27 STABILITY NUMBERS FOR NONLINEAR SYSTEMS

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The definition of Lyapunov stability is used for the introduction of stability numbers for nonlinear systems. A norm of the initial conditions $\| \boldsymbol{x}_0 \|$ and a norm of the system response $\| \boldsymbol{x}(t) \|$ are related to each other, resulting in stability numbers depending on the initial conditions. This concept presents some information on the global behaviour of the system including all types of solutions from limit cycles to strange attractors. The stability numbers characterize that part of the state space in which the motion occurs.

27.1 INTRODUCTION

The stability of motion in the sense of Lyapunov characterizes the qualitative behaviour of the equilibrium position of linear and nonlinear mechanical systems (Lyapunov 1907, 1966). The information on the stability of a system is the most fundamental one from both a theoretical and an engineering point of view. Unstable systems are not acceptable for engineering applications. Therefore, stability is a necessary requirement. For a well-designed engineering system the dynamical performance has to be evaluated, too. One approach is to provide quantitative stability information on the global dynamical behaviour of the system under consideration.

For linear systems the degree of stability or the degree of damping provides useful information on the dynamical behaviour of the system. These degrees are based on the eigenvalue distribution of the system considered (see e.g. Müller and Schiehlen 1985). Furthermore, the absolute-value criterion (absolute error – AE) is used to characterize the maximum amplitudes of some or all state variables. The absolute-value criterion depends on the initial conditions and the chosen state variables.

For nonlinear systems eigenvalues do not exist. Therefore, an extension of the degree of stability or the degree of damping is not possible. However, the absolute-value criterion may also be used for nonlinear systems. The stability numbers defined consider especially the influence of the initial conditions and include all state variables.

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The chapter is organized as follows. The equations of motion of nonlinear multi-body systems are represented in the canonical form or state space form, respectively. The stability definition of Lyapunov is extended to stability numbers of nonlinear dynamical systems. The application of this approach is demonstrated for the single and double pendulums as well as for the Van der Pol equation.

27.2 EQUATIONS OF MOTION

Multi-body systems are mechanical systems consisting of rigid bodies, constraint elements like bearings and joints, and coupling elements like springs, dampers or controlled actuators. For holonomic constraints and proportional-differential forces the equations of the motion read as

$$M(y, t)\ddot{y} + k(y, \dot{y}, t) = q(y, \dot{y}, t), \tag{1}$$

where y is the $f \times 1$ vector of generalized coordinates, $M(y, t)$ is the symmetric $f \times f$ inertia matrix, and $k(y, \dot{y}, t)$ and $q(y, \dot{y}, t)$ represent $f \times 1$ vectors of generalized Coriolis and applied forces, respectively.

In the more general case, non-holonomic constraints and proportional-integral forces result in an extended set of first-order equations of motion:

$$\begin{aligned} &M(y, z, t)\dot{z} + k(y, z, t) = q(y, z, w, t), \\ &\dot{y} = \dot{y}(y, z, t), \quad \dot{w} = w(y, z, t), \end{aligned} \tag{2}$$

where z is a $q \times 1$ vector of generalized velocities and w is a $p \times 1$ vector representing the eigendynamics of the coupling elements and, in particular, the dynamical behaviour of the actuators. The equations (2) are in the literature sometimes known as Kane's equations. For more details see Schiehlen (1984, 1986).

In addition to the mechanical representation, (1) and (2), of a multi-body system, there exists also the possibility of using the more general representation of dynamical systems in the state space, i.e.

$$\dot{x} = f(x, t), \tag{3}$$

where x means the $n \times 1$ state vector composed of generalized coordinates and velocities, and t is the time. In autonomous systems the $n \times 1$ vector function f does not depend on time.

The equations of motion presented may be automatically generated by the formalism NEWEUL described in Schiehlen (1990). NEWEUL is a software package for the dynamic analysis of mechanical systems with the multi-body system method. It deals with the computation of the symbolic equations of motion.

27.3 STABILITY NUMBERS

The dynamical equations of multi-body systems describing autonomous nonlinear oscillations are represented in a canonical form as

$$\dot{x} = f(x), \quad x(t_0) = x_0, \tag{4}$$

following on from (3). Here $\boldsymbol{x}$ is the $n \times 1$ state vector and $\boldsymbol{f}$ is an $n \times 1$ vector function not explicitly depending on time t. At initial time t_0 the initial state $\boldsymbol{x}_0$ is given. It is assumed that $\boldsymbol{f}(\boldsymbol{0}, t) = \boldsymbol{0}$ represents an equilibrium position $\boldsymbol{x} = \boldsymbol{0}$. Due to the nonlinearity of the system, there may exist additional equilibrium positions $\boldsymbol{x} = \boldsymbol{x}^*$.

The stability in the sense of Lyapunov characterizes the qualitative behaviour of the equilibrium position $\boldsymbol{x} = \boldsymbol{0}$ of the dynamical system (4). For the stability definition the absolute-value norm of a vector is used.

The time-variant norm of the $n \times 1$ state vector $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$, $t \in [t_0, \infty)$ is defined by

$$\|\boldsymbol{x}(t)\| := \max_{1 \leqslant i \leqslant n} |x_i(t)|. \tag{5}$$

The time-interval norm reads as

$$\|\boldsymbol{x}(t)\|_T := \max_{t \in [t_0, T]} \|\boldsymbol{x}(t)\| \tag{6}$$

where time T may approach infinity, $T \to \infty$. These definitions are also valid for matrices.

The dynamical system (4) is called *stable* (in the sense of Lyapunov) if for every positive $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon) > 0$ such that for all initial conditions bounded by

$$\|\boldsymbol{x}_0\| < \delta = \delta(\varepsilon) \tag{7}$$

the corresponding trajectories $\boldsymbol{x}(t)$ remain bounded for all t:

$$\|\boldsymbol{x}(t)\| < \varepsilon. \tag{8}$$

The dynamical system (4) is asymptotically stable if it is stable and for all bounded initial conditions (7) the corresponding trajectory tends to zero:

$$\lim_{t \to \infty} \|\boldsymbol{x}(t)\| = 0. \tag{9}$$

If the dynamical system (4) is not stable it is said be *unstable*.

There is a large literature on stability problems, and quite a number of textbooks, e.g. Hahn (1967) and Müller (1977). However, the stability analysis provides only a qualitative answer. For engineering applications some quantitative global information on the dynamical behaviour is of interest.

Based on the stability definitions in the sense of Lyapunov the following stability numbers are defined, see Hu (1992) and Schiehlen (1993).

The local stability number $S1$ given by

$$S1(\boldsymbol{x}_0, t_0) = \begin{cases} \dfrac{\|\boldsymbol{x}_0\|}{\|\boldsymbol{x}(t)\|_\infty} & \text{for } \boldsymbol{x}_0 \neq \boldsymbol{0}, \\ 1 & \text{for } \boldsymbol{x}_0 = \boldsymbol{0}, \end{cases} \tag{10}$$

characterizes the ratio between a given initial state $\boldsymbol{x}_0$ and the corresponding maximal displacement of the trajectory. The number $S1$ depends on $\boldsymbol{x}_0$ and t_0.

The global stability number $S2$ defined by

$$S2(r, t_0) := \min_{\mathbf{x}_0 \in \{\mathbf{x} \mid \|\mathbf{x}\| = r\}} S1(\mathbf{x}_0, t_0) \tag{11}$$

is defined for a subspace of the initial-conditions state space. The number $S2$ characterizes the maximal displacement of all trajectories starting out of the initial-conditions subspace which is by definition a hypercube with respect to the equilibrium point $\mathbf{x} = \mathbf{0}$. By definition it yields

$$0 \leqslant S1 \leqslant 1, \quad 0 \leqslant S2 \leqslant 1. \tag{12}$$

In a numerical analysis the integration interval is limited. Then, the numbers (10) and (11) have to be replaced by

$$S1_T(\mathbf{x}, t_0) := \frac{\|\mathbf{x}_0\|}{\|\mathbf{x}(t)\|_T}, \tag{13}$$

$$S2_T(r, t_0) := \min_{\mathbf{x}_0 \in \{\mathbf{x} \mid \|\mathbf{x}\| = r\}} S1_T(\mathbf{x}_0, t_0). \tag{14}$$

For autonomous systems the initial time can be chosen as $t_0 = 0$ without loss of generality.

There is s direct relation between the stability and the above-defined stability numbers. For an unstable system one may choose a series of initial conditions satisfying $\|\mathbf{x}_{0_1}\| > \|\mathbf{x}_{0_2}\| > \cdots > \|\mathbf{x}_{0_n}\|$ Then it yields

$$\lim_{n \to \infty} \|\mathbf{x}_{0_n}\| = 0 \quad \text{and} \quad \lim_{n \to \infty} S1(\mathbf{x}_{0_n}, t_0) = 0. \tag{15}$$

On the other hand, if the stability number $S1$ is limited, then

$$\|\mathbf{x}(t)\|_\infty \leqslant s\|\mathbf{x}_0\|, \text{ where } s \text{ is some positive number} \tag{16}$$

Usually the components of the state vector, $x_1(t), \ldots, x_n(t)$, have different units. For the application of the stability numbers it is necessary that all components have the same unit. This can be achieved by standardizing operations.

The stability numbers defined characterize dynamical systems of arbitrary dimension geometrically by a scalar number. Therefore, they are especially well suited for multi-body system analysis. More sophisticated geometric methods of nonlinear dynamics, like Poincaré mapping or cell mapping respectively, are usually restricted to two or at most three dimensions.

The above-defined stability numbers may be applied to linear systems, too (see Hu 1992).

27.4 APPLICATIONS TO ENGINEERING DYNAMICS

27.4.1 Single pendulum

As a first example, the single pendulum will be used, since, in addition to the numerical solution, an analytical solution is available. The equation of motion reads without units as

$$\varphi'' + \sin\varphi = 0. \tag{17}$$

After some calculation, the following results are obtained:
for $\varphi_0'^2 \leqslant 2(1 + \cos\varphi_0)$,

$$S1(\boldsymbol{x}_0) = \frac{\max(|\varphi_0|, |\varphi_0'|)}{\max\left\{\sqrt{\varphi_0'^2 + 2(1 - \cos\varphi_0)}, \arccos(\cos\varphi_0 - \varphi_0'^2/2)\right\}}, \tag{18}$$

for $\varphi_0'^2 > 2(1 + \cos\varphi_0)$,

$$S1(\boldsymbol{x}_0) = 0, \tag{19}$$

for $0 < r \leqslant r^*$,

$$S2(r) = \frac{r}{\arccos(\cos r - r^2/2)}, \tag{20}$$

for $r > r^*$

$$S2(r) = 0, \tag{21}$$

where $r^* = 1.478$ follows from the equation $r^2 = 2(1 + \cos r)$. Figure 27.1, presenting the stability number $S2$, shows clearly the instability of the equilibrium position $\boldsymbol{x} = \boldsymbol{0}$ for a sufficiently large initial angular velocity. The numerical simulation yields the same results as shown in the figure.

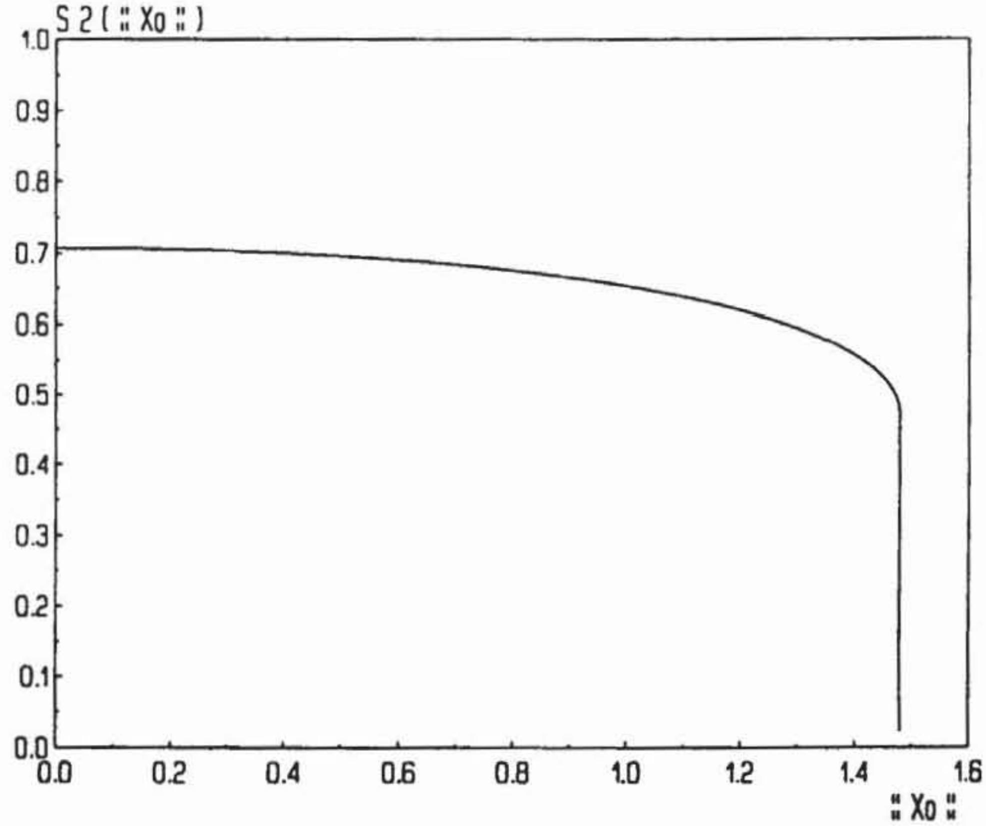

Figure 27.1 Stability number $S2$ of the single pendulum.

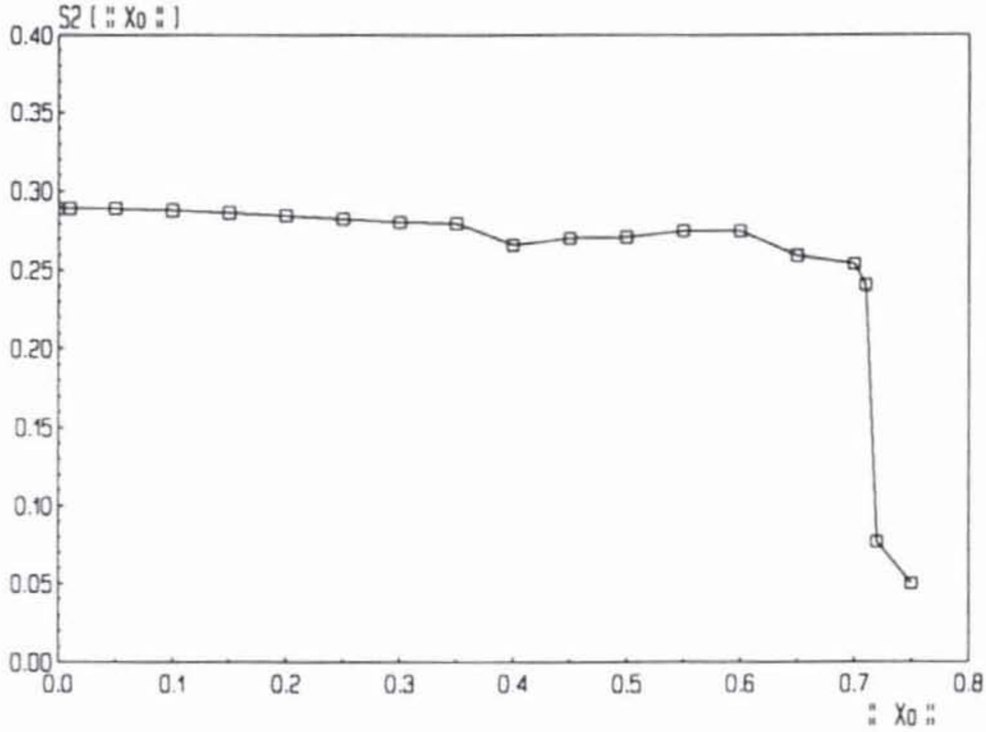

Figure 27.2 Stability number $S2$ of the double pendulum for $T = 100$.

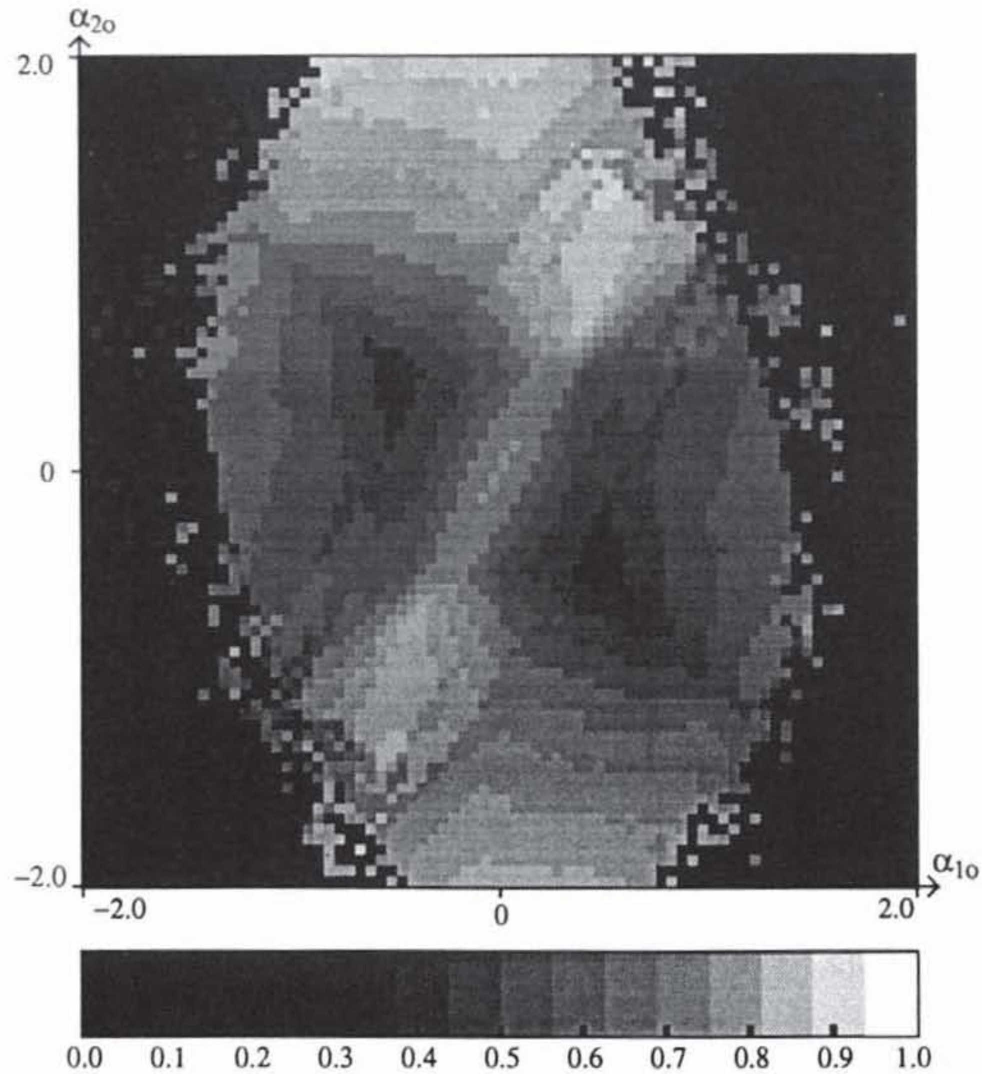

Figure 27.3 Stability number $S1_T$ of the double pendulum for $T = 100$, $\alpha_1' = \alpha_2' = 0.5$.

27.4.2 The double pendulum

The equations of the double pendulum in the standardized form result in

$$\begin{bmatrix} (1+\lambda_1)\lambda_2^2 & \lambda_2\cos(\alpha_1-\alpha_2) \\ \lambda_2\cos(\alpha_1-\alpha_2) & 1 \end{bmatrix}\begin{bmatrix} \alpha_1'' \\ \alpha_2'' \end{bmatrix}+\begin{bmatrix} \sin(\alpha_1-\alpha_2)\lambda_2^2\alpha_2'^2 \\ -\sin(\alpha_1-\alpha_2)\lambda_1^2\alpha_1'^2 \end{bmatrix}=\begin{bmatrix} (1+\lambda_1)\lambda_2\sin\alpha_1 \\ -\sin\alpha_2 \end{bmatrix}, \tag{22}$$

and may be rewritten in state-space representation, too. Then, the state vector reads as $\boldsymbol{x}=[\alpha_1,\alpha_2,\alpha_1',\alpha_2']^{\mathrm{T}}$. A thorough numerical analysis was performed by Hu (1992). For the graphical representation the software for cell mapping methods developed by Schaub (1990) was extensively used.

The first step of the analysis requires the integration of the equations of motion. Then, by variation of all initial conditions, the stability number $S2$ is obtained (see, Figure 27.2). A comparison between Figures 27.1 and 27.2 shows that the double pendulum is much more sensitive than the single pendulum to initial disturbances in the displacement.

It is interesting to analyse the double pendulum also for larger initial displacement (see Figure 27.3). It turns out that there are two clearly separated regions. The left- and right-hand dark regions represent chaotic behaviour. There is a very high sensitivity to the initial conditions. From this point of view, the boundary $\|\boldsymbol{x}_0\|=0.72$ in Figure 27.2 is due to chaotic behaviour and not to simple instability. But, from an engineering point of view, neither chaos nor instability is acceptable.

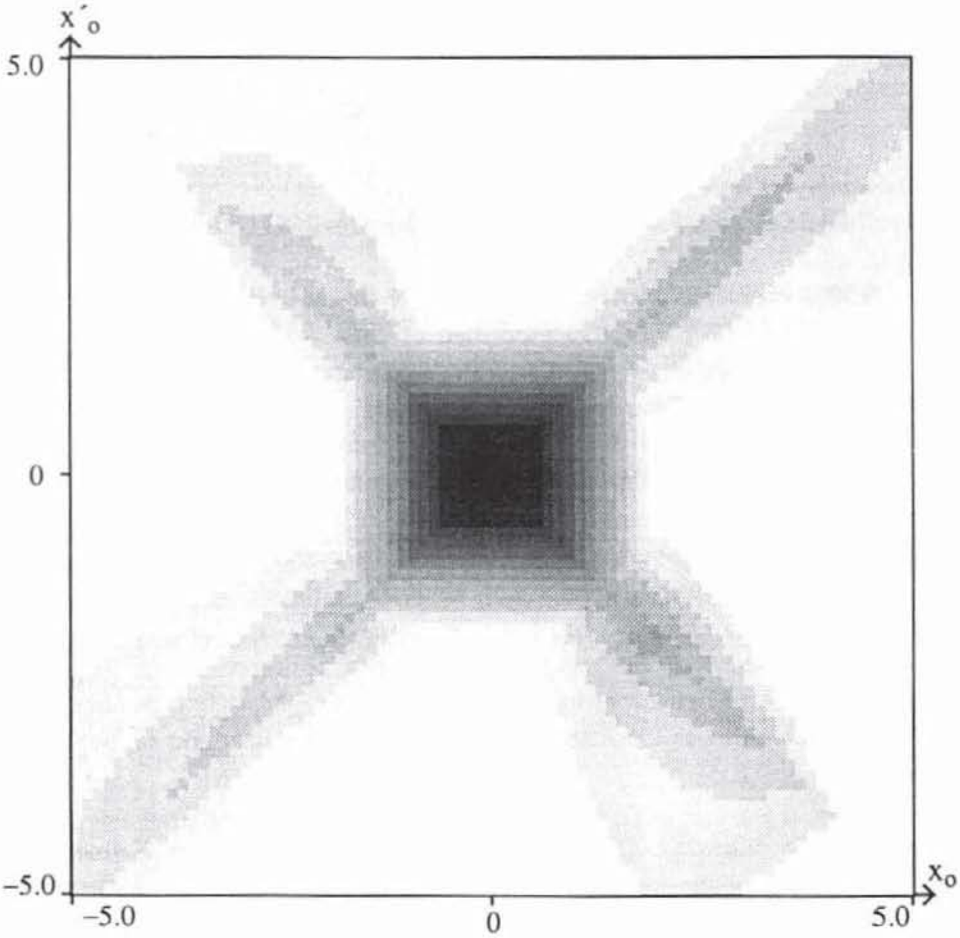

Figure 27.4 Stability number $S1_T$ of the van der Pol equation for $\mu=0.1$ and $T=200$.

27.4.3 Van der Pol equation

The stability numbers are also useful for nonlinear dissipative systems, an example of which is the Van der Pol equation:

$$\begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -\mu(x^2-1)-x \end{bmatrix}.$$

For $\mu > 0$, the equilibrium position $\boldsymbol{x} = [x, x']^{\mathrm{T}} = \boldsymbol{0}$ is unstable; on the other hand, there is a stable limit cycle. The stability number $S1_I$ shows clearly both phenomena, the instability in the origin of the state plane and the limit cycle (Figure 27.4). The grey chart is the same as in Figure 27.3.

27.5 CONCLUSION

Multi-body systems result in highly nonlinear equations of motion typical of engineering dynamics. In engineering applications only bounded motions are acceptable. The approach of stability numbers allows the systematic computation of a basin of bounded motions for systems of arbitrary dimension. Within that basin, the sensitivity of the systems considered with respect to initial conditions or parameters, respectively, may be investigated by other methods in more detail. Further, it has to be pointed out that the stability numbers are not related to the frequency of the system response, they consider the absolute value of all state variables often important in the engineering design of mechanical systems.

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