μ-Suboptimal design of a robustly performing controller for a chemical reactor

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The design of robustly performing controllers for an important class of control problems is treated. The problem is dealt with in the structured singular value (μ)-framework. The systems considered are characterized by weak nonlinearities and parametric and unstructured uncertainties. Nonlinear effects are taken into account via linearization families. Parametric uncertainty is included in a non-conservative way. Both lead to a μ-synthesis problem with repeated scalar real perturbations. As no direct solution to such a μ-synthesis problem is known to date, an iterative approach comprising both μ-analysis of the actual problem and D-K-iteration for a modified problem is shown. This approach is applied to a realistic continuous stirred-tank reactor for production of cyclopentenol to show the applicability to non-trivial MIMO process control problems.

1. Introduction

Modelling of chemical processes on the basis of balance equations and conservation laws usually leads to nonlinear state-space descriptions with uncertain chemical and physical parameters. In general, controller design for such processes has two goals: to ensure stability and to ensure good performance (e.g. disturbance attenuation, tracking) for the real controlled plant. The emphasis is on the *real plant*, meaning that these goals have to be achieved despite uncertain knowledge about the dynamical process model. These goals are called *robust stability* and *robust performance*. To design compensators that achieve these goals despite nonlinearities and general uncertainties is a very difficult and as yet unsolved problem.

We restrict our attention to a special class of chemical processes, namely reaction processes. Typically, these can be described by nonlinear differential equations, where the modelling errors are mostly due to uncertain parameters. In this paper we specifically consider the synthesis of cyclopentenol in a realistic, non-isothermal continuous stirred-tank reactor (CSTR) as a typical representative of this class.

Most industrial reactor control schemes work on the basis of linear single-input single-output PID-controllers that are tuned using trial and error procedures or heuristic tuning rules. In many cases only marginal performance can be achieved by such controllers and, as the major drawback, no assertions on robust stability and robust performance can be made. In recent years the use of a variety of more advanced schemes has been studied for the control of CSTRs (amongst others Hoskins and Himmelblau 1992, Szeifert *et al.* 1992, Soroush

and Kravaris 1992) but very few approaches can achieve guaranteed perform-ance in the face of the inevitable modelling errors. To achieve robust, i.e. guaranteed, stability and performance is the main goal of the controller design method proposed in this paper.

Very often the complications imposed by insufficient knowledge about the physical constants are more restricting than those imposed by the nonlinearities. As in the example system studied here, the nonlinearities in the operating range considered are mild enough to allow linear controllers to achieve good perform-ance. However, to ensure stability and performance robustness, the nonlineari-ties cannot simply be neglected. The idea here is to incorporate the nonlinear effects into the uncertainty description and design the controller based on the linearized system. Stability and performance are then not only demanded for the linearized system in a neighbourhood of the main operating point, but for a so-called linearization family around any equilibrium point in the operating regime; thus leading to a linear state-space system with highly structured uncertainty stemming from uncertain parameters and the inclusion of the linearization family. This general idea was introduced in the context of nonlinear controller design by Baumann and Rugh (1986). There, however, a nonlinear controller is sought that causes the controlled plant to have the same linearized dynamics around *each* equilibrium point. Here, we seek a linear controller achieving stability and performance for every (different) system that results when linearizing around each equilibrium point. For many practical applications this is sufficient for achieving robust performance for the nonlinear uncertain plant. However, it gives no guarantee for the nonlinear plant. It is possible to construct (strongly) nonlinear examples where not even the stability for the nonlinear system can be assured via linearization families.

It is shown that the control problem being considered can be set up in the structured singular value (μ) framework. The structured singular value theory (Doyle 1982, 1985) allows analysis of stability and performance of control systems in the presence of different types of structured uncertainties. This analysis procedure is called μ-analysis. The design of μ-suboptimal controllers (i.e. controllers that yield closed-loop systems for which stability and perform-ance can be guaranteed despite the structured uncertainty) is called μ-synthesis.

The advantage of μ-synthesis compared with other linear robust control schemes is twofold. Firstly, the highly structured uncertainties can be directly incorporated in the design. It is not necessary to convert them into a larger class of possible uncertainties in order to be able to treat them mathematically. Thus, no unnecessary conservatism is introduced into the design. Secondly, the μ-framework allows for analysis and controller design for robust performance, and not only for robust stability as, for example, in H_∞-optimization.

The robust control problem at hand, with a family of linearized systems with uncertain physical parameters, leads to perturbation blocks with repeated real scalars in the μ-framework. Recent results in μ-computation (e.g. Young and Doyle 1990) permit one to analyse such structures. It is, however, not possible directly to synthesize μ-suboptimal controllers at present. An iterative approach comprising both μ-analysis of the actual problem and μ-synthesis of a suitably modified problem is shown leading to μ-suboptimal controllers for perturbation blocks with repeated real scalars.

The outline of this article is as follows: a brief tutorial review of standard

μ-theory is given, followed by the formulation of the control problem with parametric uncertainties and nonlinearities in the μ-framework and its solution with an iterative scheme. By way of the example CSTR for cyclopentenol synthesis, the concepts are further illustrated, demonstrating the benefits and limitations of this approach.

2. Design method

2.1. *Review of μ-control theory*

The structured singular value (SSV or μ) was introduced to give a quantitative characterization of the effects of structured uncertainty on stability performance of linear dynamical systems (Doyle 1982, 1985).
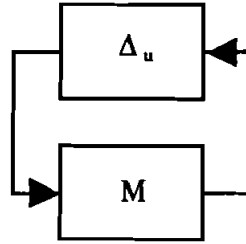
Roughly speaking, uncertainty will be called structured if there is more than one 'source' for it and if these different sources are independent of each other. A simple example of structured uncertainty is a system with two outputs where the only assumed uncertainty comes from the sensors in the outputs. If, for example, one sensor is a flow meter described by a model with an uncertain parameter and the other a temperature sensor of which the high frequency dynamics are known only within bounds, the uncertainties are clearly distinct and independent of each other.

In general, the 'real' plant is considered to be one (unknown) member of a set of possible plant models $\mathcal{G}$. This set is parametrized by (normalized) perturbations Δ_i, that describe the deviation from the nominal model $\tilde{G}$. The perturbations represent the various uncertainty sources. Exemplary, unmodelled dynamics can be represented by a perturbation Δ_i that is an arbitrary, norm bounded, complex transfer matrix satisfying $\bar{\sigma}(\Delta_i) \leq 1$ (Doyle *et al.* 1982). The nominal model $\tilde{G}$ and this perturbation Δ_i (multiplied with a dynamic uncertainty weight W_{Ui} that reflects knowledge about the frequency dependency of the uncertainty, for example) in parallel, form the set of all possible plant descriptions when Δ_i varies over all complex transfer matrices with $\bar{\sigma}(\Delta_i) \leq 1$

$$\mathcal{G} = \left\{ G: G = (\tilde{G} + W_{Ui}\Delta_i), \quad \Delta_i \in \mathfrak{RH}_\infty \text{ with } \bar{\sigma}(\Delta_i) \leq 1 \right\} \qquad (1)$$

As another example, parametric uncertainty can be represented in the obvious way by a scalar real perturbation Δ_i with $|\Delta_i| \leq 1$. It is also possible that exactly the same uncertainty appears at different 'locations' in the plant. For example, an unknown physical parameter, like the ambient pressure in a chemical plant, influences different subprocesses. This naturally results in different perturbations that are 'linked' together. They can be modelled by repeated perturbations. A more detailed treatment can, for instance, be found in Morari and Zafiriou (1989).

In order to evaluate stability and performance of the closed loop it is almost always possible to 'pull out' the uncertainty perturbations and obtain, through this rearrangement, the so-called *M-Δ*-structure shown in Fig. 1. The *M-Δ*-structure is a standardized description of the set of all possible closed loop systems (Morari and Zafiriou 1989). The nominal model of the plant, the controller and the uncertainty weights are contained in the transfer matrix *M*. The perturbation block Δ_u contains only the different uncertainty perturbations. It is clear that the *M-Δ*-structure depends strongly on the description of the uncertainties. Every 'source' of uncertainty has a contribution to Δ_u that is

Figure 1. M-Δ-structure for robust stability analysis.

independent of the others. Therefore, Δ_u will be block diagonal with one block for every perturbation

$$\Delta_u = \text{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_m\} \quad \text{with} \quad \bar{\sigma}(\Delta_i) \leq 1 \ \forall \omega \tag{2}$$

There will also be different types of uncertainty blocks in Δ_u depending on the source of the perturbation (e.g. complex transfer matrices, real scalars, repeated real scalars). For the control problem under consideration, the formulation of the M-Δ-structure is elucidated in more detail in § 2.2.

If the closed loop is given in this standard M-Δ-structure, conditions for robust stability can be stated by making use of the structured singular value. Formally the SSV of a complex matrix M with respect to a perturbation structure Δ_u is defined as

$$\mu(M, \Delta_u) = \left(\min_{\Delta \in \Delta_u} \{\bar{\sigma}(\Delta): \det(I - M\Delta) = 0\}\right)^{-1} \tag{3}$$

The structured singular value can be interpreted as the inverse of the maximum singular value of the smallest permissible perturbation, which makes the closed loop unstable (which is equivalent to $\det(I - M\Delta) = 0$). The SSV is therefore the reciprocal of a stability margin with respect to the structured uncertainty under consideration.

The following theorem follows directly from the definition of μ.

Robust stability theorem (Doyle 1982):   *Assume the nominal system M is stable, the number of unstable poles of the uncertain plant does not depend on Δ_u, and all perturbation blocks Δ_i satisfy $\bar{\sigma}(\Delta_i) \leq 1$. Then the closed loop in Fig. 1 is stable for all such perturbations Δ_u if and only if*

$$\mu(M, \Delta_u) < 1 \quad \forall \omega \tag{4}$$

The assumption on the number of unstable poles of the plant is, for example, assured if the real plant to be controlled is known to be stable, i.e. no permissible perturbation changes the stability characteristics.

In the μ-framework it is not only possible to examine a closed loop system for robust stability but also for robust performance. This is one of the main advantages of using μ. The control performance that is considered in this paper is characterized by the influence that external inputs (like disturbances and sensor noise) have on external outputs like the tracking error. In a procedure analogous to the formation of the M-Δ-structure in Fig. 1, we can construct the block diagram shown in Fig. 2(a) where the input vector v stands for the external inputs and the output vector e stands for the external outputs. The performance objective can then be expressed in the frequency domain in terms
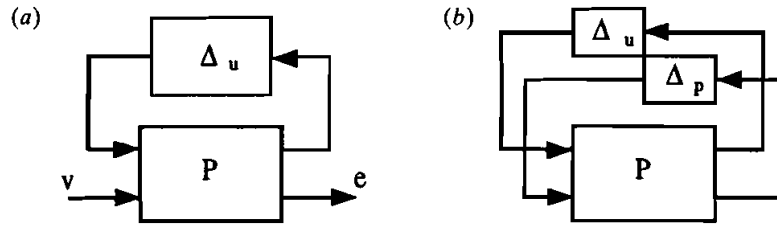
Figure 2. (*a*) Block structure for robust performance analysis; (*b*) equivalence of robust performance requirement to robust stability requirement.

of the H_∞-norm of the transfer matrix $F_u(P, \Delta_u)$ from v to e

$$\|F_u(P, \Delta_u)\|_\infty = \sup_\omega \bar\sigma(F_u(P, \Delta_u)) < 1 \tag{5}$$

Here, $F_u(P, \Delta_u)$ denotes the upper linear fractional transformation (LFT) of P and Δ_u (Doyle *et al.* 1991). For this to be a meaningful specification the use of weights on v and e is essential in order to reflect the varying frequency content of the external input and the performance specification. Such specifications are commonly used, for example, in H_∞-optimal control where Fig. 2(*a*) with $\Delta_u = 0$ represents the Standard Compensator Configuration with compensator K already included in system P. Here, transfer matrix P contains the nominal plant, controller and weights. If P is partitioned into four blocks consistent with the two sets of inputs and outputs, P_{11} can be identified to be exactly the transfer matrix M in Fig. 1. The block P_{22} is equivalent to the performance objective $F_u(P, \Delta_u)$.

It can easily be shown (Morari and Zafiriou 1989) that (5) is formally equivalent to a robust stability condition on $F_u(P, \Delta_u)$ with a fictitious perturbation Δ_p satisfying $\bar\sigma(\Delta_p) \leqslant 1$ (Fig. 2(*b*)). Examination of robust performance means to ensure that (5) holds in the presence of any permissible perturbation Δ_u. If the two perturbation blocks Δ_u and Δ_p are combined into a single block diagonal perturbation Δ

$$\Delta = \text{diag}\{\Delta_u, \Delta_p\} \tag{6}$$

then the following theorem follows immediately.

Robust performance theorem (Doyle 1982): *Assume the nominal system M is stable, the number of unstable poles of the uncertain plant does not depend on Δ_u, and all perturbation blocks Δ_i in Δ_u satisfy $\bar\sigma(\Delta_i) \leqslant 1$. Then the closed loop system in Fig. 2(a) satisfies the H_∞-performance condition $\|F_u(P, \Delta_u)\|_\infty < 1$ for all such perturbations Δ_u if and only if*

$$\mu(P, \Delta) < 1 \quad \forall\omega \tag{7}$$

The simultaneous satisfaction of robust stability and robust performance can thus be expressed as a single condition in terms of μ. From the above it is clear that robust performance implies robust stability.

The formal definition of μ is, in general, not very useful for its computation. There are, however, algorithms that allow for computation of upper and lower bounds on μ, which are usually tight for practical relevant cases (Doyle 1982). Of course, in order to test for conditions (4) and (7) the upper bound of μ has to be used. For general Δ-structures, including mixed complex, real and

repeated blocks, there has been considerable progress in computing bounds for μ during the past years (Young *et al.* 1991, Fan and Tits 1991). Computation for general structures is possible for most problems (Balas *et al.* 1991). However, computational costs increase significantly with the size of the problem and upper and lower bounds for general structures do not remain tight.

The two theorems stated above allow the analysis of the closed loop behaviour of linear systems having a given controller with respect to the effects of structured uncertainty on stability and performance (μ-analysis). The synthesis of controllers that guarantee robust stability and robust performance in the presence of structured uncertainty (μ-synthesis) is, however, a considerably more difficult question. To date, only the so-called D-K-iteration (Doyle and Chu 1985) and the μ-K-iteration (Lin *et al.* 1993) are known to solve the μ-synthesis problem for a limited class of uncertainty descriptions. The D-K-iteration is restricted to Δ-structures with pure complex and non-repeated blocks. The synthesis of μ-optimal controllers for general structures remains an open problem. In § 2.3 it is shown that the control problem considered in this paper is not suitable for standard D-K-iteration. An alternative approach to the synthesis of μ-suboptimal controllers for this case is shown.

2.2. *Problem formulation*

The purpose of this section is to set up the control problem in the μ-framework. We wish to analyse and synthesize controllers for a class of systems such that not only robust stability but also robust performance in the presence of model uncertainties are achieved. The systems considered are characterized by two features: first, they may include analytic nonlinearities and secondly, they may contain parametric uncertainties and possibly unstructured uncertainties like high frequency dynamics. Typical examples for such systems are chemical reaction processes. Models for systems of this kind are typically derived by 'modelling', as opposed to identification, i.e. they are derived by first principles or balance equations. Thus, the resulting models are nonlinear state-space descriptions containing physical parameters p, e.g. geometric quantities, kinetic constants, or enthalpies:

$$\dot{X} = f(X, U, Z, p) \tag{8}$$

$$Y = h(X, U, Z, p) \tag{9}$$

Linearization of the nonlinear equations about the main operating point leads to a linear MIMO-model in state-space:

$$\dot{x} = Ax + Bu + Ez \tag{10}$$

$$y = Cx + Du + Fz \tag{11}$$

As mentioned, the models depend on physical parameters **p** and also on stationary values of the states X_s, inputs U_s and disturbances Z_s. The elements of the state-space matrices A, B, C, D, E and F are consequently (known) functions of X_s, U_s, Z_s and **p**. Thus, for example, an element of matrix A is

$$a_{ij} = a_{ij}(X_s, U_s, Z_s, p) \tag{12}$$

Values of physical parameters are often 'uncertain', i.e. their exact value is unknown. However, they are assumed to be time-invariant and to lie within a

given range of values. For example, chemical constants determined by experiments may often contain significant inaccuracies.

In classical linear examination of robustness (Doyle and Stein 1981) the uncertainties from all sources of uncertainty are normally lumped together in one (unstructured) uncertainty. This introduces conservatism because the set of possible plants is potentially significantly enlarged.

In order to avoid unnecessary conservatism, a mathematical description of the parametric uncertainty, that retains the highly structured nature of the uncertainties, is shown below. This leads to a special M-Δ-structure, that allows analysis and design of controllers in the μ-framework.

The parameter uncertainties are taken into account in the following straightforward way. The basic idea is to write out the parameter dependence of the elements of the system matrices. The key point is that the functional relations of the state-space description, in which the uncertain parameters p appear, are retained. Assume for the time being that the parameters appear only as polynomials

$$a_{ij} = a_{ij}(p_1, p_1^2, \ldots, p_2, p_2^2, \ldots, p_1 p_2, \ldots) \tag{13}$$

Preserving the functional relations is different to just calculating bounds for each element of the state-space matrices, e.g.

$$\underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij} \tag{14}$$

For each uncertain parameter p_i there is a nominal value $\tilde{p}_i$ and an uncertainty range Δp_i, meaning that the actual value of the parameter lies somewhere in the interval $[\tilde{p}_i - \Delta p_i, \tilde{p}_i + \Delta p_i]$. This uncertainty is modelled by a scalar perturbation δ_i with $|\delta_i| \leq 1$, $\delta_i \in \mathbb{R}$ and a 'weight' Δp_i that is introduced to allow normalization of δ_i to unity:

$$p_i = \tilde{p}_i + \Delta p_i \delta_i, \quad |\delta_i| \leq 1, \quad \delta_i \in \mathbb{R} \tag{15}$$

By replacing each parameter p_i by its uncertainty description, this uncertainty model is introduced into the model equations. Using only algebraic operations, it is possible to collect all nominal values in one term for each state-space matrix element and to collect the uncertainties in additional terms. The M-Δ-structure can then be computed by separating the terms containing nominal values and uncertainty weights (leading to the M-matrix) and uncertainties (appearing in the Δ_u-matrix). Every term containing $\Delta p_i \delta_i$ gives rise to an extra entry in the perturbation matrix Δ_u. Hence, this procedure leads to repeated, real, scalar uncertainties in the Δ_u-matrix, as an uncertain parameter appears typically several times in the model equations. Nonlinear dependencies of the parameters also lead to repeated entries in the matrix Δ_u. Grouping perturbations stemming from the same source leads to a block structure where each block contains the same scalar perturbation δ_i repeatedly.

To clarify this concept, consider for instance the following simple SISO-example with one scalar state x.

Example:

$$\dot{x} = ax + bu \quad \text{with } a = \tfrac{1}{2}p^2, \quad b = 2 + p$$

$$y = cx \quad \text{with } c = 1$$

The value of the physical parameter p is merely known to lie in an interval $[\tilde{p} - \Delta p, \tilde{p} + \Delta p]$ around its nominal value $\tilde{p}$. The plant with uncertainty $p = \tilde{p} + \Delta p \delta_p$ is described as

$$\dot{x} = [\tfrac{1}{2}\tilde{p}^2]x + [2 + \tilde{p}]u + [\tilde{p}\Delta p]\delta_p x + [\tfrac{1}{2}(\Delta p)^2]\delta_p^2 x + [\Delta p]\delta_p u, \quad |\delta_p| \leqslant 1, \delta_p \in \mathbb{R}$$

$$y = x$$

where δ_p is the normalized perturbation and Δp the uncertainty 'weight'. 'Pulling out' the perturbations δ_p leads to the block diagram shown in Fig. 3(a). This results in an M-Δ-structure with a diagonal Δ_u-matrix, containing a four times repeated, real, scalar perturbation δ_p. Two of the four repeated perturbations stem from the quadratic term δ_p^2.   □

This representation of parametric uncertainties always leads to the same structure of the perturbation matrix Δ_u with several diagonal blocks $\delta_i I$

$$\Delta_u = \mathrm{diag}\{\delta_1 I, \delta_2 I, \ldots, \delta_m I\}, \quad \text{with } \delta_i \in \mathbb{R} \tag{16}$$

Figure 3. (a) Block diagram of uncertainty description of example system; (b) block-structure for robust performance analysis.

As long as uncertain parameters p_i appear in (13) only in polynomials, or general LFTs, the rearrangement into the M-Δ-structure is achieved through pure matrix algebra. Other functions however, like $\sin(p_i)$, cannot be described exactly, but can be approximated by a Taylor-series (to arbitrary precision on the parameter interval). Thus, any analytic nonlinear appearance of the parameters p in (13) can be considered up to arbitrary accuracy.

In addition to the parametric uncertainties, the nonlinear nature of system (8)–(9) can also be considered in the robustness specifications in the following way. The idea is to include not only the linearization of the (uncertain) plant around the main operating point in the set of all possible plant models $\mathscr{G}$, but also the linearizations around *any possible* equilibrium point in the operating region of interest. We desire to achieve robust stability and robust performance for all linear plants in $\mathscr{G}$. For a point (X_s, U_s) to be an equilibrium point the condition

$$0 = f_s(X_s, U_s, Z_s, p) \tag{17}$$

has to hold. The set of all equilibrium points forms, in general, a q-dimensional manifold, called the equilibrium manifold (Hauser 1991), where q is dependent on the dimensions of X, U, Z and p. The set of models that consists of linearizations around all equilibrium points is called the linearization family of (8)–(9).

When the operating point of (8)–(9) is changed (e.g. due to a set point change), the linearized dynamics around the new operating point are contained in this linearization family. The true nonlinear behaviour is then approximated by (varying) representatives of the linearization family. If the domain of operation of the system is near to the equilibrium manifold, and the behaviour of the system in the neighbourhood of the equilibrium points is accurately described by the linearization, and the transitions are not too fast, then we can expect that the approximation is of sufficient accuracy. However, there is no stability or performance guarantee for the nonlinear system (8)–(9), but only for a set of linear systems 'covering' the nonlinear system in the described way.

The Extended Linearization approach (Wang and Rugh 1987 a, 1987 b) and the Pseudo-Linearization approach (Champetier *et al.* 1984) to nonlinear control are based upon a similar idea. There, however, a nonlinear controller is designed that causes the controlled plant to have the same linearized dynamics on the whole equilibrium manifold. The positive experiences with those non-linear design techniques give rise to the assumption that, in general, linearization families constitute good approximations for weakly nonlinear systems.

The linearization family is incorporated into the uncertainty description by the following scheme. In (12) the elements of the state-space matrices depend only on fixed values of X_s, U_s and Z_s at the main operating point. Now we do not assume X_s, U_s and Z_s to be fixed anymore, but allow them to take on arbitrary values on the equilibrium manifold, bounded only by the operating region. Thus, for an element of A:

$$a_{ij} = a_{ij}(X, U, Z, p) \tag{18}$$

where X, U, Z, and p have to satisfy the equilibrium constraint

$$0 = f(X, U, Z, p) \tag{19}$$

As can immediately be seen, there are as many independent additional 'sources' of uncertainties, stemming from the linearization family, as there are states, inputs, disturbances and uncertain parameters minus the number of equilibrium-constraints in (19). In order to calculate the M-Δ-structure, the equilibrium-constraints (19) have to be incorporated into the functional dependency of the elements of the matrices A, B, C, D, E and F. In general, this calculation requires that the equilibrium manifold (19) can be parametrized. For practical applications this often proves to be very difficult and thus restricts the practical applicability. To exemplify the principal idea, assume that it is possible to solve (19) for X

$$X = \bar{f}(U, Z, p) \tag{20}$$

Replacing X in (18) by (20) yields the following dependencies of the matrix elements:

$$a_{ij} = \bar{a}_{ij}(U, Z, p) \tag{21}$$

Now the elements of the state-space matrices depend only on U, Z and p but no longer on X, with bounds given for the parameters as in (15) and for the input and disturbance coordinates U_i, Z_i according to the desired operating region

$$U_i = U_{is} + \Delta U_i \delta_{U_i}, \qquad |\delta_{U_i}| \leq 1, \, \delta_{U_i} \in \mathbb{R} \tag{22}$$

$$Z_i = Z_{is} + \Delta Z_i \delta_{Z_i}, \qquad |\delta_{Z_i}| \leq 1, \, \delta_{Z_i} \in \mathbb{R} \tag{23}$$

The resulting Δ_u-matrix has as many additional blocks with repeated real scalars as there are input and disturbance coordinates. If (19) cannot be parametrized exactly, it is often possible to find adequate approximations in the operating region considered. This is demonstrated in §3.2 for the reactor application.

The interesting feature of this consideration of nonlinearities is that it leads to the same structural representation of the uncertainties as pure parametric uncertainties. Thus, nonlinearities can be treated without too much additional effort in exactly the same way in the controller design step.

In addition to the parametric uncertainty, there are potentially other sources of model uncertainty. For instance, in many applications there is usually little knowledge about the dynamics of the system at high frequencies. Such additional non-parametric uncertainties can be modelled by a further, unstructured uncertainty block, Δ_m, e.g. as multiplicative output uncertainty (Morari and Zafiriou 1989). This leads to a perturbation matrix Δ_u as follows

$$\Delta_u = \text{diag}\{\delta_1 I, \delta_2 I, \ldots, \delta_{m-1} I, \Delta_m\}, \quad \text{with} \quad \delta_i \in \mathbb{R}, \Delta_m \in \mathbb{C}^{r \times r} \tag{24}$$

Up to now, the part of the M-Δ-structure that is related to Δ_u has been treated. The remaining part of the M-Δ-structure to be specified is the performance specification related to Δ_p (see also Fig. 3(b)). The performance is specified by the H_∞-Norm of some problem dependent closed-loop transfer matrix. For instance, this can be the weighted sensitivity matrix S (Morari and Zafiriou 1989)

$$\|W_P S\|_\infty < 1 \tag{25}$$

The overall structure is depicted in Fig. 3(b) and contains the closed loop,

including controller $K(s)$, the plant $G'(s)$ with uncertainty description and the performance specification $W_P(s)$.

2.3. Solution of the control problem in the μ-framework

In § 2.2 the M-Δ-structure needed for μ-analysis and μ-synthesis was specified. If the controller $K(s)$ is given then the method explained above allows the analysis of control loops for robust performance, using the robust performance theorem from § 2.1. It is, however, not possible to perform a μ-synthesis for M-Δ-structures containing repeated, real uncertainties, as in (24). The synthesis of μ-suboptimal controllers is performed by way of D-K-iteration (Doyle 1985, Doyle and Chu 1985), where μ-analysis, leading to so-called D-matrices, and H_∞-synthesis, making use of the D-matrices and leading to controllers K, alternate with each other. μ-analysis for systems with repeated, real uncertainties does not lead to D-matrices that are suited for H_∞-synthesis. Consequently, D-K-iteration is not possible for the problem at hand.

D-K-iteration is possible, if the set of permissible perturbations in Δ_u is enlarged in a certain way. Each real, repeated scalar $\delta_i I_{k \times k}$ in (24) is replaced by a diagonal block Δ_i consisting of independent complex scalars (Braatz and Morari 1992)
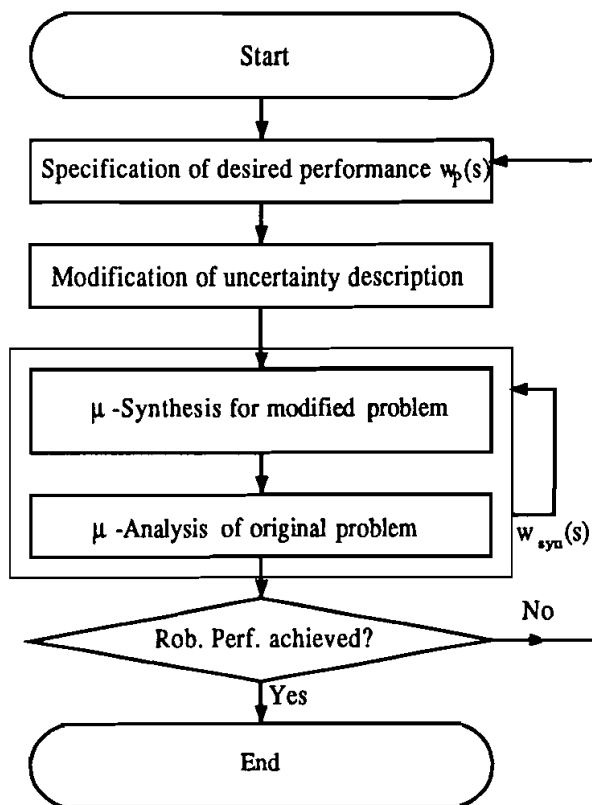
$$\Delta_i = \text{diag}\{\delta_{i1}, \delta_{i2}, \ldots, \delta_{ik}\}, \quad \text{with } \delta_{ij} \in \mathbb{C} \tag{26}$$

The D-K-iteration is performed with the modified perturbations and is followed by a μ-analysis for the original problem. As described in § 2.1, μ-analysis for systems with repeated, real perturbations Δ_u as in (24) is indeed possible. Because the modified perturbations introduce potentially significant conservatism and thus make the design task more difficult, it is quite possible that a controller that achieves robust performance cannot be computed in a single step. However, a robustly performing controller might be synthesized iteratively. It is suggested that not only modified perturbations during the synthesis but also modified weights W_{syn} be used in order to 'assist' the algorithm to compute a satisfactory controller. Figure 4 shows a flowchart of this approach to μ-synthesis. After the D-K-iteration has converged, the subsequent μ-analysis for the original problem with the original weights and perturbations determines exactly whether a satisfactory controller has been achieved. If it has not, the μ-plots give hints about the necessary modifications to the intermediary weights W_{syn} for the next synthesis. For instance, if the μ-values of the original problem and the modified problem differ in certain frequency ranges, the weights should be modified there such as to compensate for it. Braatz and Morari (1992) also suggest how heuristics may be used to modify the weights to achieve a desired effect on the controller.

It requires the intuition of the designing engineer to find, for each specific problem at hand, suitable weights. This *ad hoc* design procedure is not guaranteed to succeed and must be judged as an attempt to solve an otherwise intractable problem.

3. Application of control design method to a CSTR

In this section we show how the approach outlined in § 2 can be used to design a robustly performing controller for a realistic process control problem.

Figure 4.   Diagram of proposed iterative μ-synthesis.

3.1. *Description of control problem*

The plant considered is a continuous stirred-tank reactor (CSTR) in which cyclopentenol is produced from cyclopentadiene by acid-catalysed electrophylic hydration in aqueous solution. The initial reactant cyclopentadiene (substance A) also reacts in an unwanted parallel reaction to the by-product dicyclopentadiene (D). Furthermore, cyclopentanediol (C) is formed in an unwanted consecutive reaction from the product cyclopentenol (B). This is the so-called *Van de Vusse* reaction scheme

$$\left. \begin{array}{l} A \xrightarrow{k_1} B \xrightarrow{k_2} C \\ 2A \xrightarrow{k_3} D \end{array} \right\} \tag{27}$$

that was first studied in the context of control by Kantor (1986) and has been treated by many authors since then (Doyle III *et al.* 1992). The reaction rates are k_1, k_2 and k_3.

Only the reactant cyclopentadiene is fed into the reactor with concentration c_{A0}. The flow rate $\dot{V}$ and the volume specific heat input $\dot{q}_H$ are considered as manipulated variables. The concentration c_B of product B and the temperature ϑ can be measured directly. Figure 5 shows a schematic diagram of the system.

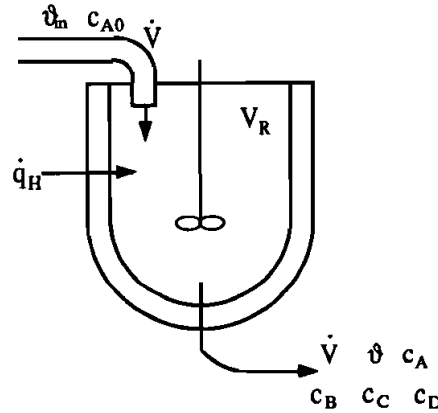The control objective is to maintain c_B at a fixed value despite variations in

Figure 5. Schematic illustration of the CSTR.

the feed concentration c_{A0}, by up to ± 12%, which are regarded as disturbances. Moreover, the controller should track set-point step changes in the product concentration c_B by up to −23% and +6%. The temperature ϑ is also controlled for reasons of safety and for transparent operation of the reactor.

The dynamics of the system are described by the following nonlinear equations:

$$\dot{c}_A = \frac{\dot{V}}{V_R}(c_{A0} - c_A) - k_1(\vartheta)c_A - k_3(\vartheta)c_A^2, \quad c_A > 0 \tag{28}$$

$$\dot{c}_B = -\frac{\dot{V}}{V_R}c_B + k_1(\vartheta)c_A - k_2(\vartheta)c_B, \quad c_B > 0 \tag{29}$$

$$\dot{\vartheta} = \frac{\dot{V}}{V_R}(\vartheta_{in} - \vartheta) - \frac{1}{c_p}(k_1(\vartheta)c_A\Delta H_{R,AB} + k_2(\vartheta)c_B\Delta H_{R,BC}$$

$$+ k_3(\vartheta)c_A^2\Delta H_{R,AD}) + \frac{\dot{q}_H}{c_p} \tag{30}$$

where the reaction rates are assumed to depend on the temperature via the Arrhenius law

$$k_i(\vartheta) = k_{i0}\exp\left(-\frac{E_i}{R(\vartheta + 273\cdot15\text{ K})}\right), \quad i = 1, 2, 3 \tag{31}$$

The collision factors k_{i0}, reaction enthalpies $\Delta H_{R,i}$ and volume specific heat capacity c_p (product of density and heat capacity) are known only within bounds, with a relative uncertainty ranging from 1·3% for c_p to 56% for $\Delta H_{R,AB}$. The activation energies E_i, gas constant R and the reactor volume V_R are known with good precision. Therefore, seven parameters are uncertain.

The manipulated variable $\dot{V}$ is confined by physical constraints and can only be decreased by 73% and increased by 86% of its nominal value. The heat input $\dot{q}_H$ is not constrained.

This problem was proposed by Klatt and Engell (1992) as a benchmark problem for controller design.

3.2. Design of a μ-suboptimal controller for the CSTR

The CSTR introduced in § 3.1 is modelled by a third-order system of nonlinear differential equations. Parametric uncertainties are brought in by seven uncertain chemophysical constants. The reactor is only weakly nonlinear in the operating region considered. The system exhibits one zero in the right half-plane of which the location varies significantly with the operating point. This results in limitations on the achievable performance (Freudenberg and Looze 1985).

The robust stability and robust performance theorems, as formulated in § 2.1, are only applicable to systems where the number of unstable poles is the same for all members of the set of possible plant models. It can be shown by physical reasoning that the CSTR is statically stable for any permissible value of the uncertain parameters (Wahl 1992).

This control example is meant to show that it is possible to solve a rather realistic process control problem with the present state of μ-methodology.

The μ-framework requires a linear model plus uncertainty description that can be rearranged into an M-Δ-structure as in Fig. 3(b). The nonlinear equations (28)–(30) can easily be linearized, using the nominal values of the parameters and the stationary values of the states $X_s = [c_{As}, c_{Bs}, \vartheta_s]$, inputs $U_s = [\dot{V}_s, \dot{q}_{Hs}]$ and disturbance $Z_s = c_{A0s}$ at the main operating point.

Regarding the uncertainty description, there are 13 uncertain parameters in the equations, stemming from uncertain physical constants ($k_{i0}, \Delta H_{R,i}$ and c_p) and from uncertain 'stationary values' (X_s, U_s and Z_s). It should be mentioned that the reaction velocities k_i comprise two different uncertainty sources: from the collision factors Δk_{i0} and from the dependence on the temperature ϑ. For the latter, the Taylor-series up to the first term is used as a (sufficiently good) approximation for the exponential function $e^{-E/(R(\vartheta+273\cdot15))}$.

The construction of the M-Δ-structure is split into two parts. The first step is to calculate the plant G' including uncertainty descriptions (corresponding to Fig. 3(a)). In the second step, the overall M-Δ-structure, including the closed-loop (with controller and performance weight), is constructed (see Fig. (3(b)).

For the first step, the uncertainty description for the uncertain parameters and stationary values, as explained in § 2.2, is used. Each uncertain parameter is replaced by its uncertainty model. This is illustrated with c_p:

$$c_p = \tilde{c}_p + \Delta c_p \delta_{c_p}, \quad |\delta_{c_p}| \leq 1, \quad \delta_{c_p} \in \mathbb{R} \tag{32}$$

The nonlinearities are also taken into account as outlined in § 2.2. An explicit parametrization of the equilibrium manifold by a function of the form (20) is not possible for the reactor at hand. A sufficiently accurate approximation to (20) can be found by approximating the exponential functions in (31), yielding an M-Δ-structure of very high complexity (perturbation matrix Δ_u with ten blocks of repeated real scalars where each block $\delta_i I$ is unfeasibly big). To restrict the complexity, a larger linearization family is found by linearizing around all points in the operating region, i.e. not only equilibrium points. Thus, the number of repeated real scalar blocks $\delta_i I$ in (24) increases to 13. However, the size of each block is considerably reduced, giving in total a μ-problem of substantially lower complexity.

By not restricting the linearization family to the equilibrium manifold,

significant conservatism is potentially introduced. For the design undertaken here, this can be justified by the benefit of the lower complexity and the satisfactory robust performance achieved with the final controller despite this conservatism. In a more serious design there is, of course, scope for improvement at this point through a (partial) consideration of (20).

To show the complexity and resultant limitations of this method of uncertainty description the differential equation for $\dot{c}_A$ is shown (the index s denotes the nominal stationary value):

$$
\begin{aligned}
\frac{dc_A}{dt} &= \left[-\frac{\dot{V}}{V_R}\bigg|_s - k_{1s} - 2k_{3s}c_{As} \right] c_A + \left[-c_{As}\frac{\partial k_1}{\partial \vartheta}\bigg|_s - c_{As}^2\frac{\partial k_3}{\partial \vartheta}\bigg|_s \right] \vartheta \\
&\quad + [c_{A0s} - c_{As}]\frac{\dot{V}}{V_R} + \left[\frac{\dot{V}}{V_R}\bigg|_s \right] c_{A0} \\
&\quad + \left[-\Delta\frac{\dot{V}}{V_R}\delta_{\dot{V}/V_R} - \frac{\partial k_1}{\partial \vartheta}\bigg|_s \Delta\vartheta\delta_\vartheta - 2k_{3s}\Delta c_A\delta_{c_A} \right] c_A \\
&\quad + \left[-\frac{\partial k_1}{\partial \vartheta}\bigg|_s \Delta c_A\delta_{c_A} - \frac{\partial k_3}{\partial \vartheta}\bigg|_s (2c_{As} + \Delta c_A\delta_{c_A})\Delta c_A\delta_{c_A} \right] \vartheta \\
&\quad + [\Delta c_{A0}\delta_{c_{A0}} - \Delta c_A\delta_{c_A}]\frac{\dot{V}}{V_R} + \left[-\Delta\frac{\dot{V}}{V_R}\delta_{\dot{V}/V_R} \right] c_{A0}
\end{aligned}
\tag{33}
$$

The first two lines contain only nominal values and correspond to the lower set of inputs and outputs of the plant description in Fig. 3(*b*). The last three lines comprise the uncertainties and correspond to the upper connections of G'.

Calculation of G' does not have to be performed by hand. Numerical computation is straightforward and is included in commercially available software packages like the Matlab-Toolbox 'μ-Tools' (Balas *et al.* 1991).

The perturbation matrix Δ_u contains 13 Δ_i-blocks, each consisting of repeated, real scalars, with size ranging from 1×1 to 11×11. This is still too large for a synthesis with reasonable computing effort. Hence, another simplification is necessary. For this, only those uncertainties are retained which have a strong effect on the I/O-behaviour of the plant. It is a rather involved optimization problem to select those uncertainties. Eight uncertainties, namely Δc_A, $\Delta\vartheta$, $\Delta\dot{V}/V_R$, Δc_{A0}, $k_1(\Delta\vartheta)$, $k_3(\Delta\vartheta)$, $\Delta\Delta H_{R,AD}$ and $\Delta\Delta H_{R,BC}$ are selected on the basis of this analysis, using an optimization tool. The final perturbation matrix Δ_u consists of eight blocks with 19 perturbations altogether. In (33), only those uncertainties are shown that remained after the selection of the most important uncertainty perturbations. Four uncertainties from the eight chosen appear in this equation.

The model structure was assumed to be correct. Consequently, no uncertain high frequency dynamics are presumed. The perturbation matrix Δ_u is of the form shown in (16).

The second step is to construct the overall M-Δ-structure (see Fig. 3(*b*)). This can also be done numerically.

Following the scheme in Fig. 4, a μ-synthesis is undertaken. The performance objective chosen is expressed in terms of the weighted output sensitivity function

$$
\|W_P S\|_\infty < 1
\tag{34}
$$

The desired performance is described by $W_P = w_{P1}I$:

$$w_{P1}(s) = \frac{s + 10}{2(s + 0\cdot1)} \tag{35}$$

The parameters of w_{P1} have the following simple interpretations.

(a) The zero of w_{P1} is related to the loop bandwidth.

(b) $1/|w_{P1}(0)|$ is the maximum steady-state tracking error for step changes.

(c) $\lim_{\omega\to\infty}|1/w_{P1}(j\omega)|$ is the maximum peak value of $\bar{\sigma}(S)$.

The weight $w_{P1}(s)$ is also chosen as the first synthesis weight $W_{syn}(s)$ and a μ-synthesis step is performed with it. As explained in § 2.2, this step consists of a D-K-iteration for the modified problem, using a Δ_u with 19 independent, complex, scalar perturbations. It is followed by a μ-analysis of the original problem, making use of a Δ_u with the eight blocks of repeated, real, scalar perturbations. The controller found does not achieve a μ-value of less than one. In particular, nominal performance is not nearly achieved. This is evident from the singular values of the sensitivity function. To stress the performance over robust stability during synthesis, a more demanding performance weight for the synthesis is chosen. This weight is specified using information from the μ-analysis above. Namely, the μ-analysis of the original problem suggests a stronger weighting of the tracking error and an increase in the bandwidth during synthesis because the μ-value of the original problem is much lower than μ of the modified problem. Thus, the maximally allowed tracking error is lowered from 0·02 to 0·005 and the loop bandwidth is increased by a factor of two. With this synthesis weight, the D-K-iteration converges after six steps to a controller that fails only marginally to meet the μ-test for the robust performance objective w_{P1}. In order to give an impression of the necessary computing effort, these six iteration steps need several hours on a VAX station 3100-M76. The μ-synthesis is not repeated but instead a slightly modified performance objective

$$w_{P2}(s) = \frac{s + 5}{5(s + 0\cdot05)} \tag{36}$$

is chosen so that robust performance with the μ-controller obtained above can be guaranteed. Figure 6(a) shows the graph of μ (upper bound only) for robust performance, robust stability and nominal performance for this performance objective. It can be seen that the condition for robust performance $(\mu(P, \Delta) < 1$ $\forall\omega)$ is satisfied, since the peak value of μ is 0·997 (at $\omega = 0$). This guarantees robust performance not worse than $1/|w_{P2}(j\omega)|$ not only for the nominal plant but also for the perturbed plant.

We have thus solved the posed problem and designed a controller that achieves robust performance for the system with parametric uncertainties, by using a modified M-Δ-structure for D-K-iteration. No robust performance *guarantee* can be given for the nonlinear plant, but due to the incorporation of nonlinear effects into the uncertainty description, the results are not limited to the neighbourhood of *one* operating point, but at least to the neighbourhood of *arbitrary* operating points.

Since standard μ-theory does not allow us to treat the posed controller design problem, a number of simplifications and approximations had to be made during problem formulation and synthesis.
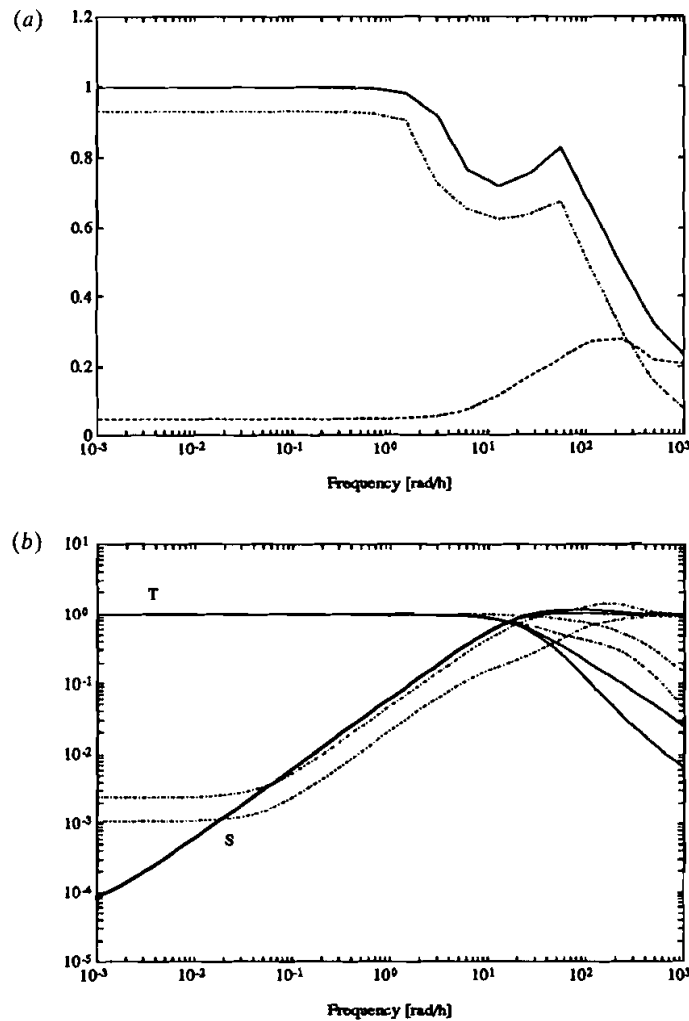
Figure 6. (a) μ for robust performance (solid curve), robust stability (dash-dotted) and nominal performance (dashed); (b) singular values of S and T for the μ-controller (dashed) and the H_∞-controller (solid).

An H_∞-controller, designed by loop-shaping with an (S, T, KS)-criterion is compared with the μ-controller. It was not attempted to achieve robustness by embedding the structured uncertainties into a specific unstructured uncertainty description. Instead, the bandwidth of the closed loop is limited in order to achieve a certain robustness for general multiplicative output uncertainty. Explicit use of the information about the uncertainties is not feasible because transforming the parametric uncertainties into a single unstructured uncertainty would lead to a highly conservative robustness condition and an extremely slow controller. The singular values of the sensitivity S and complementary sensitivity T of the H_∞-controller are shown in Fig. 6(b), together with the respective singular values of the μ-controller. The loop shapes of the H_∞-controller suggest superior dynamic behaviour compared with the μ-controller. However, σ(S) and

$\sigma(T)$ only describe nominal performance and do not give any information about robust performance. The crucial test is a μ-analysis that can also be performed for this controller with the uncertainty description formulated. The H_∞-controller has a peak value of $\mu = 1\cdot106 > 1$ (see Fig. 7). Hence, the H_∞-controller does not satisfy the robust performance objective, but is close to it. Here, the loop-shaping design of the H_∞-controller was achieved with a knowledge of the μ-controller. This information was implicitly used in the design of the H_∞-controller. There is, of course, an H_∞-controller of some order that achieves the same performance/robustness as the μ-controller. However, a great deal of effort is needed to find such an H_∞-controller compared with the effort needed for the μ-scheme proposed. This specific controller is nevertheless compared with the μ-controller to show the effects of a μ-value of greater than one.

The performance objective was posed in the frequency domain. Figures 8 to 12 show time-domain simulations of the nonlinear plant (28)–(30) with the μ- and the H_∞-controller. Figures 8 and 11 show typical step responses of c_B for different simultaneous disturbance and set-point changes. Figure 9 shows a typical response for ϑ, while Figs 10 and 12 show the necessary actions of the manipulated variables $\dot{V}$ and $\dot{q}_H$. In the former figures, the full line is the step response for the nominal plant while the other lines are responses for plants with physical parameters that are varied within their uncertainty limits. The parameters are changed in such a way that the plant strongly alters its I/O-behaviour. While the H_∞-controller is faster than the μ-controller in the nominal case, it exhibits a longer settling time in the other cases, as was expected by the μ-analysis. The behaviour of the H_∞-controlled system depends on the actual value of the uncertain parameters. The μ-controller on the other hand does not degrade its performance significantly despite perturbed parameters. It generally shows a well-damped, smooth transient response over the full range of operating conditions.
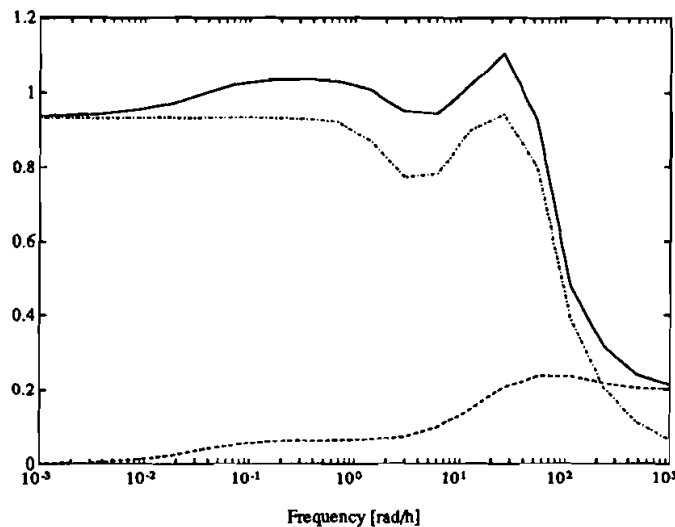

Figure 7. μ for robust performance (solid curve), robust stability (dash-dotted) and nominal performance (dashed) for the H_∞-controller.
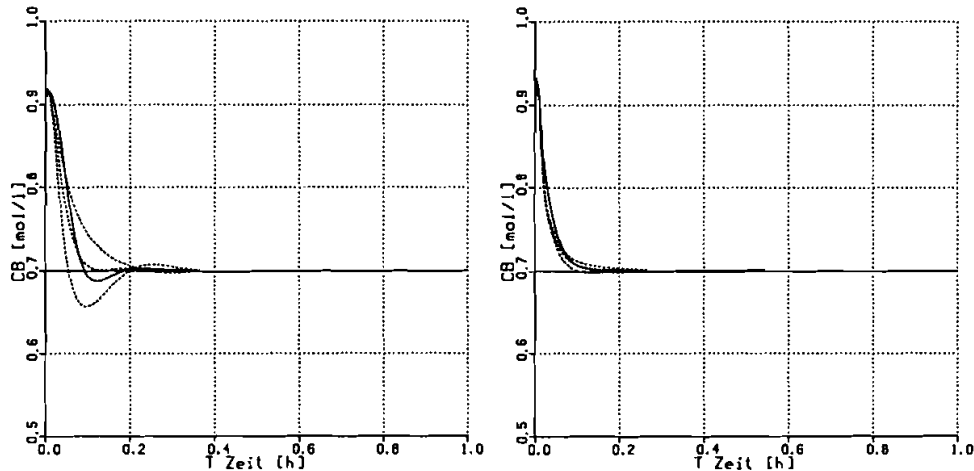
Figure 8. Step responses of $c_B(t)$ for H_∞-controller (left) and μ-controller (right). The disturbance c_{A0} is changed by $+12\%$. Simultaneously the set point $c_{B,ref}$ is changed by -23%.
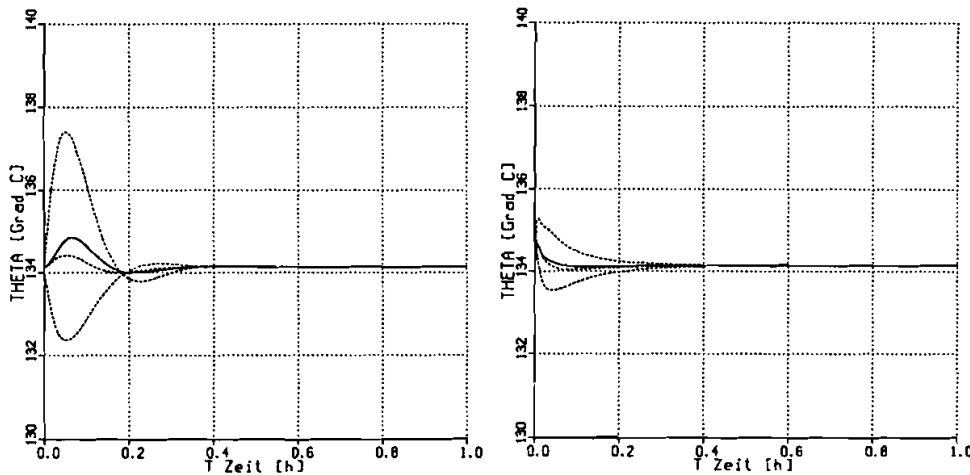

Figure 9. Step responses of $\vartheta(t)$ for H_∞-controller (left) and μ-controller (right). The disturbance/set-point changes are the same as in Fig. 8.

The nonlinear simulations confirm the results of the μ-analysis and show that the μ-controller achieves a truly robust performance.

The performance of the H_∞-controller and the μ-controller do not, however, differ much. Of course, one cannot expect the performance of the μ-controller to be as fast as that of a controller that is designed with main regard to the nominal case. But the big difference between the two controllers is that the μ-controller is guaranteed to keep its performance for all plants in the uncertainty description.

It is one advantage of the problem formulation of §2.2 that arbitrary controllers can be examined for robust performance. In the case of the H_∞-controlled plant this μ-analysis showed that robust performance is almost
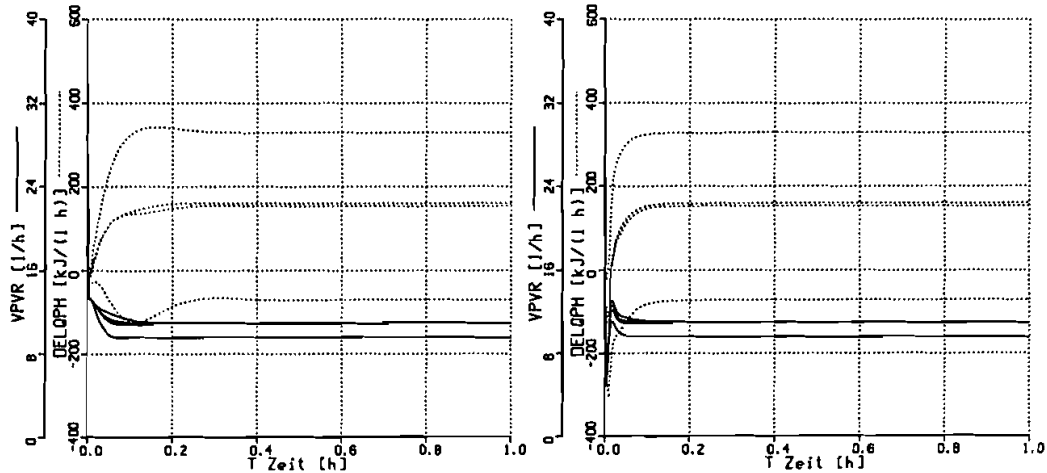
Figure 10. Step responses of $U_1(t)$ (VPVR) and $u_2(t)$ (DELQPH) for the H_∞-controller (left) and μ-controller (right). The disturbance/set-point changes are the same as in Fig. 8.
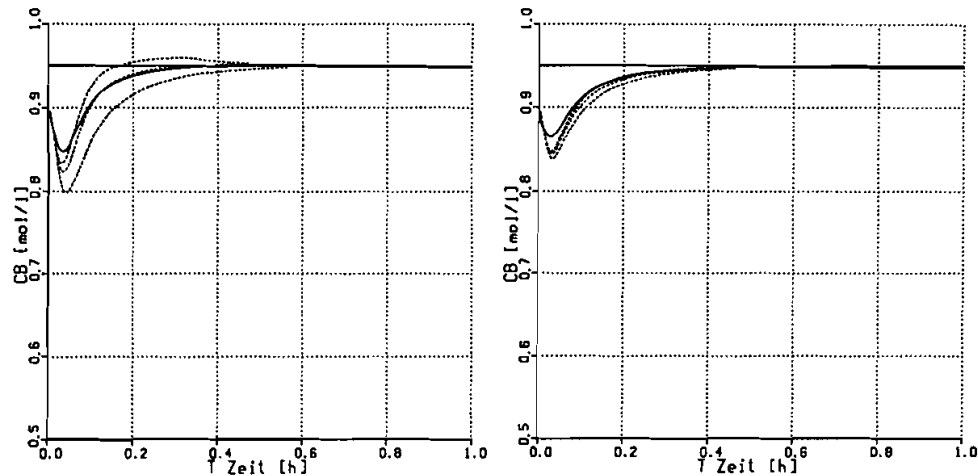

Figure 11. Step responses of $c_B(t)$ for H_∞-controller (left) and μ-controller (right). The disturbance c_{A0} is changed by -12%. Simultaneously the set point $c_{B,ref}$ is changed by $+6\%$.

achieved. It is, however, not easy to direct H_∞-controller design specifically towards achieving robust performance. With the iterative μ-synthesis method proposed this is, however, possible.

Clearly, in this work, it was not intended to design the best possible controller for the CSTR at hand. The example is mainly meant to show the applicability of the method. For a more serious design one would try to limit the conservatism that is introduced by allowing the states and inputs to vary independently despite condition (19). Additionally, the disturbance model could be included in the μ-problem formulation. Also, more iterations in the iterative synthesis scheme would lead to improvements.
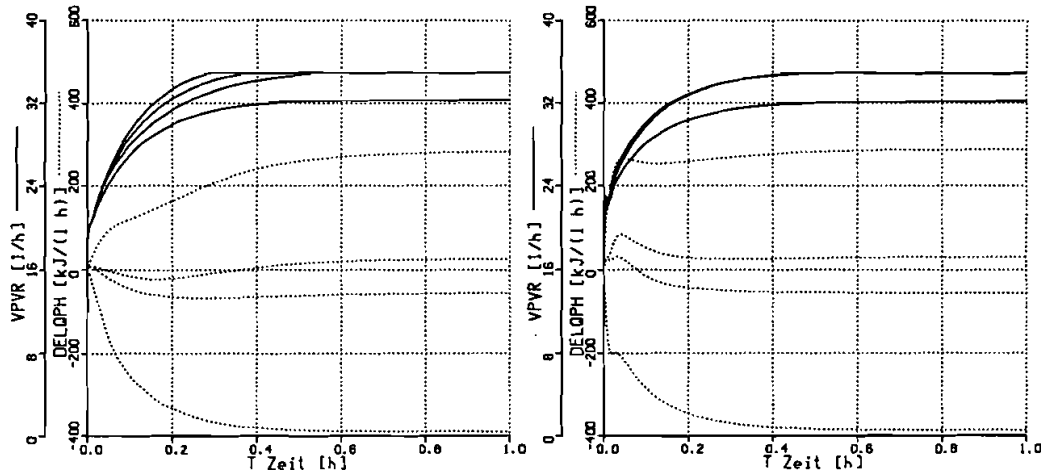
Figure 12. Step responses of $U_1(t)$ (VPVR) and $u_2(t)$ (DELQPH) for the H_∞-controller (left) and μ-controller (right). The disturbance/set-point changes are the same as in Fig. 11.

4. Conclusions

The design of robustly performing controllers for a class of practical control problems has been considered. This class appears naturally when the design is based on models obtained by linearizing nonlinear balance equations with uncertain physical parameters. Typical representatives for this class are chemical reactions.

It has been shown how parametric uncertainties and nonlinear effects can be incorporated in a non-conservative uncertainty description. The set-up is such that the problem can be treated in the structured singular value (μ)-framework. Although μ-analysis is possible, standard D-K-iteration for μ-synthesis cannot be applied. Instead a technique was shown that involves μ-analysis of the actual problem and D-K-iteration for a modified problem. By this technique, a μ-suboptimal controller can be found by an iterative procedure. It should, however, be stressed that there are cases for which the proposed scheme does not lead to satisfactory results. Experience shows, however, that satisfactory results are possible for a large class of practical problems.

The benefits and drawbacks of the method have been illustrated with a rather realistic reactor example. The main advantage of this approach is that not only stability but also satisfactory performance can be guaranteed for the perturbed plant. This has been confirmed by nonlinear simulations of the reactor. The use of μ and the proposed uncertainty description do not introduce any unnecessary conservatism and only therefore a satisfactory level of performance can be ensured for the real plant. Another advantage is that even if μ-synthesis is not feasible or not desired, the proposed μ-framework allows non-conservative analysis of the robust performance of controllers that have been found in another manner.

On the other hand, these advantages have to be paid for by a high complexity of the set-up. In particular, for inclusion of nonlinear effects in the uncertainty description the equilibrium manifold has to be parametrized and this

is often not possible. It can be circumvented in part by suitably chosen approximations. Robustness assertions on the basis of linearization families require a weakly nonlinear system, and also require that the system state is never too far from the equilibrium manifold and that the state transitions are not too fast. But even in these cases, no real guarantee can be given. Our own experience and the experiences reported in the nonlinear control literature suggests that for a large class of systems, linearization families are a good approximation for the nonlinear system and thus satisfactory results can be achieved. The substantial computational effort for μ-analysis and μ-synthesis has to be weighed against the advantage of decreased conservatism and achievement of robust performance. Also, any robust design is by nature a worst-case design. This can only be avoided by using adaptive, learning or nonlinear schemes.

The CSTR example has shown that a satisfactory robust controller design for chemical reactors is indeed possible.

ACKNOWLEDGMENT

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