Supercharacters and generalized Gelfand–Graev characters for orthogonal groups

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Vorgelegt von

Denis Pablo Weiler
aus Berlin

Hauptberichter:	Professor Dr. Meinolf Geck
Mitberichter:	Professor Dr. Richard Dipper
	Professor Dr. Nathaniel Thiem
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Institut für Algebra und Zahlentheorie der Universität Stuttgart

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Zusammenfassung

Die verallgemeinerten Gelfand–Graev Charaktere sind ein wichtiges Mittel zur Analyse von unipotenten Charakteren der endlichen Gruppen vom Lie Typ. Allerdings ist ihre Konstruktion bekannterweise mit großen Schwierigkeiten verbunden, da sie unter anderem von einer nicht eindeutig definierten unipotenten Untergruppe der zugrundeliegenden algebraischen Gruppe abhängt. Deshalb war es von großem Nutzen, dass S. Andrews und N. Thiem in der Lage waren, die verallgemeinerten Gelfand–Graev Charaktere der endlichen speziellen linearen Gruppe aus Supercharakteren der endlichen Gruppe von unitriangulären Matrizen zu erzeugen. Ein unmittelbarer Gedanke ist es, den Zusammenhang zwischen den verallgemeinerten Gelfand–Graev Charakteren und der Supercharaktertheorie auf andere endliche Gruppen vom Lie Typ auszuweiten. So wird in dieser Arbeit der Fall der endlichen speziellen orthogonalen Gruppe mit gerader Dimension und guter Charakteristik untersucht.
Diese Supercharaktertheorie der endlichen Gruppe von unitriangulären Matrizen wurde urspünglich von N. Yan eingeführt, um eine Annäherung der Klassifikation der irreduziblen Charaktere dieser Gruppe, was an sich ein wildes Problem ist, zu schaffen. In ihrem Mittelpunkt steht ein 1-Cozykel von der Gruppe der unitriangulären Matrizen in ihre Algebra. Um ihren Gebrauch auf weitere endliche Gruppen vom Lie Typ auszudehnen, haben C. A. M. André und A. M. Neto Supercharaktertheorien für die maximalen unipotenten Untergruppen der endlichen Gruppen vom Typ B_n, C_n und D_n definiert, indem sie sogenannte elementare Charaktere nutzten, die induziert von linearen Charakteren der Wurzeluntergruppen sind. Um zum Gebrauch eines 1-Cozykel zurückzukommen, hat M. Jedlitschky die Supercharaktere von C. A. M. André und A. M. Neto der speziellen orthogonalen Gruppe mit gerader Dimension und guter Charakteristik zerlegt, die zwar nicht mehr eine Supercharaktertheorie bilden, da es für sie keine zugehörige Menge der Superklassen gibt, aber die restlichen Eigenschaften der Supercharaktertheorie beibehalten. Für die Klassifikation dieser Charaktere kann eine Abwandlung einer Gram Matrix für jeden solchen Charakter definiert werden, die es nicht nur möglich macht identische Charaktere zu identifizieren, sondern die Information über ihre Irreduzierbarkeit enthält.

Während es S. Andrews und N. Thiem möglich war, die verallgemeinerten Gelfand–Graev Charaktere der endlichen speziellen linearen Gruppe direkt aus den Supercharakteren ihrer maximalen unipotenten Untergruppe zu bilden, was die aufwendige Konstruktion, die N. Kawanaka definiert hatte, umgeht, ist dasselbe für die verallgemeinerten Gelfand–Graev Charaktere der endlichen speziellen orthogonalen Gruppe mit gerader Dimension nicht möglich, da die Supercharaktere von C. A. M. André und A. M. Neto im Allgemeinenen nicht dafür geeignet sind. Allerdings ist es möglich, mithilfe der zuvor genannten Gram Matrix, die von M. Jedlitschky definierten Konstituenten dieser Supercharaktere nutzen, um die verallgemeinerten Gelfand–Graev Charaktere der endlichen speziellen orthogonalen Gruppe zu erzeugen.

Abstract

The generalized Gelfand–Graev characters defined by N. Kawanaka are an important tool for the analysis of unipotent characters of finite groups of Lie type. But their construction is notoriously difficult, as among other things it relies on a not uniquely defined unipotent subgroup of the underlying algebraic group. Therefore, it was of great benefit that S. Andrews and N. Thiem were able to construct generalized Gelfand–Graev characters for the finite special linear group from supercharacters of the finite group of unitriangular matrices. An immediate idea is to expand this connection between generalized Gelfand–Graev characters and supercharacter theory to other finite groups of Lie type. We will investigate the case of the finite special orthogonal group of even dimension in good characteristic.

This supercharacter theory of the finite group of unitriangular matrices was originally introduced by N. Yan for the purpose of approximating the classification of the irreducible characters of this group, which in itself is a wild problem. It centers around a 1-cocycle from the group of unitriangular matrices onto its algebra. Expanding their utilization to other finite groups of Lie type, C. A. M. André and A. M. Neto defined such supercharacter theories for the maximal unipotent subgroups of the finite groups of type B_n, C_n and D_n by using so-called elementary characters which are induced from linear characters of root subgroups. Reintroducing the use of a 1-cocycle M. Jedlitschky decomposed the supercharacters of C. A. M. André and A. M. Neto for the finite special orthogonal group of even dimension in good characteristic, which no longer form a supercharacter theory owing to not having an accompanying set of superclasses but still retain the other properties of a supercharacter theory. For the classification of these characters, we can define a derivation of a Gram matrix for each such character, which not only allows us to identify identical characters but also contains the information about the irreducibility of such characters.

While S. Andrews and N. Thiem were able to establish the generalized Gelfand–Graev characters of the finite special linear group directly from supercharacters of its maximal unipotent subgroup, avoiding the laborious construction defined by N. Kawanaka, the extension of the link between supercharacter theories and generalized Gelfand–Graev characters to the case of

the special orthogonal group of even dimension is not immediately possible, as the supercharacters defined by C. A. M. André and A. M. Neto in general do not fit this purpose. Yet, with the aid of the aforementioned Gram matrix, we can use the constituents of these supercharacters defined by M. Jedlitschky to produce the generalized Gelfand–Graev characters of the finite special orthogonal group.

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Introduction

For the study of representation theory of finite groups Persi Diaconis and I. Martin Isaacs [DI08] created an abstaction of the duality of irreducible characters and conjugacy classes, called a supercharacter theory. Such a supercharacter theory consists of a set $\mathcal{X}$ of characters called supercharacters, such that every irreducible character is constituent of exactly one supercharacter, and a set $\mathcal{K}$ of unions of conjugacy classes called superclasses, such that every supercharacter of the group is constant on every superclass, as well as $|\mathcal{X}| = |\mathcal{K}|$. The greatest possible supercharacter theory consists of the set irreducible characters and the set conjugacy classes, while the smallest possible supercharacter theory has the principal character together with the sum of all other irreducible characters as supercharacters and the identity element together with the set of all other group elements as superclasses.

This concept was a generalization of the supercharacter theory for $UT_n = UT_n(\mathbb{F}_q)$, the group of upper unitirangular $n \times n$ matrices over a finite field $\mathbb{F}_q$, where q is the power of some prime p, created by Carlos A. M. André [And95a] and later Ning Yan [Yan01] independently.[1] The classification of the conjugacy classes of UT_n is a known wild problem, so Yan designed a system that can approximate this classification. He used the orbit method established by Alexandre A. Kirillov [Kir62] to construct UT_n- UT_n-biorbits on both the algebra of the nilpotent upper $n \times n$ matrices $\mathfrak{ut}_n$ and its dual space $\mathfrak{ut}_n^*$, where each such biorbit is a disjoint union of adjoint orbits. For the construction of these supercharacters it is essential that both the left and right multiplication of UT_n on $\mathfrak{ut}_n$ together with the map $f : UT_n \to \mathfrak{ut}_n$ with $f(g) = g - I$ for $g \in UT_n$ define a 1-cocycle, that is UT_n acts on $\mathfrak{ut}_n$ with both $gh - I = g(h - I) + (g - I) = (g - I)h + (h - I)$ for $g, h \in UT_n$. This way the UT_n- UT_n-biorbits of $\mathfrak{ut}_n$ applied to f^{-1} are the superclasses, while the UT_n- UT_n-biorbits of $\mathfrak{ut}_n^*$ give rise to the supercharacters by taking the sum over all elements of such a UT_n- UT_n-biorbit applied to a non-trivial group homomorphism from $\mathbb{F}_q$ to $\mathbb{C}^*$.

For other finite groups of Lie type we do not have access to such constructions of biorbits, but for the other classical finite unipotent groups of type $B_n(q)$, $C_n(q)$ and $D_n(q)$ André and

[1] see also [Yan10] as well as [And95b], [And01], [And02] and [And03]

Ana M. Neto [AN06] were able to adjust André's previous work for the type $A_n(q)$ to create supercharacter theories for the other.[2] These supercharacters are products of so-called elementary characters, which in turn are induced from linear characters of root subgroups. We will focus here on the type $D_n(q)$ and the finite special orthogonal group SO_N for even $N = 2n$ in particular. The special orthogonal group SO_N depends on a symmetric bilinear form b on ${\mathbb{F}_q}^N$, which gives rise to an anti-involution $\cdot^\dagger : M_N(\mathbb{F}_q) \to M_N(\mathbb{F}_q)$ with $g^\dagger g = I$ for $g \in \mathrm{SO}_N$. If we choose this bilinear form such that $b(u, v) = u^t J_N v$ for $u, v \in {\mathbb{F}_q}^N$, where $J_N \in M_N(\mathbb{F}_q)$ is the counter-diagonal matrix

$$J_N = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix},$$

then the group of unitriangular matrices $U_N = UT_N \cap \mathrm{SO}_N$ in SO_N is a maximal unipotent group in SO_N. Scott Andrews [And15] has shown that André's and Neto's supercharacter theory is again based on UT_N-orbits in $\mathfrak{u}_N$ and $\mathfrak{u}_N^*$ respectively using the fact that the group UT_N acts on $\mathfrak{u}_N$ by $g * X = g^\dagger X g \in \mathfrak{u}_N$ for $g \in UT_N$ and $X \in \mathfrak{u}_N$.
Reintroducing the use of a 1-cocycle, Markus Jedlitschky [Jed13] decomposed the supercharacters of André and Neto for the maximal unipotent group U_N of the special orthogonal group into characters, such that they retain the property that every irreducible character is constituent of exactly one such character but do not admit a corresponding set of superclasses. This concept was later expanded upon Qiong Guo and Richard Dipper [DG15], and it will be the topic of the first part of this thesis.[3] The map $\pi : M_N(\mathbb{F}_q) \to \mathbf{v}$ that restricts a matrix to the vector space $\mathbf{v}$ of the matrices with non-zero entries only strictly above both the diagonal and counter-diagonal then admits a 1-cocycle together with the right action of UT_N on $\mathbf{v}$ defined by $V \circ g = \pi(Vg)$ for $V \in \mathbf{v}$ and $g \in UT_N$, such that we have $\pi(gh) = \pi(g) \circ h + \pi(h)$ for $g, h \in UT_N$. As π restricted to U_N is a bijection to $\mathbf{v}$, it allows for the construction of characters for the U_N-orbits in the dual space $\mathbf{v}$, which are represented by certain elements in $\mathbf{v}$ called core patterns. These characters are constructed by taking the sum over all elements of an U_N-orbit evaluated at the average value of a conjugacy class of U_N and applied to a non-trivial group homomorphism from $\mathbb{F}_q$ to $\mathbb{C}^*$. We will establish a direct link to the characters of André and Neto as they are the characters constructed in the same way over the sum of certain UT_N-orbits in $\mathbf{v}$, which are represented by elements $\mathbf{v}$ called verge patterns. Every core pattern in $\mathbf{v}$ is predi-

[2] see also [AN09] and [AN08].
[3] see also [GJD18] and [GJD19]

cated on a verge pattern in whose UT_N-orbit it is contained, and characters for core patterns based on different verge patterns are mutually orthogonal. All characters based on the same verge pattern, that is, their respective core patterns are contained in the same UT_N-orbit, are either identical or mutually orthogonal. But since the aforementioned 1-cocycle defined for a left action of UT_N on $\mathbf{v}$ does not commute with its right action, finding identical characters is not a straight forward task as it is for the unipotent group of type A_n, where supercharacters are identical if and only if they are based on the same UT_n- UT_n-biorbit. So in order to classify these characters, we will construct an $n \times n$ matrix, derived from a Gram matrix based on the bilinear form on ${\mathbb{F}_q}^N$, that defines the orthogonal group, which is constant on any U_N-orbit and congruent under the operation of a certain group to such a matrix for any identical character.

This Gram matrix has versatile applications, as it can not only be used for the classification of the characters for core patterns, but also for the construction of core patterns representing the different unipotent conjugacy classes of the finite special orthogonal group. Tony A. Springer and Robert Steinberg [SS70] have shown that classifying the unipotent conjugacy classes of an algebraic group $\overline{G}$ or the nilpotent orbits of its Lie algebra $\overline{\mathfrak{ut}}$ can be done by using the Jacobson-Morozov theorem by Nathan Jacobson and Valery V. Morozov [Jac79] if the characteristic of the field on which the group is based is either zero or a large enough prime. Here any nilpotent element $A \in \overline{\mathfrak{ut}}$ of the Lie algebra is linked with another nilpotent element $B \in \overline{\mathfrak{ut}}$ as well as an element $T \in \overline{\mathfrak{ut}}$ of the Cartan subalgebra such that there is an algebra isomorphism from the two-dimensional special linear Lie algebra $\mathfrak{sl}_2$ to the subalgebra spanned by the triple $\{A, B, T\}$. This allows for the imposition of a $\mathbb{Z}$-grading of the Lie algebra such that $\overline{\mathfrak{ut}} = \bigoplus_{z\in\mathbb{Z}} \overline{\mathfrak{ut}}_A(z)$ for subalgebras $\overline{\mathfrak{ut}}_A(z) \leq \overline{\mathfrak{ut}}$ with $A \in \overline{\mathfrak{ut}}_A(2)$. Furthermore, there is a parabolic subgroup $\overline{P}_A \leq \overline{G}$ linked to the nilpotent element of A and a descending series of normal unipotent subgroups $\overline{U}_{A,i} \trianglelefteq \overline{P}_A$ of $\overline{P}_A$ for $i \in \mathbb{N}$, where $\overline{U}_{A,i}$ is the unipotent radical of $\overline{P}_A$, such that $\mathrm{Lie}(\overline{P}_A) = \bigoplus_{z\geq 0} \overline{\mathfrak{ut}}_A(z)$ and $\mathrm{Lie}(\overline{U}_{A,i}) = \bigoplus_{z\geq i} \overline{\mathfrak{ut}}_A(z)$. A further approach was taken by Pawan Bala and Roger W. Carter [BC74] by using distinguished parabolic subgroups of Levi complements for the classification of the nilpotent orbits, which Klaus Pommerening [Pom77] has proven also holds for smaller characteristic if that characteristic is good.[4]

Nilpotent orbits of classical groups can be linked to partitions of the dimension of the group by calculating the rank of the powers of their representatives, which in turn identifies the dominance order on the set of partitions with the order on the nilpotent orbits, where an orbit is less or equal than another orbit if the first is contained in the algebraic closure of the latter. The nilpotent orbits of the Lie algebra of the general linear group $\overline{\mathrm{Gl}}_N$ are represented by

[4] see also [Pom80]

all partitions of N, whereas the nilpotent orbits of the Lie algebra of the special orthogonal group $\overline{\mathrm{SO}}_N$ are represented by partitions of N where every even element occurs with even multiplicity, unless the partition only comprises even elements, in which case there are two nilpotent orbits for each such partition. Relating this to the finite groups, the fixed points of nilpotent orbits of the Lie algebra of the general linear group are themselves nilpotent orbits of the finite algebra, while the fixed points of nilpotent orbits of the Lie algebra of the special orthogonal group split into multiple nilpotent orbits of the finite algebra depending on the number of unique odd elements in their respective partition. Representatives of these nilpotent orbits can be can be constructed from the core patterns defined by Jedlitschky by again using the aforementioned Gram matrix.

Linking the topic of supercharacter theory to another vital concept in the field of representation theory, Andrews and Nathaniel Thiem [AT17] were able to describe generalized Gelfand–Graev characters for the classical finite group of type $A_n(q)$ by the method of supercharacters of UT_{n+1}. The Gelfand–Graev character, developed by Israïl M. Gel'fand and Mark I. Graev [GG62], is the character of a finite group of Lie type induced from a character of a maximal unipotent subgroup in general position. This character is not irreducible, but all its irreducible constituents occur with multiplicity one. From this character, Noriaki Kawanaka [Kaw85] derived the generalized Gelfand–Graev characters in order to prove Veikko Ennola's [Enn63] conjecture on the irreducible characters of the finite unitary group. These are characters of a finite group of Lie type representing each unipotent conjugacy class of the corresponding algebraic group, and they are constructed by inducing a character related to each such conjugacy class from a unipotent subgroup $U_{1.5}$, situated between the groups U_1 and U_2 that is maximal such that this character is linear. Using George Lusztig's [Lus92] concept of unipotent support, Meinolf Geck and David Hézard [GH07] have shown that if the prime p is large enough the generalized Gelfand–Graev characters are up to their rank fully determined by the fixed points of the unipotent conjugacy classes of the algebraic group on which the character as well as its dual character vanishes, where the dual character is here with regard to the Alvis-Curtis-Kawanaka duality defined by both Charles W. Curtis [Cur80] and Kawanaka [Kaw81] independently. Imposing the dominance order on the set of unipotent conjugacy classes of the algebraic group, a generalized Gelfand–Graev character then vanishes on every unipotent conjugacy class that is not smaller than the conjugacy class aligned to the character, while its dual vanishes on every unipotent conjugacy class that is not larger than this conjugacy class. Here the Gelfand–Graev character, considered the character aligned to the largest unipotent conjugacy class of regular unipotent elements, vanishes on no unipotent conjugacy class, whereas its dual is zero for every unipotent element but the regular unipotent elements. The generalized Gelfand–Graev characters are notoriously hard to construct, among other things, because the unipotent group from which they are induced is not uniquely defined. This makes the Andrews' and Thiem's approach appealing, as we can circumvent this complex construction and use characters induced from the group of upper unitriangular matrices instead. It is a palpable idea to try the same method for the special orthogonal group with the supercharacters of André and Neto, but these supercharacters are in general too "large" for this cause. Yet, the characters defined by Jedlitschky, which are constituents of such supercharacters, fit this purpose. While unlike in the case of the general linear group, we cannot define the group $U_{1.5}$ from which the generalized Gelfand–Graev character is induced, but we can construct a suitable character for the group U_1 containing $U_{1.5}$. Imposing comparable restrictions on core patterns for nilpotent orbits to the "non-nesting" condition for nilpotent elements by Andrews and Thiem, we can reduce the corresponding character to the normal unipotent group U_1 and induced again to the full group SO_N we obtain the generalized Gelfand–Graev character for this nilpotent orbit up to some scalar. If we impose further restrictions on this core pattern, we can find a set of core patterns such that the sum of their characters induced to SO_N is just this character. This way we can describe the generalized Gelfand–Graev characters as sums of characters of U_N induced to SO_N over a set of core patterns, but the restrictions on the core pattern for this nilpotent orbit is so severe that there are not many choices left. Therefore, we will limit our endeavour to a singular core pattern for each nilpotent in which case the rank of the character is minimal, that is, it is as close to the rank of the corresponding generalized Gelfand–Graev character as possible.

Content of this thesis

Chapter one: The special orthogonal group

In chapter one we will outline basic results about the special orthogonal group over fields of finite characteristic. For this purpose in the first section we will give a rudimentary introduction into linear algebraic groups and their Lie algebras.
In the second section we define a bilinear form b on the vector space $\overline{\mathbb{F}}_q^N$ over the algebraically closed field $\overline{\mathbb{F}}_q$ for a "good prime" $2 \neq p \in \mathbb{N}$ such that the special orthogonal group $\overline{\mathrm{SO}_N}$ comprises the elements that preserve b and have determinant one. We choose the bilinear form b such that the group of upper triangular matrices in $\overline{\mathrm{SO}_N}$ is a Borel group and its unipotent

radical $\overline{U}_N \le \overline{\mathrm{SO}}_N$ is the group of upper unitriangular matrices, while the group of diagonal matrices in $\overline{\mathrm{SO}}_N$ is a maximal torus in this Borel group.
In the third section we introduce reduced root systems both for the general linear group GL_N and the special orthogonal group $\overline{\mathrm{SO}}_N$ for their respective maximal torus of diagonal matrices. With this we can define pattern subgroups for both groups, which are subgroups of the upper unitriangular matrix groups that are normalized by the maximal torus. A pattern subgroup of GL_N is defined by its support above the diagonal, that is the sets of positions above the diagonal of $N \times N$-matrices, on which entries can be non zero. As we will show in lemma 1.3.9, the same holds for pattern subgroups of $\overline{\mathrm{SO}}_N$, which are defined by their support above both the diagonal and counter-diagonal. As the map $x \mapsto x - I$ defines a bijection from the variety of unipotent elements in GL_N onto the variety of nilpotent elements of its Lie algebra, so does the Cayley transformation $x \mapsto (x-I)(x+I)^{-1}$ defined in lemma 1.3.12 for the variety of unipotent elements in $\overline{\mathrm{SO}}_N$, which moreover defines a bijection from the pattern subgroups of $\overline{\mathrm{SO}}_N$ onto their respective Lie algebra.
In the last section we will relate these linear algebraic groups over an algebraically closed field to their counterparts of groups over a finite field $\mathbb{F}_q$ where $q \in \mathbb{N}$ is a power of the prime p. For that purpose we use the standard Frobenius endomorphism, where its fixed points applied to $\overline{\mathrm{Gl}}_N$ and $\overline{\mathrm{SO}}_N$ are the finite linear algebraic group GL_N and the finite special orthogonal group SO_N, where the structure of the upper unitriangular matrix groups as well as the pattern subgroups is preserved.

Chapter two: Decomposition of supercharacters for SO_N

In chapter two we will classify the characters of U_N established by Jedlitschky, by extracting their information into a symmetric $n \times n$-matrix that is derived from a Gram matrix for the bilinear form b.
In the first section we will discuss the 1-cocycle of the group of upper unitiriangular matrices $UT_N \le \mathrm{GL}_N$ on the vector space $\mathbf{v}$ of $N \times N$-matrices with non-zero entries only above both the diagonal and couter-diagonal, which moves non-zero entries of $\mathbf{v}$ into positions to the right, and its "dual" group action, which moves non-zero entries of $\mathbf{v}$ into positions to the left.
In the second section we will focus on the super-character-theory of U_N introduced by André and Neto, which gives us a set of mutually orthogonal characters, each represented by an element in the nilpotent orthogonal algebra of upper triangular matrices with at most one non-zero entry per row and column, and the restriction of these representatives to $\mathbf{v}$ will be called

verge patterns. In theorem 2.2.14 we can then use the "dual" action of the 1-cocycle, defined in the previous section, to express these characters as sums over the UT_N-orbit of a verge pattern. In the third section we turn to the characters of U_N introduced by Jedlitschky, which are, as we will show in theorem 2.3.1, similarly defined as sums over U_N-orbits for the elements of the UT_N-orbit of a verge pattern. We can then immediately see that these characters are constituents of a character, defined in the previous section, if they are based on the same verge pattern. The UT_N-orbit for a verge pattern splits into different U_N-orbits and for the rest of this section we will restrict the discussion to the special case of U_N-orbits of verge patterns. These have multiple advantages, such as the stabilizer being a pattern subgroup, which we will show in lemma 2.3.2. Moreover, in theorem 2.3.6 we can show that for a verge pattern $A \in \mathbf{v}$ there is a pattern subgroup $R_A \leq U_N$ that defines all other pattern whose character is equal to the character of A.
In the last section we will generalize the previous discussion of characters of U_N-orbits for verge patterns to characters of all U_N-orbits in the UT_N-orbit of a verge pattern, which representatives are called core patterns. In lemma 2.4.2 we define a subset $D_A \subseteq UT_N$ for a verge pattern $A \in \mathbf{v}$ such that the D_A-orbit of A is the set of core patterns based on A. In theorem 2.4.6 we will then show that D_A is a set of representatives of the $(U_N, \mathrm{Stab}_{UT_N}(A))$-double coset of UT_N, which shows that indeed the core patterns are representatives of the U_N-orbits in the UT_N-orbit of a verge pattern. Returning to the subgroup R_A, defined in the previous section, in theorem 2.4.10 we will define a group action of it on D_A. Then two characters are mutually orthogonal unless they are based on the same verge pattern and their respective representatives are contained in the same orbit of this group action. Furthermore, the size of its stabilizer is equal to its inner product, which especially means that a character is irreducible if and only if its representative in D_A has a trivial stabilizer under the action of R_A. In lemma 2.4.12 we can then define a map from D_A into the set of symmetric $n \times n$-matrices, which is derived from a Gram matrix for the bilinear form b. We will then define an operation of R_A on these matrices in lemma 2.4.13, which maps one matrix to a congruent matrix, such that this action commutes with the action on D_A. This reduces the question of finding equal characters and whether they are irreducible to a task of finding congruent matrices with certain restrictions.

Chapter three: Classification of the nilpotent orbits

In chapter three we will classify the nilpotent U_N-orbits in the orthogonal algebra over both the algebraically closed field $\overline{\mathbb{F}}_q$ as well as the finite field $\mathbb{F}_q$. We can then use core patterns to produce representatives of these orbits for the finite case, which depend on the determinant of certain sub-matrices of the $n \times n$-matrix defined in the previous section.

In the first section in theorem 3.1.1 we will first define a basis for a nilpotent element in the orthogonal algebra over any field, such that the Gram matrix is a specific block matrix. Then the size of these blocks and equally the rank of all powers of this matrix determines a partition $\lambda \vdash N$ for which every even element occurs with even multiplicity, that is constant for every element of one U_N-orbit. The centralizer of a nilpotent element in the orthogonal group over the algebraically closed field then is the semidirect product of the centralizer of a one-dimensional torus, determined by the nilpotent element, acting on its unipotent radical, as we will show in theorem 3.1.6. This second centralizer then is a direct product of multiple orthogonal and symplectic groups, where their number of occurrences is determined by the number of different odd and even elements in λ respectively. Since orthogonal groups have two connected components, while symplectic groups are connected, the centralizer of the nilpotent element is connected if and only if the partition λ only contains even elements. Therefore, we will be able to show in theorem 3.1.8 that the $\overline{\mathrm{O}}_N$-orbit of a nilpotent element is also its $\overline{\mathrm{SO}}_N$-orbit, unless its partition contains only even elements, in which case the orbit splits into two unique orbits.

In the second section we will determine the nilpotent orbits over a finite field $\mathbb{F}_q$. As we will show in lemma 3.2.1, for a nilpotent element in the finite orthogonal algebra the fixed points of the $\overline{\mathrm{SO}}_N$-orbit splits into different SO_N, where their number is equal to the number of elements in the component group of the centralizer in $\overline{\mathrm{SO}}_N$. We have shown that the centralizer in $\overline{\mathrm{O}}_N$ splits into two connected components for every unique odd element in the respective partition λ, so the centralizer in $\overline{\mathrm{SO}}_N$ contains half that number. Finally, in lemma 3.2.2 we will be able to distinguish these by taking the product of the elements of the Gram matrix representing the odd elements of λ, which will stay the same under conjugation modulo $({\mathbb{F}_q}^*)^2$. Since ${\mathbb{F}_q}^*/({\mathbb{F}_q}^*)^2 \cong \mathbb{Z}/2\mathbb{Z}$, we get the desired result of splitting into two orbits for every but one unique odd element in λ.

In the third section we define centred Young diagrams for partitions $\lambda \vdash N$ for which even elements have even multiplicity. Creating a Young tableau $\mathbf{T}$ by filling such a diagram with elements $\{1, \dots N\}$ by certain rules will give us a verge pattern $A \in \mathbf{v}$. This together with a symmetric matrix $\mathbf{S}$, that is a sub-matrix of the previously defined Gram matrix, will give us a core pattern $d.A$. As we will show in theorem 3.3.5, the nilpotent matrix representing this core pattern will then fit the partition λ, while their affiliation to a certain SO_N is determined by the determinant of sub-matrices of $\mathbf{S}$. In theorem 3.3.8 we will then construct tableaus and corresponding matrices for every nilpotent orbit, which is possible in any case but for $q = 3$.

Chapter four: Core patterns for generalized Gelfand–Graev characters

In the last chapter we will tend to the topic of generalized Gelfand–Graev characters and the connection thereof with the Jedlitschky characters. First we will briefly recapitulate Kawanakas construction of generalized Gelfand–Graev characters, which are characters of SO_N induced from a unipotent subgroup $U_{1.5}$. This group is situated in the middle between the two normal unipotent subgroups U_1 and U_2 of the fixed points of a parabolic group of $\overline{\mathrm{SO}}_N$, which arise from the $\mathbb{Z}$-grading of $\mathfrak{so}_N$ with respect to a nilpotent element of $\mathfrak{so}_N$.
In the second section we will show that the Jedlitschky character for a standard core tableau, reduced to the normal subgroup U_1, is the induced character of the linear character of U_2 giving rise to the generalized Gelfand–Graev character down to a scalar. In order to do so, we first show in theorem 4.2.5 that such a reduced character is zero except on U_2, which is where we can convert the Jedlitschky character into the linear character for the generalized Gelfand–Graev character, as we will do in theorem 4.2.6. In theorem 4.2.8 we will see that such a character of U_1 induced back to U_N is a sum of Jedlitschky characters. Therefore the generalized Gelfand–Graev characters are sums of Jedlitschky characters induced to SO_N down to a scalar, which we will calculate in lemma 4.2.9.
We conclude this chapter with a description of the generalized Gelfand–Graev characters of the group SO_8 as Jedlitschky characters.

Notations

- For a field K and $m, n \in \mathbb{N}$ let $M_{m\times n}(K)$ be the set of $m \times n$-matrices over K and $M_n(K)$ be the set of quadratic $n \times n$-matrices over K.
- For a matrix $X \in M_{m\times n}(K)$ and $1 \le i \le m$, $1 \le j \le n$ let $X_{ij} \in K$ be the entry of X in the i-th row and j-th column.
- For $1 \le i \le m$ and $1 \le j \le n$ let $e_{ij} \in M_{m\times n}(K)$ be the matrix with 1 being the entry in i-th row and j-th column, while every other entry is 0.
- For $n \in \mathbb{N}$ let $[n] = \{1, 2, \ldots n\}$ be the set of natural numbers ranging from 1 to n and $[[n]] = \{(i, j) \mid 1 \le i, j \le n\}$ be the set of tuples with entries ranging from 1 to n.
- For $i, j \in \mathbb{N}$ let $\delta_{ij} \in \{0, 1\}$ be the Kronecker delta with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.
- For a prime $p \in \mathbb{N}$ let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the prime field with p elements and $\overline{\mathbb{F}}_q$ the algebraically closed field with characteristic $\mathrm{char}(\overline{\mathbb{F}}_q) = p$.
- For $1 \le i < j \le N$ with $i + j < N + 1$ and $c \in K$ let $x_{ij}(c) = I + c(e_{ij} - e_{\overline{ji}})$ be the element of a root subgroup.
- Let $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots) \vdash N$ be the partition of N, where for $i \in \mathbb{N}$ the number $m_i \in \mathbb{N}_0$ is the multiplicity with which i occurs in λ.

1 The special orthogonal group

In order to summarize general results about the finite special orthogonal group SO_N for even $N = 2n$ with $n \in \mathbb{N}$ over a finite field $\mathbb{F}_q$, where $q = p^k \in \mathbb{N}$ for some $k \in N$ is the power of a prime $p \in \mathbb{N}$, we will first introduce some basic information about linear algebraic groups. In the second section, we will introduce the special orthogonal group $\overline{\mathrm{SO}}_N$ over the algebraic closure $\overline{\mathbb{F}}_q$ of the prime field $\mathbb{F}_p$ for a prime $p \in \mathbb{N}$ as the group of $N \times N$ matrices that preserve a bilinear form b on $\overline{\mathbb{F}}_q^N$ and have determinant equal to 1. To ensure certain properties of the group, we restrict the prime p to be a "good prime", which in this case means $p \neq 2$.[1] In the third section we define root systems both for the general linear group and special orthogonal group, which allow us to define pattern subgroups of their respective groups, that is subgroups of the unipotent radical of a Borel group that are products of root subgroups. In the fourth section, we will relate these groups back to their finite counterparts. The general linear group and special linear group over the finite field $\mathbb{F}_q$ are the fixed points of the standard Frobenius endomorphism for q defined on the vector space of $2n \times 2n$ matrices over $\overline{\mathbb{F}}_q$.
In general, we will denote algebraically closed fields as well as groups and algebras over such fields with a line above, while the finite variants of those will have no line.

1.1 Linear algebraic groups

First we will state some rudimentary facts about algebraic geometry and algebraic groups following Geck's [Gec13] introduction to this topic.
Let $\overline{K}$ be an algebraically closed field. For a positive integer $n \in \mathbb{N}$ let $\overline{K}^n$ be the n-dimensional vector space over the field $\overline{K}$ and $\overline{K}[X]$ the polynomial ring of $\overline{K}$ over n independent variables $X = (X_1, \dots X_n)$. For a subset $S \subseteq \overline{K}[X]$ the subset $\mathcal{V}(S) \subseteq \overline{K}^n$ with

$$\mathcal{V}(S) = \{v \in \overline{K}^n \mid f(v) = 0 \text{ for all } f \in S\}$$

[1] cf. [SS70, 4.3, p. 178]

is called an affine variety. The ideal $I = \langle S \rangle$ generated by S gives rise to the same affine variety $\mathcal{V}(I) = \mathcal{V}(S)$. For a subset $V \subseteq \overline{K}^n$ the ideal $\mathcal{I}(V) \lhd \overline{K}[X]$ with

$$\mathcal{I}(V) = \{f \in \overline{K}[X] \mid f(v) = 0 \text{ for all } v \in v\}$$

is called vanishing ideal of V. The Zariski topology on $\overline{K}^n$ arises by defining the affine varieties as closed sets. Then for any set $V \subseteq \overline{K}^n$ its closure is $\overline{V} = \mathcal{V}(\mathcal{I}(V))$. On the other hand the radical $\sqrt{I}$ of an ideal $I \lhd \overline{K}[X]$ is the ideal

$$\sqrt{I} = \{f \in \overline{K}[X] \mid f^e \in I \text{ for some } e \in \mathbb{N}\}$$

and we have $\sqrt{I} = \mathcal{I}(\mathcal{V}(I))$.[2] Therefore, $\mathcal{I}$ and $\mathcal{V}$ define bijections between the set of radical ideals of $\overline{K}[X]$ and the set of affine varieties of $\overline{K}^n$. The ideal $\mathcal{I}(V)$ of an affine variety $V \subseteq \overline{K}^n$ is prime if and only if the variety V is irreducible in the Zariski topology.[3] The coordinate ring $\overline{K}[V]$ of V is defined to be the quotient ring $\overline{K}[V] = \overline{K}[X]/\mathcal{I}(V)$ and it is a integral domain if the variety is irreducible.

For a $\overline{K}$-algebra A and an A-module M a derivation $D : A \to M$ is a $\overline{K}$-linear map with $D(ab) = a \cdot D(b) + b \cdot D(a)$ for all $a, b \in A$ and we denote the space of derivations from A to M by $\mathrm{Der}_{\overline{K}}(A, M)$. For $1 \leq i \leq n$ let $\frac{\partial}{\partial X_i} \in \mathrm{Der}_{\overline{K}}(\overline{K}[X], \overline{K}[X])$ be the partial derivation with respect to X_i. Let $A = \overline{K}[X]/I$ be the quotient ring for some ideal $I \lhd \overline{K}[X]$, where $\overline{f} \in A$ denotes an element of the quotient ring for $f \in \overline{K}[X]$. Let M be an A-module and define $T_{A,M}$ by

$$T_{A,M} = \left\{ v = (v_1, \dots v_n) \in M^n \,\middle|\, \sum_{1 \leq i \leq n} \overline{\left(\frac{\partial}{\partial X_i} f\right)} \cdot v_i = 0 \text{ for all } f \in I \right\}.$$

$T_{A,M}$ is an A-module and there is an A-module isomorphism $\Phi : T_{A,M} \to \mathrm{Der}_{\overline{K}}(A, M)$ that maps $v \in M^n$ to the derivation D_v with

$$D_v(\overline{f}) = \sum_{1 \leq i \leq n} \overline{\left(\frac{\partial}{\partial X_i} f\right)} \cdot v_i \text{ for all } \overline{f} \in A.$$[4]

[2] cf. [Gec13, 2.1.9, p. 82]
[3] cf. [Gec13, 1.1.12, p. 6]
[4] cf. [Gec13, 1.4.3, p. 28]

For a fixed point $p \in \overline{K}^n$ and $f \in \overline{K}[X]$ we define the linear polynomial $d_p(f) \in \overline{K}[X]$ as

$$d_p(f) = \sum_{1 \leq i \leq n} \left(\frac{\partial}{\partial X_i} f(p) \right) X_i.^5$$

Definition 1.1.1. Let V be an algebraic variety and for $p \in \overline{K}^n$ let $\overline{K}_p$ be the field $\overline{K}$ with $\overline{K}[X]$-module structure defined by the evaluation of polynomials at p. Then the tangent space $T_p(V)$ of V at the point p is defined to be the $\overline{K}[V]$-module $T_{\overline{K}[V], \overline{K}_p}$ with

$$T_p(V) = \left\{ v \in \overline{K}^n \,\middle|\, d_p(f)(v) = 0 \text{ for all } f \in \mathcal{I}(V) \right\}$$

If $\mathcal{I}(V)$ is finitely generated by polynomials $f_1, \dots f_m \in \overline{K}[X]$ for $m \in \mathbb{N}$, we have

$$T_p(V) = \mathcal{V}\left(d_p(f_1), \dots d_p(f_m)\right)$$

and the map $\Phi : T_p(V) \to \mathrm{Der}_{\overline{K}}(\overline{K}[V], \overline{K}_p) : v \mapsto D_v$ is a $\overline{K}$-linear isomorphism with $D_v(\overline{f}) = d_p(f)(v)$ for $\overline{f} \in \overline{K}[V]$.[6]

Definition 1.1.2. Let $V \leq \overline{K}^m$ and $W \leq \overline{K}^n$ for $m, n \in \mathbb{N}$ be affine varieties. A map $\varphi : V \to W$ is called regular if there are polynomials $\varphi_1, \dots \varphi_n \in \overline{K}[X_1, \dots X_m]$ with

$$\varphi(v) = (\varphi_1(v), \dots \varphi_n(v))$$

for $v \in V$. Let $\varphi^* : \overline{K}[W] \to \overline{K}[V]$ be the $\overline{K}$-algebra homomorphism of the coordinate rings, defined by $\varphi^*(f)(v) = f(\varphi(v))$ for $f \in \overline{K}[W]$ and $v \in V$. Then the differential of φ at $p \in V$ is defined to be the linear map of the tangent spaces

$$d\varphi_p : \mathrm{Der}_{\overline{K}}(\overline{K}[V], \overline{K}_p) \to \mathrm{Der}_{\overline{K}}(\overline{K}[W], \overline{K}_{\varphi(p)}) : D \mapsto D \circ \varphi^*.$$

On the side of the tangent spaces the differential of φ is the map

$$d\varphi_p : T_p(V) \to T_{\varphi(p)}(W) : v \mapsto \left(d_p(\varphi_1)(v), \dots d_p(\varphi_n)(v)\right).$$

[5] cf. [Gec13, 1.4.7, p. 31]
[6] cf. [Gec13, 1.4.9, p. 32]

For an affine variety $Z \leq \overline{K}^k$ with $k \in N$ and another regular map $\psi : W \to Z$ we have $d(\psi \circ \varphi)_p = d(\psi)_{\varphi(p)} \circ d(\varphi)_p$.[7]

Definition 1.1.3. Let $M_n(\overline{K})$ be the set $(n \times n)$-matrices over $\overline{K}$. The matrix multiplication $\mu : M_n(\overline{K}) \times M_n(\overline{K}) \to M_n(\overline{K})$ is a regular map regarding $M_n(\overline{K})$ as the affine variety $\overline{K}^{n^2}$. An algebraic variety $\overline{G} \subseteq M_n(\overline{K})$ is called an algebraic group if

- it contains the identity element $I \in \overline{G}$,
- it is closed under multiplication $\mu(A, B) \in \overline{G}$ for $A, B \in \overline{G}$,
- every element of $\overline{G}$ has an inverse in $\overline{G}$ and the inverse map $\iota : \overline{G} \to \overline{G}$ is regular.

The general linear group of the vector space $\overline{K}^n$

$$\mathrm{GL}(\overline{K}^n) = \left\{ A \in M_n(\overline{K}) \mid \det(A) \neq 0 \right\}$$

is an affine open subvariety of $M_n(\overline{K})$ with respect to the determinant. Therefore, it is an algebraic variety in $\overline{K}^{n^2+1}$ and a linear algebraic group.[8] Its tangent space at the identity element is the complete set of matrices $T_I(\mathrm{GL}(\overline{K}^n)) = M_n(\overline{K})$ and is denoted by $\mathrm{gl}(\overline{K}^n)$.
Let the dimension of a linear algebraic group $\overline{G}$ be defined by the dimension of its tangent space at the identity element as a $\overline{K}$-vector space:

$$\dim \overline{G} = \dim_{\overline{K}} T_I(\overline{G})$$

Since $\overline{G}$ contains no singular points, this definition is equivalent to the Krull dimension of the coordinate ring $\overline{K}[\overline{G}]$ or the degree of the Hilbert polynomial of $\overline{K}[\overline{G}]$.[9]
For linear algebraic groups $\overline{G}, \overline{H}$ a group homomorphism $\varphi : \overline{G} \to \overline{H}$ is called a homomorphism of linear algebraic groups if it is a regular map. Then $\ker(\varphi) \leq \overline{G}$ and $\mathrm{im}(\varphi) \leq \overline{H}$ are closed subgroups and we have $\dim \overline{G} = \dim \ker(\varphi) + \dim \mathrm{im}(\varphi)$.[10]
The differential of the matrix multiplication μ and the inverse map ι at the identity are

$$d\mu_I : T(\overline{G})_I \oplus T(\overline{G})_I \to T(\overline{G})_I : (A, B) \mapsto A + B \quad \text{and} \quad d\iota_I : T(\overline{G})_I \to T(\overline{G})_I : A \mapsto -A.$$[11]

[7] cf. [Gec13, 1.4.13, p. 34]
[8] cf. [Gec13, 2.1.14, p. 83; 2.4.1, p. 99]
[9] cf. [Gec13, 1.2.15, p. 15; 1.2.18, p. 16; 1.5.2, p. 36]
[10] cf. [Gec13, 2.2.14, p. 91]
[11] cf. [Gec13, 1.5.6, p. 39]

With this we can define the adjoint representation of $\overline{G}$ and its tangent space based of the group of inner automorphisms of $\overline{G}$.

Lemma 1.1.4. *For $g \in \overline{G}$ let $Inn_g : \overline{G} \to \overline{G}$ be the inner automorphism with $Inn_g(h) = ghg^{-1}$ for $h \in \overline{G}$. Then its differential is*

$$d(Inn_g)_I : T_I(\overline{G}) \to T_I(\overline{G}) : A \mapsto gAg^{-1}.$$

Let the adjoint representation $Ad : \overline{G} \to GL(T_I(\overline{G}))$ be the derived homomorphism with $Ad(g) = d(Inn_g)_I$ for $g \in \overline{G}$. Then its differential is

$$ad = d(Ad)_I : T_I(\overline{G}) \to gl(T_I(\overline{G}))$$

with $ad(A)B = AB - BA$ for $A, B \in T_I(\overline{G})$.

Proof. Inn_g is a regular map with $\mathrm{Inn}_g(I) = I$ for any $g \in \overline{G}$. It is linear in every component, so we have $d(\mathrm{Inn}_g)_I(A) = gAg^{-1}$ for $A \in T_I(\overline{G})$. For $g, h \in \overline{G}$ we have $\mathrm{Inn}_g \circ \mathrm{Inn}_h = \mathrm{Inn}_{gh}$ and therefore $\mathrm{Ad}(gh) = d(\mathrm{Inn}_g \circ \mathrm{Inn}_h)_I = d(\mathrm{Inn}_g)_I \circ d(\mathrm{Inn}_h)_I = \mathrm{Ad}(g)\mathrm{Ad}(h)$, so Ad is a representation of $\overline{G}$.
For a fixed $B \in T_I(\overline{G})$ let $\epsilon_B : \mathrm{GL}(T_I(\overline{G})) \to T_I(\overline{G})$ with $\epsilon_B(M) = MB$ for $M \in \mathrm{GL}(T_I(\overline{G}))$. Then its differential $\epsilon'_B = d\epsilon_B : \mathrm{gl}(T_I(\overline{G})) \to T_B(T_I(\overline{G})) = T_I(\overline{G})$ is the evaluation of $\mathrm{gl}(T_I(\overline{G}))$ at B with $\epsilon'_B(N) = NB$ for $N \in \mathrm{gl}(T_I(\overline{G}))$.
$\epsilon_B \circ Ad = \mathrm{id}_{\overline{G}} B\,\iota$ is a morphism of affine varieties, so we have

$$d(\epsilon_B \circ Ad)_I(A) = d(\mathrm{id}_{\overline{G}})_I(A)B\,\iota(I) + \mathrm{id}_{\overline{G}}(I)B\,d\iota_I(A) = AB - BA$$

for $A \in T_I(\overline{G})$. Since $d(\epsilon_B \circ Ad)_I = \epsilon'_B \circ ad$, it follows that $\mathrm{ad}(A)B = AB - BA$. □

The adjoint representation ad now gives rise to a Lie algebra structure on the tangent space of $\overline{G}$ at the identity $T_I(\overline{G})$ and this Lie algebra will be denoted by $\mathrm{Lie}(\overline{G})$.

Theorem 1.1.5. *Let $\overline{G}$ be a linear algebraic group and let $[\cdot,\cdot]$ be the operation defined by*

$$[\cdot,\cdot] : T_I(\overline{G}) \times T_I(\overline{G}) \to T_I(\overline{G}) : (A, B) \mapsto ad(A)B.$$

Then $T_I(\overline{G})$ is a Lie algebra with its Lie bracket $[\cdot,\cdot]$, and we define $Lie(\overline{G}) = T_I(\overline{G})$ to be the Lie algebra of $\overline{G}$. For any homomorphism $\varphi : \overline{G} \to \overline{H}$ of linear algebraic groups $d\varphi : Lie(\overline{G}) \to Lie(\overline{H})$ is a Lie algebra homomorphism.

Proof. As we have $[A,B] = AB - BA$ for $A, B \in T_I(\overline{G})$ the bracket operation is bilinear and alternating.
Let $\overline{G}, \overline{H}$ be linear algebraic groups and $\varphi : \overline{G} \to \overline{H}$ a homomorphism of linear algebraic groups. So for $g \in \overline{G}$ we have $\varphi \circ \mathrm{Inn}_g = \mathrm{Inn}_{\varphi(g)} \circ \varphi$ and therefore $d\varphi \circ d\mathrm{Inn}_g = d\mathrm{Inn}_{\varphi(g)} \circ d\varphi$. By the definition of the adjoint representation it follows that $d\varphi\, Ad = (Ad \circ \varphi)d\varphi$ and thereby $d\varphi\, ad = (ad \circ d\varphi)d\varphi$. So we have $d\varphi([A,B]) = [d\varphi(A), d\varphi(B)]$ for $A, B \in T_I(\overline{G})$.
Since Ad itself is a homomorphism of linear algebraic groups, its differential commutes with the bracket operation, and we have $\mathrm{ad}([A,B]) = [\mathrm{ad}(A), \mathrm{ad}(B)]$ for $A, B \in T_I(\overline{G})$, where the latter bracket operates on $\mathrm{gl}(T_I(\overline{G}))$. For $C \in T_I(\overline{G})$ it follows that

$$[[A,B],C] = \mathrm{ad}([A,B])C = \mathrm{ad}(A)(\mathrm{ad}(B)C) - \mathrm{ad}(B)(\mathrm{ad}(A)C) = [A,[B,C]] - [B,[A,C]],$$

which gives us the Jacobi identity making $T_I(\overline{G})$ a Lie algebra.

□

For a linear algebraic group $\overline{G}$ let $\overline{G}^\circ \leq \overline{G}$ denote the irreducible component of $\overline{G}$ that contains the identity. Then $\overline{G}^\circ$ is a closed normal subgroup of $\overline{G}$ with finite index, and we have $\dim \overline{G}^\circ = \dim \overline{G}$.[12] Since the tangent space only depends on the identity component, we have $T_I(\overline{G}) = T_I(\overline{G}^\circ)$. For a homomorphism of linear algebraic groups $\varphi : G \to H$ we have $\varphi(\overline{G}^\circ) = \varphi(\overline{G})^\circ$.[13]

Definition 1.1.6. A a maximal closed connected solvable subgroup $\overline{B} \leq \overline{G}$ of an algebraic group is called a Borel subgroup. Any closed subgroup of $\overline{G}$ containing a Borel subgroup $\overline{B} \leq \overline{P} \leq \overline{G}$ is called a parabolic subgroup.

All Borel subgroups of an algebraic group are conjugate.[14] For a parabolic subgroup $\overline{P} \leq \overline{G}$ the quotient $\overline{G}/\overline{P}$ is a projective algebraic variety. Therefore, the projection morphism $p : \overline{G}/\overline{P} \times V \to V$ for an affine variety V is closed.[15]

Definition 1.1.7. A subgroup $\overline{T} \leq \overline{G}$ that is isomorphic to $\overline{T} \cong \overline{K}^\times \times \cdots \times \overline{K}^\times$, where $\overline{K}^\times$ denotes the multiplicative group of $\overline{K}$, is called a torus. A group isomorphic to a maximal number of copies $\overline{K}^\times$ is called a maximal torus.

[12] cf. [Gec13, 1.3.13, p. 23; 1.3.14, p. 24]
[13] cf. [Gec13, 2.2.14, p. 91]
[14] cf. [Gec13, 3.4.3, p. 149]
[15] cf. [Gec13, 3.2.7, p. 134]

Any maximal torus $\overline{T} \leq \overline{G}$ is contained in a Borel subgroup $\overline{T} \leq \overline{B} \leq \overline{G}$ and all maximal tori in a Borel subgroup are conjugated. Therefore, all maximal tori in $\overline{G}$ are conjugated.[16]

Definition 1.1.8. For a connected linear algebraic group $\overline{G}$ the unipotent radical $R_U(\overline{G})$ is defined to be the maximal closed connected unipotent subgroup of $\overline{G}$. The group $\overline{G}$ is called reductive if $R_U(\overline{G}) = 1$.

Let $\overline{G}$ be a connected linear algebraic group and $\overline{T} \leq \overline{G}$ a maximal torus. Then the Weyl Group is $W = N_G(\overline{T})/\overline{T}$, where $N_G(\overline{T})$ is the normalizer of $\overline{T}$ in $\overline{G}$ and is independent of the choice of $\overline{T}$. For a Borel subgroup $\overline{T} \leq \overline{B} \leq \overline{G}$ we have a semi direct product $\overline{B} = \overline{T} \ltimes R_U(\overline{B})$.[17]

1.2 The special orthogonal group

Let $\mathbb{F}_q$ be the field with q elements, where q is the power of a prime $p \neq 2$. Then $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$ and $\overline{\mathrm{Gl}}_n = \mathrm{GL}(\overline{\mathbb{F}}_q^n)$ the general linear group over the vector space $\overline{\mathbb{F}}_q^n$.

Definition 1.2.1. For $n \in \mathbb{N}$ and $N = 2n$ let J_N be the $N \times N$ matrix

$$J_N = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Let b be the non degenerate symmetric bilinear form with

$$b : \overline{\mathbb{F}}_q^N \times \overline{\mathbb{F}}_q^N \to \overline{\mathbb{F}}_q : (u, v) \mapsto u^t J_N v,$$

where $\cdot^t : M_N(\overline{\mathbb{F}}_q) \to M_N(\overline{\mathbb{F}}_q)$ is the involution mapping a matrix to its transposed matrix. Then $\overline{\mathrm{O}}_N = \mathrm{O}_N(\overline{\mathbb{F}}_q, b)$ denotes the orthogonal group over the algebraically closed field $\overline{\mathbb{F}}_q$ with respect to b:

$$\overline{\mathrm{O}}_N = \left\{ g \in M_N(\overline{\mathbb{F}}_q) \,\middle|\, b(gu, gv) = b(u, v) \text{ for all } u, v \in \overline{\mathbb{F}}_q^N \right\}$$

Furthermore, let the special orthogonal group be defined as $\overline{\mathrm{SO}}_N = \{g \in \overline{\mathrm{O}}_N \mid \det(g) = 1\}$.

[16] cf. [Car85, p. 16]
[17] cf. [Car85, p. 23]

$\overline{\mathrm{O}}_N$ is an linear algebraic group in $M_N(\overline{\mathbb{F}}_q)$ and comprises two connected components. The identity component $\overline{\mathrm{O}}_N^\circ \lhd \overline{\mathrm{O}}_N$ is the special orthogonal group $\overline{\mathrm{SO}}_N$.[18]

Definition 1.2.2. For $n \in \mathbb{N}$ let $[n] = \{1, \dots, n\}$ be the set of natural numbers ranging from 1 to n. For $1 \le i, j \le N$ let $e_{ij} \in M_N(\overline{\mathbb{F}}_q)$ be the matrix that is one at the (i, j)-th position and zero at all other positions. The map

$$\bar{\cdot} : [N] \to [N] : i \mapsto N + 1 - i$$

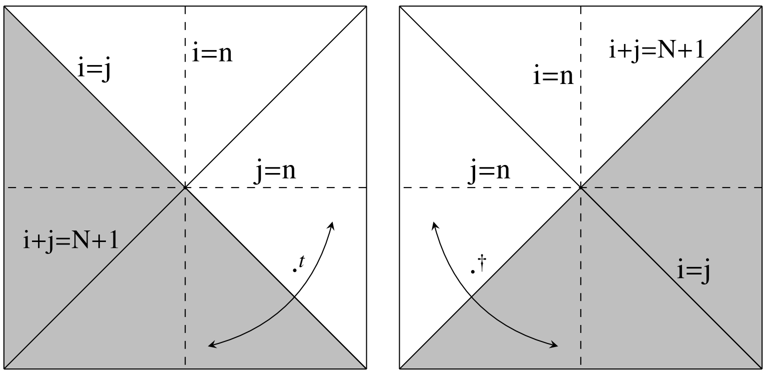
defines an involution on the set $[N]$.

Then we have $J_N = \sum_{i=1}^N e_{i\bar{i}}$, and therefore $J_N e_{ij} = e_{\bar{i}j}$ as well as $e_{ij} J_N = e_{i\bar{j}}$ for $1 \le i, j \le N$.

Definition 1.2.3. Let $\cdot^\dagger$ be the involution on $M_N(\overline{\mathbb{F}}_q)$ defined by

$$\cdot^\dagger : M_N(\overline{\mathbb{F}}_q) \to M_N(\overline{\mathbb{F}}_q) : X \mapsto J_N X^t J_N.$$

While the involution $\cdot^t$ on $M_N(\overline{\mathbb{F}}_q)$ mirrors elements $X \in M_N(\overline{\mathbb{F}}_q)$ along the diagonal with $(X^t)_{ij} = X_{ji}$ for $1 \le i, j \le N$, the above defined involution $\cdot^\dagger$ mirrors elements along the counter-diagonal with $(X^\dagger)_{ij} = X_{\bar{j}\bar{i}}$.



Lemma 1.2.4. *The Lie algebra $\overline{\mathfrak{so}}_N = Lie(\overline{SO}_N)$ of $\overline{SO}_N$ is given by*

$$\overline{\mathfrak{so}}_N = \left\{ A \in M_N(\overline{\mathbb{F}}_q) \,\middle|\, b(Au, v) = -b(u, Av) \textit{ for all } u, v \in \overline{\mathbb{F}}_q^N \right\}.$$

[18]cf. [Gec13, 1.7.8, p. 65]

Proof. Let $\overline{\mathbb{F}}_q[X]$ be the polynomial ring for $X = (X_{ij})_{1\leq i,j\leq N}$ and $f_{ij}(X) \in \overline{\mathbb{F}}_q[X]$ the polynomials for $1 \leq i \leq j \leq N$ be defined as $f_{ij}(X) = \sum_{k=1}^{N} X_{\bar{k}i}X_{kj} - \delta_{\bar{i}\bar{j}}$. For $A \in M_N(\overline{\mathbb{F}}_q)$ we then have

$$f_{ij}(A) = \sum_{k=1}^{N} A_{\bar{k}i}A_{kj} - \delta_{\bar{i}\bar{j}} = e_i^t A^t J A e_j - e_i^t e_j = b(Ae_i, Ae_j) - b(e_i, e_j),$$

where $\delta_{\bar{i}\bar{j}}$ is Kronecker delta of i and $\bar{j}$. Therefore, the vanishing ideal $\mathcal{I}(\overline{\mathrm{O}}_N)$ is generated by the $f_{ij}(X)$ for $1 \leq i \leq j \leq N$ and we have

$$d_I(f_{ij}) = X_{\bar{i}j} + X_{\bar{j}i} = e_i^t J X e_j + e_i^t X^t J e_j = b(e_i, Xe_j) + b(Xe_i, e_j).$$

It follows that $A \in \mathrm{Lie}(\overline{\mathrm{O}}_N)$ if and only if

$$d_I(f_{ij})(A) = A_{\bar{i}j} + A_{\bar{j}i} = e_i^t J A e_j + e_i^t A^t J e_j = b(e_i, Ae_j) + b(Ae_i, e_j)$$

for all $1 \leq i \leq j \leq N$. This is equivalent to $b(u, Av) + b(Au, v) = 0$ for all $u, v \in \overline{\mathbb{F}}_q^N$, which concludes the statement, since $\mathrm{Lie}(\overline{\mathrm{SO}}_N) = \mathrm{Lie}(\overline{\mathrm{O}}_N)$. □

For $X \in M_N(\overline{\mathbb{F}}_q)$ and $u, v \in \overline{\mathbb{F}}_q^N$ we have $b(Xu, v) = u^t X^t J_N v = u^t J_N X^\dagger v = b\left(u, X^\dagger v\right)$. Therefore, for $g, A \in M_N(\overline{\mathbb{F}}_q)$ with $\det g = 1$ we have $g \in \overline{\mathrm{SO}}_N$ and $A \in \overline{\mathfrak{so}}_N$ if and only if

$$g^\dagger = g^{-1} \quad \text{and} \quad A^\dagger = -A.$$

Let $\overline{B}_0 = \overline{B}_0(N, q) \leq \overline{\mathrm{Gl}}_N$ be the standard Borel group of upper triangular matrices and $\overline{T}_0 = \overline{T}_0(N, q) \leq \overline{\mathrm{Gl}}_N$ the standard maximal torus of diagonal matrices. Let W_0 the Weyl group of permutation matrices isomorphic to the symmetric group Σ_N and $\overline{U}_0 = R_U(\overline{B}_0)$ be the group of the upper unitriangular matrices with $\overline{B}_0 = \overline{T}_0 \ltimes \overline{U}_0$. Let the intersection of these groups with $\overline{\mathrm{SO}}_N$ be denoted by

$$\overline{B} = \overline{B}_0 \cap \overline{\mathrm{SO}}_N, \qquad \overline{T} = \overline{T}_0 \cap \overline{\mathrm{SO}}_N,$$

$$W = W_0 \cap \overline{\mathrm{SO}}_N \quad \text{and} \quad \overline{U} = \overline{U}_0 \cap \overline{\mathrm{SO}}_N.$$

Then $\overline{B}$ is a Borel group and $\overline{T}$ a maximal torus of $\overline{\mathrm{SO}}_N$. Furthermore, W is the Weyl group of $\overline{\mathrm{SO}}_N$, and we have $\overline{B} = \overline{T} \ltimes \overline{U}$.[19]

[19]cf. [Gec13, 1.7.8, p. 65]

The standard maximal torus $\overline{T} \leq \overline{\mathrm{SO}}_N$ is of the form:

$$\overline{T} = \left\{ \begin{pmatrix} t_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & t_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_n^{-1} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & t_1^{-1} \end{pmatrix} \middle| \, t_i \in \overline{\mathbb{F}}_q^* \text{ for all } 1 \leq i \leq n \right\}$$

The standard Borel group $\overline{B}$ is of the form:

$$\overline{B} = \left\{ \begin{pmatrix} g & gm \\ 0 & J_n g^{-t} J_n \end{pmatrix} \middle| \, g \in \overline{B}_0(n,q),\ m \in M_{n\times n}(\overline{\mathbb{F}}_q) \text{ with } J_n m + m^t J_n = 0 \right\}$$

Let $\mathfrak{S}_n$ be the symmetric group on $\{1,\dots,n\}$. For a permutation $\sigma \in \mathfrak{S}_n$ let $s_\sigma \in M_n(\overline{\mathbb{F}}_q)$ be its corresponding permutation matrix and define

$$\tau : \mathfrak{S}_n \to W : \sigma \mapsto \begin{pmatrix} s_\sigma & 0 \\ 0 & J_n s_\sigma J_n \end{pmatrix}$$

For $1 \leq i \leq n$ let $\beta_i \in W$ be the matrix corresponding to the permutation $(i, \bar{i}) \in \mathfrak{S}_N$ and for $I \subseteq \{1,\dots,n\}$ let $\beta_I = \prod_{i\in I} \beta_i$. Then the Weyl group W of $\overline{\mathrm{SO}}_N$ is the semi direct product

$$W = \tau(\mathfrak{S}_n) \ltimes \{\beta_I \,|\, I \subseteq \{1,\dots,n\} \text{ with } |I| \text{ even}\}.$$

Let $w_0 = \beta_n$, then for any $1 \leq i \leq n$ we have $\beta_i = \tau(i,n) w_0$ and W is generated by the set $\{\tau(i,i+1) \mid 1 \leq i \leq n-1\} \cup \{w_0\}$.[20]

$$w_0 = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & & 0 & 0 & 1 & 0 & & 0 \\ 0 & & 0 & 1 & 0 & 0 & & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

[20]cf. [GW09, 3.1.4, p. 132]

Springer and Steinberg [SS70, 5.3 ,p. 184] have shown that there is a faithful representation of $\overline{\mathfrak{so}}_N$ that admits a non degenerated trace form of $\overline{\mathfrak{so}}_N$.

Theorem 1.2.5. *Let $\rho : \overline{SO}_N \to GL(\overline{\mathbb{F}}_q^N)$ be the standard representation with $\rho(g)v = gv$ for $g \in \overline{SO}_N$ and $v \in \overline{\mathbb{F}}_q^N$ Then the bilinear form*

$$\tilde{\kappa} : \overline{\mathfrak{so}}_N \times \overline{\mathfrak{so}}_N \to \overline{\mathbb{F}}_q : (X, Y) \mapsto Tr(d\rho(X)d\rho(Y))$$

is non degenerated, Ad-invariant and skew with respect to ad. Furthermore, $\tilde{\kappa}$ is the trace form of the product of its arguments with $\tilde{\kappa}(X, Y) = Tr(XY)$ for $X, Y \in \overline{\mathfrak{so}}_N$.

Proof. Let $\tilde{\kappa} : \overline{\mathfrak{gl}}_N \times \overline{\mathfrak{gl}}_N \to \overline{\mathbb{F}}_q$ be the bilinear form defined by $\tilde{\kappa}(X, Y) = \mathrm{Tr}(XY)$ for $X, Y \in \overline{\mathfrak{gl}}_N$. Let $\mathfrak{m} \leq \overline{\mathfrak{gl}}_N$ be the subspace defined by

$$\mathfrak{m} = \{A \in \overline{\mathfrak{gl}}_N \mid b(Au, v) = b(u, Av) \text{ for all } u, v \in \overline{\mathbb{F}}_q^N\}$$

For $A \in \overline{\mathfrak{gl}}_N$ we have $A = \frac{1}{2}(A - A^t) + \frac{1}{2}(A + A^t)$ with $\frac{1}{2}(A - A^t) \in \overline{\mathfrak{so}}_N$ and $\frac{1}{2}(A + A^t) \in \mathfrak{m}$. Let $B \in \overline{\mathfrak{so}}_N \cap \mathfrak{m}$ then we have $b(Bu, v) = b(u, Bv) = -b(Bu, v)$ for all $u, v \in \overline{\mathbb{F}}_q^N$, which forces $B = 0$. Furthermore, we have $XY \in \overline{\mathfrak{so}}_N$ for $X \in \overline{\mathfrak{so}}_N$ and $Y \in \mathfrak{m}$ and therefore $\tilde{\kappa}(X, Y) = \mathrm{Tr}(XY) = 0$. So $\overline{\mathfrak{gl}}_N = \overline{\mathfrak{so}}_N \oplus \mathfrak{m}$ is an orthogonal direct sum with respect to $\tilde{\kappa}$, and since $\tilde{\kappa}$ is non degenerated on $\overline{\mathfrak{gl}}_N$, its restriction to $\overline{\mathfrak{so}}_N$ cannot be degenerated as well.
For the derived representation $d\rho : \overline{\mathfrak{so}}_N \to \mathfrak{gl}(\overline{\mathbb{F}}_q^N)$ we have $d\rho(X)v = Xv$ for $X \in \overline{\mathfrak{so}}_N$ and $v \in \overline{\mathbb{F}}_q^N$, so for $1 \leq i \leq N$ and $X, Y \in \overline{\mathfrak{so}}_N$ it follows that $d\rho(X)d\rho(Y)e_i = \sum_{j,k=1}^N X_{jk}Y_{ki}e_j$ and therefore $\mathrm{Tr}(d\rho(X)d\rho(Y)) = \sum_{i,k=1}^N X_{ik}Y_{ki} = \mathrm{Tr}(XY)$.
For $g \in \overline{\mathrm{SO}}_N$ and $X, Y \in \overline{\mathfrak{so}}_N$ we have $\tilde{\kappa}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = \mathrm{Tr}(g^{-1}XYg) = \mathrm{Tr}(XY) = \tilde{\kappa}(X, Y)$ and $\tilde{\kappa}$ is Ad-invariant. Finally, $\tilde{\kappa}$ is skew with respect to ad since for $X, Y, Z \in \overline{\mathfrak{so}}_N$ we have $\tilde{\kappa}(\mathrm{ad}(X)Y, Z) = \mathrm{Tr}(XYZ) - \mathrm{Tr}(YXZ) = \mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY) = \tilde{\kappa}(X, \mathrm{ad}(Y)Z)$. □

Since $\tilde{\kappa}$ is non degenerated, it follows by Carter [Car85, 5.5.1, p. 151] that $\overline{\mathrm{SO}}_N$ is a reductive group.

Definition 1.2.6. For a linear algebraic group $\overline{G}$ and its Lie algebra $\overline{\mathfrak{g}} = \mathrm{Lie}(\overline{G})$ let the centralizers of $A \in \overline{\mathfrak{g}}$ in $\overline{G}$ and $\overline{\mathfrak{g}}$ be defined as

$$C_{\overline{G}}(A) = \{g \in \overline{G} \mid gAg^{-1} = A\} \quad \text{and} \quad c_{\overline{\mathfrak{g}}}(A) = \{B \in \overline{\mathfrak{g}} \mid [A, B] = 0\}.$$

In general we have $\mathrm{Lie}(C_{\overline{G}}(A)) \leq c_{\overline{\mathfrak{g}}}(A)$, but sine $\overline{\mathfrak{so}}_N$ admits the aforementioned non degenerated bilinear form we have equality here, which can be shown in an analogous argument to Springer and Steinberg [SS70, 5.2, p. 183].

Theorem 1.2.7. *For $A \in \overline{\mathfrak{so}}_N$ we have*

$$Lie\left(C_{\overline{SO}_N}(A)\right) = c_{\overline{\mathfrak{so}}_N}(A)$$

Proof. Let $\overline{G}, \overline{H}$ be linear algebraic groups and $\varphi = (\varphi_{ij})_{1\leq i,j\leq n} : \overline{G} \to \overline{H}$ a morphism of linear algebraic groups. For $v \in \mathrm{Lie}(\ker(\varphi))$ we have $d_I(\varphi_{ij})(v) = 0$ for all $1 \leq i, j \leq n$ and therefore $d\varphi_I(v) = 0$, which concludes $\mathrm{Lie}(\ker(\varphi)) \leq \ker(d\varphi_I)$. Furthermore, we have $\mathrm{im}(d\varphi_I) \leq \mathrm{Lie}(\mathrm{im}(\varphi))$.
For a fixed $A \in \overline{\mathfrak{so}}_N$ let now $\varphi : \overline{\mathrm{Gl}}_N \to \overline{\mathfrak{gl}}_N$ with $\varphi(g) = gAg^{-1}$ for $g \in \overline{\mathrm{Gl}}_N$ be the adjoint representation of $\overline{\mathrm{Gl}}_N$ evaluated at A. Its differential is the adjoint representation of $\overline{\mathfrak{gl}}_N$ evaluated at A with $d\varphi_I(B) = [B, A]$ for $B \in \overline{\mathfrak{so}}_N$, as shwon in lemma 1.1.4. The kernel of $(d\varphi_I)$ is the centralizer of A in $\overline{\mathfrak{gl}}_N$, with $c_{\overline{\mathfrak{gl}}_N}(A) = \{B \in M_N(\overline{\mathbb{F}}_q) \mid AB = BA\}$ and the kernel of φ, the centralizer of A in $\overline{\mathrm{Gl}}_N$ with $C_{\overline{\mathrm{Gl}}_N}(A) = \{g \in c_{\overline{\mathfrak{gl}}_N}(A) \mid \det(g) \neq 0\}$, is open in $c_{\overline{\mathfrak{gl}}_N}(A)$. By Carter [Car85, p. 6] we then have $\dim \mathrm{Lie}(\ker(\varphi))) = \dim \ker(\varphi) = \dim d\varphi_I$ and therefore $\mathrm{im}(d\varphi_I) = \mathrm{Lie}(\mathrm{im}(\varphi))$ by the nsion argument.
Let now $\tilde{\varphi} : \overline{\mathfrak{so}}_N \to \overline{\mathfrak{so}}_N$ be the restriction of φ to $\overline{\mathfrak{so}}_N$. We have $\mathrm{im}(\tilde{\varphi}) \leq \mathrm{im}(\varphi) \cap \overline{\mathfrak{so}}_N$ and therefore

$$\mathrm{Lie}(\mathrm{im}(\tilde{\varphi})) \leq \mathrm{Lie}(\mathrm{im}(\varphi) \cap \overline{\mathfrak{so}}_N) = \mathrm{Lie}(\mathrm{im}(\varphi)) \cap \overline{\mathfrak{so}}_N = \mathrm{im}(d\varphi_I) \cap \overline{\mathfrak{so}}_N$$

For $B \in \overline{\mathfrak{gl}}_N$ there are $B_1 \in \overline{\mathfrak{so}}_N$ and $B_2 \in \mathfrak{m}$ with $B = B_1 + B_2$ as defined in the proof of theorem 1.2.5. Assume that $[B, A] \in \overline{\mathfrak{so}}_N$, then we have $[B_2, A] = [B, A] - [B_1, A] \in \overline{\mathfrak{so}}_N$ as well. For every $C \in \overline{\mathfrak{so}}_N$ we have $\tilde{\kappa}([B_2, A], C) = \tilde{\kappa}(B_2, [A, C]) = 0$ since $\overline{\mathfrak{so}}_N$ and $\mathfrak{m}$ are orthogonal with respect to $\tilde{\kappa}$. But $\tilde{\kappa}$ is non degenerated on $\overline{\mathfrak{so}}_N$, which forces $[B_2, A] = 0$ and we have $[B, A] = [B_1, A] \in \mathrm{im}(d\tilde{\varphi}_I)$. This shows that $\mathrm{im}(d\varphi_I) \cap \overline{\mathfrak{so}}_N = \mathrm{im}(d\tilde{\varphi}_I)$ and therefore $\mathrm{Lie}(\mathrm{im}(\tilde{\varphi})) = \mathrm{im}(d\tilde{\varphi}_I)$. By the dimension argument, we finally have $\mathrm{Lie}(\ker(\tilde{\varphi})) = \ker(d\tilde{\varphi}_I)$, which proves the claim. □

1.3 Root systems and pattern subgroups

In order to interpret pattern subgroups of $\overline{\mathrm{SO}}_N$, defined by their support on positions above both the diagonal and counter-diagonal, we must briefly introduce the concept of root systems for linear algebraic groups.

Definition 1.3.1. Let $\overline{G}$ be a connected linear algebraic group and $\overline{T} \leq \overline{G}$ a maximal torus. The group of homomorphisms $\mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*)$ is called the group of characters of $\overline{G}$ and $\mathrm{Hom}(\overline{\mathbb{F}}_q^*, \overline{T})$ the group of cocharacters of $\overline{G}$ with the group action on $\mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*)$ for $\xi_1, \xi_2 \in \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*)$ and $t \in \overline{T}$ being defined by $(\xi_1 + \xi_2)(t) = \xi_1(t)\xi_2(t)$, while the group action on $\mathrm{Hom}(\overline{\mathbb{F}}_q^*, \overline{T})$ for $\gamma_1, \gamma_2 \in \mathrm{Hom}(\overline{\mathbb{F}}_q^*, \overline{T})$ and $c \in \overline{\mathbb{F}}_q^*$ being defined by $(\gamma_1 + \gamma_2)(c) = \gamma_1(c)\gamma_2(c)$.
For $\xi \in \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*)$ and $\gamma \in \mathrm{Hom}(\overline{\mathbb{F}}_q^*, \overline{T})$ we have $\xi \circ \gamma \in \mathrm{Hom}(\overline{\mathbb{F}}_q^*, \overline{\mathbb{F}}_q^*)$ so there is $z \in \mathbb{Z}$ with $\xi \circ \gamma(c) = c^z$ for all $c \in \overline{\mathbb{F}}_q^*$. So let

$$(\cdot, \cdot) : \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*) \times \mathrm{Hom}(\overline{\mathbb{F}}_q^*, \overline{T}) \to \mathbb{Z}$$

be the non degenerated bilinear form with $(\xi, \gamma) = z$ for $\xi \circ \gamma(c) = c^z$.

Definition 1.3.2. Let $\overline{G}$ be a connected linear algebraic group, $\overline{B} \leq \overline{G}$ a Borel subgroup of $\overline{G}$ and $\overline{T} \leq \overline{B}$ a maximal torus of $\overline{G}$ contained in $\overline{B}$. The set of roots $\Phi(\overline{G}, \overline{T}) \subseteq \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*)$ of $\overline{G}$ with respect to $\overline{T}$ is a subset of the characters of $\overline{G}$ defined as

$$\Phi(\overline{G}, \overline{T}) = \left\{ \alpha \in \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*) \,\middle|\, \left\{ X \in \mathrm{Lie}(\overline{G}) \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for } t \in \overline{T} \right\} \neq (0) \right\}.$$

For $\alpha \in \Phi(\overline{G}, \overline{T})$ the weight space $\overline{\mathfrak{g}}_\alpha \leq \mathrm{Lie}(\overline{G})$ is the one dimensional subalgebra defined as $\overline{\mathfrak{g}}_\alpha = \left\{ X \in \mathrm{Lie}(\overline{G}) \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for } t \in T \right\}$. The set of positive roots $\Phi_+(\overline{G}, \overline{T}) \subseteq \Phi(\overline{G}, \overline{T})$ is the set of roots $\alpha \in \Phi(\overline{G}, \overline{T})$ for which its root subspace $\overline{\mathfrak{g}}_\alpha \leq \mathrm{Lie}(\overline{B})$ is contained in the Lie algebra of the Borel group $\overline{B}$.

The Lie algebra of $\overline{G}$ is then is the direct sum of the Lie algebra of the torus $\overline{T}$ and the root spaces for every root in $\Phi(\overline{G}, \overline{T})$.[21]

$$\mathrm{Lie}(\overline{G}) = \mathrm{Lie}(\overline{T}) \oplus \bigoplus_{\alpha \in \Phi(\overline{G}, \overline{T})} \overline{\mathfrak{g}}_\alpha$$

For the general linear group $\overline{\mathrm{Gl}}_N$ together with the torus of diagonal matrices $\overline{T}_0 \leq \overline{\mathrm{Gl}}_N$ the set of roots is defined as follows.

[21] cf. [BB69, 13.18, p. 176]

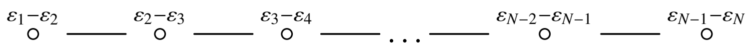
Definition 1.3.3. For $1 \le i \le N$ let $\varepsilon_i \in \mathrm{Hom}(\overline{T}_0, \overline{\mathbb{F}}_q^*)$ be defined as $\varepsilon_i(t) = t_{ii}$ for $t \in \overline{T}$. Then the sets of roots of $\overline{\mathrm{Gl}}_N$ with respect to $\overline{T}_0$ is

$$\Phi(\overline{\mathrm{Gl}}_N, \overline{T}_0) = \left\{ \pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le N \right\}.$$

For $1 \le i, j \le N$ with $i \neq j$ the root space for $\varepsilon_i - \varepsilon_j \in \Phi(\overline{\mathrm{Gl}}_N, \overline{T}_0)$ is $\overline{\mathfrak{g}}_{\varepsilon_i - \varepsilon_j} = \{ c e_{ij} \mid c \in \overline{\mathbb{F}}_q \}$.

The set of positive roots $\Phi_+(\overline{\mathrm{Gl}}_N, \overline{T}_0)$ with respect to the Borel subgroup of the upper triangular matrices $\overline{B}_0 \le \overline{\mathrm{Gl}}_N$ as well as the resulting set of simple roots $\Delta(\overline{\mathrm{Gl}}_N, \overline{T}_0)$ are

$$\Phi_+(\overline{\mathrm{Gl}}_N, \overline{T}_0) = \left\{ \varepsilon_i - \varepsilon_j \mid 1 \le i < j \le N \right\}, \qquad \Delta(\overline{\mathrm{Gl}}_N, \overline{T}_0) = \{ \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i < N \}.$$



The Dynkin diagram of the simple roots $\Delta(\overline{\mathrm{Gl}}_n, \overline{T})$

The set of roots for the special orthogonal group $\overline{\mathrm{SO}}_N$ together with the torus of diagonal matrices $\overline{T} = \overline{T}_0 \cap \overline{\mathrm{SO}}_N$ in $\overline{\mathrm{SO}}_N$ is defined as follows.

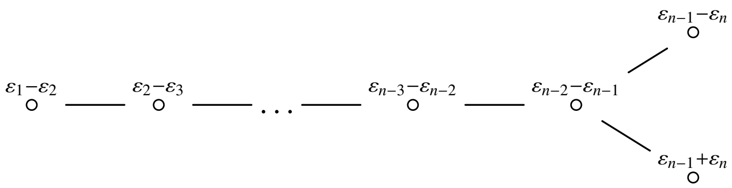
Definition 1.3.4. Let $\overline{G} = \overline{\mathrm{SO}}_N$ and $\overline{T} = \overline{T}_0 \cap \overline{\mathrm{SO}}_N$. For $1 \le i \le n$ let $\varepsilon_i \in \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_q^*)$ be defined as $\varepsilon_i(t) = t_{ii}$ for $t \in \overline{T}$. Then the sets of roots of $\overline{\mathrm{SO}}_N$ with respect to $\overline{T}$ is

$$\Phi(\overline{\mathrm{SO}}_N, \overline{T}) = \left\{ \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le n \right\}.$$

For $1 \le i, j \le n$ with $i \neq j$ the root space for $\varepsilon_i - \varepsilon_j \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ is $\overline{\mathfrak{g}}_{\varepsilon_i - \varepsilon_j} = \{ c(e_{ij} - e_{\overline{ji}}) \mid c \in \overline{\mathbb{F}}_q \}$ and for the root space for $\varepsilon_i + \varepsilon_j \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ is $\overline{\mathfrak{g}}_{\varepsilon_i + \varepsilon_j} = \{ c(e_{i\bar{j}} - e_{j\bar{i}}) \mid c \in \overline{\mathbb{F}}_q \}$.

The set of positive roots $\Phi_+(\overline{\mathrm{SO}}_N, \overline{T})$ with respect to the Borel subgroup$\overline{B} = \overline{B}_0 \cap \overline{\mathrm{SO}}_N$ as well as the resulting set of simple roots $\Delta(\overline{\mathrm{SO}}_N, \overline{T})$ are

$$\Phi_+(\overline{\mathrm{SO}}_N, \overline{T}) = \left\{ \varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n \right\}, \quad \Delta(\overline{\mathrm{SO}}_N, \overline{T}) = \{ \varepsilon_i - \varepsilon_{i+1} \mid \forall 1 \le i \le n-1 \} \cup \{ \varepsilon_{n-1} + \varepsilon_n \}.$$



The Dynkin diagram of the simple roots $\Delta(\overline{\mathrm{SO}}_N, \overline{T})$

Definition 1.3.5. Let $\overline{G}$ again be a connected linear algebraic group and $\overline{T} \leq \overline{G}$ be a a maximal torus in $\overline{G}$. For $\alpha \in \Phi(\overline{G}, \overline{T})$ let the root subgroup $\overline{U}_\alpha \leq \overline{G}$ be the one dimensional unipotent subgroup of $\overline{G}$ normalized by $\overline{T}$ such that $\mathrm{Lie}(\overline{U}_\alpha) = \overline{\mathfrak{g}}_\alpha$.

As the root systems for $\overline{\mathrm{Gl}}_N$ and $\overline{\mathrm{SO}}_N$ defined above are reduced root system, that is a root system for which every root $\alpha \in \Phi(\overline{G}, \overline{T})$ and any scalar multiple $z \in \mathbb{Z}$ such that $z\alpha \in \Phi(\overline{G}, \overline{T})$ necessitates $z \in \{-1, 1\}$, for $\alpha, \beta \in \Phi(\overline{G}, \overline{T})$, the root subgroups as well as their commutators, as given by the Chevalley commutator formula, are

$$\overline{U}_\alpha = \{I + X \mid X \in \overline{\mathfrak{g}}_\alpha\}, \qquad \left[\overline{U}_\alpha, \overline{U}_\beta\right] = \begin{cases} U_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi(\overline{G}, \overline{T}) \\ I & \text{else.} \end{cases}$$

The elements of a root subgroup $\overline{U}_\alpha \leq \overline{\mathrm{SO}}_N$ of the special orthogonal group for $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ are of the form $x_{ij}(c) := I + c(e_{ij} - e_{\overline{j}\overline{i}})$ for $c \in \overline{\mathbb{F}}_q$ and $1 \leq i, j \leq N$ with $i + j < N + 1$, where $i, j \leq n$ if $\alpha = \varepsilon_i - \varepsilon_j$ and $i \leq n < j$ or $j \leq n < i$ if $\alpha = \varepsilon_i + \varepsilon_{\overline{j}}$ or $\alpha = -(\varepsilon_{\overline{i}} + \varepsilon_j)$ and we will denominate these elements as both $x_\alpha(c) = x_{ij}(c) := I + c(e_{ij} - e_{\overline{j}\overline{i}})$ interchangeably.
Although not using the term themselves, Borel and Bass [BB69] defined pattern subgroups as subgroups of the unipotent radical $R_U(\overline{B})$ that are normalized by the maximal Torus $\overline{T}$, which is equivalent to the following definition.

Definition 1.3.6. For a reduced root system $\Phi(\overline{G}, \overline{T})$, a subset $C \subseteq \Phi_+(\overline{G}, \overline{T})$ is defined to be closed if $\alpha, \beta \in \Phi_+(\overline{G}, \overline{T})$ it follows that $\alpha + \beta \in \Phi_+(\overline{G}, \overline{T})$. For a closed subset $C \subseteq \Phi_+(\overline{G}, \overline{T})$ a pattern subgroup $\overline{H} \leq R_U(\overline{B})$ of the unipotent radical of $\overline{B}$ is defined to be the product of the root subgroups for every root in C in any order

$$\overline{H} = \prod_{\alpha \in C} \overline{U}_\alpha.$$

A pattern subgroup $\overline{H} \leq R_U(\overline{B})$ is a closed connected group normalized by $\overline{T}$ and any closed subgroup $\overline{H} \leq R_U(\overline{B})$ that is normalized by $\overline{T}$ is a pattern subgroup for the closed pattern $\Phi(\overline{H}, \overline{T}) \subseteq \Phi_+(\overline{G}, \overline{T})$.[22]
With this we can describe the pattern subgroups of the group of unitriangular matrices in both $\overline{\mathrm{Gl}}_N$ and $\overline{\mathrm{SO}}_N$ concretely as well as their relationship with their respective support.

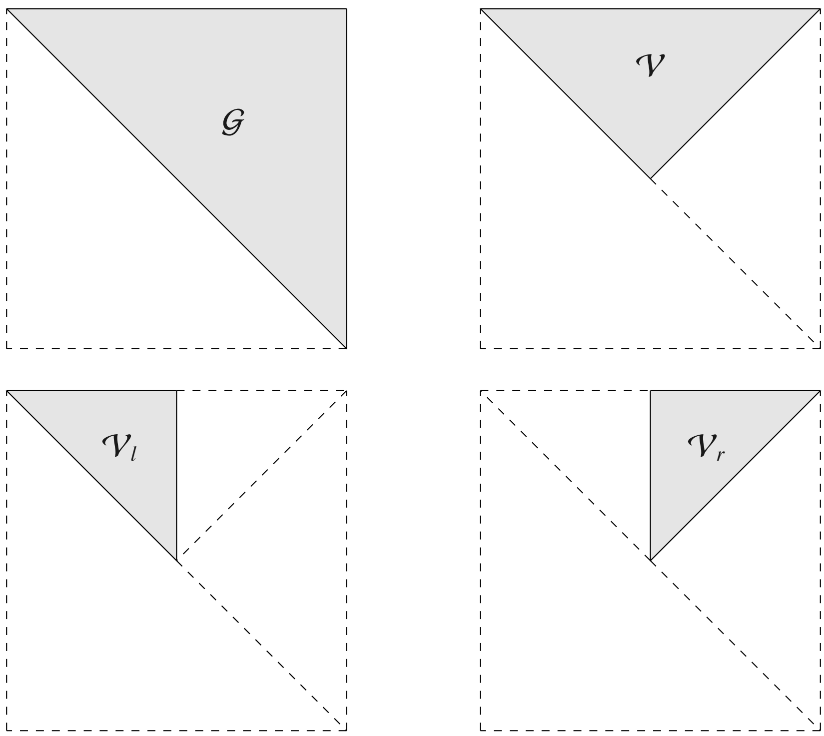
[22]cf. [BB69, p. 182]

Definition 1.3.7. For $n \in \mathbb{N}$ and $N = 2n$ let $[[N]] = \{(i, j) \mid 1 \leq i, j \leq N\}$ be the set of matrix entry coordinates of $N \times N$ matrices and for $A \in M_N(\mathbb{F}_q)$ we define its support $\operatorname{supp}(A) \subseteq [[N]]$ to be

$$\operatorname{supp}(A) = \{(i, j) \in [[N]] \mid A_{ij} \neq 0 \text{ for } 1 \leq i, j \leq N\}.$$

For any subset $\mathcal{S} \subseteq [[N]]$ let $\operatorname{supp}_{\mathcal{S}}(A) = \operatorname{supp}(A) \cap \mathcal{S}$ the restriction of $\operatorname{supp}(A)$ to $\mathcal{S}$. Furthermore, let $\mathcal{G}, \mathcal{V}, \mathcal{V}_l, \mathcal{V}_r \subseteq [[N]]$ be the subsets of $[[N]]$ defined by

$$\begin{aligned}
\mathcal{G} &= \{(i, j) \in [[N]] \,|\, 1 \leq i < j \leq N\} \\
\mathcal{V} &= \{(i, j) \in [[N]] \,|\, 1 \leq i < j \leq N \text{ with } i + j \leq N\} \\
\mathcal{V}_l &= \{(i, j) \in \mathcal{V} \,|\, 1 \leq j \leq n\} \\
\mathcal{V}_r &= \{(i, j) \in \mathcal{V} \,|\, n + 1 \leq j \leq N\}
\end{aligned}$$



Lemma 1.3.8. *A subset $\mathcal{P} \subseteq \mathcal{G}$ is called closed if for any $1 \leq i < j < k \leq N$ with $(i, j), (j, k) \in \mathcal{P}$ it follows that $(i, k) \in \mathcal{P}$. Let ρ_{GL} be the map defined as*

$$\rho_{GL} : \mathcal{G} \to \Phi_+(\overline{Gl}_N, \overline{T}_0) : (i, j) \mapsto \varepsilon_i - \varepsilon_j,$$

then for a closed subset $\mathcal{P} \subseteq \mathcal{G}$ the group $\overline{H}_{\mathcal{P}} \leq \overline{U}_0$ defined as

$$\overline{H}_{\mathcal{P}} = \left\{ g \in \overline{U}_0 \mid supp_{\mathcal{G}}(g) \in \mathcal{P} \right\}$$

is the pattern subgroup with $\rho_{GL}(\mathcal{P}) = \Phi(\overline{H}_{\mathcal{P}}, \overline{T}_0)$.

Proof. Let $\mathcal{P} \subseteq \mathcal{G}$ be closed and $\overline{H}_{\mathcal{P}} \subseteq \overline{U}_0$ be the subset of $\overline{U}_0$ such that for $g \in \overline{H}_{\mathcal{P}}$ we have $\mathrm{supp}_{\mathcal{G}}(g) \in \mathcal{P}$. For $g, h \in \overline{H}_{\mathcal{P}}$ and $1 \leq i < j \leq N$ with $(i, j) \notin \mathcal{P}$ for every $i \leq k \leq j$ it follows that $(i, k) \notin \mathcal{P}$ or $(k, j) \notin \mathcal{P}$. We then have $(gh)_{ij} = \sum_{k=i}^{j} g_{ik} h_{kj} = 0$ and therefore $(gh)_{ij} \neq 0$. It follows that $gh \in \overline{H}_{\mathcal{P}}$ and since $\overline{H}_{\mathcal{P}}$ is a closed with respect to the Zariski topology, $\overline{H}_{\mathcal{P}} \leq \overline{U}_0$ is a subgroup.
For every $t \in \overline{T}_0$ we have $t^{-1} g t \in \overline{H}_{\mathcal{P}}$, so $\overline{H}_{\mathcal{P}}$ is normalized by $\overline{T}_0$ and therefore a pattern subgroup of $\overline{U}_0$. For $1 \leq i < j \leq N$ we have $I + e_{ij} \in \overline{H}_{\mathcal{P}}$ if and only if $(i, j) \in \mathcal{P}$, and therefore $\rho_{\mathrm{GL}}(\mathcal{P}) = \Phi(\overline{H}_{\mathcal{P}}, \overline{T}_0)$ □

Lemma 1.3.9. *A subset $\mathcal{Q} \subseteq \mathcal{V}$ is called closed if for any $1 \leq i < j < k \leq N$ with $(i, j), (j, k) \in \mathcal{Q}$ it follows that $(i, k) \in \mathcal{Q}$ and for any $1 \leq i < j \leq N$ and $1 \leq k < \bar{j}$ such that $(k, \bar{j}) \in \mathcal{V}_r$ with $(i, j), (k, \bar{j}) \in \mathcal{Q}$ it follows that $(i, \bar{k}) \in \mathcal{Q}$. Let ρ_{SO} be the map defined as*

$$\rho_{SO} : \mathcal{V} \to \Phi_+(\overline{SO}_N . \overline{T}) : (i, j) \mapsto \begin{cases} \varepsilon_i - \varepsilon_j & \text{if } (i, j) \in \mathcal{V}_l \\ \varepsilon_i + \varepsilon_{\bar{j}} & \text{if } (i, j) \in \mathcal{V}_r \end{cases},$$

then for a closed subset $\mathcal{Q} \subseteq \mathcal{V}$ the group $\overline{H}_{\mathcal{Q}} \leq \overline{U}$ defined as

$$\overline{H}_{\mathcal{Q}} = \left\{ g \in \overline{U} \mid supp_{\mathcal{V}}(g) \in \mathcal{Q} \right\}$$

is the pattern subgroup with $\rho_{SO}(\mathcal{Q}) = \Phi(\overline{H}_{\mathcal{Q}}, \overline{T})$.
For a closed subset $\mathcal{P} \subseteq \mathcal{G}$ the intersection $\overline{H}_{\mathcal{P}} \cap \overline{SO}_N \leq \overline{U}$ is a pattern subgroup $\overline{H}_{\mathcal{Q}}$ of $\overline{U}$ for a closed subset $\mathcal{Q} \subseteq \mathcal{V}$ such that for $1 \leq i < j \leq N$ with $i + j < N + 1$ we have $(i, j) \in \mathcal{Q}$ if and only if $(i, j), (\bar{j}, \bar{i}) \in \mathcal{P}$.

Proof. Let $\mathcal{P} \subseteq \mathcal{G}$ be closed and $\overline{H}_{\mathcal{P}} \subseteq \overline{U}_0$. Then $\overline{H}_{\mathcal{P}} \cap \overline{\mathrm{SO}}_N$ is closed and normalized by $\overline{T} = \overline{T}_0 \cap \overline{\mathrm{SO}}_N$, and therefore a pattern subgroup of $\overline{U}$. For any $1 \leq i < j \leq N$ with $i + j < N + 1$ we have $x_{ij}(1) \in \overline{H}_{\mathcal{P}}$, and therefore $\rho_{\mathrm{SO}}(i, j) \in \Phi(\overline{H}_{\mathcal{P}} \cap \overline{\mathrm{SO}}_N, \overline{T})$ if and only if $(i, j), (\bar{j}, \bar{i}) \in \mathcal{P}$.
Let $\mathcal{Q} \subseteq \mathcal{V}$ be a closed subset. Let $\mathcal{Q}^{\dagger}, \tilde{\mathcal{Q}} \subseteq \mathcal{G}$ subsets of $\mathcal{G}$ defined by

$$\mathcal{Q}^{\dagger} = \{(i, j) \in \mathcal{G} \mid (\bar{j}, \bar{i}) \in \mathcal{Q}\}, \qquad \tilde{\mathcal{Q}} = \{(i, \bar{i}) \in \mathcal{G} \mid \exists i < k < \bar{i} : \ (i, k), (i, \bar{k}) \in \mathcal{Q}\}.$$

Clearly all three subsets $\mathcal{Q}, \mathcal{Q}^\dagger, \tilde{\mathcal{Q}} \subseteq \mathcal{G}$ are closed. Let $1 \leq i < j < l \leq N$ with $(i, j) \in \mathcal{Q}$ and $(j, l) \in \mathcal{Q}^\dagger \cup \tilde{\mathcal{Q}}$. If $l = \overline{j}$, there is a $j < k < \overline{j}$ such that $(j, k), (j, \overline{k}) \in \mathcal{Q}$ and therefore $(i, \overline{k}) \in \mathcal{Q}$ because $(j, \overline{k}) \in \mathcal{V}_r$. Since $(j, k), (i, \overline{k}) \in \mathcal{Q}$ with $(i, \overline{k}) \in \mathcal{V}_r$, it follows that $(i, \overline{j}) \in \mathcal{Q}$. If $l \neq \overline{j}$, we have $(\bar{l}, \overline{j}) \in \mathcal{Q}$ and since $(\bar{l}, \overline{j}) \in \mathcal{V}_r$, it follows that $(i, l) \in \mathcal{Q}$. By argument of symmetry, the subset $\mathcal{P} = \mathcal{Q} \cup \mathcal{Q}^\dagger \cup \tilde{\mathcal{Q}} \subseteq \mathcal{G}$ is closed and for the pattern subgroup $\overline{H}_{\mathcal{P}} \subseteq \overline{U}_0$ we have $\mathrm{supp}_{\mathcal{V}}(\overline{H}_{\mathcal{P}}) = \mathcal{P} \cap \mathcal{V} = \mathcal{Q}$, which gives us $\overline{H}_{\mathcal{Q}} = \overline{H}_{\mathcal{P}} \cap \overline{\mathrm{SO}}_N$. Since for every $(i, j) \in \mathcal{Q} \subseteq \mathcal{P}$, we have $(\overline{j}, \overline{i}) \in \mathcal{Q}^\dagger \subseteq \mathcal{P}$ it follows that $\rho_{\mathrm{SO}}(i, j) \in \Phi(\overline{H}_{\mathcal{Q}}, \overline{T})$, which proves the claim. □

Corollary 1.3.10. *Let $\overline{\mathfrak{u}} \leq \overline{\mathfrak{so}}_N$ be the Lie algebra of the unipotent radical $\overline{U}$ of the Borel subgroup $\overline{B}$ for $\overline{SO}_N$ with $\overline{\mathfrak{u}} = Lie(\overline{U})$ and $\overline{\mathfrak{u}}_0 \leq \overline{\mathfrak{gl}}_n$ be the Lie algebra of upper unitriangular matrices. Then*

$$\overline{\mathfrak{u}} = \overline{\mathfrak{u}}_0 \cap \overline{\mathfrak{so}}_N = \bigoplus_{\alpha \in \Phi_+(\overline{SO}_N, \overline{T})} \overline{\mathfrak{g}}_\alpha.$$

For a closed subset $\mathcal{Q} \subseteq \mathcal{V}$ let $\overline{\mathfrak{h}}_{\mathcal{Q}} = Lie(\overline{H}_{\mathcal{Q}}) \leq \overline{\mathfrak{u}}$ be the Lie algebra of the pattern subgroup $\overline{H}_{\mathcal{Q}} \leq \overline{SO}_N$. Then $\overline{\mathfrak{h}}_{\mathcal{Q}}$ is the pattern Lie algebra with

$$\overline{\mathfrak{h}}_{\mathcal{Q}} = \bigoplus_{\alpha \in \Phi(\overline{H}_{\mathcal{Q}}, \overline{T})} \overline{\mathfrak{g}}_\alpha = \left\{ X \in \overline{\mathfrak{u}} \mid supp_{\mathcal{V}}(g) \in \mathcal{Q} \right\}.$$

Proof. Let $\mathcal{Q} \subseteq \mathcal{V}$ be closed and $\overline{H}_{\mathcal{Q}} \leq \overline{U}$ be the corresponding pattern group for $\mathcal{Q}$, with $\overline{\mathfrak{h}}_{\mathcal{Q}} = \mathrm{Lie}(\overline{H}_{\mathcal{Q}})$. For any $\alpha \in \Phi(\overline{H}_{\mathcal{Q}}, \overline{T})$ we have $\overline{\mathfrak{g}}_\alpha \leq \overline{\mathfrak{h}}_{\mathcal{Q}}$ and by argument of dimension it follows that $\overline{\mathfrak{h}}_{\mathcal{Q}} = \bigoplus_{\alpha \in \Phi(\overline{H}_{\mathcal{Q}}, \overline{T})} \overline{\mathfrak{g}}_\alpha$. Let $X \in \overline{\mathfrak{u}}_0 \cap \overline{\mathfrak{so}}_N$ with $\mathrm{supp}_{\mathcal{V}}(X) \in \mathcal{Q}$, then we have $X = \sum_{(i,j) \in \mathcal{Q}} X_{ij}(e_{ij} - e_{\overline{j}\overline{i}})$ and therefore $X \in \bigoplus_{\alpha \in \rho_{\mathrm{SO}}(\mathcal{Q})} \overline{\mathfrak{g}}_\alpha = \overline{\mathfrak{h}}_{\mathcal{Q}}$. For $\mathcal{Q} = \mathcal{V}$ we have $\overline{H}_{\mathcal{Q}} = \overline{U}$ and $\Phi(\overline{H}_{\mathcal{Q}}, \overline{T}) = \Phi_+(\overline{\mathrm{SO}}_N, \overline{T})$, which gives us $\overline{\mathfrak{u}} = \overline{\mathfrak{u}}_0 \cap \overline{\mathfrak{so}}_N$. □

Lemma 1.3.11. *For $\mathfrak{a} \leq \overline{\mathfrak{so}}_N$ let $\mathfrak{a}^\perp$ be its orthogonal complement with respect to $\tilde{\kappa}$. The antiautomorphism consisting of matrix transposition $\cdot^t : \overline{\mathfrak{so}}_N \to \overline{\mathfrak{so}}_N : X \mapsto X^t$ defines an action on the rootspaces of $\overline{\mathfrak{so}}_N$ with respect to $\overline{T}$, such that for $\alpha \in \Phi(\overline{SO}_N, \overline{T})$ we have*

$$\overline{\mathfrak{g}}_\alpha^t = \overline{\mathfrak{g}}_{-\alpha} \quad \textit{and} \quad \overline{\mathfrak{g}}_\alpha^\perp = Lie(\overline{T}) \oplus \sum_{\substack{\beta \in \Phi(\overline{SO}_N, \overline{T}) \\ \beta \neq -\alpha}} \overline{\mathfrak{g}}_\beta.$$

Proof. For $1 \leq i < j \leq n$ we have $\overline{\mathfrak{g}}_{\varepsilon_i - \varepsilon_j} = \{ X_{ij}(c) \mid c \in \overline{\mathbb{F}}_q \}$ and $X_{ij}(c)^t = c(e_{ij}^t - e_{\overline{j}\overline{i}}^t) = X_{ji}(c)$ for $c \in \overline{\mathbb{F}}_q$. It follows that $\overline{\mathfrak{g}}_{\varepsilon_i - \varepsilon_j}^t = \overline{\mathfrak{g}}_{\varepsilon_j - \varepsilon_i} = \overline{\mathfrak{g}}_{-(\varepsilon_i - \varepsilon_j)}$. Similar, we have $X_{i\overline{j}}(c)^t = X_{\overline{i}j}(c)$ and therefore $\overline{\mathfrak{g}}_{\varepsilon_i + \varepsilon_j}^t = \overline{\mathfrak{g}}_{-(\varepsilon_i + \varepsilon_j)}$.

Let $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $\overline{\mathfrak{g}}_\alpha = \{X_{ij}(c) \mid c \in \overline{\mathbb{F}}_q\}$ for $1 \leq i, j \leq N$ with $i + j \leq N$, $i \neq j$ and $Y \in \overline{\mathfrak{so}}_N$. Then we have $\tilde{\kappa}(X_{ij}(c), Y) = c(Y_{ji} - Y_{\bar{i}\bar{j}})$ and therefore $\mathrm{Lie}(\overline{T}) \leq \overline{\mathfrak{g}}_\alpha^\perp$ as well as $\overline{\mathfrak{g}}_\beta \leq \overline{\mathfrak{g}}_\alpha^\perp$ for all $\beta \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ unless $\beta = -\alpha$. □

We now define the Cayley transformation as a Springer morphism from the variety of unipotent elements of $\overline{\mathrm{SO}}_N$ to the variety of nilpotent elements $\overline{\mathfrak{so}}_N$.[23]

Lemma 1.3.12. *Let $\overline{V} \subseteq \overline{SO}_N$ be the variety of unipotent elements of $\overline{SO}_N$ and $\overline{\mathfrak{v}} \subseteq \overline{\mathfrak{so}}_N$ the variety of nilpotent elements $\overline{\mathfrak{so}}_N$ that is*

$$\overline{V} = \left\{x \in \overline{SO}_N \mid \exists m \in \mathbb{N} : (x - I)^m = 0\right\} \quad \textit{and} \quad \overline{\mathfrak{v}} = \left\{X \in \overline{\mathfrak{so}}_N \mid \exists m \in \mathbb{N} : X^m = 0\right\}.$$

Let f be the map from the variety of unipotent elements of $\overline{SO}_N$ to variety of nilpotent elements $\overline{\mathfrak{so}}_N$ defined by

$$f : \overline{V} \to \overline{\mathfrak{v}} : x \mapsto (x - I)(x + I)^{-1}.$$

Then f is a bijection and both factors of f commute such that $f(x) = (x+I)^{-1}(x-I)$ for $x \in \overline{V}$. Furthermore, for $x, g \in \overline{V}$ we have $f(g^{-1}xg) = g^{-1}f(x)g$ and $f(x^{-1}) = -f(x)$ as well as

$$f(x) = \sum_{k\in\mathbb{N}} (-1)^{k-1} \frac{1}{2^k}(x - 1)^k$$

The restriction of f to $\overline{U}$ the group of upper unitriangular matrices in $\overline{SO}_n$ is a bijection to its Lie algebra $\overline{\mathfrak{u}}$.

Proof. Let $x \in \overline{V}$ and $m \in \mathbb{N}$ be minimal, such that $(x - I)^m = 0$. Then we have

$$\frac{1}{2}(x + I)\left(\sum_{k=0}^{m-1}(-\frac{1}{2}(x - I))^k\right) = (1 + \frac{1}{2}(x - I))\left(\sum_{k=0}^{m-1}(-\frac{1}{2}(x - I))^k\right) = I - \left(-\frac{1}{2}(x - I)\right)^m = I$$

and $(x+I)$ is invertible with $(x+I)^{-1} = \frac{1}{2}\sum_{k=0}^{m-1}(-\frac{1}{2}(x-I))^k$, so by definition of f it follows that $f(x) = \sum_{k=1}^{m-1}(-1)^{k-1}\frac{1}{2^k}(x - 1)^k$. We have $(x - I)(x + I) = x^2 - I = (x + I)(x - I)$, and therefore $(x - I)(x + I)^{-1} = (x + I)^{-1}(x - I)$. This gives us $f(x)^m = (x - I)^m(x + I)^{-m} = 0$, which shows that $f(x)$ is nilpotent. Since

$$f(x)^\dagger = J_N(x-I)^t J_N J_N(x+I)^{-t} J_N = (J_N(x-I)^t J_N)^t (J_N(x+I)^t J_N)^{-1} = (x^\dagger - I)(x^\dagger + I)^{-1} = f(x^{-1})$$

[23] cf. [SS70, 3.12, p. 229]

as well as $f(x^{-1}) = (I - x)x^{-1}(x^{-1}(I + x))^{-1} = (I - x)(I + x)^{-1} = -f(x)$, it follows that $f(x) + f(x)^\dagger = 0$ and therefore $f(x) \in \overline{\mathfrak{so}}_N$. So f is indeed a map from $\overline{V}$ to $\overline{\mathfrak{v}}$ and for any $g \in \overline{\mathrm{SO}}_N$ we have

$$f(g^{-1}xg) = g^{-1}(x - I)g\left\{g^{-1}(x + I)g\right\}^{-1} = g^{-1}(x - I)(x + I)^{-1}g = g^{-1}f(x)g.$$

Let $X \in \overline{\mathfrak{v}}$ and $m \in \mathbb{N}$ be minimal such that $X^m = 0$. Then we have $(I-X)\left(\sum_{k=0}^{m-1} X\right) = I-X^m = I$, so $(I - X)$ is invertible, and we define the map f' from the variety of nilpotent elements $\overline{\mathfrak{v}}$ to $\overline{\mathrm{Gl}}_N$ as $f'(X) = (I + X)(I - X)^{-1}$ for $X \in \overline{\mathfrak{v}}$. For $x \in \overline{V}$ we then have

$$\begin{aligned} f'(f(x)) &= \left(I + (x - I)(x + I)^{-1}\right)\left(I - (x - I)(x + I)^{-1}\right)^{-1} \\ &= ((x + I) + (x - I))\,(x + I)^{-1}(x + I)\,((x + I) - (x - I))^{-1} \\ &= (2x)\,(2I)^{-1} \\ &= x \end{aligned}$$

as well as $f(f'(X)) = X$ for $X \in \overline{\mathfrak{v}}$ by the same argument. Therefore, f is bijective with $f' = f^{-1}$.
We have $x - I \in \overline{\mathfrak{u}}_0$ as well as $\frac{1}{2}(x + I) \in \overline{U}_0$ and therefore $2(x + I)^{-1} \in \overline{U}_0$. This gives us $f(x) \in \overline{\mathfrak{u}}_0$, which proves $f(x) \in \overline{\mathfrak{u}}_0 \cap \overline{\mathfrak{so}}_N = \overline{\mathfrak{u}}$. For $X \in \overline{\mathfrak{u}}$ we have both $(I + X), (I - X) \in \overline{U}_0$ and therefore $f^{-1}(X) \in \overline{U}$, so the restriction of f to $\overline{U}_0$ is a bijection to $\overline{\mathfrak{u}}$ □

Corollary 1.3.13. *For a closed subset $Q \subseteq \mathcal{V}$ we have $f(\overline{H}_Q) \subseteq \overline{\mathfrak{h}}_Q$.*

Proof. Let $Q \subseteq \mathcal{V}$ be closed and $\overline{H}_Q \leq \overline{\mathrm{SO}}_N$ its pattern subgroup. Let $x \in \overline{H}_Q$. For any $k \in \mathbb{N}$ we have $\mathrm{supp}_{\mathcal{V}}(x^k) \subseteq \mathcal{V}$ and since $(x - I)^k = \binom{k}{i}(-1)^{k-i}x^k$, it follows that $\mathrm{supp}_{\mathcal{V}}((x - I)^k) \subseteq \mathcal{V}$ as well. We then have $\mathrm{supp}_{\mathcal{V}}(\sum_{k\in\mathbb{N}}(-1)^{k-1}\frac{1}{2^k}(x - 1)^k) \subseteq \mathcal{V}$, which by lemma 1.3.12 gives us $f(x) \in \overline{\mathfrak{h}}_Q$. □

1.4 The special orthogonal group over a finite field

We now come back to the finite special orthogonal group SO_N over the finite field $\mathbb{F}_q$ as it is the fixed points of the linear algebraic group $\overline{\mathrm{SO}}_N$ with respect to the standard Frobenius endomorphism for q.

Definition 1.4.1. The standard Frobenius endomorphism F for q is defined to be the $\mathbb{F}_q$-linear map with

$$F : \overline{\mathbb{F}}_q^n \to \overline{\mathbb{F}}_q^n : (x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q).$$

Then an algebraic variety $V \subseteq \overline{\mathbb{F}}_q^n$ is called F-stable, if $F(V) \subseteq V$ and the fixed points V^F of V are defined by

$$V^F = \{v \in V \mid F(v) = v\}$$

For another algebraic variety $W \subseteq \overline{\mathbb{F}}_q^m$ a regular map $\varphi : V \to W$ is called F-stable if we have $F \circ \varphi = \varphi \circ F$.

For an F-stable algebraic variety $V \subseteq \overline{\mathbb{F}}_q^n$ the Frobenius endormorphism F is an isomorphism on V, which is equivalent to its vanishing ideal $\mathcal{I}(V)$ being generated by a set of polynomials in $\mathbb{F}_q[X_1, \ldots X_n]$.[24] In this case the fixed points are the intersection $V^F = V \cap \mathbb{F}_q{}^n$.[25] Equally, a regular map $\varphi : V \to W$ to another algebraic variety $W \subseteq \overline{\mathbb{F}}_q^m$ is F-stable if and only if $\varphi_1, \ldots \varphi_m \in \mathbb{F}_q[X_1, \ldots X_n]$.[26]
For an algebraic group $\overline{G} \leq M_n(\overline{\mathbb{F}}_q)$ the matrix multiplication μ and the inverse map ι are F-stable and F is a group homomorphism, which is an isomorphism if $\overline{G}$ is F-stable.

Definition 1.4.2. For an algebraic group $\overline{G} \leq M_n(\overline{\mathbb{F}}_q)$ a F-stable torus $\overline{T} \leq \overline{G}$ is called maximally split if there is an F-stable Borel group $\overline{B} \leq \overline{G}$ containing $\overline{T}$.

The homomorphism $\det : \overline{\mathrm{Gl}}_N \to \overline{\mathbb{F}}_q^*$ is F-stable and therefore $\overline{\mathrm{Gl}}_N$ is an F-stable group. The fixed points $\mathrm{GL}_N = \overline{\mathrm{Gl}}_N^F$ then is the finite general linear group over the field $\mathbb{F}_q$. The Borel subgroup of upper triangular matrices $\overline{B}_0$ and its unipotent radical $\overline{U}_0$ are F-stable as well, while the group of diagonal matrices $\overline{T}_0$ is an F-stable maximally split torus. Their fixed points are the finite groups of upper triangular matrices $B_0 = \overline{B}_0^F$, upper unitriangular matrices $U_0 = \overline{U}_0^F$ and diagonal matrices $T_0 = \overline{T}_0^F$ over $\mathbb{F}_q$ respectively.
Since the vanishing ideal of $\overline{\mathrm{SO}}_N$ is generated by a set of polynomials over $\mathbb{F}_q$, it is F-stable and its fixed points

$$\mathrm{SO}_N = \overline{\mathrm{SO}}_N^F = \{g \in M_N(\mathbb{F}_q) \mid b(gu, gv) = b(u, v) \text{ for all } u, v \in \mathbb{F}_q{}^N \text{ and } \det(g) = 1\}$$

are the finite special orthogonal group over $\mathbb{F}_q$. Then the Borel group $\overline{B} = \overline{B}_0 \cap \overline{\mathrm{SO}}_N$ and its unipotent radical $\overline{U} = \overline{U}_0 \cap \overline{\mathrm{SO}}_N$ are F-stable and $\overline{T} = \overline{T}_0 \cap \overline{\mathrm{SO}}_N$ is an F-stable maximally split torus. Their respective fixed points then are the finite groups over $\mathbb{F}_q$

$$B = \overline{B}^F = B_0 \cap \mathrm{SO}_N, \qquad U = \overline{U}^F = U_0 \cap \mathrm{SO}_N \quad \text{and} \qquad T = \overline{T}^F = T_0 \cap \mathrm{SO}_N.$$

[24] cf [Gec13, 4.1.5, p. 172]
[25] cf. [Gec13, p. 170]
[26] cf. [Gec13, 4.1.7, p. 173]

For $\overline{G} = \overline{\mathrm{Gl}}_N$ or $\overline{G} = \overline{\mathrm{SO}}_N$ and a root $\alpha \in \Phi_{\overline{G},\overline{T}}$ the root subgroup $\overline{U}_\alpha$ is clearly F-stable, and we consider the fixed points $U_\alpha = \overline{U}_\alpha^F$ to be the finite root subgroups.

Lemma 1.4.3. *For closed pattern $\mathcal{P} \subseteq \mathcal{G}$ or $\mathcal{Q} \subseteq \mathcal{V}$ the pattern subgroups $\overline{H}_\mathcal{P} \leq \overline{Gl}_N$ and $\overline{H}_\mathcal{Q} \leq \overline{SO}_N$ are F-stable and their fixed points are the finite pattern groups*

$$H_\mathcal{P} = \left\{g \in U_0 \mid supp_\mathcal{G}(g) \in \mathcal{P}\right\} \quad and \quad H_\mathcal{Q} = \{g \in U \mid supp_\mathcal{V}(g) \in \mathcal{Q}\}$$

respectively. It follows that $|H_\mathcal{P}| = q^{|\mathcal{P}|}$ and $|H_\mathcal{Q}| = q^{|\mathcal{Q}|}$.

Proof. Let $\mathcal{P} \subseteq \mathcal{G}$ and $\mathcal{Q} \subseteq \mathcal{V}$ be closed pattern and let $\Phi = \Phi(\overline{H}_\mathcal{P}, \overline{T}_0)$ or $\Phi = \Phi(\overline{H}_\mathcal{Q}, \overline{T})$. For any $\alpha \in \Phi(\overline{H}_\mathcal{P}, \overline{T}_0)$ the root subgroup $\overline{U}_\alpha$ is F-stable and for its fixed points $U_\alpha = \overline{U}_\alpha^F$ we have $|U_\alpha| = q$. Since $\overline{H}_\mathcal{P} \leq \overline{\mathrm{Gl}}_N$ is a product of the root subgroups for $\Phi(\overline{H}_\mathcal{P}, \overline{T}_0)$, it is F-stable as well and $g \in \overline{H}_\mathcal{Q}^F$ if and only if $\mathrm{supp}_\mathcal{G}(g) \in \mathcal{P}$ and $g \in \mathrm{GL}_N$. Therefore, the fixed points of $\overline{H}_\mathcal{Q}$ is

$$H_\mathcal{P} = \overline{H}_\mathcal{P}^F = \left\{g \in U_0 \mid \mathrm{supp}_\mathcal{G}(g) \in \mathcal{P}\right\} = \prod_{\alpha \in \Phi(\overline{H}_\mathcal{P}, \overline{T}_0)} U_\alpha.$$

Since $|\Phi(\overline{H}_\mathcal{P}, \overline{T}_0)| = |\mathcal{P}|$, it follows that $|H_\mathcal{P}| = q^{|\mathcal{P}|}$.
For a closed pattern $\mathcal{Q} \subseteq \mathcal{V}$ and $\overline{H}_\mathcal{Q} \leq \overline{\mathrm{SO}}_N$ the same arguments apply, so we have $H_\mathcal{Q} = \overline{H}_\mathcal{Q}^F$ with $H_\mathcal{Q} = \left\{g \in U \mid \mathrm{supp}_\mathcal{G}(g) \in \mathcal{Q}\right\}$ and $|H_\mathcal{Q}| = q^{|\mathcal{Q}|}$.

□

As $\mathcal{V}$ itself is a closed pattern in $\mathcal{V}$, we have $U = \overline{H}_\mathcal{V}^F$ and therefore $|U| = q^{|\mathcal{V}|} = q^{n(n-1)}$.
The vanishing ideal of $\overline{\mathfrak{so}}_N$ is generated by a set of polynomials over $\mathbb{F}_q$, so it is F-stable and its fixed points are

$$\mathfrak{so}_N = \overline{\mathfrak{so}}_N^F = \{A \in M_N(\mathbb{F}_q) \mid b(gu, v) = -b(u, Av) \text{ for all } u, v \in {\mathbb{F}_q}^N\}.$$

The Lie algebra of upper unitriangular matrices $\overline{\mathfrak{u}}_0 \leq \overline{\mathfrak{gl}}_N$ is F-stable with $\mathfrak{u}_0 = \overline{\mathfrak{u}}_0^F$ and therefore $\overline{\mathfrak{u}} \leq \overline{\mathfrak{so}}_N$ is F-stable as well with $\mathfrak{u} = \overline{\mathfrak{u}}^F = \overline{\mathfrak{u}}_0 \cap \mathfrak{so}_N$. Furthermore, for a closed pattern $\mathcal{Q} \subseteq \mathcal{V}$ the pattern Lie algebra $\overline{\mathfrak{h}}_\mathcal{Q} \leq \overline{\mathfrak{so}}_N$ is F-stable with its fixed points being $h_\mathcal{Q} = \{X \in \mathfrak{u} \mid \mathrm{supp}_\mathcal{V}(X) \in \mathcal{Q}\}$.
As a composition of matrix products and the trace form the bilinear form $\tilde{\kappa}$ and as a composition of matrix products and matrix inverse, the Cayley transformation f are F-stable. Therefore, $\tilde{\kappa}$ defines a non degenerate bilinear form on $\mathfrak{so}_N$ and f indeed meets the requirement to be a Springer morphism.[27]

[27]cf. [SS70, 3.12, p. 229]

2 Decomposition of supercharacters for SO_N

Let $G_N = U_0 \leq \mathrm{GL}_N$ be the group of upper unitriangular matrices over the finite field $\mathbb{F}_q$ and let $U_N = U_0 \cap \mathrm{SO}_N \leq \mathrm{SO}_N$ be the group of upper unitriangular matrices in SO_N over $\mathbb{F}_q$. The subject of this chapter will be the decomposition of André–Neto supercharacters by Jedlitschky. [Jed13] While Jedlitschky primarily studied the corresponding U_N-modules, we will give an explicit description of the characters he defined, such that it becomes immediately apparent, that they are constituents of the André–Neto supercharacters. In order to prove the mutual orthogonality of these characters, we will first turn to the simple case of verge patterns and later extend this to the general case. For every such character we will then find a $n \times n$-matrix that is derived from a Gram matrix for the bilinear form b, and we can reduce the question of the classification of these characters to finding congruent matrices with certain conditions. Moreover, these matrices determine the inner product of the characters, so we can use them to find irreducible characters as well.

2.1 1-cocycle

For the construction of his super-character theory for G_N Yan [Yan10, 2.1, p. 4] utilized the fact that the map $\mu : G_N \to \mathbf{g} : g \mapsto g - I$, where $\mathbf{g} \leq M_N(\mathbb{F}_q)$ is the vector space of upper triangular matrices over $\mathbb{F}_q$ with zero diagonal, is a left and right 1-cocycle, that is a map with $\mu(gh) = \mu(g) + g\mu(h)$ and $\mu(gh) = \mu(g)h + \mu(h)$ for $g, h \in G_N$. This 1-cocycle gives rise to a monomial basis of the $\mathbb{C}G_N$-module $\mathbf{g}^*$ for the group algebra $\mathbb{C}G_N$ and the dual space $\mathbf{g}^*$ of $\mathbf{g}$. While this situation cannot be fully recreated for the case of the orthogonal group, Jedlitschky [Jed13, 2.1.11, p. 37] has shown that any right 1-cocycle of a group G and a vector space V generates such a monomial basis of the $\mathbb{C}G$-module V^*. To apply this to the group of upper unitriangular matrices in SO_N he [Jed13, 2.2.13, p. 48] defined the linear map $\pi : M_N(\mathbb{F}_q) \to \mathbf{v}$ of the restriction of a matrix in $M_N(\mathbb{F}_q)$ to the vector space $\mathbf{v} \leq M_N(\mathbb{F}_q)$ of matrices with non-zero entries only above the diagonal and counter-diagonal, which is isomorphic to the algebra of upper triangular matrices with zero on the diagonal in $\mathfrak{so}_N$.

Definition 2.1.1. For $\mathcal{G}, \mathcal{V} \subseteq [[N]]$ as defined in 1.3.7 let $\mathbf{g}, \mathbf{v} \leq M_N(\mathbb{F}_q)$ be the subsets defined by

$$\mathbf{g} = \{X \in M_N(\mathbb{F}_q) \mid \operatorname{supp}(X) \subseteq \mathcal{G}\}$$
$$\mathbf{v} = \{X \in M_N(\mathbb{F}_q) \mid \operatorname{supp}(X) \subseteq \mathcal{V}\}$$

Let π be the linear map that restricts a matrix in $M_N(\mathbb{F}_q)$ to $\mathbf{v}$ with

$$\pi : M_N(\mathbb{F}_q) \to \mathbf{v} : X \mapsto X|_{\mathbf{v}}.$$

We extend the 1-cocycle of U_N defined by Jedlitschly [Jed13, 3.1.9, p. 57] on $\mathbf{v}$ to a 1-cocycle of the full group of upper unitriangular matrices G_N on $\mathbf{v}$, where all the relevant properties prevail.

Definition/Lemma 2.1.2. *The right action $\circ$ of G_N on $\mathbf{v}$ defined as*

$$\circ : \mathbf{v} \times G_N \to \mathbf{v} : (X, g) \mapsto X \circ g = \pi(Xg)$$

is a representation of G_N on the vector space $\mathbf{v}$ and together with the map π it defines a 1-cocycle such that for $g, h \in G_N$ we have

$$\pi(gh) = \pi(g) \circ h + \pi(h).$$

Proof. For $g \in G_N$ the action $\cdot \circ g$ clearly defines an endomorphism on the vector space $\mathbf{v}$. Let now $g, h \in G_N$ and $1 \leq i < j \leq N$ with $i + j < N + 1$ and we have

$$\pi\left(e_{ij}gh\right) = \pi\left(\sum_{k=j}^{N}\sum_{l=k}^{N} g_{jk}h_{kl}e_{il}\right) = \sum_{k=j}^{N-i}\sum_{l=k}^{N-i} g_{jk}h_{kl}e_{il} = \sum_{k=i}^{N}\sum_{l=k}^{N-i} \pi\left(e_{ij}g\right)_{ik} h_{kl}e_{il} = \pi\left(\pi(e_{ij}g)h\right).$$

So for every $X \in \mathbf{v}$ by distributivity it follows that $X \circ gh = (X \circ g) \circ h$ and $\circ$ respects composition. $\square$

Corollary 2.1.3. *For $g, h \in G_N$ we have*

$$\pi(g^{-1}) = -\pi(g) \circ g^{-1} \quad \textit{and} \quad \pi(h^{-1}gh) = \pi(h) - \pi(h) \circ (h^{-1}gh) + \pi(g) \circ h$$

Proof. Let $g, h \in G_N$. Since $\pi(I) = 0$, we have $0 = \pi(gg^{-1}) = \pi(g)\circ g^{-1} + \pi(g^{-1})$ and therefore $\pi(g^{-1}) = -\pi(g)\circ g^{-1}$. It follows that

$$\pi((h^{-1}gh) = \pi(h^{-1})\circ gh + \pi(gh) = -\pi(h)\circ(h^{-1}gh) + \pi(g)\circ h + \pi(h).$$

□

The restriction of π to $\mathfrak{so}_N$ is a bijection with $\pi|_{\mathfrak{so}_N}^{-1}(X) = X - X^\dagger$ for $X \in \mathfrak{so}_N$ and with the previous corollary we can show that the restriction of π to SO_N is a bijection as well.[1]

Lemma 2.1.4. *The restriction of π to U_N is a bijection with $\pi : U_N \tilde{\rightarrow} \mathbf{v}$.*

Proof. Let $g, h \in U_N$ with $\pi(g) = \pi(h)$, so we have

$$\pi(gh^{-1})\circ h = \pi(g)\circ h^{-1}h + \pi(h^{-1})\circ h = \pi(g) - \pi(h) = 0$$

and therefore $\pi(gh^{-1}) = 0$. Let $x \in U_N$ with $\pi(x) = 0$, so for every $1 \leq i < j \leq N$ with $i + j < N + 1$ we have $x_{ij} = 0$. For $1 \leq i < j \leq N$ with $i + j \geq N + 1$ it follows that

$$0 = (xx^\dagger)_{ij} = \sum_{k=i}^{j} x_{ik}x_{\overline{jk}} = x_{ij} + x_{\overline{ji}} + \sum_{k=\bar{i}}^{\bar{j}} x_{ik}x_{\overline{jk}} = x_{ij}$$

and therefore $x = I$. So the restriction of the map π to U_N is injective. Since the number of positions in $\mathcal{V}$ are $|\mathcal{V}| = \sum_{i=1}^{n} 2(n-i) = n(n-1)$, we have $|\mathbf{v}| = q^{n(n-1)} = |U_N|$ by lemma 1.4.3. Therefore, the restriction of π to U_N is a bijection. □

Definition 2.1.5. Let κ be the non-degenerate bilinear form on $M_N(\mathbb{F}_q)$ defined as

$$\kappa : M_N(\mathbb{F}_q) \times M_N(\mathbb{F}_q) \to \mathbb{F}_q : (X, Y) \mapsto \mathrm{Tr}\,(X^t Y)$$

The restriction of κ to $\mathfrak{so}_N$ is derived from the bilinear form $\tilde{\kappa}$ as defined in theorem 1.2.5 with $\kappa(X, Y) = \tilde{\kappa}(X^t, Y)$ for $X, Y \in \mathfrak{so}_N$ and for $g \in \mathrm{SO}_N$ we have $\kappa(X, g^{-1}Yg) = \kappa(g^{-t}Xg^t, Y)$. For any $X, Y \in M_N(\mathbb{F}_q)$ we have $\kappa(X, Y) = \sum_{i,j=1}^{N} X_{ij}Y_{ij}$ and for $1 \leq i, j \leq N$ we have $\kappa(X, e_{ij}) = X_{ij}$.

Lemma 2.1.6. *For a subset $\mathcal{P} \subseteq [[N]]$ the restriction of the bilinear form κ to the pattern vector space $\mathbf{v}_\mathcal{P}$ is non-degenerate and the orthogonal complement $\mathbf{v}_\mathcal{P}^\perp$ of $\mathbf{v}_\mathcal{P}$ with respect to κ is $\mathbf{v}_\mathcal{P}^\perp = \mathbf{v}_{[[N]]\setminus\mathcal{P}}$.*

[1] [GJD19, 3.5, p. 8]

Proof. For $1 \leq i, j \leq N$ and $X \in M_N(\mathbb{F}_q)$ we have $\kappa(e_{ij}, X) = \mathrm{Tr}\left(e_{ji}X\right) = X_{ij}$ and therefore $\kappa(e_{ij}, X) = 0$ if and only if $X_{ij} = 0$. Therefore the support of the complement of the one dimensional pattern subspace $\mathbf{v}_{ij} = \mathbf{v}_{\{(i,j)\}}$ is $\mathrm{supp}(v_{ij}^{\perp}) = [[N]]\backslash\{(i,j)\}$. For $\mathcal{P} \subseteq [[N]]$ we then have $\mathrm{supp}(v_{\mathcal{P}}^{\perp}) = \bigcap_{(i,j)\in\mathcal{P}} \mathrm{supp}(v_{\{(i,j)\}}^{\perp}) = [[N]]\backslash(\bigcup_{(i,j)\in\mathcal{P}}\{(i,j)\}) = [[N]]\backslash\mathcal{P}$. Therefore, we have $\mathbf{v}_{\mathcal{P}}^{\perp} = \mathbf{v}_{[[N]]\backslash\mathcal{P}}$ and the radical of $v_{\mathcal{P}}$ is $\mathrm{rad}(\mathbf{v}_{\mathcal{P}}) = (0)$. □

Lemma 2.1.7. *Let $\vartheta : (\mathbb{F}_q, +) \to (\mathbb{C}^*, \cdot)$ be a non-trivial group homomorphism of the additive group $\mathbb{F}_q$ to the multiplicative group $\mathbb{C}^*$. For a vector space $\mathbf{w} \leq M_N(\mathbb{F}_q)$ and $X \in M_N(\mathbb{F}_q)$ we have*

$$\sum_{Y\in\mathbf{w}} \vartheta\kappa(X,Y) = \begin{cases} |\mathbf{w}| & \textit{for } X \in \mathbf{w}^{\perp} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $\mathbf{w} \leq M_N(\mathbb{F}_q)$ be a vector space and $X \in M_N(\mathbb{F}_q)$. For $X \in \mathbf{w}^{\perp}$ and $Y \in \mathbf{w}$ we have $\vartheta\kappa(X,Y) = \vartheta(0) = 1$ and therefore $\sum_{Y\in\mathbf{w}} \vartheta\kappa(X,Y) = |\mathbf{w}|$. For $X \notin \mathbf{w}^{\perp}$ there is a $Y_1 \in \mathbf{w}$ such that $\kappa(X, Y_1) \neq 0$ and since mapping $Y \in \mathbf{w}$ to $Y + Y_1$ is a bijection of $\mathbf{w}$, we have

$$\sum_{Y\in\mathbf{w}} \vartheta\kappa(X,Y) = \sum_{Y\in\mathbf{w}} \vartheta\kappa(X, Y+Y_1) = \vartheta\kappa(X,Y_1) \sum_{Y\in\mathbf{w}} \vartheta\kappa(X,Y).$$

Since $\vartheta\kappa(X,Y_1) \neq 1$, we must have $\sum_{Y\in\mathbf{w}} \vartheta\kappa(X,Y) = 0$. □

Lemma 2.1.8. *For $X, Y \in M_N(\mathbb{F}_q)$ with $\mathrm{supp}(X) \cap \mathrm{supp}(Y) \subseteq \mathcal{V}$ we have*

$$\kappa(X,Y) = \kappa(\pi(X), Y) = \kappa(X, \pi(Y))$$

Proof. Let $X, Y \in M_N(\mathbb{F}_q)$ with $\mathrm{supp}(X) \cap \mathrm{supp}(Y) \subseteq \mathcal{V}$. Then for $1 \leq i, j \leq N$ we have $X_{ij}Y_{ij} = 0$ unless $(i,j) \in \mathcal{V}$ and therefore $X_{ij}Y_{ij} = \pi(X)_{ij}Y_{ij} = X_{ij}\pi(Y)_{ij}$. It follows that

$$\kappa(X,Y) = \sum_{i,j=1}^{N} X_{ij}Y_{ij} = \kappa(\pi(X), Y) = \kappa(X, \pi(Y)).$$

□

Lemma 2.1.9. *Let "." be the left group action of G_N on $\boldsymbol{v}$ defined by*

$$\text{"."} : G_N \times \boldsymbol{v} \to \boldsymbol{v} : (g, X) \mapsto g.X = \pi(Xg^t).$$

For $X, Y \in \mathbf{v}$ and $g \in G_N$ we then have $\kappa(g.X, Y) = \kappa(X, Y \circ g)$.

Proof. Let $X, Y \in \mathbf{v}$ and $g \in G_N$. For $1 \leq i, j \leq N$ we have $(Xg^t)_{ij} = 0$ if $i + j \geq N + 1$ and $(Yg)_{ij} = 0$ if $i \geq j$, so by applying lemma 2.1.8 in the first and last step we have

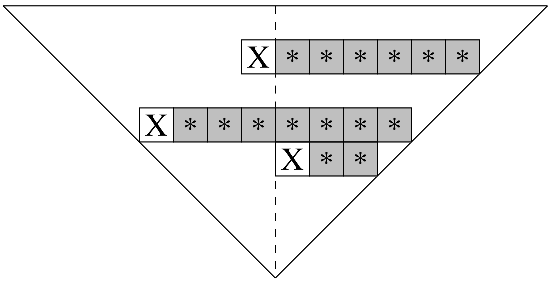
$$\kappa(g.X, Y) = \kappa(Xg^t, Y) = \operatorname{Tr}(gX^tY) = \operatorname{Tr}(X^tYg) = \kappa(X, Yg) = \kappa(X, Y \circ g).$$

Since the $\circ$ action respects multiplication for $g, h \in G_N$ and $1 \leq i < j \leq N$ with $i + j < N + 1$, we have

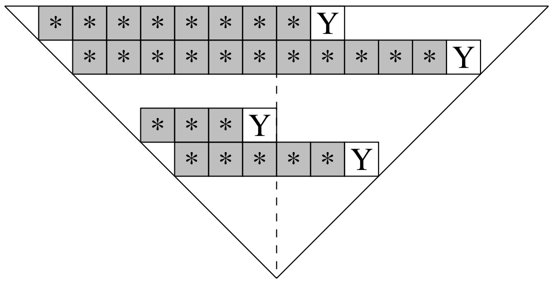
$$(g.(h.X))_{ij} = \kappa(g.(h.X), e_{ij}) = \kappa(X, (e_{ij} \circ g) \circ h) = \kappa(X, e_{ij} \circ gh) = \kappa(gh.X, e_{ij}) = (gh.X)_{ij},$$

so "." respects multiplication as well. Since $I.X = \pi(XI^t) = X$, the map "." is a group action. □

The group G_N acts through $\circ$ on $\mathbf{v}$ by changing the values of entries only to the right of non-zero entries of $\mathbf{v}$, whereas it acts through "." on $\mathbf{v}$ by changing the values of entries only to the left of non-zero entries of $\mathbf{v}$.



Possible non-zero positions (*) of $X \circ g$ for $X \in \mathbf{v}$ and $g \in G_N$



Possible non-zero positions (*) of $g.Y$ for $Y \in \mathbf{v}$ and $g \in G_N$

While Jedlitschky [Jed13, 2.2.13, p. 48] defined this action as a right group action of the opposite group G_N^{OP} on a subspace on $\mathbf{g}$, we for convenience choose to define it as a left action. With this following Jedlitschky [Jed13, 2.1.35, p. 41] for $A \in \mathbf{v}$ we can define a map $\chi_A : G_N \to \mathbb{C}$ that restricted to the stabilizer of A with respect to the group action "." is a linear character

Definition 2.1.10. Let $\vartheta : (\mathbb{F}_q, +) \to (\mathbb{C}^*, \cdot)$ be a non-trivial group homomorphism of the additive group $\mathbb{F}_q$ to the multiplicative group $\mathbb{C}^*$. Then for $A \in \mathbf{v}$ we define the map χ_A as

$$\chi_A : U_N \to \mathbb{C} : g \mapsto \vartheta\kappa(A, \pi(g)).$$

Lemma 2.1.11. *For $A, B \in \mathbf{v}$ and $g, h \in G_N$ we have*

$$\chi_A \chi_B = \chi_{A+B} \quad \text{and} \quad \chi_A(gh) = \chi_{h.A}(g)\chi_A(h).$$

Let $Stab_{G_N}(A) = \{g \in G_N \mid g.A = A\}$ be the stabilizer of A with respect to the group action ".". Then for $g, h \in Stab_{G_N}(A)$ we have $\chi_A(gh) = \chi_A(g)\chi_A(h)$ and the restriction of χ_A to $Stab_{G_N}(A)$ is a linear character.

Proof. Let $A, B \in \mathbf{v}$ and $g \in G_N$. Then we have

$$(\chi_A\chi_B)(g) = \vartheta\kappa(A, \pi(g))\,\vartheta\kappa(B, \pi(g)) = \vartheta\,(\kappa(A, \pi(g)) + \kappa(B, \pi(g))) = \vartheta\kappa(A + B, \pi(g)) = \chi_{A+B}(g).$$

For $h \in G_N$ by lemma 2.1.9 we have

$$\chi_A(gh) = \vartheta\kappa\,(A, \pi(gh)) = \vartheta\kappa\,(A, \pi(g)\circ h + \pi(h)) = \vartheta\kappa\,(h.A, \pi(g))\,\vartheta\kappa\,(A, \pi(h)) = \chi_{h.A}(g)\chi_A(h)$$

For $g, h \in \mathrm{Stab}_{G_N}(A)$ we have $h.A = A$ and therefore $\chi_A(gh) = \chi_A(g)\chi_A(h)$. □

Corollary 2.1.12. *For $A \in \mathbf{v}$ and $g \in G_N$ the complex conjugate of χ_A is*

$$\overline{\chi_A} = \chi_{-A} \quad \text{and} \quad \overline{\chi_A}(g) = \chi_{g.A}(g^{-1}).$$

For $h \in Stab_{G_N}(A)$ this means $\overline{\chi_A}(h) = \chi_A(h^{-1})$.

Proof. For any $c \in \mathbb{F}_q$ we have $\vartheta(c)\overline{\vartheta}(c) = |\vartheta(c)|^2 = 1$ and therefore $\overline{\vartheta}(c) = \vartheta(-c)$. So for $A \in \mathbf{v}$ and $g \in G_N$ we have $\overline{\chi_A}(g) = \vartheta\kappa(-A, \pi(g)) = \chi_{-A}(g)$. Since $-\pi(g) = \pi(g^{-1})\circ g$ by corollary 2.1.3, we have $\overline{\chi_A}(g) = \vartheta\kappa(A, -\pi(g)) = \vartheta\kappa(A, \pi(g^{-1})\circ g) = \vartheta\kappa(g.A, \pi(g^{-1})) = \chi_{g.A}(g^{-1})$. For $h \in \mathrm{Stab}_{G_N}(A)$ we have $h.A = A$ and therefore $\overline{\chi_A}(h) = \chi_A(h^{-1})$. □

2.2 André–Neto characters

In the development of their super-character theory André and Neto [AN06, p. 399] defined pattern subgroups $C_\alpha \leq U_N$ as well as linear characters $\chi_\alpha : C_\alpha \to \mathbb{C}$ for the positive roots $\alpha \in \Phi_+(\overline{\mathrm{SO}}_n, \overline{T})$. For $(i, j) \in \mathcal{V}$ with $\rho_{SO}(i, j) = \alpha$ as defined in lemma 1.3.9 this character is equal to the map defined in 2.1.10 with $\chi_\alpha = \chi_{ce_{ij}}$

Definition/Lemma 2.2.1. *For $1 \leq i < j \leq N$ with $i + j < N + 1$ let $\mathcal{Z}_{ij} \subseteq \mathcal{V}$ be the subset defined by*

$$\mathcal{Z}_{ij} = \{(i,k) \mid i < k \leq \min(j-1, n)\}\,.$$

Let $C_{ij} \leq U_N$ be the pattern subgroup for the closed subset $\mathcal{V} \backslash \mathcal{Z}_{ij}$. Then, for $c \in \mathbb{F}_q$ the map $\chi_{ce_{ij}}$ is a linear character of $C_{ij} \cap C_{\overline{ji}}$ defined by

$$\chi_{ce_{ij}} : C_{ij} \cap C_{\overline{ji}} \to \mathbb{C}^* : x \mapsto \vartheta\kappa(ce_{ij}, \pi(x)).$$

Proof. For $1 \leq i < j \leq N$ with $i + j < N + 1$ let $g, h \in C_{ij} \cap C_{\overline{ji}}$ and for $c \in \mathbb{F}_q$ we have

$$\chi_{ce_{ij}}(gh) = \vartheta(ce_{ij}, \pi(gh)) = \vartheta(c\pi(gh)_{ij}) = \vartheta\left(c\sum_{k=i}^{j} g_{ik}h_{kj}\right) = \vartheta\left(cg_{ii}h_{ij} + c\sum_{k=\min(j,n+1)}^{j} g_{ik}h_{kj}\right).$$

For $j \leq n$ it follows that $\chi_{ce_{ij}}(gh) = \vartheta(c(h_{ij}+g_{ij})) = \chi_{ce_{ij}}(g)\chi_{ce_{ij}}(h)$ while for $j > n$ since $h \in C_{\overline{ji}}$ we have $h_{kj} = h_{\overline{jk}} = 0$ for $n + 1 \leq k < j$ and therefore $\chi_{ce_{ij}}(gh) = \chi_{ce_{ij}}(g)\chi_{ce_{ij}}(h)$ as well, which shows that $\chi_{ce_{ij}}$ is a linear character. $\square$

Lemma 2.2.2. *For $1 \leq i < j \leq N$ with $i + j < N + 1$, $c \in \mathbb{F}_q$ and $x \in C_{ij} \cap C_{\overline{ji}}$ we have $\chi_{ce_{ij}}(x) = \vartheta\kappa\big(c(e_{ij} - e_{\overline{ji}}), f(x)\big)$ where f is the Cayley transformation as defined in lemma 1.3.12.*

Proof. Let $1 \leq i < j \leq N$, $c \in \mathbb{F}_q$ and $x \in C_{ij} \cap C_{\overline{ji}}$. By definition of C_{ij} and $C_{\overline{ji}}$ we have

$$\left((x-I)^2\right)_{ij} = \sum_{k=i+1}^{j-1} x_{ik}x_{kj} = \sum_{k=\min(j-1,n)+1}^{j-1} x_{ik}x_{kj} = 0.$$

So we have $\vartheta\kappa\left(c(e_{ij} - e_{\overline{ji}}), (x-I)^2\right) = 0$. Since we have $\kappa(ce_{ij}^\dagger, f(x)) = \kappa(ce_{ij}, f(x)^\dagger)$ and $f(x)^\dagger = -f(x)$ by lemma 1.3.12, it follows that

$$\begin{aligned}
\vartheta\kappa\left(c(e_{ij} - e_{\overline{ji}}), f(x)\right) &= \vartheta\kappa\left(c(e_{ij} - e_{ij}^\dagger), f(x)\right) \\
&= \vartheta\left(\kappa\left(ce_{ij}, f(x)\right) - \kappa\left(ce_{ij}^\dagger, f(x)\right)\right) \\
&= \vartheta\left(\kappa\left(ce_{ij}, f(x)\right) - \kappa\left(ce_{ij}, -f(x)\right)\right) \\
&= \vartheta\left(2\kappa\left(ce_{ij}, f(x)\right)\right) \\
&= \sum_{k\in\mathbb{N}} \vartheta\left((-1)^{k-1}\frac{1}{2^{k-1}}\kappa\left(ce_{ij}, (x-I)^k\right)\right) \\
&= \vartheta\kappa\left(ce_{ij}, x-I\right) \\
&= \vartheta\kappa\left(ce_{ij}, \pi(x)\right) \\
&= \chi_{ce_{ij}}(x)
\end{aligned}$$

□

André and Neto defined elementary characters $\phi_{ce_{ij}} = \mathrm{Ind}_{C_{ij}\cap C_{\overline{ji}}}^{U_N}\chi_{ce_{ij}}$ of U_N that are the induced characters of the linear characters $\chi_{ce_{ij}}$ of the pattern subgroups $C_{ij} \cap C_{\overline{ji}}$.[2] For a $B \in \mathbf{g}$ with at most one non-zero entry per row and column the André–Neto character $\phi_{\pi(B)}$ of U_N are the induced character of the linear character $\chi_{\pi(B)}$ from C_B, and they are the product of the corresponding elementary characters.[3]

Lemma 2.2.3. *For $1 \le i < j \le N$ with $i + j < N + 1$ and $c \in \mathbb{F}_q$ let $\phi_{ce_{ij}}$ be the induced character of $\phi_{ce_{ij}} = Ind_{C_{ij}\cap C_{\overline{ji}}}^{U_N}\chi_{ce_{ij}}$. For an element $B \in \mathbf{g}$ with at most one non-zero entry per row and column let $C_B = \bigcap_{(i,j)\in supp(B)} C_{ij}$ and*

$$\chi_{\pi(B)}(x) = \vartheta\kappa(\pi(B), \pi(x)) = \prod_{(i,j)\in supp_{\mathcal{V}}(B)} \chi_{ij,B_{ij}}(x)$$

for $x \in U_N$. Then the induced character $\phi_{\pi(B)}$ is the product

$$\phi_{\pi(B)} = Ind_{C_B}^{U_N}\chi_{\pi(B)} = \prod_{(i,j)\in supp_{\mathcal{V}}(B)} \phi_{ij,B_{ij}}.$$

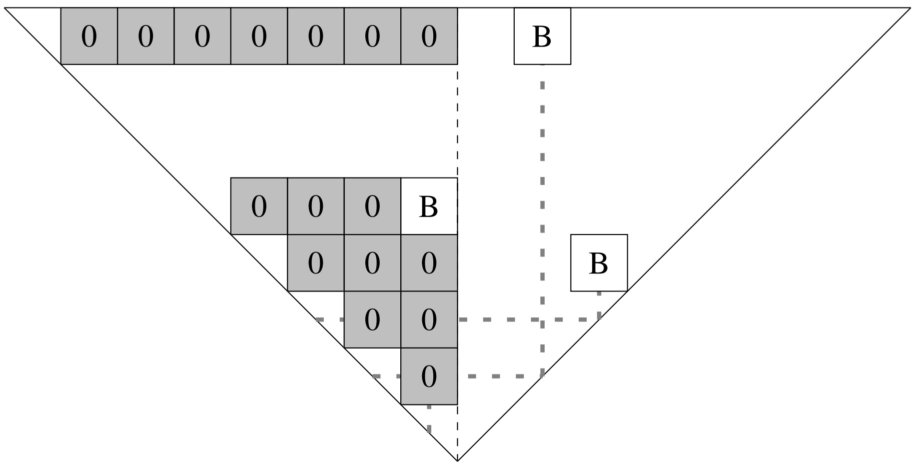
Proof. Let $B \in \mathbf{g}$ such that it has at most one non-zero entry per row and column. We have $\chi_{\pi(B)} = \prod_{(i,j)\in \mathrm{supp}_{\mathcal{V}}(B)}\chi_{ce_{ij}}$ by lemma 2.1.11.

[2] cf. [AN06, p. 399]
[3] cf. [AN06, 2.2, p. 401]

If $|\mathrm{supp}_{\mathcal{V}}(B)| = 1$, we clearly have $\phi_{\pi(B)} = \prod_{(i,j)\in\mathrm{supp}_{\mathcal{V}}(B)}\chi_{ce_{ij}}(x)$. For $m \in \mathbb{N}$ we assume that for any $B \in \mathbf{g}$ with at most one non-zero entry per row and column $|\mathrm{supp}_{\mathcal{V}}(B)| \leq m$ we have $\phi_{\pi(B)} = \prod_{(i,j)\in\mathrm{supp}_{\mathcal{V}}(B)}\chi_{ce_{ij}}(x)$ and let B be such that $|\mathrm{supp}_{\mathcal{V}}(B)| = m+1$. Let $1 \leq i < j \leq N$ with $i + j < N + 1$ such that $B_{ij} \neq 0$ and let $B' \in \mathbf{g}$ with at most one non-zero entry per row and column as well as $c \in \mathbb{F}_q$ be such that $B = B' + c(e_{ij} - e_{\overline{ji}})$. Then we have $\chi_{\pi(B')}\chi_{ce_{ij}} = \chi_{\pi(B)}$ and $C_{ij} \cap C_{\overline{ji}} \cap C_{B'} = C_B$. Since B' has no non-zero element in the i-th or $\overline{j}$-th row, we furthermore have $(C_{ij} \cap C_{\overline{ji}})C_{B'} = U_N$ and it follows that

$$\begin{aligned}
\mathrm{Ind}_{C_{B'}}^{U_N}\chi_{\pi(B')}\,\mathrm{Ind}_{C_{ij}\cap C_{\overline{ji}}}^{U_N}\chi_{ce_{ij}} &= \mathrm{Ind}_{C_{B'}}^{U_N}\left(\chi_{\pi(B')}\,\mathrm{Res}_{C_{B'}}^{U_N}\mathrm{Ind}_{C_{ij}\cap C_{\overline{ji}}}^{U_N}\chi_{ce_{ij}}\right)\\
&= \mathrm{Ind}_{C_{B'}}^{U_N}\left(\chi_{\pi(B')}\,\mathrm{Ind}_{C_B}^{C_{B'}}\mathrm{Res}_{C_B}^{C_{ij}\cap C_{\overline{ji}}}\chi_{ce_{ij}}\right)\\
&= \mathrm{Ind}_{C_{B'}}^{U_N}\mathrm{Ind}_{C_B}^{C_{B'}}\mathrm{Res}_{C_B}^{C_{ij}\cap C_{\overline{ji}}}\left(\chi_{\pi(B')}\chi_{ce_{ij}}\right)\\
&= \mathrm{Ind}_{C_B}^{U_N}\chi_{\pi(B)} = \phi_{\pi(B)}.
\end{aligned}$$

The claim then follows by induction. $\square$



Zero positions of C_B for $B \in \mathbf{g}$

In his work about super-character theories for groups defined by anti-involutions, that is an involution on a group that is also an anti-automorphism, Andrews [And15, 4,1, p. 9] develops an alternate expression of the André–Neto characters by defining a group action of G_N on $\mathfrak{so}_N$, utilizing the fact that $g^{\dagger}Xg \in \mathfrak{so}_N$ for $X \in \mathfrak{so}_N$ and $g \in G_N$.

Lemma 2.2.4. *Let $g \in G_N$ and $X \in \mathfrak{u}_N$, then $g^{\dagger}Xg \in \mathfrak{u}_N$ and $*$ defines a group action of G_N on $\mathfrak{u}_N$ by*

$$* : G_N \times \mathfrak{u}_N \to \boldsymbol{v} : (g, X) \mapsto g{*}X = \left((g^{\dagger})^t X g^t\right)|_{\mathbf{g}}.$$

Proof. For $g \in G_N$ and $X \in \mathfrak{u}_N$ we have $(g^\dagger X g)^\dagger + g^\dagger X g = g^\dagger(X^\dagger + X)g = 0$ and therefore $g^\dagger X g \in \mathfrak{so}_N$. Since $g^\dagger \in G_N$, we also have $g^\dagger X g \in \mathbf{g}$ and therefore $g^\dagger X g \in \mathbf{g} \cap \mathfrak{so}_N = \mathfrak{u}_N$. Let $g \in G_N$ and $X \in \mathfrak{u}_N$. We have $((g^\dagger)^t X g^t)^\dagger + (g^\dagger)^t X g = (g^\dagger)^t(X^\dagger + X)g^t = 0$ and therefore $(g^\dagger)^t A g^t \in \mathfrak{so}_N$. Since for $1 \le i < j \le N$ and $1 \le k, l \le N$ with $\bar{i} < \bar{k}$ and $j < l$ we have $k < i < j < l$ and therefore $(X{*}g)_{kl} = ((g^\dagger)^t A g^t)_{kl}$. For $h \in G_N$, it follows that

$$\begin{aligned}
(h{*}(g{*}X))_{ij} &= \sum_{k,l=1}^{N} h_{\overline{ik}}(X{*}g)_{kl}h_{jl} = \sum_{k=1}^{i-1}\sum_{l=j+1}^{N} h_{\overline{ik}}(X{*}g)_{kl}h_{jl} \\
&= \sum_{k=1}^{i-1}\sum_{l=j+1}^{N} h_{\overline{ik}}((g^\dagger)^t X g^t)_{kl}h_{jl} = \left((h^\dagger)^t (g^\dagger)^t X g^t h^t\right)_{ij} = \left((hg^\dagger)^t X (hg)^t\right)_{ij}, \\
&= (hg{*}X)_{ij}
\end{aligned}$$

so the operation $*$ respects composition. Since we have $I{*}X = X$ as well, $*$ is a group action. $\square$

This operation is equivalent to the operation of G_N on the dual space $\mathfrak{u}_N^*$ of $\mathfrak{u}_N$ given by $g\alpha(X) = \alpha(g^\dagger X g)$ for $\alpha \in \mathfrak{u}_N^*$, $g \in G_N$ and $X \in \mathfrak{u}_N$. For the natural isomorphism $\beta : \mathfrak{u}_N \to \mathfrak{u}_N^*$ and $Y \in \mathfrak{u}_N$ we then have

$$g\beta(X)(Y) = \beta(X)\left(g^\dagger Y g\right) = \sum_{1 \le i < j \le N} X_{ij}\left(g^\dagger Y g\right)_{ij} = \mathrm{Tr}\left(X^t g^\dagger Y g\right) = \kappa\left(\left(g^\dagger X g\right)^t, Y\right) = \kappa\left((g{*}X)^t, Y\right)$$

Jedlitschky defined a verge pattern [Jed13, 3.2.8, p. 62] as a pattern $A \in \mathbf{v}$ such that A has at most one non-zero entry in every row and column. He furthermore defined a verge pattern to be hook separated if $A - A^\dagger$ has at most one non-zero entry in every row and column.[4] As we only consider hook separated verge patterns, we will call these verge patterns and omit that they are hook separated if it is not explicitly mentioned.
Analogous to these verge patterns, we define verge matrices to be the matrices in $\overline{\mathfrak{so}}_N$ that contain at most one non-zero entry in every row and column. Every such verge matrices can be identified with a verge pattern by the bijection $\pi : \overline{\mathfrak{so}}_N \to \mathbf{v}$.

Definition 2.2.5. A element $B \in \mathbf{g}$ is called a verge matrix if B has at most one non-zero entry in every row and column. Let $\mathcal{B} \subseteq \mathfrak{so}_N$ be the subset of all verge matrices and we call $A \in \mathbf{v}$ a verge pattern if there is a $B \in \mathcal{B}$ such that $A = \pi(B)$.

[4] cf. [Jed13, 3.3.17, p. 85 / 3.3.24, p. 88]

We can now show that $\mathcal{B}$ is a set of unique representatives of the G_N-orbits in $\mathfrak{u}_N$ with respect to $*$.

Lemma 2.2.6. *For every $A \in \mathfrak{u}_N$ there is exactly one $B \in \mathcal{B}$ such that $A \in G_N * B$.*

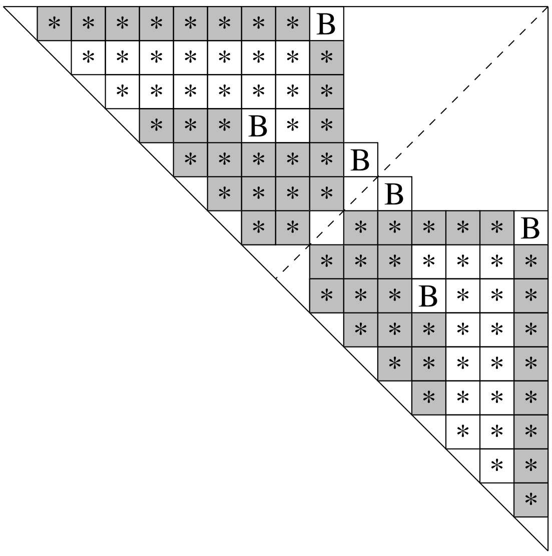
Proof. We will first show that every element in $\mathfrak{u}_N$ is contained in the orbit of some $B \in \mathcal{B}$. For $1 \leq m \leq n$ let $\mathcal{B}_m \subseteq \mathfrak{u}_N$ be the subset such that $A \in \mathfrak{u}_N$ if for every $1 \leq i < j \leq N$ with $i < m$ or $\overline{m} < j$ and $A_{ij} \neq 0$ this is the only nonzero element in the i-th row and j-th column. For $m = 1$ we have $\mathcal{B}_1 = \mathfrak{u}_N$
Let now $1 \leq m \leq n$ and $A \in \mathcal{B}_m$. If all entries in the m-th row of A are zero, then by symmetry all entries in the $\overline{m}$-th column are zero as well as for every $1 \leq i < j \leq N$ with $i < m+1$ or $\overline{m+1} < j$ and $A_{ij} \neq 0$ this is the only nonzero element in the i-th row and j-th column, so we have $A \in \mathcal{B}_{m+1}$. Conversely, let $m \leq k \leq N$ be such that $A_{mk} \neq 0$ and $A_{ml} = 0$ for all $k < l \leq N$, which is the rightmost non-zero entry in the m-th row of A. Now let $g \in G_N$ be defined as

$$g = I - \frac{1}{A_{mk}} \left(\sum_{l=m+1}^{k-1} A_{ml} e_{lk} + A_{\overline{k},\overline{l}} e_{\overline{l},\overline{m}} \right).$$

Since by assumption the k-th and $\overline{m}$-th column has no non-zero entry above the m-th row and the m-th and $\overline{k}$-th row has no non-zero entry right of the $\overline{m}$-th column, for $1 \leq i < j \leq N$ with $i < m$ or $\overline{m} < j$ we have $(g*A)_{ij} = A_{ij}$. For $1 < j \leq m$ we have $(g*A)_{mj} = A_{mj} - A_{mk} g_{jk} = 0$. For $m < i < k$ with $i \neq \overline{k}$ we have $(g*A)_{ik} = A_{ik} - g_{\overline{im}} A_{mk} = 0$, so A_{mk} is the only non-zero entry of A both in the m-th row and k-th column. Due to symmetry $A_{\overline{km}}$ is also the only non-zero entry of A both in the $\overline{k}$-th row and $\overline{m}$-th column, so $g*A \in \mathcal{B}_{m+1}$.
Since $B_n = \mathcal{B}$, we can conclude by induction that for every $A \in \mathfrak{u}_N$ there is a $g \in G_N$ such that $g*A \in \mathcal{B}$.
We will now show that the elements of $\mathcal{B}$ are unique representatives of the orbits in $\mathfrak{u}_N$. For $0 \leq m \leq n$ let $V_m \trianglelefteq G_N$ be the normal subgroup with $g \in V_m$ if $g_{ij} = 0$ for every $1 \leq i < j \leq N$ with $i < m$ or $\overline{m} < j$ and $L_m \leq G_N$ the subgroup with $g \in V_m$ if $g_{ij} = 0$ for every $1 \leq i < j \leq N$ with $i \geq m$ and $\overline{m} \geq j$, such that $G_N = L_m \ltimes V_m$. Let $S = \{g \in G_N \mid g*A = A\}$ be the stabilizer of A in G_N with respect to the operation $*$. Let now $A, B \in \mathcal{B}$ and $g \in G_N$ such that $g*A = B$. For any $1 \leq m \leq n$ let $g = lv$ with $l \in L_{m+1}$ and $v \in V_{m+1}$. We assume that $v \in S$ and that there is a $1 \leq k < \overline{m}$ such that $A_{mk} \neq 0$. Then $B_{mk} = (g*A)_{mk} = (l*A)_{mk} = A_{mk}$ as A_{mk}. Then for any $m < i < k$ we have $0 = B_{ik} = (l*A)_{ik} = A_{mk} l_{mi}$ and therefore $l_{mi} = 0$. The same way we have $0 = B_{\overline{ki}} = (l*A)_{\overline{ki}} = A_{\overline{km}} l_{\overline{im}}$ and therefore $l_{\overline{im}} = 0$. Let now $l = hw$ with $h \in L_m$ and $w \in V_m$, then for $1 \leq i < j \leq N$ we have $w_{ij} = 0$ unless $i = m$ and $j \geq k$ or $j = \overline{m}$ and $i \leq \overline{k}$. Since A_{mk}

is the only non-zero entry in the m-th row and k-th column as well as $A_{\overline{k}\overline{m}}$ is the only non-zero entry $\overline{k}$-th row and $\overline{m}$-th column, we have $w \in S$ and therefore $wv \in S$. Since $V_0 = I$ and $L_n = I$, we can conclude by induction over m that $g \in S$ and therefore $A = B$, which grants the uniqueness of the representatives $\mathcal{B}$ for the G_N-orbits. □

The G_N-orbit of $B \in \mathcal{B}$ contains matrices which have non-zero entries in positions $(j, k) \in \mathcal{V}$ if there are $1 \leq i \leq j$ and $k \leq l \leq N$ such that $B_{il} \neq 0$. As we can later show, we have $|G_N{*}B| = q^a$ for $a \in \mathbb{N}_0$ being the number of entries in $\mathcal{V}$ that are to the left or below a non-zero entry of B.



Possible non-zero positions (*) of $g{*}B$ for $B \in \mathcal{B}$ and $g \in G_N$

Analogous to Andrews [And15, p. 6] we define a normal subgroup $H \trianglelefteq G_N$ that has, excluding entries on the diagonal, non-zero entries only in the upper half of the matrix.[5] With this we can describe the character $\phi_{\pi(B)}$ for $B \in \mathcal{B}$ as a sum over the elements of the orbit $G_N{*}B$.[6]

Lemma 2.2.7. *Let $H \trianglelefteq G_N$ be the normal subgroup with*

$$H = \left\{ h \in G_N \,\middle|\, h_{ij} = 0 \textit{ for } n < i < j \leq N \right\}.$$

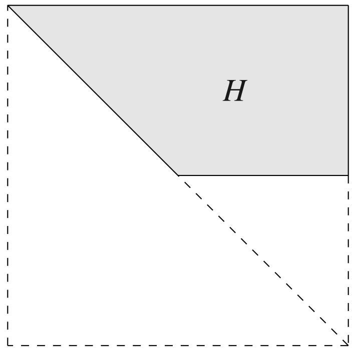
Then G_N is the product $G_N = U_N H$.

Proof. For $g \in G_N$ let $r \in H$ such that $r_{ij} = g_{ij}$ for all $1 \leq i < j \leq N$ and $i \leq n$. Let $s \in H$ be such that $s_{ij} = g_{\overline{j}\overline{i}}$ for $1 \leq i < j \leq n$ as well as $s_{kl} = 0$ for $1 \leq k \leq n < l \leq N$. We then have

[5] cf. [And15, p. 13]
[6] cf. [And15, 6.1, p. 17]

$g = s^\dagger r$ as well as $ss^\dagger = s^\dagger s$ and therefore $(s^\dagger s^{-1})^\dagger s^\dagger s^{-1} = s^{-\dagger} s s^\dagger s^{-1} s^{-\dagger} s^\dagger s s^{-1} = I$. It follows that $s^{-\dagger} s \in U_N$, which gives us $g = s^\dagger s^{-1} sr = (s^{-\dagger} s)^{-1}(sr) \in U_N H$.

□



Subgroup $H \trianglelefteq G_N$

Proposition 2.2.8. *Let $B \in \mathcal{B}$ be a verge matrix. For the orbit $B{*}H$ the set $H{*}B - B$ is a linear subspace of* $\mathbf{v}$ *with*

$$H{*}B - B = \bigoplus_{(i,k)\in supp(B)} \left\langle e_{ij} \mid i < j \le \min(k-1, n)\right\rangle_{\mathbb{F}_q}$$

and $f(C_B)$ is the orthogonal complement of $H{*}B - B$ in $\mathbf{v}$.

Proof. Let $B \in \mathcal{B}$ be a verge matrix and $h \in H$. Then for $1 \le i < j \le N$ and $1 \le l \le i$ we have $h_{\bar{i}\bar{l}} = 0$ unless $n < i$ or $l = i$. For $i + j < N + 1$, which implies $i \le n$, it follows that $(h * B)_{ij} = \sum_{l=1}^{i} \sum_{k=j}^{N} h_{\bar{i}\bar{l}} B_{lk} h_{jk} = \sum_{l=j}^{N} B_{ik} h_{jk}$ and therfore $(h * B)_{ij} = 0$ unless there is a $j \le k \le N$ such that $(i, k) \in \operatorname{supp}(B)$ and $j \le \min(k, n)$. For $j = k$ we have $(h * B)_{ij} = B_{ij}$ and therefore $(h * B - B)_{ij} = 0$. So we have $\operatorname{supp}_{\mathcal{V}}(H * B - B) = \bigcup_{(i,k)\in\operatorname{supp}(B)} \mathcal{Z}_{ik}$ as well as $\operatorname{supp}_{\mathcal{V}}(f(C_B)) = \bigcap_{(i,k)\in\operatorname{supp}(B)} \mathcal{V}\backslash\mathcal{Z}_{ik}$. It follows that $H{*}B - B$ is the orthogonal complement of $f(C_B)$ with respect to κ. □

Theorem 2.2.9. *For a verge matrix $B \in \mathcal{B}$ and $x \in U_N$ we have*

$$\phi_{\pi(B)}(x) = \frac{|H{*}B|}{|G_N{*}B|} \sum_{V\in G*B} \vartheta\kappa\left(V, f(x)\right).$$

Proof. Let $B \in \mathcal{B}$ and $x \in U_N$. By proposition 2.2.8 $H * B$ is the orthogonal complement of $f(C_B)$ with respect to κ, so by lemma 2.1.7 we have

$$\sum_{h\in H} \vartheta\kappa((h{*}B - \pi(B)), f(x)) = \begin{cases} |H| & \text{for } x \in C_B \\ 0 & \text{for } x \notin C_B. \end{cases}$$

For $x \in U_N$ by lemma 2.1.11 and 2.2.2 we have

$$\chi_{\pi(B)}(x) = \prod_{(i,j)\in \mathrm{supp}_{\mathcal{V}}(B)} \chi_{ij,B_{ij}}(x) = \prod_{(i,j)\in \mathrm{supp}_{\mathcal{V}}(B)} \vartheta\kappa(B_{ij}(e_{ij} - e_{\overline{ji}}), f(x)) = \vartheta\kappa(B, f(x)).$$

so by lemma 2.2.7 we can conclude for the induced character that

$$\begin{aligned}
\phi_{\pi(B)}(x) &= \frac{1}{|C_B|} \sum_{\substack{u\in U_N \\ u^{-1}xu\in C_B}} \vartheta\kappa\left(B, f(u^{-1}xu)\right) \\
&= \frac{1}{|C_B||H|} \sum_{u\in U_N} \vartheta\kappa\left(B, f(u^{-1}xu)\right) \sum_{h\in H} \vartheta\kappa((h{*}B - B, f(u^{-1}xu)) \\
&= \frac{1}{|C_B||H|} \sum_{u\in U_N} \sum_{h\in H} \vartheta\kappa(B + h{*}B - B, f(u^{-1}xu)) \\
&= \frac{1}{|C_B||H|} \sum_{u\in U_N} \sum_{h\in H} \vartheta\kappa(h{*}B, u^{-1}f(x)u) \\
&= \frac{1}{|C_B||H|} \sum_{u\in U_N} \sum_{h\in H} \vartheta\kappa(uh{*}B, f(x)) \\
&= \frac{|U_N|}{|C_B||G_N|} \sum_{g\in G_N} \vartheta\kappa(g{*}B, f(x)) \\
&= \frac{|U_N|}{|C_B||G_N{*}B|} \sum_{V\in G_N{*}B} \vartheta\kappa(V, f(x)).
\end{aligned}$$

Since $|U_N| = |H{*}B - B||f(C_B)| = |H{*}B||C_B|$, the claim follows. $\square$

Corollary 2.2.10. *For $B \in \mathcal{B}$ the orbits $H{*}B$ and $G_N.\pi(B)$ have the same number of elements, which is equal to the degree of their respective André–Neto character.*

$$\deg \phi_{\pi(B)} = |H{*}B| = \prod_{(i,j)\in supp(\pi(B))} q^{j-i-1}$$

Proof. Let $B \in \mathcal{B}$. Then by theorem 2.2.9 we have $\phi_{\pi(B)}(I) = \frac{|H*B|}{|G_N*B|}|G_N * B| = |H * B|$, which proves the first equatios. By lemma 2.2.3 we have

$$\deg \phi_{\pi(B)} = \prod_{(i,j)\in \mathrm{supp}(\pi(B))} \deg \phi_{B_{ij}e_{ij}} = \prod_{(i,j)\in \mathrm{supp}(\pi(B))} \frac{|U_N|}{|C_{ij} \cap C_{\overline{ji}}|}.$$

For $(i, j) \in \operatorname{supp}(\pi(B))$ with $j \leq n$ we have $\frac{|U_N|}{|C_{ij} \cap C_{\overline{ji}}|} = q^{j-i-1}$, while for $(i, j) \in \operatorname{supp}(\pi(B))$ with $j > n$ we have $\frac{|U_N|}{|C_{ij} \cap C_{\overline{ji}}|} = q^{n-i} q^{n-\overline{j}} = q^{N-i-(N+1-j)} = q^{j-i-1}$, so the last equation follows. □

Corollary 2.2.11. *For verge matrices $B, C \in \mathcal{B}$ the inner product of their Andre-Neto characters is*

$$\langle \phi_{\pi(B)}, \phi_{\pi(C)} \rangle_{U_N} = \begin{cases} \frac{|H*B|^2}{|G_N*B|} & \textit{if } B = C \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $B, C \in \mathcal{B}$ be verge matrices. Since $f(x^{-1}) = -f(x)$ for $x \in U_N$, for $V, W \in \mathfrak{u}_N$ we have $\phi_V \phi_W(x^{-1}) = \vartheta\kappa(V, f(x))\, \vartheta\kappa(W, -f(x)) = \vartheta\kappa(V - W, f(x))$ and therefore

$$\langle \phi_{\pi(B)}, \phi_{\pi(C)} \rangle_{U_N} = \frac{|H*B||H*C|}{|G_N*B||G_N*C||U_N|} \sum_{x \in U_N} \sum_{\substack{V \in G_N*B \\ W \in G_N*C}} \vartheta\kappa(U - W, f(x)).$$

When x runs through all elements of U_N then $f(x)$ runs through all elements of $\mathfrak{u}_N$, so by lemma 2.1.7 we have $\sum_{x \in U_N} \kappa(V - W, f(x)) \neq 0$ if and only if $V - W = 0$ for $V \in G_N * B$ and $W \in G_N * C$. Since by lemma 2.2.6 the verge matrices are unique representatives of the G_N-orbits in $\mathfrak{u}_N$, we must have $B = C$ in order for there to be some $U \in G_N*B$ and $W \in G_N*C$ with $V - W = 0$. If this is the case, there is exactly one $W \in G_N*C$ for every $V \in G_N*B$ such that $V - W = 0$. We then have

$$\langle \phi_{\pi(B)}, \phi_{\pi(C)} \rangle_{U_N} = \frac{|H*B|^2}{|G_N*B|^2 |U_N|} |G_N*B||U_N| = \frac{|H*B|^2}{|G_N*B|}.$$

□

Corollary 2.2.12. *Let $\rho_{U_N} : U_N \to \mathbb{C}$ be the regular representation of U_N. Then*

$$\rho_{U_N} = \sum_{B \in \mathcal{B}} \frac{|G_N*B|}{|H*B|} \phi_{\pi(B)} = \sum_{B \in \mathcal{B}} \frac{\deg \phi_{\pi(B)}}{\langle \phi_{\pi(B)}, \phi_{\pi(B)} \rangle_{U_N}} \phi_{\pi(B)}.$$

Proof. By lemma 2.2.6 every $X \in u_N$ is contained in the G_N-orbit of exactly one $B \in \mathcal{B}$, so for $u \in U_N$ we have

$$\sum_{X \in u_N} \vartheta(X, f(u)) = \sum_{B \in \mathcal{B}} \sum_{V \in G_N*B} \vartheta(V, f(u)) = \sum_{B \in \mathcal{B}} \frac{|G_N*B|}{|H*B|} \phi_{\pi(B)}(u).$$

From lemma 2.1.7 it follows that $\sum_{X \in u_N} \vartheta(X, f(u)) = |\mathfrak{u}_N| = |U_N|$ if $f(u) = 0$ and therefore $u = I$ and $\sum_{X \in u_N} \vartheta(X, f(u)) = 0$ if $u \neq I$. From corollary 2.2.11 then follows the second equation. □

As $\rho_{U_N} = \sum_{B\in\mathcal{B}} \frac{|G_N * B|}{|H * B|}\phi_{\pi(B)}$ and all André–Neto character are mutually orthogonal, every irreducible character of U_N is constituent of exactly one André–Neto character. By André and Neto [AN09, 3.6, p. 1281] there is a set of unions of conjugacy classes of SO_N, such that they form the set of superclasses for the set of supercharacters $\{\phi_{\pi(B)} \mid B \in \mathcal{B}\}$. Using the construction of Andrews [AN06, 6.1, p.6] we can define the set of superclasses as $\{K_B \mid B \in \mathcal{B}\}$ by

$$K_B = \left\{f^{-1}\left(g^{\dagger}Bg\right) \mid g \in G_N\right\},$$

for which it is obvious that every supercharacter is constant on every superclass and that the number of supercharacters and superclasses are equal.

Similar to Andrews expression of André–Neto characters we can utilize the normal subgroup $H \trianglelefteq G_N$ to describe these characters as sums over the orbit $G_N.\pi(B)$ for $B \in \mathcal{B}$.

Proposition 2.2.13. *Let $A \in \mathcal{B}$, such that $supp_{\mathcal{V}}(A) = \emptyset$. For $AH^t = \{Ah^t \mid h \in H\}$ the set $\pi(AH^t)$ is a linear subspace of* $\mathbf{v}$ *with*

$$\pi(AH^t) = \bigoplus_{(i,k)\in supp(A)} \left\langle e_{ij} \mid i < j \leq \min(k-1, n)\right\rangle_{\mathbb{F}_q}$$

and $|\pi(AH^t)| = \prod_{(i,k)\in supp(A)} q^{\min(k-1,n)-i}$. Furthermore $\pi(C_A)$ is the orthogonal complement of $\pi(AH^t)$ in $\mathbf{v}$.

Proof. Let $A \in \mathbf{m}$ with at most one non-zero entry in every row and column and $\mathrm{supp}_{\mathcal{V}}(A) = \emptyset$ as well as $h \in H$. For $1 \leq i < j \leq N$ with $i + j < N + 1$ we have $(Ah^t)_{ij} = \sum_{k=j}^{N} A_{ik}h_{jk}$, which gives us $(Ah^t)_{ij} = 0$ unless there is a $j \leq k \leq N$ such that $(i, k) \in \mathrm{supp}(A)$ and $j \leq \min(k, n)$. For $j = k$ we have $(Ah^t)_{ij} = A_{ij}$ and therefore $(Ah^t)_{ij} = 0$. So $\mathrm{supp}_{\mathcal{V}}(AH^t - A) = \bigcup_{(i,k)\in\mathrm{supp}(B)} \mathcal{Z}_{ik}$ while $\mathrm{supp}_{\mathcal{V}}(f(C_A)) = \bigcap_{(i,k)\in\mathrm{supp}(A)} \mathcal{V}\backslash\mathcal{Z}_{ik}$, and it follows that $AH^t - A$ is the orthogonal complement of $f(C_A)$ with respect to κ. As A has no more than one non-zero entry in every row, we finally have $|\pi(AH^t)| = \prod_{(i,k)\in\mathrm{supp}(A)} q^{\min(k-1,n)-i}$. □

Theorem 2.2.14. *For a verge matrix $B \in \mathcal{B}$ and $x \in U_N$ we have*

$$\phi_{\pi(B)}(x) = \frac{1}{|U_N|} \sum_{C\in G_N.\pi(B)} \sum_{u\in U_N} \vartheta\kappa\left(C, \pi(u^{-1}xu)\right).$$

Proof. Let $B \in \mathcal{B}$. Since the non-zero entries of $\pi(B)$ and $A := B - \pi(B)$ are contained in different rows and columns, we have $\pi(BH^t) - \pi(B) = (H.\pi(B) - \pi(B)) \oplus \left(\pi(AH^t)\right)$ and therefore $H.\pi(B) - \pi(B) \leq \pi\left(C_A\right)$. Furthermore, since $C_B = C_{\pi(B)} \cap C_A$ it follows that

$$\pi\left(C_A\right) = (H.\pi(B) - \pi(B)) \oplus \left(\pi\left(C_{\pi(B)}\right) \cap \pi\left(C_A\right)\right) = (H.\pi(B) - \pi(B)) \oplus \pi\left(C_B\right)$$

and $\pi\left(C_B\right)$ is the orthogonal complement of $H.\pi(B) - \pi(B)$ in $\pi\left(C_A\right)$. By lemma 2.1.8 for $x \in C_A$ we then have

$$\sum_{h \in H} \vartheta\kappa((h.\pi(B) - \pi(B)), \pi(x)) = \begin{cases} |H| & \text{for } x \in C_B \\ 0 & \text{for } x \notin C_B. \end{cases}$$

Inducing the linear character $\chi_{\pi(B)}$ of C_B to C_A, for $x \in C_A$ we then have

$$\begin{aligned} \mathrm{Ind}_{C_B}^{C_A}\chi_{\pi(B)}(x) &= \frac{1}{|C_B|} \sum_{\substack{u \in C_A \\ u^{-1}xu \in C_B}} \vartheta\kappa\left(\pi(B), \pi(u^{-1}xu)\right) \\ &= \frac{1}{|C_B||H|} \sum_{u \in C_A} \vartheta\kappa\left(\pi(B), \pi(u^{-1}xu)\right) \sum_{h \in H} \vartheta\kappa\left(h.\pi(B) - \pi(B), \pi(u^{-1}xu)\right) \\ &= \frac{1}{|C_B||H|} \sum_{u \in C_A} \sum_{h \in H} \vartheta\kappa\left(\pi(B) + h.\pi(B) - \pi(B), \pi(u^{-1}xu)\right) \\ &= \frac{1}{|C_B||H|} \sum_{u \in C_A} \sum_{h \in H} \vartheta\kappa\left(h.\pi(B), \pi(u^{-1}xu)\right). \end{aligned}$$

For the G_N-set of H-orbits $\{Hg.\pi(B) \mid g \in G_N\}$ let $S_{G_N}(H.\pi(B))$ be the stabilizer of $H.\pi(B)$. For $g \in G_N$ and $1 \leq i < k \leq N$ we have $(g.\pi(B))_{ik} = B_{ij}g_{kj}$ if there is a $i < j < k$ with $i + j < N + 1$ such that $B_{ij} \neq 0$ and $(g.\pi(B))_{ik} = 0$ otherwise. Then for $g \in S_{G_N}(H.\pi(B))$ we have $g_{kj} = 0$ if there is a $i < j < k$ with $i + j < N + 1$ and $(i, j) \in \mathrm{supp}(B)$ as well as $i \leq n < j$. So since $B \in \mathfrak{so}_N$ for every $(i, j) \in \mathrm{supp}(B)$ with $i + j > N + 1$, we have $g_{k\bar{i}} = 0$ for $n < k < \bar{i}$. For $g \in S_{U_N}(H.\pi(B)) = S_{G_N}(H.\pi(B)) \cap U_N$ we then have $g_{ik} = 0$ for $i < k \leq n$ and every $(i, j) \in \mathrm{supp}(B)$ with $i + j > N + 1$, which gives us $S_{U_N}(H.\pi(B)) = C_{B-\pi(B)}$. For $x \in U_N$ and $h \in H$ we then have $xh.\pi(B) \in H.\pi(B) = C_{\pi(B)}^{\perp}$ if and only if $x \in C_A$ and therefore

$$\sum_{w \in C_{\pi(B)}} \vartheta\kappa((xh.\pi(B) - h.\pi(B)), \pi(w)) = \begin{cases} |C_{\pi(B)}| & \text{for } x \in C_{B-\pi(B)} \\ 0 & \text{for } x \notin C_{B-\pi(B)}. \end{cases}$$

For $x \in U_N$ the character $\chi_{\pi(B)}$ induced to U_N then is

$$\begin{aligned}
\phi_{\pi(B)}(x) &= \mathrm{Ind}_{C_A}^{U_N}\mathrm{Ind}_{C_B}^{C_A}\chi_{\pi(B)}(x)\\
&= \frac{1}{|C_B||H||C_A|}\sum_{\substack{u\in U_N\\ u^{-1}xu\in C_A}}\sum_{v\in C_A}\sum_{h\in H}\vartheta\kappa\left(h.\pi(B),\pi(v^{-1}u^{-1}xuv)\right)\\
&= \frac{1}{|C_B||H|}\sum_{\substack{u\in U_N\\ u^{-1}xu\in C_A}}\sum_{h\in H}\vartheta\kappa\left(h.\pi(B),\pi(u^{-1}xu)\right)\\
&= \frac{1}{|C_B||H||C_{\pi(B)}|}\sum_{u\in U_N}\sum_{w\in C_{\pi(B)}}\sum_{h\in H}\vartheta\kappa\left(h.\pi(B),\pi(u^{-1}xu)\right)\vartheta\kappa\left(u^{-1}xuh.\pi(B)-h.\pi(B),\pi(w)\right)\\
&= \frac{1}{|C_B||H||C_{\pi(B)}|}\sum_{u\in U_N}\sum_{w\in C_{\pi(B)}}\sum_{h\in H}\vartheta\kappa\left(h.\pi(B),\pi(u^{-1}xu)+\pi(w)\circ(u^{-1}xu)-\pi(w)\right)\\
&= \frac{1}{|C_B||H||C_{\pi(B)}|}\sum_{u\in U_N}\sum_{w\in C_{\pi(B)}}\sum_{h\in H}\vartheta\kappa\left(h.\pi(B),\pi(wu^{-1}xu)-\pi(w)\right)\\
&= \frac{1}{|C_B||H||C_{\pi(B)}|}\sum_{u\in U_N}\sum_{w\in C_{\pi(B)}}\sum_{h\in H}\vartheta\kappa\left(h.\pi(B),\pi(wu^{-1}xuw^{-1})\circ w\right)\\
&= \frac{1}{|C_B||H||C_{\pi(B)}|}\sum_{u\in U_N}\sum_{w\in C_{\pi(B)}}\sum_{h\in H}\vartheta\kappa\left(wh.\pi(B),\pi(u^{-1}xu)\right).
\end{aligned}$$

Since $C_B = C_{\pi(B)} \cap C_A$ and $|\mathbf{v}| = |U_N|$, it follows by proposition 2.2.13 that

$$|C_{\pi(B)}C_A| = \frac{|C_{\pi(B)}||C_A|}{|C_B|} = |\mathbf{v}|\frac{\prod_{(i,k)\in\mathrm{supp}(B)} q^{\min(k-1,n)-i}}{\prod_{(i,k)\in\mathrm{supp}(\pi(B))\cup\mathrm{supp}(A)} q^{\min(k-1,n)-i}} = |U_N|$$

and therefore $C_{\pi(B)}C_A = U_N$. For $v \in C_A$ we have $vH.\pi(B) = H.\pi(B)$ and since $U_N H = G_N$, it follows that

$$\begin{aligned}
\phi_{\pi(B)}(x) &= \frac{1}{|C_B||H||C_{\pi(B)}||C_A|}\sum_{u\in U_N}\sum_{v\in C_A}\sum_{w\in C_{\pi(B)}}\sum_{h\in H}\vartheta\kappa\left(wvh.\pi(B),\pi(u^{-1}xu)\right)\\
&= \frac{1}{|C_B||H||U_N|}\sum_{u\in U_N}\sum_{v'\in U_N}\sum_{h\in H}\vartheta\kappa\left(v'h.\pi(B),\pi(u^{-1}xu)\right)\\
&= \frac{1}{|C_B||G_N|}\sum_{g\in G_N}\sum_{u\in U_N}\vartheta\kappa\left(g.\pi(B),\pi(u^{-1}xu)\right)\\
&= \frac{1}{|C_B||G_N.\pi(B)|}\sum_{C\in G_N.\pi(B)}\sum_{u\in U_N}\vartheta\kappa\left(C,\pi(u^{-1}xu)\right).
\end{aligned}$$

Finally, the number of elements in the orbit $G_N.\pi(B)$ is $|G_N.\pi(B)| = \prod_{(i,k)\in\operatorname{supp}(\pi(B))} q^{k-i-1}$. For every $(i,k) \in \operatorname{supp}(\pi(B))$ with $k > n$ we have $(\bar{k}, \bar{i}) \in \operatorname{supp}(A)$ with $k - n + 1 = n - \bar{k}$ and therefore $\prod_{(i,k)\in\operatorname{supp}(A)} q^{n-i} = \prod_{(i,k)\in\operatorname{supp}(\pi(B))} q^{\max(k,n+1)-i}$, so the claim holds since

$$|C_B||G_N.\pi(B)| = |U_N|\frac{\prod_{(i,k)\in\operatorname{supp}(\pi(B))} q^{k-i-1}}{\prod_{(i,k)\in\operatorname{supp}(B)} q^{\min(k-1,n)-i}} = |U_N|.$$

□

2.3 Jedlitschky characters for verge patterns

For the group algebra $\mathbb{C}U_N$ Jedlitschky [Jed13, 3.1.14, p. 58] defined a monomial basis $\{[C] \mid C \in \mathbf{v}\}$ for the $\mathbb{C}U_N$-module $\mathbf{v}$ with

$$[C] = \frac{1}{|U|}\sum_{x\in U}\overline{\chi_C(x)}x \in \mathbb{C}U$$

for $C \in \mathbf{v}$ on which U_N acts by right multiplication. For $g \in U_N$ we then have

$$[C]g = \frac{1}{|U|}\sum_{x\in U}\overline{\chi_C(x)}xg = \chi_{g^{-1}.C}(g)[g^{-1}.C],$$

because for $x \in U_N$ we have

$$\chi_C(x) = \vartheta\kappa(C, \pi(xgg^{-1})) = \vartheta\kappa(C, \pi(xg)\circ g^{-1} - \pi(g)\circ g^{-1}) = \chi_{g^{-1}.C}(xg)\overline{\chi_{g^{-1}.C}(g)}.$$

The module generated by $[C]$ is the orbit module $\mathbb{C}(U_N.C)$ with its $\mathbb{C}$-basis being $\{g.C \mid g \in U_N\}$. Let now $\psi_C : U_N \to \mathbb{C}$ be the character of this orbit module $\mathbb{C}(U_N.C)$, so by lemma 2.1.3 for $x \in U_N$ we have

$$\begin{aligned}
\psi_C(x) &= \sum_{\substack{W\in\mathcal{U}_N.C\\ x.W=W}} \chi_W(x) = \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ xg.C=g.C}} \chi_{g.C}(x)\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \chi_{g.C}(x) = \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \vartheta\kappa\,(C, \pi(x)\circ g)\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \vartheta\kappa\,(C, \pi(xg) - \pi(g))\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \vartheta\kappa\left(C, \pi(g^{-1}xg) - \pi(g^{-1})\circ xg - \pi(g)\right)\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \chi_C(g^{-1}xg)\,\vartheta\kappa\left(C, \pi(g)\circ g^{-1}xg\right)\overline{\chi_C(g)}\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \chi_C(g^{-1}xg)\,\vartheta\kappa\left(g^{-1}xg.C, \pi(g)\right)\overline{\chi_C(g)}\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \chi_C(g^{-1}xg)\chi_C(g)\overline{\chi_C(g)}\\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{g\in U_N\\ g^{-1}xg\in\mathrm{Stab}_{U_N}(C)}} \chi_C(g^{-1}xg)\\
&= \mathrm{Ind}_{\mathrm{Stab}_{U_N}(C)}^{U_N}\chi_C(x)
\end{aligned}$$

So ψ_C is the induced character of the linear character χ_C of the stabilizer $\mathrm{Stab}_{U_N}(C)$, and we can calculate the value of this character $\psi_C(x)$ for $x \in U_N$ to be the sum of all χ_W for $W \in U_N.C$ evaluated at the average value of the elements of the U_N-conjugacy class of x.

Theorem 2.3.1. *For $C \in \mathbf{v}$ let $\psi_C = Ind_{Stab_{U_N}(C)}^{U_N}\chi_C$ be the induced character from $Stab_{U_N}(C)$ to U_N of the linear character χ_C of $Stab_{U_N}(C)$. We then have*

$$\psi_C = Ind_{Stab_{U_N}(C)}^{U_N}\chi_C = \frac{1}{|U_N|}\sum_{\substack{W\in U_N.C\\ v\in U_N}} (\chi_W)^v$$

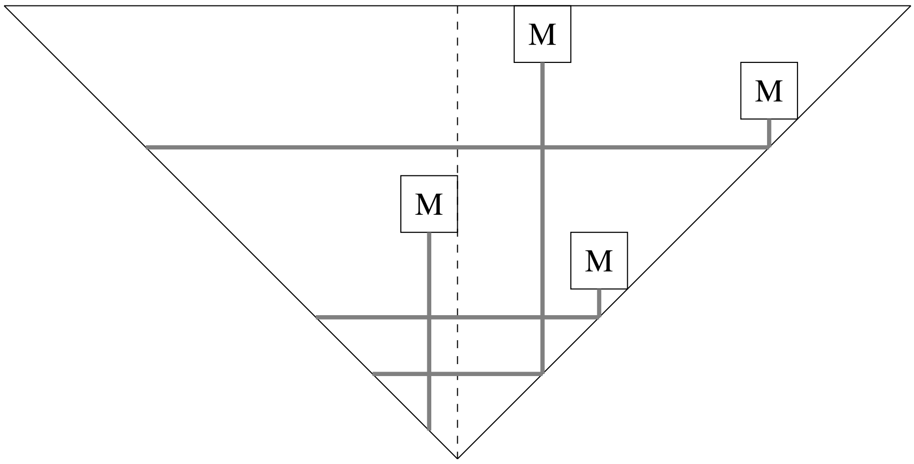
Proof. Let $C \in \mathbf{v}$ and $x \in U_N$. By lemma 2.1.3 we have

$$\begin{aligned}
\psi_C(x) &= \operatorname{Ind}_{\mathrm{Stab}_{U_N}(C)}^{U_N} \chi_C(x) \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{v \in U \\ vxv^{-1} \in \mathrm{Stab}_{U_N}(C)}} \vartheta\kappa(C, \pi(vxv^{-1})) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{v,u \in U_N} \vartheta\kappa(vxv^{-1}.C - C, \pi(u))\, \vartheta\kappa(C, \pi(vxv^{-1})) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{v,u \in U_N} \vartheta\kappa(C, \pi(u) \circ vxv^{-1})\, \vartheta\kappa(C, \pi(vxv^{-1}) - \pi(u)) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{v,u \in U_N} \vartheta\kappa(C, \pi(uvxv^{-1}) - \pi(vxv^{-1}) + \pi(vxv^{-1}) - \pi(u)) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{v,u \in U_N} \vartheta\kappa(C, \pi(uvxv^{-1}u^{-1}) \circ u) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{v,u \in U_N} \vartheta\kappa(u.C, \pi(uvxv^{-1}u^{-1})) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{v,u \in U_N} \vartheta\kappa(u.C, \pi(vxv^{-1})) \\
&= \frac{1}{|U_N|} \sum_{\substack{W \in U_N.C \\ v \in U_N}} \chi_W(vxv^{-1}).
\end{aligned}$$

□

The character $\psi_{g.A}$ for $A \in \mathbf{v}$ and $g \in G_N$ is similar to the Andre-Neto character ϕ_A as defined in theorem 2.2.14, since it is based on the same construction only restricting the sum over elements of of the G_N-orbit $G_N.A$ to the U_N-orbit $U_N g.A$.

While the stabilizer $\mathrm{Stab}_{U_N}(C)$ for any $C \in \mathbf{v}$ is in general not a pattern subgroup of U_N, this is the case for $\mathrm{Stab}_{U_N}(A)$ if $A \in \mathbf{v}$ is a verge pattern as Jedlitschky [Jed13, 3.2.30, p. 76] has shown. We consider the lower hook of a main condition, which is a non-zero position of a verge pattern, to be the positions directly below this main condition in $\mathcal{V}$ as well as their reflection at the counter-diagonal, that is the positions $(i, j) \in \mathcal{V}$ such that $(\bar{j}, \bar{i}) \in \mathcal{G}$ is directly below this main condition. As the stabilizer $\mathrm{Stab}_{U_N}(A)$ of a verge pattern comprises all elements of U_N for which their support does not intersect the lower hook of any main condition of A, we can count the elements of the stabilizer by taking the length of every lower hook and counting their intersections. Therefore, we define a set of positions $\mathcal{D}_A \subseteq \mathcal{G}$ of the intersections of the lower hooks as well as their boundary points with the counter-diagonal.



Lower hooks for a verge pattern (M)

Lemma 2.3.2. *For a verge pattern $A \in \mathbf{v}$ let $\mathcal{P} \subseteq \mathcal{G}$ and $\mathcal{Q} \subseteq \mathcal{V}$ be subsets defined by*

$$\mathcal{P} = \bigcup_{(i,k)\in supp(A)} \{(j,k) \mid i < j < k\}$$

$$\mathcal{Q} = \bigcup_{(i,k)\in supp(A)} \left\{(j,k) \mid i < j < \min(k,\bar{i})\right\} \cup \bigcup_{(i,k)\in supp(A)\cap\mathcal{V}_r} \left\{(\bar{i},j) \mid \bar{i} < j < i\right\},$$

where $\mathcal{P}$ is the set of positions below non-zero positions of A in $\mathcal{G}$, and $\mathcal{Q}$ is the set of positions of the lower hooks of A in $\mathcal{V}$. Then the stabilizers of A in G_N and U_N respectively are pattern subgroups $Stab_{G_N}(A) = H_{\mathcal{G}\setminus\mathcal{P}}$ and $Stab_{U_N}(A) = H_{\mathcal{V}\setminus\mathcal{Q}}$ such that

$$Stab_{G_N}(A) = \left\{g \in G_N \mid supp_{\mathcal{G}}(g) \cap \mathcal{P} = \emptyset\right\} \quad \text{and} \quad Stab_{U_N}(A) = \{u \in U_N \mid supp_{\mathcal{V}}(u) \cap \mathcal{Q} = \emptyset\}.$$

Let $\mathcal{D} \subseteq \mathcal{G}$ be the subset of intersections of lower hooks of A as well as their boundary points with the counter-diagonal defined by

$$\mathcal{D} = \{(i,j) \in \mathcal{G} \mid i + j \leq N+1, \exists\, 1 \leq k,l < i : (k,\bar{i}),(l,j) \in supp(A)\}.$$

Then the number of elements in the stabilizers of A in G_N and U_N respectively are

$$|Stab_{G_N}(A)| = \frac{|G_N|}{\prod_{(i,k)\in supp(A)} q^{k-i-1}} \quad \text{and} \quad |Stab_{U_N}(A)| = \frac{|U_N|\, q^{|\mathcal{D}|}}{\prod_{(i,k)\in supp(A)} q^{k-i-1}}.$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $g \in G$. We have

$$\pi(Ag^t) = \sum_{(i,k)\in \operatorname{supp}(A)} A_{ik}\pi(e_{ik}g^t) = \sum_{(i,k)\in \operatorname{supp}(A)} A_{ik} \sum_{1\le j\le k} \pi(g_{jk}e_{ij}) = \sum_{(i,k)\in \operatorname{supp}(A)} A_{ik} \sum_{i<j\le k} g_{jk}e_{ij}.$$

As no row of A contains two non-zero entries, we have $g \in \operatorname{Stab}_{G_N}(A)$ if and only if $g_{jk} = 0$ for every $(i,k) \in \operatorname{supp}(A)$ and $i < j < k$. So for $\mathcal{P} = \bigcup_{(i,k)\in\operatorname{supp}(A)} \{(j,k) \mid i < j < k\}$ by lemma 1.3.8 and 1.4.3 the stabilizer is a pattern group $\operatorname{Stab}_{G_N}(A) = H_{\mathcal{G}\setminus\mathcal{P}}$ for the closed subset $\mathcal{G}\setminus\mathcal{P} \subseteq \mathcal{G}$. We then have

$$|\operatorname{Stab}_{G_N}(A)| = q^{|\mathcal{G}|-|\mathcal{P}|} = q^{|\mathcal{G}|} \prod_{(i,k)\in\operatorname{supp}(A)} q^{-(k-i-1)} = \frac{|G_N|}{\prod_{(i,k)\in\operatorname{supp}(A)} q^{k-i-1}}.$$

Let $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{P}$ be the set of intersections of lower hooks of A as well as the set of boundary points of lower hooks of A with the counter-diagonal and let $\mathcal{Q} \subseteq \mathcal{V}$ be the sets of lower hooks of A defined by

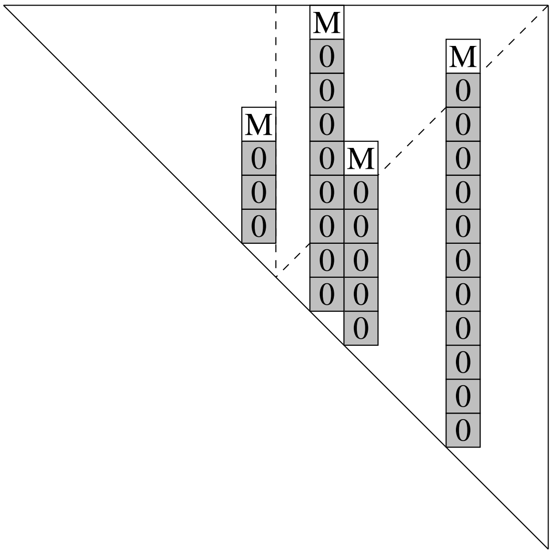
$$\begin{aligned}
\mathcal{D}_1 &= \{(i,j) \in \mathcal{G} \mid i + j < N + 1, \exists\, 1 \le k, l < i : (k, \bar{i}), (l, j) \in \operatorname{supp}(A)\} \\
\mathcal{D}_2 &= \{(i, \bar{i}) \in \mathcal{G} \mid \exists\, 1 \le k < i : (k, \bar{i}) \in \operatorname{supp}(A)\} \\
\mathcal{Q} &= \bigcup_{(i,k)\in\operatorname{supp}(A)\cap\mathcal{V}_l} \{(j,k) \mid i < j < k\} \cup \bigcup_{(i,k)\in\operatorname{supp}(A)\cap\mathcal{V}_r} \left(\left\{(j,k) \mid i < j < \bar{i}\right\} \cup \left\{(\bar{i}, j) \mid \bar{i} < j < i\right\}\right).
\end{aligned}$$

Let $\delta : \mathcal{P} \to \mathcal{V}$ be the map defined by $\delta(i,j) = (i,j)$ for $(i,j) \in \mathcal{P}$ and $i + j \le N + 1$ and $\delta(i,j) = (\bar{j}, \bar{i})$ for $(i,j) \in \mathcal{P}$ and $i + j > N + 1$. For $(i,j) \in \mathcal{P} \cap \mathcal{V}$ we have $\delta(i,j) = (i,j) \in \mathcal{Q}$ while for $(i,j) \in \mathcal{P}$ with $i + j > N + 1$ we have $\delta(i,j) = (\bar{j}, \bar{i}) \in \mathcal{Q}$. For $(i, \bar{i}) \in \mathcal{P}$ we have $\delta(i, \bar{i}) = (i, \bar{i}) \in \mathcal{D}_2$ so the image of δ is $\operatorname{im}(\delta) = \mathcal{Q} \cup \mathcal{D}_2$. For any $(i,j) \in \mathcal{P} \cap \mathcal{V}$ with $(\bar{j}, \bar{i}) \in \mathcal{P}$ there are $1 \le k < \bar{j}$ and $1 \le l < i$ such that $(k, \bar{i}), (l, j) \in \operatorname{supp}(A)$. Since $k + \bar{i} < N + 1$, we have $k < i$ and therefore $(i,j) \in \mathcal{D}_1$. It follows that the restriction of δ to $\mathcal{P}\setminus\mathcal{D}_1$ is a bijection onto $\mathcal{Q} \cup \mathcal{D}_2$. For $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ we then have

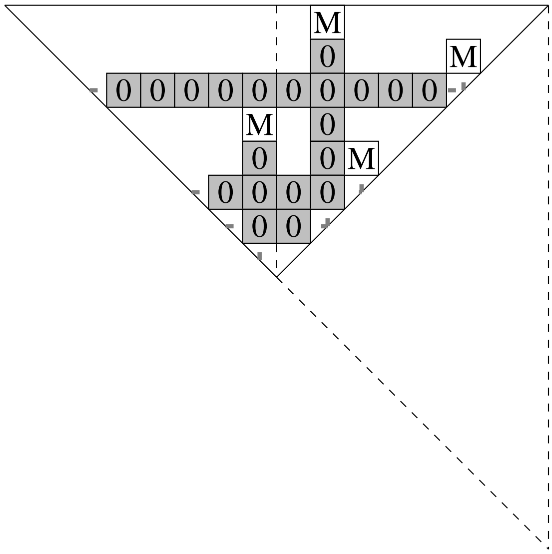
$$|\mathcal{P}| = |\mathcal{P}\setminus\mathcal{D}_1| + |\mathcal{D}_1| = |\mathcal{Q} \cup \mathcal{D}_2| + |\mathcal{D}_1| = |\mathcal{Q}| + |\mathcal{D}|,$$

because of $\mathcal{Q}\cap\mathcal{D}_2 \subseteq \mathcal{V}\cap\mathcal{D}_2 = \emptyset$. For $(i,j) \in \mathcal{V}$ we have $(i,j) \notin \mathcal{Q}$ if and only if $(i,j), (\bar{j}, \bar{i}) \notin \mathcal{P}$, so by lemma 1.3.9 as well as 1.4.3 the stabilizer of A in U_N is a pattern group in U_N with $\operatorname{Stab}_{U_N}(A) = H_{\mathcal{G}\setminus\mathcal{P}} \cap U_N = H_{\mathcal{V}\setminus\mathcal{Q}}$, and we have

$$|\operatorname{Stab}_{U_N}(A)| = q^{|\mathcal{V}|-|\mathcal{Q}|} = q^{|\mathcal{V}|-|\mathcal{P}|+|\mathcal{D}|} = |U_N| \frac{q^{|\mathcal{D}|}}{\prod_{(i,k)\in\operatorname{supp}(A)} q^{k-i-1}}. \qquad \square$$



Zero positions of the stabilizer of a verge pattern (M) in G_N



Zero positions of the stabilizer of a verge pattern (M) in U_N

Following Jedlitschky [Jed13, 3.3.21, p. 87], we calculate the inner product of two Jedlitschky characters. Using the Mackey decomposition, this inner product can be reduced to a sum of inner products of the linear characters on the intersection of their corresponding stabilizers.

Proposition 2.3.3. *For a verge pattern $A \in \mathbf{v}$ and $g \in G_N$ as well as $u \in U_N$ we have $\chi_{g.A}(x) = \chi_{ug.A}(uxu^{-1})$ for $x \in Stab_{U_N}(g.A)$.*

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $g \in G_N$. For $u \in U_N$ and $x \in \mathrm{Stab}_{U_N}(g.A)$ we have

$$\begin{aligned}
\chi_{ug.A}(uxu^{-1}) &= \vartheta\kappa\left(ug.A, \pi\left(uxu^{-1}\right)\right)\\
&= \vartheta\kappa\left(ug.A, \pi(u)\circ xu^{-1} + \pi(x)\circ u^{-1} - \pi(u)\circ u^{-1}\right)\\
&= \vartheta\kappa\left(xg.A - g.A, \pi(u)\right)\vartheta\kappa\left(g.A, \pi(x)\right)\\
&= \chi_{g.A}(x).
\end{aligned}$$

□

Theorem 2.3.4. *For verge pattern $A, B \in \mathbf{v}$ and $g, h \in G_N$ let*

$$\iota\left(g.A, h.B\right) = \frac{|Stab_{U_N}(g.A) \cap Stab_{U_N}(h.B)|}{|Stab_{U_N}(g.A)||Stab_{U_N}(h.B)|}\left\langle \chi_{g.A}, \chi_{h.B}\right\rangle_{Stab_{U_N}(g.A)\cap Stab_{U_N}(h.B)}$$

be the inner product of $\chi_{g.A}$ and $\chi_{h.B}$ on $Stab_{U_N}(g.A) \cap Stab_{U_N}(h.B)$. Then the inner product of $\psi_{g.A}$ and $\psi_{h.B}$ on U_N is

$$\left\langle \psi_{g.A}, \psi_{h.B}\right\rangle_{U_N} = \sum_{u\in U_N} \iota\left(g.A, uh.B\right).$$

Proof. Let $A, B \in \mathbf{v}$ verge pattern and $g, h \in G_N$. Furthermore, let $V = g.A$ and $W = h.B$ as well as $S = \mathrm{Stab}_{U_N}(V)$ and $T = \mathrm{Stab}_{U_N}(W)$. Let $D \subseteq U_N$ be the set of representatives of the (S,T)-double coset in U_N. Using both Frobenius reciprocity and Mackey decomposition, we have

$$\begin{aligned}
\left\langle \psi_{g.A}, \psi_{h.B} \right\rangle_{U_N} &= \left\langle \mathrm{Ind}_S^{U_N} \mathrm{Res}_S^{U_N} \chi_V,\ \mathrm{Ind}_T^{U_N} \mathrm{Res}_T^{U_N} \chi_W \right\rangle_{U_N} \\
&= \left\langle \mathrm{Res}_S^{U_N} \chi_V,\ \mathrm{Res}_S^{U_N} \mathrm{Ind}_T^{U_N} \mathrm{Res}_T^{U_N} \chi_W \right\rangle_S \\
&= \sum_{d \in D} \left\langle \mathrm{Res}_S^{U_N} \chi_V,\ \mathrm{Ind}_{S \cap dTd^{-1}}^{S} \mathrm{Res}_{S \cap dTd^{-1}}^{U_N} \chi_W^{d^{-1}} \right\rangle_S \\
&= \sum_{d \in D} \left\langle \mathrm{Res}_{S \cap dTd^{-1}}^{U_N} \chi_V,\ \mathrm{Res}_{S \cap dTd^{-1}}^{U_N} \chi_W^{d^{-1}} \right\rangle_{S \cap dTd^{-1}} \\
&= \frac{1}{|S|} \sum_{w \in S} \sum_{d \in D} \left\langle \mathrm{Res}_{S \cap wdTd^{-1}w^{-1}}^{U_N} \chi_V^{w^{-1}},\ \mathrm{Res}_{S \cap wdTd^{-1}w^{-1}}^{U_N} \chi_W^{d^{-1}w^{-1}} \right\rangle_{S \cap wdTd^{-1}w^{-1}} \\
&= \frac{1}{|S|} \sum_{w \in S} \sum_{d \in D} \left\langle \mathrm{Res}_{S \cap wdT(wd)^{-1}}^{U_N} \chi_V,\ \mathrm{Res}_{S \cap wdT(wd)^{-1}}^{U_N} \chi_{wd.W} \right\rangle_{S \cap wdT(wd)^{-1}},
\end{aligned}$$

where the last step follows from proposition 2.3.3. Since D is the set of (S,T)-double coset in U_N, we have $SD.W = U_N.W$, so every for $u \in U_N$ there are $w \in S$ and $d \in D$ such that $wd.W = u.W$. For $\tilde{w} \in S$ we have $wd.W = \tilde{w}d.W$ if and only if $\tilde{w}w^{-1} \in S \cap uTu^{-1}$. It follows that

$$\begin{aligned}
\left\langle \psi_{g.A}, \psi_{h.B} \right\rangle_{U_N} &= \sum_{u \in U_N} \frac{|S \cap uTu^{-1}|}{|S|} \frac{1}{|uTu^{-1}|} \left\langle \mathrm{Res}_{S \cap uTu^{-1}}^{U_N} \chi_V,\ \mathrm{Res}_{S \cap uTu^{-1}}^{U_N} \chi_{u.W} \right\rangle_{S \cap uTu^{-1}} \\
&= \sum_{u \in U_N} \frac{|S \cap \mathrm{Stab}_{U_N}(u.W)|}{|S||\mathrm{Stab}_{U_N}(u.W)|} \left\langle \mathrm{Res}_{S \cap \mathrm{Stab}_{U_N}(u.W)}^{U_N} \chi_V,\ \mathrm{Res}_{S \cap \mathrm{Stab}_{U_N}(u.W)}^{U_N} \chi_{u.W} \right\rangle_{S \cap \mathrm{Stab}_{U_N}(u.W)} \\
&= \sum_{u \in U_N} \iota(V, u.W)
\end{aligned}$$

□

For a verge pattern $A \in \mathbf{v}$ we now define a subgroup $R_A \leq U_N$ that acts on A by row transformation into the orbit $G_N.A$ such that for every $v \in R_A$ we have $v^tA \in G_N.A$. For another verge pattern $B \in \mathbf{v}$ and $g \in G_N$ such that $\psi_{g.B}$ is not orthogonal to ψ_A we can then show that there is a $v \in R_A$ with $v^tA = g.B$ and $\psi_{g.B} = \psi_A$. Therefore, any Jedlitschky characters is either orthogonal or equal to ψ_A. This has also been shown by Jedlitschky [Jed13, 3.3.32, p. 91] for the general case of two Jedlitschky characters, which we will do in the next section.

Lemma 2.3.5. *For a verge pattern $A \in \mathbf{v}$ let $\mathcal{R}_A \subseteq \mathcal{V}_l$ be the closed set of positions defined by $(i, j) \in \mathcal{R}_A$ for $1 \leq i < j \leq n$ if there are $j < k < l < \bar{j}$ such that $(i,k),(j,l) \in supp(A)$. Let $R_A \leq U_N$ be the pattern subgroup for $\mathcal{R}_A$, then for $v \in R_A$ there is a $g \in G_N$ such that $v^tA = g.A$. Furthermore, for $v \in R_A$, $g \in G_N$ and $u \in U_N$ we have*

$$g.(v^tA) = \pi(v^t(g.A)) \quad \textit{and} \quad \kappa\left(g.(v^tA), \pi(u)\right) = \kappa\left(vg.A, \pi(vuv^{-1})\right)$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern and let $1 \leq i < j < r \leq n$ be such that $(i, j),(j,r) \in \mathcal{R}_A$. Then there are $j < k < l < s < \bar{r}$ such that $(i,k),(j,l),(r,s) \in \mathrm{supp}(A)$, and since $k < s$, we have $(i,r) \in \mathcal{R}_A$. For all $(i, j) \in \mathcal{R}_A$ we have $j \leq n$, so it follows that $\mathcal{R}_A$ is a closed subset of $\mathcal{V}$. Let $v \in R_A$ and $1 \leq j,k \leq N$ be such that $(v^tA)_{jk} \neq 0$. Then there must be a $1 \leq i \leq N$ with $(i, j) \in \mathcal{R}_A$ and $(i,k) \in \mathrm{supp}(A)$. By definition we have $j < k$ and $j + k < N + 1$, so it follows that $v^tA \in \mathbf{v}$.
For $g \in G_N$ and $1 \leq i < j \leq N$ let $(i, j) \in \mathcal{R}_A$, so there are $j < k < l < \bar{j}$ such that $(i,k),(j,l) \in \mathrm{supp}(A)$. For $c \in \mathbb{F}_q$ we have

$$x_{ij}(c)^tA = A + cA_{ik}e_{jk} = A(I + cA_{ik}A_{jl}^{-1}e_{kl})^t = (I + cA_{ik}A_{jl}^{-1}e_{kl}).A,$$

so since R_A is a pattern group for any $v \in R_A$, there is a $g \in G_N$ with $v^tA = g.A$.
We have $g.(v^tA) = \pi(v^tAg^t)$ and $\pi(v^t(g.A)) = \pi(v^t\pi(Ag^t))$, so for $1 \leq i < j \leq N$ it follows that $(Ag^t - \pi(Ag^t))_{ij} = 0$ and therefore $\left(v^t\left(Ag^t - \pi(Ag^t)\right)\right)_{ij} = 0$. This concludes that $g.(v^tA) = \pi(v^t(g.A))$.
For $x \in U_N$, $(i, j) \in \mathcal{R}_A$ and $c \in \mathbb{F}_q$ we have

$$\pi(x_{ij}(c)u) - x_{ij}(c)\pi(u) = \pi(u) + c\sum_{m=j}^{\bar{i}-1} u_{jm}e_{im} - \pi(u) - c\sum_{m=j+1}^{\bar{j}-1} u_{jm}e_{im} = \pi(x_{ij}(c)) + c\sum_{m=\bar{j}}^{\bar{i}-1} u_{jm}e_{im},$$

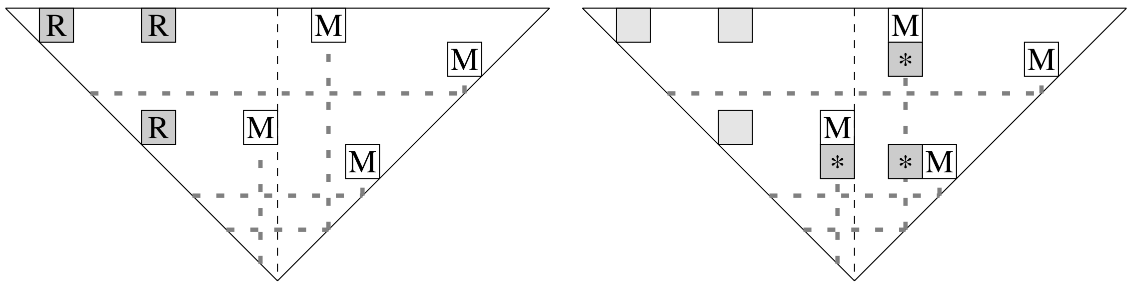
and for $g \in G_N$ it follows that

$$\begin{aligned}\kappa\left(g.(x_{ij}(c)^t A), \pi(u)\right) &= \operatorname{Tr}\left((g.A)^t x_{ij}(c)\pi(u)\right) \\ &= \kappa\left(g.A, x_{ij}(c)\pi(u)\right) \\ &= \kappa\left(g.A, \pi(x_{ij}(c)u) - \pi(x_{ij}(c))\right) - c\sum_{m=\bar{j}}^{\bar{i}-1} u_{jk}\kappa\left(g.A, e_{im}\right) \\ &= \kappa\left(g.A, \pi(x_{ij}(c)u) - \pi(x_{ij}(c))\right) - c\sum_{m=\bar{j}}^{\bar{i}-1} u_{jk}(g.A)_{im}.\end{aligned}$$

Since there are $j < k < l < \bar{j}$ such that $(i,k),(j,l) \in \operatorname{supp}(A)$, we have $(g.A)_{im} = 0$ if $m \geq \bar{j} > k$, which forces the last term in the equation above to be zero. For any $v \in R_A$ we have $\pi(vu) = \pi(vuv^{-1}v) = \pi(vuv^{-1})\circ v + \pi(v)$ and therefore

$$\kappa\left(g.(v^t A), \pi(u)\right) = \kappa\left(g.A, \pi(vu) - \pi(v)\right) = \kappa\left(g.A, \pi(vuv^{-1})\circ v\right) = \kappa\left(vg.A, \pi(vuv^{-1})\right).$$

□



Positions of $\mathcal{R}_A$ (R) in relation to the positions of a verge pattern (M)

Non zero positions of $R_A^t A$ (*) for a verge pattern $A \in \mathbf{v}$ (M)

Theorem 2.3.6. *For verge pattern $A, B \in \mathbf{v}$ and $g \in G_N$ we have $\iota(A, g.B) \neq 0$ if and only if there is a $v \in R_A$ such that $v^t A = g.B$. If this condition holds, the following statements are true as well:*

(i) $A = B$

(ii) $Stab_{U_N}(A) = Stab_{U_N}(g.B)$

(iii) $\psi_A = \psi_{g.B}$.

Proof. Let $A, B \in \mathbf{v}$ be verge pattern and $g \in G_N$ such that $\iota(A, g.B) \neq 0$. By theorem 2.2.14 the characters ψ_A and $\psi_{g.B}$ are constituents of the André–Neto characters ϕ_A and ϕ_B respectively for which we have $\langle \phi_A, \phi_B \rangle_{U_N} = 0$ unless $A = B$ by corollary 2.2.11. So since $\iota(A, g.B) \neq 0$, both characters ψ_A and $\psi_{g.B}$ must be constituents of the same Andre-Neto character, and we have $A = B$.

Let $S = \mathrm{Stab}_{U_N}(A) \cap \mathrm{Stab}_{U_N}(g.A)$. For any $u \in S$ we have

$$\begin{aligned}
\left\langle \chi_A, \chi_{g.A} \right\rangle_S &= \frac{1}{|S|} \sum_{w \in S} \vartheta\kappa\,(A - g.A, \pi(w)) \\
&= \frac{1}{|S|} \sum_{w \in S} \vartheta\kappa\,(A - g.A, \pi(wu)) \\
&= \frac{1}{|S|} \sum_{w \in S} \vartheta\kappa\,(A - g.A, \pi(w) \circ u + \pi(u)) \\
&= \frac{1}{|S|} \vartheta\kappa\,(A - g.A, \pi(u)) \sum_{w \in S} \vartheta\kappa\,(u.A - ug.A, \pi(w)) \\
&= \frac{1}{|S|} \vartheta\kappa\,(A - g.A, \pi(u)) \sum_{w \in S} \vartheta\kappa\,(A - g.A, \pi(w)) \\
&= \vartheta\kappa\,(A - g.A, \pi(u)) \left\langle \chi_A, \chi_{g.A} \right\rangle_S .
\end{aligned}$$

Since $\iota(A, g.A) \neq 0$, we have $\left\langle \chi_A, \chi_{g.A} \right\rangle_S \neq 0$, so we must have $\vartheta\kappa\,(A - g.A, \pi(u)) = 1$ and therefore $\kappa\,(A - g.A, \pi(u)) = 0$.

For $1 \leq k < l \leq N$ with $k + l < N + 1$ and $(g.A)_{kl} \neq 0$ let $1 \leq j \leq k$ be such that $(g.A)_{jl} \neq 0$ and $(g.A)_{il} = 0$ for all $1 \leq i < j$. Then (j, l) is the highest non-zero entry in the l-th column of $g.A$ and for we have $x_{jl}(1) \in \mathrm{Stab}_{U_N}(g.A)$. Since $(g.A)_{jl} \neq 0$, there is no $1 \leq m < j$ with $(m, \bar{j}) \in \mathrm{supp}(A)$. If there were no $1 \leq i \leq j$ with $(i, l) \in \mathrm{supp}(A)$, we would have $x_{jl}(1) \in \mathrm{Stab}_{U_N}(A)$ and therefore $(A - g.A)_{jl} = \kappa\big(A - g.A, \pi(x_{jl}(1))\big) = 0$, which is a contradiction because $(g.A)_{jl} \neq 0$ and $A_{jl} = 0$. So there is a $1 \leq i \leq j$ such that $(i, l) \in \mathrm{supp}(A)$, and we must have $i = j$. Therefore, for every $1 \leq k < l \leq N$ with $(g.A)_{kl} \neq 0$ there is a $1 \leq j \leq k$ such that $(j, l) \in \mathrm{supp}(A)$.

Let $\tilde{A} \in \mathbf{v}$ be the verge pattern such that $\mathrm{supp}(\tilde{A}) = \mathrm{supp}(A)$ and $\tilde{A}_{ij} = A_{ij}^{-1}$ for every $(i, j) \in \mathrm{supp}(A)$. Then, for every $(i, j) \in \mathrm{supp}_{\mathcal{V}}(\tilde{A}(g.A)^t)$ there is a $(i, k) \in \mathrm{supp}(A)$ and $(g.A)_{jk} \neq 0$. Therefore, there is a $k < l < \bar{j}$ with $(j, l) \in \mathrm{supp}(A)$ and $g_{kl} \neq 0$, so we have $\mathrm{supp}_{\mathcal{V}}(\tilde{A}(g.A)^t) \subseteq R_A$. Let $v \in U_N$ be such that $\pi(v) = \pi(\tilde{A}(g.A)^t)$, which implies $v \in R_A$. For any $1 \leq j < k \leq N$ with $j + k < N + 1$ we have $(v^t A - A)_{jk} = 0$ unless there is a $1 \leq i < j$ with $(i, k) \in \operatorname{supp}(A)$ in which case we also have $(g.A - A)_{jk} = 0$, so it follows that

$$(v^t A - A)_{jk} = v_{ij} A_{ik} = A_{ik} \left(\tilde{A}(g.A)^t\right)_{ij} = A_{ik} \tilde{A}_{ik} (g.A)_{jk} = (g.A - A)_{jk},$$

and we have $v^t A = g.A$.

To prove the reverse let now again $A, B \in \mathbf{v}$ be verge pattern and $g \in G_N$ as well as $v \in R_A$ such that $v^t A = g.B$. Let $u \in \operatorname{Stab}_{U_N}(A)$. Then we have $ug.B = u.(v^t A) = \pi(v^t(u.A)) = v^t A = g.B$ by lemma 2.3.5. Conversely, let $u \in \operatorname{Stab}_{U_N}(g.B)$. By the same lemma it follows that $u.A = \pi(v^{-t}(u.(v^t A))) = \pi(v^{-t}(ug.B)) = \pi(v^{-t}(v^t A)) = A$, so we have $\operatorname{Stab}_{U_N}(A) = \operatorname{Stab}_{U_N}(g.B)$. Again by lemma 2.3.5 for $x \in \operatorname{Stab}_{U_N}(A) = \operatorname{Stab}_{U_N}(g.B)$ we have

$$\begin{aligned}
\chi_{g.B}(x) &= \vartheta\kappa(v^t A, \pi(x)) = \vartheta\kappa(v.A, \pi(vxv^{-1})) \\
&= \vartheta\kappa(A, \pi(vx) - \pi(v)) = \vartheta\kappa(A, \pi(v) \circ x + \pi(x) - \pi(v)) \\
&= \vartheta\kappa(A,\ \pi(x))\, \vartheta\kappa(x.A - A, \pi(v)) = \vartheta\kappa(A,\ \pi(x)) \\
&= \chi_A(x)
\end{aligned}$$

and therefore $\psi_A = \operatorname{Ind}_{\operatorname{Stab}_{U_N}(A)}^{U_N} \chi_A = \operatorname{Ind}_{\operatorname{Stab}_{U_N}(g.B)}^{U_N} \chi_{g.B} = \psi_{g.B}$. From this it follows that $\iota(A, g.B) \neq 0$ immediately.

$\square$

2.4 Classification of Jedlitschky characters

We will now expand the discussion of Jedlitschky characters for verge patters to Jedlitschky characters of all U_N-orbits in the orbit $G_N.A$ for a verge pattern $A \in \mathbf{v}$. Jedlitschky [Jed13, 3.2.14, p. 65] has classified those orbits in $G_N.A$ by so called core patterns for every verge pattern $A \in \mathbf{v}$, and for an equivalent classification we will use a subset D_A based on the subset $\mathcal{D}_A \subseteq \mathcal{G}$ defined in theorem 2.3.2, such that for every $d \in D_A$ the element $d.A$ is such a core pattern and the representative of a U_N-orbit in $G_N.A$.

Definition 2.4.1. Let $A \in \mathbf{v}$ be a verge pattern and $C \in \mathbf{v}$ such that there is a $g \in G_N$ with $C = g.A$. A position $(i, j) \in \mathcal{V}$ is called a main condition of C if all entries of C right of the j-th column in the i-th row are zero, that is if $(i, j) \in \operatorname{supp}(A)$. The main conditions of the verge pattern A is therefore simply its support. A main condition is called a left or right main condition if it is contained in $\mathcal{V}_l$ or $\mathcal{V}_r$ respectively. A position $(i, j) \in \mathcal{V}_l$ is called a minor condition of C if $(i, \bar{j})$ is a main condition of C. A position $(i, j) \in \mathcal{V}_l$ is called a supplementary condition of C if it is left of a left main or minor condition and if the j-th row of A contains a minor condition. The core conditions then are the union of all main, minor and supplementary conditions. We therefore define the subsets of minor, supplementary conditions and core conditions of $\mathcal{V}$ accordingly:

$$\begin{aligned}
\mathrm{minor}(C) &= \{(i, j) \in \mathcal{V}_l \mid (i, \bar{j}) \in \mathrm{supp}(A)\} \\
\mathrm{supl}(C) &= \{(i, j) \in \mathcal{V}_l \mid \exists\, 1 \leq k \leq n : (k, \bar{j}) \in \mathrm{supp}(A), j < l < \bar{j} : (i, l) \in \mathrm{supp}(A)\} \\
\mathrm{core}(C) &= \mathrm{supp}(A) \cup \mathrm{minor}(C) \cup \mathrm{supl}(C)
\end{aligned}$$

Any such $C \in \mathbf{v}$ with $\mathrm{supp}(C) \subseteq \mathrm{core}(C) = \mathrm{core}(A)$ is called a core pattern for the verge pattern A.

Lemma 2.4.2. *For a verge pattern $A \in \boldsymbol{v}$ let $D_A \subseteq G_N$ be the subset defined by the set of positions*

$$\mathcal{D}_A = \left\{(i, j) \in \mathcal{G} \mid i + j \leq N + 1,\ \exists\, 1 \leq k, l < i :\ (k, \bar{i}), (l, j) \in supp(A)\right\},$$

with $D_A = \{g \in G_N \mid supp_{\mathcal{G}}(g) \in \mathcal{D}_A\}$. Then $\{d.A \mid d \in D_A\}$ is the set of core patterns C for which there is a $g \in G_N$ with $C = g.A$.

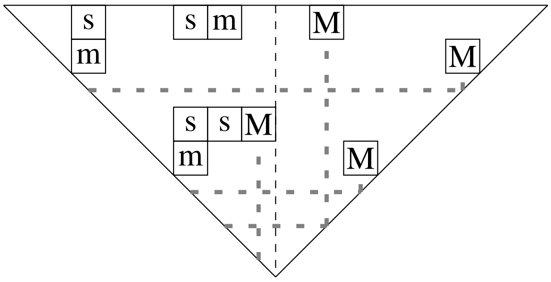
Proof. Let $A \in \mathbf{v}$ be a verge pattern. For $d \in D_A$ we have

$$\begin{aligned}
\mathrm{supp}(d.A) &\subseteq \mathrm{supp}(A) \cup \{(l, i) \in \mathcal{V} \mid (l, j) \in \mathrm{supp}(A), (i, j) \in \mathcal{D}_A\} \\
&= \mathrm{supp}(A) \cup \{(l, i) \in \mathcal{V} \mid (l, \bar{i}) \in \mathrm{supp}(A), (i, \bar{i}) \in \mathcal{D}_A\} \\
&\qquad \cup \{(l, i) \in \mathcal{V} \mid (l, j) \in \mathrm{supp}(A), (i, j) \in \mathcal{D}_A \cap \mathcal{V}\} \\
&= \mathrm{supp}(A) \cup \mathrm{minor}(d.A) \cup \mathrm{supl}(d.A) \\
&= \mathrm{core}(d.A),
\end{aligned}$$

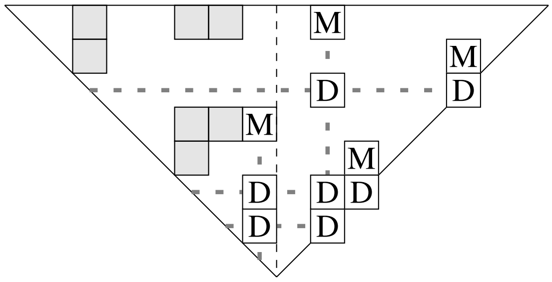
since for $(l, i) \in \mathcal{V}$ with $(l, \bar{i}) \in \mathrm{supp}(A)$ we have $(i, \bar{i}) \in \mathcal{D}_A$ by definition and for $(l, i) \in \mathcal{V}$ with $l < j < \bar{l}$ such that $(l, j) \in \mathrm{supp}(A)$ we have $(l, i) \in \mathrm{supl}(A)$ if and only if there is a $1 \leq k \leq n$ with $(k, \bar{i}) \in \mathrm{supp}(A)$ as well as $i < j < \bar{i}$, which requires $(i, j) \in \mathcal{D}_A$.
Since for $d_1, d_2 \in D_A$ with $d_1.A = d_2.A$, we have $d_1 = d_2$ as well as $|\mathrm{core}(A)| = |\mathcal{D}_A|$ it follows that

$$\{d.A \mid d \in D_A\} = \{g.A \mid g \in G_N \text{ such that } \mathrm{supp}(g.A) \subseteq \mathrm{core}(A)\}.$$

□



Core pattern with main conditions (M), minor conditions (m) and supplementary conditions (s)



Positions of $\mathcal{D}_A$ (D) in relation to the positions of a verge pattern (M)

Guo, Jedlitschky and Dipper [GJD18, 4.21, p. 20] describe a way to directly calculate the stabilizer of any core pattern. As the stabilizer is in general not a pattern subgroup of U_N, this is not straight forward as it is the case for a verge pattern. Therefore, using the Cayley transformation as defined in lemma 1.3.12, for any $g \in G_N$ we instead define a bijective map β_g on U_N such that its restriction to the stabilizer of some $B \in \mathbf{v}$ is a bijection onto the stabilizer of $g.B$. With this we will be able to reduce this problem to the case of verge patterns, where the stabilizer is a pattern subgroup of U_N.

Lemma 2.4.3. *Let f be the Cayley transformation as defined in lemma 1.3.12. For $C \in \mathbf{v}$ and $g \in G_N$ let β_g be the map defined by*

$$\beta_g : U_N \to U_N : u \mapsto f^{-1}\left(g^{-\dagger} f(u) g^{-1}\right).$$

This map restricted to $\mathrm{Stab}_{U_N}(C)$ is a bijection onto $\mathrm{Stab}_{U_N}(g.C)$. This bijection respects composition such that for $h \in G_N$ we have $\beta_g \circ \beta_h = \beta_{gh}$.

Proof. Let $C \in \mathbf{v}$ and $g \in G_N$. By Lemma 2.2.4 for every $g \in G_N$ and $X \in \mathfrak{so}_N$ we have $g^{-\dagger} X g^{-1} \in \mathfrak{so}_N$. Therefore, we have $g^{\dagger} f(x) g \in \mathfrak{so}_N$ for $x \in U_N$, which shows that $\beta_g(u) \in U_N$ for $u \in U_N$. For every $x \in U_N$ we have $f(x^{-1}) = -f(x)$ and $f(x^t) = f(x)^t$. For $u \in \mathrm{Stab}_{U_N}(C)$ we have $2(u+I)^{-1} \in G_N$ as well as $\frac{1}{2}(\beta_g(u) + I) \in G_N$. So for $x \in G_N$ defined as $x = (\beta_g(u) + I) g^{-\dagger} (u+I)^{-1}$ it follows that

$$\begin{aligned}
\beta_g(u)g.C - g.C &= \pi\left(Cg^t\left(\beta_g^t(u) - I\right)\right)\\
&= \pi\left(Cg^t f\left(\beta_g^t(u)\right)\left(\beta_g^t(u) + I\right)\right)\\
&= \pi\left(Cg^t\left(g^{-\dagger}f(u)g^{-1}\right)^t\left(\beta_g^t(u) + I\right)\right)\\
&= \pi\left(Cf(u)^t(g^{-\dagger})^t\left(\beta_g(u) + I\right)^t\right)\\
&= \pi\left(Cf(u)^t(u + I)^t x^t\right)\\
&= \pi\left(C\left(u^t - I\right)x^t\right)\\
&= \pi\left(Cu^t - C\right).x\\
&= \left(u.C - C\right).x = 0
\end{aligned}$$

and therefore $\beta_g(u) \in \mathrm{Stab}_{U_N}(C.g)$. For $h \in G_N$ and $u \in U_N$ we have

$$\beta_g \circ \beta_h(u) = f^{-1}\left(g^{-\dagger}h^{-\dagger}f(u)h^{-1}g^{-1}\right)f^{-1}\left((gh)^{-\dagger}f(u)(gh)^{-1}\right) = \beta_{gh}(u),$$

which especially means that $\beta_{g^{-1}} \circ \beta_g = \mathrm{id}_{U_N}$, so β_g is a bijection, with $\beta_{g^{-1}}$ being its inverse map. □

For $u \in U_N$ the map β_u is the conjugate map Inn_u of $\mathrm{Stab}_{U_N}(C)$, since $u^\dagger = u^{-1}$, and for $x \in U_N$ we have $\beta_u(x) = f^{-1}(uf(x)u^{-1}) = f^{-1}(f(uxu^{-1})) = uxu^{-1}$.
Dipper and Guo [DG15, 3.8, p. 10] have shown that the stabilizer of every core pattern has the same size as the stabilizer of the corresponding verge pattern. By our line of argumentation, this follows immediately from the previous lemma.

Corollary 2.4.4. *For a verge pattern $A \in \mathbf{v}$ and $g \in G_N$ we have*

$$|Stab_{U_N}(g.A)| = |Stab_{U_N}(A)| = |U_N|\frac{q^{|\mathcal{D}_A|}}{\prod_{(i,k)\in supp(A)} q^{j-i-1}}$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $g \in G_N$. Then the map $\beta_g : \mathrm{Stab}_{U_N}(A) \to \mathrm{Stab}_{U_N}(g.A)$ is a bijection, so by theorem 2.3.2 we have $|\mathrm{Stab}_{U_N}(g.A)| = |U_N|q^{|\mathcal{D}_A|}/(\prod_{(i,k)\in\mathrm{supp}(A)} q^{j-i-1})$. □

Using the bilinear form b of ${\mathbb{F}_q}^N$ as defined in 1.2.1, for any verge pattern $A \in \mathbf{v}$ we can now decompose the group G_N into a disjoint union of $(U_N, \mathrm{Stab}_{G_N}(A))$-double cosets, for which D_A is a set of representatives.

Proposition 2.4.5. *Let $A \in \mathbf{v}$ be a verge pattern and $g \in G_N$. For the bilinear form b as defined in 1.2.1 and $1 \le k, l \le n$ we have $b\left((g.A)^t e_k, (g.A)^t e_l\right) = b\left(gA^t e_k, gA^t e_l\right)$ unless there are $1 \le i, j \le N$ with $(k, i), (l, j) \in supp(A)$ such that $i \ge \bar{l}$ or $j \ge \bar{k}$.*

Proof. Let $A \in \mathbf{v}$ be a verge pattern, $g \in G_N$ and $1 \le k, l \le n$. If there is no $1 \le i, j \le N$ such that $(k, i) \in \mathrm{supp}(A)$ or $(l, j) \in \mathrm{supp}(A)$, then $(g.A)^t e_k = gA^t e_k = 0$ or $(g.A)^t e_l = gA^t e_l = 0$, and the claim holds. We assume now that there are $1 \le i, j \le N$ such that $A_{ki}, A_{lj} \neq 0$, so we have

$$b\left(gA^t e_k, gA^t e_l\right) = A_{ki}A_{lj} b\left(g e_i, g e_j\right) = A_{ki}A_{lj} \sum_{r=1}^{N} g_{ri} g_{\bar{r}j} = A_{ki}A_{lj} \sum_{r=\bar{j}}^{i} g_{ri} g_{\bar{r}j}$$

while also

$$b\left((g.A)^t e_k, (g.A)^t e_l\right) = b\left(A_{ki} \sum_{r=k+1}^{i} g_{ri} e_r, A_{lj} \sum_{s=l+1}^{j} g_{sl} e_s\right) = A_{ki}A_{lj} \sum_{r=\max(k+1,\bar{j})}^{\min(\bar{l}-1,i)} g_{ri} g_{\bar{r}j}.$$

These two expressions are equal if $i < \bar{l}$ and $k > \bar{j}$, that is $\bar{k} < j$, which proves the claim. □

Theorem 2.4.6. *Let $A \in \mathbf{v}$ be a verge pattern. The group G_N decomposes into a disjoint union of double cosets*

$$G_N = \bigcup_{d \in D_A} U_N d Stab_{G_N}(A),$$

where D_A defined in lemma 2.4.2 is the set of representatives of the $(U_N, Stab_{G_N}(A))$-double coset. The G_N-orbit $G_N.A$ of A decomposes into $|D_A|$ many disjoint U_N-orbits $U_N d.A$ for $d \in D_A$ with

$$G_N.A = \bigcup_{d \in D_A} U_N d.A$$

Let $g \in G_N$ as well as $u \in U_N$, $d \in D_A$ and $s \in Stab_{G_N}(A)$ such that $g = uds$. For $1 \le i < j \le N$ with $(i, j) \in \mathcal{D}_A$ there are $1 \le k, l \le n$ with $k \neq l$ and $(k, \bar{i}), (l, j) \in supp(A)$ such that

$$b\left((g.A)^t e_k, (g.A)^t e_k\right) = 2A_{ki}^2 d_{\bar{i}i}$$
$$b\left((g.A)^t e_k, (g.A)^t e_l\right) = A_{ki}A_{lj} d_{ij}.$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $g \in G_N$. Let $S = \mathrm{Stab}_{G_N}(A)$ be the stabilizer of A in G_N and let $d \in D_A$, $s \in S$ and $u \in U$ such that $g = uds$. Let $1 \le i < j \le N$ such that $(i, j) \in \mathcal{D}_A$, so we have $j \le \bar{i}$ and there are $1 \le k, l < n$ with $(k, \bar{i}), (l, j) \in \mathrm{supp}(A)$ as well as $l < i$. Since $l < i$ and $k < i \le \bar{j}$ by definition of $\mathcal{D}_A$, it follows by proposition 2.4.5 that

$$
\begin{aligned}
b\left((g.A)^t e_k, (g.A)^t e_l\right) &= b\left((ud.A)^t e_k, (ud.A)^t e_l\right) \\
&= b\left((udA^t e_k, udA^t e_l\right) \\
&= b\left((dA^t e_k, dA^t e_l\right) \\
&= b\left(A_{k\bar{i}} d e_{\bar{i}}, A_{lj} d e_j\right) \\
&= A_{k\bar{i}} A_{lj} \sum_{m=\bar{j}}^{\bar{i}} d_{m\bar{i}} d_{\overline{m}j}
\end{aligned}
$$

Let $\overline{j} \leq m < \overline{i}$ be such that $(m, \overline{i}) \in \mathcal{D}_A$. Then there is a $1 \leq r < n$ with $(r, \overline{m}) \in \operatorname{supp}(A)$ and we have $\overline{i} \leq \overline{m}$. But we also have $j \leq \overline{i}$ as $(i, j) \in \mathcal{D}_A$ and therefore $\overline{j} \leq m \leq i \leq \overline{j}$, which forces $i = \overline{j}$. It then follows that $b\left((g.A)^t e_k, (g.A)^t e_l\right) = A_{k\bar{i}} A_{lj}(d_{\bar{i}i} d_{ij} + d_{\bar{j}i} d_{jj}) = 2A_{k\bar{i}}^2 d_{ij}$. Conversely, if $i \neq \overline{j}$ we have $b\left((g.A)^t e_k, (g.A)^t e_l\right) = A_{k\bar{i}} A_{lj} d_{ij}$. So for any element $g \in U_N dS$ of the double coset $U_N dS$ every $b\left((g.A)^t e_k, (g.A)^t e_l\right)$ for $(i, j) \in \mathcal{D}_A$ is constant and only depends on the choice of $d \in D_A$.

Let now $d_1, d_2 \in D_A$ with $g \in U_N d_1 S \cap U_N d_2 S$. Let $s_1, s_2 \in S$ and $u_1, u_2 \in U_N$ such that $g = s_1 d_1 u_1 = s_2 d_2 u_2$. For every $(i, j) \in \mathcal{D}_A$ let $\lambda_{ij} = 2$ if $j = \overline{i}$ and $\lambda_{ij} = 1$ otherwise, so we have

$$
\lambda_{ij} A_{lj} (d_1)_{ij} = b\left((g.A)^t e_k, (g.A)^t e_l\right) = \lambda_{ij} A_{lj} (d_2)_{ij}
$$

and therefore $d_1 = d_2$. This concludes that the double cosets $U_N dS$ are pairwise disjoint for all $d \in D_A$.

By lemma 2.3.2 we have $|S|/|\operatorname{Stab}_{U_N}(A)| = |G_N|/(|D_A||U_N|)$, and by lemma 2.4.3 it follows that

$$
|U_N dS| = \frac{|U_N||S|}{|d^{-1} S d \cap U_N|} = \frac{|U_N||S|}{|\operatorname{Stab}_{U_N}(d.A)|} = \frac{|U_N||S|}{|\operatorname{Stab}_{U_N}(A)|} = \frac{|G_N|}{|D_A|}.
$$

Since the union $\bigcup_{d \in D_A} U_N dS$ is disjoint, we have $G_N = \bigcup_{d \in D_A} U_N dS$ by argument of cardinality.

For every element $g.A$ of the G_N-orbit of A with $g \in G_N$ there is a $d \in D_A$ with $g \in U_N dS$ such that $g.A$ is contained in the U_N-orbit $U_N d.A$. Since

$$
|G_N.A| = \frac{|G_N|}{|S|} = \frac{|D_A||U_N|}{|\operatorname{Stab}_{U_N}(A)|} = \sum_{d \in D_A} \frac{|U_N|}{|\operatorname{Stab}_{U_N}(A.d)|} = \sum_{d \in D_A} |U_N d.A|
$$

by argument of cardinality, the orbit $G_N.A$ decomposes into the disjoint union of U_N-orbits $G_N.A = \bigcup_{d \in D_A} U_N d.A$.

□

Corollary 2.4.7. *For a verge pattern $A \in \mathbf{v}$ the André–Neto character ϕ_A decomposes into $|D_A|$ many Jedlitschky characters with*

$$\phi_A = \sum_{d \in D_A} \psi_{d.A}.$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern. Then for $x \in U_N$ we have

$$\phi_A(x) = \frac{1}{|U_N|} \sum_{V \in G_N.A} \sum_{v \in U_N} \chi_V(vxv^{-1}) = \frac{1}{|U_N|} \sum_{d \in D_A} \sum_{W \in U_N d.A} \sum_{v \in U_N} \chi_W(vxv^{-1}) = \sum_{d \in D_A} \psi_{d.A}(x).$$

□

In proposition 2.3.3 we have already shown that for a verge pattern $A \in \mathbf{v}$, $C \in G_N.A$ and $u \in U_N$ the map β_u, that is the inner automorphism for u, applied to χ_C is $\chi_{u^{-1}.C}$. To get this result for β_g for any $g \in G_N$ we just have to consider the the maps β_d for $d \in D_A$ and since D_A is not necessarily a group, for $d \in D_A^{-1} = \{d^{-1} \mid d \in D_A\}$ as well.

Lemma 2.4.8. *Let $A \in \mathbf{v}$ be a verge pattern. For $C \in G_N.A$ and $d \in D_A$ or $d \in D_A^{-1}$ we have $\chi_C = \chi_{d.C} \circ \beta_d$.*

Proof. Let $A \in \mathbf{v}$ be a verge pattern, $C \in G_N.A$ and $d \in D_A$. For $1 \leq i < j \leq N$ with $i + j > N + 1$ we have $d_{ij} = 0$. So for the adjugate matrix we have $\mathrm{adj}(d)_{ij} = 0$ and therefore $d^{-1}_{ij} = 0$ as well.
For $x \in \mathrm{Stab}_{U_N}(C)$ we have $\frac{1}{2}(x + I).C = \frac{1}{2}(x.C + C) = C$ and $\frac{1}{2}(\beta_d(x) + I)d.C = d.C$ since $\beta_d(x) \in \mathrm{Stab}_{U_N}(d.C)$. It follows that $2(x + I)^{-1} \in \mathrm{Stab}_{G_N}(C)$ and $\frac{1}{2}(\beta_d(x) + I) \in \mathrm{Stab}_{G_N}(d.C)$. For any $u \in U_N$ we have $\pi(u) = \pi(u - I)$ and $\pi(u - I) = \pi(f(u)(x + I)) = \pi(2f(u)) \circ \frac{1}{2}(x + I)$. It follows that

$$\begin{aligned}
\chi_{d.C}(\beta_d(x)) &= \vartheta\kappa\,(d.C, \pi\,(\beta_d(x) - I)) \\
&= \vartheta\kappa\left(d.C, \pi\left(d^{-\dagger}2f(x)d^{-1}\right)\circ\frac{1}{2}\,(\beta_d(x) + I)\right) \\
&= \vartheta\kappa\left(\frac{1}{2}\,(\beta_d(x) + I)\,d.C, \pi\left(d^{-\dagger}(x - I)2(x + I)^{-1}d^{-1}\right)\right) \\
&= \vartheta\kappa\left(d.C, \pi\left(d^{-\dagger}(x - I)\right)\circ 2(x + I)^{-1}d^{-1}\right) \\
&= \vartheta\kappa\left(2(x + I)^{-1}.C, \pi\left(d^{-\dagger}(x - I)\right)\right) \\
&= \vartheta\kappa\left(C, \pi\left(d^{-\dagger}(x - I)\right)\right).
\end{aligned}$$

For $1 \leq i < k \leq N$ with $(d^{-1})_{\bar{k}\bar{i}} \neq 0$ we have $\bar{k} + \bar{i} \leq N + 1$ and therefore $\bar{i} < k$. So for $1 \leq i < j \leq N$ we have $\left(d^{-\dagger}(x - I)\right)_{ij} = x_{ij} + \sum_{k=i+1}^{j-1}(d^{-1})_{\bar{k}\bar{i}}x_{kj} = x_{ij} + \sum_{k=\max(i+1,\bar{i}+1)}^{j-1}(d^{-1})_{\bar{k}\bar{i}}x_{kj}$. If $i+j < N+1$, we have $j < \bar{i}$ and therefore $\left(d^{-\dagger}(x - I)\right)_{ij} = x_{ij}$. It follows that $\pi\left(d^{-\dagger}(x - I)\right) = \pi(x)$, which gives us

$$\chi_{d.C}(\beta_d(x)) = \vartheta\kappa\,(C, \pi(x)) = \chi_C(x)$$

for $d \in D_A$. For $d \in U_N$ with $d^{-1} \in D_A$ the same argument applies since $(d^{-1})_{ij} = 0$ for any $1 \leq i < j \leq N$ with $i + j > N + 1$. □

With this we can now shift the inner product $\iota(B, C)$ as defined in theorem 2.3.4 of the linear characters for two patterns $B, C \in \mathbf{v}$ with the same verge pattern to the previously solved case, where one of the patterns is a verge pattern. So we can generalize theorem 2.3.6 to calculate the inner product of two arbitrary Jedlitschky characters.

Proposition 2.4.9. *Let $A \in \mathbf{v}$ be a verge pattern. Let $B \in \mathbf{v}$ be such that there are $d \in D_A$ and $u \in U_N$ with $B = ud.A$. Then for $C \in \mathbf{v}$ we have $\iota(B, C) = \iota(A, (ud)^{-1}.C)$.*

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $B \in \mathbf{v}$ such that there are $d \in D_A$ and $u \in U_N$ with $B = ud.A$. By proposition 2.3.3 we have $\chi_B(x) = \chi_{d.A}(u^{-1}xu)$ for $x \in \mathrm{Stab}_{U_N}(B)$. For $C \in \mathbf{v}$ and $x \in \mathrm{Stab}_{U_N}(C)$ we also have $\chi_C(x) = \chi_{u^{-1}.C}(u^{-1}xu)$. Moreover, we have $u^{-1}\left(\mathrm{Stab}_{U_N}(B) \cap \mathrm{Stab}_{U_N}(C)\right)u = \mathrm{Stab}_{U_N}(d.A) \cap \mathrm{Stab}_{U_N}(u^{-1}.C)$.
By lemma 2.4.3 the map $\beta_{d^{-1}}$ is a bijection from $\mathrm{Stab}_{U_N}(d.A)$ to $\mathrm{Stab}_{U_N}(A)$ as well as a bijection from $\mathrm{Stab}_{U_N}(u^{-1}.C)$ to $\mathrm{Stab}_{U_N}((du)^{-1}.C)$. As the same argument holds true for the inverse map β_d of $\beta_{d^{-1}}$, the restriction of $\beta_{d^{-1}}$ to $\mathrm{Stab}_{U_N}(d.A) \cap \mathrm{Stab}_{U_N}(u^{-1}.C)$ is a bijection onto $\mathrm{Stab}_{U_N}(A) \cap \mathrm{Stab}_{U_N}((du)^{-1}.C)$. By lemma 2.4.8 we have $\chi_{d.A} = \chi_A \circ \beta_{d^{-1}}$ as well as $\chi_{u^{-1}.C} = \chi_B \circ \beta_{(du)^{-1}.C}$. It follows that

$$\begin{aligned}\langle \chi_B, \chi_C \rangle_{\mathrm{Stab}_{U_N}(B)\cap\mathrm{Stab}_{U_N}(C)} &= \langle \chi_{d.A}, \chi_{u^{-1}.C}\rangle_{u^{-1}(\mathrm{Stab}_{U_N}(ud.A)\cap\mathrm{Stab}_{U_N}(C))u}\\ &= \langle \chi_{d.A}, \chi_{u^{-1}.C}\rangle_{\mathrm{Stab}_{U_N}(d.A)\cap\mathrm{Stab}_{U_N}(u^{-1}.C)}\\ &= \left\langle \chi_A \circ \beta_{d^{-1}}, \chi_{(du)^{-1}.C}\circ\beta_{d^{-1}}\right\rangle_{\mathrm{Stab}_{U_N}(d.A)\cap\mathrm{Stab}_{U_N}(u^{-1}.C)}\\ &= \left\langle \chi_A, \chi_{(du)^{-1}.C}\right\rangle_{\beta_{d^{-1}}(\mathrm{Stab}_{U_N}(d.A)\cap\mathrm{Stab}_{U_N}(u^{-1}.C))}\\ &= \left\langle \chi_A, \chi_{(du)^{-1}.C}\right\rangle_{\mathrm{Stab}_{U_N}(A)\cap\mathrm{Stab}_{U_N}((du)^{-1}.C)}.\end{aligned}$$

Since we have $|\mathrm{Stab}_{U_N}(d.A)| = |\mathrm{Stab}_{U_N}(A)|$ and $|\mathrm{Stab}_{U_N}(C)| = |\mathrm{Stab}_{U_N}((du)^{-1}.C)|$ as well as $|\mathrm{Stab}_{U_N}(d.A)\cap\mathrm{Stab}_{U_N}(u^{-1}.C)| = |\mathrm{Stab}_{U_N}(A)\cap\mathrm{Stab}_{U_N}((du)^{-1}.C)|$, it follows from the definition in theorem 2.3.4 that $\iota(B, C) = \iota(A, (ud)^{-1}.C)$. □

Theorem 2.4.10. *For a verge pattern $A \in \mathbf{v}$ let $\bullet_A : D_A \times R_A \to D_A$ be the right action of R_A on D_A such that for $d \in D_A$ there is a $u \in U_N$ such that $d.(v^tA) = u(d \bullet_A v).A$. Let $R_A^0(d) = \{v \in R_A \mid d \bullet_A v = d\}$ be the stabilizer of $d \in D_A$ with respect to $\bullet_A$. For another verge pattern $B \in \mathbf{v}$ and $f \in D_A$ we then have*

$$\left\langle \psi_{d.A}, \psi_{f.B}\right\rangle_{U_N} = \begin{cases} |R_A^0(d)| & \text{if } A = B,\ \exists\, v \in R_A : d \bullet_A v = f\\ 0 & \text{otherwise.}\end{cases}$$

If $\left\langle \psi_{d.A}, \psi_{f.B}\right\rangle_{U_N} \neq 0$, it follows that $\psi_{d.A} = \psi_{f.B}$.

Proof. Let $A \in \mathbf{v}$ be a verge pattern, $d \in D_A$ and $v \in R_A$. By lemma 2.3.5 there is a $g \in G_N$ such that $d.(v^tA) = g.A$ and by theorem 2.4.6 there are unique $f \in D_A$ and $u \in U_N$ such that $g.A = uf.A$. Therefore the operation $\bullet_A$ is well defined, by mapping (d, v) to $d \bullet_A v = f$. Let now $v_1, v_2 \in R_A$ and $u_1, u_2, u_3 \in U_N$ such that $d.(v_1^tA) = u_1(d \bullet_A v_1).A$ as well as $(d \bullet_A v_1).(v_1^tA) = u_2((d \bullet_A v_1) \bullet_A v_2).A$ and $d.((v_1v_2)^tA) = u_3(d \bullet_A v_1v_2).A$. We have

$$\begin{aligned} u_3(d \bullet_A v_1v_2)d.A = d.((v_1v_2, d)^tA) &= \pi\left(v_2^t\left(d.(v_1^tA)\right)\right)\\ &= \pi\left(v_2^t\left(u_1(d \bullet_A v_1).A\right)\right)\\ &= u_1.\left((d \bullet_A v_1).(v_2^tA)\right)\\ &= u_1u_2((d \bullet_A v_1) \bullet_A v_2).A\end{aligned}$$

and therefore $d \bullet_A v_1v_2 = (d \bullet_A v_1) \bullet_A v_2$. So $\bullet_A$ satisfies multiplication and since $d \bullet_A I = d$, the map $\bullet_A$ is a right group action of R_A on D_A.

Let now $A, B \in \mathbf{v}$ be verge pattern and $d \in D_A$ as well as $f \in D_B$. By theorem 2.3.4 we have $\left\langle \psi_{d.A}, \psi_{f.B} \right\rangle_{U_N} \neq 0$ if and only if there is a $u \in U_N$ such that $\iota(d.A, uf.B) \neq 0$. If this condition holds, by proposition 2.4.9 we have $\iota(A, d^{-1}uf.B) \neq 0$. By theorem 2.3.6, it follows that $A = B$. Then there is a $v \in R_A$ such that $v^t A = d^{-1}uf.B$ and therefore $d.(v^t A) = uf.B$. So we have $d \bullet_A v = f$. Conversely, for every $v \in R_A$ there is a $u \in U_N$ such that $v^t A = d^{-1}u(d \bullet_A v).A$ and by theorem 2.3.6 and proposition 2.4.9 we have $\iota\,(d.A, u(d \bullet_A v).A) = \iota\left(A, d^{-1}u(d \bullet_A v).A\right) \neq 0$. By lemma 2.3.5 for $x \in U_N$ it follows that

$$\begin{aligned}
\psi_{(d \bullet_A v).A}(x) &= \frac{|\mathrm{Stab}_{U_N}((d \bullet_A v).A)|}{|U_N|^2} \sum_{w,y \in U_N} \vartheta\kappa\left(w(d \bullet_A v).A, \pi(y^{-1}xy)\right) \\
&= \frac{|\mathrm{Stab}_{U_N}(d.A)|}{|U_N|^2} \sum_{w,y \in U_N} \vartheta\kappa\left(wu^{-1}d.(v^t A), \pi(y^{-1}xy)\right) \\
&= \frac{|\mathrm{Stab}_{U_N}(d.A)|}{|U_N|^2} \sum_{w,y \in U_N} \vartheta\kappa\left(vwu^{-1}d.A, \pi(vy^{-1}xyv^{-1})\right) \\
&= \frac{|\mathrm{Stab}_{U_N}(d.A)|}{|U_N|^2} \sum_{w,y \in U_N} \vartheta\kappa\left(wd.A, \pi(y^{-1}xy)\right) \\
&= \psi_{d.A}(x).
\end{aligned}$$

For $f = (d \bullet_A v)$ and $u \in U_N$ such that $\iota\,(d.A, uf.A) \neq 0$ we have $\mathrm{Stab}_{U_N}(A) = \mathrm{Stab}_{U_N}(d^{-1}uf.A)$ by theorem 2.3.6. So by lemma 2.4.3 it follows that $\mathrm{Stab}_{U_N}(d.A) = \mathrm{Stab}_{U_N}(uf.A)$ and therefore

$$\begin{aligned}
\iota\,(d.A, uf.A) &= \frac{|\mathrm{Stab}_{U_N}(d.A) \cap \mathrm{Stab}_{U_N}(uf.A)|}{|\mathrm{Stab}_{U_N}(d.A)||\mathrm{Stab}_{U_N}(uf.A)|} \left\langle \chi_{d.A}, \chi_{uf.A} \right\rangle_{\mathrm{Stab}_{U_N}(d.A) \cap \mathrm{Stab}_{U_N}(uf.A)} \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(d.A)|} \left\langle \chi_{d.A}, \chi_{uf.A} \right\rangle_{\mathrm{Stab}_{U_N}(d.A)} \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(d.A)||U_N|} \sum_{w \in \mathrm{Stab}_{U_N}(d.A)} \vartheta\kappa\,(d.A - uf.A, \pi(w)) \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(d.A)||U_N|} \sum_{w \in \mathrm{Stab}_{U_N}(A)} \vartheta\kappa\left(A - d^{-1}uf.A, \pi(w)\right) \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(d.A)|}.
\end{aligned}$$

Here the last step follows from lemma 2.1.7 because $\mathrm{Stab}_{U_N}(A)$ is a pattern subgroup and therefore $\pi(\mathrm{Stab}_{U_N}(A)) \leq \mathbf{v}$ is a vector space with $A - d^{-1}uf.A \in \pi(\mathrm{Stab}_{U_N}(A))^{\perp}$.

For every $w \in U_N$ by theorem 2.3.6 we have $\iota(d.A, wf.A) \neq 0$ if and only if there is a $v \in R_A$ such that $v^t A = d^{-1}wf.A$ and therefore $d.(v^t A) = wf.A$. So we must have $f = (d \bullet_A v)$. Let $v_1 \in R_A$ and $u_1 \in U_N$ be fixed such that $f = (d \bullet_A v_1)$ with $d.(v_1^t A) = u_1(d \bullet_A v).A$. Then we have $d \bullet_A v_1 v^{-1} = f \bullet_A v^{-1} = d$, so $v_1 v^{-1} \in R_A^0(d)$. Let now $u_0 \in U_N$ be such that $d.((v_1 v^{-1})^t A) = u_0 d.A$. Then we have

$$d.(v_1^t A) = d.(v^t (v_1 v^{-1})^t A) = u_0 d.(v^t A) = u_0 wf.A = u_0 w u_1^{-1} d.(v_1^t A)$$

and therefore $u_0 w u_1^{-1} \in \mathrm{Stab}_{U_N}(d.A)$. Conversely, for every $v \in R_A^0(d)$ with $u_0 \in U_N$ such that $d.(v^t A) = u_0 d.A$ and $w \in u_0^{-1}\mathrm{Stab}_{U_N}(d.A)u_1$ we have

$$\iota(d.A, wf.A) = \iota\left(d.A, wu_1^{-1}d.(v_1^t A)\right) = \iota(d.A, u_0 d.(v_1^t A)) = \iota(d.A, d.((vv_1)^t A)) \neq 0.$$

Finally, by theorem 2.3.4 it follows that

$$\begin{aligned}
\left\langle \psi_{d.A}, \psi_{f.A} \right\rangle_{U_N} &= \sum_{w \in U_N} \iota(d.A, wf.B) \\
&= \sum_{v \in R_A^0(d)} \sum_{w \in u_0^{-1}\mathrm{Stab}_{U_N}(d.A)u_1} \iota(d.A, d.((vv_1)^t A)) \\
&= \sum_{v \in R_A^0(d)} \sum_{w \in u_0^{-1}\mathrm{Stab}_{U_N}(d.A)u_1} \frac{1}{|\mathrm{Stab}_{U_N}(d.A)|} \\
&= |R_A^0(d)| \frac{|u_0^{-1}\mathrm{Stab}_{U_N}(d.A)u_1|}{|\mathrm{Stab}_{U_N}(d.A)|} \\
&= |R_A^0(d)|.
\end{aligned}$$

□

This shows that the set of Jedlitschky characters are mutually orthogonal and for a verge pattern $A \in \mathbf{v}$ and $d \in D_A$ the character $\psi_{d.A}$ occurs in the decomposition of the Andre-Neto character ϕ_A with multiplicity $|R_A|$. With this we can now calculate the inner product for the Andre-Neto character as well.

Corollary 2.4.11. *For a verge pattern $A \in \mathbf{v}$ and $d \in D_A$ the inner product of the André–Neto character is*

$$\langle \psi_{d.A}, \phi_A \rangle_{U_N} = |R_A| \quad \textit{and} \quad \langle \phi_A, \phi_A \rangle_{U_N} = |D_A||R_A|.$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $d \in D_A$. For $f \in D_A$ by theorem 2.4.10 we have $\left\langle \psi_{d.A}, \psi_{f.B} \right\rangle_{U_N} \neq 0$ if and only if $f \in d \bullet_A R_A$, where $d \bullet_A R_A$ is the R_A-orbit of d. So we have

$$\langle \psi_{d.A}, \phi_A \rangle_{U_N} = \sum_{f \in D_A} \left\langle \psi_{d.A}, \psi_{f.A} \right\rangle_{U_N} = \sum_{f \in d \bullet_A R_A} |R_A^0(d)| = |R_A|,$$

where the last step follows from the orbit stabilizer theorem. We then have

$$\langle \phi_A, \phi_A \rangle_{U_N} = \sum_{d \in D_A} \langle \psi_{d.A}, \phi_A \rangle_{U_N} = \sum_{d \in D_A} |R_A| = |D_A||R_A|.$$

□

With this we can come back to the second section of this chapter, and calculate $|G_N * B|$ for any verge matrix $B \in \mathcal{B}$. By corollary 2.2.10 we have $|H * B| = \prod_{(i,j) \in \mathrm{supp}(\pi(B))} q^{j-i-1}$, so by corollary 2.2.11 it follows that

$$|G_N * B| = \frac{|H * B|^2}{\left\langle \phi_{\pi(B)}, \phi_{\pi(B)} \right\rangle_{U_N}} = \frac{1}{|D_A||R_A|} \prod_{(i,j) \in \mathrm{supp}(\pi(B))} q^{2(j-i-1)}.$$

The positions in $\mathcal{V}$ as well as on the counter-diagonal that are both below and left of a non-zero entry of B are $\mathcal{D}_{\pi(B)} \cup \{(j,k) \in \mathcal{V} \mid \exists\, 1 \leq i < j : (i,j) \in \mathcal{R}_{\pi(B)}, (i,k) \in \mathrm{main}(\pi(B))\}$, so their number is equal to $|\mathcal{D}_{\pi(B)}| + |\mathcal{R}_{\pi(B)}|$. Therefore, the number of position in $\mathcal{V}$ that are below or left of a non-zero entry of B are $a = \sum_{(i,j) \in \mathrm{supp}(\pi(B))} q^{2(j-i-1)} - |\mathcal{D}_{\pi(B)}| - |\mathcal{R}_{\pi(B)}|$ and we have $|G_N * B| = q^a$.
The question remains for which $d \in D_A$ the characters $\psi_{d.A}$ are equal and when these characters are irreducible. For this we define a symmetric matrix $S(d) \in M_n(\mathbb{F}_q)$ for every $d \in D_A$ that is derived from the Gram matrix of the bilinear form b. This matrix only depends on d and for any $d, f \in D_A$ their characters $\psi_{d.A}$ and $\psi_{f.A}$ coincide if their matrices $S(d)$ and $S(f)$ are congruent under the operation of R_A modulo some subspace of $M_n(\mathbb{F}_q)$. Furthermore, the number of $v \in R_A$ that preserve $S(d)$ under this operation determine the inner product $\langle \psi_{d.A}, \psi_{d.A} \rangle_{U_N}$.

Lemma 2.4.12. *Let $A \in \mathbf{v}$ be a verge pattern and $\mathcal{S}_A, \mathcal{Z}_A \subseteq [[n]]$ subsets with $(k,l) \in \mathcal{Z}_A$ if there are $(k,i), (l,j) \in supp(A)$ such that $i \geq \bar{l}$ or $j \geq \bar{k}$, whereas $\mathcal{S}_A = [[n]] \backslash \mathcal{Z}_A$. Let $S : D_A \to M_n(\mathbb{F}_q)$ be the map defined by*

$$S(d) = A_{\{1,\dots n\} \times \{1,\dots N\}} (d + d^\dagger - I) J_N (A_{\{1,\dots n\} \times \{1,\dots N\}})^t.$$

Then $S(d)$ for $d \in D_A$ is a symmetric matrix. For $g \in G_N$ with $u \in U_N$ and $s \in Stab_{G_N}(A)$ such that $g = uds$ and $(k,l) \in \mathcal{S}_A$ we have $S_{kl}(d) = b\left((g.A)^t e_k, (g.A)^t e_l\right)$. Let $\mathbf{z}_A \leq M_n(\mathbb{F}_q)$ be the pattern linear subspace with respect to $\mathcal{Z}_A$. Let $X \in M_n(\mathbb{F}_q)$ such that $X_{kl} = b\left((g.A)^t e_k, (g.A)^t e_l\right)$ for $1 \leq k, l \leq n$, then we have $X - S(d) \in \mathbf{z}_A$.

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $d \in D_A$. Let $g \in G_N$ be such that there are $u \in U_N$ and $s \in \mathrm{Stab}_{G_N}(A)$ with $g = uds$. We have

$$\left((d + d^\dagger - I)J_N\right)^t = J_N(d^t + (d^\dagger)^t - I) = J_N(J_N d^\dagger J_N + J_N d J_N - I) = (d + d^\dagger - I)J_N,$$

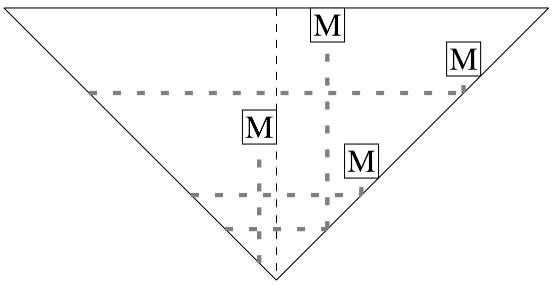
so $S(d)$ is a symmetric matrix. Let $1 \leq k, l \leq n$ be such that there is no $k < i < \bar{k}$ with $(k,i) \in \mathrm{supp}(A)$ or $l < j < \bar{l}$ with $(l,j) \in \mathrm{supp}(A)$ then we clearly have both $S_{kl}(d) = 0$ and $b\left((g.A)^t e_k, (g.A)^t e_l\right) = 0$. We now assume that there are such $k < i < \bar{k}$ and $l < j < \bar{l}$ with $(k,i), (l,j) \in \mathrm{supp}(A)$. Then we have

$$S_{kl}(d) = A_{ki}A_{lj}\left(d + d^\dagger - I\right)_{\bar{i}\bar{j}} = A_{ki}A_{lj}\begin{cases} 0 & \text{for } i + j < N + 1 \\ 1 & \text{for } i + j = N + 1 \\ 2d_{\bar{i}\bar{i}} & \text{for } i + j > N + 1 \text{ and } i = j \\ d_{\bar{i}\bar{j}} & \text{for } i + j > N + 1 \text{ and } i < j \text{ and } j < \bar{k} \\ d_{\bar{j}\bar{i}} & \text{for } i + j > N + 1 \text{ and } i > j \text{ and } i < \bar{l} \\ 0 & \text{for } i \geq \bar{l} \text{ or } j \geq \bar{k}. \end{cases}$$

For $i + j \leq N + 1$ we have $i < \bar{l}$ and $j < \bar{k}$, so by proposition 2.4.5 we have

$$b\left((g.A)^t e_k, (g.A)^t e_l\right) = b\left(gA^t e_k, gA^t e_l\right) = A_{ki}A_{lj}\sum_{r=\bar{j}}^{i} d_{ri}d_{\bar{r}l} = A_{ki}A_{lj}\delta_{i\bar{j}}.$$

Therefore, $S_{kl}(d)$ and $b\left((g.A)^t e_k, (g.A)^t e_l\right)$ coincide for the first two cases of the equation above and by theorem 2.4.6 it does so as well for the third to fifth case, which covers all possibilities for $(k,l) \in \mathcal{S}_A$. □



Main conditions (M)
of a verge pattern $A \in \mathbf{v}$

$$\begin{bmatrix} \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{Z} & \mathcal{Z} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{Z} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{Z} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \end{bmatrix}$$

Positions of $\mathcal{Z}_A$ ($\mathcal{Z}$) and $\mathcal{S}_A$ ($\mathcal{S}$)
corresponding to the verge pattern A

Lemma 2.4.13. *For a verge pattern $A \in \mathbf{v}$ let $\odot_A$ be the right action of R_A on $M_n(\mathbb{F}_q)/\mathbf{z}_A$ defined by*

$$\odot_A : M_n(\mathbb{F}_q)/\mathbf{z}_A \times R_A \to M_n(\mathbb{F}_q)/\mathbf{z}_A : X + \mathbf{z}_A \mapsto (v|_{[[n]]})^t X v|_{[[n]]} + \mathbf{z}_A.$$

Then for $d \in D_A$ and $v \in R_A$ the matrices $S(d)$ and $S(d \bullet_A v)$ as defined in 2.4.12 are congruent modulo $\mathbf{z}_A$ such that

$$S(d \bullet_A v) + \mathbf{z}_A = S(d) \odot_A v.$$

$$\begin{tikzcd} D_A \arrow[r, "\bullet_A v"] \arrow[d, "S"] & D_A \arrow[d, "S"] \\ M_n(\mathbb{F}_q)/\mathbf{z}_A \arrow[r, "\odot_A v"] & M_n(\mathbb{F}_q)/\mathbf{z}_A \end{tikzcd}$$

Proof. Let $A \in \mathbf{v}$ be a verge pattern, $Z \in \mathbf{z}_A$ as defined in 2.4.12 and $v \in R_A$. Then for $1 \leq k, l \leq n$ we have $((v|_{[[n]]})^t Z v|_{[[n]]})_{kl} = \sum_{m=1}^{k} \sum_{s=1}^{l} v_{mk} v_{sl} Z_{ms}$. Let now $1 \leq m \leq k$ and $1 \leq s \leq l$ such that $v_{mk}, v_{sl} \neq 0$ and $(m, s) \in \mathcal{Z}_A$. Then by lemma 2.4.12 there are $m < r < \overline{m}$ and $s < t < \overline{s}$ with $(m, r), (s, t) \in \mathrm{supp}(A)$ and $r \geq \overline{s}$ or $t \geq \overline{m}$. Furthermore, since $v_{mk} \neq 0$ there is a $k < i < \overline{k}$ with $(k, i) \in \mathrm{supp}(A)$ such that $r < i$ if $(m, k) \in \mathcal{R}_A$ or $r = i$ if $m = k$. In the same way, since $v_{sl} \neq 0$ there is a $l < j < \overline{l}$ with $(l, j) \in \mathrm{supp}(A)$ and $t \leq j$.
Without loss of generality, we assume that $r \geq \overline{s}$ and since $r \leq i$ as well as $s \leq l$, it follows that $i \geq r \geq \overline{s} \geq \overline{l}$, which gives us $(k, l) \in \mathcal{Z}_A$. In case of $t \geq \overline{m}$ it follows that $j \geq \overline{k}$ and therefore $(k, l) \in \mathcal{Z}_A$ in the same way. So for any $1 \leq k, l \leq n$ with $((v|_{[[n]]})^t Z v|_{[[n]]})_{kl} \neq 0$ we have $(k, l) \in \mathcal{Z}_A$ and therefore $(v|_{[[n]]})^t Z v|_{[[n]]} \in \mathbf{z}_A$. For any $X \in M_n(\mathbb{F}_q)$ and $Z \in \mathbf{z}_A$ we then

have $(v|_{[[n]]})^t(X+Z)v|_{[[n]]} - (v|_{[[n]]})^t X v|_{[[n]]} \in \mathbf{z}_A$, so the action $\odot_A$ on $M_n(\mathbb{F}_q)/\mathbf{z}_A$ is well defined. For any $v, w \in R_A$ we have $(vw)|_{[[n]]} = v|_{[[n]]}w|_{[[n]]}$, so the action $\odot_A$ respects multiplication. As $\odot_A$ respects the identity as well, it is indeed a group action of R_A on $M_n(\mathbb{F}_q)/\mathbf{z}_A$.

Let $d \in D_A$ and $1 \leq k, l \leq n$ with $(k,l) \in \mathcal{S}_A$. If there is no $k < i < \overline{k}$ with $(k,i) \in \operatorname{supp}(A)$, then there is no $1 \leq j < k$ with $(j,k) \in \mathcal{R}_A$ and for $v \in R_A$ we have $(S(d) \odot_A v)_{kl} = \sum_{i=1}^{l} S_{ki}(d)v_{il}$. But by lemma 2.4.12 for any $1 \leq i \leq l$ we have $S_{ki} = 0$ and therefore $(S(d) \odot_A v)_{kl} = S(d \bullet_A v) = 0$. If there is no $l < j < \overline{l}$ with $(l,l) \in \operatorname{supp}(A)$, we have $(S(d) \odot_A v)_{kl} = S(d \bullet_A v) = 0$ as well by the same argument.
Let now $1 \leq k, l \leq n$ with $(k,l) \in \mathcal{S}_A$ be such that there are $k < i < \overline{k}$ and $l < j < \overline{l}$ with $(k,i), (l,j) \in \operatorname{supp}(A)$. If there are $1 \leq m < k$ or $1 \leq s < l$ such that $(m,k) \in \mathcal{R}_A$ or $(s,l) \in \mathcal{R}_A$, then there is a $m < r < i \leq N$ with $(m,r) \in \operatorname{supp}(A)$ or there is a $s < t < j \leq N$ with $(s,t) \in \operatorname{supp}(A)$. Let now $(m,k) \in \mathcal{R}_A$ or $m = k$ and $(s,l) \in \mathcal{R}_A$ or $s = l$. We have $\pi((d.A)^t e_{mk})e_k = \sum_{a=k+1}^{r-1} d_{ar}A_{mr}e_a$ and $\pi((d.A)^t e_{sl})e_l = \sum_{b=l+1}^{t-1} d_{bt}A_{st}e_b$, and since $l < \overline{i} < \overline{r}$ as well as $t < j \leq i < \overline{k}$, we have

$$b\left(\sum_{a=k+1}^{r-1} d_{ar}A_{mr}e_a, \sum_{b=l+1}^{t-1} d_{bt}A_{st}e_b\right) = b\left(dA^t e_m, dA^t e_s,\right) = b\left((d.A)^t e_m, (d.A)^t e_s\right),$$

where the last step follows from proposition 1.2.1 because $t < j \leq i < \overline{r} < \overline{m}$ and $r < i < \overline{l} < \overline{s}$. For any $1 \leq a \leq b \leq n$ with $(a,b) \in \mathcal{R}_A$ or $a = b$ such that $b \neq k$ or $b \neq l$ we have $\pi((d.A)^t e_{ab})e_k = 0$ or $\pi((d.A)^t e_{ab})e_l = 0$ respectively, so for $v \in R_A$ by lemma 2.3.5 we have

$$\begin{aligned}
b\left((d.(v^tA))^t e_k, (d.(v^tA)))^t e_l\right) &= b\left(\pi\left((d.A)^t v\right) e_k, \pi\left((d.A))^t v\right) e_l\right) \\
&= \sum_{\substack{1 \leq m \leq n \\ (m,k)\in\mathcal{R}_A \vee m=k}} \sum_{\substack{1 \leq s \leq n \\ (s,l)\in\mathcal{R}_A \vee s=l}} v_{mk}v_{sl} b\left(\pi\left((d.A)^t e_{mk}\right) e_k, \pi\left((d.A))^t e_{sl}\right) e_l\right) \\
&= \sum_{\substack{1 \leq m \leq n \\ (m,k)\in\mathcal{R}_A \vee m=k}} \sum_{\substack{1 \leq s \leq n \\ (s,l)\in\mathcal{R}_A \vee s=l}} v_{mk}v_{sl} b\left((d.A)^t e_m, (d.A))^t e_s\right) \\
&= b\left((d.A)^t v e_k, (d.A))^t v e_l\right).
\end{aligned}$$

Since $1 \leq k, l \leq n$, we have $ve_k = v|_{[[n]]}e_k$ as well as $ve_l = v|_{[[n]]}e_l$ and therefore

$$S_{kl}(d \bullet_A v) = b\left((d.(v^tA))^t e_k, (d.(v^tA)))^t e_l\right) = b\left((d.A)^t v e_k, (d.A))^t v e_l\right) = \left((v|_{[[n]]})^t S(d) v|_{[[n]]}\right)_{kl}$$

by theorem 2.4.10, which gives us $S(d \bullet_A v) - S(d) \odot_A v \in \mathbf{z}_A$.

□

In general, a symmetric matrix over a field K with $\mathrm{char}(K) \neq 2$ is congruent to a diagonal matrix, that is for a symmetric matrix $S \in M_n(K)$ there is an orthogonal matrix $P \in O_n(K)$ such that $P^t S P$ is a diagonal matrix. Unfortunately this is not necessarily the case for the matrix $S(d)$ for $d \in D_A$ under the operation of $\odot_A R_A$. But under certain conditions we can still find a partial diagonalization of $S(d)$.

Lemma 2.4.14. *For a verge pattern $A \in \mathbf{v}$ and $1 \leq m \leq n$ let $F_m \subseteq [n]$ be the subset of $[n]$ such that $i \in F_m$ if $(i,m) \in \mathcal{R}_A$. If $\det(S(d)|_{F_m \times F_m}) \neq 0$, there is a $v \in R_A$ such that $S_{im}(d \bullet_A v) = 0$ for all $i \in F_m$.*
For any $v \in R_A$ we have $\det(S(d \bullet_A v)|_{F_m \times F_m}) = \det(S(d)|_{F_m \times F_m})$. Furthermore, if $\det(S(d)|_{F_m \times F_m}) \neq 0$ for all $1 \leq m \leq n$ then $\chi_{d.A}$ is irreducible.

Proof. Let $A \in \mathbf{v}$ be a verge pattern and $d \in D_A$. For $1 \leq m \leq n$ let $\det(S(d)|_{F_m \times F_m}) \neq 0$, and let $b \in \mathbb{F}_q^{|F_m|}$ be defined by $b_i = S_{F_m(i)m}(d)$ for $1 \leq i \leq |F_m|$, where $F_m(i)$ refers to the i-th entry of the ordered set F_m. Then the restriction of $S(d)$ to $F_m \cup \{m\} \times F_m \cup \{m\}$ is a block matrix of the form

$$S(d)|_{F_m \cup \{m\} \times F_m \cup \{m\}} = \begin{pmatrix} S(d)|_{F_m \times F_m} & b \\ b^t & S_{mm}(d) \end{pmatrix}.$$

Let $v \in R_A$ be such that that $\mathrm{supp}_{\mathcal{V}}(v) \subseteq F_m$ and $v_{F_m(i)m} = -\left(S(d)|_{F_m \times F_m}^{-1} b\right)_i$ for $1 \leq i \leq |F_m|$. Then we have

$$v_{F_m \cup \{m\} \times F_m \cup \{m\}} = \begin{pmatrix} I_{F_m} & -S(d)|_{F_m \times F_m}^{-1} b \\ 0 & 1 \end{pmatrix}$$

and therefore

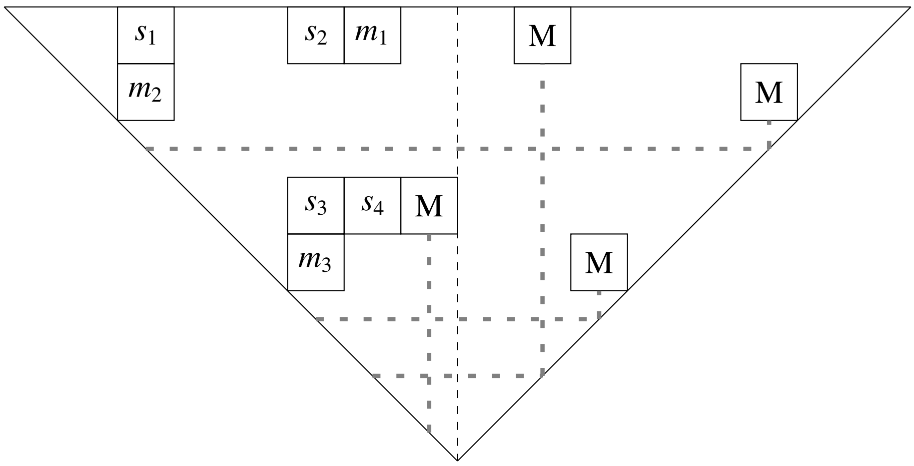
$$S(d \bullet_A v)|_{F_m \cup \{m\} \times F_m \cup \{m\}} = \begin{pmatrix} S(d)|_{F_m \times F_m} & 0 \\ 0 & S_{mm}(d) - b^t S(d)|_{F_m \times F_m}^{-1} b \end{pmatrix}.$$

For any $1 \leq i < j \leq n$ with $j \in F_m$ and $(i,j) \in \mathcal{R}_A$ we have $(i,m) \in \mathcal{R}_A$ since $(j,m) \in \mathcal{R}_A$ and therefore $i \in F_m$. It follows that for any $v \in R_A$ the restriction of $S(d \bullet_A v)$ to $F_m \times F_m$ is

$$S(d \bullet_A v)|_{F_m \times F_m} = v|_{F_m \times F_m}^t S(d)|_{F_m \times F_m} v|_{F_m \times F_m}$$

and therefore $\det\left(S(d \bullet_A v)|_{F_m\times F_m}\right) = \det\left(S(d)|_{F_m\times F_m}\right)$. Let now $d \in D_A$ be such that we have $\det(S(d)|_{F_m\times F_m}) \neq 0$ for all $1 \leq m \leq n$ and let $v \in R_A^0(d)$. For any $1 \leq m \leq n$ and $1 \leq i \leq |F_m|$ we have $0 = S_{F_m(i)m}(d \bullet_A v) - S_{F_m(i)m}(d) = \left(S(d)|_{F_m\times F_m} v_{F_m\times\{m\}}\right)_{im}$. Since $S(d)|_{F_m\times F_m}$ is invertible, it follows that $v_{F_m(i)m} = 0$ and therefore $\mathrm{supp}_{\mathcal{V}}(v) \cap F_m \times \{m\} = \emptyset$. Since this holds for any $1 \leq m \leq n$ and $\mathcal{R}_A = \bigcup_{m=1}^n F_m \times \{m\}$, we have $\mathrm{supp}_{\mathcal{V}}(v) = \emptyset$ and therefore $v = I$. By theorem 2.4.10 it then follows that $\langle \chi_{d.A}, \chi_{d.A}\rangle = |R_A^0(d)| = 1$. □

Example 2.4.15. Following our running example we define the verge pattern $A \in \mathbf{v} \subseteq M_{16}(\mathbb{F}_q)$ and the core patter $C \in \mathbf{v}$ for which A is its verge pattern as $A = e_{1,10} + e_{2,14} + e_{4,8} + e_{5,11}$ and $C = A + s_1e_{1,3} + s_2e_{1,6} + m_1e_{1,7} + m_2e_{2,3} + s_3e_{4,6} + s_4e_{4,7} + m_3e_{5,6}$ for $m_1, m_2, m_3, s_1, s_2, s_3, s_4 \in \mathbb{F}_q$.



We then have have $\mathcal{R}_A = \{(1,2), (1,5), (4,5)\}$ and for $d \in D_A$ such that $C = d.A$ we have the symmetric matrix

$$S(d) = \begin{pmatrix} 2m_1 & s_1 & 0 & s_4 & s_2 & 0 & 0 & 0 \\ s_1 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_4 & \mathbf{z} & 0 & 0 & s_3 & 0 & 0 & 0 \\ s_2 & \mathbf{z} & 0 & s_3 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

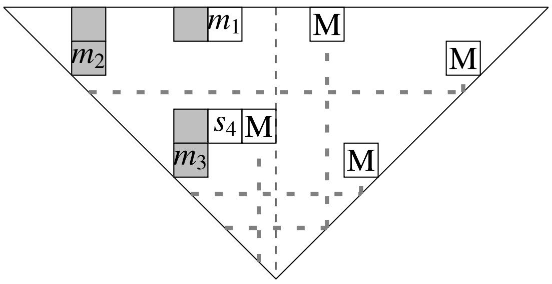
where $\mathbf{z}$ refers to irrelevant entries in positions of $\mathcal{Z}_A$ and grey positions are those, which entries can be changed by the action of R_A. The restrictions of $S(d)$ as defined in lemma 2.4.14 are

$$S(d)|_{F_2\cup\{2\}\times F_2\cup\{2\}} = \begin{pmatrix} 2m_1 & s_1 \\ s_1 & 2m_2 \end{pmatrix} \quad \text{and} \quad S(d)|_{F_5\cup\{5\}\times F_5\cup\{5\}} = \begin{pmatrix} 2m_1 & s_4 & s_2 \\ s_4 & 0 & s_3 \\ s_2 & s_3 & 2m_3 \end{pmatrix},$$

where $S(d)|_{F_2\times F_2}$ and $S(d)|_{F_5\times F_5}$ are in grey. These are then partially diagonalizable if $m_1 \neq 0$ and $s_4 \neq 0$ respectively. We can now distinguish nine different cases for the choice of entries in minor and supplementary conditions and calculate their matrices $S(d)$ and $d.A$ for their respective $d \in D_A$. We can see that there are $q^4 + 4q^3 - 5q^2 + q$ unique characters of which $q^4 - 3q^2 + 2q$ are irreducible and $4q^3 - 2q^2 - q$ are not.

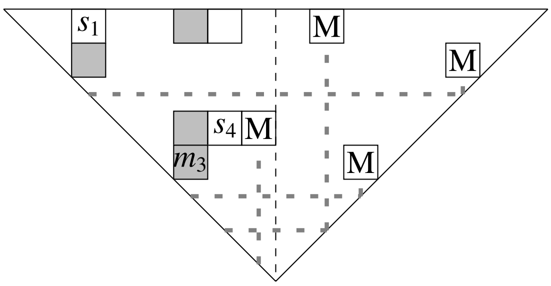
(i) For the $\mathbf{(q-1)^2q^2}$ different characters with $m_1, s_4 \in \mathbb{F}_q^*$ and $m_2, m_3 \in \mathbb{F}_q$ there are $\mathbf{q^3}$ copies of the same character with $|R_A^0(d)| = \mathbf{1}$:

$$\begin{pmatrix} 2m_1 & 0 & 0 & s_4 & 0 & 0 & 0 & 0 \\ 0 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_4 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



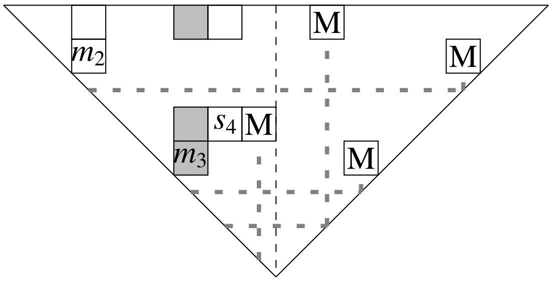
(ii) For the $\mathbf{(q-1)^2q}$ different characters with $m_1 = 0$, $s_1, s_4 \in \mathbb{F}_q^*$ and $m_3 \in \mathbb{F}_q$ there are $\mathbf{q^3}$ copies of the same character with $|R_A^0(d)| = \mathbf{1}$:

$$\begin{pmatrix} 0 & s_1 & 0 & s_4 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_4 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



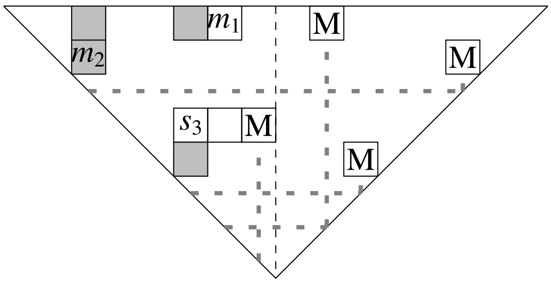
(iii) For the $(\mathbf{q}-\mathbf{1})\mathbf{q}^{\mathbf{2}}$ different characters with $m_1, s_1 = 0$, $s_4 \in \mathbb{F}_q{}^*$ and $m_2, m_3 \in \mathbb{F}_q$ there are $\mathbf{q}^{\mathbf{2}}$ copies of the same character with $|R_A^0(d)| = \mathbf{q}$:

$$\begin{pmatrix} 0 & 0 & 0 & s_4 & 0 & 0 & 0 & 0 \\ 0 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_4 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



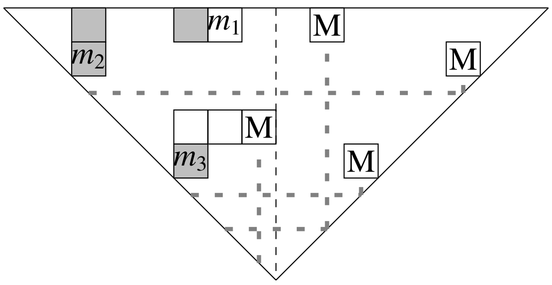
(iv) For the $(\mathbf{q}-\mathbf{1})^{\mathbf{2}}\mathbf{q}$ different characters with $s_4 = 0$, $m_1, s_3 \in \mathbb{F}_q{}^*$ and $m_2 \in \mathbb{F}_q$ there are $\mathbf{q}^{\mathbf{3}}$ copies of the same character with $|R_A^0(d)| = \mathbf{1}$:

$$\begin{pmatrix} 2m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & s_3 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & s_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



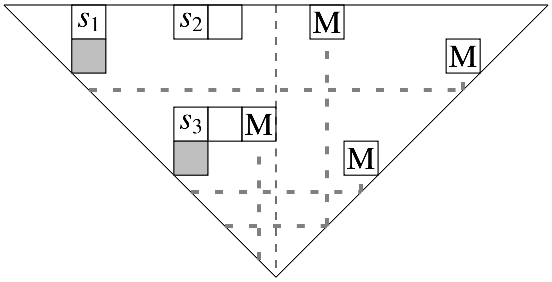
(v) For the $(\mathbf{q}-\mathbf{1})\mathbf{q}^{\mathbf{2}}$ different characters with $s_3, s_4 = 0$, $m_1 \in \mathbb{F}_q{}^*$ and $m_2, m_3 \in \mathbb{F}_q$ there are $\mathbf{q}^{\mathbf{2}}$ copies of the same character with $|R_A^0(d)| = \mathbf{q}$:

$$\begin{pmatrix} 2m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



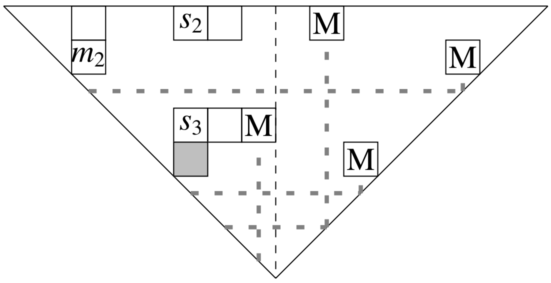
(vi) For the $(\mathbf{q}-\mathbf{1})\mathbf{q}^{\mathbf{2}}$ different characters with $m_1, s_4 = 0$ and $s_1, s_2, s_3 \in \mathbb{F}_q$ with $s_2 \neq 0$ or $s_3 \neq 0$ there are $\mathbf{q}^{\mathbf{2}}$ copies of the same character with $|R_A^0(d)| = \mathbf{q}$:

$$\begin{pmatrix} 0 & s_1 & 0 & 0 & s_2 & 0 & 0 & 0 \\ s_1 & 0 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & s_3 & 0 & 0 & 0 \\ s_2 & \mathbf{z} & 0 & s_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



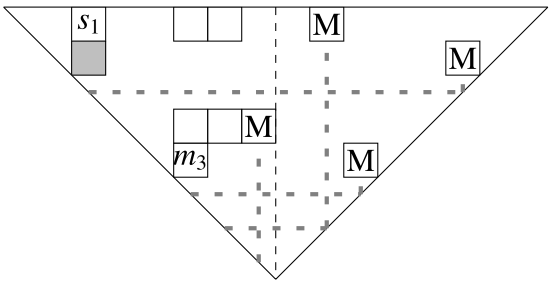
(vii) For the $(\mathbf{q}-\mathbf{1})\mathbf{q}^{\mathbf{2}}$ different characters with $m_1, s_1, s_4 = 0$ and m_2, s_2, s_3 with $s_2 \neq 0$ or $s_3 \neq 0$ there are $\mathbf{q}$ copies of the same character with $|R_A^0(d)| = \mathbf{q}^{\mathbf{2}}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & s_2 & 0 & 0 & 0 \\ 0 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & s_3 & 0 & 0 & 0 \\ s_2 & \mathbf{z} & 0 & s_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



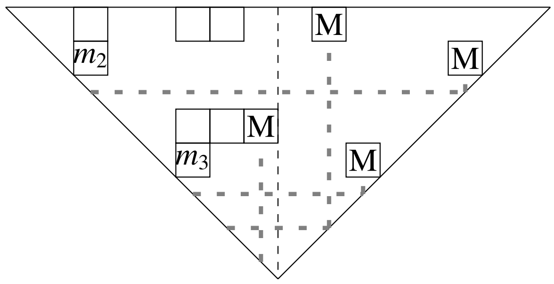
(viii) For the $(\mathbf{q}-\mathbf{1})\mathbf{q}$ different characters with $m_1, s_2, s_3, s_4 = 0$, $s_1 \in \mathbb{F}_q{}^*$ and $m_3 \in \mathbb{F}_q$ there are $\mathbf{q}$ copies of the same character with $|R_A^0(d)| = \mathbf{q}^{\mathbf{2}}$:

$$\begin{pmatrix} 0 & s_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



(ix) For the $\mathbf{q^2}$ different characters with $m_1, s_1, s_2, s_3, s_4 = 0$ and $m_2, m_3 \in \mathbb{F}_q$ there is **one** copy of the same character with $|R_A^0(d)| = \mathbf{q^3}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m_2 & 0 & \mathbf{z} & \mathbf{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{z} & 0 & 0 & 2m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



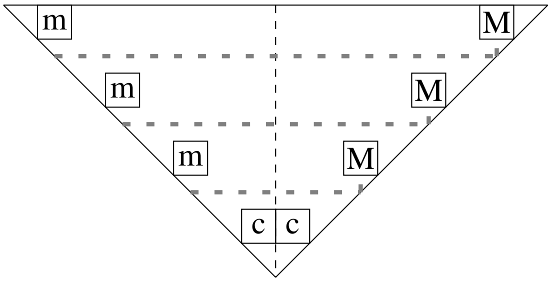
Remark 2.4.16. Jedlitschky [Jed13, 3.3.56, p. 102] conjectured that for a verge pattern $A \in \mathbf{v}$ and $d \in D_A$ there is an $a \in \mathbb{N}_0$ such that $\langle \psi_{d.A}, \psi_{d.A} \rangle_{U_N} = q^a$. We can provide a further clue towards proving this conjecture, as it will hold true if the group contained in the linear algebraic group $\overline{\mathrm{SO}}_N$, which fixed points are $R_A^0(d)$, is connected.
Let $A \in \mathbf{v}$ be a verge pattern and $d \in D_A$. Let $\overline{R}_A^0(d) \leq \overline{U}_N$ be the unipotent group defined by polynomials $f_{ij}(X) \in \mathbb{F}_q[X]$ with indeterminate $X = (X_{kl})_{1\leq k,l\leq N}$ for $(i,j) \in \mathcal{S}_A$ such that $f_{ij}(X) = \sum_{k,l=1}^{n} S_{kl}(d) X_{ki} X_{lj}$. Then we have $R_A^0(d) = \overline{R}_A^0(d)^F\ R_A^0(d)$. If $\overline{R}_A^0(d)$ is connected, the $R_A^0(d)$ is $\mathbb{F}_q$-split[7], that is it admits a subnormal series such that their quotients are isomorphic to the additive group of $\mathbb{F}_q$. It follows that $\langle \psi_{d.A}, \psi_{d.A} \rangle_{U_N} = |R_A^0(d)| = q^a$ where $a \in \mathbb{N}_0$ is the length of this series.

Every Jedlitschky character for a verge pattern $A \in \mathbf{v}$ is especially irreducible if $R_A = I$, which has already been shown by Jedlitschky [Jed13, 3.3.50, p. 100]. This is the case for the irreducible characters of U_N of maximum degree, which are, as shown by André and Neto [AN06, 6.7, p. 425], the characters for the verge pattern $A = \sum_{i=1}^{\lceil n/2 \rceil -1} M_i e_{2i-1,N-2i} + c_1 e_{n-1,n} + c_2 e_{n-1,n+1}$ for $M_1, \dots M_{\lceil n/2 \rceil -1} \in \mathbb{F}_q^*$ as well as $c_1 = c_2 = 0$ if n is odd and $c_1, c_2 \in \mathbb{F}_q$ with $c_1 c_2 = 0$ if n is even, where $\lceil n \rceil$ is $n/2$ rounded up. There are $(q-1)^{(n-1)/2}$ such verge patterns if n is odd and $(q-1)^{n/2-1}(2q-1)$ verge patterns if n is even. We then have $|\mathcal{D}_A| = \lceil n/2 \rceil - 1$ if n is odd or $c_2 = 0$ as well as $|\mathcal{D}_A| = n/2$ if n is even and $c_2 \neq 0$. Therefore, there is a total of

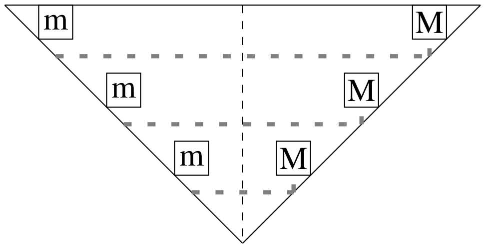
[7] cf. [Mil17, 15.57, p. 266]

$((q-1)q)^{(n-1)/2}$ irreducible characters of maximum degree if n is odd and $(q-1)^{n/2-1}q^{n/2+1}$ many if n is even. In any case for $d \in D_A$ the character $\psi_{d.A}$ has degree

$$\deg \psi_{d.A} = \prod_{i=1}^{\lceil n/2\rceil -1} q^{\overline{2i}-2i-1} = \prod_{i=1}^{\lceil n/2\rceil -1} q^{N-4i} = q^{2(\lceil n/2\rceil -1)(n-\lceil n/2\rceil)} = \begin{cases} q^{\frac{1}{2}n^2-n+\frac{1}{2}} & \text{if } n \text{ is odd} \\ q^{\frac{1}{2}n^2-n} & \text{if } n \text{ is even.} \end{cases}$$



Core pattern with main conditions (M), minor conditions (m) and optional main and minor conditions (c) for even n



Core pattern with main conditions (M) and minor conditions (m) for odd n

3 Classification of the nilpotent orbits

The nilpotent $\overline{\mathrm{Gl}}_n$-orbits in $\mathfrak{gl}_n$ over $\overline{\mathbb{F}}_q$ can be classified by the set of partitions $\{\lambda \vdash n\}$ of n such that for $A \in \mathfrak{gl}_n$ we count the rank of every power of A with $r_m = \mathrm{rank}(A^m)$ for $m \in \mathbb{N}_0$ and define the corresponding partition $\lambda \vdash n$ to be $\lambda = (1^{r_0 - r_1}, 2^{r_1 - r_2}, 3^{r_2 - r_3}, \dots)$. As conjugation with elements of $\overline{\mathrm{Gl}}_n$ preserves the rank of every power of A, we can see that the partition λ is the same for every element in the $\overline{\mathrm{Gl}}_n$-orbit of A. The nilpotent $\overline{\mathrm{SO}}_N$-orbits in $\mathfrak{so}_N$ can mostly be classified the same way by the partitions $\{\lambda \vdash N\}$, but not every partition gives rise to a nilpotent $\overline{\mathrm{SO}}_N$-orbit in $\mathfrak{so}_N$. Only partitions $\lambda \vdash N$ for which their even elements occur with even multiplicity in λ represent a nilpotent $\overline{\mathrm{SO}}_N$-orbit, and furthermore, if λ has only even elements there are two orbits for the same partition. To show this, we will examine the centralizer $C_{\overline{\mathrm{O}}_N}(A)$ of a nilpotent element $A \in \mathfrak{so}_N$ in the orthogonal group $\overline{\mathrm{O}}_N$, which has a decomposition into a semidirect product of its unipotent radical and the centralizer of a 1-dimensional torus. This second centralizer is the key to not only to show whether there are two or only one $\overline{\mathrm{SO}}_N$-orbit for a given partition, but also to classify the finite SO_N-orbits in the fixed points of the $\overline{\mathrm{SO}}_N$-orbit. This centralizer is in general not connected, but has 2^l connected components, where $l \in \mathbb{N}$ is the number of unique odd elements in λ, which gives rise to 2^{l-1} different SO_N-orbits in the fixed points of the $\overline{\mathrm{SO}}_N$-orbit. In the final section we will construct core patterns representing finite SO_N-orbits for every case unless $|\mathbb{F}_q| = 3$. Here we can use the matrix $S(d)$ as defined in the previous chapter, where the determinate of certain restrictions of $S(d)$ determine in which orbit the nilpotent matrix associated to the core pattern is contained.

3.1 Nilpotent orbits over a closed field

Let K be a field with good characteristic $\mathrm{char}(K) = p \neq 2$ that is not necessarily algebraically closed. Every nilpotent matrix in $\mathfrak{gl}(K)$ is conjugate to a Jordan matrix with eigenvalues 0, but this conjugation does not necessarily preserve the bilinear form b. Yet following Springer and

Steinberg, [SS70, 2.18, p. 259] for every nilpotent element of $\mathfrak{so}_n(K)$ we can find a basis of K^N for which it is a Jordan matrix with specific changes to the bilinear form b.
Jordan blocks of even size will occure in pairs in the Jordan decomposition. We will find a basis transformation matrix $C \in \overline{\mathrm{Gl}}_{2n}$ for the Jordan decomposition such, that the Gram matrix $C^t J_{2n} C$ is a block matrix, where for every Jordan block of odd size and every pair of Jordan blocks of even size the corresponding block in the Gram matrix $C^t J_{2n} C$ is of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a \\ 0 & 0 & \cdots & -a & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -a & \cdots & 0 & 0 \\ a & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_i(K) \text{ for } a \in K \quad \text{and} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{2i}(K)$$

respectively.
For a Jordan block of size $i \in \mathbb{N}$ let $l(i) \in \mathbb{N}_0$ the multiplicity with which they occur if i is odd and half of that number if i is even. For $1 \leq j \leq l(i)$ we then find elements $e_j^{(i)} \in K^{2n}$ respectively pairs of elements $f_j^{(i)}, g_j^{(i)} \in K^{2n}$ for odd respectively even i such, that $\{A^{i-1}e_j^{(i)}, \dots Ae_j^{(i)}, e_j^{(i)}\}$ respectively $\{A^{i-1}f_j^{(i)}, \dots Af_j^{(i)}, f_j^{(i)}\}$ and $\{A^{i-1}g_j^{(i)}, \dots Ag_j^{(i)}, g_j^{(i)}\}$ form a basis for their respective Jordan block in the Jordan decomposition, for which the aforementioned properties of the Gram matrix hold.

Theorem 3.1.1. *Let $A \in \mathfrak{so}_n(K)$ be nilpotent with $m \in \mathbb{N}$ such that $A^m = 0$ and $A^{m-1} \neq 0$. For $1 \leq i \leq m$ and $1 \leq j \leq l(i)$ with $l(i) \in \mathbb{N}$ there are $e_j^{(i)} \in K^N$ for odd i and $f_j^{(i)}, g_j^{(i)} \in K^N$ for even i with the following properties:*

(i) $A^i e_j^{(i)} = 0$ *or* $A^i f_j^{(i)} = A^i g_j^{(i)} = 0$ *respectively for all* $1 \leq i \leq m$ *and* $1 \leq j \leq l(i)$

(ii) $A^k e_j^{(i)}$ *and* $A^k f_j^{(i)}, A^k g_j^{(i)}$ *for* $1 \leq i \leq m$, $1 \leq j \leq l(i)$ *and* $1 \leq k \leq i-1$ *form a K-Basis of* K^N

(iii) $b(A^{i-1-k}e_j^{(i)}, A^k e_j^{(i)}) = (-1)^k a_j^{(i)}$ *for all* $1 \leq i \leq m$ *odd,* $1 \leq j \leq l(i)$, $1 \leq k \leq i-1$ *and some* $a_j^{(i)} \in K^\times$

(iv) $b(A^{i-1-k}f_j^{(i)}, A^k g_j^{(i)}) = (-1)^k$ *for all* $1 \leq i \leq m$ *even,* $1 \leq j \leq l(i)$ *and* $1 \leq k \leq i-1$

(v) $b(\cdot, \cdot) = 0$ *for all other pairs of elements of the aforementioned basis*

Proof. The bilinear form b satisfies $b(Av, w) + b(v, Aw) = 0$ for all $v, w \in K^N$, so we have

$$b(A^j v, w) = (-1)^j b(v, A^j w)$$

for all $j \in \mathbb{N}$. Let $V \leq K^N$ be an non-degenerate A-invariant subspace and let $i \in \mathbb{N}$ be the number such that $V \leq \ker A^i$ but $V \nleq \ker A^{i-1}$

Let i be odd. For every $v \in V$ with $A^{i-1}v \neq 0$ there is a $w \in V$ with $b(A^{i-1}v, w) \neq 0$. Assume that $b(A^{i-1}v, v) = 0$ for every $v \in V$. Then we have

$$b(A^{i-1}(v + w), (v + w)) = b(A^{i-1}v, w) + (-1)^{i-1}b(w, A^{i-1}v) = 2b(A^{i-1}v, w) \neq 0$$

for $v, w \in W_i$ with $b(A^{i-1}v, w) \neq 0$, which is a contradiction. So there is a $e_{i-1} \in V$ with $a := b(A^{i-1}e_{i-1}, e_{i-1}) \neq 0$. For $0 \leq k \leq i - 2$ we recursively define

$$e_k = e_{k+1} + rA^{i-1-k}e_{k+1}$$

for some $r \in K$. For all $k < j \leq i - 1$ we then have

$$\begin{aligned} b(A^j e_k, e_k) &= b(A^j e_{k+1}, e_{k+1}) + rb(A^{j+i-1-k}e_{k+1}, e_{k+1}) + rb(A^j e_{k+1}, A^{i-1-k}e_{k+1}) \\ &\quad + r^2 b(A^{j+i-1-k}e_{k+1}, A^{i-1-k}e_{k+1}) \\ &= b(A^j e_{k+1}, e_{k+1}) + rb(A^{j+i-1-k}e_{k+1}, e_{k+1}) + r(-1)^{i-1-k}b(A^{j+i-1-k}e_{k+1}, e_{k+1}) \\ &\quad + r^2(-1)^{i-1-k}b(A^{j+2i-2-2k}e_{k+1}, e_{k+1}) \\ &= b(A^j e_{k+1}, e_{k+1}), \end{aligned}$$

because of $j + i - 1 - k \geq i$. This especially means $b(A^{i-1}e_k, e_k) = a$ for all $0 \leq k \leq i - 2$. Therefore, we have

$$\begin{aligned} b(A^k e_k, e_k) &= b(A^k e_{k+1}, e_{k+1}) + rb(A^{i-1}e_{k+1}, e_{k+1}) + rb(A^k e_{k+1}, A^{i-1-k}e_{k+1}) \\ &\quad + r^2 b(A^{i-1}e_{k+1}, A^{i-1-k}e_{k+1}) \\ &= b(A^k e_{k+1}, e_{k+1}) + r(1 + (-1)^k)b(A^{i-1}e_{k+1}, e_{k+1}) + r^2 b(-1)^{i-1-k}(A^{2i-2-k}e_{k+1}, e_{k+1}) \\ &= b(A^k e_{k+1}, e_{k+1}) + ra(1 + (-1)^k). \end{aligned}$$

Since $b(A^k e_{k+1}, e_{k+1}) = (-1)^k b(e_{k+1}, A^k e_{k+1}) = (-1)^k b(A^k e_{k+1}, e_{k+1})$, for odd k it follows that $b(A^k e_{k+1}, e_{k+1}) = 0$. So we choose $r = -(2a)^{-1}b(A^k e_{k+1}, e_{k+1})$ and we have $b(A^k e_k, e_k) = 0$.

This way we get $e_0 \in V$ such that $b(A^{i-1}e_0, e_0) = a$ and $b(A^j e_0, e_0) = 0$ for all $0 \le j < i-1$. Let

$$U = \left\langle e_0, Ae_0, \dots A^{i-1}e_0 \right\rangle \le V$$

be the A-invariant subspace generated by e_0 with $\dim U = i$, which we call the odd i-cyclic subspace. We have $b(A^j e_0, A^{i-1-j}e_0) = (-1)^j a$ for all $0 \le j < i-1$ and these are the only combinations of basis elements of U such that the bilinear form b is non-zero. Therefore, U is not degenerate. Let $U^\perp \le V$ be the orthogonal complement of U in V. This is again a non-degenerate A-invariant subspace with $V = U \oplus U^\perp$.

Let i now be even. For every $v \in V$ we have $b(A^{i-1}v, v) = (-1)^{i-1} b(v, A^{i-1}v) = -b(A^{i-1}v, v) = 0$. So for $f_{i-1} \in V$ with $A^{i-1}f_{i-1} \neq 0$ there is a $g \in V$ linear independent of f such that $b(A^{i-1}f_{i-1}, g) \neq 0$, and we can choose $g_{i-1} = b(A^{i-1}f_{i-1}, g)^{-1}g$, so we have $b(A^{i-1}f_{i-1}, g_{i-1}) = 1$. For $0 \le k \le i-2$ we again recursively define

$$f_k = f_{k+1} + rA^{i-1-k}f_{k+1} + sA^{i-1-k}g_{k+1} \quad \text{and} \quad g_k = g_{k+1} + tA^{i-1-k}f_{k+1}$$

for some $r, s, t \in K$. We have $b(A^j f_k, f_k) = b(A^j f_{k+1}, f_{k+1})$, $b(A^j f_k, g_k) = b(A^j f_{k+1}, g_{k+1})$ as was the case for i being odd and $b(A^j g_k, g_k) = b(A^j g_{k+1}, g_{k+1})$ for all $k \le j < i-1$, so especially $b(A^{i-1}f_k, g_k) = 1$. Therefore, we have

$$\begin{aligned} b(A^k f_k, f_k) &= b(A^k f_{k+1}, f_{k+1}) + r\left(b(A^{i-1}f_{k+1}, f_{k+1}) + b(A^k f_{k+1}, A^{i-1-k}f_{k+1}\right) \\ &\qquad s\left(b(A^{i-1}g_{k+1}, f_{k+1}) + b(A^k f_{k+1}, A^{i-1-k}g_{k+1}\right) \\ &= b(A^k f_{k+1}, f_{k+1}) + s(1 + (-1)^k), \end{aligned}$$

$$\begin{aligned} b(A^k g_k, g_k) &= b(A^k g_{k+1}, g_{k+1}) + t\left(b(A^{i-1}f_{k+1}, g_{k+1}) + b(A^k g_{k+1}, A^{i-1-k}f_{k+1}\right) \\ &= b(A^k g_{k+1}, g_{k+1}) - t(1 + (-1)^k), \end{aligned}$$

$$\begin{aligned} b(A^k f_k, g_k) &= b(A^k f_{k+1}, g_{k+1}) + rb(A^{i-1}f_{k+1}, g_{k+1}) + sb(A^k f_{k+1}, A^{i-1-k}f_{k+1}) \\ &= b(A^k f_{k+1}, g_{k+1}) + r. \end{aligned}$$

For odd k we have $b(A^k f_{k+1}, f_{k+1}) = b(A^k g_{k+1}, g_{k+1}) = 0$. So we can choose $r = -b(A^k f_{k+1}, g_{k+1})$, $s = -2^{-1}b(A^k f_{k+1}, f_{k+1})$ and $t = 2^{-1}b(A^k g_{k+1}, g_{k+1})$ such that $b(A^k f_k, f_k) = b(A^k f_k, g_k) = 0$ as well as $b(A^k g_k, g_k) = b(A^k f_k, g_k) = 0$.

This way we get $f_0, g_0 \in V$ such that $b(A^{i-1}f_0, g_0) = 1$, $b(A^{i-1}f_0, f_0) = b(A^{i-1}g_0, g_0) = 0$ and $b(A^j f_0, f_0) = b(A^j f_0, g_0) = b(A^j g_0, g_0) = 0$ for all $0 \leq j < i-1$. Let

$$U = \left\langle f_0, g_0, Af_0, Ag_0, \ldots A^{i-1}f_0, A^{i-1}g_0 \right\rangle \leq V$$

be the A-invariant subspace generated by f_0, g_0 with $\dim U = 2i$, which we call the even i-cyclic subspace. We have $b(A^j f_0, A^{i-1-j}g_0) = (-1)^j$ for all $0 \leq j < i-1$ and these are the only combinations of basis elements of U such that the bilinear form b is non-zero. Therefore, U is not degenerate. Let $U^\perp \leq V$ be the orthogonal complement of U in V. This is again a non-degenerate A-invariant subspace with $V = U \oplus U^\perp$.

Let now $V_0 = K^N$. Since A is nilpotent, there is an $m \in \mathbb{N}$ such that $V_0 = \ker A^m$ but $V_0 \nleq \ker A^{m-1}$. For $k \in \mathbb{N}$ we assume that $\emptyset \neq V_k \leq K^N$ is a non-degenerate A-invariant subspace, so there is a $1 \leq i \leq m$ with $V_k \leq \ker A^i$ and $V_k \nleq \ker A^{i-1}$. Let $U \leq V_k$ be an i-cyclic subgroup. We define $V_{k+1} = U^\perp \cap V_k$, which again is a non-degenerate A-invariant subspace. Therefore, $V_k = U \oplus V_{k+1}$ is a direct sum of orthogonal non-degenerate subspaces. So we have a strictly descending chain of subspaces $K^N = V_0 > V_1 > V_2 > \ldots$, and since K^N is finite dimensional, there is a $\tilde{k} \in \mathbb{N}$ with $V_{\tilde{k}} = (0)$. So K^N is a a direct sum of $\tilde{k}$ mutually orthogonal cyclic subspaces. For $1 \leq i \leq m$ let $l(i) \in \mathbb{N}_0$ be the number of i-cyclic subspaces $U_1^{(i)}, \ldots U_{l(i)}^{(i)}$ in this sum, so we finally have

$$K^N = \bigoplus_{i=1}^{m} U_1^{(i)} \oplus \cdots \oplus U_{l(i)}^{(i)}.$$

□

For every $A \in \overline{\mathfrak{so}}_N$ we can construct as follows a one-dimensional torus of $\overline{\mathfrak{so}}_N$ that comprises diagonal matrices with respect to the basis defined above. If $F(A) = A$, the basis of $\overline{\mathbb{F}}_q^N$ corresponding to A is defined over $\mathbb{F}_q$ and the torus is F-stable.[1]

Lemma 3.1.2. *For $A \in \overline{\mathfrak{so}}_N$ let $\tau_A = \tau : \overline{\mathbb{F}}_q^* \to \overline{SO}_N$ be the homomorphism defined by*

$$\tau(x)A^k\epsilon^{(i)} = x^{1-i+2k}A^k\epsilon^{(i)}$$

for $\epsilon^{(i)} = e_j^{(i)}, f_j^{(i)}$ or $g_j^{(i)}$, $1 \leq i \leq m$, $1 \leq j \leq l(i)$, $0 \leq k \leq i-1$ and $x \in \overline{\mathbb{F}}_q^*$. Then $\overline{T}_A = \tau(\overline{\mathbb{F}}_q^*) \leq \overline{SO}_N$ is a one-dimensional torus that is F-stable if $F(A) = A$.

[1] cf. [SS70, 2.22, p. 260]

Proof. τ actually maps into $\overline{\mathrm{SO}}_N$ since

$$b(\tau(x)A^{i-1-k}\epsilon^{(i)}, \tau(x)A^k\epsilon^{(i)}) = b(x^{i-1-2k}A^{i-1-k}\epsilon^{(i)}, x^{1-i+2k}A^k\epsilon^{(i)}) = b(A^{i-1-k}\epsilon^{(i)}, A^k\epsilon^{(i)})$$

for all $\epsilon^{(i)} = e_j^{(i)}, f_j^{(i)}$ or $g_j^{(i)}$, $1 \leq i \leq m$ and $1 \leq j \leq l(i)$.
Since $\sum_{0\leq k\leq i-1} 1 - i + 2k = 0$, we also have $\prod_{0\leq k\leq i-1} x^{1-i+2k} = 1$ for all $x \in \overline{\mathbb{F}}_q$ and therefore $\det \tau(x) = 1$.
If $F(A) = A$, the basis of $\overline{\mathbb{F}}_q^N$ corresponding to A is defined over $\mathbb{F}_q$, and we have $F(\epsilon^{(i)}) = \epsilon^{(i)}$, where F is the standard Frobenius endomorphism on $\overline{\mathbb{F}}_q^N$. It follows that

$$F(\tau(x))A^k\epsilon^{(i)} = F(\tau(x)A^k\epsilon^{(i)}) = F(x^{1-i+2k}A^k\epsilon^{(i)}) = x^{(1-i+2k)q}A^k\epsilon^{(i)} = \tau(x^q)A^k\epsilon^{(i)}$$

and τ is F-stable. □

Let $F(A) = A$. Then the one-dimensional torus $\overline{T}_A$ is contained in a maximally split, F-stable torus. Following Carter [Car85, 5.7, p. 163], we will show that this defines an F-stable parabolic subgroup of $\overline{\mathrm{SO}}_N$ and a $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_N$.

Theorem 3.1.3. *There is a maximally split, F-stable torus $\overline{T}$ of $\overline{SO}_N$ with $\overline{T}_A \leq \overline{T}$. For the set of roots $\Phi(\overline{SO}_N, \overline{T})$ with respect to $\overline{T}$ let $\Delta(\overline{SO}_N, \overline{T}) \subseteq \Phi(\overline{SO}_N, \overline{T})$ be its set of simple roots for which τ is dominant. For $\alpha \in \Phi(\overline{SO}_N, \overline{T})$ let $\overline{U}_\alpha$ the corresponding root subgroup, then $\overline{P}$ is an F-stable parabolic subgroup of $\overline{SO}_N$ defined by*

$$\overline{P} = \left\langle \overline{T}, \overline{U}_\alpha \mid \alpha \in \Phi(\overline{SO}_N, \overline{T}), (\tau, \alpha) \geq 0 \right\rangle$$

with F-stable unipotent radical $\overline{U}$ and Levi-subgroup $\overline{L}$ defined by

$$\overline{L} = \left\langle \overline{T}, \overline{U}_\alpha \mid \alpha \in \Phi(\overline{SO}_N, \overline{T}), (\tau, \alpha) = 0 \right\rangle = C_{\overline{SO}_N}(\overline{T}_A)$$

$$\overline{U} = \left\langle \overline{U}_\alpha \mid \alpha \in \Phi(\overline{SO}_N, \overline{T}), (\tau, \alpha) > 0 \right\rangle.$$

Proof. Let $\overline{L} = C_{\overline{\mathrm{SO}}_N}(\overline{T}_A)$ be the centralizer of $\overline{T}_A$. As $\overline{T}_A$ is F-stable, so is $\overline{L}$. All Borel subgroups of $\overline{L}$ are conjugate, so for any Borel subgroup $\overline{B}$ there is a $g \in \overline{L}$ with $F(\overline{B}) = g^{-1}\overline{B}g$. Since the centralizer of any torus is connected, by Langs theorem there is an $x \in \overline{L}$ with $g = x^{-1}F(x)$ and

$$F(x\overline{B}x^{-1}) = F(x)F(\overline{B})F(x^{-1}) = F(x)g^{-1}\overline{B}gF(x)^{-1} = x\overline{B}x^{-1}.$$

Therefore, $\overline{L}$ has an F-stable Borel subgroup $\overline{B}_L$. Since $\overline{B}_L$ is connected and all maximal tori of $\overline{B}_L$ are conjugate, there is a maximal F-stable torus $\overline{T} \leq \overline{B}_L$ by the same argument. Because $\overline{T}$ is maximal in $\overline{L}$, we have $\overline{T}_A \leq \overline{T}$, and $\overline{T}$ is maximal torus of $\overline{\mathrm{SO}}_N$.
Let $\Phi(\overline{\mathrm{SO}}_N, \overline{T})$ be the set of roots with respect to $\overline{T}$ and let $\Delta(\overline{\mathrm{SO}}_N, \overline{T}) \subseteq \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ be a set of simple roots such that $(\tau, \alpha) \geq 0$ for all $\alpha \in \Delta(\overline{\mathrm{SO}}_N, \overline{T})$. Therefore, $\{\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T}) \mid (\tau, \alpha) > 0\}$ is a closed subset of Φ and $\overline{U}$ is a subgroup of $\overline{\mathrm{SO}}_N$. For any $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau, \alpha) > 0$ and $u \in \overline{U}_\alpha$ we have

$$\tau(c^q)^{-1} F(u) \tau(c^q) = F(\tau(c)^{-1} u \tau(c)) = F\left(I + (\alpha \circ \tau)(c)(u - I)\right) = I + (\alpha \circ \tau)(c^q)(F(u) - I),$$

so $F(\overline{U}_\alpha)$ is a root subgroup for some $\beta \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau, \beta) = (\tau, \alpha) > 0$. It follows that $F(\overline{U}_\alpha) \leq \overline{U}$, and $\overline{U}$ is F-stable.
For $x \in \overline{L}$ and $u \in \overline{U}_\alpha$ with $(\tau, \alpha) \geq 0$ we have $\tau(c)^{-1} x^{-1} u x \tau(c) = I + (\alpha \circ \tau)(c)(x^{-1} u x - I)$ for all $c \in \overline{\mathbb{F}}_q$ and $x^{-1} u x \in \overline{U}$. Therefore, $\overline{L}$ normalizes $\overline{U}$.
For $u \in \overline{U}_\alpha$ we have $u \tau(c) u^{-1} = (\alpha \circ \tau)(c) \tau(c)$ for all $c \in \overline{\mathbb{F}}_q$, so $\overline{U}_\alpha \leq C_{\overline{\mathrm{SO}}_N}(\overline{T}_A)$ if and only if $(\tau, \alpha) = 0$. Since $\overline{T} \leq C_{\overline{\mathrm{SO}}_N}(\overline{T}_A)$, it follows that $\overline{L} = \left\langle \overline{T}, \overline{U}_\alpha \mid \alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T}), (\tau, \alpha) = 0 \right\rangle$ and $\overline{P} = \overline{L} \ltimes \overline{U}$ is F-stable as it is the product of two F-stable groups.
The Borel subgroup $\overline{B}_L \leq \overline{L}$ contains either the root subgroup of a root $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau, \alpha) = 0$ or the root subgroup of its negative root and $\overline{U}$ contains the root subgroup of every root $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau, \alpha) > 0$. Therefore, the semi direct product $\overline{B} = \overline{B}_L \ltimes \overline{U}$ contains either the root subgroup of a root $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ or the root subgroup of its negative root and is Borel group with $\overline{T} \leq \overline{B}$. Moreover, $\overline{B}$ is F-stable and $\overline{T}$ is maximally split. $\overline{P}$ is a parabolic group, since $\overline{B} \leq \overline{P}$ and $\overline{U}$ is the maximal subgroup in $\overline{P}$ such that $\overline{U}_{-\alpha} \nleq \overline{U}$ for every $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $\overline{U}_\alpha \leq \overline{U}$, which concludes $\overline{U} = R_u(\overline{P})$. □

For every rootspace $\overline{\mathfrak{g}}_\alpha \leq \overline{\mathfrak{so}}_N$ with $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ we have $\tau(c) X \tau(c)^{-1} = c^{(\alpha, \tau)} X$ for all $X \in \overline{\mathfrak{u}}_\alpha$ and $c \in \overline{\mathbb{F}}_q^*$, so we can define the following $\mathbb{Z}$-grading on $\overline{\mathfrak{so}}_N$:

Lemma 3.1.4. *The one-dimensional torus $\overline{T}_A$ defines a $\mathbb{Z}$-grading on $\overline{\mathfrak{so}}_N$ by*

$$\overline{\mathfrak{g}}(k) = \{X \in \overline{\mathfrak{so}}_N \mid \tau(c) X \tau(c)^{-1} = c^k X \textit{ for all } c \in \overline{\mathbb{F}}_q^*\}.$$

For $i \in \mathbb{N}$ let $\overline{U}_i \trianglelefteq \overline{P}$ be the normal subgroups defined by $\overline{U}_i = \prod_{\alpha \in \Phi, (\tau, \alpha) \geq i} \overline{U}_\alpha$. Then the following statements hold:

(i) $\overline{\mathfrak{so}}_N = \oplus_{k\in\mathbb{Z}}\,\overline{\mathfrak{g}}(k)$

(ii) $A \in \overline{\mathfrak{g}}(2)$

(iii) $[\overline{\mathfrak{g}}(i), \overline{\mathfrak{g}}(j)] \subseteq \overline{\mathfrak{g}}(i+j)$ *for* $i, j \in \mathbb{Z}$

(iv) *Lie*$(\overline{P}) = \oplus_{k\geq 0}\,\overline{\mathfrak{g}}(k)$

(v) *Lie*$(\overline{U}) = \oplus_{k>0}\,\overline{\mathfrak{g}}(k)$

(vi) *Lie*$(\overline{L}) = \overline{\mathfrak{g}}(0)$

(vii) *Lie*$(\overline{U}_i) = \oplus_{k>0}\,\overline{\mathfrak{g}}(k)$ *for* $i \in \mathbb{N}$

The orthogonal complement of the subalgebra $\overline{\mathfrak{g}}(z)$ *for* $z \in \mathbb{Z}$ *with respect to the bilinear form* $\tilde{\kappa}$ *is* $\overline{\mathfrak{g}}(z)^\perp = \oplus_{-z\neq k\in\mathbb{Z}}\overline{\mathfrak{g}}(k)$ *and we have* $\overline{\mathfrak{so}}_N = \overline{\mathfrak{g}}(z)^t \oplus \overline{\mathfrak{g}}(z)^\perp$.

Proof. Let $\overline{T} \leq \overline{\mathrm{SO}}_N$ be the maximal torus with $\overline{T}_A \leq \overline{T}$ as defined in theorem 3.1.3. For $i \in \mathbb{N}$ and $\alpha,\beta \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau,\alpha) \geq i$ and $(\tau,\beta) \geq 0$ we have $[\overline{U}_\alpha, \overline{U}_\beta] = I$ unless $\alpha+\beta \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ in which case it follows that $[\overline{U}_\alpha, \overline{U}_\beta] = \overline{U}_{\alpha+\beta}$. Since we have $(\tau, \alpha+\beta) = (\tau,\alpha) + (\tau,\beta) \geq i$ as well as $t^{-1}\overline{U}_i t = \overline{U}_i$, the group $\overline{U}_i$ is normal in $\overline{P}$. Let $\overline{\mathfrak{g}}_\alpha$ be the root space for a root $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$. For $X \in \overline{\mathfrak{g}}_\alpha$ we have $\tau(c)^{-1}X\tau(c) = (\alpha\circ\tau)(c)X$ for all $c \in \overline{\mathbb{F}}_q$ and $X \in \overline{\mathfrak{g}}((\tau,\alpha))$. For $X \in \mathrm{Lie}(\overline{T})$ we have $\tau(c)^{-1}X\tau(c) = X$ and $X \in \overline{\mathfrak{g}}(0)$.
If $\epsilon^{(i)} = e^{(i)}_j, f^{(i)}_j$ or $g^{(i)}_j$ with $1 \leq i \leq m$, $1 \leq j \leq l(i)$, $0 \leq k \leq i-2$ and $x \in \overline{\mathbb{F}}_q^*$, we have $\tau(x)A\tau(x)^{-1}A^k\epsilon^{(i)} = \tau(x)Ax^{-1+i-2k}A^k\epsilon^{(i)} = x^{-1+i-2k}\tau(x)A^{k+1}\epsilon^{(i)} = (x^2A)A^k\epsilon^{(i)}$. As this holds for every element of the basis of $\overline{\mathbb{F}}_q^N$, it follows that $A \in \overline{\mathfrak{g}}(2)$.
For $X \in \overline{\mathfrak{g}}(i)$ and $Y \in \overline{\mathfrak{g}}(j)$ with $i, j \in \mathbb{Z}$ we have

$$\tau(x)[X,Y]\tau(x)^{-1} = [\tau(x)X\tau(x)^{-1}, \tau(x)Y\tau(x)^{-1}] = x^{i+j}[X,Y],$$

and therefore $[X,Y] \in \overline{\mathfrak{g}}(i+j)$. Since $\mathrm{Lie}(\overline{U}_\alpha) = \overline{\mathfrak{g}}_\alpha \leq \overline{\mathfrak{g}}((\tau,\alpha))$ for every $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$, the last four statements follow by an argument of dimension.
By lemma 1.3.11 for every $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ we have $\overline{\mathfrak{g}}_\alpha^\perp = \mathrm{Lie}(\overline{T}) \oplus \sum_{-\alpha\neq\beta\in\Phi(\overline{\mathrm{SO}}_N,\overline{T})} \overline{\mathfrak{g}}_\beta$. For $z \in \mathbb{Z}$ it then follows that

$$\overline{\mathfrak{g}}(z)^\perp = \left(\sum_{\substack{\alpha\in\Phi(\overline{\mathrm{SO}}_N,\overline{T})\\(\tau,\alpha)=z}} \overline{\mathfrak{g}}_\alpha\right)^\perp = \bigcap_{\substack{\alpha\in\Phi(\overline{\mathrm{SO}}_N,\overline{T})\\(\tau,\alpha)=z}} \overline{\mathfrak{g}}_\alpha^\perp = \mathrm{Lie}(\overline{T}) \oplus \bigcap_{\substack{\alpha\in\Phi(\overline{\mathrm{SO}}_N,\overline{T})\\(\tau,\alpha)=z}} \left(\sum_{-\alpha\neq\beta\in\Phi(\overline{\mathrm{SO}}_N,\overline{T})} \overline{\mathfrak{g}}_\beta\right).$$

For $\alpha, \beta \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau, \alpha) = z$ and $(\tau, \beta) \neq -z$ we have $\beta \neq -\alpha$ and therefore $\overline{\mathfrak{g}}(k) \leq \overline{\mathfrak{g}}(z)^{\perp}$ for $-z \neq k \in \mathbb{Z}$ with respect to the non-degenerate $\tilde{\kappa}$ bilinear form on $\overline{\mathfrak{so}}_N$. Conversely, for $\beta \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $(\tau, \beta) = -z$ we have $(\tau, -\beta) = z$ and therefore $\overline{\mathfrak{g}}_{\beta} \cap \overline{\mathfrak{g}}(z)^{\perp} = 0$. This gives us $\overline{\mathfrak{g}}(z)^{\perp} = \oplus_{-z \neq k \in \mathbb{Z}} \overline{\mathfrak{g}}(k)$ as well as $\overline{\mathfrak{so}}_N = \overline{\mathfrak{g}}(z)^{\perp} \oplus \overline{\mathfrak{g}}(-z)$. Again by lemma 1.3.11 for $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ we have $\overline{\mathfrak{g}}_{\alpha}^{t} = \overline{\mathfrak{g}}_{-\alpha}$. This gives us $\overline{\mathfrak{g}}(z)^{t} = \overline{\mathfrak{g}}(-z)$ and the last claim follows.

□

Example 3.1.5. Let $\overline{T}_A \leq \overline{\mathrm{SO}}_8$ be defined by the coroot $\tau(x) = \mathrm{diag}(x^2, x, x, 1, 1, x^{-1}, x^{-1}, x^{-2})$ for $x \in \overline{\mathbb{F}}_q^{*}$ and let $A \in \overline{\mathfrak{so}}_8$ be

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & 0 & 0 & 0 & 0 & -1 \\ & & & & 0 & 0 & 0 & -1 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}.$$

The following graphic shows the $\overline{\mathfrak{g}}(i)$ a root space is contained in, corresponding to the $i \in \mathbb{Z}$ of its position in the matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ -1 & 0 & 0 & 1 & 1 & 2 & 2 & 3 \\ -1 & 0 & 0 & 1 & 1 & 2 & 2 & 3 \\ -2 & -1 & -1 & 0 & 0 & 1 & 1 & 2 \\ -2 & -1 & -1 & 0 & 0 & 1 & 1 & 2 \\ -3 & -2 & -2 & -1 & -1 & 0 & 0 & 1 \\ -3 & -2 & -2 & -1 & -1 & 0 & 0 & 1 \\ -4 & -3 & -3 & -2 & -2 & -1 & -1 & 0 \end{bmatrix}$$

Springer and Steinberg [SS70, 2.23-2.27, p.260] showed that the centralizer $C_{\overline{\mathrm{O}}_N}(A)$ of a nilpotent element A is isomorphic to a semidirect product of its unipotent radical and a direct product of symplectic and orthogonal groups. Therefore, the component group of the centralizer is a direct product of cyclic groups of order two.

Theorem 3.1.6. *Let $C = C_{\overline{O}_N}(A)$ be the centralizer of A in $\overline{O}_N$ and $C_{\overline{O}_N}(\overline{T}_A, A) = C \cap C_{\overline{O}_N}(\overline{T}_A)$. Then we have $C = C_{\overline{O}_N}(\overline{T}_A, A) \ltimes R_u(C)$ and $C_{\overline{O}_N}(\overline{T}_A, A)$ is isomorphic to the direct product*

$$C_{\overline{O}_N}(\overline{T}_A, A) \cong \prod_{\substack{1 \le i \le m-1 \\ l(i)>0, i \text{ odd}}} \overline{O}_{l(i)} \times \prod_{\substack{1 \le i \le m-1 \\ l(i)>0, i \text{ even}}} \overline{Sp}_{2l(i)},$$

where $\overline{O}_{l(i)}$ is a orthogonal and $\overline{Sp}_{2l(i)}$ is a symplectic group in dimension $l(i)$ and $2l(i)$ respectively. Furthermore, the component group C/C° of the centralizer of A in $\overline{O}_N$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$ where d is the number of odd i with $l(i) > 0$.

Proof. Let $v \in \overline{\mathbb{F}}_q^N$ and let $\epsilon_j^{(i)} = e_j^{(i)}$ or $\epsilon_j^{(i)} = f_j^{(i)}$ and $\epsilon_{j+l(i)}^{(i)} = f_j^{(i)}$ for $1 \le i \le m$ and $1 \le j \le l(i)$. Then v is the sum

$$v = \sum_{\substack{1 \le i \le m, j \\ 0 \le k \le i-1}} \lambda_{i,j,k} A^k \epsilon_j^{(i)}$$

for coefficients $\lambda_{i,j,k} \in \overline{\mathbb{F}}_q$, where j runs from 1 to $l(i)$ or $2l(i)$ respectively. Let $r \in \mathbb{N}$, $s \in \mathbb{Z}$ with $s+r \le 1$ such, that $A^r v = 0$ and $\tau(c)v = c^s$ for all $c \in \overline{\mathbb{F}}_q$. We then have $\lambda_{i,j,k} = 0$ unless $r \ge i-k$. Since $\tau(c)A^k\epsilon_j^{(i)} = c^{1-i+2k}A^k\epsilon_j^{(i)}$, we must have $\lambda_{i,j,k} = 0$ unless $s = 1-i+2k$. By assumption, we have $s + r \le 0$ and therefore $\lambda_{i,j,k} = 0$ unless $k = (1-i+2k) + (i-k) - 1 \le s + r - 1 \le 0$. So since $k \ge 0$, we have $v = 0$ for $s + r \le 0$ and $v = \sum_j \lambda_{r,j,0} A^k \epsilon_j^{(r)}$ if $s + r = 1$.

The identity component C° of the centralizer $C = C_{\overline{\mathrm{O}}_N}(A)$ is contained in $\overline{\mathrm{SO}}_N$ and is therefore the identity component of $C_{\overline{\mathrm{SO}}_N}(A)$. By theorem 1.2.7 its Lie algebra is the centralizer $\mathfrak{c}_{\overline{\mathfrak{so}}_N}(A) = \mathrm{Lie}(C^\circ)$ of A in $\overline{\mathfrak{so}}_N$. For $l \in \mathbb{C}$ let $X \in \mathfrak{c}_{\overline{\mathfrak{so}}_N}(A) \cap \overline{\mathfrak{g}}(l)$, and we have

$$\tau(c)X\epsilon^{(i)} = c^l X\tau(c)\epsilon^{(i)} = c^{1-i+l}X\epsilon^{(i)} \quad \text{and} \quad A^i X\epsilon^{(i)} = XA^i\epsilon^{(i)} = 0$$

for $\epsilon^{(i)} = e_j^{(i)}, f_j^{(i)}$ or $g_j^{(i)}$ with $1 \le j \le l(i)$ and $c \in \overline{\mathbb{F}}_q$. If $l < 0$, we have $X\epsilon^{(i)} = 0$ as shown above and therefore $XA^k\epsilon^{(i)} = A^kX\epsilon^{(i)} = 0$ for $0 \le k \le i-1$. It follows that $\mathfrak{c}_{\overline{\mathfrak{so}}_N}(A) \cap \overline{\mathfrak{g}}(l) = (0)$ for $l < 0$ and $\mathfrak{c}_{\overline{\mathfrak{so}}_N}(A) \subseteq \oplus_{l \ge 0} \overline{\mathfrak{g}}(l) = \mathrm{Lie}(\overline{P})$, which gives us $C^\circ \le \overline{P}$.

For $g \in C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)$ we have $t(c)g\epsilon^{(i)} = gt(c)\epsilon^{(i)} = c^{1-i}g\epsilon^{(i)}$ for all $c \in \overline{\mathbb{F}}_q$. Since $A^i g\epsilon^{(i)} = 0$, it follows from our previous proof that $g\epsilon^{(i)} = \sum_j \lambda_j \epsilon_j^{(i)}$ for $\lambda_j \in \overline{\mathbb{F}}_q$ and $gA^k\epsilon^{(i)} = \sum_j \lambda_j A^k \epsilon_j^{(i)}$ for all $0 \le k \le i-1$. Therefore, $\overline{\mathbb{F}}_q^N$ decomposes into invariant subspaces under the action of elements of $C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)$ in the form of

$$\left\langle A^k e_1^{(i)}, A^k e_2^{(i)}, \ldots, A^k e_{l(i)}^{(i)} \right\rangle \quad \text{and} \quad \left\langle A^k f_1^{(i)}, \ldots, A^k f_{l(i)}^{(i)}, A^k g_1^{(i)}, \ldots, A^k g_{l(i)}^{(i)} \right\rangle$$

for odd i and even i respectively, where the action of g is the same for any $0 \leq k < i$.

We define the $C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)$-invariant subspace

$$V_i = \left\langle e_1^{(i)}, e_2^{(i)}, \ldots, e_{l(i)}^{(i)} \right\rangle \quad \text{or} \quad V_i = \left\langle f_1^{(i)}, \ldots, f_{l(i)}^{(i)}, g_1^{(i)}, \ldots, g_{l(i)}^{(i)} \right\rangle$$

for all odd or even $1 \leq i \leq m-1$ respectively. For every i let b_i be a non-degenerated symmetric or skew-symmetric bilinear form for odd or even i respectively defined by

$$b_i : V_i \times V_i \to \overline{\mathbb{F}}_q : (v, w) \mapsto b(A^{i-1}v, w).$$

Their Gram matrices for odd or even i are

$$M(b_i) = \begin{pmatrix} a_1^{(i)} & 0 & \ldots & 0 \\ 0 & a_2^{(i)} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_{l(i)}^{(i)} \end{pmatrix} \quad \text{and} \quad M(b_i) = \begin{pmatrix} 0 & I_{l(i)} \\ -I_{l(i)} & 0 \end{pmatrix}.$$

For $g \in C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)$ let g_i be the restriction of g to V_i, so for the bilinear form b_i we have $b_i(g_i v, g_i w) = b(A^{i-1} g_i v, g_i w) = b(g_i A^{i-1} v, g_i w) = b_i(v, w)$ for all $v, w \in V_i$. This gives us a group isomorphism

$$\phi : C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A) \to \prod_{i=1}^{m-1} O(V_i, b_i) : g \mapsto (g_1, \ldots g_{m-1}).$$

The bilinear form b_i is symmetric or skew-symmetric for odd or even i, so $O(V_i, b_i)$ is an orthogonal or a symplectic group respectively whenever $l(i) > 0$. Symplectic groups are connected and the identity component of orthogonal groups are their special orthogonal group. Therefore, the identity component $C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ$ is isomorphic to a product of symplectic and special orthogonal groups. Furthermore, as a direct product of reductive groups $C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ$ is reductive.

$C_{\overline{\mathrm{O}}_N}(\overline{T}_A) \cap C^\circ$ is the centralizer of the torus $\overline{T}_A$ in the connected group C° and is therefore connected, so we have $C_{\overline{\mathrm{O}}_N}(\overline{T}_A) \cap C^\circ = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ$. Since $C^\circ \leq \overline{P}$, it is the semidirect product $C^\circ = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ \ltimes (C^\circ \cap \overline{U})$. As $C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ$ is reductive and $\overline{U} \trianglelefteq \overline{P}$ is unipotent, we have $R_u(C^\circ) = C^\circ \cap \overline{U}$. Since $R_u(C)$ is connected and therefore a unipotent normal subgroup of C°, we have $R_u(C) \leq R_u(C^\circ)$. As every conjugate of $R_u(C^\circ)$ in C is also a maximal connected unipotent normal subgroup of C°, so by the maximality condition it must be $R_u(C^\circ)$ itself, and we have $R_u(C^\circ) \trianglelefteq C$, which gives us $R_u(C) = R_u(C^\circ)$. Therefore, we have $C^\circ = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ \ltimes R_u(C)$, with $R_u(C) = C^\circ \cap \overline{U}$.

For $g \in C$ and $t \in \overline{T}_A$ let $[t, g] = t^{-1}g^{-1}tg$ be their commutator. The set of commutators $[\overline{T}_A, g]$ is connected for all $g \in C$ with $I \in [\overline{T}_A, g]$, so $[\overline{T}_A, g] \subseteq C^\circ$. Since $C^\circ = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ \ltimes R_u(C)$ for $h \in C^\circ$ and $t \in \overline{T}_A$, we have $[h, t] \in R_u(C)$. For $g \in C$ and $t_1, t_2 \in \overline{T}_A$ we therefore have $[t_1, g][t_2, g][g, t_1t_2] = [t_1, g][g, t_1]^{t_2} = [[g, t_1], t_2] \in R_u(C)$, and it follows that $[\overline{T}_A, g]R_u(C) \leq C^\circ$ is a subgroup for any $g \in C$. Furthermore, for $h \in C^\circ$ and $t \in \overline{T}_A$ we have

$$\begin{aligned}[h, [t, g]] &= [g, t]h^{-1}[g, t]h = g^{-1}t^{-1}g[t^{-1}, h]gh^{-1}g^{-1}tgh \\ &= g^{-1}t^{-1}g[t^{-1}, h]g^{-1}t[t, gh^{-1}g^{-1}]g = [t^{-1}, h]^{g^{-1}tg}[t, gh^{-1}g^{-1}]^g \in R_u(C),\end{aligned}$$

so $[\overline{T}_A, g]R_u(C)/R_u(C) \leq Z(C^\circ/R_u(C))$, where $Z(C^\circ/R_u(C))$ is the center of the quotient group $C^\circ/R_u(C)$. As $[\overline{T}_A, g]$ is connected, so is $[\overline{T}_A, g]R_u(C)/R_u(C)$, and it is contained in the connected center of $C^\circ/R_u(C)$. Since $C^\circ/R_u(C)$ is isomorphic to a direct product of orthogonal and symplectic groups, which all have the trivial group as their connected center $Z(C^\circ/R_u(C))^\circ$ must be the trivial group as well, and it follows that $[\overline{T}_A, g] \subseteq R_u(C)$.

Following an argument of Liebeck, [LS12, 2.25, p. 28] for $g \in C$ and $t \in \overline{T}_A$ we have $g^{-1}tg = t[t, g] \in \overline{T}_A R_u(C)$, so every conjugate of $\overline{T}_A$ in C is contained in $\overline{T}_A R_u(C)$. As $R_u(C)$ is unipotent, $\overline{T}_A$ is a maximal torus in $\overline{T}_A R_u(C)$, so every conjugate of $\overline{T}_A$ in C is also conjugate in $R_u(C)$. By the generalized Frattini argument we therefore have $C = N_C(\overline{T}_A)R_u(C)$. For all $c \in \overline{\mathbb{F}}_q$ we have $\tau(c)A\tau(c)^{-1} = c^2A$, so $\overline{T}_A \cap C$ contains at most two elements. Since $[\overline{T}_A, N_C(\overline{T}_A)] \subseteq \overline{T}_A \cap C$ is connected, it follows that $[\overline{T}_A, N_C(\overline{T}_A)] = 1$. Therefore, we have $N_C(\overline{T}_A) = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)$ which gives us $C = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A) \ltimes R_u(C)$.

Symplectic groups are connected, and the component group of an orthogonal group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It follows that $C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)/C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ \cong (\mathbb{Z}/2\mathbb{Z})^d$ where d is the number of odd i with $l(i) > 0$ and since $C = C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A) \ltimes R_u(C)$, we have

$$C/C^\circ \cong {C/R_u(C)}\big/{C^\circ/R_u(C)} \cong C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)/C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)^\circ.$$

□

Corollary 3.1.7. *For the centralizer of A in $\overline{SO}_N$ we have $C_{\overline{SO}_N}(A) \leq \overline{P}$ and the component group $C_{\overline{SO}_N}(A)/C_{\overline{SO}_N}(A)^\circ$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{d-1}$ where d is the number of odd i with $l(i) > 0$, unless $l(i) = 0$ for all odd i, where $C_{\overline{SO}_N}(A) = C$ is connected.*

Proof. Since $R_u(C) \leq \overline{P}$, we have $C_{\overline{\mathrm{SO}}_N}(A) = C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A) \ltimes R_u(C) \leq \overline{P}$. Let ϕ be the isomorphism defined in theorem 3.1.6 with

$$\phi : C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A) \to \prod_{\substack{1\leq i\leq m-1\\ l(i)>0, i \text{ odd}}} \overline{\mathrm{O}}_{l(i)} \times \prod_{\substack{1\leq i\leq m-1\\ l(i)>0, i \text{ even}}} \overline{\mathrm{Sp}}_{l(i)}.$$

Let $g \in C_{\overline{\mathrm{O}}_N}(\overline{T}_A, A)$. Since $g_i \in \overline{\mathrm{O}}_{l(i)}$ or $g_i \in \overline{\mathrm{Sp}}_{l(i)}$ for odd or even i, we have $\det(g_i) \in \{\pm 1\}$ or $\det(g_i) = 1$ respectively. As g is invariant on the subspaces $A^k V_i$ for all i with $l(i) > 0$ and $0 \leq k \leq i-1$ on all of which it acts as g_i, it follows that

$$\det(g) = \prod_{\substack{1\leq i\leq m-1\\ l(i)>0}} \det(g_i)^i = \prod_{\substack{1\leq i\leq m-1\\ l(i)>0, i \text{ odd}}} \det(g_i).$$

So we have $C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A) = \ker(\det \phi^{-1})$ and $C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A)$ must be a normal subgroup of index two in C which contains half of its components. $\square$

We can now classify the nilpotent $\overline{\mathrm{SO}}_N$-orbits of $\overline{\mathfrak{so}}_N$ by constructing a map from the nilpotent $\overline{\mathrm{SO}}_N$-orbits to a subset of partitions of N, which is almost a bijection.

Theorem 3.1.8. *For $A \in \overline{\mathfrak{so}}_N$ let $\lambda_A \vdash N$ be the partition $\lambda_A = (1^{l(1)}, 2^{2l(2)}, \dots)$, where the entries $i \in \mathbb{N}$ occur with multiplicity $l(i)$ for odd i and $2l(i)$ for even i as defined in theorem 3.1.1. Then the map*

$$\left\{ \begin{matrix} \overline{SO}_N\text{-orbits of} \\ \text{nilpotent elements} \end{matrix} \right\} \to \left\{ \lambda \vdash N \,\middle|\, \begin{matrix} \text{where every even element} \\ \text{has even multiplicity} \end{matrix} \right\}$$
$$\mathcal{O}(A) \mapsto \lambda_A = \left(1^{l(1)}, 2^{2l(2)}, 3^{l(3)}, 4^{2l(4)}, \dots\right).$$

is surjective. Every partition corresponds to exactly one orbit unless it comprises only even elements, in which case it corresponds to two orbits. For such a partition the two distinct orbits of an $A \in \overline{\mathfrak{so}}_N$ are $\mathcal{O}(A)$ and $\mathcal{O}(g^{-1}Ag)$ for any $g \in \overline{O}_N$ with $\det(g) = -1$. *The conjugate partition of λ_A is $\lambda'_A = (rank(A^0) - rank(A),\ rank(A) - rank(A^2),\ rank(A^2) - rank(A^3), \dots)$.*

Proof. Let $A \in \overline{\mathfrak{so}}_N$. Clearly by definition every even element of λ_A occurs with even multiplicity. Following from the definition in theorem 3.1.1, for $k \in \mathbb{N}_0$ the rank of A^k is

$$\operatorname{rank}(A^k) = \sum_{\substack{i>k\\ i \text{ odd}}} (i-k)l(i) + \sum_{\substack{i>k\\ i \text{ even}}} (i-k)2l(i).$$

Since the rank of every power of an element of $\overline{\mathfrak{so}}_N$ is retained under the conjugation with $\overline{\mathrm{SO}}_N$, the units $l(i)$ for $i \in \mathbb{N}$ are constant on the nilpotent orbits of $\overline{\mathfrak{so}}_N$. Therefore, the map above is well defined. For odd $i \in \mathbb{N}$ we have $l(i) - 2l(i+1) = \mathrm{rank}(A^i) - \mathrm{rank}(A^{i-1})$, whereas for even $i \in \mathbb{N}$ we have $2l(i) - l(i) = \mathrm{rank}(A^i) - \mathrm{rank}(A^{i-1})$. The conjugate partition of λ_A then is

$$\begin{aligned}\lambda_A &= (2l(2) - l(1), l(3) - 2l(2), 2l(4) - l(3), \dots)\\ &= \Big(\mathrm{rank}(A^0) - \mathrm{rank}(A),\ \mathrm{rank}(A) - \mathrm{rank}(A^2),\ \mathrm{rank}(A^2) - \mathrm{rank}(A^3), \dots\Big)\end{aligned}$$

Let now $A, \tilde{A} \in \overline{\mathfrak{so}}_N$ with their respective basis of $\overline{\mathbb{F}}_q^N$ as defined in theorem 3.1.1. Let $g \in \overline{\mathrm{GL}_N}$ defined by

$$gA^k e_j^{(i)} = c_j^{(i)} \tilde{A}^k \tilde{e}_j^{(i)}, \quad gA^k f_j^{(i)} = \tilde{A}^k \tilde{f}_j^{(i)}, \quad gA^k g_j^{(i)} = \tilde{A}^k \tilde{g}_j^{(i)}$$

for $1 \leq i \leq m$, $1 \leq j \leq l(i)$, $0 \leq k \leq i-1$, with $c_j^{(i)} \in \mathbb{F}_q^\times$ such that $(c_j^{(i)})^2 \tilde{a}_j^{(i)} = a_j^{(i)}$. Since $b(gA^{i-1-k}e_j^{(i)}, gA^k e_j^{(i)}) = (c_j^{(i)})^2 b(\tilde{A}^{i-1-k}\tilde{e}_j^{(i)}, \tilde{A}^k \tilde{e}_j^{(i)}) = (c_j^{(i)})^2 \tilde{a}_j^{(i)} = a_j^{(i)}$, it follows that $g \in \overline{\mathrm{O}}_N$ with $gAg^{-1} = \tilde{A}$. Assume that $\det(g) = -1$. Is there a $z \in C$ with $\det(z) = -1$ we have $gz \in \overline{\mathrm{SO}}_N$ since $\det(gz) = 1$ and $gzA(gz)^{-1} = gAg^{-1} = \tilde{A}$. Conversely, if there is a $h \in \overline{\mathrm{SO}}_N$ with $hgAg^{-1}h^{-1} = \tilde{A}$, then $hg \in C$ and $hg \notin \overline{\mathrm{SO}}_N$ since $\det(hg) = -1$, which gives us $C \nleq \overline{\mathrm{SO}}_N$. Therefore, the $\overline{\mathrm{O}}_N$-orbit of A splits into two $\overline{\mathrm{SO}}_N$ orbits if and only if $C \leq \overline{\mathrm{SO}}_N$, which is the case if C is isomorphic to a direct product of only symplectic groups.

For $i \in \mathbb{N}$ let $J_i(0) \in \mathfrak{gl}_i$ be the Jordan block to the eigenvalue zero of dimension i. For odd i we define

$$\tilde{J}_i(0) = \begin{pmatrix} 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & -1 & & & \\ & & & & 0 & \ddots & & \\ & & & & & \ddots & -1 & \\ & & & & & & 0 \end{pmatrix} \in \overline{\mathfrak{so}}_i$$

to be the Jordan block of dimension i where the latter half of the entries 1 are replaced by -1. Let $\lambda \vdash N$ be a partition where every even element has even multiplicity. Let μ be the partition consisting of the odd elements of λ and let ν be the partition consisting of the even elements of

λ with half the multiplcity. Consider $\iota : \overline{\mathrm{SO}}_{\mu_1} \times \overline{\mathrm{SO}}_{\mu_2} \times \cdots \to \overline{\mathrm{SO}}_{|\mu|}$ to be the natural embeding, so we can define the nilpotent block matrix $A \in \overline{\mathrm{SO}}_N$ with $\lambda_A = \lambda$ by

$$A = \begin{pmatrix} J_{\nu_1}(0) & & & & & & \\ & J_{\nu_2}(0) & & & & & \\ & & \ddots & & & & \\ & & & \iota\left(\tilde{J}_{\mu_1}(0), \tilde{J}_{\mu_2}(0), \cdots\right) & & & \\ & & & & \ddots & & \\ & & & & & -J_{\nu_2}(0) & \\ & & & & & & -J_{\nu_1}(0) \end{pmatrix}.$$

□

Example 3.1.9. For $\lambda = (4^2) \vdash 8$ let $J_4(0) \times -J_4(0)$. Let $w_0 \in W$ be the generator of the Weyl group of $\overline{\mathrm{SO}}_N$ in the normal subgroup of W as defined in section 2 of chapter 1. Then A' and $A'' = w_0 A' w_0$ are two representatives of the two distinct SO_N-orbits for λ:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & 0 & 0 & 0 & 0 & -1 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & -1 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}$$

The matrices A' and A'' with the subalgebras $\overline{\mathfrak{g}}(-2)$ (grey), $\overline{\mathfrak{g}}(0)$ (white), $\overline{\mathfrak{g}}(2)$ (light-grey)

Due to the work of Gerstenhaber [Ger58] and Hesselink [Hes76] the nilpotent orbits of $\overline{\mathfrak{so}}_N$ are partially ordered by the dominance order of the corresponding partition of N. The nilpotent orbits of $\overline{\mathfrak{so}}_N$ are open and the order of inclusion of their algebraic closures is equivalent to the dominance order on the partitions of N. As such, for nilpotent elements $A, B \in \overline{\mathfrak{so}}_N$ we have $\lambda_A \leq \lambda_B$ if and only if the orbit $\mathcal{O}(A)$ is contained in the algebraic closure of $\mathcal{O}(B)$, that is $\mathcal{O}(A) \subseteq \overline{\mathcal{O}(B)}$.

Lemma 3.1.10. *For a nilpotent element $B \in \overline{\mathfrak{so}}_N$ the orbit $\mathcal{O}(B)$ is open in $\overline{\mathfrak{so}}_N$ and its closure is*

$$\overline{\mathcal{O}(B)} = \bigcup_{g \in \overline{SO}_N} g^{-1} \overline{\mathfrak{u}}_2 g.$$

$\overline{\mathcal{O}(B)}$ is a union of nilpotent orbits and for a nilpotent element $A \in \overline{\mathfrak{so}}_N$ we have $\mathcal{O}(A) \subseteq \overline{\mathcal{O}(B)}$ if and only if $\mathcal{O}(A) \cap \overline{\mathfrak{u}}_2 \neq \emptyset$.

Proof. The adjoint map $\mathrm{ad}(\cdot)B : \mathfrak{p} \to \overline{\mathfrak{u}}_2$ is surjective. Therefore, $\mathrm{Ad}(\overline{P})B$ is a dense open subset of $\overline{\mathfrak{u}}_2$. Let ψ be the continuous map

$$\psi : \overline{\mathrm{SO}}_N \times \overline{\mathfrak{so}}_N \to \overline{\mathrm{SO}}_N \times \overline{\mathfrak{so}}_N : (g, X) \mapsto (g, g^{-1}Xg).$$

Then the set $C = \{(g, X) \in \overline{\mathrm{SO}}_N \times \overline{\mathfrak{so}}_N \mid X \in g^{-1}\overline{\mathfrak{u}}_2 g\}$ is closed because $\psi^{-1}(C) = \overline{\mathrm{SO}}_N \times \overline{\mathfrak{u}}_2$. Since $\overline{P}$ is closed, so is the quotient map $\pi : \overline{\mathrm{SO}}_N \times \overline{\mathfrak{so}}_N \to \overline{\mathrm{SO}}_N/\overline{P} \times \overline{\mathfrak{so}}_N$. Furthermore, $\overline{P}$ is a parabolic subgroup of $\overline{\mathrm{SO}}_N$, so $\overline{\mathrm{SO}}_N/\overline{P}$ is a complete variety and the projection $\rho : \overline{\mathrm{SO}}_N/\overline{P} \times \overline{\mathfrak{so}}_N \to \overline{\mathfrak{so}}_N$ is also closed. $\overline{\mathfrak{u}}_2$ is fixed by $\mathrm{Ad}(x)$ for $x \in \overline{P}$, so

$$\rho \circ \pi(C) = \bigcup_{g \in \overline{\mathrm{SO}}_N/\overline{P}} g^{-1}\overline{\mathfrak{u}}_2 g = \bigcup_{g \in \overline{\mathrm{SO}}_N} g^{-1}\overline{\mathfrak{u}}_2 g$$

is closed and $\mathcal{O}(B) = \bigcup_{g \in \overline{\mathrm{SO}}_N} \mathrm{Ad}(g)B$ is its dense open subset. □

3.2 Finite nilpotent orbits

It is possible that the fixed points with respect to the Frobenius automorphism F of a nilpotent $\overline{\mathrm{SO}}_N$-orbit splits into different SO_N-orbits. As Springer and Steinberg [SS70, 2.5 - 2.7, p. 172] have proven, one can determine these based of the information of the component group of the centralizer of an element from this orbit.

Lemma 3.2.1. *All nilpotent orbits of $\overline{\mathfrak{so}}_N$ are F-stable and therefore contain a fixed point. Let $A \in \overline{\mathfrak{so}}_N$ with $F(A) = A$ be a fixed point. Then the fixed points of the orbit $\mathcal{O}(A)^F$ split into SO_N-orbits, and their number is equal to that of the elements of the component group $C_{\overline{SO}_N}(A)/C_{\overline{SO}_N}(A)^\circ$.*

Proof. Let $A \in \overline{\mathfrak{so}}_N$. Since $\mathrm{rank}(F(A)^k) = \mathrm{rank}(A^k)$ for all $k \in \mathbb{N}$, we have $\lambda_{F(A)} = \lambda_A$. So $F(A)$ and A are conjugate in $\overline{\mathrm{O}}_N$. Assume that not all elements of λ_A are even, then $F(A)$ and A are also conjugate in $\overline{\mathrm{SO}}_N$. If all elements of λ_A are even, let $B \in \overline{\mathfrak{so}}_N$ be the product of Jordan blocks corresponding to λ_A as described in the proof of theorem 3.1.8. Then B is fixed by F and contained in the $\overline{\mathrm{O}}_N$-orbit of A. For $x, y \in \overline{\mathrm{O}}_N$ we have $F(x^{-1}Bx) = y^{-1}By$ if and only if $F(x)y^{-1} \in C_{\overline{\mathrm{O}}_N}(B)$. Since $C_{\overline{\mathrm{O}}_N}(B) \leq \overline{\mathrm{SO}}_N$, we have $\det(x) = \det(y)$, so $x^{-1}Bx$ and $F(x^{-1}Bx)$ are contained in the same $\overline{\mathrm{SO}}_N$-orbit. Therefore, the $\overline{\mathrm{SO}}_N$-orbit of A is F-stable.

Let $A \in \overline{\mathfrak{so}}_N$ and $F(A) = xAx^{-1}$ for some $x \in \overline{\mathrm{SO}}_N$. Then by Langs theorem there is a $y \in \overline{\mathrm{SO}}_N$ with $x = y^{-1}F(y)$ and $F(y^{-1}Ay) = (x^{-1}F(y))^{-1}Ax^{-1}F(y) = y^{-1}Ay$ and $y^{-1}Ay$ is fixed by F, so every orbit contains a fixed point.

Let now A be a fixed point and $x \in \overline{\mathrm{SO}}_N$. Then we have

$$F(xAx^{-1}) = F(x)AF(x)^{-1} = x(x^{-1}F(x))A(x^{-1}F(x))^{-1}x^{-1},$$

and xAx^{-1} is also a fixed point if and only if $x^{-1}F(x) \in C_{\overline{\mathrm{SO}}_N}(A)$. Conversely, for any $c \in C_{\overline{\mathrm{SO}}_N}(A)$ there is a $x \in \overline{\mathrm{SO}}_N$ with $c = x^{-1}F(x)$ and xAx^{-1} is a fixed point. Therefore, the Lang map defines a bijection between the fixed points $O(A)^F$ and the centralizer $C_{\overline{\mathrm{SO}}_N}(A)$. For $x, y \in \overline{\mathrm{SO}}_N$ with $x^{-1}F(x), y^{-1}F(y) \in C_{\overline{\mathrm{SO}}_N}(A)$ the two fixed points xAx^{-1} and yAy^{-1} are conjugate in SO_N if there is a $z \in C_{\overline{\mathrm{SO}}_N}(A)$ with $F(xzy^{-1}) = xzy^{-1}$, which is the case if $z^{-1}(x^{-1}F(x))F(z) = y^{-1}F(y)$. So the F-conjugacy classes of $C_{\overline{\mathrm{SO}}_N}(A)$ correspond to the SO_N orbits $O(A)^F$.
Let now $c, d \in C_{\overline{\mathrm{SO}}_N}(A)$ with $cd^{-1} \in C_{\overline{\mathrm{SO}}_N}(A)^\circ$. There is a $x \in \overline{\mathrm{SO}}_N$ with $d = x^{-1}F(x)$, and we have $F(x)C_{\overline{\mathrm{SO}}_N}(A)^\circ F(x)^{-1} = xdC_{\overline{\mathrm{SO}}_N}(A)^\circ d^{-1}x^{-1} = xC_{\overline{\mathrm{SO}}_N}(A)^\circ x^{-1}$ because $C_{\overline{\mathrm{SO}}_N}(A)^\circ \trianglelefteq C_{\overline{\mathrm{SO}}_N}(A)$. Since $C_{\overline{\mathrm{SO}}_N}(A)^\circ$ is connected and F-stable, the same is the case for $xC_{\overline{\mathrm{SO}}_N}(A)^\circ x^{-1}$. Now we have $xaF(x)^{-1} = xab^{-1}x^{-1} \in xC_{\overline{\mathrm{SO}}_N}(A)^\circ x^{-1}$, so there is a $y \in xC_{\overline{\mathrm{SO}}_N}(A)^\circ x^{-1}$ with $xaF(x)^{-1} = y^{-1}F(y)$. It follows that

$$a = x^{-1}y^{-1}F(y)F(x) = (x^{-1}yx)^{-1}x^{-1}F(x)F(x^{-1}yx) = (x^{-1}yx)^{-1}bF(x^{-1}yx)$$

with $x^{-1}yx \in C_{\overline{\mathrm{SO}}_N}(A)$. Therefore, there is a bijection between the F-conjugacy classes of $C_{\overline{\mathrm{SO}}_N}(A)$ and $C_{\overline{\mathrm{SO}}_N}(A)/C_{\overline{\mathrm{SO}}_N}(A)^\circ$.[2]
Due to corollary 3.1.7 we have $C_{\overline{\mathrm{SO}}_N}(A)/C_{\overline{\mathrm{SO}}_N}(A)^\circ \cong C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A)/C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A)^\circ$. This component group is a direct product of cyclic groups of order two and therefore abelian, so the F-conjugacy classes of $C_{\overline{\mathrm{SO}}_N}(A)/C_{\overline{\mathrm{SO}}_N}(A)^\circ$ are precisely its elements. □

Lemma 3.2.2. *Let $A \in \mathfrak{so}_N$ be conjugate in $\overline{SO}_N$ with its respective basis of $\mathbb{F}_q{}^N$ as defined in theorem 3.1.1. Let d be the number of odd $1 \leq i \leq m$ with $l(i) > 0$ and let Δ be the map defined by:*

$$\begin{aligned} \Delta : O(A)^F &\to (\mathbb{F}_q{}^\times/(\mathbb{F}_q{}^\times)^2)^d \\ A &\mapsto \left(\prod_{1\leq j\leq l(1)} a_j^{(1)} \mod (\mathbb{F}_q{}^*)^2, \prod_{1\leq j\leq l(3)} a_j^{(3)} \mod (\mathbb{F}_q{}^*)^2, \dots \right) \end{aligned}$$

[2]cf. [SS70, 2.6, p. 173]

If two elements $A, B \in \mathfrak{so}_N$ are conjugate in $\overline{SO}_N$, they are also conjugate in SO_N if and only if $\Delta(A) = \Delta(B)$.

Proof. Let $A, \tilde{A} \in \mathfrak{so}_N$ be conjugate in $\overline{\mathrm{SO}}_N$ with their respective basis of ${\mathbb{F}_q}^N$ as defined in theorem 3.1.1 and let $g \in \overline{\mathrm{GL}}_N$ as in the proof of theorem 3.1.8 defined by

$$gA^k e_j^{(i)} = c_j^{(i)} \tilde{A}^k \tilde{e}_j^{(i)}, \quad gA^k f_j^{(i)} = \tilde{A}^k \tilde{f}_j^{(i)}, \quad gA^k g_j^{(i)} = \tilde{A}^k \tilde{g}_j^{(i)}$$

for $1 \leq i \leq m$, $1 \leq j \leq l(i)$, $0 \leq k \leq i-1$, with $c_j^{(i)} \in {\mathbb{F}_q}^\times$ such that $(c_j^{(i)})^2 \tilde{a}_j^{(i)} = a_j^{(i)}$.
For $1 \leq i \leq m$ odd, $1 \leq j \leq l(i)$ and $0 \leq k \leq i-1$ we have

$$g^{-1} F(g) A^k e_j^{(i)} = g^{-1} (c_j^{(i)})^q \tilde{A}^k \tilde{e}_j^{(i)} = (c_j^{(i)})^{q-1} A^k e_j^{(i)}$$

with $(c_j^{(i)})^2 \tilde{a}_j^{(i)} = a_j^{(i)}$. By construction, we have $g^{-1}F(g) \in C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A)$ and for any odd $1 \leq i \leq m$ it follows that

$$\det((g^{-1}F(g))_i) = \prod_{j=1}^{l(i)} ((c_j^{(i)})^{q-1})^i = \left(\prod_{j=1}^{l(i)} c_j^{(i)} \right)^{q-1} = \left(\frac{\prod_{j=1}^{l(i)} a_j^{(i)}}{\prod_{j=1}^{l(i)} \tilde{a}_j^{(i)}} \right)^{\frac{q-1}{2}}.$$

Since $x^{\frac{q-1}{2}} \in \{-1, 1\}$ for $x \in \mathbb{F}_q$, it follows that $\det((g^{-1}F(g))_i) = 1$ if and only if

$$\prod_{j=1}^{l(i)} a_j^{(i)} \equiv \prod_{j=1}^{l(i)} \tilde{a}_j^{(i)} \mod ({\mathbb{F}_q}^*)^2.$$

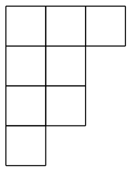
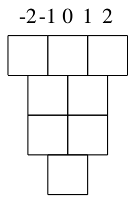
By corollary 3.1.7 we have $g^{-1}F(g) \in C_{\overline{\mathrm{SO}}_N}(\overline{T}_A, A)^\circ$ if and only if $\det((g^{-1}F(g))_i) = 1$ for all odd $1 \leq i \leq m$, so the claim follows by theorem 3.2.1. □

3.3 Construction of core patterns for nilpotent orbits

Now we will define verge patterns for certain Young tableaux called verge tableaux. These together with a matrix derived from the matrix $S(d)$ as defined in lemma 2.4.12, determining minor and supplementary conditions, give rise to a core pattern, and will be called core tableau. These core pattern represent the different nilpotent orbits of $\mathfrak{so}_N$, and unless the base field has three elements we will be able to construct a core tableau for every nilpotent orbit.

Definition 3.3.1. Let $\lambda \vdash N$ be a partition of N. Let D_λ be the standard Young diagram and let Z_λ be the Young diagram D_λ with centered rows. We consider the middle column of Z_λ the zero column, and we label all colums of the left or the right of the zero column in increments of half columns with consecutive negative or positive integers respectively. Furthermore, $n_z \in \mathbb{N}_0$ will denote the length of the z column for $z \in \mathbb{Z}$, where due to the symmetry of the Young diagram we have $n_z = n_{-z}$, and $\tilde{n}_z = \sum_{i<z} n_z$ is the sum of all n_z left of the z column.

For the partition $\lambda = (3, 2, 2, 1) \vdash 8$ we have the following Young diagrams D_λ and the centered Young diagram Z_λ with rows ranging from -2 to 2.

$D_\lambda =$  $Z_\lambda =$ 

Lemma 3.3.2. *Let $\lambda \vdash N$ be a partition with $\lambda = (1^{l_1}, 2^{l_2}, \dots)$ for $l_j \in \mathbb{N}_0$ and $j \in \mathbb{N}$, where l_j is the number of elements of length j in λ. Let n_i for $z \in \mathbb{Z}$ be the length of the z column of the Young diagram Z_λ as defined in 3.3.1. Then we have $l_j = n_{j-1} - n_{j+1}$ and it follows that*

$$\lambda' = (n_0 + n_1, n_1 + n_2, n_2 + n_3 \dots),$$

where λ' is the conjugate partition of λ.

Proof. For $j \in \mathbb{N}$ every row of length j of Z_λ has elements in the columns ranging from $-j+1$ to $j-1$ incrementing by 2. So every row of odd length has all elements in only even columns and every row of even length has all elements in only odd columns. So for $i \in \mathbb{N}_0$ even n_i is the number of odd elements of λ greater or equal to $i+1$, and for $i \in \mathbb{N}$ odd n_i is the number of even elements of λ greater or equal to $i+1$. So for $j \in N$ the number of elements of λ of length j is $l_j = n_{j-1} - n_{j+1}$ and for $k \in \mathbb{N}$ we have $\sum_{j \geq k} l_j = n_{k-1} + n_k$, which proves the last point. □

Definition 3.3.3. We call a Young tableau $\mathbf{T}$ of the Young diagram Z_λ a verge tableau of λ if the following conditions are met.

(i) For $i < j$ all entries in the i column are less then the entries in the j column, unless all elements of λ are even, in which case $n+1$ can be contained in the -1 column and n in the 1 column.

(ii) For $i \in \mathbb{N}$ and even $0 \neq j \in \mathbb{Z}$ if a is the entry contained in i-th row and column j with $j \neq 0$ then $\overline{a}$ is the entry contained in i-th row and column $-j$.

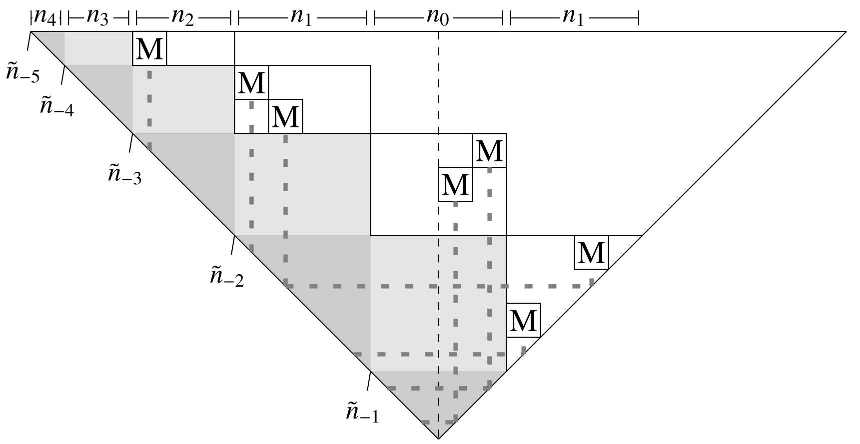
(iii) For $i \in \mathbb{N}$ and odd $j \in \mathbb{Z}$, where the i-th row has odd length, if a is the entry contained in i-th row and column k then $\overline{a}$ is the entry contained in j-th row and column $-k$.

(iv) The entry $a > n$ in the 0 column is positioned above $\overline{a}$ in the 0 column.

If all entries of λ are even and $n+1$ is contained in the -1 column, while n is contained in the 1 column, we call $\mathbf{T}$ a secondary even verge tableau, corresponding to the distinction of the two different orbits for the partition λ as defined in theorem 3.1.8.
We define the the verge pattern $A_{\mathbf{T}} \in \mathbf{v}$ with respect to the verge tableau $\mathbf{T}$ to be the pattern with $(A_{\mathbf{T}})_{ij} = 1$ for $1 < i < j \leq N$ and $i + j < N + 1$ if i and j are contained in the same row and consecutive columns in the tableau $\mathbf{T}$ and $(A_{\mathbf{T}})_{ij} = 0$ otherwise.

1	4	14	21	24
2	7	17	22	
3	8	18	23	
5	13	20		
6	12	19		
9	15			
10	16			
11				

Verge tableau $\mathbf{T}$ for $\lambda = (5, 4^2, 3^2, 2^2, 1)$



Verge pattern $A_{\mathbf{T}}$ for the verge tableau $\mathbf{T}$ together with the postions $\tilde{n}_z \in [N]$ for $z \in \mathbb{Z}$ and spans $n_z \in \mathbb{N}$ for $z \in \mathbb{N}_0$ of blocks as defined in 3.3.1

Lemma 3.3.4. *For a tableau* $\mathbf{T}$ *let* $\boldsymbol{S} \in M_{n_2}(\mathbb{F}_q)$ *be a symmetric matrix fulfilling the following conditions:*

(i) For $1 \leq a \leq n_2$ *we have* $\boldsymbol{S}_{aa} = 0$ *if the entry in in the a-th row and* 0 *column is less than or equal to n.*

(ii) For $1 \leq a, b \leq n_2$ *let i be the entry in in the a-th row and* 0 *column and j be the entry in in the b-th row and* 0 *column. Then we have* $\boldsymbol{S}_{ab} = \boldsymbol{S}_{ba} = 1$ *if* $i = \overline{j}$.

(iii) For $1 \leq a, b \leq n_2$ *let i be the entry in in the a-th row and* 0 *column and j be the entry in in the a-th row and* 0 *column. Then we have* $\boldsymbol{S}_{ab} = \boldsymbol{S}_{ba} = 0$ *if* $i + j < N + 1$.

Let $A_{\boldsymbol{T}} \in \boldsymbol{v}$ *be the verge pattern for* $\boldsymbol{T}$ *and let* $d \in D_A$ *be given by the matrix* $S(d) \in M_n(\mathbb{F}_q)$ *as defined in lemma 2.4.12 with*

$$S_{ij}(d) = \begin{cases} S_{ab} & \textit{if } i,j \textit{ are contained in the } -2 \textit{ column of } \boldsymbol{T}\textit{, where } i,j \textit{ are contained in the} \\ & a\textit{-th and } b\textit{-th row respectively,} \\ 1 & \textit{if } k+l = N+1\textit{, where } k \textit{ and } l \textit{ are the entries in the same row and in the} \\ & \textit{next column to the right of } i \textit{ and } j \textit{ respectively,} \\ 0 & \textit{otherwise.} \end{cases}$$

The core pattern $C_{\boldsymbol{T}} \in \boldsymbol{v}$ with $C_{\boldsymbol{T}} = d.A_{\boldsymbol{T}}$ is then defined by its core conditions with possible non-zero entries:

(i) $(C_{\boldsymbol{T}})_{jk} = 1$ *if the entries* $1 < j,k \le N$ *with* $j+k \le N$ *are contained in the same row and consecutive columns of* $\mathbf{T}$. *These are the major positions of* $C_{\boldsymbol{T}}$.

(ii) $(C_{\boldsymbol{T}})_{jk} = \frac{1}{2}\boldsymbol{S}_{aa}$ *if for* $1 < j,k \le n$ *the entries* $j, \overline{k}$ *are contained in the* a*-th row and* j *is contained in the* -2 *column while* $\overline{k}$ *is contained in the* 0 *column of* $\mathbf{T}$. *These are the minor positions of* $C_{\boldsymbol{T}}$.

(iii) $(C_{\boldsymbol{T}})_{jk} = \boldsymbol{S}_{ab}$ *if for* $1 < j,k \le n$ *the entry* j *is contained in the* a*-th row and* -2 *column while* $\overline{k}$ *is contained in the* b*-th row and* 0 *column, while the entry in the* k*-th row and* 0 *column is less than the entry in the* b*-th row and* 0 *column. These are the supplementary positions of* $C_{\boldsymbol{T}}$.

All other entries of $C_{\boldsymbol{T}}$, including the entries corresponding to the remaining minor and supplementary conditions, are set to be zero.

Proof. Let $\mathbf{T}$ be a verge tableau and $A_{\mathbf{T}} \in \mathbf{v}$ the corresponding verge pattern as defined in 3.3.3. Let $d \in D_A$ such that for $(k,l) \in \mathcal{D}_A$ with $1 \le i,j \le k$ such that $(i,\overline{k}),(j,l) \in \operatorname{supp}(A_{\mathbf{T}})$ we have

$$d_{kl} = \begin{cases} \mathbf{S}_{ab} & \text{if } \tilde{n}_{-3} < i,j \le \tilde{n}_{-2}, i \ne j \text{ and } i,j \text{ are the entries in the} \\ & a\text{-th and } b\text{-th row of } \mathbf{T} \text{ respectively,} \\ \frac{1}{2}\mathbf{S}_{aa} & \text{if } \tilde{n}_{-3} < i = j \le \tilde{n}_{-2} \text{ and } i \text{ is the entry in the } a\text{-th row of } \mathbf{T}, \\ 0 & \text{otherwise.} \end{cases}$$

Let now $C_{\mathbf{T}} = d.A_{\mathbf{T}}$. Then for $1 \le j < k \le N$ with $j+k < N+1$ we have $(C_{\mathbf{T}})_{jk} = (A_{\mathbf{T}})_{jk}$ if $(A_{\mathbf{T}})_{jk} \ne 0$ and $(C_{\mathbf{T}})_{jk} = (A_{\mathbf{T}})_{jl}d_{kl} = d_{kl}$ if $(j,l) \in \operatorname{supp}(A)_{\mathbf{T}}$ and $(k,l) \in \mathcal{D}_A$. Then there

is a $1 \le i \le k$ with $(i, \overline{k}) \in \operatorname{supp}(A_{\mathbf{T}})$ and we have $(C_{\mathbf{T}})_{jk} = \frac{1}{2}\mathbf{S}_{aa}$ if $i = j$ as well as $j, \overline{k}$ are contained in the a-th row and j is contained in the -2 and 0 column of $\mathbf{T}$ respectively, whereas $(C_{\mathbf{T}})_{jk} = \mathbf{S}_{ab}$ if $i \neq j$ if i, j are contained in the -2 column and a-th, b-th row, while $\overline{k}, l$ are contained in the 0 column and a-th, b-th row, respectively.
Now the matrix $S(d) \in M_n(\mathbb{F}_q)$ by lemma 2.4.12 is determined by the subsequent conditions. For $1 \le i, j \le n$ we have $S_{ij}(d) = 0$ unless there are $i < k < \overline{i}$ and $j < l < \overline{j}$ with $(i, k), (j, l) \in \operatorname{supp}(A)$ as well as $k + l \ge N + 1$ and $k < \overline{j}$, $l < \overline{i}$. If this is the case and $k + l = N + 1$, we have $S_{ij}(d) = 1$. If $k > n$, that is $2k > N + 1$, we have $S_{ii}(d) = 2d_{k\overline{k}} = \mathbf{S}_{aa}$ if i is the entry in the -2 column and a-th row of $\mathbf{T}$, while $S_{ii}(d) = 2d_{k\overline{k}} = 0$ otherwise. If $k + l > N + 1$ and $i \neq j$, we also have $k \neq l$. We assume due to the symmetry of both $S(d)$ and $\mathbf{S}$ without loss of generality that $k < l$. Then we have $S_{ij}(d) = 2d_{k\overline{l}} = \mathbf{S}_{ab}$ if i, j are entries in the -2 column and a-th, b-th row of $\mathbf{T}$ and $S_{ij}(d) = 2d_{k\overline{l}} = 0$ otherwise.
This shows that this construction of both the core pattern $C_{\mathbf{T}} = d.A_{\mathbf{T}}$ as well as its matrix $S(d) \in M_n(\mathbb{F}_q)$ arises consistently out of the given matrix $\mathbf{S} \in M_{n_{-2}}$.

□

$$\begin{pmatrix} 2m_1 & s_1 & s_2 \\ s_1 & 2m_2 & 1 \\ s_2 & 1 & 0 \end{pmatrix}$$

The matrix $\mathbf{S} \in M_3(\mathbb{F}_q)$ for the verge tableau $\mathbf{T}$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2m_1 & s_1 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s_1 & 2m_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The matrix $S(d) \in M_{12}(\mathbb{F}_q)$ for $d \in D_A$ with $C_{\mathbf{T}} = d.A_{\mathbf{T}}$

If $\lambda \vdash N$ is a partition comprising only even elements, then $\mathbf{S}$ is a 0×0-matrix and the core pattern $C_{\mathbf{T}} \in \mathbf{v}$ is solely determined by the tableau $\mathbf{T}$ with $C_{\mathbf{T}} = A_{\mathbf{T}}$.
A verge tableau $\mathbf{T}$ together with a matrix $\mathbf{S} \in M_{n_{-2}}(\mathbb{F}_q)$ as defined in lemma 3.3.4 will be called a core tableau $(\mathbf{T}, \mathbf{S})$ if $\mathbf{S}$ affords the conditions in the following theorem. Every core tableau then corresponds to a core pattern in $\mathbf{v}$ and for every nilpotent orbit of $\mathfrak{so}_N$ we have a

core tableau such that the corresponding core pattern is a representative of the nilpotent orbit unless SO_N is defined over the field $\mathbb{F}_q$ with $q = 3$.

Theorem 3.3.5. *Let $\lambda \vdash N$ be a partition of N where ever even element has even multiplicity and let $(\mathbf{T}, \boldsymbol{S})$ be a core tableau. Let $C_{\boldsymbol{T}} \in \boldsymbol{v}$ be the core pattern defined by $\mathbf{T}$ and $\boldsymbol{S}$ and let $X = C_{\boldsymbol{T}} - C_{\boldsymbol{T}}^{\dagger} \in \mathfrak{so}_N$ be the corresponding matrix in the orthogonal algebra. For $i \in \mathbb{N}$ with $n_{2i} > 0$ we define d_i by*

$$d_i = (-1)^{(n_{2i}-n_{2i+2})} \frac{\det(\boldsymbol{S}|_{\{1,\dots n_{2i}\}\times\{1,\dots n_{2i}\}})}{\det(\boldsymbol{S}|_{\{1,\dots n_{2i+2}\}\times\{1,\dots n_{2i+2}\}})}$$

where $\boldsymbol{S}|_{\{1,\dots n_{2i}\}\times\{1,\dots n_{2i}\}}$ is the restriction of $\boldsymbol{S}$ to $\{1,\dots n_{2i}\} \times \{1,\dots n_{2i}\}$ and where we consider $\det(\boldsymbol{S}|_{\{1,\dots n_{2i+2}\}\times\{1,\dots n_{2i+2}\}} = 1$ if $n_{2i+2} = 0$. Additionally, we define $d_0 = (-1)^{(n_{2i}-n_2)} \det(J_N)/\det(\boldsymbol{S})$. If $d_i \neq 0$ for all $i \in \mathbb{N}$ with $n_{2i} \neq 0$, then X is a representative of the nilpotent $\overline{SO_N}$-orbit corresponding to λ as defined in theorem 3.1.8. Furthermore, X is a representative of the nilpotent SO_N-orbit corresponding to

$$\left(d_0 \mod (\mathbb{F}_q{}^*)^2, d_1 \mod (\mathbb{F}_q{}^*)^2, \dots\right) \in \mathbb{F}_q{}^\times/(\mathbb{F}_q{}^\times)^2 \times \mathbb{F}_q{}^\times/(\mathbb{F}_q{}^\times)^2 \times \dots$$

as defined in lemma 3.2.2. Furthermore, we have $\prod_{i\in\mathbb{N}_0} d_i \equiv (-1)^{n_0/2} \mod (\mathbb{F}_q{}^*)^2$.
The coroot of the one-dimensional torus $\tau_X : \overline{\mathbb{F}}_q^* \to T$ for X as defined in lemma 3.1.2 for $1 \leq i \leq N$ is determined by $(\tau_X(c))_{ii} = c^z$ where $z \in \mathbb{Z}$ is the column in which i is contained in the tableau $\boldsymbol{T}$.

Proof. Let $(\mathbf{T}, \mathbf{S})$ be a core tableau for a partition $\lambda \vdash N$ that contains odd elements. Let $A, C \in \mathbf{v}$ be the corresponding verge and core tableaux for $(\mathbf{T}, \mathbf{S})$. By definition 3.3.4 we have $(i, j) \in \mathrm{supp}(A)$ if and only if i and j are contained in consecutive columns of $\mathbf{T}$. Furthermore, (i, j) can only be a minor or supplementary position if j is contained in the 0 column of $\mathbf{T}$. So if j is not contained in the 0 row, we have $(i, j) \in \mathrm{supp}(A)$ if and only if i and j are contained in the same row and consecutive columns. Then (i, j) is a main position and therefore $X_{ij} = 1$. Are i and j contained in the same row and consecutive columns of $\mathbf{T}$ with $i + j > N + 1$ and i is not contained in the 0 column of $\mathbf{T}$, then by definition $\bar{j}$ and $\bar{i}$ are contained in the same row and consecutive columns of $\mathbf{T}$ as well and $(\bar{j}, \bar{i})$ is a main position of A. Then we have $X_{ij} = -A_{\bar{j}\bar{i}} = -1$. So if i and j are not contained in the 0 column, we have $X_{ij} \neq 0$ if and only if i and j are contained in the same row and consecutive columns of $\mathbf{T}$. It then follows

$$Xe_j = e_i \text{ if } i + j < N + 1 \quad \text{and} \quad Xe_j = -e_i \text{ if } i + j > N + 1$$

For $r, s \in \mathbb{N}$ let i be the entry in the -2 column and r row of $\mathbf{T}$ and j be the entry in the -2 column and s row of $\mathbf{T}$. By definition $\bar{j}$ is the entry in the 2 column and s row and have

$$(X^2 e_{\bar{j}})_i = -(XJ_N X^t J_N e_{\bar{j}})_i = -(XJ_N X^t e_j)_i = -\sum_{k=1}^{N} X_{ik} X_{j\bar{k}} = -\sum_{k=n-n_0/2}^{n+n_0/2} A_{ik} A_{j\bar{k}}.$$

The last step follows since $X_{ik}, X_{j\bar{k}} \neq 0$ requires k to be contained in the 0 column of $\mathbf{T}$, which shows that $i + k \leq n$ and $\bar{j} + k \leq n$. Let l be the entry in the 0 column and r row of $\mathbf{T}$ and m be the entry in the 0 column and s row of $\mathbf{T}$. Since $(X^2 e_{\bar{j}})_i = (X^2 e_{\bar{i}})_j$, the rows can be interchanged and we assume $l \geq m$ without loss of generality. We consider the following cases:

(i) $l > m > n$: The only non zero position in the i-th row of X right of $(i, \bar{l})$ is (i, l) and the only non zero position in the j-th row of X right of $(j, \overline{m})$ is (i, m). Since $\overline{m} > \bar{l}$, we have $(X^2 e_{\bar{j}})_i = A_{i,\overline{m}} A_{j,m} = A_{i,\overline{m}} = \mathbf{S}_{rs}$ because $(i, \overline{m})$ is a supplementary position of A.

(ii) $l = m > n$: We have $i = j$ and the only non zero position in the i-th row of X right of $(i, \bar{l})$ is (i, l) and therefore $(X^2 e_{\bar{j}})_i = A_{i,\bar{l}} A_{i,l} + A_{i,l} A_{i,\bar{l}} = 2A_{i,\bar{l}} = \mathbf{S}_{rr}$ since $(i, \bar{l})$ is a minor position of A.

(iii) $l > n \geq m$, $\bar{l} > m$: The only non zero position in the i-th row of X right of $(i, \bar{l})$ is (i, l) and the j-th row has no non zero position right of (j, m). Since $\bar{l} > m$, we have $(X^2 e_{\bar{j}})_i = A_{i,l} A_{j,\bar{l}} = A_{j,\bar{l}} = \mathbf{S}_{rs}$ because $(j, \bar{l})$ is a supplementary position of A.

(iv) $l > n \geq m$, $\bar{l} = m$: The only non zero position in the i-th row of X right of $(i, \bar{l})$ is (i, l) and the j-th row has no non zero position right of $(j, \bar{l})$. Therefore, we have $(X^2 e_{\bar{j}})_i = A_{i,l} A_{j,m} = 1 = \mathbf{S}_{rs}$ since (j, m) is a left major position of A.

(v) $l > n \geq m > \bar{l}$: The only non zero position in the i-th row of X right of $(i, \bar{l})$ is (i, l) and the j-th row has no non zero position right of (j, m). Since $m > \bar{l}$, we have $(X^2 e_{\bar{j}})_i = 0 = \mathbf{S}_{rs}$.

(vi) $n > l \geq m$: The i-th row has no non zero position right of (i, l) and the j-th row has no non zero position right of (j, m). Therefore, we have $(X^2 e_{\bar{j}})_i = 0 = \mathbf{S}_{rs}$.

So is i the entry in the r-th row and -2 column and $\bar{j}$ the entry in the s-th row and 2 column of $\mathbf{T}$ we have $(X^2 e_{\bar{j}})_i = -\mathbf{S}_{rs}$.

For $z \in \mathbb{Z}$ we define $V_z = \{e_{z_1}, e_{z_2}, \dots\}$, where z_i is the entry in the i-th row and z column for $i \in \mathbb{N}$, to be the subspace of $\mathbb{F}_q{}^N$ representing the z column of $\mathbf{T}$ with $\mathbb{F}_q{}^N = \bigoplus_{z \in \mathbb{Z}} V_z$ and $V_z \cong V_{-z}$. Then X acts on these subspaces in the following ways:

(i) For $z > 2$ and $e_i \in V_z$ with $1 \leq i \leq n_z$ we have $Xe_i = -e_i \in V_{z-2}$.

(ii) For $z = 2$ and $e_i \in V_2$ with $1 \leq i \leq n_z$ we have $Xe_i \in V_0$ and $X^2 e_i = -\mathbf{S}e_i \in V_{-2}$.

(iii) For $z = 1$ and $e_i \in V_1$ with $1 \leq i \leq n_z$ we have $Xe_i = e_i \in V_{-1}$ if the sum of the entries in the i-th row and -1 and 1 column is less than or equal to n, whereas we have $Xe_i = -e_i \in V_{-1}$ otherwise.

(iv) For $z > 2$ and $e_i \in V_z$ with $1 \leq i \leq n_{z-2}$ we have $Xe_i = e_i \in V_{z-2}$ and for $n_{z-2} < i \leq n_z$ we have $Xe_i = 0$.

So for odd $z \geq 1$ and $e_i \in V_z$ with $1 \leq i \leq n_z$ we have

$$X^z e_i = (-1)^{\frac{z-1}{2}} \delta e_i \in V_{-z} \quad \text{and} \quad X^{z+1} e_i = 0$$

with $\delta = 1$ if the sum of the entries in the i-th row and -1 and 1 column is less than or equal to n and $\delta = -1$ otherwise. For even $z \geq 2$ and $e_i \in V_z$ with $1 \leq i \leq n_z$ we have

$$X^z e_i = (-1)^{\frac{z}{2}} \mathbf{S}|_{\{1,\dots n_z\} \times \{1,\dots n_z\}} e_i \in V_{-z} \quad \text{and} \quad X^{z+1} e_i = 0.$$

Since $\det(\mathbf{S}|_{\{1,\dots n_i\} \times \{1,\dots n_i\}}) \neq 0$ was a precondition for $\mathbf{S}$, we have $X^k V_z = V_{-z}$, which gives us $\dim(X^k V_z) = \dim(V_z)$ for $k \leq z$. For $k > z$ we have $X^{2(k-z)} V_{2k-z} \leq V_z$ and therefore $V_{z-2k} = X^{2k-z} V_{2k-z} \leq X^k V_z$ as well as $\dim(X^k V_z) = \dim(V_{2k-z})$. So for $k \in \mathbb{N}$ we have

$$\operatorname{rank}(X^k) = \sum_{z \in \mathbb{Z}} \dim(X^k V_z) = \sum_{z \geq k} \dim(V_z) + \sum_{z < k} \dim(V_{2k-z}) = \sum_{z \geq k} n_z + \sum_{z < k} n_{2k-z} = n_k + 2 \sum_{z > k} n_z,$$

which shows that $\operatorname{rank}(X^k) - \operatorname{rank}(X^{k+1}) = n_k + n_{k+1}$. By theorem 3.1.8 and lemma 3.3.2 it follows that the orbit $O(X)$ corresponds to the partition λ.

For $z \in \mathbb{Z}$, $1 \leq i \leq n_z$ let a be the entry in the i-th row and z column of Z_λ and $\overline{a}$ the entry in the j-th row and $-z$ column. For $e_i \in V_z$ we then have $J_N e_i = e_j \in V_{-z}$ and by dimension $J_N V_z = V_{-z}$. Moreover, if z is even we have $J_N e_i = e_i \in V_{-z}$ since a and $\overline{a}$ are both contained in the same row of Z_λ. So for $v \in V_z$ and $w \in \mathbb{F}_q^N$ if $b(w, v) = w^t J_N v \neq 0$ we must have $w \in V_{-z}$ and therefore $\operatorname{rad}(V_z) = \bigoplus_{k \in \mathbb{Z}, k \neq -z} V_k$.

By theorem 3.1.8 and lemma 3.3.2, for even $z \geq 0$ there are basis elements $e_j^{(z+1)} \in \overline{\mathbb{F}}_q^N$ with $1 \leq j \leq n_z - n_{z+2}$ as defined in theorem 3.1.1 with $b(e_j^{(z+1)}, X^z e_j^{(z+1)}) = a_j^{(z+1)}$ for some

$a_j^{(z+1)} \in \mathbb{F}_q$. Since $X^{z+1}e_j^{(z+1)} = 0$, we have $e_j^{(z+1)} \in \bigoplus_{k\leq z} V_k$. Let $\tilde{e}_j^{(z+1)} \in V_z$ such that $\hat{e}_j = e_j^{(z+1)} - \tilde{e}_j^{(z+1)} \in \bigoplus_{k<z} V_k$. We then have $X^z\hat{e}_j \in \bigoplus_{k<-z} V_k$ and it follows that

$$b\left(e_j^{(z+1)}, X^z e_j^{(z+1)}\right) = b\left(\tilde{e}_j^{(z+1)}, X^z\tilde{e}_j^{(z+1)}\right) + 2b\left(\tilde{e}_j^{(z+1)}, X^z\hat{e}_j\right) + b\left(\hat{e}_j, X^z\hat{e}_j\right) = b\left(\tilde{e}_j^{(z+1)}, X^z\tilde{e}_j^{(z+1)}\right),$$

which concludes $b\left(\tilde{e}_j^{(z+1)}, X^z\tilde{e}_j^{(z+1)}\right) = a_j^{(z+1)}$. Therefore the $\tilde{e}_j^{(z+1)} \in V_z$ are basis elements as defined in theorem 3.1.1.

For even $z \geq 2$, $k \geq 0$ and $1 \leq j \leq n_{z+2k} - n_{z+2k+2}$ we have $X^k\tilde{e}_j^{(z+2k+1)} \in V_z$, and these elements are a basis of V_z. Let $P \in \mathrm{GL}(V_z)$ be the transformation matrix with $Pe_i = X^k\tilde{e}_{i-n_{z+2k+2}}^{(z+2k+1)}$for $1 \leq i \leq n_z$ with $k > 0$ such that $n_{z+2k+2} < i \leq n_{z+2k}$. For $e_i, e_j \in V_z$ with $1 \leq i, j \leq n_z$ we have $J_N e_i = e_i \in V_{-z}$ and therefore

$$b(e_i, X^z e_j) = (J_N e_i)^t X^z e_j = (-1)^{\frac{z}{2}} e_i^t \mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}} e_j = (-1)^{\frac{z}{2}} \left(\mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}}\right)_{ij}.$$

By theorem 3.1.1, we have $b(P_z e_i, X^z P_z e_j) = 0$ for all $1 \leq i, j \leq n_z$ with $i \neq j$ and

$$b(P_z e_i, X^z P_z e_i) = b\left(X^k\tilde{e}_{i-n_{z+2k+2}}^{(z+2k+1)}, X^z X^k\tilde{e}_{i-n_{z+2k+2}}^{(z+2k+1)}\right) = (-1)^k b\left(\tilde{e}_{i-n_{z+2k+2}}^{(z+2k+1)}, X^{z+2k}\tilde{e}_{i-n_{z+2k+2}}^{(z+2k+1)}\right) = (-1)^k a_{i-n_{z+2k+2}}^{(z+2k+1)},$$

while $k > 0$ such that $n_{z+2k+2} < i \leq n_{z+2k}$ and therefore

$$(-1)^{\frac{z}{2}} \left(P_z^t \mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}} P_z\right)_{ij} = \delta_{ij}(-1)^k a_{i-n_{z+2k+2}}^{(z+2k+1)}$$

where δ_{ij} is the Kronecker delta for i and j. Let $D_z \in \mathrm{GL}(\mathbb{F}_q{}^{n_z})$ now be the diagonal matrix with $(D_z)_{ii} = (-1)^k a_{i-n_{z+2k+2}}^{(z+2k+1)}$ for $1 \leq i \leq n_z$ with $k > 0$ such that $n_{z+2k+2} < i \leq n_{z+2k}$, so we have $D_z = (-1)^{\frac{z}{2}n_z} \left(P_z^t \mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}} P_z\right)$.

The same statement also holds for $z+2$ and we have $D_{z+2} = (-1)^{n_{z+2}} D_z|_{\{1,\dots n_{z+2}\}\times\{1,\dots n_{z+2}\}}$, so it follows that

$$\begin{aligned}
\prod_{j=1}^{n_z-n_{z+2}} a_j^{(z+1)} &= (-1)^{n_{z+2}} \frac{\det(D_z)}{\det(D_{z+2})} \\
&= (-1)^{\frac{z}{2}n_z - \frac{z+2}{2}n_{z+2}+n_{z+2}} \frac{\det\left(P_z^t \mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}} P_z\right)}{\det\left((P_{z+2})^t\, \mathbf{S}|_{\{1,\dots n_{z+2}\}\times\{1,\dots n_{z+2}\}} P_{z+2}\right)} \\
&= (-1)^{\frac{z}{2}(n_z-n_{z+2})} \frac{\det\left(\mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}}\right)}{\det\left(\mathbf{S}|_{\{1,\dots n_{z+2}\}\times\{1,\dots n_{z+2}\}}\right)} \left(\frac{\det(P_z)}{\det(P_{z+2})}\right)^2 \\
&= \qquad d_{\frac{z}{2}} \qquad \left(\frac{\det(P_z)}{\det(P_{z+2})}\right)^2.
\end{aligned}$$

If $n_{z+2} = 0$, we have $\prod_{j=1}^{n_z} a_j^{(z+1)} = (-1)^{\frac{z}{2}n_z} \det(\mathbf{S}|_{\{1,\dots n_z\}\times\{1,\dots n_z\}}) \det(P_z)^2 = d_{\frac{z}{2}} \det(P_z)^2$, so for any $z \geq 2$ it follows that

$$d_{\frac{z}{2}} \equiv \prod_{j=1}^{n_z-n_{z+2}} a_j^{(z+1)} \mod \left(\mathbb{F}_q{}^*\right)^2.$$

By construction of the basis in theorem 3.1.1 its Gram matrix comprises diagonal entries that are the same as the diagonal entries of D_0 and blocks of 2×2-matrices with determinante 1. We therefore have $\det(D_0) \equiv \det(J_N) \mod (\mathbb{F}_q{}^*)^2$. Let $c \in \mathbb{F}_q{}^*$ be such that $\det(D_0) = c^2\det(J_N)$ and since n_0, that is the number of all odd elements in λ, is even we have

$$\prod_{j=1}^{n_0-n_2} a_j^{(1)} = (-1)^{n_2} \frac{\det(D_0)}{\det(D_2)} = (-1)^{n_0-n_2} \frac{\det(J_N)\, c}{\det(\mathbf{S})\det(P_{z+2})^2} = d_0 \left(\frac{c}{\det(P_{z+2})}\right)^2.$$

Therefore, $d_i \equiv \prod_{j=1}^{n_{2i}-n_{2(i+1)}} a_j^{(2i+1)} \mod \left(\mathbb{F}_q{}^*\right)^2$ holds for all $i \in \mathbb{N}_0$ and claim holds by lemma 3.2.2.

Again by theorem 3.1.8 and lemma 3.3.2 for odd $z \geq 0$ there are basis elements $f_j^{(z+1)}, g_j^{(z+1)} \in \overline{\mathbb{F}}_q^N$ with $1 \leq j \leq (n_z - n_{z+2})/2$ as defined in theorem 3.1.1 with $b(f_j^{(z+1)}, X^z g_j^{(z+1)}) = -1$. Since $X^{z+1} f_j^{(z+1)} = X^{z+1} g_j^{(z+1)} = 0$, we have $f_j^{(z+1)}, g_j^{(z+1)} \in \bigoplus_{k\leq z} V_k$. Let $\tilde{f}_j^{(z+1)}, \tilde{g}_j^{(z+1)} \in V_z$ such that $\hat{f}_j = f_j^{(z+1)} - \tilde{f}_j^{(z+1)} \in \bigoplus_{k<z} V_k$ as well as $\hat{g}_j = g_j^{(z+1)} - \tilde{g}_j^{(z+1)} \in \bigoplus_{k<z} V_k$. It follows that $X^z\hat{f}_j, X^z\hat{g}_j \in \bigoplus_{k<-z} V_k$ and therefore

$$\begin{aligned}
b\left(f_j^{(z+1)}, X^z g_j^{(z+1)}\right) &= b\left(\tilde{f}_j^{(z+1)}, X^z \tilde{g}_j^{(z+1)}\right) + b\left(\tilde{f}_j^{(z+1)}, X^z \hat{g}_j\right) + b\left(\hat{f}_j^{(z+1)}, X^z \tilde{g}_j\right) + b\left(\hat{f}_j, X^z \hat{g}_j\right) \\
&= b\left(\tilde{f}_j^{(z+1)}, X^z \tilde{g}_j^{(z+1)}\right),
\end{aligned}$$

which concludes $b\left(\tilde{f}_j^{(z+1)}, X^z\tilde{g}_j^{(z+1)}\right) = -1$, and the $\tilde{f}_j^{(z+1)}, \tilde{g}_j^{(z+1)} \in V_z$ are basis elements as defined in theorem 3.1.1. These basis elements then give rise to the linear subspaces V_z with

$$\begin{aligned} V_z &= \bigoplus_{k\in\mathbb{N}_0} \left\langle X^k\tilde{e}_j^{(z+1+2k)} \mid 1 \leq j \leq l(z+1+2k)\right\rangle \text{ for even } z \text{ and} \\ V_z &= \bigoplus_{k\in\mathbb{N}_0} \left\langle X^k\tilde{f}_j^{(z+1+2k)}, X^k\tilde{g}_j^{(z+1+2k)} \mid 1 \leq j \leq l(z+1+2k)\right\rangle \text{ for odd } z. \end{aligned}$$

Let $\tau_X : \overline{\mathbb{F}}_q^* \to \overline{\mathrm{SO}}_N$ be the coroot that gives rise to the one-dimensional torus $T_X \leq \overline{\mathrm{SO}}_N$ as defined in lemma 3.1.2 with $\tau_X(c)X^k\tilde{e}_j^{(z+1)} = c^{z-2k}X^k\tilde{e}_j^{(z+1)}$ as well as $\tau_X(c)X^k\tilde{f}_j^{(z+1)} = c^{z-2k}X^k\tilde{f}_j^{(z+1)}$ and $\tau_X(c)X^k\tilde{f}_j^{(z+1)} = c^{z-2k}X^k\tilde{f}_j^{(z+1)}$. Let $1 \leq i \leq N$ be an entry in the z column of $\mathbf{T}$. Then $e_i \in V_z$ and we have $\tau_X(c)e_i = c^z e_i$. Therefore, $\tau_X(c)$ is the diagonal matrix with $(\tau_X(c))_{ii} = c^z$. □

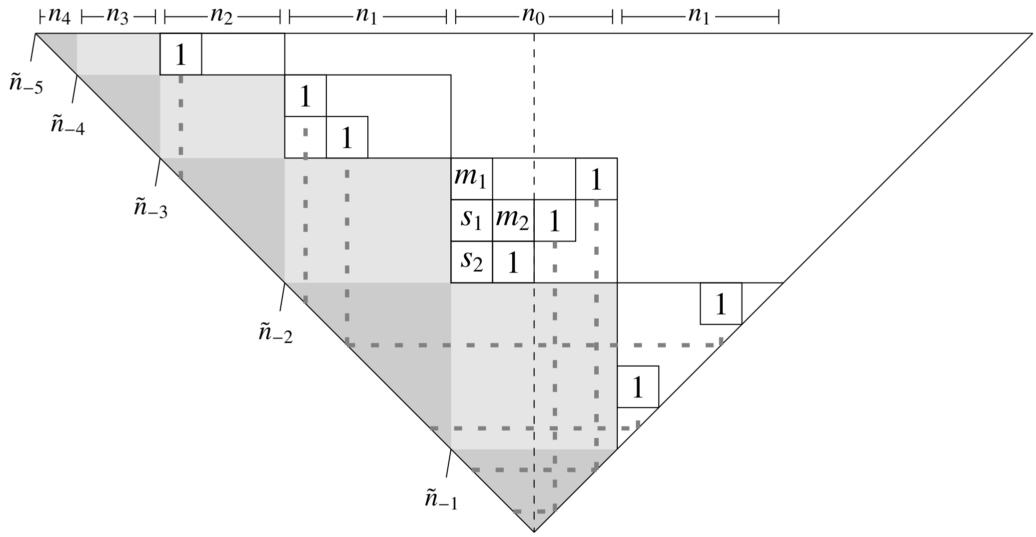
Corollary 3.3.6. *Let $\lambda \vdash N$ be a partition where every even element has even multiplicity and let $(\mathbf{T}, \boldsymbol{S})$ be a core tableau for λ. Let $\tau_{\boldsymbol{T}} : \overline{\mathbb{F}}_q^* \to T$ be the coroot of the one-dimensional torus as defined in lemma 3.1.2. For the $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_N$ as defined in theorem 3.1.3 we have $\overline{\mathfrak{g}}(z) \leq \mathfrak{u}_N$ for $z > 0$ and $\overline{U}_m \leq \overline{U}_N$ for $m \in \mathbb{N}$.*

Proof. Let $\lambda \vdash N$ be a partition where every even element has even multiplicity and let $(\mathbf{T}, \mathbf{S})$ be a core tableau for λ. Let $1 \leq i, j \leq N$ with $i + j < N + 1$ such that for $a, b \in \mathbb{Z}$ being the columns of $\mathbf{T}$ in which i and j are contained respectively we have $a < b$. Then for $c \in \overline{\mathbb{F}}_q$ we have $\tau_{\mathbf{T}}^{-1}(c)(e_{ij} - e_{\overline{ji}})\tau_{\mathbf{T}}(c) = c^{b-a}(e_{ij} - e_{\overline{ji}})$ and $e_{ij} - e_{\overline{ji}} \in \overline{\mathfrak{g}}(b - a)$. Since $a < b$, we have $i < j$ by definition of $\mathbf{T}$ unless λ is the secondary partition with only even elements as well as $a = -1$, $b = 1$ and $(i, j) = (n, n + 1)$. But then we would have $i + j = N + 1$, which contradicts the assumption So the root $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ with $\overline{\mathfrak{g}}_\alpha = \{c(e_{ij} - e_{\overline{ji}}) \mid c \in \overline{\mathbb{F}}_q\}$ is a positive root. Therefore, $\overline{\mathfrak{g}}(z) \leq \overline{\mathfrak{so}}_N$ for $z > 0$ is a sum of root spaces for positive roots and $\overline{U}_m \leq \overline{\mathrm{SO}}_N$ for $m \in \mathbb{N}$ is a product of root subgroups for positive roots. □

Example 3.3.7. For $\lambda = (5, 4^2, 3^2, 2^2, 1) \vdash 24$ and $m_1 \in \mathbb{F}_q{}^*$, $m_2, s_1, s_2 \in \mathbb{F}_q$ such that $\frac{s_2}{m_1}(s_1 - m_2) \neq 1$ we define the core tableau $(\mathbf{T}, \mathbf{S})$ as

$$\mathbf{T} = \begin{array}{l} \boxed{1}\boxed{4}\boxed{14}\boxed{21}\boxed{24} \\ \quad\boxed{2}\boxed{7}\boxed{17}\boxed{22} \\ \quad\boxed{3}\boxed{8}\boxed{18}\boxed{23} \\ \qquad\boxed{5}\boxed{13}\boxed{20} \\ \qquad\boxed{6}\boxed{12}\boxed{19} \\ \qquad\quad\boxed{9}\boxed{15} \\ \qquad\quad\boxed{10}\boxed{16} \\ \qquad\qquad\boxed{11} \end{array} \qquad \text{and} \qquad \mathbf{S} = \begin{pmatrix} 2m_1 & s_1 & s_2 \\ s_1 & 2m_2 & 1 \\ s_2 & 1 & 0 \end{pmatrix},$$

so we have $\det(\mathbf{S}_{\{1\}\times\{1\}}) = 2m_1$ and $\det(\mathbf{S}) = 2s_2(s_1 - m_2) - 2m_1$. Since $\det(J_N) = (-1)^n$, this results in $d_0 = -1/(2s_2(s_1-m_2)-2m_1)$, $d_1 = 1-\frac{s_2}{m_1}(s_1-m_2)$ and $d_2 = -2m_1$ with the following core pattern $C_{\mathbf{T}}$:



We finally can show that for every there is a core tableau $(\mathbf{T}, \mathbf{S})$ for which its core pattern gives rise to a representative of this orbit unless SO_N is based on the field $\mathbb{F}_q$ with $q = 3$.

Theorem 3.3.8. *Let $|\mathbb{F}_q| > 3$. For every nilpotent SO_N orbit $\mathcal{O}$ of $\mathfrak{so}_N$ there is a core pattern $C \in \mathcal{V}$ with $C - C^\dagger \in \mathcal{O}$.*

Proof. Let $\mathcal{O}$ be a nilpotent SO_N orbit of $\mathfrak{so}_N$ and let $\overline{\mathcal{O}}$ be the $\overline{\mathrm{SO}}_N$ orbit of $\mathfrak{so}_N$ with $\mathcal{O} \subseteq \overline{\mathcal{O}}^F$. Let $\lambda \vdash N$ be the partition corresponding to $\overline{\mathcal{O}}$ as defined in theorem 3.1.8. We then define the Young tableau $\mathbf{T}$ acording to definition 3.3.3 with the following entries for $j \in \mathbb{Z}$:

(i) For $1 \le i \le n_j$ the entry in the i-th row and j column is $\frac{N-n_0}{2} - \sum_{k=1}^{j} n_k + i$ if $j < 0$.

(ii) For $1 \le i \le n_0$ the entry in the i-th row and 0 column is $\frac{N+n_0}{2} + 1 - i$.

(iii) For $1 \le i \le n_j$ the entry in the i-th row and j column is $\frac{N+n_0}{2} + \sum_{k=1}^{j} n_k + 1 - i$ if $j > 0$ is even.

(iv) For odd $1 \le i \le n_j$ the entry in the i-th row and j column is $\frac{N+n_0}{2} + \sum_{k=1}^{j} n_k - i$ if $j > 0$ is odd.

(v) For even $1 \le i \le n_j$ the entry in the i-th row and j column is $\frac{N+n_0}{2} + \sum_{k=1}^{j} n_k + 2 - i$ if $j > 0$ is odd.

Clearly the entries in consecutive rows are increasing and since the entries in the 0 column are decreasing, every entry $a > n$ in the 0 column is placed above $\overline{a}$. Is the length of the i-th row odd, then all entries in the i-th row are contained in even columns and the sum of its entries in the j and $-j$ column is

$$\left(\frac{N-n_0}{2} - \sum_{k=1}^{j} n_k + i\right) + \left(\frac{N+n_0}{2} + \sum_{k=1}^{j} n_k + 1 - i\right) = N + 1.$$

Let i be odd. Is the length of the i-th row even, then the length of the $i+1$-th row is also even since even elements of λ occure with even multiplicity and all entries in the i-th and $i+1$-th row are contained in odd columns.Therefore, the sum of the entry in the i-th row and j column and the entry in the $i+1$-th row and $-j$ column is

$$\left(\frac{N-n_0}{2} - \sum_{k=1}^{j} n_k + i\right) + \left(\frac{N+n_0}{2} + \sum_{k=1}^{j} n_k + 2 - (i+1)\right) = N + 1,$$

while the sum of the entry in the $i+1$-th row and j column as well as the entry in the i-th row and $-j$ column is

$$\left(\frac{N-n_0}{2} - \sum_{k=1}^{j} n_k + i + 1\right) + \left(\frac{N+n_0}{2} + \sum_{k=1}^{j} n_k - i\right) = N + 1.$$

So $\mathbf{T}$ fulfills all conditions of definition 3.3.3 and for $a_1, \dots a_m \in \mathbb{F}_q{}^*$ and $b_1, \dots b_{m-1} \in \mathbb{F}_q$ with $m = \frac{n_0}{2}$ we define $A, B \in \mathrm{M}_m(\mathbb{F}_q)$ where A is the diagonal matrix with diagonal $\{a_1, \dots a_m\}$ and B is given by

$$B_{ij} = \begin{cases} 1 & \text{if } i + j = m + 1 \\ b_i & \text{if with } i + j = m \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i, j \leq m$. We then define the matrix $\tilde{S} \in \mathrm{M}_{n_0}(\mathbb{F}_q)$

$$\tilde{S} = \begin{pmatrix} a_1 & 0 & & \cdots & & & b_{m-1} & 1 \\ 0 & a_2 & & & & \iddots & 1 & 0 \\ & & \ddots & & b_1 & \iddots & & \\ \vdots & & & a_m & 1 & & & \\ & & b_1 & 1 & & & & \vdots \\ 0 & \iddots & \iddots & & & & & \\ b_{m-1} & 1 & & & & & & \\ 1 & 0 & & & \cdots & & & 0 \end{pmatrix},$$

which is a block-matrix made from the blocks

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & & b_{m-1} & 1 \\ & \iddots & \iddots & \\ b_1 & \iddots & & \\ 1 & & & 0 \end{pmatrix} \quad \text{such that} \quad \tilde{S} = \begin{pmatrix} A & B \\ B^t & 0 \end{pmatrix}.$$

The restriction $\tilde{S}|_{\{1,\dots n_1\}\times\{1,\dots n_1\}} = \mathbf{S}$ of $\tilde{S}$ then fulfils the conditions defined in lemma 3.3.4, where have $\tilde{S} = \mathbf{S}$ if $n_0 = n_2$, which is the case if the tableau $\mathbf{T}$ has no row with only one entry. For $1 \leq i \leq n_0$ the entry in the i-th row and 0 column of $\mathbf{T}$ is $\frac{N+n_0}{2} + 1 - i$. So we have $\frac{N+n_0}{2} + 1 - i > \frac{N}{2}$ if and only if $i \leq m$ for which $\tilde{S}_{ii} = a_i \neq 0$ holds. For $1 \leq i, j \leq n_0$ the sum of the entries in the i-th and j-th row and 0 column of $\mathbf{T}$ is $N + n_0 + 2 - (i + j)$. We then have $N + n_0 + 2 - (i + j) > N + 1$ if and only if $(i + j) > n_0 + 1$ for which $\tilde{S}_{ij} = 0$ holds as well as $N + n_0 + 2 - (i + j) = N + 1$ if and only if $(i + j) = n_0 + 1$ for which $\tilde{S}_{ij} = 1$ holds. Therefore, $\tilde{S}$ fulfills all conditions of Lemma 3.3.4.

For $1 \leq k \leq m$ let $\delta_k = \det\left(\tilde{S}|_{\{1,\dots k\}\times\{1,\dots k\}}\right)$ be the determinant of the restriction of $\tilde{S}$ to the square $\{1, \dots k\} \times \{1, \dots k\}$. For $k \leq m$ the restriction $\tilde{S}|_{\{1,\dots k\}\times\{1,\dots k\}}$ is a restriction of A, so we have $\delta_k = \prod_{i=1}^{k} a_i$. Since A is invertible for $k > m$, the determinant δ_k can be reduced to the product of the determinant of the blocks. So for $k' = k - m$ we have

$$\begin{aligned}\delta_k = \det\left(\tilde{S}|_{\{1,\dots m+k'\}\times\{1,\dots m+k'\}}\right) &= \det\begin{pmatrix} A & B|_{\{1,\dots m\}\times\{1,\dots k'\}} \\ (B|_{\{1,\dots m\}\times\{1,\dots k'\}})^t & 0\end{pmatrix} \\ &= \det\begin{pmatrix} A & 0 \\ 0 & -(B|_{\{1,\dots m\}\times\{1,\dots k'\}})^t A^{-1} B|_{\{1,\dots m\}\times\{1,\dots k'\}}\end{pmatrix} \\ &= (-1)^{k'}\det(A)\det\left((B|_{\{1,\dots m\}\times\{1,\dots k'\}})^t A^{-1} B|_{\{1,\dots m\}\times\{1,\dots k'\}}\right).\end{aligned}$$

For $1 \leq i, j \leq m$ we have $B_{ij} = 0$ unless $i + j \in \{m, m + 1\}$. So for $1 \leq i, j \leq k'$ it follows that

$$\left((B|_{\{1,\dots m\}\times\{1,\dots k'\}})^t A^{-1} B|_{\{1,\dots m\}\times\{1,\dots k'\}}\right)_{ij} = \sum_{l=1}^{m} \frac{B_{li}B_{lj}}{A_{ll}} = \begin{cases} \frac{1}{a_{m+1-i}} + \frac{b_i^2}{a_{m-i}} & \text{if } i = j \neq m \\ \frac{1}{a_1} & \text{if } i = j = m \\ \frac{b_i}{a_{m-i}} & \text{if } i + 1 = j \\ \frac{b_j}{a_{m-j}} & \text{if } i = j + 1 \end{cases}.$$

For $k = m + 1$, that is $k' = 1$, we have $\delta_{m+1} = -\left(\frac{1}{a_m} + \frac{b_1^2}{a_{m-1}}\right)\delta_m$, and for $m + 2 \leq k \leq n_0 - 1$ we can calculate δ_k by cofactor expansion as

$$\delta_k = -\left(\frac{1}{a_{m+1-k'}} + \frac{b_{k'}^2}{a_{m-k'}}\right)\delta_{k-1} + \frac{b_{k'-1}^2}{a_{m+1-k'}^2}\delta_{k-2}.$$

These recursively defined equations are fulfilled for $m + 1 \leq k \leq n_0 - 1$ by

$$\delta_k = -\frac{b_{k'}^2}{a_{m-k'}}\delta_{k-1} + \prod_{i \leq m-k'} a_i.$$

For $1 \leq k \leq n_0 - 1$ let $c_k \in \mathbb{F}_q{}^*$. For $m + 1 \leq k \leq n_0 - 1$ we define the sets $X \subseteq \mathbb{F}_q{}^*$ and $Y \subseteq \mathbb{F}_q$ with

$$X = \{c_k x^2 | x \in \mathbb{F}_q{}^*\} \qquad \text{and} \qquad Y = \left\{ \frac{1}{a_{m-k'}}\left(\frac{a_{m-k'}}{\delta_{k-1}}\prod_{i \leq m-k'} a_i - y^2\right) \middle| \, y \in \mathbb{F}_q \right\}.$$

Assume that $\frac{a_{m-k'}}{\delta_{k-1}}\prod_{i \leq m-k'} a_i \notin (\mathbb{F}_q{}^*)^2$. We then have $0 \notin Y$ and since $|X| = |(\mathbb{F}_q{}^*)^2| = \frac{q-1}{2}$ and $|Y| = \frac{q-1}{2} + 1$, we have $|X| + |Y| = q > |\mathbb{F}_q{}^*|$, which gives us $X \cap Y \neq \emptyset$.

Let now $\frac{a_{m-k'}}{\delta_{k-1}}\prod_{i \leq m-k'} a_i \in (\mathbb{F}_q{}^*)^2$ and we define $y_0 \in \mathbb{F}_q{}^*$ such that $y_0^2 = \frac{a_{m-k'}}{\delta_{k-1}}\prod_{i \leq m-k'} a_i$. We then have $\frac{y_0^2}{a_{m-k'}} \in (\mathbb{F}_q{}^*)^2 \cap Y$ if $a_{m-k'} \in (\mathbb{F}_q{}^*)^2$ and $\frac{y_0^2}{a_{m-k'}} \in Y \backslash (\mathbb{F}_q{}^*)^2$ if $a_{m-k'} \notin (\mathbb{F}_q{}^*)^2$. Let now $x_0 \in \mathbb{F}_q{}^*$ with $x_0 \neq -1$ and $x_0 \notin (\mathbb{F}_q{}^*)^2$. Since the characteristic of the field $\mathbb{F}_q$ is required to be good, we cannot have either $q = 2$ or $q = 4$, whereas we excluded $q = 3$ for this theorem.

Therefore, such x_0 exists since for $|\mathbb{F}_q| \geq 5$ we have $|{\mathbb{F}_q}^*\backslash({\mathbb{F}_q}^*)^2| \geq 2$. For $y = \frac{x_0-1}{x_0+1}y_0$ we then have

$$\frac{1}{a_{m-k'}}\left(y_0^2 - \left(\frac{x_0-1}{x_0+1}y_0\right)^2\right) = \frac{y_0^2}{a_{m-k'}}\left(1 - \frac{(x_0-1)^2}{(x_0+1)^2}\right) = \frac{y_0^2}{a_{m-k'}}\frac{4x_0}{(x_0+1)^2} = \frac{x_0}{a_{m-k'}}\left(\frac{2y_0}{x_0+1}\right)^2.$$

If $a_{m-k'} \in ({\mathbb{F}_q}^*)^2$, we have $\frac{x_0}{a_{m-k'}} \in Y\backslash({\mathbb{F}_q}^*)^2$, while if $a_{m-k'} \notin ({\mathbb{F}_q}^*)^2$, we have $\frac{x_0}{a_{m-k'}} \in ({\mathbb{F}_q}^*)^2 \cap Y$. It follows that $Y \cap ({\mathbb{F}_q}^*)^2 \neq \emptyset$, yet we have $Y \not\subseteq ({\mathbb{F}_q}^*)^2 \cup \{0\}$, which gives us $X \cap Y \neq \emptyset$. So there are $x_k \in {\mathbb{F}_q}^*$ and $y_k \in \mathbb{F}_q$ such that

$$c_k x_k^2 = -\frac{y_k^2}{a_{m-k'}} + \frac{1}{\delta_{k-1}}\prod_{i \leq m-k'} a_i = \frac{1}{\delta_{k-1}}\left(-\frac{y_k^2}{a_{m-k'}}\delta_{k-1} + \prod_{i \leq m-k'} a_i\right).$$

Let now $b_{k'} = y_k$, so we have $\frac{\delta_k}{\delta_{k-1}} = c_k x_k^2$. Let $\delta_0 = 1$ and for $1 \leq k \leq m$ let $a_k = c_k$, so we have $\frac{\delta_k}{\delta_{k-1}} = c_k$. For the matrix $\tilde{S}$ defined in this way it follows that

$$\frac{\delta_k}{\delta_{k-1}} \equiv c_k \mod \left({\mathbb{F}_q}^*\right)^2$$

So for $i \in \mathbb{N}$ the d_i as defined in theorem 3.3.5 are

$$d_i \equiv (-1)^{n_{2i}-n_{2i+2}} \prod_{k=n_{2i+2}+1}^{n_{2i}} c_k \mod \left({\mathbb{F}_q}^*\right)^2.$$

Since $\prod_{i\in\mathbb{N}_0} d_i \equiv (-1)^m \mod ({\mathbb{F}_q}^*)^2$ by theorem 3.3.5, we have $\delta_m = (-1)^m$ we find suitable $c_k \in {\mathbb{F}_q}^*$ for every possible combination of d_i, which proves the claim. □

4 Core patterns for generalized Gelfand–Graev characters

Andrews and Thiem [AT17] used the supercharacter theory of G_N, that is obtained by the left and right operation of G_N on $\mathbf{m}$, introduced by André [And02] to construct generalized Gelfand–Graev characters. These were originally developed by Kawanaka [Kaw85] and are characters induced from a unipoent group $U_{1.5}$ that is situated in the middle between the unipoent groups U_1 and U_2 arising from the $\mathbb{Z}$-grading of $\mathfrak{so}_N$ for a nilpotent element defined in lemma 3.1.4. Using the supercharacter theory of André and Neto discussed in the second section of the third chapter, the same can be done for the special orthogonal group SO_N, but only for a small selection of generalized Gelfand–Graev characters. Here the decomposition of the André–Neto characters into Jedlitschky characters makes such a construction for all generalized Gelfand–Graev characters of SO_N possible. We can use the characters for the patterns, which we introduced in the previous section for the nilpotent orbits, reduced to the unipotent radical U_1 of the parabolic subgroup associated to their respective nilpotent orbits. These characters of U_1 induced to the whole group SO_N then are the generalized Gelfand–Graev characters up to some scalar. We can then illustrate this result by calculating the Jedlitschky characters that give rise to the generalized Gelfand–Graev characters of SO_8.

4.1 Generalized Gelfand–Graev characters

Following Kawanaka's [Kaw85] construction of generalized Gelfand–Graev characters we will calculate the induced character of U_N of the linear character $\xi_A^{\sim}$ of the intermediary group $U_{1.5}$ for $A \in \mathfrak{u}_2$ that gives rise to the generalized Gelfand–Graev character $\gamma_A = \mathrm{Ind}_{U_{1.5}}^{\mathrm{SO}_N} \xi_A^{\sim}$.
For this we will first discuss the interaction of the Cayley transformation defined in lemma 1.3.12 with the $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_N$ for a nilpotent element $A \in \mathfrak{so}_N$ as defined in lemma 3.1.4.

Definition 4.1.1. Let $\overline{T} \leq \overline{\mathrm{SO}_n}$ be the maximal torus of diagonal matrices in $\overline{\mathrm{SO}_n}$ as defined in section 1.2. For a nilpotent element $A \in \mathfrak{so}_N$ let $\tau = \tau_A : \overline{\mathbb{F}}_q^* \to \overline{T}$ be the cocharacter defined in lemma 3.1.2. Let $\overline{P} \leq \overline{\mathrm{SO}}_n$ be the parabolic subgroups that arise from the F-stable one dimensional torus $\tau(\overline{\mathbb{F}}_q) \leq \overline{T}$ according to theorem 3.1.3. For $z \in \mathbb{Z}$ let $\mathfrak{g}(z) \leq \mathfrak{so}_N$ be the fixed points of the subalgebras of $\overline{\mathrm{SO}}_N$ defined in 3.1.4 that form a $\mathbb{Z}$-grading with

$$\mathfrak{so}_N = \bigoplus_{z\in\mathbb{Z}} \mathfrak{g}(z).$$

For $m \in \mathbb{N}$ let $U_m \leq \mathrm{SO}_N$ be the fixed points of the descending series of unipotent subgroups of $\overline{P}$ defined in 3.1.4, where U_1 is the fixed points of unipotent radicals of $\overline{P}$. Let $\overline{L}' \leq \overline{P}$ be the Levi complement of $\overline{U}_1$ in $\overline{P}$ such that $\overline{P} = \overline{L}' \ltimes \overline{U}_1$ with $\mathrm{Lie}(\overline{L}') = \overline{\mathfrak{g}}(0)$.

The bilinear form κ relates to the bilinear form $\tilde{\kappa}$ on $\mathfrak{so}_N$ by $\kappa(X, Y) = \tilde{\kappa}(X^t, Y)$ for $X, Y \in \mathfrak{so}_N$. Therefore, by lemma 3.1.4 the subalgebras $\mathfrak{g}(z)$ for $z \in \mathbb{Z}$ are non degenerate with respect to κ and their orthogonal complements are $\mathfrak{g}(z)^\perp = \bigoplus_{z\neq k\in\mathbb{Z}} \mathfrak{g}(k)$.
We first discuss the interactions of the Cayley transformation defined in lemma 1.3.12 with the $\mathbb{Z}$-grading on $\overline{\mathfrak{so}}_N$.[1]

Proposition 4.1.2. *For $i, j \geq 1$ let $u \in \overline{U}_i$, $v \in \overline{U}_j$ and $X \in \overline{\mathfrak{g}}(j)$. Then the following statements hold:*

(i) $f(\overline{U}_i) = \overline{\mathfrak{u}}_i$

(ii) $f(uv) - f(u) - f(v) \in \overline{\mathfrak{u}}_{i+j}$

(iii) $f(uvu^{-1}v^{-1}) - 2[f(u), f(v)] \in \overline{\mathfrak{u}}_{i+j+1}$

(iv) $uXu^{-1} - X - 2[f(u), X] \in \overline{\mathfrak{u}}_{2i+j}$

Proof. For $k \in \mathbb{N}$ let $\overline{\mathfrak{g}}'(k)$ be the extension of $\overline{\mathfrak{g}}(k)$ to $\overline{\mathfrak{gl}}_N$

$$\overline{\mathfrak{g}}'(k) = \left\{ X \in \overline{\mathfrak{gl}}_N \,\middle|\, \tau(c)X\tau(c)^{-1} = c^k X \text{ for all } c \in \overline{\mathbb{F}}_q^* \right\}$$

and let $\overline{\mathfrak{u}}_k' = \bigoplus_{l\geq k} \overline{\mathfrak{g}}'(l)$ be the extension of $\overline{\mathfrak{u}}(k)$ to $\overline{\mathfrak{gl}}_N$. Let $x \in U_\alpha$ for $\alpha \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ be an element of a root subgroup. Then we have $f(x) = \frac{1}{2}(x-1)$ and $f(x) \in \overline{\mathfrak{g}}((\tau, \alpha))$. Let $u \in \overline{U}_i$ for $i \geq 1$, then there is a $m \in \mathbb{N}$ and $x_k \in U_{\alpha_k}$ for $1 \leq k \leq m$ with $\alpha_k \in \Phi(\overline{\mathrm{SO}}_N, \overline{T})$ such that $(\tau, \alpha_k) \geq i$, for which we have $x = \prod_{k=1}^m x_k$. Let $I = \{1, \ldots, m\}$, then it follows that

$$u - 1 = \left(\prod_{k=1}^m x_k\right) - 1 = \sum_{l=1}^m \sum_{\substack{J\subseteq I \\ |J|=l}} \prod_{k\in J} (x_k - 1) = \sum_{l=1}^m \sum_{\substack{J\subseteq I \\ |J|=l}} \prod_{k\in J} 2f(x_k).$$

[1] cf. [Kaw85, 1.2.1, p. 179]

For $J \subseteq I$ we have $\tau(c)(\prod_{k\in J} 2f(x_k))\tau(c)^{-1} = \prod_{k\in J}\tau(c)2f(x_k)\tau(c)^{-1} = \prod_{k\in J} c^{(\tau,\alpha_k)}2f(x_k)$ for all $c \in \overline{\mathbb{F}}_q$ and $\prod_{k\in J} 2f(x_k) \in \sum_{k\in J}\overline{\mathfrak{g}}'((\tau,\alpha_k))$. Since $(\tau,\alpha_k) \geq i$ for all $1 \leq k \leq m$, it follows that $u-1 \in \overline{\mathfrak{u}}'_i$. Furthermore, we have $(u-1)^l \in \overline{\mathfrak{u}}'_{li}$ for all $l \in \mathbb{N}$ and therefore $f(u) \in \overline{\mathfrak{u}}'_i \cap \overline{\mathfrak{so}}_N = \overline{\mathfrak{u}}_i$. By argument of dimension we then must have $f(\overline{U}_i) = \overline{\mathfrak{u}}_i$.
Let now $u \in \overline{U}_i$ and $v \in \overline{U}_j$ for $i, j \geq 1$. Then any product of factors $(u-1)$ and $(v-1)$ is contained in $\overline{\mathfrak{u}}'_{i+j}$. For $k \in \mathbb{N}$ we have

$$(uv-1)^k = ((u-1)(v-1)+(u-1)+(v-1))^k = (u-1)^k+(v-1)^k+\sum \begin{array}{l}\text{products with at least one}\\ (u-1) \text{ and } (v-1) \text{ as factors}\end{array}$$

and therefore $(uv-1)^k - (u-1)^k - (v-1)^k \in \overline{\mathfrak{u}}_{i+j}$. This concludes that

$$f(uv) - f(u) - f(v) = \sum_{k\in N}(-1)^{k-1}\frac{1}{2^k}\left((uv-1)^k - (u-1)^k - (v-1)^k\right) \in \overline{\mathfrak{u}}_{i+j}.$$

Since $u^{-1} \in \overline{U}_i$ and $v^{-1} \in \overline{U}_j$ with $i, j \geq 1$, we at least have $(vu)^{-1} - 1 \in \overline{\mathfrak{u}}'_1$ and

$$\begin{aligned} uvu^{-1}v^{-1} - 1 &= (uv - vu)u^{-1}v^{-1} = ((u-1)(v-1) - (v-1)(u-1))u^{-1}v^{-1} \\ &= \underbrace{((u-1)(v-1) - (v-1)(u-1))}_{\in\overline{\mathfrak{u}}'_{i+j}}\underbrace{((vu)^{-1} - 1)}_{\in\overline{\mathfrak{u}}'_1} + \underbrace{(u-1)(v-1) - (v-1)(u-1)}_{\in\overline{\mathfrak{u}}'_{i+j}}. \end{aligned}$$

For $k \geq 2$ it follows that $(uvu^{-1}v^{-1} - 1)^k \in \overline{\mathfrak{u}}'_{i+j+1}$ and therefore

$$f(uvu^{-1}v^{-1}) - \frac{1}{2}((u-1)(v-1) - (v-1)(u-1)) \in \overline{\mathfrak{u}}'_{i+j+1}.$$

Furthermore, we have $(u-1)^k(v-1)^l$, $(v-1)^l(u-1)^k \in \overline{\mathfrak{u}}'_{i+j+1}$ if $k \geq 2$ or $l \geq 2$ and therefore $[f(u), f(v)] - \frac{1}{4}((u-1)(v-1) - (v-1)(u-1)) \in \overline{\mathfrak{u}}'_{i+j+1}$, which proves the third claim.
Let now $X \in \overline{\mathfrak{g}}(j)$. Then we have

$$\begin{aligned} uXu^{-1} - X &= (u-1)X(u^{-1} - 1) + (u-1)X + X(u^{-1} - 1) \\ &= \underbrace{(u-1)X(u^{-1} - 1)}_{\in\overline{\mathfrak{u}}'_{2i+j}} + \underbrace{(u-1)X - X(u-1)}_{\in\overline{\mathfrak{u}}'_{i+j}} - \underbrace{X(u-1)(u^{-1} - 1)}_{\in\overline{\mathfrak{u}}'_{2i+j}} \end{aligned}$$

and therefore $uXu^{-1} - X - ((u-1)X - X(u-1)) \in \overline{\mathfrak{u}}'_{2i+j}$. The last claim is then concluded by

$$[f(u), X] - \frac{1}{2}((u-1)X - X(u-1)) \in \overline{\mathfrak{u}}'_{2i+j}. \qquad \square$$

With this proposition, for $A \in \mathfrak{u}_2$ we can define a linear character ξ_A on U_2 comparable to the linear character giving rise to the basic characters as defined in lemma 2.2.2.[2]

Definition/Lemma 4.1.3. *For $A \in \mathfrak{u}_2$ let ξ_A be the linear character of U_2 defined by*

$$\xi_A : U_2 \to \mathbb{C} : x \mapsto \vartheta\kappa(A, f(x)).$$

Proof. Let $A \in \mathfrak{u}_2$ and $x, y \in U_2$. Then by proposition 4.1.2 there is a $B \in \mathfrak{u}_4$ such that $f(xy) = f(x) + f(y) + B$ and we have

$$\xi_A(xy) = \vartheta\kappa(A, f(xy)) = \vartheta\kappa(A, f(x))\,\vartheta\kappa(A, f(y))\,\vartheta\kappa(A, B) = \xi_A(x)\xi_A(y),$$

since $\mathfrak{u}_4^\perp = \bigoplus_{k<4} \mathfrak{g}(k)$. It follows that $\kappa(A, B) = 0$. □

Now we can define the unipotent subgroup $U_{1.5} \leq \mathrm{SO}_N$ that is situated in the middle between U_1 and U_2 such that ξ_A extends to a linear character of $U_{1.5}$.[3]

Lemma 4.1.4. *There exists a linear subspace $\mathfrak{s} \leq \mathfrak{g}(1)$ with $|\mathfrak{s}| = \sqrt{|\mathfrak{g}(1)|}$ and $\kappa(A, [X, Y]) = 0$ for all $X, Y \in s$. Then the subgroup $U_{1.5} \leq SO_N$ of SO_N is defined by*

$$U_{1.5} = f^{-1}(\mathfrak{u}_2 + \mathfrak{s})$$

with $U_2 \leq U_{1.5} \leq U_1$ and $|U_1 : U_{1.5}| = |U_{1.5} : U_2|$. Furthermore, ξ_A can be extended to a linear character $\xi_A^\sim$ of $U_{1.5}$.

Proof. Let s be a skew symmetric bilinear form of $\mathfrak{g}(1)$ defined by

$$s(X, Y) = \kappa(A, [X, Y])$$

for $X, Y \in \mathfrak{g}(1)$. Let $X \in \mathfrak{g}(1)$ with $s(X, Y) = 0$ for all $Y \in \mathfrak{g}(1)$. Due to the cyclic property of the trace form, we have $\kappa([A^t, X]^t, Y) = \kappa(A, [X, Y]) = 0$. Since $[A^t, X] \in \mathfrak{g}(-1)$ and κ is non degenerated, we must have $[A^t, X] = 0$ and therefore $[A, X^t] = -[A^t, X]^t = 0$. It follows that $X^t \in \mathfrak{c}_{\mathfrak{so}_N}(A) \cap \mathfrak{g}(-1)$, but by theorem 3.1.6 we have $\mathfrak{c}_{\overline{\mathfrak{so}}_N}(A) \cap \overline{\mathfrak{g}}(i) = (0)$ and therefore $\mathfrak{c}_{\mathfrak{so}_N}(A) \cap \mathfrak{g}(i) = (0)$ for $i < 0$, which gives us $X = 0$. So s is non degenerated and since it is also skew symmetric the linear subspace $\mathfrak{g}(1)$ must have even dimension and there is a linear subspace $\mathfrak{s}$ with $|\mathfrak{s}| = \sqrt{|\mathfrak{g}(1)|}$ and $\mathfrak{s} = \mathfrak{s}^\perp$.

[2] cf. [Kaw85, p. 179]
[3] cf. [Kaw85, 1.2.4, p. 179] and [Kaw85, 1.3.2, p. 180]

We clearly have $U_2 \subseteq U_{1.5} \subseteq U_1$ with $|U_1 : U_{1.5}| = \sqrt{|\mathfrak{g}(1)|} = |U_{1.5} : U_2|$. For $u, v \in U_{1.5}$ it follows by proposition 4.1.2 (ii) that $f(uv) - f(u) - f(v) \in \mathfrak{u}_2$ and therefore $uv \in U_{1.5}$. By the same point of this lemma we have $uvu^{-1}v^{-1} \in U_2$ and $\xi_A(uvu^{-1}v^{-1}) = \vartheta\kappa(A, 2[f(u), f(v)]) = 1$ by point (iii). Therefore, we can extend ξ_A to a linear character of $U_{1.5}$. □

This allows for the definition of the generalized Gelfand–Graev characters γ_A of SO_N for nilpotent $A \in \mathfrak{so}_N$ as induced characters from the unipotent subgroups $U_{1.5}$, which are equal to the character induced from U_2 up to a scalar.[4]

Definition 4.1.5. The generalized Gelfand Graev character γ_A for $A \in \mathfrak{so}_N$ is the induced character of the linear character $\xi_A^{\sim}$ of $U_{1.5}$

$$\gamma_A = \mathrm{Ind}_{U_{1.5}}^{\mathrm{SO}_N} \xi_A^{\sim}(u).$$

Lemma 4.1.6. *For $A \in \mathfrak{u}_2$ the generalized Gelfand Graev character γ_A is constituent of the induced character of SO_N of the linear character ξ_A of U_2 with*

$$\gamma_A = \frac{1}{\sqrt{|\mathfrak{g}(1)|}} Ind_{U_2}^{SO_N} \xi_A.$$

Proof. For $u \in U_1$ let $f(u) = X + Y$ with $X = f(u)|_{\mathfrak{g}(1)}$ and $Y = f(u)|_{u(2)}$. By proposition 4.1.2 (iv) we have $gYg^{-1} - Y - 2[f(g), Y] \in u_4$ for $g \in U_1$ and therefore $gYg^{-1} - Y \in u_3$. Furthermore, we have $gXg^{-1} - X - 2[f(g), X] \in u_3$ and $gXg^{-1} - X \in \mathfrak{u}_2$, so $gug^{-1} \in U_{1.5}$ if and only if $X \in \mathfrak{s}$. Is X such an element it follows that

$$\begin{aligned}
\mathrm{Ind}_{U_{1.5}}^{U_1} \xi_A^{\sim}(u) &= \frac{1}{|U_{1.5}|} \sum_{g \in U_1} \vartheta\kappa(A, gXg^{-1} + Y) \\
&= \frac{1}{|U_{1.5}|} \sum_{g \in U_1} \vartheta\kappa(A, X + Y)\vartheta\kappa(A, gXg^{-1} - X) \\
&= \frac{1}{|U_{1.5}|} \xi_A^{\sim}(u) \sum_{g \in U_1} \vartheta\kappa(A, 2[f(g), X]) \\
&= \frac{1}{|U_{1.5}|} \xi_A^{\sim}(u) \sum_{g \in U_1} \vartheta\kappa([A, X^t]^t, 2f(g)).
\end{aligned}$$

Here g runs through all elements of U_1, so by proposition 4.1.2 (i) $f(g)$ runs through all elements of $\mathfrak{u}_1$. Since $[A, X^t]^t \in g(-1)$, we have $\sum_{g \in U_1} \vartheta\kappa([A, X^t]^t, 2f(g)) = 0$ unless $[A, X^t] = 0$. But since $X^t \in g(-1)$ this requires $X = 0$. As $|U_1|/|U_{1.5}| = \sqrt{|\mathfrak{g}(1)|}$ we finally have

[4]cf. [Kaw85, 1.3.4, p. 181] and [Kaw85, 1.3.6, p. 181]

$$\mathrm{Ind}_{U_{1.5}}^{U_1}\xi_A^{\sim}(u) = \begin{cases} \sqrt{|\mathfrak{g}(1)|}\xi_A(u) & \text{for } u \in U_2 \\ 0 & \text{for } u \in U_1 \backslash U_2. \end{cases}$$

□

4.2 Induced Jedlitschky characters

Reducing a Jedlidtschky character for certain tableaux to the unipotent radical U_1 and inducing it to SO_N produces a scalar multiple of the generalized Gelfand–Graev character for the nilpotent matrix corresponding to core pattern of the respective Jedlidtschky character. So we will first describe the conditions a tableau has to meet in order for the core pattern that arises from it to fulfil such a role.

For a verge tableau $\mathbf{T}$ let again $n_z \in \mathbb{N}_0$ be the length of the z column for $z \in \mathbb{Z}$, and $\tilde{n}_z = \sum_{i<z} n_z$ is the sum of all n_z left of the z column as defined in 3.3.1.

Definition 4.2.1. For a partition $\lambda \vdash N$, for which every even entry has even multiplicity and a verge tableau $\mathbf{T}$ for λ we call the verge tableau $\mathbf{T}$ standard if the following conditions are met:

(i) For every row $r < 0$ the entries are in increasing order such that for $1 \leq a < b \leq n_r$ if i is the entry in the a-th row and j the entry in the a-th row in the r column we have $i < j$.

(ii) For the center row the entries are in decreasing order such that for $1 \leq a < b \leq n_0$ if i is the entry in the a-th row and j the entry in the a-th row in the 0 column we have $i > j$.

(iii) For odd $1 \leq a \leq n_1$, where i is the entry in the -1 column and j the entry in the 1 column in the a-th row, we have $i + j = N$, unless λ is the secondary partition with only even elements, in which case this statement must hold for the tableau $\mathbf{T}$ exchanging the entries n and $n + 1$ instead.

(iv) For even $1 \leq b \leq n_1$, where i is the entry in the -1 column and j the entry in the 1 column in the b-th row, we have $i + j = N + 2$, unless λ is the secondary partition with only even elements, in which case this statement must hold for the tableau $\mathbf{T}$ exchanging the entries n and $n + 1$ instead.

A core tableau $(\mathbf{T}, \mathbf{S})$ for a standard tableau $\mathbf{T}$ will be called standard as well and for the rest of this chapter we will consider the verge tableau $\mathbf{T}$ to be standard.

For a core tableau $(\mathbf{T}, \mathbf{S})$ let $\tau = \tau_{\mathbf{T}} : \overline{\mathbb{F}}_q^* \to \overline{T}$ be the cocharacter with $\tau_{\mathbf{T}}(x)_{ii} = x^z$ for $1 \leq i \leq N$ and $z \in \mathbb{Z}$, where i is contained in the z column of $\mathbf{T}$ as defined theorem 3.3.5. Let $A = A_T \in \mathbf{v}$

be the verge pattern and $C = C_T \in \mathbf{v}$ be the core pattern for $(\mathbf{T}, \mathbf{S})$. Then $C - C^\dagger \in \mathfrak{so}_N$ is nilpotent, and we can define a corresponding $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_n$ according to lemme 3.1.4. The $\mathbb{Z}$-grading of $\mathfrak{so}_n$ and the ensuing subgroups of SO_N as defined in lemma 4.1.1 for a core tableau aligns with the group of upper unitriangular matrices U_N in a way such that $U_k \leq U_N$ for all $k \in \mathbb{N}$. Furthermore, the ”.”-operation preserves the vector spaces $\pi(\mathfrak{g}(m)) \leq \mathbf{v}$ for $m \in \mathbb{N}_0$ to some extent.

Lemma 4.2.2. *For a core pattern $(\boldsymbol{T}, \boldsymbol{S})$ let $C \in \boldsymbol{v}$ be the corresponding core pattern. For $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$ let $\mathfrak{g}(m) \leq \mathfrak{so}_N$ and $U_k \leq SO_N$ be the fixed points of the $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_N$ for the nilpotent element $C - C^\dagger \in \mathfrak{so}_N$ as defined in 4.1.1. For $k \in \mathbb{N}$ we then have $U_k \leq U_N$ and for $L = L' \cap U_N$, where $L' \leq SO_N$ fixed points of this Levi complement, we have $U_N = L \ltimes U_1$. For $m \in \mathbb{N}_0$ the vector space $\pi(\mathfrak{g}(m)) \leq \boldsymbol{v}$ is*

$$\pi(\mathfrak{g}(m)) = \left\langle e_{ij} \,\middle|\, \begin{array}{c} 1 \leq i < j \leq N, i + j < N + 1, j \text{ is contained in the } z + m \text{ column of } \boldsymbol{T}, \\ \text{where } z \in \mathbb{Z} \text{ is the column in which } i \text{ is contained.} \end{array} \right\rangle$$

Then $\boldsymbol{v} = \bigoplus_{m \in \mathbb{N}_0} \pi(\mathfrak{g}(m))$ defines a $\mathbb{N}_0$ grading on $\boldsymbol{v}$. For $k \in \mathbb{N}$, $g \in U_k$ and $X \in \pi(\mathfrak{g}(m))$ we have $g.X - X \in \bigoplus_{i=0}^{m-k} \pi(\mathfrak{g}(m))$ and for $h \in L$ we have $h.X \in \pi(\mathfrak{g}(m))$.

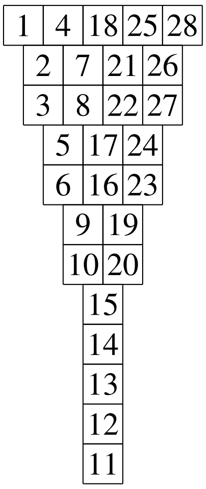
Proof. Let $(\mathbf{T}, \mathbf{S})$ be a verge pattern and $C \in \mathbf{v}$ the corresponding core pattern. By theorem 3.3.5 the coroot of the one-dimensional torus $\tau = \tau_{C-C^\dagger} : \overline{\mathbb{F}}_q^* \to T$ for $C - C^\dagger$ is the subgroup of diagonal matrices with $(\tau_{C-C^\dagger}(c))_{ii} = c^z$ where $z \in \mathbb{Z}$ is the column in which i is contained in the tableau $\mathbf{T}$. Let $1 \leq i, j \leq N$ with $i + j < N + 1$ and let $a, b \in \mathbb{Z}$ such that i and j are contained in the a and b column of $\mathbf{T}$ respectively. Then if $a < b$ we have $i < j$ since all elements in the a column are smaller than the elements in the b column unless $\mathbf{T}$ is the tableau for the secondary partition with only even elements, where $n+1$ is contained in the -1 column while n is contained in the 1 column, in which case we have $n + n + 1 \nless N + 1$. So for $c \in \mathbb{F}_q$ we have $x_{ij}(c) \in U_N$ and therefore $U_k \leq U_N$ for $k \in \mathbb{N}$.
Let now $1 \leq i, j \leq N$ with $i+j < N+1$. For $c \in \overline{\mathbb{F}}_q$ we have $\tau^{-1}(c)(e_{ij} - e_{\overline{ji}})\tau(c) = c^{b-a}(e_{ij} - e_{\overline{ji}})$ and $e_{ij} - e_{\overline{ji}} \in \mathfrak{g}(b - a)$, which gives us $e_{ij} \in \pi(\mathfrak{g}(b - a))$.
Let $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $X \in \pi(\mathfrak{g}(m))$. Let $1 \leq i < j \leq N$ and $c \in \mathbb{F}_q$ such that $I + ce_{ij} \in U_k$. Then there is a $l \geq k$ with $\tau^{-1}(f)(ce_{ij})\tau(f) = f^l(ce_{ij})$ for $f \in \overline{\mathbb{F}}_q$ and we have

$$\begin{aligned} \tau^{-1}(f)\left(X(I + ce_{ij})^t - X\right)\tau(f) &= \tau^{-1}(f)\left(Xce_{ij}^t\right)\tau(f) \\ &= \tau^{-1}(f)X\tau(f)\left(\tau(f)ce_{ij}\tau^{-1}(f)\right)^t \\ &= f^{m-l}Xce_{ij}^t. \end{aligned}$$

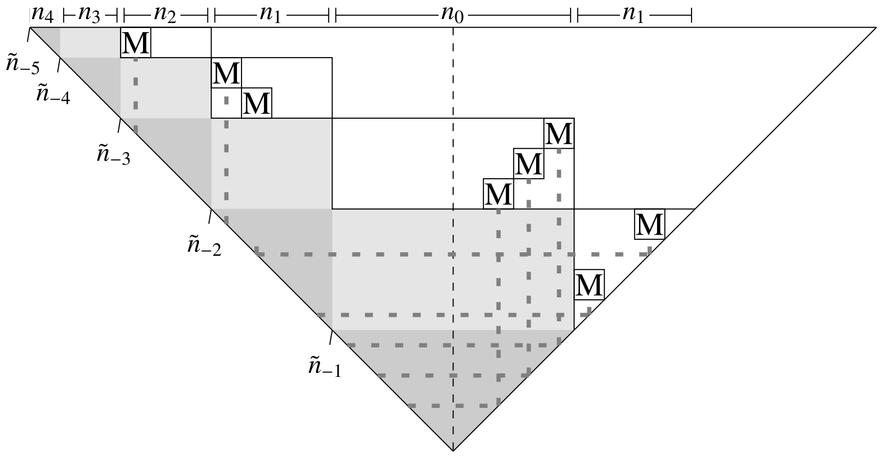
Therefore, we have $(I+ce_{ij}).X-X\in\pi(\mathfrak{g}(m-l))$ if $m-l\geq 0$ or $(I+ce_{ij}).X-X=0$ if $m-l<0$. Furthermore, for any $Y\in\bigoplus_{j=0}^{m-k}\pi(\mathfrak{g}(j))$ there is a $0\leq j\leq m-k$ with $Y_j\in\pi(\mathfrak{g}(j))$ such that $Y=\sum_{j=0}^{m-k}Y_j$ and we have

$$(I+ce_{ij}).(X+Y)-X=(I+ce_{ij}).X-X+\sum_{j=0}^{m-k}(I+ce_{ij}).Y_j\in\bigoplus_{i=0}^{m-k}\pi(\mathfrak{g}(i)).$$

As $g\in U_k$ is a product of such elementary factors $I+ce_{ij}$ we get $g.X-X\in\bigoplus_{i=1}^{m-k}\pi(\mathfrak{g}(m))$ by induction. Conversely, any element $h\in L$ is a product of root elements $I+ce_{ij}$ such that $(I+ce_{ij}).X-X\in\pi(\mathfrak{g}(m))$, so by induction we have $h.X-X\in\pi(\mathfrak{g}(m))$ and therefore $h.X\in\pi(\mathfrak{g}(m))$. $\square$



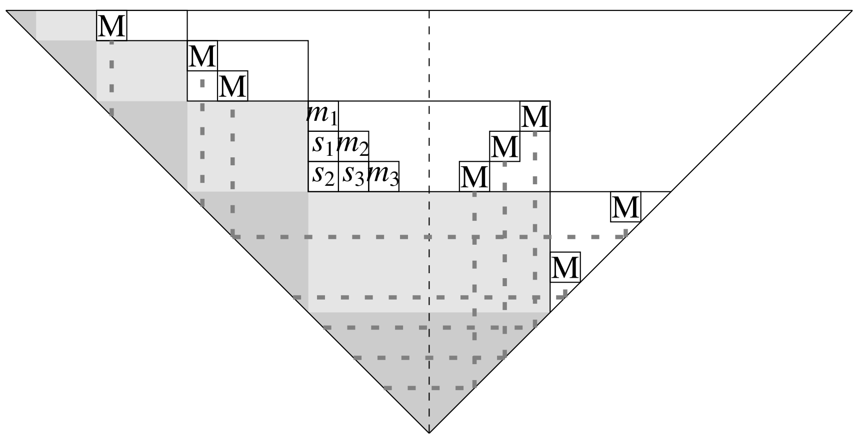
Standard tableau $\mathbf{T}$ for $\lambda=(5,4^2,3^2,2^2,1^5)\vdash 28$



Verge pattern $A_{\mathbf{T}}\in\mathbf{v}$ for $\mathbf{T}$ together with the subspaces $\pi(\mathfrak{g}(1))\leq\mathbf{v}$ (lightgray) and $\pi(\mathfrak{g}(0))\leq\mathbf{v}$ (darkgray)

$$\begin{pmatrix} m_1 & s_1 & s_2\\ s_1 & m_2 & s_3\\ s_2 & s_3 & m_3\end{pmatrix}$$

Matrix $\mathbf{S}\in M_3(\mathbb{F}_q)$



Core pattern $C_{\mathbf{T}}\in\mathbf{v}$ for $(\mathbf{T},\mathbf{S})$

In order to calculate the value of the Jedlitschky characters for standard core tableaux we first must decompose the subspaces $\pi(\mathfrak{g}(m))$ for $m \in \{0, 1\}$. For this we define the pattern group $H_A \leq U_N$ arising from the positions of the horizontal part of the lower hooks for a verge pattern $A \in \mathbf{v}$.

Lemma 4.2.3. *For a verge pattern $A \in \mathbf{v}$ let $H_A \leq U_N$ be the pattern subgroup of the horizontal part of the lower hooks, such that*

$$H_A = \left\{ u \in U_N \;\middle|\; supp_{\mathcal{V}}(u) \subseteq \bigcup_{(i,k) \in supp(A) \cap \mathcal{V}_r} \left\{ (\bar{i}, j) \mid \bar{i} < j < i \right\} \right\}.$$

Let $C \in \mathbf{v}$ be a core pattern and $d \in D_A$ such that $C = d.A$. Then for every $u \in U_N$ we have $u^{-1} H_A u \cap Stab_{U_N}(C) = I$. Moreover, for $x \in H_A$ and $g \in G_N$ we have $\pi(x) \circ g \in \pi(H_A)$. The G_N-orbit of A intersects the space of the horizontal part of the lower hooks in $\mathbf{v}$ trivially with $(G_N.A - A) \cap \pi(H_A) = 0$.

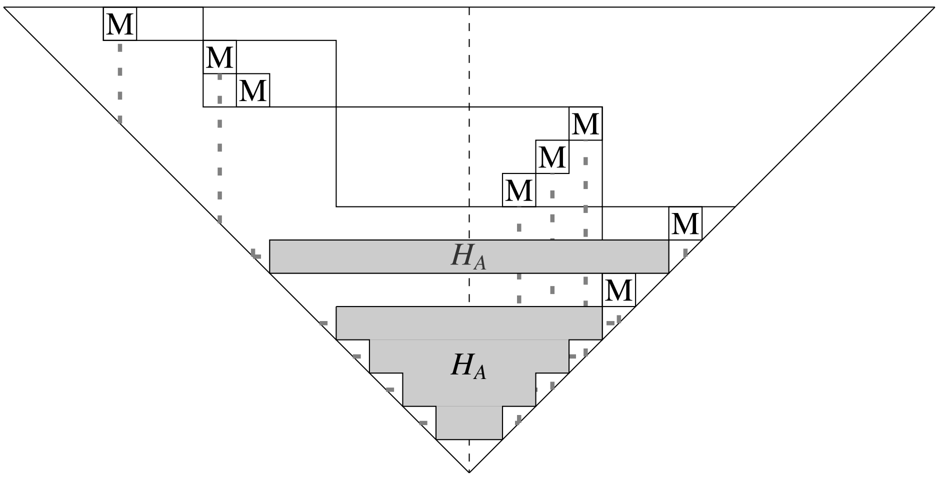
Proof. Let $A \in \mathbf{v}$ be a verge pattern and $\mathcal{H}_A \subseteq \mathcal{V}$ be the set of positions of lower hooks of A, such that $\mathcal{H}_A = \bigcup_{(i,k) \in \mathrm{supp}(A) \cap \mathcal{V}_r} \{(\bar{i}, j) \mid \bar{i} < j < i\}$. Then as H_A is the pattern subgroup for the closed subset $\mathcal{H}_A \subseteq \mathcal{V}$ and by theorem 2.3.2 we have $\mathrm{supp}_{\mathcal{V}}(\mathrm{Stab}_{U_N}(A)) \cap \mathcal{H}_A = \emptyset$. By corollary 1.3.13 the subspaces $f(H_A), f(\mathrm{Stab}_{U_N}(A)) \leq \mathfrak{u}_N$ for the Cayley transformation f are such that $\mathrm{supp}_{\mathcal{V}}(f(H_A)) \subseteq \mathcal{H}_A$ and $\mathrm{supp}_{\mathcal{V}}(f(\mathrm{Stab}_{U_N}(A))) \cap \mathcal{H}_A = \emptyset$. Let $g \in G_N$ and $X \in g^{\dagger} f(H_A) g \cap f(\mathrm{Stab}_{U_N}(A))$. Then there is a $Y \in f(H_A)$ such that $X = g^{\dagger} Y g$ and for $1 \leq i < l \leq N$ we have $X_{il} = \sum_{j=i}^{N} \sum_{k=1}^{l} g_{\overline{ji}} g_{kl} Y_{jk}$. If $\mathrm{supp}_{\mathcal{V}}(Y) \neq \emptyset$, then there is a position $(i, l) \in \mathrm{supp}(Y)$, such that there is no position in $\mathrm{supp}(Y)$ that is left or below of (i, l), that is for all $i \leq j < k \leq l$ with $(j, k) \neq (i, l)$ we have $(j, k) \notin \mathrm{supp}(Y)$. Then $X_{il} = Y_{il} \neq 0$, which is a contradiction to $X \in f(\mathrm{Stab}_{U_N}(A))$. So we have $\mathrm{supp}_{\mathcal{V}}(Y) = \emptyset$ and therefore $X = Y = 0$ as well as $g^{\dagger} f(H_A) g \cap f(\mathrm{Stab}_{U_N}(A)) = 0$. As f is a bijection, it follows that

$$\beta_{g^{-1}}(H_A) \cap \mathrm{Stab}_{U_N}(A) = f^{-1}\left(g^{\dagger} f(H_A) g \cap f(\mathrm{Stab}_{U_N}(A))\right) = f^{-1}(0) = I,$$

where $\beta_{g^{-1}}$ is the map on U_N defined in lemma 2.4.3 with $\beta_{g^{-1}}(H_A) = f^{-1}(g^{\dagger} f(H_A) g)$. Let now $d \in D_A$ and $u \in U_N$. Then $\beta_{(u^{-1}d)^{-1}} = \beta_{d^{-1}} \circ \beta_u$, where β_u is the inner automorphism of u on U_N and $\beta_d : \mathrm{Stab}_{U_N}(A) \to \mathrm{Stab}_{U_N}(d.A)$ is a bijection. So we have

$$u^{-1} H_A u \cap \mathrm{Stab}_{U_N}(d.A) = \beta_d \left(\beta_{(u^{-1}d)^{-1}}(H_A) \cap \mathrm{Stab}_{U_N}(A)\right) = \beta_d(I) = I.$$

Let $x \in H_A$ and $g \in G_N$. For $1 \le i < j \le N$ with $i+j < N+1$ we have $(\pi(x) \circ g)_{ij} = \sum_{k=i+1}^{j} x_{ik} g_{kj}$ and therefore $(i, j) \in \mathrm{supp}((\pi(x) \circ g)$ if there is a $i < k < \bar{i}$ with $(i,k) \in \mathrm{supp}_{\mathcal{V}}(x)$. It follows that $(i, j) \in \mathrm{supp}_{\mathcal{V}}(H_A)$ and therefore $\pi(x) \circ g \in \pi(H_A)$.
Let now $X \in (G_N.A - A) \cap \pi(H_A)$. If there are $1 \le i < j \le N$ with $i + j < N + 1$ such that $X_{ij} \neq 0$, then there is a $j < l < \bar{i}$ such that $(i,l) \in \mathrm{supp}(A)$. But since $(i, j) \in \mathcal{H}_A$ there is a $1 \le k < \bar{i}$ with $(\bar{i},k) \in \mathrm{supp}(A)$, which contradicts A being a verge pattern as defined in 2.2.5 since we would have had $(i,l), (i,\bar{k}) \in \mathrm{supp}(A - A^{\dagger})$. It follows that $X = 0$ and therefore $(G_N.A - A) \cap \pi(H_A) = 0$. □



Subgroup $H_A \le U_N$ for the lower hooks of the verge pattern A

Now we can calculate value of the Jedlitschky characters for standard core tableaux for elements of the normal subgroup $U_1 \trianglelefteq U_N$, where the value is zero unless the element is contained in U_2.

Proposition 4.2.4. *For a standard verge tableau $\mathbf{T}$ let $A \in \mathbf{v}$ be the verge pattern for $\mathbf{T}$. Then $\pi(\mathfrak{g}(1))$ decomposes into the direct sum of mutually orthogonal linear subspaces with*

$$\pi(\mathfrak{g}(1)) = (G_N.A - A \cap \pi(\mathfrak{g}(1))) \oplus (\pi(H_A) \cap \pi(\mathfrak{g}(1)))\,.$$

Proof. Let $\mathbf{T}$ be a standard verge tableau and $A \in \mathbf{v}$ the verge pattern for $\mathbf{T}$. We assume that $\mathfrak{g}(1) \neq 0$, and therefore the partition $\lambda \vdash N$ for the tableau $\mathbf{T}$ contains both odd and even elements. For $1 \le i < j \le N$ with $i + j < N + 1$ such that $e_{ij} \in \pi(\mathfrak{g}(1))$ there is an $a < 0$ such that i is contained in the a column and j is contained in the $a + 1$ column of $\mathbf{T}$. There is a $i < k \le N$ that is contained in the $a + 2$ column of $\mathbf{T}$ in the same row as i. If $i + k < N + 1$, we have $(i, k) \in \mathrm{supp}(A)$ and since λ does not only contain even elements and j is contained in the $a+1$ column while k is contained in the $a+2$ column, we also have $j < k$. Then we have $A + e_{ij} = (I + e_{jk}).A$ and $e_{ij} \in G_N.A - A$. If $i + k > N + 1$, we have $(\bar{k}, \bar{i}) \in \mathrm{supp}(A)$ and therefore $e_{ij} \in \pi(H_A)$. It follows that $\pi(\mathfrak{g}(1)) \leq (G.A - A) + \pi(H_A)$, and by lemma 4.2.3 we have $\pi(\mathfrak{g}(1)) = (G.A - A) + \pi(H_A)$. Since $(G.A - A)$ and $\pi(H_A)$ are distinct pattern linear subspaces of $\mathbf{v}$, they are mutually orthogonal by lemma 2.1.6. □

Theorem 4.2.5. *For a standard core tableau $(\boldsymbol{T}, \boldsymbol{S})$ let $C \in \boldsymbol{v}$ be the corresponding core pattern. Then for $x \in U_1$ we have $\psi_C(x) = 0$ unless $x \in U_2$.*

Proof. Let $(\mathbf{T}, \mathbf{S})$ be a standard core tableau and let $A, C \in \mathbf{v}$ be the corresponding verge and core pattern respectively, for which $d \in D_A$ is such that $C = d.A$. For $W \in \pi(\mathfrak{g}(1)) \cap G_N.A - A$ we have $C + W \in G_N.A$, and there are $d_W \in D_A$ and $u \in U_N$ such that $ud_W.A = C + W$. We can then calculate the matrix $S(d_W) \in M_n(\mathbb{F}_q)$, which was defined in lemma 2.4.12:

(i) For $1 < i \leq \tilde{n}_{-3}$ and $1 \leq j \leq N$ we have $(C+W)_{ik} = 0$ for all $k > \tilde{n}_{-1}$ and $(C+W)_{jl} = 0$ for all $l \geq \tilde{n}_1$. Since $\tilde{n}_{-1} + \tilde{n}_1 = N$, it follows that $S_{ij}(d_W) = b\left((C+W)^t e_i, (C+W)^t e_j\right) = 0$, which especially means $S_{ij}(d_W) = S_{ij}(d)$.

(ii) For $\tilde{n}_{-3} < i, j \leq \tilde{n}_{-2}$ we have $W_{ik} = W_{jk} = 0$ for all $k \geq \tilde{n}_{-1}$ and $C_{il} = C_{jl} = 0$ for all $l \geq \tilde{n}_1$. Again since $\tilde{n}_{-1} + \tilde{n}_1 = N$ it follows that

$$S_{ij}(d_W) = b\left((C+W)^t e_i, (C+W)^t e_j\right) = b\left(C^t e_i + W^t e_i, C^t e_j + W^t e_j\right) = b\left(C^t e_i, C^t e_j\right) = S_{ij}(d).$$

(iii) For $\tilde{n}_{-3} < i \leq \tilde{n}_{-2}$ and $\tilde{n}_{-2} < j \leq \tilde{n}_{-1}$ we have $W_{ik} = 0$ unless $\tilde{n}_{-2} \leq k \leq \tilde{n}_{-1}$ and $C_{il} = W_{jl} = 0$ unless $\tilde{n}_{-1} \leq l \leq \tilde{n}_0$ as well as $C_{jm} = 0$ unless $\tilde{n}_0 \leq m \leq \tilde{n}_1$. This gives us $b\left(C^t e_i, C^t e_j\right) = 0$, $b\left(W^t e_i, C^t e_j\right) = 0$ and $b\left(W^t e_i, W^t e_j\right) = 0$, and it follows that

$$S_{ij}(d_W) = b\left((C+W)^t e_i, (C+W)^t e_j\right) = b\left(C^t e_i, W^t e_j\right).$$

(iv) For $\tilde{n}_{-2} < i, j \leq \tilde{n}_{-1}$ we assume without loss of generality that $i \leq j$. Since for k being in the same row as i of and the 1 column of $\mathbf{T}$, we have $i + k = N$ and $\bar{k} = i + 1$. If $j = \bar{k}$, we have $X_{jl} = 0$ for all $1 \leq l \leq N$ since $j + \bar{i} = N + 2$, and therefore A has no non-zero entry in the j-th row. For $j > i + 1$ we have $j > \bar{k}$ and therefore $(i, j) \in \mathcal{Z}_A$. So $(i, j) \in \mathcal{S}_A$ if and only if $i = j$ since the same argument holds for the case of $i \geq j$. We have $W_{ik} = 0$

unless $\tilde{n}_{-1} \leq k \leq \tilde{n}_0$ and $C_{il} = 0$ unless $\tilde{n}_0 \leq l \leq \tilde{n}_1$, and therefore $b\left(C^t e_i, C^t e_i\right) = 0$ as well as $b\left(C^t e_i, W^t e_i\right) = b\left(W^t e_i, C^t e_i\right) = 0$. This gives us

$$S_{ii}(d_W) = b\left((C+W)^t e_i, (C+W)^t e_i\right) = b\left(W^t e_i, W^t e_i\right)$$

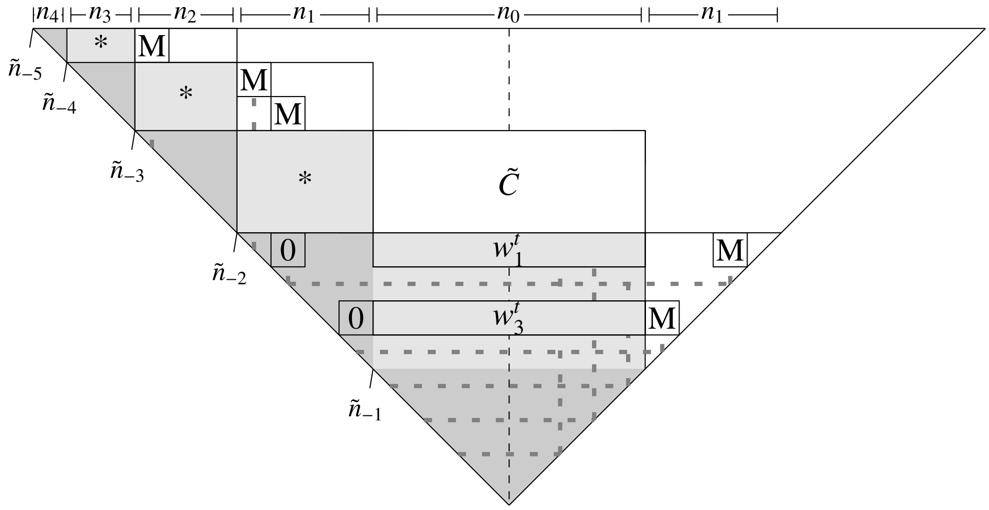
and $S_{ij}(d_W) = S_{ij}(d) = 0$ for every $i \neq j$ with $(i, j) \in \mathcal{S}_A$.

(v) For $\tilde{n}_{-1} < i \leq n$ and $1 \leq j \leq n$ for k being in the same row as i of and the 2 column of **T** we have $i + k \geq N + 1$, so A has no non-zero entry in the i-th row and we have $S_{ij}(d_W) = S_{ij}(d) = 0$.

Let $\tilde{C} \in M_{n_{-2} \times n_0}(\mathbb{F}_q)$ and $Z \in M_{n_{-2}}(\mathbb{F}_q)$ be the restrictions of the core pattern C and the matrix $S(d) \in M_n(\mathbb{F}_q)$ respectively with

$$\tilde{C} = C|_{\{\tilde{n}_{-3},\ldots\tilde{n}_{-2} \times \tilde{n}_{-1},\ldots\tilde{n}_0\}} \quad \text{and} \quad Z = S(d)|_{\{\tilde{n}_{-3},\ldots\tilde{n}_{-2}\}}.$$

Then we have $\tilde{C} J_{n_0} \tilde{C}^t = Z$ by theorem 3.3.5 and $\det(Z) \neq 0$ as a prerequisite of theorem 3.3.5. For $1 \leq k \leq n_{-1}$ let $w_k \in \mathbb{F}_q^{\,n-2}$ be the vector with $(w_k)_i = W_{\tilde{n}_{-2}+k, \tilde{n}_{-1}+i}$ for $1 \leq i \leq n_0$. For $\tilde{n}_{-3} < i \leq \tilde{n}_{-2}$ and $\tilde{n}_{-2} < j \leq \tilde{n}_{-1}$ we then have $b\left(C^t e_i, W^t e_j\right) = \tilde{C} w_{j-\tilde{n}_{-2}}$, while for $\tilde{n}_{-2} < i \leq \tilde{n}_{-1}$ we have $b\left(W^t e_i, W^t e_i\right) = w^t_{i-\tilde{n}_{-2}} J_{n_0} w_{i-\tilde{n}_{-2}}$.



The pattern $C + W$ for $W \in \pi(\mathfrak{g}(1))$

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&&&&&0&&0&0&0&0&0\\
0&0&0&&Z&&\tilde{C}w_1&0&\tilde{C}w_3&0&0&0&0&0\\
0&0&0&&&&&0&&0&0&0&0&0\\
0&0&1&&(\tilde{C}w_1)^t&&w_1^t J_{n_0} w_1&0&\mathbf{z}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&&(\tilde{C}w_3)^t&&\mathbf{z}&0&w_3^t J_{n_0} w_3&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

The matrix $S(d_W)$ for $W \in \pi(\mathfrak{g}(1))$

Let $v_W \in R_A$ be defined by $(v_W)_{ij} = 0$ for $1 \leq i < j \leq n$ unless for $\tilde{n}_{-2} < i \leq \tilde{n}_{-1}$ and $\tilde{n}_{-1} < j \leq \tilde{n}_0$ if there is a $j < l < \overline{j}$ with $(j, l) \in \operatorname{supp}(A)$, in which case let $(v_W)_{ij} = (-Zw_{i-\tilde{n}_{-2}})_{ij}$. As described in lemma 2.4.14 the element v_W partially diagonalizes the matrix $S(d_W)$. That is, for every $1 \leq k \leq n_{-1}$ for which there is a $k + \tilde{n}_{-2} < l < \overline{k + \tilde{n}_{-2}}$ with $(k + \tilde{n}_{-2}, l) \in \operatorname{supp}(A)$ the restrictions of $S(d_W)$ and $S(d_W \bullet_A v_W)$ to $M_k = \{\tilde{n}_{-2}+1, \ldots \tilde{n}_{-1}, k+\tilde{n}_{-2}\}\times\{\tilde{n}_{-2}+1, \ldots \tilde{n}_{-1}, k+\tilde{n}_{-2}\}$ are

$$S(d_W)|_{M_k} = \begin{pmatrix} Z & w_k \\ w_k^t & w_k^t J_{n_0} w_k \end{pmatrix} \text{ and } S(d_W \bullet_A v_W)|_{M_k} = \begin{pmatrix} Z & 0 \\ 0 & w_k^t J_{n_0} w_k - (\tilde{C}w_k)^t Z^{-1} \tilde{C} w_k \end{pmatrix}.$$

Moreover, for every $1 \leq i \leq n$ and $\tilde{n}_{-1} < j \leq n_{-0}$ for which there is a $j < l < \overline{j}$ with $(j, l) \in \operatorname{supp}(A)$ we have for the following cases:

(i) $S_{ik}(d_W) = 0$ for $\tilde{n}_{-2} < k \leq n_{-1}$ if $i \leq \tilde{n}_{-2}$ and therefore $S_{ij}(d_W \bullet_A v_W) = S_{ij}(d) = 0$

(ii) $S_{ik}(d_W) = 0$ for $\tilde{n}_{-2} < k \leq n_{-1}$ if $\tilde{n}_{-1} < i \leq \tilde{n}_0$ such that there is no $i < l < \overline{i}$ with $(i, l) \in \operatorname{supp}(A)$ and therefore $S_{ij}(d_W \bullet_A v_W) = S_{ij}(d) = 0$

(iii) $(i, j) \in \mathcal{Z}_A$ if $\tilde{n}_{-1} < i \leq \tilde{n}_0$ with $i \neq k$ such that there is a $i < l < \bar{i}$ with $(i, l) \in \mathrm{supp}(A)$

(iv) $S_{ik}(d_W) = 0$ for $\tilde{n}_{-2} < k \leq n_{-1}$ if $i > \tilde{n}_0$ and therefore $S_{ij}(d_W \bullet_A v_W) = S_{ij}(d) = 0$

(v) $S_{jj}(d_W \bullet_A v_W) = w^t_{j-\tilde{n}_{-1}} J_{n_0} w_{j-\tilde{n}_{-1}} - (\tilde{C} w_{j-\tilde{n}_{-1}})^t Z^{-1} \tilde{C} w_{j-\tilde{n}_{-1}}$

For $k = j - \tilde{n}_{-1}$ let now $X = \tilde{C}^t Z^{-1} \tilde{C} \in M_{n_{-2}}(\mathbb{F}_q)$. We then have $\tilde{C} J_{n_0} X = (\tilde{C} J_{n_0} \tilde{C}^t) Z^{-1} \tilde{C} = \tilde{C}$ and therefore $\tilde{C}(J_{n_0} X - I) = 0$. But since $\tilde{C}$ has maximal rank, it follows that $J_{n_0} X - I = 0$ and therefore $X = J_{n_0}$. We then have

$$S_{jj}(d_W \bullet_A v_W) = w^t_{j-\tilde{n}_{-1}} J_{n_0} w_{j-\tilde{n}_{-1}} - w^t_{j-\tilde{n}_{-1}} (\tilde{C}^t Z^{-1} \tilde{C}) w_{j-\tilde{n}_{-1}} = 0 = S_{jj}(d),$$

which concludes that $S(d_W \bullet_A v_W) = S(d)$ and by theorem 2.4.10 we have $\psi_{C+W} = \psi_C$ for every $W \in \pi(\mathfrak{g}(1)) \cap G_N.A - A$.

Let now $\mathbf{w} = \pi(\mathfrak{g}(1)) \cap G_N.A - A$, $x \in U_1$ and $g \in U_N$. Let $X_1 \in \pi(\mathfrak{g}(1))$ and $X_2 \in \pi(\mathfrak{u}_2)$ such that $\pi(x) \circ g = X_1 + X_2$. Since $\pi(\mathfrak{g}(1)) = \mathbf{w} \oplus (\pi(H_A) \cap \pi(\mathfrak{g}(1)))$ is a direct sum of orthogonal complements in $\pi(\mathfrak{g}(1))$ by proposition 4.2.4 with $\kappa(W, X_2) = 0$ for all $W \in \mathbf{w}$, by lemma 2.1.7 we have

$$\sum_{W \in \mathbf{w}} \vartheta\kappa(W, \pi(x) \circ g) = \sum_{W \in \mathbf{w}} \vartheta\kappa(W, X_1) = \begin{cases} |\mathbf{w}| & \text{for } X_1 \in \pi(H_A) \\ 0 & \text{otherwise.} \end{cases}$$

We have $X_2 \circ g^{-1} \in \pi(\mathfrak{u}_2)$ and for $X_1 \in \pi(H_A)$ we have $X_1 \circ g^{-1} \in \pi(H_A)$ by lemma 4.2.3. So if $\sum_{Y \in \mathbf{w}} \vartheta\kappa(Y, \pi(x) \circ g) \neq 0$ for some $g \in U_N$, there is a decomposition for $\pi(x)$ with $\pi(x) = X_1' + X_2'$ such that $X_1' = X_1 \circ g^{-1} \in \pi(H_A)$ and $X_2' = X_2 \circ g^{-1} \in \pi(\mathfrak{u}_2)$.
By theorem 2.4.10 we now have $\psi_{C+W} = \psi_C$ for every $W \in \mathbf{w}$ and for $x \in U_1$ it follows that

$$\begin{aligned}
\psi_C(x) &= \frac{1}{|\mathbf{w}|} \sum_{W \in \mathbf{w}} \psi_{C+W}(x) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)||\mathbf{w}|} \sum_{W \in \mathbf{w}} \sum_{u,g \in U_N} \vartheta\kappa\left(g.(W + C), \pi(u^{-1}xu)\right) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)||\mathbf{w}|} \sum_{u,g \in U_N} \vartheta\kappa\left(g.C, \pi(u^{-1}xu)\right) \left(\sum_{W \in \mathbf{w}} \vartheta\kappa\left(g.W, \pi(u^{-1}xu)\right) \right) \\
&= \frac{1}{|U_N||\mathrm{Stab}_{U_N}(C)||\mathbf{w}|} \sum_{u,g \in U_N} \vartheta\kappa\left(g.C, \pi(u^{-1}xu)\right) \left(\sum_{W \in \mathbf{w}} \vartheta\kappa\left(W, \pi(u^{-1}xu) \circ g\right) \right).
\end{aligned}$$

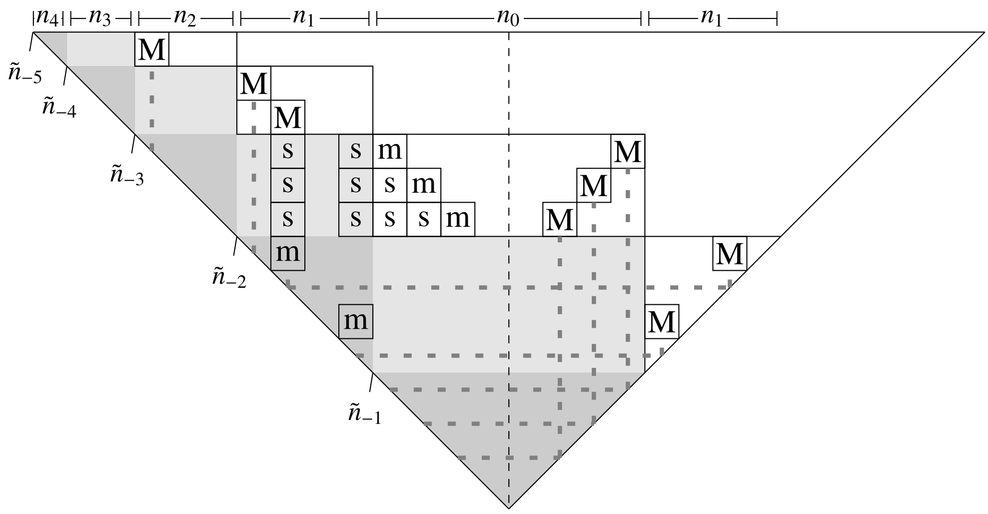
So if $\psi_C(x) \neq 0$, there is a $u \in U_N$ such that for $y = u^{-1}xu$ there are $Y_1 \in \pi(H_A)$ and $Y_2 \in \pi(\mathfrak{u}_2)$ with $\pi(y) = Y_1 + Y_2$. Let $y_2 \in U_2$ be such that $\pi(y_2) = Y_2$. Then we have

$\pi(yy_2^{-1}) \circ y_2 = \pi(y) - \pi(y_2) = Y_1 \in \pi(H_A)$ and therefore $\pi(yy_2^{-1}) \in \pi(H_A)$ by lemma 4.2.3. For $y_1 = yy_2^{-1}$ we then have $y_1 \in H_A$ and $y = y_1 y_2$. If $\psi_C(x) = \mathrm{Ind}_{\mathrm{Stab}_{U_N}(C)}^{U_N} \chi_C(x) \neq 0$, there is a $v \in U_N$ such that $v^{-1}yv \in \mathrm{Stab}_{U_N}(C)$ and we have

$$0 = (v^{-1}yv).C - C = (v^{-1}y_1v)(v^{-1}y_2v).C - C = (v^{-1}y_1v).\Big((v^{-1}y_2v).C - C\Big) + (v^{-1}y_1v).C - C.$$

Furthermore, we have $v^{-1}y_2v \in U_2$ since $U_2 \trianglelefteq U_1$, which gives us $(v^{-1}y_2v).C - C \in \pi(\mathfrak{g}(0))$ as well as $(v^{-1}y_1v).\Big((v^{-1}y_2v).C - C\Big) \in \pi(\mathfrak{g}(0))$ since $C \in \pi(\mathfrak{g}(2))$. But by lemma 4.2.3 we also have $v^{-1}y_1v \notin \mathrm{Stab}_{U_N}(C)$. It then follows that $(v^{-1}y_1v).C - C \neq 0$, which necessitates $(v^{-1}y_1v).C - C \in \pi(\mathfrak{g}(0))$. As $C \in \pi(\mathfrak{g}(2))$, it follows that $v^{-1}y_1v \in U_2$, which gives us $y_1 \in U_2$ as well as $y = y_1y_2 \in U_2$ and finally $x = uyu^{-1} \in U_2$.

□



Core pattern $C \in \mathbf{v}$ with minor conditions in $\pi(\mathfrak{g}(0))$ and supplementary conditions in $\pi(\mathfrak{g}(1))$

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&m&s&s&s&0&s&0&0&0&0&0\\
0&0&0&s&m&s&s&0&s&0&0&0&0&0\\
0&0&0&s&s&m&s&0&s&0&0&0&0&0\\
0&0&1&s&s&s&m&0&\mathbf{z}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&s&s&s&\mathbf{z}&0&m&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Matrix $S(d) \in M_n(\mathbb{F}_q)$ for the core pattern with positions relating to $\pi(\mathfrak{g}(0))$ (dark grey) and $\pi(\mathfrak{g}(1))$ (light grey)

With this result we can relate the Jedlitschky character for a standard core tableau to the generalized Gelfand–Graev character by reducing it to the normal subgroup U_1 and inducing it back to U_N. Inducing this character further to the whole group SO_N then results in the generalized Gelfand–Graev character up to a scalar.

Theorem 4.2.6. *For a standard core tableau $(\boldsymbol{T}, \boldsymbol{S})$ let $C \in \boldsymbol{v}$ be the corresponding core pattern. For $\pi(\mathfrak{g}(2))$ let $\xi_{W-W^\dagger} : U_2 \to \mathbb{C}$ be the linear character of U_2 as defined in 4.1.3 with $\xi_{W-W^\dagger}(x) = \vartheta\kappa(W - W^\dagger, f(x))$ for $x \in U_2$. We then have*

$$Ind_{U_1}^{U_N} Res_{U_1}^{U_N} \psi_C = \frac{|U_N.C|}{|\mathfrak{g}(1)|} Ind_{U_2}^{U_N} \xi_{C-C^\dagger}$$

Proof. Let $(\mathbf{T}, \mathbf{S})$ be a standard core tableau and let $C \in \mathbf{v}$ be the corresponding core pattern. Since $U_2 \trianglelefteq U_1 \trianglelefteq U_N$ for $x \in U_N$, we have $v^{-1}xv \in U_2$ for any $v \in U_N$ if and only if $x \in U_2$ and by theorem 4.2.5 it follows that

$$\mathrm{Ind}_{U_1}^{U_N} \mathrm{Res}_{U_1}^{U_N} \psi_C(g) = \frac{1}{|U_1|} \sum_{\substack{v \in U_N \\ v^{-1}gv \in U_2}} \psi_C(v^{-1}gv) = \begin{cases} \frac{|U_N|}{|U_1|}\psi_C(g) & \text{if } g \in U_2 \\ 0 & \text{if } g \in U_N \backslash U_2. \end{cases}$$

Since $C \in \pi(\mathfrak{g}(2))$, we have $h.C \in \pi(\mathfrak{g}(2))$ for $h \in L$ and $wh.C - h.C \in \bigoplus_{z<2} \pi(\mathfrak{g}(z))$ for $w \in U_1$ by lemma 4.2.2. Furthermore, for $x \in U_2$ and $v \in U_n$ we have $\kappa((wh.C - h.C)^t, \pi(v^{-1}xv)) = 0$ since $\pi(v^{-1}xv) \in \bigoplus_{z \geq 2} \pi(\mathfrak{g}(z))$ and it follows that

$$
\begin{aligned}
\mathrm{Ind}_{U_1}^{U_N} \mathrm{Res}_{U_1}^{U_N} \psi_C(x) &= \frac{1}{|\mathrm{Stab}_{U_N}(C)||U_1|} \sum_{u,v \in U} \vartheta\kappa(u.C, \pi(vxv^{-1})) \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)||U_1|} \sum_{v \in U} \sum_{\substack{h \in L \\ w \in U_1}} \vartheta\kappa((wh.C)^t, \pi(vxv^{-1})) \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)||U_1|} \sum_{v \in U} \sum_{h \in L} \left(\vartheta\kappa(h.C, \pi(vxv^{-1})) \sum_{w \in U_1} \vartheta\kappa(wh.C - h.C, \pi(vxv^{-1})) \right) \\
&= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{v \in U} \sum_{h \in K} \vartheta\kappa(h.C, \pi(vxv^{-1})).
\end{aligned}
$$

For $x \in U_2$ let $m \in \mathbb{N}$ be the lowest integer such that $(x - I)^m = 0$. By lemma 1.3.12 we have $f(x) = \sum_{k=1}^{m-1} (-1)^{k-1} \frac{1}{2^k} (x-1)^k$ and therefore $\pi(x) - 2\pi(f(x)) = \sum_{k=2}^{m-1} (-1)^k \frac{1}{2^{k-1}} \pi((x-1)^k) \in \pi(\mathfrak{u}_4)$. So for $W \in \pi(\mathfrak{g}(2))$ we have

$$
\kappa\left(W^t, \pi(x)\right) = \kappa\left(W^t, 2\pi(f(x))\right) = \kappa\left(W^t, 2f(x)\right) = \kappa\left(W^t, f(x) - f(x)^\dagger\right) = \kappa\left(W - W^\dagger, f(x)\right).
$$

Let $\xi_{W-W^\dagger} : U_2 \to \mathbb{C}$ be now the linear character with $\xi_{W-W^\dagger}(x) = \vartheta\kappa(W - W^\dagger, f(x))$. Since $U_2 \trianglelefteq U$ for $x \in U_N$, we have $\mathrm{Ind}_{U_2}^{U_N} \xi_{W-W^\dagger}(x) = 0$ unless $x \in U_2$ in which case it follows that $\mathrm{Ind}_{U_2}^{U_N} \xi_{W-W^\dagger}(x) = 1/|U_2| \sum_{v \in U_N} \xi_{W-W^\dagger}(vxv^{-1})$. Equally, we have $\mathrm{Ind}_{U_1}^{U_N} \mathrm{Res}_{U_1}^{U_N} \psi_C(x) = 0$ unless $x \in U_2$. So for $x \in U_2$ we have

$$
\begin{aligned}
\mathrm{Ind}_{U_1}^{U_N} \mathrm{Res}_{U_1}^{U_N} \psi_C(x) &= \frac{1}{|\mathrm{Stab}_{U_N}(C)|} \sum_{\substack{u \in L \\ v \in U_N}} \vartheta\kappa(u.C - (u.C)^\dagger, f(vxv^{-1})) \\
&= \frac{|U_2|}{|\mathrm{Stab}_{U_N}(C)|} \sum_{u \in L} \mathrm{Ind}_{U_2}^{U_N} \xi_{u.C-(u.C)^\dagger}(x).
\end{aligned}
$$

Unless λ is the secondary partition with only even elements, for $m \in \mathbb{N}_0$ let $L_m \leq L$ be the pattern subgroup of L defined by

$$
L_m = \left\{ g \in L \mid g_{ij} = 0 \text{ for } (i,j) \in \mathcal{V} \text{ with } i \leq \tilde{n}_{-(m+1)} \text{ or } \tilde{n}_{-m} < j \right\}.
$$

These subgroups contain the individual blocks giving rise to the block matrices that are the elements of L where $L = \prod_{m\in\mathbb{N}_0} L_m$ is a direct product, and we have $L_0 \cong U_{n_0}$ as well as $L_m \cong G_{n_m}$ for $m > 0$. Furthermore, for $m \in \mathbb{N}$ and $W \in \pi(\mathfrak{g}(2))$ let $W^{(m)} \in \mathbf{v}$ be the matrix such that for $1 \leq i < j \leq m$ with $i + j < N + 1$ we have $W^{(m)}_{ij} = W_{ij}$ if $\tilde{n}_{-(m+1)} < i \leq \tilde{n}_{-m}$ and $\tilde{n}_{-(m-1)} < j \leq \tilde{n}_{-(m-2)}$, whereas we have $W^{(m)}_{ij} = 0$ otherwise. Conversely, if λ is the secondary partition with only even elements, we define $L_m \leq L$ for $1 \neq m \in \mathbb{N}_0$ the same way, while we have

$$L_1 = \left\{g \in L \mid g_{ij} = 0 \text{ for } (i, j) \in \mathcal{V} \text{ with } i \leq \tilde{n}_{-2} \text{ or } \tilde{n}_{-1} + 1 < j \text{ or } j = n\right\}.$$

Again, these subgroups give rise to the direct product $L = \prod_{m\in\mathbb{N}_0} L_m$ with $L_1 \cong G_{n_1}$. Analogously, for $W \in \pi(\mathfrak{g}(2))$ we define $W^{(m)} \in \mathbf{v}$ the same way for $m \geq 2$, while for $1 \leq i < j \leq m$ with $i + j < N + 1$ we have $W^{(1)}_{ij} = W_{ij}$ if $\tilde{n}_{-2} < i \leq n$ and $n - 1 < j \leq \tilde{n}_1$ with $j \neq n + 1$, whereas we have $W^{(1)}_{ij} = 0$ otherwise. In any of these cases we have $W = \sum_{m\in\mathbb{N}} W^{(m)}$ as well as $L_k \leq \mathrm{Stab}_{U_N}(W^{(m)})$ for all $k \in \mathbb{N}_0$ with $m \neq |k - 2|$. Furthermore, for $u \in L$ and $W \in \pi(\mathfrak{g}(2))$ we have $\mathrm{supp}(Wu^t) \subseteq \mathcal{V}$ and therefore $u.W = Wu^t$. So for $u \in L$ and $u_m \in L_m$ for $m \in \mathbb{N}_0$ such that $u = \prod_{m\in\mathbb{N}_0} u_m$ we have

$$u.C = u_1.C^{(1)} + \sum_{m\geq 2} u_{m-2}.C^{(m)} = C^{(1)}u_1^t + \sum_{m\geq 2} C^{(m)}u_{m-2}^t.$$

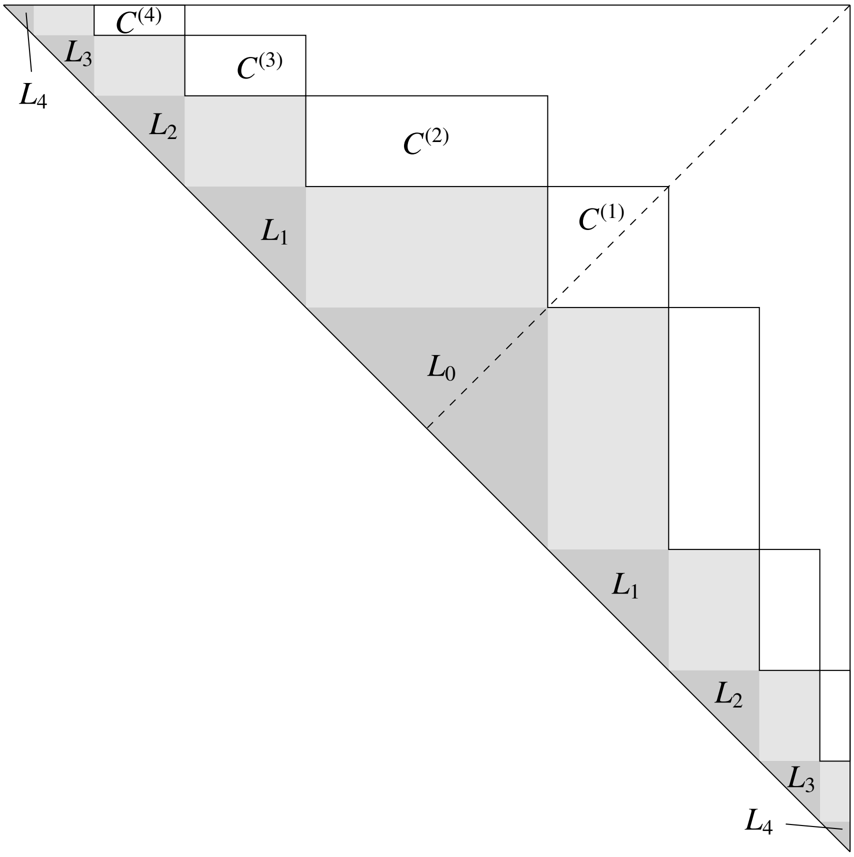
For the left multiplication of transposed elements of L for $m \in \mathbb{N}$ we have $v^tC^{(m)} = C^{(m)}$ if $v \in L_k$ for $m \neq k \in N_0$.

For $m = 1$ the pattern $C^{(1)} \in \mathbf{v}$ is a verge pattern with $C^{(1)} = \sum_{k=1}^{n_1/2} e_{\tilde{n}_{-2}+2k-1,\tilde{n}_2-2(k-1)}$, so $\mathrm{Stab}_{L_1}(C^{(1)})$ is a pattern subgroup with $u \in \mathrm{Stab}_{L_1}(C^{(1)})$ if for $(i, j) \in \mathcal{V}$ we have $u_{ij} = 0$ with $i - \tilde{n}_{-2} \in \{2, 4, \dots n_1\}$ and $1 \leq j \leq N$. Then for the pattern subgroup $\tilde{L}_1 \leq L_1$ defined by

$$\tilde{L}_1 = \left\{u \in L_1 \mid u_{ij} = 0 \text{ for } (i, j) \in \mathcal{V} \text{ with } i - \tilde{n}_{-2} \in \{1, 3, \dots n_1 - 1\}, 1 \leq j \leq N\right\}$$

we have $L_1 = \tilde{L}_1\mathrm{Stab}_{L_1}(C^{(1)})$ and therefore $L_1.C^{(1)} = \tilde{L}_1.C^{(1)}$. Furthermore, for $\tilde{u} \in \tilde{L}_1$ we have $\tilde{u}^tC^{(1)} = C^{(1)}$. So for $u \in L_1$ there is a $\tilde{u} \in \tilde{L}_1$ such that $C^{(1)}u^t = C^{(1)}\tilde{u}^t = (\tilde{u}^{-1})^tC^{(1)}\tilde{u}^t$.

For $m \geq 3$ the pattern $C^{(m)} \in \mathbf{v}$ is again a verge pattern with $C^{(m)} = \sum_{k=1}^{n_m} e_{\tilde{n}_{-(m+1)}+k,\tilde{n}_{-m}+k}$ with diagonal entries shifted $n_{m-1} + n_m - 1$ entries above the main diagonal. So for $u \in L_{m-2}$ there is a $v(u) \in L_m$ with $v(u)_{\tilde{n}_{-(m+1)}+i,\tilde{n}_{-(m+1)}+j} = u_{\tilde{n}_{-(m-1)}+i,\tilde{n}_{-(m-1)}+j}$ for $1 \leq i < j \leq n_m$ such that $C^{(m)}u^t = v(u)^tC^{(m)}$.



Decomposition of $C \in \pi(\mathfrak{g}(2))$ and direct product of L

Let now $u \in L$ and $u_m \in L_m$ for $m \in \mathbb{N}_0$ such that $u = \prod_{m \in \mathbb{N}_0} u_m$. Let $v_0 = u_0$, $v_2 = I$ and $v_1 \in L_1$ be such that $C^{(1)} u_1^t = (v_1^{-1})^t C^{(1)} v_1^t$ as described above. For $m \geq 3$ we define $v_m \in L_m$ recursively by $v_m^{-1} = v(v_{m-2}^{-1} u_{m-2})$ and we have

$$\begin{aligned}
u.C &= C^{(1)}u_1^t + \sum_{m\geq 2} C^{(m)}u_{m-2}^t \\
&= (v_1^{-1})^t C^{(1)} v_1^t + \sum_{m\geq 2} C^{(m)}u_{m-2}^t \\
&= (v_1^{-1})^t \left(C^{(1)} + v_1^t \left(\sum_{m\geq 2} C^{(m)} u_{m-2}^t \right) (v_1^{-1})^t \right) v_1^t \\
&= (v_1^{-1})^t \left(C^{(1)} + C^{(2)}u_0^t + C^{(3)}(v_1^{-1}u_1)^t + C^{(4)}u_2 + \sum_{m\geq 4} C^{(m)}u_{m-2}^t \right) v_1^t \\
&= (v_1^{-1})^t \left(C^{(1)} + (v_0^{-1})^t C^{(2)} v_0^t + (v_3^{-1})^t C^{(3)} + (v_4^{-1}) t C^{(4)} \sum_{m\geq 4} C^{(m)}u_{m-2}^t \right) v_1^t \\
&= \left(\prod_{k=0}^{4} v_k^{-1} \right)^t \left(C^{(1)} + C^{(2)} + C^{(3)} + C^{(4)} + \left(\sum_{m\geq 4} C^{(m)}u_{m-2}^t \right) (v_3^{-1})^t (v_4^{-1})^t \right) \left(\prod_{k=0}^{4} v_k \right)^t \\
&= \left(\prod_{k\in\mathbb{N}_0} v_k^{-1} \right)^t \left(\sum_{m\in\mathbb{N}} C^{(m)} \right) \left(\prod_{k\in\mathbb{N}_0} v_k \right)^t .
\end{aligned}$$

For $v = \prod_{k\in\mathbb{N}_0} v_k \in L$ we then have $u.C = (v^{-1})^t C v^t$. So for $x \in U_2$ this gives us

$$\begin{aligned}
\xi_{u.C-(u.C)^\dagger}(x) &= \vartheta\kappa\left(u.C - (u.C)^\dagger, f(x)\right) \\
&= \vartheta\kappa\left((v^{-1})^t C v^t - \left((v^{-1})^t C v^t\right)^\dagger, f(x)\right) \\
&= \vartheta\kappa\left((v^{-1})^t \left(C - C^\dagger\right) v^t, f(x)\right) \\
&= \vartheta\kappa\left(\left(C - C^\dagger\right), v^{-1} f(x) v\right) \\
&= \vartheta\kappa\left(\left(C - C^\dagger\right), f\left(v^{-1}xv\right)\right) \\
&= \xi_{C-C^\dagger}\left(v^{-1}xv\right).
\end{aligned}$$

For $x \in U_N$ we then have $\mathrm{Ind}_{U_2}^{U_N}\xi_{u.C-(u.C)^\dagger}(x) = \mathrm{Ind}_{U_2}^{U_N}\xi_{C-C^\dagger}(v^{-1}xv) = \mathrm{Ind}_{U_2}^{U_N}\xi_{C-C^\dagger}(x)$, which gives us

$$\mathrm{Ind}_{U_1}^{U_N}\mathrm{Res}_{U_1}^{U_N}\psi_C = \frac{|U_2|}{|\mathrm{Stab}_{U_N}(C)|}\sum_{u\in L}\mathrm{Ind}_{U_2}^{U_N}\xi_{u.C-(u.C)^\dagger} = \frac{|L||U_2|}{|\mathrm{Stab}_{U_N}(C)|}\sum_{u\in L}\mathrm{Ind}_{U_2}^{U_N}\xi_{C-C^\dagger}.$$

Since $U_N = L \trianglelefteq U_1$ as well as $|U_1| = |\mathfrak{g}(1)||U_2|$, we have $\frac{|L||U_2|}{|\mathrm{Stab}_{U_N}(C)|} = \frac{|U_N|}{|\mathfrak{g}(1)||\mathrm{Stab}_{U_N}(C)|} = \frac{|U_N.C|}{|\mathfrak{g}(1)|}$, and the claim follows.

□

Since the generalized Gelfand–Graev character for $C - C^\dagger$ is $\gamma_{C-C^\dagger} = \frac{1}{\sqrt{|\mathfrak{g}(1)|}}\mathrm{Ind}_{U_2}^{\mathrm{SO}_N}\xi_{C-C^\dagger}$, it can also be expressed as the Jedlitschky character reduced to U_1 and induced back to whole group SO_N with

$$\gamma_{C-C^\dagger} = \frac{|U_N.C|}{\sqrt{|\mathfrak{g}(1)|}}\mathrm{Ind}_{U_1}^{\mathrm{SO}_N}\mathrm{Res}_{U_1}^{U_N}\psi_C.$$

These Jedlitschky characters reduced to U_1 and induced back to U_N are again sums Jedlitschky characters over all core pattern, which are restricted to $\pi(\mathfrak{g}(0)) + \pi(\mathfrak{g}(1))$ equal to C. This entails not only a sum over the different minor conditions for the same verge pattern, but it can necessitate new main conditions if the 0 column of $\mathbf{T}$ is too big. If $\mathbf{T}$ is the tableau for the secondary partition with only even elements, it can even result in main conditions being shifted and replaced by a minor condition.

Proposition 4.2.7. *For a standard verge tableau $\boldsymbol{T}$ let $A \in \boldsymbol{v}$ be the verge pattern for $\boldsymbol{T}$. If $n_0 > 2n_2 + 1$, let $K_{\frac{1}{2}n_0-n_2} \leq U_N$ be the pattern subgroup defined by*

$$K_{\frac{1}{2}n_0-n_2} = \left\{g \in U_N \mid g_{ij} = 0\, \text{for}\; i \leq n - \frac{1}{2}n_0 + n_2\; \text{or}\, j \geq n + \frac{1}{2}n_0 - n_2\right\}.$$

Then $K_{\frac{1}{2}n_0-n_2} \cong U_{n_0-2n_2}$ and $\pi(\mathfrak{g}(0))$ decomposes into the direct sum of mutually orthogonal linear subspaces with

$$\pi(\mathfrak{g}(0)) = \begin{cases} \pi\left(G.A - A \cap \pi(\mathfrak{g}(0))\right) \oplus \left(\pi(H_A) \cap \pi(\mathfrak{g}(0))\right) & \text{if}\; n_0 \leq 2n_2,\\ \pi\left(G.A - A \cap \pi(\mathfrak{g}(0))\right) \oplus \left(\pi(H_A) \cap \pi(\mathfrak{g}(0))\right) \oplus \pi(K_{\frac{1}{2}n_0-n_2}) & \text{if}\; n_0 > 2n_2,\\ \pi\left(G.A - A \cap \pi(\mathfrak{g}(0))\right) \oplus \left(\pi(H_A) \cap \pi(\mathfrak{g}(0))\right) & \text{if}\; n_0 = 0\\ & \text{and}\; n+1\; \text{is a entry in the}\; 1\; \text{column of}\; \boldsymbol{T},\\ \pi\left(G.A - A \cap \pi(\mathfrak{g}(0))\right) \oplus \left(\pi(H_A) \cap \pi(\mathfrak{g}(0))\right) \oplus \langle e_{n-1,n+1}\rangle & \text{if}\; n_0 = 0\\ & \text{and}\; n+1\; \text{is a entry in the}\; -1\; \text{column of}\; \boldsymbol{T}. \end{cases}$$

Proof. Let $\mathbf{T}$ be a standard verge tableau and $A \in \mathbf{v}$ the verge pattern for $\mathbf{T}$. Unless $n+1$ is a entry in the -1 column, for every $a \in \mathbb{Z}$ and $1 \leq i, j \leq N$, such that i is contained in the a column and j is contained in the $a+1$ column, we have $i < j$. Conversely, if $n+1$ is a entry in the -1 column, the same holds for every case but $i = n+1$ and $j = n$, where n is contained in the 1 column of $\mathbf{T}$.

Let the partition $\lambda \vdash N$ for the tableau $\mathbf{T}$ be such that it contains at least one odd element. For $1 \leq i < j \leq N$ with $i + j < N + 1$ such that $e_{ij} \in \pi(\mathfrak{g}(0))$ there is an $a \leq 0$ such that i and j are contained in the a column of $\mathbf{T}$. If $a < 0$, there is a $i < k \leq N$ that is contained in the $a + 2$ column of $\mathbf{T}$ in the same row as i. If $i + k < N + 1$, we have $(i, k) \in \operatorname{supp}(A)$, and since j is contained in the $a + 1$ column while k is contained in the $a + 2$ column, we also have $j < k$. Then $A + e_{ij} = (I + e_{jk}).A$ and $e_{ij} \in G_N.A - A$. If $i + k > N + 1$, we have $(\bar{k}, \bar{i}) \in \operatorname{supp}(A)$ and therefore $e_{ij} \in \pi(H_A)$.
Let $n_0 \leq 2n_2$. For i and j being contained in the 0 column of $\mathbf{T}$ we have $\bar{i} > n \geq \tilde{n}_0 - n_2$ since $i \leq n$ and $\tilde{n}_0 = n + \frac{1}{2}n_0$. As $\bar{i}$ is contained in the 0 column and above the $n_2 + 1$-th row of $\mathbf{T}$, there is a $1 \leq k < \bar{i}$ in the -2 column and the same row as $\bar{i}$, and we have $(k, \bar{i}) \in \operatorname{supp}(A)$, which gives us $e_{ij} \in \pi(H_A)$. It follows that $\pi(\mathfrak{g}(0)) \leq (G.A - A) + \pi(H_A)$, and the first claim follows by lemma 4.2.3.
Let now $n_0 > 2n_2$ and let $V \trianglelefteq U_N$ such that $U_N = K_{\frac{1}{2}n_0 - n_2} \ltimes V$. Then we have the decomposition $\mathbf{v} = \pi(K_{\frac{1}{2}n_0 - n_2}) \oplus \pi(V)$ and $e_{ij} \in \pi(V)$ if and only if $i \leq n - \frac{1}{2}n_0 + n_2$. For i and j being contained in the 0 column of $\mathbf{T}$ with $i \leq n - \frac{1}{2}n_0 + n_2$ we have $\bar{i} \geq n + 1 + \frac{1}{2}n_0 - n_2 = \tilde{n}_0 - n_2 + 1$ since $\tilde{n}_0 = n + \frac{1}{2}n_0$. As $\bar{i}$ is contained in the 0 column and above the $n_2 + 1$-th row of $\mathbf{T}$, there is a $1 \leq k < \bar{i}$ in the -2 column and the same row as $\bar{i}$, and we have $(k, \bar{i}) \in \operatorname{supp}(A)$, which gives us $e_{ij} \in \pi(H_A)$. So for every $1 \leq i < j \leq N$ with $i + j < N + 1$ and $i \leq n - \frac{1}{2}n_0 + n_2$ we have either $e_{ij} \in G_N.A - A$ or $e_{ij} \in \pi(H_A)$ and therefore $\pi(\mathfrak{g}(0)) \cap \pi(V) \leq (G.A - A) + \pi(H_A)$. Conversely, if $i > n - \frac{1}{2}n_0 + n_2$ we have $e_{ij} \notin G_N.A - A$ since $i < n$, and therefore i is contained below the n_2-th row in $\mathbf{T}$. Moreover, we have $\bar{i} < n + 1 + \frac{1}{2}n_0 - n_2 = \tilde{n}_0 - n_2 + 1$, so $\bar{i}$ is contained below the n_2-th row in $\mathbf{T}$ as well, and we have $e_{ij} \notin \pi(H_A)$. It follows that $\pi(H_A) \cap \pi(K_{\frac{1}{2}n_0 - n_2}) = 0$ and $(G_N.A - A) \cap \pi(K_{\frac{1}{2}n_0 - n_2}) = 0$, which gives us

$$\pi(\mathfrak{g}(0)) = \pi(k) \oplus (G_N.A - A \cap \pi(\mathfrak{g}(0))) \oplus (\pi(H_A) \cap \pi(\mathfrak{g}(0)))$$

by lemma 4.2.3 since $\pi(K_{\frac{1}{2}n_0 - n_2}) \leq \pi(\mathfrak{g}(0))$, which proves the second claim.
Let now the partition $\lambda \vdash N$ for the tableau $\mathbf{T}$ be such that it contains only even elements and $n + 1$ is contained in the 1 column of $\mathbf{T}$. For $1 \leq i < j \leq N$ with $i + j < N + 1$ such that $e_{ij} \in \pi(\mathfrak{g}(0))$ there is a $a \leq -1$ such that i and j are contained in the a column of $\mathbf{T}$. There is a $i < k \leq N$ that is contained in the $a + 2$ column of $\mathbf{T}$ in the same row as i. If $i + k < N + 1$, we have $(i, k) \in \operatorname{supp}(A)$ and since j is contained in the $a + 1$ column while k is contained in the $a + 2$ column, we also have $j < k$. Then $A + e_{ij} = A.(I + e_{jk})$ and $e_{ij} \in G_N.A - A$. If $i + k > N + 1$,

we have $(\bar{k}, \bar{i}) \in \mathrm{supp}(A)$, and therefore $e_{ij} \in \pi(H_A)$. It follows that $\pi(\mathfrak{g}(0)) \leq (G.A - A) + \pi(H_A)$ and the third claim follows by lemma 4.2.3.

Is $n+1$ is contained in the -1 column of $\mathbf{T}$ then $n-1$ and n are contained in the -1 and 1 column and n_1-1-th row, while $n+1$ and $n+2$ are contained in the -1 and 1 column and n_1-th row respectively. For $1 \leq i < j \leq N$ with $i+j < N+1$ and $i \neq n-1$ such that $e_{ij} \in \pi(\mathfrak{g}(0))$ there is an $a \leq -1$ such that i and j are contained in the a column of $\mathbf{T}$. There is a $i < k \leq N$ that is contained in the $a+2$ column of $\mathbf{T}$ in the same row as i. If $i+k < N+1$, we have $(i,k) \in \mathrm{supp}(A)$ and since j is contained in the $a+1$ column while k is contained in the $a+2$ column we have $j < k$ if and only if $k \neq n$, which is the case if $i \neq n-1$. Then $A + e_{ij} = A.(I + e_{jk})$ and $e_{ij} \in G_N.A - A$. If $i+k > N+1$, we have $(\bar{k}, \bar{i}) \in \mathrm{supp}(A)$, and therefore $e_{ij} \in \pi(H_A)$. Let now $i = n-1$ and $n-1 < j \leq N$ with $n-1+j < N+1$, which forces $j = n+1$ since n is contained in the 1 column of $\mathbf{T}$. We have $e_{n-1,n+1} \notin G_N.A - A$ since n is contained in the 1 column in same row as $n-1$ and therefore $(n-1, n) \in \mathrm{supp}(A)$. Moreover, $n+1$ is contained in the -1 column in the same row as $\bar{i} = n+2$, which is contained in the 1 column, and we have $(n+1)+(n+2) = N+3$, so there is no $1 \leq k < n+1$ such that $(k, n+1) \in \mathrm{supp}(A)$. We therefore have $e_{n-1,n+1} \notin \pi(H_A)$, which gives us

$$\pi(\mathfrak{g}(0)) = \langle e_{n-1,n+1} \rangle \oplus (G_N.A - A \cap \pi(\mathfrak{g}(0))) \oplus (\pi(H_A) \cap \pi(\mathfrak{g}(0)))$$

by lemma 4.2.3 and concludes the last claim. Since all summands in these direct sums are distinct pattern linear subspaces of $\mathbf{v}$, they are mutually orthogonal by lemma 2.1.6. □

Theorem 4.2.8. *For a partition $\lambda \vdash N$ and $i \in \mathbb{N}$ let $m_i \in \mathbb{N}_0$ be the multiplicity with which i occurs in λ such that $\lambda = (1^{m_1}, 2^{m_2}, \dots)$. For $i \in \mathbb{N}$ let m_{2i} be even and let $(\mathbf{T}, \boldsymbol{S})$ be the standard core tableau for λ. Let $A \in \boldsymbol{v}$ and $C \in \boldsymbol{v}$ be the corresponding verge and core patterns for $(\mathbf{T}, \boldsymbol{S})$ respectively. Let $\boldsymbol{m}_0 = \left\langle e_{ij} \in \pi(\mathfrak{g}(0)) \mid (i,j) \in minor(A) \right\rangle$ the space of minor conditions in $\pi(\mathfrak{g}(0))$. If $m_1 > 2 + \sum_{i \in \mathbb{N}} m_{2i+1}$, let $K = K_{\frac{1}{2}(m_1 - \sum_{i\in\mathbb{N}} m_{2i+1})} \leq U_N$ be defined as in proposition 4.2.7, and let $\mathcal{B}_K$ be the set of verge matrices of K as defined in 2.2.5. Let*

$\tilde{\psi}_C$ the induced character of ψ_C reduced to U_1 multiplied with a scalar $f_{\boldsymbol{T}} \in \mathbb{N}$ such that $\tilde{\psi}_C = f_{\boldsymbol{T}} Ind_{U_1}^{U_N} Res_{U_1}^{U_N} \psi_C$. Then we have

$$\tilde{\psi}_C = \begin{cases} \sum_{M \in \boldsymbol{m}_0} \psi_{C+M} & \text{if } \exists i \in \mathbb{N} : m_{2i-1} \neq 0 \text{ and } m_1 \leq 2 + \sum_{i \in \mathbb{N}} m_{2i+1} \\ & \text{ or } \forall i \in \mathbb{N} : m_{2i-1} = 0 \text{ and } \lambda \text{ is primary} \\ \sum_{M \in \boldsymbol{m}_0} \sum_{\substack{B \in \pi(\mathcal{B}_K) \\ f \in D_B}} \frac{\deg \psi_{d.B}}{|R_B|} \psi_{C+M+B} & \text{if } \exists i \in \mathbb{N} : m_{2i-1} \neq 0 \text{ and } m_1 > 2 + \sum_{i \in \mathbb{N}} m_{2i+1} \\ q \sum_{M \in \boldsymbol{m}_0} \psi_{C+M} & \text{if } \forall i \in \mathbb{N} : m_{2i-1} = 0, \lambda \text{ is secondary and } m_2 = 0 \\ \sum_{M \in \boldsymbol{m}_0} \sum_{c \in \mathbb{F}_q} \psi_{C+M+ce_{n-1,n+1}} & \text{if } \forall i \in \mathbb{N} : m_{2i-1} = 0, \lambda \text{ is secondary and } m_2 \neq 0. \end{cases}$$

The scalar $f_{\boldsymbol{T}} \in \mathbb{N}$ then is defined as follows and $\tilde{\psi}_C$ has corresponding degree:

$$\begin{array}{lll} f_{\boldsymbol{T}} = \frac{|\boldsymbol{m}_0|}{|L|} & \deg \tilde{\psi}_C = |\boldsymbol{m}_0| \deg \psi_C & \text{if } \exists i \in \mathbb{N} : m_{2i-1} \neq 0 \text{ and } m_1 \leq 2 + \sum_{i \in \mathbb{N}} m_{2i+1} \\ & & \text{or } \forall i \in \mathbb{N} : m_{2i-1} = 0 \text{ and } \lambda \text{ is primary partition} \\ f_{\boldsymbol{T}} = \frac{|\boldsymbol{m}_0||K|}{|L|} & \deg \tilde{\psi}_C = |\boldsymbol{m}_0||K| \deg \psi_C & \text{if } \exists i \in \mathbb{N} : m_{2i-1} \neq 0 \text{ and } m_1 > 2 + \sum_{i \in \mathbb{N}} m_{2i+1} \\ f_{\boldsymbol{T}} = \frac{q|\boldsymbol{m}_0|}{|L|} & \deg \tilde{\psi}_C = q|\boldsymbol{m}_0| \deg \psi_C & \text{if } \forall i \in \mathbb{N} : m_{2i-1} \neq 0, \lambda \text{ is secondary} \end{array}$$

Proof. Let $\lambda \vdash N$ be a partition, where every even element has even multiplicity. Let $(\mathbf{T}, \mathbf{S})$ be the standard core tableau for λ. Let $A \in \mathbf{v}$ the verge pattern and $C \in \mathbf{v}$ the core pattern for $(\mathbf{T}, \mathbf{S})$ as well as $d \in D_A$ such that $C = d.A$. Let $K \leq U_N$ and $V \trianglelefteq U_N$ with $U_N = K \ltimes V$ defined by

$$K = \begin{cases} K_{\frac{1}{2}(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1})} & \text{if } \exists i \in \mathbb{N} : m_{2i-1} \neq 0 \text{ and } m_1 > \sum_{i \in \mathbb{N}} m_{2i+1} \\ \left\{x_{n-1,n+1}(c) \mid c \in \mathbb{F}_q\right\} & \text{if } \forall i \in \mathbb{N} : m_{2i-1} = 0, \lambda \text{ secondary and } m_2 \neq 0 \\ I & \text{otherwise,} \end{cases}$$

$$V = \begin{cases} V_{\frac{1}{2}(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1})} & \text{if } \exists i \in \mathbb{N} : m_{2i-1} \neq 0 \text{ and } m_1 > \sum_{i \in \mathbb{N}} m_{2i+1} \\ \{g \in U_N \mid g_{n-1,n+1} = 0\} & \text{if } \forall i \in \mathbb{N} : m_{2i-1} = 0, \lambda \text{ secondary and } m_2 \neq 0 \\ U_N & \text{otherwise.} \end{cases}$$

Here for the even number $m_1 - \sum_{i \in \mathbb{N}} m_{2i+1}$ the group $K_{\frac{1}{2}(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1})} \leq U_N$ isomorphic to $U_{\frac{1}{2}(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1})}$ is as defined in proposition 4.2.7, whereas $V_{\frac{1}{2}(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1})} \trianglelefteq U_N$ is its complementary normal subgroup defined by

$$V_{\frac{1}{2}(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1})} = \left\{ g \in U_N \mid g_{ij} = 0 \text{ for } i > n - \frac{1}{2}\left(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1}\right), j < n + \frac{1}{2}\left(m_1 - \sum_{i \in \mathbb{N}} m_{2i+1}\right) \right\}.$$

For the tableau $\mathbf{T}$ we then have $n_0 = m_1 + \sum_{i\in\mathbb{N}} m_{2i+1}$ and $n_2 = \sum_{i\in\mathbb{N}} m_{2i+1}$, which gives us $\frac{1}{2}n_0 - n_2 = \frac{1}{2}(m_1 - \sum_{i\in\mathbb{N}} m_{2i+1})$.

We will first show that the characters described above for the different choices of K are zero for any $x \in U_N$ unless we have $x \in V$

(i) Let λ be the partition such that there is an $i \in \mathbb{N}$ with $m_{2i-1} \neq 0$ and $m_1 \leq 2 + \sum_{i\in\mathbb{N}} m_{2i+1}$ or that for all $i \in \mathbb{N}$ we have $m_{2i-1} = 0$, where λ is a primary partition. We then have $K = K_1 = I$ if $m_1 = 2 + \sum_{i\in\mathbb{N}} m_{2i+1}$ and $K = I$ otherwise by definition as well as $V = U_N$ and the claim follows immediately.

(ii) Let λ be the partition such that there is an $i \in \mathbb{N}$ with $m_{2i-1} \neq 0$ and $m_1 > 2 + \sum_{i\in\mathbb{N}} m_{2i+1}$. Then for $n_K = \frac{1}{2}(m_1 - \sum_{i\in\mathbb{N}} m_{2i+1})$ we have $K = K_{n_K}$. Let $\tilde{C} \in \mathbf{v}$ be a core pattern for the verge pattern A and $\tilde{B} \in \pi(K)$ a core pattern in $\pi(K)$. Since for every $1 \leq i \leq n$ and $n - n_K < j \leq n + n_K$ we have $\tilde{C}_{ij} = 0$, for $k \in K$ it follows that $k.\tilde{C} = \tilde{C}$ and therefore $K \leq \mathrm{Stab}_{U_N}(\tilde{C})$. Conversely, for $1 \leq i \leq n - n_K$ and $1 \leq j \leq N$ we have $\tilde{B}_{ij} = 0$, which for $v \in V$ gives us $v.\tilde{B} = \tilde{B}$ and therefore $V \leq \mathrm{Stab}_{U_N}(\tilde{B})$. Since $U_N = K \ltimes V$, it follows that $U_N = \mathrm{Stab}_{U_N}(\tilde{B})\mathrm{Stab}_{U_N}(\tilde{C})$. For the intersection of stabilizers we have $\mathrm{Stab}_{U_N}(\tilde{B}) \cap \mathrm{Stab}_{U_N}(\tilde{C}) \subseteq \mathrm{Stab}_{U_N}(\tilde{B} + \tilde{C})$. Furthermore, we have $U_N.\tilde{C} \subseteq \pi(V)$ and $U_N.\tilde{B} \subseteq \pi(K)$, which since $\pi(V) \cap \pi(K) = 0$ gives us $U_N.\tilde{C} \cap U_N.\tilde{B} = 0$, and $\mathrm{Stab}_{U_N}(\tilde{B} + \tilde{C}) = \mathrm{Stab}_{U_N}(\tilde{B}) \cap \mathrm{Stab}_{U_N}(\tilde{C})$ follows. Now we can calculate the product of characters as

$$\begin{aligned}
\psi_{\tilde{B}}\,\psi_{\tilde{C}} &= \mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B})}\chi_{\tilde{B}}\,\mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B})}\chi_{\tilde{C}} \\
&= \mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B})}\left(\chi_{\tilde{B}}\,\mathrm{Res}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B})}\mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{C})}\chi_{\tilde{C}}\right) \\
&= \mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B})}\left(\chi_{\tilde{B}}\,\mathrm{Ind}^{\mathrm{Stab}_{U_N}(\tilde{B})}_{\mathrm{Stab}_{U_N}(\tilde{B}+\tilde{C})}\mathrm{Res}^{\mathrm{Stab}_{U_N}(\tilde{C})}_{\mathrm{Stab}_{U_N}(\tilde{B}+\tilde{C})}\chi_{\tilde{C}}\right) \\
&= \mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B})}\mathrm{Ind}^{\mathrm{Stab}_{U_N}(\tilde{B})}_{\mathrm{Stab}_{U_N}(\tilde{B}+\tilde{C})}\mathrm{Res}^{\mathrm{Stab}_{U_N}(\tilde{C})}_{\mathrm{Stab}_{U_N}(\tilde{B}+\tilde{C})}\left(\chi_{\tilde{B}}\,\chi_{\tilde{C}}\right) \\
&= \mathrm{Ind}^{U_N}_{\mathrm{Stab}_{U_N}(\tilde{B}+\tilde{C})}\chi_{\tilde{B}+\tilde{C}} = \psi_{\tilde{B}+\tilde{C}}.
\end{aligned}$$

For the regular character of the complement K of the normal subgroup $V \trianglelefteq U_N$, extended to the whole group U_N we have $\mathrm{Inf}^{U_N}_K \rho_K = \sum_{B\in\pi(\mathcal{B}_K)} \frac{\deg\phi_B}{\langle\phi_B,\phi_B\rangle_{U_N}}\phi_B$ by corollary

2.2.12 . For $f \in D_B$ we have $\deg \phi_B = \deg \psi_{d.B} |D_B|$ by theorem 2.3.2 as well as $\langle \phi_B, \phi_B \rangle_{U_N} = |D_B||R_B|$ by corollary 2.4.11, so by corollary 2.4.7 it follows that

$$\mathrm{Inf}_K^{U_N} \rho_K = \sum_{B \in \pi(\mathcal{B}_K)} \sum_{f \in D_B} \frac{\deg \psi_{d.B}}{|R_B|} \psi_{d.B}.$$

Let now $x \in U_N$. We have $\mathrm{Inf}_K^{U_N} \rho_K(x) = 0$ unless $x \in V$ in which case we have $\mathrm{Inf}_K^{U_N} \rho_K = |K|$. It follows that

$$\sum_{M \in \mathbf{m}_0} \sum_{\substack{B \in \pi(\mathcal{B}_K) \\ f \in D_B}} \frac{\deg \psi_{d.B}}{|R_B|} \psi_{d.B+\tilde{C}}(x) = \mathrm{Inf}_K^{U_N} \rho_K \sum_{M \in \mathbf{m}_0} \psi_{C+M}(x) = \begin{cases} |K| \sum_{M \in \mathbf{m}_0} \psi_{C+M}(x) & \text{for } x \in V \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Let λ be the partition such that for all $i \in \mathbb{N}$ we have $m_{2i-1} = 0$, where λ is the secondary partition and $m_2 = 0$. We then have $K = \{x_{n-1,n+1}(c) \mid c \in \mathbb{F}_q\}$. For any $x \in U_N$ we have $x_{n,n+1} = \frac{1}{2} \sum_{i=n}^{n+1} x_{\bar{i},n+1} x_{i,n+1} = \frac{1}{2}(x^\dagger x)_{n,n+1} = 0$, so for $x, y \in U_N$ it follows that

$$(xy)_{n-1,n+1} = \sum_{i=n-1}^{n+1} x_{n-1,i} y_{i,n+1} = x_{n-1,n+1} + y_{n-1,n+1}.$$

For $x, u \in U_N$ we then have $(u^{-1}xu)_{n-1,n+1} = x_{n-1,n+1}$ so $V = \{g \in U_N \mid g_{n-1,n+1} = 0\} \trianglelefteq U_N$ is indeed a normal subgroup. Since n is contained in the -1 column and last row of the tableau for the primary counterpart of λ, the same is true for $n+1$ in the tableau $\mathbf{T}$ for the secondary partition λ. As $m_2 = 0$, there is no row of $\mathbf{T}$ with less than four elements, so there is a $1 \leq k < n$ that is contained in the -3 column of $\mathbf{T}$ in the same row as $n+1$, and we have $(k, n+1) \in \mathrm{supp}(A)$. Since $C_{k,n} = 0$ and $C_{k,n+1} = A_{k,n+1} = 1$ for $g \in \mathrm{Stab}_{U_N}(C)$, we have $g_{n-1,n+1} = (g.C)_{k,n-1} - C_{k,n-1} = 0$ and therefore $\mathrm{Stab}_{U_N}(C) \leq V$. For $x \in U_N$ we then have $\psi_C(x) = \mathrm{Ind}_{\mathrm{Stab}_{U_N}(C)}^{U_N} \chi_C = 0$ unless $x \in V$.

(iv) Let λ be the partition such that for all $i \in \mathbb{N}$ we have $m_{2i-1} = 0$, where λ is the secondary partition and $m_2 \neq 0$. As for the previous case we have $K = \{x_{n-1,n+1}(c) \mid c \in \mathbb{F}_q\}$ and again $n+1$ is contained in the -1 column and last row of $\mathbf{T}$. But since $m_2 \neq 0$, the last row contains only elements of which $n+1$ is the first one. So there is no $1 \leq k < n$ with $(k, n+1) \in \mathrm{supp}(A)$. For $c \in \mathbb{F}_q$ let $C^{(c)} = C + ce_{n-1,n+1} \in \mathbf{v}$ be the core pattern for the verge pattern $A - e_{n-1,n} + ce_{n-1,n+1}$ turning $(n-1, n)$ into a minor condition and $(n-1, n+1)$

into a major condition. For any $g \in U_N$ we have $g.e_{n-1,n+1} - e_{n-1,n+1} = g_{n,n+1}e_{n-1,n} = 0$ and therefore $g.C^{(c)} - C^{(c)} = g.C - C$ as well as $\mathrm{Stab}_{U_N}(C^{(c)}) = \mathrm{Stab}_{U_N}(C)$. So for $x \in U_N$ we have

$$\begin{aligned}
q^{-1} \sum_{c\in\mathbb{F}_q} \psi_{C^{(c)}}(x) &= \frac{1}{q|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{c\in\mathbb{F}_q} \sum_{g,u\in U_N} \vartheta\kappa\left(g.C^{(c)}, \pi(u^{-1}xu)\right) \\
&= \frac{1}{q|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{c\in\mathbb{F}_q} \sum_{g,u\in U_N} \vartheta\kappa\left(g.C + ce_{n-1,n+1}, \pi(u^{-1}xu)\right) \\
&= \frac{1}{q|U_N||\mathrm{Stab}_{U_N}(C)|} \sum_{g,u\in U_N} \vartheta\kappa\left(g.C, \pi(u^{-1}xu)\right) \sum_{c\in\mathbb{F}_q} \vartheta\left(c\left(u^{-1}xu\right)_{n-1,n+1}\right) \\
&= q^{-1}\psi_C(x) \sum_{c\in\mathbb{F}_q} \vartheta\left(cx_{n-1,n+1}\right) = 0
\end{aligned}$$

unless $x_{n-1,n+1} = 0$ and therefore $x \in V$, in which case we have $q^{-1} \sum_{c\in\mathbb{F}_q} \psi_{C^{(c)}}(x) = \psi_C(x)$.

For $W \in \pi(\mathfrak{g}(0)) \cap G_N.A - A$ we have $C + W \in G_N.A$. and there are $d_W \in D_A$ and $u \in U_N$ such that $uf.A = C + W$. We can then calculate the matrix $S(d_W) \in M_n(\mathbb{F}_q)$, where we assume that $1 \leq i \leq j \leq n$ without loss of generality due to the symmetry of $S(d_w)$:

(i) For $1 < i \leq \tilde{n}_{-2}$ and $1 \leq j \leq N$ we have $W_{ik} = 0$ for all $k > \tilde{n}_{-2}$ as well as $W_{jl} = 0$ for all $l > \tilde{n}_{-1}$ and $C_{ik} = 0$ for all $k > \tilde{n}_1$, $C_{jl} = 0$ for all $l \geq \tilde{n}_2$. Since $\tilde{n}_{-2} + \tilde{n}_2 = N$, we have $b\left(W^t e_i, (C + W)^t e_j\right) = 0$, and since $\tilde{n}_1 + \tilde{n}_{-1} = N$ we have $b\left(C^t e_i, W^t e_j\right) = 0$ as well. It follows that

$$S_{ij}(d_W) = b\left((C + W)^t e_i, (C + W)^t e_j\right) = b\left(C^t e_i, C^t e_j\right) = S_{ij}(d).$$

(ii) For $\tilde{n}_{-2} < i \leq j \leq \tilde{n}_{-1}$ let $i < k < \bar{i}$ and $j < l < \bar{j}$ be the entries in the same row as i and j of and the 1 column of $\mathbf{T}$. For all $1 \leq m \leq N$ we have $C_{im} = W_{im} = 0$ as well as $C_{jm} = W_{jm} = 0$ and therefore $S_{ij}(d_W) = S_{ij}(d) = 0$ unless we have $i + k < N + 1$ and $j + l < N + 1$ respectively. In this case we have $i + k = N$ and $j + l = N$, which gives us $\bar{k} = i + 1$ and $\bar{l} = j + 1$. Since for all $1 \leq m \leq N$ we have $C_{im} = 0$ and $C_{jm} = 0$ unless $m = k$ and $m = l$ respectively, it follows that $b\left(C^t e_i, W^t e_j\right) = W_{j,i+1}$ and $b\left(W^t e_i, C^t e_j\right) = W_{i,j+1}$. If $i \neq j$, we have $i + 1 = \bar{k} \leq j$ and therefore $(i, j) \in \mathcal{Z}_A$. So for any $(i, j) \in \mathcal{S}_A$ with $S_{ij}(d_W) \neq 0$ we have $i = j$. Furthermore, since $W_{im} = 0$ for all $\tilde{n}_{-1} < m \leq N$ with $\tilde{n}_{-1} \leq n$ or, in case when λ is the secondary partition with only even entries, since $W_{im} = 0$ for $m \in \{n, n+2, \dots N\}$, we have $b\left(W^t e_i, W^t e_i\right) = 0$. Finally, since $b\left(C^t e_i, C^t e_i\right) = 0$ as well, it follows that

$$S_{ij}(d_W) = b\Big((C+W)^t e_i, (C+W)^t e_j\Big) = 2W_{i,i+1}.$$

(iii) For $\tilde{n}_{-1} < i \leq n$ and $1 \leq j \leq n$ for k being in the same row as i of and the 2 column of $\mathbf{T}$ we have $i + k \geq N + 1$, so A has no non-zero entry in the i-th row, and we have $S_{ij}(d_W) = S_{ij}(d) = 0$.

Let now $\mathbf{w} = \pi(\mathfrak{g}(0)) \cap G_N.A - A$. For every $1 \leq i < l \leq n$ such that $(i, l) \in \operatorname{minor}(C)$ and $e_{il} \in \pi(\mathfrak{g}(0))$ we have $\tilde{n}_{-2} < i \leq \tilde{n}_{-1}$ and $l = i + 1$. So for every $W \in \mathbf{w}$ there is a $M \in \mathbf{m}_0$ with $M_{i,i+1} = W_{i,i+1}$ for all $1 \leq i \leq n$ for which there is a $i < k < \bar{i}$ with $(i, k) \in \operatorname{supp}(A)$, and we have $\psi_{C+W} = \psi_{C+M}$ by theorem 2.4.10. If now W runs through all elements of $\mathbf{w}$, then M runs through all elements of $\mathbf{m}_0$ with multiplicity $|\mathbf{w}|/|\mathbf{m}_0|$, and we have

$$\sum_{W \in \mathbf{w}} \psi_{C+W} = \frac{|\mathbf{w}|}{|\mathbf{m}_0|} \sum_{M \in \mathbf{m}_0} \psi_{C+M}.$$

Let $x \in V$ and $g \in U_N$. Let $X_1 \in \pi(\mathfrak{g}(0))$ and $X_2 \in \pi(\mathfrak{u}_1)$ such that $\pi(x) \circ g = X_1 + X_2$. Since $\pi(\mathfrak{g}(0)) \cap \pi(V) = \mathbf{w} \oplus (\pi(H_A) \cap \pi(\mathfrak{g}(0)) \cap \pi(V))$ is a direct sum of orthogonal complements in $\pi(\mathfrak{g}(1))$ by proposition 4.2.7 with $\kappa\left(W, X_2\right) = 0$ for all $W \in \mathbf{w}$, by lemma 2.1.7 we have

$$\sum_{W \in \mathbf{w}} \vartheta\kappa\left(W, \pi(x) \circ g\right) = \sum_{W \in \mathbf{w}} \vartheta\kappa\left(W, X_1\right) = \begin{cases} |\mathbf{w}| & \text{for } X_1 \in \pi(H_A) \\ 0 & \text{otherwise.} \end{cases}$$

We have $X_2 \circ g^{-1} \in \pi(\mathfrak{u}_1)$, and for $X_1 \in \pi(H_A)$ we have $X_1 \circ g^{-1} \in \pi(H_A)$ by lemma 4.2.3. So if $\sum_{Y \in \mathbf{w}} \vartheta\kappa\left(Y, \pi(x) \circ g\right) \neq 0$ for some $g \in U_N$, there is a decomposition for $\pi(x)$ with $\pi(x) = X_1' + X_2'$ such that $X_1' = X_1 \circ g^{-1} \in \pi(H_A)$ and $X_2' = X_2 \circ g^{-1} \in \pi(\mathfrak{u}_1)$.

By theorem 2.4.10 we now have $\psi_{C+W} = \psi_C$ for every $W \in \mathbf{w}$, and for $x \in U_1$ we have

$$\begin{aligned}\sum_{M\in\mathbf{m}_0} \psi_{C+M}(x) &= \frac{|\mathbf{m}_0|}{|\mathbf{w}|}\sum_{W\in\mathbf{w}} \psi_{C+W}(x)\\ &= \frac{|\mathbf{m}_0|}{|U_N||\mathrm{Stab}_{U_N}(C)||\mathbf{w}|}\sum_{W\in\mathbf{w}}\sum_{u,g\in U_N} \vartheta\kappa\left(g.(W+C),\pi(u^{-1}xu)\right)\\ &= \frac{|\mathbf{m}_0|}{|U_N||\mathrm{Stab}_{U_N}(C)||\mathbf{w}|}\sum_{u,g\in U_N} \vartheta\kappa\left(g.C,\pi(u^{-1}xu)\right)\left(\sum_{W\in\mathbf{w}}\vartheta\kappa\left(g.W,\pi(u^{-1}xu)\right)\right)\\ &= \frac{|\mathbf{m}_0|}{|U_N||\mathrm{Stab}_{U_N}(C)||\mathbf{w}|}\sum_{u,g\in U_N} \vartheta\kappa\left(g.C,\pi(u^{-1}xu)\right)\left(\sum_{W\in\mathbf{w}}\vartheta\kappa\left(W,\pi(u^{-1}xu)\circ g\right)\right).\end{aligned}$$

So if $\psi_C(x) \neq 0$, there is a $u \in U_N$ such that for $y = u^{-1}xu$, there are $Y_1 \in \pi(H_A)$ and $Y_2 \in \pi(\mathfrak{u}_1)$ with $\pi(y) = Y_1 + Y_2$. Let $y_2 \in U_2$ be such that $\pi(y_2) = Y_2$. Then we have $\pi(yy_2^{-1}) \circ y_2 = \pi(y) - \pi(y_2) = Y_1 \in \pi(H_A)$ and therefore $\pi(yy_2^{-1}) \in \pi(H_A)$ by lemma 4.2.3. For $y_1 = yy_2^{-1}$ we then have $y_1 \in H_A$ and $y = y_1y_2$. If $\psi_C(x) = \mathrm{Ind}_{\mathrm{Stab}_{U_N}(C+M)}^{U_N}\chi_C(x) \neq 0$, there is a $v \in U_N$ such that $v^{-1}yv \in \mathrm{Stab}_{U_N}(C)$, and we have

$$0 = (v^{-1}yv).C - C = (v^{-1}y_1v)(v^{-1}y_2v).C - C = (v^{-1}y_1v).\left((v^{-1}y_2v).C - C\right) + (v^{-1}y_1v).C - C.$$

Furthermore, we have $v^{-1}y_2v \in U_1$ since $U_1 \trianglelefteq V$ and because of $C \in \pi(\mathfrak{g}(2))$, we have $(v^{-1}y_2v).C - C(v^{-1}y_2v).C - C \in \pi(\mathfrak{g}(1))$ and therefore

$$(v^{-1}y_1v).\left((v^{-1}y_2v).C - C\right) \in \pi(\mathfrak{g}(0) + \mathfrak{g}(1)).$$

But we also have $v^{-1}y_1v \notin \mathrm{Stab}_{U_N}(C + M)$ by lemma 4.2.3 and therefore $(v^{-1}y_1v).C - C \neq 0$, which forces $(v^{-1}y_1v).C - C \in \pi(\mathfrak{g}(0) + \mathfrak{g}(1))$. As $C \in \pi(\mathfrak{g}(2))$, it follows that $v^{-1}y_1v \in U_1$, which gives us $y_1 \in U_1$ as well as $y = y_1y_2 \in U_1$ and finally $x = uyu^{-1} \in U_1$.

Let now $\tilde{\psi}_C = \tilde{f}\frac{|\mathbf{m}_0|}{|L|}\mathrm{Ind}_{U_1}^{U_N}\mathrm{Res}_{U_1}^{U_N}\psi_C$ be the induced character of ψ_C reduced to U_1 multiplied with some scalar $f = \tilde{f}\frac{|\mathbf{m}_0|}{|L|}$ for $\tilde{f} \in \mathbb{N}$ and since $U_1 \trianglelefteq U_N$ for $x \in U_N$, we have

$$\tilde{\psi}_C(x) = \frac{\tilde{f}|\mathbf{m}_0|}{|L||U_1|}\sum_{\substack{u\in U_N\\ u^{-1}xu\in U_1}} \psi_C(u^{-1}xu) = \begin{cases}\frac{\tilde{f}|\mathbf{m}_0||U_N|}{|L||U_1|}\psi_C(x) & \text{if } x \in U_1\\ 0 & \text{if } x \in U_N\backslash U_1.\end{cases}$$

Since $U_N = L \ltimes U_1$ for $x \in U_1$, we have $\tilde{\psi}_C(x) = \tilde{f}|\mathbf{m}_0|\psi_C(x)$. For $K \leq U_N$ and $V \leq U_N$ as defined above for the different cases, we have $\sum_{M\in\mathbf{m}_0} \psi_{C+M}(x) = |\mathbf{m}_0|\psi_C(x)$ for $x \in U_1$ and $\sum_{M\in\mathbf{m}_0} \psi_{C+M}(x) = 0$ for $x \in V\backslash U_1$. So we can determine $\tilde{\psi}_C$ for these different cases:

(i) Let λ be the partition such that there is an $i \in \mathbb{N}$ with $m_{2i-1} \neq 0$ and $m_1 \leq \sum_{i\in\mathbb{N}} m_{2i+1}$ or that for all $i \in \mathbb{N}$ we have $m_{2i-1} = 0$, where λ is the primary partition. We then have $K = I$ as well as $V = U_N$, and for $\tilde{f} = 1$ it follows that

$$\tilde{\psi}_C = \sum_{M\in\mathbf{m}_0} \psi_{C+M}$$

with $\deg\tilde{\psi}_C = |\mathbf{m}_0| \deg\psi_C$.

(ii) Let λ be the partition such that there is an $i \in \mathbb{N}$ with $m_{2i-1} \neq 0$ and $m_1 > \sum_{i\in\mathbb{N}} m_{2i+1}$. Then we have $K = K_{n_K}$, and for $\tilde{f} = |K|$ it follows that

$$\tilde{\psi}_C = \mathrm{Inf}_K^{U_N}\rho_K \sum_{M\in\mathbf{m}_0} \psi_{C+M} = \sum_{M\in\mathbf{m}_0} \sum_{\substack{B\in\pi(\mathcal{B}_K)\\ f\in D_B}} \frac{\deg\psi_{d.B}}{|R_B|}\psi_{d.B+\tilde{C}}(x)$$

with $\deg\tilde{\psi}_C = |\mathbf{m}_0||K| \deg\psi_C$.

(iii) Let λ be the partition such that for all $i \in \mathbb{N}$ we have $m_{2i-1} = 0$, where λ is the secondary partition and $m_2 = 0$. We then have $K = \{x_{n-1,n+1}(c) \mid c \in \mathbb{F}_q\}$, and for $\tilde{f} = q$ it follows that

$$\tilde{\psi}_C = q\sum_{M\in\mathbf{m}_0} \psi_{C+M}$$

with $\deg\tilde{\psi}_C = q|\mathbf{m}_0| \deg\psi_C$.

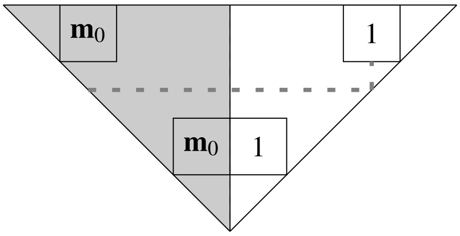
(iv) Let λ be the partition such that for all $i \in \mathbb{N}$ we have $m_{2i-1} = 0$, where λ is the secondary partition and $m_2 \neq 0$. We then have $K = \{x_{n-1,n+1}(c) \mid c \in \mathbb{F}_q\}$, and for $\tilde{f} = 1$ it follows that

$$\tilde{\psi}_C = \sum_{M\in\mathbf{m}_0} \sum_{c\in\mathbb{F}_q} \psi_{C+M+ce_{n-1,n+1}}$$

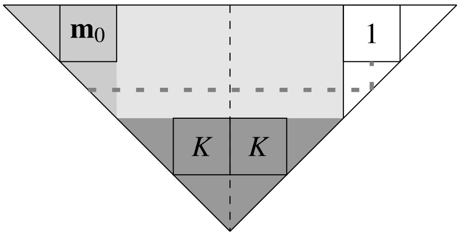
with $\deg\tilde{\psi}_C = q|\mathbf{m}_0| \deg\psi_C$.

□

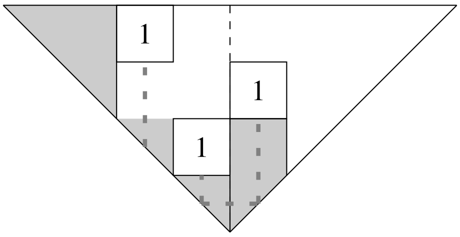
The character $\tilde{\psi}_C$ for the four different variations of λ:



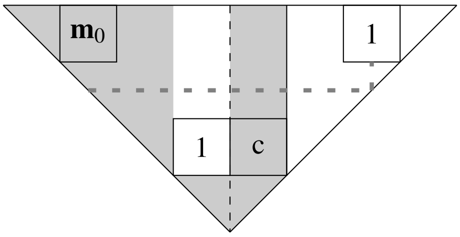
$\lambda = (2^4)' \vdash 8$:
$\tilde{\psi}_C = \sum_{M \in \mathbf{m}_0} \psi_{C+M}$



$\lambda = (2^2, 1^4) \vdash 8$:
$\tilde{\psi}_C = \sum_{M \in \mathbf{m}_0} \sum_{\substack{B \in \pi(\mathcal{B}_K) \\ f \in D_B}} \frac{\deg \psi_{d.B}}{|R_B|} \psi_{C+M+B}$



$\lambda = (4^2)'' \vdash 8$:
$\tilde{\psi}_C = q\, \psi_M$



$\lambda = (2^4)'' \vdash 8$:
$\tilde{\psi}_C = \sum_{M \in \mathbf{m}_0} \sum_{c \in \mathbb{F}_q} \psi_{C+M+ce_{n-1,n+1}}$

What remains is to calculate the scalar that differentiates the induced character of SO_N of the character $\tilde{\psi}_C$ of U_N from the generalized Gelfand–Graev character for C. In order to do so, we have to calculate the size of the orbit $U_N.C$.

Lemma 4.2.9. *For a standard core tableau $(\boldsymbol{T}, \boldsymbol{S})$ let $f(\boldsymbol{T}) \in \mathbb{N}$ be defined as*

$$f(\boldsymbol{T}) = \frac{1}{2}\left(n_3\left(n_2 + \frac{1}{2}n_3\right) + \sum_{m \geq 4} n_m(n_{m-1} + n_m - 1)\right).$$

Let $C \in \boldsymbol{v}$ be the core pattern for $(\boldsymbol{T}, \boldsymbol{S})$. Then we have

$$\mathit{Ind}_{U_N}^{SO_N} \tilde{\psi}_C = q^{f(\boldsymbol{T})} \gamma_{C-C^\dagger}.$$

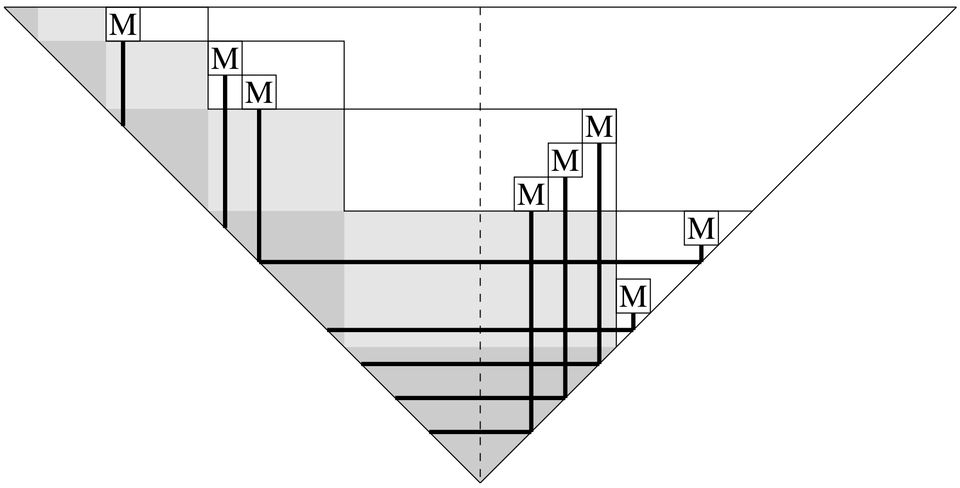
Proof. Let $(\mathbf{T}, \mathbf{S})$ be a standard core tableau and $C \in \mathbf{v}$ the corresponding core pattern. By theorem 2.3.2 we have

$$|U_N.C| = \frac{|U_N|}{|\mathrm{Stab}_{U_N}(C)|} = \frac{|U_N|}{|\mathrm{Stab}_{U_N}(A)|} = \frac{\prod_{(i,j) \in \mathrm{supp}(A)} q^{j-i-1}}{q^{|\mathcal{D}_A|}}.$$

For $1 \leq i < j \leq N$ with $i + j < N + 1$ we have $(i, j) \in \mathrm{supp}(A)$ if and only if there is a $m \in \mathbb{N}$ such that i and j be the entries in the same row and the $-m$ or $-m + 2$ column respectively of the tableau $\mathbf{T}$. If $m \geq 2$, for every i in the $-m$ column, we have $i + j < N + 1$, where j is the entry in the same row and $-m + 2$ column, while if $m = 1$ there are $n_1/2$ entries i in the -1 row such that $i + j < N + 1$.

For $m \geq 3$, as $\mathbf{T}$ is the standard tableau, we have $j = i + n_m + n_{m-1}$, while for $m = 2$, as the entries in the -2 column are ascending and the entries in the 0 column are descending, we have $j = i + n_0 + n_1 + n_2 - 2k + 1$ where $1 \leq k \leq n_2$ is the row of $\mathbf{T}$ in which i and j are contained. For $m = 1$ we have $i + j = N$ if i and j are contained in a odd row $2k - 1$ for $1 \leq k \leq n_1/2$. Then, we have $i = \tilde{n}_{-2} + 2k - 1$ as well as $j = N - \tilde{n}_{-2} - 2k + 1 = i + 2(n - \tilde{n}_{-2}) - 4k + 2 = 2n_1 + n_0 - 4k + 2$ since $n = \tilde{n}_{-1} + \frac{1}{2}n_0 = \tilde{n}_{-2} + n_1 + \frac{1}{2}n_0$. It follows that

$$\begin{aligned}\sum_{(i,j)\in\mathrm{supp}(A)} (j\text{-}i\text{-}1) &= \left[\sum_{k=1}^{n_1/2} 2n_1 + n_0 - 4k + 1\right] + \left[\sum_{k=1}^{n_2} n_0 + n_1 + n_2 - 2k\right] + \left[\sum_{m\geq 3}\sum_{k=1}^{n_m} n_{m-1} + n_m - 1\right] \\ &= \left[\frac{1}{2}n_1(n_1 - 1) + \frac{1}{2}n_0 n_1\right] + [n_2(n_0 + n_1 - 1)] + \left[\sum_{m\geq 3} n_m(n_{m-1} + n_m - 1)\right].\end{aligned}$$



Positions of the lower hooks of A, whose number is equal to $\sum_{(i,j)\in\mathrm{supp}(A)} (j - i - 1)$

Let now $1 \leq k < l \leq N$ with $k + l \leq N + 1$ such that $(k, l) \in \mathcal{D}_A$. Then by definition there are $1 \leq i, j < k$ with $(i, \overline{k}), (j, l) \in \mathrm{supp}(A)$. So $\overline{k}$ must be either contained in the 0 or 1 column of $\mathbf{T}$. If $\overline{k}$ is contained in the 1 column, then k is contained in the -1 column. Since $j < k < l$ we must have either that j and l are contained in the -3 and -1 column in the same row below the entry k in the -3 column, j and l are contained in the -2 and 0 column in any row or j and l are contained in the -1 and 1 column, in which case we must have $i = j$ and $\overline{k} = l$. For $1 \leq r \leq n_1/2$ let $\overline{k}$ be the entry in the $2r - 1$-th row. Then since the entries in the -1 column are ascending, we have $k = \tilde{n}_{-2} + 2r - 1$, so there are $\min(n_3 - 2r, 0) + n_2 + 1$ many $1 \leq l \leq N$ such that $(k, l) \in \mathcal{D}_A$.

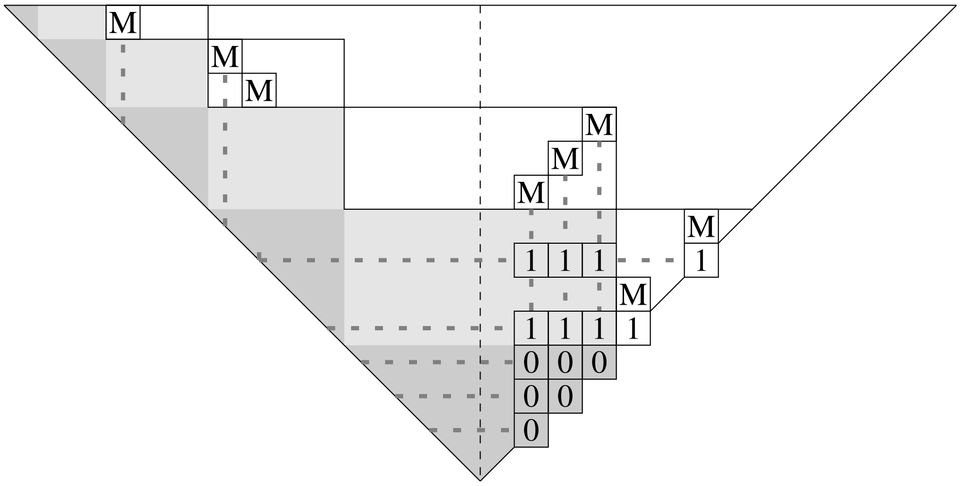
If $\overline{k}$ is contained in the 0 column, then k is contained in the -2 column. Since $j < k \leq l$, we must have that j and l are contained in the -2 and 0 column with $k < l \leq \overline{k}$. Since j is in the same row as l in the -2 column and the entries in the 0 column are descending, we also must have $l > \tilde{n}_{-1} + n_0 - n_2 = \tilde{n}_0 - n_2$ and therefore $l \in \{\min(\tilde{n}_0 - n_2, k) + 1, \dots \overline{k}\}$. For $1 \leq r \leq \min(n_2, n_0/2)$ let $\overline{k}$ be the entry in the r-th row. Then we have $\overline{k} = \tilde{n}_0 - r + 1$ and $k = \tilde{n}_0 - n_0 + r$. Therefore, for $r > n_0 - n_2$ we have $k > \tilde{n}_0 - n_2$, and there are $\tilde{k} - k = n_0 + 1 - 2r$ many $1 \leq l \leq N$ such that $(k, l) \in \mathcal{D}_A$. Conversely, for $r \leq n_0 - n_2$ we have $k \leq \tilde{n}_0 - n_2$, and there are $\tilde{k} - (\tilde{n}_0 - n_2) = n_2 + 1 - r$ many $1 \leq l \leq N$ such that $(k, l) \in \mathcal{D}_A$.

There are $n_1/2$ many $1 \leq i < k \leq n$ such that $(i, \overline{k}) \in \operatorname{supp}(A)$ and $\overline{k}$ is contained in the 1 column of $\mathbf{T}$, while there are $\min(n_2, n_0/2)$ many $1 \leq i < k \leq n$ such that $(i, \overline{k}) \in \operatorname{supp}(A)$ and $\overline{k}$ is contained in the 0 column of $\mathbf{T}$. For $n_2 \leq n_0/2$ we have $n_2 \leq n_0 - n_2$, and it follows that

$$|\mathcal{D}_A| = \sum_{r=1}^{n_1/2} (\min(n_3 - 2r, 0) + n_2 + 1) + \sum_{r=1}^{n_2} (n_2 + 1 - r) = \frac{1}{4} n_3(n_3 - 2) + \frac{1}{2} n_1 n_2 + \frac{1}{2} n_1 + \frac{1}{2} n_2(n_2 + 1).$$

Conversely, for $n_2 > \frac{1}{2} n_0$ we have $\frac{1}{2} n_0 > n_2 - n_0$, and it follows that

$$\begin{aligned}
|\mathcal{D}_A| &= \sum_{r=1}^{n_1/2} (\min(n_3 - 2r, 0) + n_2 + 1) + \sum_{r=1}^{n_0 - n_2} (n_2 + 1 - r) + \sum_{r=n_0-n_2+1}^{n_0/2} (n_0 + 1 - 2r) \\
&= \left[\frac{1}{4} n_3(n_3 - 2) + \frac{1}{2} n_1 n_2 + \frac{1}{2} n_1\right] + \left[(n_0 - n_2)(n_2 + 1) - \frac{1}{2}(n_0 - n_2)(n_0 - n_2 + 1)\right] + \left[(\frac{1}{2} n_0 - n_2)^2\right] \\
&= \left[\frac{1}{4} n_3(n_3 - 2) + \frac{1}{2} n_1 n_2 + \frac{1}{2} n_1\right] + \left[\frac{1}{2} n_2(n_2 + 1) + (\frac{1}{2} n_0 - n_2) - 2(\frac{1}{2} n_0 - n_2)^2\right] + \left[(\frac{1}{2} n_0 - n_2)^2\right] \\
&= \left(\frac{1}{4} n_3(n_3 - 2) + \frac{1}{2} n_1 n_2 + \frac{1}{2} n_1 + \frac{1}{2} n_2(n_2 + 1)\right) - \left((\frac{1}{2} n_0 - n_2)(\frac{1}{2} n_0 - n_2 - 1)\right).
\end{aligned}$$



Positions of $\mathcal{D}_A$ for the 0 column (0) and 1 column (1) of $\mathbf{T}$

Let $k = (\frac{1}{2}n_0 - n_2)(\frac{1}{2}n_0 - n_2 - 1)$ if $n_2 \leq \frac{1}{2}n_0$ and $k = 0$ otherwise. We define $f(\mathbf{T}) \in \mathbb{N}$ to be $f(\mathbf{T}) = \sum_{(i,j)\in\mathrm{supp}(A)}(j-i-1) - |\mathcal{D}_A| + k + \frac{1}{2}n_1 - |\mathrm{supp}_{\mathcal{V}}(L)| - \frac{1}{2}|\mathrm{supp}_{\mathcal{V}}(\mathfrak{g}(1))|$. By lemma 4.2.2 we have $|\mathrm{supp}_{\mathcal{V}}(\mathfrak{g}(1))| = \sum_{m\in\mathbb{N}} n_m n_{m-1}$ and $|\mathrm{supp}_{\mathcal{V}}(L)| = \frac{1}{4}n_0(n_0-2) + \sum_{m\in\mathbb{N}} \frac{1}{2}n_m(n_m-1)$. Then it follows that

$$
\begin{aligned}
f(\mathbf{T}) = {} & \frac{1}{2}n_1(n_1-1) + \frac{1}{2}n_0n_1 + n_2(n_0+n_1-1) - \left(\frac{1}{4}n_3(n_3-2) + \frac{1}{2}n_1n_2 + \frac{1}{2}n_1 + \frac{1}{2}n_2(n_2+1)\right) \\
& + \left((\frac{1}{2}n_0 - n_2)(\frac{1}{2}n_0 - n_2 - 1)\right) + \left(\sum_{m\geq 3} n_m(n_{m-1}+n_m-1)\right) \\
& + \frac{1}{2}n_1 - |\mathrm{supp}_{\mathcal{V}}(L)| - \frac{1}{2}|\mathrm{supp}_{\mathcal{V}}(\mathfrak{g}(1))| \\
= {} & \frac{1}{2}\left(n_1(n_1-1) + n_0n_1 + n_2(n_2-1) + n_1n_2\right) - \frac{1}{4}n_3(n_3-2) + n_0n_2 - n_2(n_2+1) \\
& + (\frac{1}{2}n_0 - n_2)(\frac{1}{2}n_0 - n_2 - 1) + \sum_{m\geq 3} n_m(n_{m-1}+n_m-1) - |\mathrm{supp}_{\mathcal{V}}(L)| - \frac{1}{2}|\mathrm{supp}_{\mathcal{V}}(\mathfrak{g}(1))| \\
= {} & \frac{1}{4}n_0(n_0-2) + \sum_{m\geq 1}\frac{1}{2}\left(n_m(n_m-1) + n_mn_{m-1}\right) - \frac{1}{4}n_3(n_3-2) + \sum_{m\geq 3}\frac{1}{2}(n_m(n_{m-1}+n_m-1)) \\
& - |\mathrm{supp}_{\mathcal{V}}(L)| - \frac{1}{2}|\mathrm{supp}_{\mathcal{V}}(\mathfrak{g}(1))| \\
= {} & \frac{1}{2}\left(\left(\sum_{m\geq 3} n_m(n_{m-1}+n_m-1)\right) - \frac{1}{2}n_3(n_3-2)\right) \\
= {} & \frac{1}{2}\left(n_3\left(n_2 + \frac{1}{2}n_3\right) + \sum_{m\geq 4} n_m(n_{m-1}+n_m-1)\right).
\end{aligned}
$$

We consider now the character $\tilde{\psi}_C$ as defined in theorem 4.2.7 and the factor $f \in \mathbb{N}$ for which we have $\tilde{\psi}_C = f_{\mathbf{T}} \operatorname{Ind}_{U_1}^{U_N} \operatorname{Res}_{U_1}^{U_N} \psi_C$. Let therefore again $m_i \in \mathbb{N}_0$ for $i \in \mathbb{N}$ be such that $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

(i) Let λ not be the secondary partition with only even elements, and let $m_1 \leq 2 + \sum_{i \in \mathbb{N}} m_{2i+1}$. Since $n_2 = \sum_{i \geq 2} m_{2i-1}$ and $n_0 = \sum_{i \geq 1} m_{2i-1}$, we have $\frac{1}{2} n_0 - n_2 = \frac{1}{2}(m_1 - \sum_{i \geq 2} m_{2i-1}) \leq 1$ and therefore $k = 0$ both if $\frac{1}{2} n_0 - n_2 = 1$ or $\frac{1}{2} n_0 < n_2$ by definition. The space of minor conditions in $\pi(\mathfrak{g}(0))$, denominated as $\mathbf{m}_0$ in theorem 4.2.7 has $|\mathbf{m}_0| = q^{n_1/2}$ elements because it arises from the positions $(i, j) \in \mathcal{V}$ where i and j are positions in the same row in the -1 and 1 column of $\mathbf{T}$. So we have $f_{\mathbf{T}} = \frac{|\mathbf{m}_0|}{|L|} = \frac{q^{\frac{1}{2} n_1 + k}}{|L|}$.

(ii) Let λ be such that it contain odd elements, that is there is an $i \in \mathbb{N}$ with $m_{2i-1} \neq 0$, and let $m_1 > 2 + \sum_{i \in \mathbb{N}} m_{2i+1}$. Since $n_2 = \sum_{i \geq 2} m_{2i-1}$ and $n_0 = \sum_{i \geq 1} m_{2i-1}$, we then have $\frac{1}{2} n_0 - n_2 = \frac{1}{2}(m_1 - \sum_{i \geq 2} m_{2i-1}) > 1$, which gives us $|K| = U_{n_0 - 2n_2} = q^{\frac{1}{4}(n_0 - 2n_2)(n_0 - 2n_2 - 2)} = q^k$. Again we have $|\mathbf{m}_0| = q^{n_1/2}$ and therefore $f_{\mathbf{T}} = \frac{|\mathbf{m}_0|}{|L|} = \frac{q^{\frac{1}{2} n_1 + k}}{|L|}$.

(iii) Let λ be the secondary partition with only even elements. We have $(n-1, n) \in \operatorname{supp}(A)$, where $n-1$ and n are contained in the -1 and 1 column of $\mathbf{T}$, replacing the minor condition that has this position in the pattern for the primary counterpart of λ. So we have $|\mathbf{m}_0| = q^{n_1/2-1}$. Since both $n_0 = n_2 = 0$ we also have $k = 0$ and therefore $f_{\mathbf{T}} = \frac{q|\mathbf{m}_0|}{|L|} = \frac{q^{\frac{1}{2} n_1 + k}}{|L|}$.

For $f(\mathbf{T}) \in \mathbb{N}$ as defined above it then follows that

$$f_{\mathbf{T}} \frac{|U_N.C|}{|\mathfrak{g}(1)|} = \frac{q^{\frac{1}{2} n_1 + k}}{|L|} \frac{\prod_{(i,j) \in \operatorname{supp}(A)} q^{j-i-1}}{|D_A||\mathfrak{g}(1)|} = \frac{q^{f(\mathbf{T})}}{\sqrt{|\mathfrak{g}(1)|}},$$

and for the character $\tilde{\psi}_C$ we have

$$\tilde{\psi}_C = f_{\mathbf{T}} \operatorname{Ind}_{U_1}^{U_N} \operatorname{Res}_{U_1}^{U_N} \psi_C = f_{\mathbf{T}} \frac{|U_N.C|}{|\mathfrak{g}(1)|} \operatorname{Ind}_{U_2}^{U_N} \xi_{C-C^\dagger} = q^{f(\mathbf{T})} \frac{1}{\sqrt{|\mathfrak{g}(1)|}} \operatorname{Ind}_{U_2}^{U_N} \xi_{C-C^\dagger}.$$

The claim then follows from lemma 4.1.6. □

While the value of $f(\mathbf{T})$ can get big quickly, for smaller dimensions it stays rather small. Is the partition $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ of which $\mathbf{T}$ is based on such that $m_i = 0$ for all even $i \geq 0$ and there is at most one $m_i \neq 0$ for all odd $i \geq 5$ with $m_i = 1$, that is $\sum_{i \geq 2} m_{2i+1} \leq 1$, we even have $f(\mathbf{T}) = 0$.

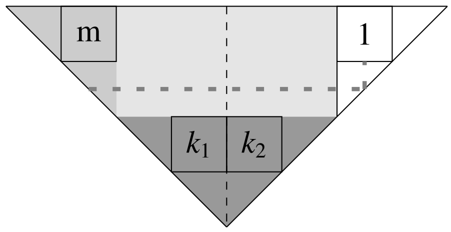
4.3 Generalized Gelfand–Graev characters of SO_8

We can now construct the eleven characters $\tilde{\psi}_C$ for U_8 for the different partitions $\lambda \vdash 8$ such that $\gamma_{C-C^\dagger} = q^{-f(\mathbf{T})}\mathrm{Ind}_{U_N}^{\mathrm{SO}_N}\tilde{\psi}_C$ is the corresponding generalized Gelfand–Graev character of SO_8.

(i) Partition: $\lambda = (2^2, 1^4)$, Core pattern: $C = e_{17}$

1	7
2	8
3	
4	
5	
6	

Tableau $\mathbf{T}$



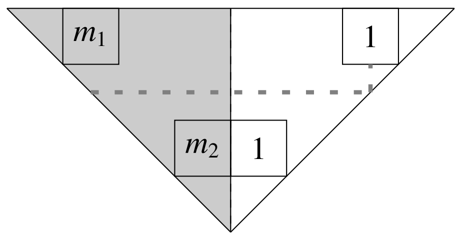
Core pattern $C \in \mathbf{v}$ for $\mathbf{T}$

$$\tilde{\psi}_C = \sum_{m \in \mathbb{F}_q} \sum_{k_1, k_2 \in \mathbb{F}_q} \psi_{C + me_{12} + k_1 e_{34} + k_2 e_{35}}, \qquad f(\mathbf{T}) = 0$$

(ii) Partition: $\lambda = (2^4)'$, Core pattern: $C = e_{17} + e_{35}$

1	7
2	8
3	5
4	6

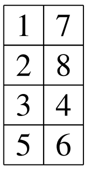
Tableau $\mathbf{T}$



Core pattern $C \in \mathbf{v}$ for $\mathbf{T}$

$$\tilde{\psi}_C = \sum_{m_1, m_2 \in \mathbb{F}_q} \psi_{C + m_1 e_{12} + m_2 e_{34}}, \qquad f(\mathbf{T}) = 0$$

(iii) Partition: $\lambda = (2^4)''$, Core pattern: $C = e_{17} + e_{34}$



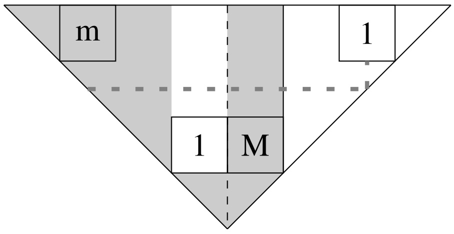
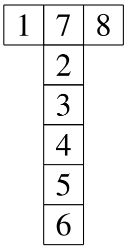


Tableau **T** Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \sum_{m,M \in \mathbb{F}_q} \psi_{C+me_{12}+Me_{35}}, \qquad f(\mathbf{T}) = 0$$

(iv) Partition: $\lambda = (3, 1^5)$, Core pattern: $C = e_{17} + ae_{13}$ for $a \in \mathbb{F}_q{}^*$



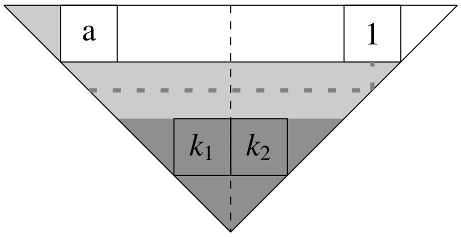
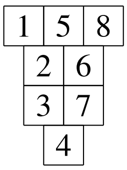


Tableau **T** Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \sum_{k_1,k_2 \in \mathbb{F}_q} \psi_{C+k_1e_{34}+k_2e_{35}}, \qquad f(\mathbf{T}) = 0$$

(v) Partition: $\lambda = (3, 2^2, 1)$, Core pattern: $C = e_{15} + e_{26} + ae_{14}$ for $a \in \mathbb{F}_q{}^*$



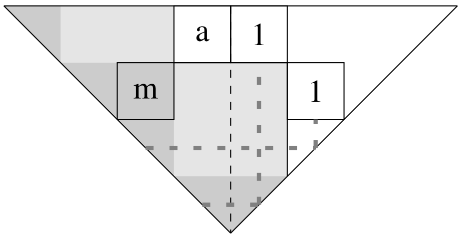


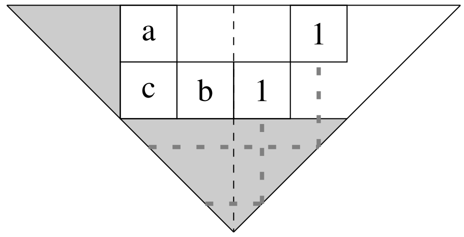
Tableau **T** Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \sum_{m \in \mathbb{F}_q} \psi_{C+me_{23}}, \qquad f(\mathbf{T}) = 0$$

(vi) Partition: $\lambda = (3^2, 1^2)$, Core pattern: $C = e_{16} + e_{25} + ae_{13} + be_{24} + ce_{23}$ for $a \in \mathbb{F}_q{}^*$, $b, c \in \mathbb{F}_q$ with $ab \neq c^2$

1	6	8
2	5	7
	4	
	3	

Tableau **T**



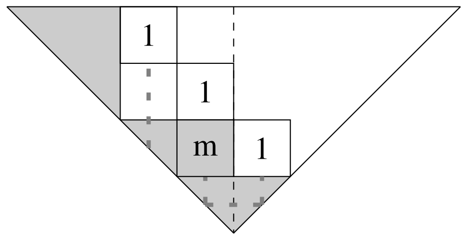
Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \psi_C, \qquad f(\mathbf{T}) = 0$$

(vii) Partition: $\lambda = (4^2)'$, Core pattern: $C = e_{13} + e_{24} + e_{35}$

1	3	5	7
2	4	6	8

Tableau **T**



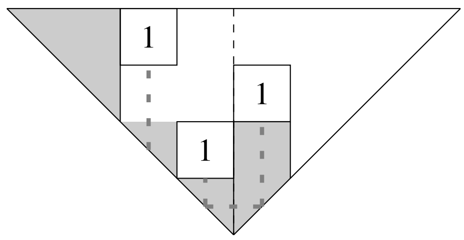
Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \sum_{m \in \mathbb{F}_q} \psi_{C+me_{34}}, \qquad f(\mathbf{T}) = 1$$

(viii) Partition: $\lambda = (4^2)''$, Core pattern: $C = e_{13} + e_{25} + e_{34}$

1	3	4	7
2	5	6	8

Tableau **T**



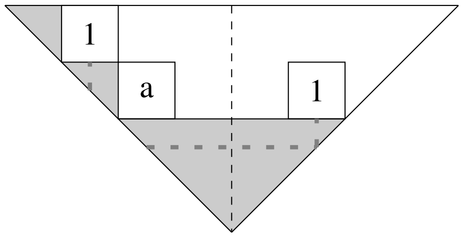
Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = q\psi_C, \qquad f(\mathbf{T}) = 1$$

(ix) Partition: $\lambda = (5, 1^3)$, Core pattern: $C = e_{12} + e_{26} + ae_{23}$ for $a \in \mathbb{F}_q{}^*$

1	2	6	7	8
		5		
		4		
		3		

Tableau **T**



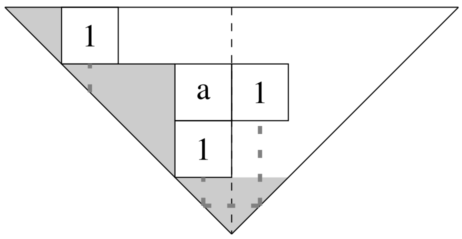
Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \psi_C, \qquad f(\mathbf{T}) = 0$$

(x) Partition: $\lambda = (5, 3)$, Core pattern: $C = e_{12} + e_{25} + e_{34} + ae_{24}$ for $a \in \mathbb{F}_q{}^*$

1	2	5	7	8
	3	4	6	

Tableau **T**



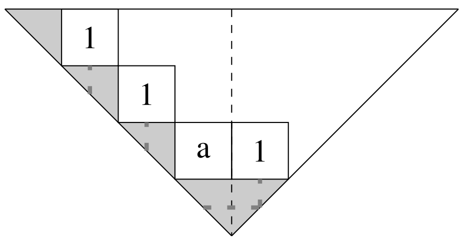
Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \psi_C, \qquad f(\mathbf{T}) = 0$$

(xi) Partition: $\lambda = (7, 1)$, Core pattern: $C = e_{12} + e_{23} + e_{35} + ae_{34}$ for $a \in \mathbb{F}_q{}^*$

1	2	3	5	6	7	8
			4			

Tableau **T**



Core pattern $C \in \mathbf{v}$ for **T**

$$\tilde{\psi}_C = \psi_C, \qquad f(\mathbf{T}) = 0$$

Nomenclature

"."	Left group action of G_N on $\mathbf{v}$, page 42
$(\mathbf{T}, \mathbf{S})$	Core tableau, page 110
$*$	Left group action of G_N on $\mathfrak{u}_N$, page 47
β_g	Bijection between stabilizer, page 69
$\cdot^\dagger$	Involution on $M_N(\overline{\mathbb{F}}_q)$, page 24
χ_A	Map with $\chi_A(g) = \vartheta\kappa(A, \pi(g))$, page 43
$\circ$	Right action of G_N on $\mathbf{v}$, page 40
$\Delta(\overline{G}, \overline{T})$	Set of simple roots of $\overline{G}$ for a torus $\overline{T}$, page 29
γ_A	Generalized Gelfand–Graev character, page 126
κ	Trace form with a transposed argument, page 41
$\langle \cdot, \cdot \rangle$	Inner product of characters, page 53
$\overline{\mathfrak{g}}(z)$	Direct summand for $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_N$, page 95
$\mathcal{B}$	Set of verge matrices, page 48
$\mathcal{G}$	Set of positions above the diagonal, page 31
$\mathcal{O}(A)$	Orbit of nilpotent element of $\overline{\mathfrak{so}}_N$, page 101
$\mathcal{V}$	Set of positions above both the diagonal and counter-diagonal, page 31
$\mathcal{V}_l$	Set of positions above the diagonal and left of the middle, page 31
$\mathcal{V}_r$	Set of positions above the counter-diagonal and right of the middle, page 31

$\mathfrak{c}_{\mathfrak{g}}(A)$ Stabilizer of A in $\mathfrak{g}$, page 27

$\overline{U}_\alpha$ Root subgroup, page 31

$\overline{U}_i$ Normal unipotent subgroup for $\mathbb{Z}$-grading of $\overline{\mathfrak{so}}_N$, page 95

$\overline{\mathfrak{g}}_\alpha$ Root space, page 29

$\Phi(\overline{G}, \overline{T})$ Set of roots of $\overline{G}$ for a torus $\overline{T}$, page 29

$\Phi_+(\overline{G}, \overline{T})$ Set of positive roots of $\overline{G}$ for a torus $\overline{T}$, page 29

$\phi_{\pi(B)}$ André–Neto character, page 46

π Restriction of a matrix to its entries above both the diagonal and counter-diagonal, page 40

ψ_C Jedlitschky character, page 58

$\mathrm{Stab}_G(A)$ Stabilizer of A in G with respect to ".", page 60

$\mathbf{g}$ Linear space with non-zero entries only above both the diagonal, page 40

$\mathbf{T}$ Verge tableau, page 107

$\mathbf{v}$ Linear space with non-zero entries only above both the diagonal and counter-diagonal, page 40

$\tilde{\kappa}$ Trace form, page 27

$\tilde{n}_z$ Length of the columns of a centered tableau to the left of a column, page 106

ϑ Non-trivial group homomorphism from $\mathbb{F}_q$ to $\mathbb{C}$, page 43

b Orthogonal group defining bilinear form on $\overline{\mathbb{F}}_q^N$, page 23

$C_G(A)$ Stabilizer of A in G, page 27

D_A Subset determining core patterns, page 68

F Standard Frobenius endomorphism for q, page 36

f Cayley transformation, page 35

G_N	Group of of unitriangular matrices, page 39
H_A	Subgroup of the horizontal part of lower hooks, page 130
H_C	Pattern group for a closed set of positions, page 31
J_N	Counter-diagonal identity matrix, page 23
n_z	Length of the column of a centered tableau, page 106
R_A	Group acting by row transformations on verge pattern, page 63
$R_A^0(d)$	Stabilizer of $d \in D_A$ in R_A with respect to $\bullet_A$, page 75
$S(d)$	Gram matrix determining equal characters, page 78
U_N	Group of of unitriangular matrices in SO_N, page 39
$U_{1.5}$	Unipotent subgroups between U_1 and U_2, page 125
core	Core positions of a pattern, page 67
deg	Degree of a character, page 52
minor	Minor positions of a pattern, page 67
rank	Rank of matrix, page 101
supl	Supplementary positions of a pattern, page 67
supp	Support of a matrix, page 31
Tr	Trace of a matrix, page 27

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