BIFURCATION ANALYSIS OF DYNAMIC SYSTEMS WITH CONTINUOUSLY PIECEWISE LINEARITY

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The periodic response and associated bifurcations of a harmonically driven oscillator with continuously piecewise linearity has received great attention during the past decade. The oscillator serves as a widely used model to describe various mechanical systems with clearances, elastic constraints etc. Even if the 'corners' of the piecewise-linear restoring force in the oscillator make the system non-smooth on the interface between two linear regions, the bifurcation analysis published up to now is still based on the theory of smooth dynamic systems (e.g. Natsiavas 1990; Kleczka *et al.* 1992). The validity of such an extention is an open problem.

To avoid the difficulty caused by the non-smoothness in bifurcation analysis, it is very natural to replace the non-smooth restoring force by a smoothed one under certain approximation. The smoothed model proposed here is based on a locally defined polynomial in the neighbourhood of the restoring-force corner. It is proved that the polynomial can be tangential to both sides of the corner in any required order of smoothness and that the maximal error can be less than any desired quantity. One may expect, therefore, that the bifurcation analysis to the smoothed system give approximate results for the original system. The numerical simulations did support the expectation in most cases, but sometimes new phenomena were observed, which cannot be explained using the theory of smoothed dynamic systems.

In order to understand the non-smoothness effect of the restoring force on the system behaviour, the analytical properties of the Poincaré map for a continuously piecewise-linear system are studied in detail. According to Filippov (1988), it is proved that the second derivative of a Poincaré map suddenly jumps on the interface between two linear regions in state space and that the jumping amplitude becomes very large if the slope of the restoring force has a high jump or the trajectory passes through the interface at low velocity. By means of the Lyapunov–Schmidt procedure, the Poincaré map of a dissipative system near a fixed point can be reduced to a truncated one-dimensional normal form with a control parameter λ:

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$$p(y, \lambda) = p_y(0, 0)y + p_\lambda(0, 0)\lambda + \frac{1}{2}\begin{cases} p_{yy}(0^+, 0)y^2, & y \geq 0, \\ p_{yy}(0^-, 0)y^2, & y < 0 \end{cases} \tag{1}$$

Moreover, the normal form for the smoothed system can be approximately written as

$$q(y, \lambda) = q_y(0, 0)y + q_\lambda(0, 0)\lambda + \tfrac{1}{2} q_{yy}(0, 0)y^2 + \tfrac{1}{6} q_{yyy}(0, 0)y^3, \tag{2}$$

where

$$\left.\begin{aligned} q_y(0, 0) &= p_y(0, 0), \\ q_\lambda(0, 0) &= p_y(0, 0), \\ q_{yy}(0, 0) &= \tfrac{1}{2}(p_{yy}(0^+, 0) + p_{yy}(0^-, 0)), \\ q_{yyy}(0, 0) &= \frac{9}{8\delta}(p_{yy}(0^+, 0) - p_{yy}(0^-, 0)), \end{aligned}\right| \tag{3}$$

and δ is the radius of the smoothed range.

Now the simplest case is considered, where the normal form of the smoothed system yields the condition of saddle–node bifurcation, i.e.

$$q_y(0, 0) = 1, \quad q_\lambda(0, 0) \neq 0, \quad q_{yy}(0, 0) \neq 0. \tag{4}$$

From equation (1), one finds that the fixed point undergoes a bifurcation due to $p_y(0, 0) = 1$, but only if the inequality $p_{yy}(0^+, 0)p_{yy}(0^-, 0) > 0$ is valid does the bifurcation look like a 'saddle–node type', i.e. a pair of fixed points with opposite stabilities disappears suddenly through the bifurcation. In the case of, $p_{yy}(0^-, 0)p_{yy}(0, 0) < 0$, the bifurcation looks like a 'hysteresis type with co-dimension 1', changing the number of fixed points and their stabilities not at all. But the non-zero codimension of the bifurcation usually results in a jump of the bifurcation curve in practice. This is a local bifurcation caused by the non-smoothness of the restoring force and has not been reported before to the authors' knowledge.

To verify the theoretical results, numerical simulations were performed for a piecewise-linear softening oscillator under harmonic excitation. Usually the smoothed system provides a good approximation to the original system both in response calculation and in bifurcation analysis. But if the original system undergoes a 'hysteresis-type' bifurcation, the standard bifurcation analysis of the smoothed system fails to offer correct information. In fact, the term y^3 in the normal-form equation (2) takes an important role in such a case, as the coefficient $q_{yyy}(0, 0)$ is very large.

All in all, the bifurcation analysis based on the smoothed system can be applied to the continuously piecewise-linear system, but with great care.

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