3/2} - C_2\varepsilon E_s^{3/2} \\ &\qquad - C_2^2\varepsilon^{1/2} E_u^2 - C_2^2\varepsilon^{1/2} E_s^2 - C_2^3 E_u^{5/2} - C_2^3 E_s^{5/2} - C_3\|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{\ell_1^2} \\ &\geq \mu\varepsilon^{3/2} E_u - \mu\varepsilon^{3/2} E_s - C_3\|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{\ell_1^2} \\ &\geq \frac{1}{2}\mu\varepsilon^{3/2} E, \end{aligned}$$

with constants $C_i > 0$ for $i = 1, \dots, 5$ under the assumptions

$$C_1 \varepsilon^{1/2} \leq \mu/4, \tag{3.42}$$

$$C_2 E_u^{1/2} \leq \mu \varepsilon^{1/2}/4, \tag{3.43}$$

$$C_2 E_s^{1/2} \leq \mu \varepsilon^{1/2}/4, \tag{3.44}$$

$$C_3 \|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{\ell_1^2} \leq \mu \varepsilon^{3/2} E/2. \tag{3.45}$$

We define

$$t_* = \inf\{t : E^{1/2}(t) \geq \mu \varepsilon^{1/2}/(4C_2)\}.$$

Note that the failure of the DNLS approximation will happen on an $\mathcal{O}(|\ln(\varepsilon)|/\varepsilon^{3/2})$ time scale. For the purpose of this section, it is sufficient to take an $\mathcal{O}(\varepsilon^{-\eta})$ time scale with a $\eta \in (3/2, 2)$ chosen appropriately. Theorem 3.6.1 follows, if we prove $t_* \leq \varepsilon^{-\eta}$. In other words, if the assumptions (3.43) and (3.44) are not satisfied for a $t \in [0, \varepsilon^{-\eta}]$, we are done. Thus, in the following, we assume that (3.43) and (3.44) are satisfied.

The assumption (3.42) can be easily satisfied by choosing ε_0 sufficiently small. Further, we show that (3.45) can be satisfied. According to the conditions in Theorem 3.6.1, we have that $E^{1/2}(0) = \mathcal{O}(\varepsilon^n)$ and thus,

$$C_3 \|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{\ell_1^2} \leq \mu \varepsilon^{3/2} E(0) \tag{3.46}$$

is satisfied for

$$\sup_{t \in [0, \varepsilon^{-\eta}]} \|\mathrm{Res}(\varepsilon^{1/2}\Psi)\|_{\ell_1^2} = \mathcal{O}(\varepsilon^{2n+2}).$$

Since the procedure for estimating the residual was already sketched in Section 3.2, we abstain from recalling this; we just note that the residual can be made arbitrarily small by adding higher order terms to the DNLS approximation, cf. [SU17, §11.2]. Thus, by continuity with respect to the time variable t, (3.46) implies that (3.45) is also satisfied for all $t > 0$ in a neighborhood of $t = 0$. By repeating this procedure for an increasing t, one can extend this neighborhood such that the assumption (3.45) holds for all $t \in [0, t_*]$. Hence, under the assumptions (3.42)–(3.45), for all $t \in [0, t_*]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E \geq \frac{1}{2} \mu \varepsilon^{3/2} E.$$

Therefore, we can conclude

$$E(t) \geq E(0) e^{\frac{1}{2}\mu\varepsilon^{3/2} t}. \tag{3.47}$$

By construction and continuity, we then have $E_u(t) \geq E_s(t)$. On the one hand, from (3.43)–(3.44), it follows that

$$E^{1/2}(t) = \mathcal{O}(\varepsilon^{1/2}).$$

On the other hand, from (3.47) and $E^{1/2}(0) = \mathcal{O}(\varepsilon^n)$, it follows that

$$E^{1/2}(t) = e^{\frac{1}{4}\mu\varepsilon^{3/2}t}\mathcal{O}(\varepsilon^n).$$

These relations both hold for

$$t = \mathcal{O}\left((n-1/2)\,|\ln(\varepsilon)|/\varepsilon^{3/2}\right).$$

We choose $\eta \in (3/2, 2)$ sufficiently big such that $t \ll \varepsilon^{-\eta}$ holds for $\varepsilon \ll 1$ sufficiently small. Since $t \in [0, t_*]$, this contradicts the assumptions (3.43)–(3.44). Thus, $t_* \leq \varepsilon^{-\eta}$ holds. This completes the proof of Theorem 3.6.1.

3.7 Discussion

We close this chapter by discussing the justification of the DNLS equation (3.5) for the Klein-Gordon equation (3.2) in case of solutions which are analytic in a strip of the complex plane, instead of solutions in Sobolev spaces. The idea of considering such solutions is based on a Cauchy–Kowalevskaya-like method and was explained in [Sch98] and carried out in [DHSZ16]. We refer to [Sch19, HS22a] for further applications of this method.
In [HS22a], the DNLS approximation was justified for a Klein-Gordon model with a cubic nonlinearity considering solutions in Gevrey spaces. They used similar methods of [KN86, Sch96] for the justification of the KdV approximation. Gevrey spaces G^s_σ for $\sigma, s \geq 0$ are defined by $G^s_\sigma = \mathcal{X}^s_{\vartheta_\sigma}$ with $\vartheta_\sigma(k) = \exp(\sigma(|k|+1))$ where

$$\mathcal{X}^s_{\vartheta_\sigma} = \{u : \mathbb{R} \to \mathbb{C} : \; \|u\|_{\mathcal{X}^s_{\vartheta_\sigma}} := \|\vartheta_\sigma(\cdot)(1 + |\cdot|^{2s})^{1/2}\hat{u}(\cdot)\|_{L^2} < \infty\}.$$

For initial conditions in G^{s+1}_σ, the DNLS approximation is initially in $M^s_\sigma = \mathcal{X}^s_{\vartheta_{\sigma/\varepsilon}}$ with weight $\vartheta_\sigma(k) = \exp(-\sigma \inf_{m\in\mathbb{Z}} |k - mk_0|)$. In [Sch98, DHSZ16, HS22a], the idea is to make the width of analyticity linearly smaller in time, i.e., we replace σ in the M^s_σ-norm by

$$\sigma(t) = \sigma_0/\varepsilon - \eta\varepsilon^{3/2}t$$

for $\sigma_0 > 0$ and $\eta = \sigma_0/T_0$. Since we require that the resonances are bounded away from any integer multiple of the basic wave number k_0, this generates some damping with an exponential rate, and hence, allows to regain missing powers of ε. In general, this method leads to a restriction in time, i.e., the approximation result holds for $t \in [0, T_1/\varepsilon^2]$ with a $T_1 \in (0, T_0]$. It is the purpose of this section to answer the question whether these methods also apply to the setting considering the Klein-Gordon equation with quadratic and quartic terms. For local existence and uniqueness of the DNLS equation (3.5) in Gevrey spaces, we refer to [HS22a]

whose proof is based on the one given in [Kat75]. We note that the DNLS equation contains an additional $A_1|A_1|^4$-term in the nonlinearity but the proof goes the same way.
We remark that, in [HS22a], the problems arising from the totally resonant and second order resonant terms were already solved. For the totally resonant terms, we again use energy estimates. For the second order resonant terms, we exploit the fact that in lowest order the system near the wave numbers $k = \pm k_0$ is given by the DNLS equation. The exponential localization of the solutions in Fourier space allows us to use the derivative in front of the nonlinearity and to come to the correct time scale. Since in the quadratic terms no resonances occur, the only question is how to handle the resonances arising at the fourth order terms. As these resonances are bounded away from any integer multiple of the basic wave number k_0, see Section 3.4.3, one can use the method from [DHSZ16] to control them. However, instead of using the variation of constants formula, we have to use energy estimates since the totally resonant terms also have to be controlled.
According to [HS22a], there is a restriction in time coming from the second order resonances in the cubic terms. However, in the following we show that there is no further restriction in time arising from the resonances at the fourth order terms. Note again that the resonances in the fourth order terms are bounded away from any integer multiple of the basic wave number k_0. By nonlinear interaction, the solutions of the original system have a Fourier mode distribution which is strongly localized at integer multiples of k_0. Consequently, the modes associated to the resonant wave numbers are exponentially small. In detail, following the course of [DHSZ16], due to the spatial scaling of order $\mathcal{O}(\varepsilon^{-1})$, these modes are initially of order $\mathcal{O}(e^{-\sigma_0/\varepsilon})$ for a $\sigma_0 > 0$ independent of $0 < \varepsilon \ll 1$. At the same time, since the resonances arise at terms of order $\mathcal{O}(\varepsilon^{3/2})$, these modes grow with some exponential rate of order $\mathcal{O}(e^{\sigma_1 \varepsilon^{3/2} t})$ for a $\sigma_1 > 0$ independent of $0 < \varepsilon \ll 1$. Hence, these modes are less than $\mathcal{O}(\varepsilon^2)$ for all $t \in [0, \frac{\sigma_0}{2\sigma_1}\varepsilon^{-5/2}]$. Since this is no restriction in the natural $\mathcal{O}(\varepsilon^{-2})$ time scale of the DNLS approximation, the modes associated to the resonant wave numbers are less than $\mathcal{O}(\varepsilon^2)$ for all $t \in [0, T_0/\varepsilon^2]$. In total, there is a restriction in time when we handle the resonant cubic terms but there is no further restriction in time when we handle the resonant fourth order terms given that these resonances are bounded away from any integer multiple of the basic wave number k_0.
Finally, we can conclude that the methods from [Sch98, DHSZ16, HS22a] combined can be applied to the setting of this chapter without further problems. We formulate the following theorem.

Theorem 3.7.1. *Let $s_A \geq 6$, $\sigma_0 > 0$, and $A_1 \in C([0, T_0], G^{s_A}_{\sigma_0})$ be a solution of the DNLS equation* (3.5). *Then, there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$, and $C > 0$ such that* *for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* u *of the Klein-Gordon model* (3.2) *such that*

$$\sup_{t \in [0, T_1/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| u(x,t) - \varepsilon^{1/2} \psi_{DNLS}(x,t) \right| \leq C \varepsilon^{3/2},$$

where $\varepsilon^{1/2}\psi_{DNLS}$ *is given by* (3.3).

Remark 3.7.2. Theorem 3.7.1 holds regardless of whether the resonances are stable or not. Hence, if we discard the quartic terms in the Klein-Gordon model, the result remains unchanged.

Bibliography

[AA88] Hélène Added and Stéphane Added. Equations of Langmuir turbulence and nonlinear Schrödinger equation: smoothness and approximation. *J. Funct. Anal.*, 79(1):183–210, 1988.

[Agr01] G. P. Agrawal. *Solitons and nonlinear wave equations.* Academic Press, 2001.

[AS81] Mark J. Ablowitz and Harvey Segur. *Solitons and the inverse scattering transform*, volume 4 of *SIAM Studies in Applied Mathematics.* 1981.

[BBC96] Luc Bergé, Brigitte Bidégaray, and Thierry Colin. A perturbative analysis of the time-envelope approximation in strong Langmuir turbulence. *Phys. D*, 95(3-4):351–379, 1996.

[BP20] Hajer Bahouri and Galina Perelman. Global well-posedness for the derivative nonlinear Schrödinger equation. *Inventiones Mathematicae*, 2020.

[BSS20] Simon Baumstark, Guido Schneider, and Katharina Schratz. Effective numerical simulation of the Klein–Gordon–Zakharov system in the Zakharov limit. In *Mathematics of Wave Phenomena*, pages 37–48, Cham, 2020. Springer International Publishing.

[BSSZ20] Simon Baumstark, Guido Schneider, Katharina Schratz, and Dominik Zimmermann. Effective slow dynamics models for a class of dispersive systems. *J. Dynam. Differential Equations*, 32(4):1867–1899, 2020.

[BSTU06] Kurt Busch, Guido Schneider, Lasha Tkeshelashvili, and Hannes Uecker. Justification of the nonlinear Schrödinger equation in spatially periodic media. *Z. Angew. Math. Phys.*, 57(6):905–939, 2006.

[CE90] P. Collet and Jean-Pierre Eckmann. The time dependent amplitude equation for the Swift-Hohenberg problem. *Communications in Mathematical Physics*, 132, 07 1990.

[CEGT04] T. Colin, G. Ebrard, G. Gallice, and B. Texier. Justification of the Zakharov model from Klein-Gordon–wave systems. *Comm. Partial Differential Equations*, 29(9-10):1365–1401, 2004.

[CKS+02] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness result for Schrödinger equations with derivative. *SIAM Journal on Mathematical Analysis*, 2002.

[CLPS99] S. Champeaux, D. Laveder, T. Passot, and P. L. Sulem. Remarks on the parallel propagation of small-amplitude dispersive Alfvénic waves. *Nonlinear Processes in Geophysics*, 6(3/4):169–178, 1999.

[Cra85] Walter Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Communications in Partial Differential Equations*, 10(8):787–1003, 1985.

[DHSZ16] Wolf-Patrick Düll, Alina Hermann, Guido Schneider, and Dominik Zimmermann. Justification of the 2D NLS equation for a fourth order nonlinear wave equation - quadratic resonances do not matter much in case of analytic initial conditions. *J. Math. Anal. Appl.*, 436(2):847–867, 2016.

[DSS16] Markus Daub, Guido Schneider, and Katharina Schratz. From the Klein–Gordon–Zakharov system to the Klein–Gordon equation. *Math. Methods Appl. Sci.*, 39(18):5371–5380, dec 2016.

[DSW16] Wolf-Patrick Düll, Guido Schneider, and C. Eugene Wayne. Justification of the nonlinear Schrödinger equation for the evolution of gravity driven 2D surface water waves in a canal of finite depth. *Arch. Ration. Mech. Anal.*, 220(2):543–602, 2016.

[Due17] Wolf-Patrick Duell. Justification of the nonlinear Schrödinger approximation for a quasilinear Klein-Gordon equation. *Commun. Math. Phys.*, 355(3):1189–1207, 2017.

[Dül12] Wolf-Patrick Düll. Validity of the Korteweg-de Vries approximation for the two-dimensional water wave problem in the arc length formulation. *Comm. Pure Appl. Math.*, 65(3):381–429, 2012.

[GW16] Zihua Guo and Yifei Wu. Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}(\mathbb{R})$. *Discrete and Continuous Dynamical Systems*, 37(1):257–264, 2016.

[HITW15] John K. Hunter, Mihaela Ifrim, Daniel Tataru, and Tak Kwong Wong. Long time solutions for a Burgers-Hilbert equation via a modified energy method. *Proc. Amer. Math. Soc.*, 143(8):3407–3412, 2015.

[HO92] Nakao Hayashi and Tohru Ozawa. On the derivative nonlinear Schrödinger equation. *Physica D: Nonlinear Phenomena*, 55(1):14–36, 1992.

[HS18] Tobias Haas and Guido Schneider. Failure of the N-wave interaction approximation without imposing periodic boundary conditions. *Journal of Applied Mathematics and Mechanics*, (2018/41), nov 2018.

[HS22a] Max Heß and Guido Schneider. A robust way to justify the Derivative NLS approximation. *Journal of Applied Mathematics and Physics*, 2022(50), 2022.

[HS22b] Max Heß and Guido Schneider. The validity of the Derivative NLS approximation for systems with cubic nonlinearities. *CRC 1173 Preprint*, 2022(58), 2022.

[JLPS19] Robert Jenkins, Jiaqi Liu, Peter Perry, and Catherine Sulem. The Derivative Nonlinear Schrödinger equation: Global well-posedness and soliton resolution. *Quarterly of Applied Mathematics*, 78:33–73, 2019.

[Kal88] L.A. Kalyakin. Asymptotic decay of a one-dimensional wave-packet in a nonlinear dispersive medium. *Math. USSR, Sb.*, 60(2):457–483, 1988.

[Kat75] Tosio Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Archive for Rational Mechanics and Analysis*, 58:181–205, 1975.

[KN78] David J. Kaup and Alan C. Newell. An exact solution for a derivative nonlinear Schrödinger equation. *J. Mathematical Phys.*, 19(4):798–801, 1978.

[KN86] Tadayoshi Kano and Takaaki Nishida. A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves. *Osaka Journal of Mathematics*, 23:389–413, 1986.

[Mjø76] Einar Mjølhus. On the modulational instability of hydromagnetic waves parallel to the magnetic field. *Journal of Plasma Physics*, 16:321–334, 12 1976.

[MN02] Nader Masmoudi and Kenji Nakanishi. From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations. *Math. Ann.*, 324(2):359–389, 2002.

[MN05] Nader Masmoudi and Kenji Nakanishi. From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation. *J. Hyperbolic Differ. Equ.*, 2(4):975–1008, 2005.

[MN08] Nader Masmoudi and Kenji Nakanishi. Energy convergence for singular limits of Zakharov type systems. *Invent. Math.*, 172(3):535–583, 2008.

[MN10] Nader Masmoudi and Kenji Nakanishi. From the Klein-Gordon-Zakharov system to a singular nonlinear Schrödinger system. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 27(4):1073–1096, 2010.

[MOMT76] Koji Mio, Tatsuki Ogino, Kazuo Minami, and Susumu Takeda. Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *Journal of the Physical Society of Japan*, 41(1):265–271, 1976.

[Osb10] A. R. Osborne. *Nonlinear ocean waves and the inverse scattering transform*, volume 97 of *International Geophysics Series*. Elsevier/Academic Press, Boston, MA, 2010.

[OT92] Tohru Ozawa and Yoshio Tsutsumi. The nonlinear Schrödinger limit and the initial layer of the Zakharov equations. *Differential Integral Equations*, 5(4):721–745, 1992.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[Pel11] D. E. Pelinovsky. *Localization in periodic potentials. From Schrödinger operators to the Gross-Pitaevskii equation.* London Mathematical Society Lecture Note Series 390. Cambridge: Cambridge University Press. x, 398 p., 2011.

[Sch94a] Guido Schneider. Error estimates for the Ginzburg-Landau approximation. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 45:433–457, 1994.

[Sch94b] Guido Schneider. A new estimate for the Ginzburg-Landau approximation on the real axis. *Journal of Nonlinear Science*, 4:23–34, 1994.

[Sch95] Guido Schneider. Validity and limitation of the Newell-Whitehead equation. *Math. Nachr.*, 176:249–263, 1995.

[Sch96] Guido Schneider. Limits for the Korteweg-de Vries-approximation. *Z. Angew. Math. Mech.*, 76:341–344, 1996.

[Sch98] Guido Schneider. Justification of modulation equations for hyperbolic systems via normal forms. *NoDEA, Nonlinear Differ. Equ. Appl.*, 5(1):69–82, 1998.

[Sch05] Guido Schneider. Justification and failure of the nonlinear Schrödinger equation in case of non-trivial quadratic resonances. *J. Differ. Equations*, 216(2):354–386, 2005.

[Sch19] Guido Schneider. The Zakharov limit of Klein–Gordon–Zakharov like systems in case of analytic solutions. *Appl. Anal.*, 100(12):2659–2667, 2019.

[SH77] Jack Swift and Pierre Hohenberg. Hydrodynamic fluctuations at the convective instability. *Physical Review A*, 15:319–328, 1977.

[SH94] Y. Shi and J. E. Hearst. The Kirchhoff elastic rod, the nonlinear Schrödinger equation, and DNA supercoiling. *J. Chem. Phys.*, 101(6):5186–5200, 1994.

[SS99] Catherine Sulem and Pirre-Louis Sulem, editors. *Langmuir Oscillations*, pages 245–262. Springer New York, New York, NY, 1999.

[SSZ15] Guido Schneider, Danish Ali Sunny, and Dominik Zimmermann. The NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions. *J. Dyn. Differ. Equations*, 27(3-4):1077–1099, 2015.

[SU17] Guido Schneider and Hannes Uecker. *Nonlinear PDEs: a dynamical systems approach.* American Mathematical Society, 2017.

[SW86] Steven H. Schochet and Michael I. Weinstein. The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence. *Comm. Math. Phys.*, 106(4):569–580, 1986.

[SW00] Guido Schneider and C. Eugene Wayne. The long-wave limit for the water wave problem I. The case of zero surface tension. *Communications on Pure and Applied Mathematics*, 53(12):1475–1535, 2000.

[SW02] Guido Schneider and Clarence Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Archive for Rational Mechanics and Analysis*, 162:247–285, 01 2002.

[Tak99] Hideo Takaoka. Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. *Advances in Differential Equations*, 4(4):561 – 580, 1999.

[Tex07] Benjamin Texier. Derivation of the Zakharov equations. *Arch. Ration. Mech. Anal.*, 184(1):121–183, 2007.

[TF80] Masayoshi Tsutsumi and Isamu Fukuda. On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem. *Funkc. Ekvacioj, Ser. Int.*, 23:259–277, 1980.

[TW12] Nathan Totz and Sijue Wu. A rigorous justification of the modulation approximation to the 2D full water wave problem. *Commun. Math. Phys.*, 310(3):817–883, 2012.

[Wu15] Yifei Wu. Global well-posedness on the derivative nonlinear Schrödinger equation. *Anal. PDE*, 8(5):265–271, 2015.

[Zak68] V.E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Sov. Phys. J. Appl. Mech. Tech. Phys*, 4:190–194, 1968.

[Zak72] V. E. Zakharov. Collapse of Langmuir Waves. *Soviet Journal of Experimental and Theoretical Physics*, 35:908, 1972.