Quantum fluctuations in one-dimensional supersolids

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(Received 1 December 2022; revised 3 March 2023; accepted 12 June 2023; published 9 August 2023)

In one dimension, quantum fluctuations prevent the appearance of long-range order in a supersolid, and only quasi-long-range order can survive. We derive this quantum critical behavior and study its influence on the superfluid response and properties of the solid. The analysis is based on an effective low-energy description accounting for the two coupled Goldstone modes. We find that the quantum phase transition from the superfluid to the supersolid is shifted by quantum fluctuations from the position where the local formation of a solid structure takes place. For current experimental parameters with dipolar atomic gases, this shift is extremely small and cannot be resolved yet, i.e., current observations in experiments are expected to be in agreement with predictions from mean-field theory based on the extended Gross-Pitaevskii formalism.

DOI: 10.1103/PhysRevResearch.5.033092

I. INTRODUCTION

A remarkable property of quantum fluctuations is that they strongly influence spontaneous symmetry breaking in one-dimensional systems. Especially, it is well established that one-dimensional superfluids only exhibit quasi-long-range order with a characteristic algebraic decay [1–4]. Nevertheless, the latter can support a superfluid flow across a weak impurity [5,6]. Recent experiments with weakly interacting dipolar Bose gases have observed the appearance of a supersolid phase in elongated one-dimensional geometries [7–9]. Such a supersolid state breaks the translational symmetry giving rise to a solid structure as well as the U(1) symmetry for the superfluid and combines the characteristic properties of both [10,11]. In this paper, we study the influence of quantum fluctuations and the appearance of quasi-long-range order for such one-dimensional supersolid phases in the thermodynamic limit.

For a long time, the search for a supersolid state of matter focused on systems with an averaged particle number per lattice site close to 1 [11–13]. Remarkably, recent experiments with dysprosium atoms are in a complementary regime with a large amount of particles per lattice site [7–9]. So far, the theoretical description of these experiments is based on mean-field theory within the extended Gross-Pitaevskii formalism. It includes the leading beyond-mean-field correction within the local-density approximation [14,15] and accounts for quantum fluctuations modifying short-range correlations; the behavior of this contribution has been studied for superfluids in tight traps [16–18]. These analyses are in good agreement with the experimental observations in elongated trap geometries and also predict the stability of the supersolid phase in the thermodynamic limit for a one-dimensional geometry [19–22].

Quantum fluctuations will in addition prevent the appearance of long-range order and modify the characteristic properties of this supersolid state. This effect has been intensely studied for superconducting thin wires: In addition to the appearance of quasi-long-range order, the quantum nucleation of phase slips provides dissipation even in the superconducting phase at zero temperature and gives rise to an algebraic current-voltage characteristic [23–28]. Such an algebraic current-voltage characteristic still gives rise to a superconducting phase as the linear resistivity vanishes. However, quantum fluctuations can eventually drive a quantum phase transition from the superconductor to a state with finite resistivity for increasing influence of the phase fluctuations [23–28]. For a Galilei invariant superfluid, quantum nucleation of phase slips can only appear at an impurity, but it similarly gives rise to dissipation and eventually drives a quantum phase transition [5,6,29,30]. In analogy, the phonon mode in a solid gives rise to a similar effective low-energy description, and therefore quantum fluctuations also strongly affect the properties of a solid [31].

In this paper, we analyze how quantum fluctuations affect the characteristic properties of a supersolid in a one-dimensional geometry. The analysis is based on the effective low-energy theory for a supersolid with many particles within a lattice site [32–34] and allows for the derivation of the algebraic behavior of the characteristic correlation functions. The superfluid is defined by the ability of the system to sustain a dissipationless particle flow across a weak impurity, i.e., the absence of a linear relation between flow and pressure [5,6]. In analogy, the solid character is defined by the ability of the system to drag the solid structure with a moving impurity. We find that the quantum phase transition from the superfluid to the supersolid is shifted: The formation of a solid structure takes place first, while the supersolid phase only appears for sufficient correlations between these local solid structures. However, for current experimental setups with dipolar quantum gases, this shift is extremely small and beyond current experimental resolution. In addition, we find that correlations only decay very slowly, such that mean-field theory within the extended Gross-Pitaevskii approach is expected to provide an accurate description for current experimental parameters in elongated traps. Finally, we study the disappearance of the supersolid phase for increasing correlations. This paper is ordered as follows: In Sec. II we discuss the underlying low-energy theory. The phase boundaries between superfluid, supersolid, and solid are derived in Sec. III. Finally, in Sec. IV we discuss the connection of the effective parameters in the low-energy theory to the experimental setting with dysprosium atoms.

II. LOW-ENERGY THEORY

We start with the effective low-energy description of a one-dimensional supersolid consisting of weakly interacting bosons with a large number of atoms per lattice site. Then, the bosonic field operator can be written as $\psi(\boldsymbol{x}) = \sqrt{\rho(\boldsymbol{x})}\, e^{i\varphi(\boldsymbol{x})}$ with the phase field φ and the density field

$$\rho(\boldsymbol{x}) = [n + \delta n(\boldsymbol{x})] f\left(x - \frac{d}{2\pi} u(\boldsymbol{x})\right), \tag{1}$$

while we introduced the notation $\boldsymbol{x} = (x, t)$ for the space-time coordinate. Here, $f(x) = f(x+d)$ is a periodic function with period d and normalized to $\int_0^d dx\, f(x)/d = 1$; it accounts for the local formation of a solidlike structure by droplets, while the displacement field $u(\boldsymbol{x})$ allows for fluctuations in the position of these droplets. Note that n denotes the averaged density with nd particles within each droplet, while $\delta n(\boldsymbol{x})$ describes local density fluctuations. The low-energy behavior is then captured by the effective Lagrangian for the slowly varying fields φ, δn, and u [32–34],

$$\begin{aligned}\mathcal{L} = &- \hbar \delta n \partial_t \varphi - \frac{\kappa}{2}(\delta n)^2 - \frac{\lambda'}{2}(\partial_x u)^2 - \xi' \delta n \partial_x u \\ &+ \frac{\hbar^2 n}{2m}\left[\frac{n_{\rm L}}{n}\left(\frac{md}{2\pi\hbar}\partial_t u - \partial_x \varphi\right)^2 - (\partial_x\varphi)^2\right].\end{aligned} \tag{2}$$

The second line corresponds to the kinetic energy, where the term $\partial_t u$ accounts for the velocity of the droplet at position x, while the superfluid exhibits a reduced superfluid density $n_{\rm s} \equiv n - n_{\rm L}$ due to the formation of a local solidlike structure [10]. Furthermore, the first line includes the conventional coupling between the phase field and the density in a superfluid as well as an expansion of the interaction energy to second order in the slowly varying fields with parameters κ, λ', and ξ'. Note that these parameters can be conveniently derived within mean-field theory [21], and the stability of the system naturally requires $\kappa\lambda' - (\xi')^2 > 0$. In the weakly interacting regime, we also require $\hbar^2 n/(m\kappa) \gg 1$, i.e., the kinetic energy of the superfluid is much larger than the interaction energy.

The Lagrangian in Eq. (2) describes a strong coupling between the Bogoliubov mode of the superfluid and the phonon mode of a solid, and it gives rise to two linear sound modes accounting for the two broken symmetries. Furthermore, we find the current conservation $\partial_t \delta n = -\partial_x(j_{\rm s} + j_{\rm n})$ with the normal and superfluid current $j_{\rm n} = (n_{\rm L} d/2\pi)\partial_t u$ and $j_{\rm s} = (\hbar n_{\rm s}/m)\partial_x\varphi$. For $n_{\rm L}/n \to 0$ the solid structure disappears. Consequently, $\xi' \to 0$ and $\lambda' \to 0$, and we recover the effective low-energy description of a superfluid. For $n_{\rm L}/n \to 1$, we obtain the theory of phonons in a solid with the compressibility $(n^2\kappa + 4\pi^2\lambda'/d^2 - 4\pi n\xi'/d)/m$.

In the following, it is convenient to switch to a Hamiltonian description of the low-energy quantum theory,

$$\begin{aligned}H = \frac{\hbar}{2\pi}\int dx \Bigg[&v_J \begin{pmatrix}\partial_x\varphi \\ \partial_x w\end{pmatrix} \mathbf{M}_J \begin{pmatrix}\partial_x\varphi \\ \partial_x w\end{pmatrix} \\ &+ v_N \begin{pmatrix}\partial_x\vartheta \\ \partial_x u\end{pmatrix} \mathbf{M}_N \begin{pmatrix}\partial_x\vartheta \\ \partial_x u\end{pmatrix}\Bigg],\end{aligned} \tag{3}$$

where $-\hbar\partial_x\vartheta/\pi$ and $-\hbar\partial_x w/\pi$ denote the conjugate variables to φ and u, respectively. We have introduced the two velocities $v_J = \hbar\pi n/m$ and $v_N = \kappa/\pi\hbar$, with $v_N/v_J \ll 1$ in the weakly interacting regime. The matrices $\mathbf{M}_J$ and $\mathbf{M}_N$ take the form

$$\mathbf{M}_J = \begin{pmatrix} 1 & -\beta \\ -\beta & \beta^2/\gamma \end{pmatrix}, \quad \mathbf{M}_N = \begin{pmatrix} 1 & \xi \\ \xi & \lambda \end{pmatrix}, \tag{4}$$

with $\beta = 2/nd$, $\gamma = n_{\rm L}/n$, and the dimensionless parameters $\lambda = \lambda'\pi^2/\kappa$ and $\xi = \xi'\pi/\kappa$. Since $\gamma \leqslant 1$, $\mathbf{M}_J$ is positive semidefinite. The stability in the thermodynamic limit requires $\mathbf{M}_N$ to be positive semidefinite as well, i.e., $\lambda - \xi^2 \geqslant 0$. Being conjugate variables, the canonical commutation relations read

$$[\partial_x\vartheta(\boldsymbol{x}), \varphi(\boldsymbol{y})] = i\pi\delta(\boldsymbol{x} - \boldsymbol{y}) = [\partial_x u(\boldsymbol{x}), w(\boldsymbol{y})]. \tag{5}$$

It is possible to diagonalize this Hamiltonian into two uncoupled sound modes by the transformation

$$\begin{pmatrix}\phi_+ \\ \phi_-\end{pmatrix} = \mathbf{Q}\begin{pmatrix}\varphi \\ w\end{pmatrix}, \quad \begin{pmatrix}\theta_+ \\ \theta_-\end{pmatrix} = (\mathbf{Q}^{-1})^T\begin{pmatrix}\vartheta \\ u\end{pmatrix}. \tag{6}$$

The construction of this canonical transformation is presented in the Appendix, and we obtain the Hamiltonian

$$H = \frac{\hbar}{2\pi}\int dx \sum_{\sigma\in\{+,-\}} v_\sigma[(\partial_x\phi_\sigma)^2 + (\partial_x\theta_\sigma)^2] \tag{7}$$

with the two sound velocities

$$v_\pm^2 = \frac{v_J v_N}{2}[\alpha \pm \sqrt{\alpha^2 - 4\beta^2(\lambda - \xi^2)(1-\gamma)/\gamma}] \tag{8}$$

and $\alpha = 1 - 2\xi\beta + \beta^2\lambda/\gamma$. The Hamiltonian in Eq. (7) allows us to derive the behavior of correlation functions at long distances [2]. We obtain quasi-long-range off-diagonal order for the superfluid as well as quasi-long-range diagonal order for the solid,

$$\begin{aligned}\langle\psi(x)\psi^\dagger(0)\rangle &= n\left(\frac{\zeta}{|x|}\right)^{A/2}, \\ \langle\rho(x)\rho(0)\rangle - n^2 &= -\frac{C}{2\pi^2|x|^2} + \eta\cos\frac{2\pi x}{d}\left(\frac{\zeta}{|x|}\right)^{B/2}.\end{aligned} \tag{9}$$

Here, we introduced the nonuniversal parameter η and the short-distance cutoff ζ, while the algebraic decay is determined by the canonical transformation via $A = ((\mathbf{Q}^{-1})_{11})^2 + ((\mathbf{Q}^{-1})_{12})^2$, $B = (\mathbf{Q}_{12})^2 + (\mathbf{Q}_{22})^2$, and $C = (\mathbf{Q}_{11})^2 + (\mathbf{Q}_{21})^2$.

III. PHASE DIAGRAM

In the following, we study the quantum phase transitions in the system. The characteristic property of a superfluid is that it can sustain a superfluid flow, while in a solid a moving localized impurity can drag the solid structure along; a supersolid exhibits both of these properties, i.e., it can sustain a superfluid flow while a moving impurity drags the solid structure along. These conditions provide critical values for the algebraic correlations above and will be studied in the following.

A. Superfluid-supersolid transition

We start with the parameters in the superfluid close to the formation of a solidlike structure, i.e., $\gamma \ll 1$ with $A \sim \sqrt{v_N/v_J} \ll 1$. This condition is sufficient to sustain a superfluid flow (see below). Therefore we first study the transition into the supersolid for increasing γ, i.e., stronger local solidlike structure. A local impurity at position x_0 is described by an external potential $V_I \approx g\delta(x - x_0)$ and provides a contribution to the low-energy Hamiltonian

$$H_{\rm I} = \int dx \rho(x) V_I(x) \sim g_u \cos\left(u(x_0) + 2\pi x_0/d\right), \qquad (10)$$

where we expanded the local solid structure $f(x)$ in Eq. (1) into a Fourier series. Note that the impurity can provide additional terms when taking the discrete nature of particles into account [2], but these do not become relevant before superfluidity is lost (see below). The low-energy description then reduces to a coupled boundary sine-Gordon model [35–40]. The term in Eq. (10) is irrelevant for $B > 4$, and therefore the system does not feel the presence of the impurity in the low-energy regime. In turn, the term becomes relevant for $B < 4$ and pins $u(x_0)$ to the minimum of the cosine. Varying the position x_0 of the impurity now results in a change in u, which shifts the local solid structure of the system with the impurity. Hence the system exhibits a solid character for $B < 4$. In our dimensionless units, B is given by

$$B = \beta^2 \frac{v_J}{(v_+ + v_-)}\left[\frac{1}{\gamma} + \sqrt{\frac{1-\gamma}{\gamma\beta^2(\lambda - \xi^2)}}\,\right] \sim \beta^2 \sqrt{\frac{v_J}{v_N}} \frac{1}{\sqrt{\gamma\beta^2\lambda}} \quad \text{for } \gamma \to 0. \qquad (11)$$

In Fig. 1 the critical line $B = 4$ of the quantum phase transition separating the superfluid from the supersolid is shown for different values of ξ. The parameters $v_J/v_N = 1.0 \times 10^7$ and $\beta = 2.1 \times 10^{-4}$ are fixed to realistic values derived within mean-field theory using an extended Gross-Pitaevskii approach for an experimentally realistic setup (see below). The transition always takes place at a finite and nonvanishing value of γ, i.e., the transition is shifted from the local formation of a solid structure at $\gamma = 0$.

It is important to note that the local formation of droplets can act as a source for the nucleation of quantum phase slips. Even in the superfluid as well as in the supersolid, such phase slips will give rise to a small dissipation and an algebraic behavior between the pressure difference ΔP for sustaining the particle current I with $\Delta P \sim I^{2/A-2}$; this behavior is in analogy to thin superconducting wires [23].

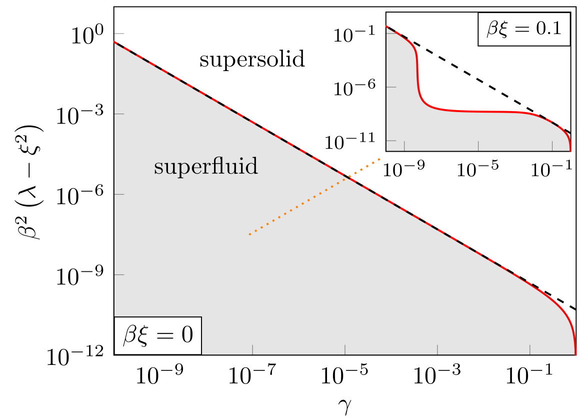

FIG. 1. Critical line $B = 4$ (red solid line) as a function of γ and $\beta^2(\lambda - \xi^2)$ at $\beta\xi = 0$ for fixed $v_J/v_N \approx 1.0 \times 10^7$ and $\beta \approx 2.1 \times 10^{-4}$. The black dashed line shows the asymptotic behavior of B for $\gamma \to 0$. In the gray-shaded region, $B > 4$ and the system does not feel the impurity (superfluid), while in the white region $B < 4$ and the perturbation becomes relevant (supersolid). The orange dotted line shows the path across the phase transition for experimentally realistic parameters. In the inset, we fix $\beta\xi = 0.1$ while v_J/v_N and β remain unchanged.

B. Supersolid-solid transition

Next we analyze the quantum phase transition from the supersolid into a solid, which appears for $\gamma \to 1$. One can understand this transition as in this regime, the different droplets of the solid structure are only connected by a very weak superfluid link and essentially give rise to a Josephson junction between each droplet. Note that such a Josephson junction can only support a superfluid flow if $A < 1$ [35]. For experimentally realistic setups with $v_N/v_J \ll 1$ and the asymptotic behavior $A \sim \frac{1}{\sqrt{1-\gamma}}\sqrt{\frac{v_N}{v_J}}$ for $\gamma \to 1$, we find indeed that this transition can only appear for $\gamma \approx 1$. This simple criterion provides an upper bound on the transition from the supersolid into the solid. However, this transition can be preempted at commensurate fillings with an integer number of particles within each droplet, i.e., $nd = 2/\beta \in \mathbb{N}$. Then, the microscopic interaction between the particles also generates a term [2]

$$H_{\rm M} = g_M \iint dx\, dt \cos\left(2\vartheta(\boldsymbol{x}) + 2u(\boldsymbol{x})/\beta\right), \qquad (12)$$

which becomes relevant for $D = (\mathbf{Q}_{11} + \mathbf{Q}_{12}/\beta)^2 + (\mathbf{Q}_{21} + \mathbf{Q}_{22}/\beta)^2 < 2$ and pins the number of particles in each droplet to the integer value $2/\beta$. It describes the quantum phase transition into a Mott insulator with an excitation gap for adding or removing a particle from a droplet. However, the droplets can still fluctuate in position giving rise to a phononic sound mode characteristic for a solid. Since $D = A^{-1}$ for $\gamma \to 1$, the Mott transition at commensurate fillings occurs earlier than the transition of a single Josephson junction.

IV. EXPERIMENTAL PARAMETERS

In the following, we demonstrate how to obtain the effective parameters of our low-energy theory close to the superfluid-to-supersolid phase transition within a mean-field study. The setup we focus on is based on dysprosium atoms. We consider a reduced three-dimensional model of weakly interacting dipolar bosons of mass m in a transverse harmonic trap of oscillator length $l_\perp$ within mean-field theory. Our approach is based on a recent mean-field study [21], where also the excitation spectrum has been studied in detail. For the transverse wave function we use the ansatz $\psi(y,z) = \frac{1}{\sqrt{\pi}\sigma} e^{-(\nu y^2+z^2/\nu)/\sigma^2}$ and determine the variational parameters σ and ν by minimizing the ground-state energy. In addition, we include quantum fluctuations in local-density approximation (extended mean-field theory), which is crucial for the stability of the supersolid state. These quantum fluctuations only influence the short-range correlations and provide a correction to the mean-field ground-state energy. They do not influence spontaneous symmetry breaking in the one-dimensional system as mean-field theory inherently assumes the presence of order. For a transverse harmonic trapping with length $l_\perp = 200a_s$ and a density $n \approx 11.931/a_s$, the extended mean-field formalism predicts a second-order quantum phase transition from the superfluid to the supersolid state, where the local formation of a structure appears at $\varepsilon_{\rm dd}^* = 1.34$ [21]; here, a_s denotes the s-wave scattering length, and $\varepsilon_{\rm dd}$ denotes the relative dipolar interaction strength. In the supersolid regime, we use the mean-field ansatz

$$\phi(x) = \frac{\sqrt{n}}{\sqrt{1+\sum_{j=1}^{\infty}\frac{\Delta_j^2}{2}}}\left(1+\sum_{j=1}^{\infty}\Delta_j \cos[j\, k_s x]\right) \tag{13}$$

for the longitudinal direction, with the order parameters Δ_j, the wave vector k_s of the modulation, and the one-dimensional density n. We write the energy as

$$\begin{aligned} E = E_t(\sigma,\nu) &- \int dx\, \phi^*(x)\frac{\hbar^2\nabla^2}{2m}\phi(x) \\ &+ \frac{1}{2}\int dx\, dx'\, V(x-x')|\phi(x')|^2|\phi(x)|^2 \\ &+ \frac{2}{5}\tilde{\gamma}\int dx\, |\phi(x)|^5, \end{aligned} \tag{14}$$

where $E_t = \frac{N\hbar^2}{4ml_\perp^2}(\frac{1}{\nu}+\nu)(\frac{1}{\sigma^2}+\sigma^2)$ is the energy contribution of the transverse confinement of the N particles and $V(x)$ is the effective one-dimensional (1D) dipolar interaction potential in real space, which we obtain by integrating out the transverse degrees of freedom of the three-dimensional dipolar interaction potential with dipolar strength $\varepsilon_{\rm dd}$ and scattering length a_s, assuming the shape $\psi(y,z)$ for the transverse wave function [41]. The last term in Eq. (14) takes into account fluctuations within the local-density approximation, and $\tilde{\gamma}$ controls the strength of these fluctuations. The ground state is obtained by minimizing Eq. (14) with respect to Δ_l and k_s as well as σ and ν. From the ground state we can extract all relevant quantities of our model. Within mean-field theory, Leggett's upper bound for the superfluid fraction [10] is completely saturated [42],

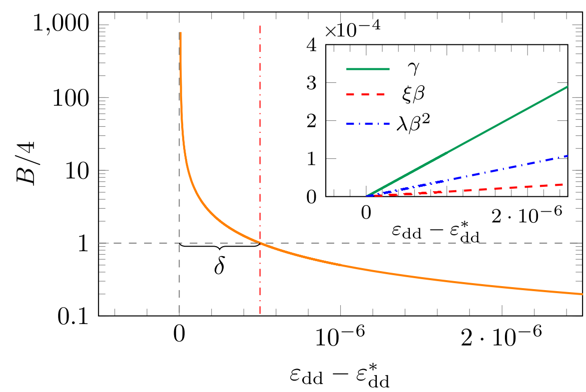

FIG. 2. Superfluid-to-supersolid transition for increasing dipolar strength. We show $B/4$ as a function of $\varepsilon_{\rm dd}-\varepsilon_{\rm dd}^*$, where $\varepsilon_{\rm dd}^* = 1.34$ marks the local formation of droplets. The system transitions to the supersolid at $B/4 = 1$ (red vertical line), which is shifted compared with the local formation of a solid structure at $\gamma = 0$ (gray vertical line). The inset shows the system parameters γ, $\beta\xi$, and $\beta^2\lambda$ used to calculate $B/4$ as a function of $\varepsilon_{\rm dd}$ for a harmonic transverse trapping with oscillator length $l_\perp = 200a_s$ and one-dimensional density $n = 11.931/a_s$.

$$\frac{1}{f_s} = \frac{k_s^2}{4\pi^2}\left(\int_0^{2\pi/k_s} dx|\phi(x)|^2\right)\left(\int_0^{2\pi/k_s}\frac{dx}{|\phi(x)|^2}\right), \tag{15}$$

and we obtain $\gamma = 1 - f_s$. The parameters κ, ξ', and λ' of our model characterize the effective potential of the supersolid. By considering the ground state of Eq. (14) for a fixed chemical potential μ, we can connect variations of the wave vector k_s and the density n to the parameters κ, ξ', and λ'. Variations in the density δn are directly connected to κ, while a variation in the wave vector k_s is connected to λ, since a linear displacement field $\partial_x u = \delta a = \text{const}$ changes the periodicity of the system, $k_s \to k_s - \delta a = k_s + \delta k_s$. We expand $E - \mu N$ up to second order in δn and δk_s,

$$\begin{aligned} \frac{ml_\perp^2}{\hbar^2}\frac{(E-\mu N)a_s}{L} = C_0 &+ C_\kappa(a_s\delta n)^2 + C_\lambda(l_\perp\delta k_s)^2 \\ &+ C_\xi(a_s\delta n)(l_\perp\delta k_s), \end{aligned} \tag{16}$$

and obtain the expansion coefficients C_κ, C_λ, and C_ξ numerically. Comparing the Hamiltonian in Eq. (3) with the expansion in Eq. (16) yields the dimensionless parameters

$$\lambda = \frac{\pi^2 l_\perp^2 C_\lambda}{a_s^2 C_\kappa}, \quad \xi = -\frac{\pi l_\perp C_\xi}{2a_s C_\kappa}, \quad \beta = \frac{k_s}{n\pi}, \quad \frac{v_J}{v_N} = \frac{\pi^2 n l_\perp^2}{2a_s C_\kappa}. \tag{17}$$

For the parameters $a_s/l_\perp = 1/200$ and $n \approx 11.931/a_s$, $v_J/v_N \approx 1.0\times 10^7$ and $\beta \approx 2.1\times 10^{-4}$ are approximately constant in the vicinity of the extended mean-field transition at $\varepsilon_{\rm dd}^* = 1.34$. In the inset of Fig. 2, we show the parameters γ, ξ, and β as a function of $\varepsilon_{\rm dd}-\varepsilon_{\rm dd}^*$. As predicted above, we find that ξ and λ vanish for $\gamma \to 0$. Then, the behavior of the parameter $B/4$ is shown in Fig. 2 as a function of $\varepsilon_{\rm dd}-\varepsilon^*_{\rm dd}$. Indeed, we find that the quantum phase transition to the supersolid is shifted by quantum fluctuations from the position where a local solid structure forms at $\varepsilon^*_{\rm dd}=1.34$. However, this region is extremely small with $\delta\approx 5\times 10^{-7}$ and beyond current control on the experimental parameters. Therefore the transition from the superfluid into the supersolid phase is extremely well described by mean-field theory within the extended Gross-Pitaevskii approach for current experimental setups.

V. CONCLUSION

In conclusion, we have studied the influence of quantum fluctuations on a one-dimensional supersolid and determined both the quasi-long-range diagonal and off-diagonal order. The quantum phase transition from the superfluid to the supersolid is shifted by quantum fluctuations from the position where a local solid structure forms. Furthermore, the quantum nucleation of phase slips provides a weak dissipation with an algebraic behavior between the pressure difference and the particle flow. However, for current experimental parameters for dysprosium atoms with many atoms per lattice site, these effects are extremely weak, and therefore the supersolid is accurately described by the extended Gross-Pitaevskii equation.

ACKNOWLEDGMENT

This work is supported by the German Research Foundation (DFG) within FOR2247 under Bu2247/1-2. This publication was funded by the German Research Foundation (DFG) grant "Open Access Publication Funding / 2023-2024 / University of Stuttgart" (512689491).

APPENDIX: CONSTRUCTION OF THE TRANSFORMATION MATRIX

In this Appendix we derive the transformation which decouples the Hamiltonian in Eq. (3) of the main text into two modes with sound velocity $v_\pm$. This can be achieved if we require that the transformation $\mathbf{Q}$ fulfills the equations

$$(\mathbf{Q}^{-1})^T[v_J\mathbf{M}_J]\mathbf{Q}^{-1}=\begin{pmatrix}v_+ & 0\\ 0 & v_-\end{pmatrix},$$
$$\mathbf{Q}[v_N\mathbf{M}_N]\mathbf{Q}^T=\begin{pmatrix}v_+ & 0\\ 0 & v_-\end{pmatrix}. \tag{A1}$$

From these equations it also follows that $\mathbf{Q}$ has to fulfill

$$\mathbf{Q}[v_Nv_J\mathbf{M}_N\mathbf{M}_J]\mathbf{Q}^{-1}=\begin{pmatrix}v_+^2 & 0\\ 0 & v_-^2\end{pmatrix}, \tag{A2}$$

meaning that it diagonalizes the matrix $\mathbf{M}_N\mathbf{M}_J$ with eigenvalues $v_\pm^2$. We can therefore construct $\mathbf{Q}$ by finding the eigenvectors of $\mathbf{M}_N\mathbf{M}_J$. These are

$$\mathbf{p}_\sigma=\frac{1}{N_\sigma}\left(\frac{1}{2}[1-\beta^2\lambda/\gamma+\sigma\sqrt{\Delta}],\,\xi-\beta\lambda\right)^T, \tag{A3}$$

with $\sigma\in\{+,-\}$ and $\Delta=(1-2\xi\beta+\beta^2\lambda/\gamma)^2-4\beta^2(\lambda-\xi^2)(1-\gamma)/\gamma$ and yet-undetermined constants N_σ. The matrix is then defined as $\mathbf{Q}^{-1}=(\mathbf{p}_+\ \mathbf{p}_-)$. The still-undetermined constants N_σ must now be used to make sure that Eq. (A1) is fulfilled. We obtain that

$$N_\pm^2=\pm\sqrt{\Delta}\frac{v_\pm^2-v_Nv_J\beta^2\lambda(1-\gamma)/\gamma}{v_Nv_\pm}, \tag{A4}$$

which fully determines the transformation. A bit of calculation leads to a somewhat compact form

$$\mathbf{Q}^{-1}=\sqrt[4]{\frac{v_N}{v_J}}\begin{pmatrix}s_1^+\sqrt{\frac{\gamma\hat{v}_+^2-\beta^2(\lambda-\xi^2)}{\gamma\hat{v}_+\sqrt{\Delta}}} & s_1^-\sqrt{\frac{\beta^2(\lambda-\xi^2)-\gamma\hat{v}_-^2}{\gamma\hat{v}_-\sqrt{\Delta}}}\\ s_2^+\sqrt{\frac{\lambda\hat{v}_+^2-(\lambda-\xi^2)}{\hat{v}_+\sqrt{\Delta}}} & s_2^-\sqrt{\frac{(\lambda-\xi^2)-\lambda\hat{v}_-^2}{\hat{v}_-\sqrt{\Delta}}}\end{pmatrix},$$

with $s_i^\pm=\mathrm{sgn}((\mathbf{p}_\pm)_i)$ and $\hat{v}_\sigma=v_\sigma/\sqrt{v_Jv_N}$. The inverse is then given by

$$\mathbf{Q}=\sqrt[4]{\frac{v_J}{v_N}}\begin{pmatrix}\sqrt{\frac{\gamma\hat{v}_+^2-\beta^2\lambda(1-\gamma)}{\gamma\hat{v}_+\sqrt{\Delta}}} & -s_1^-s_2^-\beta\sqrt{\frac{\hat{v}_+^2-(1-\gamma)}{\gamma\hat{v}_+\sqrt{\Delta}}}\\ -\sqrt{\frac{\beta^2\lambda(1-\gamma)-\gamma\hat{v}_-^2}{\gamma\hat{v}_-\sqrt{\Delta}}} & s_1^+s_2^+\beta\sqrt{\frac{(1-\gamma)-\hat{v}_-^2}{\gamma\hat{v}_-\sqrt{\Delta}}}\end{pmatrix}.$$

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