A Collection of Probability Integrals

– Applied using Matlab

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This paper presents a collection of probability integrals often used in optimization problems. All the formulas are applied using Matlab with the source code given. It may be noted that all the formulas were intensively tested and seem to be correct, however, this cannot be guaranteed.

Contents

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1. Introduction

Using probability integrals, e. g. in optimization problems, is less computing time consuming if the integrals are solved analytically rather than numerically. Often a simple normal distribution of a probability variable χ is used (cf. Eqn. 1). However, already the first analytical integration gets complex and can only be solved using the so called error function as described in the next section (cf. Eqn. 5). But what happens if the integral of the error function needs to be calculated, i. e. if it is necessary to calculate double or even triple integrals of a probability distribution? And what happens if even a convolution is considered? The main aspect of finding a solution in these cases is to solve the integrals sequentially. Resulting in very long and rather complex formulas (you may think different if you are a mathematician, though). Anyway, it may be useful to have a collection of the most relevant probability integrals and their representation at hand. This paper tries to provide this collection by summarizing some of these formulas in a structured way, cf. Tab. 1.

Most of the calculations needed for the derivation of the formulas in this collection are based on the web-based formula compendium by Wolfram Research. However, these formulas are sometimes to specific and only representations in Mathematica and XML are given, but none in Matlab. Some other formulas are not available at all. However, if a basic formula of one of the probability integrals in this collection can be found on the website of Wolfram Research, a unique number in Tab. 1 is cited. The number can be used to find the corresponding formula on the website (http://functions.wolfram.com/).

Table 1: Probability Integrals in this Collection

	Formula (to be integrated)	Equation Number	Matlab Code Line	Wolfram Research Number	Wolfram Research Assumption
1	$\exp(-(ax+c)^2+bx+d)$	8	80	–	–
2	$x\exp(-(ax+c)^2+bx+d)$	11	95	–	–
3	$x^2\exp(-(ax+c)^2+bx+d)$	14	118	–	–
4	$\operatorname{erf}(ax+c)$	17	143	06.25.21.0001.01	–
5	$x\operatorname{erf}(ax+c)$	19	152	06.25.21.0006.01	$c=0$
6	$x^2\operatorname{erf}(ax+c)$	21	163	–	–
7	$\exp(bx+d)\operatorname{erf}(ax+c)$	23	174	06.25.21.0011.01	$c,d=0$
8	$x\exp(bx+d)\operatorname{erf}(ax+c)$	27	190	06.25.21.0017.01	$c,d=0$
9	$x^2\exp(bx+d)\operatorname{erf}(ax+c)$	31	209	06.25.21.0018.01	$c,d=0$
10	$\operatorname{erfc}(ax+c)$	35	232	06.27.21.0001.01	–
11	$x\operatorname{erfc}(ax+c)$	37	240	06.27.21.0006.01	$c=0$
12	$x^2\operatorname{erfc}(ax+c)$	39	253	–	–
13	$\exp(bx+d)\operatorname{erfc}(ax+c)$	41	266	06.27.21.0011.01	$c,d=0$
14	$x\exp(bx+d)\operatorname{erfc}(ax+c)$	45	282	06.27.21.0016.01	$c,d=0$
15	$x^2\exp(bx+d)\operatorname{erfc}(ax+c)$	49	301	06.27.21.0017.01	$c,d=0$

2. Probability Integrals

In many optimization problems a normal distribution with mean $\mathrm{E}_1[x] = \mu$ and variance $\mathrm{VAR}_1[x] = \sigma^2$ is used.

$$f_1(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2\right) \tag{1}$$

The integration of the normal distribution is possible using the so called error function. The error function equals twice the integral of a normalized normal distribution ($\mu = 0$ and $\sigma^2 = 1$) between 0 and $\sqrt{2}x$. The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp\left(-t^2\right)\, dt \tag{2}$$

Some specific values and the symmetry of the error function are given as follows:

$$\begin{array}{ll} \text{Specific values:} & \operatorname{erf}(0) = 0,\ \operatorname{erf}(\infty) = 1 \text{ and } \operatorname{erf}(-\infty) = -1 \\ \text{Symmetry:} & \operatorname{erf}(-x) = -\operatorname{erf}(x) \end{array} \tag{3}$$

While the error function is well suited to provide the probability that the parameter of interest (probability variable χ) is within a specific range, the complementary error function provides the probability that the parameter is outside that range.

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_x^\infty \exp\left(-t^2\right)\, dt \tag{4}$$

Using these definitions it can be derived that the integral of the normal distribution in Eqn. 1 is given by

$$F_1(x|\mu,\sigma) = \mathrm{P}[\chi \le x|\mu,\sigma] = \int\limits_{-\infty}^x f_1(x)\, dx = \frac{1}{2} \operatorname{erfc}\left(-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right) \tag{5}$$

Still the question arises, how the integral may be calculated if the integration limits are different, say x_1 and x_2? How do integrals of higher order look like? And what happens if a convolution of two distributions, say a normal distribution f_1 like Eqn. 1 and e. g. an Erlang distribution f_2 given as a special case of a Gamma distribution with $a = 0$ and $b > 0$ is used? With mean $\mathrm{E}_2[u] = 2b$ and variance $\mathrm{VAR}_2[u] = 2b^2$ this distribution is given as

$$f_2(u|a,b) = \frac{1}{b^a \Gamma(a)} u^{a-1} \exp\left(-\frac{u}{b}\right) \quad \text{with special case} \quad f_2(u|0,b) = \begin{cases} 0 & \text{if } u \le 0, \\ \frac{1}{b^2} u \exp\left(-\frac{u}{b}\right) & \text{if } u > 0. \end{cases} \tag{6}$$

Generally a convolution of two functions, where the mean and variance are defined as derived in Appendix A, may be written as

$$f(x) = \int\limits_{-\infty}^{\infty} f_1(x-u)\, f_2(u)\, du \tag{7}$$

Obviously, in these cases more complex integrals need to be solved. In the following subsections this collection tries to answer these questions. The formulas are given as indefinite integrals, however the Matlab code in Section 3 is developed for definite integrals.

2.1. Exponential Function

$\boxed{\textbf{Case 1}}$ $\longrightarrow$ cf. Line 80 ff of the Matlab code.

Eqn. 8 is defined only if $a \neq 0$.

$$\int \exp\left(-\left(ax+c\right)^2+bx+d\right)\,dx = \frac{\sqrt{\pi}}{2a}\exp\left(\frac{b^2}{4a^2}-\frac{bc}{a}+d\right)\operatorname{erfc}\left(ax+c-\frac{b}{2a}\right) \tag{8}$$

Eqn. 9 is a special case of Eqn. 8 if $a = 0 \wedge b \neq 0$.

$$\int \exp\left(-c^2+bx+d\right)\,dx = \frac{1}{b}\exp\left(-c^2+bx+d\right) \tag{9}$$

Eqn. 10 is a special case of Eqn. 8 if $a, b = 0$.

$$\int \exp\left(-c^2+d\right)\,dx = x\exp\left(-c^2+d\right) \tag{10}$$

2.1.1. Involving Power Function

$\boxed{\textbf{Case 2}}$ $\longrightarrow$ cf. Line 95 ff of the Matlab code.

Eqn. 11 is defined only if $a \neq 0$.

$$\begin{aligned}&\int x\exp\left(-\left(ax+c\right)^2+bx+d\right)\,dx\\&=-\frac{1}{2a^2}\exp\left(-\left(ax+c\right)^2+bx+d\right)-\sqrt{\pi}(2ac-b)\exp\left(\frac{b^2}{4a^2}-\frac{bc}{a}+d\right)\operatorname{erfc}\left(ax+c-\frac{b}{2a}\right)\end{aligned} \tag{11}$$

Eqn. 12 is a special case of Eqn. 11 if $a = 0 \wedge b \neq 0$.

$$\int x\exp\left(-c^2+bx+d\right)\,dx = \frac{1}{b^2}(bx-1)\exp\left(-c^2+bx+d\right) \tag{12}$$

Eqn. 13 is a special case of Eqn. 11 if $a, b = 0$.

$$\int x\exp\left(-c^2+d\right)\,dx = \frac{x^2}{2}\exp\left(-c^2+d\right) \tag{13}$$

$\boxed{\textbf{Case 3}}$ $\longrightarrow$ cf. Line 118 ff of the Matlab code.

Eqn. 14 is defined only if $a \neq 0$.

$$\begin{aligned}
&\int x^2 \exp\left(-\left(ax+c\right)^2 + bx + d\right)\,dx \\
&= -\frac{1}{4a^4}\big(2a(ax-c)+b\big)\exp\left(-\left(ax+c\right)^2+bx+d\right) \\
&\quad + \sqrt{\pi}\left(4a^2c^2 - 4abc + b^2 + 2a^2\right)\exp\left(\frac{b^2}{4a^2} - \frac{bc}{a} + d\right)\operatorname{erfc}\left(ax + c - \frac{b}{2a}\right)
\end{aligned} \tag{14}$$

Eqn. 15 is a special case of Eqn. 14 if $a = 0 \land b \neq 0$.

$$\int x^2 \exp\left(-c^2 + bx + d\right)\,dx = \frac{1}{b^3}\left(b^2x^2 - 2bx + 2\right)\exp\left(-c^2 + bx + d\right) \tag{15}$$

Eqn. 16 is a special case of Eqn. 14 if $a, b = 0$.

$$\int x^2 \exp\left(-c^2 + d\right)\,dx = \frac{x^3}{3}\exp\left(-c^2 + d\right) \tag{16}$$

2.2. Error Function

$\boxed{\textbf{Case 4}}$ $\longrightarrow$ cf. Line 143 ff of the Matlab code.

Eqn. 17 is defined only if $a \neq 0$.

$$\int \operatorname{erf}(ax+c)\,dx = \frac{1}{a\sqrt{\pi}}\left(\sqrt{\pi}(ax+c)\operatorname{erf}(ax+c) + \exp\left(-(ax+c)^2\right)\right) \tag{17}$$

Eqn. 18 is a special case of Eqn. 17 if $a = 0$.

$$\int \operatorname{erf}(c)\,dx = x\operatorname{erf}(c) \tag{18}$$

2.2.1. Involving Power Function

$\boxed{\textbf{Case 5}}$ $\longrightarrow$ cf. Line 152 ff of the Matlab code.

Eqn. 19 is defined only if $a \neq 0$.

$$\begin{aligned}
&\int x\operatorname{erf}(ax+c)\,dx \\
&= \frac{1}{4a^2\sqrt{\pi}}\left(\sqrt{\pi}\left(2\left(a^2x^2 - c^2\right) - 1\right)\operatorname{erf}(ax+c) + 2(ax-c)\exp\left(-(ax+c)^2\right)\right)
\end{aligned} \tag{19}$$

Eqn. 20 is a special case of Eqn. 19 if $a = 0$.

$$\int x \operatorname{erf}(c)\, dx = \frac{x^2}{2} \operatorname{erf}(c) \tag{20}$$

$\boxed{\textbf{Case 6}}$ $\longrightarrow$ cf. Line 163 ff of the Matlab code.

Eqn. 21 is defined only if $a \neq 0$.

$$\begin{aligned} &\int x^2 \operatorname{erf}(ax + c)\, dx \\ &= \frac{1}{6a^3\sqrt{\pi}} \left(\sqrt{\pi} \left(2\left(a^3x^3 - c^3\right) + 3c\right) \operatorname{erf}(ax + c) + 2\left(a^2x^2 - acx + c^2 + 1\right) \exp\left(-(ax + c)^2\right) \right) \end{aligned} \tag{21}$$

Eqn. 22 is a special case of Eqn. 21 if $a = 0$.

$$\int x^2 \operatorname{erf}(c)\, dx = \frac{x^3}{3} \operatorname{erf}(c) \tag{22}$$

2.2.2. Involving Exponential Function

$\boxed{\textbf{Case 7}}$ $\longrightarrow$ cf. Line 174 ff of the Matlab code.

Eqn. 23 is defined only if $a, b \neq 0$.

$$\begin{aligned} &\int \exp(bx + d) \operatorname{erf}(ax + c)\, dx \\ &= \frac{1}{b} \left(\exp\left(\frac{b^2}{4a^2} - \frac{bc}{a} + d \right) \operatorname{erfc}\left(ax + c - \frac{b}{2a} \right) + \exp(bx + d) \operatorname{erf}(ax + c) \right) \end{aligned} \tag{23}$$

Eqn. 24 is a special case of Eqn. 23 if $a = 0 \wedge b \neq 0$.

$$\int \exp(bx + d) \operatorname{erf}(c)\, dx = \ldots \text{use Eqn. 9 with } c = 0 \text{ and multiply with } \operatorname{erf}(c) \tag{24}$$

Eqn. 25 is a special case of Eqn. 23 if $a \neq 0 \wedge b = 0$.

$$\int \exp(d) \operatorname{erf}(ax + c)\, dx = \ldots \text{use Eqn. 17 and multiply with } \exp(d) \tag{25}$$

Eqn. 26 is a special case of Eqn. 23 if $a, b = 0$.

$$\int \exp(d) \operatorname{erf}(c)\, dx = \ldots \text{use Eqn. 18 and multiply with } \exp(d) \tag{26}$$

2.2.3. Involving Exponential and Power Function

$\boxed{\textbf{Case 8}}$ $\longrightarrow$ cf. Line 190 ff of the Matlab code.

Eqn. 27 is defined only if $a, b \neq 0$.

$$\begin{aligned}
&\int x \exp(bx+d) \operatorname{erf}(ax+c)\, dx \\
&= \left(-\frac{1}{b^2} + \frac{1}{2a^2} - \frac{c}{ab}\right) \exp\left(\frac{b^2}{4a^2} - \frac{bc}{a} + d\right) \operatorname{erfc}\left(ax + c - \frac{b}{2a}\right) \\
&\quad + \frac{1}{ab\sqrt{\pi}} \exp\left(-(ax+c)^2 + bx + d\right) + \left(\frac{bx-1}{b^2}\right) \exp(bx+d) \operatorname{erf}(ax+c)
\end{aligned} \tag{27}$$

Eqn. 28 is a special case of Eqn. 27 if $a = 0 \wedge b \neq 0$.

$$\int x \exp(bx+d) \operatorname{erf}(c)\, dx = \ldots \text{use Eqn. 12 with } c = 0 \text{ and multiply with } \operatorname{erf}(c) \tag{28}$$

Eqn. 29 is a special case of Eqn. 27 if $a \neq 0 \wedge b = 0$.

$$\int x \exp(d) \operatorname{erf}(ax+c)\, dx = \ldots \text{use Eqn. 19 and multiply with } \exp(d) \tag{29}$$

Eqn. 30 is a special case of Eqn. 27 if $a, b = 0$.

$$\int x \exp(d) \operatorname{erf}(c)\, dx = \ldots \text{use Eqn. 20 and multiply with } \exp(d) \tag{30}$$

$\boxed{\textbf{Case 9}}$ $\longrightarrow$ cf. Line 209 ff of the Matlab code.

Eqn. 31 is defined only if $a, b \neq 0$.

$$\begin{aligned}
&\int x^2 \exp(bx+d) \operatorname{erf}(ax+c)\, dx \\
&= \left(\frac{8a^4 - 2a^2b^2 + b^4}{4a^4b^3} + \frac{2c}{ab^2} - \frac{c}{a^3} + \frac{c^2}{a^2b}\right) \exp\left(\frac{b^2}{4a^2} - \frac{bc}{a} + d\right) \operatorname{erfc}\left(ax + c - \frac{b}{2a}\right) \\
&\quad + \left(\frac{x}{ab\sqrt{\pi}} + \frac{b^2 - 4a^2 - 2abc}{2a^3b^2\sqrt{\pi}}\right) \exp\left(-(ax+c)^2 + bx + d\right) \\
&\quad + \left(\frac{(bx-1)^2 + 1}{b^3}\right) \exp(bx+d) \operatorname{erf}(ax+c)
\end{aligned} \tag{31}$$

Eqn. 32 is a special case of Eqn. 31 if $a = 0 \wedge b \neq 0$.

$$\int x^2 \exp(bx+d) \operatorname{erf}(c)\, dx = \ldots \text{use Eqn. 15 with } c = 0 \text{ and multiply with } \operatorname{erf}(c) \tag{32}$$

Eqn. 33 is a special case of Eqn. 31 if $a \neq 0 \wedge b = 0$.

$$\int x^2 \exp(d) \operatorname{erf}(ax+c)\, dx = \ldots \text{use Eqn. 21 and multiply with } \exp(d) \tag{33}$$

Eqn. 34 is a special case of Eqn. 31 if $a, b = 0$.

$$\int x^2 \exp(d) \operatorname{erf}(c)\, dx = \ldots \text{use Eqn. 22 and multiply with } \exp(d) \tag{34}$$

2.3. Complementary Error Function

$\boxed{\textbf{Case 10}}$ $\longrightarrow$ cf. Line 232 ff of the Matlab code.

Eqn. 35 is defined only if $a \neq 0$.

$$\int \operatorname{erfc}(ax+c)\, dx = \frac{1}{a\sqrt{\pi}} \Big(\sqrt{\pi}(ax+c) \operatorname{erfc}(ax+c) - \exp\left(-(ax+c)^2\right) \Big) \tag{35}$$

Eqn. 36 is a special case of Eqn. 35 if $a = 0$.

$$\int \operatorname{erfc}(c)\, dx = x \operatorname{erfc}(c) \tag{36}$$

2.3.1. Involving Power Function

$\boxed{\textbf{Case 11}}$ $\longrightarrow$ cf. Line 240 ff of the Matlab code.

Eqn. 37 is defined only if $a \neq 0$.

$$\begin{aligned} &\int x \operatorname{erfc}(ax+c)\, dx \\ &= -\frac{1}{4a^2\sqrt{\pi}} \Big(\sqrt{\pi}(-2a^2x^2) \operatorname{erfc}(ax+c) + 2(ax-c) \exp\left(-(ax+c)^2\right) \\ &\quad - \left(2c^2(\sqrt{\pi}+1) \operatorname{erf}(ax+c)\right) \Big) \end{aligned} \tag{37}$$

Eqn. 38 is a special case of Eqn. 37 if $a = 0$.

$$\int x \operatorname{erfc}(c)\, dx = \frac{x^2}{2} \operatorname{erfc}(c) \tag{38}$$

$\boxed{\textbf{Case 12}}$ $\longrightarrow$ cf. Line 253 ff of the Matlab code.

Eqn. 39 is defined only if $a \neq 0$.

$$\begin{aligned} &\int x^2 \operatorname{erfc}(ax+c)\,dx \\ &= -\frac{1}{6a^3\sqrt{\pi}}\Big(\sqrt{\pi}(-2a^3x^3)\operatorname{erfc}(ax+c) + 2(a^2x^2 - acx + c^2 + 1)\exp\left(-(ax+c)^2\right) \\ &\quad + (3c\sqrt{\pi} + 2c^3\sqrt{\pi})\operatorname{erf}(ax+c)\Big) \end{aligned} \tag{39}$$

Eqn. 40 is a special case of Eqn. 39 if $a = 0$.

$$\int x^2 \operatorname{erfc}(c)\,dx = \frac{x^3}{3}\operatorname{erfc}(c) \tag{40}$$

2.3.2. Involving Exponential Function

$\boxed{\textbf{Case 13}}$ $\longrightarrow$ cf. Line 266 ff of the Matlab code.

Eqn. 41 is defined only if $a, b \neq 0$.

$$\begin{aligned} &\int \exp(bx+d)\operatorname{erfc}(ax+c)\,dx \\ &= \frac{1}{b}\left(-\exp\left(\frac{b^2}{4a^2} - \frac{bc}{a} + d\right)\operatorname{erfc}\left(ax + c - \frac{b}{2a}\right) + \exp(bx+d)\operatorname{erfc}(ax+c)\right) \end{aligned} \tag{41}$$

Eqn. 42 is a special case of Eqn. 41 if $a = 0 \wedge b \neq 0$.

$$\int \exp(bx+d)\operatorname{erfc}(c)\,dx = \ldots \text{use Eqn. 9 with } c = 0 \text{ and multiply with } \operatorname{erfc}(c) \tag{42}$$

Eqn. 43 is a special case of Eqn. 41 if $a \neq 0 \wedge b = 0$.

$$\int \exp(d)\operatorname{erfc}(ax+c)\,dx = \ldots \text{use Eqn. 35 and multiply with } \exp(d) \tag{43}$$

Eqn. 44 is a special case of Eqn. 41 if $a, b = 0$.

$$\int \exp(d)\operatorname{erfc}(c)\,dx = \ldots \text{use Eqn. 36 and multiply with } \exp(d) \tag{44}$$

2.3.3. Involving Exponential and Power Function

$\boxed{\textbf{Case 14}}$ $\longrightarrow$ cf. Line 282 ff of the Matlab code.

Eqn. 45 is defined only if $a, b \neq 0$.

$$\begin{aligned}
&\int x \exp(bx+d)\,\mathrm{erfc}(ax+c)\,dx \\
&\quad = -\left(-\frac{1}{b^2}+\frac{1}{2a^2}-\frac{c}{ab}\right)\exp\left(\frac{b^2}{4a^2}-\frac{bc}{a}+d\right)\mathrm{erfc}\left(ax+c-\frac{b}{2a}\right) \\
&\qquad -\frac{1}{ab\sqrt{\pi}}\exp\left(-(ax+c)^2+bx+d\right)+\left(\frac{bx-1}{b^2}\right)\exp(bx+d)\,\mathrm{erfc}(ax+c)
\end{aligned} \tag{45}$$

Eqn. 46 is a special case of Eqn. 45 if $a = 0 \wedge b \neq 0$.

$$\int x \exp(bx+d)\,\mathrm{erfc}(c)\,dx = \ldots \text{use Eqn. 12 with } c=0 \text{ and multiply with } \mathrm{erfc}(c) \tag{46}$$

Eqn. 47 is a special case of Eqn. 45 if $a \neq 0 \wedge b = 0$.

$$\int x \exp(d)\,\mathrm{erfc}(ax+c)\,dx = \ldots \text{use Eqn. 37 and multiply with } \exp(d) \tag{47}$$

Eqn. 48 is a special case of Eqn. 45 if $a, b = 0$.

$$\int x \exp(d)\,\mathrm{erfc}(c)\,dx = \ldots \text{use Eqn. 38 and multiply with } \exp(d) \tag{48}$$

$\boxed{\textbf{Case 15}}$ $\longrightarrow$ cf. Line 301 ff of the Matlab code.

Eqn. 49 is defined only if $a, b \neq 0$.

$$\begin{aligned}
&\int x^2 \exp(bx+d)\,\mathrm{erf}(ax+c)\,dx \\
&\quad = -\left(\frac{8a^4-2a^2b^2+b^4}{4a^4b^3}+\frac{2c}{ab^2}-\frac{c}{a^3}+\frac{c^2}{a^2b}\right)\exp\left(\frac{b^2}{4a^2}-\frac{bc}{a}+d\right)\mathrm{erfc}\left(ax+c-\frac{b}{2a}\right) \\
&\qquad -\left(\frac{x}{ab\sqrt{\pi}}+\frac{b^2-4a^2-2abc}{2a^3b^2\sqrt{\pi}}\right)\exp\left(-(ax+c)^2+bx+d\right) \\
&\qquad +\left(\frac{(bx-1)^2+1}{b^3}\right)\exp(bx+d)\,\mathrm{erfc}(ax+c)
\end{aligned} \tag{49}$$

Eqn. 50 is a special case of Eqn. 49 if $a = 0 \wedge b \neq 0$.

$$\int x^2 \exp(bx+d)\,\mathrm{erfc}(c)\,dx = \ldots \text{use Eqn. 15 with } c=0 \text{ and multiply with } \mathrm{erfc}(c) \tag{50}$$

Eqn. 51 is a special case of Eqn. 49 if $a \neq 0 \wedge b = 0$.

$$\int x^2 \exp(d)\,\mathrm{erfc}(ax + c)\,dx = \ldots \text{use Eqn. 39 and multiply with } \exp(d) \tag{51}$$

Eqn. 52 is a special case of Eqn. 49 if $a, b = 0$.

$$\int x^2 \exp(d)\,\mathrm{erfc}(c)\,dx = \ldots \text{use Eqn. 40 and multiply with } \exp(d) \tag{52}$$

3. Matlab Source Code

```
function y = ierfunclib(func,x1,x2,a,b,c,d)
%
% Copyright:
%       (c) IER University of Stuttgart - Derk Jan Swider (March 19, 2003)
%
% This library contains analytical solutions for the following integrals:
%
%  No  | func (string) | Description
%  '01' : 'exp'         : y = \int_{x_1}^{x_2} exp(-(a*x+c)^2+b*x+d) dx
%  '02' : 'xexp'        : y = \int_{x_1}^{x_2} x*exp(-(a*x+c)^2+b*x+d) dx
%  '03' : 'x2exp'       : y = \int_{x_1}^{x_2} x^2*exp(-(a*x+c)^2+b*x+d) dx
%  '04' : 'erf'         : y = \int_{x_1}^{x_2} erf(a*x+c) dx
%  '05' : 'xerf'        : y = \int_{x_1}^{x_2} x*erf(a*x+c) dx
%  '06' : 'x2erf'       : y = \int_{x_1}^{x_2} x^2*erf(a*x+c) dx
%  '07' : 'experf'      : y = \int_{x_1}^{x_2} exp(b*x+d)*erf(a*x+c) dx
%  '08' : 'xexperf'     : y = \int_{x_1}^{x_2} x*exp(b*x+d)*erf(a*x+c) dx
%  '09' : 'x2experf'    : y = \int_{x_1}^{x_2} x^2*exp(b*x+d)*erf(a*x+c) dx
%  '10' : 'erfc'        : y = \int_{x_1}^{x_2} erfc(a*x+c) dx
%  '11' : 'xerfc'       : y = \int_{x_1}^{x_2} x*erfc(a*x+c) dx
%  '12' : 'x2erfc'      : y = \int_{x_1}^{x_2} x^2*erfc(a*x+c) dx
%  '13' : 'experfc'     : y = \int_{x_1}^{x_2} exp(b*x+d)*erfc(a*x+c) dx
%  '14' : 'xexperfc'    : y = \int_{x_1}^{x_2} x*exp(b*x+d)*erfc(a*x+c) dx
%  '15' : 'x2experfc'   : y = \int_{x_1}^{x_2} x^2*exp(b*x+d)*erfc(a*x+c) dx
%
% Input:
%       func    : String identifying the integral to be calculated
%       x1      : Lower integration limit
%       x2      : Upper integration limit
%       a       : Parameter multiplied with x
%       b       : Parameter multiplied with x
%       c       : Parameter added to multiplication of a with x
%       d       : Parameter added to multiplication of b with x
%
% Output:
%       y       : Calculated value of the selected integral
%

% Bug-Fix to prevent some calculation failures in Matlab
%
```

```
if abs(a) < 10.*eps, a = 0; ,end
if abs(b) < 10.*eps, b = 0; ,end

% Selection of the function to be calculated
%
switch func
    case {'01','exp'}
        y = int_exp(x1,x2,a,b,c,d);
    case {'02','xexp'}
        y = int_xexp(x1,x2,a,b,c,d);
    case {'03','x2exp'}
        y = int_x2exp(x1,x2,a,b,c,d);
    case {'04','erf'}
        y = int_erf(x1,x2,a,b,c,d);
    case {'05','xerf'}
        y = int_xerf(x1,x2,a,b,c,d);
    case {'06','x2erf'}
        y = int_x2erf(x1,x2,a,b,c,d);
    case {'07','experf'}
        y = int_experf(x1,x2,a,b,c,d);
    case {'08','xexperf'}
        y = int_xexperf(x1,x2,a,b,c,d);
    case {'09','x2experf'}
        y = int_x2experf(x1,x2,a,b,c,d);
    case {'10','erfc'}
        y = int_erfc(x1,x2,a,b,c,d);
    case {'11','xerfc'}
        y = int_xerfc(x1,x2,a,b,c,d);
    case {'12','x2erfc'}
        y = int_x2erfc(x1,x2,a,b,c,d);
    case {'13','experfc'}
        y = int_experfc(x1,x2,a,b,c,d);
    case {'14','xexperfc'}
        y = int_xexperfc(x1,x2,a,b,c,d);
    case {'15','x2experfc'}
        y = int_x2experfc(x1,x2,a,b,c,d);
    otherwise
        error(['''',func,''' is an undefined function.']);
end

function y = int_exp(x1,x2,a,b,c,d)
if b ~=0 & a ~= 0
    y = sqrt(pi)./(2.*a).* ...
        exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).*( ...
        erfc(a.*x1+c-b./(2.*a)) -...
        erfc(a.*x2+c-b./(2.*a)) ...
        );
elseif b ~= 0 & a == 0
    y = 1./b.*(exp(b.*x2-c.^2+d)-exp(b.*x1-c.^2+d));
elseif b == 0 & a ~= 0
    y = (sqrt(pi).*exp(d))./(2.*a).*(erfc(a.*x1+c)-erfc(a.*x2+c));
elseif b == 0 & a == 0
    y = exp(-c.^2+d).*(x2-x1);
end
```

```
function y = int_xexp(x1,x2,a,b,c,d)
if b ~= 0 & a ~= 0
    y = - 1./(4.*a.^3).*( ...
        ( 2.*a.*exp(-a.^2.*x2.^2-(2.*a.*c-b).*x2-c.^2+d) - ...
          2.*a.*exp(-a.^2.*x1.^2-(2.*a.*c-b).*x1-c.^2+d) ) + ...
        (sqrt(pi).*(2.*a.*c-b)).*exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).*( ...
        erfc(a.*x1+c-b./(2.*a)) - ...
        erfc(a.*x2+c-b./(2.*a)) ) );
elseif b ~= 0 & a == 0
    y = 1./(b.^2).*(...
        (b.*x2-1).*exp(b.*x2-c.^2+d) - ...
        (b.*x1-1).*exp(b.*x1-c.^2+d) );
elseif b == 0 & a ~= 0
    y = - 1./(2.*a.^2).*( (...
        exp(-a.^2.*x2.^2-2.*a.*c.*x2-c.^2+d) - ...
        exp(-a.^2.*x1.^2-2.*a.*c.*x1-c.^2+d) ) + ...
        sqrt(pi).*c.*exp(d).* ( ...
        erfc(a.*x1+c) - ...
        erfc(a.*x2+c) ) );
elseif b == 0 & a == 0
    y = 1./2.*exp(-c.^2+d).*(x2.^2-x1.^2);
end

function y = int_x2exp(x1,x2,a,b,c,d)
if b ~= 0 & a ~= 0
    y = -1./(8.*a.^5).*(...
        ((4.*a.^3.*x2-4.*a.^2.*c+2.*a.*b).*...
         exp(-(a.*x2+c).^2+b.*x2+d)-...
         (4.*a.^3.*x1-4.*a.^2.*c+2.*a.*b).*...
         exp(-(a.*x1+c).^2+b.*x1+d))+...
        sqrt(pi).*(-4.*a.^2.*c.^2+4.*a.*b.*c-b.^2-2.*a.^2).*...
        exp((b.^2)./(4.*a.^2)-b.*c./a+d).*(...
        erfc(a.*x1+c-b./(2.*a))-...
        erfc(a.*x2+c-b./(2.*a)) ) );
elseif b ~= 0 & a == 0
    y = 1./(b.^3).*(...
        exp(-c.^2+b.*x2+d).*(b.^2.*x2.^2-2.*b.*x2+2) - ...
        exp(-c.^2+b.*x1+d).*(b.^2.*x1.^2-2.*b.*x1+2) );
elseif b == 0 & a ~= 0
    y = -1./(4.*a.^3).*(...
        ((2.*a.*x2*-2.*c).*exp(-(a.*x2+c).^2+b.*x2+d)-...
        (2.*c.^2+1).*sqrt(pi).*exp(d).*erf(a.*x2+c))-...
        ((2.*a.*x1*-2.*c).*exp(-(a.*x1+c).^2+b.*x1+d)-...
        (2.*c.^2+1).*sqrt(pi).*exp(d).*erf(a.*x1+c)) );
elseif b == 0 & a == 0
    y = 1./3.*exp(-c.^2+d).*(x2.^3-x1.^3);
end

function y = int_erf(x1,x2,a,b,c,d)
if a ~= 0
    y = 1./(sqrt(pi).*a) .*( ...
        (sqrt(pi).*(a.*x2+c).*erf(a.*x2+c)+exp(-(a.*x2+c).^2))-...
        (sqrt(pi).*(a.*x1+c).*erf(a.*x1+c)+exp(-(a.*x1+c).^2)) );
elseif a == 0
    y = erf(c).*(x2-x1);
```

```
end

function y = int_xerf(x1,x2,a,b,c,d)
 if a ~= 0
     y = 1./(4.*sqrt(pi).*a.^2).*(...
          (sqrt(pi).*(2.*a.^2.*x2.^2-2.*c.^2-1).*erf(a.*x2+c)-...
           sqrt(pi).*(2.*a.^2.*x1.^2-2.*c.^2-1).*erf(a.*x1+c))+...
          ((2.*a.*x2-2.*c).*exp(-(a.*x2+c).^2)-...
           (2.*a.*x1-2.*c).*exp(-(a.*x1+c).^2)) );
 elseif a == 0
     y = 1./2.*erf(c).*(x2.^2-x1.^2);
end

function y = int_x2erf(x1,x2,a,b,c,d)
 if a ~= 0
     y = 1./(6.*sqrt(pi).*a.^3).*(...
          (sqrt(pi).*(2.*a.^3.*x2.^3+2.*c.^3+3.*c).*erf(a.*x2+c)-...
           sqrt(pi).*(2.*a.^3.*x1.^3+2.*c.^3+3.*c).*erf(a.*x1+c))+...
          (2.*(a.^2.*x2.^2-a.*c.*x2+c.^2+1).*exp(-(a.*x2+c).^2)-...
           2.*(a.^2.*x1.^2-a.*c.*x1+c.^2+1).*exp(-(a.*x1+c).^2)) );
 elseif a == 0
     y = 1./3.*erf(c).*(x2.^3-x1.^3);
end

function y = int_experf(x1,x2,a,b,c,d)
 if b ~= 0 & a ~= 0
     y = 1./b.*( ...
              (exp(b.*x2+d).*erf(a.*x2+c)- ...
               exp(b.*x1+d).*erf(a.*x1+c))- ...
              exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).*( ...
              erfc(a.*x1+c-b./(2.*a))-...
              erfc(a.*x2+c-b./(2.*a)) ) );
 elseif b ~= 0 & a == 0
     y = erf(c).*int_exp(x1,x2,0,b,0,d);
 elseif b == 0 & a ~= 0
     y = exp(d).*int_erf(x1,x2,a,0,c,0);
 elseif b == 0 & a == 0
     y = exp(d).*int_erf(x1,x2,0,0,c,0);
end

function y = int_xexperf(x1,x2,a,b,c,d)
 if b ~= 0 & a ~= 0
     y = ((1./(b.^2)-1./(2.*a.^2)+c./(a.*b)).*...
              exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).*(...
          erfc(a.*x1+c-b./(2.*a)) - ...
          erfc(a.*x2+c-b./(2.*a)) ) + ...
          1./(a.*b.*sqrt(pi)).* (...
          exp(b.*x2+d-(a.*x2+c).^2) - ...
          exp(b.*x1+d-(a.*x1+c).^2) ) + ...
          (x2./b-1./(b.^2)).*exp(b.*x2+d).*erf(a.*x2+c) - ...
          (x1./b-1./(b.^2)).*exp(b.*x1+d).*erf(a.*x1+c) );
 elseif b ~= 0 & a == 0
     y = erf(c).*int_xexp(x1,x2,0,b,0,d);
 elseif b == 0 & a ~= 0
     y = exp(d).*int_xerf(x1,x2,a,0,c,0);
```

```
elseif b == 0 & a == 0
    y = exp(d).*int_xerf(x1,x2,0,0,c,0);
end

function y = int_x2experf(x1,x2,a,b,c,d)
if b ~= 0 & a ~= 0
    y = ((b.*x2-1).^2+1)./(b.^3).*exp(b.*x2+d).*erf(a.*x2+c)-...
        ((b.*x1-1).^2+1)./(b.^3).*exp(b.*x1+d).*erf(a.*x1+c)+ ...
        exp(b.*x2+d-(a.*x2+c).^2).*...
        (x2./(a.*b.*sqrt(pi))+(b.^2-4.*a.^2-2.*a.*b.*c)./...
            (2.*a.^3.*b.^2.*sqrt(pi))) - ...
        exp(b.*x1+d-(a.*x1+c).^2).*...
        (x1./(a.*b.*sqrt(pi))+(b.^2-4.*a.^2-2.*a.*b.*c)./...
            (2.*a.^3.*b.^2.*sqrt(pi))) - ...
        ((8.*a.^4-2.*a.^2.*b.^2+b.^4)./(4.*a.^4.*b.^3)+...
            (2.*c)./(a.*b.^2)-c./(a.^3)+((c./a).^2)./b).* ...
        exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).* (...
        erfc(a.*x1+c-b./(2.*a))-...
        erfc(a.*x2+c-b./(2.*a)));
elseif b ~= 0 & a == 0
    y = erf(c).*int_x2exp(x1,x2,0,b,0,d);
elseif b == 0 & a ~= 0
    y = exp(d).*int_x2erf(x1,x2,a,0,c,0);
elseif b == 0 & a == 0
    y = exp(d).*int_x2erf(x1,x2,0,0,c,0);
end

function y = int_erfc(x1,x2,a,b,c,d)
if a ~= 0
    y = 1./a .*((a.*x2+c).*erfc(a.*x2+c)-exp(-(a.*x2+c).^2)./sqrt(pi))-...
        1./a .*((a.*x1+c).*erfc(a.*x1+c)-exp(-(a.*x1+c).^2)./sqrt(pi));
elseif a == 0
    y = erfc(c).*(x2-x1);
end

function y = int_xerfc(x1,x2,a,b,c,d)
if a ~= 0
    y = -1./(4.*sqrt(pi).*a.^2).*(...
        (sqrt(pi).*(-2.*a.^2.*x2.^2).*erfc(a.*x2+c)-...
         sqrt(pi).*(-2.*a.^2.*x1.^2).*erfc(a.*x1+c))+...
        (2.*(a.*x2-c).*exp(-(a.*x2+c).^2)-...
         2.*(a.*x1-c).*exp(-(a.*x1+c).^2))-...
        ((sqrt(pi)+2.*c.^2.*sqrt(pi)).*erf(a.*x2+c)-...
         (sqrt(pi)+2.*c.^2.*sqrt(pi)).*erf(a.*x1+c)) );
elseif a == 0
    y = 1./2.*erfc(c).*(x2.^2-x1.^2);
end

function y = int_x2erfc(x1,x2,a,b,c,d)
if a ~= 0
    y = -1./(6.*sqrt(pi).*a.^3).*(...
        (sqrt(pi).*(-2.*a.^3.*x2.^3).*erfc(a.*x2+c)-...
         sqrt(pi).*(-2.*a.^3.*x1.^3).*erfc(a.*x1+c))+...
        (2.*(a.^2.*x2.^2-a.*c.*x2+c.^2+1).*exp(-(a.*x2+c).^2)-...
         2.*(a.^2.*x1.^2-a.*c.*x1+c.^2+1).*exp(-(a.*x1+c).^2))+...
```

```
        ((3.*c.*sqrt(pi)+2.*c.^3.*sqrt(pi)).*erf(a.*x2+c)-...
         (3.*c.*sqrt(pi)+2.*c.^3.*sqrt(pi)).*erf(a.*x1+c)) );
elseif a == 0
    y = 1./3.*erfc(c).*(x2.^3-x1.^3);
end

function y = int_experfc(x1,x2,a,b,c,d)
if b ~= 0 & a ~= 0
    y  = 1./b.* ( ...
        (exp(b.*x2+d).*erfc(a.*x2+c) - ...
         exp(b.*x1+d).*erfc(a.*x1+c))+ ...
        exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).* ( ...
        erfc(a.*x1+c-b./(2.*a))-...
        erfc(a.*x2+c-b./(2.*a)) ) );
elseif b ~= 0 & a == 0
    y = erfc(c).*int_exp(x1,x2,0,b,0,d);
elseif b == 0 & a ~= 0
    y = exp(d).*int_erfc(x1,x2,a,0,c,0);
elseif b == 0 & a == 0
    y = exp(d).*int_erfc(x1,x2,0,0,c,0);
end

function y = int_xexperfc(x1,x2,a,b,c,d)
if b ~= 0 & a ~= 0
    y = (-(1./(b.^2)-1./(2.*a.^2)+c./(a.*b)).*...
            exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).*(...
        erfc(a.*x1+c-b./(2.*a)) - ...
        erfc(a.*x2+c-b./(2.*a)) ) - ...
        1./(a.*b.*sqrt(pi)).* (...
        exp(b.*x2+d-(a.*x2+c).^2) - ...
        exp(b.*x1+d-(a.*x1+c).^2) ) + ...
        (x2./b-1./(b.^2)).*exp(b.*x2+d).*erfc(a.*x2+c) - ...
        (x1./b-1./(b.^2)).*exp(b.*x1+d).*erfc(a.*x1+c) );
elseif b ~= 0 & a == 0
    y = erfc(c).*int_xexp(x1,x2,0,b,0,d);
elseif b == 0 & a ~= 0
    y = exp(d).*int_xerfc(x1,x2,a,0,c,0);
elseif b == 0 & a == 0
    y = exp(d).*int_xerfc(x1,x2,0,0,c,0);
end

function y = int_x2experfc(x1,x2,a,b,c,d)
if b ~= 0 & a ~= 0
    y = (((b.*x2-1).^2+1)./(b.^3).*exp(b.*x2+d).*erfc(a.*x2+c)-...
         ((b.*x1-1).^2+1)./(b.^3).*exp(b.*x1+d).*erfc(a.*x1+c) )- ...
        (exp(b.*x2+d-(a.*x2+c).^2).*...
         (x2./(a.*b.*sqrt(pi))+(b.^2-4.*a.^2-2.*a.*b.*c)./...
            (2.*a.^3.*b.^2.*sqrt(pi))) - ...
         exp(b.*x1+d-(a.*x1+c).^2).*...
         (x1./(a.*b.*sqrt(pi))+(b.^2-4.*a.^2-2.*a.*b.*c)./...
            (2.*a.^3.*b.^2.*sqrt(pi))) ) + ...
        ((8.*a.^4-2.*a.^2.*b.^2+b.^4)./(4.*a.^4.*b.^3)+...
            (2.*c)./(a.*b.^2)-c./(a.^3)+((c./a).^2)./b).* ...
        exp((b.^2)./(4.*a.^2)-(b.*c)./a+d).* (...
         erfc(a.*x1+c-b./(2.*a))-...
```

```
        erfc(a.*x2+c-b./(2.*a)) );
elseif b ~= 0 & a == 0
    y = erfc(c).*int_x2exp(x1,x2,0,b,0,d);
elseif b == 0 & a ~= 0
    y = exp(d).*int_x2erfc(x1,x2,a,0,c,0);
elseif b == 0 & a == 0
    y = exp(d).*int_x2erfc(x1,x2,0,0,c,0);
end
```

A. Convolution

In this Appendix first the expected value or mean and then the variance or squared standard deviation of a convolution is derived.

The expected value or mean of a convolution

$$\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_1 (x - u) f_2 (u)\ du\, dx = \mathrm{E}[x] \tag{53}$$

By substitution of $z = x - u$ and therefore $dz = dx$ follows

$$\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (z + u) f_1 (z) f_2 (u)\ du\, dz \tag{54}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[z f_1 (z) f_2 (u) + u f_1 (z) f_2 (u) \right] du\, dz \tag{55}$$

$$= \int_{-\infty}^{\infty} z f_1 (z)\ dz \int_{-\infty}^{\infty} f_2 (u)\ du + \int_{-\infty}^{\infty} f_1 (z)\ dz \int_{-\infty}^{\infty} u f_2 (u)\ du \tag{56}$$

$$= \int_{-\infty}^{\infty} z f_1 (z)\ dz + \int_{-\infty}^{\infty} u f_2 (u)\ du \tag{57}$$

This finally gives

$$\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_1 (x - u) f_2 (u)\ du\, dx = \mathrm{E}_1 [z] + \mathrm{E}_2 [u] \tag{58}$$

The variance or squared standard deviation of a convolution

$$\sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu)^2 f_1(x-u) f_2(u) \, du \, dx = \mathrm{E}\left[(x-\mu)^2\right] \tag{59}$$

This may be rewritten as

$$\begin{aligned}\sigma^2 = &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_1(x-u) f_2(u) \, du \, dx \\ &- 2\mu \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_1(x-u) f_2(u) \, du \, dx}_{=\mathrm{E}_1[z]+\mathrm{E}_2[u]=\mu, \text{ cf. Eqn. 58}} + \mu^2 \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x-u) f_2(u) \, du \, dx}_{=1}\end{aligned} \tag{60}$$

By substitution of $z = x - u$ and therefore $dz = dx$ this gives

$$\sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (z+u)^2 f_1(z) f_2(u) \, du \, dz - \mu^2 \tag{61}$$

$$= \int_{-\infty}^{\infty} z^2 f_1(z) \, dz \int_{-\infty}^{\infty} f_2(u) \, du + 2 \int_{-\infty}^{\infty} z f_1(z) \, dz \int_{-\infty}^{\infty} u f_2(u) \, du + \int_{-\infty}^{\infty} f_1(z) \, dz \int_{-\infty}^{\infty} u^2 f_2(u) \, du - \mu^2 \tag{62}$$

$$= \int_{-\infty}^{\infty} z^2 f_1(z) \, dz + 2\mathrm{E}_1[z] \, \mathrm{E}_2[u] + \int_{-\infty}^{\infty} u^2 f_2(u) \, du - \mu^2 \tag{63}$$

And with $\mu^2 = (\mathrm{E}_1[z] + \mathrm{E}_2[u])^2$, cf. Eqn. 58, finally follows

$$\sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu)^2 f_1(x-u) f_2(u) \, du \, dx = \mathrm{E}_1\left[z^2\right] - \mathrm{E}_1[z]^2 + \mathrm{E}_2\left[u^2\right] - \mathrm{E}_2[u]^2 \tag{64}$$