Observability Analysis and Observer Design for Controlled Population Dynamics

Beobachtbarkeitsanalyse und Beobachterentwurf für Populationsmodelle mit Eingang

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Abstract

The diploma thesis studies the design of nonlinear observers with exactly linear error dynamics via transformation or immersion into an appropriate observer canonical form. A model for the dynamics of two interacting species, which was derived as a generalisation of the predator–prey model by Lotka and Volterra, is used as benchmark system for this design. In particular, an additional control input is modelled in three ways.

As observability of the system is a necessary condition for observer design, the methods for an observability analysis are presented and applied to the model. After that, the theoretical basics of the observer design methods are described and used for the design of an observer with exactly linear error dynamics, with regard to the results from the observability analysis. The observers are designed as Luenberger observers with output injection for the uncontrolled system and with input/output injection for the controlled systems.

Some new studies concern the invariance properties of the nonlinear observers on a state region which is relevant for the system. For this purpose, the notation of invariant observers is introduced, which guarantee a global observation of the system on the relevant region. Based on the considered observer canonical form, this notation helps to develop some general methods how to design such observers.

Zusammenfassung

Die Diplomarbeit untersucht den Entwurf nichtlinearer Beobachter mit exakt linearer Fehlerdynamik durch Transformation oder Immersion in eine geeignete Beobachternormalform. Als Benchmark–System für diesen Entwurf wird ein Modell zweier interagierender Populationen verwendet, das sich aus einer Verallgemeinerung des Räuber–Beute–Modells von Lotka und Volterra ergibt. Insbesondere werden drei Möglichkeiten betrachtet, einen Eingang zu diesem System hinzuzufügen.

Da die Beobachtbarkeit des Systems eine notwendige Voraussetzung für den Entwurf eines Beobachters ist, werden zuerst die Methoden zur Beobachtbarkeitsanalyse vorgestellt und auf das Modell angewandt. Danach werden die theoretischen Grundlagen der Beobachterentwurfsverfahren erläutert, welche unter Berücksichtigung der Ergebnisse aus der Beobachtbarkeitsuntersuchung für den Entwurf eines Beobachters mit exakt linearer Fehlerdynamik verwendet werden. Die Beobachter werden als Luenberger–Beobachter mit Ausgangsaufschaltung für das autonome System und mit Eingangs–/Ausgangsaufschaltung für die Systeme mit Eingang entworfen.

Neue Fragestellungen betreffen die Invarianzeigenschaften der entworfenen Beobachter auf einem für das System relevanten Zustandsbereich. Hierfür wird die Notation der invarianten Beobachter eingeführt, die eine globale Beobachtung des Systems im relevanten Bereich garantieren. Basierend auf der verwendeten Beobachternormalform werden mit Hilfe dieser Notation einige allgemeine Methoden entwickelt, mit denen solche Beobachter entwerfen werden können.

Deutsche Kurzfassung:
Beobachtbarkeitsanalyse und Beobachterentwurf für Populationsmodelle mit Eingang

Entstehung

Die vorliegende Arbeit entstand während eines sechsmonatigen Erasmus–Aufenthalts am Institut National des Sciences Appliquées (INSA) in Rouen im Wintersemester 2004/05. Der Aufenthalt wurde von der Friedrich–Ebert–Stiftung und von der Europäischen Union im Rahmen des Erasmus–Programms gefördert.

Die Arbeit wurde von Prof. Witold Respondek betreut, der am INSA Rouen Direktor des Laboratoire de Mathématiques ist. Prof. Respondek beschäftigt sich dort in Lehre und Forschung mit der mathematischen Regelungstheorie. Einige in dieser Arbeit verwendete Methoden sind das Resultat seiner Forschungsarbeiten. An der Universität Stuttgart wurde die Arbeit von Prof. Dr.-Ing. Dr.h.c. Michael Zeitz betreut. Einige wertvolle Hinweise, insbesondere für den Beobachterentwurf durch Immersion, stammen von MdC. Philippe Jouan, der an der Universität Rouen in der mathematischen Fakultät arbeitet.

Fragestellung

Die Arbeit behandelt den Entwurf von Beobachtern für nichtlineare dynamische Populationsmodelle. Dabei wird angenommen, dass in einem System von zwei interagierenden Populationen nur eine Populationsdichtefunktion gemessen wird. Der Beobachter soll mit Hilfe der Messwerte einen Schätzwert für die Dichten beider Populationen liefern. Das Beobachtungsproblem ist in Abbildung 1 dargestellt.

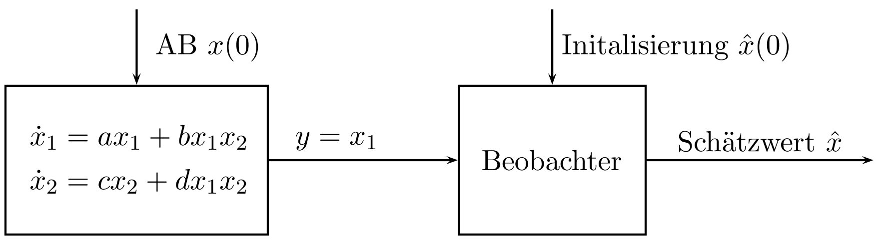

Abbildung 1: Beobachtungsproblem für das autonome Populationsmodell

Die betrachteten Modelle basieren auf dem Räuber–Beute–Modell von Lotka und Volterra. Als Erweiterung zum Grundmodell werden noch weitere Interaktionsmuster zwischen den Populationen untersucht, nicht nur das Räuber–Beute–Verhalten. Dies wird durch unterschiedliche Wahl der Modellparameter ermöglicht. Die Modelle dienen als Benchmark–System, allerdings sind die untersuchten Methoden auch auf allgemeinere Systeme anwendbar. Auch die neuen Fragestellungen dieser Arbeit sind so allgemeingültig wie möglich gehalten und werden beispielhaft für die betrachteten Populationsmodelle untersucht.

Für die Populationsmodelle werden drei verschiedene Varianten betrachtet, um Eingangssignale zu berücksichtigen, die beispielsweise einen menschlichen Eingriff beschreiben können. Diese Varianten haben unterschiedliche biologische Hintergründe und sind für den Beobachterentwurf von unterschiedlicher Qualität. Für jede dieser Varianten soll ein Beobachter konstruiert werden.

Als erste notwendige Bedingung für einen Beobachterentwurf wird die Beobachtbarkeit der Populationsmodelle ohne und mit Eingang untersucht. Das Ziel hierbei ist, Parameterwerte und Mannigfaltigkeiten im Zustandsraum zu finden, für die das System seine Beobachtbarkeit verliert. Mit diesen Ergebnissen werden geeignete Annahmen für das System formuliert, um einen Beobachter zu entwerfen.

Für den nichtlinearen Beobachterentwurf werden Verfahren verwendet, die eine Konvergenz des Schätzwertes gegen den realen Systemzustand auf Basis einer exakt linearen Fehlerdynamik gewährleisten. Diese etablierten Verfahren werden auf Seite 5 dieser Zusammenfassung beschrieben.

Aus der Anwendung der Beobachterverfahren auf das Populationsmodell ergibt sich eine weitere interessante Fragestellung. Die verwendeten Methoden liefern lokale Ergebnisse, das heißt, die erhaltenen Beobachter sind nur in einer normalerweise nicht näher bekannten Umgebung eines Referenzpunktes gültig. Für das Populationsmodell ist ein Bereich in der reellen Ebene bekannt, den der Systemzustand nicht verlässt, da nur positive Populationsdichten betrachtet werden. Die Frage ist, ob und wie es möglich ist, einen Beobachter zu entwerfen, der global in diesem Bereich gültig ist und im besten Fall auch noch Schätzwerte liefert, die den gleichen biologischen Beschränkungen unterliegen wie der reale Systemzustand. Hierfür werden Invarianzeigenschaften von nichlinearen Beobachtern definiert und untersucht.

Beobachtbarkeit des Populationsmodells

Die Beobachtbarkeitsuntersuchung basiert auf der Notation der Ununterscheidbarkeit zweier Anfangsbedingungen sowie auf der von Hermann und Krener [1977] verwendeten Beobachtbarkeits–Normalform nichtlinearer Systeme. Aus der Beobachtbarkeitsanalyse ergibt sich, dass das autonome Modell für fast alle Parameterwerte beobachtbar ist. Der Zustandsbereich, in dem das Modell beobachtbar ist, wird durch die Achse $\{x_1 = 0\}$ begrenzt; für strikt positive Populationsdichten ist das Populationsmodell global beobachtbar.

Für Systeme mit Eingang lässt sich eine Beobachtbarkeitsanalyse leicht nach dem

Ansatz von Gauthier und Bornard [1981] durchführen. Dabei ist zu beachten, dass es Eingangsignale geben kann, für die das System nicht beobachtbar ist. Dieser Fall tritt bei der dritten untersuchten Modellvariante mit Eingang auf und erschwert den Beobachterentwurf. Für die ersten zwei Modellvarianten ergeben sich durch das Eingangssignal keine Einschränkungen der Beobachtbarkeit, d.h. sie sind gleichförmig beobachtbar.

Verwendete Entwurfsmethoden für das autonome System

Die untersuchten Verfahren für den Entwurf eines Beobachters autonomer Systeme basieren auf den Ergebnissen von Bestle und Zeitz [1983] sowie Krener und Isidori [1983]. Dabei bringt man das System durch eine Zustandstransformation in eine Darstellung, für die ein Luenberger–Beobachter mit Ausgangsaufschaltung entworfen wird, dessen Konvergenz durch eine lineare Eigenwertvorgabe gewährleistet werden kann.

Die von Krener und Isidori angegebenen Voraussetzungen für die benötigte Zustandstransformation sind recht restriktiv, und auf das betrachtete Populationsmodell lässt sich das Verfahren in dieser Form nicht anwenden. Deshalb werden drei Ansätze untersucht, diese Transformation zu erweitern und so die Voraussetzungen etwas abzuschwächen. Mit allen drei Ansätzen kann ein Beobachter für das autonome Populationsmodell entworfen werden.

Die erste Erweiterung verwendet zusätzlich zur Zustandstransformation noch eine Ausgangstransformation und wurde von Krener und Respondek [1985] eingehend untersucht. Dieses Verfahren ist für das betrachtete Populationsmodell besonders geeignet, da durch die verwendete Ausgangstransformation $\tilde{y} = \ln y$ der Bereich, in dem das System beobachtbar ist, auf die gesamte reelle Ebene abgebildet wird.

Als zweiter Ansatz wird zusätzlich zur Zustandstransformation noch eine Zeitskalierung verwendet, wie von Respondek, Pogromsky und Nijmeijer [2004] eingeführt. Dieser Ansatz basiert auf der Idee, dass durch eine Multiplikation der Dynamikgleichung des Systems mit einem positiven skalaren Wert nicht die Form und Richtung der Trajektorien verändert wird, sondern nur die jeweilige Zeitparametrierung.

Der dritte Ansatz wurde von Jouan [2003] entwickelt. Dabei wird die Systemdarstellung, in der ein Luenberger–Beobachter mit Ausgangsaufschaltung entworfen werden kann, durch eine Immersion des Systems in einen Zustandsraum höherer Dimension erreicht. Da Jouan auch eine Ausgangstransformation verwendet, kann seine Methode als eine Erweiterung zum Beobachterentwurf durch Ausgangs– und Zustandstransformation angesehen werden. Für das Populationsmodell ist die Immersion äquivalent zur Zustands– und Ausgangstransformation, da eine Koordinatentransformation des Zustandes als eine Immersion in ein System gleicher Ordnung angesehen werden kann.

Die Entwurfsmethoden für das autonome Populationsmodell lassen sich leicht auf die erste Modellvariante mit Eingang erweitern, scheitern jedoch für die zweite und dritte Variante. Für diese müssen andere Ansätze verwendet werden, die auf Seite 7 beschrieben werden.

Invarianzeigenschaften von Beobachtern

Die Wichtigkeit von Invarianzeigenschaften wird am Beispiel des Beobachters veranschaulicht, der durch Zustandstransformation und Zeitskalierung für das Populationsmodell entworfen wird. Dieser Beobachter ist nicht invariant auf dem Beobachtbarkeitsbereich $\{x_1 > 0\}$ des Systems, das heißt, der Zustand des Beobachters kann diesen Bereich verlassen. Damit entweicht der Schätzwert $\hat{x}$ ins Unendliche, was beispielsweise eine Regelung durch Rückführung des Schätzwertes unmöglich macht.

Eng verknüpft mit dieser Problematik ist die Vererbung der Invarianzeigenschaft des Systems auf den Beobachter: Ein Zustandsbereich, der für das beobachtete System invariant ist, soll diese Eigenschaft an den Beobachter weitervererben. Für das Populationsmodell ist ein solcher Bereich beispielsweise die Menge der positiven Populationsdichten.

Um einen allgemeinen Einblick in Invarianzeigenschaften zu erhalten, werden im Kapitel 6 einige Definitionen eingeführt, die das Problem mathematisch formulieren und für weitere Untersuchungen hilfreich sind. Das erste Problem, die Invarianz auf einem Beobachtbarkeitsbereich, ist leichter zu behandeln als die Vererbung von Invarianz. Da es außerdem für beide Probleme noch keine Ergebnisse in Bezug auf nichtlineare Systeme gibt, wird in dieser Arbeit hauptsächlich das erste Problem betrachtet.

Beim ersten Lösungsansatz werden die lineare Fehlerdynamik und die Form des Beobachtbarkeitsbereiches verwendet, um die Invarianz durch eine Initialisierungsstrategie für den Beobachter zu erreichen. Dieser Ansatz wird verwendet, um für das Populationsmodell sowie ein weiteres Beispiel einen Beobachter zu konstruieren, der invariant auf dem Beobachtbarkeitsbereich des jeweiligen Systems ist.

Für den zweiten Ansatz werden die Invarianzeigenschaften von Beobachtern reduzierter Ordnung untersucht. Hierbei muss die sogenannte Formbedingung an den Beobachtbarkeitsbereich gestellt werden, die im Wesentlichen besagt, dass der Beobachtbarkeitsbereich eine abgeschlossene Menge enthalten muss, die der Zustand des beobachteten Systems nicht verlässt und die von Hyperebenen im Zustandsraum, die senkrecht zur Achse des Systemausgangs sind, begrenzt wird. Unter dieser Voraussetzung ist ein Beobachter reduzierter Ordnung invariant. Dieses Resultat gilt grundsätzlich für Systeme beliebiger Ordnung, allerdings gibt es mit zunehmender Ordnung weniger Systeme, welche die Formbedingung für den Beobachtbarkeitsbereich erfüllen.

Das Populationsmodell erfüllt die Formbedingung und somit lässt sich ein invarianter Beobachter reduzierter Ordnung entwerfen. Für spezielle Modellparameter erbt dieser Beobachter auch die Invarianzeigenschaften des Systems für positive Populationsdichten.

Der dritte Ansatz basiert auf dem Luenberger–Beobachter vollständiger Ordnung mit Ausgangsaufschaltung. Mit Hilfe des Nagumo–Theorems (1942, aus [Blanchini, 1999]) kann das Invarianzproblem auf den Rand des betrachteten Bereiches eingeschränkt werden. Durch eine Modifikation der Dynamik des Luenberger–Beobachters, die nur am Rand des relevanten Bereiches wirksam ist, wird eine Invarianz der Beobachterdynamik erreicht. Allerdings wird dadurch die Dynamik des Beobachtungsfehlers nichtlinear und eine Konvergenz kann bisher noch nicht bewiesen werden. Für praktische Anwendungen wird allerdings häufig ausreichen, dass in vielen Fällen eine Konvergenz direkt aus dem simulierten Verlauf des Schätzwertes sichtbar ist — eine Eigenschaft des entworfenen Beobachters, die hier *Sichtbarkeit der Konvergenz* genannt wird. Die Anwendung dieses Ansatzes wird am Beispiel des autonomen Populationsmodells illustriert.

Beobachterentwurf für die Systeme mit Eingang

Für die zweite und dritte Variante des Populationsmodells mit Eingang kann mit der reinen Zustandstransformation und den oben aufgeführten Erweiterungen kein Beobachter mit exakt linearer Fehlerdynamik entworfen werden. Jedoch ist ein Entwurf mit dem Ansatz von Keller [1987] möglich, der basierend auf der Beobachtbarkeitsnormalform von Zeitz [1984] eine eingangsabhängige Zustandstransformation verwendet. Für die dritte Variante, die nicht gleichförmig beobachtbar ist, müssen bestimmte Eingangssignale ausgeschlossen werden, aber ansonsten lässt sich die Methode gut anwenden. Der entscheidende Nachteil ist dabei, dass die Transformation und die Dynamik des Beobachters von Zeitableitungen des Eingangssignals abhängig sind. Wenn diese nicht analytisch zur Verfügung stehen, sondern online berechnet werden müssen, führt dies zu Schätzfehlern und durch numerische Ungenauigkeiten eventuell sogar zu Instabilität. Dies ist besonders für eine Regelung mit Rückführung des Schätzwertes problematisch, da die Ableitungen des Eingangssignals nicht mehr analytisch berechnet werden können, falls Ableitungen höherer als erster Ordnung für den Beobachter benötigt werden.

Um dieses Problem zu vermeiden, wird für die zweite Modellvariante mit Eingang ein Beobachter mit einer bilinearen Fehlerdynamik entworfen, der nur von der Eingangsgröße direkt, nicht aber von deren Zeitableitungen abhängig ist. Dies wird durch die bereits für das autonome Modell verwendete Zustands– und Ausgangstransformation ermöglicht, in der das System eine bis auf Eingangs– und Ausgangsaufschaltung bilineare Form annimmt. Mit einer eingangsabhängigen Beobachterverstärkung kann dann eine Konvergenz des Schätzwertes gegen den realen Systemzustand erreicht werden, was mit einer eigens für diesen Fall konstruierten Ljapunov–Funktion gezeigt wird.

Ausblick

Der Entwurf nichtlinearer Beobachter mit linearer Fehlerdynamik ist bereits seit ca. 20 Jahren etabliert, jedoch immer noch Gegenstand aktueller Forschungen, wie neuere Publikationen aus diesem Bereich zeigen [Jouan, 2003, Respondek et al., 2004]. Eine weitergehende Fragestellung ist die gleichzeitige Anwendung und Zusammenführung der verschiedenen Erweiterungen, die in dieser Arbeit verwendet werden, in einem einheitlichen Ansatz. Eine entsprechende Beobachternormalform lässt sich leicht formulieren, aber noch sind keine Bedingungen für ein nichtlineares System bekannt, welche die Existenz der zugehörigen Transformation garantieren. Ein solcher vereinheitlichter Ansatz würde jedoch die Klasse der transformierbaren Systeme erweitern.

Bei der Untersuchung der Invarianzeigenschaften von Beobachtern mit Ausgangsaufschaltung und linearer Fehlerdynamik wird das globale Verhalten dieser Beobachter in einem für die Systemdynamik relevanten Bereich betrachtet. In dieser Arbeit werden eine mathematische Notation für diese Betrachtungsweise eingeführt und erste Ergebnisse für den Beobachterentwurf abgeleitet. Allerdings beinhalten diese noch eine restriktive Formbedingung an den relevanten Zustandsbereich des Systems und für den Beobachter vollständiger Ordnung kann die Konvergenz noch nicht analytisch nachgewiesen werden.

Zu den Invarianzeigenschaften besteht sicherlich noch Untersuchungsbedarf, insbesondere da es auch für den praktischen Einsatz der entworfenen Beobachter relevant ist. Eine interessante Fragestellung hierbei ist, wie invariante Beobachter für Systeme entworfen werden können, welche die Formbedingung nicht erfüllen. Ein Ergebnis hierzu würde auch das Problem der Vererbung von Invarianz vereinfachen. Die eingeführte Notation sollte noch auf Systeme mit Eingang erweitert werden, um auch in diesem Fall invariante Beobachter entwerfen zu können.

Contents

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1 Introduction

When solving control engineering problems, it is often necessary to know the state of a dynamical system. Most of the modern control design methods, especially for nonlinear systems, use a state feedback as the controller. Knowing the system state is also important for surveillance of a technical system, either by a human or automatically. But in most applications, it is very difficult or even impossible to measure the entire state of the system: either because applying sensors for all states would require too much effort, or because there are no methods to measure a state variable in realtime. Thus the problem of observer design is how to get an estimate for the state of a dynamical system from the knowledge of its input and output signals.

The observability problem is closely related to observer design. From a mathematical viewpoint, the question of observability is whether a given input and output signal determine uniquely the state trajectory for the system. In a control engineering context, observability is a necessary condition to solve the observer design problem. It will be impossible to design an observer for a system where the given input and output signal can correspond to several different state trajectories.

In this work, we are going to study both the observability and the observer design problem for a Lotka–Volterra model, which describes the evolution of the population densities of two interacting species. This model is rather simple and yet very well suited to study several methods in control theory. Our work is based on well known results from nonlinear observer theory. A short overview of relevant literature is given below. As most observer design methods are local methods, some new results were obtained for the question on how to globalise these design methods.

For linear systems, the observability problem was solved by Kalman (see e.g. [Kailath, 1980]). First important results for nonlinear systems were obtained by Hermann and Krener [1977], who gave a sufficient condition for local observability. Gauthier and Bornard [1981] found a class of systems which are observable for any input signal. Their result is quite important for control applications, where the input is directly computed by the controller, usually without regarding whether the system is observable with this input or not. A different approach for controlled systems can be found in the work of Zeitz [1984], where also derivatives of the input signal are taken into account.

If a dynamical system is found to be observable, one can start thinking about how to design an observer for this system. In nonlinear control theory, the observer is usually constructed as another dynamical system which takes the input and output signals of the observed system as its input and gives an estimation for the state of the observed system as output. With this approach, the observer has its own state, which evolves according to a differential equation. Then the observer design problem is reduced to designing the dynamical behaviour of the observer state and the output function of the observer.

The observer design problem for linear systems was solved by Luenberger (see e.g. [Kailath, 1980]), and many design methods for nonlinear observers are based on his ideas. In fact, the first main approach of observer design we are going to present reduces the observer design problem for a nonlinear system to designing a linear observer with Luenberger's method.

We can distinguish two main approaches for the observer design for nonlinear systems in literature over the past twenty years. The first approach, which we will concentrate on in this work, designs the dynamics of the observer to be linearisable with respect to the observer state. Then, stability of the observer error and convergence of the observer estimation to the state of the observed system is achieved simply by linear pole placement, as in the Luenberger observer.

This method is based on the work of Bestle and Zeitz [1983] and Krener and Isidori [1983]. A complete solution based on their ideas has been given by Krener and Respondek [1985] for a certain class of MIMO–systems. In the late 80s, an important extension to these results was made by Keller [1987], who introduced the use of input derivatives to facilitate the observer design for controlled systems. Another result is due to Zeitz [1987], who's extended Luenberger observer is based on the same coordinate transformation as the previous solutions, but can be seen already as predecessor of the high–gain observers presented below.

In the last decade, more extensions to the linearisation approach were proposed. An generalisation of Keller's design is the completely generalised input–output injection suggested by Plestan [1995]. Further important results include the time scaling done by Respondek et al. [2004], where the observer error dynamics are linearised by using a different time scale, and the immersion into higher–dimensional systems for which one can design an observer with linearisable error dynamics, as suggested by Jouan [2003].

A slightly different approach, but which uses essentially the same idea of linearisable error dynamics, is the observer design method first introduced by Kazantzis and Kravaris [1998]. Actually, they quit the extension of the Luenberger observer and directly design an observer that has linear dynamics with respect to the observer state, but nonlinear dynamics with respect to the observer input and a nonlinear output function. The class of systems this method can be applied to was then enlarged by Andrieu and Praly [2004], who gave theoretical results on the possibility of such an observer design.

The second main approach in observer design is the use of high–gain principles. The basic idea of this approach is to dominate the nonlinear behaviour of the system by applying high gains to a slightly modified Luenberger observer. Convergence is then usually proven by giving a quadratic Ljapunov function, as normally used for linear systems. The first results for this design principle were obtained by Gauthier et al. [1992] and extended afterwards by several authors.

Our work is structured as follows. First, in chapter 2, we introduce the population model which is going to be used, and do some analysis of the system. Several variants of the basic model are obtained by adding a control term in different ways. Then, the first major part of the text is dedicated to observability analysis of the system. In chapter 3, the observability analysis is done by local methods based on the results of Hermann and Krener [1977]. Since the system dynamics are rather simple, a global analysis is also possible, which is done in chapter 4.

The second part of the text deals with observer design. We concentrate on observers which achieve convergence by linearisable error dynamics. After a short general introduction to observer design, we present three observer design methods which aim to construct observers with linearisable error dynamics in chapter 5. These methods are applied to the basic uncontrolled population model, and simulations of the designed observers are done.

A major problem of nonlinear observers is that they are usually designed locally, i.e. we will have to assume that the initial condition of the observed system and the initial estimation given by the observer are sufficiently close to each other. However, the methods give no hints on what exactly is "sufficiently close", and so this question has to be investigated further in the engineering process of designing an observer for a given control problem. Of course, in many technical applications the initial condition of the system is well known, e.g. for an airplane before take–off. But there are other applications, especially in biology, where the initial condition is only known to be in some subset of the state space. Then we would like the observer to work, whenever the initial estimation is also an element of this subset.

This is the problem we are going to address in chapter 6. We introduce some notation to deal with this problem and give some new results to solve it in specific cases. The population model we are studying will be one of these cases for some parameter configurations, and thus we are using it to illustrate our results.

Finally, chapter 7 deals with observer design for the controlled variants of the population model. We present the theoretical background of the applied methods and do simulations using the observers designed by these methods.

2 The population model

The model of the population dynamics considered in this work describes the evolution of two interacting species. It is based upon the predator–prey model introduced by Lotka and Volterra. The model is slightly more general in the sense that some choices of model parameters may represent other forms of interaction between the two species than just the predator–prey relation.

Furthermore, we will add a control input to the model in different ways and see how this extension influences observability and observer design for the system.

2.1 The uncontrolled system

The mathematical model for the uncontrolled system is given by the equations

$$LV: \qquad \begin{aligned} \dot{x}_1 &= ax_1 + bx_1x_2 \\ \dot{x}_2 &= cx_2 + dx_1x_2 \\ y &= x_1, \end{aligned} \tag{2.1}$$

where $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2$ is the vector containing the two population densities and $a, c \in \mathbb{R}$ are parameters describing the development of the corresponding population if the population of the other species is 0, whereas $b, d \in \mathbb{R}$ are coefficients describing the interaction between the two species. Only the first species can be measured, which is expressed by the output equation $y = x_1$.

The model is given in the standard form of many dynamical systems, i.e.

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned}$$

and may therefore be worked with using standard methods, which are, among others, summarised by Isidori [1995] or Nijmeijer and van der Schaft [1990]

2.1.1 A biological interpretation

Depending on the sign of the four parameters, the model dynamics will be different. Actually, there are three cases in which we can find a biological interpretation of the model (2.1) as shown in table 2.1.

A symbiosis means that the two species each gain profit in terms of population density by interacting with each other, whereas they will die out if there is no interaction.

$b, d > 0;\ a, c < 0$	symbiosis
$b, d < 0;\ a, c > 0$	competition
$a, d > 0;\ b, c < 0$ or vice versa	predator and prey

Table 2.1: Biological interpretation of different parameter values

Contrary, in a competition, both species would grow exponentially without the presence of the other, but the interaction between them hinders their growth or even reduces the population size.

The most interesting case is the one where one species profits from interaction and will even die out without it, whereas the other could grow exponentially if its population size was not reduced by interaction with the other species. This is the predator–prey configuration originally introduced by Lotka and Volterra and we will see why we consider it the most interesting in the next section.

Since the system output is always x_1, we can consider two different predator–prey configurations: One where the prey is measured, and the other where the system output is the predator population.

In biological terms, the natural state space of the system is the closed set $\mathbb{R}^2_{0+} = \{\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mid x_1 \geq 0 \text{ and } x_2 \geq 0\}$, since population sizes cannot be negative. But for mathematical interest, the state space will usually be chosen as $\mathbb{R}^2$. Sometimes the open subset $\mathbb{R}^2_+ = \{\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mid x_1 > 0 \text{ and } x_2 > 0\}$ — which is an invariant manifold for the system — will be considered as state space, admitting only positive population sizes.

2.1.2 Equilibrium points and stability

If the parameters $a, b, c, d \in \mathbb{R} \setminus \{0\}$, the system has two equilibrium points. The first one is the origin $x_e^{(1)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathrm{T}}$, the second one is $x_e^{(2)} = \begin{pmatrix} -\frac{c}{d} & -\frac{a}{b} \end{pmatrix}^{\mathrm{T}}$.

An interesting fact to note is that in the cases stated in table 2.1, where the model equations really have a biological sense, the second equilibrium $x_e^{(2)}$ lies in the manifold $\mathbb{R}^2_+$ and therefore in the biological meaningful subset of the state space $\mathbb{R}^2$.

To investigate stability, let us consider the linear approximation of the system at the two equilibrium points. We get

$$\frac{\partial f}{\partial x} = \begin{pmatrix} a + bx_2 & bx_1 \\ dx_2 & c + dx_1 \end{pmatrix}$$

and thus

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{x_e^{(1)}} &= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \\ \left.\frac{\partial f}{\partial x}\right|_{x_e^{(2)}} &= \begin{pmatrix} 0 & -\frac{bc}{d} \\ -\frac{ad}{b} & 0 \end{pmatrix}. \end{aligned}$$

At $x_e^{(1)}$, the eigenvalues of the linearised system are a and c and the stability of the nonlinear system can be determined depending on these parameters if they are not equal to zero.

At $x_e^{(2)}$, the characteristic equation of the dynamics matrix is $s^2 - ac = 0$. The nonlinear system is unstable if $ac > 0$, since its linearisation has a positive and a negative eigenvalue. This is the case for both the symbiosis and the competition parameter configurations. If $ac < 0$, the eigenvalues are both on the imaginary axis and stability cannot be determined by linearisation.

The global dynamics of the model (2.1) will be studied in the next section, which will also allow us to deduce global stability properties.

2.1.3 Phase portraits of biological parameter configurations

To get a better understanding of the system dynamics for different parameter configurations, we will study the phase portraits for the cases listed in table 2.1.

For all phase portraits, we chose model parameters with absolute value 1, and changed only the sign to reflect the biological cases given in table 2.1. The resulting phase portraits are shown in figure 2.1 on the next page.

All configurations considered here have the same two equilibrium points. The dynamics of the symbiosis and competition cases have already been studied by their linear approximation at the equilibrium points. The global dynamics are not very different from putting these local behaviours together.

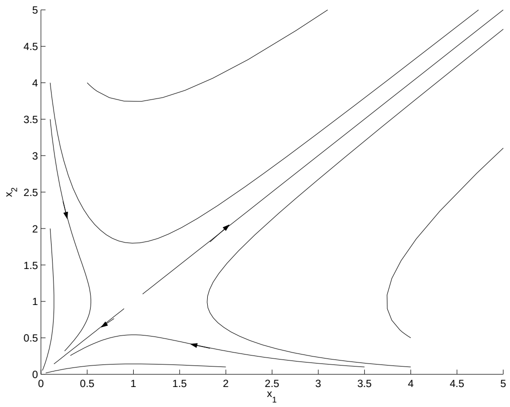
For the predator–prey case, we see that the second equilibrium is a center, the periodic cycles cover the whole invariant submanifold $\mathbb{R}^2_+$.

2.2 Models for a controlled system

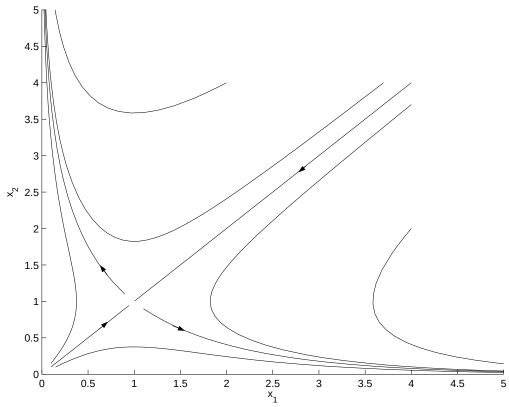
Control can be introduced in several ways to the system modelled by equation (2.1). Our first approach is the one considered also by Jakubczyk and Respondek [2004], where an input signal u is added with different coefficients to both population growth rates. This produces the dynamics

$$LV_1: \qquad \begin{aligned} \dot{x}_1 &= ax_1 + bx_1x_2 + eu \\ \dot{x}_2 &= cx_2 + dx_1x_2 + fu \\ y &= x_1. \end{aligned} \tag{2.2}$$

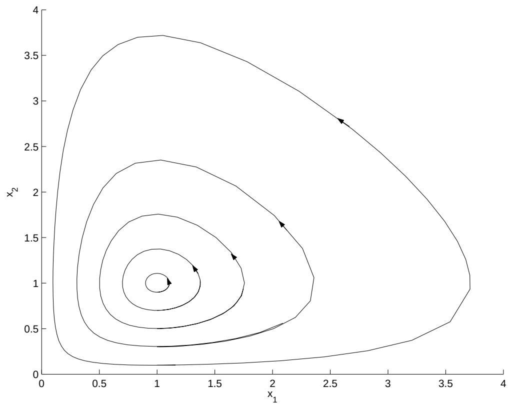
In the biological sense, we just add or remove a given number of individuals per time interval to or from both populations. The weakness of this model is obvious: It allows for negative population sizes. Thus we do not except this model to have a relevant biological interpretation. Nevertheless, it will be considered in this work since it is rather easy to handle with respect to observation.



(a) Symbiosis



(b) Competition



(c) Predator and prey

Figure 2.1: Phase portraits for different parameter configurations

An improved model in the biological sense is given by

$$LV_2: \quad \begin{aligned} \dot{x}_1 &= ax_1 + bx_1x_2 + ex_1u \\ \dot{x}_2 &= cx_2 + dx_1x_2 + fx_2u \\ y &= x_1. \end{aligned} \tag{2.3}$$

A constant input signal u would just change the growth rates of the two populations. The effect of this change is determined by the coefficients e and f.

As an example, consider the populations of two different kinds of bacterias living in a biological reactor. The input signal u models a chemical added to the reactor and influencing the growth rate of the bacterias, and the coefficients e and f model the numerical change of the growth rate caused by the chemical.

The third controlled system considered in this work is

$$LV_3: \qquad \begin{aligned} \dot{x}_1 &= ax_1 + bx_1x_2 + ex_1x_2u \\ \dot{x}_2 &= cx_2 + dx_1x_2 + fx_1x_2u \\ y &= x_1. \end{aligned} \tag{2.4}$$

Here, a constant input signal changes the rate of interaction between the two species, and the change is again modelled with the parameters e and f.

The three controlled systems under consideration are control affine, they can be written as

$$LV_i: \qquad \begin{aligned} \dot{x} &= f(x) + g_i(x)u \\ y &= h(x). \end{aligned}$$

System analysis and observer design is usually easier to accomplish for a control affine system than for a more general nonlinear one.

In the controlled case, the analysis of equilibrium points and their stability depending on model parameters is much more difficult than in the uncontrolled case. One could do such an analysis using bifurcation theory. In fact, the model LV_1 has been used by Jakubczyk and Respondek [2004] as an example to illustrate several control bifurcations.

Usually one searches for equilibrium points assuming a constant input u_e. Then the position of equilibria will in general depend on the input, and one gets curves of equilibrium points which are parametrised by the input u_e.

For the models LV_2 and LV_3, these curves can be computed easily. The equilibrium $x_e^{(1)} = \begin{pmatrix} 0 & 0 \end{pmatrix}$ is the same for both models and does not depend on u_e. For LV_2, we get additionally $x_e^{(2)} = \begin{pmatrix} -\frac{c+fu_e}{d} & -\frac{a+eu_e}{b} \end{pmatrix}^{\mathrm{T}}$, and for LV_3 we have $x_e^{(2)} = \begin{pmatrix} -\frac{c}{d+fu_e} & -\frac{a}{b+eu_e} \end{pmatrix}^{\mathrm{T}}$.

Further analysis concerning the stability of these equilibrium points is beyond the scope of this work. It involves an active research issue in bifurcations of nonlinear control systems.

Part I

Observability

Introduction to observability

This chapter introduces basic concepts of observability which will be applied both in the local and in the global observability analysis done in the following two chapters.

We consider a control affine dynamical system given by the equations

$$\Sigma : \quad \begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2.5}$$

with $x \in X$, a smooth n–dimensional manifold. Usually we will consider X as an open subset of $\mathbb{R}^n$. The input $u(\cdot)$ is chosen as an element of a set of admissible input functions, which we will denote by $\mathcal{U}$. It will usually be the set of all piecewise constant functions. Other choices would be a set of Lebesgue–integrable functions such as $L^\infty(\mathbb{R})$ or $L^2(\mathbb{R})$.

For this system, we can define the *output* and *state response maps*, which apply to each initial condition $x(0) = x_0 \in X$ and $u(\cdot) \in \mathcal{U}$ the corresponding output function $y(t, x_0, u(\cdot))$, respectively the state trajectory $x(t, x_0, u(\cdot))$, for $t \in [0, \delta_{x_0}]$, the maximal interval where a solution of the system Σ exists for the initial condition x_0.

When observing a system, we measure the output function $y(t)$. In this work, we consider only the case where also the input $u(t)$ is known.

The question of observability is whether one can deduce the initial state $x(0)$ from the knowledge of these two functions, and by using the vector fields f, g and the system output mapping h. It is equivalent to know either the initial state $x(0)$ or the complete state trajectory $x(t, x(0), u(\cdot))$, since we can compute the state trajectory from the initial condition and the input using the vector fields f and g.

To determine whether a system is observable, the concept of indistinguishability will be used.

Definition 1. Two initial conditions x, $\tilde{x}$ of the system Σ are called *indistinguishable through the input* $u(\cdot) \in \mathcal{U}$, if

$$\forall t \geq 0 : y(t, x, u(\cdot)) = y(t, \tilde{x}, u(\cdot)), \tag{2.6}$$

for $t \in [0, \min\{\delta_x, \delta_{\tilde{x}}\}]$.

The definition of indistinguishability depends on the input, i.e. two states may be indistinguishable for one choice of u, while they are distinguishable for another input.

We will denote by $I_x = \{\tilde{x} \in \mathbb{R}^n \mid \tilde{x} \text{ and } x \text{ are indistinguishable}\}$ the set of all points which are indistinguishable from x, in particular x itself. As our definition of indistinguishability depends on the input $u(\cdot)$, so will the set I_x.

Using the concept of indistinguishability, we can now state our definition of observability, which was introduced by Hermann and Krener [1977].

Definition 2. The system Σ is called *observable*, if there exists an input function $u(\cdot) \in \mathcal{U}$ such that for any $x \in X$, we have $I_x = \{x\}$.

By this definition, a system is observable if there is at least one input which distinguishes any two initial states. Concerning observation, this property might be too weak for practical use, as it will not always be possible to apply the input which makes the system observable. Therefore, we will give another definition for systems which are observable for any input.

Definition 3. The system Σ is called *uniformly observable*, if for any $u(\cdot) \in \mathcal{U}$ and $x \in X$, we have $I_x = \{x\}$.

The problem of observability is thus the question of injectivity of the output response map $y(t, x_0, u(\cdot))$. If this map is injective for at least one input u, then we say that the system is observable. If it is injective for all inputs, then the system is said to be uniformly observable.

3 Local observability analysis

For the concept of local observability, we require only local injectivity of the output response map for the system Σ. We will say that the system is locally observable, if any two initial conditions which are close to each other are distinguishable, while there may be other initial conditions which far from each other and are indistinguishable.

3.1 Theoretical Background

The method used for the observability analysis done here has been introduced by Hermann and Krener [1977]. Their main result concerning observability is recalled here.

Definition 4. The system Σ is called *weakly locally observable at* $x_0 \in X$, if there exists a neighbourhood U of x_0 and an input $u(\cdot) \in \mathcal{U}$, such that the set $I_{x_0} \cap U = \{x_0\}$.

The above definition is rather weak, in the sense that the following definition gives a stronger condition on local observability, and the theorem of Hermann and Krener [1977] on local observability stated below still holds with this stronger condition. Hermann and Krener [1977].

Definition 5. The system Σ is called *strongly locally observable at* $x_0 \in X$, if there exists a neighbourhood U of x_0 such that the system Σ restricted to U is observable by definition 2.

It is clear that strong local observability implies weak local observability. Sometimes we will omit the term “strongly” from our notion, and throughout this text local observability will mean strong local observability.

The system Σ is called *(weakly) locally observable*, if it is so at every $x \in X$.

To check for local observability, the observability codistribution is constructed.

Definition 6. The observability codistribution $\mathcal{H}$ for the system Σ is defined as

$$\mathcal{H} = \text{span}\,\{dL_{\tau_k} \ldots L_{\tau_1} h \mid k \geq 0, \tau_k \in \{f, g\}\}\,. \tag{3.1}$$

We use the following theorem, which is due to Hermann and Krener [1977]. It gives an easily checkable sufficient condition on strong local observability.

Theorem 1 (Local observability). *If* $\dim \mathcal{H}(x) = n$*, then the system* Σ *is strongly locally observable at* x*.*

3.1.1 The observability canonical form

For locally observable systems, there may exist local canonical coordinates in which the system takes a specific form. This form was established by Gauthier and Bornard [1981] for control affine systems, but we start with the uncontrolled case (see also Isidori [1995], Respondek [2001]).

An uncontrolled system in observability canonical form is given by the equations

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n \\ \dot{\xi}_n &= \rho(\xi) \\ y &= \xi_1. \end{aligned} \tag{3.2}$$

The function $\rho(\xi)$ contains the characteristic nonlinearity of the system, as the other part of the dynamics is just a chain of integrators and hence linear. If the system can be transformed to the observability form, then it is observable. If this transformation is only local, then we will also have only local observability.

The condition for the existence of the observability canonical form is similar to the one for local observability, but it is stronger regarding the order of output derivatives the observability codistribution may use.

Theorem 2. *An uncontrolled system Σ can be transformed to the observability canonical form locally around x, if and only if*

$$\dim \operatorname{span} \{L_f^k h(x) \mid 0 \leq k \leq n-1\} = n. \tag{3.3}$$

The condition used in this theorem is rather important for observer design, it will be referred to as the *observability rank condition* in further chapters.

The transformation which brings the system to the observability canonical form is given by

$$\xi = \Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}, \tag{3.4}$$

with the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$. The condition from theorem 2 guarantees that Φ is a local diffeomorphism and can thus be used for a local coordinate transformation.

Observability of controlled systems usually depends on the choice of the control input. A system which is observable for any input is called uniformly observable. In the next definition, we will give the local equivalent of definition 3.

Definition 7. The system Σ is called *locally uniformly observable at $x_0 \in X$*, if there exists a neighbourhood U of x_0 such that the system Σ restricted to U is uniformly observable.

In the control affine case, the observability canonical form is defined by the equations

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 && +g_1(\xi_1)u \\
\dot{\xi}_2 &= \xi_3 && +g_2(\xi_1, \xi_2)u \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n && +g_{n-1}(\xi_1, \dots, \xi_{n-1})u \\
\dot{\xi}_n &= \rho(\xi) && +g_n(\xi_1, \dots, \xi_n)u \\
y &= \xi_1.
\end{aligned}
\tag{3.5}
$$

The following theorem, which was established by Gauthier and Bornard [1981] gives conditions on local uniform observability which can be checked easily.

Theorem 3 (Local uniform observability). *The two following conditions are equivalent:*

(i) The system Σ admits the form given by equation (3.5) *locally around x.*

(ii) The conditions

- $\dim \operatorname{span}\{dh(x), \dots, dL_f^{n-1}(x)\} = n$ *and*
- $[D_j, g] \subset D_j$, *where* $D_j = \operatorname{kern}\{dh(x), \dots, dL_f^{j-1}(x)\},\ 1 \le j \le n-1$

are both satisfied.

If these conditions are satisfied, then the system Σ is locally uniformly observable at x.

3.2 Analysis of the uncontrolled population model

To check for local observability of the uncontrolled Lotka–Volterra model as defined in equation (2.1), let us construct the observability codistribution $\mathcal{H}$. We start with the subdistribution $\mathcal{H}_2$ of $\mathcal{H}$ where we take output derivatives up to order 1 into account. The output and its first time derivative are

$$
\begin{aligned}
L_f^0 h(x) &= x_1 \\
L_f^1 h(x) &= ax_1 + bx_1x_2,
\end{aligned}
$$

and thus

$$\mathcal{H}_2 = \operatorname{span}\{\begin{pmatrix}1 & 0\end{pmatrix},\ \begin{pmatrix}a + bx_2 & bx_1\end{pmatrix}\}.$$

Equivalently, we might consider the matrix

$$H_2 = \begin{pmatrix} 1 & 0 \\ a + bx_2 & bx_1 \end{pmatrix}.$$

The observability codistribution has dimension 2 if $x_1 \neq 0$ and $b \neq 0$, since in that case already $\mathcal{H}_2$ has dimension 2.

But for local observability, also higher derivatives of the output may be used and the remaining cases have to be studied further. Let us first consider the case where $b = 0$. We then have by induction $L_f^k h(x) = a^k x_1$ for any $k \geq 0$. The observability codistribution is thus $\mathcal{H}(x) = \operatorname{span}\{\begin{pmatrix}1 & 0\end{pmatrix}\}$ and we conclude $\dim \mathcal{H}(x) = 1$ for all $x \in \mathbb{R}^2$.

In the case where $x_1 = 0$, the dimension of $\mathcal{H}$ can be computed in a similar way. For small x_1 and $k \geq 1$, we have $L_f^k h(x) = a^k x_1 + x_2 \mathcal{O}(x_1)$, where $\mathcal{O}(\cdot)$ denotes "order of". This can be proven by induction:

$$\begin{aligned}
L_f L_f^k h(x) &= \begin{pmatrix} a^k + x_2 & \mathcal{O}(x_1) \end{pmatrix} \begin{pmatrix} ax_1 + bx_1x_2 \\ cx_2 + dx_1x_2 \end{pmatrix} \\
&= a^{k+1}x_1 + x_2 \Big(a^k bx_1 + bx_1x_2 + (c + dx_1)\mathcal{O}(x_1) \Big) \\
&= a^{k+1}x_1 + x_2\mathcal{O}(x_1) \\
&= L_f^{k+1} h(x).
\end{aligned}$$

We have then $dL_f^k h(x) = \begin{pmatrix} a^k + x_2\phi(x) & \mathcal{O}(x_1) \end{pmatrix}$ and the dimension of $\mathcal{H}(x)$ is 1 for $x_1 = 0$.

We can then conclude that the Lotka–Volterra–Model is strongly locally observable at every point in $\mathbb{R}^2$ where $x_1 \neq 0$, if the parameter $b \neq 0$.

For $b \neq 0$, the Lotka–Volterra model even satisfies the condition given in theorem 2 at any $x \in \mathbb{R}^2$ where $x_1 \neq 0$. The system can hence be transformed to the observability canonical form given by equation (3.2). For the Lotka–Volterra–Model, the required coordinate transformation $\xi = \Phi(x)$ is

$$\begin{aligned}
\xi_1 &= h(x) &&= x_1 \\
\xi_2 &= L_f h(x) &&= ax_1 + bx_1x_2.
\end{aligned} \tag{3.6}$$

The resulting dynamics in ξ–coordinates are

$$\begin{aligned}
\dot{\xi}_1 &= L_f h\left(\Phi^{-1}(\xi)\right) = \xi_2 \\
\dot{\xi}_2 &= L_f^2 h\left(\Phi^{-1}(\xi)\right) = \frac{\xi_2^2}{\xi_1} + c\xi_2 + d\xi_1\xi_2 - ac\xi_1 - ad\xi_1^2.
\end{aligned} \tag{3.7}$$

The characteristic nonlinearity $\rho(\xi)$ given by $L_f^2 h\left(\Phi^{-1}(\xi)\right)$ is of special interest for some observer design methods. For the Lotka–Volterra model, we have

$$\rho(\xi) = \frac{\xi_2^2}{\xi_1} + c\xi_2 + d\xi_1\xi_2 - ac\xi_1 - ad\xi_1^2.$$

Note that this expression is a polynomial of second order with respect to ξ_2. It is also easy to see that $\rho(\xi)$ is undefined for $\xi_1 = 0$, which is coherent with the analysis carried out above, where we found that the Lotka–Volterra model cannot be transformed to the observability canonical form if $x_1 = 0$.

The set $X_2 = \{\left(\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right) \mid x_2 \in \mathbb{R}\}$, where the Hermann–Krener rank condition is not satisfied, is actually a submanifold which is invariant under system dynamics. If the initial condition is an element of this submanifold, the second population will evolve due to its own dynamics without any interaction and cannot be observed, since the measurement will be 0 for all times.

If $b = 0$, the dynamics of the first population do not depend on the second population. If there is any interaction between the two species (which would mean $d \neq 0$), this does not influence the first population and so the second one cannot be observed by measuring the first one.

A more formal analysis of these effects will be done in chapter 4, "Global Observability analysis".

3.3 The controlled population models

Recall from section 2.2 that we consider the controlled population models to be given by the dynamics

$$LV_i : \quad \begin{aligned} \dot{x} &= f(x) + g_i(x)u \\ y &= h(x), \end{aligned}$$

with different input vector fields g_i depending on the individual model.

To check for uniform local observability of these models, theorem 3 on page 14 will be applied. Note that neither the observability codistribution nor the distributions D_j used in the conditions of this theorem depend on the input direction g and thus need only be calculated once for all the models under consideration.

As discussed in the previous section, the observability codistribution satisfies the Hermann–Krener rank condition, if and only if $x_1 \neq 0$ and $b \neq 0$. We will assume this to hold in the following analysis.

Next we need to compute the distributions D_i as defined in theorem 3 on uniform local observability. We get

$$D_1 = \text{kern}\,\{\begin{pmatrix}1 & 0\end{pmatrix}\} = \text{span}\,\{\begin{pmatrix}0\\1\end{pmatrix}\},$$

The next step has to be done for each model individually, since it uses the input vector fields g_i. Let us start with the model LV_1, where

$$[D_1, g_1] = \{0\} \subset D_1.$$

The model LV_1 is thus uniformly observable for any point $x \in \mathbb{R}^2$, under the assumption $x_1 \neq 0$ and $b \neq 0$.

Considering LV_2, we have

$$[D_1, g_2] = \text{span}\{\left(\begin{smallmatrix}0\\f_2\end{smallmatrix}\right)\} \subset D_1,$$

and we conclude that the model LV_2 is also locally uniformly observable under the assumptions made above.

For the model LV_3, we compute

$$[D_1, g_3]\,(x) = \text{span}\{\left(\begin{smallmatrix}e_3x_1\\f_3x_1\end{smallmatrix}\right)\}. \tag{3.8}$$

$[D_1, g_3]$ is not contained in D_1, and thus we can conclude that the model is not uniformly observable at any $x \in \mathbb{R}^2$. The reason for this result may be seen in better detail in chapter 4, where a global observability analysis is done.

4 Global observability analysis

In this chapter, a global observability analysis of the uncontrolled and the three controlled Lotka–Volterra models is done. Precisely, we are interested in the indistinguishability sets partitioning the state space $\mathbb{R}^n$. This will allow us to find a restriction of the system to a subset of $\mathbb{R}^n$ such that the restricted system is globally observable. This subset will be composed of indistinguishability sets each containing only one single point.

4.1 Methodology of the analysis

To carry out our analysis, we make use of the observability mapping q applying to each state $x(t)$ and input value $u(t)$ the corresponding output $y(t)$ and its time derivatives up to some order m.

Definition 8. The mapping

$$q_m : \begin{cases} \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m \\ (x, u) \mapsto \left(y, \dot{y}, \ldots, \overset{(m-1)}{y} \right) \end{cases} \tag{4.1}$$

is called the observability mapping of the system Σ as given by equation (2.5).

The observability mapping allows us to give a condition on indistinguishability which may be easier to verify than the original definition, especially if the output of the system cannot be computed analytically.

Proposition 1. *If, for some $m \in \mathbb{N}$, we have $q_m(x, u) \neq q_m(\tilde{x}, u)$, then the two states $x, \tilde{x}$ are distinguishable.*

Proof. The proof is done by the negated statement. Suppose that two states $x, \tilde{x}$ are indistinguishable. Then, by definition, $y(t, x, u(\cdot)) = y(t, \tilde{x}, u(\cdot))$ for all times t and inputs $u(\cdot)$. This implies $\overset{(k)}{y}(t, x, u(t)) = \overset{(k)}{y}(t, \tilde{x}, u(t))$ for all integers k and input values $u(t)$ and we get $q_m(x, u) = q_m(\tilde{x}, u)$. □

In some special cases, it may be possible to compute the output $y(t, x, u(\cdot))$ analytically, e.g. due to certain parameter values or the existence of a lower dimensional, invariant submanifold. In these cases, it is often easy to decide for indistinguishability using the computed output function.

To construct indistinguishability sets, it is sometimes more convenient to assume that the output of two initial states is the same and to deduce conditions these initial states must satisfy. If this can be done analytically, we can directly compute the indistinguishability sets. This approach will be applied to the controlled population models.

4.2 The uncontrolled model

The global observability analysis is now carried for the uncontrolled Lotka–Volterra model given by equation (2.1).

Let us first consider the case where $b = 0$. For any initial condition $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we observe the output $y(t) = x_1 e^{at}$. The output does not depend on the second component x_2 of the initial condition. It follows that the indistinguishability sets are

$$I_c = \{x \in \mathbb{R}^2 | x_1 = c\}, \quad c \in \mathbb{R},$$

which are lines parallel to the x_2–axis. In particular, we cannot find a set containing only zero–dimensional indistinguishability sets, i.e. the system cannot be restricted to an open subset of $\mathbb{R}^2$ such that the restricted system becomes observable.

In the case where $b \neq 0$, we can observe that any two points $x, \tilde{x}$ are indistinguishable, provided that $x_1 = 0$ and $\tilde{x}_1 = 0$, since the output will be $y(t,x) = y(t,\tilde{x}) = 0$ for all t. It is furthermore obvious that a point x with $x_1 \neq 0$ can be distinguished from another point $\tilde{x}$ with $\tilde{x}_1 = 0$, since we have already $h(x) \neq h(\tilde{x})$, or equivalently $y(0,x) \neq y(0,\tilde{x})$ to be consistent with our notation. One indistinguishability set is thus

$$I_0 = \{x \in \mathbb{R}^2 | x_1 = 0\}.$$

Let us now consider two points x, $\tilde{x}$ where $x_1 \neq 0$ and $\tilde{x}_1 \neq 0$. We compute the observability mapping with time derivatives up to order 1 as

$$q_2(x) = (x_1, ax_1 + bx_1x_2).$$

As we assumed $b \neq 0$, $x_1 \neq 0$ and $\tilde{x}_1 \neq 0$, we can conclude

$$q_2(x) \neq q_2(\tilde{x}) \Leftrightarrow x \neq \tilde{x}.$$

Any point x with $x_1 \neq 0$ is thus only indistinguishable from itself.

As a conclusion, we get as indistinguishability set containing a given state space point x the set

$$I_x = \left\{\tilde{x} \in \mathbb{R}^2 \mid \tilde{x}_1 = 0 \text{ if } x_1 = 0;\ \tilde{x} = x \text{ otherwise}\right\}. \tag{4.2}$$

Under the modest assumption that $b \neq 0$, we have found two connected regions which are globally observable, namely the left and the right open half plane where $x_1 < 0$ respectively $x_1 > 0$. Note that also the union of these regions contains only distinguishable points, and thus the system restricted to the union of the two open half planes would be observable.

Nevertheless, the unconnectivity of this set may lead to problems for observer design, so we will keep these two regions apart for further work. In fact, the observer design will generally only be done for the right half plane, where $x_1 > 0$. This coincides well with the biological background, provided we assume the presence of the species counted with the variable x_1. Any design for the left half plane should work equivalently, but is not considered in this work, since negative populations do not occur in biology.

4.3 The controlled model

In this section, we will determine the indistinguishability sets for the three controlled Lotka–Volterra models. Note that these sets will in general depend on the choice of the system input u, as two points might be distinguishable for one control input u, while they are indistinguishable for another input. If a system is observable for all possible inputs, then it is called uniformly observable, as stated in definition 3.

The models analysed here were introduced in section 2.2 on page 6 and are referred to as LV_i with $i \in \{1, 2, 3\}$.

4.3.1 Analysis of the model LV_1

Let us first consider the model LV_1. We consider two cases, the first one with the model parameter $b = 0$ and the second case with $b \neq 0$.

If $b = 0$, the equation for x_1 is linear and we can compute its solution and thus the system output for an initial condition $x(0)$ as

$$y(t) = \exp(at)x_1(0) + e \int_0^t \exp(a(t - \tau))u(\tau)d\tau.$$

The initial state $x_2(0)$ does not appear here and hence does not influence the system output. The indistinguishability sets are the same as in the uncontrolled case with $b = 0$, i.e. they are lines parallel to the x_2–axis as given in equation (4.2).

Let us now assume $b \neq 0$. The approach here is to suppose that the output of two initial conditions is the same and to deduce conditions on these initial conditions from this assumption. We take two initial states x and $\tilde{x}$ and assume that

$$\forall t : y(t, x, u(\cdot)) = y(t, \tilde{x}, u(\cdot)).$$

From this, we conclude directly $x_1 = \tilde{x}_1$.

Continuing with the fist time derivative of the output gives us the equation

$$x_1 x_2 = \tilde{x}_1 \tilde{x}_2, \tag{4.3}$$

from which we can conclude $x_2 = \tilde{x}_2$ if $x_1 \neq 0$. Otherwise, we can derive this equation further and get

$$x_2(ax_1+bx_1x_2+eu)+x_1(cx_2+dx_1x_2+fu) = \tilde{x}_2(a\tilde{x}_1+b\tilde{x}_1\tilde{x}_2+eu)+\tilde{x}_1(c\tilde{x}_2+d\tilde{x}_1\tilde{x}_2+fu),$$

and, for $x_1 = 0$, we get

$$eux_2 = eu\tilde{x}_2.$$

If both $e \neq 0$ and $u \neq 0$, we can conclude $x_2 = \tilde{x}_2$, otherwise the two points x and $\tilde{x}$ might not be the same. Note that the case where any of these two conditions is false is actually the uncontrolled case, at least with respect to x_1. If we suppose $e \neq 0$ and $u(t) \neq 0$ for some time t, any point x is only indistinguishable from itself and thus the model LV_1 is globally observable.

4.3.2 Analysis of the model LV_2

For the analysis of the model LV_2, we consider again two cases depending on the model parameter b. If $b = 0$, the output and its time derivatives never depend on the state x_2 and the indistinguishability sets are the same as in equation (4.2).

Otherwise, for the case where $b \neq 0$, we assume that the output $y(t, x, u(\cdot)) = y(t, \tilde{x}, u(\cdot))$ for all times t with two initial states x and $\tilde{x}$. Again we get directly $x_1 = \tilde{x}_1$.

The first time derivate gives the equation $x_1x_2 = \tilde{x}_1\tilde{x}_2$, from which we get $x_2 = \tilde{x}_2$ if $x_1 \neq 0$. In the case where $x_1 = 0$, we have to consider further derivatives. Computing

$$\begin{aligned}\frac{d}{dt}(x_1x_2) &= (ax_1 + bx_1x_2 + ex_1u)x_2 + x_1(cx_2 + dx_1x_2 + fx_2u)\\ &= x_1x_2(a + c + eu + fu + dx_1 + bx_2),\end{aligned}$$

we can conclude by induction that the term x_1x_2 is a factor in each of its own time derivatives. It follows that in the case $x_1 = 0$, we might have $x_2 \neq \tilde{x}_2$ for any input u. The indistinguishability sets for the system LV_2 are thus the same as in the uncontrolled case. Note that these sets do not depend on the input u. As in the uncontrolled case, it is possible to restrict the system to one of the open half planes of the original state space $\mathbb{R}^2$. The restricted system is then uniformly observable.

4.3.3 Analysis of the model LV_3

Let us now consider the model LV_3 with two initial states x and $\tilde{x}$. We assume that the output $y(t, x, u(\cdot)) = y(t, \tilde{x}, u(\cdot))$ for all times t. As for the other models, we conclude immediately $x_1 = \tilde{x}_1$. Computing the first time derivative of the output yields

$$\begin{gathered}ax_1 + bx_1x_2 + ex_1x_2u = a\tilde{x}_1 + b\tilde{x}_1\tilde{x}_2 + e\tilde{x}_1\tilde{x}_2u \quad \text{or}\\ (b + eu)x_1x_2 = (b + eu)\tilde{x}_1\tilde{x}_2.\end{gathered}$$

To obtain additional information, we may derive this term further and get

$$\frac{d}{dt}((b + eu)x_1x_2) = x_1x_2\left((b + eu)(a + c + dx_1 + fux_1 + bx_2 + eux_2) + e_3\dot{u}\right).$$

Similar to the previously analysed model, if $x_1 = 0$, we do not have additional information from any further time derivatives. Any two points x and $\tilde{x}$ where $x_1 =$

$\tilde{x}_1 = 0$ are indistinguishable for any input. Otherwise, if either $e = 0$ and $b \neq 0$ or $u \neq -\frac{b}{e}$, we can distinguish x_2 from $\tilde{x}_2$.

If $b + eu = 0$ for all times (which implies that u is constant), we cannot distinguish x_2 from $\tilde{x}_2$.

Thus we conclude that any two points x and $\tilde{x}$, where $x_1 = \tilde{x}_1 \neq 0$ are indistinguishable if and only if both $e = 0$ and $b = 0$ or $u = -\frac{b}{e}$. The indistinguishability sets of the system depend on the input. If the input is $u = -\frac{b}{e}$, then any two points x, $\tilde{x}$ where $x_1 = \tilde{x}_1$ are indistinguishable. For any other input u, we have the same indistinguishability sets as in the previous section for the model LV_2.

The system can be restricted to one of the open half planes such that the restricted system is observable for nearly any input, but it is not uniformly observable. There is exactly one input function for which the system is not observable, which is the constant input $u = -\frac{b}{e}$.

Part II

Observer design

Introduction to nonlinear observer design

Consider a dynamical system described by the equations

$$\Sigma : \quad \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{4.4}$$

with $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$ and f, g smooth vector fields on $\mathbb{R}^n$.

Since only the system output y and input u are known or available for measurement, the goal of observer design is to build another dynamical system which gives an estimate on the system state x using y and u from the observed system. We will denote the estimation with $\hat{x}$. Using the observer state $\hat{z}$, such an observer can be written with the equations

$$\begin{aligned} \dot{z} &= F(z, y, u) \\ \hat{x} &= H(z, y). \end{aligned} \tag{4.5}$$

Often z will be used to represent coordinates for the system which are useful for observer design. Then the use of the notion $\hat{z}$ for the observer state indicates that it is actually an estimate for the system state in z–coordinates. In general, the observer can be of different dimension compared to the observed system, i.e. the internal state $z \in \mathbb{R}^m$.

Usually, one demands two properties on a dynamical system to be an observer: Tracking of and convergence to the state of the observed system. These properties are defined in the following list.

(i) *Tracking.* If the initial estimation is right, i.e. we have $\hat{x}(0) = x(0)$, then the observer tracks the state of the observed system such that

$$\forall t \geq 0 : \hat{x}(t) = x(t).$$

(ii) *Convergence.* The estimation converges to the state of the observed system:

$$\lim_{t \to \infty} (\hat{x}(t) - x(t)) = 0.$$

For most observer design techniques, one demands exponential convergence, which implies both convergence and boundedness and is even stronger.

(iv) *Exponential convergence.* The estimation converges exponentially to the state of the observed system, i.e.

$$\exists C, \delta > 0 \, \forall t \geq 0 : \|\hat{x}(t) - x(t)\| \leq Ce^{-\delta t}.$$

With these utilities, we can give our formal definition of an observer.

Definition 9. An *observer* for the system Σ is a dynamical system as described by equation (4.5) which has the properties (i) and (ii) from above.

The observer is called *exponential*, if it has the properties (i) and (iv) from above.

The observer output function H from equation (4.5) is quite general. In fact, we often have only $\hat{x} = H(z)$, where the estimation depends just on the observer state. We will refer to this setup as a *full order observer*.

Contrarily, some observers might use the system output y directly to compute the estimation $\hat{x}$. This will only be reasonable if the output measurement can be done without or with negligible noise. In such a case, we will typically have $h(\hat{x}) = h(x)$ and we will consider this to be a *reduced order observer*. Note that in general, the reduced order observer might be of larger dimension than the observed system. But if m is the dimension of the full order observer, then the reduced order observer is usually of dimension $m - 1$.

To satisfy the tracking property, observers use a simulative part in their dynamics, which just implements the dynamics of the observed system using the estimation of the observer as system state. Convergence may then be guaranteed by adding an adjustive term depending on the measureable estimation error. This is what the classical Luenberger observer for linear system does, and most nonlinear observers have adopted that scheme. For the linear system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx, \end{aligned}$$

one uses the observer $\dot{\hat{x}} = A\hat{x} + G(C\hat{x} - y)$ with the simulative term $A\hat{x}$ and the adjustive term $G(C\hat{x} - y)$, involving the observer gain G and the difference between the estimation of y and its real value.

5 Observers with linearisable error dynamics

Nonlinear observers with linear error dynamics were proposed independently by Krener and Isidori [1983] and Bestle and Zeitz [1983]. The approach is to transform a general nonlinear system by a local diffeomorphism to a system which is linear up to output injection (i.e. up to terms depending on the output only). The output of the real system is then injected into the simulative part of the observer dynamics. The original system given by the equations

$$\Sigma : \quad \begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \tag{5.1}$$

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, we want to find a local coordinate transformation $z = \Phi(x)$ such that the transformed system becomes

$$\begin{aligned} \dot{z} &= Az + k(y) \\ y &= Cz \end{aligned} \tag{5.2}$$

with an observable pair (A, C) and a function $k : \mathbb{R} \to \mathbb{R}^n$.

Using these coordinates, an observer can be designed as

$$\begin{aligned} \dot{\hat{z}} &= A\hat{z} + k(y) + G(C\hat{z} - y) \\ \hat{x} &= \Phi^{-1}(\hat{z}) \end{aligned} \tag{5.3}$$

with the internal observer state $\hat{z} \in \mathbb{R}^n$ and a suitable gain G. The gain can be chosen by considering the observer error e in z–coordinates

$$e = \hat{z} - z.$$

The dynamics of the observer error are computed as

$$\dot{e} = (A + GC)e. \tag{5.4}$$

These dynamics are linear, and since (A, C) is observable, the error e can be made to converge exponentially to 0 with an appropriate choice of G by placing the eigenvalues of $(A + GC)$ in the left half complex plane.

For this approach, it is generally sufficient to choose A and C in the so called observer canonical form where the Kalman observability matrix becomes the identity matrix of dimension n. The matrices A and C in observer canonical form are

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{5.5}$$

and

$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}. \tag{5.6}$$

This choice of A and C is equivalent to any other observable pair, since we can just apply a further linear transformation to the nonlinear transformation Φ to get other matrices which are not in canonical form. A detailed proof can be found in [Isidori, 1995, p. 204].

Necessary and sufficient conditions on a nonlinear system for the existence of a transformation into observer canonical form have been given by Krener and Isidori [1983] and are summarised in [Isidori, 1995]. The first necessary condition is of course local observability of the system and even stronger; we need the observability rank condition (3.3), or existence of the observability canonical form. Further conditions make use of a vector field τ which is defined by

$$\begin{aligned} L_\tau h(x) &= 0 \\ L_\tau L_f h(x) &= 0 \\ &\vdots \\ L_\tau L_f^{n-2} h(x) &= 0 \\ L_\tau L_f^{n-1} h(x) &= 1. \end{aligned} \tag{5.7}$$

Note that τ is unique, if the observability rank condition is satisfied. It can be computed by solving a linear equation using the $n \times n$ observability matrix H, whose k–th row is given by $dL_f^{k-1} h(x)$. The observability rank condition implies that H is invertible, and with the definition above, τ is the unique solution of

$$H\,\tau = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}^{\mathrm{T}}.$$

The following theorem established by Krener and Isidori [1983] then gives conditions on the existence of a linearising transformation.

Theorem 4 (Observer canonical form). *The system Σ can be transformed to the observer canonical form given by equation* (5.2) *locally around x if and only if*

(i) the observability rank condition is satisfied, i.e.

$$\dim \operatorname{span} \{dh(x), dL_f h(x), \ldots, dL_f^{n-1} h(x)\} = n \tag{5.8}$$

(ii) the frame $\{\mathrm{ad}_f^i\tau\}$ *commutes, i.e. for all* i, j *where* $0 \leq i < j \leq n-1$*, we have* $[\mathrm{ad}_f^i\tau, \mathrm{ad}_f^j\tau] = 0$.

The second condition guarantees the resolvability of the partial differential equation for the inverse coordinate transformation Φ^{-1}, which is

$$\frac{\partial\Phi^{-1}}{\partial z} = \Big((-1)^{n-1}\mathrm{ad}_f^{n-1}\tau(x) \quad \dots \quad -\mathrm{ad}_f\tau(x) \quad \tau(x)\Big)_{x=\Phi^{-1}(z)}. \tag{5.9}$$

By solving this p.d.e., one can find the transformation of the given system to observer canonical form.

Example 1 *(The Lotka–Volterra model).* Consider the uncontrolled Lotka–Volterra model given by equation (2.1). We are looking for a transformation of this model into observer canonical form. To this end, we use the observability matrix

$$H = \begin{pmatrix} 1 & 0 \\ a + bx_2 & bx_1 \end{pmatrix}$$

and compute the vector field τ as

$$\tau(x) = \begin{pmatrix} 0 \\ \frac{1}{bx_1} \end{pmatrix}.$$

Now we need to check condition (ii) from theorem 4. In our case, this amounts to

$$[\tau, \mathrm{ad}_f\tau]\,(x) = \left[\begin{pmatrix} 0 \\ \frac{1}{bx_1} \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{a+c+dx_1+bx_2}{bx_1} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ -\frac{2}{bx_1^2} \end{pmatrix}.$$

The second condition of theorem 4 is not satisfied and the Lotka–Volterra model cannot be transformed into a system which is linear up to output injection.

The conditions for the observer canonical form (5.2) are in fact rather restrictive and the method is not applicable to our problem, as we have seen in the preceding example. Therefore, we will consider several methods which enhance the above approach to relax the conditions under which the desired transformation is possible.

Remark 1. The use of the linear state to output mapping C is retained in all the methods considered here. The computation of the necessary state space transformation is facilitated by this approach, since we are able to make use of the vector field τ for computations. But it imposes severe restrictions on the class of system that can be handled. A broader approach is to just take the nonlinear output mapping resulting from the linearising transformation as $y = h \circ \Phi^{-1}(z)$. This has been proposed by Kazantzis and Kravaris [1998] and conditions for the existence of a linearising transformation were further investigated by Andrieu and Praly [2004].

The Kazantzis–Kravaris observer will not be used in this work, as the linearisation for the uncontrolled model can already be done with the methods described in the following sections. They are easier to handle computationally than the approach used by Kazantzis and Kravaris and as their prerequisites are met, we see no use in applying a more complicated approach.

5.1 Observer design via output transformation

The choice of the output function in the observer canonical form (5.2) can be changed to extend the class of systems that can be transformed to the desired form. This is done by not only applying a state space transformation, but also a transformation of the output. This approach was deeply investigated by Krener and Respondek [1985] for multi output systems. We will restrict ourselves to single output systems, since this is sufficient for our problem.

5.1.1 Finding a suitable output transformation

We use essentially the same observer canonical form as in the previous section, just not for the real output y, but for the transformed output $\tilde{y}$.

$$\begin{aligned}\dot{z} &= Az + k(\tilde{y}) \\ \tilde{y} &= Cz.\end{aligned} \tag{5.10}$$

The transformed output $\tilde{y}$ is obtained from the real output y by a nonlinear, diffeomorphic output transformation Ψ, such that

$$\tilde{y} = \Psi(y). \tag{5.11}$$

As in the previous section, we intend to use a state space transformation $z = \Phi(x)$ which also has to be computed.

With respect to the theory presented at the beginning of this chapter, it remains to check if it is possible to find an output transformation Ψ such that the partially transformed system

$$\begin{aligned}\dot{x} &= f(x) \\ \tilde{y} &= \Psi(h(x))\end{aligned} \tag{5.12}$$

satisfies the conditions of theorem 4. Furthermore, if such an output transformation exists, we are of course interested in how to compute it. We will only give necessary conditions here. Sufficient conditions are obtained by applying theorem 4 to the partially transformed system.

The first necessary condition is the existence of the observability canonical form (3.2). That given, let us carry out the transformation of the system to observability canonical form[1] and continue our considerations based upon these coordinates which we will denote by ξ.

Next, we give conditions on the form of the characteristic nonlinearity $\rho(\xi)$ appearing in the observability canonical form (3.2). To this end, we use a different notation of polynomial degree than usual. Let us consider a polynomial in ξ with coefficients that are smooth functions of ξ_1. The degree of the basic monomial ξ_i is defined as $i - 1$.

To get a reasonable interpretation for this definition, consider e.g. ξ_1, which is of degree 0. With each time derivative, the degree increases by 1. Hence we get e.g.

[1] The method used here is sometimes called two–step transformation [Keller, 1987] because a given system is first transformed to observability and then to observer canonical form.

the degree of ξ_2 — the first time derivative of ξ_1 — as 1 and so forth. The degree of any product of basic monomials is the sum of the degrees of the basic monomials, e.g. the degree of the product $\xi_i\xi_j$ is $j + i - 2$. The degree of a given polynomial is the maximum of degrees of the monomials the polynomial is composed of.

The following proposition, which was given by Krener and Respondek [1985], uses this notation of degree to state necessary conditions on the existence of a transformation into observer canonical form.

Proposition 2. *If the system Σ can be transformed to observer canonical form* (5.10)*, then the characteristic nonlinearity $\rho(\xi)$ is a polynomial of degree equal to or less than n with smooth functions of ξ_1 as coefficients.*

In the next step, to find a suitable output transformation, we shall use the following theorem which was also given by Krener and Respondek [1985].

Theorem 5 (Output transformation). *Consider the system Σ from eq.* (5.1) *in observability canonical form. If it can be transformed to the observer canonical form* (5.10)*, then the output transformation Ψ satisfies the linear differential equation*

$$\Psi'' = -\frac{1}{n}\frac{\partial^2 \rho}{\partial \xi_n \partial \xi_2}\Psi'. \tag{5.13}$$

Some remarks on this theorem are in order.

Remark 2. The necessary condition given by proposition 2 guarantees that the differential equation (5.13) lives on the output space and does not contain a coefficient depending on ξ_i, where $2 \leq i \leq n$. Otherwise it would be impossible to find a solution.

Remark 3. The original formulation of the theorem is slightly different. First, it was given for multi output systems and has been simplified here to be applied to single output systems only. Second, the differential equation (5.13) was originally formulated for the inverse of the output transformation Ψ. If we denote this inverse as $\chi = \Psi^{-1}$, the original formulation for a single output system is

$$\frac{\partial}{\partial y}\chi' = \frac{1}{n}\frac{\partial^2 \rho}{\partial \xi_n \partial \xi_2}\chi'.$$

But χ takes the transformed output $\tilde{y}$ as an argument, and since the terms in the differential equation depend on the original output $y = \xi_1$, we have to reformulate the equation for Ψ using the relation $\chi' = (\Psi')^{-1}$. We thus compute

$$\frac{\partial}{\partial y}\chi' = \frac{\partial}{\partial y}\frac{1}{\Psi'} = -\frac{\Psi''}{(\Psi')^2},$$

which leads to the formulation used above, allowing us to directly compute Ψ from the differential equation (5.13).

This transformation for single output systems was also investigated by Keller [1987], who gave explicit differential equations which need to be solved in order to find the transformation to observer canonical form for second and third order systems in the controlled case.

Once we know an output transformation Ψ which satisfies the necessary condition, we will apply theorem 4 to the partially transformed system (5.12) to find the state space transformation which will give the observer canonical form.

5.1.2 Application to the uncontrolled Lotka–Volterra model

The theory outlined in the previous section is now applied to the uncontrolled Lotka–Volterra model as given by equation (2.1). With the assumption that $b \neq 0$, we want to design a local observer for for the region where $x_1 > 0$; the state space of the system will thus be restricted to $\{x \in \mathbb{R}^2 \mid x_1 > 0\}$.

With this restriction, we can consider the system in observability canonical coordinates, the system is hence described by (see equation (3.7))

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \frac{\xi_2^2}{\xi_1} + c\xi_2 + d\xi_1\xi_2 - ac\xi_1 - ad\xi_1^2 \\
y &= \xi_1.
\end{aligned}$$

The condition of proposition 2 is satisfied, since the characteristic nonlinearity is a polynomial of degree 2 in ξ_2, and its coefficients are smooth functions of ξ_1. To find the output transformation Ψ, we compute

$$\frac{\partial^2 \rho}{\partial \xi_2^2} = \frac{2}{\xi_1} = \frac{2}{y}.$$

The differential equation for Ψ is then

$$\Psi'' = -\frac{1}{y}\Psi',$$

and a solution is

$$\Psi(y) = \ln y.$$

Applying the output transformation $\tilde{y} = \ln y$, we obtain the partially transformed system

$$\begin{aligned}
\dot{x}_1 &= ax_1 + bx_1x_2 \\
\dot{x}_2 &= cx_1 + dx_1x_2 \\
\tilde{y} &= \ln x_1.
\end{aligned}$$

This system was represented in x–coordinates, because we want to find the transformation $z = \Phi(x)$ from the original to the observer canonical coordinates, forgetting about the intermediate transformation to the observability canonical form, which was only used to compute the output transformation.

Next, we will apply the linearisation method developed by Krener and Isidori as stated in theorem 4 to the partially transformed system. First, we have to compute the vector field τ for this system. The observability matrix is

$$H = \begin{pmatrix} \frac{1}{x_1} & 0 \\ * & b \end{pmatrix},$$

and, using equation (5.7), we obtain

$$\tau(x) = \begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix},$$

Checking the second condition of theorem 4 gives

$$[\tau, \mathrm{ad}_f \tau] = \left[\begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix}, \begin{pmatrix} -x_1 \\ -\frac{c+dx_1}{b} \end{pmatrix} \right] = 0$$

and all conditions of theorem 4 are satisfied.

This implies that a state space transformation into observer canonical form is possible. To compute the required state space transformation $z = \Phi(x)$, we have to solve the partial differential equation (5.9) for the inverse transformation, which is in our case

$$\frac{\partial \Phi^{-1}}{\partial z} = \begin{pmatrix} \Phi_1^{-1}(z) & 0 \\ \frac{c+d\Phi_1^{-1}(z)}{b} & \frac{1}{b} \end{pmatrix}.$$

The p.d.e. has the solution

$$\Phi^{-1}(z) = \begin{pmatrix} e^{z_1} \\ \frac{z_2 + cz_1 + de^{z_1}}{b} \end{pmatrix},$$

with all integration constants chosen as 0.

We invert this mapping to get the transformation Φ as

$$\Phi(x) = \begin{pmatrix} \ln x_1 \\ bx_2 - c\ln x_1 - dx_1 \end{pmatrix}. \tag{5.14}$$

The system dynamics in z–coordinates are computed as

$$\dot{z} = \frac{\partial \Phi(x)}{\partial x} f(x) \bigg|_{x=\Phi^{-1}(z)} = Az + k(z_1),$$

where A is in observer canonical form and k is given by

$$k(z_1) = \begin{pmatrix} a + de^{z_1} + cz_1 \\ -ac - ade^{z_1} \end{pmatrix}$$

The system output is $\tilde{y} = z_1$, or, in the original output space coordinate system, $y = e^{z_1}$.

A nonlinear observer for this system is implemented as

$$\begin{aligned} \dot{\hat{z}} &= A\hat{z} + k(\ln y) + G\left(\hat{z}_1 - \ln y\right) \\ \hat{x} &= \Phi^{-1}(\hat{z}), \end{aligned} \tag{5.15}$$

where $G = \begin{pmatrix} g_1 & g_2 \end{pmatrix}^{\mathrm{T}}$ is the 2×1 gain matrix which can be chosen to assign the poles of the observation error dynamics.

With the observation error $e = \hat{z} - z$, the error dynamics are

$$\dot{e} = (A + GC)e,$$

and its eigenvalues can be assigned using the design parameter G by standard pole placement. In our case, where the order of the system is 2, to set the eigenvalues λ_1 and λ_2 for the observer error dynamics we choose

$$G = \begin{pmatrix} \lambda_1 + \lambda_2 \\ -\lambda_1\lambda_2 \end{pmatrix}.$$

Sometimes it is more convenient to consider directly the dynamics of the estimated state $\hat{x}$. They are computed from $\dot{\hat{x}} = \frac{\partial \Phi^{-1}}{\partial z}\dot{\hat{z}}$, which gives for the observer designed above

$$\begin{aligned} \dot{\hat{x}}_1 &= a\hat{x}_1 + b\hat{x}_1\hat{x}_2 - d\hat{x}_1(\hat{x}_1 - y) - c\hat{x}_1(\ln\hat{x}_1 - \ln y) + g_2\hat{x}_1(\ln\hat{x}_1 - \ln y) \\ \dot{\hat{x}}_2 &= c\hat{x}_2 + d\hat{x}_1\hat{x}_2 + \frac{1}{b}\left(d(a-c) - d^2\hat{x}_1\right)(\hat{x}_1 - y) \\ &\quad - \frac{1}{b}(cd\hat{x}_1 + c^2)(\ln\hat{x}_1 - \ln y) + \frac{1}{b}(g_1 + cg_2 + dg_2\hat{x}_1)(\ln\hat{x}_1 - \ln y). \end{aligned} \tag{5.16}$$

With these dynamics, three different terms can be distinguished: The simulative term is the same as in the dynamics of the observed system, just using the estimated state instead of the real state. Then we have two error terms depending on both the direct estimation error $\hat{x}_1 - y$ and the error $\ln\hat{x}_1 - \ln y$ in the transformed output space: The first one, where the gains g_1 and g_2 do not appear, is the output injection term, and the second one is the adjustive term depending on the gains g_1 and g_2.

Simulation results

The observer designed in the previous section has been implemented for the Lotka–Volterra–Model using different model parameters.

For the first simulation, we use the parameter configuration (A) which is

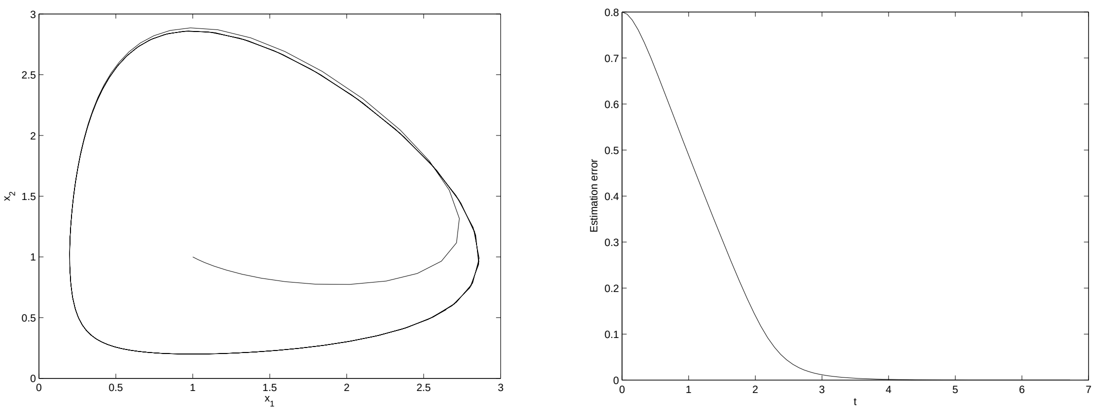
$$(A) \qquad \begin{aligned} a &= 1 \qquad & b &= -1 \\ c &= -1 \qquad & d &= 1. \end{aligned} \tag{5.17}$$

The nontrivial equilibrium point for the system with these parameters is $x_e^{(2)} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Initial conditions were chosen as $x(0) = \left(\begin{smallmatrix} 1 \\ 0.2 \end{smallmatrix}\right)$ for the real system and $\hat{x}(0) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ for the estimation of the observer, corresponding to $\hat{z}(0) = \binom{0}{-2}$ for the internal observer state.

With the above parameter values, we are considering a predator prey configuration for the Lotka–Volterra model. By the analysis carried out in chapter 2, we know that we have$x_1(t) > 0$ as required, if the starting value $x_1(0) > 0$. Moreover, for any starting value $x(0)$, there is an $\epsilon > 0$ such that $x_1(t) > \epsilon$ for all t. That property is very important for the observer, since the manifold $\{x_1 = 0\}$ is the border of the state space we consider and approaching it can be critical concerning the observation of the system.

The observer gain G was chosen as $G = \binom{-4}{-4}$ to place both poles of the observer error dynamics at -2.

The results of the simulation are shown in figure 5.1. The trajectory of the state estimation is shown in subfigure (a). It approaches the orbit of the real system state within the first half period of the system cycle. The norm of the estimation error in x–coordinates, which is plotted in subfigure (b), decays asymptotically to 0. Due to the nonlinear coordinate transformation, the dynamics of the error are nonlinear here. This can be seen in the figure, as the convergence of the estimation error is not of the exponential form one would have for a second order linear system with an eigenvalue of multiplicity two.



(a) Orbit of the estimation

(b) Norm of the estimation error

Figure 5.1: Results of simulation 1 for the observer via output transformation using parameter set (A).

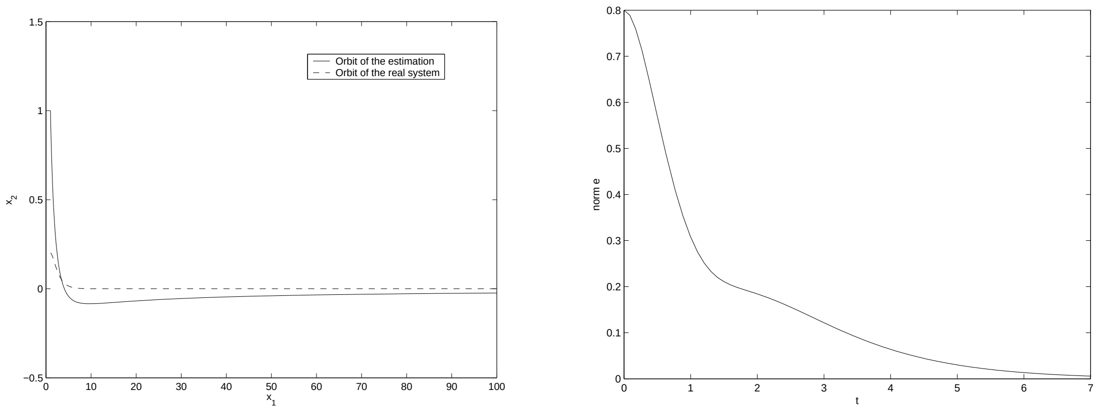
In the second simulation, the parameters are taken from configuration (B) which is

$$(B) \qquad \begin{matrix} a = 1 & b = -1 \\ c = 1 & d = -1. \end{matrix} \tag{5.18}$$

The equilibrium point stays the same, but we get a different dynamic behaviour.

Biologically, there is competition between the two species. There is one stable and one unstable submanifold at the nontrivial equilibrium point.

The second simulation uses the same initial conditions for both the system and the estimation of the observer as the first simulation. Results are displayed in figure 5.2. Here the nonlinear characteristic of the convergence of the estimation error in original coordinates is even more visible than in the first simulation. Furthermore, we note that although the set $\mathbb{R}^2_+$ was invariant for the original system, it clearly is not invariant for the observer, as we get estimates where $\hat{x}_2 < 0$. A natural question here is whether we can get estimates where $\hat{x}_1 < 0$. In this special case, the answer is rather easy, but since it is of more interest in the general case, we will deal with this question separately in chapter 6.



(a) Orbit of the estimation

(b) Norm of the estimation error

Figure 5.2: Results of simulation 2 for the observer via output transformation using parameter set (B).

Furthermore, for the parameter configuration (B), there are initial conditions such that the system will approach the border of the restricted state space as the time t goes to infinity. We may expect that this can pose problems for the observer. Indeed, the simulation runs into numerical problems, since the observer uses $\ln y$ as input. When y approaches the axis where we loose observability, it does so approximately like $\exp(-\exp t)$ and thus comes close to 0 really fast. The simulation stops after a short time, because numerically, the observer tries to compute $\ln 0$.

5.1.3 Extension to systems with control inputs

Let us now consider a system with an affine control input u, i.e. a system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{5.19}$$

There are basically two approaches to extend the method used previously to controlled systems. However, they are quite similar in that they both try to put the input into the measurable nonlinear part k of the observer canonical form, such that the input u does not appear in the error equation (5.4). In reference to the term *output injection*, this approach is called *input–output injection* [Plestan, 1995].

In the work of Krener and Respondek [1985], the problem is solved in two steps. First, a transformation of the uncontrolled part of the system into observer canonical form is computed, if this is possible. In the second step, the same transformation is applied to the controlled system. What we would like to find is a transformed system of the form

$$\begin{aligned} \dot{z} &= Az + k(\tilde{y}, u) \\ \tilde{y} &= Cz \end{aligned} \tag{5.20}$$

such that we can use essentially the same observer as in the uncontrolled case, just taking also the input into account.

Since the original system is control affine, the transformed system will also be control affine. We get the desired form if and only if the input vector field g only depends on the output in transformed coordinates, i.e. we have

$$\frac{\partial \Phi}{\partial x} g(x) = \gamma(\tilde{y}). \tag{5.21}$$

Obviously, we will get this form only in rare cases, but if we get it, the observer for the controlled system can be designed by just extending the observer for the uncontrolled system by the appropriate terms.

Let us now apply the input–output injection to the controlled Lotka–Volterra models introduced in section 2.2. For the first model LV_1, the input vector field is $g(x) = \begin{pmatrix} e & f \end{pmatrix}^{\mathrm{T}}$. Under the transformation Φ from equation (5.14), it becomes

$$\left(\frac{\partial \Phi}{\partial x} g(x) \right)_{x_1 = \exp(\tilde{y})} = \begin{pmatrix} e \exp(-\tilde{y}) \\ -ce \exp(-\tilde{y}) - de + bf \end{pmatrix} = \gamma(\tilde{y}).$$

The condition from equation (5.21) is satisfied and the observer designed previously can be extended straightforward to the model LV_1 by adding the term $\gamma(\tilde{y})u$ to its dynamics, which now become

$$\dot{\hat{z}} = A\hat{z} + k(\ln y) + \gamma(\ln y)u + G(\hat{z}_1 - \ln y).$$

This will result in the same error dynamics as for the uncontrolled case.

However, the other models LV_2 and LV_3 are of course more interesting, as their input vector fields are nonconstant. Concerning the model LV_3, where we have $g(x) = \begin{pmatrix} ex_1x_2 & fx_1x_2 \end{pmatrix}^{\mathrm{T}}$, it is obvious that this cannot be linearised by input–output injection. We found previously that LV_3 is not uniformly observable. The property of uniform observability is invariant under state space and output transformations, and since the observer canonical form (5.20) is uniformly observable, we cannot transform the system LV_3 to this form.

Let us now consider the model LV_2, where $g(x) = \begin{pmatrix} ex_1 & fx_2 \end{pmatrix}^{\mathrm{T}}$. This system was found to be uniformly observable. The transformed input vector field is computed as

$$\frac{\partial \Phi}{\partial x} g(x) = \begin{pmatrix} e \\ -ce - dex_1 + bfx_2 \end{pmatrix}. \tag{5.22}$$

The unmeasured state x_2 is contained here with the coefficient bf and thus the input vector field does not transform to the form $\gamma(\tilde{y})$, if $f \neq 0$ ($b \neq 0$ is already needed for observability reasons). In this case, the input–output injection cannot be used to design an observer with linear error dynamics for the system LV_2. If the transformation Φ found for the uncontrolled system is applied to the system LV_2, we get a system which is bilinear up to input–output injection. Only if $f = 0$, the observer design by input–output injection can be applied to the system LV_2. But since this is a rather special case, we will not implement it here.

The second approach to design observers with linear error dynamics for controlled systems which we will study is the one introduced by Keller [1987]. His method will be applied to the model LV_2 in section 7.1 on page 84.

5.2 Observer design via time scaling

In this section, another extension to the observer canonical form (5.2) is studied. The idea of this method is to transform the system to the form

$$\begin{aligned} \dot{z} &= s(y)\,(Az + k(y)) \\ y &= Cz, \end{aligned} \tag{5.23}$$

where the function $s : \mathbb{R} \to \mathbb{R}_+$ can be interpreted as a time scaling applied to the dynamics of the system.

Suppose we have a transformation $z = \Phi(x)$ transforming the original system (5.1) to the form given above. The observer is then designed as

$$\begin{aligned} \dot{\hat{z}} &= s(y)\,(A\hat{z} + k(y) + G(C\hat{z} - y)) \\ \hat{x} &= \Phi^{-1}(\hat{z}). \end{aligned} \tag{5.24}$$

The observer error $e = \hat{z} - z$ evolves according to the dynamics

$$\dot{e} = s(y)\,(A + GC)\,e.$$

When introducing the scaled time $d\theta = s(y)dt$, we get the dynamics

$$\frac{de}{d\theta} = (A + GC)e.$$

Both error dynamics are equivalent in the sense that they have the same trajectories, which are just parametrised by different times. The scaling is required to be positive, that is $s(y) > 0$ for all y, such that the direction of trajectories is retained. Then, asymptotically stable equilibria of the second dynamics are also asymptotically stable for the first dynamics, so that we can guarantee convergence of the observer (5.24) by linear eigenvalue assignment for the matrix $(A + GC)$.

The remaining question is how to find a transformation to the canonical form (5.23). This has been studied by Respondek et al. [2004]. Their result gives sufficient and necessary conditions on the existence of such a transformation and also shows how to compute the function s and the transformation Φ.

The first necessary condition is the existence of the observability canonical form as given in theorem 2 on page 13. If that is satisfied, we define the vector field τ as in equation (5.7). Due to the observability rank condition, τ is unique.

The next condition states that there needs to exist a smooth function λ such that

$$dL_\tau L_f^n h = l_n \lambda dL_f h \bmod \operatorname{span}\{dh\}, \tag{5.25}$$

where $l_n = \frac{n(n-1)}{2} + 1$. If this condition holds, we can construct the time scaling $s = \exp\sigma$, where σ is defined as the solution of the equations

$$L_{ad_f^j \tau}\sigma = \begin{cases} 0 & \text{if } 0 \le j \le n-2 \\ (-1)^{n-1}\lambda & \text{if } j = n-1. \end{cases}$$

Using the observability canonical form, this system of partial differential equations can be transformed to the ordinary differential equation

$$\frac{d\sigma}{dy} = \lambda(y). \tag{5.26}$$

By construction, the time scaling s is always positive, as required for stability.

Using the time scaling s, we can then compute the scaled vector fields $\bar{f} = \frac{1}{s} f$ and $\bar{\tau} = s^{n-1}\tau$, which are used in the conditions of the main theorem of Respondek et al. [2004], which is stated next.

Theorem 6 (Time Scaling). *Assuming the observability rank condition to hold, the system Σ from equation* (5.1) *can be transformed locally to the form* (5.23)*, if and only if the following conditions are satisfied locally:*

(i) There exists a smooth function λ satisfying equation (5.25)*.*

(ii) For the vector fields $\bar{f}$ and $\bar{\tau}$ as defined above, we have

$$\left[\mathrm{ad}_{\bar{f}}^i \bar{\tau}, \mathrm{ad}_{\bar{f}}^j \bar{\tau}\right] = 0 \quad \textit{for } 0 \le i < j \le n-1.$$

Once these conditions are satisfied and one has computed the time scaling to apply, the state space transformation is defined by the same partial differential equation as in the standard linearisation procedure by Krener and Isidori given in equation (5.9), but for the time scaled system with the vector fields $\bar{f}$ and $\bar{\tau}$ instead of the original vector fields f and τ. We get thus

$$\frac{\partial \Phi^{-1}}{\partial z} = \Big((-1)^{n-1}\mathrm{ad}_{\bar{f}}^{n-1}\bar{\tau}(x) \quad \ldots \quad -\mathrm{ad}_{\bar{f}}\bar{\tau}(x) \quad \bar{\tau}(x) \Big)_{x=\Phi^{-1}(z)}. \tag{5.27}$$

5.2.1 Application to the uncontrolled Lotka–Volterra model

We will now apply the observer design by time scaling to the uncontrolled Lotka–Volterra model (2.1). The vector field τ has already been computed earlier as

$$\tau(x) = \begin{pmatrix} 0 \\ \frac{1}{bx_1} \end{pmatrix}.$$

Additionally, to compute the function λ, we need to know the Lie derivatives

$$\begin{aligned} L_f h(x) &= ax_1 + bx_1x_2 \\ L_\tau L_f^2 h(x) &= 2a + c + dx_1 + 2bx_2. \end{aligned}$$

We now search a function λ satisfying equation (5.25), which writes in this case

$$\begin{pmatrix} d & 2b \end{pmatrix} = 2\lambda(x) \begin{pmatrix} a + bx_2 & bx_1 \end{pmatrix} \mod \operatorname{span}\{\begin{pmatrix} 1 & 0 \end{pmatrix}\}.$$

This implies $\lambda(x) = 1/x_1$. λ actually depends only on $y = x_1$ and we can write $\lambda(y) = 1/y$.

The time scaling s is computed using σ, the solution of the ordinary differential equation (5.26), which is in our case

$$\frac{d\sigma}{dy} = \frac{1}{y}. \tag{5.28}$$

A solution of this equation is $\sigma(y) = \ln y$ and then the time scaling is $s(y) = y$. The choice of another integration constant for σ would result in a multiplicative constant applied to the time scaling. This might be interesting to adjust the rate of convergence, but we will not consider it further here.

The next step is to check if a state space transformation of the system to the observer canonical form (5.23) is possible. This is done using item (ii) in theorem 6. We compute the scaled vector fields as

$$\begin{aligned} \bar{f} = \frac{1}{s} f &= \begin{pmatrix} a + bx_2 \\ c\frac{x_2}{x_1} + dx_2 \end{pmatrix} \\ \bar{\tau} = s\tau &= \begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix}. \end{aligned}$$

The expression $\left[\bar{\tau}, \operatorname{ad}_{\bar{f}}\bar{\tau}\right]$ is 0, and condition (ii) is satisfied. Thus it is possible to find a state space transformation $z = \Phi(x)$ such that the system takes the form of equation (5.23) in z–coordinates. The inverse of this transformation is given by the partial differential equation

$$\frac{\partial \Phi^{-1}}{\partial z} = \begin{pmatrix} -\operatorname{ad}_{\bar{f}(x)}\bar{\tau}(x) & \bar{\tau}(x) \end{pmatrix}_{x=\Phi^{-1}(z)} = \begin{pmatrix} 1 & 0 \\ \frac{c + d\Phi_1^{-1}(z)}{b\Phi_1^{-1}(z)} & \frac{1}{b} \end{pmatrix}.$$

The inverse transformation is thus computed as

$$\Phi^{-1}(z) = \begin{pmatrix} z_1 \\ \frac{1}{b}\left(z_2 + c\ln z_1 + dz_1\right) \end{pmatrix},$$

and the transformation is

$$\Phi(x) = \begin{pmatrix} x_1 \\ bx_2 - dx_1 - c\ln x_1 \end{pmatrix}.$$

where the output function of the transformed system is $y = h\left(\Phi^{-1}(z)\right) = z_1$.

Using the time scaling $d\theta = s\,dt$, we get the system dynamics in transformed coordinates

$$\begin{aligned} \dot{z} &= \frac{dz}{d\theta}\frac{d\theta}{dt} \\ &= \left(\frac{\partial\Phi(x)}{\partial x}\bar{f}(x)\right)_{x=\Phi^{-1}(z)} s(y) \\ &= y\begin{pmatrix} a + z_2 + dz_1 + c\ln z_1 \\ -\frac{ac+adz_1}{z_1} \end{pmatrix}, \\ y &= Cz, \end{aligned}$$

which is the form given by (5.23), where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ k(y) &= \begin{pmatrix} a + dy + c\ln y \\ -\frac{ac+ady}{y} \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 \end{pmatrix}. \end{aligned}$$

The observer is then designed as

$$\begin{aligned} \dot{\hat{z}} &= y\left(A\hat{z} + k(y) + G(C\hat{z} - y)\right) \\ \hat{x} &= \Phi^{-1}(\hat{z}). \end{aligned} \tag{5.29}$$

The gain G is chosen to place the poles of the linear error dynamics in scaled time

$$\frac{de}{d\theta} = (A + GC)e$$

in the same manner as for the observer design via output transformation.

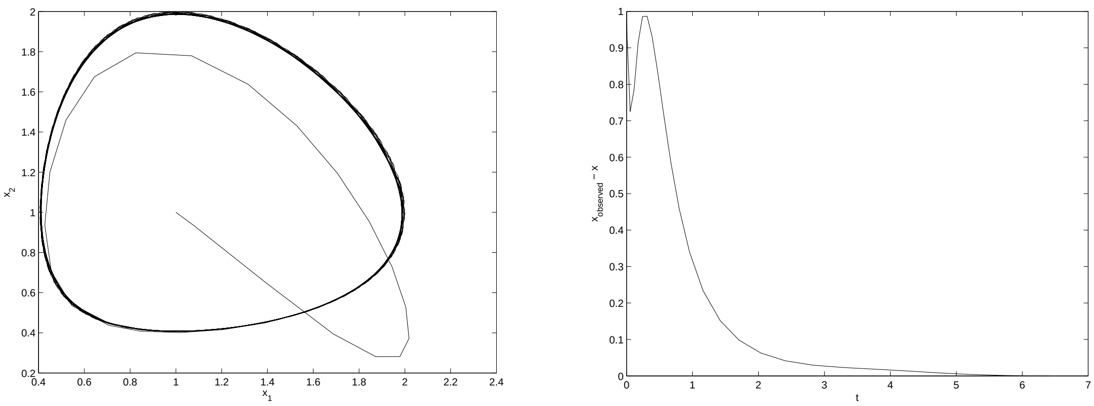
To compare the observers obtained by the two different methods applied so far, let us compute the dynamics of the estimation $\hat{x}$ of the time scaling observer. When implementing the observer, we will use the z–coordinate system, but the dynamics may be compared only in original x–coordinates, since the observer canonical coordinates are different for the two design methods. For the estimation $\hat{x}$, we get the dynamics

$$\begin{aligned}\dot{\hat{x}}_1 &= ay + by\hat{x}_2 - yc(\ln\hat{x}_1 - \ln y) - y(d-g_1)(\hat{x}_1 - y)\\ \dot{\hat{x}}_2 &= c\hat{x}_2\frac{y}{\hat{x}_1} + dy\hat{x}_2 - \frac{1}{b}\left(d + \frac{c}{\hat{x}_1}\right)(\ln\hat{x}_1 - \ln y)\\ &\quad + \frac{1}{b}\left(d^2y + \frac{(a+d)c}{\hat{x}_1} - yg_1 - yg_2\right)(\hat{x}_1 - y).\end{aligned} \tag{5.30}$$

The main difference between the dynamics of the observer estimation given by the time scaling observer and the output transformation observer (5.16) is the term simulating the observed system. With the output transformation, the simulative term was computed only from the estimation $\hat{x}$, whereas the time scaling observer uses mainly the measurement y instead of the estimated state $\hat{x}_1$. However, since the tracking and exponential convergence properties are satisfied locally as well, this difference should not pose any problems.

Simulation results

The observer (5.29) has been implemented for simulation with the model parameters $a = d = 1$ and $b = c = -1$, the initial state for the system $x_0 = \binom{2}{1}$ and the initial observer estimation $\hat{x}_0 = \binom{1}{1}$. The observer gain has been chosen as $G = \begin{pmatrix}-4 & -4\end{pmatrix}^{\mathrm{T}}$ such that the eigenvalues of the observer error matrix $A + GC$ are $(-2, -2)$. The simulation gave the results displayed in figure 5.3. The first subfigure shows the trajectory of the observer estimation $\hat{x} = \Phi^{-1}(\hat{z})$, the second one the norm of the estimation error in x–coordinates, $\|\hat{x} - x\|$. The orbit of the real system is the limit cycle the estimation trajectory in subfigure (a) approaches to.

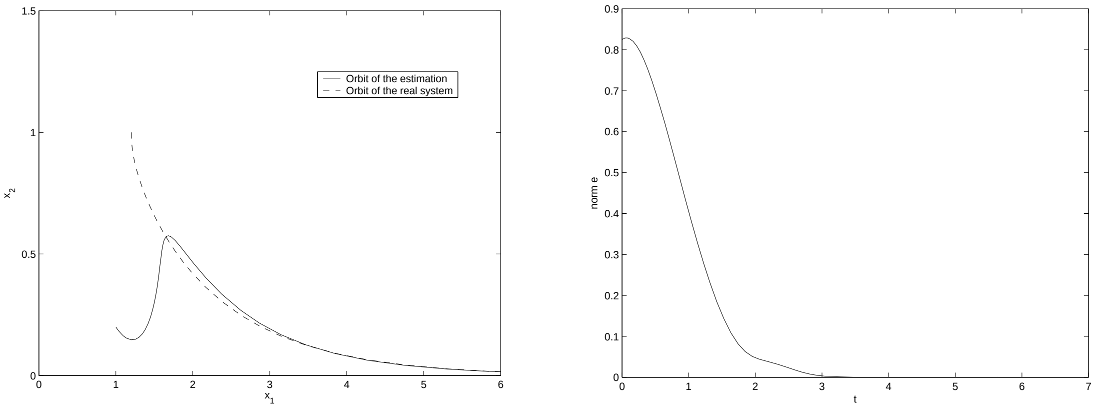


(a) Orbit of the estimation

(b) Norm of the estimation error

Figure 5.3: Results of simulation 1 for the observer via time scaling

The second simulation was done with the same observer gain G, but with a different parameter set, where $a = c = 1$ and $b = d = -1$. Biologically, this is a competition setting. The initial condition of the system was chosen as $x(0) = \left(\begin{smallmatrix}1.2\\1\end{smallmatrix}\right)$, whereas the initial estimate was $\hat{x}(0) = \left(\begin{smallmatrix}1\\0.2\end{smallmatrix}\right)$. The results are displayed in figure 5.4.



(a) Orbit of the estimation

(b) Norm of the estimation error

Figure 5.4: Results of simulation 2 for the observer via time scaling

For the second simulation, we see that the estimate converges rather fast — in comparison to the output transformation observer (see figure 5.2 on page 35) — to the state of the real system, although the same gain was used. This is due to the time scaling $s(y) = y$, which becomes very large here, leading to a fast convergence. Conversely, we expect a worse performance if y becomes small. However, due to the required coordinate transformation which is singular for $y = 0$, this poses problems for both observers.

5.2.2 Application to controlled Lotka–Volterra models

The approach to extend the time scaling design to the controlled models uses the same concept as done with the output transformation (see section 5.1.3 on page 35). We use the state space transformation which transforms the uncontrolled system into observer canonical form and apply it to the controlled system. What we would like to get in transformed coordinates is

$$\begin{aligned} \dot{z} &= s(y)(Az + k(y) + \gamma(y)u) \\ y &= Cz. \end{aligned} \tag{5.31}$$

If the controlled system takes this form, the observer can simply be designed as an appropriate extension of the observer for the uncontrolled system, i.e. we put

$$\begin{aligned} \dot{\hat{z}} &= s(y)\left(A\hat{z} + k(y) + \gamma(y)u + G(C\hat{z} - y)\right) \\ \hat{x} &= \Phi^{-1}(\hat{z}). \end{aligned} \tag{5.32}$$

This is only possible if the transformed input vector field does only depend on the system output y. As we do a time scaling, we have to consider the time scaled input vector field

$$\bar{g} = \frac{1}{s} g.$$

To transform the input vector field, we apply the coordinate transformation Φ. The condition for existence of the canonical form (5.31) then becomes

$$\frac{\partial \Phi}{\partial x} \bar{g}(x) = \gamma(y).$$

We are going to check this condition for the three different models introduced in section 2.2 on page 6. First, the Jacobian matrix of the transformation Φ used in the time scaling design is

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} 1 & 0 \\ -d - \frac{c}{x_1} & b \end{pmatrix}.$$

For the model LV_1, we have $g(x) = \left(\begin{smallmatrix} e \\ f \end{smallmatrix}\right)$ and thus $\bar{g}(x) = \frac{1}{x_1}\left(\begin{smallmatrix} e \\ f \end{smallmatrix}\right)$. It transforms to observer canonical coordinates as

$$\frac{\partial \Phi}{\partial x} \bar{g}(x) = \begin{pmatrix} \frac{e}{x_1} \\ \frac{1}{x_1^2}(-ce - dex_1 + bfx_1) \end{pmatrix} = \gamma(x_1),$$

and thus the observer for this system is constructed by just adding the term $\gamma(y)u$ to the dynamics of the observer for the uncontrolled system as in equation (5.32).

In the model LV_2, the input vector field is $g(x) = \left(\begin{smallmatrix} ex_1 \\ fx_2 \end{smallmatrix}\right)$, and it is changed by time scaling to $\bar{g}_x = \left(\begin{smallmatrix} e \\ fx_2/x_1 \end{smallmatrix}\right)$. By Φ, it transforms to

$$\frac{\partial \Phi}{\partial x} \bar{g}(x) = \begin{pmatrix} e \\ \frac{1}{x_1}(-ce - dex_1 + bfx_2) \end{pmatrix}.$$

If $f \neq 0$, the transformed vector field cannot be represented in the form $\gamma(x_1)$, hence it is not possible to use the above observer design for the controlled model LV_2 in this case. Only if $f = 0$, it is possible to extend the observer design by time scaling for the model LV_2.

For the model LV_3, we can use the same argument as with the observer design via output transformation. The model LV_3 is not uniformly observable, whereas any system in the observer canonical form (5.31) is uniformly observable. However, state space transformations do not change uniform observability, hence there is now state space transformation which puts the system LV_3 to the form (5.31).

5.3 Immersion into linearisable systems

In this section, the method of linearisation up to output injection by immersion into a higher order dynamical system is applied to the design of an observer for the Lotka–Volterra model. This method was introduced by Jouan [2003].

5.3.1 Theoretical background of the immersion method

Consider a control affine dynamical system of the form

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{5.33}$$

with $x \in X \subset \mathbb{R}^d$, $y \in I \subset \mathbb{R}$ and $u \in U \subset \mathbb{R}$.

The goal of the method is to find an immersion (τ, Ψ) of the original system Σ into another system Σ_i for which the observer design problem can be solved easily. An immersion is a pair formed by the C^∞ mapping τ, which maps the state space X of the original system onto the state space of the system Σ_i it is immersed in, and the diffeomorphism Ψ which maps the original output set I onto the new output set $\Psi(I)$. We require for an immersion that the output mappings for both the original system Σ and the system Σ_i are equivalent, i.e. if Σ is initialised with x_0 and Σ_i with $\tau(x_0)$, then we find Ψ–equivalent outputs in the sense that

$$\Psi \circ h(x(t)) = h_i(z(t))$$

for any input function $u(t)$, where h_i is the output function of Σ_i and z its state variable.

Note that if both systems are uniformly observable, then equivalence of output trajectories implies that also state trajectories are mapped on the corresponding state trajectories of the other system by τ.

The form of the system we want to immerse the original system in is given by the equations

$$\begin{aligned} \dot{z} &= Az + \varphi(Cz) + \gamma(Cz)u \\ \tilde{y} &= Cz \end{aligned} \tag{5.34}$$

where $z = \tau(x) \in \mathbb{R}^n$ is the state variable for the system and $\tilde{y} = \Psi(y)$ its output. Furthermore, we require the pair of matrices (A, C) to be observable. If such an immersion is possible, the system Σ from (5.33) is said to be LIS[2].

Conditions for the existence of such an immersion are given by the main theorem of Jouan [2003] for the uncontrolled case:

Theorem 7 (Immersion into observer canonical form). *The system Σ with $g = 0$ is LIS if and only if there exist a diffeomorphism $\Psi \in \mathcal{C}^\infty(I, \Psi(I))$, an integer n and n functions $\varphi_1, \ldots, \varphi_n \in \mathcal{C}^\infty(\Psi(I), \mathbb{R})$ such that*

$$L_f^n(\Psi \circ h) = L_f^{n-1}(\varphi_1 \circ \Psi \circ h) + \ldots + L_f(\varphi_{n-1} \circ \Psi \circ h) + \varphi_n \circ \Psi \circ h. \tag{5.35}$$

Remark 4. Note that if a pair (τ, Ψ) is an immersion of the uncontrolled system Σ as given in equation (5.33) with $g = 0$ into observer canonical form, then necessarily

(i) $\tau_1 = \Psi \circ h$ and

[2] for the french *linéarisable par injection de sortie*

(ii) $\tau_{k+1} = L_f \tau_k - \varphi_k \circ (\Psi \circ h)$ for $k = 1, \dots, n-1$

This allows to compute the mapping τ required for the immersion, once we know the functions Ψ and φ_i from equation (5.35).

The control affine case as considered above is also treated in Jouan [2003] with the following theorem:

Theorem 8 (Immersion of a controlled system). *The system Σ is LIS if and only if the following conditions are satisfied:*

(i) The system Σ with $g = 0$ is LIS.

(ii) Denoting by (τ, Ψ) with $\tau = (\tau_1, \dots, \tau_n)$ the immersion of the uncontrolled system in a LIS system in canonical form, there exist n smooth functions γ_i, $i = 1, \dots, n$ such that

$$L_g \tau_i = \gamma_i \circ (\Psi \circ h). \tag{5.36}$$

5.3.2 Application to the uncontrolled Lotka–Volterra model

We will now search an immersion of the Lotka–Volterra model (2.1) into a system of the form (5.34), where we can design an observer by the same approach as in the previous sections.

It is most interesting to find an immersion of the Lotka–Volterra model (2.1) into a higher dimensional system in observer canonical form such that the mapping Ψ in equation (5.35) is the identity. We already know from section 5.1 that using an output transformation $\Psi(y) = \ln y$, the Lotka–Volterra model can be transformed to a linear system up to an input injection by a simple coordinate transformation, which can be seen as an immersion into a LIS system of dimension 2. A natural question might be whether it is possible to immerse the Lotka–Volterra model into a LIS system without using an output transformation Ψ. We have already seen at the beginning of this chapter that there is no state space transformation of the Lotka–Volterra model to observer canonical form without output transformation.

Unfortunately, we find that also an immersion into a system of the form (5.34) is not possible for $\Psi = \mathrm{Id}$.

Proposition 3. *If $\Psi = \mathrm{Id}$, then for any integer n, there exist no functions $\varphi_1, \dots \varphi_n$ such that equation* (5.35) *holds for the Lotka–Volterra model in observability canonical form as given by equation* (3.7)*.*

Proof. The notion $P_k(\xi_2)$ is used throughout this proof for an unspecified polynomial in ξ_2 of degree k or less. Note that two polynomials in ξ_2 of degree k will both be considered as $P_k(\xi_2)$ even if they are not equal.

Let us first compute $L_f^n h(\xi)$. We have $L_f^2 h(\xi) = \frac{\xi_2^2}{\xi_1} + P_1(\xi_2)$, and for $k \geq 2$,

$$L_f^k h(\xi) = \frac{\xi_2^k}{\xi_1^{k-1}} + P_{k-1}(\xi_2),$$

which is proved by induction:

$$\begin{aligned}
L_f^{k+1}h(\xi) &= \left(-\tfrac{(k-1)\xi_2^k}{\xi_1^k} + P_{k-1}(\xi_2),\quad \tfrac{k\xi_2^{k-1}}{\xi_1^{k-1}} + P_{k-2}(\xi_2)\right) f(\xi) \\
&= -\frac{(k-1)\xi_2^{k+1}}{\xi_1^k} + \frac{k\xi_2^{k+1}}{\xi_1^k} + P_k(\xi_2) \\
&= \frac{\xi_2^{k+1}}{\xi_1^k} + P_k(\xi_2)
\end{aligned}$$

Let us now compute the $L_f^k(\varphi_i \circ h)$ for $k \geq 0$ which appear in equation (5.35). We get $L_f^0(\varphi_i \circ h) = \varphi_i(\xi_1) := D_i^0$, $L_f^1(\varphi_i \circ h) = \varphi_i'\xi_2 := D_i^1\xi_2$ and for $k \geq 2$ by induction $L_f^k(\varphi_i \circ h) = D_i^k\xi_2^k + P_{k-1}(\xi_2)$, as

$$\begin{aligned}
L_f^{k+1}(\varphi_i \circ h) &= \left((D_i^k)'\xi_2^k + P_{k-1}(\xi_2),\quad kD_i^k\xi_2^{k-1} + P_{k-2}(\xi_2)\right) f(\xi) \\
&= ((D_i^k)' + \frac{kD_i^k}{\xi_1})\xi_2^{k+1} + P_k(\xi_2) \\
&= D_i^{k+1}\xi_2^{k+1} + P_k(\xi_2)
\end{aligned}$$

where D_i^k is an expression of ξ_1, $\varphi_i(\xi_1)$ and its derivatives up to order k and any D_i^k could be calculated by induction.

In the end, with equation (5.35) this leads to

$$\frac{\xi_2^n}{\xi_1^{n-1}} + P_{n-1}(\xi_2) = D_1^{n-1}\xi_2^{n-1} + P_{n-2}(\xi_2).$$

Regardless of n, there is no D_1^{n-1} and thus no function $\varphi_1(\xi_1)$ which can satisfy this equation. □

Note that actually the linearisation method with state space and output transformation used in section 5.1 is also covered by the immersion method used in this section. One simply chooses to immerse the observed system in a LIS system of the same dimension. We will thus apply the method of immersion with some suitable output transformation Ψ to obtain a 2–dimensional observer canonical form of the uncontrolled Lotka–Volterra model. We expect the result to be the same as obtained previously, maybe up to integration constants.

First we will compute the necessary output transformation Ψ. This is done by using equation (5.35). We get

$$L_f^2(\Psi \circ h) = (\Psi'' + \frac{\Psi'}{\xi_1})\xi_2^2 + \alpha_1(\xi_1)\xi_2 + \alpha_2(\xi_1).$$

We know already from the proof of proposition 3 that the right hand side of equation (5.35) is a polynomial of degree 1 in ξ_2 for the Lotka–Volterra model. Thus Ψ has to satisfy the differential equation

$$\Psi'' + \frac{\Psi'}{\xi_1} = 0.$$

A solution for this equation is $\Psi(\xi_1) = \ln \xi_1$, as expected the same result as obtained in section 5.1.

Calculating again equation (5.35) with $\Psi(\xi_1) = \ln \xi_1$ to find the corresponding functions α_i, we get

$$\begin{aligned} L_f^2 \Psi(\xi_1) &= c\frac{\xi_2}{\xi_1} + d\xi_2 - ac - ad\xi_1 \\ L_f(\varphi_1 \circ \Psi)(\xi_1) &= \varphi_1' \frac{\xi_2}{\xi_1} \end{aligned}$$

and we have to find φ_1, φ_2 such that

$$\frac{c}{\xi_1}\xi_2 + d\xi_2 - ac - ad\xi_1 = \frac{\varphi_1'}{\xi_1}\xi_2 + \varphi_2(\Psi(\xi_1)). \tag{5.37}$$

Note that $\frac{d(\varphi_1 \circ \Psi)}{d\xi_1} = \varphi_1' \frac{d\Psi}{d\xi_1} = \varphi_1' \frac{1}{\xi_1}$. We obtain equations for the functions φ_i by coefficient comparison of ξ_2 in equation (5.37). Thus $\varphi_1 \circ \Psi$ has to satisfy the differential equation

$$\frac{d(\varphi_1 \circ \Psi)}{d\xi_1} = c\frac{1}{\xi_1} + d.$$

The solution to this equation is

$$\varphi_1 \circ \Psi(\xi_1) = d\xi_1 + c \ln \xi_1 + \alpha$$

with an integration constant α. For φ_2 we get directly

$$\varphi_2 \circ \Psi(\xi_1) = -ac - ad\xi_1.$$

As expected, this coincides well with the result found by output and state space transformation in section 5.1, where

$$\begin{aligned} \varphi_1(y) &= dy + c \ln y + a \\ \varphi_2(y) &= -ac - ady. \end{aligned}$$

5.3.3 Application to a controlled Lotka–Volterra model

We consider the controlled Lotka–Volterra model LV_2 defined by equation (2.3). Its dynamics in observability canonical form are given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + e\xi_1 u \\ \dot{\xi}_2 &= \frac{\xi_2^2}{\xi_1} + c\xi_2 + d\xi_1\xi_2 - ac\xi_1 - ad\xi_1^2 + ((f+e)\xi_2 - af\xi_1)u \\ y &= \xi_1. \end{aligned} \tag{5.38}$$

Note that written in the form $\dot{\xi} = f(\xi) + g(\xi)u$, one gets

$$g(\xi) = \begin{pmatrix} e\xi_1 \\ (f+e)\xi_2 - af\xi_1 \end{pmatrix}. \tag{5.39}$$

As found in section 5.1.3, for $f \neq 0$, this system is not linear up to input and output injection by state space and output transformations. Hence we are assuming $f \neq 0$ and are looking for an immersion of the model LV_2 into a higher dimensional system in observer canonical form using theorem 8, i.e. we are searching an integer n and functions Ψ, φ_i and $\gamma_i,\ i = 1, \dots, n$ such that equations (5.35) and (5.36) hold.

We have $h(\xi) = \xi_1$. Thus we have to satisfy

$$\begin{aligned} \gamma_i \circ \Psi(\xi_1) &= L_g \tau_i(\xi) \\ &= \begin{pmatrix} \partial_1 \tau_i, & \partial_2 \tau_i \end{pmatrix} g(\xi) \\ &= e\xi_1 \partial_1 \tau_i + (f+e)\xi_2 \partial_2 \tau_i - af\xi_1 \partial_2 \tau_i. \end{aligned}$$

Deriving this equation with respect to ξ_2 yields

$$e\xi_1 \partial_2 \partial_1 \tau_i + (f+e)\xi_2 \partial_2^2 \tau_i + (f+e)\partial_2 \tau_i - af\xi_1 \partial_2^2 \tau_i = 0.$$

This equation has to be satisfied for all τ_i we want to use in the immersion. It holds trivially, if $\partial_2 \tau_i = 0$, but we need $\partial_2 \tau_i \neq 0$ for at least one τ_i.

We are now searching for a Ψ to use in the immersion. Recall that τ_1 is given by $\tau_1(\xi) = \Psi(\xi_1)$ and thus satisfies the equation found above trivially. Continuing, we get

$$\begin{aligned} \tau_2(\xi) &= L_f \tau_1(\xi) - \varphi_1 \circ \Psi(\xi_1) \\ &= \Psi'(\xi_1)\xi_2 - \varphi_1 \circ \Psi(\xi_1). \end{aligned}$$

Since Ψ has to be a local diffeomorphism and thus $\Psi' \neq 0$, the condition found above for τ_i does not hold trivially for τ_2. In fact, one gets $\partial_2 \tau_2 = \Psi'$, $\partial_2^2 \tau_2 = 0$ and $\partial_2 \partial_1 \tau_2 = \partial_1 \partial_2 \tau_2 = \Psi''$. Putting all this into our condition for τ_2, we get

$$e\xi_1 \Psi'' + (f+e)\Psi' = 0,$$

whatever φ_1 we intend to use. Recall that we assumed $f \neq 0$. If $e = 0$, an admissible Ψ cannot be found. Otherwise, by substituting $\delta = \Psi'$, we get the linear differential equation

$$\delta' = -\delta \frac{f+e}{e\xi_1}$$

which is solved by $\delta(\xi_1) = \tilde{k}\xi_1^{-(f+e)/e}$ and thus

$$\Psi(\xi_1) = k_1 \xi_1^{-\frac{f}{e}} + k_2.$$

We conclude that the output transformation Ψ found here is necessary to immerse the controlled Lotka–Volterra model into observer canonical form. Furthermore, this is only possible if the uncontrolled system can be immersed into observer canonical form. This is what we will try to do now.

Let us use the following statement which will be proved by induction. For any $m \geq 1$ we have

$$L_f^m \Psi(\xi_1) = k_1 \left(-\frac{f}{e}\right)^m \xi_1^{-\frac{f}{e}-m} \xi_2^m + P_{m-1}(\xi_2).$$

Proof. For $m = 1$, we have $L_f\Psi(\xi_1) = \left(-k_1\frac{f}{e}\xi_1^{-\frac{f}{e}-1},\ \ 0\right) f(\xi) = -k_1\frac{f}{e}\xi_1^{-\frac{f}{e}-1}\xi_2$ and the statement holds. Assume now that the equation to prove holds for some $m \geq 0$. Then

$$\begin{aligned} L_f^{m+1}\Psi(\xi_1) &= k_1\left(-\frac{f}{e}\right)^m \left(-\left(\tfrac{f}{e}-m\right)\xi_1^{-\frac{f}{e}-m-1}\xi_2^m + P_{m-1}(\xi_2),\quad m\xi_1^{-\frac{f}{e}-m}\xi_2^{m-1} + P_{m-2}(\xi_2)\right) f(\xi) \\ &= k_1\left(-\frac{f}{e}\right)^m \left(-\left(\frac{f}{e}-m\right)\xi_1^{-\frac{f}{e}-m-1}\xi_2^{m+1} + P_m(\xi_2) + m\xi_1^{-\frac{f}{e}-m-1}\xi_2^{m+1} + P_m(\xi_2)\right) \\ &= k_1\left(-\frac{f}{e}\right)^{m+1} \xi_1^{-\frac{f}{e}-m-1}\xi_2^{m+1} + P_m(\xi_2). \end{aligned}$$

□

Therefore $L_f^n\Psi$ is a polynomial of degree n with respect to ξ_2. Furthermore, $L_f^k(\varphi_i \circ \Psi)$ is also polynomial of degree k with respect to ξ_2. This fact does not change from the previously studied case where $\Psi = \mathrm{Id}$, as one can always set $\varphi_i = \tilde{\varphi}_i \circ \Psi^{-1}$ and search for appropriate $\tilde{\varphi}_i$.

Thus the system cannot be immersed into observer canonical form, as on the right hand side of equation (5.35), we get a polynomial of degree n in ξ_2 and on the left hand side only a polynomial of degree $n-1$ in ξ_2, and there are no functions φ_i which can satisfy this equation.

5.3.4 Conclusions for the immersion method

The observer design by immersion into a higher dimensional system may be seen as an extension to the transformation into observer canonical form by state space and output transformation as treated in section 5.1. The output transformation is in fact the special case of an immersion into an equal dimensional system.

For the example of the Lotka–Volterra model, in both the uncontrolled and controlled case, we have seen that the two approaches are actually equivalent. Both allow to transform the uncontrolled system to observer canonical form. For the controlled model LV_2, we found that neither the immersion approach nor the observer design by state space and output transformation lead to an observer for this system.

However, some examples showing that the class of transformable systems is really extended are already known. One such example has been given by Back and Seo [2004], where the characteristic nonlinearity contains a square root of the unmeasured states. Such systems cannot be handled by only state space and output transformation, due to proposition 2 on page 30 [Krener and Respondek, 1985]. An immersion into a higher dimension has been given by Back and Seo [2004].

So it remains to study how the class of systems which can be transformed to a linear up to output injection form is extended by the immersion approach. Since the conditions of theorem 7 cannot be checked easily for a given system, this question is not yet answered. A characterisation was given by Back and Seo [2004] as resolvability of a system of differential equations, but in the general case, this cannot be checked easily either. We know that the immersion is an extension, but do not yet know how far it reaches.

6 Invariance properties of nonlinear observers

This chapter introduces a problem which may arise in observing systems which are only observable on a subset of their state space. Usually one will then restrict the system to this observable subset and design an observer for it. However, it is not guaranteed that the observer cannot leave the set it was designed for, and if it does leave, the estimated state might become undefined, as we will show in some examples. We will refer to this problem as the invariance problem for the observer, because what we actually want the set for which the observer is designed to be invariant under the dynamics of the observer.

A closely related problem appears when the observed system has an invariant set. When designing an observer for the system, we might want this set to be also invariant for the observer. This property will be referred to as inheritance of invariance, since an observer with this property inherits the invariant set from the system.

Although these problems appear in the Lotka–Volterra model, they are of more general type, and we will therefore develop general methods to deal with them. The Lotka–Volterra model will be our primary example, but some other examples are given as well.

6.1 Introduction to observer invariance

For many dynamical systems, there is an invariant subset of the state space, such that if the system is initialised in this set, its state remains there for all future times. Consider e.g. compartmental systems, which are derived from some material conservation law, where the amount of material is assumed to be always positive. The invariant subset is then $\mathbb{R}^n_+ = \left\{ \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n \mid x_i > 0,\ i = 1, \dots, n \right\}$. The design of observers inheriting this invariance property has been intensively studied for linear compartmental systems by van den Hof [1998].

For linear systems, this property is mainly desirable for physical reasons, such that the estimated state fulfils the same physical restriction as the real system state. For nonlinear systems, such a property may become crucial to guarantee that the estimated state is well defined in original coordinates for all times. There may be hypersurfaces in the state space where the system looses observability. In the observer design, it is import to guarantee that the observer cannot cross these surfaces and thus try to estimate a state where the system would actually loose observability.

We will only deal with the observer design methods presented in chapter 5, where the observer error dynamics are linearisable. Nevertheless, the problems considered here may also appear with observer design methods based on a transformation to observability canonical form such as the high gain observer introduced by Gauthier et al. [1992].

The first example will illustrate the invariance problem, i.e. the problem of an observer leaving the region where the system is observable.

Example 2 *(The Lotka–Volterra model).* Let us consider the Lotka–Volterra model (2.1) and the observer (5.29) designed via time scaling in section 5.2.

The observer was designed using a coordinate transformation $z = \Phi(x)$ given by

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= bx_2 - c\ln x_1 - dx_1.
\end{aligned}
$$

When designing this observer, we assumed that $x_1 > 0$. Actually, the observer was intended to work locally around some point x where $x_1 > 0$. But the transformation Φ is a global diffeomorphism on the set $O = \{x \in \mathbb{R}^2 \mid x_1 > 0\}$[1], and so we used the observer as global observer for the set O. This was quite reasonable, because the system dynamics are invariant on the set $\mathbb{R}^2_+$, which is a subset of O.

The condition for the observer to work is that its internal state $\hat{z}$ stays inside the set we designed the observer for. By applying the transformation Φ to the set O, we get the set $\Phi(O) = \{z \in \mathbb{R}^2 \mid z_1 > 0\}$. If the observer state $\hat{z}$ leaves the set $\Phi(O)$, the inverse transformation $\Phi^{-1}(\hat{z})$ is not defined and we are not able to compute an estimate $\hat{x}$ for the state of the system.

For this example, we will provoke the effect of the observer leaving the set it was designed for, to get a first idea of the problem resulting therefrom. Let us consider the observer dynamics when the observer state $\hat{z}$ approaches the border of the set $\Phi(O)$. By computing the sign of $\dot{\hat{z}}_1$ when $\hat{z}_1$ is close to this border, we will be able to decide under what conditions it is possible that the state $\hat{z}$ leaves the set $\Phi(O)$. Precisely, we have

$$
\lim_{\hat{z}_1 \to 0} \dot{\hat{z}}_1 = y\left(\hat{z}_2 + a + dy + c\ln y - g_1 y\right).
$$

If this expression can become negative, then the state $\hat{z}$ can leave the set $\Phi(O)$. The condition for this behaviour is thus

$$
\hat{z}_2 - g_1 y < -(a + dy + c\ln y),
$$

where $g_1 < 0$, which is required for convergence of the observer, and $y > 0$ due to the dynamics of the system. We are going to study a predator–prey parameter configuration for the Lotka–Volterra model, where we have $a, d > 0$ and $b, c < 0$.

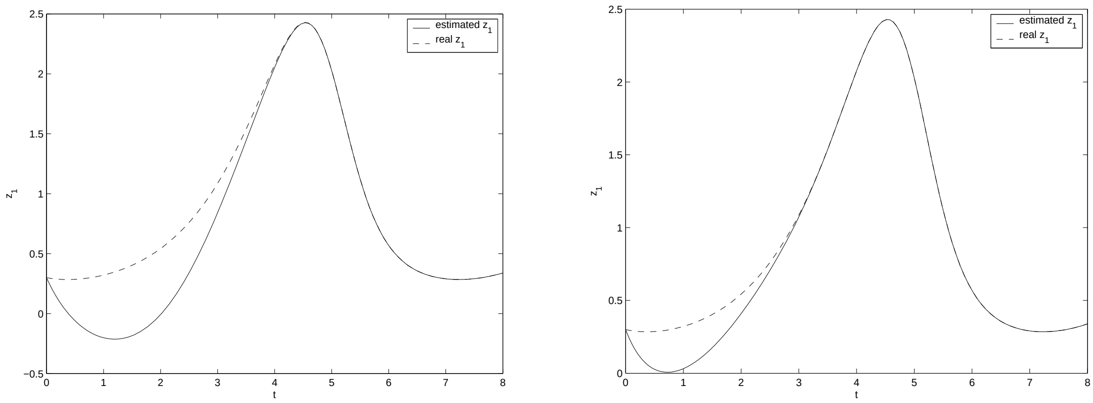
By Φ, we have $z_2 = bx_2 - c\ln x_1 - dx_1$. For x_1 close to zero or for large x_2, z_2 will become large negative. We can presume that the observer state leaves the set $\Phi(O)$ if we have small values for $\hat{z}_2$, and if in particular the initial value for $\hat{z}_2$ is too small

[1] The notation O is based on the term *observability region* which will be defined later.

or, respectively, the initial value for $\hat{x}_2$ too large. With a large absolute value for the gain g_1, we might be able to compensate a bad choice of the initial value for $\hat{z}_2$.

So let us try a simulation of the observer with an initial condition of the system as $x(0) = \begin{pmatrix} 0.3 & 1.3 \end{pmatrix}^{\mathrm{T}}$ and an initial estimation for the observer as $\hat{x}(0) = \begin{pmatrix} 0.3 & 4.5 \end{pmatrix}^{\mathrm{T}}$, which is equivalent to $\hat{z}(0) = \begin{pmatrix} 0.3 & -5.32 \end{pmatrix}^{\mathrm{T}}$, where the value for $\hat{z}_2(0)$ is rather small. Note that the observer was initialised quite reasonably with $h(\hat{x}(0)) = h(x(0))$. The same model parameters as for the first simulation in section 5.2.1 have been used, i.e. $a = d = 1$, $b = c = -1$. Simulations were done for two different observer gains, the first one with $G_1 = \begin{pmatrix} -4 & -4 \end{pmatrix}^{\mathrm{T}}$ and the second $G_2 = \begin{pmatrix} -16 & -8 \end{pmatrix}^{\mathrm{T}}$.

The resulting trajectories of the observer state $\hat{z}_1(t)$ together with the trajectories of the system state $z_1(t)$ are shown in figure 6.1.

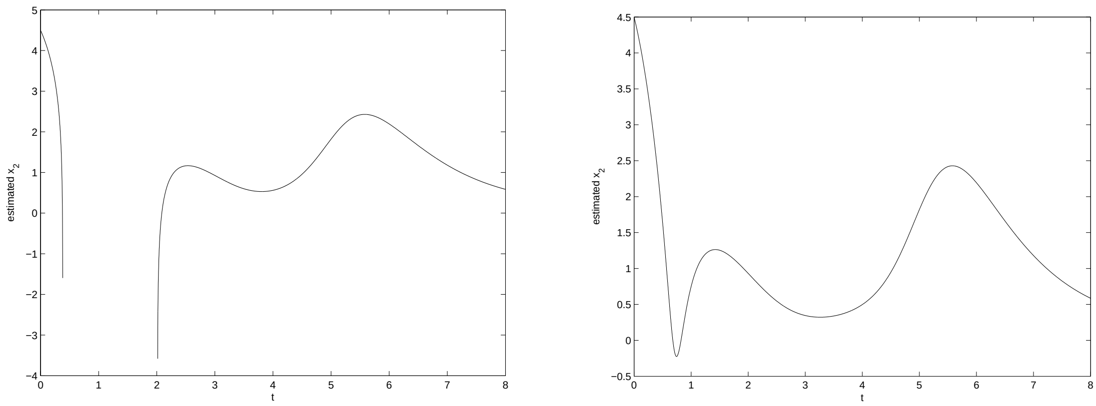


(a) using $G_1 = \begin{pmatrix} -4 & -4 \end{pmatrix}^{\mathrm{T}}$ (b) using $G_2 = \begin{pmatrix} -16 & -8 \end{pmatrix}^{\mathrm{T}}$

Figure 6.1: Observer and system state for the time–scaling observer with different gains

For the first simulation using the observer gain G_1, we have $\hat{z}_1(t) < 0$ for the time interval of approximately $0.5 < t < 2$. This implies $\hat{z}(t) \notin \Phi(O)$. The observer state $\hat{z}$ leaves $\Phi(O)$ for the given time interval. As a consequence, $\hat{x}_2(t)$ is undefined during this interval, as the transformation $\hat{x}_2 = \Phi_2^{-1}(\hat{z})$ includes $\ln \hat{z}_1$. Contrarily, if $G = \begin{pmatrix} -16 & -8 \end{pmatrix}^{\mathrm{T}}$, the higher gain can compensate for the bad choice of $\hat{x}_2(0)$ and we have $\hat{z}_1(t) > 0$ for all t. But if the gain is too high, the observer might of course not work well due to measurement noise.

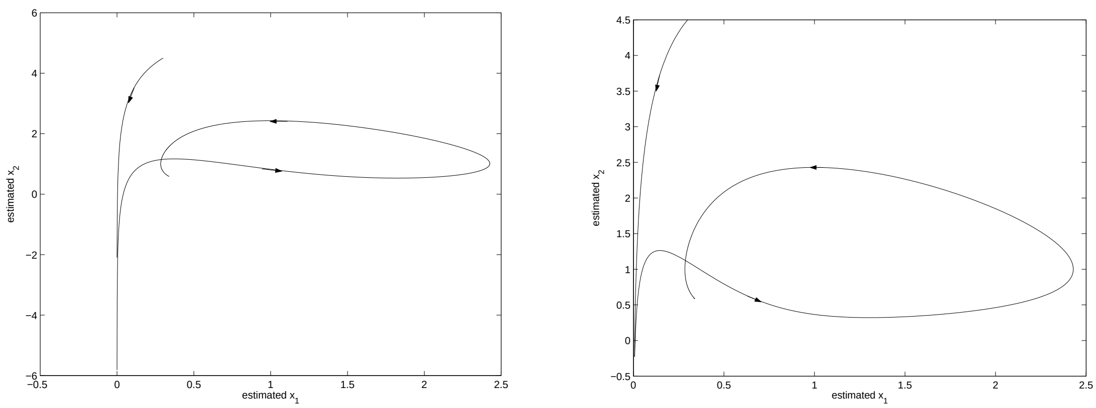
One might get the impression that the plots shown in figure 6.1 both look more or less harmless. The plot of $\hat{x}_2(t)$ in figure 6.2 — at least for t where it exists — shows that the two cases are rather very different. For the observer with smaller gain G_1, where the results are shown in figure 6.2(a), the estimation $\hat{x}_2(t)$ tends to $-\infty$ as t tends to the border of the interval where $\hat{x}_2(t)$ does not exist. It actually seems like the estimation would escape to infinity in finite time. But as $\hat{z}(t)$ returns to the set $\Phi(O)$ where $\hat{x}$ is defined, after some time we get again estimates for $\hat{x}_2(t)$.



(a) using $G_1 = \begin{pmatrix}-4 & -4\end{pmatrix}^{\mathrm{T}}$ (b) using $G_2 = \begin{pmatrix}-16 & -8\end{pmatrix}^{\mathrm{T}}$

Figure 6.2: Estimation for x_2 of the time–scaling observer with different gains G

The phase plane trajectories in figure 6.3 give also interesting results: We can see in subfigure (a) that $\hat{x}_2(t)$ tends to $-\infty$ as $\hat{x}_1(t)$ approaches 0. Though we know that $\hat{x}_1(t) = \hat{z}_1(t)$ takes negative values, these do not really appear in the phase plane plots of the estimation, as the second value $\hat{x}_2(t)$ is not defined for these times.



(a) using $G_1 = \begin{pmatrix}-4 & -4\end{pmatrix}^{\mathrm{T}}$ (b) using $G_2 = \begin{pmatrix}-16 & -8\end{pmatrix}^{\mathrm{T}}$

Figure 6.3: State–space plots of the estimation with different gains G

It is clear that the behaviour encountered here is highly undesirable for an observer. Using a formalised approach to this problem, we will search strategies for observer design and conditions on the observed system such that the problem of estimates escaping to infinity can be avoided.

First, let us introduce the notations used throughout this chapter.

6.1.1 Basic Notations

We consider a single output autonomous system given by the equations

$$\Sigma : \begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \tag{6.1}$$

with $x \in X \subset \mathbb{R}^n$, $y \in \mathbb{R}$ and the initial condition $x(0) = x_0 \in X_0 \subset X$. We assume that the solution $x(t, x_0)$ exists for all $t \geq 0$ and denote $y(t) = h(x(t, x_0))$.

Since the problem encountered in example 2 on page 51 is related to invariance of sets for the dynamics Σ, we will base our analysis on the notion of invariance widely used in system theory. A good overview of results and applications concerning invariant sets in control can be found in [Blanchini, 1999].

There is usually a distinction between positive and negative invariance, where general invariance is given if a set is both positively and negatively invariant. Since we need only positive invariance in this work, an invariant set will actually be defined as a positively invariant set.

Definition 10 (Invariant set). A connected set $X_0 \subset \mathbb{R}^n$ is called *invariant* for the system Σ from equation (6.1), if

$$x_0 \in X_0 \Rightarrow \forall t > 0 : x(t, x_0) \in X_0. \tag{6.2}$$

The notion of invariance is now extended to systems with inputs, as the observer takes the output of the observed system as input and we need to consider invariance properties for the observer. Since the input of an observer cannot be chosen, but is given by the observed system, we will use the term excited dynamics rather than controlled dynamics. The input u is assumed to belong to the set $\mathcal{U} : \mathbb{R} \to U$ of all differentiable functions, with U being a segment of $\mathbb{R}$.

Definition 11 (Uniformly invariant set). A connected set $X_0 \subset \mathbb{R}^n$ is called *uniformly invariant* for the excited dynamics $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$ (with respect to u), if

$$x_0 \in X_0 \Rightarrow \forall t > 0 \, \forall u \in \mathcal{U} : x(t, x_0, u(\cdot)) \in X_0. \tag{6.3}$$

Next, let us recall some results for invariant sets which can be found in more detail in [Blanchini, 1999]. The basic theorem, proven by Nagumo in 1942, requires the definition of a tangent cone to a set. We denote the border[2] of the set X as $\partial X = \overline{X} \setminus \operatorname{int} X$.

Definition 12 (Tangent cone). *(Boulingand, 1932)* Let $X \subset \mathbb{R}^n$ be a convex and closed set. The tangent cone to X in $x \in \partial X$ is the set

$$\mathcal{C}_X(x) = \left\{ z \in \mathbb{R}^n \,\middle|\, \lim_{h \to 0} \frac{\operatorname{dist}(x + hz, X)}{h} = 0 \right\}.$$

[2] This is the border in the set theoretic sense, not in the differential geometric one.

The tangent cone to X in a point x on the border contains all vectors which point inside or are tangent to X. If the border of X is smooth, then the tangent cone is a halfspace in the considered border point. Using the notion of the tangent cone, we are ready to state the basic theorem on invariant sets.

Theorem 9 (Nagumo, 1942). *A closed and convex set $X_0 \subset X$ is invariant for the system Σ from equation* (6.1)*, if and only if*

$$\forall x \in \partial X_0 : f(x) \in \mathcal{C}_X(x). \tag{6.4}$$

The theorem has an illustrative geometrical interpretation. In fact, it says that a set X_0 is invariant, if and only if the vector field f, which describes the evolution of the system, points to the inside of or is tangent to the set X_0. Furthermore, it is sufficient to check this condition on the border of the set X_0.

The Nagumo theorem is easily extended to give conditions on uniform invariance of a set.

Corollary 1. *A closed and convex set $X_0 \subset X$ is uniformly invariant for the excited dynamics $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$ if and only if*

$$\forall x \in \partial X_0 \, \forall u \in U : f(x, u) \in \mathcal{C}_X(x). \tag{6.5}$$

To handle the problems of invariance and inheritance of invariance, we need to use a more precise definition of an observer than the one given in definition 9. The dynamics we use for the observer are the same, i.e. the observer as a dynamical system is described by the equations

$$\hat{\Sigma} : \quad \begin{cases} \dot{\hat{z}} = F(\hat{z}, y) \\ \hat{x} = H(\hat{z}, y). \end{cases} \tag{6.6}$$

The observer state $\hat{z}$ evolves on a subset of $\mathbb{R}^d$, i.e. we have $\hat{z} \in \hat{Z} \subset \mathbb{R}^d$. F is a vector field on $\hat{Z}$ which is parametrised by the output $y \in h(X) \subset \mathbb{R}$ of the system Σ from equation (6.1). As we have seen in example 2, the problem of estimates $\hat{x}$ escaping to infinity is due to the fact that the mapping H is not defined for all values of $\hat{z} \in \hat{Z}$. We will thus consider another set $\hat{Z}_0 \subset \hat{Z}$ and assume that H is a mapping from $\hat{Z}_0 \times \mathbb{R}$ in X. Since a corresponding observer state $\hat{z}$ should exist for any state x of Σ, we assume that $H(\hat{Z}_0, h(X)) = X$. Furthermore, the observer has to be initialised such that it gives also an initial estimation; the dynamics $\hat{\Sigma}$ are initialised at $\hat{z}(0) = \hat{z}_0 \in \hat{Z}_0$.

In the end, the pair $(\hat{\Sigma}, \hat{Z}_0)$ will be associated to the system Σ from equation (6.1). The following definition makes a statement about when such a pair is said to be an observer for Σ.

Definition 13. The pair $(\hat{\Sigma}, \hat{Z}_0)$ is called an *observer* for the system Σ if

(i) $\hat{Z}_0$ is uniformly invariant for $\hat{\Sigma}$ with respect to y.

(ii) $\forall x_0 \in X_0 \, \forall \hat{z}_0 \in \hat{Z}_0 : x_0 = H(\hat{z}_0, h(x_0)) \Rightarrow \forall t > 0 : x(t, x_0) = \hat{x}(t, \hat{z}_0, y(\cdot))$

(iii) $\forall x_0 \in X_0\, \forall \hat{z}_0 \in \hat{Z}_0 : (\hat{x}(t, \hat{z}_0, y(\cdot)) - x(t, x_0)) \to 0$ as $t \to \infty$

The difference to other definitions of observers is the set $\hat{Z}_0$, which is mainly used in condition (i) of the definition above. This condition implies that the estimate $\hat{x}$ exists for all times, regardless of which output y is actually coming from the observed system Σ. The other conditions are just the tracking and convergence property introduced on page 24, but with a formulation which is aware of the initial condition $\hat{z}_0$ of the observer dynamics being an element of $\hat{Z}_0$.

When a pair $(\hat{\Sigma}, \hat{Z}_0)$ for the system Σ is given, and we intend to use it as an observer, but do not know yet if it is indeed an observer by the definition above, we will call it a prospective observer for Σ.

The next definition describes what we mean by an invariant set for a prospective observer. To avoid confusion with an invariant set for the original system as introduced in definition 10, we will say that the prospective observer is invariant on a given set, meaning that if both the initial system state and the initial estimation of the prospective observer are in this set, then the estimation stays in this set for all future times. We denote $H^{-1}(\tilde{X}, y) = \{\hat{z} \in \hat{Z}_0 \mid H(\hat{z}, y) \in \tilde{X}\}$, which is the preimage of the set $\tilde{X}$ under the mapping H, parametrised by $y \in \mathbb{R}$.

Definition 14 (Observer invariance). The pair $(\hat{\Sigma}, \hat{Z}_0)$ is called invariant on the set $\tilde{X} \subset X$, if

$$\forall x_0 \in \tilde{X} : \hat{z}_0 \in H^{-1}(\tilde{X}, y(0)) \Rightarrow \forall t > 0 : \hat{z}(t, \hat{z}_0, y(\cdot)) \in H^{-1}(\tilde{X}, y(t)). \tag{6.7}$$

If H does not explicitly depend on y, then the definition can be simplified; it is then equivalent to say that $H^{-1}(\tilde{X})$ is uniformly invariant for the dynamics $\hat{\Sigma}$ with respect to y.

6.1.2 Formalisation of the inheritance of invariance

Since the problem of inheriting invariance is more natural to consider, we start with this one. Suppose that the set X_0 of initial conditions is invariant for the system Σ. If we have a prospective observer $(\hat{\Sigma}, \hat{Z}_0)$, we will say that it inherits invariance properties from Σ with respect to X_0, if the estimation $\hat{x}$ is always an element of X_0, provided that the initial estimation $\hat{x}(0)$ is in X_0. Then we have the following simple statement.

Proposition 4 (Inheritance of invariance). *Assume that the set X_0 is invariant for the system Σ and let $(\hat{\Sigma}, \hat{Z}_0)$ be an observer for Σ. If this observer is invariant on X_0, then*

$$\forall \hat{z}_0 \in \hat{Z}_0 : \hat{x}(0, \hat{z}_0, y(0)) \in X_0 \Rightarrow \forall t > 0 : \hat{x}(t, \hat{z}_0, y(\cdot)) \in X_0.$$

This statement formalises what we actually want for invariance: Whenever the observer gives an initial estimation which is an element of the invariant set X_0, then the estimation is an element of this set for all times.

The main difference between the definition of observer invariance and the conclusion in the proposition above is that the definition does not need to assume the existence of the estimation for all times. If we have invariance by definition 14, then the estimation $\hat{x}$ is always defined, as $H^{-1}(\tilde{X}, y)$ is always a subset of $\hat{Z}_0$, and this leads directly to the formulation in proposition 4.

6.1.3 Formalisation of the invariance problem for observers

To solve the invariance problem, we want the set for which the observer is designed to be uniformly invariant. In the following, we will look more closely at the set we use the observer for. We will define this set as an observability region for the system Σ. Of course the system should be observable when restricted to this set, such that a global observer for this set can exist.

Definition 15 (Observability region). A connected set $O \subset X$ is called an *observability region* of the system Σ, if the system considered on O is globally distinguishable while its trajectories are still in O, i.e. the condition

$$\begin{gathered} \forall x_0, \tilde{x}_0 \in O \, \exists T > 0 : x(t, x_0), x(t, \tilde{x}_0) \in O \;\, \forall t \in [0, T] \text{ and} \\ \exists t \in [0, T] : h(x(t, x_0)) \neq h(x(t, \tilde{x}_0)) \end{gathered} \tag{6.8}$$

is satisfied.

Global observability is of course not a sufficient condition for the existence of an observer. As we want to observe the system Σ on O, we will have to make further assumptions concerning the existence of an observer. This will be done later when we actually consider special observer design methods and study their invariance properties.

In example 2 on page 51, the observability region O was the open set $\{x \in \mathbb{R}^2 \mid x_1 > 0\}$. The system lost observability on the border of O, and as a consequence the mapping H we used as observer output mapping was undefined there. As the observer state $\hat{z}$ could leave the image of the observability region, the estimation $\hat{x}$ got undefined. Actually, the time–scaling "observer" we used for global observation did not satisfy the first condition from definition 13.

However, if a prospective observer $(\hat{\Sigma}, \hat{Z}_0)$ is invariant on an observability region O, then the estimation $\hat{x}(t)$ will always be defined, since the preimage $H^{-1}(O, y)$ is a subset of $\hat{Z}_0$ for all possible system outputs y.

Of course, if the pair $(\hat{\Sigma}, \hat{Z}_0)$ is to be an observer for Σ and invariant on an observability region O, the state x of the system must not leave the observability region. This will be guaranteed in the following sections by assuming that the set X_0 of initial conditions is invariant for Σ and $X_0 \subset O$.

If the set X_0 is invariant for the system Σ and a subset of an observability region O of Σ, then a prospective observer which inherits system invariance properties for X_0 is also invariant on O. But in the case where O and X_0 are different, it may be easier to design an observer which is invariant on O, but does not inherit system invariance properties for X_0. So these two problems will be considered separately, even though they are in fact closely related.

6.2 Invariance and observers with linearisable error dynamics

In this section, we will consider two of the observer design methods used in chapter 5, namely the observer design by coordinate transformation together with either output transformation or time scaling.

We will assume the existence of an open observability region O for the system Σ, and restrict the state space of the system to $X = O$. To assure the existence of a unique solution for all times, we will suppose that the set X_0 of initial conditions for the system is invariant and a subset of O. This is not a very severe restriction, because if the system could leave the observability region, then chances to find a global observer would be very low.

Throughout the rest of this chapter, we will assume that the system can be transformed to an observer canonical form globally on O, allowing for the construction of a global observer — under additional conditions which we are going to study.

To simplify the notation, the output transformation and the time scaling will be handled in a unified way, though they are actually not applied at the same time. The common approach used by these two methods is to transform the system Σ from equation (6.1) to the observer canonical form given by

$$\begin{aligned} \dot{z} &= s(Cz)\,(Az + k(Cz)) \\ \tilde{y} &= Cz, \end{aligned} \tag{6.9}$$

with $z \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$, (A, C) in observer canonical form and k a mapping from $\mathbb{R}$ in $\mathbb{R}^n$.

Together with the assumption on the observability region, we will assume that the system Σ can be transformed to the observer canonical form (6.9) globally on O. Precisely, we have the following two basic assumptions.

(A1) For the system Σ, there exist an open observability region O and a closed invariant set $X_0 \subset O$. The initial system state $x(0) = x_0$ is an element of X_0.

(A2) There exist global diffeomorphisms $\Phi : O \to \Phi(O)$, $\Psi : h(O) \to C \circ \Phi(O)$ and a function $s : C \circ \Phi(O) \to \mathbb{R}_+$ which transform Σ to the observer canonical form (6.9) by setting $z = \Phi(x)$ and $\tilde{y} = \Psi(y)$.

To design an observer for the system Σ on O, let us construct a suitable pair $(\hat{\Sigma}, \hat{Z}_0)$. The dynamics are chosen as the standard canonical form observer, such that we get

$$\hat{\Sigma} : \quad \begin{cases} \dot{\hat{z}} = s(\tilde{y})\,(A\hat{z} + k(\tilde{y}) + G(C\hat{z} - \tilde{y})) \\ \hat{x} = \Phi^{-1}(\hat{z}) \end{cases} \tag{6.10}$$

with $\hat{z} \in \mathbb{R}^n$ and $\tilde{y} = \Psi(y)$. The observer gain G is such that the matrix $(A + GC)$ is asymptotically stable. To complete the construction of the prospective observer, we set $\hat{Z}_0 = \Phi(O)$.

In this case, the observer state $\hat{z}$ is directly linked to the system in the sense that it is an estimate for the state z of Σ in observer canonical coordinates. This allows us to consider the observation error[3] $e = \hat{z} - z$, which evolves with the dynamics $\dot{e} = (A + GC)e$ and thus converges to 0.

Our first result gives conditions on the constructed pair to be an observer.

Theorem 10 (Invariant observer). *Assume (A1) and (A2). The pair* $(\hat{\Sigma}, \Phi(O))$ *with* $\hat{\Sigma}$ *from* (6.10) *is an observer for* Σ *if and only if it is invariant on* O*.*

Proof. (Necessity) Assume $(\hat{\Sigma}, \Phi(O))$ is an observer for Σ. Then, by item (i) of definition 13, $\Phi(O)$ is uniformly invariant for the observer dynamics $\hat{\Sigma}$. Since the output H of $\hat{\Sigma}$ does not depend on the system output y, this is equivalent to the pair $(\hat{\Sigma}, \Phi(O))$ being invariant on O.

(Sufficiency) We are going to show that the three conditions listed in definition 13 are satisfied, provided the assumptions in the theorem hold.

$(\hat{\Sigma}, \Phi(O))$ being invariant on O implies that $\Phi(O)$ is uniformly invariant for the observer dynamics $\hat{\Sigma}$ and condition (i) holds. Now consider the dynamics for the observer error $e = \hat{z} - z$, which evolves with the dynamics $\dot{e} = (A + GC)e$ and thus converges asymptotically to 0.

Furthermore, the mapping Φ is a diffeomorphism and $e(0) = 0$ if and only if $\hat{x}(0) = x_0$. Then, by the linear dynamics for e, $e(t) = 0$ for all $t \geq 0$ and condition (ii) holds.

For any initial conditions $\hat{z}_0$ and x_0, we have $e(t) \to 0$ as $t \to \infty$. This implies $\hat{x}(t) \to x(t)$ as $t \to \infty$ and condition (iii) holds. □

This result is nice because it reduces the property of the pair $(\hat{\Sigma}, \hat{Z}_0)$ being a global observer for Σ on the observability region O to invariance of this pair on O. But in general, the invariance property cannot be checked easily for a given system and a prospective observer. There is one special case where we have invariance automatically, which leads to the following corollary of theorem 10.

Corollary 2. *Assume (A1) and (A2). If* $\Phi(O) = \mathbb{R}^n$*, then* $(\hat{\Sigma}, \mathbb{R}^n)$ *is an observer for* Σ*.*

Proof. With theorem 10, it remains to proof that $\mathbb{R}^n$ is invariant under the dynamics $\hat{\Sigma}$, i.e. that the observer state $\hat{z}$ cannot escape to infinity in finite time. By assumption on the system Σ, its solution $x(t)$ and thus, by the diffeomorphism Φ, also the transformed trajectory $z(t)$ exist for all times $t \geq 0$. With the linear dynamics for the observer error e, we have $e(t) \in \mathbb{R}^n$ for all t and thus also $\hat{z}(t) = z(t) + e(t) \in \mathbb{R}^n$ for all times $t \geq 0$. □

Example 3. Consider the uncontrolled Lotka–Volterra model given by equation 2.1. If we assume an initial condition $x(0)$ such that $x_1(0) > 0$, then the system satisfies assumption (A1) with $O = \{x \in \mathbb{R}^2 \mid x_1 > 0\}$.

For the observer design via output transformation in section 5.1, we transformed the system to observer canonical form using the diffeomorphisms

$$\Phi(x) = \begin{pmatrix} \ln x_1 \\ bx_2 - c \ln x_1 - dx_1 \end{pmatrix} \tag{6.11}$$

[3] whereas the estimation error is $\hat{x} - x$

and

$$\Psi(y) = \ln y. \tag{6.12}$$

We have $\Phi(O) = \mathbb{R}^2$ and Φ and Ψ are global diffeomorphisms. Thus, by corollary 2, the observer dynamics (5.15) designed in section 5.1 together with the set $\hat{Z}_0 = \mathbb{R}^2$ are an observer for the Lotka–Volterra model.

In example 2, the problem of the estimation being undefined persisted only for a bounded time interval. In fact, we get the general result that, if considered after some time, observation works fine if the assumptions we made at the beginning are satisfied and the system state stays separated from the border of the observability region.

Proposition 5 (Noninstantaneous observation). *Assume (A1) and (A2). Assume further that* $\exists \delta > 0 : \operatorname{dist}(\partial O, \partial X_0) > \delta$. *Then the pair* $(\hat{\Sigma}, \Phi(O))$ *is an observer if considered after some time, i.e.* $\forall x_0 \in X_0 \forall \hat{z}_0 \in \Phi(O) \exists T > 0$ *such that the conditions*

(i) $\forall t > T : \hat{z}(t, \hat{z}_0, y(\cdot)) \in \Phi(O)$

(ii) $x_0 = H(\hat{z}_0, h(x_0)) \Rightarrow \forall t > 0 : x(t, x_0) = \hat{x}(t, \hat{z}_0, y(\cdot))$

(iii) $(\hat{x}(t, \hat{z}_0, y(\cdot)) - x(t, x_0)) \to 0$ *as* $t \to \infty$

hold.

Items (ii) and (iii) are the same as in definition 13 for an observer. Only the first property changed, implying now that the estimations $\hat{x}$ are in general only defined after a certain time T.

Proof. Again we use the dynamics of the observer error $e = \hat{z} - z$ which are given by $\dot{e} = (A + GC)e$ and are asymptotically stable. Then condition (ii) follows from the fact that $e(0) = 0$ implies $e(t) = 0$ for all t.

Furthermore, $\hat{z}$ converges to the real system state z, i.e. we have $\forall \epsilon > 0 \, \exists T > 0 \, \forall t > T : \|\hat{e}(t)\| < \delta$. In particular, we can choose $\epsilon < \delta$ from the assumption in the proposition. Then we have immediately $\hat{z}(t) \in \Phi(0)$ for all $t > T$ and condition (i) holds. This implies also that $\hat{x}(t)$ exists for all $t > T$. With $e(t) \to 0$ for $t \to \infty$, we get also $\hat{x}(t) \to x(t)$ as $t \to \infty$. □

Proposition 5 shows that the invariance problem for observers with linearisable error dynamics is just an intermediate problem if considered on a longer time span. Nevertheless, the unboundedness of the estimation will usually pose problems if e.g. the estimation is to be used as input for a state controller.

6.3 Observer invariance via initialisation strategy

We have seen in the previous section that the pair $(\hat{\Sigma}, \hat{Z}_0)$ designed by transformation of the system Σ to observer canonical form will only be an observer, if it is invariant on the observability region O. But up to now, we do not have a way to check a prospective observer for invariance.

In this section, we are going to introduce an approach which may sometimes guarantee invariance for a prospective observer designed by either time scaling or output transformation. To do this, we will use an *initialisation strategy* for the observer. An initialisation strategy is a mapping $\sigma : \mathbb{R} \to \mathbb{R}^n$, such that the initial state of the observer is chosen as a function of the initial output from the observed system. Precisely, we have the following definition.

Definition 16 (Initialization strategy). The prospective observer $(\hat{\Sigma}, \hat{Z}_0)$ for Σ is said to be *initialised according to the initialisation strategy* $\sigma : \mathbb{R} \to \mathbb{R}^n$, if

$$\hat{z}(0) = \sigma(h(x_0)). \tag{6.13}$$

In some cases, including the Lotka–Volterra model, it is possible to guarantee invariance on the considered observability region by initialising the observer according to some well–chosen initialisation strategy. However, since we do not have a method to do this for a general system, we will just give a short introduction and concentrate rather on the application of this approach to the Lotka–Volterra model and to another example which we are going to present at the end of this section.

In general, the effect of the prospective observer leaving the observability region it was designed for can be related to the linearisable error dynamics for the observer error $e = \hat{z} - z$. Since the system state z is always an element of the invariant set $\Phi(X_0) \subset \Phi(O)$, the prospective observer is not invariant on O if the error can be such that $z + e = \hat{z} \notin \Phi(O)$.

To find out how to avoid that the error can reach such a state, let us study the dynamics for the observation error. We are restricting ourselves to the case $n = 2$, because this is sufficient for our needs. But the method introduced here might also be applied to higher order systems.

The dynamics of the observation error e are for a second order system

$$\dot{e} = (A + GC)e = \begin{pmatrix} g_1 & 1 \\ g_2 & 0 \end{pmatrix} e.$$

Assigning the eigenvalues λ_1 and λ_2 gives the equation

$$\dot{e} = \begin{pmatrix} \lambda_1 + \lambda_2 & 1 \\ -\lambda_1\lambda_2 & 0 \end{pmatrix} e. \tag{6.14}$$

The corresponding eigenvectors are then

$$v_1 = \begin{pmatrix} 1 \\ -\lambda_2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -\lambda_1 \end{pmatrix}, \tag{6.15}$$

where v_1 is the eigenvector corresponding to λ_1 and v_2 to λ_2.

Since we want the observer error dynamics to be asymptotically stable, there are strong constraints on the eigenvalues. To achieve a convergence of the observation error which is at least as fast as desired by some positive parameter γ, we need $\operatorname{Re}\lambda_1 < -\gamma$ and $\operatorname{Re}\lambda_2 < -\gamma$. The eigendirections of the error dynamics must thus both be straight lines with positive derivatives when considered in the planar state space of e. The resulting dynamics are visualised in figure 6.4 for eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. The origin is a stable node. The direction closer to the e_1–axis corresponds to the faster mode of the error dynamics.

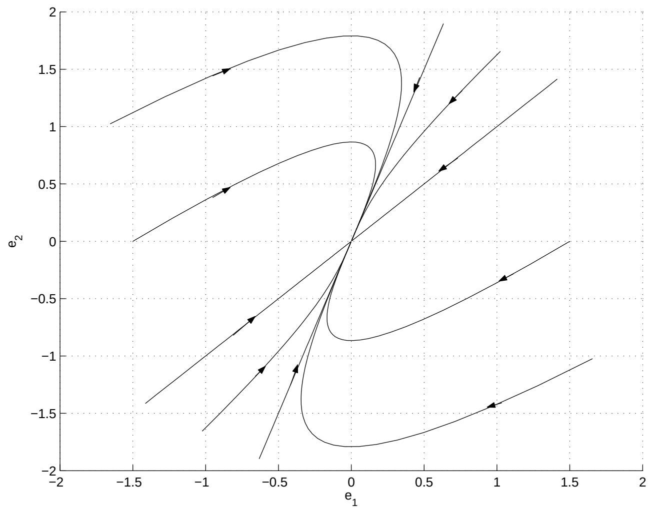


Figure 6.4: Trajectories of the observation error e

By an initialisation strategy $\hat{z}(0) = \sigma(h(x_0))$, it is possible to choose the value of $e_1(0) = \hat{z}_1(0) - z_1(0)$. Using the dynamics of the error in figure 6.4, we are searching an initialisation strategy such that the observation error will always stay in some — yet to specify — desirable subset of the plane $\mathbb{R}^2$. As this step depends highly on the properties of the observed system Σ, we will illustrate our method directly for the following two examples.

6.3.1 Initialisation of the time scaling observer for the Lotka–Volterra model

In this section, we apply an initialisation strategy to the observer for the Lotka–Volterra model (2.1) designed via time scaling as done in section 5.2. The observer dynamics $\hat{\Sigma}$ are defined by equation (5.24). It is based on a coordinate transformation $z = \Phi(x)$ defined by

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= bx_2 - dx_1 - c\ln x_1. \end{aligned}$$

We assume that the observer gain G has been chosen such that the eigenvalues λ_1 and λ_2 are negative real values and $\lambda_1 \leq \lambda_2$. This observer was already used in example 2 to illustrate the concept of invariance for observers.

The observability region we want to use the observer for is $O = \{x \in \mathbb{R}^2 \mid x_1 > 0\}$. We will only consider the predator–prey case, as with other parameter configurations the system state may converge to the border of O, which poses numerical problems for the observation. We then have the properties $a > 0$, $b < 0$, $c < 0$ and $d > 0$ for the model parameters.

We choose a closed invariant subset X_0 of the biological state space $\mathbb{R}^2_+$ as set of system initial conditions. Then we have $\forall t \geq 0 : z_1(t) > 0$, and the observer will be invariant on the observability region O, if

$$\forall t \geq 0 : e_1(t) = \hat{z}_1(t) - z_1(t) \geq 0. \tag{6.16}$$

This condition is quite conservative. But the system dynamics cannot easily be related to the dynamics of the observation error. If for some time t we have $\hat{z}_1(t) < z_1(t)$ and the system dynamics are faster than the dynamics of the observation error, then it might be possible to get $\hat{z}_1 < 0$ and the prospective observer would not be invariant.

From the error dynamics in figure 6.4, we see that allowing only an error satisfying equation (6.16) is equivalent to the conditions

$$\begin{aligned} &e_1(0) \geq 0 \\ &e_2(0) \geq -\lambda_2 e_1(0). \end{aligned}$$

By an initialisation strategy $\hat{z}(0) = \sigma(h(x_0))$, we can choose only the first component $e_1(0)$ of the initial observer error. Let us set $e_1(0) = 0$. Then the second condition becomes $e_2(0) \geq 0$. At this point, we can make use of the invariant set X_0 we chose before. Due to this set, we have $x_2(0) > 0$. Using the coordinate transformation Φ and, as assigned previously, $\hat{x}_1(0) = x_1(0)$, the condition on $e_2(0)$ becomes $b(\hat{x}_2(0) - x_2(0)) \geq 0$, or, since $b < 0$, $\hat{x}_2(0) \leq x_2(0)$. This can always be achieved by setting $\hat{x}_2(0) = 0$.

In the end, we find the initialisation strategy

$$\begin{aligned} \hat{z}_1(0) &= h(x_0) \\ \hat{z}_2(0) &= -dh(x_0) - c\ln(h(x_0)). \end{aligned}$$

If our prospective observer for the Lotka–Volterra model is initialised according to this strategy, then it is invariant on O and, using theorem 10, an observer by definition 13.

The approach used here works well as long as an exact measurement of the output y at the time $t = 0$ is available.

If only a noisy measurement of y is available, the observer problem becomes more complicated. The noise of the measurement will directly influence the observation error dynamics, which will become nonlinear. Statements about convergence of the estimated state cannot be made without further assumptions on the nonlinear parts of the observed system.

Therefore, in this context only a noisy measurement of the initial system output $y(0)$ will be assumed, supposing that afterwards the measurement is good enough to give essentially the linear error dynamics considered above.

It is clear that with the influence of measurement noise, the initialisation strategy has to be of the form $\hat{z}(0) = \sigma(y(0))$, where $y(0)$ is the initial noisy measurement for $h(x_0)$. In fact we will compute an initial value for $\hat{x}(0)$, and the initial value for $\hat{z}(0)$ is then given by $\hat{z}(0) = \Phi(\hat{x}(0))$.

Assume that $|y(0) - h(x_0)| \leq \eta$, where η is a known upper bound of the measurement noise. $e_1(0) \geq 0$ can be assured by setting $\hat{x}_1(0) = y(0) + \eta$, which will give $0 \leq e_1(0) \leq 2\eta$. Using the coordinate transformation Φ, the condition on $e_2(0)$ gives then

$$\hat{x}_2(0) \leq \frac{1}{b}\left((d - \lambda_2)e_1(0) - c\ln\frac{x_1(0)}{\hat{x}_1(0)}\right) + x_2(0).$$

Denote $\tilde{\eta}$ as the actual measurement noise, i.e. $y(0) = h(x_0) + \tilde{\eta}$. Then the equation above can be rewritten as

$$\hat{x}_2(0) \leq F(\tilde{\eta}) + x_2(0)$$

with $F(\tilde{\eta}) = \frac{1}{b}\left((d - \lambda_2)(\tilde{\eta} + \eta) - c\ln\left((\hat{x}_1(0) - \tilde{\eta} - \eta)/\hat{x}_1(0)\right)\right)$. This condition is satisfied for all $x_2(0) > 0$ and $\tilde{\eta}$ with $|\tilde{\eta}| \leq \eta$ by setting

$$\hat{x}_2(0) = \min_{-\eta \leq \tilde{\eta} \leq \eta} F(\tilde{\eta}).$$

The second derivative of F is $F'' = -\frac{c}{b}\hat{x}_1(0)(\hat{x}_1(0) - \tilde{\eta} - \eta)^{-2} < 0$. Thus F takes its minimum on the border and we have

$$\min_{-\eta \leq \tilde{\eta} \leq \eta} F(\tilde{\eta}) = \min\{F(-\eta), F(\eta)\}$$

where $F(\eta) = \frac{1}{b}\left(2(d - \lambda_2)\eta - c\ln\left((\hat{x}_1(0) - 2\eta)/\hat{x}_1(0)\right)\right)$ and $F(-\eta) = 0$. Depending on the value of $\hat{x}_1(0)$ and on the model parameters, we will choose either $\hat{x}_2(0) = F(\eta)$ or $\hat{x}_2(0) = 0$, depending on which one is smaller. It is thus possible that $\hat{x}_2(0)$ is chosen negative with this initialisation strategy and then $\hat{x}(0) \notin \mathbb{R}^2_+$. This is not ideal in the biological sense, but it guarantees observer invariance. Note that this initialisation depends on the slower mode λ_2 of the assigned observation error dynamics. The faster this mode is, the more negative will be the initialisation of $\hat{x}_2(0)$.

6.3.2 Example with an observer via output transformation

In this section, we will give an example for the invariance problem with a dynamic system which can be transformed to observer canonical form via output transformation.

Consider the system given by

$$\Sigma : \quad \begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = -x_2^2 + a x_1 x_2 \\ \quad y = x_1 \end{cases} \tag{6.17}$$

with a parameter $a \in \mathbb{R}$. The system has an observability region

$$O = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 > 0 \right\}$$

and an invariant submanifold $X_0 = \mathbb{R}^2_+ \subset O$. The initial condition of the system is $x(0) = x_0 \in X_0$. The system will be restricted to O, and we want to design an observer for the restricted system.

The system can be transformed to observer canonical form by the state space diffeomorphism

$$\Phi : \begin{cases} O \to Z \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1^2 \\ x_1^2 x_2 - \frac{a}{3}x_1^3 \end{pmatrix} \end{cases}, \tag{6.18}$$

where $Z = \Phi(O) = \left\{ \begin{pmatrix} z_1 & z_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2 \mid z_1 > 0 \right\}$, and the output transformation $\tilde{y} = \Psi(y)$ defined by

$$\tilde{y} = \frac{1}{2}y^2. \tag{6.19}$$

In these coordinates, the system dynamics are given by

$$\begin{aligned} \dot{z}_1 &= z_2 + \frac{a}{3}(2z_1)^{\frac{3}{2}} \\ \dot{z}_2 &= 0 \\ \tilde{y} &= z_1. \end{aligned} \tag{6.20}$$

We design the dynamics for the prospective observer as

$$\hat{\Sigma} : \begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 + \frac{a}{3}y^3 + g_1\left(\hat{z}_1 - \frac{1}{2}y^2\right) \\ \dot{\hat{z}}_2 = g_2\left(\hat{z}_1 - \frac{1}{2}y^2\right) \\ \hat{x} = \Phi^{-1}(\hat{z}) \end{cases} \tag{6.21}$$

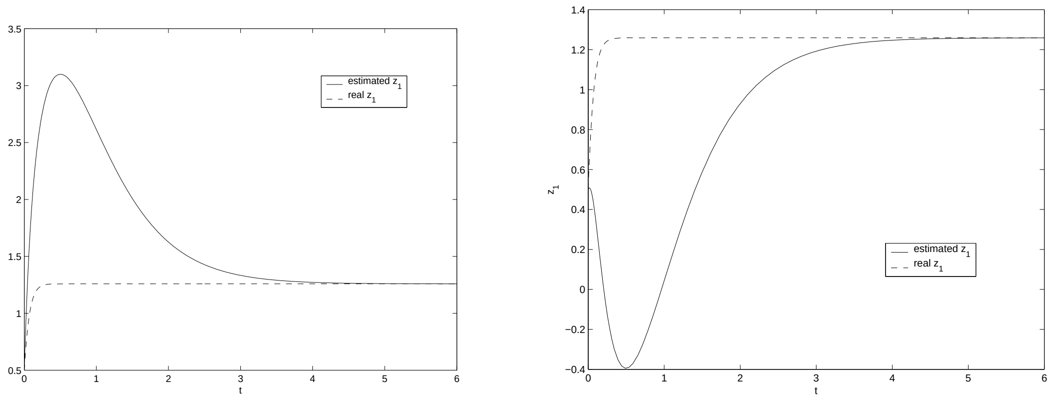
with an observer gain $g_1 < 0$ and $g_2 < 0$. We have then the prospective observer $(\hat{\Sigma}, \Phi(O))$.

The observability region is the same as in the previous example, and for our initialisation strategy, we will set $\hat{z}_1(0) = \frac{1}{2}(h(x_0))^2$, such that $e_1(0) = 0$. For invariance then $\hat{z}_2(0) \geq z_2(0)$ is needed.

Using the transformation Φ from equation (6.18), if $e_1(0) = 0$, this is equivalent to $\hat{x}_2(0) \geq x_2(0)$. If an upper bound $\bar{x}_2$ for $x_2(0)$ is known, setting the initial estimate to this value will guarantee observer invariance.

We will check our results in a numerical simulation. The parameter a is set to $a = -10$. We assume that we have the bound $\bar{x}_2 = 20$ on the second state x_2. The initial state of the system is set to $x(0) = \begin{pmatrix} 1 & 10 \end{pmatrix}^{\mathrm{T}}$.

We will compare two simulations: In the first simulation, the observer was initialised according to the initialisation strategy $\hat{x}_1(0) = y(0)$ and $\hat{x}_2(0) = \bar{x}_2$. For the second simulation, we did not use this strategy, but rather the opposite initialisation with $\hat{x}_1(0) = y(0)$ and $\hat{x}_2(0) = 1$. The observer was implemented with the gain $G = \begin{pmatrix} -4 & -4 \end{pmatrix}^{\mathrm{T}}$. Figure 6.5 shows the resulting trajectories of the system state z_1 and the observer state $\hat{z}_1$.



(a) Observer with initialisation strategy (b) Observer without initialisation strategy

Figure 6.5: Simulation results for the trajectory of $\hat{z}_1$

The difference between the two simulations is clear: In the first simulation, we have $\hat{z}_1(t) > 0$ for all t. With the applied initialisation strategy, the pair $(\hat{\Sigma}, \Phi(O))$ is invariant on O and thus an observer. In the second simulation, we did not use this initialisation strategy, and the observer left the region where the estimation $\hat{x}$ is defined, as we have $\hat{z}_1(t) < 0$ for some time interval.

6.4 Invariance of reduced order observers

When designing an observer for the system Σ from equation (6.1), we have seen that invariance on an observability region is a reasonable required property for this observer. Up to now, we do not have general conditions for this property. Initialisation strategy may help sometimes, but we have no general approach how to apply this approach. Moreover, we need to have an initial measurement of the system output without or with low noise.

If the system output can be measured without noise, then it is reasonable to use a reduced order observer, which will use directly the measurement y to compute the estimation of the system state. If the system output is a component of the state, e.g. $y = x_1$, then a reduced order observer will give $\hat{x}_1 = y$.

First, we will study how to design reduced order observers. We assume that the system can be transformed to observer canonical form (6.9) on an observability region O and do the construction of the reduced order observer based on these coordinates. Afterwards, we will give conditions for invariance of the designed observer dynamics and apply them to some examples.

6.4.1 Design of reduced order observers

For a single output system of order n, the dynamics of the reduced order observer will be of order $n-1$.

Let us first study the design of reduced order observers for linear systems. As the nonlinear systems we consider are actually linear up to output–injection, the design can be generalised easily, as done with the full order Luenberger observer.

Linear reduced order observers

Consider a single output linear system in observer canonical form:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} k_1 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ k_n & 0 & \cdots & \cdots & 0 \end{pmatrix} x, \\ y &= x_1 \end{aligned} \tag{6.22}$$

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. We transform this system to a system of reduced order using the $n-1$ state variables

$$x_i^* = x_i + l_i x_1 \qquad i = 2, \ldots, n, \tag{6.23}$$

where the real numbers l_i can be chosen arbitrarily.

The dynamics of the reduced system are computed from the original system dynamics as

$$\begin{aligned} \dot{x}_i^* &= l_i x_2^* + x_{i+1}^* - l_i l_2 x_1 - l_{i+1} x_1 + k_i x_1 + l_i k_1 x_1 \qquad i = 2, \ldots, n-1 \\ \dot{x}_n^* &= l_n x_2^* - l_n l_2 x_1 + k_n x_1 + l_n k_1 x_1. \end{aligned} \tag{6.24}$$

The observer uses the x^*–coordinates for its state and is designed as

$$\begin{aligned} \dot{\hat{x}}_i^* &= l_i \hat{x}_2^* + \hat{x}_{i+1}^* - l_i l_2 y - l_{i+1} y + k_i y + l_i k_1 y \qquad i = 2, \ldots, n-1 \\ \dot{\hat{x}}_n^* &= l_n \hat{x}_2^* - l_n l_2 y + k_n y + l_n k_1 y \\ \hat{x}_1 &= y \\ \hat{x}_i &= \hat{x}_i^* - l_i y, \qquad i = 2, \ldots, n. \end{aligned}$$

Note that the measurement y is directly used to compute the estimation $\hat{x}$ of the reduced order observer.

The dynamics for the observation error $e = (\hat{x}_i^* - x_i^*)_{i=2,\ldots,n}$ are computed as

$$\dot{e} = Le,$$

with the $(n-1)\times(n-1)$ matrix L given by

$$L = \begin{pmatrix} l_2 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ l_n & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{6.25}$$

The poles of the error dynamics can be placed easily by choosing suitable l_i. If the error dynamics are asymptotically stable, then the estimation $\hat{x}$ of the reduced order observer converges exponentially to the state x of the linear system.

Generalisation for linear up to output injection systems

Consider the nonlinear system Σ from equation (6.1). We will make the same assumptions as in section 6.2, that is we will assume (A1) and (A2) from page 58.

Then, the system Σ is transformed to observer canonical form by setting $z = \Phi(x)$ and $\tilde{y} = \Psi(y)$, such that we obtain for its dynamics

$$\begin{aligned} \dot{z} &= s(z_1)\,(Az + k(z_1)) \\ \tilde{y} &= z_1, \end{aligned} \tag{6.26}$$

with $z \in \mathbb{R}^n$ and $\tilde{y} \in \mathbb{R}$. The only difference compared to the linear case is that the input dependent terms are now nonlinear. As these terms did not appear in the transformation to the reduced order system, we do this reduction in the nonlinear case in exactly the same way, i.e. we put

$$z_i^* = z_i + l_i z_1, \qquad i = 2, \ldots, n, \tag{6.27}$$

where the real numbers l_i can be chosen arbitrarily.

These variables evolve with the dynamics

$$\Sigma^* : \quad \begin{cases} \dot{z}_i^* = s(z_1)\left(l_i z_2^* + z_{i+1}^* - l_i l_2 z_1 - l_{i+1} z_1 + k_i(z_1) + l_i k_1(z_1)\right) & i = 2, \ldots, n-1 \\ \dot{z}_n^* = s(z_1)\left(l_n z_2^* - l_n l_2 z_1 + k_n(z_1) + l_n k_1(z_1)\right). \end{cases}$$

The observer uses exactly the same dynamics for its state $\hat{z}^* \in \mathbb{R}^{n-1}$, where the state z_1 appearing in the dynamics is replaced by the transformed output $\tilde{y}$ from the system Σ. To get the output equation for the observer, we first define the extended observer state $\hat{z}$ as

$$\begin{aligned} \hat{z}_1 &= \tilde{y} \\ \hat{z}_i &= \hat{z}_i^* - l_i \tilde{y}, \qquad i = 2, \ldots, n, \end{aligned} \tag{6.28}$$

Using the extended state, we compute the observer output mapping H as

$$H(\hat{z}^*, \tilde{y}) = \Phi^{-1}(\hat{z}). \tag{6.29}$$

The dynamics for the prospective reduced order observer can thus be written as

$$\hat{\Sigma}^* : \quad \begin{cases} \dot{\hat{z}}^* = s(\tilde{y})\,(L\hat{z}^* + B(\tilde{y})) \\ \hat{x} = H(\hat{z}^*, \tilde{y}), \end{cases} \tag{6.30}$$

with $\hat{z}^* \in \mathbb{R}^{n-1}$, the matrix L from equation (6.25) and B collecting all terms depending on z_1 from the reduced system dynamics Σ^*.

Considering the observer error $e = \hat{z}^* - z^*$, we get the dynamics

$$\dot{e} = s(\tilde{y})Le. \tag{6.31}$$

These dynamics can be made asymptotically stable by an appropriate choice of the numbers l_i.

6.4.2 Results concerning invariance

We are now going to design a pair $(\hat{\Sigma}^*, \hat{Z}_0^*)$ which is an observer by definition 13 for the system Σ, using the reduced order observer dynamics $\hat{\Sigma}^*$ from the previous section. The condition we will require from the system is in fact very similar to the one of corollary 2 on page 59, but due to the reduced order it is considerably weakened.

First, let us define a set of a special form which will be used for all results in the remainder of this chapter. For some constants $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$, we define the set $M_{\alpha\beta} = \{x \in \mathbb{R}^n \mid \alpha \leq h(x) \leq \beta\}$. This is the set of all system states whose corresponding output is an element of the segment $[\alpha, \beta]$ of $\mathbb{R}$.

Then we have the following result.

Theorem 11. *For the system Σ from equation* (6.1), *assume (A1) and (A2) from page 58. If there exist $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ such that $X_0 \subset M_{\alpha\beta} \subset O$, then the pair $(\hat{\Sigma}^*, \mathbb{R}^{n-1})$ is an observer for Σ.*

Remark 5. The invariance property for the observer is nearly automatic, as we use $\hat{Z}_0 = \mathbb{R}^{n-1}$. The condition on the set $M_{\alpha\beta}$ then becomes important, because we need to assure that the observer output mapping H is really globally defined on $\mathbb{R}^{n-1}$.

Proof. We will first show that the pair $(\hat{\Sigma}^*, \mathbb{R}^{n-1})$ is well defined, where it remains to show that H is defined for all $\hat{z}^* \in \mathbb{R}^{n-1}$ and $y \in h(X_0)$.

Note that for the system Σ, with the diffeomorphisms Φ and Ψ from (A1), we have $z_1 = \tilde{y} = \Psi \circ h \circ \Phi^{-1}(z)$ for any $z \in \Phi(O)$. If we transform the set $M_{\alpha\beta}$ by Φ, since $M_{\alpha\beta} \subset O$, we get $\Phi(M_{\alpha\beta}) = \{z \in \Phi(O) \mid \alpha \leq h \circ \Phi^{-1}(z) \leq \beta\}$. $M_{\alpha\beta} \subset O$ implies further that $\alpha, \beta \in h(O)$, so we can apply Ψ to the inequality defining the set $\Phi(M_{\alpha\beta})$ and get $\Phi(M_{\alpha\beta}) = \{z \in \Phi(O) \mid \Psi(\alpha) \leq z_1 \leq \Psi(\beta)\}$.

$X_0 \subset M_{\alpha\beta}$ implies that $\forall t \geq 0 : y(t) \in [\alpha, \beta]$ or equivalently $\tilde{y}(t) \in [\Psi(\alpha), \Psi(\beta)]$. Now consider the extended state $\hat{z}$ defined by equation (6.28). We have $\hat{z}_1(t) = \tilde{y}(t)$ and thus $\hat{z}(t) \in \Phi(M_{\alpha\beta})$ for all $t \geq 0$, $\hat{z}^* \in \mathbb{R}^{n-1}$ and $y \in h(X_0)$.

The assumption $M_{\alpha\beta} \subset O$ implies $\Phi(M_{\alpha\beta}) \subset \Phi(O)$, and since $H(\hat{z}^*, \tilde{y}) = \Phi^{-1}(\hat{z}(t))$ and $\hat{z}(t) \in \Phi(O)$,, the observer output mapping H is well defined.

Now we are ready to prove that the conditions of definition 13 hold with the assumptions made in the theorem. Let us consider the observer error $e^* = \hat{z}^* - z^*$. It evolves with the linear, asymptotically stable dynamics $\dot{e}^* = Le^*$. Using the same argumentation as in the proof of corollary 2 on page 59, we get that $\mathbb{R}^{n-1}$ is invariant for the observer dynamics $\hat{\Sigma}^*$ and condition (i) holds.

Note that $e^* = 0$ is equivalent to $\hat{x} = x$. By the linear dynamics for e^*, we have $e^*(0) = 0 \Rightarrow \forall t \geq 0 : e^*(t) = 0$, and thus $\hat{x}(0) = x_0 \Rightarrow \hat{x}(t) = x(t)$ and condition (ii) holds. We have also $e^*(t) \to 0$ as $t \to \infty$. This is equivalent to $(\hat{x}(t) - x(t)) \to 0$ as $t \to \infty$ and condition (iii) holds. □

The condition of theorem 11 is easy to check for a given nonlinear system. If it is satisfied, a global observer of reduced order can be designed. In the following section, we will apply this design to the uncontrolled Lotka–Volterra model, where the observer canonical form is obtained by time scaling, and another third order system where we use output transformation to get the observer canonical form.

6.4.3 Examples

The Lotka–Volterra model

We consider the Lotka–Volterra model (2.1) and its transformation to observer canonical form with time scaling. This was already done in section 5.2 and used in example 2 to introduce the invariance problem.

The model has the observability region $O = \{x \in \mathbb{R}^2 \mid x_1 > 0\}$. If we consider only the predator–prey parameter configurations, then we have a closed invariant set $X_0 \subset O$, such that furthermore there exists $\delta > 0$ with $X_0 \subset M_{\delta\infty} \subset O$.

Previously, we found a global diffeomorphism Φ on O such that with the coordinate transformation $z = \Phi(x)$, the system dynamics become

$$\begin{aligned} \dot{z} &= z_1 \begin{pmatrix} z_2 + a + dz_1 + c \ln z_1 \\ -\frac{ac+adz_1}{z_1} \end{pmatrix}, \\ y &= z_1. \end{aligned}$$

Let us now compute the reduced order dynamics for the system. Setting $z^* = z_2 + lz_1$, we get

$$\dot{z}^* = y\left(lz^* - l^2 y - \frac{ac + ady}{y} + l(a + dy + c\ln y)\right). \tag{6.32}$$

The observer dynamics are then designed as

$$\hat{\Sigma}^* : \quad \begin{cases} \dot{\hat{z}}^* = y\left(l\hat{z}^* - l^2 y - \dfrac{ac + ady}{y} + l(a + dy + c\ln y)\right) \\ \hat{x} = \Phi^{-1}(y, \hat{z}^* - ly) \end{cases} \tag{6.33}$$

with $\hat{z}^* \in \mathbb{R}$.

The dynamics for the observation error $e = \hat{z}^* - z^*$ are linearisable by the time scaling $d\theta = ydt$ and we obtain

$$\frac{de}{d\theta} = le.$$

Choosing $l < 0$ will guarantee observer convergence.

All conditions from theorem 11 are satisfied, and thus the pair $(\hat{\Sigma}^*, \mathbb{R})$ is indeed an observer for the Lotka–Volterra model. In particular we cannot get the effect encountered in example 2, that the estimation $\hat{x}$ becomes undefined for some time interval.

Let us consider the dynamics of $\hat{x}$, the estimated state in original coordinates, as obtained by the dynamics of the observer (6.33). For $\hat{x}_1 = y$ we get directly the measured system state, as desired for a reduced order observer. But for the unmeasured system state x_2, we get

$$\dot{\hat{x}}_2 = cx_2 + dx_1x_2 + lx_1(\hat{x}_2 - x_2). \tag{6.34}$$

The state x_2 enters the observer dynamics even though it cannot be measured. This is due to the fact that in deriving $\hat{x}$ from equation (6.33) we obtain derivatives of y and thus the state x_2 appears in the equation above. Contrarily, the observer itself as given in equation (6.33) does not use any derivatives of the system output.

Note also that the typical observer form with a simulation term and an error correction term is clearly visible in equation (6.34). In fact, the simulation is done using the dynamics from the real system, the estimation itself enters only into the corrective term. Furthermore, we get the interesting result that even in original coordinates, the dynamics of the estimation error $\hat{x}_2 - x_2$ are linearisable by time scaling. In fact, the dynamics of the estimation error are exactly the same as those of e, as

$$\frac{d}{dt}(\hat{x}_2 - x_2) = ly(\hat{x}_2 - x_2).$$

This is probably due to the fact that the state space transformation Φ used to transform the system to observer canonical form is affine in x_2.

By theorem 11, we know that the reduced order observer we designed is invariant on the observability region O. A natural question to pose is if it also does inherit invariance properties with respect to $\mathbb{R}^2_+$. Using the Nagumo theorem 9, we will check this by considering the dynamics of the observer at the border of $\mathbb{R}^2_+$. We already know that $\hat{x}_1 = y > 0$, so only the border $\hat{x}_2 = 0$ has to be checked.

Let us use equation (6.34) for the dynamics of $\hat{x}_2$ obtained above to analyse the uniform invariance of $\mathbb{R}^2_+$. The vector field driving the observer at the border of $\mathbb{R}^2_+$ where $\hat{x}_2 = 0$ can be calculated as

$$\dot{\hat{x}}_2|_{\hat{x}_2=0} = cx_2 + dx_1x_2 - lx_1x_2. \tag{6.35}$$

If $\dot{\hat{x}}_2|_{\hat{x}_2=0} > 0$, we have $\hat{x}_2(t) > 0$ and thus $\hat{x}(t) \in \mathbb{R}^2_+$ for all times t, assuming a reasonable initialisation of the observer such that $\hat{x}(0) \in \mathbb{R}^2_+$.

Considering the design parameter l, this can be rewritten as the condition

$$l < \frac{c}{x_1} + d. \tag{6.36}$$

Recall that we need $l < 0$ for convergence of the observer, which does not pose any problems here. In the case where $c > 0$, inheritance of invariance is easily achieved by setting $l < \min(0, d)$. But in the case where $c > 0$, inheritance of invariance can only be guaranteed if a lower bound greater than 0 is known for x_1. This might not always be the case. However, invariance properties can be improved in this case by choosing a higher observer gain $-l$, meaning that we get inheritance of invariance for a larger set of initial conditions of the system.

Simulations have been done mainly to illustrate the concept of invariance for the reduced order observer designed above. This is the reason why usually rather small gains $-l$ have been chosen, as the problem of observer invariance gets less interesting with higher observer gains. Nevertheless, we will give an example of a reduced order observer with rather high gain which does not inherit invariance properties with respect to $\mathbb{R}_+^2$ of the system.

The settings for the different simulations are given in the following listing, together with some remarks on the scope of the example. The resulting trajectories of the estimation are displayed in figure 6.6. The caption beneath each subfigure refers each phase plot to the corresponding setting for the simulation listed below.

(a) This setting illustrates the smooth convergence of the reduced order observer for favourable initial conditions of the system.

Parameters:	$a = 1$, $b = -1$, $c = -1$, $d = 1$
Initialisation:	$x_1(0) = 0.8$, $x_2(0) = 0.3$, $\hat{x}_2(0) = 1$
Observer gain:	$l = -1$

(b) Here the inheritance of invariance found above, if $c > 0$, is illustrated. The chosen initial condition is rather representative for this case. As the observer without the correction term follows the real system dynamics, the state $\hat{x}_2$ would become negative in the beginning of this simulation if l was closer to 0.

Parameters:	$a = -1$, $b = 1$, $c = 1$, $d = -1$
Initialisation:	$x_1(0) = 2.5$, $x_2(0) = 1.1$, $\hat{x}_2(0) = 0$
Observer gain:	$l = -1$

(c) This simulation shows that the observer does not inherit invariance properties with respect to $\mathbb{R}_+^2$ even for rather high gains if $c < 0$.

Parameters:	$a = 1$, $b = -1$, $c = -1$, $d = 1$
Initialisation:	$x_1(0) = 0.2$, $x_2(0) = 2$, $\hat{x}_2(0) = 0$
Observer gain:	$l = -4$

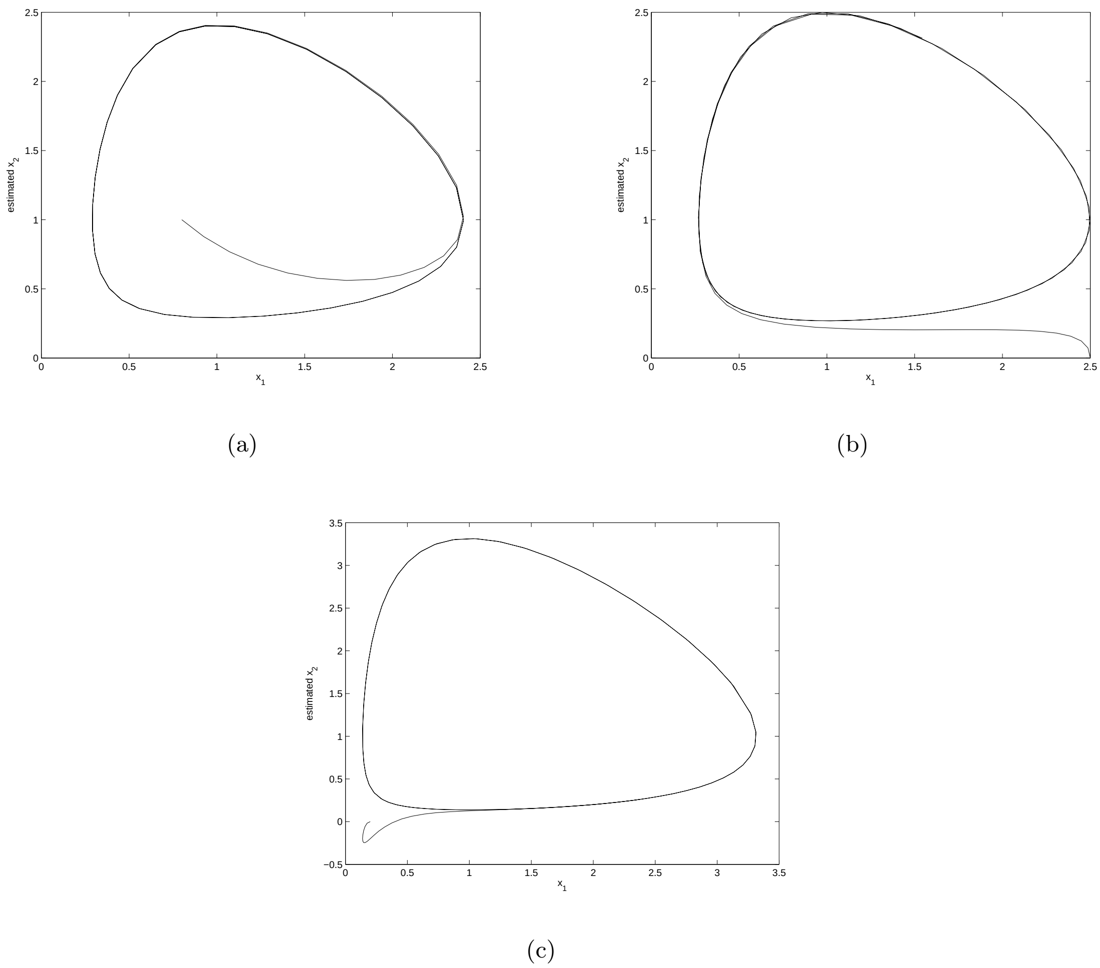


Figure 6.6: Estimation trajectories of the reduced order observer for the Lotka–Volterra model

A third order system

The Lotka–Volterra model (2.1) is a very simple example for a reduced order observer, because it is of second order and thus the observer is only first order. Therefore, we will give another example with a system of third order to which our theory can be applied. Consider the system given by

$$\Sigma : \quad \begin{cases} \dot{x}_1 = e^{-x_1} x_2 \\ \dot{x}_2 = -e^{-x_1} x_2 + x_3 - 2 \\ \dot{x}_3 = -x_2 + 1 \\ \ \ y = x_1 \end{cases} \tag{6.37}$$

with $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^3$.

A short analysis of the system dynamics gives the conclusion that we have a rotation around the point $(1,2)$ in the x_2–x_3 plane, with an additional nonlinear term resulting in (maybe asymptotic) stability when projected on this plane. x_1 is obtained by integrating x_2 with some nonlinear, strictly positive factor.

Although a unique solution for all times may not exist for all initial conditions of $\mathbb{R}^3$, we conclude the existence of an invariant set containing at least the axis $(x_1, 1, 2)$ and some neighbourhood around it. We will use this set as the set of initial conditions X_0 for Σ. The larger x_1, the larger we can set the neighboorhood around the axis of rotation which is included in this set.

Since all states can be computed consecutively from the output y and its time derivatives, the system is globally observable, i.e. we have an observability region $O = \mathbb{R}^3$.

Moreover, we can transform the system to observer canonical form by setting $z = \Phi(x)$ and $\tilde{y} = \Psi(y)$, where

$$\begin{aligned} z_1 &= e^{x_1} \\ z_2 &= x_2 + x_1 \\ z_3 &= x_3 + e^{x_1} \end{aligned} \tag{6.38}$$

and

$$\tilde{y} = e^y. \tag{6.39}$$

The mappings Φ and Ψ are global diffeomorphisms.

In z–coordinates, the system dynamics become

$$\begin{aligned} \dot{z}_1 &= z_2 - \ln z_1 \\ \dot{z}_2 &= z_3 - z_1 - 2 \\ \dot{z}_3 &= 1 \\ \tilde{y} &= z_1 \end{aligned} \tag{6.40}$$

and are thus linear up to output injection.

The two basic assumptions (A1) and (A2) from page 58 are satisfied, and we can apply some of our results to this system.

Let us first design the dynamics for a prospective full order observer as

$$FOO: \quad \begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 - y + g_1(\hat{z}_1 - e^y) \\ \dot{\hat{z}}_2 = \hat{z}_3 - e^y - 2 + g_2(\hat{z}_1 - e^y) \\ \dot{\hat{z}}_3 = 1 + g_3(\hat{z}_1 - e^y) \\ \hat{x} = \Phi^{-1}(\hat{z}) \end{cases} \tag{6.41}$$

with $\hat{z} \in \mathbb{R}^3$ and the observer gain $G = \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix}^{\mathrm{T}}$ is chosen such that the dynamics of the observer error $e = \hat{z} - z$ are asymptotically stable.

The image of the observability region is $\Phi(O) = \mathbb{R}_+ \times \mathbb{R}^2$, and we will use the prospective full order observer $(FOO, \Phi(O))$. By theorem 10, this pair is an observer for the system Σ, if $\Phi(O)$ is uniformly invariant for the dynamics FOO. But this condition is not easy to check for a given system, and for the system Σ it is in fact not true, as we will see later in the results of the simulation. So $(FOO, \Phi(O))$ is not an observer for Σ by our definition. Nevertheless, we get of course noninstantaneous observation by proposition 5.

A reduced order observer based on z–coordinates is obtained by setting

$$\begin{aligned} z_2^* &= z_2 + l_2 z_1 \\ z_3^* &= z_3 + l_3 z_1. \end{aligned}$$

We obtain the system dynamics for z_2^* and z_3^* as

$$\begin{aligned} \dot{z}_2^* &= l_2 z_2^* + z_3^* - z_1 - 2 + l_3 z_1 - l_2^2 z_1 - l_2 \ln z_1 \\ \dot{z}_3^* &= l_3 z_2^* + 1 - l_3 l_2 z_1 - l_3 \ln z_1. \end{aligned}$$

The dynamics for the prospective reduced order observer are now designed as

$$ROO: \quad \begin{cases} \dot{\hat{z}}_2^* = l_2 z_2^* + z_3^* - e^y - 2 - l_3 e^y - l_2^2 e^y - l_2 y \\ \dot{\hat{z}}_3^* = l_3 z_2^* + 1 - l_3 l_2 e^y - l_3 y \\ \hat{x}_1 = y \\ \hat{x}_2 = \hat{z}_2^* - l_2 e^y - y \\ \hat{x}_3 = \hat{z}_3^* - l_3 e^y - e^y, \end{cases} \tag{6.42}$$

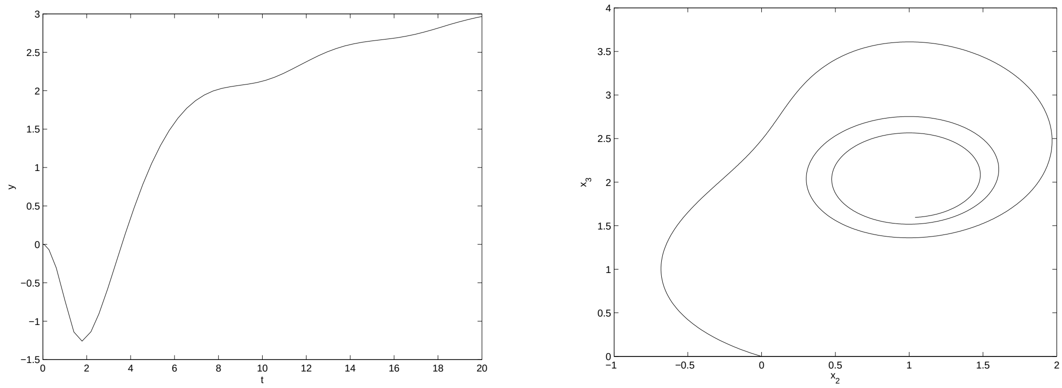
with $\hat{z}^* = \begin{pmatrix} \hat{z}_2^* & \hat{z}_3^* \end{pmatrix} \in \mathbb{R}^2$. The observer parameters l_2 and l_3 are chosen to assure convergence of the observer error $\hat{z}^* - z^*$ to 0.

Let us check the condition of theorem 11. We have $O = \mathbb{R}^3 = M_{\infty\infty}$, and so $X_0 \subset M_{\infty\infty} \subset \mathbb{R}^3$. By theorem 11, the pair $(ROO, \mathbb{R}^2)$ is an observer for Σ.

Let us now illustrate the theoretical statements obtained for this example with some simulation results. First, we give some idea of the system dynamics by showing the system trajectory of a simulation with initial condition $x(0) = (0,0,0)^{\mathrm{T}}$ in figure 6.7. The same initial condition for the system was used in all simulations.

For the implementation of the observer dynamics, we have chosen the gain $G = \begin{pmatrix} -3.3 & -3.6 & -1.3 \end{pmatrix}^{\mathrm{T}}$ for the full order observer and $l_2 = -2$, $l_3 = -1$ for the reduced order observer. Both observers where initialised such that $\hat{x}(0) = (0,-2,-3)^{\mathrm{T}}$. Figure 6.9 gives the time plots of the estimation obtained for the two observer dynamics FOO and ROO. Subfigure (a) does not contain the estimation of the reduced order observer because it is identical to the state $x_1(t)$ which is measured for observation.

The unboundedness of the estimation in original coordinates from the full order observer can be clearly seen in these plots. This is due to the event that $\hat{z}_1$ approaches 0 and even becomes negative for a short time interval. During this time interval, the observer can give no estimates for the states x_1 and x_2, as the transformation to calculate them would involve $\ln \hat{z}_1$. The time plot for $\hat{z}_1(t)$ is displayed in figure 6.8, and it shows the passage of the full order observer state $\hat{z}_1$ below 0.



(a) System output $y(t) = x_1(t)$

(b) Phase plot for x_2, x_3

Figure 6.7: System state trajectory

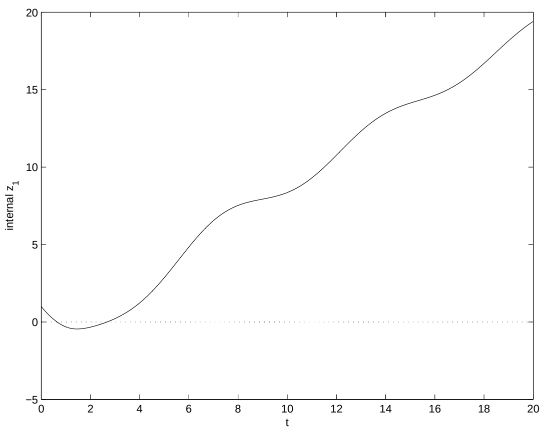
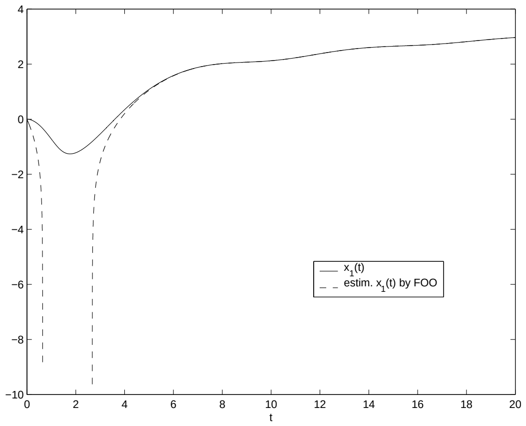
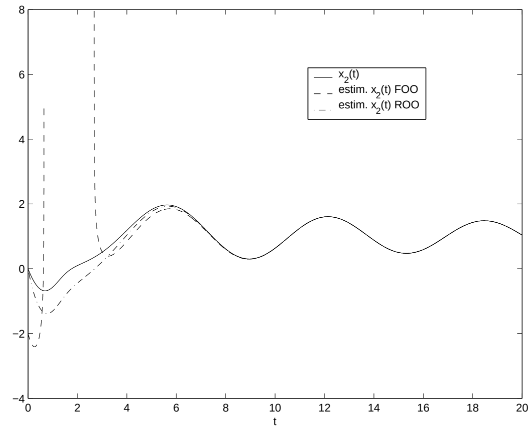


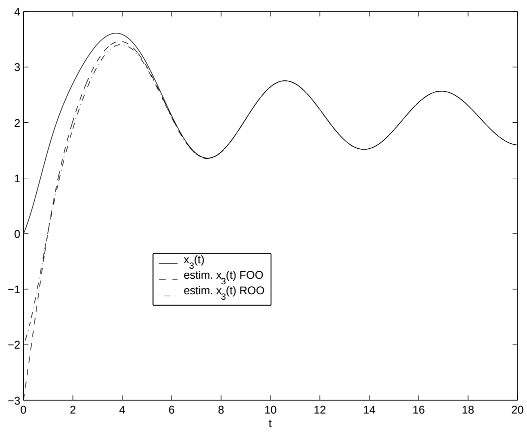
Figure 6.8: Time plot of the state $\hat{z}_1$ of the full order observer dynamics FOO



(a) Time plots of x_1 and estimation of FOO



(b) Time plots of x_2 and estimation of FOO and ROO



(c) Time plots of x_3 and estimation of FOO and ROO

Figure 6.9: Estimations of FOO and ROO for the system Σ

6.5 Invariant full order observers

In this section, we will try to design a full order observer which is invariant on an observability region of the system Σ from equation (6.1). In the previous section, we encountered the restricting condition on the existence of a specific set $M_{\alpha\beta}$ to guarantee invariance of the reduced order observer. In what follows it will turn out that this condition is quite important for observer invariance, also for full order observers designed by output injection.

We will assume that (A1) and (A2) from page 58 hold for the system Σ. Then we have an observability region O containing the invariant set of initial conditions X_0.

The system Σ can then be transformed to the observer canonical form (6.9) by the state space transformation $z = \Phi(x)$ and the output transformation $\tilde{y} = \Psi(y)$. Based on these coordinates, we can design observer dynamics via a standard Luenberger approach. However, we do not have conditions for invariance of the observability region under these dynamics.

From the Nagumo theorem, we know that invariance of closed sets depends only on the system dynamics at the border of the considered set. Therefore we will change the observer dynamics (6.10) obtained by the Luenberger approach at the border of the observability region to guarantee its invariance.

For a given set $M \in \mathbb{R}^n$ and a small parameter $\epsilon > 0$, we choose a continous function

$$\gamma : M \to [0.1] \tag{6.43}$$

satisfying the conditions

(i) $\gamma(x) = 1$ if $x \in \partial M$

(ii) $\gamma(x) = 0$ if $\operatorname{dist}(x, \partial M) \geq \epsilon$.

We will also make use of the set $M_{\alpha\beta}$ defined in section 6.4 on page 69.

Next we want to design full order observer dynamics such that some set M is invariant for the observer dynamics. To this end, consider the dynamics based on the Luenberger observer (6.10)

$$\hat{\Sigma}: \quad \begin{cases} \dot{\hat{z}} = s(\tilde{y})\,(A\hat{z} + (1-\gamma(\hat{x}))k(\tilde{y}) + \gamma(\hat{x})k(C\hat{z}) + G(C\hat{z} - \tilde{y})) \\ \hat{x} = \Phi^{-1}(\hat{z}), \end{cases} \tag{6.44}$$

with $\hat{z} \in \mathbb{R}^n$ and the function γ as defined above for the set M and a small ϵ. The function γ is used to change the dynamics at the border of M, when compared to the standard observer (6.10). In fact, when the estimated state approaches the border of M, then we tend to replace the output–injection $k(\tilde{y})$ by only simulating the dynamics of the system Σ, using $k(C\hat{z})$.

With the dynamics $\hat{\Sigma}$, we have a result on inheritance of invariance.

Lemma 1 (Inheritance of invariance). *Assume (A1) and (A2) from page 58. The function γ from equation 6.43 is designed for the set X_0 from (A1) and some small $\epsilon > 0$. Then the pair $(\hat{\Sigma}, \Phi(O))$ inherits system invariance properties with respect to X_0, if there exist $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ such that $X_0 = M_{\alpha\beta}$.*

Remark 6. There are two properties of the dynamics $\hat{\Sigma}$ as defined above that we are using in this proposition: First, let us consider the dynamics without the observer adjustment term $G(C\hat{z} - \tilde{y})$. Then, if the estimation $\hat{x}$ approaches the border of the set X_0, the observer dynamics just simulate the dynamics of the system Σ. Since X_0 is invariant for Σ, it is so also for the observer dynamics $\hat{\Sigma}$.

Now let us take also the adjustment term $G(C\hat{z} - \tilde{y})$ into account. By the Nagumo theorem, the set $\Phi(X_0)$ is invariant for $\hat{\Sigma}$, if and only if the vector $G(C\hat{z} - \tilde{y})$ points inside or is tangent to the set $\Phi(X_0)$. To assure stability of the matrix $(A + GC)$, all components of G must be negative. Then the direction of the adjustment term is mainly given by the sign of the first component $\hat{z}_1 - \tilde{y}$ of the observer error. We use the condition on the form of X_0 to allow only for one sign at the border of X_0.

Proof. $(\hat{\Sigma}, \Phi(O))$ inheriting system invariance properties with respect to X_0 is equivalent to the property that the set $\Phi(X_0)$ is uniformely invariant for the dynamics $\hat{\Sigma}$. For our proof, we will apply the Nagumo theorem. To this end, let us consider the vector field F of the dynamics $\hat{\Sigma}$ at the border of $\Phi(X_0)$. For any $\hat{z} \in \partial\Phi(X_0)$, we have $\gamma(\Phi^{-1}(\hat{z})) = 1$ and thus $F(\hat{z}, \tilde{y}) = s(\tilde{y})\,(A\hat{z} + k(\hat{z}_1) + G(\hat{z}_1 - \tilde{y}))$. Since $s(\tilde{y})$ is a positive real number, it is not relevant for the question whether $F(\hat{z}, \tilde{y})$ is in the tangent cone to $\Phi(X_0)$ at $\hat{z}$ or not. Hence we will omit it and study only $\bar{F}(\hat{z}, \tilde{y}) = A\hat{z} + k(\hat{z}_1) + G(\hat{z}_1 - \tilde{y})$.

Then, let us consider the term $A\hat{z} + k(\hat{z}_1)$. Under the coordinate transformation $x = \Phi^{-1}(z)$, this vector transforms to $\Phi_*^{-1}(A\hat{z} + k(\hat{z}_1)) = f(\Phi^{-1}(\hat{z}))$. By the assumption that X_0 is invariant for Σ, $f(\Phi^{-1}(\hat{z}))$ is in the tangent cone to X_0 at $\Phi^{-1}(\hat{z})$. Being in the tangent cone or not is invariant under diffeomorphism, and thus $A\hat{z} + k(\hat{z}_1)$ is in the tangent cone to $\Phi(X_0)$ at $\hat{z}$.

Now let us consider the term $G(\hat{z}_1 - \tilde{y})$ and assume that we have α, β such that $X_0 = M_{\alpha\beta}$. Under the diffeomorphism Φ, this set is transformed to $\Phi(X_0) = \{z \in \Phi(O) \mid \Psi(\alpha) \leq z_1 \leq \Psi(\beta)\}$ (see the proof of theorem 11 on page 69). This set has at most two borders, if both $\alpha, \beta \in \mathbb{R}$. For the border point $\hat{z}$, we have either $\hat{z}_1 = \tilde{\alpha} = \Psi(\alpha)$ or $\hat{z}_1 = \tilde{\beta} = \Psi(\beta)$.

By the form of the border $\partial\Phi(X_0)$, a vector is in the tangent cone at $\hat{z}$ if its first component is positive or zero, when $\hat{z}_1 = \tilde{\alpha}$, respectively negative or zero when $\hat{z}_1 = \tilde{\beta}$. Furthermore, since we assumed X_0 to be invariant for the system, we have always $\tilde{\alpha} < \tilde{y} < \tilde{\beta}$.

We have chosen the gain G such that the matrix $A + GC$ is asymptotically stable. Using the Hurwitz criterion, this implies that all components of G, in particular the first one, are negative. Then, if $\hat{z}_1 = \tilde{\alpha}$, we have $\hat{z}_1 - \tilde{y} < 0$ and the first component of $G(\hat{z}_1 - \tilde{y})$ is positive. In the other case where $\hat{z}_1 = \tilde{\beta}$, we have $\hat{z}_1 - \tilde{y} > 0$ and the first component of $G(\hat{z}_1 - \tilde{y})$ is negative. In both cases, $G(\hat{z}_1 - \tilde{y})$ is in the tangent cone to $\Phi(X_0)$ at $\hat{z}$.

Both summands of $\bar{F}(\hat{z}, \tilde{y})$ are in the tangent cone, and thus also $\bar{F}(\hat{z}, \tilde{y})$ is in the tangent cone of $\Phi(X_0)$ at $\hat{z}_0$ for any $\tilde{y} \in \Psi \circ h(X_0)$. By the Nagumo theorem, $\Phi(X_0)$ is uniformly invariant for $\hat{\Sigma}$. □

Using the same approach as for the inheritance of invariance, we will try to render the pair $(\hat{\Sigma}, \Phi(O))$ invariant on O. The next theorem gives our results for this approach.

Theorem 12 (Invariance on O). *For the system Σ from equation (6.1), assume (A1) and (A2) from page 58. Assume further that for O and X_0 from (A1), $\exists\delta > 0$: $\operatorname{dist}(\partial O, \partial X_0) > \delta$ and that there exist $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ such that the closure $\overline{O}$ of the observability region O satisfies $\overline{O} = M_{\alpha\beta}$ and is invariant for Σ. Then the pair $(\hat{\Sigma}, \Phi(O))$, with γ for O and some $\epsilon < \delta$, is invariant on O.*

Furthermore, if for some initial conditions $x_0 \in X_0$ and $\hat{z}_0 \in \Phi(O)$, there exists a time $T > 0$ such that $\forall t > T : \gamma(\hat{x}(t, \hat{z}_0, y(\cdot))) = 0$, then the estimation $\hat{x}$ of the pair $(\hat{\Sigma}, \Phi(O))$ converges to the state x of Σ for these initial conditions.

Remark 7. First, we do not know if the pair $(\hat{\Sigma}, \Phi(O))$ is actually an observer for the system Σ. The conditions (i) and (ii) of definition 13 are satisfied, but we have no proof for convergence. However, the second statement in the theorem can be interpreted as visibility of convergence: As long as the observer state $\hat{z}$ is such that the value $\gamma(\Phi^{-1}(\hat{z})) = 0$, the observer state will converge. If this value is 0 for a long time, we can suppose that the estimation $\hat{x}$ has come close to the system state x. Note that this condition depends entirely on the state of the dynamics $\hat{\Sigma}$ and thus can be checked during observation without further knowledge of the system state.

Proof. Using the same argumentation as in the proof of lemma 1, we get the statement that the closed set $\operatorname{cl}(\Phi(O))$ is uniformly invariant for $\hat{\Sigma}$. Moreover, with the assumption that the distance of the borders of O and X_0 is larger than δ, we have $\exists\tilde{\delta} > 0 \, \forall \tilde{y} \in \Psi \circ h(X_0), \hat{z} \in \partial\Phi(O) : |\hat{z}_1 - \tilde{y}| > \tilde{\delta}$. This implies that the term $G(\hat{z}_1 - \tilde{y})$ is always in the interior of the tangent cone to $\Phi(O)$ at $\hat{z} \in \partial\Phi(O)$. Then also $F(\hat{z}, \tilde{y})$ is in the interior of the tangent cone for any $\hat{z} \in \partial\Phi(O)$ and $\tilde{y} \in \Psi \circ h(X_0)$. This means that the dynamics $\hat{\Sigma}$ have no trajectories which are tangent to the border $\partial\Phi(O)$ and that the open set $\Phi(O)$ is uniformly invariant for $\hat{\Sigma}$. Then the pair $(\hat{\Sigma}, \Phi(O))$ is invariant on O.

For the second statement, let us consider the observer error $e = \hat{z} - z$, where $z = \Phi(x)$. The dynamics for the observer error are given by

$$\dot{e} = (A + GC)e + \gamma(\hat{x}(t, \hat{z}_0, y(\cdot)))\,(k(\hat{z}_1) - k(\tilde{y}))\,.$$

If $\exists T > 0$ such that $\forall t > T : \gamma(\hat{x}(t, \hat{z}_0, y(\cdot))) = 0$, the error has linear, asymptotically stable dynamics after the time T. This implies that the estimation $\hat{x}(t, \hat{z}_0, y(\cdot))$ converges to $x(t, x_0)$ as $t \to \infty$. □

Although we have no proof for convergence, we conjecture that for most applications the pair $(\hat{\Sigma}, \Phi(O))$ will be an observer for Σ. The dynamics of the observer error $e = \hat{z} - z$ are of course nonlinear, but they are so only if the estimation $\hat{x}$ is in a small neighbourhood of the region we design the observer for. Otherwise we have just linear, asymptotically stable dynamics. In proposition 5, we noticed that the problem of observer invariance is only intermediate. The use of the function γ in the dynamics designed in this section can be seen in the same manner: If everything works fine, the observer should only intermediately have $\gamma(\hat{x}) > 0$.

6.5.1 Application to the Lotka–Volterra model

We will apply the approach developed in the previous section to the Lotka–Volterra model which is transformed to the observer canonical form via time scaling and state space diffeomorphism. This transformation has already been used in section 6.4.3 and we will base the work of this section on the results obtained there.

The observability region we consider is $O = \{x \in \mathbb{R}^2 \mid x_1 > 0\}$. If we consider only predator–prey parameter configurations for the model, then we have a closed invariant set X_0 such that $\exists \delta > 0 : \operatorname{dist}(\partial O, \partial X_0) > \delta$.

Furthermore, the closure $\bar{O}$ of the observability region is invariant for the Lotka–Volterra model 2.1 and we have $\bar{O} = M_{0\infty}$. All conditions of theorem 12 are satisfied.

To get the dynamics $\hat{\Sigma}$, we have yet to choose the function γ. We will choose a positive $\epsilon < \operatorname{dist}(\partial O, \partial X_0)$ and define γ as

$$\gamma(\hat{x}) = \begin{cases} 1 - \dfrac{\hat{x}_1}{\epsilon} & \text{if } \hat{x}_1 < \epsilon \\ 0 & \text{if } \hat{x}_1 \geq \epsilon. \end{cases} \tag{6.45}$$

Then the dynamics $\hat{\Sigma}$ we are going to use are given by

$$\begin{aligned} \dot{\hat{z}}_1 &= y\left(\hat{z}_2 + (1-\gamma(\hat{x}))(a + dy + c\ln y) + \gamma(\hat{x})(a + d\hat{z}_1 + c\ln\hat{z}_1) + g_1(\hat{z}_1 - y)\right) \\ \dot{\hat{z}}_2 &= y\left((1-\gamma(\hat{x}))\frac{ac + ady}{y} + \gamma(\hat{x})\frac{ac + ad\hat{z}_1}{\hat{z}_1}\right) \\ \hat{x}_1 &= \hat{z}_1 \\ \hat{x}_2 &= \frac{1}{b}(\hat{z}_2 + d\hat{z}_1 + c\ln\hat{z}_1). \end{aligned} \tag{6.46}$$

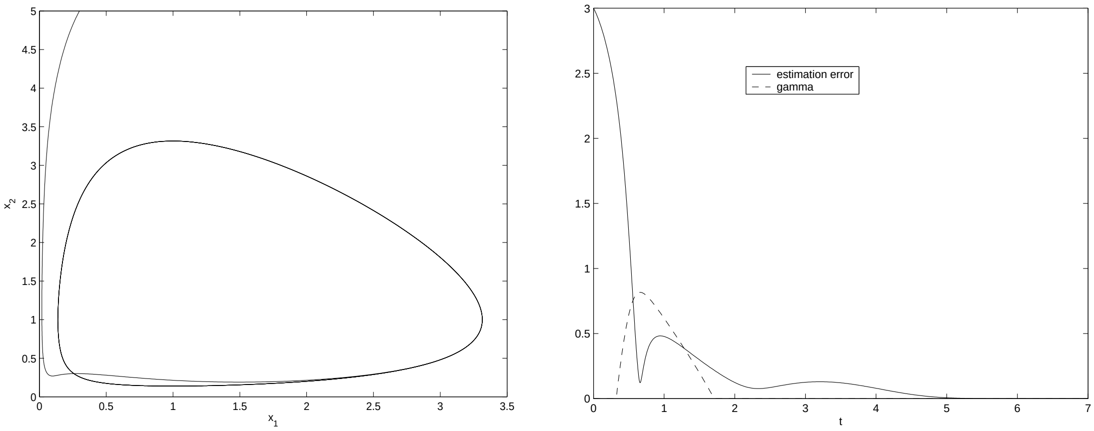
We have not proven that the estimation of the pair $(\hat{\Sigma}, \Phi(O))$ converges to the real system state, but we will give some simulation results which illustrate well the behaviour of these dynamics and for which the estimation does converge.

The model parameters were chosen as $a = d = 1$ and $b = c = -1$. The system was initialized at $x_0 = \begin{pmatrix} 0.2 & 2 \end{pmatrix}^{\mathrm{T}}$, while the observer was initialised such that its initial estimation was $\hat{x}(0) = \begin{pmatrix} 0.2 & 5 \end{pmatrix}^{\mathrm{T}}$. For the original observer designed by time scaling, this initialisation would be even more problematic than the one we used in example 2 on page 51, where we introduced the invariance problem. We will see that the dynamics designed in the previous section work as an observer in this case.

With the chosen initial condition x_0, we get $x_1(t) > 0.1$ for all t, and thus a reasonable choice for the parameter ϵ in the function γ is $\epsilon = 0.1$.

For the first simulation, the observer gain was set to $G = \begin{pmatrix} -4 & -4 \end{pmatrix}^{\mathrm{T}}$ to get the eigenvalues $(-2, -2)$ for the linear part of the error dynamics. Figure 6.10 shows the results of the numerical simulation. The trajectory of the estimation $\hat{x}$ is shown in subfigure (a). It comes quite close to the critical border of the observability region, but it does not reach or even cross it (respectively $\hat{x}_2$ does not tend to $-\infty$ as we have seen in example 2). Subfigure (b) shows the estimation error, which converges to 0.

Furthermore, subfigure (b) shows also the trajectory of the function $\gamma(\hat{x}(t))$ during the simulation. During the time intervall where the estimation $\hat{x}$ is close to the border of the observability region, $\gamma(\hat{x}(t))$ takes a value larger than 0. During this time the convergence behaviour is not very good, but this is only an intermediate problem.



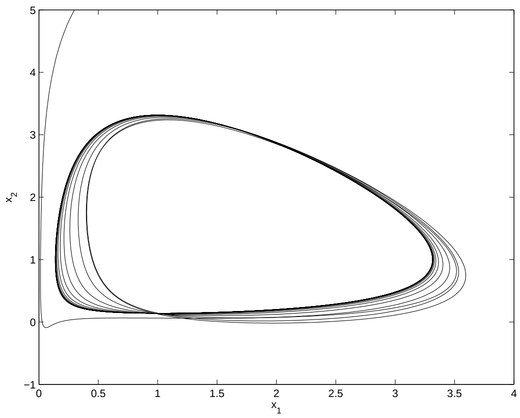
(a) Trajectory of the estimation $\hat{x}$

(b) Estimation error $\|\hat{x}(t) - x(t)\|$ and values of $\gamma(\hat{x}(t))$

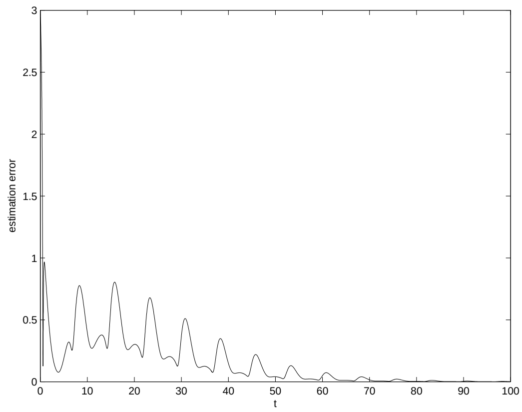
Figure 6.10: Invariant full order observer for $G = (-4 \;\; -4)^{\mathrm{T}}$

The second simulation shall show that the proposed observer dynamics converge also for very small gains, to illustrate that we do not rely on a high–gain principle if we have convergence, but rather on the fact that the error dynamics become linear if the estimation does not come close to the border of the observability region. We have therefore chosen $G = \begin{pmatrix} -0.2 & -0.01 \end{pmatrix}$, such that the eigenvalues for the linear part of the error dynamics are $(-0.1, -0.1)$. The results of this simulation are shown in figure 6.11. Due to the small gain, the estimation converges very slowly, it is near 0 only after the time $t = 90$. However, the value of $\gamma(\hat{x}(t))$ is different from 0 only at the very beginning of the simluation, during the intervall $0.3 < t < 1.6$. So the oscillating behaviour of the estimation error in subfigure (b) is not due to the extension we made to the standard Luenberger observer in this section, but does also appear where the dynamics of the observer error $\hat{z} - z$ are actually linear.

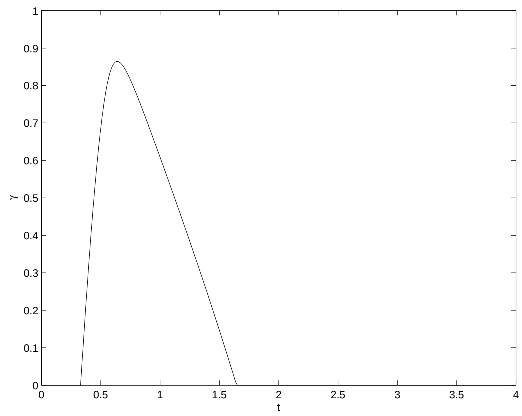
From subfigure (a), we can also see that the observer is invariant on the observability region O, as stated by theorem 12. However, it does not inherit invariance properties with respect to $\mathbb{R}^2_+$, as we get estimations where $\hat{x}_2 < 0$.



(a) Trajectory of the estimation $\hat{x}$



(b) Estimation error $\|\hat{x}(t) - x(t)\|$



(c) Values of $\gamma(\hat{x}(t))$

Figure 6.11: Invariant full order observer for $G = (-0.2 \;\; -0.01)^{\mathrm{T}}$

7 Observers for the controlled Lotka–Volterra models

7.1 The Keller observer for the model LV_2

In section 5.1, we designed an observer for the Lotka–Volterra model using state space and output transformation. This worked well for the uncontrolled model, but it failed for the controlled models LV_2 and LV_3 (see section 2.2).

The goal of this section is to design an observer with linearisable error dynamics for the controlled system LV_2, using the approach of Keller [1987]. That means we want to find a coordinate transformation depending on the original coordinates x and the input u with some of its time derivatives up to order $n-1$, such that the system takes the observer canonical form proposed by Keller [1987], which is for $n = 2$

$$\begin{aligned} \dot{z}_1 &= z_2 + k_1(y, u, \dot{u}) \\ \dot{z}_2 &= k_2(y, u, \dot{u}, \ddot{u}) \\ y &= \Psi^{-1}(z_1). \end{aligned} \tag{7.1}$$

The system LV_2 was found to be uniformly observable for any input, if we assume $b \neq 0$ and $x_1 > 0$. Following Zeitz [1984], we can transform the system to observability canonical form by taking consecutive time derivatives of the output as coordinates. These coordinates will depend on the original coordinates x and on the input u and some of its time derivatives up to order $n-1$. Of course the input has to be sufficiently smooth for this approach. Denoting this transformation as $\xi = \Phi(x, u, \dot{u})$ we obtain

$$\begin{aligned} \xi_1 = y \quad &= x_1 \\ \xi_2 = \dot{y} \quad &= ax_1 + bx_1x_2 + ex_1u. \end{aligned} \tag{7.2}$$

For the Lotka–Volterra model, the transformation actually does not depend on input derivatives, such that also the functions k_i used in the observer canonical form (7.1) have reduced dependencies on input derivatives. We will hence use $k_1(y, u)$ and $k_2(y, u, \dot{u})$ for the observer canonical form.

Assuming $b \neq 0$, the transformation Φ is a global diffeomorphism on the set $\{x \in \mathbb{R}^2 \mid x_1 > 0\}$, and is used to transform the system LV_2 to ξ–coordinates on this set. Note that this transformation is valid for any input u.

The dynamics in ξ–coordinates are computed as

$$\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= \frac{\xi_2^2}{\xi_1} + (c + d\xi_1 + fu)\xi_2 - (a+eu)d\xi_1^2 - (ac + ecu + fua)\xi_1 - feu^2\xi_1 + e\dot{u}\xi_1.\end{aligned} \tag{7.3}$$

The right hand side of the last line, which is the second time derivative of the output y, is also called the characteristic nonlinearity of the system and will be denoted by $\rho(\xi, u, \dot{u})$.

When transforming the general observer canonical form (7.1) to the observability canonical form, one obtains equations for the unknowns k_1, k_2 and Ψ by coefficient comparison of the respective characteristic nonlinearities. To this end, we derive the equation $\Psi(\xi_1) = z_1$, which describes the output of the observer canonical form, two times and obtain

$$\Psi''(\xi_1)\xi_2^2 + \Psi'(\xi_1)\rho(\xi, u, \dot{u}) = k_2(\xi_1, u, \dot{u}) + \frac{\partial k_1}{\partial \xi_1}\xi_2 + \frac{\partial k_1}{\partial u}\dot{u}. \tag{7.4}$$

From this characteristic equation one can deduce that the transformation to observer canonical form is possible if and only if the characteristic nonlinearity ρ is a polynomial of second degree in ξ_2 with coefficients as functions of the output ξ_1 and the input u and its derivatives [see Keller, 1987]. By comparison of coefficients with respect to ξ_2 we obtain the equations determining the unknowns as

$$\begin{aligned}\Psi''(\xi_1) + \Psi'(\xi_1)\frac{1}{\xi_1} &= 0\\ \frac{\partial k_1}{\partial \xi_1}(\xi_1, u) &= \Psi'(\xi_1)\left(c + d\xi_1 + fu\right)\\ k_2(\xi_1, u, \dot{u}) + \frac{\partial k_1}{\partial u}(\xi_1, u)\dot{u} &= \Psi'(\xi_1)\left(-(a+eu)d\xi_1 - ac - ecu - fua - feu^2 + e\dot{u}\right).\end{aligned}$$

The first equation for Ψ can be solved by separation of variables, the second for k_1 by simple integration and the third is an algebraic equation for k_2. We chose integration constants such that in the uncontrolled case, where $e = f = 0$, the solution is the same as in the observer design done by output transformation in section 5.1. The result is

$$\begin{aligned}\Psi(\xi_1) &= \ln \xi_1\\ k_1(\xi_1, u) &= c\ln\xi_1 + d\xi_1 + fu\ln\xi_1 + a\\ k_2(\xi_1, u, \dot{u}) &= -ac - d(a+eu)\xi_1 - (ec + af)u - feu^2 + (e - f\ln\xi_1)\dot{u}.\end{aligned}$$

The observer canonical form (7.1) has now been computed, but not the transformation from coordinates ξ to z or from x to z. This transformation is needed to reconstruct the observer estimate $\hat{x}$; it can be computed by considering the consecutive differential equations the observer canonical form consists of. We get thus

$$\begin{aligned}z_1 &= \Psi(\xi_1) && = \ln\xi_1\\ z_2 &= \dot{z}_1 - k_1(\xi_1, u) && = \frac{\xi_2}{\xi_1} - (c\ln\xi_1 + d\xi_1 + fu\ln\xi_1 + a).\end{aligned}$$

Inverting this transformation and concatenating it with the inverse of the transformation Φ into observability canonical form gives the coordinate change from z to x as

$$\begin{aligned} x_1 &= \exp(z_1) \\ x_2 &= \frac{1}{b}\left(z_2 - a - eu - cz_1 - d\exp(z_1) - fuz_1\right). \end{aligned}$$

The observer is now constructed by input–output injection and a standard Luenberger adjustment term. We get thus

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + k_1(y,u) + g_1(\hat{z}_1 - \ln y) \\ \dot{\hat{z}}_2 &= k_2(y,u,\dot{u}) + g_2(\hat{z}_1 - \ln y) \\ \hat{x} &= H(\hat{z},u) \end{aligned} \tag{7.5}$$

with the observer state $\hat{z} \in \mathbb{R}^2$, the inputs to the observer $y, u, \dot{u} \in \mathbb{R}$ and the gain $G = \begin{pmatrix} g_1 & g_2 \end{pmatrix}^{\mathrm{T}}$. The dynamics for the observer error $e = \hat{z} - z$ are

$$\dot{e} = (A + GC)e \tag{7.6}$$

with A and C in observer canonical form. These dynamics are stabilised by pole placement through an appropriate gain G.

7.1.1 Simulation results

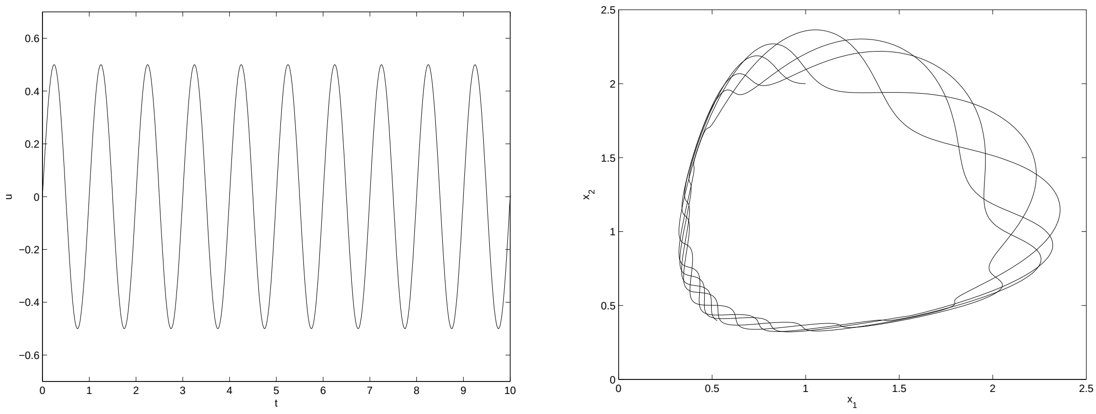
For all simulations, the gain vector is chosen as $G = \begin{pmatrix} -4 & -4 \end{pmatrix}^{\mathrm{T}}$, giving eigenvalues of the error dynamics at $(-2,-2)$. The model parameters are taken from the set (A) in equation (5.17), i.e. $a = d = 1$ and $b = c = -1$. These parameters were already used for the simulation of the observer obtained via output transformation, they describe a predator–prey configuration where x_1 is the prey.

The initial condition of the system is chosen as $x(0) = \begin{pmatrix} 1 & 2 \end{pmatrix}^{\mathrm{T}}$, whereas the initial observer estimate is $\hat{x}(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^{\mathrm{T}}$. As input function we choose

$$u(t) = 0.5\sin(2\pi t).$$

In this first simulation, the observer was directly supplied with the analytically computed derivative of the input, i.e. we use $\dot{u}(t) = \pi\cos(2\pi t)$ and did not do a numerical derivation of the input signal during the simulation. The input and the resulting state trajectory of the controlled system are shown in figure 7.1. The resulting trajectory of the estimate and the estimation error are displayed in figure 7.2. As expected from the theoretical results, the estimation error converges asymptotically to 0, .

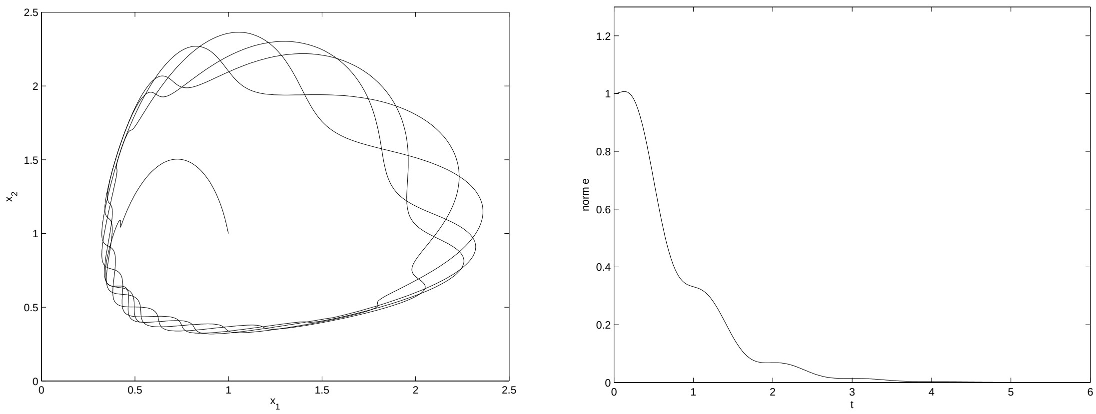
For the second simulation, the same setting was used. The only change is that the derivative $\dot{u}$ of the input was not provided directly, but rather computed numerically during the simulation. The resulting trajectory of the estimation error $\|\hat{x}(t) - x(t)\|$ is quite unpleasant: The error does not converge asymptotically to 0, but there stays a small noise, which is certainly due to numerical inaccuracy in computing the derivative



(a) Input $u(t)$ for the system LV_2

(b) State trajectory of the system LV_2

Figure 7.1: Setting for the simulation of the Keller observer



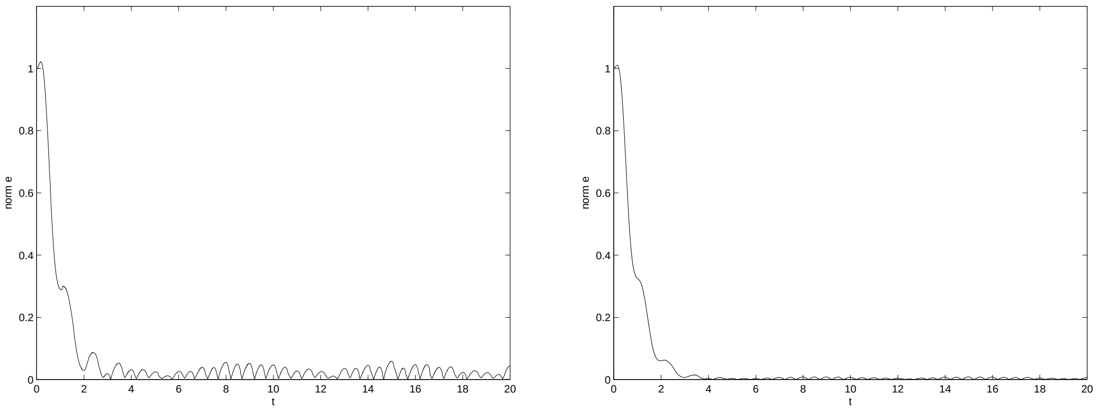
(a) Estimation trajectory

(b) Norm of the observation error

Figure 7.2: Simulation results using the Keller observer

of the input. The simulation is done by a Dormand–Prince integration method, which is of variable step size, and we chose a relative error tolerance for the integration of 1e-5. For comparison, the same simulation was repeated with a different relative error tolerance of 1e-7. The noise in the error remains, but is now less than with the broader tolerance. The trajectories of the estimation error for the two simulations are shown in figure 7.3.

When applying the Keller observer method, one will of course try to provide analytical derivatives of the input to the observer, to avoid the noise due to numerical differentiation. This will usually be possible if the input is known in advance, e.g. in



(a) Relative error tolerance of 1e-5

(b) Relative error tolerance of 1e-7

Figure 7.3: Error noise due to numerical input differentiation

feed forward control problems. Another important case is the one of static estimation feedback, where we apply a control law of the form

$$u(t) = \alpha(\hat{x}(t)). \tag{7.7}$$

We will first study the case where we have an observer which depends only on the first time derivative of the input, as we had in the Lotka–Volterra model handled above. The observer dynamics are thus of the form $\dot{\hat{z}} = F(\hat{z}, u, \dot{u})$. Using the observer output equation $\hat{x} = H(\hat{z}, u)$ and denoting $\tilde{\alpha} = \alpha \circ H$, we get

$$u(t) = \tilde{\alpha}(\hat{z}(t), u(t)).$$

The time derivative of u is then given as

$$\dot{u}(t) = \frac{\partial \tilde{\alpha}}{\partial \hat{z}} F(\hat{z}(t), u(t), \dot{u}(t)) + \frac{\partial \tilde{\alpha}}{\partial \hat{u}} \dot{u}(t).$$

We have thus obtained two nonlinear equations for $u(t)$ and $\dot{u}(t)$ which have to be solved online for each time t. If we have only one derivative $\dot{u}$ to consider, this might still be possible. But let us now consider the case where the observer depends also on higher derivatives of the input. Consider e.g. the case where the dynamics of the observer take the form $\dot{\hat{z}} = F(\hat{z}, u, \dot{u}, \ddot{u})$. The equation to solve for $u(t)$ remains the same, but the second equation for $\dot{u}(t)$ now depends on $\ddot{u}(t)$, the equation for $\ddot{u}(t)$ will depend on $\dddot{u}(t)$ and so forth. It will not be possible to compute the derivatives of u analytically just from the control law (7.7), and thus one will tend to use numerical derivatives of the input.

In practical applications, the inaccuracy of the observer estimation due to numerical input differentiation is not necessarily annoying, since we always have an estimation error due to measurement noise, model inaccuracy and external disturbances. However, the dependence on input derivatives is certainly a drawback for the Keller observer design method.

7.2 An observer with time–varying linear error dynamics for the model LV_2

The observer designed via the Keller method for the model LV_2 has the weakness that it depends on derivatives of the input and thus is not useable if we allow for inputs which are not differentiable. Based on the observation that the system LV_2 is affine with respect to the unmeasured state x_2 when transformed into observer canonical form via output transformation, we will design an observer for this system with time variant linear error dynamics. The observer gain which will guarantee asymptotic stability is computed using a quadratic Ljapunov function. We will have to use light input restrictions, assuming the input to be bounded by a previously known value.

We work on the model LV_2 defined in equation (2.3). Using the state space transformation $z = \Phi(x)$ defined as

$$\begin{aligned} z_1 &= \ln x_1 \\ z_2 &= bx_2 - c \ln x_1 - dx_1 \end{aligned} \tag{7.8}$$

which was obtained by the observer design via output transformation in section 5.1, the model is transformed to

$$\begin{aligned} \dot{z}_1 &= z_2 + a + d \exp z_1 + cz_1 + eu \\ \dot{z}_2 &= -ac - ad \exp z_1 + fuz_2 + (cfz_1 - ce + d(f-e) \exp z_1)u \\ y &= \exp z_1 \end{aligned}$$

We can represent the dynamics of this system as

$$\begin{aligned} \dot{z} &= A(u)z + k(z_1) + \delta(z_1)u \\ y &= \Psi^{-1}(Cz) \end{aligned} \tag{7.9}$$

with an input–dependent and thus time–varying matrix

$$A(u) = \begin{pmatrix} 0 & 1 \\ 0 & fu \end{pmatrix}$$

and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

The observer for the system LV_2 is constructed as

$$\begin{aligned} \dot{\hat{z}} &= A(u)\hat{z} + k(\Psi(y)) + \delta(\Psi(y))u + G(u)(\hat{z}_1 - \Psi(y)) \\ \hat{x} &= \Phi^{-1}(\hat{z}) \end{aligned} \tag{7.10}$$

with a suitable input–dependent gain matrix $G(u)$ which will be designed in the following section.

Let us first consider the dynamics for the observer error $e = \hat{z} - z$. We get

$$\dot{e} = (A(u) + G(u)C)e. \tag{7.11}$$

Note that due to the structure of A and C, it is impossible to choose a G such that the error dynamics do not depend on u. Such an approach would require a further linear coordinate transformation using the Ackermann formula. But this would import input derivatives into the dynamics, which we would like to avoid.

Instead, we will directly construct a Ljapunov function and use it to compute an input–dependent gain G which renders the error dynamics (7.11) asymptotically stable.

7.2.1 Design of the observer gain G

Taking a constant, symmetric positive definite matrix W, we will use the Ljapunov function V for the observer error defined by

$$V(e) = \langle e, We \rangle , \tag{7.12}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product.

The time–derivate of V along the error trajectories is

$$\begin{aligned} \dot{V}(e) &= \langle \dot{e}, We \rangle + \langle e, W\dot{e} \rangle \\ &= \langle e, ((A+GC)^{*}W + W(A+GC))e \rangle . \end{aligned}$$

To guarantee asymptotic stability, the matrix $\tilde{A} := (A+GC)^{*}W + W(A+GC)$ must be negative definite for all admissible input values u.

For a gain $G = \begin{pmatrix} g_1 & g_2 \end{pmatrix}^{\mathrm{T}}$, we have

$$(A+GC) = \begin{pmatrix} g_1 & 1 \\ g_2 & fu \end{pmatrix} \tag{7.13}$$

and the matrix W is

$$W = \begin{pmatrix} w_1 & w_3 \\ w_3 & w_2 \end{pmatrix}. \tag{7.14}$$

The positive definiteness of W gives the conditions

$$\begin{aligned} w_1 w_2 - w_3^2 &> 0 \\ w_1 + w_2 &> 0. \end{aligned}$$

It can be immediately seen that this implies both $w_1 > 0$ and $w_2 > 0$.

The gain G must be chosen such that the eigenvalues of the matrix $\tilde{A}$ have negative real part. This can be done by assigning a characteristic polynomial to $\tilde{A}$, i.e. we require $cP(\tilde{A}) = \det(sI - \tilde{A}) = s^2 + p_1 s + p_0$, where

$$p_1 > 2\gamma \quad \text{and} \quad p_0 > \gamma^2 \tag{7.15}$$

with $\gamma > 0$ to keep away from the critical case. Computing the characteristic polynomial of $\tilde{A}$, we see that this does only hold if

$$\begin{aligned} p_0 =& - w_1^2 - 2g_2 w_1 w_2 + 4fuw_1 w_2 g_1 - g_2^2 w_2^2 - (g_1 + fu)^2 w_3^2 + 4g_2 w_3^2 \\ &+ 2(g_1 - fu)(w_1 - g_2 w_2) w_3 \end{aligned} \tag{7.16}$$

$$p_1 = - 2(fuw_2 + w_3 + g_1 w_1 + g_2 w_3). \tag{7.17}$$

For simplicity, let us denote $\bar{u} = fuw_2 + w_3$ in the following considerations.

Solving equation (7.17) for g_2 gives

$$g_2 = \frac{-p1 - 2\bar{u} - 2g_1 w_1}{2w_3}.$$

Using this result in equation (7.16), we get a quadratic equation for g_1 with the solutions

$$g_1 = \frac{2fuw_3^2 + 2w_1 w_3 - w_2(p_1 + 2\bar{u}) \pm 2\sqrt{-w_3^2(p_0 + 2\bar{u}p_1 + 4\bar{u}^2)}}{2(w_1 w_2 - w_3^2)}.$$

g_1 will only be real if $p_0 + 2\bar{u}p_1 + 4\bar{u}^2 < 0$. To assure this, we put

$$\begin{aligned} p_1 &= -\frac{\gamma^2 + 2\bar{u}^2}{\bar{u}} \\ p_0 &= 2\gamma^2 + \bar{u}, \end{aligned} \tag{7.18}$$

such that the condition for g_1 being real becomes $\bar{u} < 0$.

If we assume $\gamma \geq 1$, the conditions from equation (7.15) will be satisfied, provided that $-\gamma^2 < \bar{u}$. Putting the two conditions on $\bar{u}$ together yields

$$-\gamma^2 < \bar{u} < 0. \tag{7.19}$$

This condition will lead to restrictions on the inputs we can handle.

The gain G is now computed by replacing p_1 and p_2 with the expressions from (7.18) in the equations for g_1 and g_2, such that we obtain

$$\begin{aligned} g_1 &= \frac{2fuw_3^2 + 2w_1 w_3 + w_2^2 \gamma^2 \bar{u}^{-1} + 2w_3\sqrt{-\bar{u}}}{2(w_1 w_2 - w_3^2)} \\ g_2 &= \frac{\gamma^2 - 2g_1 w_1 \bar{u}}{2w_3 \bar{u}}. \end{aligned} \tag{7.20}$$

The last thing to do is to choose the design parameters γ and W satisfying the conditions found above. This choice must be done by using previously known bounds for the input u. One chooses a $\gamma \geq 1$, and w_2 and w_3 must then be chosen such that $-\gamma^2 < \bar{u} < 0$ for all possible values of u. Finally one chooses w_1 such that W is positive definite.

The bounds for u should be chosen rather conservatively, because the gain will become very large if $\bar{u}$ approaches 0. This should be avoided to allow for better suppression of measurement noise.

7.2.2 Simulation results

The observer (7.10) designed in the previous section was implemented for simulation. The model parameters have been set to

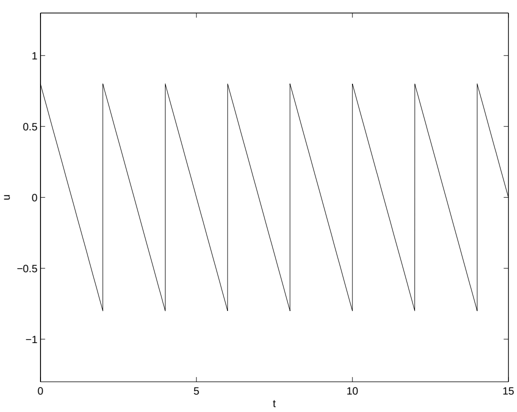
$$\begin{aligned} a &= 1, & b &= -1, & e &= 1, \\ c &= -1, & d &= 1, & f &= 1. \end{aligned} \tag{7.21}$$

For all simulations, the input is assumed to satisfy $|u| < 1$. Furthermore, we choose $\gamma = 2$. The condition (7.19) is satisfied by setting $w_2 = 1$ and $w_3 = -2$, which assures that $-3 < \bar{u} < -1$. Finally, we set $w_1 = 6$ such that W is positive definite. The system was initialised with $x(0) = \begin{pmatrix} 2 & 1 \end{pmatrix}^{\mathrm{T}}$, while the initial estimate of the observer was set to $\hat{x}(0) = \begin{pmatrix} 2.2 & 2.6 \end{pmatrix}^{\mathrm{T}}$. This does not satisfy $h(x(0)) = h(\hat{x}(0))$, but the initial output and the first component of the initial estimate are yet close together.

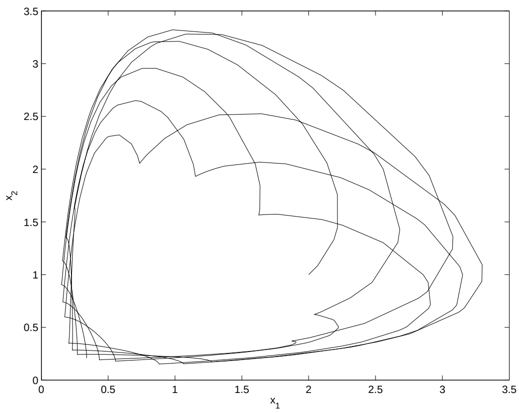
To make use of the fact that the observer is supposed to work with any input staying within the specified bounds, we chose essentially nonsmooth input functions. The first simulation was done with a sawtooth input, i.e. we used u defined by

$$u(t) = \begin{cases} 0.8 - 0.8t & \text{for } 0 \leq t < 2 \\ u(t \bmod 2) & \text{otherwise.} \end{cases} \tag{7.22}$$

The input function and the resulting state trajectory of the system are displayed in figure 7.4. The discontinuities in the input are clearly visible in the resulting state space trajectory.



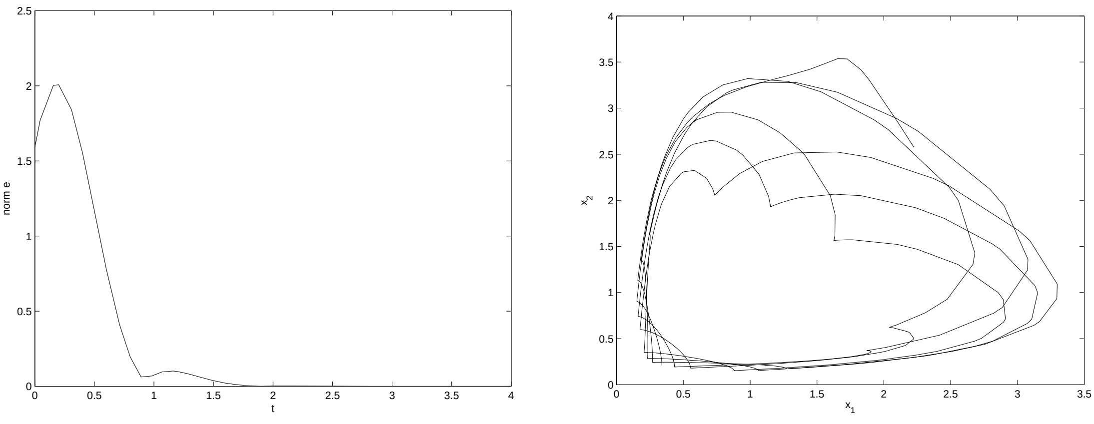
(a) Sawtooth input function



(b) System state trajectory

Figure 7.4: Setting for simulation 1 of the input–varying observer

The results of the observation are displayed in figure 7.5. Subfigure (a) shows the norm of the estimation error, $\|\hat{x}(t) - x(t)\|$, which converges to 0 as expected. The discontinuities do not appear in the plot of the estimation error because the first discontinuity is at $t = 2$, where the error has already diminished below the scale used in the plot.



(a) Norm of the estimation error

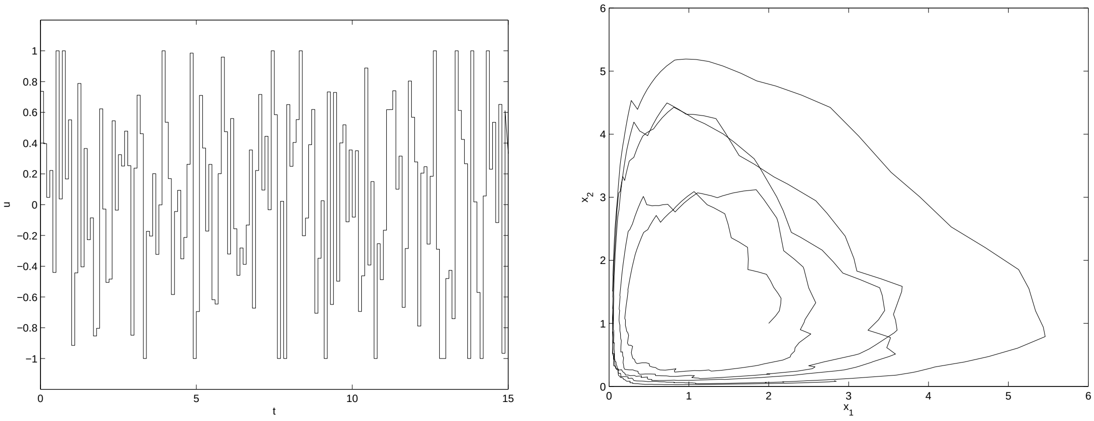
(b) Trajectory of the estimated state

Figure 7.5: Results of simulation 1 of the input–varying observer

For the second simulation, the input is produced by a random number generator with a sample time of 0.1. The input which was actually produced and the resulting state trajectory are shown in figure 7.6. We have not analysed stability of the system with respect to the input, but in this simulation the state variables stayed within reasonable bounds. The results of this simulation are shown in figure 7.7. Here the discontinuities in the input have a slight influence on the observation error, but it still converges to 0.

The two preceding simulations were rather harmless, because the general system properties remained unchanged by the input. Consider now the same parameter set, with the only difference that $f = 2$. Note that the input may now change the system dynamics dramatically. For example, apply the constant input $u = 0.7$ and the equation for the second species becomes $\dot{x}_2 = 0.4x_2 + x_1x_2$. The system has obviously become unstable in the submanifold $\mathbb{R}^2_+$, where the population x_1 will die out and the population x_2 grows exponentially.

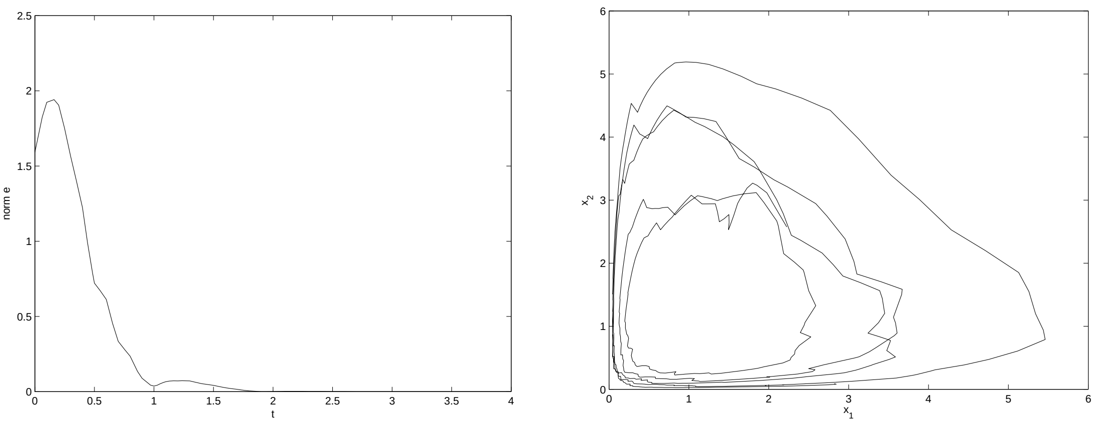
For the observer design, we leave $\gamma = 2$, but choose now $w_2 = 0.7$ and $w_3 = -2.5$ to take the possibly larger value of fu into account. Positive definiteness of W is assured by setting $w_1 = 8$. The sine function $u(t) = 0.8\sin(2\pi t)$ has been applied as input for this simulation. The resulting state space trajectory and the norm of the observation error are shown in figure 7.8. These results show that the observer will work even in the case when the input may change the stability properties of the system.



(a) Randomly generated input

(b) System state trajectory

Figure 7.6: Setting for simulation 2 of the input–varying observer



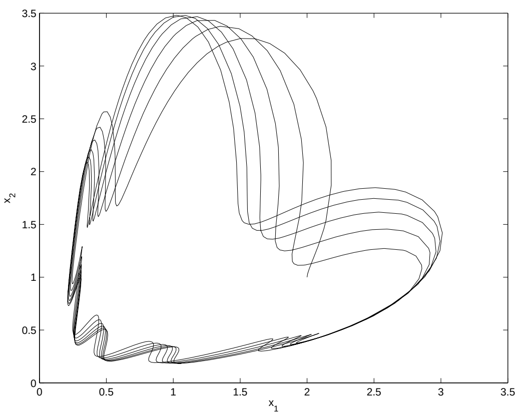
(a) Norm of the estimation error

(b) Trajectory of the estimated state

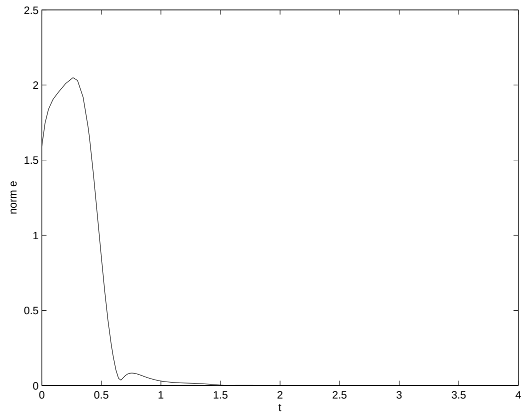
Figure 7.7: Results of simulation 2 of the input–varying observer

7.3 The Keller observer for the model LV_3

The transformation to the observer canonical from proposed by Keller [1987] has already been applied successfully to the controlled Lotka–Volterra model LV_2. It turns out that it is also applicable to the model LV_3 (2.4). However, since LV_3 is not uniformly observable, the transformation which is used to bring the system to observer canonical form will be undefined for some input values and we will have to exclude these inputs.



(a) System state trajectory



(b) Norm of the estimation error

Figure 7.8: Results of simulation 3 of the input–varying observer

For the design of the observer, the state space of the system will be restricted to $\{x \in \mathbb{R}^2 \mid x_1 > 0\}$, such that the system is globally observable for inputs different from $u = -\frac{b}{e}$.

First, we transform the system to the observability canonical form introduced by Zeitz [1984], where consecutive time derivatives of the output are taken as state coordinates. This transformation will not only depend on the original coordinates, but also on the input and possibly on its time derivatives. We compute the transformation

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= ax_1 + (b+eu)x_1x_2. \end{aligned} \tag{7.23}$$

The vector $\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}^{\mathrm{T}}$ may be used as state, if $b+eu \neq 0$. Thus we will have to exclude all input functions where $u(t) = -\frac{b}{e}$ for some t. Note that this is a more severe restriction than the one that was found in the observability analysis. The system is actually observable for any input which takes not constantly the value $-\frac{b}{e}$, but the observer design done here does not work for any input function only crossing this value.

The system LV_3 in observability canonical coordinates ξ writes

$$\begin{aligned} \dot{\xi}_1 =& \xi_2 \\ \dot{\xi}_2 =& \frac{\xi_2^2}{\xi_1} + \left(c + d\xi_1 + fu\xi_1 + \frac{e\dot{u}}{b+eu}\right)\xi_2 \\ & - ac\xi_1 - ad\xi_1^2 - afu\xi_1^2 - \frac{ea\dot{u}}{b+eu}\xi_1 \\ y =& \xi_1. \end{aligned} \tag{7.24}$$

This representation depends not only on the input u, but also on its time derivative $\dot{u}$. Moreover, the dynamics are singular for the bad input value $u = -b/e$. As in section 7.1, we will denote the right hand side of the equation for $\dot{\xi}_2$ as the characteristic nonlinearity $\rho(\xi, u, \dot{u})$ of the system.

As the derivative $\dot{u}$ appears already in the coefficient of ξ_2 in the characteristic nonlinearity, we will have to use input time derivatives up to order 2 in the observer canonical form. More precisely, the observer canonical form will write as

$$\begin{aligned} \dot{z}_1 &= z_2 + k_1(y, u, \dot{u}) \\ \dot{z}_2 &= k_2(y, u, \dot{u}, \ddot{u}) \\ y &= \Psi^{-1}(z_1). \end{aligned} \tag{7.25}$$

To determine the unknowns k_1, k_2 and Ψ in these equations, we transform the observer canonical form (7.25) to the observability canonical form and compare the characteristic nonlinearity obtained for the observer canonical form to the one from the observed system. It is done the same way as in section 7.1, but this time we have to consider more input derivatives. Deriving the equation $\Psi(\xi_1) = z_1$ obtained from the output of the observer canonical form (7.25) two times yields

$$\Psi''(\xi_1)\xi_2^2 + \Psi'(\xi_1)\rho(\xi, u, \dot{u}, \ddot{u}) = k_2(\xi_1, u, \dot{u}, \ddot{u}) + \frac{\partial k_1}{\partial \xi_1}\xi_2 + \frac{\partial k_1}{\partial u}\dot{u} + \frac{\partial k_1}{\partial \dot{u}}\ddot{u}. \tag{7.26}$$

This equation is polynomial of degree 2 with respect to ξ_2, where the coefficients are functions of ξ_1, u, $\dot{u}$ and $\ddot{u}$. A coefficient comparison with ρ taken from (7.24) leads to the three equations

$$\begin{aligned} \Psi'' + \Psi'\frac{1}{\xi_1} &= 0 \\ \frac{\partial k_1}{\partial \xi_1} &= \Psi'\left(c + d\xi_1 + fu\xi_1 + \frac{e\dot{u}}{b+eu}\right) \\ k_2 + \frac{\partial k_1}{\partial u}\dot{u} + \frac{\partial k_1}{\partial \dot{u}}\ddot{u} &= \Psi'\left(-ac\xi_1 - ad\xi_1^2 - afu\xi_1^2 - \frac{ea\dot{u}}{b+eu}\xi_1\right). \end{aligned}$$

Consecutively solving these equations, we get

$$\begin{aligned} \Psi(\xi_1) =& \ln\xi_1 \\ k_1(\xi_1, u, \dot{u}) =& \, a + c\ln\xi_1 + d\xi_1 + fu\xi_1 + \frac{e\dot{u}}{b+eu}\ln\xi_1 \\ k_2(\xi_1, u, \dot{u}, \ddot{u}) =& -ac - ad\xi_1 - f(au + \dot{u})\xi_1 - \frac{ea\dot{u}}{b+eu} + \left(\frac{e\dot{u}}{b+eu}\right)^2 \ln\xi_1 - \frac{e\ddot{u}}{b+eu}\ln\xi_1, \end{aligned}$$

where we chose integration constants such that the functions k_1 and k_2 are the same as in the uncontrolled case (see 5.1 on page 29) for the parameters $e = f = 0$.

The transformation from x to z–coordinates is computed as

$$\begin{aligned} z_1 &= \Psi(h(x)) && = \ln x_1 \\ z_2 &= \dot{z}_1 - k_1(y, u, \dot{u}) && = (b + eu)x_2 - c\ln x_1 - dx_1 - fux_1 - \frac{e\dot{u}}{b+eu}\ln x_1. \end{aligned}$$

The inverse of this transformation will be used for output mapping H of the observer. We obtain

$$\begin{aligned} x_1 &= \exp z_1 \\ x_2 &= \frac{z_2 + cz_1 + d\exp z_1}{b+eu} + \frac{e\dot{u}}{(b+eu)^2}\ , \end{aligned}$$

which will be denoted as $x = H(z, u, \dot{u})$.

The observer for the model LV_3 is now implemented as

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + k_1(y, u, \dot{u}) + g_1(\hat{z}_1 - \ln y) \\ \dot{\hat{z}}_2 &= k_2(y, u, \dot{u}, \ddot{u}) + g_2(\hat{z}_1 - \ln y) \\ \hat{x} &= H(\hat{z}, u, \dot{u}). \end{aligned} \tag{7.27}$$

Considering the observer error $e = \hat{z} - z$, the error dynamics are

$$\dot{e} = (A + GC)e,$$

with A and C in observer canonical form (5.5) respective (5.6) and the gain $G = \begin{pmatrix} g_1 & g_2 \end{pmatrix}^{\mathrm{T}}$. By choosing a suitable G, we can place the poles of the error dynamics arbitrarily.

7.3.1 Simulation results

The observer designed above is implemented using a gain vector $G = \begin{pmatrix} -4 & -4 \end{pmatrix}^{\mathrm{T}}$, to place both poles of the error dynamics at -2. The same model parameters as in section 7.1 are chosen, i.e. we have $a = d = 1$, $b = c = -1$ and $e = f = 1$. With these parameter values, the singular input is $u = 1$. The system is initialised with $x(0) = \begin{pmatrix} 1 & 2 \end{pmatrix}^{\mathrm{T}}$, while the initial estimate of the observer is set to $\hat{x}(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^{\mathrm{T}}$.

As we have already seen with the Keller observer for the model LV_2 in section 7.1, it is important to feed the analytical and not numerical derivatives of the input to the observer. For the model LV_3, where the observer uses also the second time derivative of the input, it becomes even more crucial and thus the analytical derivative of the input has been used for all simulations, although we know that this may not be possible in practical applications.

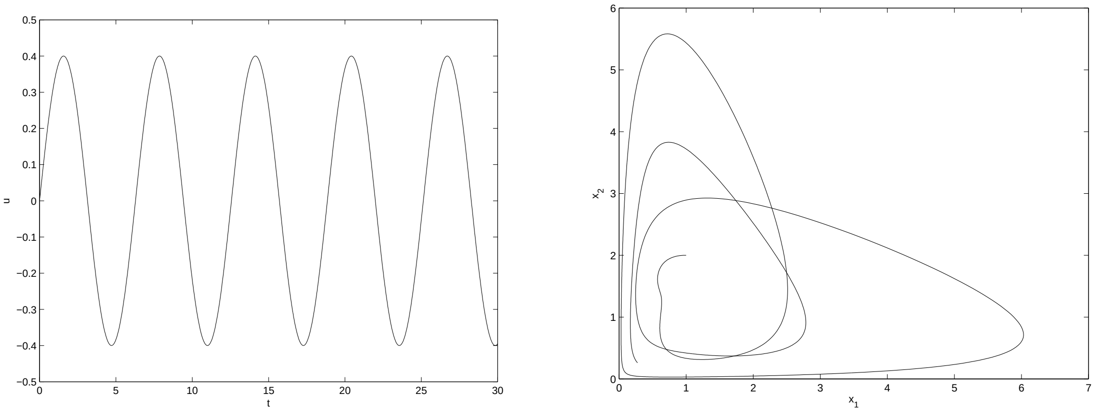
The simulation was done using the sine input function

$$u(t) = \alpha \sin t$$

with different values for the amplitude α. To ensure that the input does not take the bad value, we need to put $|\alpha| < 1$.

The input function $u(t)$ and the resulting state space trajectory of the system LV_3 with $\alpha = 0.4$ are shown in figure 7.9.

Figure 7.10 shows the plots of the estimation error $\hat{x}(t) - x(t)$ using three different values for α, which were chosen from $\{0.4, 0.75, 0.9\}$. Clearly the convergence in x–coordinates is much better for the smaller values of α. Although the error dynamics

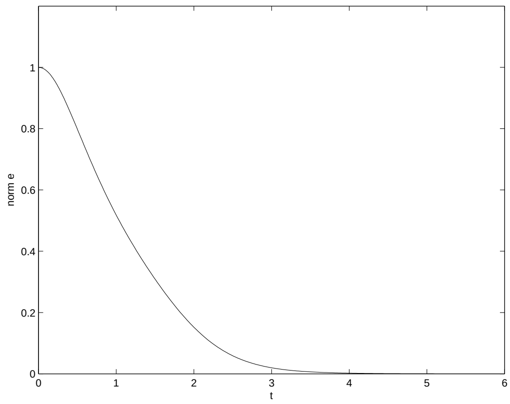


(a) Input function $u(t)$ (b) State space trajectory

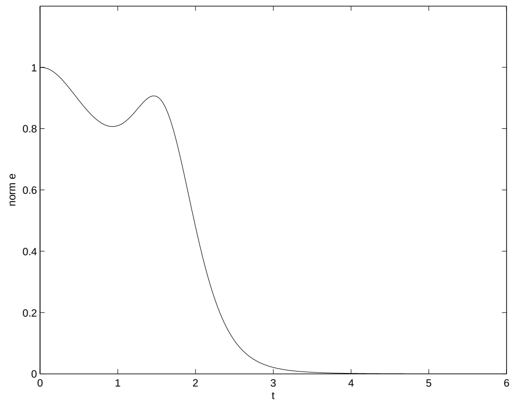
Figure 7.9: Setting for the simulation of the Keller observer for LV_3

in observer canonical coordinates do not depend on the input, the dynamics of the estimation error in original coordinates do, because the transformation depends on the input. The convergence of the observer becomes worse if the input approaches the singular value, as the coordinate transformation will also approach its singularity point.

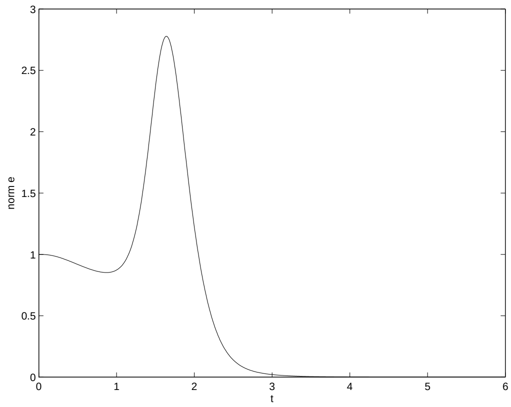
This can be seen especially from subfigure (c) in figure 7.10, where the estimation error becomes very high at the same time were the input comes close to its singular value.



(a) Estimation error for $\alpha = 0.4$



(b) Estimation error for $\alpha = 0.75$



(c) Estimation error for $\alpha = 0.9$

Figure 7.10: Estimation error for the simulation of the Keller observer for LV_3

8 Conclusions

In this work, we applied the theory of nonlinear observability analysis and observer design to a model of a biological system, which describes the evolution of two interacting species. Based on the Lotka–Volterra model with generalised parameters, three different input vector fields were considered to extend the system by a control affine input. Two of these input vector fields have a biological background, only the simplest input vector field we considered has not.

In the first part of the text, we studied observability properties of the population model. We noticed that the system is not globally observable, but it is observable when restricted to an observability region, a subset of the state plane. Moreover, the set of states which are actually reasonable for the biological system is invariant for the system dynamics, when considering biologically meaningful input vector fields or the uncontrolled model, and this set is a subset of the observability region we found. These results gave rise to the construction of an observer, which should be used to observe the system while it evolves on the biological subset we considered.

Concerning the theory of observability, the basic definition using the concept of indistinguishability is well suited to study dynamical systems and can directly be applied to simple systems where it is possible to compute an analytical solution. Furthermore, the local methods based on the approach introduced by Hermann and Krener [1977] were found to be well suited for a local analysis of nonlinear systems like the population model we worked with. Especially the observability canonical form is a powerful tool. It may also give some insights into the observability of a given system on a global scale, if one can estimate the region where the observability canonical form is defined by one transformation mapping.

For the population model we studied, the results from the global observability analysis done in chapter 4 can be linked to the conditions on existence of the observability canonical form found in chapter 3.

The results of Gauthier and Bornard [1981] were very useful for the analysis of the controlled system versions. In particular, we found the third controlled model to be rendered unobservable by a certain input signal. Thus this model can be expected to be more involved concerning the design of an observer.

For the observer design in the second part, we concentrated on design methods which construct linear error dynamics for the observer, and which place the poles of these dynamics by linear eigenvalue assignment, such that they become asymptotically stable. So convergence of the observer estimation is actually based on linear system theory in these design methods.

The approaches we considered all use a transformation of a nonlinear system to observer canonical form, based on the work of Krener and Isidori [1983] and Bestle and Zeitz [1983]. Three design methods were presented and applied to the population model in chapter 5. The first method we used does the transformation to canonical form by state space and output transformation and was introduced by Krener and Respondek [1985]. The second approach, which is based on the results of Respondek et al. [2004], uses a state space transformation and a positive time scaling to put the system in observer canonical form. The third approach finally immerses the observed system into a manifold of higher dimension, as introduced by Jouan [2003].

All three design methods succeeded in constructing an observer for the uncontrolled population model. The state space and output transformation was actually found to be equivalent to the immersion in this case, because we did not have to use a manifold of higher dimension to get the observer canonical form, but it was sufficient to immerse the system in the same dimension. The output transformation and the time scaling gave different state space transformations and different observer dynamics, but apart from problems with invariance we found later in chapter 6, these methods could be used equivalently for observer design[1].

Concerning the application of the three design methods to the controlled versions of the population model, we found that they could only be applied to the simplest input vector field we used. It is of course not astonishing that the methods could not be applied to the third controlled model we introduced, since this one is not uniformly observable. But the second controlled model is uniformly observable, and yet none of the treated methods is able to give an observer for this model. The reason for this inability is that the methods used in chapter 5 are intended mainly for uncontrolled systems. For controlled systems they will only work if the input vector field is compatible.

Further research concerning these methods should be able to find a unified way to apply all these methods. The goal would be to find an observer canonical form which is obtained by applying all the transformations we considered in chapter 5 at the same time. We assumed the existence of a canonical form obtained by output and state space transformation as well as time scaling in chapter 6, but we do not know which class of nonlinear systems can be transformed this way. A future research task will thus be to find the class of systems which can be transformed to observer canonical form using these transformations together. Since the classes of systems which can be transformed by only one of these methods are different, unifying these transformations will clearly enlarge the class of systems we can apply this observer design to.

The design methods presented in chapter 5 are local design methods. But the transformations we used give observer canonical coordinates which represent the population model globally on the biological meaningful subset. Hence we tried to observe the system globally on this subset. However, the introductory example of chapter 6 showed that such an approach to global observer design might fail in some cases.

This observation led us to introduce the notion of invariant observers, which we did in chapter 6. After establishing our notation, we searched a way how to guarantee that

[1] The problems concerning invariance are in general not tied to the use of a time scaling, they can also appear with output and state space transformation, as we saw in some examples of chapter 6.

an observer with linearisable error dynamics can be used globally on an observability region of a nonlinear system. Our first result is the use of initialisation strategies for the observer. We did not propose a general algorithm for doing this, but showed for two examples how to achieve global observation on an observability region when an initial measurement can be used to initialise the observer. The second result uses reduced order observers, which are designed on the basis of the full order observer canonical form. Invariance of reduced order observers is guaranteed, when the observability region we consider satisfies certain conditions. Our third result introduces invariant full order observers, which are obtained by a small extension to the standard canonical form observer. Though we did not prove convergence of the proposed observer, it should yet be useful in technical observer applications due to the property we called visibility of convergence.

The problem we studied and partially solved in chapter 6 is usually only of intermediate nature, but nevertheless it is quite important for control applications, especially when the observer estimation is directly used by a state controller to compute the input which is then applied to the system. Thus we think that further research should be devoted to this problem to find eventually a broader solution for it. Additionally, an extension of the problem formulation to controlled systems would be desirable.

In chapter 7, we dealt with the controlled versions of the population models, where the input vector field has a biological meaning and for which we did not find an observer using the methods of chapter 5. To get an observer for the two versions of the population model under consideration, we used the approach introduced by Keller [1987], where the system is also transformed to an observer canonical form, but in this method the transformation does also depend on the input signal. Even the system version which is not uniformly observable could be treated with this method. However, the transformation was singular for the bad input and thus we had to exclude some input values. A drawback of the Keller design is that it depends also on derivatives of the input signal. If analytical input derivatives cannot be fed to the observer, as when using an estimation feedback controller, there are ususally strong numerical problems for the observer implementation.

To overcome this problem, we designed an observer for the second controlled version of the population model, which was based on the standard canonical form observer, but used an input dependent gain. Using a Ljapunov function we designed explicitly for this case, we could prove convergence of the observer, provided that we know any a priori bound of the input values. We illustrated that our observer works with essentially nonsmooth input signals in several simulations. However, we have not derived a general method for observer design, so the approach we developed in this section remains a solution which is only applicable to systems of the considered form.

In summary, the established theory on observability and observer design for nonlinear systems is well suited for our biological benchmark model. The problems we encountered when trying to construct a global observer have been dealt with using our results on observer invariance. Based on the theory for observers with linearisable error dynamics, we could handle all of the three ways how we added a control input to the basic population.

Bibliography

V. Andrieu and L. Praly. Remarks on the existence of a Kazantzis–Kravaris/Luenberger observer. In *43rd IEEE Conference on Decision and Control*, 2004.

J. Back and J. H. Seo. A constructive algorithm for system immersion into nonlinear observer form. In *Preprints of NOLCOS*, 2004.

D. Bestle and M. Zeitz. Canonical form observer design for non–linear time–varying systems. *International Journal of Control*, 38:419–443, 1983.

F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

E. Bullinger. Dynamik nichttechnischer Systeme. Lecture notes, Universität Stuttgart, 2004.

J. P. Gauthier and G. Bornard. Observability for any $u(t)$ of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 26:922–926, 1981.

J. P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems. applications to bioreactors. *IEEE Transactions on Automatic Control*, 37:875–880, 1992.

R. Hermann and A. J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22:728–740, 1977.

A. Isidori. *Nonlinear Control Systems*. Springer–Verlag Berlin Heidelberg New York, third edition, 1995.

B. Jakubczyk and W. Respondek. On bifurcations of quadratic planar control systems. In *Preprints of NOLCOS*, 2004.

P. Jouan. Immersion of nonlinear systems into linear systems modulo output injection. *SIAM Journal of Control and Optimization*, 41:1756–1778, 2003.

T. Kailath. *Linear Systems*. Prentice–Hall, Englewood Cliff New Jersey, 1980.

N. Kazantzis and C. Kravaris. Nonlinear observer design using Ljapunov's auxiliary theorem. *System & Control Letters*, 34:241–247, 1998.

H. Keller. Nonlinear observer design by transformation into a generalized observer canonical form. *International journal of control*, 46:1915–1930, 1987.

A. J. Krener and A. Isidori. Linearization by output injection and nonlinear observers. *Systems & Control Letters*, 3:47–52, 1983.

A. J. Krener and W. Respondek. Nonlinear observers with linearizable error dynamics. *SIAM Journal of Control and Optimization*, 23:197–216, 1985.

J. Lunze. *Regelungstechnik 1*. Springer–Verlag Berlin, third edition, 2001.

H. Nijmeijer and A. J. van der Schaft. *Nonlinear dynamical control systems*. Springer Verlag, New York, 1990.

F. Plestan. *Linéarisation par injection d'entrée–sortie généralisée et synthèse d'observateurs*. PhD thesis, Ecole Centrale de Nantes, 1995.

W. Respondek. Introduction to geometric nonlinear control; linearization, observability, decoupling. Lectures given at the Summer School on Mathematical Control Theory, Trieste, 2001.

W. Respondek. Automatique nonlinéaire. Lecture notes, INSA Rouen, 2005a.

W. Respondek. Calcul différentiel. Lecture notes, INSA Rouen, 2005b.

W. Respondek, A. Pogromsky, and H. Nijmeijer. Time scaling for observer design with linearizable error dynamics. *Automatica*, 40:277–285, 2004.

L. Råde and B. Westergreen. *Springers Mathematische Formeln*. Springer–Verlag Berlin Heidelberg New York, third edition, 2000. Übersetzt und bearbeitet von P. Vachenauer.

J. M. van den Hof. Positive linear observers for linear compartmental systems. *SIAM Journal of Control and Optimization*, 36:590–608, 1998.

F. Verhulst. *Nonlinear differential equations and dynamical systems*. Springer–Verlag Berlin Heidelberg New York, second edition, 1996.

M. Zeitz. Observability canonical (phase-variable) form for non-linear time-variable systems. *International Journal of Systems Science*, 15, 1984.

M. Zeitz. The extended Luenberger observer for nonlinear systems. *System & Control Letters*, 9:149–156, 1987.

M. Zeitz. Regelung nichtlinearer Systeme. Lecture notes, Universität Stuttgart, 2004.