# Intrinsic Deformation Analysis of the Earth Surface Based on 3-Dimensional Displacement Fields Derived from Space Geodetic Measurements 

Doctoral thesis accepted by the
Faculty of Civil Engineering and Surveying Science of the University of Stuttgart

Von der Fakultät für
Bauingenieur- und Vermessungswesen der Universität Stuttgart zur Erlangung der Würde eines Doktor-Ingenieurs (Dr.-Ing.) genehmigte Dissertation

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October 2000

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Day of submission - Tag der Einreichung: 3.05.2000
Day of examination - Tag der mündlichen Prüfung: 16.10.2000

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#### Abstract

Lagrangian and Eulerian deformation tensors are a key tool in the study of deformation. Although many methods have been proposed to calculate deformation tensor fields of the Earth surface, only few refer to the real surface of the Earth. Most of these methods formulate the problem on reference surfaces such as projection plane or sphere and consequently their results suffer from possible effects of incompleteness in the mathematical models of projections. The surface deformation tensors and their associated invariants are critical for a meaningful study of deformations and kinematics of the Earth. Moreover, their geodetic estimates are crucial as initial values for geophysical models as well as quantifying potential seismic activities. Here we present a method of differential geometry that allows deformation analysis of the real surface of the Earth on its own rights for a more reliable and accurate estimate of the surface deformation measures. The method takes advantage of the simplicity of the 2-dimensional spaces versus 3-dimensional spaces without losing or neglecting information and effect of the third dimension in the final results. The dissertation describes analytical modelling, derivation and implementation of the surface deformation measures based on the proposed method with particular attention to the formulation and implementation of the tensors of linearized rotation and change of curvature in Earth deformation studies. Finally the method is applied to a real data set of space geodetic positions and displacement vectors. This application reveals capabilities and strengths of the developed mathematical models of the suggested method.


## Zusammenfassung

Lagrangesche und Eulersche Deformationstensoren sind ein wesentliches Werkzeug für die Untersuchung von Deformationen. Obwohl eine Vielzahl von Methoden zur Berechnung von Deformationstensorfeldern für die Erdoberfläche existiert, beziehen sich nur wenige von ihnen auf die wahre Oberfläche der Erde. Die meisten dieser Methoden formulieren das Problem bezüglich einer Referenzfläche, wie eine Projektionsebene oder Kugel, und folglich leiden ihre Ergebnisse unter möglichen Effekten, die durch die Unvollkommenheiten der mathematischen Modelle der Projektionen verursacht werden. Die Oberflächendeformationstensoren und die mit ihnen verbundenen Invarianten sind wesentlich für eine aussagekräftige Untersuchung von Deformationen und Bewegungen der Erdoberfläche. Zusätzlich dazu sind ihre geodätischen Schätzwerte unverzichtbar als Anfangswerte für geophysikalische Modelle und die Quantifizierung möglicher seismischer Ereignisse. In dieser Arbeit wird eine Methode aus der Differentialgeometrie gezeigt, die eine Deformationsanalyse der wahren Erdoberfläche ermöglicht und so zu einer verlässlicheren und genaueren Bestimmung von Oberflächendeformationsgrößen führt. Die Methode nutzt die Vorteile aus, die sich aus der Einfachheit zweidimensionaler gegenüber dreidimensionalen Räumen ergeben, ohne die Informationen und Effekte der dritten Dimension in den Endergebnissen zu verlieren oder zu vernachlässigen. Die Dissertation beschreibt die analytische Modellbildung, die Herleitung und die Implementierung der Oberflächendeformationsmaße, basierend auf der vorgeschlagenen Methode. Dabei wird vor allem die Formulierung und Implementierung von Tensoren der linearisierten Rotation und Krümmungsänderung, wie sie bei Deformationsuntersuchungen der Erde vorkommen, beachtet. Schließlich wird die Methode auf einen realen Datensatz von räumlichen geodätischen Positionen und Verschiebungsvektoren angewandt. Diese Anwendung zeigt die Möglichkeiten und Stärken der mathematischen Modelle der vorgeschlagenen Methode auf.

## Acknowledgments

I would like to express my sincere gratitude to my supervisor, Prof. Dr. Erick W. Grafarend, for introducing this fascinating topic to me and his permanent scientific supports throughout the time of my Ph.D. study at the Department of Geodesy and GeoInformatic, University of Stuttgart. This dissertation would not have been completed without his encouragements and guidelines.

My hearty thanks also go to Prof. Dr. Heck and Prof. Dr. Kleusberg for reviewing the work with great care and providing many helpful comments and suggestions that further improved this dissertation. Special mention must also be made of my colleagues Dr. J. Engels and Dr. J. Dambeck whose scientific supports in continuum mechanics and theory of manifolds provided me with the theoretical basis for further works described herein. I am also indebted to Dr. Ch. Schaefer who assisted me not only on computer programming and numerical aspects of the study, but also in affairs of my daily life. We had many hours of valuable discussions in various aspects of the work. I am thankful to all members of the Department for their hospitality and for creating a friendly and conducive environment which helped in making my social as well as professional experiences delightful. Grateful acknowledgment is made to the Ministry of Culture and Higher Education of Iran for sponsoring this Ph.D. study. Finally, I would like to dedicate this dissertation to my wife for all her unsparing support, patience and understanding.

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## Chapter 1

## Introduction

Geodesy has worked well and proved that measurement and representation of geodynamic phenomena such as crustal motion, Earth rotation and Earth tides, are stated as one of its main goals. In particular, the study of geometrical aspects of these geodynamic phenomena falls within the realm of geodesy. Crustal deformation analysis based on classical geodetic measurements has been subject of a large number of studies in geodesy for many years. In recent years, space geodetic techniques have provided a new, more accurate, and reliable source of information for geodetic positioning which is used to detect and quantify deformations of the Earth surface. Advents of these techniques have changed dramatically the rules of crustal deformation analysis. Moreover, it is not difficult to foresee, thanks to the rapid progresses of the space geodetic techniques, that position accuracy and density of point distribution in geodetic networks which play a key role in Earth deformation studies will increase rapidly in the not too distant future. These practical progresses ask for new methodologies in Earth surface deformation investigations that can take advantage of this invaluable source of information. Thus, the theoretical principles and methods of the analysis have to be reconsidered and renewed. Based on this point, with the aim to achieve a better representation of the kinematics of the Earth surface and to contribute to a deeper understanding of the geodynamic processes involved, this study has been carried out.

Generally speaking, deformation is the alteration of form and shape of a material. Its scientific treatment is linked to mechanics as a field of physical science which refers bodies of idealized properties reflecting the characteristic response of the material to applied loads on the basis of experience and observation. In mechanics, the bodies of idealized properties are called models. The continuous medium is one of the most successful models widely used in mechanics. Two, idealized, main properties are assumed for the model. Firstly, the geometrical space available is continuously filled by the material with the molecular structure neglected but not forgotten. Secondly, the mathematical relations, describing the mechanical state and change of the state of the model, are expressed by tensor functions of the position vector and time, and the functions together with their derivatives of adequate orders are continuous. Continuum mechanics as a branch of the science of mechanics has been designed to study the state and change of state, namely deformation, velocity distribution and wave propagation, of deformable materials (gases, liquids and non-rigid solid bodies) by means of the continuum model [A. C. Eringen (1962, 1967), L. E. Malvern (1969), G. E. Mase (1970), G. Beda et al. (1995)].

The principal subject of geodesy is the study of the size, shape, and gravity field of the Earth as well as the time variations of all the above. The relevant concepts and mathematical tools of deformation analysis developed in continuum mechanics have found manifold applications in geodesy even in geodetic problems which are not directly related to deformation of a material body. In mechanics, the analysis is done by establishing a one-to-one correspondence between two different states of the deformable body for comparison of its altered geometrical characteristics. In order to apply the mathematical tools outside the realm of mechanics, it is only
necessary that geometrical characteristics of physical or even abstract entities under investigation be brought into a one-to-one correspondence in the same way. A direct application of the developed concepts and tools of continuum mechanics in geodesy can be seen in the Earth deformation studies which clearly correspond to the treatment of material deformable bodies in continuum mechanics. The deformations of the Earth due to body tide, tidal loading and attraction, plate tectonic motion and so on, have been subjected to extensive investigations from theoretical and practical points of view based on terrestrial and extra-terrestrial geodetic observations. The study of temporal variations and deformations of the gravity field of the Earth is another example of applications of deformation analysis in geodesy. In this case, the change of the gravity vector field in the vicinity of a fixed point in geometry space can be regarded as deformation. Alternatively, the problem can be formulated by studying the change of geometric positions identified by the same value of the gravity vector or any other appropriate gravity field parameters with respect to two different states of gravity field. As the third alternative, the problem can be studied by determining the changes of the both geometric and gravity characteristics, as well as their interrelations, in the neighborhood of a point on the Earth surface. Some examples of the application can be seen in the works of E. W. Grafarend (1978), E. Osada (1980), A. Dermanis et al. (1983b), A. Filaretou (1986), E. Livieratos (1987), B. Heck (1985, 1986). Another interesting application is the comparison of two forms of a geodetic network resulting from its adjustment by different subsets of the available observations. In this case, deformation measures are used to analyze the impact of incompatible observations or detect blunders in the network [K. Thapa (1980), P. Vaníček et al. (1981)]. A. Fotiou and E. Liveratos (1984) apply deformation analysis techniques in network inconsistency studies when two sets of coordinates of the same network referred to different datums are compared.

In continuum mechanics, deformation is not regarded by itself but mainly in connection with the underlying forces, namely stress-strain relations. E. W. Grafarend (1977) applies the concept and derives stress-strain relations, specially of local, homogenous and isotropic type, in geodetic networks. It should be mentioned that in case of abstract entities with a non-material nature, the physical meaning of deformation varies depending on each particular application. A comprehensive discussion of existing or possible future applications of deformation analysis in geodesy and geodynamics was given by A. Dermanis and E. Liveratos (1983b) with special emphasis to the study of crustal deformation, deformation of the gravity field and gravity field related deformations of the Earth.

Continuum mechanics is divided into different subdomains. Some of these subdomains such as kinematics, which deals with the displacement and deformation and study the time and space dependent tensor fields of continua or basic principals, describing the general physical theorems and laws applied to continua independently of their material characteristics, refer to every continua. Besides these subdomains of general theory, there exist special continuum mechanical theories for formulation and solution of boundary condition and initial value problems of special continua of idealized geometry under different external impacts on the basis of various methods and laws, like fluid mechanics, theory of elasticity, theory of plasticity and shell theory.

A shell as an idealized continuum model, is a three-dimensional material body of which one dimension, namely the thickness, is much smaller than the two other dimensions. Hence, a shell can be regarded as a surface-like body. The theory of shell is destined to describe the three-dimensional behaviour of a deformable body of this type by means of surface fields in a two-dimensional manner ,e.g. W. B. Kraetzig (1971), P. M. Naghdi (1972), W. Pietraszkiewicz (1977), W. Olszak (1980). Any unique mapping from three- to two-dimensional space is incompatible with our experience. Thus, the goal of dimensional reduction in shell theory can only be achieved in an approximate sense. The replacement of three-dimensional mathematical models of shell by two-dimensional ones is carried out by defining a reference surface called middle surface, that is, the surface which is equally far from both the outer surfaces of shell, and transforming all basic three-dimensional mechanical equations such that they remain functions of the two surface coordinates of the middle surface. Thus, we restrict the deformations of the shell to surface deformations and end up with the two-dimensional shell equations. In fact,
the thinner the shell the better is the approximation of its three-dimensional behaviour by two-dimensional quantities describing the deformation of its middle surface. Therefore, an exact theory of surface deformation based on differential geometry of surface builds the main theoretical foundation of shell theory. The concept and developed mathematical tools of shell theory have found wider applications in civil, mechanical, architectural, aeronautical and marine engineering in design and study of surface-like, man-made structures. In geodesy, applications of mathematical methods of surface deformation analysis can be seen in map projection studies. Interesting works have been done to the present time for the study of deformations induced when original figures on a sphere or an ellipsoid, as two-dimensional Riemann manifolds, are mapped on a plane, as a twodimensional Euclidean space, see for examples the works of B. H. Chovitz (1979), V. Hojovec and L. Jokl (1981), A. Dermanis and E. Livieratos (1983a), A. Dermanis et al. (1983a). An extensive and detailed review of surface deformation measures with application to the optimal universal transverse Mercator projection is given by E. W. Grafarend (1995). A. Dermanis et al. (1983b) utilize the surface deformation analysis for studying mappings of geoid to ellipsoid. A possible application of techniques of surface deformation analysis into a traditional geodetic problem, namely the surface mapping of a rotational ellipsoid onto a triaxial counterpart is shown by M. Amalvict and E. Livieratos (1988).

The mathematical tools and concepts of deformation analysis, developed in continuum mechanics, have since the late 1920's been applied by geoscientists to Earth deformation studies based on geodetic measurements. The repeated observations of geodetic networks, within a convenient interval, have become an important source of information for the investigation of the contemporary kinematics of the Earth surface in seismic areas and along the plate tectonic boundaries. The first studies of this type appear in literature in the works of Japanese seismologists T. Terada and N. Miyabe (1929), and C. Tsuboi (1930), who developed computational and graphical methods of strain determination from station coordinates of a horizontal geodetic network. Since then, great efforts have been made by the geodetic community to improve the analysis methods from both practical and theoretical aspects.

Following the classical separation of traditional geodetic techniques, namely triangulation and trilateration versus levelling, deformation of the Earth surface has been separated into horizontal and vertical components and has been treated individually. The main reason for this conventional separate treatment is due to the separately available horizontal and vertical networks in classical geodesy. For the study of horizontal crustal deformations, the two-dimensional plane deformation measures are considered for the description of the geometric alterations in the positions of the projections of the surface points onto the reference plane. When analyzing crustal deformations in a local geodetic network, namely in a local scale, a common horizontal reference plane is assumed for the perpendicular projection of the network points. In case of the analysis in a regional scale, a map projection plane is considered as a reference plane and the network points are projected to the plane by means of mathematical models of the selected map projection system. There is an extensive literature concerning the two-dimensional plane deformation analysis of the crust using classical geodetic results, e.g. W. Baarda (1975), T. Harada and M. Shimura (1978), E. Livieratos (1979), D. Schneider (1982), P. Wellman (1983), W. I. Reilly (1989), R. Chen (1991), J. Kakkuri and R. Chen (1992), J. Pagarete et al. (1998).

Space geodesy has changed the rules of the game of positioning radically. Thanks to the space geodetic techniques, such as GPS, VLBI, SLR and DORIS, three-dimensional positions of network points, containing both horizontal and vertical components, can be determined with high precision, enough to be used as an accurate and reliable source of information in Earth deformation studies. The great number of studies of this type using displacement fields derived from repeated observations of space geodetic networks indicates how valuable and important role the space geodetic techniques play in present and future states of geodynamics. Despite the ability of space geodesy to provide three-dimensional displacement fields, the crustal deformation studies are still carried out in horizontal and vertical components separately, e.g. Y. Bock and S. Shimada (1989), R. E. Reilinger et al. (1997a, 1997b), P. Tregoning et al. (1998), H. -G. Kahle et al. (1998), P. G. Clarke et al.
(1998), T. Kato et al. (1998), C. DeMets and T. H. Dixon (1999). The main reason of the separation is claimed to be the non-sufficient accuracy of height component of point position due to unresolved modeling errors such as the antenna phase center variations, path delays caused by atmospheric variations, and loading effects of the ocean and atmosphere. Hence contrary to horizontal positions, there are still systematic effects that do not cancel out in data processing steps and degrade the accuracy of the vertical positions which are determined from extra-terrestrial observations.

Regarding the fact that in reality, crustal motions and deformations are of three-dimensional nature, purely horizontal and purely vertical deformations do not exist. In the last two decades, some efforts have been made to formulate the problem in three-dimensional space. J. Zaiser (1984) computes the displacement field, the strain field and the rotation field in the context of arbitrary shaped geodetic networks and three-dimensional finite elements. A curvilinear three-dimensional finite element method is introduced by E. W. Grafarend (1986) for representation of local strain and local rotation tensors in terms of ellipsoidal, Gauss-Krüger or UTM coordinates. A study of the estimability-invariance characteristics of deformation parameters obtained through the finite element method, by using a dimension free approach with results that can be immediately specialized to three or two dimensions, has been carried out by A. Dermanis and E. W. Grafarend (1993).

However, existing methods of Earth deformation analysis suffer from some weaknesses and difficulties. Major problems existing in practical application of these methods are summarized as follows:

- Real crustal motions and deformations are of three-dimensional nature. Modeling the problems connected with deformations in three dimensions by computing separately the two-dimensional plane deformations and vertical motions can't portray the real state of crustal deformations.
- Deformation parameters are used as initial values for geophysical models. For example, an application of geodetic strain tensor in computation of seismic moment rates and analysis of earthquake potential budget of the study area are shown by $S . N$. Ward (1998a, 1998b). The planar deformation parameters, referring to the reference plane, can't be used as initial values for geophysical models. They have to be referred to the real surface of the Earth.
- The two-dimensional plane deformation analysis of the crust is limited to investigation of the alteration of the metric characteristics of the crust. In other words, the analysis allows us to bring the metric tensors of the two states of the body into a one-to-one correspondence. The Earth surface deformations can't be completely specified by the change of the metric tensor of the surface.
- The three-dimensional methods of Earth deformation analysis lose the simplicity of computations in twodimensional spaces. Moreover interpretation of the result of the analysis, namely three-dimensional deformation tensors and particularly invariants associated with them, is not an easy task.
- The geodetic measurements are connected to the surface of the Earth and are of a surface nature. A threedimensional deformation study of the whole Earth based on only displacement fields derived from surface geodetic measurements, neglecting the deformation measurements related to the interior of the Earth, can't reflect the real situation of the Earth deformations.

These facts indicate the need for reevaluation of the theoretical foundations of the Earth deformation analysis methods. Regarding these disadvantages and difficulties and also the fact that we have only surface geodetic measurements in our hands, it seems that a surface approach in Earth surface deformation analysis based on three-dimensional displacement fields is an appropriate solution. In other words, an approach that keeps the simplicity of computations in two-dimensional spaces, includes both vertical and horizontal components of the Earth deformations, and refers to the real surface of the Earth will be able to resolve the problems of the existing methods. Moreover, the space geodetic techniques can now provide accurate and dense geodetic data that can
model the geometry of the real surface of the Earth at a level of accuracy convenient for developing an approach of Earth deformation analysis referring to the real surface of the Earth. The first fundamental study of geodetic surface deformation analysis has been performed by S. Heitz and worked out in detail by his Ph.D. student Y. Altiner $(1996,1999)$ who developed a method of analytical surface deformation analysis of the Earth's crustal movements.

This dissertation presents an analytical formulation and implementation of a method of Earth surface deformation analysis referring to the real surface of the Earth. We benefit from the mathematical models and tools of surface deformation analysis in shell theory to develop appropriate models of the analysis applicable to deformation studies of the Earth surface. The Earth surface is considered as a two-dimensional Riemann manifold, namely a curved surface, embedded in a three-dimensional Euclidean space. Thus, deformation of the surface can be completely specified by the change of the first and second fundamental tensors, namely metric tensor and curvature tensor, of the surface. Special emphasis is given to definition of proper invariants of the introduced surface deformation tensors with meaningful physical interpretations. The main contributions of this study are,

- Introduction of intrinsic surface deformation analysis as a standard approach in shell theory to the realm of the Earth deformation studies in geodesy
- New mathematical formulations for the tensor of change of curvature of the Earth surface as a function of the difference vector of the unit normal vectors in addition to the displacement vector for both the extrinsic and intrinsic approaches in surface deformation analysis.

The new formulation produces meaningful numerical results for the tensor of change of curvature and its associated invariants, and allows us to apply it as a powerful surface deformation measure in studies of the current kinematics of the Earth surface. The role and placement of the intrinsic approach in geometrical modelling of Earth surface deformations has been shown using a diagram in Figure 1-1.

The surface deformation analysis in shell theory is developed on the fundamental mathematical foundations of type differential geometry and tensor analysis. The basic definitions and principles of the foundation within the scope necessary for this study are recapitulated in Chapter 2. Particularly, the main concepts of the theory of differential manifolds are reviewed. Tensors take a prominent role in the mathematical models of the study. Thus, the notions of covariant and contrvariant components of the tensor fields are introduced.

Chapter 3 begins with an introduction into the concept of surface deformation within the notion of Riemann manifolds. The general theories of the methods of the extrinsic and intrinsic surface deformation analysis are developed and various types of Lagrangian and Eulerian surface deformation measures are defined. The defined measures are described in terms of the displacement vector and the difference vector of unit normal vectors. The linearized theory of surface deformation analysis is presented with emphasis to the notion of linearized surface rotation tensor. The Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) is introduced as a measure of surface deformation and formulated in terms of the displacement vector and the difference vector of the unit normal vectors. The associated invariants of the surface deformation tensors such as surface dilatation, linearized rotation around the normal and changes of the mean and Gaussian curvatures, with certain physical meanings are discussed in the last section of the chapter.

The general theory of the extrinsic and intrinsic surface deformation analysis is tailored and applied to study surface deformations of the real surface of the Earth in Chapter 4. A Gaussian representation of the Earth surface in terms of the geodetic coordinates with respect to the reference ellipsoid is assumed. Analytical formulation of the mathematical tools of the extrinsic and intrinsic surface deformation analysis is developed for this special case. An important step for application of the intrinsic approach in Earth surface deformation analysis is the conversion of space Cartesian components of the displacement vector, resulting from data processing of geodetic measurements, onto the surface curvilinear components. We conclude the chapter with the development of


Figure 1-1: Methods of geometrical modelling of Earth surface deformations in geodesy
exact and approximate methods of the conversion.
Chapter 5 reviews the main causes of the Earth deformation in the first section. The second section deals with a brief introduction of the space geodetic techniques that can provide us with dense and accurate displacement fields of the Earth surface. In order to compute the surface deformation tensors and consequently their associated invariants, it is necessary to know the continuous field of displacement, for evaluation of its partial derivatives, at every point of the surface or, at least, in the neighborhood of specific points where surface deformation tensors are to be calculated. Geodetic observations and displacement fields derived from them are usually discrete. In such a case, the displacement field and its partial derivatives have to be approximated numerically. The finite element method provide the necessary tools to achieve the goal. A review of the method and the role that it plays in the context of the intrinsic and extrinsic surface deformation analysis of the Earth surface, are treated in the third section of the chapter.

The efficiency of the developed method for geometrical modelling of the Earth surface deformations is demonstrated in Chapter 6 by analysis of a real data set. The position and displacement rates of the data set are dense
and accurate enough to be utilized in the numerical part of the study. The European and Mediterranean areas, which are selected for the analysis of the capabilities of the intrinsic approach, are known as an extraordinary natural laboratory for the study of geodynamics processes. Abundance of the preexisting deformation studies in the region enable us to compare our numerical results with independent studies. We will investigate the links between various patterns of the surface deformation measures with geophysical and seismological evidences of the area to judge the validity of our numerical results.

Chapter 7 concludes the study. In the chapter, the main contributions of the study are summarized. The advantages and the main features of the application of the intrinsic versus extrinsic approach in Earth deformation analysis will be critically discussed.

## Chapter 2

## Theory of Manifolds

As mentioned previously, the concept of deformation will be presented in the next chapter based on the notion of differentiable manifolds. Thus, the basic definitions and principles of the theory of manifolds, within the scope necessary for this study, are recapitulated here. In fact, this chapter introduces the mathematical language of the thesis. A more comprehensive treatment of the theory may be found in standard textbooks, e.g. T. J. Willmore (1972), N. Prakash (1981), D. Martin (1991), and A. Visconti (1992).

From the point of view of structure mathematics the theory of manifolds is developed on the fundamental structures of type algebra and topology. Adequate treatments of the materials of the structures required for the theory of manifolds are provided by T. A. Whitelaw (1983), E. M. Patterson (1959) and W. A. Sutherland (1975).

### 2.1 Differentiable Manifolds

Generally speaking, a manifold is a topological space which is locally Euclidean. This means that a shortsighted observer at any point on it would regard his neighborhood as flat. The differentiable manifold is the one which the existence of a unique tangent space is guaranteed at each point on it. Even in this elementary definition of the differentiable manifolds, there are a few fundamental mathematical terms which have to be defined.

## Hausdorff Topological Space:

A topological space $\mathbb{M}$ is called a Hausdorff (separated) topological space if for any two distinct points $x$ and $y$ of $\mathbb{M}$, there exist separated neighborhoods $U_{\mathbb{M}}(x)$ and $U_{\mathbb{M}}(y)$, open sets of $\mathbb{M}$, that do not intersect each other, i.e. $U_{\mathbb{M}}(x) \cap U_{\mathbb{M}}(y)=\emptyset$.

## Homeomorphism and Diffeomorphism:

The notion of homeomorphism plays an essential role in the theory of manifold. It can be defined as a bicontinuous bijection $\phi$ between two open sets. This means that $\phi$ and its inverse $\phi^{-1}$ are both continuous bijective mappings. Two topological spaces are said to be equivalent if they are homeomorphic with respect to each other. If the mappings $\phi$ and $\phi^{-1}$ are differentiable mapping of class $C^{k}, \phi$ will be called a $C^{k}$ diffeomorphism.

## Manifold:

A Hausdorff topological space $\mathbb{M}$ is called an $n$-dimensional topological manifold, if to any point $x \in \mathbb{M}$ there exists a homeomorphism $\phi$ mapping an open neighborhood $U_{\mathbb{M}}(x)$ onto an open set $\phi\left(U_{\mathbb{M}}\right)$ of $n$-dimensional real vector space $\mathbb{R}^{n}$. Thus, $\mathbb{M}$ is supposed to be locally equivalent to $\mathbb{R}^{n}$. The dimension of $\mathbb{M}$ is defined as the dimension $n$ of $\mathbb{R}^{n}$ and the space will be denoted by $\mathbb{M}^{n}$.

## Chart and Atlas of a Manifold:

A manifold $\mathbb{M}^{n}$ was defined as a Hausdorff topological space with further provision that every point $x$ of it has a neighborhood $U(x)$ homeomorphic to an open set in $\mathbb{R}^{n}$, i.e. $\phi: U(x) \mapsto \mathbb{R}^{n}$. The pair $(U(x), \phi)$ is called an n-dimensional chart on $\mathbb{M}^{n}$ and $U(x)$ is the domain of the chart. It is almost impossible to find a single chart covering the whole of a given manifold $\mathbb{M}^{n}$, in which case it is necessary to introduce a family of charts whose domains take together $\mathbb{M}^{n}$ entirely. A family $\Phi=\left(U_{i}, \phi_{i}\right)_{i \in I}$ of charts on $\mathbb{M}^{n}, I$ being an index set, is called an atlas of $\mathbb{M}^{n}$ if the domains $U_{i}$ of $\phi_{i}$ cover $\mathbb{M}^{n}$ completely, i.e. $\mathbb{M}^{n}=\cup_{i \in I} U_{i}$. Furthermore, $\Phi$ is said to be a $C^{k}$ atlas if for each two of these charts $\left(U_{i}, \phi_{i}\right)$ and $\left(U_{j}, \phi_{j}\right)$ on the manifold, with a non-empty intersection $U_{i} \cap U_{j} \notin \emptyset$, the mapping $\phi_{j} \circ \phi_{i}^{-1}$ is a $C^{k}$ diffeomorphism of $\phi_{i}\left(U_{i} \cap U_{j}\right)$ onto $\phi_{j}\left(U_{i} \cap U_{j}\right)$. In such a case, ( $U_{i}, \phi_{i}$ ) and $\left(U_{j}, \phi_{j}\right)$ will be compatible charts.

## Differentiable Manifolds:

A $C^{k}$ differentiable manifold of dimension $n$ is a topological n -dimensional manifold with a complete ( or maximal) $C^{k}$ atlas of charts $\left(U_{i}, \phi_{i}\right)$, defined over it. The term complete atlas is given to a $C^{k}$ atlas on a manifold if any chart, which is compatible to each chart $\left(U_{i}, \phi_{i}\right)$, is itself contained in the atlas. A well-known example of differentiable manifolds is the real vector space $\mathbb{R}^{n} . \mathbb{R}^{n}$ is a manifold covered by the single chart $(U, \phi)$, where $U=\mathbb{R}^{n}$ and $\phi$ is the identity mapping.

### 2.2 Coordinates of Points on a Manifold

A way to describe the geometry of an $n$-dimensional manifold $\mathbb{M}^{n}$ is assuming the manifold as an embedded submanifold of a higher dimension manifold, usually $\mathbb{M}^{n+1}$, which is called the hypersurface of $\mathbb{M}^{n} \subset \mathbb{M}^{n+1}$. The embedding manifold is usually assumed to be an Euclidean space $\mathbb{E}^{n+1}$. The most convenient way of describing Euclidean spaces is by means of Cartesian orthonormal coordinate systems. Thus, a point $p \in \mathbb{M}^{n}$ is parametrized in the frame $\left\{o, \mathbf{i}_{1}, \mathbf{i}_{2}, \ldots, \mathbf{i}_{n+1}\right\}$ of the embedding space $\mathbb{E}^{n+1}$ by its coordinates $x^{k}$,

$$
\begin{equation*}
o p=\mathbf{x}=x^{k} \mathbf{i}_{k} \quad k=1, \ldots, n+1 \tag{2-1}
\end{equation*}
$$

$x^{k}$ are called space Cartesian coordinates of point $p \in \mathbb{M}^{n}$. Thus, each point on the manifold $\mathbb{M}^{n}$ corresponds to a vector $\mathbf{x}$ as a position vector.

In accordance with the summation convention, when an index appears in a product term once as a subscript and once as a superscript, e.g. Equation (2-1), then unless the contrary is stated, the index is given all its possible values and the resulting terms added together. Hereafter, the summation convention is applied to all the statements.

Besides the embedding space $\mathbb{E}^{n+1}$ and the embedded manifold $\mathbb{M}^{n}$, one more space is involved here: a real vector space $\mathbb{R}^{n}$ which is the image of $\mathbb{M}^{n}$ under the homeomorphism $\phi$, as a chart of the manifold. Any point
$p \in \mathbb{M}^{n}$ is consequently considered either as a point $\mathbb{E}^{n+1}$ and is parametrized by its coordinates $\left(x^{1}, \ldots, x^{n+1}\right)$, or as a point of $\mathbb{M}^{n}$ and its image in $\mathbb{R}^{n}$ through the chart $(U, \phi)$ which is parametrized by its local curvilinear coordinates $\left(q^{1}, \ldots, q^{n}\right)$. In order to express the fact that $p$ belong to $\mathbb{M}^{n}$, it is supposed that every single space coordinate $x^{k}$ is a function of $n$ curvilinear coordinates $\left(q^{1}, \ldots, q^{n}\right)$. Therefore, a parametric representation of the manifold $\mathbb{M}^{n}$ is given by,

$$
\begin{equation*}
x^{k}=x^{k}\left(q^{1}, \ldots, q^{n}\right) \quad k=1, \ldots, n+1 \tag{2-2}
\end{equation*}
$$

The point $p$, as a point of Euclidean manifold $\mathbb{E}^{n+1}$, can be also parametrized by space curvilinear coordinates $\left(q^{1}, \ldots, q^{n+1}\right)$ through the chart representation of $\mathbb{E}^{n+1}$.

### 2.3 Tensor Fields

Tensor fields, as differentiable entities associated with any manifold, play an important role in determining the geometry of a manifold. The fact that a tensor equation is true in all coordinate systems if it is true in one, makes tensors so useful not only in the theory of manifold but also in many other disciplines. A comprehensive treatment of tensor analysis can be read in the standard textbooks such as J. L. Synge and A. Schild (1949), J. A. Schouten (1951, 1954).

A tensor field on a manifold may be viewed either as a multilinear mapping from Cartesian product of vector spaces to the real or as a set of special type of functions, namely coordinates of the tensor, which obey certain given transformation laws. The first approach is an index-free approach as opposed to the second, which leans heavily on usage of indices. Selecting the second approach as a more familiar one to physicists, we restrict ourselves to defining a tensor in index notation instead of invariant notation.

The n-dimensional manifold $\mathbb{M}^{n}$ can be parametrized locally by a set of curvilinear coordinates $\left(q^{1}, \ldots, q^{n}\right)$. Introducing another set of curvilinear coordinates $\left(q^{\prime 1}, \ldots, q^{\prime n}\right)$ on the manifold, the point transformation between the two coordinate systems is given by

$$
\begin{equation*}
q^{\prime i}=q^{\prime i}\left(q^{1}, \ldots, q^{n}\right), \quad i=1, \ldots, n \tag{2-3}
\end{equation*}
$$

The inverse transformation $\left(q^{\prime 1}, \ldots, q^{\prime n}\right) \rightarrow\left(q^{1}, \ldots, q^{n}\right)$ exists if the Jacobian $\left|\frac{\partial q^{\prime i}}{\partial q^{j}}\right|$ does not vanish. Various kinds of tensor fields can be introduced according to the defined coordinate transformation.

## A Scalar Field ( Tensor of Type(0,0) ):

It corresponds to a real function $f$ of $n$ variables $q^{k}$ such that if $\left(q^{1}, \ldots, q^{n}\right)$ is changed into $\left(q^{\prime 1}, \ldots, q^{\prime n}\right)$

$$
\begin{equation*}
\left(q^{1}, \ldots, q^{n}\right) \rightarrow\left(q^{\prime 1}, \ldots, q^{\prime n}\right) \Rightarrow f^{\prime}\left(q^{\prime 1}, \ldots, q^{\prime n}\right)=f\left(q^{1}, \ldots, q^{n}\right) \tag{2-4}
\end{equation*}
$$

Hence, the value of the scalar field $f$ at a point on the manifold does not depend on the choice of the curvilinear coordinate system.

## A Contravariant Vector Field (Tensor of Type(1,0) ):

At each point $p \in \mathbb{M}^{n}$, there is a tangent space $\mathbb{T}_{p} \mathbb{M}^{n}$ of dimension $n$. By assigning an element $\mathbf{v}$ of $\mathbb{T}_{p} \mathbb{M}^{n}$ to each point $p \in \mathbb{M}^{n}$, we obtain a contravariant vector field over $\mathbb{M}^{n}$.

$$
\begin{equation*}
\mathbf{v}:=v^{i} \mathbf{a}_{i} \quad i=1, \ldots, n \tag{2-5}
\end{equation*}
$$

where $\mathbf{a}_{i}$ are base vectors of $\mathbb{T}_{p} \mathbb{M}^{n}$. Thus, a contravariant vector field may be given by its $n$ coordinates $v^{i}$ which are functions of curvilinear coordinates $\left(q^{1}, \ldots, q^{n}\right)$ and obey the following transformation law.

$$
\begin{equation*}
\left(q^{1}, \ldots, q^{n}\right) \rightarrow\left(q^{\prime 1}, \ldots, q^{\prime n}\right) \Rightarrow v^{\prime i}\left(q^{\prime 1}, \ldots, q^{\prime n}\right)=\frac{\partial q^{\prime i}}{\partial q^{j}} v^{j}\left(q^{1}, \ldots, q^{n}\right) \tag{2-6}
\end{equation*}
$$

## A Covariant Vector Field (Tensor of Type (0,1) ):

At each point $p \in \mathbb{M}^{n}$, besides the tangent space $\mathbb{T}_{p} \mathbb{M}^{n}$, its dual space $\mathbb{T}_{p}^{*} \mathbb{M}^{n}$ also of dimension $n$ can be considered. A covariant vector field over $\mathbb{M}^{n}$ assigns an element $\mathbf{v}$ of $\mathbb{T}_{p}^{*} \mathbb{M}^{n}$ to each point $p \in \mathbb{M}^{n}$.

$$
\begin{equation*}
\mathbf{v}:=v_{i} \mathbf{a}^{i}, \quad i=1, \ldots, n \tag{2-7}
\end{equation*}
$$

where, $\mathbf{a}^{i}$ are base vectors of the dual space $\mathbb{T}_{p}^{*} \mathbb{M}^{n}$. Similar to the contravariant vector field, the coordinates of covariant vector $\mathbf{v}$ satisfy the transformation law which is given as

$$
\begin{equation*}
\left(q^{1}, \ldots, q^{n}\right) \rightarrow\left(q^{\prime 1}, \ldots, q^{\prime n}\right) \Rightarrow v_{i}^{\prime}\left(q^{\prime 1}, \ldots, q^{\prime n}\right)=\frac{\partial q^{j}}{\partial q^{\prime i}} v_{j}\left(q^{1}, \ldots, q^{n}\right) \tag{2-8}
\end{equation*}
$$

## A Tensor Field of Type (r,s):

By taking tensor products of the tangent space $\mathbb{T}_{p} \mathbb{M}^{n}$ and its dual space $\mathbb{T}_{p}^{*} \mathbb{M}^{n}$, we can define a tensor field of any desired covariant and contravariant type. Hence, an $r$ times contravariant and $s$ times covariant tensor field $\boldsymbol{T}_{s}^{r}$ can be defined as

$$
\begin{equation*}
\boldsymbol{T}_{s}^{r}:=t_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{r}} \mathbf{a}_{i_{1}} \otimes \ldots \otimes \mathbf{a}_{i_{r}} \otimes \mathbf{a}^{j_{1}} \otimes \ldots \otimes \mathbf{a}^{j_{s}} \tag{2-9}
\end{equation*}
$$

The transformation laws between the coordinates of the tensor with respect to two related base vectors of $\mathbb{T}_{p} \mathbb{M}^{n}$ can be given as

Techniques that constitute tensor algebra, namely sums, products, contractions, etc. or properties of tensors such as symmetry, skew-symmetry, ..., can be well defined in terms of components of tensors via an index notation.

### 2.4 Riemann Manifolds

Before proceeding with the definition of Riemann manifolds, we have to introduce two more elementary concepts, namely tangent space and metric tensor field.

## Tangent Vector Space to a Manifold:

The tangent vector space is defined here in an extrinsic manner. The n-dimensional differentiable manifold $\mathbb{M}^{n}$ is assumed to be embedded in $\mathbb{E}^{n+1}$. Point $p \in \mathbb{M}^{n}$ with position vector $\mathbf{x}$, Equation (2-1), is considered on the
manifold. The infinitesimal vector

$$
\begin{equation*}
d \mathbf{x}=\frac{\partial \mathbf{x}}{\partial q^{i}} d q^{i} \tag{2-11}
\end{equation*}
$$

is an element of an affine n -dimensional vector space, called tangent space to $\mathbb{M}^{n}$ at $p$ and denoted by $\mathbb{T}_{p} \mathbb{M}^{n}$. The associated vector space $\mathbb{T}_{p} \mathbb{M}^{n}$ has a basis consisting of the covariant vectors $\frac{\partial \mathbf{x}}{\partial q^{2}}, i=1, \ldots, n$.

## Metric Tensor:

Assuming every point $p$ on the differentiable n-dimensional manifold $\mathbb{M}^{n}$ possesses a mapping $\boldsymbol{A}: \mathbb{T}_{p} \mathbb{M}^{n} \times$ $\mathbb{T}_{p} \mathbb{M}^{n} \mapsto \mathbb{R}$ which satisfies
i) $\boldsymbol{A}(\mathbf{v}, \mathbf{w})=\boldsymbol{A}(\mathbf{w}, \mathbf{v})$
ii) $\boldsymbol{A}\left(\alpha_{1} \mathbf{v}_{1}+\alpha_{2} \mathbf{v}_{2}, \mathbf{w}\right)=\alpha_{1} \boldsymbol{A}\left(\mathbf{v}_{1}, \mathbf{w}\right)+\alpha_{2} \boldsymbol{A}\left(\mathbf{v}_{2}, \mathbf{w}\right)$
iii) $\boldsymbol{A}\left(\mathbf{v}, \alpha_{1} \mathbf{w}_{1}+\alpha_{2} \mathbf{w}_{2}\right)=\alpha_{1} \boldsymbol{A}\left(\mathbf{v}, \mathbf{w}_{1}\right)+\alpha_{2} \boldsymbol{A}\left(\mathbf{v}, \mathbf{w}_{2}\right)$
iv) $\boldsymbol{A}(\mathbf{v}, \mathbf{v}) \geq 0$
v) $\boldsymbol{A}(\mathbf{v}, \mathbf{v})=0 \Rightarrow \mathbf{v}=0$
for all $\mathbf{v}, \mathbf{w} \in \mathbb{T}_{p} \mathbb{M}^{n}$. The mapping $\boldsymbol{A}$ as a symmetric, bilinear, positive definite tensor field of type $(0,2)$ over $\mathbb{M}^{n}$ is called the metric tensor or first fundamental tensor of the surface. The metric tensor $\boldsymbol{A}$ gives rise to an inner product on each tangent space $\mathbb{T}_{p} \mathbb{M}^{n}$ of $\mathbb{M}^{n}$. Thus, the inner product $<\mathbf{v}, \mathbf{w}>$ is simply defined to be $\boldsymbol{A}(\mathbf{v}, \mathbf{w})$. The differentiable manifold equipped with such an inner product, is named Riemann manifold. The metric tensor field $\boldsymbol{A}\left(q^{1}, \ldots, q^{n}\right)$ can be represented by a $n$ by $n$ symmetric matrix of its covariant coordinates,

$$
\begin{equation*}
\boldsymbol{A}=\left[a_{i j}\right]_{n \times n} \tag{2-12}
\end{equation*}
$$

with elements $a_{i j}\left(q^{1}, \ldots, q^{n}\right)$ that are real and given by

$$
\begin{equation*}
a_{i j}=<\frac{\partial \mathbf{x}}{\partial q^{i}}, \frac{\partial \mathbf{x}}{\partial q^{j}}> \tag{2-13}
\end{equation*}
$$

## Covariant Derivatives:

The problem of defining the derivative of a tensor field on an n-dimensional manifold can be solved by introducing an operator of differentiation such as exterior differentiation or Lie derivatives [K. Yano (1957)]. These operators have some limitations : the exterior differentiation is only applicable to differential forms and Lie derivatives depend on field vectors in the neighborhood of the point of differentiation [D. Martin (1991)]. The covariant derivative is given as an operator of differentiation on the manifold which is free of such limitations.

Assuming the contravariant vector field $\mathbf{v}\left(q^{1}, \ldots, q^{n}\right)$ over the manifold $\mathbb{M}^{n}$ with the contravariant coordinates $v^{i}$, one can obtain partial derivative of the vector field as

$$
\begin{align*}
\frac{\partial \mathbf{v}}{\partial q^{k}} & =\frac{\partial}{\partial q^{k}}\left(v^{i} \mathbf{a}_{i}\right)=\frac{\partial v^{i}}{\partial q^{k}} \mathbf{a}_{i}+v^{i} \frac{\partial \mathbf{a}_{i}}{\partial q^{k}} \\
& =\left(\frac{\partial v^{i}}{\partial q^{k}}+\Gamma_{k l}^{i} v^{l}\right) \mathbf{a}_{i} \tag{2-14}
\end{align*}
$$

where the $\Gamma_{k l}^{i}$ are three-index functions of curvilinear coordinates $\left(q^{1}, \ldots, q^{n}\right)$ called affine connection coefficients. They describe an affine connection field whose components are the affine connection coefficients. The equation 2-14 may be abbreviated by

$$
\begin{equation*}
\frac{\partial \mathbf{v}}{\partial q^{k}}=\left.v^{i}\right|_{k} \mathbf{a}_{i} \tag{2-15}
\end{equation*}
$$

$\left.v^{i}\right|_{k}$ are coordinates of a tensor of contravariant order 1 and covariant order 1 . They define the covariant partial derivative of a contravariant vector field as

$$
\begin{equation*}
\left.v^{i}\right|_{k}:=\frac{\partial v^{i}}{\partial q^{k}}+\Gamma_{k j}^{i} v^{j} \tag{2-16}
\end{equation*}
$$

Similarly, the covariant partial derivative of a covariant vector field $\mathbf{v}\left(q^{1}, \ldots, q^{n}\right)$ will be a covariant tensor of order 2 . The coordinates of the tensor, namely $V_{i \mid k}$, are obtained by using $\mathbf{v}=v_{i} \mathbf{a}^{i}$ as follows

$$
\begin{equation*}
\frac{\partial \mathbf{v}}{\partial q^{k}}=v_{i \mid k} \mathbf{a}^{i} \tag{2-17}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{i \mid k}:=\frac{\partial v_{i}}{\partial q^{k}}-\Gamma_{i k}^{j} v_{j} . \tag{2-18}
\end{equation*}
$$

In the particular case of Riemann manifold of dimension $n$ without torsion, with symmetric affine connection, i.e. $\Gamma_{j k}^{i}=\Gamma_{k j}^{i}$, the connection coefficients are named the Christoffel symbols of the second kind, due to E. B. Christoffel (1869). They can be uniquely determined as functions of the coordinates of the covariant metric tensor $\boldsymbol{A}$ defined over the manifold.

$$
\begin{equation*}
\Gamma_{j k}^{i}=\frac{1}{2} a^{i l}\left(\frac{\partial a_{j l}}{\partial q^{k}}+\frac{\partial a_{k l}}{\partial q^{j}}-\frac{\partial a_{j k}}{\partial q^{l}}\right) \tag{2-19}
\end{equation*}
$$

$a^{i l}$, as the contravariant coordinates of $\boldsymbol{A}$, are specified by the following property

$$
\begin{equation*}
a^{i k} a_{k j}=a_{i k} a^{k j}=\delta_{j}^{i}=\delta_{i}^{j} \tag{2-20}
\end{equation*}
$$

### 2.5 Geometry of 2-dimensional Riemann Manifolds

In this section the main objective is to present a concise introduction to the geometry of an ordinary surface. The surface is assumed to be a 2-dimensional Riemann manifold $\mathbb{M}^{2}$ embedded in a 3-dimensional Euclidean space $\mathbb{E}^{3}$. After introducing the notions of normal vector field and fundamental forms of the surface, we proceed with defining some of the geometric invariants of the surface which will be of importance in our considerations on surface deformation.

## Unit Normal Vector:

Considering the parametric representation of the surface in terms of surface curvilinear coordinates $\left(q^{1}, q^{2}\right)$, Equation (2-2), there exists a vector field $\mathbf{n}\left(q^{1}, q^{2}\right)$ on $\mathbb{M}^{2}$ at least locally, if not globally, such that at every point $p \in \mathbb{M}^{2}$

$$
\begin{equation*}
<\mathbf{n}_{p}, \mathbf{x}>=0 \tag{2-21}
\end{equation*}
$$

for every $\mathbf{x} \in \mathbb{T}_{p} \mathbb{M}^{2}$. The vector field $\mathbf{n}$ is called unit normal vector field if $\mathbf{n}$ is assumed to be a unit vector, $<\mathbf{n}, \mathbf{n}>=1$. The unit normal vector at point $p$ with surface curvilinear coordinates $\left(q^{1}, q^{2}\right)$ will be obtained as

$$
\begin{equation*}
\mathbf{n}_{p}=\frac{\mathbf{a}_{1} \times \mathbf{a}_{2}}{\left\|\mathbf{a}_{1} \times \mathbf{a}_{2}\right\|} \tag{2-22}
\end{equation*}
$$

where $\mathbf{a}_{1}=\frac{\partial \mathbf{x}}{\partial q^{1}}$ and $\mathbf{a}_{2}=\frac{\partial \mathbf{x}}{\partial q^{2}}$ are base vectors of the 2-dimensional tangent vector space $\mathbb{T}_{p} \mathbb{M}^{2}$ and $\wedge$ denotes vector product.

## The Gaussian Moving Frame:

The Gaussian moving frame in 3-dimensional space is constructed using three linearly independent vectors, namely the unit normal vector $\mathbf{n}$ and two tangent base vectors $\mathbf{a}_{1}$ and $\mathbf{a}_{2}$, associated to any point on the surface. The moving frame plays an essential role in deformation analysis of surfaces based on the intrinsic approach discussed in the next chapter.

## First and Second Fundamental Forms:

The function $\mathbf{I}\left(q^{1}, q^{2}\right)$, defined as

$$
\begin{equation*}
\mathbf{I}\left(q^{1}, q^{2}\right):=<d \mathbf{x}, d \mathbf{x}>=a_{\alpha \beta} d q^{\alpha} d q^{\beta} \tag{2-23}
\end{equation*}
$$

is a quadratic form which is called the first fundamental form of the surface. The coefficients $a_{\alpha \beta}$ are coordinates of a surface symmetric tensor of type ( 0,2 ), named the first fundamental tensor of the surface. It can be easily checked that $a_{\alpha \beta}$ are coordinates of the metric tensor of the surface. The first fundamental form $\mathbf{I}$ is invariant with respect to coordinate transformations. In fact, I depends only on the surface and not on any particular representation of the surface.

The second fundamental form II is defined as,

$$
\begin{equation*}
\mathbf{I I}\left(q^{1}, q^{2}\right):=-<d \mathbf{n}, d \mathbf{x}>=b_{\alpha \beta} d q^{\alpha} d q^{\beta} \tag{2-24}
\end{equation*}
$$

where, the coefficients $b_{\alpha \beta}$ are given by

$$
\begin{align*}
b_{\alpha \beta} & =-<\frac{\partial \mathbf{n}}{\partial q^{\alpha}}, \frac{\partial \mathbf{x}}{\partial q^{\beta}}>  \tag{2-25}\\
& =-<\frac{\partial \mathbf{n}}{\partial q^{\alpha}}, \mathbf{a}_{\beta}>=<\mathbf{n}, \frac{\partial \mathbf{a}_{\alpha}}{\partial q^{\beta}}>
\end{align*}
$$

$b_{\alpha \beta}$ are known as the coordinates of a surface symmetric tensor Bof type $(0,2)$, which is known as the second fundamental tensor of the surface. The second fundamental form II is invariant under a coordinate transformation in the same sense that the first fundamental form $\mathbf{I}$ is invariant. It should be noted that $\mathbf{I I}$ remains invariant as long as the coordinate transformation preserves the direction of $\mathbf{n}$. Otherwise the second fundamental formII changes its sign.

## Gaussian and Mean Curvatures:

Having the covariant coordinates of the first fundamental tensor $a_{\alpha \beta}$ and the second fundamental tensor $b_{\alpha \beta}$ of the surface, Gaussian curvature $k$ and mean curvature $h$ can be determined as two geometric invariants associated with these tensors.

$$
\begin{equation*}
k\left(q^{1}, q^{2}\right):=\frac{\operatorname{det}\left(b_{\alpha \beta}\right)}{\operatorname{det}\left(a_{\alpha \beta}\right)} \tag{2-26}
\end{equation*}
$$

$$
\begin{equation*}
h\left(q^{1}, q^{2}\right):=\frac{1}{2} a^{\alpha \beta} b_{\alpha \beta} \tag{2-27}
\end{equation*}
$$

$\operatorname{det}\left(a_{\alpha \beta}\right)$ and $\operatorname{det}\left(b_{\alpha \beta}\right)$ denote the determinants of the matrices formed by the covariant coordinates of the metric tensor and curvature tensor, respectively.

Gaussian curvature is unaffected by change of sign of the unit normal vector while the mean curvature reflects the change. This significant invariance property of the Gausssian curvature function $k\left(q^{1}, q^{2}\right)$, besides its invariant nature with respect to a change of surface coordinates, makes it the most appropriate tool to determine the geometry of the surface.

## Gauss-Weingarten Equations:

The Gauss-Weingarten equations are partial differential equations for surfaces that play a role somewhat analogous to the role of the Frenet equations for space curves. They express the derivatives of the tangent base vectors $\mathbf{a}_{1}, \mathbf{a}_{2}$ and unit normal vector $\mathbf{n}$ with respect to surface coordinates $q^{1}, q^{2}$ as linear combinations of these vectors. In fact, they present a decomposition of the vectors $\frac{\partial \mathbf{a}_{\alpha}}{\partial q^{\beta}}, \frac{\partial \mathbf{n}}{\partial q^{\alpha}}$, as second order derivatives of the position vector $\mathbf{x}\left(q^{1}, q^{2}\right)$, in the Gaussian moving frame.

The first group of the partial differential equations to be introduced is due to C. F. Gauss (1827).


The coefficients $\Gamma_{\alpha \gamma}^{\beta}$ and $b_{\alpha \beta}$ are already known as the Christoffel symbols of the second kind and the covariant coordinates of the curvature tensor of the surface, respectively.

The second group of differential equations involving the derivatives of the unit normal vector $\mathbf{n}$ are given by the so-called Weingarten differential equations. The equations introduced for the first time by J. Weingarten (1861).

$$
\begin{gather*}
\text { Weingarten Differential Equations of a Surface } \\
d \mathbf{n}=-b_{\alpha \gamma} a^{\gamma \beta} d q^{\alpha} \mathbf{a}_{\beta} \quad \longleftrightarrow \quad \frac{\partial \mathbf{n}}{\partial q^{\alpha}}=-b_{\alpha \gamma} a^{\gamma \beta} \mathbf{a}_{\beta} \quad \alpha, \beta, \gamma=1,2 \tag{2-29}
\end{gather*}
$$

The Weingarten equations 2-29 may be abbreviated by

$$
\begin{equation*}
\frac{\partial \mathbf{n}}{\partial q^{\alpha}}=-b_{\alpha}^{\beta} \mathbf{a}_{\beta} \tag{2-30}
\end{equation*}
$$

where, $-b_{\alpha}^{\beta}$ are coordinates of a mixed tensor $\boldsymbol{C}$ of type $(1,1)$, namely contravariant order 1 and covariant order 1, which is called Gaussian curvature tensor of the surface. General eigenvalue problem of the pair $(B, A)$, namely $\operatorname{det}(\boldsymbol{B}-\lambda \boldsymbol{A})=0$, or special eigenvalue problem of the pair $(C, I)$, namely $\operatorname{det}(\boldsymbol{C}-\lambda \boldsymbol{I})=0$, lead us the
principal curvatures $\kappa_{1}$ and $\kappa_{2}$ of the surface as eigenvalues of the tensor $\boldsymbol{B}$ or $\boldsymbol{C}[\mathrm{E} . \mathrm{W}$. Grafarend (1995)].

Gaussian Curvature Tensor of a Surface

$$
\begin{equation*}
\boldsymbol{C}=-\boldsymbol{B} \boldsymbol{A}^{-1} \longleftrightarrow c_{\alpha}^{\beta}=-b_{\alpha \gamma} a^{\gamma \beta} \tag{2-31}
\end{equation*}
$$

The mean curvature $h\left(q^{1}, q^{2}\right)$ and Gaussian curvature $k\left(q^{1}, q^{2}\right)$ are two important geometrical quantities in theory of surfaces because of their invariant property. They are defined as the average and the product of the principal curvatures, respectively.

$$
\begin{gather*}
h\left(q^{1}, q^{2}\right)=\frac{1}{2}\left(\kappa_{1}+\kappa_{2}\right)  \tag{2-32}\\
k\left(q^{1}, q^{2}\right)=\kappa_{1} \kappa_{2} \tag{2-33}
\end{gather*}
$$

## Chapter 3

## Surface Deformation Analysis

By a surface deformation we understand the changes in the characteristics of the geometry of the surface, namely length of the line segments and angles included by them, curvature, and so on during the motion of the surface. Surface deformation is always analyzed on the basis of comparisons of the differential invariants I, II, and III of the deforming surface between two chosen states labelled as the reference versus current state. Hence, we denote the deforming surface as reference surface and current surface in these two different states.

Two different ways of describing deformation may be used in general: Lagrangian or Eulerian. In the Lagrangian portray, the geometry of the reference surface, defined by the first and second fundamental tensors of the surface, is supposed to be known and all the tensor fields defined over the surface are connected to the geometrical points of the surface in the reference state. We can then speak of Lagrangian tensor fields. In Lagrangian portray, a coordinate system which is defined in connection with the reference surface, is called material coordinate system and coordinates of geometrical points in such a system are referred to as material coordinates. In the Eulerian portray, the geometry of the surface in the current state is assumed to be known. Hence, the Eulerian tensor fields of the surface are expressed in terms of coordinates of geometrical points of the current surface. In Eulerian portray, coordinate systems related to the current state are called spatial coordinate systems. Thus, in this case spatial coordinates are used in place of material coordinates. In this chapter, relations valid for arbitrary smooth deformation of a surface are treated in both Lagrangian and Eulerian portrays.

Following the conventions of continuum mechanics, all the material coordinates, coordinates of tensor fields and indices are printed in capital letters wherever they are connected to the reference surface and given in Lagrangian portray. Small letters are used for all the notations in Eulerian portray and spatial coordinates of the current surface. If Roman letters are used as an index, they will assume the values of $1,2,3$. An index printed in Greek letter will take only values of 1,2 . Hence, Roman indices will refer to the space coordinates, namely to coordinates which cover 3-dimensional Euclidean space, while Greek indices will be assumed for the surface coordinates.

### 3.1 The Concept of Surface Deformation

Here, the notion of surface deformation is discussed with reference to Riemann manifolds introduced in chapter 2. Let there be given the left 2-dimensional Riemann manifold $\mathbb{M}_{l}^{2}$ and the right 2-dimensional Riemann manifold $\mathbb{M}_{r}^{2}$. We start from these two 2-dimensional Riemann manifolds $\left\{\mathbb{M}_{l}^{2}, A_{\Lambda \Theta}\right\}$ and $\left\{\mathbb{M}_{r}^{2}, a_{\lambda \theta}\right\}$, with standard metric tensor $\boldsymbol{A}_{l}=\left[A_{\Lambda \Theta}\right]=\left[A_{\Theta \Lambda}\right]$ and $\boldsymbol{A}_{r}=\left[a_{\lambda \theta}\right]=\left[a_{\theta \lambda}\right]$ both symmetric and positive-definite, which represent the reference surface and current surface, respectively. The open subsets $U_{l} \subset \mathbb{M}_{l}^{2}$ and $u_{r} \subset \mathbb{M}_{r}^{2}$


Figure 3-1: The fundamental commutative digram
are covered by charts $\left\{\Phi, U_{l}\right\}$ and $\left\{\phi, u_{r}\right\}$. Such charts are constituted by surface curvilinear (local) coordinates $\left\{Q^{1}, Q^{2}\right\} \in \Phi\left(U_{l}\right) \subset \mathbb{R}^{2}$ and $\left\{q^{1}, q^{2}\right\} \in \phi\left(u_{r}\right) \subset \mathbb{R}^{2}$ over open sets $\Phi\left(U_{l}\right)$ and $\phi\left(u_{r}\right)$. Figure 3-1 illustrates the fundamental commutative diagram that governs the descriptive elements once we transform from the left Riemann manifold $\mathbb{M}_{l}^{2}$ onto the right Riemann manifold $\mathbb{M}_{r}^{2}$.

The mapping $\underline{f}: \mathbb{M}_{l}^{2} \mapsto \mathbb{M}_{r}^{2}$ and its local representation $\bar{f}: \Phi\left(U_{l}\right) \mapsto \phi\left(u_{r}\right)=\phi \circ \underline{f} \circ \Phi^{-1}$ are assumed to be a homeomorphism. A system of classification, based upon mapping equations $\bar{f}$ from a left chart to a right chart, namely of type isoparametric, identical charts, conformal, equiareal, equidistant, cocircular, geodesic, harmonic, and the general case, has been given by E. W. Grafarend (1982). The mapping $\underline{f}$ is called deformation.

### 3.2 Surface Deformation Measures

A comparative analysis of the metric tensors of the two manifolds under comparison is the standard way for the description of deformation in continuum mechanics [G. Beda et al. (1995), p.18; D. B. Macvean (1968), p.158]. A comprehensive review of various local as well as global multiplicative and additive measures of surface deformation, based on comparison of the metric tensors of the two parametrized surfaces, is given in E. W. Grafarend (1995). In addition to the metric tensors, a comparative analysis of the second fundamental tensors of the reference and current surfaces is considered as a way of describing surface deformation in shell theory. In this study, we concentrate on the most common measures of surface deformation which are derived from the first and second fundamental tensors of the two surfaces and some certain invariants of these derived measures.

### 3.2.1 Cauchy-Green deformation tensor

Referring to the fundamental commutative diagram of Figure 3-1, the homeomorphism $\underline{f}$ and its inverse $\underline{f}^{-1}$ will be represented by the chart mapping $q^{\lambda}\left(Q^{\Lambda}\right)$ and $Q^{\Lambda}\left(q^{\lambda}\right)$, respectively. The two point tensors $\boldsymbol{J}_{l}$ and $\boldsymbol{J}_{r}$ defined as,

$$
\begin{equation*}
\boldsymbol{J}_{l}=J_{\Lambda}^{\lambda} \mathbf{a}_{\lambda} \otimes \mathbf{A}^{\Lambda} \quad \text { versus } \quad \boldsymbol{J}_{r}=j_{\lambda}^{\Lambda} \mathbf{A}_{\Lambda} \otimes \mathbf{a}^{\lambda} \tag{3-1}
\end{equation*}
$$

where

$$
\begin{equation*}
J_{\Lambda}^{\lambda}:=\frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \quad j_{\lambda}^{\Lambda}:=\frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \tag{3-2}
\end{equation*}
$$

are termed the deformation gradients in the Lagrangian versus Eulerian portray. The coordinates of the deformation gradients determine the elements of the Jacobi matrices of the mapping $\bar{f}$ and its inverse ${ }^{\text {a }} r f^{-1}$.

Assuming the two Riemann manifolds $\left\{\mathbb{M}_{l}^{2}, A_{\Lambda \Theta}\right\}$ and $\left\{\mathbb{M}_{r}^{2}, a_{\lambda \theta}\right\}$ as embedded submanifolds of two different 3-dimensional Euclidean space $\mathbb{E}_{l}^{3}$ and $\mathbb{E}_{r}^{3}$, the first fundamental forms $\mathbf{I}_{l}$ of $\mathbb{M}_{l}^{2}$ and $\mathbf{I}_{r}$ of $\mathbb{M}_{r}^{2}$ in surface local coordinates of the manifolds are specified by

$$
\begin{equation*}
\mathbf{I}_{l}=A_{\Lambda \Theta}\left(Q^{\Phi}\right) d Q^{\Lambda} d Q^{\Theta} \quad \text { versus } \quad \mathbf{I}_{r}=a_{\lambda \theta}\left(q^{\phi}\right) d q^{\lambda} d q^{\theta} \tag{3-3}
\end{equation*}
$$

The left versus right Cauchy-Green deformation tensor are introduced in Box 3-1 as a multiplicative measure of deformation. The tensors are positive-definite, symmetric tensors of type ( 0,2 ).

Box 3-1: Left versus right Cauchy-Green deformation tensor
Lagrangian portray
Eulerian portray

$$
\begin{aligned}
\mathbf{I}_{r} & =a_{\lambda \theta} d q^{\lambda} d q^{\theta} \\
& =a_{\lambda \theta} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}} d Q^{\Lambda} d Q^{\Theta}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_{l} & =A_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta} \\
& =A_{\Lambda \Theta} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}} d q^{\lambda} d q^{\theta}
\end{aligned}
$$

## Left Cauchy-Green deformation tensor

$$
\boldsymbol{C}_{l}=C_{\Lambda \Theta}\left(Q^{\Phi}\right) \mathbf{A}^{\Lambda} \otimes \mathbf{A}^{\Theta}
$$

versus

## Right Cauchy-Green deformation tensor

$$
\boldsymbol{C}_{r}=c_{\lambda \theta}\left(q^{\phi}\right) \mathbf{a}^{\lambda} \otimes \mathbf{a}^{\theta}
$$

where

$$
\begin{array}{rlrl}
C_{\Lambda \Theta}\left(Q^{\Phi}\right): & =a_{\lambda \theta}\left(Q^{\Phi}\right) \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}} & c_{\lambda \theta}\left(q^{\phi}\right):=A_{\Lambda \Theta}\left(q^{\phi}\right) \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}} \\
& =a_{\lambda \theta}\left(Q^{\Phi}\right) J_{\Lambda}^{\lambda} J_{\Theta}^{\theta} & & =A_{\Lambda \Theta}\left(q^{\phi}\right) j_{\lambda}^{\Lambda} j_{\theta}^{\Theta}
\end{array}
$$

By means of the left Cauchy-Green deformation tensor we have succeeded to represent the first fundamental tensor (metric tensor) of the current surface in terms of the material coordinates of the reference surface.

Similarly, the right Cauchy-Green deformation tensor portrays the first fundamental tensor of the reference surface in the spatial coordinates of the current surface.

### 3.2.2 Euler-Lagrange deformation tensor of the first kind (Tensor of change of metric)

For the description of surface deformations, we can also look at the difference between corresponding first fundamental forms of the deforming surface at the reference- and current state. The difference $\mathbf{I}_{r}-\mathbf{I}_{l}$ leads us to the definition of a well-known additive measure of deformation called Euler-Lagrange deformation tensor of the first kind or tensor of change of metric. The Euler-Lagrange deformation tensor(I) is also known widely as strain tensor. Box 3-2 introduces left versus right Euler-Lagrange deformation tensor of the first kind.

Box 3-2: Left versus right Euler-Lagrange deformation tensor of the first kind (Lagrangian versus Eulerian tensor of change of metric)

> Lagrangian portray

Eulerian portray

$$
\begin{aligned}
\mathbf{I}_{r}-\mathbf{I}_{l} & =a_{\lambda \theta} d q^{\lambda} d q^{\theta}-A_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta} \\
& =a_{\lambda \theta} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}} d Q^{\Lambda} d Q^{\Theta}-A_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta} \\
& =\left(a_{\lambda \theta} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}}-A_{\Lambda \Theta}\right) d Q^{\Lambda} d Q^{\Theta} \\
& =2 E_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_{r}-\mathbf{I}_{l} & =a_{\lambda \theta} d q^{\lambda} d q^{\theta}-A_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta} \\
& =a_{\lambda \theta} d q^{\lambda} d q^{\theta}-A_{\Lambda \Theta} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}} d q^{\lambda} d q^{\theta} \\
& =\left(a_{\lambda \theta}-A_{\Lambda \Theta} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}}\right) d q^{\lambda} d q^{\theta} \\
& =2 e_{\lambda \theta} d q^{\lambda} d q^{\theta}
\end{aligned}
$$

Left Euler-Lagrange deformation tensor(I)

$$
\boldsymbol{E}_{l}=E_{\Lambda \Theta}\left(Q^{\Phi}\right) \mathbf{A}^{\Lambda} \otimes \mathbf{A}^{\Theta} \quad \text { versus }
$$

Right Euler-Lagrange deformation tensor(I)

$$
\boldsymbol{E}_{r}=e_{\lambda \theta}\left(q^{\phi}\right) \mathbf{a}^{\lambda} \otimes \mathbf{a}^{\theta}
$$

where

$$
\begin{aligned}
E_{\Lambda \Theta}\left(Q^{\Phi}\right) & :=\frac{1}{2}\left(a_{\lambda \theta}\left(Q^{\Phi}\right) \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}}-A_{\Lambda \Theta}\left(Q^{\Phi}\right)\right) & e_{\lambda \theta}\left(q^{\phi}\right) & :=\frac{1}{2}\left(a_{\lambda \theta}\left(q^{\phi}\right)-A_{\Lambda \Theta}\left(q^{\phi}\right) \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}}\right) \\
& =\frac{1}{2}\left(C_{\Lambda \Theta}-A_{\Lambda \Theta}\right) & & =\frac{1}{2}\left(a_{\lambda \theta}-c_{\lambda \theta}\right)
\end{aligned}
$$

The left Euler-Lagrange deformation tensor of the first kind $\boldsymbol{E}_{l}$, namely Lagrangian strain tensor, is sometimes associated with the names of Green and St. Venant while the right Euler-Lagrange deformation tensor of the first kind $\boldsymbol{E}_{r}$, namely Eulerian strain tensor, is associated with the names Almansi and Hamel [K. Wilmanski (1998)]. The symmetric deformation tensors $\boldsymbol{E}_{l}$ and $\boldsymbol{E}_{r}$ are powerful tools in studying deformations. In surface deformation analysis, they allow us a pointwise illustration of alteration of the metric properties of the deforming surface.

Apart from the Cauchy-Green and Euler-Lagrange(I) deformation tensors used to describe the changes in the geometry of the deforming body induced by the deformation, it is often convenient in continuum mechanics to employ other equivalent deformation measures. Table 3-1 collects the most common deformation tensors and their definitions appearing in various applications in continuum mechanics.

Table 3-1: The most common deformation tensors and their definitions [D. B. Macvean (1968)]

| Name | Symbol | Definition |
| :---: | :---: | :---: |
| Left Euler-Lagrange(I) | $\boldsymbol{E}_{1}$ | $\frac{1}{2}\left(\boldsymbol{C}_{l}-\boldsymbol{A}_{l}\right)$ |
| Right Euler-Lagrange(I) | $\boldsymbol{E}_{2}$ | $\frac{1}{2}\left(\boldsymbol{A}_{r}-\boldsymbol{C}_{r}\right)$ |
| Hencky | $\boldsymbol{E}_{3}$ | $\frac{1}{2} \ln \left(\boldsymbol{C}_{l}\right)$ |
| Hencky | $\boldsymbol{E}_{4}$ | $\frac{1}{2} \ln \left(\boldsymbol{C}_{r}\right)$ |
| Left Cauchy-Green | $\boldsymbol{E}_{5}$ | $\boldsymbol{J}_{l}^{T} \boldsymbol{A}_{r} \boldsymbol{J}_{l}$ |
| Right Cauchy-Green | $\boldsymbol{E}_{6}$ | $\boldsymbol{J}_{r}^{T} \boldsymbol{A}_{l} \boldsymbol{J}_{r}$ |
| Left stretch | $\boldsymbol{E}_{7}$ | $\boldsymbol{J}_{l}=\boldsymbol{R} \boldsymbol{E}_{7}$ ( Polar decomposition of the left Jacobi matrix) |
| Right stretch | $\boldsymbol{E}_{8}$ | $\boldsymbol{J}_{r}=\boldsymbol{R} \boldsymbol{E}_{8}$ ( Polar decomposition of the right Jacobi matrix) |
| Piola | $\boldsymbol{E}_{9}$ | $\boldsymbol{C}_{l}^{-1}$ |
| Finger | $\boldsymbol{E}_{10}$ | $\boldsymbol{C}_{r}^{-1}$ |
| Karni-Reiner | $\boldsymbol{E}_{11}$ | $\frac{1}{2}\left(\boldsymbol{A}_{l}-\boldsymbol{C}_{l}^{-1}\right)$ |
| Karni-Reiner | $\boldsymbol{E}_{12}$ | $\frac{1}{2}\left(\boldsymbol{C}_{r}^{-1}-\boldsymbol{A}_{r}\right)$ |

### 3.2.3 Euler-Lagrange deformation tensor of the second kind (Tensor of change of curvature)

In surface deformation analysis, as another additive measure of surface deformation we can take into account the difference between the second fundamental forms of the deforming surface at the reference- and current state. The additive comparison of the second fundamental forms $\mathbf{I I}_{l}$ of $\mathbb{M}_{l}^{2}$ and $\mathbf{I I}_{r}$ of $\mathbb{M}_{r}^{2}$ leads us to definition of Euler-Lagrange deformation tensor of the second kind or tensor of change of curvature, introduced in Box 3-3. The Euler-Lagrange deformation tensors of the first- and second kind (tensor of change of metric and the tensor of change of curvature) are considered as the two basic measures of surface deformation in literature of shell theory, e.g. W. Pietraszkiewicz (1977), L. J. Ernst (1981).

Box 3-3: Left versus right Euler-Lagrange deformation tensor of the second kind (Lagrangian versus Eulerian tensor of change of curvature)

> Lagrangian portray

$\mathbf{I} \mathbf{I}_{r}-\mathbf{I I}_{l}=b_{\lambda \theta} d q^{\lambda} d q^{\theta}-B_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta}$
$=b_{\lambda \theta} d q^{\lambda} d q^{\theta}-B_{\Lambda \Theta} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}} d q^{\lambda} d q^{\theta}$
$=\left(b_{\lambda \theta}-B_{\Lambda \Theta} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}}\right) d q^{\lambda} d q^{\theta}$
$=k_{\lambda \theta} d q^{\lambda} d q^{\theta}$

Left Euler-Lagrange deformation tensor(II)

$$
\boldsymbol{K}_{l}=K_{\Lambda \Theta}\left(Q^{\Phi}\right) \mathbf{A}^{\Lambda} \otimes \mathbf{A}^{\Theta}
$$ versus

Right Euler-Lagrange deformation tensor(II)

$$
\begin{aligned}
& =b_{\lambda \theta} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}} d Q^{\Lambda} d Q^{\Theta}-B_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta} \\
& =\left(b_{\lambda \theta} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}}-B_{\Lambda \Theta}\right) d Q^{\Lambda} d Q^{\Theta} \\
& =K_{\Lambda \Theta} d Q^{\Lambda} d Q^{\Theta}
\end{aligned}
$$

$\boldsymbol{K}_{r}=k_{\lambda \theta}\left(q^{\phi}\right) \mathbf{a}^{\lambda} \otimes \mathbf{a}^{\theta}$
where

$$
K_{\Lambda \Theta}\left(Q^{\Phi}\right):=b_{\lambda \theta}\left(Q^{\Phi}\right) \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}}-B_{\Lambda \Theta}\left(Q^{\Phi}\right)
$$

$$
k_{\lambda \theta}\left(q^{\phi}\right):=b_{\lambda \theta}\left(q^{\phi}\right)-B_{\Lambda \Theta}\left(q^{\phi}\right) \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}}
$$

It should be noted that the definition of the Euler-Lagrange deformation tensor(I) (strain tensor), Box 3-2, is
generally accepted in the literature of shell theory. However, the definition of the Euler-Lagrange deformation tensor(II) (tensor of change of curvature) varies depending on the applications. One important difference comes from a sign convention adopted. Besides, any functionally independent combination of the strain tensor $\boldsymbol{E}_{l} / \boldsymbol{E}_{r}$ and second fundamental tensor $\boldsymbol{B}_{l} / \boldsymbol{B}_{r}$ may be chosen as a measure for the surface curvature changes [W. Pietraszkiewicz (1977)].

### 3.3 Surface Deformation Measures and the Displacement Vector

We recall the assumption that the reference- and current surface are considered as two 2-dimensional Riemann manifolds $\mathbb{M}_{l}^{2}$ and $\mathbb{M}_{r}^{2}$ embedded in two different 3-dimensional Euclidean spaces $\mathbb{E}_{l}^{3}$ and $\mathbb{E}_{r}^{3}$. On the reference surface, the place in the embedding space of a generic point is given by the placement vector $\mathbf{X}\left(Q^{\Lambda}\right)$. After deformation the place in space of the same point is given by a new placement vector $\mathbf{x}\left(q^{\lambda}\right)$. Referring to Figure $3-2$, the displacement vector $\mathbf{u}$ is defined as

$$
\begin{equation*}
\mathbf{u}:=\mathbf{t}+\mathbf{x}-\mathbf{X} \tag{3-4}
\end{equation*}
$$

where the translation vector $\mathbf{t}$ serves to locate the origin $o$ of the spatial coordinate system $o x^{1} x^{2} x^{3}$ with respect to origin $O$ of the material coordinate system $O X^{1} X^{2} X^{3}$.


Figure 3-2: The dispacement vector, the reference- and the current surface in a commutative diagram

### 3.3.1 Surface deformation measures as functions of the displacement vector

For practical application of the theory, it is more convenient that measures of deformation be described in terms of the displacement vector $\mathbf{u}$. Moreover, various approximate theories in continuum mechanics and particularly shell theory are developed by dropping or approximating nonlinear terms in relations of the deformation measures expressed as functions of the displacement vector. Thus, we proceed with expressions of the surface deformation measures as functions of the displacement vector.

The covariant base vectors

$$
\begin{equation*}
\mathbf{A}_{\Lambda}:=\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}} \quad \text { versus } \quad \mathbf{a}_{\lambda}:=\frac{\partial \mathbf{x}}{\partial q^{\lambda}} \tag{3-5}
\end{equation*}
$$

span the tangent space $\mathbb{T}_{Q} \mathbb{M}_{l}^{2}$ and $\mathbb{T}_{q} \mathbb{M}_{r}^{2}$ of the reference- and current surface at $Q$ and $q$, respectively. Their inner products lead to the covariant coordinates of the metric tensors $\boldsymbol{A}_{l}$ and $\boldsymbol{A}_{r}$ of the left and right manifolds, respectively.

$$
\begin{equation*}
A_{\Lambda \Theta}:=<\mathbf{A}_{\Lambda}, \mathbf{A}_{\Theta}>\quad \text { versus } \quad a_{\lambda \theta}:=<\mathbf{a}_{\lambda}, \mathbf{a}_{\theta}> \tag{3-6}
\end{equation*}
$$

Similar to the metric tensors $\boldsymbol{A}_{l}$ and $\boldsymbol{A}_{r}$, left and right Cauchy-Green deformation tensor can be defined as scalar product of new base vectors $\mathbf{C}_{\Lambda}\left(Q^{\Phi}\right)$ and $\mathbf{c}_{\lambda}\left(q^{\phi}\right)$ which are given as

$$
\begin{align*}
\mathbf{C}_{\Lambda}\left(Q^{\Phi}\right) & :=\mathbf{a}_{\lambda} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}}=\frac{\partial \mathbf{x}}{\partial q^{\lambda}} \frac{\partial q^{\lambda}}{\partial Q^{\Lambda}}=\frac{\partial \mathbf{x}}{\partial Q^{\Lambda}}  \tag{3-7}\\
\mathbf{c}_{\lambda}\left(q^{\phi}\right) & :=\mathbf{A}_{\Lambda} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}}=\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}} \frac{\partial Q^{\Lambda}}{\partial q^{\lambda}}=\frac{\partial \mathbf{X}}{\partial q^{\lambda}}
\end{align*}
$$

Thus, left versus right Cauchy-Green deformation tensor can be written in terms of the new base vectors as

$$
\begin{equation*}
C_{\Lambda \Theta}=C_{\Theta \Lambda}:=<\mathbf{C}_{\Lambda}, \mathbf{C}_{\Theta}>\quad \text { versus } \quad c_{\lambda \theta}=c_{\theta \lambda}:=<\mathbf{c}_{\lambda}, \mathbf{c}_{\theta}> \tag{3-8}
\end{equation*}
$$

To derive Cauchy-Green and Euler-Lagrange deformation tensors in terms of the displacement vector, we consider the relations of these tensors using the scalar product of the base vectors $C_{\Lambda \Theta} / c_{\lambda \theta}$, Equation (3-8), and the definition of the displacement vector as difference of the placement vectors $\mathbf{X} / \mathbf{x}$, Equation (3-4). Box 3-4 highlights the main steps of the derivation for these deformation tensors in Lagrangian versus Eulerian portray.

It should be noted that the derived expressions for the deformation tensors $\boldsymbol{C}_{l} / \boldsymbol{C}_{r}$ and $\boldsymbol{E}_{l} / \boldsymbol{E}_{r}$ in Box 3-4, are general and exact formulae without any approximation being applied to extract them. Another important result is that these deformation tensors are insensitive to the translation vector $\mathbf{t}$. Hence, the translation vector $\mathbf{t}$ is not considered in our computations any more.

Analogous to Cauchy-Green deformation tensors and Euler-Lagrange deformation tensors of the first kind, it is more adequate to express the Euler-Lagrange deformation tensors of the second kind (tensor of change of curvature) as functions of the displacement vector. Unfortunately, similar relations for the tensor of change of curvature happen to be more complicated, see for example the work of L. J. Ernst (1980). To overcome this problem and obtain less sophisticated relations, E. Stein (1980) takes into account another difference vector called difference vector of the unit normal vectors in addition to the displacement vector. Having the unit

Box 3-4: The left versus right Cauchy-Green deformation tensor ( $\boldsymbol{C}_{l}$ versus $\boldsymbol{C}_{r}$ ), and the left versus right Euler-Lagrange geformation tensor(I) $\left(\boldsymbol{E}_{l}\right.$ versus $\left.\boldsymbol{E}_{r}\right)$ as functions of the displacement vector u

## Lagrangian portray

## Coordinates of the left Cauchy-Green

## deformation tensor:

$$
\begin{aligned}
C_{\Lambda \Theta}:= & <\mathbf{C}_{\Lambda}, \mathbf{C}_{\Theta}> \\
= & <\frac{\partial \mathbf{x}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{x}}{\partial Q^{\Theta}}> \\
= & <\frac{\partial(\mathbf{u}+\mathbf{X}-\mathbf{t})}{\partial Q^{\Lambda}}, \frac{\partial(\mathbf{u}+\mathbf{X}+\mathbf{t})}{\partial Q^{\Theta}}> \\
= & <\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>+<\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>+ \\
& <\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>+<\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>
\end{aligned}
$$

Coordinates of the left Euler-Lagrange deformation tensor(I):

$$
\begin{aligned}
E_{\Lambda \Theta}:= & \frac{1}{2}\left(C_{\Lambda \Theta}-A_{\Lambda \Theta}\right) \\
= & \frac{1}{2}\left(<\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>+<\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>+\right. \\
& \left.<\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>\right)
\end{aligned}
$$

Eulerian portray

Coordinates of the right Cauchy-Green deformation tensor:

$$
\begin{aligned}
c_{\lambda \theta}:= & <\mathbf{c}_{\lambda}, \mathbf{c}_{\theta}> \\
= & <\frac{\partial \mathbf{X}}{\partial q^{\lambda}}, \frac{\partial \mathbf{X}}{\partial q^{\theta}}> \\
= & <\frac{\partial(\mathbf{x}-\mathbf{u}+\mathbf{t})}{\partial q^{\lambda}}, \frac{\partial(\mathbf{x}-\mathbf{u}+\mathbf{t})}{\partial q^{\theta}}> \\
= & <\frac{\partial \mathbf{u}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>-<\frac{\partial \mathbf{u}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>- \\
& <\frac{\partial \mathbf{x}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>+<\frac{\partial \mathbf{x}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>
\end{aligned}
$$

## Coordinates of the right Euler-Lagrange

 deformation tensor(I):$$
\begin{aligned}
e_{\lambda \theta}: & =\frac{1}{2}\left(a_{\lambda \theta}-c_{\lambda \theta}\right) \\
= & \frac{1}{2}\left(<\frac{\partial \mathbf{u}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>+<\frac{\partial \mathbf{x}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>-\right. \\
& \left.<\frac{\partial \mathbf{u}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>\right)
\end{aligned}
$$

normal vectors $\mathbf{N}$ of the reference surface and $\mathbf{n}$ of the current surface, the difference vector of the unit normal vectors $\mathbf{w}$ can be defined as

$$
\begin{equation*}
\mathbf{w}:=\mathbf{n}-\mathbf{N} \tag{3-9}
\end{equation*}
$$

Figure 3-3 shows the role of the vector in linking the reference- and current surface in a commutative diagram. The difference vector of the unit normal vectors is used to formulate Lagrangian and Eulerian tensors of change of curvature as functions of the difference vectors $\mathbf{u}$ and $\mathbf{w}$ in less complicated features. Box 3-5 summarizes the main steps towards this goal.

As can be seen in Box 3-5, we end up with the expressions of the left and right Euler-Lagrange deformation tensor(I) (Lagrangian and Eulerian tensors of change of curvature) as inner products of the displacement vector and the difference vector of the unit normal vectors. Thanks to the use of the difference vector of the unit normal vectors, the expressions of the left and right Euler-Lagrange deformation tensors of the second kind are derived in a less sophisticated manner. Here, the difference vectors $\mathbf{u}$ and $\mathbf{w}$ appeared in the final relations of the deformation tensors in an invariant notation. They will be decomposed to coordinates in the following sections.


Figure 3-3: The difference vector of the unit normal vectors, the reference- and the current surface in a commutative diagram

Box 3-5: Left versus right Euler-Lagrange deformation tensor(II) (Tensor of change of curvature) in terms of the displacement vector $\mathbf{u}$ and the difference vector of the unit normal vectors $\mathbf{w}$

Coordinates of the left Euler-Lagrange deformation tensor(II):

$$
\begin{aligned}
K_{\Lambda \Theta} & :=\frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}} b_{\lambda \theta}-B_{\Lambda \Theta} \\
= & -\frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}}<\frac{\partial \mathbf{n}}{\partial q^{\lambda}}, \mathbf{a}_{\theta}>-B_{\Lambda \Theta} \\
= & -\frac{\partial q^{\lambda}}{\partial Q^{\Lambda}} \frac{\partial q^{\theta}}{\partial Q^{\Theta}}<\frac{\partial(\mathbf{w}+\mathbf{N})}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>-B_{\Lambda \Theta} \\
= & -<\frac{\partial(\mathbf{w}+\mathbf{N})}{\partial Q^{\Lambda}}, \frac{\partial(\mathbf{X}+\mathbf{u})}{\partial Q^{\Theta}}>-B_{\Lambda \Theta} \\
= & -<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>-<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>- \\
& <\frac{\partial \mathbf{N}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>
\end{aligned}
$$

Eulerian portray

## Coordinates of the right Euler-Lagrange

 deformation tensor(II):$$
\begin{aligned}
k_{\lambda \theta} & :=b_{\lambda \theta}-\frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}} B_{\lambda \theta} \\
& =b_{\lambda \theta}+\frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}}<\frac{\partial \mathbf{N}}{\partial Q^{\Lambda}}, \mathbf{A}_{\Theta}> \\
= & b_{\lambda \theta}-\frac{\partial Q^{\Lambda}}{\partial q^{\lambda}} \frac{\partial Q^{\Theta}}{\partial q^{\theta}}<\frac{\partial(\mathbf{w}-\mathbf{n})}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}> \\
= & b_{\lambda \theta}-<\frac{\partial(\mathbf{w}-\mathbf{n})}{\partial q^{\lambda}}, \frac{\partial(\mathbf{x}-\mathbf{u})}{\partial q^{\theta}}> \\
= & -<\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>+<\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>- \\
& <\frac{\partial \mathbf{n}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>
\end{aligned}
$$

### 3.3.2 Surface deformation measures (Extrinsic approach)

We derived the expressions of the various surface deformation measures as functions of the difference vectors $\mathbf{u}$ and $\mathbf{w}$ where they are considered in an invariant notation. For practical applications of the theory, we should take into account the decomposition of the difference vectors in convenient coordinate systems. This is the point of departure for two different approaches called extrinsic approach versus intrinsic approach. In the extrinsic approach, a class of coordinate systems are considered which are defined in relation to the embedding spaces.

Here, the embedding spaces of reference- and current surface are assumed to be 3-dimensional Euclidean spaces. A way of describing Euclidean spaces is by means of Cartesian orthonormal coordinate systems. The space Cartesian coordinates of a point with the placement vector $\mathbf{X}$ in the reference state and $\mathbf{x}$ in the current state with respect to the orthonormal fixed frames $\left\{\mathbf{I}_{1}, \mathbf{I}_{2}, \mathbf{I}_{3}\right\}$ and $\left\{\mathbf{i}_{1}, \mathbf{i}_{2}, \mathbf{i}_{3}\right\}$ are given by

$$
\begin{equation*}
\mathbf{X}=X^{K} \mathbf{I}_{K} \quad \text { versus } \quad \mathbf{x}=x^{k} \mathbf{i}_{k} \tag{3-10}
\end{equation*}
$$

The displacement vector can also be decomposed in these Cartesian coordinate systems as

$$
\begin{equation*}
\mathbf{u}=U^{K} \mathbf{I}_{K} \quad \text { versus } \quad \mathbf{u}=u^{k} \mathbf{i}_{k} \tag{3-11}
\end{equation*}
$$

where $U^{K}$ and $u^{k}$ are titled space Cartesian coordinates of the displacement vector in Lagrangian- and Eulerian portray, respectively. Box 3-6 includes the expressions of the Cauchy-Green- and Euler-Lagrange(I) deformation tensors as functions of the space Cartesian coordinates of the displacement vector.

Box 3-6: Left versus right Cauchy-Green deformation tensor ( $\boldsymbol{C}_{l}$ versus $\boldsymbol{C}_{r}$ ) and left versus right Euler-Lagrange deformation tensor(I) ( $\boldsymbol{E}_{l}$ versus $\left.\boldsymbol{E}_{r}\right)$ as functions of the space Cartesian coordinates of the displacement vector $U^{K} / u^{k}$

> Lagrangian portray

Eulerian portray

Coordinates of the left Cauchy-Green deformation tensor:

$$
\begin{aligned}
C_{\Lambda \Theta}= & A_{\Lambda \Theta}+\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}}+\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial X^{I}}{\partial Q^{\Theta}}+ \\
& \frac{\partial X^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}}
\end{aligned}
$$

Coordinates of the left Euler-Lagrange
deformation tensor(I):

$$
\begin{aligned}
E_{\Lambda \Theta}= & \frac{1}{2}\left(\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial X^{I}}{\partial Q^{\Theta}}+\frac{\partial X^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}}+\right. \\
& \left.\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}}\right)
\end{aligned}
$$

## Coordinates of the right Cauchy-Green

 deformation tensor:$$
\begin{gathered}
c_{\lambda \theta}=a_{\lambda \theta}+\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}}-\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial x^{i}}{\partial q^{\theta}}- \\
\frac{\partial x^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}}
\end{gathered}
$$

## Coordinates of the right Euler-Lagrange deformation tensor(I):

$$
\begin{aligned}
e_{\lambda \theta}= & \frac{1}{2}\left(\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial x^{i}}{\partial q^{\theta}}+\frac{\partial x^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}}-\right. \\
& \left.\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}}\right)
\end{aligned}
$$

Similarly, left and right Euler-Lagrange deformation tensors of the second kind (Lagrangian and Eulerian tensors of change of curvature) can be introduced in terms of space Cartesian coordinates of the displacement vector and space Cartesian coordinates of the difference vector of the unit normal vectors, Box 3-7. The Space Cartesian coordinates of the difference vector $\mathbf{w}$ are

$$
\begin{equation*}
\mathbf{w}:=W^{K} \mathbf{I}_{K} \quad \text { versus } \quad \mathbf{w}:=w^{k} \mathbf{i}_{k} \tag{3-12}
\end{equation*}
$$

and its partial derivatives with respect to the surface curvilinear coordinates $Q^{\Lambda} / q^{\lambda}$ are given by

$$
\begin{equation*}
\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}=\frac{\partial W^{K}}{\partial Q^{\Lambda}} \mathbf{I}_{K} \quad \text { versus } \quad \frac{\partial \mathbf{w}}{\partial q^{\lambda}}=\frac{\partial w^{k}}{\partial q^{\lambda}} \mathbf{i}_{k} \tag{3-13}
\end{equation*}
$$

Box 3-7: Left vesus right Euler-Lagrange deformation tensor of the second kind (Lagrangian versus Eulerian tensor of change fo curvature) as functions of the space Cartesian coordinates of the displacement vector $\mathbf{u}$ and the difference vector of the unit normal vectors $\mathbf{w}$

Lagrangian portray

Coordinates of the left Euler-Lagrange deformation tensor(II):

$$
\begin{aligned}
K_{\Lambda \Theta}= & -<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>-<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>- \\
& <\frac{\partial \mathbf{N}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}> \\
= & B_{\Lambda}^{\Phi} \frac{\partial X^{K}}{\partial Q^{\Phi}} \frac{\partial U^{K}}{\partial Q^{\Theta}}-\frac{\partial W^{K}}{\partial Q^{\Lambda}} \frac{\partial X^{K}}{\partial Q^{\Theta}}-\frac{\partial W^{K}}{\partial Q^{\Lambda}} \frac{\partial U^{K}}{\partial Q^{\Theta}}
\end{aligned}
$$

Eulerian portray

## Coordinates of the right Euler-Lagrange

 deformation tensor(II):versus $k_{\lambda \theta}=-<\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>+<\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>-$
$<\frac{\partial \mathbf{n}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>$
$=b_{\lambda}^{\phi} \frac{\partial x^{k}}{\partial q^{\phi}} \frac{\partial u^{k}}{\partial q^{\theta}}-\frac{\partial w^{k}}{\partial q^{\lambda}} \frac{\partial x^{k}}{\partial q^{\theta}}+\frac{\partial w^{k}}{\partial q^{\lambda}} \frac{\partial u^{k}}{\partial q^{\theta}}$

Instead of Cartesian coordinates, the embedding spaces in the reference- and current state can be parametrized locally by means of space curvilinear coordinates $Q^{K}(K=1,2,3)$ and $q^{k}(k=1,2,3)$ through the chart representations of the Euclidean spaces. In extrinsic approach, the surface deformation tensors can be expressed as functions of the space curvilinear coordinates of the difference vectors $\mathbf{u}$ and $\mathbf{w}$ as well. The space curvilinear coordinates of a vector are those coordinates which are obtained by the decomposition of the vector in the 3dimensional orthogonal moving frames which are established by means of the triads of the covariant base vectors

$$
\begin{equation*}
\mathbf{G}_{K}:=\frac{\partial \mathbf{X}}{\partial Q^{K}} \quad \text { versus } \quad \mathbf{g}_{k}:=\frac{\partial \mathbf{x}}{\partial q^{k}} \tag{3-14}
\end{equation*}
$$

at each point of the embedding spaces with the space curvilinear coordinates $Q^{K}$ and $q^{k}$. The covariant and contravariant curvilinear coordinates of the displacement vector, referred to the triads of base vectors $\mathbf{G}_{K} / \mathbf{g}_{k}$ and their reciprocal (contravariant) base vectors $\mathbf{G}^{K} / \mathbf{g}^{k}$, are given by

$$
\begin{equation*}
\mathbf{u}=\bar{U}^{K} \mathbf{G}_{K}=\bar{U}_{K} \mathbf{G}^{K} \quad \text { versus } \quad \mathbf{u}=\bar{u}^{k} \mathbf{g}_{k}=\bar{u}_{k} \mathbf{g}^{k} \tag{3-15}
\end{equation*}
$$

where the space curvilinear coordinates are barred in order to distinguish them from the space Cartesian coordinates.

The partial derivatives of the placement vectors $\mathbf{X} / \mathbf{x}$ and the displacement vector $\mathbf{u}$ with respect to surface curvilinear coordinates $Q^{\Lambda} / q^{\lambda}$, given in invariant notation in Box 3-4, can be rewritten for the placement vectors as

$$
\begin{align*}
\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}} & =\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial \mathbf{X}}{\partial Q^{K}} & \frac{\partial \mathbf{x}}{\partial q^{\lambda}} & =\frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial \mathbf{x}}{\partial q^{k}} \\
& =\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \mathbf{G}_{K} & & =\frac{\partial q^{k}}{\partial q^{\lambda}} \mathbf{g}_{k} \tag{3-16}
\end{align*}
$$

and for the displacement vector as

$$
\begin{align*}
\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}} & =\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial \mathbf{u}}{\partial Q^{K}} & \frac{\partial \mathbf{u}}{\partial q^{\lambda}} & =\frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial \mathbf{u}}{\partial q^{k}} \\
& =\left.\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \bar{U}^{J}\right|_{K} \mathbf{G}_{J}=\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \bar{U}_{J \mid K} \mathbf{G}^{J} & & =\left.\frac{\partial q^{k}}{\partial q^{\lambda}} \bar{u}^{j}\right|_{k} \mathbf{g}_{j}=\frac{\partial q^{k}}{\partial q^{\lambda}} \bar{u}_{j \mid k} \mathbf{g}^{j},
\end{align*}
$$

where $\left.\bar{U}^{J}\right|_{K} /\left.\bar{u}^{j}\right|_{k}$ and $\bar{U}_{J \mid K} / \bar{u}_{j \mid k}$ denote the covariant derivatives of the displacement vector in terms of its contravariant and covariant curvilinear coordinates.

$$
\begin{array}{lll}
\left.\bar{U}^{J}\right|_{K}=\frac{\partial \bar{U}^{J}}{\partial Q^{K}}+\bar{U}^{L} \Gamma_{L K}^{J} & \text { versus } & \left.\bar{u}^{j}\right|_{k}=\frac{\partial \bar{u}^{j}}{\partial q^{k}}+\bar{u}^{l} \Gamma_{l k}^{j} \\
\bar{U}_{J \mid K}=\frac{\partial \bar{U}_{J}}{\partial Q^{K}}-\bar{U}_{L} \Gamma_{J K}^{L} & \text { versus } & \bar{u}_{j \mid k}=\frac{\partial \bar{u}_{j}}{\partial q^{k}}-\bar{u}_{l} \Gamma_{j k}^{l} \tag{3-19}
\end{array}
$$

Regarding Equations (3-16) and (3-17), Cauchy-Green and Euler-Lagrange(I) deformation tensors can be expressed as functions of the space curvilinear coordinates of the displacement vector. Box 3-8 summarizes the final results.

Box 3-8: Left versus right Cauchy-Green deformation tensor ( $\boldsymbol{C}_{l}$ versus $\boldsymbol{C}_{r}$ ) and left versus right Euler-Lagrange deformation tensor (I) ( $\boldsymbol{E}_{l}$ versus $\left.\boldsymbol{E}_{r}\right)$ as functions of the space curvilinear coordinates of the displacement vector $\bar{U}^{K} / \bar{u}^{k}$ and $\bar{U}_{K} / \bar{u}_{k}$

Lagrangian portray
Coordinates of the left first fundamental tensor:

$$
A_{\Lambda \Theta}=\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} G_{K J}
$$

## Coordinates of the left Cauchy-Green deformation tensor:

$$
\begin{aligned}
C_{\Lambda \Theta} & =\frac{\partial Q^{K}}{\partial Q^{\Lambda}}\left(\left.\bar{U}^{J}\right|_{K} \bar{U}_{J \mid L}+\bar{U}_{K \mid L}+\bar{U}_{L \mid K}+G_{K L}\right) \frac{\partial Q^{L}}{\partial Q^{\Theta}} \\
& =\frac{\partial Q^{K}}{\partial Q^{\Lambda}}\left(\left.\bar{U}^{J}\right|_{K} \bar{U}_{J \mid L}+\bar{U}_{K \mid L}+\bar{U}_{L \mid K}\right) \frac{\partial Q^{L}}{\partial Q^{\Theta}}+A_{\Lambda \Theta}
\end{aligned}
$$

Eulerian portray
Coordinates of the right first fundamental tensor:

$$
a_{\lambda \theta}=\frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{j}}{\partial q^{\theta}} g_{k j}
$$

## Coordinates of the right Cauchy-Green deformation tensor:

$$
\begin{aligned}
c_{\lambda \theta} & =\frac{\partial q^{k}}{\partial q^{\lambda}}\left(\left.\bar{u}^{j}\right|_{k} \bar{u}_{j \mid l}-\bar{u}_{k \mid l}-\bar{u}_{l \mid k}+g_{k l}\right) \frac{\partial q^{l}}{\partial q^{\theta}} \\
& =\frac{\partial q^{k}}{\partial q^{\lambda}}\left(\left.\bar{u}^{j}\right|_{k} \bar{u}_{j \mid l}-\bar{u}_{k \mid l}-\bar{u}_{l \mid k}\right) \frac{\partial q^{l}}{\partial q^{\theta}}+a_{\lambda \theta}
\end{aligned}
$$

Box 3-8(Contd.): Left versus right Cauchy-Green deformation tensor ( $\boldsymbol{C}_{l}$ versus $\boldsymbol{C}_{r}$ ) and left versus right Euler-Lagrange deformation tensor (I) ( $\boldsymbol{E}_{l}$ versus $\boldsymbol{E}_{r}$ ) as functions of the space curvilinear coordinates of the displacement vector $\bar{U}^{K} / \bar{u}^{k}$ and $\bar{U}_{K} / \bar{u}_{k}$

Coordinates of the left Euler-Lagrange deformation tensor(I):

## Coordinates of the right Euler-Lagrange deformation tensor(I):

$$
\begin{aligned}
E_{\Lambda \Theta} & =\frac{1}{2}\left(C_{\Lambda \Theta}-A_{\Lambda \Theta}\right) \\
& =\frac{1}{2} \frac{\partial Q^{K}}{\partial Q^{\Lambda}}\left(\bar{U}_{K \mid L}+\bar{U}_{L \mid K}+\left.\bar{U}^{J}\right|_{K} \bar{U}_{J \mid L}\right) \frac{\partial Q^{L}}{\partial Q^{\Theta}}
\end{aligned}
$$

$$
e_{\lambda \theta}=\frac{1}{2}\left(a_{\lambda \theta}-c_{\lambda \theta}\right)
$$

$$
=\frac{1}{2} \frac{\partial q^{k}}{\partial q^{\lambda}}\left(\bar{u}_{k \mid l}+\bar{u}_{l \mid k}-\left.\bar{u}^{j}\right|_{k} \bar{u}_{j \mid l}\right) \frac{\partial q^{l}}{\partial q^{\theta}}
$$

The partial derivatives $\frac{\partial Q^{K}}{\partial Q^{\Lambda}} / \frac{\partial q^{k}}{\partial q^{\lambda}}$ are the coordinates of the Jacobi matrices of the coordinate transformations from surface curvilinear coordinates $Q^{\Lambda} / q^{\lambda}$ to the space curvilinear coordinates $Q^{K} / q^{k}$.

To complete this section, Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) has to be evaluated in terms of the space curvilinear coordinates of the displacement vector and the difference vector of the unit normal vectors. The partial derivatives of $\mathbf{w}$ with respect to the surface curvilinear coordinates $Q^{\Lambda} / q^{\lambda}$ can be written in terms of covariant derivatives of contravariant and covariant space curvilinear coordinates of $\mathbf{w}$ as

$$
\begin{align*}
\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}} & =\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial \mathbf{w}}{\partial Q^{K}} & \frac{\partial \mathbf{w}}{\partial q^{\lambda}} & =\frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial \mathbf{w}}{\partial q^{k}}  \tag{3-20}\\
& =\left.\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \bar{W}^{J}\right|_{K} \mathbf{G}_{J}=\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \bar{W}_{J \mid K} \mathbf{G}^{J} & & =\left.\frac{\partial q^{k}}{\partial q^{\lambda}} \bar{w}^{j}\right|_{k} \mathbf{g}_{j}=\frac{\partial q^{k}}{\partial q^{\lambda}} \bar{w}_{j \mid k} \mathbf{g}^{j}
\end{align*}
$$

Final results of the formulations have been collected in Box 3-9.

Box 3-9: Left versus right Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) as functions of the space curvilinear coordinates of the displacement vector $\mathbf{u}$ and the difference vector of the unit normal vectors w

Lagrangian portray

Coordinates of the left Euler-Lagrange deformation tensor(II):

$$
\begin{array}{cc}
\text { Lagrangian portray } & \begin{array}{c}
\text { Eulerian portray } \\
\text { rdinates of the left Euler-Lagrange } \\
\text { rmation tensor(II): }
\end{array} \\
\begin{array}{cc}
K_{\Lambda \Theta}=-<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>-<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}>- & \begin{array}{l}
\text { Coordinates of the right Euler-Lagrange } \\
\text { deformation tensor(II): }
\end{array} \\
<\frac{k_{\lambda \theta}=}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}> & <\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>+<\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}>- \\
=-\frac{\partial Q^{I}}{\partial Q^{\Lambda}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} \bar{W}_{P \mid I}<\mathbf{G}^{P}, \mathbf{G}_{J}>- & =\frac{\partial \mathbf{n}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}> \\
\left.\frac{\partial Q^{I}}{\partial Q^{\Lambda}} \frac{\partial Q^{K}}{\partial Q^{\Theta}} \bar{W}_{P \mid I} \bar{U}^{M}\right|_{K}<\mathbf{G}^{P}, \mathbf{G}_{M}>+ & \frac{\partial q^{j}}{\partial q^{\theta}} \bar{w}_{p \mid i}<\mathbf{g}^{p}, \mathbf{g}_{j}>+ \\
\left.\frac{\partial q^{i}}{\partial q^{\lambda}} \frac{\partial q^{k}}{\partial q^{\theta}} \bar{w}_{p \mid i} \bar{u}^{m}\right|_{k}<\mathbf{g}^{p}, \mathbf{g}_{m}>+
\end{array}
\end{array}
$$

Box 3-9(Contd.):Left versus right Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) as functions of the space curvilinear coordinates of the displacement vector $\mathbf{u}$ and the difference vector of the unit normal vectors $\mathbf{w}$

$$
\begin{array}{rlr}
B_{\Lambda}^{\Phi} \frac{\partial Q^{L}}{\partial Q^{\Phi}} \frac{\partial Q^{K}}{\partial Q^{\Theta}} \bar{U}_{M \mid K}<\mathbf{G}_{L}, \mathbf{G}^{M}> & b_{\lambda}^{\phi} \frac{\partial q^{l}}{\partial q^{\phi}} \frac{\partial q^{k}}{\partial q^{\theta}} \bar{u}_{m \mid k}<\mathbf{g}_{l}, \mathbf{g}^{m}> \\
=B_{\Lambda}^{\Phi} \frac{\partial Q^{I}}{\partial Q^{\Phi}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} \bar{U}_{I \mid J}-\frac{\partial Q^{I}}{\partial Q^{\Lambda}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} \bar{W}_{J \mid I}- & =b_{\lambda}^{\phi} \frac{\partial q^{i}}{\partial q^{\phi}} \frac{\partial q^{j}}{\partial q^{\theta}} \bar{u}_{i \mid j}-\frac{\partial q^{i}}{\partial q^{\lambda}} \frac{\partial q^{j}}{\partial q^{\theta}} \bar{w}_{j \mid i}+ \\
& \left.\frac{\partial Q^{I}}{\partial Q^{\Lambda}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} \bar{W}_{K \mid I} \bar{U}^{K}\right|_{J} & \left.\frac{\partial q^{i}}{\partial q^{\lambda}} \frac{\partial q^{j}}{\partial q^{\theta}} \bar{w}_{k \mid i} \bar{u}^{k}\right|_{j}
\end{array}
$$

### 3.3.3 Surface deformation measures (Intrinsic approach)

In the previous section, surface deformation analysis was presented based on the extrinsic approach where all the surface deformation measures were investigated and formulated from the viewpoint of the embedding spaces. Unlike the extrinsic approach, the intrinsic approach formulates the surface deformation measures in connection to the geometry of the deforming surface in its own right. In the intrinsic approach, the surface deformation measures are written alternatively in terms of surface curvilinear coordinates of the displacement vector and the difference vector of the unit surface normal vectors. The surface curvilinear coordinates of a vector are obtained by the decomposition of the vector in the Lagrangian and Eulerian Gaussian surface moving frames. The Gaussian surface moving frames are built by the triads of the two tangent base vectors $\mathbf{A}^{\Lambda} / \mathbf{a}^{\lambda}$ and the unit normal vector $\mathbf{N} / \mathbf{n}$, with

$$
\begin{equation*}
\mathbf{N}:=\frac{\mathbf{A}_{1} \times \mathbf{A}_{2}}{\left\|\mathbf{A}_{1} \times \mathbf{A}_{2}\right\|}=N^{K} \mathbf{I}_{K} \quad \text { versus } \quad \mathbf{n}:=\frac{\mathbf{a}_{1} \times \mathbf{a}_{2}}{\left\|\mathbf{a}_{1} \times \mathbf{a}_{2}\right\|}=n^{k} \mathbf{i}_{k} \tag{3-21}
\end{equation*}
$$

being the third base vectors of the frames, at every point on the reference- and current surface.


Figure 3-4: The Gaussian surface moving frames of the reference- and current surface
In such a case, the displacement vector can be decomposed in the Lagrangian and Eulerian three-dimensional surface moving frames as
$\mathbf{u}=\bar{U}^{\Lambda} \mathbf{A}_{\Lambda}+\bar{U}^{3} \mathbf{N}$
$\mathbf{u}=\bar{U}_{\Lambda} \mathbf{A}^{\Lambda}+\bar{U}^{3} \mathbf{N}$
$\mathbf{u}=\bar{u}^{\lambda} \mathbf{a}_{\lambda}+\bar{u}^{3} \mathbf{n}$
versus
$\mathbf{u}=\bar{u}_{\lambda} \mathbf{a}^{\lambda}+\bar{u}^{3} \mathbf{n}$

The contravariant coordinates $\bar{U}^{\Lambda} / \bar{u}^{\lambda}$ and $\bar{U}^{3} / \bar{u}^{3}$ or their covariant counterparts $\bar{U}_{\Lambda} / \bar{u}_{\lambda}$ and $\bar{U}_{3} / \bar{u}_{3}$ are called surface curvilinear coordinates of the displacement vector. It should be noted that because of the normality of the unit normal vector $\mathbf{N} / \mathbf{n}$ and its orthogonality to the tangent base vectors, the unit normal vector and its reciprocal vector are equal and consequently there is no difference between the contravariant and covariant coordinates of $\mathbf{u}$ with respect to $\mathbf{N} / \mathbf{n}$, i.e. $\bar{U}^{3}=\bar{U}_{3}$ and $\bar{u}^{3}=\bar{u}_{3}$.

Considering the above decomposition of the displacement vector, Equation (3-22), the partial derivatives of the vector with respect to the surface coordinates $Q^{\Lambda} / q^{\lambda}$ can be given as follows

$$
\left.\left.\begin{array}{rlrl}
\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}} & =\frac{\partial\left(\bar{U}^{\Theta} \mathbf{A}_{\Theta}\right)}{\partial Q^{\Lambda}}+\frac{\partial\left(\bar{U}^{3} \mathbf{N}\right)}{\partial Q^{\Lambda}} & \text { versus } & \frac{\partial \mathbf{u}}{\partial q^{\lambda}}
\end{array}\right)=\frac{\partial\left(\bar{u}^{\theta} \mathbf{a}_{\theta}\right)}{\partial q^{\lambda}}+\frac{\partial\left(\bar{u}^{3} \mathbf{n}\right)}{\partial q^{\lambda}}\right)
$$

where the surface tensors $U_{\Lambda}^{\Theta} / u_{\lambda}^{\theta}$ and $U_{\Lambda}^{3} / u_{\lambda}^{3}$ are considered to have the form of,

$$
\begin{equation*}
U_{\Lambda}^{\Theta}:=\left.\bar{U}^{\Theta}\right|_{\Lambda}-\bar{U}^{3} B_{\Lambda}^{\Theta} \quad \text { versus } \quad u_{\lambda}^{\theta}:=\left.\bar{u}^{\theta}\right|_{\lambda}-\bar{u}^{3} b_{\lambda}^{\theta}, \tag{3-24}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{\Lambda}^{3}:=\frac{\partial \bar{U}^{3}}{\partial Q^{\Lambda}}+\bar{U}^{\Theta} B_{\Lambda \Theta} \quad \text { versus } \quad u_{\lambda}^{3}:=\frac{\partial \bar{u}^{3}}{\partial q^{\lambda}}+\bar{u}^{\theta} b_{\lambda \theta} \tag{3-25}
\end{equation*}
$$

To achieve the above results for partial derivatives of $\mathbf{u}$, we have taken into account the Gauss-Weingarten equations introduced in chapter 2,

$$
\begin{array}{rlr}
\frac{\partial \mathbf{A}_{\Theta}}{\partial Q^{\Lambda}}:=\Gamma_{\Lambda \Theta}^{\Phi} \mathbf{A}_{\Phi}+B_{\Lambda \Theta} \mathbf{N} & \text { versus } & \frac{\partial \mathbf{a}_{\theta}}{\partial q^{\lambda}}:=\Gamma_{\lambda \theta}^{\phi} \mathbf{a}_{\phi}+b_{\lambda \theta} \mathbf{n} \\
\frac{\partial \mathbf{N}}{\partial Q^{\Lambda}}:=-B_{\Lambda \Theta} \mathbf{A}^{\Theta}=-B_{\Lambda}^{\Theta} \mathbf{A}_{\Theta} & \text { versus } & \frac{\partial \mathbf{n}}{\partial q^{\lambda}}:=-b_{\lambda \theta} \mathbf{a}^{\theta}=-b_{\lambda}^{\theta} \mathbf{a}_{\theta} . \tag{3-27}
\end{array}
$$

The surface covariant derivatives of the surface curvilinear contravariant coordinates of the displacement vector are given as

$$
\begin{equation*}
\left.\bar{U}^{\Lambda}\right|_{\Theta}:=\bar{U}_{, \Theta}^{\Lambda}+\Gamma_{\Theta \Phi}^{\Lambda} \bar{U}^{\Phi} \quad \text { versus }\left.\quad \bar{u}^{\lambda}\right|_{\theta}:=\bar{u}_{, \theta}^{\lambda}+\Gamma_{\theta \phi}^{\lambda} \bar{u}^{\phi} \tag{3-28}
\end{equation*}
$$

With similar computations, it can be also proved that

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}=U_{\Theta \Lambda} \mathbf{A}^{\Theta}+U_{3 \Lambda} \mathbf{N} \quad \text { versus } \quad \frac{\partial \mathbf{u}}{\partial q^{\lambda}}=u_{\theta \lambda} \mathbf{a}^{\theta}+u_{3 \lambda} \mathbf{n} \tag{3-29}
\end{equation*}
$$

where

$$
\begin{equation*}
U_{\Theta \Lambda}:=\bar{U}_{\Theta \mid \Lambda}-\bar{U}^{3} B_{\Theta \Lambda} \quad \text { versus } \quad u_{\theta \lambda}:=\bar{u}_{\theta \mid \lambda}-\bar{u}^{3} b_{\theta \lambda} \tag{3-30}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{3 \Lambda}=U_{\Lambda}^{3} \quad \text { versus } \quad u_{3 \lambda}=u_{\lambda}^{3} \tag{3-31}
\end{equation*}
$$

Now, we can derive expressions of the Cauchy-Green and Euler-Lagrange(I) deformation tensors as functions of the surface curvilinear coordinates of the displacement vector. Box 3-10 highlights the main steps of the computations.

Box 3-10: Left versus right Cauchy-Green deformation tensor ( $\boldsymbol{C}_{l}$ versus $\boldsymbol{C}_{r}$ ) and left versus right Euler-Lagrange deformation tensor(I) ( $\boldsymbol{E}_{l}$ versus $\boldsymbol{E}_{r}$ ) as functions of the surface curvilinear coordinates of the displacement vector

## Lagrangian portray

## Coordinates of the left Cauchy-Green

 deformation tensor:$$
C_{\Lambda \Theta}=<\left(\mathbf{u}_{, \Lambda}+\mathbf{A}_{\Lambda}\right),\left(\mathbf{u}_{, \Theta}+\mathbf{A}_{\Theta}\right)>
$$

$$
=<\left(U_{\Phi \Lambda} \mathbf{A}^{\Phi}+U_{3 \Lambda} \mathbf{N}+\mathbf{A}_{\Lambda}\right),\left(U_{\Psi \Theta} \mathbf{A}^{\Psi}+\right.
$$

$$
\left.U_{3 \Theta} \mathbf{N}+\mathbf{A}_{\Theta}\right)>
$$

$$
=U_{\Phi \Lambda} U_{\Psi \Theta} A^{\Phi \Psi}+U_{\Phi \Lambda} \delta_{\Theta}^{\Phi}+U_{\Psi \Theta} \delta_{\Lambda}^{\Psi}+
$$

$$
U_{3 \Lambda} U_{3 \Theta}+A_{\Lambda \Theta}
$$

$$
=U_{\Lambda}^{\Psi} U_{\Psi \Theta}+U_{\Theta \Lambda}+U_{\Lambda \Theta}+U_{3 \Lambda} U_{3 \Theta}+A_{\Lambda \Theta}
$$

Coordinates of the left Euler-Lagrange deformation tensor:

$$
\begin{aligned}
E_{\Lambda \Theta} & =\frac{1}{2}\left(C_{\Lambda \Theta}-A_{\Lambda \Theta}\right) \\
& =\frac{1}{2}\left(U_{\Lambda \Theta}+U_{\Theta \Lambda}+U_{\Lambda}^{\Psi} U_{\Psi \Theta}+U_{3 \Lambda} U_{3 \Theta}\right)
\end{aligned}
$$

Eulerian portray

## Coordinates of the right Cauchy-Green deformation tensor:

$$
\begin{aligned}
c_{\lambda \theta}= & <\left(\mathbf{a}_{\lambda}-\mathbf{u}_{, \lambda}\right),\left(\mathbf{a}_{\theta}-\mathbf{u}_{, \theta}\right)> \\
= & <\left(\mathbf{a}_{\lambda}-u_{\phi \lambda} \mathbf{a}^{\phi}-u_{3 \lambda} \mathbf{n}\right),\left(\mathbf{a}_{\theta}-\right. \\
& \left.u_{\psi \theta} \mathbf{a}^{\psi}-u_{3 \theta} \mathbf{n}\right)>
\end{aligned}
$$

$$
=u_{\phi \lambda} u_{\psi \theta} a^{\phi \psi}-u_{\phi \lambda} \delta_{\theta}^{\phi}-u_{\psi \theta} \delta_{\lambda}^{\psi}+
$$

$$
u_{3 \lambda} u_{3 \theta}+a_{\lambda \theta}
$$

$$
=u_{\lambda}^{\psi} u_{\psi \theta}-u_{\theta \lambda}-u_{\lambda \theta}+u_{3 \lambda} u_{3 \theta}+a_{\lambda \theta}
$$

## Coordinates of the right Euler-Lagrange deformation tensor:

$$
\begin{aligned}
e_{\lambda \theta} & =\frac{1}{2}\left(a_{\lambda \theta}-c_{\lambda \theta}\right) \\
& =\frac{1}{2}\left(u_{\lambda \theta}+u_{\theta \lambda}-u_{\lambda}^{\psi} u_{\psi \theta}-u_{3 \lambda} u_{3 \theta}\right)
\end{aligned}
$$

Again, we have to consider the surface curvilinear coordinates of the difference vector of the unit normal vectors in addition to the surface curvilinear coordinates of the displacement vector to obtain expressions of the left and right Euler-Lagrange deformation tensors of the second kind (tensors of change of curvature). The surface curvilinear coordinates of $\mathbf{w}$ of contravariant and covariant types are defined by

$$
\begin{array}{rlrl}
\mathbf{w} & :=\bar{W}^{\Lambda} \mathbf{A}_{\Lambda}+\bar{W}^{3} \mathbf{N} & \mathbf{w} & :=\bar{w}^{\lambda} \mathbf{a}_{\lambda}+\bar{w}^{3} \mathbf{n} \\
& =\bar{W}_{\Lambda} \mathbf{A}^{\Lambda}+\bar{W}^{3} \mathbf{N} & &  \tag{3-32}\\
& & \bar{w}_{\lambda} \mathbf{a}^{\lambda}+\bar{w}^{3} \mathbf{n}
\end{array}
$$

The coordinates of the left and right Euler-Lagrange deformation tensors of the second kind (tensor of change of curvature) $\boldsymbol{K}_{l} / \boldsymbol{K}_{r}$ can now be evaluated as functions of the surface curvilinear coordinates $\left(\bar{W}_{\Lambda}, \bar{W}^{3}\right) /\left(\bar{w}_{\Lambda}, \bar{w}^{3}\right)$ and the surface curvilinear coordinates of the displacement vector, Box 3-11.

Box 3-11: Left versus right Euler-Lagrange deformation tensor of the second kind (Lagrangian versus Eulerian tensor of change of curvature) in terms of the surface curvilinear coordinates of the displacement vector $\mathbf{u}$ and the difference vector of the unit normal vectors $\mathbf{w}$

Lagrangian portray
Eulerian portray

Coordinates of the left Euler-Lagrange deformation tensor(II):

$$
\begin{array}{rlrl}
K_{\Lambda \Theta}= & -<\left(\frac{\partial \mathbf{w}}{\partial Q^{\Theta}}+\frac{\partial \mathbf{N}}{\partial Q^{\Theta}}\right),\left(\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}}+\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}\right)>-B_{\Lambda \Theta} & k_{\lambda \theta}=b_{\lambda \theta}-<\left(\frac{\partial \mathbf{w}}{\partial q^{\theta}}-\frac{\partial \mathbf{n}}{\partial q^{\theta}}\right),\left(\frac{\partial \mathbf{x}}{\partial q^{\lambda}}-\frac{\partial \mathbf{u}}{\partial q^{\lambda}}\right)> \\
= & -<\left(W_{\Phi \Theta} \mathbf{A}^{\Phi}+W_{3 \Theta} \mathbf{N}+\frac{\partial \mathbf{N}}{\partial Q^{\Theta}}\right),\left(\mathbf{A}_{\Lambda}+\right. & & =b_{\lambda \theta}-<\left(w_{\phi \theta} \mathbf{a}^{\phi}+w_{3 \theta} \mathbf{n}-\frac{\partial \mathbf{n}}{\partial q^{\theta}}\right),\left(\mathbf{a}_{\lambda}-\right. \\
& \left.U_{\Psi \Lambda} \mathbf{A}^{\Psi}+U_{3 \Lambda} \mathbf{N}\right)>-B_{\Lambda \Theta} & & \left.u_{\psi \lambda} \mathbf{a}^{\psi}-u_{3 \lambda} \mathbf{n}\right) \\
= & B_{\Theta}^{\Psi} U_{\Psi \Lambda}-W_{\Lambda \Theta}-W_{\Theta}^{\Psi} U_{\Psi \Lambda}-W_{3 \Theta} U_{3 \Lambda} & & =b_{\theta}^{\psi} u_{\psi \lambda}-w_{\lambda \theta}+w_{\theta}^{\psi} u_{\psi \lambda}+w_{3 \theta} u_{3 \lambda}
\end{array}
$$

Coordinates of the right Euler-Lagrange deformation tensor(II):
where

$$
\begin{array}{ll}
W_{\Lambda}^{\Theta}:=\left.\bar{W}^{\Theta}\right|_{\Lambda}-B_{\Lambda}^{\Theta} \bar{W}^{3} & w_{\lambda}^{\theta}:=\left.\bar{w}^{\theta}\right|_{\lambda}-b_{\lambda}^{\theta} \bar{w}^{3} \\
W_{\Lambda \Theta}:=\bar{W}_{\Lambda \mid \Theta}-B_{\Theta \Lambda} \bar{W}^{3} & w_{\lambda \theta}:=\bar{w}_{\lambda \mid \theta}-b_{\theta \lambda} \bar{w}^{3} \\
W_{3 \Lambda}:=B_{\Lambda}^{\Theta} \bar{W}_{\Theta}+\bar{W}_{, \Lambda}^{3} & w_{3 \lambda}:=b_{\lambda}^{\theta} \bar{w}_{\theta}+\bar{w}_{, \lambda}^{3}
\end{array}
$$

### 3.4 Linearized Theory of Surface Deformation Analysis

All the expressions of the surface deformation tensors derived so far are non-linear formulae referred to the space and surface coordinates of the displacement vector and the difference vector of the unit normal vectors. They form a basis for various approximate theories in different applications of the surface deformation analysis. The linear expressions of the surface deformation tensors in terms of the displacement vector and the difference vector of the unit normal vectors can be determined by dropping the nonlinear terms in the non-linear mathematical relations of the previous section for surface deformation measures, Box 3-6 through Box 3-11. The developed linearized theory of surface deformation analysis performs a theoretical base for the widely used so-called infinitesimal approach in deformation analysis. For more details on the infinitesimal approach in deformation analysis, we refer to A. C. Eringen (1962). In addition to the simplicity of the formulae in linearized theory, another advantage of the approach is the determination of a skew-symmetric tensor, the so-called linearized surface rotation tensor $\tilde{\boldsymbol{R}}_{l} / \tilde{\boldsymbol{R}}_{r}$, which represents the infinitesimal rotation of a small neighborhood of the point in question with respect to its original orientation. Box $3-12$ reviews the final expressions of the linearized Cauchy-Green deformation tensors $\tilde{\boldsymbol{C}}_{l} / \tilde{\boldsymbol{C}}_{r}$ and the linearized Euler-Lagrange deformation tensors of the first kind $\tilde{\boldsymbol{E}}_{l} / \tilde{\boldsymbol{E}}_{r}$ as well as the linearized surface rotation tensors $\tilde{\boldsymbol{R}}_{l} / \tilde{\boldsymbol{R}}_{r}$ in the cases that space Cartesian or space curvilinear coordinates of the displacement vector are known, namely the extrinsic approach.

Box 3-12: The linearized surface deformation tensors $\tilde{\boldsymbol{C}}_{l} / \tilde{\boldsymbol{C}}_{r}$ and $\tilde{\boldsymbol{E}}_{l} / \tilde{\boldsymbol{E}}_{r}$, and the linearized surface rotation tensors $\tilde{\boldsymbol{R}}_{l} / \tilde{\boldsymbol{R}}_{r}$ in the extrinsic approach

$$
\begin{aligned}
& \text { Lagrangian portray } \\
& \text { Eulerian portray } \\
& \text {-Coordinates of the linearized surface deformation tensors } \\
& \text { in terms of the space Cartesian coordinates of } u \\
& \tilde{C}_{\Lambda \Theta}=A_{\Lambda \Theta}+\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial X^{I}}{\partial Q^{\Theta}}+\frac{\partial X^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}} \quad \quad \tilde{c}_{\lambda \theta}=a_{\lambda \theta}-\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial x^{i}}{\partial q^{\theta}}-\frac{\partial x^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}} \\
& \tilde{E}_{\Lambda \Theta}=\frac{1}{2}\left(\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial X^{I}}{\partial Q^{\Theta}}+\frac{\partial X^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}}\right) \quad \quad \tilde{e}_{\lambda \theta}=\frac{1}{2}\left(\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial x^{i}}{\partial q^{\theta}}+\frac{\partial x^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}}\right) \\
& \tilde{R}_{\Lambda \Theta}=\frac{1}{2}\left(\frac{\partial U^{I}}{\partial Q^{\Lambda}} \frac{\partial X^{I}}{\partial Q^{\Theta}}-\frac{\partial X^{I}}{\partial Q^{\Lambda}} \frac{\partial U^{I}}{\partial Q^{\Theta}}\right) \quad \quad \tilde{r}_{\lambda \theta}=\frac{1}{2}\left(\frac{\partial u^{i}}{\partial q^{\lambda}} \frac{\partial x^{i}}{\partial q^{\theta}}-\frac{\partial x^{i}}{\partial q^{\lambda}} \frac{\partial u^{i}}{\partial q^{\theta}}\right) \\
& \text {-Coordinates of the linearized surface deformation tensors } \\
& \text { in terms of the space curvilinear coordinates of } u \\
& \tilde{C}_{\Lambda \Theta}=A_{\Lambda \Theta}+\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{L}}{\partial Q^{\Theta}}\left(\bar{U}_{K \mid L}+\bar{U}_{L \mid K}\right) \\
& \tilde{c}_{\lambda \theta}=a_{\lambda \theta}-\frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{l}}{\partial q^{\theta}}\left(\bar{u}_{k \mid l}+\bar{u}_{l \mid k}\right) \\
& =A_{\Lambda \Theta}+\frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{L}}{\partial Q^{\Theta}}\left(\frac{\partial \bar{U}_{K}}{\partial Q^{L}}+\frac{\partial \bar{U}_{L}}{\partial Q^{K}}-\right. \\
& =a_{\lambda \theta}-\frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{l}}{\partial q^{\theta}}\left(\frac{\partial \overline{u_{k}}}{\partial q^{l}}+\frac{\partial \bar{u}_{l}}{\partial q^{k}}-\right. \\
& \left.2 \Gamma_{K L}^{M} \bar{U}_{M}\right) \\
& \left.2 \Gamma_{k l}^{m} \bar{u}_{m}\right)
\end{aligned}
$$

Box 3-12(Contd.): The linearized surface deformation tensors $\tilde{\boldsymbol{C}}_{l} / \tilde{\boldsymbol{C}}_{r}$ and $\tilde{\boldsymbol{E}}_{l} / \tilde{\boldsymbol{E}}_{r}$, and the linearized surface rotation tensors $\tilde{\boldsymbol{R}}_{l} / \tilde{\boldsymbol{R}}_{r}$ in the extrinsic approach

$$
\begin{array}{rlrl}
\tilde{E}_{\Lambda \Theta}= & \frac{1}{2} \frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{L}}{\partial Q^{\Theta}}\left(\bar{U}_{K \mid L}+\bar{U}_{L \mid K}\right) & \tilde{e}_{\lambda \theta} & =\frac{1}{2} \frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{l}}{\partial q^{\theta}}\left(\bar{u}_{k \mid l}+\bar{u}_{l \mid k}\right) \\
= & \frac{1}{2} \frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{L}}{\partial Q^{\Theta}}\left(\frac{\partial \bar{U}_{K}}{\partial Q^{L}}+\frac{\partial \bar{U}_{L}}{\partial Q^{K}}-\right. & & \frac{1}{2} \frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{l}}{\partial q^{\theta}}\left(\frac{\partial \overline{u_{k}}}{\partial q^{l}}+\frac{\partial \bar{u}_{l}}{\partial q^{k}}-\right. \\
& \left.2 \Gamma_{K L}^{M} \bar{U}_{M}\right) & \left.2 \Gamma_{k l}^{m} \bar{u}_{m}\right) \\
\tilde{R}_{\Lambda \Theta}= & \frac{1}{2} \frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{L}}{\partial Q^{\Theta}}\left(\bar{U}_{K \mid L}-\bar{U}_{L \mid K}\right) & \tilde{r}_{\lambda \theta} & =\frac{1}{2} \frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{l}}{\partial q^{\theta}}\left(\bar{u}_{k \mid l}-\bar{u}_{l \mid k}\right) \\
= & \frac{1}{2} \frac{\partial Q^{K}}{\partial Q^{\Lambda}} \frac{\partial Q^{L}}{\partial Q^{\Theta}}\left(\frac{\partial \bar{U}_{K}}{\partial Q^{L}}-\frac{\partial \bar{U}_{L}}{\partial Q^{K}}\right) & & =\frac{1}{2} \frac{\partial q^{k}}{\partial q^{\lambda}} \frac{\partial q^{l}}{\partial q^{\theta}}\left(\frac{\partial \overline{u_{k}}}{\partial q^{l}}-\frac{\partial \bar{u}_{l}}{\partial q^{k}}\right)
\end{array}
$$

Box 3-13 presents expressions of the above linearized tensors in the intrinsic approach where the surface curvilinear coordinates of the displacement vector are known. In this case, the linearized deformation tensors will be linear functions of the surface coordinates $\bar{U}^{\Lambda} / \bar{u}^{\lambda}$ and $\bar{U}^{3} / \bar{u}^{3}$. Note the simple expressions of the linearized Euler-Lagrange deformation tensors of the first kind and particularly the linearized rotation tensors in the intrinsic approach in comparison to the extrinsic approach.

Box 3-13: The linearized surface deformation tensors $\tilde{\boldsymbol{C}}_{l} / \tilde{\boldsymbol{C}}_{r}$ and $\tilde{\boldsymbol{E}}_{l} / \tilde{\boldsymbol{E}}_{r}$, and the linearized surface rotation tensors $\tilde{\boldsymbol{R}}_{l} / \tilde{\boldsymbol{R}}_{r}$ in the intrinsic approach

$$
\begin{array}{rlrl} 
& \text { Lagrangian portray } & & \text { Eulerian portray } \\
\tilde{C}_{\Lambda \Theta}=A_{\Lambda \Theta}+U_{\Lambda \Theta}+U_{\Theta \Lambda} & & \tilde{c}_{\lambda \theta} & =a_{\lambda \theta}-u_{\lambda \theta}-u_{\theta \lambda} \\
= & A_{\Lambda \Theta}+\bar{U}_{\Lambda \mid \Theta}+\bar{U}_{\Theta \mid \Lambda}-2 B_{\Lambda \Theta} \bar{U}^{3} & & a_{\lambda \theta}-\bar{u}_{\lambda \mid \theta}-\bar{u}_{\theta \mid \lambda}+2 b_{\lambda \theta} \bar{u}^{3} \\
= & A_{\Lambda \Theta}+\frac{\partial \bar{U}_{\Lambda}}{\partial Q^{\Theta}}+\frac{\partial \bar{U}_{\Theta}}{\partial Q^{\Lambda}}-2 \Gamma_{\Lambda \Theta}^{\Phi} \bar{U}_{\Phi}- & & a_{\lambda \theta}-\frac{\partial \bar{u}_{\lambda}}{\partial q^{\theta}}-\frac{\partial \bar{u}_{\theta}}{\partial q^{\lambda}}+2 \Gamma_{\lambda \theta}^{\phi} \bar{u}_{\phi}+ \\
& 2 B_{\Lambda \Theta} \bar{U}^{3} & & =\frac{1}{2}\left(a_{\lambda \theta}-\tilde{c}_{\lambda \theta}\right) \\
\tilde{E}_{\Lambda \Theta}\left(\frac{\partial \bar{u}_{\lambda}}{\partial q^{\theta}}+\frac{\partial \bar{u}_{\theta}}{\partial q^{\lambda}}\right)-\left(\Gamma_{\lambda \theta}^{\phi} \bar{u}_{\phi}+\right. \\
=\frac{1}{2}\left(\tilde{C}_{\Lambda \Theta}-A_{\Lambda \Theta}\right) & \tilde{e}_{\lambda \theta}\left(\frac{\partial \bar{U}_{\Lambda}}{\partial Q^{\Theta}}+\frac{\partial \bar{U}_{\Theta}}{\partial Q^{\Lambda}}\right)-\left(\Gamma_{\Lambda \Theta}^{\Phi} \bar{U}_{\Phi}+\right. & & \left.b_{\lambda \theta} \bar{u}^{3}\right) \\
& \left.B_{\Lambda \Theta} \bar{U}^{3}\right) & & \frac{1}{2}\left(\bar{u}_{\lambda \mid \theta}-\bar{u}_{\theta \mid \lambda}\right) \\
\tilde{R}_{\Lambda \Theta}=\frac{1}{2}\left(\bar{U}_{\Lambda \mid \Theta}-\bar{U}_{\Theta \mid \Lambda}\right) & & \frac{1}{2}\left(\frac{\partial \bar{u}_{\lambda}}{\partial q^{\theta}}-\frac{\partial \bar{u}_{\theta}}{\partial q^{\lambda}}\right)
\end{array}
$$

We can also define the linearized Euler-Lagrange deformation tensors of the second kind (linearized tensors of change of curvature by dropping nonlinear terms in relations of the tensors in Box 3-7 and 3-9. The linearization can be done nicely due to expressions of the tensors in terms of the coordinates of the difference vector of the unit normal vectors in addition to the coordinates of the displacement vector. This should be considered as another main advantage of the use of the difference vector $\mathbf{w}$ in the expressions of the Euler-Lagrange deformation tensors of the second kind. Box 3-14 includes the final results of the linearization of the left versus right EulerLagrange deformation tensor of the second kind (Lagrangian versus Eulerian tensor of change of curvature) in case that the tensors are functions of the space Cartesian and space curvilinear coordinates of $\mathbf{u}$ and $\mathbf{w}$, namely in the extrinsic approach.

Box 3-14: Left versus right linearized Euler-Lagrange deformation tensor of the second kind (Lagrangian versus Eulerian linearized tensor of change of curvature) in the extrinsic approach

## Lagrangian portray

Left linearized Euler-Lagrange deformation tensor(II) as function of the space Cartesian coordinates of $\mathbf{u}$ and $\mathbf{w}$ :

$$
\begin{aligned}
\tilde{K}_{\Lambda \Theta} & =-<\frac{\partial \mathbf{w}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{X}}{\partial Q^{\Theta}}>-<\frac{\partial \mathbf{N}}{\partial Q^{\Lambda}}, \frac{\partial \mathbf{u}}{\partial Q^{\Theta}}> \\
& =B_{\Lambda}^{\Phi} \frac{\partial X^{K}}{\partial Q^{\Phi}} \frac{\partial U^{K}}{\partial Q^{\Theta}}-\frac{\partial W^{K}}{\partial Q^{\Lambda}} \frac{\partial X^{K}}{\partial Q^{\Theta}}
\end{aligned}
$$

Left linearized Euler-Lagrange deformation tensor(II) as function of the space curvilinear coordinates of $\mathbf{u}$ and $\mathbf{w}$ :

$$
\tilde{K}_{\Lambda \Theta}=B_{\Lambda}^{\Phi} \frac{\partial Q^{I}}{\partial Q^{\Phi}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} \bar{U}_{I \mid J}-\frac{\partial Q^{I}}{\partial Q^{\Lambda}} \frac{\partial Q^{J}}{\partial Q^{\Theta}} \bar{W}_{J \mid I}
$$

Eulerian portray

Right linearized Euler-Lagrange deformation tensor(II) as function of the space Cartesian coordinates of $\mathbf{u}$ and $\mathbf{w}$ :

$$
\begin{aligned}
\tilde{k}_{\lambda \theta} & =-<\frac{\partial \mathbf{w}}{\partial q^{\lambda}}, \frac{\partial \mathbf{x}}{\partial q^{\theta}}>-<\frac{\partial \mathbf{n}}{\partial q^{\lambda}}, \frac{\partial \mathbf{u}}{\partial q^{\theta}}> \\
& =b_{\lambda}^{\phi} \frac{\partial x^{k}}{\partial q^{\phi}} \frac{\partial u^{k}}{\partial q^{\theta}}-\frac{\partial w^{k}}{\partial q^{\lambda}} \frac{\partial x^{k}}{\partial q^{\theta}}
\end{aligned}
$$

Right linearized Euler-Lagrange deformation tensor(II) as function of the space curvilinear coordinates of $\mathbf{u}$ and $\mathbf{w}$ :

$$
\tilde{k}_{\lambda \theta}=b_{\lambda}^{\phi} \frac{\partial q^{i}}{\partial q^{\phi}} \frac{\partial q^{j}}{\partial q^{\theta}} \bar{u}_{i \mid j}-\frac{\partial q^{i}}{\partial q^{\lambda}} \frac{\partial q^{j}}{\partial q^{\theta}} \bar{w}_{j \mid i}
$$

We complete this section by introducing the Left and right linearized Euler-Lagrange deformation tensor of the second kind in case of the intrinsic approach. The relations of Box 3-15 show efficiency of the intrinsic approach in leading us to the compact and simplified relations of the tensors after linearization with respect to the surface curvilinear coordinates of the difference vectors $\mathbf{u}$ and $\mathbf{w}$.

Box 3-15: Left versus right linearized Euler-Lagrange deformation tensor of the second kind (Lagrangian versus Eulerian linearized tensor of change of curvature) in the intrinsic approach


Left linearized Euler-Lagrange deformation tensor(II) as function of the surface curvilinear coordinates of $\mathbf{u}$ and $\mathbf{w}$ :

$$
\begin{aligned}
\tilde{K}_{\Lambda \Theta} & =B_{\Lambda}^{\Psi} U_{\Psi \Theta}-W_{\Lambda \Theta} \\
& =B_{\Lambda}^{\Psi} \bar{U}_{\Psi \mid \Theta}-\bar{W}_{\Lambda \mid \Theta}+\bar{U}^{3}\left(B_{\Lambda \Theta}-B_{\Lambda}^{\Psi} B_{\Psi \Theta}\right)
\end{aligned}
$$

Eulerian portray

Right linearized Euler-Lagrange deformation tensor(II) as function of the surface curvilinear coordinates of $\mathbf{u}$ and $\mathbf{w}$ :

$$
\begin{aligned}
\tilde{k}_{\lambda \theta} & =b_{\lambda}^{\psi} u_{\psi \theta}-w_{\lambda \theta} \\
& =b_{\lambda}^{\psi} \bar{u}_{\psi \mid \theta}-\bar{w}_{\lambda \mid \theta}+\bar{u}^{3}\left(b_{\lambda \theta}-b_{\lambda}^{\psi} b_{\psi \theta}\right)
\end{aligned}
$$

The linearized surface deformation tensors, which are introduced in this section, are the most appropriate deformation measures in infinitesimal approach where

$$
\begin{equation*}
\left\|\frac{\partial \mathbf{u}}{\partial Q^{\Lambda}}\right\| \ll\left\|\frac{\partial \mathbf{X}}{\partial Q^{\Lambda}}\right\| \quad \text { versus } \quad\left\|\frac{\partial \mathbf{u}}{\partial q^{\lambda}}\right\| \ll\left\|\frac{\partial \mathbf{x}}{\partial q^{\lambda}}\right\| \tag{3-33}
\end{equation*}
$$

However, for arbitrarily large displacement, different deformation measures will be needed to describe finite strain and finite rotations of the deforming surface.

### 3.5 Associated Invariants of the Surface Deformation Tensors

The coordinates of the surface deformation tensors introduced in Section 3.4 depend on the surface coordinates and consequently the moving reference frame used for the decomposition of the displacement vector and the difference vector of the unit surface normal vectors. Thus, we have to look for scalar functions of the elements of the surface deformation tensors which are invariant with respect to the change of surface coordinates. Moreover, these associated invariants should have evident physical interpretations to be of any use. We should emphasize here that in contrast to the classical 2-dimensional plane deformation analysis, where for example the coordinates of the strain tensor have direct physical interpretations, we can't proceed in this way because of the curvilinear nature of the methods of the analysis.
E. W. Grafarend (1995) treats the method of general eigenvalue problem for the pair of two second order tensors, one symmetric and one symmetric and positive-definite, to determine eigenvalues associated with the deformation and strain tensors. The general eigenvalue problem is equivalent to the standard problem of the simultaneous diagonalization of two symmetric matrices in matrix algebra. Taking into account the matrix representation of the metric tensors of the reference- and current surface and left and right Cauchy-Green deformation tensors

$$
\begin{array}{rlr}
\boldsymbol{A}_{l}=\left[A_{\Lambda \Theta}\right]_{2 \times 2} & \text { versus } & \boldsymbol{A}_{r}=\left[a_{\lambda \theta}\right]_{2 \times 2} \\
\boldsymbol{C}_{l}=\left[C_{\Lambda \Theta}\right]_{2 \times 2} & \text { versus } & \boldsymbol{C}_{r}=\left[c_{\lambda \theta}\right]_{2 \times 2} \tag{3-35}
\end{array}
$$

for the pair of positive-definite, symmetric matrices $\left\{\boldsymbol{C}_{l}, \boldsymbol{A}_{l}\right\}$ or $\left\{\boldsymbol{C}_{r}, \boldsymbol{A}_{r}\right\}$ a simultaneous diagonalization is obtained from the general eigenvalue problem. Then, the associated eigenvalues are determined by

$$
\begin{array}{rlrl}
\Lambda_{1,2}^{2}= & \frac{1}{2}\left\{\operatorname{tr}\left(\boldsymbol{C}_{l} \boldsymbol{A}_{l}^{-1}\right) \pm\right. & \lambda_{1,2}^{2} & =\frac{1}{2}\left\{\operatorname{tr}\left(\boldsymbol{C}_{r} \boldsymbol{A}_{r}^{-1}\right) \pm\right. \\
& \left.\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{C}_{l} \boldsymbol{A}_{l}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{C}_{l} \boldsymbol{A}_{l}^{-1}\right)}\right\} & & \text { versus } \\
& & \left.\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{C}_{r} \boldsymbol{A}_{r}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{C}_{r} \boldsymbol{A}_{r}^{-1}\right)}\right\}
\end{array}
$$

where, $t r$ and det denote trace and determinant of the corresponding matrices, respectively. The eigenvalues are usually numbered according to the inequalities $\Lambda_{2}^{2}<\Lambda_{1}^{2}$ and $\lambda_{2}^{2}<\lambda_{1}^{2}$. They are positive real numbers for symmetric, positive-definite second order tensors $\boldsymbol{C}_{l}$ and $\boldsymbol{C}_{r}$. The square roots of the eigenvalues $\Lambda_{1,2}^{2} / \lambda_{1,2}^{2}$ are interpreted physically as the maximum and minimum values of stretch (dilatation factor or length distortion) at the points with the surface curvilinear coordinates $Q^{\Theta} / q^{\theta}$ on the reference- or current surface.

$$
\begin{equation*}
\Lambda^{2}\left(Q^{\Theta}\right):=\frac{d s^{2}}{d S^{2}} \quad \text { versus } \quad \lambda^{2}\left(q^{\theta}\right):=\frac{d S^{2}}{d s^{2}} \tag{3-37}
\end{equation*}
$$

It is a well-known fact in matrix algebra that eigenvalues are invariant quantities independent of the selected coordinate system. Thus, principal stretches are convenient scalar invariants associated with the left and right Cauchy-Green deformation tensors.

With reference to the general eigenvalue problem we experienced for the left and right Cauchy-Green deformation tensors, we arrive at the general eigenvalue problem for the pair of symmetric matrices $\left\{\boldsymbol{E}_{l}, \boldsymbol{A}_{l}\right\}$ or $\left\{\boldsymbol{E}_{r}, \boldsymbol{A}_{r}\right\}$, where

$$
\begin{equation*}
\boldsymbol{E}_{l}=\left[E_{\Lambda \Theta}\right]_{2 \times 2} \quad \text { versus } \quad \boldsymbol{E}_{r}=\left[e_{\lambda \theta}\right]_{2 \times 2} \tag{3-38}
\end{equation*}
$$

The eigenvalues of the left and right Euler-Lagrange deformation tensors of the first kind $\boldsymbol{E}_{l}$ and $\boldsymbol{E}_{r}$ can be determined analogously as

$$
\begin{align*}
\Lambda_{1,2}^{\prime}= & \frac{1}{2}\left\{\operatorname{tr}\left(\boldsymbol{E}_{l} \boldsymbol{A}_{l}^{-1}\right) \pm\right. & \quad \lambda_{1,2}^{\prime} \quad & =\frac{1}{2}\left\{\operatorname{tr}\left(\boldsymbol{E}_{r} \boldsymbol{A}_{r}^{-1}\right) \pm\right. \\
& \left.\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{E}_{l} \boldsymbol{A}_{l}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{E}_{l} \boldsymbol{A}_{l}^{-1}\right)}\right\} & & \left.\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{E}_{r} \boldsymbol{A}_{r}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{E}_{r} \boldsymbol{A}_{r}^{-1}\right)}\right\} . \tag{3-39}
\end{align*}
$$

Unlike the eigenvalues of the Cauchy-Green deformation tensors which are positive due to positive-definite property of the deformation tensors, the eigenvalues of the Euler-Lagrange deformation tensors can be negative or positive. The invariant quantities $\left(\Lambda_{1}^{\prime}, \Lambda_{2}^{\prime}\right) \operatorname{versus}\left(\lambda_{1}^{\prime}, \lambda_{2}^{\prime}\right)$ are called the Lagrangian versus Eulerian principal strains. A deformation portrait with a positive principal strain is referred to as extension, while that with a negative principal strain as compression. Two well-known associated invariants of the left versus right EulerLagrange deformation tensors of the first kind (Lagrangian versus Eulerian strain tensors are defined in terms of principal strains. They are surface dilatation

$$
\begin{equation*}
\Delta:=\Lambda_{1}^{\prime}+\Lambda_{2}^{\prime}=\operatorname{tr}\left(\boldsymbol{E}_{l} \boldsymbol{A}_{l}^{-1}\right) \quad \text { versus } \quad \delta:=\lambda_{1}^{\prime}+\lambda_{2}^{\prime}=\operatorname{tr}\left(\boldsymbol{E}_{r} \boldsymbol{A}_{r}^{-1}\right) \tag{3-40}
\end{equation*}
$$

and surface maximum shear strain

$$
\begin{array}{rlrl}
\Gamma & :=\Lambda_{1}^{\prime}-\Lambda_{2}^{\prime} & & \gamma:=\lambda_{1}^{\prime}-\lambda_{2}^{\prime} \\
& =\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{E}_{l} \boldsymbol{A}_{l}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{E}_{l} \boldsymbol{A}_{l}^{-1}\right)} & & \\
& & =\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{E}_{r} \boldsymbol{A}_{r}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{E}_{r} \boldsymbol{A}_{r}^{-1}\right)} \tag{3-41}
\end{array}
$$

Surface dilatation represents the isotropic part and surface maximum shear strain the anisotropic part of deformation in the infinitesimal vicinity of the point of interest. They are both point functions with the following physical interpretation: $\Delta / \delta$ is the areal change per unit area which is positive for an increase in area, and $\Gamma / \gamma$ is the shear across the direction of its maximum value which has always positive sign. When the linearized Cauchy-Green deformation tensors $\tilde{\boldsymbol{C}}_{l} / \tilde{\boldsymbol{C}}_{r}$ and the the linearized Euler-Lagrange deformation tensors of the first kind $\tilde{\boldsymbol{E}}_{l} / \tilde{\boldsymbol{E}}_{r}$ are considered, their corresponding associated invariants may be determined in the same fashion as the exact tensors.

For the linearized surface rotation tensor $\tilde{\boldsymbol{R}}_{l} / \tilde{\boldsymbol{R}}_{r}$, the associated invariant $\Phi / \phi$, titled linearized rotation around the normal is introduced [W. Pietraszkiewicz (1977)] with

$$
\begin{equation*}
\Phi:=\frac{1}{2} \epsilon^{\Lambda \Theta} \tilde{R}_{\Lambda \Theta} \quad \text { versus } \quad \phi:=\frac{1}{2} \epsilon^{\lambda \theta} \tilde{r}_{\lambda \theta} \tag{3-42}
\end{equation*}
$$

and $\epsilon^{\Lambda \Theta} / \epsilon^{\lambda \theta}$ being the contravariant coordinates of the surface alternation tensor given by

$$
\begin{array}{rlrl}
\epsilon^{12}=-\epsilon^{21}=\frac{1}{\sqrt{\operatorname{det}\left(\boldsymbol{A}_{l}\right)}} & \epsilon^{12}=-\epsilon^{21}=\frac{1}{\sqrt{\operatorname{det}\left(\boldsymbol{A}_{r}\right)}}  \tag{3-43}\\
\epsilon^{11}=\epsilon^{22}=0 & \text { versus } & \epsilon^{11}=\epsilon^{22}=0
\end{array}
$$

The linearized rotation around the normal can be interpreted as the third component of the linearized rotation vector along the unit normal vector to the surface. Once, we made use of the property to map the skewsymmetric tensors onto a vector. The uniqueness of the unit normal vector $\mathbf{N} / \mathbf{n}$ assures the invariant property of $\Phi / \phi$.

Analogous to the Cauch-Green deformation tensors and Euler-Lagrange deformation tensors of the first kind, the general eigenvalue problem can be applied to the pair of symmetric matrices $\left\{\boldsymbol{K}_{l}, \boldsymbol{A}_{l}\right\}$ or $\left\{\boldsymbol{K}_{r}, \boldsymbol{A}_{r}\right\}$ to obtain the eigenvalues of the left or right Euler-Lagrange deformation tensors of the second kind (tensor of change of curvature), named principal curvature differences,

$$
\begin{equation*}
\boldsymbol{K}_{l}=\left[K_{\Lambda \Theta}\right]_{2 \times 2} \quad \text { versus } \quad \boldsymbol{K}_{r}=\left[k_{\lambda \theta}\right]_{2 \times 2} . \tag{3-44}
\end{equation*}
$$

Thus, the principal curvature differences are achieved as result of simultaneous diagonalization of the left versus right Euler-Lagrange deformation tensor of the second kind (Lagrangian versus Eulerian tensor of change of curvature) along with their corresponding metric tensors,

$$
\begin{array}{rlrl}
\Lambda_{1,2}^{\prime \prime}= & \frac{1}{2}\left\{\operatorname{tr}\left(\boldsymbol{K}_{l} \boldsymbol{A}_{l}^{-1}\right) \pm\right. & \lambda_{1,2}^{\prime \prime} & =\frac{1}{2}\left\{\operatorname{tr}\left(\boldsymbol{K}_{r} \boldsymbol{A}_{r}^{-1}\right) \pm\right. \\
& \left.\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{K}_{l} \boldsymbol{A}_{l}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{K}_{l} \boldsymbol{A}_{l}^{-1}\right)}\right\} & &  \tag{3-45}\\
& \left.\sqrt{\left\{\operatorname{tr}\left(\boldsymbol{K}_{r} \boldsymbol{A}_{r}^{-1}\right)\right\}^{2}-4 \operatorname{det}\left(\boldsymbol{K}_{r} \boldsymbol{A}_{r}^{-1}\right)}\right\} .
\end{array}
$$

We can look at the sum $\Lambda_{1}^{\prime \prime}+\Lambda_{2}^{\prime \prime}$ or the difference $\Lambda_{1}^{\prime \prime}-\Lambda_{2}^{\prime \prime}$ of Lagrangian principal curvature differences or their Eulerian counterparts as invariant measures of surface deformation. However, in differential geometry we learn about two well-known scalar invariants connected to the second fundamental tensor of the surface, namely mean curvature $H / h$ and Gaussian curvature $K / k$, Box $3-16$. Therefore, the differences of the Gaussian or mean curvatures of the current- and reference surface are considered as more appropriate surface deformation measures. They find a gentle physical interpretation relevant to sinking and rising regions. We will discuss about it more later when the practical applications of the theory are treated.

Box 3-16: The change of the mean- and Gaussian curvature in the Lagrangian versus Eulerian portray

```
Lagrangian portray
```

Eulerian portray

The change of mean curvature:

$$
\begin{aligned}
h-H & =\frac{1}{2}\left[a^{\lambda \theta} b_{\lambda \theta}-A^{\Lambda \Theta} B_{\Lambda \Theta}\right] & h-H & :=\frac{1}{2}\left[a^{\lambda \theta} b_{\lambda \theta}-A^{\Lambda \Theta} B_{\Lambda \Theta}\right] \\
& =\frac{1}{2}\left[\left(A^{\Lambda \Theta}+2 E^{\Lambda \Theta}\right)\left(B_{\Lambda \Theta}+K_{\Lambda \Theta}\right)-A^{\Lambda \Theta} B_{\Lambda \Theta}\right] & & =\frac{1}{2}\left[a^{\lambda \theta} b_{\lambda \theta}-\left(a^{\lambda \theta}-2 e^{\lambda \theta}\right)\left(b_{\lambda \theta}-k_{\lambda \theta}\right)\right] \\
& =\frac{1}{2}\left[A^{\Lambda \Theta} K_{\Lambda \Theta}+2 E^{\Lambda \Theta} B_{\Lambda \Theta}+2 E^{\Lambda \Theta} K_{\Lambda \Theta}\right] & & =\frac{1}{2}\left[a^{\lambda \theta} k_{\lambda \theta}+2 e^{\lambda \theta} b_{\lambda \theta}-2 e^{\lambda \theta} k_{\lambda \theta}\right]
\end{aligned}
$$

The change of Gaussian curvature:

$$
\begin{aligned}
k-K & :=\frac{1}{2}\left(\epsilon^{\lambda \phi} \epsilon^{\theta \psi} b_{\lambda \theta} b_{\phi \psi}-\epsilon^{\Lambda \Phi} \epsilon^{\Theta \Psi} B_{\Lambda \Theta} B_{\Phi \Psi}\right) & k-K & :=\frac{1}{2}\left(\epsilon^{\lambda \phi} \epsilon^{\theta \psi} b_{\lambda \theta} b_{\phi \psi}-\epsilon^{\Lambda \Phi} \epsilon^{\Theta \Psi} B_{\Lambda \Theta} B_{\Phi \Psi}\right) \\
& =\frac{\operatorname{det}\left(b_{\lambda \theta}\right)}{\operatorname{det}\left(a_{\lambda \theta}\right)}-\frac{\operatorname{det}\left(B_{\Lambda \Theta}\right)}{\operatorname{det}\left(A_{\Lambda \Theta}\right)} & & =\frac{\operatorname{det}\left(b_{\lambda \theta}\right)}{\operatorname{det}\left(a_{\lambda \theta}\right)}-\frac{\operatorname{det}\left(B_{\Lambda \Theta}\right)}{\operatorname{det}\left(A_{\Lambda \Theta}\right)} \\
& =\frac{\operatorname{det}\left(B_{\Lambda \Theta}+K_{\Lambda \Theta}\right)}{\operatorname{det}\left(A_{\Lambda \Theta}+2 E_{\Lambda \Theta}\right)}-\frac{\operatorname{det}\left(B_{\Lambda \Theta}\right)}{\operatorname{det}\left(A_{\Lambda \Theta}\right)} & & =\frac{\operatorname{det}\left(b_{\lambda \theta}\right)}{\operatorname{det}\left(a_{\lambda \theta}\right)}-\frac{\operatorname{det}\left(b_{\lambda \theta}-k_{\lambda \theta}\right)}{\operatorname{det}\left(a_{\lambda \theta}-2 e_{\lambda \theta}\right)}
\end{aligned}
$$

## Chapter 4

## The Earth Surface Deformation Analysis

The general theory of the extrinsic versus intrinsic surface deformations analysis, developed in Chapter 3, is formulated here for the particular case of deformations of the Earth surface. By the Earth surface, we mean the realistic (topographic or physical) surface of the Earth with a mathematical description. The surface is assumed to be a star-shaped orientable, smooth surface (no sharp point, edges or self-interaction), a 2dimensional Riemann manifold which is being isometrically embedded into a 3-dimensional Euclidean space with the mass center of the Earth as its origin. The Earth surface is called star-shaped if the mapping of the surface onto the Earth reference surface is one-to-one. The geometry of the Earth surface in the reference or current surfaces has to be known as a parametrized curved surface. In this chapter, we focus on the Lagrangian portray of the surface deformation measures where we assume the first and second fundamental tensors of the Earth surface as two known surface tensor fields. The parameterization of the Earth surface in the reference state should be done by means of appropriate surface coordinates. This will be the topic of the first section of this chapter. The remaining sections deal with the formulation of the surface deformation measures for the Earth surface as a deforming surface.

### 4.1 Surface-normal Coordinates

The embedding space $\mathbb{E}^{3}$ can be covered by a chart of space curvilinear coordinates $Q^{J},(J=1,2,3)$. In geodesy, the space curvilinear coordinates are usually defined by means of an Earth-fixed, 3-dimensional coordinate system, whose geometry is a good regional or global approximation of the geometry of the Earth gravity field. Such coordinate systems are preferred for representing points on the Earth surface. They are generally defined based on a geodetic reference surface, which roughly represents an equipotential surface near the Earth surface. As a restriction for the surface, the points on the Earth surface are assigned to their corresponding points on a geodetic reference surface using a one-to-one mapping. Projection by means of surface normals is often selected for the purpose of the mapping from the Earth surface onto this geodetic reference surface. Thus, a class of space curvilinear coordinates, so-called surface-normal coordinates, is introduced. A comprehensive introduction of the surface-normal coordinates and their geometric principals from a geodetic standpoint is provided by S. Heitz (1985).

The geodetic reference surface, which gives a good mean approximation of the equipotential surfaces near the Earth surface, is considered as a surface with a well known geometry. The geodetic reference surface
is parametrized by means of surface curvilinear coordinates $Q^{\Lambda}(\Lambda=1,2)$. The surface-normal coordinates $Q^{J},(J=1,2,3)$ of a point on the Earth surface consist of the surface coordinates $Q^{\Lambda}$ of the foot point of its normal to the reference surface and the height $H$ of the point above the reference surface, namely the distance along the surface normal.

$$
\begin{equation*}
\left(Q^{1}, Q^{2}, Q^{3}\right):=\left(Q^{1}, Q^{2}, H\right)=\left(Q^{\Lambda}, H\right) \tag{4-1}
\end{equation*}
$$

## Geodetic coordinates

Geodetic coordinates are a special type of surface-normal coordinates for which the geodetic reference surface is referred to as an ellipsoid of revolution, i.e. a biaxial ellipsoid. In this case, the surface-normal coordinate $H$ is named ellipsoidal height. If the ellipsoidal (geographic) longitude $L$ and ellipsoidal (geographic) latitude $B$ play the role of surface coordinates of the ellipsoid of revolution, the geodetic coordinates are named geographically geodetic coordinates or geographic coordinates. Hence, the geographic coordinates of a point on the Earth surface is given by

$$
\begin{equation*}
\left(Q^{\Lambda}, H\right)=(L, B, H) \tag{4-2}
\end{equation*}
$$

The geographically geodetic coordinates are of main interest in this study. Hereafter, we refer to them as simply geodetic coordinates.

According to E. W. Grafarend and P. Lohse (1991), and E. W. Grafarend and J. Engels (1992), a Gaussian representation of the Earth surface in terms of the geodetic coordinates with respect to the reference ellipsoid $\mathbb{E}_{A_{1}, A_{2}}^{2}$ is

$$
\begin{align*}
\mathbf{X}(L, B)= & \mathbf{I}_{1} X^{1}(L, B)+\mathbf{I}_{2} X^{2}(L, B)+\mathbf{I}_{3} X^{3}(L, B) \\
= & \mathbf{I}_{1}\left[\frac{A_{1}}{\sqrt{1-E^{2} \sin ^{2} B}}+H(L, B)\right] \cos L \cos B+ \\
& \mathbf{I}_{2}\left[\frac{A_{1}}{\sqrt{1-E^{2} \sin ^{2} B}}+H(L, B)\right] \sin L \cos B+  \tag{4-3}\\
& \mathbf{I}_{3}\left[\frac{A_{1}\left(1-E^{2}\right)}{\sqrt{1-E^{2} \sin ^{2} B}}+H(L, B)\right] \sin B
\end{align*}
$$

where

$$
\begin{equation*}
E:=\frac{\sqrt{A_{1}^{2}-A_{2}^{2}}}{A_{1}} \tag{4-4}
\end{equation*}
$$

is called first relative eccentricity in terms of semi-major axis $A_{1}$ and semi-minor axis $A_{2}$ of $\mathbb{E}_{A_{1}, A_{2}}^{2}$. Here, the geodetic longitude $L$ and the geodetic latitude $B$ serve as surface coordinates of the Earth surface. In this case, the geodetic height $H$ is considered as a function of surface curvilinear coordinates $L$ and $B$ and not as an independent coordinate. In order to apply this representation, Equation (4-3), in deformation studies of the Earth surface, the theory of differential geometry requests at least $C^{2}$ continuity. Namely, the embedding functions $X^{K}(L, B)$ and $H(L, B)$ should enjoy at least continuity up to the second derivatives.

### 4.2 Extrinsic Deformation Analysis of the Earth Surface

In chapter 3, we derived the expressions of Lagrangian surface deformation tensors as functions of space Cartesian coordinates of the displacement vector and the difference vector of unit normal vectors, Box 3-6 and Box 3-7.

We also obtained the linearized relations of these tensors, Box 3-12 and Box 3-14. According to Equation (4-3), the partial derivatives $\frac{\partial X^{K}}{\partial Q^{\Lambda}}$ which are necessary in this case, are acquired as

$$
\begin{align*}
& \frac{\partial X^{1}}{\partial Q^{1}}=\frac{\partial X^{1}}{\partial L}=\cos B\left[H_{L} \cos L-(N+H) \sin L\right] \\
& \frac{\partial X^{2}}{\partial Q^{1}}=\frac{\partial X^{2}}{\partial L}=\cos B\left[H_{L} \sin L+(N+H) \cos L\right] \\
& \frac{\partial X^{3}}{\partial Q^{1}}=\frac{\partial X^{3}}{\partial L}=H_{L} \sin B \\
& \frac{\partial X^{1}}{\partial Q^{2}}=\frac{\partial X^{1}}{\partial B}=\cos L\left[H_{B} \cos B-(M+H) \sin B\right]  \tag{4-5}\\
& \frac{\partial X^{2}}{\partial Q^{2}}=\frac{\partial X^{2}}{\partial B}=\sin L\left[H_{B} \cos B-(M+H) \sin B\right] \\
& \frac{\partial X^{3}}{\partial Q^{2}}=\frac{\partial X^{3}}{\partial B}=\left[H_{B} \sin B+(M+H) \cos B\right]
\end{align*}
$$

$H_{L}$ and $H_{B}$ stand for partial derivatives of the height function with respect to the surface coordinates $L$ and $B$. The normal curvature radius $N$ and meridional curvature radius $M$ of the reference ellipsoid are given as follows

$$
\begin{align*}
& N(B):=\frac{A_{1}}{\left(1-E^{2} \sin ^{2} B\right)^{\frac{1}{2}}}  \tag{4-6}\\
& M(B):=\frac{A_{1}\left(1-E^{2}\right)}{\left(1-E^{2} \sin ^{2} B\right)^{\frac{3}{2}}} \tag{4-7}
\end{align*}
$$

Besides the above partial derivatives, we also need to have the partial derivatives $\frac{\partial U^{I}}{\partial Q^{\Lambda}}$ and $\frac{\partial W^{I}}{\partial Q^{\Lambda}}$ and the coordinates of the mixed tensor $B_{\Lambda}^{\Phi}$ that will be discussed later.

The second method of the extrinsic deformation analysis was presented by formulating the surface deformation measures in terms of space curvilinear coordinates of the difference vectors $\mathbf{u}$ and $\mathbf{w}$, Box 3-8 and Box 3-9. In Lagrangian portray, the method asks for the coordinates of the Jacobi matrix of the coordinate transformation from surface curvilinear coordinates $Q^{\Lambda}$ to the space curvilinear coordinates $Q^{K}$. The Jacobi matrix can be derived for the particular case for which the geodetic coordinates serve as the space curvilinear coordinates and geodetic longitude and geodetic latitude $(L, B)$ as the surface coordinates.

$$
\left[J_{\Lambda}^{K}\right]=\left[\frac{\partial Q^{K}}{\partial Q^{\Lambda}}\right]=\left[\begin{array}{cc}
1 & 0  \tag{4-8}\\
0 & 1 \\
H_{L} & H_{B}
\end{array}\right]
$$

In addition to the coordinates of the Jacobian matrix, we also need the analytical expressions of the space Christoffel symbols $\Gamma_{J K}^{I}$ for computing the surface deformation measures of the Earth surface. They have been computed and listed in Box 4-2 for the geodetic coordinates $(L, B, H)$.

Box 4-1: The covariant and contravariant coordinates of the embedding space metric tensor for the geodetic coordinates $(L, B, H)$
-The covariant coordinates of the metric tensor of the embedding space $\mathbb{E}^{3}$

$$
\left[G_{I J}(L, B, H)\right]=\left[\begin{array}{ccc}
(N+H)^{2} \cos ^{2} B & 0 & 0 \\
0 & (M+H)^{2} & 0 \\
0 & 0 & 1
\end{array}\right]
$$

-The contravariant coordinates of the metric tensor of the embedding space $\mathbb{E}^{3}$

$$
\begin{aligned}
{\left[G^{I J}(L, B, H)\right] } & =\left[G_{I J}(L, B, H)\right]^{-1} \\
& =\left[\begin{array}{ccc}
\frac{1}{(N+H)^{2} \cos ^{2} B} & 0 & 0 \\
0 & \frac{1}{(M+H)^{2}} & 0 \\
0 & 0 & 1
\end{array}\right]
\end{aligned}
$$

-Partial derivatives of $G_{I J}$ with respect to $(L, B, H)$

$$
\begin{aligned}
\frac{\partial G_{11}}{\partial L} & =0 \\
\frac{\partial G_{11}}{\partial B} & =-(N+H)(M+H) \sin 2 B \\
\frac{\partial G_{11}}{\partial H} & =2(N+H) \cos ^{2} B \\
\frac{\partial G_{22}}{\partial L} & =0 \\
\frac{\partial G_{22}}{\partial B} & =2 M_{B}(M+H) \\
\frac{\partial G_{22}}{\partial H} & =2(M+H) \\
\frac{\partial G_{33}}{\partial L} & =\frac{\partial G_{33}}{\partial B}=\frac{\partial G_{33}}{\partial H}=0
\end{aligned}
$$

Box 4-2: Space Christoffel symbols $\Gamma_{J K}^{I} / \Gamma_{j k}^{i}$ as functions of the geodetic coordinates $(L, B, H)$

$$
\begin{aligned}
& \Gamma_{I J}^{K}=\frac{1}{2} G^{K L}\left(\frac{\partial G_{L I}}{\partial Q^{J}}+\frac{\partial G_{L J}}{\partial Q^{I}}-\frac{\partial G_{I J}}{\partial Q^{L}}\right) \\
& \Gamma_{11}^{1}=\Gamma_{22}^{1}=\Gamma_{23}^{1}=\Gamma_{33}^{1}=0 \\
& \Gamma_{12}^{1}=\frac{-(M+H) \tan B}{(N+H)} \\
& \Gamma_{13}^{1}=\frac{1}{(N+H)} \\
& \Gamma_{11}^{2}=\frac{(N+H) \sin 2 B}{2(M+H)} \\
& \Gamma_{12}^{2}=\Gamma_{13}^{2}=\Gamma_{33}^{2}=0 \\
& \Gamma_{22}^{2}=\frac{M_{B}}{(M+H)} \\
& \Gamma_{23}^{2}=\frac{1}{(M+H)} \\
& \Gamma_{12}^{3}=\Gamma_{13}^{3}=\Gamma_{23}^{3}=\Gamma_{33}^{3}=0 \\
& \Gamma_{11}^{3}=-(N+H) \cos ^{2} B \\
& \Gamma_{22}^{3}=-(M+H)
\end{aligned}
$$

where

$$
M_{B}:=\frac{3 A_{1} E^{2}\left(1-E^{2}\right) \sin B \cos B}{\left(1-E^{2} \sin ^{2} B\right)^{\frac{5}{2}}}
$$

Having the partial derivatives $\frac{\partial \bar{U}_{K}}{\partial Q^{J}}$ and $\frac{\partial \bar{W}_{K}}{\partial Q^{J}}$ and also the space Christoffel symbols, the space covariant derivatives $\bar{U}_{K \mid J}$ and $\bar{W}_{K \mid J}$ can be computed. Consequently, the surface deformation measures of the Earth surface can be determined based on the second procedure of the extrinsic approach. The linearized surface deformation tensors and the linearized surface rotation tensor have a less complicated form, Box 3-12. The analytical expressions of the coordinates of the tensors have been calculated explicitly as functions of the geodetic coordinates in Box 4-3.

Box 4-3: The left linearized Cauchy-Green deformation tensor $\tilde{\boldsymbol{C}}_{l}$, the left linearized Euler-Lagrange deformation tensor $\tilde{\boldsymbol{E}}_{l}$, and the Left linearized rotation tensor $\tilde{\boldsymbol{R}}_{l}$ of the Earth surface in terms of the space curvilinear coordinates of the displacement vector (Extrinsic approach)
-Coordinates of the left Cauchy-Green deformation tensor:

$$
\begin{aligned}
\tilde{C}_{11}= & 2\left[\frac{\partial \bar{U}_{1}}{\partial L}+H_{L}\left(\frac{\partial \bar{U}_{1}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial L}\right)+H_{L}^{2} \frac{\partial \bar{U}_{3}}{\partial H}-H_{L} \frac{2 \bar{U}_{1}}{(N+H)}-(N+H)\left(\frac{\sin 2 B}{2(M+H)} \bar{U}_{2}-\right.\right. \\
& \left.\left.\bar{U}_{3} \cos ^{2} B\right)\right]+H_{L}^{2}+(N+H)^{2} \cos ^{2} B \\
\tilde{C}_{12}=\tilde{C}_{21}= & \frac{\partial \bar{U}_{1}}{\partial B}+\frac{\partial \bar{U}_{2}}{\partial L}+H_{L}\left(\frac{\partial \bar{U}_{2}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial B}\right)+H_{L} H_{B}+H_{B}\left(\frac{\partial \bar{U}_{1}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial L}\right)+2 H_{L} H_{B} \frac{\partial \bar{U}_{3}}{\partial H}+ \\
& 2\left[\frac{\bar{U}_{1}\left[(M+H) \tan B-H_{B}\right]}{(N+H)}-\frac{\bar{U}_{2} H_{L}}{(M+H)}\right] \\
\tilde{C}_{22}= & 2\left[\frac{\partial \bar{U}_{2}}{\partial B}+H_{B}\left(\frac{\partial \bar{U}_{2}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial B}\right)+H_{B}^{2} \frac{\partial \bar{U}_{3}}{\partial H}-\left(\frac{M_{B}+2 H_{B}}{(M+H)}\right) \bar{U}_{2}+(M+H) \bar{U}_{3}\right]+ \\
& H_{B}^{2}+(M+H)^{2}
\end{aligned}
$$

-Coordinates of the left Euler-Lagrange deformation tensor of the first kind:

$$
\begin{aligned}
\tilde{E}_{11}= & \frac{\partial \bar{U}_{1}}{\partial L}+H_{L}\left(\frac{\partial \bar{U}_{1}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial L}\right)+H_{L}^{2} \frac{\partial \bar{U}_{3}}{\partial H}-H_{L} \frac{2 \bar{U}_{1}}{(N+H)}-(N+H)\left(\frac{\sin 2 B}{2(M+H)} \bar{U}_{2}-\right. \\
& \left.\bar{U}_{3} \cos ^{2} B\right) \\
\tilde{E}_{12}=\tilde{E}_{21}= & \frac{1}{2}\left[\frac{\partial \bar{U}_{1}}{\partial B}+\frac{\partial \bar{U}_{2}}{\partial L}+H_{L}\left(\frac{\partial \bar{U}_{2}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial B}\right)+H_{B}\left(\frac{\partial \bar{U}_{1}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial L}\right)\right]+H_{L} H_{B} \frac{\partial \bar{U}_{3}}{\partial H}+ \\
& {\left[\frac{\bar{U}_{1}\left[(M+H) \tan B-H_{B}\right]}{(N+H)}-\frac{\bar{U}_{2} H_{L}}{(M+H)}\right] } \\
\tilde{E}_{22}= & \frac{\partial \bar{U}_{2}}{\partial B}+H_{B}\left(\frac{\partial \bar{U}_{2}}{\partial H}+\frac{\partial \bar{U}_{3}}{\partial B}\right)+H_{B}^{2} \frac{\partial \bar{U}_{3}}{\partial H}-\left(\frac{M_{B}+2 H_{B}}{(M+H)}\right) \bar{U}_{2}+(M+H) \bar{U}_{3}
\end{aligned}
$$

-Coordinates of the Left linearized rotation tensor:

$$
\begin{gathered}
\tilde{R}_{11}=\tilde{R}_{22}=0 \\
\tilde{R}_{12}=-\tilde{R}_{21}=\frac{1}{2}\left[\frac{\partial \bar{U}_{1}}{\partial B}-\frac{\partial \bar{U}_{2}}{\partial L}+H_{B}\left(\frac{\partial \bar{U}_{1}}{\partial H}-\frac{\partial \bar{U}_{3}}{\partial L}\right)+H_{L}\left(\frac{\partial \bar{U}_{3}}{\partial B}-\frac{\partial \bar{U}_{2}}{\partial H}\right)\right]
\end{gathered}
$$

### 4.3 Intrinsic Deformation Analysis of the Earth Surface

Taking into account the parameterization of the Earth surface in terms of the geodetic longitude $L$ and geodetic latitude $B$ as surface coordinates Equation (4-3), the geometric quantities of the Earth surface in the reference state can be identified.

- Covariant tangent base vectors of the Earth surface $\mathbf{A}_{\Lambda}$ :

$$
\begin{gather*}
\mathbf{A}_{1}=\frac{\partial \mathbf{X}(L, B)}{\partial L}=\begin{array}{c}
\mathbf{I}_{1} \cos B\left[H_{L} \cos L-(N+H) \sin L\right]+\mathbf{I}_{2} \cos B\left[H_{L} \sin L+\right. \\
(N+H) \cos L]+\mathbf{I}_{3} \sin B H_{L}
\end{array} \\
 \tag{4-9}\\
\mathbf{A}_{2}=\frac{\partial \mathbf{X}(L, B)}{\partial B}=\begin{array}{c}
\mathbf{I}_{1} \cos L\left[H_{B} \cos B-(M+H) \sin B\right]+\mathbf{I}_{2} \sin L\left[H_{B} \cos B-\right. \\
(M+H) \sin B]+\mathbf{I}_{3}\left[H_{B} \sin B+(M+H) \cos B\right]
\end{array}
\end{gather*}
$$

- Covariant coordinates of the metric tensor of the Earth surface $A_{\Lambda \Theta}$ :

$$
\begin{align*}
A_{11}(L, B) & =H_{L}^{2}+(N+H)^{2} \cos ^{2} B \\
A_{12}(L, B)=A_{21}(L, B) & =H_{L} H_{B}  \tag{4-10}\\
A_{22}(L, B) & =H_{B}^{2}+(M+H)^{2}
\end{align*}
$$

- Determinant of the covariant metric tensor $A$ :

$$
\begin{align*}
A & =\operatorname{det}\left(A_{\Lambda \Phi}\right)  \tag{4-11}\\
& =H_{L}^{2}(M+H)^{2}+(N+H)^{2} \cos ^{2} B\left[(M+H)^{2}+H_{B}^{2}\right]
\end{align*}
$$

- Contravariant coordinates of the metric tensor of the surface $A^{\Lambda \Phi}$ :

$$
\begin{align*}
A^{11}(L, B) & =\frac{A_{22}}{A} \\
& =\frac{\left[H_{B}^{2}+(M+H)^{2}\right]}{H_{L}^{2}(M+H)^{2}+(N+H)^{2} \cos ^{2} B\left[(M+H)^{2}+H_{B}^{2}\right]} \\
A^{12}(L, B)=A^{21}(L, B) & =-\frac{A_{12}}{A} \\
& =\frac{-H_{L} H_{B}}{H_{L}^{2}(M+H)^{2}+(N+H)^{2} \cos ^{2} B\left[(M+H)^{2}+H_{B}^{2}\right]}  \tag{4-12}\\
A^{22}(L, B) & =\frac{A_{11}}{A} \\
& =\frac{\left[H_{L}^{2}+(N+H)^{2} \cos ^{2} B\right]}{H_{L}^{2}(M+H)^{2}+(N+H)^{2} \cos ^{2} B\left[(M+H)^{2}+H_{B}^{2}\right]}
\end{align*}
$$

- Contravariant tangent base vectors of the topographic surface $A^{\Lambda}$ :

$$
\begin{align*}
\mathbf{A}^{1}= & A^{1 \Lambda} \mathbf{A}_{\Lambda} \\
= & \frac{1}{A}\left\{(M+H)^{2} \cos B\left[H_{L} \cos L-(N+H) \sin L\right]+H_{L} H_{B} \cos L \sin B(M+H)-\right. \\
& \left.H_{B}^{2} \cos B \sin L(N+H)\right\} \mathbf{I}_{1}+\frac{1}{A}\left\{(M+H)^{2} \cos B\left[H_{L} \sin L+(N+H) \cos L\right]+\right. \\
& \left.H_{L} H_{B} \sin L \sin B(M+H)+H_{B}^{2} \cos B \cos L(N+H)\right\} \mathbf{I}_{2}+ \\
& \frac{1}{A}\left\{(M+H)^{2} \sin B H_{L}-H_{L} H_{B}(M+H) \cos B\right\} \mathbf{I}_{3}  \tag{4-13}\\
\mathbf{A}^{2}= & A^{2 \Lambda} \mathbf{A}_{\Lambda} \\
= & \frac{1}{A}\left\{(N+H)^{2} \cos ^{2} B \cos L\left[H_{B} \cos B-(M+H) \sin B\right]+H_{L} H_{B} \cos B \sin L(N+H)-\right. \\
& \left.H_{L}^{2} \cos L(M+H) \sin B\right\} \mathbf{I}_{1}+\frac{1}{A}\left\{(N+H)^{2} \cos { }^{2} B \sin L\left[H_{B} \cos B-(M+H) \sin B\right]-\right. \\
& \left.H_{L} H_{B}(N+H) \cos L \cos B-H_{L}^{2} \sin L(M+H) \sin B\right\} \mathbf{I}_{2}+\frac{1}{A}\left\{H_{L}^{2}(M+H) \cos B+\right. \\
& \left.(N+H)^{2} \cos ^{2} B\left[H_{B} \sin B+(M+H) \cos B\right]\right\} \mathbf{I}_{3}
\end{align*}
$$

Taking a closer look at the relations of the tensors of change of curvature in intrinsic and extrinsic approaches, Box 3-9 and Box 3-11, and their corresponding linearized tensors, Box $3-14$ and Box $3-15$, one notes the need for the covariant coordinates of the second fundamental tensor $B_{\Lambda \theta}$ of the reference surface. The covariant coordinates of the second fundamental tensor of the Earth surface have been computed explicitly in terms of the surface coordinates $(L, B)$ and listed in Box 4-4.

Box 4-4: Covariant coordinates of the second fundamental tensor $B_{\Lambda \Phi}$ of the Earth surface as functions of (L, B)

$$
\begin{aligned}
B_{11}(L, B)= & \frac{1}{\sqrt{A}} \cos B\left\{H_{L L}(N+H)(M+H)-2 H_{L}^{2}(M+H)-(N+H)^{2} \cos B \sin B H_{B}-\right. \\
& \left.\cos ^{2} B(N+H)^{2}(M+H)\right\} \\
B_{12}(L, B)= & B_{21}(L, B) \\
= & \frac{1}{\sqrt{A}}\left\{\cos B(N+H)\left[H_{B L}(M+H)-H_{B} H_{L}\right]+H_{L}(M+H)[\sin B(M+H)-\right. \\
& \left.\left.H_{B} \cos B\right]\right\} \\
B_{22}(L, B)= & \frac{1}{\sqrt{A}} \cos B(N+H)\left\{H_{B B}(M+H)-2 H_{B}^{2}-H_{B} M_{B}-(M+H)^{2}\right\}
\end{aligned}
$$

The coordinates of the mixed tensor $B_{\Lambda}^{\Phi}$ can be obtained by the rule of raising indices applied to the second fundamental tensor $B_{\Lambda \Theta}$ as

$$
\begin{equation*}
B_{\Lambda}^{\Phi}=A^{\Phi \Theta} B_{\Lambda \Theta} \tag{4-14}
\end{equation*}
$$

The surface Christoffel symbols of the second kind $\Gamma_{\Lambda \Theta}^{\Phi}$ are another important geometrical quantity of the intrinsic surface deformation analysis. They enter in all the relations of the surface deformation measures in the intrinsic approach via the surface covariant derivatives, Equation (3-28). The surface Christoffel symbols of the second kind of the Earth surface have been computed and included in Box $4-5$ along the partial derivatives of the covariant coordinates of the metric tensors with respect to $(L, B)$, necessary in the computations.

Box 4-5: Christoffel symbols of the second kind $\Gamma_{\Phi \Theta}^{\Lambda}$ computed for the Earth surface

- Partial derivatives of the covariant coordinates of the metric tensor $\frac{\partial A_{\Lambda \Phi}}{\partial Q^{\Theta}}$ :

$$
\begin{aligned}
& \frac{\partial A_{11}}{\partial L}=2 H_{L}(N+H) \cos ^{2} B+2 H_{L} H_{L L} \\
& \frac{\partial A_{11}}{\partial B}=2 H_{B}(N+H) \cos ^{2} B+2 H_{L} H_{B}-2 \sin B \cos B(N+H)(M+H) \\
& \frac{\partial A_{12}}{\partial L}=H_{L L} H_{B}+H_{B L} H_{L} \\
& \frac{\partial A_{12}}{\partial B}=H_{B B} H_{L}+H_{L B} H_{B} \\
& \frac{\partial A_{22}}{\partial L}=2 H_{L}(M+H)+2 H_{B} H_{B L} \\
& \frac{\partial A_{22}}{\partial B}=2\left(M_{B}+H_{B}\right)(M+H)+2 H_{B} H_{B B}
\end{aligned}
$$

- Surface Christoffel symbols of the second kind of the Earth surface $\Gamma_{\Phi \Theta}^{\Lambda}$ :

$$
\begin{gathered}
\Gamma_{\Phi \Theta}^{\Lambda}=\frac{1}{2} A^{\Lambda \Psi}\left(A_{\Psi \Phi, \Theta}+A_{\Psi \Theta, \Phi}-A_{\Phi \Theta, \Psi}\right) \\
\Gamma_{11}^{1}=\frac{1}{A}\left\{H_{L}(M+H)^{2}\left[(N+H) \cos ^{2} B+H_{L L}\right]+H_{L} H_{B}(N+H) \cos B\left[2 H_{B} \cos B-\sin B(M+H)\right\}\right. \\
\Gamma_{12}^{1}=\frac{1}{A}\left\{\left[H_{B}^{2}+(M+H)^{2}\right]\left[H_{B}(N+H) \cos ^{2} B-\sin B \cos B(N+H)(M+H)\right]+\right. \\
\left.H_{L}(M+H)\left[H_{L B}(M+H)-H_{L} H_{B}\right]\right\} \\
\Gamma_{22}^{1}=\frac{1}{A} H_{L}(M+H)\left\{(M+H)\left[H_{B B}-(M+H)\right]-H_{B}\left(2 H_{B}+M_{B}\right)\right\}
\end{gathered}
$$

Box 4-5 (Contd.): Christoffel symbols of the second kind $\Gamma_{\Phi \Theta}^{\Lambda}$ of the Earth surface

$$
\begin{aligned}
\Gamma_{11}^{2}= & \frac{1}{A}(N+H) \cos B\left\{H_{B}(N+H) \cos B\left[H_{L L}-(N+H) \cos ^{2} B\right]-2 H_{L}^{2} H_{B} \cos B+\right. \\
& \left.\sin B(M+H)\left[H_{L}^{2}+(N+H)^{2} \cos ^{2} B\right]\right\} \\
\Gamma_{12}^{2}= & \frac{1}{A}\left\{H_{L}(M+H)\left[H_{L}^{2}+(N+H)^{2} \cos ^{2} B+H_{B} \sin B \cos B(N+H)\right]+\right. \\
& \left.(N+H) H_{B} \cos ^{2} B\left[(N+H) H_{B L}-H_{L} H_{B}\right]\right\} \\
\Gamma_{22}^{2}= & \frac{1}{A}\left\{(M+H)\left(2 H_{B}+M_{B}\right) H_{L}^{2}+(N+H)^{2} \cos ^{2} B\left[H_{B} H_{B B}+\left(M_{B}+H_{B}\right)(M+H)\right]\right\}
\end{aligned}
$$

where

$$
H_{L L}=\frac{\partial^{2} H(L, B)}{\partial L^{2}} \quad H_{B B}=\frac{\partial^{2} H(L, B)}{\partial B^{2}} \quad H_{L B}=H_{B L}=\frac{\partial^{2} H(L, B)}{\partial L \partial B}
$$

We specified the explicit relations of most of the geometrical quantities and the terms which are involved in the extrinsic and intrinsic deformation measures of the Earth surface. The only terms that have not been discussed yet, are the partial derivatives of the coordinates of the difference vector functions $\mathbf{u}$ and $\mathbf{w}$ with respect to the surface and space curvilinear coordinates. In practical applications, these vector valued functions rarely exist analytically. The values of these functions, given at nodal points of geodetic networks on the Earth surface, are the only available information. Thus, we are confronted with the problem of propagating the vector valued functions from nodal point values of coordinates of the difference vector functions in order to estimate the necessary partial derivatives. In the same manner, the scalar valued function $H(L, B)$ has to be propagated from geodetic height values of nodal points in order that the partial derivatives of the function can be approximated.

### 4.4 Transformations of Space and Surface coordinates of a Vector

The space geodetic techniques provide us with Cartesian coordinates of nodal points at different time epochs on the Earth surface. The observed coordinates of a point are related to space Cartesian coordinates of the displacement vector of the point $\left(U^{1}, U^{2}, U^{3}\right)$ as follows

$$
\begin{align*}
& U^{1}=x^{1}(t)-X^{1}\left(t_{0}\right) \\
& U^{2}=x^{2}(t)-X^{2}\left(t_{0}\right)  \tag{4-15}\\
& U^{3}=x^{3}(t)-X^{3}\left(t_{0}\right)
\end{align*}
$$

where $\left[x^{1}(t), x^{2}(t), x^{3}(t)\right]$ and $\left[X^{1}\left(t_{0}\right), X^{2}\left(t_{0}\right), x^{3}\left(t_{0}\right)\right]$ are point Cartesian coordinates observed at time epoch $t$ and $t_{0}$ in a common reference fixed frame. The developed theory of the extrinsic versus intrinsic deformation
analysis of the Earth surface in the previous sections asks for space and surface curvilinear coordinates of the displacement vector. Therefore, conversion of the observed Cartesian coordinates into the space and surface curvilinear coordinates will be the subject of this section.

### 4.4.1 Conversion of the space Cartesian coordinates into the space curvilinear coordinates

The placement vector $\mathbf{X}$ of a point on the Earth surface can be represented in terms of the geodetic coordinates $(L, B, H)$ as

$$
\begin{align*}
\mathbf{X}(L, B, H) & =\mathbf{I}_{1} X^{1}(L, B, H)+\mathbf{I}_{2} X^{2}(L, B, H)+\mathbf{I}_{1} X^{3}(L, B, H) \\
& =\mathbf{I}_{1}(N+H) \cos L \cos B+\mathbf{I}_{2}(N+H) \sin L \cos B+\mathbf{I}_{3}\left[N\left(1-E^{2}\right)+H\right] \sin B \tag{4-16}
\end{align*}
$$

This representation of the placement vector with respect to the fixed frame $\left\{\mathbf{I}_{1}, \mathbf{I}_{2}, \mathbf{I}_{3}\right\}$ is used to derive the covariant base vectors of the geodetic coordinate system in the following manner

$$
\begin{align*}
& \mathbf{G}_{1}=\frac{\partial \mathbf{X}(L, B, H)}{\partial L}=-\mathbf{I}_{1}(N+H) \sin L \cos B+\mathbf{I}_{2}(N+H) \cos L \cos B \\
& \mathbf{G}_{2}=\frac{\partial \mathbf{X}(L, B, H)}{\partial B}=-\mathbf{I}_{1}(M+H) \cos L \sin B-\mathbf{I}_{2}(M+H) \sin L \sin B+\mathbf{I}_{3}(M+H) \cos B  \tag{4-17}\\
& \mathbf{G}_{3}=\frac{\partial \mathbf{X}(L, B, H)}{\partial H}=\mathbf{I}_{1} \cos L \cos B+\mathbf{I}_{2} \sin L \cos B+\mathbf{I}_{3} \sin B .
\end{align*}
$$

The contravariant base vectors $\mathbf{G}^{I}$ are computed by means of the covariant base vector $\mathbf{G}_{I}$ and contravariant coordinates of the metric tensor $G^{I J}$, presented in Box 4-1, as

$$
\begin{align*}
& \mathbf{G}^{1}=G^{1 J} \mathbf{G}_{J}=-\mathbf{I}_{1} \frac{\sin L}{(N+H) \cos B}+\mathbf{I}_{2} \frac{\cos L}{(N+H) \cos B} \\
& \mathbf{G}^{2}=G^{2 J} \mathbf{G}_{J}=-\mathbf{I}_{1} \frac{\cos L \sin B}{(M+H)}-\mathbf{I}_{2} \frac{\sin L \sin B}{(M+H)}+\mathbf{I}_{3} \frac{\cos B}{(M+H)}  \tag{4-18}\\
& \mathbf{G}^{3}=G^{3 J} \mathbf{G}_{J}=\mathbf{I}_{1} \cos L \cos B+\mathbf{I}_{2} \sin L \cos B+\mathbf{I}_{3} \sin B
\end{align*}
$$

With respect to the orthogonal, non-normalized moving frames $\left\{\mathbf{G}_{1}, \mathbf{G}_{2}, \mathbf{G}_{3}\right\}$ or $\left\{\mathbf{G}^{1}, \mathbf{G}^{2}, \mathbf{G}^{3}\right\}$, the space curvilinear coordinates of the displacement vector of covariant type can be derived as

$$
\begin{align*}
\bar{U}_{K} & =<\mathbf{u}, \mathbf{G}_{K}> \\
& =<\left(U^{J} \mathbf{I}_{J}\right), \mathbf{G}_{K}>=<\mathbf{I}_{J}, \mathbf{G}_{K}>U^{J} \tag{4-19}
\end{align*}
$$

Thus, the conversion of the space Cartesian coordinates $U^{J}$ into the curvilinear coordinates $\bar{U}_{K}$ has been derived. In matrix representation, the final form of the conversion is written as follows

$$
\left[\begin{array}{c}
\bar{U}_{1}  \tag{4-20}\\
\bar{U}_{2} \\
\bar{U}_{3}
\end{array}\right]_{\text {space }}=\left[\begin{array}{ccc}
-(N+H) \cos B \sin L & (N+H) \cos B \cos L & 0 \\
-(M+H) \sin B \cos L & -(M+H) \sin B \sin L & (M+H) \cos B \\
\cos B \cos L & \cos B \sin L & \sin B
\end{array}\right]\left[\begin{array}{c}
U^{1} \\
U^{2} \\
U^{3}
\end{array}\right]
$$

Similarly, the space curvilinear coordinates of the displacement vector of contravariant type are computed from the Cartesian coordinates,

$$
\begin{align*}
& \bar{U}^{K}=<\mathbf{u}, \mathbf{G}^{K}> \\
& \bar{U}^{K}=<\left(U^{J} \mathbf{I}_{J}\right), \mathbf{G}^{K}>=<\mathbf{I}_{J}, \mathbf{G}^{K}>U^{J} \tag{4-21}
\end{align*}
$$

and the conversion into contravariant coordinates $\bar{U}^{K}$ is specified in matrix representation as

$$
\left[\begin{array}{c}
\bar{U}^{1}  \tag{4-22}\\
\bar{U}^{2} \\
\bar{U}^{3}
\end{array}\right]_{\text {space }}=\left[\begin{array}{ccc}
\frac{-\sin L}{(N+H) \cos B} & \frac{\cos L}{(N+H) \cos B} & 0 \\
\frac{-\sin B \cos L}{(M+H)} & \frac{-\sin B \sin L}{(M+H)} & \frac{\cos B}{(M+H)} \\
\cos B \cos L & \cos B \sin L & \sin B
\end{array}\right]\left[\begin{array}{c}
U^{1} \\
U^{2} \\
U^{3}
\end{array}\right]
$$

As can be seen from the Equations (4-20) and (4-22), the coefficient matrices of the conversions are presented in terms of the geodetic coordinates. Hence, the conversions request the geodetic coordinates to be available. While transformation of the geodetic coordinates onto the Cartesian coordinates can be carried out straightforwardly, as in Equation (4-16), the inverse transformation raises certain difficulties. The inverse transformation is usually done by successive approximations in an iterative process [W. A. Heiskanen and H. Moritz (1967)], or using closed form solutions [M. K. Paul (1973); K. M. Borkowski (1989)]. We should not forget the presence of the two parameters semi-major axis $A_{1}$ and relative eccentricity $E$ in all the above transformations which reflect the role of the reference ellipsoid of revolution in final results.

The left and right Euler-Lagrange deformation tensors of the second kind (Lagrangian and Eulerian tensors of change of curvature) require that the coordinates of the difference vector of unit normal vectors $\mathbf{w}$ as well as the coordinates of the displacement vector $\mathbf{u}$, Box $3-7$, Box $3-9$ and Box $3-11$ be specified. The space Cartesian coordinates $W^{J}$ of the difference vector of the unit normal vectors at a point on the Earth surface can be determined by means of Cartesian coordinates of the unit normal vectors

$$
\begin{align*}
& W^{1}=N^{1}-n^{1} \\
& W^{2}=N^{2}-n^{2}  \tag{4-23}\\
& W^{3}=N^{3}-n^{3} .
\end{align*}
$$

The Cartesian coordinates $N^{J}$ and $n^{j}$ can be determined in terms of geodetic longitudes and geodetic latitudes of the point at the reference- and current state, respectively. Box $4-6$ presents the final derivations.

Box 4-6: Space Cartesian coordinates of the unit normal vectors $\mathbf{N}(L, B)$ and $\mathbf{n}(l, b)$

$$
\begin{array}{cc}
\text { The Reference Surface } & \text { The Current Surface } \\
N^{1}(L, B)=\frac{1}{\sqrt{A}}\left\{\sin L(M+H) H_{L}+\cos B \cos L(N+\right. & n^{1}(l, b)=\frac{1}{\sqrt{a}}\left\{\sin l(m+h) h_{l}+\cos b \cos l(n+\right. \\
\left.H)\left[\sin B H_{B}+\cos B(M+H)\right]\right\} & \left.h)\left[\sin b h_{b}+\cos b(m+h)\right]\right\} \\
N^{2}(L, B)=\frac{1}{\sqrt{A}}\left\{\operatorname { c o s } B \operatorname { s i n } L ( N + H ) \left[\sin B H_{B}+\right.\right. & n^{2}(l, b)=\frac{1}{\sqrt{a}}\left\{\operatorname { c o s } b \operatorname { s i n } l ( n + h ) \left[\sin b h_{b}+\right.\right. \\
\left.\cos B(M+H)]-\cos L(M+H) H_{L}\right\} & \left.\cos b(m+h)]-\cos l(m+h) h_{l}\right\} \\
& \\
N^{3}(L, B)=\frac{1}{\sqrt{A}}\{\cos B(N+H)[\sin B(M+H)- & n^{3}(l, b)=\frac{1}{\sqrt{a}}\{\cos b(n+h)[\sin b(m+h)- \\
\left.\left.H_{B} \cos B\right]\right\} & \left.\left.h_{b} \cos b\right]\right\}
\end{array}
$$

Having the space Cartesian coordinates of $\mathbf{w}$, they can be converted into the space or surface curvilinear coordinates using the convenient methods of conversions, established in the remainder of this section.

### 4.4.2 Conversion of the space Cartesian coordinates into the surface curvilinear coordinates

The intrinsic approach is based on the surface curvilinear coordinates of the displacement vector as well as the difference vector of the unit normal vectors. Thus, we have to establish a sort of conversion from space Cartesian coordinates, which are usually available, into the surface curvilinear coordinates, which are requested. The procedure here is the same as conversion into the space curvilinear coordinates, discussed in Section 4.4.1. The surface curvilinear coordinates of the displacement vector of type covariant can be obtained as orthogonal projections of the displacement vector onto the covariant base vectors of the reference surface.

$$
\begin{align*}
\bar{U}_{\Lambda} & =<\mathbf{u}, \mathbf{A}_{\Lambda}>  \tag{4-24}\\
& =<\left(U^{J} \mathbf{I}_{J}\right), \mathbf{A}_{\Lambda}>=<\mathbf{I}_{J}, \mathbf{A}_{\Lambda}>U^{J}
\end{align*}
$$

The out of surface coordinate $U_{3}$ of the displacement vector is derived by inner product of the displacement vector and the unit normal vector of the surface $\mathbf{N}$

$$
\begin{align*}
\bar{U}_{3}=\bar{U}^{3} & =<\mathbf{u}, \mathbf{N}> \\
& =<\left(U^{J} \mathbf{I}_{J}\right), \mathbf{N}>=<\mathbf{I}_{J}, \mathbf{N}>U^{J}  \tag{4-25}\\
& =N^{1} U^{1}+N^{2} U^{2}+N^{3} U^{3}
\end{align*}
$$

The covariant base vectors $\mathbf{A}_{\Lambda}$ have been represented with respect to the fixed frame $\left\{\mathbf{I}_{1}, \mathbf{I}_{2}, \mathbf{I}_{3}\right\}$, Equation
(4-9). The representation of the unit normal vector $\mathbf{N}$ with respect to this frame can be established as follows,

$$
\begin{align*}
\mathbf{N}(L, B)= & \mathbf{I}_{1} N^{1}(L, B)+\mathbf{I}_{2} N^{2}(L, B)+\mathbf{I}_{3} N^{3}(L, B) \\
= & \frac{1}{\sqrt{A}}\left\{\sin L(M+H) H_{L}+\cos B \cos L(N+H)\left[\sin B H_{B}+\cos B(M+H)\right]\right\} \mathbf{I}_{1}+ \\
& \frac{1}{\sqrt{A}}\left\{\cos B \sin L(N+H)\left[\sin B H_{B}+\cos B(M+H)\right]-\cos L(M+H) H_{L}\right\} \mathbf{I}_{2}+  \tag{4-26}\\
& \frac{1}{\sqrt{A}}\left\{\cos B(N+H)\left[\sin B(M+H)-H_{B} \cos B\right]\right\} \mathbf{I}_{3} .
\end{align*}
$$

Finally, the conversion of the Cartesian coordinates into the surface curvilinear coordinates can be shown in matrix representation as

$$
\left[\begin{array}{l}
\bar{U}_{1}  \tag{4-27}\\
\bar{U}_{2} \\
\bar{U}_{3}
\end{array}\right]=\left[\begin{array}{lll}
\cos B\left[H_{L} \cos L-(N+H) \sin L\right] & \cos B\left[H_{L} \sin L+(N+H) \cos L\right] & \sin B H_{L} \\
\operatorname{surface} & \sin L\left[H_{B} \cos B-(M+H) \sin B\right] & {\left[H_{B} \sin B+\left(M+H^{\prime} \cos B\right]\right.} \\
\cos L\left[H_{B} \cos B-(M+H) \sin B\right] & U^{1} \\
\frac{1}{\sqrt{A}}\left\{\sin L(M+H) H_{L}+\right. & \frac{1}{\sqrt{A}}\left\{\operatorname { c o s } B \operatorname { s i n } L ( N + H ) \left[\sin B H_{B}+\right.\right. & \frac{1}{\sqrt{A}}\{\cos B(N+H)[\sin B(M+ \\
\cos B \cos L(N+H)\left[\sin B H_{B}+\right. & \cos B(M+H)]-\cos L(M+ \\
\cos B(M+H)]\} & \left.H) H_{L}\right\} & \left.\left.H)-H_{B} \cos B\right]\right\}
\end{array}\right]
$$

Analogous to the covariant surface curvilinear coordinates of the displacement vector, their contravariant counterparts can be computed. In this case, the coefficient matrix will be much more complicated because of the form of contravariant base vectors $\mathbf{A}^{\Lambda}$, Equation (4-13).

$$
\begin{align*}
\bar{U}^{\Lambda} & =<\mathbf{u}, \mathbf{A}^{\Lambda}> \\
& =<\left(U^{J} \mathbf{I}_{J}\right), \mathbf{A}^{\Lambda}>=<\mathbf{I}_{J}, \mathbf{A}^{\Lambda}>U^{J} \tag{4-28}
\end{align*}
$$

The conversion in matrix representation will be written as follows

$$
\left[\begin{array}{lll}
\frac{1}{A}\left\{( M + H ) ^ { 2 } \operatorname { c o s } B \left[H_{L} \cos L-\right.\right. & \frac{1}{A}\left\{( M + H ) ^ { 2 } \operatorname { c o s } B \left[H_{L} \sin L+\right.\right. &  \tag{4-29}\\
(N+H) \sin L]+H_{L} H_{B} & (N+H) \cos L]+H_{L} H_{B} & \frac{1}{A}\left\{(M+H)^{2} \sin B H_{L}-\right. \\
\cos L \sin B(M+H)- & \sin L \sin B(M+H)+ & \left.H_{L} H_{B}(M+H) \cos B\right\} \\
\left.H_{B}^{2} \cos B \sin L(N+H)\right\} & \left.H_{B}^{2} \cos B \cos L(N+H)\right\} & \\
& & \frac{1}{A}\left\{H_{L}^{2}(M+H) \cos B+\right. \\
\frac{1}{A}\left\{( N + H ) ^ { 2 } \operatorname { c o s } ^ { 2 } B \operatorname { c o s } L \left[H_{B}\right.\right. & \frac{1}{A}\left\{( N + H ) ^ { 2 } \operatorname { c o s } ^ { 2 } B \operatorname { s i n } L \left[H_{B}\right.\right. & \left.\left.(M+H) \cos ^{2} B\right]\right\}\left[H_{B} \sin B+\right. \\
\cos B-(M+H) \sin B]+ & \cos B-(M+H) \sin B]- & U U^{2} \\
H_{L} H_{B} \cos B \sin L(N+H)- & H_{L} H_{B}(N+H) \cos L \cos B- & U^{1} \\
\left.H_{L}^{2} \cos L(M+H) \sin B\right\} & \left.H_{L}^{2} \sin L(M+H) \sin B\right\} & \frac{1}{\sqrt{A}}\{\cos B(N+H)[\sin B(M+ \\
& & \left.\left.H)-H_{B} \cos B\right]\right\}
\end{array}\right]
$$

### 4.4.3 Conversion of the space curvilinear coordinates into the surface curvilinear coordinates

In the same manner, we can obtain the conversion of the curvilinear coordinates into the surface curvilinear coordinates. The conversion is developed for the case of transformation from space contravariant coordinates into the surface covariant and contravariant coordinates. Again, the coordinates of the coefficient matrices can be written as inner products of the base vectors of the space moving frame $\left\{\mathbf{G}_{1}, \mathbf{G}_{2}, \mathbf{G}_{3},\right\}$ and the surface moving frames $\left\{\mathbf{A}^{1}, \mathbf{A}^{2}, \mathbf{N}\right\}$ and $\left\{\mathbf{A}_{1}, \mathbf{A}_{2}, \mathbf{N}\right\}$

$$
\begin{align*}
\bar{U}_{\Lambda} & =<\mathbf{u}, \mathbf{A}_{\Lambda}>  \tag{4-30}\\
& =<\left(\bar{U}^{J} \mathbf{G}_{J}\right), \mathbf{A}_{\Lambda}>=<\mathbf{G}_{J}, \mathbf{A}_{\Lambda}>\bar{U}^{J}
\end{align*}
$$

The out of surface component of the displacement vector $U_{3}$ is derived by the inner product of the displacement vector and the unit normal vector of the surface $\mathbf{N}$

$$
\begin{align*}
\bar{U}_{3}=\bar{U}^{3} & =<\mathbf{u}, \mathbf{N}> \\
& =<\left(\bar{U}^{J} \mathbf{G}_{J}\right), \mathbf{N}>=<\mathbf{G}_{J}, \mathbf{N}>\bar{U}^{J} . \tag{4-31}
\end{align*}
$$

The coefficient matrix in case of transformation from space contravariant coordinates onto surface covariant coordinates will be written as

$$
\left[\begin{array}{c}
\bar{U}_{1}  \tag{4-32}\\
\bar{U}_{2} \\
\bar{U}_{3}
\end{array}\right]_{\text {Surface }}=\left[\begin{array}{ccc}
(N+H)^{2} \cos ^{2} B & 0 & H_{L} \\
0 & (M+H)^{2} & H_{B} \\
\frac{-(N+H)(M+H) H_{L} \cos B}{\sqrt{A}} & \frac{-(N+H)(M+H) H_{B} \cos B}{\sqrt{A}} & \frac{(N+H)(M+H) \cos B}{\sqrt{A}}
\end{array}\right]\left[\begin{array}{c}
\bar{U}^{1} \\
\bar{U}^{2} \\
\bar{U}^{3}
\end{array}\right]_{\text {Space }}
$$

In the same manner, the space contravariant coordinates can be converted onto the surface contravariant coordinates

$$
\begin{align*}
\bar{U}^{\Lambda} & =<\mathbf{u}, \mathbf{A}^{\Lambda}> \\
& =<\left(\bar{U}^{J} \mathbf{G}_{J}\right), \mathbf{A}^{\Lambda}>=<\mathbf{G}_{J}, \mathbf{A}^{\Lambda}>\bar{U}^{J} . \tag{4-33}
\end{align*}
$$

The conversion in matrix representation has the final form as

The transformations from the space curvilinear coordinates into the surface curvilinear coordinates can be useful in the linearized theory of deformation where we assume the displacement vector and the coordinates of its gradients are small. In such a case, the space curvilinear coordinates of the displacement vector of a point on the Earth surface can be written with sufficient accuracy as

$$
\left[\begin{array}{c}
\bar{U}^{1}  \tag{4-35}\\
\bar{U}^{2} \\
\bar{U}^{3}
\end{array}\right]_{\text {space }} \approx\left[\begin{array}{c}
l(t)-L\left(t_{0}\right) \\
b(t)-B\left(t_{0}\right) \\
h(t)-H\left(t_{0}\right)
\end{array}\right]
$$

where $\left(L\left(t_{0}\right), B\left(t_{0}\right), H\left(t_{0}\right)\right)$ and $(l(t), b(t), h(t))$ are geodetic coordinates of the material point observed at time epoch $t_{0}$ and $t$, respectively. Thus, having the geodetic coordinates of a point on the Earth surface in the reference and current states, space curvilinear contravariant coordinates of the displacement vector of the point can be obtained in the first order approximation from the difference of these coordinates. Then, the space curvilinear coordinates can be converted into the surface curvilinear coordinates of the displacement vector, by means of Equations (4-32) and (4-34), which are requested in the intrinsic approach of the surface deformation theory.

### 4.4.4 Conversion of the space Cartesian coordinates into the surface curvilinear coordinates based on the least square solution

As an alternative approach to the problem of the conversion of the space Cartesian coordinates of the displacement vector into its surface curvilinear coordinates in a particular case of small displacements, the generalized inverse method can be applied. Taking into account the Gaussian representation of a surface $\mathbf{X}\left(Q^{1}, Q^{2}\right)$, the differential of the vector valued function $\mathbf{X}$ is defined as,

$$
\begin{equation*}
d \mathbf{X}:=\frac{\partial \mathbf{X}}{\partial Q^{1}} d Q^{1}+\frac{\partial \mathbf{X}}{\partial Q^{2}} d Q^{2} \tag{4-36}
\end{equation*}
$$

or in terms of Cartesian coordinates as,

$$
\begin{equation*}
d X^{J}:=\frac{\partial X^{J}}{\partial Q^{1}} d Q^{1}+\frac{\partial X^{J}}{\partial Q^{2}} d Q^{2} \tag{4-37}
\end{equation*}
$$

Then, the matrix representation of the differential can be written in the form,

$$
\left[\begin{array}{l}
d X^{1}  \tag{4-38}\\
d X^{2} \\
d X^{3}
\end{array}\right]=\left[\begin{array}{ll}
\frac{\partial X^{1}}{\partial Q^{1}} & \frac{\partial X^{1}}{\partial Q^{2}} \\
\frac{\partial X^{2}}{\partial Q^{1}} & \frac{\partial X^{2}}{\partial Q^{2}} \\
\frac{\partial X^{3}}{\partial Q^{1}} & \frac{\partial X^{3}}{\partial Q^{2}}
\end{array}\right]\left[\begin{array}{c}
d Q^{1} \\
d Q^{2}
\end{array}\right] \quad \text { or } \quad \boldsymbol{Y}=\boldsymbol{A} \boldsymbol{X}
$$

In the language of the least square theory, we encounter the following problem: Find the vector of unknown parameters $\boldsymbol{X}$ from the vector of observations $\boldsymbol{Y}$ subject to $\operatorname{rank} \boldsymbol{A}=2, \boldsymbol{Y} \in \mathbb{R}^{3 \times 1}, \boldsymbol{A} \in \mathbb{R}^{3 \times 2}, \boldsymbol{X} \in \mathbb{R}^{2 \times 1}$. The least squares solution of this overdetermined problem based on $\|\boldsymbol{Y}-\boldsymbol{A} \boldsymbol{X}\|=$ min., is given by means of left inverse $\boldsymbol{A}_{L}^{-1}$. The reader interested in an in-depth treatment of the generalized inverse techniques is referred to E. W. Grafarend and B. Schaffrin (1993); K. R. Koch (1999).

$$
\begin{equation*}
\widehat{\boldsymbol{X}}=\boldsymbol{A}_{L}^{-1} \boldsymbol{Y}=\left(\boldsymbol{A}^{T} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{T} \boldsymbol{Y} \tag{4-39}
\end{equation*}
$$

As application of this theory, in a special case of small displacements the space Cartesian coordinates of the displacement vector $U^{J}$ can be considered as the differentials $d X^{J}$ and its surface curvilinear coordinates $\bar{U}^{\Lambda}$ as differentials $d Q^{\Lambda}$. Thus, the solution of the transformation from the space Cartesian coordinates onto the surface curvilinear coordinates is introduced.

$$
\left[\begin{array}{c}
\bar{U}^{1}  \tag{4-40}\\
\bar{U}^{2}
\end{array}\right]_{\text {Surface }} \approx\left(\boldsymbol{A}^{T} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{T}\left[\begin{array}{c}
U^{1} \\
U^{2} \\
U^{3}
\end{array}\right]_{\text {Space }}
$$

In this case, the coefficient matrix $\boldsymbol{A}$ can be specified based on the geodetic longitude and geodetic latitude as the surface coordinates.

$$
\boldsymbol{A}=\left[\begin{array}{cc}
\frac{\partial X^{1}}{\partial L} & \frac{\partial X^{1}}{\partial B}  \tag{4-41}\\
\frac{\partial X^{2}}{\partial L} & \frac{\partial X^{2}}{\partial B} \\
\frac{\partial X^{3}}{\partial L} & \frac{\partial X^{3}}{\partial B}
\end{array}\right]
$$

## Chapter 5

# The Earth Surface Deformations and Displacement Fields 

This chapter deals with the practical aspects of the Earth surface deformation analysis. The developed theory of the surface deformation analysis for the Earth surface in Chapter 4 requests a known displacement field. Geodesy as a discipline of geosciences which treats the geometrical modelling of the Earth surface, provides us valid knowledge of these displacement fields thanks to the advances in the space geodetic techniques. Displacement is understood as an effect of (visco-) elastic deformations of the Earth in response to time-varying surface loads and other causes. We start with considering the main causes of the Earth temporal deformations and proceed with the role of space geodesy in geometrical modelling of the Earth surface. The last section discusses the finite element method as a mathematical tool for propagating the discrete displacement fields derived from space geodetic measurements.

### 5.1 The Earth's Deformations

The Earth's deformations can be classified as secular, periodic or episodic in a temporal scale and as global, regional or local in a spatial scale. Our knowledge of these deformations varies considerably depending on the nature of the causes and time duration of observational records. In geodesy, we are interested in the effects of these deformations on horizontal and vertical movements of the earth's surface. According to P. A. Cross et al. (1987), the major causes of Earth's deformations include tidal forces, Earth rotation and polar motion, man-made activities, crustal loading and unloading, plate tectonic and episodic movements, Figure 5-1.

The periodic tidal forces directly influence the Earth's gravity field and deform the shape of the Earth as a non-rigid body. The most impressive effect of tidal forces is the tidal variation of sea water masses, known as ocean tide. The ocean tide creates secondary tidal deformations of the Earth's surface, called ocean loading effect. The tidal force deforms the Earth in the global scale. The maximum deformation in the radial direction is as large as 48 centimeters [P. Vaníček and E. Krakiwsky (1982)]. These effects can be modelled mathematically because their origin has been understood well enough. They need to be taken into account when an accuracy of 1 mm is desired in determining station coordinates. The polar motion affects the shape of the Earth surface at the cm level. Taking into account the coordinate accuracy achievable by space geodetic techniques nowadays, the station coordinates are recommended to be corrected for polar motion deformation. Variation in the Earth rotation rate causes displacements that are below the $m m$ level and can be neglected. D. D. McCarthy (1996) gives technical information and mathematical models of the corrections to station coordinates due to solid Earth and ocean loadings and polar motion effects. E. W. Grafarend (1986) proposes a two-step procedure


Figure 5-1: Main causes of Earth surface deformations
to decompose the displacement field into a global and a local part. The global part due to the tidal and polar motion effects are represented in terms of vector spherical harmonics and are filtered from the observed displacements.

The man-made activities cause deformations mostly in local scale. The major cause is withdrawal of ground water or oil. An example is the subsidence of the Ekofisk oil field in the North Sea. Here an annual subsidence of several decimeters has been observed [ R. Johnsen (1985)]. The Earth surface is composed of lithospheric plates of average density $\rho=2.67 \mathrm{~g} / \mathrm{cm}^{3}$ varying in thickness from 10 to 80 km . The lighter solidified plates float on the denser, viscous mantle material made by heat and pressure. There is a tendency to use the term crust for the top 10 to 30 km of the lithospheric plates. The crust is affected by various geophysical loads which cause deformations of the Earth surface. The Earth response to a loading or unloading process depends on the size and temporal behaviour of the load and the rheology of the lithosphere and the mantle. The most significant examples of unloading regional deformations are the postglacial rebounds in Fennoscandia and Canada. The Fennoscandian Shield is presently undergoing crustal uplift with a maximum rate of about $10 \mathrm{~mm} / \mathrm{yr}$ [J. Kakkuri (1997)].

Today, there is no doubt that the old assumption of a rigid Earth surface is no longer acceptable in the light of the new dynamic concept of plate tectonics and experimental evidences. For the past three decades, the theory of plate tectonics has provided the geodynamic framework for the geosciences. Its fundamental postulate is that the Earth lithosphere is broken up into a finite number of quasi-rigid plates moving relative to one another. The relative current motions between the assumed-rigid plates are described by global plate motion models. These global models have been determined by inverting geological plate motion data such as transform fault azimuth, spreading rates and earthquake slip vectors observed at plate boundaries. The first plate motion models were presented by J.B.Minster and T. H. Jordan (1978) and C. G. Chase (1978). Many new high-quality plate motion models have become available since the publications of these models. The new data have been used to determine improved global models, for example NUVEL-1 [C. DetMets et al. (1990)] and its successor NUVEL1A [C. DetMets et al. (1994)]. These models can explain the large-scale features of plate kinematics. Major deformations take place only in the comparatively narrow zones near the plate boundaries. Consequently, a large number of intense earthquakes occur near these zones. On the other hand, there is a low level of seismicity in the interior of plates. Figure 5-2 shows the plate boundaries on a map of the world. A global seismicity distribution is shown in Figure 5-3 for earthquakes with magnitude $\geq 5$ on Richter's scale.


Figure 5-2: Map of global major tectonic plate boundaries


Figure 5-3: Map of global seismicity distribution $\mathrm{M} \geq 5$ (1973-2000)

A comparison of these two figures confirms that boundaries of the global tectonic plates are specified by narrow zones of high seismic activity. Four principal types of interaction between plates are distinguished by plate tectonic studies [C. Lomnitz (1975), E. V. Artyushkov (1983)]. They are subduction, strike-slip, spreading and collision boundaries. Space geodesy plays an important role in plate tectonic studies as it provides the precise geometrical information necessary for plate motion determinations. Space geodetic techniques have now approached the level of precision of global plate models and measured plate motions over a period of the last few years. Thus, the space geodetic measurements are used to confirm the validity of plate motion models determined from geological data, see for example R. G. Williamson et al (1989), D. E. Smith et al. (1989), R. E. Reilinger et al. (1997b); D. S. MacMillan and C. Ma (1999); C. DeMets and T. H. Dixon (1999).

As long as the plates move, elastic strain energy is accumulated at the boundary zones. After a certain time interval dependent on various factors, the accumulated energy will suddenly be released in the form of earthquakes, often without any warning. The episodic deformations related to earthquakes can be as large as several meters. In recent years, earthquake prediction as an interdisciplinary research has been considered a serious scientific topic in countries with high risk of earthquakes like U.S.A and Japan. Again, space geodetic measurements contribute the main source of geometrical information on the temporal deformations of the Earth surface in earthquake investigations [P. J. Clarke et al. (1998); S. N. Ward (1998a); T. Kato et al. (1998); T. Sagiya and W. Thatcher (1999)]. The geodetic measurements can quantify potential seismic activities even on faults that are unknown, too slowly slipping or too deeply buried to be studied by conventional geological or seismological methods.

The theory of plate tectonics assumes a rigid behaviour of the plates and therefore fails to account for observed intraplate deformations. The assumption also restricts the theory's applicability because no real material is absolutely rigid. But what makes the theory applicable is that smaller deformations and lower level of seismic activities happen in plate interiors in comparison with the plate boundaries. Geophysicists have already attacked the issue by dividing the Earth surface into two domains, nearly rigid plate interiors and deforming plate boundary regions. Deformations in each domain are treated differently. Plate interiors, which show very limited permanent deformation induced by boundary forces, are generally treated as elastic plates deforming in response to tectonic loads. Plate boundary regions, where significant permanent deformations usually occur, should be studied taking inelastic deformations into account [S. Wdowinski (1998)].

### 5.2 Geodetic Displacement fields

Geodesy has functioned well and proved that measurement and representation of geodynamic phenomena such as Earth tides, polar motion and crustal motion, is stated as one of its major goals. The study of geometrical aspects of these geodynamic phenomena fall within the realm of geodesy. Repeated positioning of geodetic network benchmarks using either conventional or satellite-based geodetic techniques has allowed differential displacements to be measured at the surface of the Earth over time scales of a few to one hundred years. About seventy years ago, Japanese seismologists T. Terada and N. Miyabe (1929) used the geodetic displacements of a re-observed horizontal network to investigate strain field patterns of the area. Since that time, a huge number of investigations has been carried out in various aspects of Earth deformation analysis based on geodetic measurements. Specially, over the last three decades, considerable efforts have been made by geodesists to develop new methodologies and techniques for deformation studies as well as new theories and algorithms for network design and analysis of geodetic observations.

It has been a common practice to study the horizontal and vertical components of the displacement vectors separately. The main reason for this divided treatments was the separately available horizontal and vertical observation data in classical geodesy. In reality though, purely horizontal or purely vertical Earth surface
movements do not exist. This separation was the main weakness of classical geodesy. Another weakness of classical geodesy was that the accuracy of the point positioning techniques for global, regional and, even more so, for local deformation studies was not high enough. These two limitations restricted applications of the geodetic measurements in Earth deformation studies. It should be mentioned that despite their relative poor precision, classical geodetic observations can provide valuable information about broad-scale tectonic deformations of the Earth surface. Their lack of precision is balanced by the time span of the observations. J. Pagarete et al. (1998) have analyzed sets of classical geodetic observations carried out in Acores triple junction spanning about six decades in combination with GPS observations to establish a tectonic model of the area.

In the last two decades, Space geodetic methods have overcome these restrictions and opened the door for geodesy to join geology and seismology in geodynamics and earth science researches. Space geodetic measurements form, for the first time, a quantitative link between historical seismology, fault geology and deformation modelling. The current capabilities of space geodesy allow us to perform geodetic observations with an accuracy which is significantly higher than typical deformations of the Earth surface occurring on global to regional or even local scales.

Three international services in IAG dealing with space geodesy should be mentioned here. They are the International Earth Rotation Service (IERS), the International GPS Service (IGS), and the Commission on International Coordination of Space Techniques for Geodesy and Geodynamics (CSTG). The IERS and IGS are service-type organization, while CSTG is a joint commission of the IAG and of the committee on Space Research (COSPAR). Information concerning IERS activities and related topics is given in the series of technical notes, e.g. D. D. McCarthy (1996). The IGS central bureau also publishes annual reports on IGS activities which can be downloaded via ftp://igscb.jpl.nasa.gov/igscb/resource/pubs/. As mentioned previously, 3-dimensional displacements can be monitored with a rapidly improving accuracy based on space geodetic techniques. At present, there are four most widely used techniques in space geodesy, namely VLBI, SLR, DORIS and GPS. The status of these techniques is reviewed here briefly.

VLBI (Very Long Baseline Interferometry) is a geometric space technique. The technique measures the time difference between the arrival at two Earth-based antennas of a radio wavefront emitted by a distant quasar. Using large numbers of time difference measurements from many quasars observed with a global network of antennas, VLBI determines the inertial reference frame defined by the quasars and simultaneously the precise positions of the antennas. The time difference measurements are precise to a few picoseconds. Thus, VLBI determines the relative positions of the antennas to a few millimeters. Since the antennas are fixed to the Earth, their locations track the instantaneous orientation of the Earth in the inertial reference frame. Relative changes in the antenna locations from a series of measurements can indicate tectonic plate motion, regional deformation, and local uplift or subsidence. The technique is unique in its ability to define an inertial reference frame and to measure the Earth's orientation in this frame. In the past, the focus in VLBI was exclusively on horizontal movements. Recently, more attention has been drawn to the vertical components [T. M. VanDam and T. A. Herring (1994)]. However, the error of vertical components of VLBI displacement vectors is still about three times that of the horizontal components and considerable efforts are being devoted to improve the accuracy. A significant source of random and systematic error in VLBI is the neutral atmosphere, which slows down the incoming radio waves and causes an excess "delay" of the radio signal. Main improvements for VLBI are to be expected in the observing methods. Troposphere modelling is also planned. Most VLBI stations are now equipped with a permanent GPS receiver on site. Comparison of the results from VLBI and GPS observations are likely to improve further the accuracy of VLBI measurements by use of GPS-derived troposphere parameters for VLBI solutions. Precision of VLBI station position for a one-day session is now as good as 1 mm in the horizontal and 3 mm in the vertical.

SLR (Satellite Laser Ranging) targets are satellites equipped with corner cubes or retroreflectors. The observable is the round-trip pulse time-of-flight to the satellite. This round trip time-of-flight of an ultra-short (a few
picosecond) laser pulse from a ground station to a satellite is certainly the most straightforward and accurate observable of all the competing space geodetic techniques. The current global SLR network consists of over forty five stations. During the past three decades, this network has evolved into a powerful source of data for studies of the solid Earth and its ocean and atmospheric sub-systems. Of all the space geodetic techniques, SLR suffers the least from propagation delay in the atmosphere because optical frequencies are relatively insensitive to the two most dynamic components of the atmospheric refraction delay, namely the ionosphere and water vapor distribution. Another advantage of SLR is the simplicity and low cost of the space segment. Despite all the aforementioned advantages that make the technique the preferred approach for Earth deformation studies, it presently suffers from some negative features. SLR is not an all-weather method, and the current global coverage of SLR stations is far from optimum. Future improvements are expected by a better distribution of stations, improved handling of the atmospheric corrections, and the deployment of a new generation of satellite targets supporting millimeter accuracy ranging. Some of the scientific results derived from SLR include detection and monitoring of tectonic plate motion, crustal deformation, Earth rotation, and polar motion, establishment and maintenance of the International Terrestrial Reference System (ITRS), and detection and monitoring of post-glacial rebound and subsidence.

DORIS (Doppler Orbitography and Radio positioning Integrated by Satellite) is a radio tracking system developed by the French National Space Agency, Center National d'Etudes Spatiales (CNES). The main reason of this development was to get an efficient system for the precise tracking of low orbiting remote sensing satellites which require precise orbits. Among the various applications of the DORIS observations, utilization of the coordinates and velocities of the DORIS global network stations in the definition of global reference frames and Earth surface deformation studies is of interest to this study.

GPS (Global Positioning System) has certainly provided the most important source of information in geodynamics projects. For most scientific applications, GPS is used as an interferometric technique, namely highest accuracies are obtainable only if the differences of the original phase observables are analyzed. The principles of GPS in observation and processing are much the same as VLBI. In this sense, the difference is VLBI absolute feature because of no orbit modelling requirement in VLBI data processing phase. GPS uses microwave frequencies. Hence, it is fully exposed to the so-called wet tropospheric delay which is very difficult to predict on the necessary accuracy level. The models of tropospheric delay are getting more complex. The more complex the model is, the better the observable may be represented. In high-accuracy level GPS projects, SLR is usually used as a calibration tool for testing tropospheric delay models. Currently, a large fraction of GPS observations originate from campaigns with episodic occupations of stations. But there is a clear trend for more continuously operating GPS networks, e.g. S. Miyazaki et al. (1997). A major improvement in high-accuracy GPS applications is expected due to advents in next generation of receivers and antennas, several improvements in the atmosphere modelling, and better coverage of global GPS networks. Thus, it is predictable that GPS measurements will be the main source to acquire spatially and temporally dense 3-dimensional displacement fields in near future.

The precise definition and realization of reference frames is a key factor for Earth deformation studies. These studies require the adoption of consistent reference frames whose uncertainty in realization is less than the level of the signals assigned to the deformation processes. A reference frame is fundamentally defined by an origin of a coordinate system, three orthogonal axes, and a scale. The realization of such a reference frame is via the 3-dimensional coordinates of points on the Earth surface [K. Lambeck (1988)]. Space geodetic techniques such as VLBI, SLR, DORIS and GPS play the main role in definition and realization of terrestrial reference frames by providing high accuracy 3-dimensional station coordinates. IERS has produced a series of terrestrial reference frames from a combination of all submitted solutions of these space geodetic techniques, such as ITRF92, ITRF94, ITRF96 and ITRF97. A complete description of the definition of the IERS reference systems and their realizations is given in D. D. McCarthy (1996). The organization is currently working on a new reference frame,
named ITRF2000. Data processing procedures of space geodetic data usually produce a free-network estimate of Cartesian coordinates of network sites that is loosely oriented with respect to any arbitrary terrestrial reference frame. The coordinates of this free network adjustment represent the locations of the sites with respect to each other, and the displacement rates (velocities) show the deformation of the network as a result of tectonic movements of the Earth surface. Then, the free network is aligned with an ITRF coordinate system by sevenparameter Helmert transformations with reference to a global network with known ITRF coordinates, see for example P. Tregoning et al. (1998). IGS has a global network of GPS stations with known coordinates and velocities in ITRF framework. The network includes 227 stations as of February 2000 according to the IGS homepage http://igscb.jpl.nasa.gov, Figure 5-4 .


Figure 5-4: Current global network of GPS stations supporting the IGS (February 2000)

### 5.3 Finite Element Approach

The models of the intrinsic and extrinsic deformation analysis of the Earth surface, developed in Chapter 4, demand partial derivatives of the height function and vector-valued displacement function with respect to the surface coordinates. The ideal form of these functions should be of areal nature and continuous in spatial and temporal domains, and continuously differentiable. Unfortunately, geodetic techniques have not provided us with the functions in the ideal form so far. Typical geodetic observables are discrete functions in time and space. Consequently, the height and displacement vectors, deduced from the geodetic data, are of discrete nature. Assuming that a sufficient number of appropriately distributed discrete data are available, continuous information in space and time has to be estimated by computing best interpolation or approximation of the unknown functions over the given discretizations. The determination of interpolation or approximation functions is a basic subject of applied mathematics and numerical analysis and has been treated there in a great variety.

The Finite element method has been introduced as a powerful and widely used numerical technique which deals
with the problem. It is defined as a computer-aided mathematical technique for obtaining approximate numerical solutions to the abstract equations of calculus that predict the response of physical systems subjected to external influences [D. S. Burnett (1987)]. This introductory definition of the method identifies the broad spectrum of its applications in areas of engineering, science and applied mathematics. Here, we provide a summary of how the method generally works. The domain of the problem is divided into smaller regions or subdomains, called elements. Adjacent elements touch without overlapping, and there are no gaps between the elements. The shape of the elements is intentionally made as simple as possible, such as triangle and quadrilaterals in 2-dimensional domains, and tetrahedra and pentahedra in 3-dimensions. The process of partitioning a domain into a set of elements, namely mesh generation, is nowadays an automated procedure to a high degree by means of suitable computer programs. In each element, the element equations which are usually algebraic equations, are replaced instead of the governing equations. The much simpler element equations will be an approximation of the governing equations which are often unknown or too difficult to be handled. Two features are here specially noteworthy. First, the element equations are algebraically identical for all elements. Second, the derivation of element equations is usually straightforward because of the simple geometry of the element. Thus, a good approximation may be obtained with only a few algebraic equations since the element covers only a small part of the entire domain. Then, the terms in the element equations are numerically evaluated for each element in the mesh, a process best performed on computers. Depending on the type of applications, the resulting numbers can be assembled into a much larger set of algebraic equations to characterize the response of the entire system to loading conditions. The final operation displays the solutions in tabular, graphical, pictorial form, or other physically meaningful quantities which might be derived from the solution and also represented.

The mathematical structure of the finite element method is identified by three principal operations that are present in every finite element analysis, namely construction of a trial solution, application of an optimization criterion and estimation of accuracy. The finite element method seeks a trial solution as an approximate solution of the unknown function, which only approximately satisfies the governing equation(s) and boundary conditions. The trial solution in the form of a finite sum of functions is given as

$$
\begin{equation*}
\tilde{U}\left(Q^{J} ; A\right)=F_{0}\left(Q^{J}\right)+A_{1} F_{1}\left(Q^{J}\right)+A_{2} F_{2}\left(Q^{J}\right)+\ldots+A_{N} F_{N}\left(Q^{J}\right) \tag{5-1}
\end{equation*}
$$

Here $Q^{J}$ represents all the independent variables in the problem. The functions $F_{I}\left(Q^{J}\right)$ are known functions, called trial functions or basis functions. The trial function $F_{0}\left(Q^{J}\right)$ is not multiplied by any parameter. Its purpose is to satisfy some or all of the boundary conditions. From a practical point of view, it is important to use the trial functions that are algebraically as simple as possible and also easy to work with, such as polynomials or trigonometric functions. The coefficient $A_{1}, A_{2}, \ldots, A_{N}$ are unknown parameters.

Each unknown parameter can assume an infinity of possible values. Hence, there is infinity of possible solutions for $\tilde{U}\left(Q^{J} ; A\right)$. The purpose of the optimizing criterion is to determine specific numerical values for each of the undetermined parameters $A_{I}$ and consequently the optimum (a best) trial solution. There are two types of optimizing criteria, namely methods of weighted residuals and Ritz variational method. Methods of weighted residuals are applicable when the governing equations are differential equations. They seek to minimize an expression of error in the differential equation and not the unknown function itself. Some of the most popular methods of weighted residuals are the collocation method, the least-squares method, and the Galerkin method. In the variational method, we look for a minimum or an extremum in some physical quantities, such as energy, relevant to the unknown functions. The final operation is estimation of accuracy. We would like to get some idea of the closeness of the approximate solution to the exact solution. For a more comprehensive treatment of the subject, we refer the reader to D. S. Burnett (1987), and D. L. Logan (1993).

In finite element analysis, the domain of the problem is partitioned into elements and the approximate solution is determined for each element. A question naturally arises concerning the continuity of the solutions at the
interelement boundary points. This is a critical problem in the step of assembly of elements. Similar to continuity definition of functions, the interelement continuity can be of different classes from $C^{0}$ to $C^{\infty}$. The problem is handled by definition of interelement boundary conditions. These conditions must be satisfied by the element solutions on the boundaries between elements.

The nodes and elements are collectively referred to as a mesh. The process of defining the size, shapes and locations of the elements, and assigning numbers to each nodes and element is called mesh generation. A planar triangle is the most easy-going geometry for an element in 2-dimensional finite element analysis. We use a procedure of mesh generation based on triangular elements or a triangulation algorithm, which was devised originally by A. K. Cline and R. J. Renka (1984) and modified by S. W. Bova and G. F. Carey (1992) to handle boundaries. It yields what is known as a Delaunay triangulation, one in which the triangles formed are as near equilateral as possible for the given positions of irregularly spaced nodes. Figure $5-5$ shows the results of Delaunay triangulation among a set of VLBI stations in the Northern America in the plane of chart of geodetic longitude and latitude.

## Optimal Delaunay triangulation of the VLBI sites



Figure 5-5: Result of Delaunay triangulation among a set of VLBI stations

The finite element method has found manifold applications in geodesy and particularly in Earth deformation studies because a geodetic network can be viewed as a typical example for a set of irregularly shaped finite elements in 2- or 3-dimensions. Some examples of the recent works are E. W. Grafarend (1986), S. Zhao (1994), R. Klees (1995), P. Lundgren et al. (1995), M. Gölke et al. (1996), O. Lesne et al. (1998), Y. Vanbrabant et al. (1999).

In realm of Earth deformation analysis based on the extrinsic or intrinsic approaches, mathematical tools include partial derivatives of Cartesian or curvilinear components of displacement vector and difference vector of unit normal vectors with respect to space or surface curvilinear coordinates. These functions are usually unknown and their partial derivatives have to be estimated numerically. The main computational steps for deriving the linearized deformation tensors and the linearized rotation tensor based on the extrinsic and intrinsic approaches are depicted in Figure 5-6. The flowchart shows the role of two- and three-dimensional finite element methods in numerical estimations of the unknown functions and their relevant partial derivatives.


Figure 5-6: The role of finite element method in deriving the linearized deformation tensors based on the extrinsic versus intrinsic approach

## Chapter 6

## Present-day Surface Deformation Patterns of the European and Mediterranean Area


#### Abstract

The efficiency of the proposed methodology in Chapter 4 for geometrical modelling of surface deformation is demonstrated here by analysis of a real data set. The selected region, namely European and Mediterranean area, is known as an extraordinary natural laboratory for the study of seismotectonic processes. Abundance of deformation studies in the region will allow test of the developed models in real applications. This area is geologically and geophysically as well as geodetically one of the best studied regions on the Earth surface. The research interest encompasses the past 100 year and consequently a huge number of publications exist addressing local and regional geodynamic processes. Various surface deformation patterns of the area are computed and compared with the results of these studies. We investigate the links between the numerical results and geophysical and seismological evidences for the area of interest which encloses a wide region from the Atlantic ocean in the west to the Black sea in the east, and from Fennoscandia in the north as far as to the northern border of the African plate in the south. Besides the special geodynamic features, local and regional networks of space geodetic stations are already available or presently being developed in the area, suitable for surface deformation studies. These networks can benefit from the existence of the stations of global reference frames such as those produced by the IERS and IGS which are fortunately dense and accurate in Europe as illustrated in Figure 5-4. Many fixed stations in this region are fiducial stations of IGS and IERS global networks and consequently their tracking history is remarkable for quality and quantity of data. Thus, the space geodetic data of the region will be accurate enough and reliable for the goals of this study.


### 6.1 The Tectonic and Geophysical Settings

The assessment and interpretation of the geodetic results for detection of possible spatial displacements and the Earth surface deformation studies have to be integrated with realistic tectonic and geophysical models for the region. The Earth surface deformations of the European and Mediterranean area can be most readily explained by tectonic forces associated with spreading at the mid-Atlantic Ridge and the northward motion of the African plate relative to the Eurasian plate. The recent major tectonic processes occur within the large-scale kinematic framework of active sea-floor spreading in the Atlantic ocean and the African-Eurasian convergence boundary in the Mediterranean sea, Figure 6-1.


Figure 6-1: Geodynamic settings of Europe and adjacent areas

The higher spreading rate in the South Atlantic ( $40 \mathrm{~mm} / \mathrm{yr}$ ) as compared to that in the North Atlantic $(20 \mathrm{~mm} / \mathrm{yr})$ leads to a gradual counterclockwise rotation of the African plate resulting in a north-northwestwarddirected push against Eurasia, which in turn leads to a lithospheric shortening of 5 to $6 \mathrm{~mm} / \mathrm{yr}$ increasing to $40 \mathrm{~mm} / \mathrm{yr}$ in active subduction zones [D. F. Argus et al. (1989)]. With northwest-southeastward-oriented spreading in the North Atlantic the whole region is expected to be under compression, particularly the Mediterranean area to a large extent. As far as the Mediterranean in concerned, the collision of Eurasia, Africa and Arabia plates make the tectonic of the area very puzzling. The seismic map of the region shown in Figure 6-2 clearly shows a high seismic activity due to relatively strong tectonic forces that govern the compression zone. Moreover, a wide-spread intraplate seismicity takes place in the region which illustrates that the plate collision zone is complex and not sharply defined.

The European and Mediterranean area can be divided into three main sub-regions with distinct geodynamic features, namely western Europe, northern Europe and the Alpine-Mediterranean sub-regions. Within western Europe, roughly covering the area between $45^{\circ} \mathrm{N}$ to $55^{\circ} \mathrm{N}$ latitude and $-10^{\circ} \mathrm{W}$ to $20^{\circ} \mathrm{E}$ longitude, a weak seismic activity is observed. The area is characterized as a field of compressional tectonics. A generalized stress map of Europe [B. Müller et al. (1992)] indicates a generally northwest-southeast uniform orientation for the maximum compressive horizontal principal stress in western Europe.

The northern European sub-region is defined as the area lying at latitudes greater than $55^{\circ} \mathrm{N}$. The stress map of Europe shows that the NW-SE orientation of the maximum compressive horizontal stress, which generally prevails in western Europe, is not as consistent in northern Europe. The Fennoscandian (Norway, Finland and Sweden) part of this area is characterized by thick lithosphere ( $110-170 \mathrm{~km}$ ) and low thermal heat flow $\left(<50 \mathrm{~mW} / \mathrm{m}^{2}\right)$. It is presently undergoing recent crustal uplift as a result of postglacial rebound. The observed land uplift rates, Figure 6-3, vary from $9 \mathrm{~mm} / \mathrm{yr}$ to $-1 \mathrm{~mm} / \mathrm{yr}$ with maximum uplift in the northern Baltic Sea [J. Kakkuri (1997)]. There are geological and geophysical evidences that suggest stresses created from ridge


Figure 6-2: Seismicity of the European and Mediterranean area
push of the Mid-Atlantic Ridge as the dominant source of intraplate deformations in the region, e.g. P. Wu (1998), O. Stephansson (1988). Figure 6-4 shows the spatial distribution of recent earthquakes in northern Europe which indicates little correlation with the center of postglacial rebound. Most of larger earthquakes (magnitude $>4$ ) are distributed along the coastal regions while the interior is relatively nonseismic. The seismic map, Figure 6-4 has been created using a catalogue of earthquakes in northern Europe [T. Ahjos and M. Uski (1992)], updated by the institute of Seismology, University of Helsinki.

Present-day 3-dimensional displacement fields in and around the former glacial area, derived from space geodetic techniques, have stimulated new interest in the contemporary deformations associated with the present postglacial rebound, e.g. H. -G. Scherneck et al. (1998), T. S. James and A. Lambert (1993). The present status of postglacial rebound measurements in Fennoscandia has been summarized in H. -P. Plag et al. (1998). The existence of more than 50 permanent GPS stations covering the area, is a promising news for a more complete understanding of current surface deformation fields in northern Europe.

The Alpine-Mediterranean region marks a broad transformation zone between the African, Arabian and Eurasian plates. The region is expected to be largely under compression. It is characterized as a region of intensive seismic activity, Figure 6-2. The region has been interpreted as an assemblage of microplates trapped between the Eurasian and African plates [C. Gasparini et al. (1985)]. Tectonic evolution of this region is strongly affected by the convergence of the microplates. In spite of the overall compression between African and Eurasian plates, the Alpine-Mediterranean area encompasses large sedimentary basins that have in the past and even now experienced major extension such as the western Mediterranean, Aegean and Panonian basins [J. F. Dewey (1988)]. The displacement rates obtained so far for the region present a complex nature of the tectonics. In Eastern Mediterranean, many of the estimated motions from SLR data, are very different from the model predictions for the major plates [G. Bianco et al. (1998)]. The Aegean Sea and Western Anatolia area is bounded by the North Anatolian fault to the north and the East Anatolian fault to the east, and a convergent boundary, namely the extension of Hellenic Trench, to the south and west. The Hellenic Trench results from the northward motion of Africa relative to Eurasia, Figure 6-1. The stress data for the Aegean Sea and Western Anatolia compiled by B. Müller et al. (1992) show a dominant N-S extension. It has been confirmed by geodetic deformation studies of the area based on space geodetic data, e.g. H. -G. Kahle et al. (1998).


Figure 6-3: The present day observed post glacial land uplift in Fennoscandia [J. Kakkuri (1997), J. Kakkuri and Z. T. Wang (1998)]


Figure 6-4: Seismicity of Northern Europe (1950-2000)

The most recent tectonic feature and the most likely tectonic unit to influence the current deformation field in Europe is the Alpine orogenic belt, Figure 6-1. The Alps have formed as a results of collision between the African plate and Eurasian plate. Currently, the central Alps are estimated to be rising with an average speed of about $2 m m / y r$ [E. Gubler et al. (1984), G. Bada et al. (1999)].

### 6.2 Space Geodetic Data

Figure 6-5 represents the sites of ITRF97 solution across the European and Mediterranean area. The ITRF97 solution includes positions, displacement rates (velocities) and uncertainties of about 390 stations around the world. The solution is based on a combination of a selected set of individual solutions submitted to the IERS Central Bureau in 1998 and some past data sets. The individual solutions selected for the ITRF97 analysis are 4 VLBI, 5 SLR, 6 GPS, 3 DORIS and one multi-technique (SLR+DORIS) solutions [C. Boucher et al. (1999)]. Most of the individual solutions have data time spans longer than 4 years, Table 61. The site positions, displacement rates, and uncertainties can be downloaded free from the internet site (http://lareg.ensg.ign.fr/ITRF/ITRF97/). Table 6-1 gives the quality analysis of the ITRF97 results which is based more specifically on global residuals per solution.


Figure 6-5: Sites of ITRF97 in the European and Mediterranean area

Table 6-1: Selected solutions for the ITRF97 analysis and their global residuals

| No. | Technique | Data time span <br> $[\mathrm{yr}]$ | Number of <br> stations | Reference <br> epoch | Position RMS <br> $[\mathrm{mm}]$ | Velocity RMS <br> $[\mathrm{mm} / \mathrm{yr}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | VLBI | $1979-98$ | 129 | $97: 001$ | 3.3 | 0.7 |
| 2 | VLBI | $85-97$ | 49 | $93: 001$ | 2.5 | 0.6 |
| 3 | VLBI | $79-94$ | 107 | $93: 001$ | 5.8 | 1.1 |
| 4 | VLBI | $79-98$ | 110 | $97: 001$ | 2.9 | 0.6 |
| 5 | SLR | $86-98$ | 76 | $93: 001$ | 11.9 | 2.2 |
| 6 | SLR | $76-98$ | 129 | $93: 001$ | 11.7 | 2.5 |
| 7 | SLR | $83-97$ | 72 | $93: 001$ | 19.2 | 3.3 |
| 8 | SLR | $80-97$ | 38 | $86: 182$ | 3.8 | 0.7 |
| 9 | SLR | $93-98$ | 51 | $93: 001$ | 5.2 | 2.8 |
| 10 | GPS | $93-98$ | 139 | $95: 314$ | 3.7 | 1.6 |
| 11 | GPS | $94-97$ | 40 | $98: 001$ | 5.3 | 0.5 |
| 12 | GPS | $96-98$ | 67 | $97: 074$ | 1.9 | 1.8 |
| 13 | GPS | $93-97$ | 76 | $97: 001$ | 8.7 | 2.9 |
| 14 | GPS | $91-98$ | 84 | $96: 001$ | 3.6 | 2.2 |
| 15 | GPS | $95-98$ | 145 | $98: 001$ | 2.1 | 2.8 |
| 16 | DORIS | $93-96$ | 54 | $93: 001$ | 26.9 | 10.4 |
| 17 | DORIS | $93-98$ | 63 | $93: 001$ | 27.9 | 5.5 |
| 18 | DORIS | $90-97$ | 69 | $94: 001$ | 32.1 | 10.7 |
| 19 | SLR + DORIS | $85-96$ | 143 |  | 10.8 | 4.4 |

Besides the accuracy, the main feature of ITRF97 displacement rates is that they are free of any tectonic plate model assumption. This feature makes the data adequate for this study. The European and Mediterranean area is covered by nearly 100 sites of ITRF97 solution. We make use of this sufficiently dense space geodetic data to obtain various surface deformation patterns of the area. Figure 6-6 plots representative horizontal displacement rates in a western Europe-fixed frame. In this map, an averaged displacement vector in western Europe is subtracted from all the ITRF97 displacement vectors to draw the vectors in a western Europe-fixed frame. Unlike the northern European sites in Scandinavia, the western European stations north of the Alps do not show significant horizontal motions. In the Mediterranean area, the horizontal displacement vectors behave completely different. Moving westward across Turkey to the southern Italy, the horizontal displacement rates are detected with large magnitude, over $30 \mathrm{~mm} / \mathrm{yr}$, apparently in irregular and different directions. As can be seen, the plot of horizontal displacement vectors does not give much information about the surface deformation fields especially in high active seismicity regions such as the Mediterranean area. This justifies why we have to look for some other analysis and representation procedures of the Earth surface deformations.

Figure 6-7 shows the vertical components of the displacement rates of ITRF97 sites in the European and Mediterranean area. The upward and downward arrows indicate positive and negative vertical movements in real scale, respectively. Similar to the sketch of the horizontal components of the displacement rates in Figure $6-6$, the sketch of the vertical components can not be of much use for interpretation of the current kinematics of the area, particularly relevant to existence of rising and sinking regions.

## Horizontal Displacement Rates in a Western Europe Fixed Frame



Figure 6-6: Horizontal displacement rates of ITRF97 sites in a Western Europe-fixed frame

Vertical Component of the Displacement Rates [mm/yr]


Figure 6-7: Vertical displacement rates of ITRF97 sites

### 6.3 Data Processing Strategy

The main objective of this study is to obtain patterns of the Earth surface deformation parameters for the area, covered by the selected sites of ITRF97 solution. The analysis procedure is divided into four main computational phases: conversion of space coordinates to surface coordinates, 2-dimensional finite element method, left linearized Euler-Lagrange deformation tensor of the first and second kinds and invariants associated with these deformation tensors, and graphical representation of the numerical results. From theoretical point of view, both the developed intrinsic and extrinsic approaches turn out the same results for the surface deformation tensors and consequently their associated invariants. In practice, there are some numerical limitations and difficulties related to the application of the extrinsic approach. Particularly, the extrinsic approach based on space curvilinear coordinates of the displacement vector requests 3 -dimensional finite element method for numerical estimation of necessary partial derivatives of the mathematical models. Moreover, we have special interest in investigation of the efficiency of the intrinsic approach, as the standard method of surface deformation analysis in shell theory, in Earth surface deformation analysis. The results of this investigation clarify applicability of the intrinsic approach to the Earth deformation studies and let us take more advantages of existing mathematical models and relations of shell theory in our field.

To put into practice the intrinsic surface deformation analysis, in the first phase the space Cartesian coordinates of ITRF97 sites $\left(X^{1}, X^{2}, X^{3}\right)$ are transformed onto geodetic coordinates $(L, B, H)$ for the whole test area. The geodetic coordinates are computed with respect to GRS80 reference ellipsoid ( $A_{1}=6378137.0 \mathrm{~m}, E^{2}=$ 0.00669438003 ), recommended for the transformation of ITRF Cartesian coordinates onto geodetic coordinates [D. D. McCarthy (1996)]. The Cartesian coordinates of the displacement rate vectors are also converted onto the surface curvilinear coordinates according to the developed mathematical models in Section 4.4.2. Now, the Earth surface of the test area has been covered by the chart of the geodetic longitude and geodetic latitude as surface curvilinear coordinates. The mathematical models of the left Euler-Lagrange deformation tensors of the first and second kinds (Lagrangian strain tensor and Lagrangian tensor of change of curvature), in the intrinsic approach, requires that partial derivatives of some unknown functions such as $H(L, B), \bar{U}^{\Lambda}(L, B)$ and $\bar{W}^{\Lambda}(L, B)$ be provided. We apply the 2-dimensional finite element approach to find approximate solutions of these unknown functions. The planar triangular element is selected and Delaunay triangulation method is utilized for mesh generation in the plane of chart of the geodetic coordinates as described in Section 5.3. Figure 6-8 depicts the results of optimal Delaunay triangulation among the points on the chart of the area.

Optimal Delaunay Triangulation of ITRF97 Data Set


Figure 6-8: Optimal Delaunay triangulation of ITRF97 sites across the European and Mediterranean area

We consider the element trial solution to be a linear 2-dimensional polynomial in our straight-sided triangular-shaped elements.

$$
\begin{equation*}
\tilde{U}(L, B ; A)=A_{1}+A_{2} L+A_{3} B \tag{6-1}
\end{equation*}
$$

The coefficients $A_{1}, A_{2}, A_{3}$ are determined for each unknown function, namely surface curvilinear coordinates of $\mathbf{u}$ and $\mathbf{w}$ and height, based on the nodal values of the function. Clearly, the inter-element boundary condition is satisfied by the element trial solution $F(L, B)$ on the boundaries between triangular elements. In fact, this choice of a triangular shape and the placement of the interpolation points at the nodes achieve the desired interelement continuity. The interelement continuity will be of class $C^{0}$ in this case. Thus, the element trial solution will be continuous but not its partial derivatives. It should be noted that unlike the Cartesian coordinates of the displacement vectors which are input data of the analysis, the Cartesian coordinates of the difference vector of the unit normal vectors $W^{I}$ have to be determined at nodal points. Having the estimated values of the partial derivatives of the height functions $H(L, B)$ and $h(l, b)$ of each element, the space Cartesian coordinates of the unit normal vectors $N(L, B)$ and $n(l, b)$, and consequently $W^{I}$ can be computed based on the developed mathematical models in Box 4-6 at nodal points. The remaining computational steps for quantifying the partial derivatives of the surface functions $W^{\Lambda}(L, B)$ and $W^{3}$ is similar to the surface curvilinear coordinates of the displacement vector field.

In the third phase, the left linearized Euler-Lagrange deformation tensors of the first and second kinds are quantified at the geometrical center of each element using the results of the finite element analysis of the second phase. These quantities are utilized to the computations of the scalar invariants connected to these two deformation tensors, introduced in Section 3.5. In the last step of this phase, the scalar invariants are refined from any outliers that may have resulted due to any unrealistic estimation of displacement rates in ITRF97 solution or weak geometry of the triangular elements.

The numerical results have to be represented in an appropriate way for any further interpretations and comparisons. We used the GMT (Generic Mapping Tools) graphical software package [P. Wessel and W. H. F. Smith (1998)] to map the surface deformation patterns. The main advantages of the software, for our application, are its ability in contouring the input data and representing the patterns in adequate map projections. Albers Conic map projection is often utilized to create the plots. The results are presented in Figure 6-9 to Figure 6-18. The projection, developed by H. C. Albers (1773-1833) of Germany in 1805, is generally used to map regions of large east-west extent. It is a conic, equal-area projection, in which parallels are unequally spaced arcs of concentric circles. The parallels are more closely spaced toward north and south edges of the map. Meridians are equally spaced radii about a common center, and cut the parallels at right angles. The two standard parallels are free of angular and scale distortion. The distortion will be constant along any parallel and small for the area between the two standard parallels.

### 6.4 Analysis and Discussion of the Results

Various deformation patterns, based on intrinsic deformation analysis of the ITRF displacement rates, are compared with the geophysical and geological deformation evidences of the area. The analysis and interpretation of the surface deformation patterns of the area are done by taking into account the geological and geophysical information to avoid any sort of the misleading interpretations. We first discuss the patterns of the invariants associated with the left Euler-Lagrange deformation tensor of the first kind (Lagrangian strain tensor), namely surface maximum shear strain and surface dilatation. They are well known and often used for interpretation of the strain tensors. Then, we analyze the patterns of the associated invariants of linearized surface rotation tensor and the Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature), namely linearized rotation around the surface normal, Mean- and Gaussian curvature difference. These invariants are less familiar and new in the Earth deformation investigations as surface deformation measures.

### 6.4.1 Left Euler-Lagrange deformation tensor of the first kind (Lagrangian Strain tensor)

Surface maximum shear strain and surface dilatation were introduced as two associated invariants of surface strain tensor in Section 3.5. Here, we study patterns of these invariants as well as another associated invariant of the strain tensor, the so-called maximum geodetic surface strain. Figure 6-9 maps maximum geodetic surface strain rate for the European and Mediterranean area in units of $10^{-8} / y r$. The maximum geodetic surface strain rate is the largest eigenvalue, in absolute value, of rate of the Lagrangian surface strain tensor, $\max \left[\left|\Lambda_{1}^{\prime}(L, B)\right|,\left|\Lambda_{2}^{\prime}(L, B)\right|\right]$. S. N. Ward (1998a, 1998b) offers the maximum geodetic strain rate as a good representation of total deformation where only horizontal motions are available or accurate. It should be noted that Figure 6-9 does not provide contours for maximum geodetic strain rates in equal contour intervals because the rates vary by a factor of 100 across the area. The values of the contour lines are considered in the manner such that the whole area of the study is covered by contour lines. This strategy is considered for all the contour maps of this section. Thus, we can quantify the surface deformation measures in any particular region of our maps.

## Maximum Geodetic Surface Strain Rate $\times 10^{-8} / \mathrm{yr}$



Figure 6-9: Maximum geodetic surface strain rate

The pattern of surface maximum shear strain rate in units of $10^{-8} / y r$ is shown in Figure 6 -10. A comparison between Figure 6-9 and Figure 6-10 shows similarity between the two maps. The surface maximum shear strain rates have greater values than the maximum geodetic strain rates which can be expected due to their mathematical formulations. Similar to the map of horizontal displacement rates Figure 6-6, the surface maximum shear strain rates increase southward from a continent low values $0.75 \times 10^{-8} / \mathrm{yr}$ in the western Europe to high values $7 \times 10^{-8} / y r$ across northern Turkey and Aegean sea. Unlike western Europe, high values about $9 \times 10^{-8} / y r$ are observed over the southwestern Fennoscandia.

Surface Maximum Shear Strain Rate x $10^{-8} / \mathrm{yr}$


Figure 6-10: Surface maximum shear strain rate
Northern Italy is another region with high surface maximum shear strain rates peak beyond $8 \times 10^{-8} / \mathrm{yr}$. As can be seen, regions in vicinity of plate boundaries are mapped with significantly larger values of the surface maximum shear strain rates. Surface maximum shear strain represents the anisotropic part of deformation and is considered as a key deformation measure in understanding physical processes of the Earth surface and earthquake prediction studies. A comparison can be made between the pattern of surface maximum shear strain and the seismic map of the Mediterranean area, Figure 6-2, and also the seismic map of the northern

Europe, Figure 6-4. The agreement of the geodetic pattern and seismic maps, namely peaks of the pattern at regions with high seismic activity, confirms the key role of the surface maximum shear strain rate in earthquake studies as well as the validity of the surface maximum shear strain rate pattern which are obtained based on the intrinsic Earth surface deformation analysis.

Figure 6-11 illustrates the patterns of surface dilatation rates in units of $10^{-7} / y r$ over the European and Mediterranean area. The surface dilatation represents the isotropic part of surface deformation. The negative values of surface dilation indicate areas which are under a compressional strain regime. Its positive values are observed at areas with extensional strain regime. In general, the pattern clearly reveals extensive regions under high compressional or extensional strain regimes in vicinity of plate boundaries across Italy and eastern Mediterranean.

Surface Dilatation Ratex $10^{-7} \mathrm{yr}$


Figure 6-11: Surface dilatation rate

There are no significant values of surface dilatation over western Europe. A very interesting pattern of surface dilatation rate can be seen across the northern Europe, particularly in the southwestern Fennoscandia. The pattern in this area indicates that although most of central and eastern Fennoscandia are characterized as areas of extension because of post glacial land uplift, the western coastal regions are under compression due to the ridge-push forces at Mid-Atlantic. The stress data for the Aegean sea and the western Turkey compiled by B. Müller et al. (1992) come mainly from neotectonics and seismology. These data suggest strong north-south extension in the Aegean sea and the western Turkey. A number of further studies based on geodetic data has verified the north-south extension of the region, e.g. H. -G. Kahle et al. (1998), The pattern of surface dilatation rate for these area, $40^{\circ} N$ and $25^{\circ} \mathrm{E}$, derived from space geodetic data, confirms obviously the existence of the dominant extensional strain regime. West and south of this area, the central Turkey and eastern Mediterranean are to a large extent under compression due to north-northwestward directed push of the African and Arabian plates against the Eurasian plate.

### 6.4.2 Left surface linearized rotation tensor

The linearized rotation around the normal is defined in Section 3.5 as an associated invariant of linearized surface rotation tensor. Unlike the surface maximum shear strain and surface dilatation, the surface rotation tensor has rarely been considered as a deformation measure in Earth deformation studies. Here, we investigate which significant role the linearized rotation around the normal can play as an invariant deformation measure in Earth deformation studies. Figure 6-12 contours absolute values of rates of the linearized rotation around the normal $|\Phi|$ in units of $10^{-8} r a d / y r$ for the European and Mediterranean area. In general, the pattern of the linearized rotation around the normal is similar to the pattern of surface maximum shear strain, Figure 6-10. Peaks as high as $3 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$ are monitored in the regions of high active seismicity such as the northern Italy, southwestern Scandinavia and east Mediterranean. The correlation between the pattern of absolute values of rates of the linearized rotation around the normal and the seismic map of the area proves that the linearized rotation around the normal as a deformation measure can play a valuable role in earthquake investigations. Moreover, the pattern in the eastern Mediterranean area presents more detailed information in comparison to the pattern of surface maximum shear strain for the same area.

The pattern of the rates of the linearized rotation around the normal in units of $10^{-8} \mathrm{rad} / \mathrm{yr}$ is illustrated in Figure 6-13. In this map, rates of the linearized rotation around the normal with positive values are referred to as clockwise rotations, shown by the red contour lines, with negative values as counterclockwise rotations, shown by the blue contour lines. The pattern uncovers some significant signals of the current kinematics of the area. In northern Europe, two adjacent zones of high surface rotation are observed. They display a clockwise rotation of nearly $3 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$ for the western zone and a counterclockwise rotation of about $-2 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$ for the eastern zone. The zones can be linked to preweakened tectonic belts of the area and the ridge-push forces at the Mid-Atlantic which are perpendicular to the shore lines in Scandinavia. Rotations of these tectonic belts can be responsible for the high seismicity of the region.

Another zone of high values of surface rotation is obviously monitored over the Anatolian microplate in Turkey. The shape of contour lines of the zone proves the existence of the microplate in this region and suggests a counterclockwise rotation of about $-4 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$ for this microplate. The Arabian plate is moving in a north-northwest direction relative to Eurasia at a rate of about $25 \mathrm{~mm} / \mathrm{yr}$. The African plate is also moving in a northward direction relative to Eurasia at a rate of about $10 \mathrm{~mm} / \mathrm{yr}$. Differential motion between Africa and Arabia ( $\sim 15 m m / y r$ ) is thought to cause the apparent westward extrusion of the Anatolian plate [D. P. McKenzie (1970)]. R. E. Reilinger et al. (1997a) have estimated a counterclockwise rotation of $1.3^{\circ} \pm 0.1 / \mathrm{Myr}$,
or $2.3 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$, for the Anatolian microplate based on a data set of GPS measurements across the eastern Mediterranean. The GPS-derived Euler pole of the microplate, located at $29^{\circ} N 33^{\circ} \mathrm{E}$, is very close to the peak of the contour lines of the pattern of linearized rotation around the normal over this area.

## Absolute Value of rate of the linearized rotation around the normalx $10^{-8} \mathrm{rad} / \mathrm{yr}$



Figure 6-12: Absolute values of rates of the linearized rotation around the normal

Moving westward in the Mediterranean, the pattern exhibits a zone of high values of surface rotation over Greece. It suggests existence of a subregion with a clockwise rotation of about $3 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$ in this area. Some independent investigations support this idea. F. Horner and R. Freeman (1983) reported geological evidence of a clockwise rotation of $38^{\circ}$ in northwestern Greece since 30 million years ago, namely a rotation rate of about $2.2 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$. Moving upward from this region, a small zone with a counterclockwise rotation is found in Northern Italy. The noticeable behaviour of contourlines has been verified independently to be due to the underlying kinematics of the region. A.Caporali and S.Martin (2000) have concluded the existence of such a zone with a counterclockwise rotation relative to a stable continental Europe in Northern Italy from analysis of

## Rate of the linearized rotation around the normal $\times 10^{-8} \mathrm{rad} / \mathrm{yr}$



Figure 6-13: rates of the linearized rotation around the normal
the observations of permanent GPS stations along the flanks of the Alps. These geodetic interpretations have been supported by other investigations based on geologic and structural data, e.g. A. Castellarian et al. (1992), G. Renner and D. Slejko (1994).

In southwestern Europe, another region with a clockwise rotation of about $2 \times 10^{-8} \mathrm{rad} / \mathrm{yr}$ is located over Spain. Taking into account the northwestward motion of African plate relative to Eurasian plate, the clockwise rotation of this subregion of Eurasian plate can be justified. Moreover, the seismic and tectonic maps of the area indicate that a different deformation regime rules over Spain which distinguishes the motion of this subregion from the rest of the Eurasian plate.

### 6.4.3 Left Euler-Lagrange deformation tensor of the second kind (Lagrangian tensor of change of curvature)

As mentioned before, the use of the Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) as a deformation measure is new in Earth deformation studies. We decided to first look at the patterns of difference and sum of eigenvalues of the tensor. The aim was to evaluate the patterns for any trends and meaningful results connected to the eigenvalues. Figure 6-14 shows the pattern of the difference of eigenvalues $\Lambda_{\max }^{\prime \prime}-\Lambda_{\min }^{\prime \prime}$ for the quantified Lagrangian tensor of the change of curvature over the European and Mediterranean area. According to its definition, the difference is a positive quantity like the maximum shear strain. Thus, the pattern includes only contourlines of positive

Rate of the Tensor of Change of Curvature:Difference of eigenvalues $\Lambda_{\text {max }}^{\prime \prime} \Lambda_{\text {min }}^{\prime \prime}$


Figure 6-14: Rate of the left Euler-Lagrange deformation tensor of the second kind (Lagrangian tensor of change of curvature): $\Lambda_{\max }^{\prime \prime}-\Lambda_{\min }^{\prime \prime}$

Rate of the Tensor of Change of Curvature: sum of eigenvalues $\Lambda_{\text {max }}^{\prime \prime}+\Lambda_{\text {nin }}^{\prime \prime}$


Figure 6-15: Rate of the left Euler-Lagrange deformation tensor of the second kind (Lagrangian tensor of change of curvature): $\Lambda_{\max }^{\prime \prime}+\Lambda_{\min }^{\prime \prime}$
and zero values.In general, the pattern is very similar to the pattern of surface maximum shear strain rates in Figure 6-10. The pattern clearly uncovers the expected regions of highly active surface deformations. The southern Scandinavia and northern Italy are covered with two obvious peaks of contour lines. The values $\Lambda_{\max }^{\prime \prime}-\Lambda_{\min }^{\prime \prime}$ generally increase southwestward from nearly zero values in western Europe to a rainbow of contour lines of significantly larger values over the central and east Mediterranean. The very interesting feature of the pattern is a peak of high values around the triple junction of Arabian, African and Eurasian plates in the southeast of the map where a high active zone of deformations can be expected. Unlike the difference of eigenvalues $\Lambda_{\max }^{\prime \prime}-\Lambda_{\min }^{\prime \prime}$, the sum of eigenvalues can be negative or positive quantities like surface dilatation.

The sum of the eigenvalues of the Lagrangian tensor of change of curvature is contoured for the test area in Figure 6-15. The map contains some clear trends over the active deformation regions. The patterns of difference and sum of the eigenvalues proves that the coordinates of the tensor of change of curvature, derived from the intrinsic deformation analysis of the ITRF97 data, possess some signals of deformation processes of the test area. This motivates the computation of values of the associated invariants of the tensor and study of their patterns.

Mean- and Gaussian curvature differences were introduced as scalar invariants associated with the tensor of change of curvature in Section 3.5. Positive and negative values of these invariants are connected to relative motion in direction of the normal to the surface, namely rising and lowering regions on the deforming surface. This will be the main characteristic of patterns of Mean- and Gaussian curvature differences. They are used to investigate possible vertical deformations of the surface. The rate of mean curvature difference in units of $10^{-14} / m y r$ is shown in Figure 6-16. In this map, negative values of rate of Mean curvature difference are

## Rate of Mean Curvature Difference x $10^{-14} / \mathrm{m}$ yr



Figure 6-16: Rate of the Mean curvature difference
referred to as rising regions, shown with the blue contour lines, and positive values as sinking regions, shown with the red contour lines. The pattern reveals some significant signals of the current vertical deformations of the area. In northern Europe, the most interesting signal of the map is relevant to undergoing postglacial land uplift of Fennoscandia. The shape of contour lines is in a general similarity with the map of the observed land uplift of the area in Figure 6-3. The peaks of the maximum land uplift and the maximum Mean curvature difference are located closely in the north of Baltic Sea.

In a similar fashion, Figure 6-17 maps the rate of Gaussian curvature difference in units of $10^{-21} / \mathrm{m}^{2} y r$. Both the maps show nearly the same pattern of rising and sinking regions in the European and Mediterranean area, though the map of Gaussian curvature difference apparently behaves more stable than the map of mean curvature difference. Another interesting result of these patterns is the evidence of a sinking area in the North

## Rate of Gaussian Curvature Differencex $10^{-21} / \mathrm{m}^{2} \mathrm{yr}$



Figure 6-17: Rate of the Gaussian curvature difference
sea, among United Kingdom, Norway, Denmark and Germany. The sinking area can be related to the oil extraction activities of the area. The pattern of rate of Gaussian curvature difference is portrayed in a larger scale for the specific region of northern Europe in Figure 6-18. The disagreement between the pattern of Gaussian curvature difference and the map of observed land uplift over Finland and the southern Baltic sea can be traced back to lack of necessary accuracy of vertical components of the displacement rates, or appropriate coverage of the stations in the eastern Fennoscandia.

In the Mediterranean area the pattern of rate of Gaussian curvature difference, mapped in a larger scale in Figure 6-19, shows clearly a rising region in Aegean sea and a sinking region across the Pannonian basin and the northeastern Italy. These rising and sinking regions can be connected to large sedimentary basins existing in these areas. The shape of contour lines of the sinking region in the northeastern Italy is approximately similar

## Rate of Gaussian Curvature Differencex $10^{21 /} / \mathrm{m}^{2}$ yr



Figure 6-18: Rate of Gaussian curvature difference in Fennoscandian area

## Rate of Gaussian Curvature Differencex $10^{-21} / \mathrm{m}^{2}$ yr



Figure 6-19: Rate of Gaussian curvature difference in Mediterranean area
to the shape of Pannonian basin in Figure 6-1. G. Bada et al. (1999) present a comprehensive review of the recent crustal deformation and geodynamics of the Pannonian basin and its surroundings. This study confirms that distinct areas inside the Pannonian basin are still subsiding. Unfortunately, assessment and interpretation of rising and sinking fields of the patterns, based on geological and geophysical evidences, are very difficult. Except the postglacial land uplift phenomena, which has been under intensive investigations and relatively well-known, most of deformation studies focus on horizontal motions and only little is known about possible vertical deformations in the European and Mediterranean area.

Figure 6-20 shows pattern of vertical components of the displacement rates in the European and Mediterranean area. The figure contours the vertical component for the area in units of millimeter as obtained from difference of geodetic heights of each site. Thus, contours of positive values are connected to rising regions whereas contours of negative values are related to sinking regions. In the northern Europe, it is very difficult to conclude
existence of the rising region. There is no consistency between the pattern and the real situation of the vertical deformations of the area. The shape of the contour lines of the northern Europe and the contour lines of the land uplift of Fennoscandian area in Figure 6-3 can not stick together.

A comparison of the patterns of vertical components of the displacement rates in Figure 6-20 and the Gaussian curvature differences in Figure 6-17 indicates capability of the pattern of Gaussian curvature differences in describing current kinematics of the area in vertical direction. In other words, spatial pattern of Gaussian curvature differences presents more reliable and accurate portray of the existing sinking and rising regions of the study area. This can be considered as a unique ability of the Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) and its associated invariants in comparison to what can be extracted in this context from the vertical components alone.

## Pattern of vertical component of displacement rates [mm/yr]



Figure 6-20: Pattern of vertical components of the displacement rates in units of $m m / y r$

## Chapter 7

## Summary and Conclusions


#### Abstract

This chapter summarizes the main contributions and results of this study. The advantages and the characteristics of the proposed method of deformation analysis of the Earth surface are critically reviewed. Having recognized the absence of any method of surface deformation analysis which has been developed based on the geometry of the real surface of the Earth and the incompleteness of the existing mathematical models for analysis in geodetic literature, this study has presented the development and the implementation of the intrinsic deformation analysis of the Earth surface. The developed models of the analysis were established upon the observed displacement vectors on the Earth surface without referring or projecting the vectors on any reference surfaces.

Tensor analysis was extensively applied to the mathematical formulation of the method. Lagrangean and Eulerian descriptions of various deformation tensors were introduced as measures of surface deformations in Chapter 3. The description of the tensors obtained as functions of space Cartesian and space curvilinear components of the displacement vector $\mathbf{u}$ and the difference vector of the unit normal vectors $\mathbf{w}$ in the extrinsic approach, and as functions of surface curvilinear components of $\mathbf{u}$ and $\mathbf{w}$ in the intrinsic approach. Particular attention was given to definitions and derivations of the linearized rotation tensor and Euler-Lagrange deformation tensor of the second kind (tensor of change of curvature) according to literature of shell theory. The elegance of this treatment was clearly demonstrated by the formulation of the Euler-Lagrange deformation tensor of the second kind in terms of the difference vector of the unit normal vector as well as the displacement vector. The introduction of the difference vector $\mathbf{w}$ yielded transparent and compact mathematical expressions for the Euler-Lagrange deformation tensor of the second kind in both the extrinsic and intrinsic approaches. The treatment was also advantageous as far as computer programming and practical applications were concerned. It allowed us to quantify Euler-Lagrange deformation tensor of the second kind based only on the first order partial derivatives of $\mathbf{u}$ and $\mathbf{w}$, instead of higher order partial derivatives of the displacement vector. The independence of the numerical computations from estimation of these higher order derivatives clearly improved the numerical accuracies of the computations and achieved meaningful results for the Euler-Lagrange deformation tensor of the second kind.


The appropriate invariants with specific physical meanings were defined for all the introduced deformation tensors of the study in Section 3.5. The considerable role of these invariants was obvious in graphical representations of spatial variations of the deformation tensors fields. The graphical representation of a scalar field is an easier task in comparison to a tensor field. Moreover the spatial patterns of the invariants eased physical interpretation and assessment of the information contained in the deformation tensor fields especially the comparison of the information with the geodynamic and seismic maps of the studied area.

Assuming a Gaussian representation of the Earth surface in terms of the geodetic coordinates, the mathematical
models of the extrinsic and intrinsic approaches were set up for the parameterized Earth surface in Chapter 4. The coordinates of the deformation tensors were obtained as explicit functions of the space Cartesian and curvilinear components of $\mathbf{u}$ and $\mathbf{w}$ in the extrinsic approach, and of the surface curvilinear components of $\mathbf{u}$ and $\mathbf{w}$ in the intrinsic approach. A key step for the method of intrinsic analysis was the conversion of the Cartesian coordinates of $\mathbf{u}$ and $\mathbf{w}$ into their surface curvilinear coordinates. The exact relations as well as the approximate technique of the conversion were established and discussed in Section 4.4. The exact relations were developed based on the theory of shifters in tensor analysis whereas the approximate technique was set up based on the method of least squares solution.

As a real case study, a series of tests using displacement rates and positions of ITRF97 solution was performed to investigate the efficiency of the proposed method of the Earth surface deformation analysis in Chapter 6. The study made use of the space geodetic displacement field available in the European and Mediterranean area. The data set included sufficiently dense and accurate data for the aims of this study. Moreover, the study area, rich in various tectonic and geodynamic processes, was one of the best studied regions on the Earth from geological, geophysical and geodetic points of view. A huge number of publications exist addressing geodynamic problems and contribute to the better understanding of various aspects of kinematics of the area. The numerical results of the tests of the geodetic data based on the developed models of the intrinsic approach were graphically displayed as various spatial patterns. A comparison of the patterns with the geological and geophysical evidences of the area indicated how well the patterns were able to reveal different geodynamical features of the region. One of the main goals of this research was an investigation about the possible applications of the linearized rotation tensor and Euler-Lagrange deformation tensor of the second kind in Earth deformation studies. The meaningful and reliable patterns of the linearized rotation around the normal revealed that the linearized surface rotation tensor contained valuable information. A comparison of this pattern with the surface maximum shear strain pattern shows the ability of the linearized rotation to reveal more details of the underlying geodynamics of the area. It yields a better insight into the tectonic structure and possible rotations of specified or unspecified subplates with respect to each other. One of the most interesting results of this research is the spatial pattern of Gaussian curvature difference as an associated invariant of the Euler-Lagrange deformation tensor of the second kind. The power of the patterns in detecting and addressing the on-going sinking and rising regions showed that the tensor was able to provide significant information about the vertical changes of the Earth surface in the area. Reliable estimates of vertical motions could be extracted from the patterns because of their stable behaviours. The stability of the numerical results of the tensor could be due to its differential nature which filtered out small irregularities in the discrete displacement field and presented more meaningful patterns for vertical deformations. Thanks to the Euler-Lagrange deformation tensor of the second kind, the surface method preserved the simplicity of 2-dimensional spaces while it nicely modelled vertical deformations.

We should not forget the significant role of the 2-dimensional finite element method in the determination of an approximate solution for the unknown functions. Reliable estimates of the necessary partial derivatives of the mathematical models based on the finite element triangulation lead us to convincing numerical results. The method was able to extract valuable information from a 3-dimensional discrete displacement field. The spatial pattern presented different aspects of governing deformation regimes of the study area. Thus, the proposed method, as a well developed and efficient tool of the Earth surface deformation analysis, equips us with a new way for a deeper understanding of the present day kinematics and strain regime of the Earth surface. A deeper knowledge of spatial variations in the deformation fields of the Earth surface will certainly provide a key contribution to establishing a link between the determined kinematics and the dynamics of the Earth's interior.

This study has indicated the applicability of the developed concepts and mathematical tools of shell theory in Earth deformation studies. This attempt opened the door for geodesy to join other scientific disciplines which have been taking advantages of these tools in shell theory. Further studies could be carried out to improve the performance and numerical results of the model of finite element method used in this study. An example
could be use of curved triangular elements, with the sides of the elements are chosen along geodesic lines on the deforming surface, instead of planar ones which might further improves accuracy of the phase of unknown functions modelling in the intrinsic approach. There exist a great number of studies concerning the application and efficiency of different methods of finite element in shell theory.

Research could be undertaken into establishment of stress-strain relations and functional representation of strain energy as further steps in Earth surface deformation analysis based on the theoretical foundation provided herein. Further work should be undertaken to formulate uncertainty measure and assess the statistical significance of coordinates of surface deformation tensors and their associated invariants. Although the individual coordinates of the deformation tensors are linear functions of displacement vector, their associated invariant are not. The formulation of general eigenvalue problem introduces non-linearity in the associated invariants. Fortunately, a convenient theoretical basis has been provided in this direction by the works of P. Xu and E. W. Grafarend (1996) and, P. Xu (1999).

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