Toric Cohiggs Bundles

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Zur Ehre der Immaculata

Confiteo

Viele Stunden saß ich hier

Und viele Stunden saß ich nicht

Viele Fehler waren hier

Manche Fehler sind entwicht

Des alles bekenne ich mich

Des Rühmes weiche ich ab

Denn dies ist wahrlich

nur wenn Gott es gab

Danksagung

Geehrter Leser,

falls du hier einen Fehler finden solltest, darfst du meinen Namen nennen, denn ich bin dafür ganz und gar verantwortlich. Falls du in den nächsten Seiten etwas lernen solltest, bedenke nicht meiner. Bedenke Gottes, der die Mathematik schon durchgedacht hat, bevor er die Welt erschaffen hat. Bedenke meiner zwei Eltern Carmem und Julio, die meine ersten Lehrer waren. Bedenke meines Bruders Vinícius, meiner Schwester Julianne und meiner ganzen Familie, die mich immer gefördert haben und meiner Neugier Raum gegeben haben. Bedenke meiner Lehrer, die mich der Sprachen der Mathematik, des Englischen und des Deutschen mächtig gemacht haben. Bedenke meines Professors und Betreuers, Herrn Doktor Frederik Witt, der mich viele Jahre lang geduldet hat und mich durch viele Schwierigkeiten und manche Enttäuschungen geleitet hat. Und wenn das alles nicht genug ist, bedenke auch, dass du Teil davon bist, denn ein Satz, der nicht gelesen wird, bedeutet letztendlich auch nichts.

Stuttgart, zur Maria Lichtmess 2024 *Anderson Gama*

„Nunc dimíttis sérvum túum Dómine“
– Símeon –

TORIC COHIGGS BUNDLES

Anderson Luis Gama

ZUSAMMENFASSUNG

In der vorliegenden Arbeit studieren wir den Modulraum von torischen pre-Cohiggs-Bündeln. Dieser ist der Raum von Paaren $(\mathscr{E}, \Phi)$*, wobei* $\mathscr{E}$ *ein torisches Bündel über einer torischen Varietät* X *und* Φ *ein Morphismus* $\Phi : \mathscr{E} \to \mathscr{E} \otimes TX$ *ist. Um die Existenz des Modulraums zu beweisen, müssen wir einen Rahmen dazu stellen, das ist ein Isomorphismus zwischen* $\mathbb{C}^r$ *und der generischen Faser* $\mathsf{E} := \mathscr{E}_{x_0}$*. Ähnlich zu Paynes Ergebnissen für torische Bündeln finden wir einen feinen Modulraum von eingerahmten pre-Cohiggs-Bündeln. Außerdem, falls es einen Quotienten dieses Raumes durch den Rahmenwechsel gibt, dann ist er ein grober Modulraum von pre-Cohiggs-Bündeln. Dazu können wir auch die Integrabilitätsbedingung* $\Phi \wedge \Phi = 0$ *stellen und bekommen einen Modulraum von Cohiggs-Bündeln. Diese Räume sind jedoch nicht separiert. In manchen Fällen liefert sogar die Einschränkung auf stabile torische Bündeln keinen separierten Raum. Dies können wir in Beispielen zeigen, wo der Modulraum sich als Raum projektiver Konfigurationen darstellen lässt. Für diese ist der Chow Quotient bekannt und daher auch ein Stabilitätsbedingung vorhanden. Wir können auch zeigen, dass der von Altmann und Witt definierte Higgsbereich [AW21] auch über stabilen Bündeln nicht konstant ist. Insgesamt können wir schließen, dass torische pre-Cohggs Bündeln viel komplexer sind als zunächst angenommen. Die Theorie ist von nicht-linearem Charakter, und eine einfache kombinatorische Beschreibung scheint unwahrscheinlich.*

TORIC COHIGGS BUNDLES

Anderson Luis Gama

Abstract

In this work, we concentrate on describing the moduli space of toric pre-Cohiggs bundles. That is the space of pairs $(\mathscr{E}, \Phi)$*, where* $\mathscr{E}$ *is a toric bundle over a toric variety* X *and* Φ *is a morphism* $\Phi : \mathscr{E} \to \mathscr{E} \otimes TX$*. To prove the existence of a moduli space, we add a frame, that is an isomorphism between* $\mathbb{C}^r$ *and* $\mathsf{E} := \mathscr{E}_{x_0}$*, the fibre over a generic point* $x_0 \in X$*. Just like Payne for toric bundles, we find that the moduli space of framed pre-Cohiggs bundles is fine and if the quotient by frame change exists, it is a coarse moduli space of toric pre-Cohiggs bundles. We can also add the integrability condition* $\Phi \wedge \Phi = 0$ *and get a moduli space of Cohiggs bundles. However, these spaces are non separated. In some cases this is also true even when restricting ourselves to stable bundles — this can be shown in examples, where the moduli space is reduced to a space of projective configurations, for which the Chow quotient is well known. We can also show the Higgs range defined by Altmann and Witt [AW21] is not constant over stable bundles. As a conclusion toric pre-Cohiggs bundles are much more complex than originally expected; the theory has a non-linear characterister and a simple combinatorial description is unlikely.*

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INTRODUCTION

The Bigger Picture

Where we discuss the context where this work is situated, the motivation and meaning behind the results we achieved.

Well-Known Classification Problems

Before addressing the object of this dissertation in the next subsection, we present a quick review of classic Higgs bundles, as we shall compare our case against it.The classification of geometric objects has been a goal of mathematicians for millennia. Theaetetus classified all platonic solids†; the projective space classifies all subvector spaces of dimension one; the fundamental group classifies all closed paths up to homotopy. Riemann was probably one of the first to use the term Moduli to denote a geometric space classifying other geometric objects — in his case Riemann surfaces.The word 'Modul' comes from Latin [EucIX] *and means measurement, so a moduli space is a measuring space. The search for moduli space is a vast field of research. One example of particular importance is the classification of vector bundles over a fixed curve. From a differential point of view, a vector bundle is given by its rank and degree; however, from a holomorphic or an algebraic point of view, one can create continuous deformations of vector bundles. The search for a moduli space to classify those is a frustrating problem since some families are such that they jump from one bundle to another while being constant everywhere else. This makes a classifying geometric space impossible. However, there is a satisfying solution; the bundles that allow this jumping phenomenon are of a particular kind, which came to be described as unstable. Removing those allows – in some cases – the construction of a classifying space.*

For the case of complex vector bundles over a curve of genus equal to or greater than 2, this construction is done via a method first developed by Mumford — this is now known as GIT quotient, an abbreviation of 'geometric invariant theory'. It happens that a certain class of bundles — more commonly known as semi-stable bundles — can be represented as the quotient of its global sections. Moreover, if the degree of the bundle is big enough — which can be assumed without loss of generality — then the global sections form a vector space of constant dimension. This means that by choosing a basis of global sections — let us call this a frame — we can identify the vector bundle with a point in the Quot scheme. The Quot Scheme is a projective variety classifying all the quotient morphisms of a given free sheaf; it is even a fine moduli space, the best kind of classifying space one can find. Of course, not all quotients form a semi-stable bundle; only a subset of those. In Addition, although it does not change the vector bundle, the choice of a framing does change the quotient morphism, so it corresponds to a different point in the Quot scheme. That means the classifying space of (unframed) vector bundles is the quotient of a *subvariety of the Quot scheme by the changing of frames. There is an action of the projective linear group in the Quot scheme representing this change of frames. The work of Mumford assures us there exists a good quotient of the semistable locus and even a geometric quotient of the stable locus. This geometric quotient is a coarse moduli space, which is not as good of a classification as a fine moduli space, but still a 1-to-1 relation. A good introduction to this topic is the work of Victoria Hoskins [Hos15]*

The moduli space of Higgs bundles is an expansion of this, introduced by [Hit87]. It classifies pairs $(\mathbb{E}, \Phi)$*, where* Φ *is a morphism from* $\mathbb{E}$ *to* $\mathbb{E} \otimes TX$*. The construction of the classifying space provided by Simpson*[†] *follows a similar route as above: using the universal family of the Quot scheme, one can create a scheme of framed pairs. The change of frame is an action of* $\mathsf{GL}_r(\mathbb{C})$*. Considering the* Git *quotient of this action provides the coarse moduli space of stable (unframed) Higgs bundles. For a fixed bundle* $\mathbb{E}$ *the set of Higgs fields* Φ *is a finite vector space and if* $\mathbb{E}$ *is stable, the dimension of this vector space is a fixed number. That means the space of (unframed) stable Higgs bundles is itself a vector bundle over the space of (unframed) stable vector bundles. For the semistable locus, the picture is more complex. The stability condition itself depends on the Higgs fields, so it happens that unstable bundles can turn semi-stable for some Higgs fields. In other words, the Higgs field acts as a stabilising force. For more on this construction, we recommend the work of Nitin Nitsure [Nit91].*

[†] [Sim92]

There are several variations of this construction. This includes a work on Cohiggs Bundles by Rayan [Ray11]. Here there morphisms go from $\mathbb{E}$ *to* $\mathbb{E} \otimes T^*X$ *— instead of* $\mathbb{E} \otimes TX$ *— therefore the name. While the study of Higgs bundles most restrict itself to surfaces of genus greater or equal 2, there are several examples of Cohiggs bundles for surfaces of negative genus. This is a natural choice for us, since this includes toric* varieties. In fact Rayan's work already presents a construction of the moduli space of Cohiggs Bundles, so we know this to exist. However bundles over toric varieties is not the same as toric bundles, which is a different question.

Toric Bundles

The topic of our work is dual to the more classical objects. It presents some important differences, but also some similarities. I even added the term 'framed' in the above description, as so to make the relationship more visible. The advantage of studying toric bundles, as we will define later, is the multitude of concrete examples. The combinatorial description of toric bundles goes back to Klyachko [Kly90]. In that work, he first shows how a toric bundle can be described by a collection of filtrations in the generic fiber[†] see 1.17. Every ray of the fan defining the toric variety gives rise to a filtration. Klyachko put his results as an equivalence of categories, between toric bundles and collections of compatible filtrations of a vector space[†] see 1.21. For itself, this does not describe a moduli space; to achieve this Payne in [Pay07] changed the combinatorial description. It turns out that a filtration can be described by two data: a multiset representing the index where the dimension jumps and a flag containing the vector subspaces present in the filtration. The first data is discrete and encodes the toric Chern class of the toric bundle[†] see 1.28 and the second data varies in a product of Grassmannians. The compactibility condition present in Klyachko's work reflects a restriction of the dimension of intersections[†] see 1.34. This means the set of framed toric bundles of a given Chern class is described as a subvariety of a product of Grassmannians[†] see 1.40.

The coarse moduli space of unframed toric bundles is then the geometric quotient of the framed space[†] see 1.47. This quotient however may not *exist. Payne did not go this far, but we can compare his construction with the Quot scheme construction we explained above. In both cases, there is a fine moduli space for framed objects, although what we call a frame is not the same in both cases — one is a basis in the space of global sections and the other a basis on the generic fibre. Also notice that to go from vector bundles to the Quot scheme one already needs semistability, while Payne needs at first no stability concept. Another significant difference is that* Git *is a natural way of creating the quotient in the first case. One should mention that by itself* Git *strongly depends on the choice of an ample bundle — or equivalently to that of a linearization of the action. However, for the Quot scheme, there is a natural choice and the stability condition deriving from it can be expressed in an intrinsic property concerning the rank and degree of sub-bundles.*

It looks more like a happy accident that the semi stability needed to translate the problem to the Quot scheme is also the semi stability provided by this natural choice of an ample bundle. This could not be further from the truth for toric bundles: one can easily construct an example where the quotient can be translated into the case of $\mathbb{C}^*$ *acting on* $\mathbb{C}^2$ *by hyperboles. There are two main kinds of linearizations one can choose in this example: one renders the* x*-axis unstable, and for the other one, the* y*-axis is unstable. The toric bundles corresponding to points in the* x*-axis and in the* y*-axis are symmetric, no intrinsic distinction can be made between them. So even though we know from* Git *that there are open sets such that the geometric quotient exists, to construct a moduli space of unframed toric bundles properly, we will need to consider more intrinsic quotients, for instance, the Chow quotient*[†]*.* see a.23

Pre-Cohiggs Bundles

However, we are getting ahead of ourselves. We are not simply interested in toric bundles, but in toric Cohiggs bundles. Those were studies in [AW21] and [BDPR21]. Those works do not seak a moduli spaces description like in Payne, but follow Klyashko's formalism more closely. The difference between [AW21] and [BDPR21] is that [AW21] does not require the Cohiggs fields to be homogeneous. Since this does not preserve the integrability condition $\Phi \wedge \Phi = 0$*, it is more* see 2.6 *fitting to speak of pre-Cohiggs fields*[†]*. In analysing the deformation of Cohiggs bundles and extending Payne's results we also follow this approach. This means we are mostly focused on pre-Cohiggs bundles. As it turns out for the toric case there is no need for the integrability condition; a bounded problem can already be archived by requiring* see 2.12 X *to be complete*[†]*.*

Constructing the moduli space of framed pre-Cohiggs bundles is not particullary difficult since Payne's work provides a fine moduli space for toric bundles. This means we can use the universal family and construct a universal coherent sheaf of pre-Cohiggs fields. The moduli space of toric pre-Cohiggs bundles is the scheme associated with see 2.9 *this sheaf*[†]*. The price we pay is that the resulting space is not as concrete. We do not have a description as a subset of a product of Grassmannians or something like that. We also just get a scheme and not a variety. This is not easy to remediate. As we learned the moduli space is often not even separated. Since the moduli space is constructed as the scheme representing a coherent sheaf this would have to be locally free for the moduli space to be separated. This is not the case since examples show the dimension of the space of pre-Cohiggs fields is not* see 3.3 *constant*[†]*.*

Our Main Result

The main contribution of our work is in providing a better understanding of how this dimension varies. In fact, since the universal sheaf of pre-Cohiggs fields is coherent, the dimension of the fibres varies as a low upper semicontinuous function. This means the set of degrees admitting a pre-Cohiggs field is minimal — and therefore constant — on an open set. This was expected, because the experience working with moduli problems shows that they often contain extremely wild objects on non-generic sets. We would, however, have expected stable bundles to be well-behaved and in the beginning, this looked to be the case. Though GIT-stability is non-intrinsic†, one can consider the Chow quotient instead and for at least one group of examples there is a nice criteria for describing the unframed moduli space†. For instance, one can calculate that for the examples presented by [AW21] for Fano surfaces that the Higgs range is constant for 'Chow-stable' bundles†. This is, however, not always the case. Finding counter examples is quite difficult and involves carefully choosing a toric variety and corresponding toric bundle and even more care if we want the varieties to be smooth. We were nevertheless able to find 3 interesting examples. One shows in rank two that the dimension of the space of pre-Cohiggs fields is not constant, even when considering Chow stable bundles†. A second shows the same is also true for Cohiggs fields, i.e. fields satisfying the integrability condition $\Phi \wedge \Phi = 0$†. The third example in rank three shows the Higgs range as defined by Altmann and Witt is not constant over Chow stable bundles†. What these examples indicate is, first of all, how complex the theory of toric pre-Cohiggs bundles is. The moduli space of pre-Cohiggs bundles is a scheme and generally not a variety. Even well-behaved toric bundles can present wild Cohiggs fields, meaning the Cohiggs fields are a destabilising force. Combinatorial invariants, like the ones defined in [AW21] are of little use as a classifying tool.

see 1.53
see 3.13
see 3.18
see 3.15
see 3.16
see 3.17

Structure of this work

In this work, we presume the reader is familiar with basic concepts of modern geometry, in special algebraic geometry: scheme, sheaves, (co)homology etc. Not necessary, but highly recommended is some experience with toric varieties and their quotients [KSZ91], so the examples will be much easier to understand. For everything else, we try to provide the relevant concepts and references for further study. We work fully over $\mathbb{C}$. Some of our work could, in theory, also be done for other fields. However, we would eventually need to switch back to the complex numbers. Every chapter is organized in the following manner: first, an introduction informally covering the content of the chapter. One could, for instance, just read the introduction of every chapter and get a pretty good idea of the work, deciding later to explore the details. After the chapter introduction, we give the formal definition, proofs and examples in a meaningful and progressing sequence. Every chapter also has a set of notes that are placed as appendixes to the work; those could include definitions of some concepts used, observations of smaller relevance, lemmas and technical proofs of theorems in the chapter. The endnotes do not have a meaningful order and are not made to be read separately, but rather referenced in the text — the reader may want to use a bookmark to quickly skip from chapter to endnote. Sides notes are used as referencing mechanism to enable both quick read and detailed study, we vastly use side notes as a referencing mechanism. A superscripted † always signals a note on the side of the line. Side notes are either a reference to the literature or of the form 'see x.xx', where x indicates the chapter and xx a definition / proposition / observation etc. If x is a letter, it is found in the endnotes; otherwise, one finds it in the main part of the corresponding chapter.

This work is organized as follows: The first chapter contains a recapitulation of the work of Klaychko and Payne in describing the moduli space toric bundles. In the second chapter we apply Payne's work to the problem of toric Cohiggs bundles; although partially based on the work of Altmann and Witt, the moduli space itself is an original construction. Finally the third and last chapter is an in-depth original study of the stability of toric bundles and pre-Cohiggs fields. We also add to the end notes a small revision of important technical tools, including a mostly categorical exposition on moduli problem and moduli spaces and an introduction to GIT and Chow quotients.

FIRST CHAPTER

Moduli of Toric Bundles

Where we discuss Klyachko's classification of toric vector bundles and Payne's construction of the moduli space thereof.

In this section, we concentrate on presenting the moduli space of toric vector bundles as introducted by [Pay07] as an extension of Klyachko's classification theorem [Kly90]. However, before that, we must explain what a toric variety is. A simple definition† is that of a variety X with algebraic torus $(\mathbb{C}^*)^n \subset X$ as an open dense subset such that the action of the torus on itself extend to X. This definition, however, is much less important than how one constructs toric varieties: they can be fully described by a fan — i.e., a collection of convex cones — in a lattice N, *dual to the space of characters of the torus, which induces a lattice we denote* M. *Each one of the cones in* N *defines an affine variety, and the fact that those cones intersect in common faces translates into gluing data for these affine varieties. The result is an abstract variety†. The*

see 1.2
see 1.1

advantage here is that several of the questions concerning the action of see b.4 *the torus T in such a variety X can be reduced to the fan*[†]*. So algebraic and geometric questions can be translated into combinatorial ones, which are easier to calculate given enough computational power. An easy example is that the set of T-Orbits are given by the cones in the* see b.4 *fan*[†]*. Another example is that toric varieties are smooth if and only* see b.7 *if all cones are spanned by a subset of an integral basis*[†]*. For further study of toric varieties we recommend the book by Fulton [Ful93].*

An extension of this phenomenon occures when considering a vector bundle $\mathscr{E}$ over the variety X, with a compatible action of T. In this case, we also have a similar conclusion: the bundle $\mathscr{E}$ — called a toric bundle — can be described by specific vector spaces associated with every ray in the fan. We shall present this briefly now: a section of see 1.13 *$\mathscr{E}$ is decomposable in eigensections of the T-action*[†] *and, since X con-* see 1.14 *tains a dense orbit, their value at a single generic point*[†] *x_0 uniquely defines the eigensections — the rest being given by translating. However, not all values define sections since translations may create a poll, rendering the section non-regular. Therefore for every open set U and eigenvalue $u \in \mathsf{M}$ the set of values defining eigensections of $\mathscr{E}|_U$ is a sub-vector spaces of $\mathsf{E} := \mathscr{E}|_{x_0}$.*

As it turns out, we must only consider the open sets defined by rays — i.e., cones of dimension one — and decreasing filtrations of E. These filtrations are not independent but subject to a compatibility condi- see 1.19 *tion*[†]*. Klyachko's classification theorem*[†] *states that toric bundles are* see 1.21 *uniquely given by any such collection of filtrations, satisfying the compatibility condition. Even further, this establishes an equivalence of categories.*

[Pay07] *Payne's work*[†] *consists of proving the same theorem for deformations of toric bundles, so that we may have a moduli space. However, the formalism of filtrations is not useful here since it has far too much redundant information: E is a finite vector space, so the filtration must* *be constant almost everywhere, except for a few indexes where the dimension jumps. Therefore the same information can be provided by the finite set of subvector spaces and the indexes where the dimension jumps†. The compatibility condition of Klyachko translates into how the subspaces corresponding to different rays may intersect; it gives the dimension of the intersection. This restriction is called the rank condition†. The advantage of this formalism is that the indexes constitute discrete data, while the subspaces can vary in a Grassmannian. Therefore toric bundles with a fixed frame† can be classified by a subspace of a product of Grassmannians†.*

see 1.32
see 1.34
see 1.39
see 1.40

At this point, the only thing one must prove to get a moduli space is that deformations† of toric bundles correspond to subspaces of this classification space. This is better expressed in an isomorphism of moduli functors. Payne's classification theorem† states precisely the existence of such isomorphism for the framed case†. For the unframed case, the moduli space is the quotient by the $\mathsf{GL}_r(\mathbb{C})$*-action if such quotient exists†.*

see 1.44
see 1.46
see 1.45
see 1.47

To prove all of this is the objective of this chapter. We shall now dive into the details. First, a small introduction to toric varieties and toric bundles introduced by Klyachko. Then Payne's formalism of flags, finishing off with the definition of the moduli problems and the proof of the main theorem†. We also provide — at the risk of being repetitive — examples whenever possible to assist in understanding.

see 1.46

Toric Bundels

Our first step is to set some notation about our fixed toric variety.

1.1) Construction of a toric variety: Let N be a lattice of finite rank — i.e. $\mathsf{N} \cong \mathbb{Z}^n$ — and $\mathsf{M} = \hom(N, \mathbb{Z})$ its dual lattice. By σ we commonly denote a strongly convex rational polyhedral cone in $\mathsf{N}_{\mathbb{R}} := \mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$. Then its dual cone $\check{\sigma}^{\dagger}$ defines an affine variety $U_\sigma = \mathsf{spec}(\mathbb{C}[\check{\sigma} \cap \mathsf{M}])$. If σ is part of a fan Σ — i.e., a collection of cones closed under taking intersections and faces — then there are inclusions

see b.1

$$U_\sigma \hookleftarrow U_{\sigma \cap \sigma'} \hookrightarrow U_{\sigma'}$$

defining gluing data. This data constructs a variety $X = X(\Sigma)$ — called a *toric variety* — which has a canonical action of the torus $\mathsf{T} := \mathsf{spec}(\mathbb{C}[\mathsf{M}])$, free on a dense open set of X.

We can also take the above construction as a definition since it is equivalent to the standard definition, as the following proposition states.

1.2) Proposition: Any irreducible variety X containing the torus T as a zariski open set, such the action of T on itself extend to an algebraic action of T on X is defined by a fan Σ as in 1.1.

For the proof: [CLS11, Corollary 3.1.8]

1.3) Observation: For simplification, we would like to fix an inclusion $\mathsf{T} \hookrightarrow X$, so that we consider T as a dense subset of X and points in T — for instance, the identity $x_0 := \mathbb{1} \in \mathsf{T}$ — as points in X.

As we are going to be using the lattice M *and* N *a lot, we should add a small comment on its notation.*

1.4) Notation: Even though the fan Σ is in N, most of the algebraic calculation takes place in the lattice M. It has two interpretations: First, $\mathsf{M} = \hom(\mathsf{T}, \mathbb{C}^*)$, that is, M is *the lattice of characters of* T. At the same time $\mathsf{M} = \mathsf{N}^* := \hom(\mathsf{N}, \mathbb{Z})$, that is, M is the *set of integral linear functions on* N. This creates a notation conundrum since the first is usually regarded as a multiplicative and the second as an additive group. To work around this problem, we write χ_u for the character defined by an element $u \in \mathsf{N}^* = \mathsf{M}$. So when using simple letters, the elements of M are additive. In contrast, when using χ they are multiplicative. The lattice N is always additive since — as already mentioned — its main use is for defining the vector space $\mathsf{N}_{\mathbb{R}}$, where our cones form a fan. We also do often work with the dual cones in $M_{\mathbb{R}}$, but they do not create a fan there.

Regarding notation, it should also be mentioned that by definition the faces of any polyhedral cone in a fan are also in that fan. Therefore we define in Σ the relation $\sigma' \preceq \sigma$, whenever σ' is a face of σ. We will also work a lot with *rays*, i.e. one-dimensional cones. We normally denote them by the letter ρ, instead of σ, and by $\Sigma(1)$ we denote the set of rays in the fan Σ.

Let us now see how we may construct the projective space as a toric variety.

1.5) Example: Consider $X = \mathbb{P}^n$ and $\mathsf{T} = (\mathbb{C}^*)^n$ with the inclusion $\mathsf{T} \hookrightarrow X$ given by $(t_1, \ldots, t_n) \mapsto [1 : t_1 : \cdots : t_n]$. There are characters $\chi_{ij}([x_0 : \cdots : x_n]) = x_i/x_j$, for $[x_0 : \cdots : x_n] \in \mathsf{T}$. Then for the open set $U_j = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_j \neq 0\}$ we conclude that

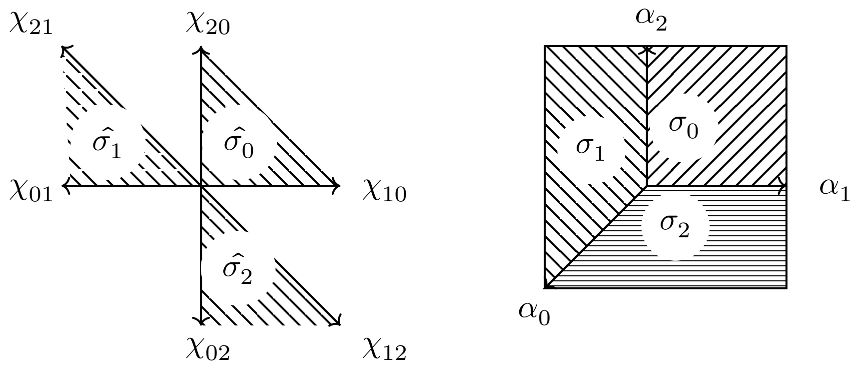
$$\mathscr{O}_X(U_j) = \mathbb{C}[\chi_{0j}, \ldots, \chi_{nj}].$$

That means every U_j corresponds to the dual cone

$$\check{\sigma}_j = \text{span}\{\chi_{ij},\ 0 \leq i \leq n\} \subset M_{\mathbb{R}} := \mathsf{M} \otimes_{\mathbb{Z}} \mathbb{R}.$$

One can quickly check that the cones $\{\sigma_j\}$ induce a fan on $\mathsf{N}_{\mathbb{R}}$. Of course, we also have open sets $U_{ij} = U_i \cap U_j$, which correspond to the common faces between σ_i and σ_j and are also part of the fan. One dimensional cones, i.e. rays, are spanned by primitive generators α_k defined by the condition $\langle \alpha_k, \chi_{ij} \rangle = \delta_{ki}$. The following diagram shows the fan and the dual cones of $\mathbb{P}^2$ in the two- dimensional plane.

Notice the open sets U_j are symmetrical in the projective plane, but the inclusion $\mathsf{T} \hookrightarrow X$ is not, which is reflected in the asymmetry of the diagram.



Quod erat faciendum

Now we turn to a central object of our study:

1.6) Toric Bundle: A toric bundle $\mathscr{E}$ over a toric variety X is a vector bundle $\pi : \mathbb{E} \to X$ with an algebraic equivariant action of the torus T — i.e., $\pi(t \cdot v) = t \cdot \pi(v)$ for all $v \in \mathbb{E}$ and $t \in \mathsf{T}$.

Together with this definition we shall also present the most trivial form of a toric bundle. Actually, this is a local model for toric bundles, as we will see later.

1.7) Affine Toric Line Bundles: Given a cone σ and a $u \in \mathsf{M}$, we denote by $\mathscr{L}_{[u]} = U_\sigma \times \mathbb{C}$ the toric line bundle given by $t \cdot (x, v) = (t \cdot x, \chi_u(t) \cdot v)$ — as the notation suggests this only depends on the equivalence class $[u] \in \mathsf{M}_\sigma := \mathsf{M}/(\sigma^\perp \cap \mathsf{M})^\dagger$, which correspond to an integral linear function on σ. *see b.2*

1.8) Proposition: All toric line bundles over U_σ are of the form $\mathscr{L}_{[u]}$ for some $[u] \in \mathsf{M}_\sigma$.

For the proof: [CLS11, Prop 4.2.2]

Before proceeding, let us illustrate that $\mathscr{L}_{[u]}$ only depends on $[u]$.

1.9) Example: Consider the toric variety $\mathbb{P}^2$ and the one-dimensional cone — that is, the ray — $\sigma = \sigma_{01} := \sigma_0 \cap \sigma_1{}^\dagger$. Then $\sigma_{01}^\perp = \{\lambda\chi_{01} \in M_\mathbb{R} : \lambda \in \mathbb{R}\}$ and M_σ can be represented by the integer powers $\{\chi_{20}{}^i\}$ as well as for the powers $\{\chi_{21}{}^i\} = \{(\chi_{20}\chi_{01})^i\}$. Let $u = \chi_{20}$, then $\mathscr{L}_{[u]}$ is the vector bundle $U_{01} \times \mathbb{C}$ over the open set $U_{01} = \{[x_0 : x_1 : x_2] : x_0, x_1 \neq 0\}$ with the toric action given by: *see 1.5*

$$(t_1, t_2) \cdot ([x_0 : x_1 : x_2], z) \mapsto ([x_0 : x_1 t_1 : x_2 t_2], t_2 z)$$

where

$$(t_1, t_2) \in \mathbb{C}^{*2} \cong \{[1 : t_1 : t_2] \in \mathbb{P}^2 : t_1, t_2 \neq 0\}$$

and $([x_0 : x_1 : x_2], z) \in U_{01} \times \mathbb{C}$. Had we chosen $u' = \chi_{21}$, then $[u] = [u']$ but the toric action is given by:

$$(t_1, t_2) \cdot ([x_0 : x_1 : x_2], z) \mapsto ([x_0 : x_1 t_1 : x_2 t_2], t_1^{-1} t_2 z).$$

Both these actions are isomorphic over U_{01}. To see that, let $\psi : U_{01} \times \mathbb{C} \to U_{01} \times \mathbb{C}$ be the vector bundle isomorphism given by $([x_0 : x_1 : x_2], z) \mapsto ([x_0 : x_1 : x_2], \frac{x_0}{x_1} z)$. Then the following diagram commutes

$$\begin{array}{ccc} ([x_0 : x_1 : x_2], z) & \longmapsto & ([x_0 : x_1 : x_2], \frac{x_0}{x_1} z) \\ \downarrow & & \downarrow \\ ([x_0 : t_1 x_1 : t_2 x_2], t_2 z) & \longmapsto & ([x_0 : t_1 x_1 : t_2 x_2], \frac{t_2 x_0}{t_1 x_1} z) \end{array}$$

where the horizontal arrows are the isomorphism ψ and the vertical arrows are the actions of ${\mathbb{C}^*}^2$ induced by u at the left and by u' at the right. So we see the bundle only depends on the equivalence class $[u] = [u']$. *Quod erat faciendum*

A famous theorem by Gubeladze states that an arbitrary vector bundle on an affine toric variety is trivial. The following proposition proves this for toric vector bundles.

1.10) PROPOSITION: Every toric vector bundle on an affine toric variety splits equivariantly as a sum of toric line bundle. The underlying line bundles are geometrically trivial as topological vector bundles.

For the proof: b.14

It immediately follows that:

1.11) COROLLARY: There is a natural bijection between finite multisets† $\mathbb{u}(\sigma)$ of elements of M_σ and isomorphism classes of toric vector

see b.3

bundles on U_σ, which is given by:

$$\mathbb{u}(\sigma) \mapsto \bigoplus_{u \in \mathbb{u}(\sigma)} \mathscr{L}_{[u]}.$$

Since any (toric) bundle can be restricted to a collection of bundles over affine spaces U_σ, we may associate for every toric bundle $\mathscr{E}$ a collection $\Psi := \{\mathbb{u}(\sigma)\}_{\sigma \in \Sigma}$ such that

$$\mathscr{E}|_{U_\sigma} = \bigoplus_{u \in \mathbb{u}(\sigma)} \mathscr{L}_{[u]}.$$

These multisets are compatible in the sense that for $\tau = \sigma \cap \sigma'$ it follows that

$$\mathbb{u}(\tau) = \mathbb{u}(\sigma)|_\tau = \mathbb{u}(\sigma')|_\tau$$

where elements of $\mathbb{u}(\sigma)$ are interpreted as integral linear functions on $\sigma^\dagger$. see 1.4

How to calculate the multiset we will learn as this Chapter proceeds. For now we just present an example without proof.

1.12) Example: The tangent space of the projective space is a toric bundle. The table below shows the multisets associated to it. The elements of M are written as pairs $(a, b) := {\chi_{10}}^a {\chi_{20}}^{b\dagger}$. The brackets [ ] represent the equivalence class in M_σ for the respective cone. see 1.5

σ	$\mathbb{u}(\sigma)$
span$\{\alpha_1\}$	$\{[(0,0)], [(1,0)]\}$
span$\{\alpha_1, \alpha_2\}$	$\{(1,0), (0,1)\}$
span$\{\alpha_2\}$	$\{[(0,0)], [(0,1)]\}$
span$\{\alpha_2, \alpha_0\}$	$\{(-1,1), (-1,0)\}$
span$\{\alpha_0\}$	$\{[(0,0)], [(0,-1)]\}$
span$\{\alpha_0, \alpha_1\}$	$\{(1,-1), (0,-1)\}$

Notice that in this example the elements do not repeat, so the multisets are actually just normal sets. Also notice when passing from a maximal cone to a ray, one just takes the equivalence class. However, for passing from the ray to a maximal cone a representant must be choosen and the representant is not the same for every cone.

Quod erat faciendum

The collection of multisets is a discrete invariant and, therefore, the first step toward classifying general toric bundles. The full classification, however, is more complicated. To achieve that we need some preparatory work.

1.13) DECOMPOSITION IN T-EIGENSPACES: Let $s \in \Gamma(X, \mathscr{E})$ be a section of the toric bundle. A $t \in \mathsf{T}$ acts on this section by $(t \cdot s)(x) = t(s(t^{-1}x))$. This action induces a decomposition into T-eigenspaces

$$\Gamma(X, \mathscr{E}) = \bigoplus_{u \in \mathsf{M}} \Gamma(X, \mathscr{E})_u$$

see 1.4 where for $s \in \Gamma(X, \mathscr{E})_u$ we have $t \cdot s = \chi_u(t)s^{\dagger}$. Conversely, this eigenspace decomposition of the module of sections determines the action of T in $\mathscr{E}$.

For the proof: [Kan75, Proposition 3.4]

These eigensections are, however, uniquely determined by their value at the identity in T — actually in any point of the dense orbit of X. Let us examine this in more detail:

1.14) EVALUATION AT THE IDENTITY: Two sections in the eigenspace
see 1.3 $\Gamma(X, \mathscr{E})_u$ that agree at the identity $x_0 \in \mathsf{T}^{\dagger}$ must by translation agree

on T. Since T is dense, they also agree on all of X. So the evaluation at x_0 gives an injection $\Gamma(X,\mathscr{E})_u \hookrightarrow \mathsf{E}$, where $u \in \mathsf{M}$ and $\mathsf{E} := \mathscr{E}_{x_0}$ is the fiber over x_0. We denote by $\mathsf{E}_u^\sigma \subset \mathsf{E}$ the image of $\Gamma(U_\sigma,\mathscr{E})_u$ under evaluation at x_0.

Now for $u' \in \check{\sigma} \cap M$ the character $\chi_{u'}$ is a well-defined regular function over U_σ and multiplication by it gives a canonical linear map $\Gamma(U_\sigma,\mathscr{E})_u \to \Gamma(U_\sigma,\mathscr{E})_{u-u'}$. This multiplication commutes with the evaluation at x_0, giving rise to an inclusion $\mathsf{E}_u^\sigma \subset \mathsf{E}_{u-u'}^\sigma$.

This point can be difficult to understand at first, so let us see an example that will follow us throughout this section.

1.15) Example: Let $X = \mathbb{P}^2$ and consider the toric bundle $\mathscr{E} = \mathscr{L}_{\chi_{10}} \oplus \mathscr{L}_{\chi_{20}} \oplus \mathscr{L}_{\chi_{12}}$. We can calculate the T-Eigenspaces of $\mathscr{E}$: Let $s := (s_0, s_1, s_2) \in \Gamma(\mathscr{E}, X)$ be a section of $\mathscr{E}$, then for instance we have $(t \cdot s_0)(x) = \chi_{10}(t)s_0(t^{-1}x) = t_1 s_0(t^{-1}x)$. So s is in the T-Eigenspace $\Gamma(X,\mathscr{E})_u$ if and only if

$$s(t) = s(t \cdot x_0) = (\chi_{-u}(t)t_1 s_0(x_0), \chi_{-u}(t)t_2 s_1(x_0), \chi_{-u}(t)\frac{t_1}{t_2} s_2(x_0)).$$

We can see that since T is dense in X, the value of s in the whole of X is uniquely given by its value at the identity $x_0 = [1 : 1 : 1]$.

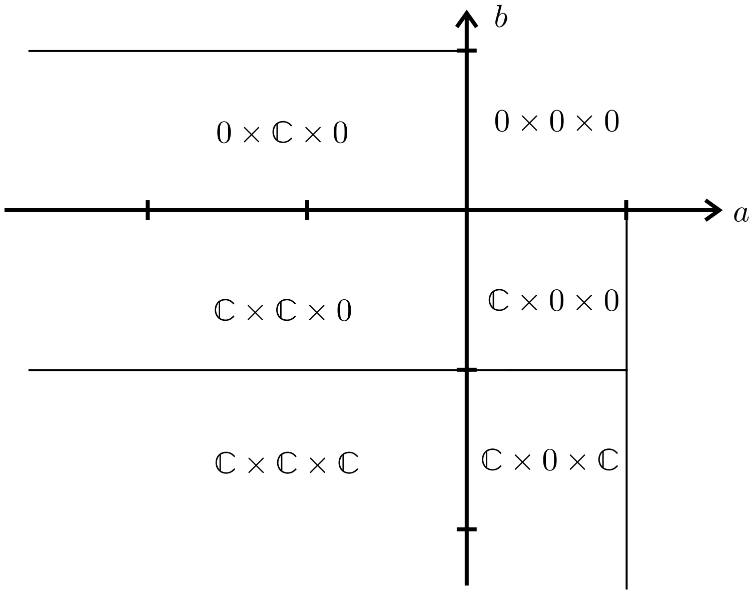
Now consider the restriction of $\mathscr{E}$ to the open set U_0[†]. Since $\sigma_0^\perp = \{0\}$ see 1.5
we have $\mathsf{M}_{\sigma_0} = \mathsf{M}$, which will make things easier for a while. In this case, our lattice is spanned by χ_{10} and χ_{20}, so we may write $\chi_u = {\chi_{10}}^a {\chi_{20}}^b$. For s to be well-defined in U_0 independent of the value at x_0 the conditions $a \le 0$ and $b \le -1$ must hold. Otherwise, if for instance $a = 1$ then

$$\lim_{t_1 \to 0} t \cdot x_0 = [1 : 0 : t_2] \in U_0,$$

but

$$\lim_{t_1 \to 0} s_1(t \cdot x_0) = \infty.$$

So it follows that $s_1(x_0) = 0$ unless $a \leq 0$ and $b \leq 1$. With similar arguments unless $a \leq 1$ and $b \leq -1$ it follow that $s_2(x_0) = 0$ and unless $b \leq 0$ that $s_0(x_0) = 0$. The following graphic illustrates $\mathsf{E}_u^{\sigma_1}$ in dependence of a and b.



In this example, $\check{\sigma}$ is the positive span of χ_{10} and χ_{20}, therefore we see in the figure that moving down or left, we have canonical inclusions.

Quod erat faciendum

We can express this in a more general way

1.16) EXAMPLE: Consider the line bundle $\mathscr{L}_{[u]}$ over U_σ. Then the constant section is an eigensection with eigenvalue u and $\chi_{u'}$ is a well-defined section of $\mathscr{L}_{[u]}$ over U_σ if and only if $u' \in \check{\sigma}$. Therefore there exists a non zero v-eigensection if and only if $v = u - u'$ for

$u' \in \check{\sigma} \cap \mathsf{M}$. That is to say

$$\mathsf{E}_v^\sigma = \begin{cases} \mathbb{C} & \text{if } u - v \in \check{\sigma} \cap M \\ 0 & \text{otherwise} \end{cases}$$

Quod erat faciendum

A particular case of the evaluation at identity arrives by considering rays, as we see in the following.

1.17) Filtration on E: For a character u' orthogonal to the cone σ — i.e., $u' \in \sigma^\perp$ — the inclusion $\mathsf{E}_u^\sigma \subset \mathsf{E}_{u-u'}^\sigma$ is an isomorphism since $-u' \in \sigma^\perp$. Therefore E_u^σ only depends on the class $[u] \in \mathsf{M}_\sigma$. For a ray $\rho \in \Sigma(1)$ one can identify the equivalence class with the product $i = \langle u, v_\rho \rangle \in \mathbb{Z}$, where v_ρ is the primitive generator of ρ. In this case, we write $\mathsf{E}^\rho(i)$ for E_u^ρ and get a decreasing filtration:

$$\mathsf{E} \supset \cdots \supset \mathsf{E}^\rho(i-1) \supset \mathsf{E}^\rho(i) \supset \mathsf{E}^\rho(i+1) \supset \cdots \supset 0$$

Let us see this in our example

1.18) Example: Consider once more $X = \mathbb{P}^2$ and the toric bundle $\mathscr{E} = \mathscr{L}_{\chi_{10}} \oplus \mathscr{L}_{\chi_{20}} \oplus \mathscr{L}_{\chi_{12}}$, but this time over $U_{01} = U_0 \cap U_1$. The corresponding cone σ_{01} is the ray between σ_0 and σ_1, with primitive generator α_2 given by the condition $\langle \alpha_2, \chi_{ij} \rangle = \delta_{2i}$. That is, α_2 measures the exponent of x_2 in the character. Since x_2 is the only coordinate that can be zero in U_{10}, the question whether a given value of $s(x_0)$ yields an eigensection over U_{10} depends only on the limit $x_2 \to 0$. In terms of Example 1.15 there are no constraints on a anymore, since the limit points $[1 : 0 : 1]$ and $[0 : 1 : 0]$ are not in U_{01} — so the the orbit of $\mathbb{C}^* \times \{1\}$ is open in U_{01} . We conclude that characters u and u' with the same exponent for x_2 yield the same space $\mathsf{E}_{[u]}^{\sigma_{01}}$. In the example above, we called this exponent b, and the constraints on b give the filtration:

$$\begin{array}{ccccccccccc} \dots & \supset & \mathbb{C}\times\mathbb{C}\times\mathbb{C} & \supset & \mathbb{C}\times\mathbb{C}\times 0 & \supset & 0\times\mathbb{C}\times 0 & \supset & 0 & \supset & \dots \\ & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \\ \dots & \supset & \mathsf{E}^{\sigma_{10}}(-1) & \supset & \mathsf{E}^{\sigma_{10}}(0) & \supset & \mathsf{E}^{\sigma_{10}}(1) & \supset & \mathsf{E}^{\sigma_{10}}(2) & \supset & \dots \end{array}$$

This filtration is also the one we get from 1.15 by choosing a small enough value for a. This is no coincidence. *Quod erat faciendum*

The filtrations we get in this way are not independent, but rather compatible with each other for every cone the rays span. This gives us a critical condition.

1.19) Compatibility Condition: Let $\mathscr{E}$ be a toric bundle over $X(\Sigma)$. Then the filtrations $\{\mathsf{E}^\rho(i)\}$ satisfy the following compatibility condition: For every cone $\sigma \in \Sigma$ there is a decomposition

$$\mathsf{E} = \bigoplus_{[u]\in \mathsf{M}_\sigma} \mathsf{E}_{[u]},$$

such that for every ray $\rho \preceq \sigma$ and integers $i \in \mathbb{Z}$ we have

$$\mathsf{E}^\rho(i) = \sum_{\langle [u], v_\rho\rangle \geq i} \mathsf{E}_{[u]}$$

where $\sum$ denotes the sum of subspaces.

For the proof: From 1.8 we know that $\mathscr{E}|_{U_\sigma}$ is isomorphic to the sum of line bundles $\bigoplus_{u\in\mathfrak{u}(\sigma)} \mathscr{L}_{[u]}$. With 1.16 we conclude that there is a decomposition $\mathsf{E} \sim \bigoplus_{[u]\in\mathsf{M}_\sigma} \mathbb{C}^{\#u}$, where $\#u$ is the multiplicity of $[u]$ in $\mathfrak{u}(\sigma)$. So set

$$\mathsf{E}_{[u]} := \mathbb{C}^{\#u}.$$

Now restricting the direct sum of line bundles to the open set U_ρ and again using 1.16 we get the equation for $\mathsf{E}^\rho(i)$, as needed.

We can see this clearly in our example.

1.20) Example: We continue with example 1.15. The filtrations for the ray $\sigma_{10} = \langle \alpha_2 \rangle$ were already calculated in 1.18. The filtration for $\sigma_{20} = \langle \alpha_1 \rangle$ is

$$\begin{array}{ccccccccc} \dots & \supset & \mathbb{C}\times\mathbb{C}\times\mathbb{C} & \supset & \mathbb{C}\times 0\times\mathbb{C} & \supset & 0 & \supset & \dots \\ & & \shortparallel & & \shortparallel & & \shortparallel & & \\ \dots & \supset & \mathsf{E}^{\sigma_{20}}(0) & \supset & \mathsf{E}^{\sigma_{20}}(1) & \supset & \mathsf{E}^{\sigma_{20}}(2) & \supset & \dots \end{array}$$

This we can read off directly from the figure in 1.15 for $b \ll 0$. We further notice in the figure that the intersection $\mathsf{E}^{\sigma}_{u} \cap (\mathbb{C}\times 0\times 0) = \{0\}$ if and only if $a > 1$ and $b > 0$. On the other hand, the intersection with $0\times\mathbb{C}\times 0$ is trivial if and only if $a > 0$ and $b > 1$ while the intersection with $0\times 0\times\mathbb{C}$ is trivial if and only if $a > 1$ and $b > -1$. We therefore may set

$$\mathsf{E}_{[\chi_{10}]} = \mathbb{C}\times 0\times 0,\;\; \mathsf{E}_{[\chi_{20}]} = 0\times\mathbb{C}\times 0,\;\; \mathsf{E}_{[\chi_{10}\chi_{20}{}^{-1}]} = 0\times 0\times\mathbb{C}$$

and $\mathsf{E}_{[u]} = 0\times 0\times 0$ otherwise. It is straightforward to check the compatibility condition for the rays σ_{10} and σ_{20}.

We leave it for the reader to calculate the filtration for σ_{12}.

Quod erat faciendum

This condition allows us to classify toric bundles according to the following theorem.

1.21) Klyachko's Classifications Theorem: A toric vector bundle is classified by a vector space with filtrations† $(\mathsf{E}, \{\mathsf{E}^{\rho}(i)\}_{\rho\in\Sigma(1)})$ satisfying the compatibility condition†. Moreover, the category of toric vector bundles on $X(\Sigma)$ is naturally equivalent to the category of finite-dimensional $\mathbb{C}$-vector spaces E with filtrations $\{\mathsf{E}^{\rho}(i)\}$ satisfying the compatibility condition.

see b.8
see 1.19

Here we provide proof for that, following mostly [Pay07].

1.22) PROOF OF 1.21: In one direction the equivalence is given by associating to $\mathscr{E}$ the fiber $\mathsf{E} = \mathscr{E}_{x_0}$ over the identity, together with its filtrations $\{\mathsf{E}^\rho(i)\}$ for every ray $\rho \in \Sigma(1)$. A morphism of (toric) vector bundles $f : \mathscr{E} \to \mathscr{F}$ maps the fibers linearly. Hence it induces a linear map $\psi_f : \mathsf{E} \to \mathsf{F}$. The map ψ_f preserves T-Eigenspaces, because it maps $\Gamma(U_\sigma, \mathscr{E})_u$ to $\Gamma(U_\sigma, \mathscr{F})_u$ for every cone $\sigma \in \Sigma$ and $u \in \mathsf{M}$ and in particular for every ray $\rho \in \Sigma(1)$. Hence ψ_f takes $\{\mathsf{E}^\rho(i)\}$ into $\{\mathsf{F}^\rho(i)\}$ for every $i \in \mathbb{Z}$ as required.

On the other hand, consider a vector space with filtrations $(\mathsf{E}, \{\mathsf{E}^\rho(i)\})$ satisfying the compatibility condition. From the filtrations, we can reconstruct E^σ_u for every cone $\sigma \in \Sigma$ and degree $u \in \mathsf{M}$:

$$\mathsf{E}^\sigma_u = \bigcap_{\rho \preceq \sigma} \mathsf{E}^\rho(\langle u, v_\rho \rangle).$$

From that we get natural inclusions $\mathsf{E}^\sigma_u \hookrightarrow \mathsf{E}^\sigma_{u-u'}$ for $u' \in \check{\sigma} \cap \mathsf{M}$.

With this information, we want to recreate the sheaf structure. For that, define

$$\mathsf{E}^\sigma = \bigoplus_{u \in \mathsf{M}} \mathsf{E}^\sigma_u$$

and therefore there is a natural inclusion map:

$$\bigoplus_{u \in \mathsf{M}} \mathsf{E}^\sigma_u \hookrightarrow \bigoplus_{u \in \mathsf{M}} \mathsf{E}^\sigma_{u-u'}.$$

This map induces the structure of multiplication by $\chi_{u'}$. That means E^σ has a natural $\mathbb{C}[U_\sigma]$-module structure, where $\mathbb{C}[U_\sigma] = \mathbb{C}[\check{\sigma} \cap \mathsf{M}]$. There is also an action of T on E^σ, where $t \in \mathsf{T}$ acts on E^σ_u by multiplication with the eigenvalue $\chi_u(t) \in \mathbb{C}^*$. The $\mathbb{C}[U_\sigma]$-module E^σ induces a quasicoherent sheaf $\widetilde{\mathsf{E}^\sigma}$ on U_σ torically isomorphic to

$$\bigoplus_{[u]\in \mathsf{M}_\sigma} \mathscr{L}_{[u]} \otimes \mathsf{E}_{[u]}.$$

In particular is $\widetilde{\mathsf{E}^\sigma}$ locally free and defines a toric bundle $\mathscr{E}^\sigma$. Since in the compatibility condition $\mathsf{E} = \bigoplus_{[u]\in\mathsf{M}_\sigma} \mathsf{E}_{[u]}$ it follows that $\mathsf{E}^\rho(i) = \mathsf{E}$ for $i \ll 0$. Therefore the natural inclusion $\mathsf{E}^{\sigma'}_u \subset \mathsf{E}^\sigma_{u-u'}$ for $\sigma' \preceq \sigma$ and $u' \in \sigma'^\perp$ induces a toric isomorphism:

$$\mathscr{E}^\sigma|_{U_{\sigma'}} \cong \mathscr{E}^{\sigma'}.$$

The intersection of two cones σ_1 and σ_2 is always a common face σ' belonging to both, so this gives an isomorphism $\mathscr{E}^{\sigma_1}|_{U_{\sigma'}} \cong \mathscr{E}^{\sigma_2}|_{U_{\sigma'}}$. One can also verify the cocycle condition since the intersection of three cones is also a common face. Therefore, these isomorphisms constitute gluing data and give rise to a toric bundle $\mathscr{E}$. Using the same process one shows that a morphism of vector bundles with fibrations gives rise to a corresponding equivariant morphism of toric bundles. The functor so defined is inverse to the functor $\mathscr{E} \mapsto (\mathsf{E}, \{\mathsf{E}^\rho(i)\})$ up to a natural isomorphism. So we have an equivalence of categories.

Quod erat demonstrandum

The following example may clarify the proof.

1.23) Example: As an exercise, let us go through the proof 1.22 again with example 1.15. But this time, we consider $\mathscr{E} = \mathscr{L}_{\chi_{10}} \oplus \mathscr{L}_{\chi_{20}} \oplus \mathscr{L}_{\chi_{12}}$ as a toric bundle over $\mathbb{P}^2$. The underlying vector bundle is trivial, and the toric action is given by multiplication with $\chi_{10}(t)$, $\chi_{20}(t)$, or $\chi_{12}(t)$ respectively. We have three rays: ρ_0, ρ_1 and ρ_2 spanned by α_0, α_1 and α_2 respectively. We may also call these rays σ_{12}, σ_{02} and σ_{01} in reference to the maximal cones, to which they are a common face. We already showed how to calculate the filtrations and the compatibility condition. The three filtrations may be given as a table:

	$i \leq -1$	$i = 0$	$i = 1$	$i \geq 2$
ρ_0	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$0 \times 0 \times \mathbb{C}$	$0 \times 0 \times 0$	0
ρ_1	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathbb{C} \times 0 \times \mathbb{C}$	0
ρ_2	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathbb{C} \times \mathbb{C} \times 0$	$0 \times \mathbb{C} \times 0$	0

We have also shown how to calculate $\mathsf{E}_{[u]}$: it is 0 almost everywhere, except for three $[u]$'s corresponding to each factor of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$. We also give those in the table below: for every cone and every factor, we give the corresponding u where it appears. The u's are given as pairs (a, b), just as before.

	$\mathbb{C} \times 0 \times 0$	$0 \times \mathbb{C} \times 0$	$0 \times 0 \times \mathbb{C}$
σ_0	$(1, 0)$	$(0, 1)$	$(1, -1)$
σ_1	$(1, 0)$	$(0, 1)$	$(1, -1)$
σ_2	$(1, 0)$	$(0, 1)$	$(1, -1)$

For the rays, there are several u's in the same equivalence class. We give the defining equations in a, b or $c := -a - b$.

	$\mathbb{C} \times 0 \times 0$	$0 \times \mathbb{C} \times 0$	$0 \times 0 \times \mathbb{C}$
$\rho_1 = \sigma_{02}$	$a = 1$	$a = 0$	$a = 1$
$\rho_2 = \sigma_{01}$	$b = 0$	$b = 1$	$b = -1$
$\rho_0 = \sigma_{12}$	$c = -1$	$c = -1$	$c = 0$

We have thus fully described the vector space with filtrations $(\mathsf{E}, \{\mathsf{E}^{\rho}(i)\})$ we get from this example — including the compatibility condition. One may easily calculates that the filtration $\mathsf{E}^{\sigma_0}_u$ we calculated in 1.15 can be reconstructed by

$$\mathsf{E}^{\sigma_0}_u = \mathsf{E}^{\rho_1}(a) \cap \mathsf{E}^{\rho_2}(b)$$

and therefore the module E^{σ_0} constructed in the proof is

$$\mathsf{E}^{\sigma_0} = \left(\bigoplus_{(a,b)\in\mathbb{Z}^2} \mathsf{E}^{\rho_1}(a) \cap \mathsf{E}^{\rho_2}(b) \right) \subset \mathbb{Z}^2 \times \mathbb{C}^3.$$

Here multiplication by $\chi_{u'}$ is given by translating $(a,b) \mapsto (a-a', b-b')$, which is well-defined since $a', b' \geq 0$ for $u' = (a', b') \in \check{\sigma} \cap \mathsf{M}$. The action of $\mathsf{T} = {\mathbb{C}^*}^2$ on E^{σ_0} is given by

$$(t_1, t_2), (v_0, v_1, v_2)_{(a,b)} \mapsto t_1^a t_2^b (v_0, v_1, v_2)_{(a,b)}$$

where $(t_1, t_2) \in \mathsf{T}$, $(a,b) \in \mathbb{Z}^2$, and $(v_0, v_1, v_2) \in \mathbb{C}^3$.

Since E^σ_u is the image of the inclusion $\Gamma(U_\sigma, \mathscr{E})_u \hookrightarrow \mathsf{E}$ given by the evaluation at identity and $\Gamma(U_\sigma, \mathscr{E}) = \bigoplus_{u\in M} \Gamma(U_\sigma, \mathscr{E})_u$ there is a natural bijection $\Gamma(U_\sigma, \mathscr{E}) \cong \mathsf{E}^\sigma$. Since $\chi_{u'}(x_0) = 1$ for all u', multiplication does not change the image of $\Gamma(U_\sigma, \mathscr{E}) \hookrightarrow \mathsf{E}$, but it does map the T-eigenspaces $\Gamma(U_\sigma, \mathscr{E})_u \to \Gamma(U_\sigma, \mathscr{E})_{u-u'}$. Therefore, it is clear that the bijection above agrees with the $\mathbb{C}[U_\sigma]$-module structure. This isomorphism of modules induces an isomorphism of sheaves.

Since $U_\sigma = \mathsf{spec}(\mathbb{C}[U_\sigma])$, this gives rise to a second isomorphism of sheaves

$$\widetilde{\mathsf{E}^{\sigma_0}} \cong [\mathscr{L}_{\chi_{10}} \otimes \mathbb{C} \times 0 \times 0] \oplus [\mathscr{L}_{\chi_{20}} \otimes 0 \times \mathbb{C} \times 0] \oplus [\mathscr{L}_{\chi_{10}{\chi_{20}}^{-1}} \otimes 0 \times 0 \times \mathbb{C}],$$

i.e. $\widetilde{\mathsf{E}^{\sigma_0}} \cong \mathscr{L}_{\chi_{10}} \oplus \mathscr{L}_{\chi_{20}} \oplus \mathscr{L}_{\chi_{10}{\chi_{20}}^{-1}}$ and that is exactly what we began with, since $\chi_{10}{\chi_{20}}^{-1} = \chi_{12}$. We also get the same sheaf for the other cones, which reflects the fact that the bundle $\mathscr{E}$ is topologically trivial.

Quod erat faciendum

In order to consider examples that are not topologically trivial, we shall present some more complex examples without further proof.

1.24) Example[†]: Let X be a generic toric variety with fan Σ. For every *[Kly90, example 2.3]* ray $\rho \in \Sigma(1)$ the affine space $U_\rho \subset X$ let D_ρ be the closure of the orbit defined by $\rho^\dagger$. We may consider the invertible sheaf $\mathscr{O}_X(D_\rho)$ defined by the divisor D_ρ. In this sheaf, sections are rational functions, such that they have at most an order one pole in D_ρ. That results in more sections than $\mathscr{O}_X$, since we just weaken the conditions for a section to be regular. This moves the trivial fibration to the right.

see b.4

$$\mathsf{E}^\rho(i) = \left\{ \begin{array}{ll} \mathbb{C} & \text{if } i \leq 0 \\ 0 & \text{if } i \geq +1 \end{array} \right\} \subset \mathbb{C} = \mathsf{E}.$$

More generally, let $D = \sum_\rho \lambda_\rho D_\rho$ be a divisor, then the inversible sheaf $\mathscr{O}_X(D)$ is encoded by

$$\mathsf{E}^\rho(i) = \left\{ \begin{array}{ll} \mathbb{C} & \text{if } i \leq \lambda_\rho \\ 0 & \text{if } i \geq \lambda_\rho + 1 \end{array} \right\} \subset \mathbb{C} = \mathsf{E}.$$

The tangent space TX also has a canonical action of T and the fiber at x_0 is canonicaly isomorphic to $\mathsf{N}_\mathbb{C}$. The filtration is then given by

$$(TX|_{x_0})^\rho(i) = \left\{ \begin{array}{ll} N_\mathbb{C} & \text{if } i \leq 0 \\ \text{span}(\rho) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{array} \right\} \subset \mathsf{N}_\mathbb{C} =: \mathsf{E}.$$

Dual to that, the cotangent space T^*X has fiber at x_0 isomorphic to $\mathsf{M}_\mathbb{C}$ and filtrations

$$(T^*X|_{x_0})^\rho(i) = \left\{ \begin{array}{ll} M_\mathbb{C} & \text{if } i \leq -1 \\ \rho^\perp & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{array} \right\} \subset \mathsf{M}_\mathbb{C} =: \mathsf{E}.$$

Quod erat faciendum

Toric Chern Class

In this section, we calculate the equivariant Chern class of a toric vector bundle. It is essential since it fixes the discrete part of the filtrations and leaves the continuous data free. But we will see this in detail later. For now, we make some definitions.

1.25) Definition: Given a fan Σ, we associate its ring of integral piecewise polynomial functions $PP^*(\Sigma)$. That is the ring of functions $f : \bigcup_{\sigma\in\Sigma} \sigma \to \mathbb{R}$, such that f is polynomial with integral coefficients over every cone. Alternatively f associates to every cone $\sigma \in \Sigma$ an element $f(\sigma) := f|_\sigma \in \mathsf{Sym}(\mathsf{M}_\sigma)$, such that they agree at intersections

$$f(\sigma)|_{\sigma'} = f(\sigma') \text{ for every } \sigma' \preceq \sigma.$$

This ring gives us a concrete representation of the Chow cohomology ring, as stated below.

1.26) Proposition: The equivariant Chow cohomology ring $A^*_{\mathsf{T}}(X)$ is naturally isomorphic to $PP^*(\Sigma)$. The isomorphism is given by

$$u \mapsto c^1_{\mathsf{T}}(\mathscr{L}_{[u]})$$

taking the linear function $u \in \mathsf{M}$ to the first equivariant chern class of the line bundle $\mathscr{L}_{[u]}$.

For the proof: [Payo5]

The second definition we need is the following.

1.27) Definition: Fix $r \in \mathbb{N}$ and let $\mathbb{u}(\sigma)$ be a multiset of size r of linear functions $f \in \mathsf{M}_\sigma$ for every $\sigma \in \Sigma$, just as in 1.11. Denote by $\Psi := \{\mathbb{u}(\sigma)\}_{\sigma\in\Sigma}$ a collection of such multisets, agreeing at intersections, i.e.:

$$\mathbb{u}(\sigma') = \mathbb{u}(\sigma)|_{\sigma'}$$

for $\sigma' \preceq \sigma$. Let $c_i(\Psi)$ be the piecewise polynomial function, whose restriction to σ is $e_i(\mathbb{u}(\sigma))$, the ith elementary symmetric function in the multiset of linear functions $\mathbb{u}(\sigma)$. We define

$$c(\Psi) = 1 + c_1(\Psi) + \cdots + c_r(\Psi)$$

This brings us directly to the following proposition:

1.28) Proposition [Pay05]: The equivariant total Chern class of a toric vector bundle $\mathscr{E}$ is $c_{\mathsf{T}}(\mathscr{E}) = c(\Psi_{\mathscr{E}})$, where by $\Psi_{\mathscr{E}}$ we denote the collection of compatible multisets of linear functions determined by a toric vector bundle $\mathscr{E}$.

For the proof: Since the restriction of $\mathscr{E}$ to U_σ is

$$\mathscr{E}|_{U_\sigma} = \bigoplus_{[u]\in\mathbb{u}(\sigma)} \mathscr{L}_{[u]}$$

and since the first Chern class of $\mathscr{L}_{[u]}$ corresponds to the linear function $[u]$, the proposition follows from the naturality of the isomorphism $A^*_{\mathsf{T}}(X) \cong PP^*(\Sigma)$

We can therefore calculate the Chern class explicitly:

see 1.23

1.29) Example: Consider the same example as before[†], that is $\mathbb{u}(\sigma) =$

$\{\chi_{10}, \chi_{20}, \chi_{12}\}$ for every cone. Writing x for χ_{10} and y for χ_{20}, we can easily calculate

$$c_{\mathsf{T}}(\mathscr{E}) = 1 + 2x + x^2 + xy - y^2 + x^2y - xy^2$$

since $\chi_{12} = \chi_{x-y}$. *Quod erat faciendum*

1.30) Observation: The collection of multisets $\Psi_{\mathscr{E}}$ is itself sometimes called the toric Chern class of $\mathscr{E}$, since by 1.28 describing it as multisets is easier than calculating the polynomial itself and in the end, it makes no practical difference since

$$\Psi = \Psi' \Leftrightarrow c(\Psi) = c(\Psi').$$

For the proof: The $\Rightarrow$ direction is evident, we prove the $\Leftarrow$ direction: Consider the function on a variable X.

$$c_X(\Psi) = X^r + c_1(\Psi)X^{r-1} + \cdots + c_r(\Psi).$$

Since every $c_i(\Psi)$ is a piecewise polynomial of degree i, the above is the homogenisation of $c(\Psi)$ and therefore uniquely defined by it. Now c_i are the symetrical function, so over every σ this is equal to the Newton polynom

$$c_X(\Psi)|_{U_\sigma} = \prod_{u \in \mathbb{u}(\sigma)} (X + u),$$

so the multiset $\mathbb{u}(\sigma)$ gives the roots of $c_X(\Psi)$ — to be more precise the additive inverse of the roots. Since the roots of $c_X(\Psi)$ are unique, we get from $c(\Psi) = c(\Psi')$ that $\mathbb{u}(\sigma) = \mathbb{u}(\sigma)'$ for every σ, that is $\Psi = \Psi'$.

Flags and rank conditions

We now turn to the moduli space. The classification of toric bundles given by Klyachko's formalism cannot be easily described by a moduli space since writing it via filtrations is too redundant. Payne's idea to overcome that is to express a filtration through a finite set of subvector spaces and a finite set of indexes. The purpose of this section is to introduces Payne's formalism.

1.31) Definition: Let $\mathscr{E}$ be a toric bundle over $X(\Sigma)$. For every ray $\rho \in \Sigma(1)$ we reduce the filtration $\{\mathsf{E}^{\rho}(i)\}$ to a partial flag[†] of E by removing repeating subspaces. Let $Fl(\rho)$ be this flag. We also denote by $J(\rho) \subset \{0, 1, \dots, r\}$ the set of dimensions present in $Fl(\rho)$ and by $\mathscr{Fl}_{J(\rho)}(\mathsf{E})$ the set of flags with given dimensions. That is:
see b.10

$$Fl(\rho) \in \mathscr{Fl}_{J(\rho)}(\mathsf{E})$$

This formalism is equivalent to the filtrations provided by Klyachko, as stated below.

1.32) Proposition: Klyachko's classification theorem implies that $\mathscr{E}$ is determined up to isomorphism by the data $(\Psi_{\mathscr{E}}, Fl(\rho))$, i.e., by its collection of multisets[†] and a flag of E for every ray.
see b.9

For the proof: It is possible to reconstruct the filtrations $\{\mathsf{E}^{\rho}(i)\}$ since for a ray ρ the multiset $\mathbb{u}(\rho)$ tells us exactly at which i's there is a jump in the dimension of $\{\mathsf{E}^{\rho}(i)\}$ and the multiplicity of elements gives the size of the jump[†]. The flag $Fl(\rho)$, on the other hand, provides the spaces, but not where the jump occurs. Both information together construct the filtration.
see 1.19

Note in the following example that the flag data is much more compact to write.

1.33) Example: In the case of the ray ρ_1 in example 1.23 we have

$$Fl(\rho_1) := 0 \subset (\mathbb{C} \times 0 \times \mathbb{C}) \subset (\mathbb{C} \times \mathbb{C} \times \mathbb{C}) = \mathsf{E}$$

and if we allow the index $i = \langle u, \alpha_1 \rangle$ to represent $[u] \in \mathsf{M}_{\rho_1}$ we have

$$\mathbb{u}(\rho_1) = \{1, 1, 0\}.$$

Once again we can see that the flag provides the subvector spaces of the filtration and the multiset contains the index where the dimension jumps. The fact that 1 has double multiplicity in $\mathbb{u}(\rho_1)$ is, of course, redundant because the jump from 0 to $(\mathbb{C} \times 0 \times \mathbb{C})$ is of two dimensions. *Quod erat faciendum*

The price that we pay for considering flags is that the compatibility condition is challenging to express.

1.34) Rank condition: Let

$$\Delta_\rho(j) = \max\{i \in \mathbb{Z} | \dim \mathsf{E}^\rho(i) \geq j\},$$

that is the last index before the filtration $\mathsf{E}^\rho(i)$ drops below dimension j — for $j = 0$ we put $\Delta_\rho(j) = \infty$ and $\mathsf{E}^\rho(\infty) = 0$. The function Δ depends exclusively on the multiset $\mathbb{u}(\rho)$ since from the compatibility conditions† it follows that $\dim \mathsf{E}^\rho(i) = \#\{[u] \in \mathbb{u}(\rho) | \langle [u], v_\rho \rangle \geq i\}$, see 1.19
where the $[u]$'s are counted with multiplicity. Now let $Fl(\rho)_j$ be the j-dimensional subspace of E in the flag $Fl(\rho)$ for $j \in J(\rho)$. Then for any collection of rays $\rho_1, \dots, \rho_s$ of a cone σ, and for any set of dimension $j_1, \dots, j_s$ such that $j_l \in J(\rho_l)$, we have† see b.11

$$\dim \bigcap_{l=1}^{s} Fl(\rho_l)_{j_l} = \# \left\{ \substack{[u]\in \mathbb{u}(\sigma) \text{ such that } \langle [u], v_{\rho_l}\rangle \geq \Delta_{\rho_l}(j_l) \\ \text{for } 1\leq l\leq s} \right\}.$$

This condition is called *rank condition* since it is related to the rank of the natural map

$$\mathsf{E} \to \prod_{l=1}^{s} \mathsf{E}/Fl(\rho_l)_{j_l}$$

whose kernel is precisely $\bigcap_{l=1}^{s} Fl(\rho_l)_{j_l}$.

Since the dimension of the intersection is an upper semicontinuous
see b.12
function in the product of Grassmannians†, the rank conditions corresponds to the intersection of certain Zariski open and closed subsets of the following product of Grassmannians:

$$\prod_{\rho\in\Sigma(1)} \left[\prod_{j\in J(\rho)} \mathsf{Gr}(j, \mathsf{E}) \right].$$

The condition for a point in $\prod_{j\in J(\rho)} \mathsf{Gr}(j, \mathsf{E})$ to define a flag is also closed, namely, the vector-subspaces must contain each other. That is, the set of admissible flags for a given collection Ψ of multisets is a quasi-projective subvariety of the above product of Grassmannians. We denote by $\mathfrak{M}_{\Psi}^{fr}$ this subvariety.

This formalism also allows us to easily see an important property.

1.35) Proposition: Over toric varieties of dimension two the compatibility condition is always satisfied, that means any set of filtrations defines a toric vector bundle.

For the proof: Consider the formalism of 1.27 and see the collection of multisets Ψ as a collection of piecewise linear functions on the fan. In this light the collection $\Psi(1) := \{\mathbb{u}(\rho)\}_{\rho\in\Sigma(1)}$ is a collection of linear functions over the rays. Therefore the information in Ψ is just an interpolation of $\Psi(1)$ to the whole fan. Essentially it is a bijection $\mathbb{u}(\rho) \to \mathbb{u}(\rho')$ such that $u \mapsto u'$ if and only if there is a $v \in \mathbb{u}(\sigma)$ for which $u = v|_\rho$ and $u' = v|_{\rho'}$. This is well defined since the multisets in Ψ agree in intersections. For dimension greater than 2 a cone may be spanned by as many rays as wanted. A linear interpolation of the functions on those rays may not exist. However for dimension two, a full-dimensional cone is spanned by just 2 rays, therefore it is always possible to find a linear interpolation. In other words, for any bijection $u \mapsto u'$ we can always construct a v respecting the condition above.

That means the following: given a set of filtrations, one can extract the collection $\{\mathbb{u}(\rho)\}_{\rho\in\Sigma(1)}$. From the intersection of the flags, one reads out which functions to interpolate. For instance, if the vector spaces with the lowest dimension for both flags agree, this would mean the linear function corresponding to the greatest index must be interpolated together. This interpolation is always possible, therefore resulting in a valid Ψ and the rank condition is satisfied. If the rank condition is satisfied, so is the compatibility condition†. see 1.41

1.36) Observation: Proposition 1.35 does not mean the the collection Ψ is given by its restriction to the rays. That would be false, it is not difficult to find of examples of distinct Ψ's with the same restriction to the rays. In terms of Chern classes we can express it as follows: different flags may lie on different Chern classes, but always lie on a Chern class. For dimension greater 2 some flags don't belong to any Chern class at all.

There is also an alternative proof for the proposition. As explained in [AW21] on smooth surfaces if we drop the compatibility condition from Klyachko's formalism we get to classification of toric sheafs. Those are reflexiv sheafs and reflexiv sheafs over smooth surfaces are always locally free, therefore they are also toric bundles and the compatibility condition must apply.

The variety of admissible flags may be quite complicated for higher ranks. However, upon the choice of a hermitian metric for rank strict smaller than 4 all relevant Grassmannians are isomorphic to projective spaces. In this case, we may write it down quite easily.

1.37) EXAMPLE: Let us see how example 1.23 fits with the rank conditions. One can read off the collection Ψ of multisets directly from the tables in 1.23. From that we get $J(\rho_0) = \{0, 1, 3\}$, $J(\rho_1) = \{0, 2, 3\}$, $J(\rho_2) = \{0, 1, 2, 3\}$. In this case, there are only five combinations of rays and dimensions where the rank condition is not trivial, that is, the dimension of the intersection is not defined by the dimension of the flags. Those are

ρ	j	ρ'	j'	dim $Fl(\rho)_j \cap Fl(\rho')_{j'}$
ρ_0	1	ρ_1	2	1
ρ_0	1	ρ_2	1	0
ρ_0	1	ρ_2	2	0
ρ_1	2	ρ_2	1	0
ρ_1	2	ρ_2	2	1

One can see that the result of the first line is the greatest possible dimension for the intersection, so this is a closed condition. The other lines give the smallest possible dimension. Those are open conditions. The set of flags satisfying this table is a subset of a product of Grassmannians. Actually, if we fix a hermitian product in $\mathbb{C}^3$, it is a subset of $(\mathbb{P}_2)^4$, where the first two coordinates give the flags $Fl(\rho_0)_1$ and $Fl(\rho_2)_1$ and the last two coordinates give the orthogonal spaces $Fl(\rho_1)_2^{\perp}$ and $Fl(\rho_2)_2^{\perp}$. Let $([v_0], [v_1], [v_2], [v_3])$ be those coordinates. Then the previous table translates into

$$\begin{gathered}\langle v_0, v_2 \rangle = 0 \\ \langle v_0, v_1 \rangle^2 - |v_0|^2 |v_1|^2 \neq 0 \\ \langle v_0, v_3 \rangle \neq 0 \\ \langle v_2, v_1 \rangle \neq 0 \\ \langle v_2, v_3 \rangle^2 - |v_2|^2 |v_3|^2 \neq 0\end{gathered}$$

In order for $[v_1]$ and $[v_2]$ to define a flag $Fl(\rho_2)_1 \subset Fl(\rho_2)_2$ they need to be orthogonal. So the set of flags defining a toric bundle with the collection of multisets $\Psi_{\mathscr{E}} = \Psi$ is the intersection of the variety given by $\langle v_1, v_3 \rangle = 0$ with the 1 closed and the 4 open sets defined by the polynomial equations above. *Quod erat faciendum*

Let us also consider a second example

1.38) Example: Over $\mathbb{P}^2$ let $\Psi = \Psi_{T\mathbb{P}^2}{}^{\dagger}$, which means we are interested in toric bundles with the same Chern class of the tangent bundle $T\mathbb{P}^{2\dagger}$. For the rays we have $\mathbb{u}(\rho_0) = \mathbb{u}(\rho_1) = \mathbb{u}(\rho_2) = \{0, 1\}$; for the maximal cones — using the (a, b) notation — we have see b.9 see 1.24

$$\begin{gathered}\mathbb{u}(\sigma_0) = \{(1,0),(0,1)\} \\ \mathbb{u}(\sigma_1) = \{(1,-1),(0,-1)\} \\ \mathbb{u}(\sigma_2) = \{(-1,0),(-1,1)\}\end{gathered}$$

Therefore, our flags are given by just three lines in $\mathsf{N}_{\mathbb{C}} \cong \mathbb{C}^2$, where the rank condition means they do not coincide. From that we conclude that $\mathfrak{M}_{\Psi}^{fr}$ is the open subset of $(\mathbb{P}^1)^3$ given by points $([x], [y], [z])$ with $[x]$, $[y]$ and $[z]$ distinct from one another. This is the space of three distinct lines in $\mathbb{C}^2$. We can localize $T\mathbb{P}^2$ in this space by the point

$$(\mathrm{span}(\rho_0), \mathrm{span}(\rho_1), \mathrm{span}\rho_2,)$$

provided we fix the framing to be the canonical isomorphism $T_{x_0}\mathbb{P}^2 \cong \mathsf{N}_{\mathbb{C}} \cong \mathbb{C}^2$.

It is straightforward to generalize this result for higher dimensions: for $\Psi = \Psi_{T\mathbb{P}^n}$ we conclude that $\mathfrak{M}_\Psi^{fr}$ is the space of $n+1$ distinct lines in $\mathbb{C}^n$.

Quod erat faciendum

Considering the equations defining the flags as points in a product of Grassmannian depends on fixing an isomorphism between E *and* $\mathbb{C}^r$. *Therefore let us add this in a definition.*

1.39) Definition: A *framed toric bundle* on X is a toric bundle $\mathscr{E}$ together with a framing isomorphism $\mathsf{E} \to \mathbb{C}^r$. A *morphism of framed toric vector bundles* is an equivariant morphism of toric vector bundles that is compatible with the framing — i.e., the following diagram commutes

$$\begin{array}{ccc} \mathsf{E} := \mathscr{E}_{x_0} & \longrightarrow & \mathsf{E}' := \mathscr{E}'_{x_0} \\ \downarrow & & \downarrow \\ \mathbb{C}^r & = & \mathbb{C}^r \end{array}$$

With this extra constraint, we translate Klyachko's classification theorem into the flag framework.

1.40) Proposition: The map $\mathscr{E} \mapsto \{Fl(\rho)\}$ gives a bijection between the set of isomorphism classes of framed toric vector bundles on X with equivariant total Chern class $c(\Psi)$ and the set of $\mathbb{C}$-points of *see 1.34* $\mathfrak{M}_\Psi^{fr\dagger}$.

For the proof: This map's injectivity follows from Klyachko's classification theorem[†]. For surjectivity, we can reconstruct the filtrations using the multiset $\mathbb{u}(\rho)$ and the flags. Now lemma 1.41 says that the rank condition implies the compatibility condition. Therefore we can apply Klyachko's classification theorem in the other direction.

see 1.21

To complete the proof, we need to prove the following lemma.

1.41) Lemma: The rank condition implies the compatibility condition. In other words: Let $\mathbb{u}(\sigma) \subset \mathsf{M}_\sigma$ be a multiset of size r and $(\mathsf{E}, \{\mathsf{E}^\rho(i)\})$ be a vector space of dimension r together with a collection of filtrations indexed by the rays of σ such that, for any set of integers $\{i_\rho\}$

$$\dim\left(\bigcap_\rho \mathsf{E}^\rho(i_\rho)\right) = \#\{[u] \in \mathbb{u}(\sigma) | \langle [u], v_\rho\rangle \geq i_\rho \text{ for all } \rho\}.$$

Then there is a splitting $\mathsf{E} = \bigoplus_{[u]\in\mathbb{u}(\sigma)} \mathsf{E}_{[u]}$ such that

$$\mathsf{E}^\rho(i) = \sum_{\langle [u], v_\rho\rangle \geq i} \mathsf{E}_{[u]}$$

for all ρ and all i.

For the proof: b.15

Since the framing introduces an extra $\mathsf{GL}_r(\mathbb{C})$ freedom, we get this immediate corollary from the theorem.

1.42) Corollary: There is a natural bijection between the set of isomorphism classes of toric vector bundles on X with equivariant total Chern class $c(\Psi)$ and the set of $\mathsf{GL}_r(\mathbb{C})$-Orbits of $\mathfrak{M}_\Psi^{fr}$.

Until now, we just worked with single toric bundles, and the results we got were bijections. In order to describe the moduli space as a geometric object — for instance, as a scheme — we need to consider how toric bundles can deform. So let us add this to our context.

1.43) Notation: In this subsection, let S be a scheme of finite type over $\mathbb{C}$ and T_S be the relative torus $\mathsf{T} \times S$. Likewise, let X_S be the product $X \times S$. We fix a collection $\Psi := \{\mathbb{u}(\sigma)\}_{\sigma\in\Sigma}$ of multisets of integral linear functions of size r such that $\mathbb{u}(\sigma') = \mathbb{u}(\sigma)|_{\sigma'}$ for $\sigma' \preceq \sigma$.

With this language we define a deformation, i.e., a family, of toric bundles and the moduli functors resulting from it:

1.44) Definition: An *S-family of toric vector bundles on X* is a vector bundle $\mathscr{E}$ on X_S with an action of the relative torus T_S compatible with the action on X_S. We say that such a family has total Chern class $c(\Psi)$ if

$$c_{\mathsf{T}}(\mathscr{E}|_{X\times s}) = c(\Psi)$$

for every geometric point $s \in S$.

Likewise, a morphism of S-families is a morphism of vector bundles over X_S, which is compatible with the action of T_S. This defines a moduli functor $\mathcal{M}_\Psi : \mathcal{S}ch \to \mathcal{S}et$ given by

$$\mathcal{M}_\Psi(S) = \left\{ \substack{\text{isomorphism classes of S families of toric vector} \\ \text{bundles on X with equivariant total Chern class } c(\Psi)} \right\}$$

We also define the same for the framed case.

1.45) Definition: An *S-family of framed toric vector bundles on X* is a
see b.16 family of toric vector bundles with an isomorphism $\mathscr{E}|_{x_0\times S} \cong \mathscr{O}_S^{\oplus r\dagger}$. A

morphism of such families is a morphism compatible with the framing. This defines a functor $\mathcal{M}_{\Psi}^{fr} : \mathcal{S}ch \to \mathcal{S}et$, given by

$$\mathcal{M}_{\Psi}^{fr}(S) = \left\{ \begin{matrix} \text{isomorphism classes of S families of framed toric vector} \\ \text{bundles on X with equivariant total Chern class } c(\Psi) \end{matrix} \right\}$$

To say that the classifying space we found before is a fine moduli space, we must prove the following.

1.46) Theorem: There is a natural isomorphism of functors

$$\mathcal{M}_{\Psi}^{fr} \cong \hom(\_\_, \mathfrak{M}_{\Psi}^{fr})$$

For the proof: 1.50

1.47) Corollary: A scheme $\mathfrak{M}_{\Psi}$ is a coarse moduli space for $\mathcal{M}_{\Psi}$ if and only if it is a good quotient of $\mathfrak{M}_{\Psi}^{fr}$ for the action of $\mathsf{GL}_r(\mathbb{C})$ and its set of points is bijective to the set of $\mathsf{GL}_r(\mathbb{C})$-Orbits of $\mathfrak{M}_{\Psi}^{fr}$. This is the case mainly if $\mathsf{PGL}_r(\mathbb{C})$ acts freely on $\mathfrak{M}_{\Psi}^{fr}$.

For the proof: The set of equivalence classes of toric vector bundles is bijective to the set of orbits in $\mathfrak{M}_{\Psi}^{fr}$. This is Klyachko Classification Theorem†. Therefore we must only prove the respective universal properties. Lemma b.13 establishes that a $\mathsf{GL}_r(\mathbb{C})$-invariant morphism is equivalent to a functor $\mathcal{M}_{\Psi} \to h^{\mathfrak{M}_{\Psi}}$. Under this correspondence, the universal property of a categorical quotient† translates directly into the universal property of a coarse moduli space. It is just a question of comparing definitions and using the Yoneda lemma.

see 1.21

see a.11a

Before we can prove the theorem, we must introduce the flag and rank formalism for families.

1.48) Family of filtrations: Let $\mathscr{E}$ be a S-family of toric vector bundles. Over every affine variety U_σ there is a splitting in $\mathscr{O}_S$-Modules:

$$(\pi_S)_*\mathscr{E}|_{U_\sigma \times S} \cong \bigoplus_{u \in \mathsf{M}} \mathsf{E}_u^\sigma$$

where T_S acts in E_u^σ by χ_u. The restriction to $x_0 \times S$ gives an injection $\mathsf{E}_u^\sigma \hookrightarrow \mathsf{E} := \mathscr{E}|_{x_0 \times S}$, whose image depends only on the class of $[u] \in \mathsf{M}_\sigma$. The image of this inclusion is a subvector bundle $\mathsf{E}_u^\sigma \to S$ of the vector bundle $\mathsf{E} \to S$; in the case of framed families $\mathsf{E} \simeq \mathbb{C}^n \times S$.

We also get from this a rank condition.

1.49) Proposition: The category of S-families of toric vector bundles on X is naturally equivalent to the category of vector bundles E on S with a collection of decreasing bundle filtrations $\{\mathsf{E}^\rho(i)\}$ indexed by the rays of Σ, satisfying the following rank condition.

For each cone $\sigma \in \Sigma$, there is a multiset $\mathbb{u}(\sigma) \subset \mathsf{M}_\sigma$ such that, for any rays $\rho_1, \dots, \rho_s$ of σ and integers $j_1, \dots, j_s$, the vector bundle $\mathsf{E}^{\rho_1}(i_1) \cap \dots \cap \mathsf{E}^{\rho_s}(i_s)$ is of rank equal to

$$\#\{[u] \in \mathbb{u}(\sigma) \mid \langle [u], v_{\rho_l} \rangle \geq \Delta_{\rho_j}(j_l) \text{ for } 1 \leq l \leq s\}$$

For the proof: b.17

Finally, we can prove the main theorem:

1.50) Proof of 1.46: Let $\mathscr{E}$ be an S-family of toric vector bundles on X with equivariant total Chern class $c(\Psi)$, and let $\phi : \mathscr{E}|_{x_0 \times S} \to \mathscr{O}_S^{\oplus r}$ be a framing of $\mathscr{E}$. For each ray ρ in $\Sigma\,(1)$, let $J(\rho) \subset \{0, \dots, r\}$ be the set of ranks of bundles $\mathsf{E}^\rho(i)$ appearing in the filtrations associated to $\mathscr{E}$ and let $Fl(\rho)$ be the flag consisting of precisely those subbundles. Then it follows from 1.49 that $\mathscr{E} \to \{Fl(\rho)\}$ gives a bijection between $\mathcal{M}_\Psi^{fr}(S)$ and the set of collections of partial flags in $\prod_\rho \mathcal{Fl}_{J(\rho)}(\mathscr{O}_S^{\oplus r})$ satisfying the rank conditions, which is canonically identified with $\hom(S, \mathfrak{M}_\Psi^{fr})$ since the Grassmannian is also a fine moduli space.

Quod erat demonstrandum

Before ending, we should revisit our examples.

1.51) Example: We calculated in 1.37 the variety $\mathfrak{M}_\Psi^{fr}$ for example 1.23. However, $\mathsf{PGL}_3(\mathbb{C})$ does not act freely in general, for instance, $v_0 = v_3$, $v_3 \perp v_2$ and $v_2 = v_1$ is a valid configuration where the action is not free. So there may be no general coarse moduli space for $\mathcal{M}_\Psi$.

On the other hand for example 1.38, $\mathsf{PGL}_2(\mathbb{C})$ does act freely on $\mathfrak{M}_\Psi^{fr}$. Moreover, it's an easy linear algebra exercise that $\mathsf{PGL}_2(\mathbb{C})$ can map any three distinct lines of $\mathbb{C}^2$ onto any other three distinct lines. Therefore the coarse moduli space for $\mathcal{M}_\Psi$ is a single point $\{[TX]\}$.

For higher dimension, there is no good quotient. Any set of $n+1$ lines in a generic position can be mapped by $\mathsf{GL}_n(\mathbb{C})$ to any other such set. There are other orbits, but this one is a dense orbit, so the only possible good quotient is the point. However, if all lines fall in the same two-dimensional plane, the orbit of $\mathsf{GL}_n(\mathbb{C})$ is closed in the moduli space, and for four or more lines there is more than one such orbit. Since the fibers of a good quotient must contain exactly one closed orbit, the quotient $\mathfrak{M}_\Psi^{fr} \to \{*\}$ is not a good one. This means, there is no coarse moduli space for this problem. *Quod erat faciendum*

Let us also present an example where the calculation of the unframed moduli space is not that trivial

1.52) Example: We calculate the moduli space of toric bundles over
see 1.30 $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the same Chern Class† as the tangential space — i.e., $\Psi = \Psi_{TX}$. In this case the toric variety is given by the fan Σ spanned by the vectors $\{e_1, e_2, -e_1, -e_2\}$, where $\{e_1, e_2\}$ is a $\mathbb{Z}$-basis of N. Let $\rho_1 = -\rho_{-1}$ and $\rho_2 = -\rho_{-2}$ be the 4 rays in Σ. Those span 4 maximal cones

$$\sigma_{++} = \text{span}\{e_1, e_2\}$$
$$\sigma_{-+} = \text{span}\{-e_1, e_2\}$$
$$\sigma_{--} = \text{span}\{-e_1, -e_2\}$$
$$\sigma_{+-} = \text{span}\{e_1, -e_2\}$$

From 1.24 we can read off $\Psi = \{\mathbb{u}(\sigma)\}_{\sigma\in\Sigma}$ and write it as follows

$$\mathbb{u}(\rho) = \{0, 1\} \text{ for all rays } \rho$$
$$\mathbb{u}(\sigma_{ij}) = \{(ie_1, 0), (0, je_2)\} \text{ for } i, j \in \{+, -\}$$

That means the flags are

$$0 \subsetneq R_1 \subsetneq \mathbb{C}^2 \qquad 0 \subsetneq R_2 \subsetneq \mathbb{C}^2$$
$$0 \subsetneq R_{-1} \subsetneq \mathbb{C}^2 \qquad 0 \subsetneq R_{-2} \subsetneq \mathbb{C}^2$$

So the moduli space is given by the space of 4 lines $\{R_1, R_2, R_{-1}, R_{-2}\}$ in $\mathbb{C}^2$. The compatibility condition here is satisfied, if for every maximal cone the 2 corresponding lines do not coincide. In a way similar to 1.38 the moduli space $\mathfrak{M}_\Psi^{fr}$ is the open subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by $R_1 \neq R_2$, $R_1 \neq R_{-2}$, $R_2 \neq R_{-1}$ and $R_{-1} \neq R_{-2}$. The tangent space $T_{\mathbb{P}^1\times\mathbb{P}^1}$ is given by $R_1 = R_{-1}$ and $R_2 = R_{-2}$. In fact in a standard choice of frame it's filtrations are

$$0 \subsetneq \text{span}(e_1) \subsetneq \mathbb{C}^2 \qquad 0 \subsetneq \text{span}(e_2) \subsetneq \mathbb{C}^2$$
$$0 \subsetneq \text{span}(e_1) \subsetneq \mathbb{C}^2 \qquad 0 \subsetneq \text{span}(e_2) \subsetneq \mathbb{C}^2$$

The space $\mathfrak{M}_{\Psi}^{fr}$ has an action of $\mathsf{GL}_2(\mathbb{C})$ given by change of frame. We want to calculate this quotient. With a change of frame we can always set

$$R_1 = \mathbb{C} \times \{0\} \text{ and } R_2 = \{0\} \times \mathbb{C},$$

i.e. we map them into the coordinate axis. Now the compatibility condition states that R_{-1} must not fall in the y-axis and R_{-2} not in the x-axis. Therefore we can write

$$R_{-1} = \text{span}((1, z_1)) \text{ and } R_{-2} = \text{span}((z_2, 1))$$

where z_1 and z_2 are in $\mathbb{C}$. Therefore we can represent any filtration by the tuple $(z_1, z_2) \in \mathbb{C}^2$. This is however still not the quotient. We do have fixed axes, but we can still modify the framing by acting with diagonal maps. This means we reduced the action of $\mathsf{GL}_2(\mathbb{C})$ to a $(\mathbb{C}^*)^2$-action on $\mathbb{C}^2$. On our tuples (z_1, z_2) this action looks like

$$(a, b)(z_1, z_2) = (\frac{b}{a} z_1, \frac{a}{b} z_2).$$

By writing $t = \frac{b}{a}$ we can further reduce this to an action of $\mathbb{C}^*$ via

$$t(z_1, z_2) = (tz_1, t^{-1} z_2).$$

Since $R_{-1} \neq R_{-2}$ the orbit of $(1, 1)$ is not in the moduli space, otherwise there is a quotient

$$\mathbb{C}^2 \setminus \{z_1 z_2 = 1\} \to \mathbb{C} \setminus \{1\}$$

given by the product $(z_1, z_2) \mapsto z_1 z_2$. This is a good quotient; however, it is not a coarse moduli space according to 1.47 since there is not a 1-to-1 relation between orbits and points in the quotient. In particular the fiber of the origin $0 \in \mathbb{C}$ is the union of 3 orbits: the two coordinate axes and the origin. That means those toric bundles where the flags are not in a general position. Restricted to the generic bundles the quotient

$$(\mathbb{C}^*)^2 \setminus \{z_1 z_2 = 1\} \to \mathbb{C} \setminus \{1\}$$

is a coarse moduli space, which can be proven with a small modification of 1.47.

Quod erat faciendum

This previous example, although simple, is extremely important. It discards any hopes of a meaningful GIT *theory of toric bundles:*

1.53) OBSERVATION: We see that the quotient in example 1.52 is pretty similar to a.25. Actually if we replace $\mathbb{P}^1 \times \mathbb{P}^1$ by its blow-up, we get a toric surface whose fan is spanned by 5 rays. The resulting quotient is exactly a.25. We therefore can apply its conclusion to the moduli space of toric bundles: there are several choices of a linearisation, non of which is really canonical. Futhermore, the choices are symmetrical, so any GIT concept of stability is necessarily not intrinsic. This is not the case for instance in the theory of algebraic vector bundles over a curve since in there the stability of a vector bundle is indeed intrinsic.

SECOND CHAPTER

Toric Cohiggs Bundles

Where we examine the classification of toric Cohiggs bundles and construct the moduli space for them.

In this chapter, we introduce the study of Cohiggs fields in toric bundles. Those are similar to classic Higgs fields $\mathscr{E} \to \mathscr{E} \otimes T^*X$. However, instead of the cotangent bundle, we take the tangent bundle $\mathscr{E} \to \mathscr{E} \otimes TX$. While using the cotangent bundle is the classical approach, when working with complete toric varieties, Higgs fields are always nilpotent and in many prominent cases trivial. There are, however, multiple examples for Cohiggs fields [AW21, Remark 9]. Since we will be working with the tangent space we will assume X is smooth to avoid technical difficulties, however, a generalisation to non-smooth toric varities seems to be possible.

Like sections in $\Gamma(X, \mathscr{E})$ — with which we worked previously — Cohiggs fields can also be decomposed by degrees in M since $\mathscr{E} \otimes TX$ has a canonical toric structure. This allows us to apply Klyanchko's classification of toric morphisms. It states that an equivariant — i.e

degree zero — toric morphism is given by a morphism of vector spaces
see 1.21 with fibrations†. Morphism with different degrees can also be classified by simply shifting the filtrations†. see 2.1 Since the filtrations of TX are quite simple†, see 1.24 we may easily describe such conditions for Cohiggs fields: a map $\phi : \mathsf{E} \to \mathsf{E} \otimes \mathsf{N}_{\mathbb{C}}$ induces a field $\mathscr{E} \to \mathscr{E} \otimes TX$ of degree $u \in \mathsf{M}$ if and only if:

$$\langle \phi(\mathsf{E}^{\rho}(i)), s \rangle \subset \left\{ \begin{array}{ll} \mathsf{E}^{\rho}(i + \langle u, v_{\rho} \rangle) & \text{if } s \in \rho^{\perp} \\ \mathsf{E}^{\rho}(i - 1 + \langle u, v_{\rho} \rangle) & \text{if } s \notin \rho^{\perp} \end{array} \right\}$$

for all $s \in \mathsf{M}$, all rays $\rho \in \Sigma(1)$ and all $i \in \mathbb{Z}$.

In order to agree with the nomeclature of the classical theory, we also impose on proper Cohiggs fields the integrability condition $\Phi \wedge \Phi = 0$. This non-linear condition does not communicate well with the toric structure. Therefore we will focus our study first on fields $\Phi : \mathscr{E} \to \mathscr{E} \otimes TX$ as above. We call those pre-Cohiggs fields and pairs $(\mathscr{E}, \Phi)$ pre-Cohiggs bundles.

We already know from the general theory that there must be a moduli space for pre-Cohiggs bundles under some restrictions. The toric case is, however, considerably more manageable than the general one. We may follow two approaches, one is building a moduli space of
see 2.9 morphisms of vector spaces with filtrations† and by assuming com-
see 2.12 pleteness we build a moduli space of toric pre-Cohiggs bundles†. A second approach is to use the universal family of the fine moduli space of framed toric bundles $\mathfrak{M}^{fr}_{\Psi}$. Considering this universal family as a sheaf $\mathscr{E}$ over $\mathfrak{M}^{fr}_{\Psi} \times X$ we build a 'universal framed pre-Cohiggs sheaf'
see 2.19 $\mathscr{H}om(\mathscr{E}, \mathscr{E} \otimes \pi_X^* TX)^{\dagger}$. Now, this is a coherent flat sheaf, so if we assume X is complete it can be represented by a scheme. More specifically we get here a linear scheme over $\mathfrak{M}^{fr}_{\Psi}$, which can be seen as a
see 2.16 'vector bundle with varying rank'†. This (linear) scheme is automatically a fine moduli space for the problem of framed toric pre-Cohiggs
see 2.15 bundles†.

The moduli spaces for related problems follow from this one. For instance, the problem of unframed pre-Cohiggs has a coarse moduli space if and only if the framed case has a quotient by $\mathsf{GL}_r(\mathbb{C})$*†. There are sub-moduli spaces given by fixing a degree* $u \in \mathsf{M}$*. These spaces induce a canonical decomposition of the general moduli space†. On the other hand, since the integrability condition* $\Phi \wedge \Phi = 0$ *can be expressed in algebraic terms, it is likewise not difficult to show that the moduli space of (framed or unframed) Cohiggs bundles is a sub-scheme. This however is no longer a linear scheme†.*

see 2.20
see 2.21
see 2.23

Those constructions are mostly standard techniques; it is, however, still worth going through the details.

Toric Cohiggs Bundels

Before we tackle our specific case, let us review Klyachko's classification of toric morphisms.

2.1) Morphism of Toric Vector Bundles: Let $\mathscr{E}$ and $\mathscr{F}$ denote two toric vector bundles over $X(\Sigma)$ and E and F represent the fibers over the identity†. Then T acts on a homomorphism of vector bundles $\Phi : \mathscr{E} \to \mathscr{F}$ by $t(\Phi(t^{-1}v))$. The characters of T are eigenvalues for this action and create a decomposition

see 1.14

$$\hom(\mathscr{E}, \mathscr{F}) = \bigoplus_{u \in \mathsf{M}} \hom(\mathscr{E}, \mathscr{F})_u$$

where $\hom(\mathscr{E},\mathscr{F})_u$ is the eigenspace for the character χ_u. For an eigenmorphism with degree u — i.e. $\Phi \in \hom(\mathscr{E},\mathscr{F})_u$ — one may easily check that $\Phi(tv) = \chi_u(t^{-1})\, t \cdot \Phi(v)$. For instance, equivariant homomorphisms are the ones with degree $0 \in \mathsf{M}$. †

Those are the morphisms in the category of toric bundles, see 1.21.

Similar to the sections $s \in \Gamma(X,\mathscr{E})$ in 1.14, an eigenmorphism $\Phi \in \hom(\mathscr{E},\mathscr{F})_u$ is uniquely defined by its value at the identity of the torus— that is the linear map $\phi : \mathsf{E} \to \mathsf{F}$. The character χ_u gives therefore an inclusion $\hom(\mathscr{E},\mathscr{F}) \subset \hom(\mathsf{E},\mathsf{F}) \otimes \mathbb{C}[\mathsf{M}]$.

On the other hand, if $s \in \Gamma(U_\sigma,\mathscr{E})_u$ is of degree u and $\Phi \in \hom(\mathscr{E},\mathscr{F})_{u'}$ is of degree u', then the section $\Phi(s(x))$ in $\Gamma(U_\sigma,\mathscr{F})$ has degree $u + u'$. So it follows directly from 1.16 that a linear map $\phi : \mathsf{E} \to \mathsf{F}$ defines an eigenmorphism with eigenvalue χ_u if and only if

$$\phi(\mathsf{E}^\rho(i)) \subset \mathsf{F}^\rho(i + \langle u, v_\rho\rangle) \qquad (*)$$

Be aware of some important notation choices:

2.2) Observation: In some places in the literature — most importantly for us in [AW21] — the opposite notation is used: a homomorphism is said to have degree r if $\Phi(tv) = \chi_r(t)t\Phi(v)$. The consequence being that eq. (*) in the definition above gets a negative sign: $\phi(\mathsf{E}^\rho(i)) \subset \mathsf{F}^\rho(i - \langle u, v_\rho\rangle)$. We will continue to use the notation established above.

Now let us calculate this condition for the example of pre-Cohiggs fields — a formal definition will be given later.

2.3) Example: We can calculate from 2.1 the condition for a linear map $\mathsf{E} \to \mathsf{E}\otimes\mathsf{N}_\mathbb{C}$ to induce a morphism $\mathscr{E} \to \mathscr{E}\otimes TX$ of degree u. Let X be smooth and $\mathscr{F} = \mathscr{E} \otimes TX$. By letting $t \in \mathsf{T}$ act in both coordinates — which is well defined since it preserves the tensor properties —

one gets a natural toric action on $\mathscr{F}$. The fiber over the identity is $\mathsf{E}\otimes\mathsf{N}_{\mathbb{C}}$ and if v and v' are eigensection in $\mathscr{E}$ and TX with degrees u and u', then $v\otimes v'$ is an eigensection of $\mathscr{F}$ with degree $u+u'$. Therefore the filtration $\mathsf{F}^{\rho}(i)$ is given by the sum of tensor $\mathsf{E}^{\rho}(j)\otimes\mathsf{N}^{\rho}_{\mathbb{C}}(j')$ with $i=j+j'$. One concludes from 1.24 that

$$\mathsf{F}^{\rho}(i)=\mathsf{E}^{\rho}(i)\otimes\mathsf{N}_{\mathbb{C}}+\mathsf{E}^{\rho}(i-1)\otimes\mathrm{span}(\rho).$$

Now we may decompose an element $\phi\in\hom(\mathsf{E},\mathsf{E}\otimes\mathsf{N}_{\mathbb{C}})=\mathsf{End}(\mathsf{E})\otimes\mathsf{N}_{\mathbb{C}}$ into

$$\phi=\sum_{s}\phi_s\otimes s^*$$

where the sum goes over a set $\{s\}$ forming a basis of M and the s^*'s are the dual basis. Preferentially we may also write $\phi_s=(\mathrm{id}_{\mathsf{E}}\otimes s)\circ\phi\in\mathsf{End}(\mathsf{E})$ for any integral functional $(s:\mathsf{N}_{\mathbb{C}}\to\mathbb{C})$ in M. Either way we get the following

2.3A) Assertion: A $\mathbb{C}$-linear map $\phi:\mathsf{E}\to\mathsf{E}\otimes\mathsf{N}_{\mathbb{C}}$ induces a homogeneous field $\mathscr{E}\to\mathscr{E}\otimes TX$ of degree u if and only if the associated contraction ϕ_s satisfy

$$\phi_s(\mathsf{E}^{\rho}(i))\subset\left\{\begin{array}{ll}\mathsf{E}^{\rho}(i+\langle u,v_\rho\rangle) & \text{if } s\in\rho^{\perp}\\ \mathsf{E}^{\rho}(i+\langle u,v_\rho\rangle-1) & \text{if } s\notin\rho^{\perp}\end{array}\right\}\subset\mathsf{E}^{\rho}(i+\langle u,v_\rho\rangle)$$

for all $s\in\mathsf{M}$, rays $\rho\in\Sigma(1)$ and $i\in\mathbb{Z}$.

Quod erat faciendum

We have previously mentioned the importance of describing Klaychko's compatibility conditions with a finite number of equations. This is also the case here.

2.4) OBSERVATION: The conditions on 2.3a are finite. For every ray $\rho \in \Sigma(1)$ let $S_\rho \subset \mathsf{M}$ be a basis of $M_{\mathbb{C}}$ including a basis of $\rho^\perp$. Then one must only verify the condition for $s \in S_\rho$ and for a finite number of indeces:

$$\phi_s(Fl(\rho)_j) \subset \left\{ \begin{array}{ll} Fl(\rho)_{\nabla_\rho(\langle w_j+u,v_\rho\rangle)} & \text{if } s \in \rho^\perp \\ Fl(\rho)_{\nabla_\rho(\langle w_j+u,v_\rho\rangle-1)} & \text{if } s \notin \rho^\perp \end{array} \right\} \subset Fl(\rho)_{\nabla_\rho(\langle w_j-u,v_\rho\rangle)}$$

where

$$\nabla_\rho(i) = \dim \mathsf{E}^\rho(i)$$

see 1.34

and $w_j \in \mathsf{M}$ is such that $\langle w_j, v_\rho\rangle = \Delta_\rho(j)^\dagger$. One verifies that the map

$$j \mapsto \nabla_\rho(\langle w_j + u, v_\rho\rangle)$$

or respectively

$$j \mapsto \nabla_\rho(\langle w_j + u, v_\rho\rangle - 1)$$

is non decreasing. This must be the case since this is just a re-writing of 2.3a.

For a fixed ray these conditions create a staircase matrix, as the lemma below shows.

2.5) LEMMA: Let

$$\{0\} = \mathsf{V}_0 \subsetneq \cdots \subsetneq \mathsf{V}_k = \mathsf{V}$$

be a partial flag of a finite dimensional vector space V. For every index j between 0 and k, let j' be another index such that the map $j \mapsto j'$ is non-decreasing. Then the dimension of

$$W = \{f \in \hom(V,V) : f(V_j) \subset V_{j'}, \forall j \in \{0,\cdots,k\}\}$$

depends only on the set of dimensions $\{\dim V_j\}_{j\in\{0,\cdots,k\}}$ and on the mapping $j \mapsto j'$, but not on the flag itself.

For the proof: Choose a basis $(v_1, \cdots, v_k)$ for V such that

$$V_j = \text{span}\{v_1, \cdots, v_{n_j}\}$$

where $n_j = \dim V_j$. Then $f \in W$ if and only if $f(v_i) \in V_{j'}$ for all $i \in \{n_j + 1, \cdots, n_{j+1}\}$ and all $j \in \{0, \cdots, k-1\}$. In other words, the map f written in the basis $(v_1, \cdots, v_n)$ is a staircase matrix, whose form is given by the mapping $j \mapsto j'$ and the dimensions $\{\dim V_j\}_{j\in\{0,\cdots,k\}}$. The dimension of W is given by the form of this matrix or more precisely by

$$\dim W = \sum_{j=0}^{k-1} (n_{j+1} - n_j) \dim V_{j'}$$

Before concluding this section, we introduce a proper nomenclature.

2.6) (Pre-)Cohiggs field: A *pre-Cohiggs field* in a toric bundle $\mathscr{E}$ is a morphism of vector bundles $\Phi : \mathscr{E} \to \mathscr{E} \otimes TX$. If Φ is in $\text{hom}(\mathscr{E}, \mathscr{E} \otimes TX)_u$, we say that Φ is a *homogeneous pre-Cohiggs field of degree* u. We drop the prefix 'pre-' if the field satisfies the *integrability condition* $\Phi \wedge \Phi = 0$, where $\wedge$ is the exterior power in the tangent space TX. The toric vector bundle $\mathscr{E}$, together with a (pre-)Cohiggs field Φ is a *toric (pre-)Cohiggs bundle*.

Notice that the integrability condition is a non-toric feature coming from the general theory of Higgs bundles [Sim92]. Therefore the following is not surprising.

2.7) Observation: A pre-Cohiggs bundle Φ can be separated in its homogeneous parts $\Phi = \sum_{u \in M} \Phi_u$. But the same does not apply for proper Cohiggs fields: the homogenous parts of a Cohiggs field may not satisfy the integrability condition, and likewise, a sum of homogenous Cohiggs fields need not be a Cohiggs field, merely a pre-Cohiggs field.

Morphisms of Vector Spaces with Filtrations

This section addresses the moduli problem of pre-Cohiggs bundles from the point of view of vector spaces with filtrations. We must therefore work with framed bundles.

2.8) Framed Pre-Cohiggs Bundles: A toric pre-Cohiggs bundle is said to be *framed* if the underlying toric bundle is framed. Likewise, a *family of framed pre-Cohiggs bundles* over a scheme S is a family $\mathscr{E}_S$ of framed toric bundles together with a pre-Cohiggs field $\mathscr{E}_S \to \mathscr{E}_S \otimes \pi_X^* TX$, where $\pi_X : X \times S \to X$ is the projection. Such a family is *homogeneous of degree* u if the pre-Cohiggs field induced on $\mathscr{E}_s$ is of degree u for every geometric point $s \in S$.

The conditions established in 2.3a can be written as a kernel, allowing us to construct a fine moduli space thereof. The techniques of the proof will be useful later.

2.9) THEOREM: Fix a collection Ψ of multisets. There exists a fine moduli space $(\mathfrak{N}_\Psi^{fr})_u$ for the moduli problem of framed homogeneous pre-Cohiggs bundles† of degree $u \in \mathsf{M}$ and rank r. It comes with a natural projection $(\mathfrak{N}_\Psi^{fr})_u \to \mathfrak{M}_\Psi^{fr}$, such that over an open set, this has the structure of a vector bundle. *see 2.8*

2.10) PROOF OF 2.9: Consider the vector bundle

$$\mathsf{E}_{\mathfrak{M}} := \mathfrak{M}_\Psi^{fr} \times \mathbb{C}^r$$

over $\mathfrak{M}_\Psi^{fr}$. The fine moduli space $\mathfrak{M}_\Psi^{fr}$ of framed toric bundles is constructed as a subscheme of a product of partial flag varieties;

$$\mathfrak{M}_\Psi^{fr} \subset \prod_{\rho\in\Sigma(1)} \left[\prod_{j\in J(\rho)} \mathsf{Gr}(j, \mathbb{C}^r) \right].$$

Remember that a multiset $\mathbb{u}(\rho) \in \Psi$ contains the indexes of a filtration, where a jump in dimension of the filtration occures†. The multiplicity of an elements is the size of that jump. From this information we can construct $J(\rho)$ — that is the dimensions contained on a filtration — and *see 1.32*

$$i = \Delta_\rho(j)$$

which denotes the last index before the dimention jumps below j†. A point in $\mathfrak{M}_\Psi^{fr}$ can be represented as a set of subvector spaces, to be precise we write it as $(\mathsf{E}^\rho(i))_{\rho,i}$ where ρ varies over all rays in $\Sigma(1)$ and $i = \Delta_\rho(j)$ for j varying over $J(\rho)$†. Therefore there exists a tau- *see 1.34* *see 2.4* tological bundle $Fl_{\mathfrak{M}}$ over $\mathfrak{M}^{fr}_{\Psi}$. Its fibers are

$$Fl_{\mathfrak{M}}|_{(\mathsf{E}^{\rho}(i))_{\rho,i}} = \bigoplus_{\rho,i} \mathsf{E}^{\rho}(i).$$

Now for every ray $\tilde{\rho}$ and index $\tilde{j}$ there is a canonical morphism $Fl_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}}$, mapping

$$\bigoplus_{\rho,i} \mathsf{E}^{\rho}(i) \mapsto \mathsf{E}^{\tilde{\rho}}(\Delta_{\tilde{\rho}}(\tilde{j})) \subset \mathsf{E}.$$

We denote by $(Fl(\tilde{\rho})_{\tilde{j}})_{\mathfrak{M}}$ the image of this morphism, this is a vector bundle over $\mathfrak{M}^{fr}_{\Psi}$ with fiber

$$(Fl(\tilde{\rho})_{\tilde{j}})_{\mathfrak{M}}|_{(\mathsf{E}^{\rho}(i))_{\rho,i}} = \mathsf{E}^{\tilde{\rho}}(\Delta_{\tilde{\rho}}(\tilde{j})).$$

That is the vector space of the filtration associated to the ray $\tilde{\rho}$ and with dimension $\tilde{j}$.

We want to express the condition in 2.4 using these bundles: for that fix a ray $\tilde{\rho}$, an element $s \in \mathsf{M}$ and for a fixed $\tilde{j}$ let

$$\tilde{j}' = \left\{ \begin{array}{ll} \nabla_{\tilde{\rho}}(\langle w_{\tilde{j}} + u, v_{\tilde{\rho}}\rangle) & \text{if } s \in \tilde{\rho}^{\perp} \\ \nabla_{\tilde{\rho}}(\langle w_{\tilde{j}} + u, v_{\tilde{\rho}}\rangle - 1) & \text{if } s \notin \tilde{\rho}^{\perp} \end{array} \right\}.$$

Since M is the dual of N there is a map $\langle s, \_\rangle : \mathsf{N}_{\mathbb{C}} \to \mathbb{C}$, from which we get a morphism:

$$\mathrm{id} \otimes \mathrm{id} \otimes \langle s, \_\rangle : \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^{*}_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}} \to \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^{*}_{\mathfrak{M}}$$

where the tensor $\otimes \mathsf{N}_{\mathbb{C}}$ is shorthand for $\otimes(\mathfrak{M}^{fr}_{\Psi} \times \mathsf{N}_{\mathbb{C}})$. On the other hand there is an inclusion $\zeta : (Fl(\tilde{\rho})_{\tilde{j}})_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}}$, from which we get a dual morphism

$$\mathrm{id} \otimes \zeta^{*} : \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^{*}_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}} \otimes (Fl(\tilde{\rho})_{\tilde{j}})^{*}_{\mathfrak{M}}.$$

Since $Fl(\tilde{\rho})_{\tilde{j}'}$ is a subvector bundle, there is also a quotient $\eta : \mathsf{E}_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}}/(Fl(\tilde{\rho})_{\tilde{j}'})_{\mathfrak{M}}$, from which we build

$$\eta \otimes \mathrm{id} : \mathsf{E}_{\mathfrak{M}} \otimes (Fl(\tilde{\rho})_{\tilde{j}})^*_{\mathfrak{M}} \to \left(\mathsf{E}_{\mathfrak{M}}/(Fl(\tilde{\rho})_{\tilde{j}'})_{\mathfrak{M}}\right) \otimes (Fl(\tilde{\rho})_{\tilde{j}})^*_{\mathfrak{M}}.$$

The morphism $\eta \otimes \zeta^* \otimes s$ is the composition of these morphisms

$$\mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}} \to \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}} \otimes (Fl(\tilde{\rho})_{\tilde{j}})^*_{\mathfrak{M}} \to \left(\mathsf{E}_{\mathfrak{M}}/(Fl(\tilde{\rho})_{\tilde{j}'})_{\mathfrak{M}}\right) \otimes (Fl(\tilde{\rho})_{\tilde{j}})^*_{\mathfrak{M}}.$$

Over a point $(\mathsf{E}^{\rho}(i))_{\rho,i} \in \mathfrak{M}^{fr}_{\Psi}$, this induces the following linear maps between the fibers

$$\mathbb{C}^r \otimes \mathbb{C}^{r*} \otimes \mathsf{N}_{\mathbb{C}} \to \mathbb{C}^r \otimes \mathbb{C}^{r*} \to \mathbb{C}^r \otimes \mathsf{E}^{\tilde{\rho}}(\tilde{i})^* \to \left(\mathbb{C}^r/\mathsf{E}^{\tilde{\rho}}(\tilde{i}')\right) \otimes \mathsf{E}^{\tilde{\rho}}(\tilde{i})^*$$

where $\tilde{i} = \Delta_{\tilde{\rho}}(\tilde{j})$ and $\tilde{i}' = \Delta_{\tilde{\rho}}(\tilde{j}')$. An arbitrary linear map $\phi_s : \mathsf{E}_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}}$ is in the kernel of $\eta \otimes \zeta^* \otimes s$ if and only if it satisfies the conditions

$$\phi_s(Fl(\tilde{\rho})_{\tilde{j}}) \subset Fl(\tilde{\rho})_{\tilde{j}'}$$

coming from 2.4, which for the fibers means

$$\phi_s(\mathsf{E}^{\tilde{\rho}}(\tilde{i})) \subset \mathsf{E}^{\tilde{\rho}}(\tilde{i}').$$

This is not difficult to see, $\langle s, \_ \rangle$ is just the same as the mapping $\phi \mapsto \phi_s$, applying the dual ζ^* restricts the domain to $Fl(\tilde{\rho})_{\tilde{j}}$ and η is the quotient of $Fl(\tilde{\rho})_{\tilde{j}'}$. So if the linear map is in the kernel, it means that $Fl(\tilde{\rho})_{\tilde{j}}$ maps into $Fl(\tilde{\rho})_{\tilde{j}'}$.

Now consider this for a flag, that is the intersection of the kernels over all j's. The flag $Fl(\rho)$ may itself vary wildly, but it remains a flag, so on a suitable basis matrixes representing the kernel of this morphism are just stair-case matrixes — see 2.5. That is to say, the rank of this map is constant. Therefore the kernel is a subvector bundle of $\mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}}$. Now let S_ρ be a basis of $M_{\mathbb{C}}$ such that it contains a basis of $\rho^\perp$ and let $(\mathfrak{N}_\Psi^{fr})_u$ be the finite intersection of these kernels for every ray $\rho \in \Sigma(1)$, and every $s \in S_\rho$ as explained in 2.4:

$$(\mathfrak{N}_\Psi^{fr})_u := \bigcap_{\substack{\rho\in\Sigma(1)\\ s\in S_\rho \\ j\in J(\rho)}} \ker \eta_{\rho,s,j} \otimes \zeta^*_{\rho,s,j} \otimes s \subset \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}}.$$

This is our candidate for a fine moduli space, since it contains the subset of morphisms $\phi : \mathsf{E}_{\mathfrak{M}} \to \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}}$ which respect all conditions of 2.4 and therefore define a homogeneous pre-Cohiggs field.

Now let us see what this means for a family $(\mathscr{E}_S, \Phi : \mathscr{E}_S \to \mathscr{E}_S \otimes \pi_X^* T^*X)$. Since $\mathfrak{M}_\Psi^{fr}$ is a fine moduli space, every family of pre-Cohiggs bundles induces a scheme-morphism $f : S \to \mathfrak{M}_\Psi^{fr}$ and just like in 2.1 we can restrict the family of pre-Cohiggs fields to $\{\mathbb{1}\} \times S$, where $\mathbb{1}$ is the identity of the torus T and

$$\mathsf{E}_S = S \times \mathsf{E}.$$

This results in a section ϕ of $\mathsf{E}_S \otimes \mathsf{E}_S^* \otimes \mathsf{N}_{\mathbb{C}}$. Since $\mathsf{E}_S \cong f^*\mathsf{E}_{\mathfrak{M}}$, this means ϕ is a section of $f^*(\mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}})$. So we can lift f to a morphism $\hat{f} : S \to (\mathfrak{N}_\Psi^{fr})_u$ by pushing ϕ forward:

$$\hat{f} : \quad S \xrightarrow{\ \phi\ } f^*(\mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}}) \longrightarrow (\mathfrak{N}_\Psi^{fr})_u \subset \mathsf{E}_{\mathfrak{M}} \otimes \mathsf{E}^*_{\mathfrak{M}} \otimes \mathsf{N}_{\mathbb{C}}$$

where the second arrow is the natural morphism arising from the pullback- It must fall into $(\mathfrak{N}_\Psi^{fr})_u$ because Φ is a homogeneous morphism between toric bundle. This constructs a mapping

$$(\mathscr{E}_S, \Phi) \mapsto (f, \phi) \mapsto \hat{f}.$$

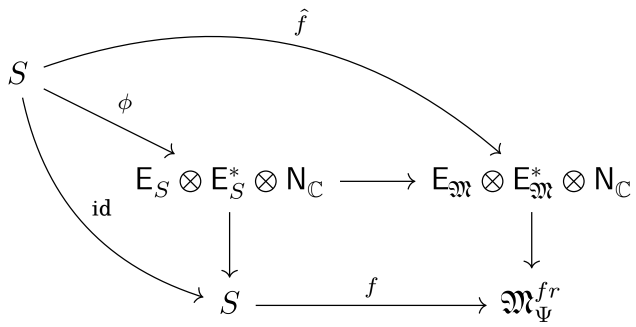
On the other hand, the fibers of $(\mathfrak{N}_\Psi^{fr})_u$ over $\mathfrak{M}_\Psi^{fr}$ are a subspace of $\mathsf{E}\otimes\mathsf{E}^*\otimes\mathsf{N}_\mathbb{C}$, so any morphism $\hat{f}$ from S to $(\mathfrak{N}_\Psi^{fr})_u$ induces also a morphism $\mathsf{E}_S \to \mathsf{E}_S\otimes\mathsf{N}_\mathbb{C}$. By translating this morphism via the action of T, we can extend it to an eigenmorphism of eigenvalue χ_u. Since the conditions of 2.3a are satisfied this induces a morphism $\Phi : \mathscr{E}_S \to \mathscr{E}_S\otimes\mathsf{N}_\mathbb{C}$ of degree u, which turns the family of toric bundles $\mathscr{E}_S$ into a family of pre-Cohiggs bundles of degree u. So there is also a mapping:

$$\hat{f} \mapsto (f,\phi) \mapsto (\mathscr{E}_S,\Phi).$$

These processes define functors between the category of S-points of $(\mathfrak{N}_\Psi^{fr})_u$ and the category of S-families of framed homogeneous toric pre-Cohiggs bundles. Up to a natural isomorphism they are inverse to one another, as we shall show:

Let $\mathcal{M}$ be the moduli functor of framed toric bundles of total Chern class $c(\Psi)$ and $\mu : \mathcal{M} \Rightarrow h^{\mathfrak{M}_\Psi^{fr}}$ the isomorphism of functors given by 1.46. Let $\mathcal{N}$ be the moduli functor of homogeneous framed pre-Cohiggs bundles of total Chern class $c(\Psi)$. We have to show the above process defines an isomorphism $\nu : \mathcal{N} \Rightarrow h^{(\mathfrak{N}_\Psi^{fr})_u}$. This means there is a bijection between the set of morphisms $S \to (\mathfrak{N}_\Psi^{fr})_u$ and equivalence classes of S-families of framed pre-Cohiggs bundles. This bijection agrees with pullbacks — in simple words $g^*\circ\nu_S = \nu_T\circ g^*$, where g is a morphism $T\to S$.

Therefore let us begin with a family of framed pre-Cohiggs bundle $(\mathscr{E}_S,\Phi)$. Extract from it $f=\mu_S(\mathscr{E}_S)$ and ϕ by restricting Φ to E_S. The lift of f via ϕ is a morphism $\hat{f} : S \to (\mathfrak{N}_\Psi^{fr})_u$ as wanted. On the other hand, any such $\hat{f}$ is by inclusion also a morphism $S\to \mathsf{E}_\mathfrak{M}\otimes\mathsf{E}_\mathfrak{M}^*\otimes\mathsf{N}_\mathbb{C}$. Push this down to $f : S\to\mathfrak{M}_\Psi^{fr}$. Next we can pull $\hat{f}$ back via f to form a section of $\mathsf{E}_S\otimes\mathsf{E}_S^*\otimes\mathsf{N}_\mathbb{C}$ — this is done by applying the universal property of the pullback, in other words $\phi = (\mathrm{id}_S,\hat{f})$. The diagram below illustrates the situation.



Now f defines via μ a framed toric bundle $\mathscr{E}_S$, and ϕ is extended to a pre-Cohiggs field Φ. Since μ is an isomorphism, the toric bundle we end up with is isomorphic to the one we started. Also the field Φ is uniquely determined by ϕ — its evaluation at the identity — and the pullback and pushforward operations are inverse to each another up to isomorphism. Notice that on this processes the frame is preserved by pushing it forward and backwards together with the bundle. Any bundle isomorphism that may arise from the pullbacks and pushforwards is not a change of frame, but an isomorphism of framed bundles. This means we end up within the same equivalence class of homogeneous framed pre-Cohiggs bundles as wanted.

To show the bijection is compatible with pullbacks just introduce a morphism $g : T \to S$ in the diagram above. That the bundles agree is clear, since μ is an isomorphism of functors. That the fields must agree means that the pullback by $g \circ f$ is equivalent to consecutive pullbacks — which is a known property of pullbacks. In other words $g^*\phi = (g^*\mathrm{id}_S, g^*\hat{f}) = (\mathrm{id}_T, \widehat{f \circ g})$. Therefore we proved that $(\mathfrak{N}_{\Psi}^{fr})_u$ is a fine moduli space, as wanted.

$(\mathfrak{N}^{fr}_{\Psi})_u$ was created as the intersection of vector bundles over $\mathfrak{M}^{fr}_{\Psi}$. The dimension of vector bundle intersections is an upper semicontinuous function†, so over an open set the dimension is minimal and $(\mathfrak{N}^{fr}_{\Psi})_u$ is a vector bundle. *Quod erat demonstrandum*

see b.12

In the general case, the space of (non-homogeneous) pre-Cohiggs fields may be infinite-dimensional. That means the problem is not bounded, and there is no moduli space. To get a bounded problem, an extra assumption is needed, for instance completeness.

2.11) Proposition: We call a $u \in \mathsf{M}$ admissible if there exists a nonzero pre-Cohiggs field of degree u. Over a complete toric variety there is only a limited number of admissible degrees. To be more concrete, let N_ρ be the maximum length of the filtration E^ρ for a ray $\rho \in \Sigma(1)$, that is to say, there exists an index $l(\rho)$ such that $\mathsf{E}^\rho(l(\rho)) = \mathsf{E}$, but $\mathsf{E}^\rho(l(\rho) + N_\rho) = 0$. Then

$$\{u \in \mathsf{M} : \quad u \text{ is admissible}\} \subset \{r \in \mathsf{M} : \langle r, v_\rho \rangle \leq N \text{ for all } \rho \in \Sigma(1)\}$$

For the proof: [AW21, Proposition 19]

2.12) Corollary: Let X be a complete toric variety, then there is a fine moduli space of framed pre-Cohiggs bundles of rank r and equivariant total Chern class $c(\Psi)$.

For the proof: If X is complete, we can limit ourselves to a finite number of u's in view of 2.11†. Similarly to pre-Cohiggs fields, a family of pre-Cohiggs fields is decomposable into homogeneous parts. So for a fine moduli space, we can simply set

see c.1

$$\mathfrak{N}^{fr}_{\Psi} = \bigoplus_u (\mathfrak{N}^{fr}_{\Psi})_u$$

where $\bigoplus$ is the direct sum of vector bundles.

We already know every component $(\mathfrak{N}_{\Psi}^{fr})_u$ is a fine moduli space, we need just to generalize it to the direct sum. So start with a framed pre-Cohiggs family $\mathscr{E}_S$ and Φ. Decompose Φ into its homogeneous parts — consider only u's that admit a non-trivial pre-Cohiggs field. Since X is complete and $c(\mathscr{E}_S) = c(\Psi)$ this is a finite collection of homogeneous framed pre-Cohiggs bundles $(\mathscr{E}, \Phi_u)$. Via 2.9 every one of those defines a morphism $S \to (\mathfrak{N}_{\Psi}^{fr})_u$. We sum those morphism together and get a morphism $S \to \mathfrak{N}_{\Psi}^{fr}$, which is possible since it is a finite sum. These steps are all invertable operations, so in order to go back, starting with a morphism $S \to \mathfrak{N}_{\Psi}^{fr}$, use the projections to the coordinates to create morphisms $S \to (\mathfrak{N}_{\Psi}^{fr})_u$. Now 2.9 gives us S-families of homogeneous framed pre-Cohiggs bundles. Sum the homogeneous pre-Cohiggs fields into a single field — which is possible since this is a finite sum. We end up with a family of framed pre-Cohiggs bundles $(\mathscr{E}_S, \Phi)$. This is isomorphic to the original family since taking sums and projection are inverse operations by the universal property of fiber products. So we have a bijection between the set of morphisms $S \to (\mathfrak{N}_{\Psi}^{fr})$ and equivalence classes of S-families of framed pre-Cohiggs bundles. For every homogeneous component
see 2.9
we have already shown this bijection is compatible with pullbacks† and it is a general property that pullbacks commute with finite sums, therefore the whole bijection is compatible with pullbacks. We conclude that $\mathfrak{N}_{\Psi}^{fr}$ is a fine moduli space.

To show that the completeness condition above is necessary, we give a counterexample by considering affine toric varieties.

2.13) Example: Let X be an affine toric variety. For a given $u \in \mathsf{M}$ let $\mathscr{E}$ be the toric bundle $\mathscr{L}_{[u]}$; in 1.16 we already calculated the filtration thereof. Since the maximal cone defining X is convex, there is a $u' \in \mathsf{M}$, such that $\langle u', v_\rho \rangle << 0$ for every ray ρ in the fan Σ. Therefore the conditions in 2.3a are always satisfied by any map $\phi : \mathsf{E} \to \mathsf{E} \otimes \mathsf{N}_{\mathbb{C}}$. Hence there are non-trivial pre-Cohiggs field of degree u'. Moreover, there are infinitely many such u', for instance, any positive multiple of one thereof. That means the space of pre-Cohiggs fields on $\mathscr{L}_{[u]}$ is infinite-dimensional.

By reading the proof of [AW21, Proposition 19] one sees that it depends only on the non-convexity of Σ to bound the number of u's with non-trivial pre-Cohiggs fields. On the other hand, the example above shows the non-convexity of Σ is not only sufficient, but also a necessary condition for that. So we may expand 2.12 to the following

> 2.13A) Assertion: Let X be a toric variety, then there is a fine moduli space of framed pre-Cohiggs bundles if and only if the support of the fan Σ is not convex.

Quod erat faciendum

As the title of this section suggests, the moduli spaces we constructed are more precisely described as moduli spaces of 'pre-Cohiggs-morphisms of vector spaces with filtrations'. It is only according to Klaychko's classification that we may call them moduli spaces of pre-Cohiggs bundles. However, this problem can also be addressed in a more direct way, as we do in the next section.

Moduli Space of Toric Cohiggs Bundles

Before we can state and prove our theorems, let us expand the nomenclature.

2.14) Cohiggs Bundle: In 2.8 we defined framed pre-Cohiggs bundles and their families. We expand this definition and say such a bundle or such a family is instead *unframed* if we drop the framing from the underlying toric bundle/family. Instead of writing 'unframed', we may just drop the qualifier 'framed'. Likewise, we drop the qualifier 'pre-' if the pre-Cohiggs field satisfies the integrability condition $\Phi \wedge \Phi = 0$.

We first present an alternative proof for the moduli space of framed pre-Cohiggs bundles. This time, however, we make a direct proof without using homogeneity.

2.15) Theorem: Let X be complete. Then there is a fine moduli space for the moduli problem of framed toric pre-Cohiggs bundles. That is — fixing a collection Ψ of multisets — the functor given by

$$S \mapsto \left\{ \begin{matrix} \text{isomorphism classes of S families of framed toric} \\ \text{pre-Cohiggs bundles on X with equivariant total Chern class } c(\Psi) \end{matrix} \right\}$$

is representable by a scheme $\mathfrak{N}_\Psi^{fr}$. Furthermore there is a canonical projection $\mathfrak{N}_\Psi^{fr} \to \mathfrak{M}_\Psi^{fr}$, given by taking the underlying framed toric vector bundle, such that over an open set this projection has the structure of a vector bundle.

The proof of theorem is also key to understanding the subsequent proofs. It depends strongly on the following technical lemma.

2.16) Lemma: Let Y be a reduced scheme, X a variety and $\mathscr{E}$ a coherent sheaf over $Y \times X$, flat over Y. Let $\omega_{\mathscr{E}}$ be the functor

$$\begin{array}{rccc} \omega_{\mathscr{E}}: & \mathcal{S}ch_Y & \to & \mathcal{S}et \\ & (S \xrightarrow{f} Y) & \mapsto & \Gamma(S, (\pi_S)_*(f \times \mathrm{id}_X)^*\mathscr{E}) \end{array}$$

from the category of Y-schemes to the category of sets. There exists a Y-scheme E, representing this functor, i.e. $\omega_{\mathscr{E}}$ is isomorphic to the functor of points $h^{E\dagger}$. Furthermore the scheme E is what one calls a linear Y-scheme [Nit05, p. 16], so that a functor isomorphism exists also in the category of groups – in addition to the category of sets. As a consequence, if $(\pi_Y)_*\mathscr{E}$ is locally free of rank n, the scheme E is a vector bundle of rank n over Y. see a.4

For the proof: [Nit91, Lemma 3.5]

Before adressing the theorem, let us give an example of the lemma, to help the reader to understand how it can be used.

2.17) Example: It is a well known fact that vector bundles over an algebraic variety Y correspond to locally free sheaves. Let $\mathscr{E}$ be a locally free sheaf over Y, applying the lemma 2.16 with $X = \{*\}$ gives us exactly the vector bundle E corresponding to this sheaf. To see that notice that morphisms $\hat{f} : S \to E$ correspond to sections of f^*E, where $f : S \to Y$ is $\pi_Y \circ \hat{f}$. Also notice that

$$\Gamma(S, (\pi_S)_*(f \times \mathrm{id}_X)^*\mathscr{E}) = \Gamma(S, f^*\mathscr{E}).$$

Now replace S by a subset $U \subset Y$. In this case $f^*\mathscr{E}$ is just the sheaf $\mathscr{E}$ restricted to U. So we have that sections $U \to E$ correspond to elements of $\Gamma(U, \mathscr{E})$. In other words sections of E correspond to sections of $\mathscr{E}$, so E is the vector space corresponding to the sheaf $\mathscr{E}$.

Quod erat faciendum

2.18) LEMMA: Let $\mathscr{E}$ be a coherent sheaf over a reduced local Noetherian scheme X. For every point $x \in X$ we can calculate the dimension of the fiber $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$, where $\mathscr{E}_x$ is the stalk over x and $\mathfrak{m}_x$ the maximal ideal of X in x. If this dimension is constant, than $\mathscr{E}$ is locally free.

For the proof: We just need a local proof. So let $\mathscr{E}$ be a coherent sheaf over an affine scheme given by a Noetherian ring. This sheaf is therefore given by a finitely generated module [Uen97, prop 4.27]. As stated by [EE95, Ex 20.13], a finitely generated module over a reduced ring and with constant rank is a projective module. We however work over $\mathbb{C}$, so the ring is a polynomial ring and by the famous theorem of Quillen–Suslin this must be a free module.

2.19) PROOF OF 2.15: The moduli space of framed toric bundles $\mathfrak{M}_\Psi^{fr}$ is a fine moduli space. That means there is a universal family $\mathscr{E}$, which is essentially a vector bundle over $X \times \mathfrak{M}_\Psi^{fr}$. From that, consider the sheaf of homomorphisms

$$\mathscr{C}\!\mathit{ohig} := \mathscr{H}\!\mathit{om}(\mathscr{E}, \mathscr{E} \otimes \pi_X^* TX)$$

where $\pi_X : X \times \mathfrak{M}_\Psi^{fr} \to X$ is the standard projection. $\mathscr{E}$ and TX are coherent sheaves, and therefore, the sheaf of homomorphisms is also coherent. Furthermore, $\mathscr{E}$ is flat by definition since it is the sheaf of a family of vector bundles and the pullback π_X^* always creates a flat sheaf since the projection π_X is a flat morphism. Therefore $\mathscr{Cohig}$ is also flat over $\mathfrak{M}_\Psi^{fr}$ because the tensor product and taking duals preserve flatness. Of course as algebraic varieties X and $\mathfrak{M}_\Psi^{fr}$ are reduced. That means all conditions of 2.16 are satisfied and there is a scheme $\mathfrak{N}_\Psi^{fr}$ representing $\omega_{\mathscr{Cohig}}$, i.e. there is an isomorphism of functors between $h^{\mathfrak{N}_\Psi^{fr}}$ and the functor

$$\omega_{\mathscr{Cohig}} : (S \xrightarrow{f} \mathfrak{M}_\Psi^{fr}) \mapsto \Gamma(S, (\pi_S)_*(f \times \mathrm{id}_X)^* \mathscr{Cohig}).$$

Now we analyse what this means. Write

$$\mathscr{E}_f := (f \times \mathrm{id}_X)^* \mathscr{E}$$

and remember that $\Gamma(S, (\pi_S)_* \_\_) = \Gamma(S \times X, \_\_)$. With that rewrite the right side of the functor:

$$\begin{aligned} \Gamma(S, (\pi_S)_*(f \times \mathrm{id}_X)^* \mathscr{Cohig}) &= \Gamma(S \times X, (f \times \mathrm{id}_X)^* \mathscr{Cohig}) = \\ = \Gamma(S \times X, \mathscr{E}_f \otimes \mathscr{E}_f \otimes \pi_X^* TX) &= \mathrm{hom}(\mathscr{E}_f, \mathscr{E}_f \otimes \pi_X^* TX) \end{aligned}$$

where π_X^* is the pullback to $S \times X$. This means the set on the right is the set of pre-Cohiggs fields on $\mathscr{E}_f$. However since $\mathscr{E}$ is an universal family, any S-family of framed toric bundle is of the form $\mathscr{E}_f$ for some f.

In order to conclude that the functor $\omega_{\mathscr{Cohig}}$ is also the moduli functor of framed pre-Cohiggs bundles there is one small technical detail we need to adress: the functor $\omega_{\mathscr{Cohig}}$ takes place on the category of $\mathfrak{M}_\Psi^{fr}$-schemes, not on the category of schemes, so we have to rewrite it to:

$$S \mapsto \{S \xrightarrow{f} \mathfrak{M}_\Psi^{fr}\} \mapsto^{\omega_{\mathscr{Cohig}}} \bigcup_{f \in \{S \to \mathfrak{M}_\Psi^{fr}\}} \Gamma(S, (\pi_S)_*(f \times \mathrm{id}_X)^* \mathscr{Cohig})$$

where $\{S \xrightarrow{f} \mathfrak{M}_\Psi^{fr}\}$ is the set of all morphism $S \xrightarrow{f} \mathfrak{M}_\Psi^{fr}$. Using the same natural associations as before we may write

$$\bigcup_{f\in\{S\to\mathfrak{M}_\Psi^{fr}\}} \Gamma(S,(\pi_S)_*(f\times \mathrm{id}_X)^*\mathscr{Cohig}) = \bigcup_{f\in\{S\to\mathfrak{M}_\Psi^{fr}\}} \hom(\mathscr{E}_f,\mathscr{E}_f\otimes\pi_X^*TX).$$

That means

$$\omega_{\mathscr{Cohig}}(S) = \left\{\text{pre-Cohiggs fields on } \mathscr{E}_f \text{ for some } f : S \to\to \mathfrak{M}_\Psi^{fr}\right\}.$$

Now families of toric pre-Cohiggs bundles are just families of toric bundles with a pre-Cohiggs field, so rewriten in the category of schemes, the functor $\omega_{\mathscr{Cohig}}$ is isomorphic to

$$S \mapsto \left\{\begin{matrix}\text{isomorphism classes of S families of toric pre-Cohiggs}\\ \text{bundles on X with equivariant total Chern class } c(\Psi)\end{matrix}\right\}.$$

Since X is complete and properness is stable under base change the projection $\pi_{\mathfrak{M}_\Psi^{fr}}$ is a proper morphism. So the pushforward $(\pi_{\mathfrak{M}_\Psi^{fr}})_*\mathscr{Cohig}$ is a coherent sheaf over $\mathfrak{M}_\Psi^{fr}$. Since $\mathfrak{M}_\Psi^{fr}$ is noetherian we conclude[†] the rank of $(\pi_{\mathfrak{M}_\Psi^{fr}})_*\mathscr{Cohig}$ is upper semi-continuous. It is also minimal and constant in an open set of $\mathfrak{M}_\Psi^{fr}$. Therefore $(\pi_{\mathfrak{M}_\Psi^{fr}})_*\mathscr{Cohig}$ is locally free[†] and the linear scheme $\mathfrak{N}_\Psi^{fr}$ is a vector bundle over this open set.

[Har10, Theorem III 12.7.2]

see 2.18

To prove this consider a inclusion $U \overset{f}{\hookrightarrow} \mathfrak{M}_\Psi^{fr}$. Then $\Gamma(U, f^*\_\_) = \Gamma(U,\_\_)$ and $f\times \mathrm{id}_X$ is also the inclusion $U\times X \hookrightarrow \mathfrak{M}_\Psi^{fr}\times X$, so we may calculate

$$\begin{aligned}\omega_{\mathscr{Cohig}}(f) &= \Gamma(U,(\pi_U)_*(f\times\mathrm{id}_X)^*\mathscr{Cohig}) = \\ &= \Gamma(U\times X,(f\times\mathrm{id}_X)^*\mathscr{Cohig}) = \Gamma(U\times X,\mathscr{Cohig}) = \\ &= \Gamma(U,(\pi_{\mathfrak{M}_\Psi^{fr}})_*\mathscr{Cohig})\end{aligned}$$

On the other hand, for some open set $U \overset{f}{\hookrightarrow} \mathfrak{M}_\Psi^{fr}$ the set $h^{\mathfrak{N}_\Psi^{fr}}(f)$ is by definition the set of morphisms $U \to \mathfrak{N}_\Psi^{fr}$ over f, which is the same as the set of sections over U. Since $\omega_{\mathscr{C}ohig}$ is isomorphic to $h^{\mathfrak{N}_\Psi^{fr}}$ we conclude

$$\Gamma(U, \mathfrak{N}_\Psi^{fr}) = h^{\mathfrak{N}_\Psi^{fr}}(f) = \omega_{\mathscr{C}ohig}(f) = \Gamma(U, (\pi_{\mathfrak{M}_\Psi^{fr}})_* \mathscr{C}ohig)$$

so $(\pi_{\mathfrak{M}_\Psi^{fr}})_* \mathscr{C}ohig$ is the sheaf of sections of $\mathfrak{N}_\Psi^{fr}$. This completes the proof. *Quod erat demonstrandum*

Similarly to the case of unframed toric bundles, a coarse moduli space exists if and only if it is a quotient of the framed case:

2.20) Corollary: Let X be complete. A scheme $\mathfrak{N}_\Psi$ is a coarse moduli space for the moduli functor of unframed toric pre-Cohiggs bundles — defined in the obvious way, by removing the frame from 2.15 — if and only if it is a good quotient of $\mathfrak{N}_\Psi^{fr}$ by the action of $\mathsf{GL}_r(\mathbb{C})$ and its set of points is bijective to the set of $\mathsf{GL}_r(\mathbb{C})$-Orbits of $\mathfrak{N}_\Psi^{fr}$.

For the proof: This follows the same argument as 1.47. Just as previously, any two frames can be mapped into one another by the action of $\mathsf{GL}_r(\mathbb{C})$. Therefore the orbits of this action on $\mathfrak{N}_\Psi^{fr}$ are in a 1-to-1 correspondence with the equivalence classes of unframed toric pre-Cohiggs bundles. The rest of the proof is the same, as it follows from general arguments involving quotients and fine moduli spaces, which are independent of the underlying functor.

We should now also verify that the decomposition found in 2.12 still holds.

2.21) Proposition: Let X be complete. For every $u \in \mathsf{M}$, there exists a fine moduli space $(\mathfrak{N}_{\Psi}^{fr})_u$ of framed toric pre-Cohiggs bundles of degree u, which are linear schemes over $\mathfrak{M}_{\Psi}^{fr}$ and such that there is a decomposition

$$\mathfrak{N}_{\Psi}^{fr} = \bigoplus_{u \in \mathsf{M}} (\mathfrak{N}_{\Psi}^{fr})_u.$$

Here, the direct sum here is defined by the fiber product over $\mathfrak{M}_{\Psi}^{fr}$, similar to the direct sum of vector bundles.

Likewise, if there exists a moduli space of unframed toric pre-Cohiggs bundles, there is also a decomposition

$$\mathfrak{N}_{\Psi} = \bigoplus_{u \in \mathsf{M}} (\mathfrak{N}_{\Psi})_u$$

where $(\mathfrak{N}_{\Psi})_u$ is a quotient of $(\mathfrak{N}_{\Psi}^{fr})_u$ by the $\mathsf{GL}_r(\mathbb{C})$ action.

2.22) Proof of 2.21: There is a natural action of T in the sheaf $\mathscr{H}om(\mathscr{E}, \mathscr{E} \otimes \pi_X^* TX)$, where $\mathscr{E}$ is the universal family of framed toric vector bundles. Therefore this is a graded sheaf and there is a decomposition [Dem70, p. i.4.7.3]

$$\mathscr{H}om(\mathscr{E}, \mathscr{E} \otimes \pi_X^* TX) = \bigoplus_{u \in \mathsf{M}} \mathscr{H}om(\mathscr{E}, \mathscr{E} \otimes \pi_X^* TX)_u.$$

Set $(\mathfrak{N}_{\Psi}^{fr})_u$ to be the linear scheme representing every component. In a way similar to 2.15 those are moduli space of the homogeneous problem. Since X is complete only finitely many u's give rise to non-trivial components, so we can replace $\bigoplus_{u \in \mathsf{M}}$ by a finite sum. This sum moves smoothly into a sum of functors $\omega_{\mathscr{C}ohig} = \oplus \omega_{\mathscr{C}ohig_u}$, where

$\omega_{\mathscr{Cohig}}$ is the same functor as in 2.19. Here $\oplus$ means the sum in the category of sets, that is, the cartesian product. That is to say $\mathfrak{N}_\Psi^{fr}$ is the linear scheme representing the functor

$$\omega_{\mathscr{Cohig}} : (S \xrightarrow{f} \mathfrak{M}_\Psi^{fr}) \mapsto \bigtimes_{u \in \mathsf{M}} \Gamma(S, (\pi_S)_*(f \times \mathrm{id}_X)^* \mathscr{Cohig}_u).$$

This is the same decomposition one sees for the functor of points of a sum of $\mathfrak{M}_\Psi^{fr}$-schemes

$$h^{\mathfrak{N}_\Psi^{fr}} : (S \xrightarrow{f} \mathfrak{M}_\Psi^{fr}) \mapsto \bigtimes_{u \in \mathsf{M}} \{S \to (\mathfrak{N}_\Psi^{fr})_u\}$$

which is just a long way of saying there is also a decomposition

$$\mathfrak{N}_\Psi^{fr} = \bigoplus_{u \in \mathsf{M}} (\mathfrak{N}_\Psi^{fr})_u.$$

Since the degree of a pre-Cohiggs field is a discrete invariant and depends only on the field, the factors are also fine moduli spaces — just rewrite 2.19 for homogeneous pre-cohiggs bundles. The degree is also invariant under the $\mathsf{GL}_r(\mathbb{C})$ action, and therefore if there is a quotient as before†, it also decomposes into a direct sum of coarse moduli spaces see 2.20

$$\mathfrak{N}_\Psi = \bigoplus_{u \in \mathsf{M}} (\mathfrak{N}_\Psi)_u.$$

Quod erat demonstrandum

The next moduli space we are interested in are proper Cohiggs bundles, that is when the field satisfy the integrability condition.

2.23) Proposition: There is a subscheme $\mathfrak{H}_\Psi^{fr} \subset \mathfrak{N}_\Psi^{fr}$, which is a fine moduli space of framed toric Cohiggs bundles. The $\mathsf{GL}_r(\mathbb{C})$-quotient thereof — if it exists — is a coarse moduli space of unframed toric Cohiggs bundles.

For the proof: Let $\mathscr{E}$ be the universal family of toric bundles over $\mathfrak{M}^{fr}_{\Psi}$. There is a morphism of coherent sheafs

$$\_\wedge\_ : \begin{array}{ccc} (\mathscr{H}om(\mathscr{E},\mathscr{E}\otimes\pi_X^*TX))^{\otimes 2} & \to & \mathscr{H}om(\mathscr{E},\mathscr{E}\otimes\pi_X^*TX^{\wedge 2}) \\ \Phi\otimes\Phi' & \to & \Phi\wedge\Phi' \end{array}$$

This defines a scheme morphism between $\mathfrak{N}^{fr}_{\Psi}\otimes_{\mathfrak{M}^{fr}_{\Psi}}\mathfrak{N}^{fr}_{\Psi}$ and the scheme representing the coherent sheaf $\mathscr{H}om(\mathscr{E},\mathscr{E}\otimes\pi_X^*TX^{\wedge 2})$. The moduli space of framed toric Cohiggs bundles is the intersection of the diagonal $\Phi=\Phi'$ and the pre-image of the zero section of the second scheme. The subscheme we get from this corresponds to the points with $\Phi\wedge\Phi=0$. It is straightforward to show this is also a fine moduli space: fix Ψ and let $\mathscr{N}$ be the moduli functor of framed pre-Cohiggs bundles:

$$\mathscr{N}: S\mapsto\left\{\begin{array}{c}\text{isomorphism classes of S families of framed toric}\\ \text{pre-Cohiggs bundles on X with equivariant total Chern class } c(\Psi)\end{array}\right\}$$

and consider also the functor of points of $\mathfrak{N}^{fr}_{\Psi}$

$$h^{\mathfrak{N}^{fr}_{\Psi}}: S\mapsto\{S\to\mathfrak{N}^{fr}_{\Psi}\}.$$

These two functors are isomorphic according to 2.15, i.e. there exists an isomorphism of functors ν. Now let $\mathfrak{H}^{fr}_{\Psi}$ be the subscheme of $\mathfrak{N}^{fr}_{\Psi}$ we described above. By the process explained in the proof of 2.15 any map $S\to\mathfrak{H}^{fr}_{\Psi}$ corresponds to a family of pre-Cohiggs bundles, which in this case also satisfies $\Phi\wedge\Phi=0$. Therefore, it is a family of Cohiggs bundles. In the other direction any family of Cohiggs bundles corresponds to a morphism $S\to\mathfrak{H}^{fr}_{\Psi}\subset\mathfrak{N}^{fr}_{\Psi}$. This means the bijection ν_S restricts to a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes of S families of framed toric}\\ \text{Cohiggs bundles on X with equivariant total Chern class } c(\Psi)\end{array}\right\}\leftrightarrow\{S\to\mathfrak{H}^{fr}_{\Psi}\}.$$

So the isomorphism of the functors above restricts to an isomorphism between

$$S \mapsto \left\{ \begin{matrix} \text{isomorphism classes of S families of framed toric} \\ \text{Cohiggs bundles on X with equivariant total Chern class } c(\Psi) \end{matrix} \right\}$$

and the functor of points

$$h^{\mathfrak{H}_\Psi^{fr}} : S \mapsto \{S \to \mathfrak{H}_\Psi^{fr}\}.$$

The condition $\Phi \wedge \Phi$ is independent of the framing. Therefore the second part of the proposition follows from the same arguments used before†.

see 2.20

THIRD CHAPTER

DISCRETE INVARIANTS

Where we take a further look at stable toric bundles and analyse how pre-Cohiggs fields vary on the moduli space.

In the previous chapter, we proved the existence of a moduli space of framed toric (pre-)Cohiggs bundles. Now we are interested in describing those spaces, specifically how the space of pre-Cohiggs fields varies in the moduli space. For that Altmann and Witt† have defined some invariants which we desire to analyse further: as we have seen before, the pre-Cohiggs fields of a given degree form a finite-dimensional vector space. The dimension of this space is the first discrete invariant, called multiplicity of thr degree. The second invariant is obtained by considering which degrees admit non-trivial pre-Cohiggs fields. This is called the Higgs range†. We already encountered this indirectly when we noticed the moduli space of framed pre-Cohiggs bundles may not exist†. We need some condition on X, for instance, complete-

[AW21]

see 3.2

see 2.13

see 2.12 *ness, so we get a bounded moduli problem[†]. This corresponds to a bounded Higgs range — otherwise, the moduli space would be infinite-dimensional. The last invariant comes from considering the extremal degrees which are 'present' in a given pre-Cohiggs field. This is called* see 3.1 *the Higgs polytope[†], which is necessarily a sub-polytope of the Higgs range.*

From our construction of the moduli space we can extract some properties of those invariants previously unknown to Altmann and Witt. For instance, the conditions for the existence of a pre-Cohiggs field are see 2.3a *kernel conditions[†]. This means the multiplicity of a degree varies as* see 3.4 *an upper-semi-continuous function[†]. Moreover, the Higgs range varies* see 3.5 *in the same way, and therefore it is minimal in a Zariski open set[†]. On the other hand, the presence of a given degree in a pre-Cohiggs field* see 3.6 *is an open condition[†]. So the Higgs polytope is maximal in an open* see 3.7 *set[†]. At the same time, GIT theory provides us with an open subset of stable toric bundles, so that a moduli space of unframed toric bundles exists. The hope would be that those sets coincide.*

In their paper Altmann and Witt calculated the Higgs range and corresponding multiplicities for the Fano surfaces — to be precise for the toric bundle defined by the tangent space of the Fano surfaces. While calculating example 1.52 we noticed the tangent bundle is not necessarily a stable point in the moduli space. We then recalculated the Higgs range for the corresponding generic points. While doing so we noticed that the existence of pre-Cohiggs fields is essentially restricted see 2.3a *by the flags[†]. Whenever flags coincide the restriction weakens — just one component of the field became restricted and the other was free. We also found out that this condition — namely flags not coinciding — can in some cases be used to identify the generic points in the Chow quotient3.13. So we may observe that for the Fano surfaces in the work of Altmann and Witt the Higgs range of stable bundles is constant.*

This is an important finding since this is a precondition for the moduli space of pre-Cohiggs bundles to be separated. However, it is not true in general. Finding a counterexample is tricky and somewhat artificial. We present three of those in increasing order of complexity. The first† *shows that the Higgs multiplicity is not constant over stable bundles. The second*† *shows that the moduli space of stable Cohiggs bundles is not separated. And finally, the third*† *shows a case of stable bundles with non-constant Higgs range. For the reader who do not wish to read through the examples, we explain what is happening in an analogous circumstance. The conditions in 2.3a describe when a Higgs field is toric and always look like* $f(V) \subset W$ *for some linear spaces* V *and* W*. Consider also* $f : \mathbb{C}^2 \to \mathbb{C}^2$ *the condition* $f([1:0]) \subset [1:0]$, $f([0:1]) \subset [0:1]$ *and* $f([1:1]) \subset [1:2]$*. Now consider a fourth condition* $f([3:2]) \subset [3:\alpha]$ *depending on a parameter* α*. Normally there is no non-trivial solution for these four equations. However for* $\alpha = 4$ *a numerical coincidence occurs:* $[3/3 : 2/\alpha] = [1/1 : 1/2]$*. As a result, there are non-zero solutions. All spaces continue to be distinct, but this does not prevent them from creating linearly dependent conditions. This is because in the end the conditions involve the product* $V^* \otimes W$*, so they are by nature non-linear.*

see 3.15
see 3.16
see 3.17

There are two lessons to be taken from this chapter. First, the moduli space of toric Cohiggs bundles tends to be not separated, and there does not seem to be a good way of preventing that. So a general theory must be made in the language of schemes and sheaves — varieties are not enough. Second, when analyzing the stability of toric pre-Cohiggs bundles one must not only consider the underlying toric bundles. In the classical theory the Higgs field is a stabilizing force the toric world pre-Cohiggs fields are more like a destabilizing force. A third possibility would be to further analyze toric Fano varieties.

Combinatorial Invariants

First, we define the invariants proposed by Altmann and Witt, which we want to explore in this section.

3.1) Definition: Let Φ be a pre-Cohiggs field. We define the *support* of Φ to be the set of degrees where it has a non-zero component

$$\text{supp } \Phi = \{u \in \mathsf{M} : \Phi_u \neq 0\}.$$

The *Higgs polytope* is the convex hull of the support

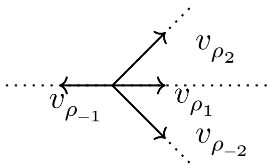
$$\nabla\Phi = \text{conv supp } \Phi$$

3.2) Definition: For a toric bundle $\mathscr{E}$ let $V_u(\mathscr{E})$ be the subspace of $\text{hom}(\mathsf{E}, \mathsf{E} \otimes \mathsf{N}_{\mathbb{C}})$ satisfying 2.3a, that is, the space of $\mathbb{C}$-linear maps defining a pre-Cohiggs field of degree u. The dimension of $V_u(\mathscr{E})$ is the *Higgs multiplicity* of a point $u \in M$. Admissible points are the ones with multiplicity greater than zero and the convex hull of these points is the *Higgs range*, denoted by

$$H(\mathscr{E}) := \text{conv } \{u \in \mathsf{M} : V_u(\mathscr{E}) \neq 0\}.$$

The first question that comes to mind is whether these invariants are constant in a moduli space. We can clearly deny that, as shown below.

3.3) Example: The following example shows that the Higgs range is not constant in general. Let Σ be the complete fan defined by the primitive vectors $(1, 0)$, $(-1, 0)$, $(1, 1)$ and $(1, -1)$ in $\mathbb{R}^2$ and X be the
see b.7 smooth[†] toric variety defined by this fan.

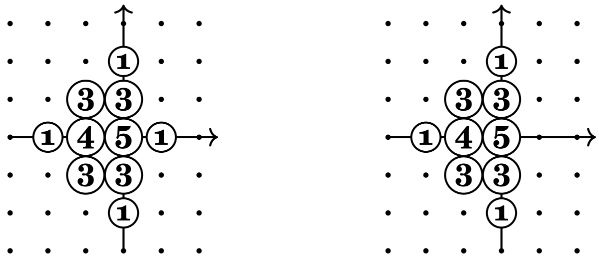


We consider a family of rank 2 bundles with the following filtrations, where z varies over $\mathbb{C} =: \mathbb{A}^1$:

$$\begin{array}{lcccccccc} & & i=0 & & i=1 & & i=2 & \\ \mathsf{E}^{\rho_1}: & \cdots \supset & \mathbb{C}^2 & \supset & 0 & \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_{-1}}: & \cdots \supset & \mathbb{C}^2 & \supset & \langle (0,1) \rangle & \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_2}: & \cdots \supset & \mathbb{C}^2 & \supset & \langle (1,0) \rangle & \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_{-2}}: & \cdots \supset & \mathbb{C}^2 & \supset & \langle (1,z) \rangle & \supset & 0 & \supset \cdots \end{array}$$

Let Ψ be the collection of multisets defined by the filtrations above with $z = 0$. The moduli space $\mathfrak{M}^{fr}_{\Psi}$ is analogous to example 1.52. That means $\mathfrak{M}^{fr}_{\Psi}$ is given by 3 points in $\mathbb{P}^1$. The compatibility condition requires that the points correspondig to the rays ρ_{-1} and ρ_2 — i.e $[0 : 1]$ and $[1 : 0]$ — do not coincide, which is the case. The same is also valid for ρ_{-1} and ρ_{-2} — i.e $[0 : 1] \neq [1 : z]$ — which is also always the case. For ρ_1 there is no compatibility condition, since the filtration E^{ρ_1} is trivial. And since the rays ρ_2 and ρ_{-2} do not have a common cone, there is no compatibility condition between the points $[1 : 0]$ and $[1 : z]$.

The (framed) bundle corresponding to the filtrations above is then identified by 3 points in the projective space: $R_2 = [1 : 0]$, $R_{-1} = [0 : 1]$ and $R_{-2} = [1 : z]$. This defines a curve in $\mathfrak{M}^{fr}_{\Psi}$, that is a family of framed toric vector bundles. Applying 2.4 we may calculate the Higgs range. [AW21] provides a simplified way to proceed in this calculations, however on such small dimensions brute force verification of the conditions in 2.3a is still possible by hand. The diagram below shows the dimension of $V_u(\mathscr{E})$ for all $u \in \mathsf{N}$, depending on the whether $z = 0$ or $z \neq 0$ — no number implies dimension 0.



For our case the most relevant degree is $u = (1,0)$. Therefore we calculate $V_u(\mathscr{E})$ explicitly. First of all, ρ_{-1} is not relevant in this case, since $\langle u, v_{\rho_{-1}}\rangle < 0$ and the filtration has length 1. Now since $\langle u, v_{\rho_1}\rangle = 1$ and the filtration E^{ρ_1} is trivial, we conclude that $\phi_{(0,1)} = 0$. On the other hand, there is no restriction on $\phi_{(1,0)}$ because $\langle u, v_{\rho_1}\rangle - 1 = 0$. Now take ρ_2, applying 2.4 gives

$$\phi_{(1,-1)}(\mathbb{C}^2) \subset \langle (1,0)\rangle \text{ and } \phi_{(1,-1)}(\langle (1,0)\rangle) = 0.$$

That means $\phi_{(1,-1)}$ is nilpotent with kernel $\langle (1,0)\rangle$. However $\phi_{(1,-1)} = \phi_{(1,0)}$ since $\phi_{(0,1)} = 0$. So $\phi_{(1,0)}$ is nilpotent with kernel $\langle (1,0)\rangle$. The same calculation with ρ_{-2} gives kernel $\langle (1,z)\rangle$. For $z \neq 0$ both those condition together implies $\phi_{(1,0)} = 0$ and therefore there is no non-trivial pre-Cohiggs field with degree $(1,0)$. For $z = 0$ on the other hand:

$$V_{(1,0)}\mathscr{E}|_{z=0} = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

Quod erat faciendum

If the Higgs range is not constant, then we should try to understand how it varies. The following observation will provide an initial answer to this question.

3.4) Proposition: Let $\mathscr{E}$ be a family of (framed) toric bundles over a scheme S and $u \in \mathsf{M}$ be a degree, then the dimension $\dim V_u(\mathscr{E}|_s)$ of the vector space of u-homogeneous pre-Cohiggs fields on $\mathscr{E}|_s$ is an upper-semi-continuous function of $s \in S$. In particular, there is a Zariski open set of S, such that the dimension is minimal.

For the proof: This is essentially already implicit in 2.10, however, let us make this more explicit. As a geometric object, $\mathscr{E}$ is just a vector bundle over $S \times X$. Restricting it to $S \times \{x_0\}$ gives us a vector bundle E over S. By the classification theorem†, homogeneous pre-Cohiggs fields over $\mathscr{E}$ are given by morphisms of vector bundles $\phi : \mathsf{E} \to \mathsf{E} \otimes (\mathsf{N}_\mathbb{C})_S$, where $(\mathsf{N}_\mathbb{C})_S$ is just the trivial bundle $\mathsf{N}_\mathbb{C} \times S$. This morphism must satisfy some condition, which, as we saw in 2.10 can be written in terms of a kernel. The dimension of a kernel is semi-upper-continuous in the Zariski topology†, therefore the statement of the proposition follows.

see 2.3a

see d.1

Now we apply this proposition to the Higgs range.

3.5) Corollary: Let X be a complete toric variety, S an irreducible scheme and $\mathscr{E}$ an S-family of (framed) toric bundles over X, then there is a nonempty Zariski open set of S such that the Higgs range is minimal, meaning it is contained as a polytope in the Higgs range of $\mathscr{E}_s$ for any $s \in S$.

For the proof: Since X is complete, we already know that the Higgs range is bounded. Furthermore, in 2.11 we found a concrete bound. For every degree $u \in \mathsf{M}$ inside that bound, pick the open set such that $\dim V_u(\mathscr{E}|_s)$ is minimal. The intersection of all these sets is the desired Zariski open set and it is not empty because S is irreducible.

We can do this same analysis for the Higgs polytope, as follows.

3.6) Proposition: Let $(\mathscr{E}, \Phi)$ be an S-family of pre-Cohiggs bundles over a toric variety X, S irreducible and a fixed degree $u \in \mathsf{M}$. Then the set

$$\{s \in S : \quad (\Phi_u)|_s \neq 0\}$$

is a Zariski open — possibly empty — set of S.

For the proof: Just as in 3.4, a pre-Cohiggs field of degree u is equivalent to a morphism $\phi : \mathsf{E} \to \mathsf{E} \otimes (\mathsf{N}_{\mathbb{C}})_S$ of the vector bundle E over S. The condition $(\Phi_u)|_s \neq 0$ is the same as $\dim \ker(\phi|_s) < \dim \mathsf{E}$, which ends the proof since the dimension of a kernel is upper-semi-
see d.1 continuous†.

3.7) Corollary: Let S be irreducible and $(\mathscr{E}, \Phi)$ be an S-family of pre-Cohiggs bundles over a complete toric variety X. Then there is a nonempty Zariski open set of S, such that the Higgs polytope $\nabla(\Phi|_s)$ is maximal, meaning it contains the Higgs polytope of any other $s \in S$.

see 2.11 *For the proof:* Since X is complete, the Higgs range is bounded†. For every u in the Higgs range, pick either the open set

$$\{s \in S : \dim \ker(\Phi_u)|_s < \dim \mathsf{E}\}$$

or the whole S if that is empty. The intersection of these sets gives the desired open set.

3.8) Observation: Corollaries 3.5 and 3.7 are probably valid also for incomplete varieties, but one needs more complex arguments to limit the number of open sets in the intersection.

3.9) Corollary: Let X be a complete variety, then if the moduli space of pre-Cohiggs bundles $\mathfrak{N}_{\Psi}^{fr}$ is irreducible, the Higgs range is constant.

For the proof: There is a projection $\mathfrak{N}_{\Psi}^{fr} \to \mathfrak{M}_{\Psi}^{fr}$, so if $\mathfrak{N}_{\Psi}^{fr}$ is irreducible than $\mathfrak{M}_{\Psi}^{fr}$ is also irreducible. So we can apply 3.7 and 3.5 to the universal families of this spaces. Meaning there are non empty open sets of $\mathfrak{N}_{\Psi}^{fr}$ such that the Higgs polytope is maximal and the underlying Higgs range is minimal. If the moduli space is irreducible, the intersection of those two sets is also not empty. So there are points where the Higgs range is minimal and the Higgs polytope is maximal. However, over a complete variety for every Higgs range we can construct a pre-Cohiggs field so that the the Higgs polytope and the Higgs range coincide†. Since the Higgs range always contains the Higgs polytope, we conclude the Higgs range cannot be bigger than the minimal Higgs range, otherwise the corresponding Higgs polytope would be bigger than the maximal one. Therefore it must be constant.

see 2.7

3.10) Observation: Knowing if the moduli space $\mathfrak{N}_{\Psi}^{fr}$ is irreducible is probably not easy while calculating the Higgs range is much more feasible. So a pratical version of the corollary is: if the Higgs range is not constant, then the moduli space is not irreducible.

Projective Configurations

see 3.3
see 3.10
see 1.53

We have already shown in an example† that the Higgs range is not constant in general and therefore the moduli space is not irreducible†. This is not particularly surprising since we are considering all kinds of toric bundles. One could hope, however, that the Higgs range is constant for well-behaved bundles. Hier, the concept of stable bundles comes to mind. Unfortunately, another example† already showed this concept is not intrinsic to the bundle and depends strongly on the choice of a linearisation. The alternative is to consider generic bundles — in the sense of the Chow quotient. For one special case, this can be reduced to a quotient known in the literature as moduli of projective configurations. It gives rise to the following concept:

3.11) Linearly generic: Let $(p_1, \cdots, p_k)$ be a collection of points in $\mathbb{C}^{N+1}$. We say that these points are *linearly generic* if any subset $(\{p_{i_1}, \cdots, p_{i_m}\})$ spans the largest possible space. In other words any such subset is linearly independent if $m \leq N+1$. Obviously this condition is invariant under scalar multiplication, therefore we use the same definition for collections of k points in $\mathbb{P}^N$. This is equivalent to Kapranovs notion of 'in general position' used in [Kap93, Proposition 2.1.4].

3.12) Example: The following collection of points in $\mathbb{C}^3$ is not linearly generic

$$(1,0,0)$$
$$(0,1,0)$$
$$(0,0,1)$$
$$(1,1,1)$$
$$(1,1,2)$$

because the last three vectors are linearly dependent. However, replacing the last vector by $(-1,1,2)$ gives a linearly generic set.

Quod erat faciendum

3.13) THEOREM: Consider collections of multisets Ψ such that the flags are all given by one-dimensional subspaces, that is, $0 \subsetneq \mathbb{C} \subsetneq \mathsf{E}$ — for instance for rank 2 this is always the case if we ignore the trivial filtrations $0 \subsetneq \mathsf{E}$. Let $\mathfrak{M}_{\Psi}^{fr^{gen}}$ be the submoduli space given by linearly generic flags — that is, the flags define a linearly generic collection of points in $\mathbb{P}^{r-1}$. Then the set $\mathfrak{M}_{\Psi}^{fr^{gen}}$ satisfies the property in a.22, meaning it is in the generic part of the Chow quotient†. Therefore the geometric quotient $\mathfrak{M}_{\Psi}^{fr^{gen}}/\mathsf{GL}_r(\mathbb{C})$ exists and is an open subset of the Chow quotient $\mathfrak{M}_{\Psi}^{fr}//^{\mathrm{ch}}\mathsf{GL}_r(\mathbb{C})$. see a.22

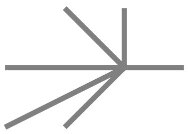
For the proof: Under the conditions of the theorem, the moduli space of unframed toric pre-Cohiggs bundles corresponds to the moduli space of projective configuration. The Chow quotient of this problem is already known and the generic part corresponds to our definition of linearly generic. We present in d.5 a complete proof based on [Kap93, Proposition 2.1.7].

3.14) OBSERVATION: Considering how the moduli space of toric bundles is composed of a product of Grassmannians, it would be extremely interesting to have a version of the previous theorem for Grassmannians instead of projective spaces. One could try to expand the concept of linearly generic† for subvector spaces of higher dimension. One may eventually demand not only for the span to be maximal but also the dimension of intersections to be minimal — after all, those are distinct conditions†. The homology of the Grassman- see 3.11 see d.3

nian can be described by Schubert cycles, which involve constraining the dimension of the inversection of a vector space with some gene-
[GH14, Ch. 1.5] ric total flag†. However, constraining the dimension is equivalent to constraining the rank of a matrix. For rank greater than 1 this is a non-linear problem, so generalizing the above proof is not trivial. As far as we know this is still an open problem.

Generic Bundles

Now that 3.13 gives us a characterization of generic bundles, we may start checking how the Higgs range behaves in relation to them. For see 3.18 *simple examples, it actually behaves quite well†, so that one could hope to find a correspondence between the generic bundles and the open set described in 3.4. However more elaborated and somewhat artificial examples show this is not the case. Begin by considering the following.*



3.15) Example: Consider the 2-dimensional fan spanned by the following primitive generators: $\rho_1 = (0,1)$, $\rho_2 = (1,0)$, $\rho_3 = (-1,-1)$, $\rho_4 = (-1,1)$, $\rho_5 = (-2,-1)$ and $\rho_6 = (-1,0)$. Now consider the following filtrations, for the rays 1 to 6, respectively:

$$\begin{array}{lccccc} & i=0 & i=1 & i=2 & \\ \mathsf{E}^{\rho_1}: \cdots \supset & \mathbb{C}^2 \supset & [\ 1: \ \ 0] \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_2}: \cdots \supset & \mathbb{C}^2 \supset & [\ 0: \ \ 1] \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_3}: \cdots \supset & \mathbb{C}^2 \supset & [\ 1: -1] \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_4}: \cdots \supset & \mathbb{C}^2 \supset & [\ 1: \ \ 1] \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_5}: \cdots \supset & \mathbb{C}^2 \supset & [\ 1: -2] \supset & 0 & \supset \cdots \\ \mathsf{E}^{\rho_6}: \cdots \supset & \mathbb{C}^2 \supset & [\ 1: \sqrt{2}] \supset & 0 & \supset \cdots \end{array}$$

We want to show that the toric bundles defined by this data is stable. However, the Higgs multiplicity at the origin — i.e. $\dim V_{(0,0)}\mathscr{E}$ — is 3, while the generic multiplicity at the origin is 2. This means the Higgs multiplicity is not constant on the locus of generic bundles.

Let the flags of this bundle be generated by the following vectors $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1,-1)$, $v_4 = (1,1)$, $v_5 = (1,-2)$ and $v_6 = (1,\sqrt{2})$. We are interested in calculating what is the dimension of the space of pre-Cohiggs fields of degree 0. This is a straightforward application of 2.3a. We nonetheless present the computations here for better understanding.

Since $\langle u, v_\rho \rangle$ is always zero, for every ray there is just one relevant constraint — namely $\phi_{s_i}(v_i) \subset \operatorname{span}\{v_i\}$ where s_i is orthogonal to $\rho_i{}^\dagger$. In this case this condition can be better expressed as $\phi_{s_i}(v_i) \cdot \hat{v}_i = 0$, where $\hat{v}_i$ is orthogonal to v_i — for the sake of notation fix $\hat{v}_i$ to be the 90° clockwise rotation of v_i. We can represent this constraint as linear equations on the entries of a matrix representing ϕ_{s_i}. The coefficients of this equation are given by the matrix product $\hat{v}_i{}^T v_i{}^\dagger$. This means that in the example of the first filtration the condition

$$\phi_{s_i}(1,0) \subset \langle (1,0) \rangle$$

see 2.4

from simple linear algebra if y and x are line vectors and A is a square matrix then $yAx^T = 0$ if and only if $A \cdot y^T x = 0$, where the dot $\cdot$ is the elementwise product of matrixes is equivalent to

$$\phi_{s_i} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \phi_{s_i} \cdot \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = 0.$$

That means we know how to write the constraints of ϕ_{s_i}. In order to compare those we need a common coordinate system, meaning we need to remove the dependency on s_i. Use that $\mathsf{M} = \mathsf{N} = \mathbb{Z}^2$ and pick $s_i = \hat{\rho}_i$ — all other relevant s_i are a multiple of this one and yield the same restriction. Using the formula

$$\phi = \sum_s \phi_s \otimes s^*$$

a simple change of basis yields $\phi_s = a\phi_{(1,0)} + b\phi_{(0,1)}$, where $s = (a, b)$. That means we can represent the restriction $\phi_{s_i}(v_i) \subset \text{span}\{v_i\}$ on the matrixes $\phi_{(1,0)}$ and $\phi_{(0,1)}$ by multiplying the matrix $\hat{v_i}^T v_i$ by the coefficient of s_i. For the first filtration $\rho_1 = (0, 1)$, so we choose the orthogonal degree $s_i = (1, 0)$ and the equation looks like the following

$$\phi_{s_i} \cdot \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \phi_{(1,0)} \cdot \left[1 \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] + \phi_{(1,0)} \cdot \left[0 \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] = 0$$

Likewise every ray gives us a pair of matrixes as the coefficients of a linear system on $(\phi_{(1,0)}, \phi_{(0,1)})$. The coefficients we get for the filtrations 1 to 6 are as follows:

$$C_1 : \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C_2 : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$C_3 : \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$C_4 : \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$C_5 : \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -4 & 8 \\ -2 & 4 \end{pmatrix}$$

$$C_6 : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} & 2 \\ -1 & -\sqrt{2} \end{pmatrix}$$

Notice that C_1 to C_4 are linearly independent. However, $C_5 = C_4 - 3C_3 - 6C_2 + 3C_1$ and C_6 is independent from the other because it is the only one containing $\sqrt{2}$. So we have a system of 8 variables and 5 equations. We conclude the space of pre-Cohiggs fields of degree zero has dimension 3. This is but a coincidence using that $s_5 = v_5$. A small variation on v_5 makes the system of 6 equations linearly independent. That is to say, generically in the moduli space containing this bundle the space of pre-Cohiggs fields of degree zero has dimension 2.

We can easily calculate the moduli space of toric bundles. We proceed in a way similar to 1.52. For the framed case $\mathfrak{M}_\Psi^{fr}$ is a subset of $((\mathbb{P}^1)^6)$ — since flags are given by one-dimensional subsets of $\mathbb{C}^2$. The compatibility condition just states that neighboring flags shall not coincide — which is the case in our example. For the unframed case, we must calculate the quotient by a change of basis. We can fix two of the flags to be the coordinate axis. Of course, we can still scale the axis, which is a linear action on $(\mathbb{P}^1)^4$:

$$\begin{aligned} \lambda \cdot ([x_1 : y_1], [x_2 : y_2], [x_3 : y_3], [x_4 : y_4]) = \\ = ([\lambda x_1 : y_1], [\lambda x_2 : y_2], [\lambda x_3 : y_3], [\lambda x_4 : y_4]) \end{aligned} .$$

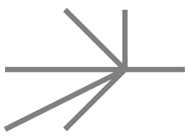
That means the unframed moduli space $\mathfrak{M}_\Psi$ is a subspace of the quotient of $(\mathbb{P}^1)^4$ by $\mathbb{C}^*$. To simplify the calculation consider the case that the third, fourth, fifth and sixth flags do not coincide with the coordinate axis. This is the subset $(\mathbb{C}^*)^4 \subset (\mathbb{P}^1)^6$ and we can parametrise the flags by its inclination in relation to the coordinate axis.

The quotient $(\mathbb{C}^*)^4/\mathbb{C}^*$ is isomorphic to $(\mathbb{C}^*)^3$. This is a geometric quotient, meaning those points are stable for an adimissable GIT quotient. The bundle defined by the filtrations above fits this criteria and is stable.

That shows the multiplicity $\dim V_u(\mathscr{E})$ is not constant even over stable points. *Quod erat faciendum*

The previous example is basically the worst that can happen in rank 2, there is just not enough space in such a low rank. For further examples, we need to increase the rank, as in the following.

3.16) Example: We wish to modify example 3.16 to show the same behaviour is also observable for Cohiggs fields. That means an example satisfying the integrability condition $\Phi \wedge \Phi = 0$. Normally Cohiggs fields do not form a vector space — the condition is not linear — the phenomenon is however still valid: when the multiplicity $\dim V_u(\mathscr{E})$ jumps down, it means there are pre-Cohiggs fields that cannot be deformed parallel to a deformation of the underlying toric vector bundle. The same could also happen for fields satisfying the integrability condition. In the following example the Cohiggs fields actually do create a vector space, but this is not necessary.



see b.7

Consider the 2-dimensional fan spanned by the following primitive generators: $\rho_1 = (1,0)$, $\rho_2 = (0,1)$, $\rho_3 = (-1,-1)$, $\rho_4 = (-1,1)$, $\rho_5 = (-1,0)$ and $\rho_6 = (-2,-1)$. Over the corresponding smooth[†] toric variety we consider a toric vector bundle of rank 3 given by the following filtrations, for rays 1 to 6 respectively:

$$
\begin{array}{lllll}
 & i=0 & i=1 & i=2 & i=3 \\
\mathsf{E}^{\rho_1}:\cdots\supset & \mathbb{C}^3\supset & (\ \ 1,\ \ 0,\ \ 0)^{\perp}\supset & [\ \ 0:\ \ 0:\ \ 1]\supset & 0 \quad \supset\cdots \\
\mathsf{E}^{\rho_2}:\cdots\supset & \mathbb{C}^3\supset & (\ \ 0,\ \ 0,\ \ 1)^{\perp}\supset & [\ \ 1:\ \ 0:\ \ 0]\supset & 0 \quad \supset\cdots \\
\mathsf{E}^{\rho_3}:\cdots\supset & \mathbb{C}^3\supset & (\ \ 4,-4,\ \ 1)^{\perp}\supset & [\ \ 1:\ \ 2:\ \ 4]\supset & 0 \quad \supset\cdots \\
\mathsf{E}^{\rho_4}:\cdots\supset & \mathbb{C}^3\supset & (\ \ 1,\ \ 2,\ \ 1)^{\perp}\supset & [\ \ 1:-1:\ \ 1]\supset & 0 \quad \supset\cdots \\
\mathsf{E}^{\rho_5}:\cdots\supset & \mathbb{C}^3\supset & (\ \ 3,-1,\ \ 1)^{\perp}\supset & [\ \ 2:\ \ 1:-5]\supset & 0 \quad \supset\cdots \\
\mathsf{E}^{\rho_6}:\cdots\supset & \mathbb{C}^3\supset & (-7,-3,\ \ 1)^{\perp}\supset & [-1:\ \ 3:\ \ 2]\supset & 0 \quad \supset\cdots
\end{array}
$$

where by $[a : b : c]$ we mean the one-dimensional space spanned by (a, b, c) and by $(a, b, c)^{\perp}$ the two-dimensional space composed of vectors orthogonal to (a, b, c) — with respect to the standard hermetian product. This may be an unusual way of writting a filtration, but those are the informations we will need in the calculations bellow.

Now consider the pre-Cohiggs fields of degree $u = (1, 0)$. We go step by step and apply 2.3a just to the first filtration. For this we have $\langle u, v_{\rho_1}\rangle = 1$. The resulting ϕ we represent as a pair of matrixes $(\phi_{(1,0)}, \phi_{(0,1)})$, just as in the example before. It must look like this:

$$
\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}
$$

where by $*$ we mean there is no restriction on this coordinate, it may vary freely. For the second ray we have $\langle u, v_{\rho_2}\rangle = 0$. So adding this restriction results in pre-Cohiggs fields looking like this:

$$
\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}.
$$

For the ray 3 to 5 we have $\langle u, v_{\rho_n} \rangle = -1$. Therefore the only conditions left are $\phi_{(1,-1)}(1,2,4) \in (4,-4,1)^\perp$, $\phi_{(1,1)}(1,-1,1) \in (1,2,1)^\perp$ and $\phi_{(0,1)}(1,1,-2) \in (3,-1,1)^\perp$. As in the example before we present those as a pair of matrixes with coefficients — for instance the first matrix we calculate is

$$\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 16 \\ -4 & -8 & -16 \\ 1 & 2 & 4 \end{pmatrix}$$

the relevant degree s for this filtration is $(1,-1)$, therefore the cooeficients of the equation $\phi_{(1,-1)}(1,2,4) \in (4,-4,1)^\perp$ are

$$C_3 : \begin{pmatrix} \underline{4} & 8 & 16 \\ -4 & \underline{-8} & -16 \\ 1 & 2 & \underline{4} \end{pmatrix}, \begin{pmatrix} -4 & -8 & -16 \\ \underline{4} & 8 & 16 \\ -1 & \underline{-2} & -4 \end{pmatrix}$$

where we underlined the relevant coefficients in view of the considerations above. For the remaining equations we get

$$C_4 : \begin{pmatrix} \underline{1} & -1 & 1 \\ 2 & \underline{-2} & 2 \\ 1 & -1 & \underline{1} \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ \underline{2} & -4 & 3 \\ 1 & \underline{-1} & 1 \end{pmatrix}$$

$$C_5 : \begin{pmatrix} \underline{0} & 0 & 0 \\ 0 & \underline{0} & 0 \\ 0 & 0 & \underline{0} \end{pmatrix}, \begin{pmatrix} 6 & 3 & -15 \\ \underline{-2} & -1 & 5 \\ 2 & \underline{1} & -5 \end{pmatrix}$$

The last ray does not impose any restriction on pre-Cohiggs fields of degree $(1,0)$ because $\langle u, v_\rho \rangle = -2$ for this ray.

Now consider which Cohiggs fields of degree $(1,0)$ are possible in this configuration. When writing the equation $\phi \wedge \phi = 0$ in terms of $(\phi_{(1,0)}, \phi_{(0,1)})$ it means both matrixes must commute. Commuting matrixes have the same Jordan normal form on the same basis. So in view of the restrictions provided by the first and second filtrations any solution of the integrability condition must have either $\phi_{(1,0)} = 0$ or $\phi_{(0,1)} = 0$. The following is one such solution that also satisfy conditions C_3 to C_5:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This however depends strongly on the fact that the underlined diagonals $(\ \ 4 \ -8 \ \ \ 4)$ of C_3 and $(\ \ 1 \ -2 \ \ \ 1)$ of C_4 are a multiple of one another. If this was not the case only the multiple of the identity in the first matrix would be possible solutions. This is the case for any generic deformation of the third and fourth filtrations, however the Cohiggs fields above cannot be deformed accordingly since it would need to jump into a multiple of the identity. So this is the example we are looking for.

It is only left to show this is a stable toric bundle. The moduli space $\mathfrak{M}_{\Psi}^{fr}$ containing this toric vector bundle is given by 6 full flags in $\mathbb{C}^3$. This means 6 pairs of a plane and a line in that plane. Normally one would represent that as a subset of a product of projective spaces $\mathbb{P}^2$ and Grassmannians $\mathsf{Gr}(2, \mathbb{C}^3)$. However, calculating the quotient like this is challenging. One may be tempted to use an isomorphism $\mathsf{Gr}(2, \mathbb{C}^3) \simeq \mathbb{P}^2$. However, this does not preserve the action of $\mathsf{GL}_3(\mathbb{C})$, since the inner product is not invariant. Instead, we use a trick for this specific case: in $\mathbb{C}^3$ a generic pair of two planes and two lines contained in such planes can be described by the two lines and the line given by the intersection of the two planes. In other words if $\cdots V_1 \supset V_2 \cdots$ and $\cdots V_3 \supset V_4 \cdots$ are generic filtrations, then we can write $(V_1 = \text{span}\{V_2, V_1 \cap V_3\})$ and $(V_3 = \text{span}\{V_4, V_1 \cap V_3\})$. So instead of describing the moduli space by (V_1, V_2, V_3, V_4) we choose to describe it by $(V_2, V_4, V_1 \cap V_3)$. This has the advantage that it only consists of lines — i.e. points in the projective space— and it preserves the action of $\mathsf{GL}_3(\mathbb{C})$. Of course, for that to work the lines and the planes must be distinct, which is our case.

This means we can map an open set of $\mathfrak{M}^{fr}_{\Psi}$ containing our toric bundle into $(\mathbb{P}^2)^9$. Now according to 3.13 the generic points of this quotient are given by configurations in $(\mathbb{P}^2)^9$ such that any 3 lines — from the 9 defining a point — are linearly independent. Let us locate our vector space in this configuration and see if it is generic. The first 6 lines are already given by the flags: $[0:0:1]$, $[1:0:0]$, $[1:2:4]$, $[1:-1:1]$, $[2:1:-5]$ and $[-1:3:2]$. The exact choice of which planes to intersect is irrelevant, as long as we keep it consistent. We therefore make the opportunistic choice of intersecting the first with the fourth plane, the second with the third and the sixth with the fifth. The 3 missing lines are therefore calculated by the products $(1,0,0) \times (1,2,1)$, $(0,0,1) \times (4,-4,1)$ and $(-7,-3,1) \times (3,-1,1)$. So the vector bundle is identified in $(\mathbb{P}^2)^9$ by:

$$\begin{array}{l} [\;\;0 : \;\;0 : \;\;1] \\ [\;\;1 : \;\;0 : \;\;0] \\ [\;\;1 : \;\;2 : \;\;4] \\ [\;\;1 : -1 : \;\;1] \\ [\;\;2 : \;\;1 : -5] \\ [-1 : \;\;3 : \;\;2] \\ [\;\;0 : -1 : \;\;2] \\ [\;\;1 : \;\;1 : \;\;0] \\ [-2 : 10 : 16] \end{array}$$

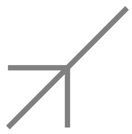
With some patience one may show that any three of those are linearly independent. In other words all subdeterminantes of the corresponding 9x3-matrix are non zero. This means our toric vector bundle is

see 3.13

in the generic Chow quotient† and therefore stable in all admissable linearisations[†]. *Quod erat faciendum* see a.29

Of course by itself this does not prove the Higgs range is not constant. For that, we need to choose our filtrations with even more care:

3.17) EXAMPLE: The purpose of this example is to expand the previous ones. It shows the Higgs range is not constant, even when considering stable toric bundles. Instead of writing the example down from the start, we choose to show how to construct it. We eventually need to restrict the pre-Cohiggs fields to zero. So let us first remove unnecessary degrees of freedom. For that let our first ray be spanned by $\rho_1 = (1,1)$ and the corresponding filtration:



$$\mathsf{E}^{\rho_1} : \cdots \supset \underset{i=0}{\mathbb{C}^3} \supset \underset{i=1}{(1,0,0)^\perp} \supset \underset{i=2}{(1,0,0)^\perp} \supset \underset{i=3}{[0{:}0{:}1]} \supset \underset{i=4}{0} \supset \cdots$$

Now we are interested in pre-Cohiggs fields of degree $u = (1,2)$. For the first rays this means $\langle u, v_\rho \rangle = 3$. From 2.3a it follows that ϕ must have the following form, where stars $*$ mean this coordinate can vary freely:

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, differently from before we choose the basis $(1,1)$ and $(1,-1)$ rather than the standard one. So the first matrix is $\phi_{(1,1)}$ and the second $\phi_{(1,-1)}$. We choose the second ray to be $\rho_2 = (-1,0)$ with filtrations:

$$\mathsf{E}^{\rho_2} : \cdots \supset \underset{i=0}{\mathbb{C}^3} \supset \underset{i=1}{(-2,1,1)^\perp} \supset \underset{i=2}{(-2,1,1)^\perp} \supset \underset{i=3}{[1{:}1{:}1]} \supset \underset{i=4}{0} \supset \cdots$$

Since $\langle u, v_\rho \rangle = -1$ this filtration gives 3 restrictions based on 2.4. First, for $s = (0,1)$ the only condition is $\phi_{(0,1)}(1,1,1) \subset (-2,1,1)^\perp$ since $s \perp \rho$. However for $s = (1,0)$ this does not apply so we have two conditions: $\phi_{(1,0)}(1,1,1) \subset (-2,1,1)^\perp$ and $\phi_{(1,0)}((-2,1,1)^\perp) \subset$

$(-2,1,1)^{\perp}$. To express this last condition better, just choose a vector in $(-2,1,1)^{\perp}$ — for instance $(0,1,-1)$ — and in the view of the second condition we can replace this third by $\phi_{(1,0)}(0,1,-1) \subset (-2,1,1)^{\perp}$. We should therefore consider the following equations

$$\phi_{(0,1)}(1,1,1) \cdot (-2,1,1) = 0$$

$$\phi_{(1,0)}(1,1,1) \cdot (-2,1,1) = 0$$

$$\phi_{(1,0)}(0,1,-1) \cdot (-2,1,1) = 0$$

Translating these conditions into the non-standard basis $\phi_{(1,1)}$ and $\phi_{(1,-1)}$ results in the following matrixes — in view of the first filtration we underline the important coefficients, the other ones are irrelevant and provided just for information:

$$C_2 : \frac{1}{2}\begin{pmatrix} -2 & -2 & -2 \\ \underline{1} & 1 & 1 \\ \underline{1} & 1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$C_3 : \frac{1}{2}\begin{pmatrix} -2 & -2 & -2 \\ \underline{1} & 1 & 1 \\ \underline{1} & 1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -2 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$C_4 : \frac{1}{2}\begin{pmatrix} 0 & -2 & 2 \\ \underline{0} & 1 & -1 \\ \underline{0} & 1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

For the third ray we choose $\rho_3 = (0,-1)$ with filtrations:

$$\mathsf{E}^{\rho_3} : \cdots \supset \underset{i=0}{\mathbb{C}^3} \supset \underset{i=1}{(3,1,\alpha)^{\perp}} \supset \underset{i=2}{(3,1,\alpha)^{\perp}} \supset \underset{i=3}{[1{:}-2{:}-\alpha^{-1}]} \supset \underset{i=4}{0} \supset \cdots$$

In this case $\langle u, v_\rho \rangle = -2$ so we have just one condition $\phi_{(0,1)}(1, -2, -\alpha^{-1}) \subset (3, 1, \alpha)^\perp$. This translates into the following matrixes:

$$C_5 : \frac{1}{2} \begin{pmatrix} 3 & -6 & -3\alpha^{-1} \\ \underline{1} & -2 & -\alpha^{-1} \\ \underline{\alpha} & -2\alpha & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -3 & 6 & 3\alpha^{-1} \\ -1 & 2 & \alpha^{-1} \\ -\alpha & 2\alpha & 1 \end{pmatrix}.$$

Notice that for $\alpha = 1$ all underlined coefficients are the same, therefore there are still non-trivial solutions for these conditions; for instance the following is a solution inducing a Cohiggs field of degree $(1, 2)$ — we remember the first matrix is $\phi_{(1,1)}$ and the second $\phi_{(1,-1)}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This means dim $V_{(1,2)}(\mathscr{E}) = 1$. However, for $\alpha \neq 1$ this is not the case and only the trivial solution exists, so dim $V_{(1,2)}(\mathscr{E}) = 0$.

If $u = (1, 2)$ were in the center of the Higgs range this would not change the range itself. So we must also show it is a corner: For $u = (a, b)$ with $a + b > 3$ there is no non-trivial pre-Cohiggs field. This follows from the first ray alone. Now consider $u = (a, b)$ with $a + b = 3$ and $a \leq 0$. For the first ray we still have $\langle u, v_\rho \rangle = 3$, so the Cohiggs fields must have the same form as above:

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, for the second ray $\langle u, v_\rho \rangle \geq 0$ which means $\phi_{(1,0)}(1,1,1) \subset [1:1:1]$. These two conditions imply that ϕ must be trivial. These two cases $a+b > 3$ or $a+b = 3$ with $a \leq 0$ surround the point $(1,2)$ from one side, so this is a vertex of the Higgs range for $\alpha = 1$. For $\alpha \neq 0$, it is outside of the Higgs range.

Finally, to show that the Higgs range is not constant even for stable points, we need to show that the toric bundle for $\alpha = 1$ is stable. To apply the trick presented in the previous examples we need an even number of rays. So we add a fourth ray with some generic filtration so that it does not change any of the considerations above. For instance pick $\rho_4 = (-1,-1)$ with filtration:

$$\mathsf{E}^{\rho_4} : \cdots \supset \underset{i=0}{\mathbb{C}^3} \supset \underset{i=1}{(3,3,-1)^\perp} \supset \underset{i=2}{[1:0:3]} \supset \underset{i=3}{0} \supset \cdots$$

see b.7

which is a good choice since it preserves the smoothness† of X and because $\langle (1,2), v_\rho \rangle = -3$, it does not create any restriction on the Cohiggs fields we analysed above. With this choice we get that the toric bundle is represented by the following points in $\mathbb{P}^2$

$$\begin{array}{ll}
[\,0: 0: 1] & \\
[\,1: 1: 1] & \\
[\,1: -2: -1] & \\
[\,1: 0: 3] & \\
[\,0: 1: -1] & = (1,0,0)^\perp \cap (-2,1,1)^\perp \\
[\,2: -3: -3] & = (3,1,1)^\perp \cap (3,3,-1)^\perp
\end{array}$$

Simple calculations show that any 3 of the 6 lines above are linearly independent. It follows also from 3.13 that our toric bundle is in the generic Chow open set for $\alpha = 1$ and is therefore stable†. This completes the example of a Higgs range that is not constant over stable points.

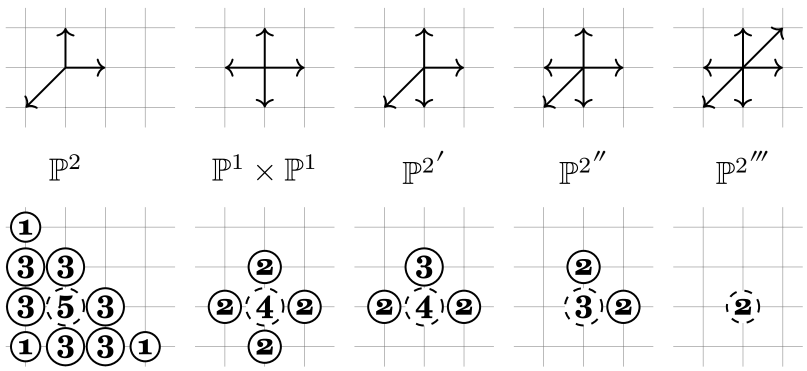
see a.29

Quod erat faciendum

Fortunately, not all news are bad news. There are examples where the moduli space works nicely. We present one such example considering the Fano surfaces studied by [AW21].

3.18) EXAMPLE: We have already calculated the moduli space of toric bundles for the component containing the tangential bundle of the projective space $\mathbb{P}^{2\dagger}$ and $\mathbb{P}^1 \times \mathbb{P}^{1\dagger}$. These are two examples of Fano surfaces. In dimension 2 there are a total of 5 Fano surfaces†. The full list is shown in the image below.

see 1.38
see 1.52
[Øb07]



Now for every Fano surface X consider the moduli space containing the tangential space — i.e., $\Psi = \Psi_{TX}$. Calculating the framed moduli space for these surfaces is not difficult and follows analogously to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. All filtrations look like

$$\cdots \supset \mathbb{C}^2 \supset \mathbb{C} \supset 0 \supset \cdots$$

so the flags are all given by 1-dimensional subvector space of $\mathbb{C}^2$, so the moduli space of framed toric bundles is a product of $\mathbb{P}^1$ — with some diagonals removed for the compatibility conditions.

The image above shows the Higgs multiplicity of generic points in $\mathfrak{M}^{fr}_{\Psi}$. This is different from the multiplicity calculated by [AW21] for two reasons: first they only considered trace zero pre-Cohiggs fields. Second and more importantly, they calculated it only for the tangent space $TX \in \mathfrak{M}_{\Psi}^{fr}$; we, on the other hand, are interested in generic toric bundles with same Chern class as the tangent space. For the case of the projective space however, the space $T\mathbb{P}^2$ is generic, so our calculation does agree with [AW21, Fig. 6] up to the different convention[†] from 2.1, that causes the triagle to appear upside down. The generic Higgs range can either be calculated by hand, or the following heuristic can by applied: the maximal possible multiplicity is 8. From that we remove multiplicities based on each ray, the calculation behind these are similar to the examples above. If $\langle u, v_\rho \rangle < 0$, there is no dimension drop. If $\langle u, v_\rho \rangle = 0$ the multiplicity drops one dimension. If $\langle u, v_\rho \rangle = 1$ the multiplicity drops 4 dimensions — three for the case $s \in \rho^\perp$ and one for the case $s \notin \rho^\perp$. If $\langle u, v_\rho \rangle = 2$ the dimension already drops by 7. If $\langle u, v_\rho \rangle > 2$ there exists no non-trivial pre-Cohiggs field. This however just works if we consider the restrictions coming from different rays are linearly independent. We assert this is the case for linearly generic flags. That is to say, the phenomenon described in 3.15 does not occur here.

see 2.2

We would like to quickly explain why this is the case, but proper proofs can only be provided by calculating it. The following are just some observations we made during these calculations. We must focus on $u = (0, 0)$. For other degrees the dimension drops quickly, so there is not much space for dependencies to manifest and it is also easy to calculate the multiplicity by hand. We also can discard any surface with less than 5 rays: speaking in the language of 3.15 the coefficient matrixes of three rays — with linearly generic flags — are always independent. Matrixes of the form $\hat{v_i}^T v_i$ are always linearly dependent on four rays, however, since the rays are not all equal they will become independent when considering both matrices $\phi_{(1,0)}$ and $\phi_{(0,1)}$. That means there are just two important cases to analyze. We take the one with 6 rays and therefore most likely to create dependencies. We can always make a change of basis so that two of the flags are the coordinate axes and a third is the diagonal $[1 : 1]$. We can therefore write a generic case as follows:

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & a^2 \\ 1 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 1 \\ -b^2 & -b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -c & -c^2 \\ 1 & c \end{pmatrix}.$$

In order for these conditions to be linearly dependent it must happen that

$$a(-1+a) = c(-1+bc)$$

$$(-1+a) = b(-1+bc).$$

Dividing the two equations — which can be done since the flags are all distinct — we get $a = \frac{c}{b}$. Replace a in the equations and then isolate cb^2 in the second one. Apply this to the first equation to reduce the degree of b. Eventually one gets that $b = 1$, so the fifth flag must agree with the first. This is not possible under linearly generic conditions. We conclude the following:

3.18A) Conclusion: For toric bundles over Fano surfaces such that $\Psi = \Psi_{TX}$ the Higgs multiplicity is constant over linearly generic points. Since the framed moduli space is the space representing a
see 2.18
locally free sheaf† it has naturally the structure of a vector bundle

$$\mathfrak{N}^{\mathrm{fr}}|_{\mathfrak{M}^{\mathrm{fr}^{\mathrm{gen}}}} \to \mathfrak{M}^{\mathrm{fr}^{\mathrm{gen}}}.$$

Also the existence of a coarse moduli space $\mathfrak{M}_{\Psi}^{fr^{gen}} // \mathsf{GL}_r(\mathbb{C})$ of unframed generic toric bundles follows from the Chow quotient con-
see d.4
struction plus 3.13. Using the 'Lemme de Decente'† we conclude there exists a coarse moduli space $\mathfrak{N}_{\Psi}^{fr^{gen}} // \mathsf{GL}_r(\mathbb{C})$ of unframed generic pre-Cohiggs bundles — for that we must only show the stabilizer in $\mathsf{GL}_r(\mathbb{C})$ acts trivially: in our case, the stabilizer of a generic toric bundles are the multiples of the identity and they act on ϕ via conjugation, therefore trivially.

Quod erat faciendum

3.19) Observation: Notice how in 3.18a we used the existence of a coarse moduli space of unframed toric bundles to conclude the existence of a coarse moduli space of unframed toric pre-Cohiggs bundles, which is a quite nice property. However, also notice that using the 'Lemme de Decente' in 3.18a is necessary. If the stabilizer does not act trivially the moduli space of unframed toric pre-Cohiggs bundles may not exist. This is the case for instance if Ψ is such that all flags are trivial — i.e. $\{0\} \subsetneq \mathsf{E}$ without intermediary steps. In this case $\mathfrak{M}_{\Psi}^{fr}$ is a single point. The change of frame still acts on ϕ via conjugation. The quotient, that is the equivalence classes of toric pre-Cohiggs bundles, corresponds to the Jordan normal form of the pre-Cohiggs field. But a family of pre-Cohiggs fields can easily jump between normal forms. This is a jumping phenomen, so there can be no coarse moduli space whatsoever†. see a.10

END NOTES

A Technical Tools

We begin with a definition:

A.1) Moduli Problem: A *moduli problem* in a category $\mathcal{C}$ is given by:

- For all objects $S \in \mathcal{C}$ a set $\mathcal{A}_S$ and an equivalence relation $\sim_S$ in this set
- For all morphisms $f : T \to S$ a function $\mathcal{A}_f : \mathcal{A}_S \to \mathcal{A}_T$

so that the following properties hold:

- For the identity we have $\mathcal{A}_{\mathrm{id}_S} = \mathrm{id}_{\mathcal{A}_S}$
- From $F \sim_S G$ it follows $\mathcal{A}_f(F) \sim_T \mathcal{A}_f(G)$
- For $f : T \to S$, $g : S \to R$ and every element $F \in \mathcal{A}_R$ it holds that $\mathcal{A}_{g \circ f}(F) \sim_T \mathcal{A}_f \circ \mathcal{A}_g(F)$

Elements of the set $\mathcal{A}_S$ are normally called *families over* S, *families parametrized by* S or S*-families*. If $\{*\}$ is a terminal object of the category $\mathcal{C}$, then one may restrict every family to points: that is, given a family $F \in \mathcal{A}_S$ and a point $s : \{*\} \to S$, one may consider the element $\mathcal{A}_s(F) \in \mathcal{A}_{\{*\}}$. In this way S-families are seen as variations parametrized by S of elements in $\mathcal{A}_{\{*\}}$, this explains the nomeclature.

For a family F over S and a point $s : \{*\} \to S$ we use F_s to denote the pullback $\mathcal{A}_s(F)$.

Verifying all the conditions above may be tiresome. Instead we will look to work inside an special case: a variety of configurations with the action of a group of symmetries. Therefore, to some extent, it is more important to understand the following example.

A.2) Example – Groupoids and Equivalences: The simplest example of a moduli problem is a set C with the action of a group G. In this case we name G the *symmetry group* of the *configuration space* C. This defines a moduli problem in $\mathcal{S}et$ — the category of sets. To show that let $\mathcal{A}_S = \{S \to C\}$ be the set of maps from S to C. Two maps a and a' are equivalent, if there is a map $g : S \to \mathsf{G}$ so that $a(s) = a'(s) \cdot g(s)$. This defines the needed equivalence relation and $\mathcal{A}_f$ is the canonical pullback of maps: $a \mapsto a \circ f$.

A pair (C, G) composed of a set and its symmetry group can also be seen as a groupoid: elements of C are the objects while morphisms are arrows $m \xrightarrow{g} gm$ with $m \in C$ and $g \in \mathsf{G}$. Therefore we can use of the language of groupids to speak of moduli problems. This has the disadvantage that other structures of C — for instance a topology etc. — are neglected. In the literature, moduli problems in $\mathcal{S}et$ are also called naive moduli problems for this reason [Hos15]. However, there is a way around: let us suppose that C is an object in a locally small category $\mathfrak{C}$, for instance the category of varieties. We can define a moduli problem just as above, with the extra conditions that the morphisms $S \to C$ must be in $\mathfrak{C}$ instead of any general map between sets. *Quod erat faciendum*

In any case, whether the moduli problem is described with the categorical definition or when possible with a groupoid, for us the important part is the moduli functor.

A.3) LEMMA – MODULI FUNCTOR: Any moduli problem defines a functor $\mathcal{M} : \mathfrak{C} \to \mathcal{S}et$ given by:

$$\mathcal{M}(S) = \mathcal{A}_S/\sim_S$$

$$\mathcal{M}(f : T \to S) = \mathcal{A}_f/\sim_S : \mathcal{M}(S) \to \mathcal{M}(T)$$

A.4) FUNCTOR OF POINTS: There exists a covariant functor $h : C \to \mathcal{P}sh(C)$, which naturally associates every object to a pre-sheaf . This is defined in the following way:

- For an object $C \in \mathfrak{C}$ is h^C the contravariant functor $\mathfrak{C} \to \mathcal{S}et$ given by:

$$h^C(C') := \hom(C', C)$$

$$h^C(f : C' \to C'') := (g \mapsto g \circ f)$$

- For a morphism $f : C \to C'$, is $h^f : h^C \Rightarrow h^{C'}$ a morphism of functors given by the mappings $h^f_{C''} := g \mapsto f \circ g$ from $h^C(C'')$ to $h^{C'}(C'')$

This functor h is called the *functor of points*.

A.5) FINE MODULI SPACE: A moduli functor $\mathcal{M}$ is said to admit a *fine moduli space*, if there is a $M \in \mathfrak{C}$ and a functor isomorphism $\nu : \mathcal{M} \Rightarrow h^M$, where h^M is the functor of points[†] of M. Such a fine moduli space always admits a so-called *universal family*. This is any element F_M in the equivalence class given by $\nu^{-1}_M(\mathrm{id}_M)$. see a.4

A.6) Observation: A universal family F_M has the following property: for any family $F \in \mathcal{A}_S$ there exists a morphism $f : S \to M$ so, that $\mathcal{A}_f(F_M) \sim_S F$.

One may notice that a fine moduli space is something like a complete solution for a moduli problem, since $\mathcal{M}$ is completely represented by M. However, this may be much too optimistic for most cases and therefore we must loosen our conditions.

A.7) Coarse Moduli Space: A *coarse moduli space* for a moduli functor $\mathcal{M}$ is an $M \in \mathfrak{C}$ with a functor morphism $\nu : \mathcal{M} \Rightarrow h^M$, satisfying the properties

- $\nu_{\{*\}} : \mathcal{M}(\{*\}) \to h^M(\{*\})$ is a bijection, where $\{*\}$ is a terminal object in $\mathfrak{C}$ — considering such an object exists.
- For all $N \in \mathfrak{C}$ and functor morphisms $\nu' : \mathcal{M} \Rightarrow h^N$ there is a unique morphism $f : M \to N$, that factorizes ν' as $\nu' = h^f \circ \nu$.

In this case, we may not have an universal family, but we still have uniqueness.

A.8) Proposition: The coarse moduli space is unique up to isomorphism.

That means both coarse and fine moduli spaces are unique. If they exist, they must be equal, since a fine moduli space is also coarse. The next proposition will demonstrate the conditions under which the converse is true.

A.9) Proposition: Let (M, ν) be a coarse moduli space for a functor $\mathcal{M}$. This is a fine moduli space, if the following properties hold.

- There exists an universal family F_M over M such, that $\nu_{F_M} = \mathrm{id}_M$
- If F and G are two families over S, then $F \sim_S G$ if and only if $\nu_S([F]) = \nu_S([G])$

The following question comes from the other perspective: is there always a (coarse) moduli space? The answer is clearly no.

A.10) Lemma – Jumping Phenomen: Let $\mathcal{M}$ be a moduli functor in the category of complex algebraic varieties. Assume there is a family F over $\mathbb{C}$ with the property that $F_z \sim F_1$ for all $z \neq 0$ but $F_0 \nsim F_1$. Then for any M and functor morphism $\nu : \mathcal{M} \Rightarrow h^M$, it results that $\nu_{\mathbb{C}}([F]) : \mathbb{C} \to M$ is constant. Therefore there can be no coarse moduli space for $\mathcal{M}$.

For the proof: Let $f = \nu_{\mathbb{C}}(F)$. For every $z : \mathsf{spec}(\mathbb{C}) \to \mathbb{C}$ we have $f \circ z = \nu_{\mathsf{spec}(\mathbb{C})}(F_z)$. Therefore it holds for $z \neq 0$, that $F_z = F_1 \in \mathcal{M}(\mathsf{spec}(\mathbb{C}))$. That means $f|_{\mathbb{C}^*}$ is constant.

Since the points of M are closed and f is a morphism, f must be constant over the whole of $\mathbb{C}$. That would violate the bijectivity of $\nu_{\mathsf{spec}(\mathbb{C})} : \mathcal{M}(\mathsf{spec}(\mathbb{C})) \to h^M(\mathsf{spec}(\mathbb{C}))$ since $F_1 \nsim F_0$ in $\mathcal{M}(\mathsf{spec}(\mathbb{C}))$ and should not be mapped to the same point in M.

A.11) Quotient: Let X and Y be two schemes, with a morphism of schemes $p : X \to Y$, and a group G acting on X. We present some different way we may say Y is a quotient of X according to [Hos15, sections 3.4 and 3.5].

A.11A) Categorical Quotient: Y is a *categorical quotient* if $p : X \to Y$ is a G-invariant universal morphism. This means for any other G-invariant morphism of schemes $q : X \to Z$ there is one and only one $h : Y \to Z$ such that $q = h \circ p$.

A.11B) Good Quotient: The scheme Y is a *good quotient*, if the following holds:

- p is G-invariant.

Notice the second condition is redundant, it follows from the third and fourth conditions. We included it for better understanding.

- p is surjective.†
- Functions on Y correspond to G-invariant functions on X. That means $p^* : \mathscr{O}_Y(U) \to \mathscr{O}_X(p^{-1}(U))^{\mathsf{G}}$ is a ring isomorphism for every open subset $U \subset Y$.
- $p(W)$ is closed if $W \subset X$ is closed and G-invariant.
- If $W_1, W_2 \subset X$ are closed and G-invariant with $W_1 \cap W_2 = \emptyset$ it follows that $p(W_1) \cap p(W_2) = \emptyset$
- p is an affine morphism — i.e pre-images of affine opens are affine

A.11C) Geometric Quotient: Y is a *geometric quotient* if it is a good quotient and the fibers of p are orbits — that is to say $p(x) = p(x') \Leftrightarrow \mathsf{G} \cdot x = \mathsf{G} \cdot x'$.

A.11D) Remark: Good quotients are categorical quotients, and therefore so are geometric quotients [Hos15, Proposition 3.30].

For better understanding consider the following example: Finding a good quotient is a way to turn the topological quotient X/G — which always exists — into a scheme or even in most cases into an algebraic variety. With this in mind, let us fix the following notation.

A.12) Notation: Let X be a complex variety with the action of a group G. We use $X//\mathsf{G}$ to denote the good quotient†, if it exists; the notation X/G is used, when it is a geometric quotient. Pure topological quotients play no role here, so there is no risk of confusion. The notation is also appropriate since for the geometric quotient every fiber of the morphism $X \to X/\mathsf{G}$ is an orbit of G. That is, as a topological space the geometric quotient matches the topological quotient.

see a.11

This already allows us to state the main result:

A.13) GIT's Fundamental Theorem: Let G be a reductive group, acting on an irreducible variety $X \subset \mathbb{P}^N$ via linear maps . Then there are certain open sets $X^s \subset X^{ss} \subset X^{\dagger}$ and a good quotient see a.30

$$p : X^{ss} \to X^{ss}//\mathsf{G},$$

which can be restricted to form a geometric quotient:

$$p : X^s \to X^s/\mathsf{G}.$$

Further, X^s/G is an open set on $X^{ss}//\mathsf{G}$.

It is also possible to show that $X^{ss}//\mathsf{G}$ is always quasi-projective. If X is projective, then the quotient is also projective.

For the proof: [Dol03, Section 8.2]

This theorem is extremely useful to find moduli spaces under the condition that the space X has the following key property:

A.14) Configuration spaces and local universal families: Let $\mathcal{A}$ be a moduli problem in the category of the schemes. A family F over the scheme S is a *local universal family*, if for all other families G over schemes T and for all points $t \in T$, a neighborhood U of t exists with a morphism $f : U \to S$ such that:

$$G|_U \sim_U \mathcal{A}_f(F)$$

The scheme S will be called a *configuration space* for the moduli problem $\mathcal{A}$.

With this condition, the next proposition gives us a formula for finding moduli spaces.

A.15) PROPOSITION: Let $\mathcal{A}$ be a moduli problem with a local universal family F over a scheme X. Further let an algebraic group G act on X with the following property: two points s and t are in the same G-orbit if and only if $F_s \sim F_t$. Under this condition the geometric quotient X/G, if it exists, is also a coarse moduli space for $\mathcal{A}$.

For the proof: [Hos15, Proposition 3.35]

For groupoids – of finite dimension – this proposition always applies since:

A.16) OBSERVATION: If a moduli problem is given by the action of
see a.2 G on $X^\dagger$, then X is a configuration space with the identity map id : $X \to X$ as its local universal family and the conditions of a.15 are automaticaly satisfied.

We now turn our attention to a different, but somewhat related, form of quotient. For that, we must explain the Chow variety, beginning with the Chow form.

[KSZ91] A.17) CHOW FORM†: Let $Y \subset \mathbb{P}^N$ be a k-dimensional irreducible variety of degree d. Denote by $\tilde{Y}$ its cone in $\mathbb{C}^{N+1}$ and by $Z(Y)$ the set

$$\{V \in \mathsf{Gr}(N-k, \mathbb{C}^{N+1}) : V \cap \tilde{Y} \neq \{0\}\}.$$

This set forms a hypersurface of degree d in the Grassmannian. That means that $Z(Y)$ is defined by a polynomial $R_Y \in B_d$, where B_d is the homogenous part of degree d in the coordinate ring of $\mathsf{Gr}(N-k, \mathbb{C}^{N+1})$. We call R_Y the *Chow Form* of Y, which is unique up to multiplication by a scalar.

This would, however, be of little importance without the following proposition, which assures a variety is uniquely given by its Chow form.

A.18) Proposition: The mapping $Z : Y \mapsto Z(Y)$ is injective

For the proof: [GKZ94, Corollary 2.6]

This can be generalized for algebraic cycles.

A.19) Chow form of a cycle: A k-dimensional *algebraic cycle* in $\mathbb{P}^N$ is a formal linear combination $C = \sum_i m_i C_i$, where i takes values on some finite set, the C_i's are k-dimensional irreducible subvarieties and the coefficients m_i are integers.

We can expand the definition of the Chow form for cycles:

$$R_C = \prod_i R_{C_i}^{m_i} \in B_d$$

where $d = \deg(C) := \sum_i m_i \deg(C_i)$.

A.20) Family of effective algebraic cycles: An effective algebraic cycle, that is, a cycle with positive coefficients, can also be represented by a scheme. Furthermore, a relative scheme $Y \to S$, such that over every point $s \in S$ the fiber Y_s has constant degree, defines a *family of effective algebraic cycles*. Together with the isomorphism of schemes and the standard pullback of schemes, this describes a moduli problem, as one can see by verifying the properties of a.1.

[GKZ94] **A.21) Chow Variety**†: The *Chow variety* is a variety parameterizing cycles of dimension k and degree d in $\mathbb{P}^N$. Formally it can be defined to be the subset $\mathsf{Chow}_{k,d}(\mathbb{P}^N) \subset \mathbb{P}(B_d)$, given by the Chow forms of such cycles. Furthermore this variety is a fine moduli space to the moduli problem of effective algebraic cycles.

see a.33 We can also define it for every quasi-projective variety by restricting ourselves to cycles that are contained in said variety†. When it is clear from the context, we may drop the references for dimension, degree and projective space, referring to the Chow variety simply as Chow.

Well-definedness: To show that the Chow variety defined above is a moduli space is not a trivial task. The theorem was first proven by Chow and van der Waerden, whence its name. A complete proof can be found in [GKZ94, Section 4.1].

A.22) Proposition: Let X be an irreducible variety with an action of an algebraic group G. There exists an invariant Zariski open subset $U \subset X$, so that for $x \in U$ the closure of orbits $\overline{\mathsf{G} \cdot x}$ have constant homology in $\mathsf{H}_*(X, \mathbb{Z})$.

The Chow Quotient is then the closure of the image of this generic set in the Chow variety:

see a.33 **A.23) Chow Quotient:** Let X be an irreducible variety with the action of an algebraic group G. Let U be an open subset of X, so that the closure of orbits $G \cdot x$ have constant homology δ. The mapping $x \mapsto \overline{\mathsf{G} \cdot x} \in \mathsf{Chow}_\delta(X)^\dagger$ gives a quotient $U \to U/\mathsf{G}$. The closure of this quotient in $\mathsf{Chow}_\delta(X)$ we call the *Chow Quotient* of X by G and denote it by

$$X//^{\mathrm{ch}}\mathsf{G} := \overline{U/\mathsf{G}} \subset \mathsf{Chow}_\delta.$$

This is independent of the choice of U.

Well-definedness: The map $U \to U/\mathsf{G}$ is a geometric quotient. This derives from the Chow variety being a fine moduli space: for that create a variety Y with a morphism $Y \to U$ such that for $x \in U$ the fiber $Y_x = \overline{\mathsf{G} \cdot x}$. This is a family of cycles, therefore it is represented by a morphism $U \to \mathsf{Chow}$. The fibers of this morphism are individual orbits $\mathsf{G} \cdot x$, so restricting the morphism to the image U/G gives the quotient.

It is important to mention that despite its name, the Chow quotient is not a quotient as commonly understood.

A.24) COROLLARY: The Chow quotient is not a quotient in itself, there is in general no morphism $X \to X//^{\mathrm{ch}}\mathsf{G}$, not even a set theoretic map. However, there is a fibering $Y \to X//^{\mathrm{ch}}\mathsf{G}$, given by restricting the universal family of the Chow variety . This is sometimes called the Chow family or the Chow fibering.

A.25) EXAMPLE: In order to understand the Chow quotient, it is useful to keep the following example in mind: the group $\mathbb{C}^*$ acts on $\mathbb{C}^3$ by

$$t \cdot (x, y, z) = (tx, ty, t^{-1}z).$$

The Chow quotient is isomorphic to $(\mathbb{C}^2)'$ — the blow-up of $\mathbb{C}^2$ at the origin — as the following indicates: The generic orbits are hyperboles and they cross the plane $z = 1$ in exactly one point. So we can conclude there is a geometric quotient $U \to \mathbb{C}^2 \setminus \{0\}$, where U is the 3-dimensional space without the z-axis and the xy-plane. In the limit to the origin, however, a hyperbole degenerates into the sum of two lines, one is the z-axis and the other is a line by the origin on the xy-plane. So for non-generic points there is no geometric quotient, since points in the z-axis are present in more than one fiber — namely those fibers distinguish themselves by the line in xy-plane, whence the blow up at the origin. To achieve a Chow fibering we also need a blow-up of $\mathbb{C}^3$ at the z-axis. That is to say the Chow family looks like

$$Y = (\mathbb{C}^2)' \times \mathbb{C} \to (\mathbb{C}^2)' = \mathbb{C}^3//^{\text{ch}}\mathbb{C}^*.$$

Now let us consider the GIT quotients of this example. There are mainly two relevant linearizations, one rendering the z-axis as unstable and another rendering the xy-plane unstable. Generic orbits intersect the plane $z = 1$ in exactly one point, therefore the generic quotient is $\mathbb{C}^2 \setminus \{0\}$. In the first linearization, the orbits in the xy-plane are semistable and parameterizable by a projective line $\mathbb{P}^1$, which means we need to glue a projective line into the origin — i.e. perform a blow-up. The semistable quotient is therefore $(\mathbb{C}^2)'$. On the second linearization, the quotient is $\mathbb{C}^2$ since the z-axis is a stable orbit and must thus map to a single point. In both cases, the Chow quotient maps to the GIT quotient: the identity $(\mathbb{C}^2)' \to (\mathbb{C}^2)'$ in the first case and the blow-down $(\mathbb{C}^2)' \to \mathbb{C}^2$ in the second case. We say that the Chow quotient dominates the GIT quotient.

Since this is such a simple example, one of the GIT quotients is equal to the Chow quotient. This is not always the case, for instance with one extra dimension $t \cdot (x, y, z, w) = (tx, ty, t^{-1}z, t^{-1}w)$, the Chow quotient has 2 blow-ups, each one collapsing into a GIT quotient.

Quod erat faciendum

A.26) THEOREM: The Chow quotient dominates all GIT quotients, that is to say, for any choice of linearization there is always a regular birational morphism $X//^{\text{ch}}\mathsf{G} \to X^{ss}//\mathsf{G}$.

For the proof: As originally proven in [Kap93, Theorem 0.4.3], this consists of two steps, first showing there are semistable orbits in the Chow fibers† and then showing the map so defined is a morphism†. The first step rests on the symplectic description of stability while the second step follows from some calculations with the Chow form.

see a.27
see a.28

A.27) Lemma: For any cycle C in the Chow quotient $X//^{\mathrm{ch}}\mathsf{G}$ there are semistable orbits in its support and they are all equivalent, meaning they map to the same point in the Git quotient.

A.28) Lemma: The map $p : X//^{\mathrm{ch}}\mathsf{G} \to X^{ss}//\mathsf{G}$ — taking a cycle C to the point $X^{ss}//\mathsf{G}$ representing any of the semistable orbits in its support — is a morphism of algebraic varieties.

The following corollary shows that the generic points used in the construction of the Chow quotient are some kind of super-stable points.

A.29) Corollary: For an admissable linearization† the points in U are all stable and the morphism in a.26 is an isomorphism $U//^{\mathrm{ch}}\mathsf{G} \simeq U//\mathsf{G}$, meaning the Chow quotient and the Git quotient agree over the generic points.

see a.32

For the proof: This follows from the symplectic stability criterion†. For generic points the cycle $\overline{\mathsf{G}\cdot x}$ have one big orbit and several smaller ones on the edges — remember stable orbits are closed on X^{ss}, but not necessarily on X. Admissable linearization are exactly the ones with momentum charge in the interior of $\Omega(\overline{\mathsf{G}\cdot x}) = \Omega(X)$. From the continuity of Ω it follows that the momentum charge is in $\Omega(\mathsf{G}\cdot x)$. This means this orbit is stable, therefore the only semistable orbit in this Chow fiber. But the fiber is generic, so the orbit $\mathsf{G}\cdot x$ is present in no other fiber. This means the morphism in a.26 is a bijection between varities and therefore an isomorphism.

see a.32

A.30) Stability Criterium: Let $X \subset \mathbb{P}^N$ be a variety with a linear action of G, that means G acts via a representation $\rho : \mathsf{G} \to \mathsf{GL}_{N+1}(\mathbb{C})$. Let $x \in \mathbb{C}^{N+1}$ be a representant of $[x] \in X$. By definition, a point $[x]$ is *semi stable* if and only if $0 \notin \overline{\mathsf{G} \cdot x}$. In addition to that, if for every 1-parameter subgroup $\mathbb{C}^* \hookrightarrow \mathsf{G}$ the orbit $\mathbb{C}^* \cdot x$ is closed in $\mathbb{C}^{N+1}$, then $[x]$ is *stable*. This is equivalent to the following numerical condition:

Let $\lambda : \mathbb{C}^* \to \mathsf{G}$ be a 1-parameter subgroup. The representation ρ turns it into a 1-parameter subgroup of $\mathsf{GL}_{N+1}(\mathbb{C})$. Therefore it must be diagonalizable[†] — i.e. there is a basis where the action is given by

[Wil19]

$$\lambda(t)x = (t^{m_0}x_0, \cdots, t^{m_N}x_N).$$

From that define $\mu([x], \lambda) := \min\{m_i : x_i \neq 0\}$. The point $[x]$ is semi stable if and only if $\mu([x], \lambda) \leq 0$ for all λ and stable if $\mu([x], \lambda) < 0$. Since every 1-parameter subgroup has an inverse λ^{-1} this means that stable points most always act with positive and negative degrees — i.e.

$$m_0 \lneq \cdots \leq 0 \leq \cdots \lneq m_n$$

For the proof: [Dol03, Theorem 9.1]

A.31) Symplectic Stability Criterion: The stability criterium in a.30 is equivalent to the following symplectic version. Let K be the compact real form of G, for instance if $\mathsf{G} = (\mathbb{C}^*)^n$, then $\mathsf{K} = (\mathbb{S}^1)^n$. A linearization is equivalent to the choice of an ample line bundle L on X with an action of G. By fixing a K-invariant Hermitian metric on it, we get a momentum map

$$\Omega : X \to \mathfrak{k}$$

from the variety X to the Lie algebra of K. In this context, a point x is semi stable if and only if $0 \in \Omega(\overline{\mathsf{G} \cdot x})$ and it is stable if and only if $0 \in \Omega(\mathsf{G} \cdot x)$.

There is nothing significant about the zero here, actually any point in $\mathfrak{k}$ defines a different stability criterion, as we will now explain: let r be a rational point in $\mathfrak{k}$, then its integral multiple mr is identifiable with a character ξ of G. We can twist the G action of the bundle L^m by the character $-\xi$. In terms of moment maps, this represents a new map $\Omega'(x) = m\Omega(x) - mr$. That means a point x is stable in relation to the twisted bundle $L^m(-\xi)$ if and only if r is in $\Omega(\mathsf{G} \cdot x)$, or respectively semi stable if the closure is taken.

For the proof: [Hu05, section 2.1]

A.32) Observation: Let C be a Chow fiber, that is, the support of the cycle corresponding to a point in the Chow quotient, and by Ω the momentum map as in a.31, then

$$\Omega(C) = \Omega(X).$$

This was first proven for generic points by Atiyah in [Ati82] and than completed in the general case in [Hu05, prop 3.3].

So by the symplectic stability criterion above, if the Git quotient is not empty — i.e. $r \in \Omega(X)$ — generic points are always semistable. Further, if r is in the interior of $\Omega(X)$, generic points are stable. In accordance with the literature, we call such quotients *admissable*. In his doctoral thesis, Yi Hu remarked that r is in the boundary of $\Omega(X)$ if and only if the corresponding Git quotient degenerates, meaning its dimension is smaller than $\dim X - \dim \mathsf{G}$ [Hu09, Sec 1.3]. He was, however, just working in the case of G being a torus, we believe this is also valid for the general case, but were not able to track down a reference.

A.33) THE CHOW VARIETY OF AN ABSTRACT VARIETY: In order to define the Chow variety of an abstract quasi-projective variety, one must fix the homology of the cycles, instead of the degree. This is because for any embedding $f : X \to \mathbb{P}^N$, cycles of constant homology are mapped to cycles of constant degree via the push forward $f_* : \mathsf{H}_*(X, \mathbb{Z}) \to \mathsf{H}_*(\mathbb{P}^N, \mathbb{Z})$. So we have $\mathsf{Chow}_\delta(X)$ as a subvariety of $\mathsf{Chow}_{f_*(\delta)}(\mathbb{P}^n)$.

B Notes on the First Chapter

B.1) Definition: Let σ be a cone in a vector space V. We define the dual cone and orthogonal space as follows:

$$\check{\sigma} := \{w \in \mathsf{V}^* : \langle v, w \rangle \geq 0, \forall v \in \sigma\}$$

$$\sigma^\perp := \{w \in \mathsf{V}^* : \langle v, w \rangle = 0, \forall v \in \sigma\}$$

where V^* is the dual space of V and $\langle \_, \_ \rangle$ is the natural pairing between V and V^*.

B.2) Proposition: The lattice M is the character lattice of $\mathsf{T} := \mathsf{spec}(\mathbb{C}[\mathsf{M}])$. So there is a canonical equivalence between the set of integral linear functions on $\mathsf{N}_\mathbb{R}$ and the set of linear characters of T. For a cone σ in $\mathsf{N}_\mathbb{R}$

$$\mathsf{M}_\sigma := \mathsf{M}/(\sigma^\perp \cap \mathsf{M})$$

is thus the group of integral linear function on σ, where $\sigma^\perp$ denotes the orthogonal linear space[†].

see b.1

B.3) Definition: A multiset is a set where the elements have an associated multiplicity; it is a set where an element can be present more than once.

B.4) THEOREM: There is a bijective correspondence between cones σ in a fan Σ and T-orbits in the corresponding toric variety $X(\Sigma)$, given by

$$\sigma \mapsto O_\sigma \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap \mathsf{M}, \mathbb{C}^*).$$

This correspondence is such that the dimension of the orbit is complementary to the dimension of the cone. For instance

$$\{0\} \mapsto O_{\{0\}} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathsf{M}, \mathbb{C}^*) = \mathsf{T}.$$

Moreover, there are decompositions

$$U_\sigma = \bigcup_{\sigma' \preceq \sigma} O_{\sigma'}$$

$$\overline{O_\sigma} = \bigcup_{\sigma \preceq \sigma'} O_{\sigma'}$$

where the closure $\overline{O_{\sigma'}}$ is the same in both classical and Zariski topologies.

For the proof: [CLS11, Theorem 3.2.6]

see b.4

B.5) COROLLARY: From the decomposition in the theorem above[†], we get that O_σ is the minimal T-orbit in U_σ, which is necessarily closed since

$$\overline{O_\sigma} \cap U_\sigma = O_\sigma.$$

Over the whole of X, there are several of these orbits, one for every maximal cone.

B.6) Corollary: Every non-empty closed and T-invariant subset of X must contain a closed minimal orbit[†], since it is the finite union of orbits, and the closure of every orbit contains an orbit of smaller dimension. Therefore an induction on the dimension provides an easy proof. see b.5

B.7) Theorem: A toric variety $X(\Sigma)$ is smooth if and only if every cone in Σ is generated by a subset of a $\mathbb{Z}$-Basis of N.

For the proof: [CLS11, Theorem 3.1.19]

B.8) Vector Space with Filtrations: Let E be a $\mathbb{C}$-vector space. We equip E with a collection of decreasing filtrations $\{\mathsf{E}^\rho(i)\}$ for every ray $\rho \in \Sigma(1)$ and call it a *vector space with filtrations*. Given a second vector space with filtration $(\mathsf{F}, \{\mathsf{F}^\rho(i)\}_{\rho\in\Sigma(1)})$, we define a *morphism of vector spaces with filtrations* as a linear map $\psi : \mathsf{E} \to \mathsf{F}$, that maps $\mathsf{E}^\rho(i)$ into $\mathsf{F}^\rho(i)$ for every ray ρ and every $i \in \mathbb{Z}$. This provide us with a category.

B.9) Notation: Given a toric bundles $\mathscr{E}$ we denote by $\Psi_{\mathscr{E}} = \{\mathfrak{u}(\sigma)\}_{\sigma\in\Sigma}$ the collection of multisets given by 1.8 so that for all cones σ in the fan Σ

$$\mathscr{E}|_{U_\sigma} \cong \bigoplus_{[u]\in\mathfrak{u}(\sigma)} \mathscr{L}_{[u]}.$$

As a collection of functions over N, the multisets agree in intersections, so for $\sigma' \preceq \sigma$

$$\mathfrak{u}(\sigma') = \mathfrak{u}(\sigma)|_{\sigma'}$$

since

$$\bigoplus_{[u]\in\mathfrak{u}(\sigma)} \mathscr{L}_{[u]}|_{U_{\sigma'}} \cong \bigoplus_{[u]\in\mathfrak{u}(\sigma)|_{\sigma'}} \mathscr{L}_{[u]}.$$

B.10) Definition: A *flag* over a vector space V is a set of sub-vector spaces constructing a sequence

$$\{0\} = \mathsf{V}_0 \subsetneq \cdots \subsetneq \mathsf{V}_k = \mathsf{V}.$$

We say this is a *complete flag* if there is a sub-vector space for every dimension between 0 and the rank of V and a *partial frag* otherwise.

see 1.19
see 1.34

B.11) Proposition: The compatibility condition† implies the rank condition†.

For the proof: The compatibility condition states that

$$\mathsf{E}^{\rho}(i) = \sum_{\langle [u], v_\rho \rangle \geq i} \mathsf{E}_{[u]},$$

therefore

$$\bigcap_{l=1}^{s} \mathsf{E}^{\rho_l}(i_l) = \sum_{\langle [u], v_{\rho_l} \rangle \geq i_l \ \forall l} \mathsf{E}_{[u]}.$$

And since the dimension of $\mathsf{E}_{[u]}$ is the multiplicity of $[u]$ in $\mathbb{u}(\sigma)$ we get

$$\dim \bigcap_{l=1}^{s} \mathsf{E}^{\rho_l}(i_l) = \#\{[u] \in \mathbb{u}(\sigma) | \langle [u], v_{\rho_l} \rangle \geq i_l \ \forall l\}.$$

Furthermore $Fl(\rho_l)_{j_l} = \mathsf{E}^{\rho_l}(\Delta_{\rho_j}(j_l))$, which completes the proof.

B.12) Proposition: The dimension of the intersection defines an upper semi-continuous regular function on the product of Grassmannians.

For the proof: Let $\mathsf{Gr}_1 = \mathsf{Gr}(n_1, \mathsf{V})$ and $\mathsf{Gr}_2 = \mathsf{Gr}(n_2, V)$ be two Grassmannians with universal families $\mathscr{Gr}_1$ and $\mathscr{Gr}_2$, respectively — i.e. the sheaves of sections of the tautological bundles. Then on the product $\mathsf{Gr}_1 \times \mathsf{Gr}_2$ we can consider the locally free sheaf $\pi_1^* \mathscr{Gr}_1 \oplus \pi_2^* \mathscr{Gr}_2$ and the morphism of sheaves

$$\begin{array}{ccc} \pi_1^* \mathscr{Gr}_1 \oplus \pi_2^* \mathscr{Gr}_2 & \to & \mathscr{O}_{\mathsf{Gr}_1 \times \mathsf{Gr}_2} \otimes \mathsf{V} \\ (v_1, v_2) & \mapsto & v_1 - v_2 \end{array}$$

The kernel of this morphism is the sheaf of sections on the intersection of the tautological bundles. To see this let $W_1 \in \mathsf{Gr}_1$ and $W_2 \in \mathsf{Gr}_2$. Since $v_1 - v_2 = 0$ it follows that $v_1(W_1) = v_2(W_2)$, however, $v_1(W_1) \in W_1$ and $v_2(W_2) \in W_2$. So $v = v_1 = v_2$ takes value on $W_1 \cap W_2$. The kernel of a morphism of coherent sheaves is coherent, and therefore the dimension of the fibers is upper semi-continuous [Har10, II 5.7 and III 12.8]. This completes the proof since the fiber — i.e. the stalk — of a kernel sheaf is the kernel of the morphism between stalks, that means it is the kernel of

$$\begin{array}{ccc} W_1 \times W_2 & \to & V \\ (w_1, w_2) & \mapsto & w_1 - w_2 \end{array}$$

which we already saw is isomorphic to $W_1 \cap W_2$. This easily extends to a finite product of multiple Grassmannians.

B.13) Lemma: In the context of 1.47 let $\mathfrak{M}_\Psi^{fr}$ be a fine moduli space for $\mathcal{M}_\Psi^{fr}$. Then a scheme $\mathfrak{M}_\Psi$ has a $\mathsf{GL}_r(\mathbb{C})$-invariant morphism of schemes $\mathfrak{M}_\Psi^{fr} \to \mathfrak{M}_\Psi$ if and only if there is a functor $\mathcal{M}_\Psi \to h^{\mathfrak{M}_\Psi}$.

For the proof: Consider a $\mathsf{GL}_r(\mathbb{C})$-invariant morphism $\mathfrak{M}_\Psi^{fr} \to \mathfrak{M}_\Psi$ and pick a family of toric bundles over S. Locally — that is, over an affine open set — we can choose a frame, giving us a locally framed family of toric bundles. Since $\mathfrak{M}_\Psi^{fr}$ is a fine moduli space, there are local morphisms from S to $\mathfrak{M}_\Psi^{fr}$. These morphisms descend to $\mathfrak{M}_\Psi$, and they must agree in intersections since different frames fall in the same $\mathsf{GL}_r(\mathbb{C})$-Orbit. This means those local morphisms glue to a global morphism $S \to \mathfrak{M}_\Psi$. This induces a functor $\mathcal{M}_\Psi \to h^{\mathfrak{M}_\Psi}$.

On the other hand, consider we have such a functor. The functor $h^{\mathfrak{M}_\Psi^{fr}}$ is isomorphic to $\mathcal{M}_\Psi^{fr}$ and removing the frame gives a forgetful functor $\mathcal{M}_\Psi^{fr} \to \mathcal{M}_\Psi$. Combining all those functors gives a functor $h^{\mathfrak{M}_\Psi^{fr}} \to h^{\mathfrak{M}_\Psi}$, and the Yoneda lemma provides a scheme morphism $\mathfrak{M}_\Psi^{fr} \to \mathfrak{M}_\Psi$, which must be $\mathsf{GL}_r(\mathbb{C})$-invariant by construction.

B.14) PROOF OF 1.10: A short proof is provided by [Pay07, Proposition 2.2]: Let $\mathcal{E}$ be a toric vector bundle on U_σ and x_σ a point in the minimal T-Orbit $O_\sigma \subset U_\sigma$[†]. Choose T-eigensections $s_1, \dots, s_r$ such that $\{s_i(x_\sigma)\}$ form a basis of the fiber $\mathcal{E}_{x_\sigma}$. Such a set exists since over an affine set there are enough sections, and actions of a torus over a vector space are always diagonalisable[†]. These sections form a basis for every point since the set of points where they do not is closed, T-Invariant and does not include x_σ, therefore it must be empty[†]. As these are eigensections, every s_i spans a toric line bundle that must be isomorphic[†] to $\mathcal{L}_{[u_i]}$, for some u_i — which may or may not be the eigenvalue to s_i. We get $\mathcal{E} \cong \bigoplus_i \mathcal{L}_{[u_i]}$.

see b.5

[Kly90, Proposition 2.1.1]

see b.6

see 1.8

Quod erat demonstrandum

B.15) PROOF OF 1.41: We may assign to $\mathbb{u}(\sigma)$ a partial order, where $[u] \geq [u']$ if and only if — as a function over σ — the difference $[u] - [u']$ is nonnegative. By moving down throughout $\mathbb{u}(\sigma)$ with respect to this order, we can inductively choose $\mathsf{E}_{[u]}$ in the following matter: If $[u]$ is maximal, then the multiplicity of $[u]$ in $\mathbb{u}(\sigma)$ is exactly $\dim \bigcap_\rho \mathsf{E}^\rho(\langle u, v_\rho \rangle)$. So we can set

$$\mathsf{E}_{[u]} = \bigcap_{\rho \text{ ray of } \sigma} \mathsf{E}^\rho(\langle u, v_\rho \rangle).$$

Starting from such a maximal $[u]$ and moving down, the index $\langle u, v_\rho \rangle$ decreases. That means we are moving left in the filtrations, so the intersection above increases. In particular, $\dim \bigcap_\rho \mathsf{E}^\rho(\langle u, v_\rho \rangle)$ is given by $\#\{[u'] \in \mathbb{u}(\sigma) | [u'] \geq [u]\}$. Therefore, let $\mathsf{E}_{[u']}$ be fixed by the induction hypothesis for every $[u'] > [u]$. We have

$$\sum_{[u'] > [u]} \mathsf{E}_{[u']} \subset \bigcap_{\rho \text{ ray of } \sigma} \mathsf{E}^\rho(\langle u, v_\rho \rangle).$$

Choose also $\mathsf{E}_{[u]}$ to be any subspace of $\bigcap_\rho \mathsf{E}^\rho(\langle u, v_\rho \rangle)$ complementary to $\sum_{[u'] < [u]} \mathsf{E}_{[u']}$. Independently of our choices, we get at the end of the process a direct sum $\mathsf{E} = \oplus \mathsf{E}_{[u]}$, since all filtrations eventually stabilize at E — to see that choose $i_\rho \ll 0$. Finally we check that for any ray ρ and integer i

$$\mathsf{E}^\rho(i) = \sum_{\langle u, v_\rho \rangle \geq i} \mathsf{E}_{[u]}. \tag{*}$$

For that, we apply the convexity of the cone σ: find an $[u]$ such that $\langle u, v_\rho \rangle = i$ and $\langle u, v_{\rho'} \rangle \ll 0$ for all other rays then $\bigcap_{\rho'} \mathsf{E}^{\rho'}(\langle u, v_{\rho'} \rangle) = \mathsf{E}^\rho(i)$. Such $[u]$ may not be in $\mathbb{u}(\sigma)$, but there is a $[u] \in \mathbb{u}(\sigma)$ just before the dimension of the intersection jumps down. Use this $[u]$, and eq. (*) follows. *Quod erat demonstrandum*

B.16) Observation: [Pay07] requires for framed families that $\mathsf{E} := \mathscr{E}|_{x_0 \times S}$ be a trivial bundle while for unframed families it does not. Since we do not know of any fine moduli space for the unframed case, the precise definition of unframed families is not relevant. However, for the framed case it is; the bundle E must be trivial, since those are the families we get via pullbacks of the universal family.

B.17) Proof of 1.49: The equivalence of categories is in one direction given by associating to an S-family the vector bundle $\mathsf{E} := \mathscr{E}|_{x_0 \times S}$ and the filtration $\mathsf{E}^\rho(i)$ assigned according to 1.48 by the image of E^ρ_u, where $i := \langle u, v_\rho \rangle$. Since the rank is defined over any single geometric point $s \in S$ and the multiset $\mathbb{u}(\sigma)$ is constant over all S, the rank condition† follows from the classification theorem† for $\mathscr{E}|_{X \times s}$. A morphism of S-families preserves the relative torus action, and therefore restriction to $\mathscr{E}|_{x_0 \times S}$ gives a morphism of vector bundles with filtrations.

see 1.41
see 1.21

On the other hand, let $(\mathsf{E}, \{\mathsf{E}^\rho(i)\})$ be a vector bundle over S with filtrations satisfying the rank conditions for our fixed collection of multisets. Define the vector bundle on S

$$\mathsf{E}^\sigma_u = \bigcap_{\rho \text{ ray of } \sigma} \mathsf{E}^\rho(\langle u, v_\rho \rangle)$$

see 1.21

for any $\sigma \in \Sigma$ and $u \in \mathsf{M}$. Construct as previously† a $\mathbb{C}[U_\sigma \times S]$-module

$$\mathsf{E}^\sigma = \bigoplus_{u \in \mathsf{M}} \mathsf{E}^\sigma_u$$

where multiplication by $\chi_{u'}$ is given by the natural inclusion $\mathsf{E}^\sigma_u \subset \mathsf{E}^\sigma_{u-u'}$. This induces a T_S-equivariant sheaf $\tilde{\mathsf{E}}^\sigma$ on $U_\sigma \times S$ and these sheaves glue together to form an S-family of toric vector bundles $\mathscr{E}$ on X. A morphism of vector bundles with filtrations induces a morphism of S-families of toric vector bundles by the same arguments as before. The functor so defined is inverse to the functor $\mathscr{E} \to (\mathsf{E}, \{\mathsf{E}^{\rho}(i)\})$ up to natural isomorphism. This results in the desired equivalence of categories. *Quod erat demonstrandum*

C Notes on the Second Chapter

C.1) Observation: Altmann and Witt proved using the convexity of the fan, that for degrees far away from the origin there is no pre-Cohiggs field [AW21, Proposition 19]. The proof even gives a concrete bound. They do not use this formalism, but it is evident in the proof that this bound does not depend on the flag, just on the multiset given by the toric bundle. This means the same proposition is also valid for families of toric bundles with constant toric Chern class.

D Notes on the Third Chapter

D.1) Proposition: Let E and F be two complex vector bundles over a scheme X, by this we mean two linear X-schemes corresponding to some locally free coherent sheaf over X according to 2.16. Let $q : E \to F$ be a morphism of vector bundles, i.e. a X-morphism between E and F which is linear on the fibers. Then $\dim\{v \in E_x : q(v) = 0\}$ is an upper-semi-continuous function on the Zariski topology of X.

D.2) Proof of d.1: According to 2.16 we may formulate the questions as a statement on sheaves: applying ν to an open subset $U \hookrightarrow X$ provides a mapping between sections of F and maps $U \to F$, those can be pulled back to $U \to E$ via q and therefore correspond to sections of E. This means q is a sheaf-morphism of the corresponding locally free coherent sheaves of E and F. Let $\ker q$ be kernel-sheaf. The statement is, therefore, that the dimension of the fiber $\ker q|_x$ is upper-semi-continuous.

As a kernel sheaf of coherent sheaves $\ker q$ is also a coherent sheaf and the identity $f : X \to X$ is trivially a flat morphism, therefore [Har10, III Theorem 12.8] states that $\dim H^{(i)}(X_x, (\ker q)_x)$ is upper-semi-continuous, where $H^{(i)}$ is the i-th right derived functors of the functor of sections. However $(\ker q)_x$ is just the stalk over the point $x := X_x$ consisting of the fiber $\ker q|_x$. Since

$$H^0(X_x, (\ker q)_x) = \Gamma(X_x, (\ker q)_x) = \ker q|_x$$

the proposition follows by considering $i = 0$. *Quod erat demonstrandum*

D.3) Observation: To see that maximal span and minimal intersection are distinct conditions, consider the following two examples: First, three lines through the origin in $\mathbb{C}^3$, all falling in the same 2-dimensional plane. The intersection of any subset of those is always 0, therefore minimal. However, they do not span $\mathbb{C}^3$. The second example are three planes through the origin in $\mathbb{C}^3$, all distinct but containing the same line. The intersection of the 3 planes is of dimension one, therefore not minimal since the minimal intersection is a single point. However, the planes are distinct, so any two of those already span $\mathbb{C}^3$.

D.4) Lemme de Descente: Let X be a variety with an action of G, and X/G a geometric quotient. Let E be a vector bundle over X with a transitive action of G. Then there is a vector bundle E/G over X/G if and only if the stabilizer G_x of a point $x \in X$ in a closed G-orbit acts trivially on the fiber E_x.

For the proof: [DN89, Théorème 2.3]

D.5) Proof of 3.13: On the conditions of the theorem, the space of framed toric bundles is a product of projective spaces $(\mathbb{P}^{r-1})^n$: for every ray ρ the filtration is defined by a 1-dimensional subvector space V_ρ on $\mathbb{C}^r$. A change of frame acts as $\mathsf{GL}_r(\mathbb{C})$ on V_ρ. The space of unframed toric bundles is nothing else than the space of projective configurations $(\mathbb{P}^{r-1})^n//\mathsf{GL}_r(\mathbb{C})$. This is a well-known example, found for instance in [Hu08] and [HKT19]. For the proof, we follow [Kap93, Proposition 2.1.7]. To arrive at the conclusion, we only need to show that the closure of orbits in $\mathfrak{M}_\Psi^{fr^{gen}}$ has constant homology. We will even be able to exactly calculate this homology for $n > r+1$ — in the opposite case $n \leq r+1$ there is only one generic orbit as seen in 1.51 and the theorem is trivial. This means we need to only prove the following:

D.5A) Assertion: Let $[m_i]$ be the homology class of the projective space $\mathbb{P}^{m_i}$ inside $\mathbb{P}^{r-1}$, then according to the Künneth formula the elements $[m_1] \otimes \cdots \otimes [m_n]$ form a basis of the graded homology of $(\mathbb{P}^{r-1})^n$. The closure of a $\mathsf{GL}_r(\mathbb{C})$-Orbit in $\mathfrak{M}_\Psi^{fr^{gen}}$ — for $n > r+1$ — has homology class given by

$$\delta = \sum_{m_1+\cdots+m_n=r^2+1} [m_1] \otimes \cdots \otimes [m_n].$$

For linearly generic points the $\mathsf{GL}_r(\mathbb{C})$-stabilizers are just multiples of the identity. To see that just consider for instance how the only linear transformations fixing $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ are diagonal matrixes. From those the only ones fixing $[1:1:1]$ are the multiple of the identity. This reflects the fact that $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$ are linearly generic. On the other hand, $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:0]$ are not linearly generic and therefore have non-trivial stabilizers. Any set of linearly generic points can be reduced to an example like this by just choosing r of those points to form a basis.

The dimension of an orbit is the dimension of the group minus the dimension of the stabilizer. Therefore for points in $\mathfrak{M}_\Psi^{fr^{gen}}$ the orbits have dimension r^2-1. So the closure of the orbit has homology class in $\mathrm{H}_{2(r^2-1)}((\mathbb{P}^{r-1})^n)$, that means it must be of the form

$$\sum a_{m_1,\cdots,m_n} [m_1] \otimes \cdots \otimes [m_n]$$

where $\sum_{i=1}^{n} m_i = r^2 - 1$. That makes the formula in the assertion plausible. We only have to prove that the coefficient $a_{m_1,\cdots,m_n}$ of every basis element is indeed 1. From general homology theory we know that we can count the coefficients by intersecting with the dual of a basis element, which is a generic linear space of respective codimension. That is to say, for a point $x \in \mathfrak{M}_{\Psi}^{fr^{gen}}$ the intersection of $\overline{\mathsf{GL}_r(\mathbb{C}) \cdot x}$ with a generic $L_1 \times \cdots \times L_n$ should be exactly one point, where L_i is a linear subspace of $\mathbb{P}^{r-1}$ with codimension m_i. To do that we first use a linear argument to show the intersection is at most one point. Then we count dimensions and use 3.11 to show it must be exactly one point.

The intersections between $\overline{\mathsf{GL}_r(\mathbb{C}) \cdot x}$ and $L_1 \times \cdots \times L_n$ are equivalent to finding a $r \times r$ matrix g with the property

$$g(x_i) \in L_i \text{ for every } i \tag{*}$$

where $x = (x_1, \cdots, x_n)$ — for notation's sake we do not differentiate between the linear subspace $L_i \subset \mathbb{C}^r$ and its image in the projective space $\mathbb{P}^{r-1}$. These equations form a linear system, the matrix g has r^2 variables and every equation $g(x_i) \in L_i$ adds m_i restrictions to this system. One way to see that is to write

$$g(x_i) = b_1 v_1 + \cdots + b_{r-m_i} v_{r-m_i}$$

where the v's form a basis of L_i. This equation adds r linear equalities to the system, but also $r - m_i$ new unknowns — the coefficients $b_1, \cdots b_{r-m_i}$. That means the whole system has r^2 variables and $r^2 - 1$ restrictions, and the x_i's are linearly generic†. So there is at least a 1-dimensional subvector space of solutions for g. However for dimension reasons and generality of $L_1 \times \cdots \times L_n$ the intersection with $\overline{\mathsf{GL}_r(\mathbb{C}) \cdot x}$ must be a discrete set of points in $(\mathbb{P}^{r-1})^n$ (Bertini's the-

see 3.11

orem). So the set of solutions must be 1-dimensional. So the cooeficients of $m_1 \otimes \cdots \otimes m_n$ is at most one. However if the solutions we get for g are degenerated matrixes it could happen that $g(x_i) = 0$ and this would not correspond to a point in $\mathbb{P}^{r-1}$. So we are still not finished.

To conclude that g is not degenerated, we must invert our logic. Instead of acting with $\mathsf{GL}_r(\mathbb{C})$ in $\mathfrak{M}_\Psi^{fr}$, we act in the space of tuples $(L_1, \cdots, L_n)$. Let Π be this space — which is a product of Grassmanians — and let Π_Y be the subspace such that $x_i \in L_i$. If the matrix g is degenerated, it means that the union of $\mathsf{GL}_r(\mathbb{C})$-orbits of points in Π_Y is not dense in Π. To see that start with a $(L'_1, \cdots, L'_n) \in \Pi_Y$. If g is degenerated, $(gL'_1, \cdots, gL'_n)$ is not generic since

$$\sum_i gL'_i \subset g(\mathbb{C}^r) \subsetneq \mathbb{C}^r.$$

If we count dimensions we get

$$\sum_i \dim L_i = nr - r^2 - 1$$

but $n > r + 1$ so a generic tuple must satisfy

$$\sum_i L_i = \mathbb{C}^r.$$

That means even though we have a solution $g(x_i) \in L_i$ for generic but fixed L_i's, we cannot find $(L'_1, \cdots, L'_n) \in \Pi_Y$ such that $(L_1, \cdots, L_n)$ is in the closure of its orbit. The codimension of Π_Y in Π is $r^2 - 1$, since

$$L_i \in \mathsf{Gr}(r - m_i, \mathbb{C}^r)$$

and the locus $x_1 \in L_1$ is isomorphic to $\mathsf{Gr}(r - m_i - 1, \mathbb{C}^{r-1})$,. Hence, counting the dimension of this Grassmannians we get to codimension m_i and again $\sum_i m_i = r^2 - 1$. That means a degenerated matrix g implies that the generic tuple $(L'_1, \cdots, L'_n) \in \Pi_Y$ has a $\mathsf{PGL}_r(\mathbb{C})$-stabilizer with positive dimension. We will show this is not the case.

For this we use that x is linearly generic. Let us choose L_i's, with $x_i \in L_i$, such that the stabilizer is trivial. For every L_i the equation $gL_i = L_i$ reduces the stabilizer in $m_i(r - m_i)$ dimensions. Since $\sum m_i = r^2 - 1$ even in the worse case $m_i = 1$ the system is still overdetermined. To be more concrete, in this case one could pick L_i to be the span of a $r - 1$ subset from $\{x_1, \cdots, x_n\}$ and also pick the x_j's to always minimise the dimension of any intersection $\hat{L} = L_{j_1} \cap L_{j_2} \cap \cdots \cap L_{j_{r-1}}$, this is possible because $\{x_1, \cdots, x_n\}$ is linear generic. There are more than $r + 1$ such $\hat{L}$'s and they are all one-dimensional. The only possible stabilizer of $(L_1, \cdots, L_n)$ are the multiples of the identity.

This means the g provided as solution to the linear system is invertible and provides one single intersection point between $\overline{\mathsf{GL}_r(\mathbb{C}) \cdot x}$ and $L_1 \times \cdots \times L_n$.

Quod erat demonstrandum

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