REDUCTION OF NONHOLONOMIC SYSTEMS

W. SCHIEHLEN AND W. EISMANN

Institute B of Mechanics, University of Stuttgart, D-70550 Stuttgart, Germany

ABSTRACT

The positional degrees of freedom of a mechanical system are by nonholonomic constraints further reduced to a smaller number of motional degrees of freedom. It is shown how the corresponding equations of motion can be reduced to a minimal number using generalized coordinates and generalized velocities. The theoretical results are applied to an actively controlled vehicle with stiff tires. One scalar control variable provides full contollability of the position of the vehicle moving on a plane surface. The control strategy is found for stationary and instationary motions.

INTRODUCTION

Nonholonomic systems have been studied in dynamics for a long time, see Neimark and Futaev[1]. In principle, the equations of motion may be obtained by two approaches: i) application of D'Alembert's principle or Lagrange's equations of the second kind for the evaluation of the equations considering the holonomic constraints only and adding Lagrangian multipliers to the equations to represent the nonholonomic constraints or ii) application of Jourdain's principle considering the nonholonomic constraints immediately. Jourdain's principle, see e. g. Bahar[2], Schiehlen[3], also referred to as Kane's equations, see e. g. Kane[4], results in a minimal number of ordinary differential equations (ODEs) while the Lagrangian multiplier approach leads to a maximal number of differential algebraical equations (DAEs). However, the Lagrangian multipliers may be eliminated using the nonholonomic constraints as shown, e. g., by Wittenburg[5].

More recently nonholonomic systems found increasing interest in applied mathematics. Geometrical methods proved to be most efficient in modeling and analysing mechanical systems, Marsden, Montgomery and Ratiu[6], Krishnaprasad, Yang and Dayawansa[7], Yang[8]. In particular the reduction of the equations of motion to minimal Riemannian spaces is an important topic.

In this paper, the reduction of nonholonomic systems will be treated from a mechanical point of view. Then, the motion of a nonlinearily controlled vehicle with nonholonomic constraints will be discussed in detail. Simulation results present an illustrative overview on the dynamical behaviour of such a vehicle.

EQUATIONS OF MOTION

Nonholonomic systems like skaters and service vehicles can be modeled properly as multibody systems for dynamical analysis. The theoretical background is today available from a number of textbooks authored e. g. by Roberson and Schwertassek[9], Nikravesh[10], Haug[11] and Shabana[12]. The state-of-the-art is also presented at a series of IUTAM/IAVSD symposia and NATO Advanced Study Institutes, documented in the corresponding proceedings, see, e. g., Magnus[13], Slibar and Springer[14], Haug[15], Kortüm and Schiehlen[16], Bianchi and Schiehlen[17], Kortüm and Sharp[18], Haug[19].

The method of multibody systems is based on a finite set of elements such as rigid bodies and/or particles, bearings, joints and supports, springs and dampers, active force and/or position actuators.

The multibody system model has to be described mathematically by equations of motions. The general theory for holonomic and nonholonomic systems will be presented using a minimal number of generalized variables for a unique representation of the motion.

KINEMATICS OF MULTIBODY SYSTEMS

According to the free body diagram of a mechanical system, firstly, all constraints are omitted and the system of p bodies holds $6p$ degrees of freedom. The position of the system is given relative to the inertial frame by the 3×1-translation vector

$$r_i = \left[r_{i1}\ r_{i2}\ r_{i3}\right]^T, \quad i = 1(1)p \quad , \tag{1}$$

of the center of mass C_i and the 3×3-rotation tensor

$$S_i = S_i\left(\alpha_i, \beta_i, \gamma_i\right) \quad , \ i = 1(1)p \quad , \tag{2}$$

written down for each body. The rotation tensor S_i depends on three angles $\alpha_i, \beta_i, \gamma_i$ and corresponds with the direction cosine matrix relating the inertial frame I and the body-fixed frame i to each other. The $3p$ translational coordinates and the $3p$ rotational coordinates (angles) can be summarized in a $6p \times 1$-position vector

$$x = \left[r_{11}\ r_{12}\ r_{13}\ r_{21}\ \cdots\ \alpha_p\ \beta_p\ \gamma_p\right]^T. \tag{3}$$

Eqs. (1) and (2) read now

$$r_i = r_i\,(x)\,,\ S_i = S_i\,(x) \quad . \tag{4}$$

Secondly, the q holonomic, rheonomic constraints are added to the mechanical system given explicitly by

$$x = x(y,t) \ , \tag{5}$$

where the $f \times 1$-position vector

$$y = \left[y_1\ y_2\ y_3\ \ \cdots\ y_f\right]^T \tag{6}$$

is used summarizing the f generalized coordinates of the system. The number of generalized coordinates corresponds to the number of positional degrees of freedom, $f = 6p - q$, with respect to the system's position. Then, translation and rotation of each body follow from (4) and (5) as

$$r_i = r_i\,(y,t)\,,\ S_i = S_i\,(y,t) \quad , \tag{7}$$

and the velocities are found by differentiation with respect to the inertial frame:

$$v_i = \dot{r}_i = \frac{\partial r_i}{\partial y}\dot{y} + \frac{\partial r_i}{\partial t} = J_{Ti}(y,t)\,\dot{y} + \bar{v}_i(y,t), \tag{8}$$

$$\omega_i = \dot{s}_i = \frac{\partial s_i}{\partial y}\dot{y} + \frac{\partial s_i}{\partial t} = J_{Ri}(y,t)\,\dot{y} + \bar{\omega}_i(y,t) \tag{9}$$

The $3 \times f$–Jacobian matrices J_{Ti} and J_{Ri} defined by (8) and (9) characterize the virtual translational and rotational displacement of the system, respectively. They are also needed later for the application of d'Alembert's principle. The infinitesimal 3×1–rotation vector s_i used in (9) follows analytically from the corresponding infinitesimal skew–symmetrical 3×3–rotation tensor. However, the matrix J_{Ri} can also be found by a geometrical analysis of the angular velocity vector ω_i with respect to the angles $\alpha_i, \beta_i, \gamma_i$, see e. g. Schiehlen[3].

The accelerations are obtained by a second differentiation with respect to the inertial frame:

$$a_i = J_{Ti}(y,t)\,\ddot{y} + \frac{\partial v_i}{\partial y}\dot{y} + \frac{\partial v_i}{\partial t}, \tag{10}$$

$$a_i = J_{Ri}(y,t)\,\ddot{y} + \frac{\partial \omega_i}{\partial y}\dot{y} + \frac{\partial \omega_i}{\partial t} \tag{11}$$

For scleronomic constraints the partial time–derivatives in (8), (9) and (10), (11) vanish.

Thirdly, the r nonholonomic, rheonomic constraints, especially due to rigid wheels, are introduced explicitly by

$$\dot{y} = \dot{y}(y,z,t) \tag{12}$$

with the $g \times 1$–velocity vector

$$z(t) = \begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_g \end{bmatrix}^T \tag{13}$$

summarizing the g generalized velocities of the system. The number of generalized velocities characterizes the number of motional degrees of freedom, $g = f - r$, with respect to the system's velocity. From (8), (9) and (12) the translational and rotational velocity of each body follow immediately as

$$v_i = v_i(y,z,t) \quad , \quad \omega_i = \omega_i(y,z,t) \tag{14}$$

The accelerations are found again by differentiation with respect to inertial frame:

$$a_i = \frac{\partial v_i}{\partial z}\dot{z} + \frac{\partial v_i}{\partial y}\dot{y} + \frac{\partial v_i}{\partial t} = L_{Ti}(y,z,t)\,\dot{z} + \dot{\bar{v}}_i(y,z,t), \tag{15}$$

$$a_i = \frac{\partial \omega_i}{\partial z}\dot{z} + \frac{\partial \omega_i}{\partial y}\dot{y} + \frac{\partial \omega_i}{\partial t} = L_{Ri}(y,z,t)\,\dot{z} + \dot{\bar{\omega}}_i(y,z,t) \tag{16}$$

Here, the $3 \times g$–matrices L_{Ti} and L_{Ri} describe the virtual translational and rotational velocity of the system needed also for the application of Jourdain's principle. Further, it has to be mentioned that the partial time–derivatives vanish in (15), (16) for scleronomic systems.

NEWTON–EULER EQUATIONS

For the application of Newton's and Euler's equations to multibody systems the free body diagram has to be used again. Now the rigid bearings and supports are replaced by adequate constraint forces and torques as discussed later in this section.

Newton's and Euler's equations read for each body in the inertial frame

$$m_i \dot{v}_i = f_i^e + f_i^r, \quad i = 1(1)p \ , \tag{17}$$

$$I_i \, \dot{\omega}_i + \tilde{\omega}_i \, I_i \, \omega_i = l_i^e + l_i^r \qquad , \quad i = 1(1)p \ . \tag{18}$$

The inertia is represented by the mass m_i and the 3×3–inertia tensor I_i with respect to the center of mass C_i of each body. The external forces and torques in (17) and (18) are composed by the 3×1–applied force vector f_i^e and torque vector l_i^e due to springs, dampers, actuators, weight etc. and by the 3×1–constraint force vector f_i^r and torque vector l_i^r. All torques are related to the center of mass C_i. The applied forces and torques, respectively, depend on the motion by different laws and they may be coupled to the constraint forces and torques in the case of friction.

The constraint forces and torques originate from the reactions in joints, bearings, supports or wheels. They can be reduced by distribution matrices to the generalized constraint forces. The number of the generalized constraint forces is equal to the total number of constraints $(q + r)$ in the system. Introducing the $(q + r) \times 1$–vector of generalized constraint forces

$$g = \begin{bmatrix} g_1 \ g_2 \ g_3 \ \cdots \ g_{q+r} \end{bmatrix}^T \tag{19}$$

and the $3 \times (q + r)$–distribution matrices

$$F_i = F_i \ (y, z, t) \, , \ L_i = L_i \ (y, z, t) \tag{20}$$

it turns out

$$f_i^r = F_i \, g \quad , \quad l_i^r = L_i \, g \, , \ i = 1(1)p \ , \tag{21}$$

for each body. The constraint forces or the distribution matrices, respectively, can be found analytically or they are derived by geometrical analysis.

The ideal applied forces and torques depend only on the kinematical variables of the system, they are independent of the constraint forces. Ideal applied forces are due to the elements of multibody systems, and further actions on the system, e. g. gravity. The forces may be characterized by proportional, differential and/or integral behavior.

Proportional forces are given by the system's position and timefunctions

$$f_i^e = f_i^e(x, t) \, . \tag{22}$$

E. g., conservative spring and weight forces as well as purely time–varying control forces are proportional forces.

Proportional–Differential forces depend on the position and the velocity:

$$f_i^e = f_i^e(x, \dot{x}, t) \, . \tag{23}$$

A parallel spring–dashpot configuration is a typical example for this kind of forces.

Proportional – Integral forces are a function of the position and integrals of the position:

$$f_i^e = f_i^e(x, w, t) \ , \ \dot{w} = \dot{w}(x, w, t) \ , \tag{24}$$

where the $p \times 1$ – vector w describes the position integrals. E. g., serial spring – damper configurations and the eigendynamics of actuators result in proportional – integral forces. In vehicle systems proportional – integral forces appear, e. g., with modern engine mounts for simultaneous noise and vibration reduction. The same laws hold also for ideal applied torques.

In the case of nonideal constraints with sliding friction or contact forces, respectively, the applied forces are coupled with the constraint forces.

The Newton – Euler equations of the complete system are summarized in matrix notation by the following vectors and matrices. The inertia properties are written in the $6p \times 6p$ – diagonal matrix

$$\overline{M} = diag\{m_1 E \ m_2 E \ \cdots \ I_1 \ \cdots \ I_p\} \ , \tag{25}$$

where the 3×3 – identity matrix E is used. The $6p \times 1$ – force vectors q^c, q^e, q^r representing the coriolis forces, the ideal applied forces and the constraint forces, respectively, are given by the following scheme,

$$\overline{q} = \left[f_1^T \ f_2^T \ \cdots \ l_1^T \ \cdots \ l_p^T\right]^T \ . \tag{26}$$

Further, the $6p \times f$ – matrix $\overline{J}$ and $6p \times g$ – matrix $\overline{L}$ as well as the $6p \times (q + r)$ – distribution matrix $\overline{Q}$ are introduced as global matrices, e. g.,

$$\overline{J} = \left[J_{T1}^T \ J_{T2}^T \ \cdots \ J_{R1}^T \ \cdots \ J_{Rp}^T\right]^T \tag{27}$$

Now, the Newton – Euler equations can be represented as follows for holonomic systems in the inertial frame

$$\overline{M} \, \overline{J} \, \ddot{y} + \overline{q}^c(y, \dot{y}, t) = \overline{q}^e(y, \dot{y}, t) + \overline{Q} g \tag{28}$$

and for nonholonomic systems

$$\overline{M} \, \overline{L} \, \dot{z} + \overline{q}^c(y, z, t) = \overline{q}^e(y, z, t) + \overline{Q} g \tag{29}$$

If the nonholonomic constraints are omitted, e. g. $z = \dot{y}$, eq. (29) reduces to (28), showing a close relation between both representations.

EQUATIONS OF MOTION

The Newton – Euler equations are combined algebraical and differential equations and the question arises if they can be separated for solution into purely algebraical and differential equations. There is a positive answer given by the dynamical principles. In a first step, the system's motion can be found by integration of the separated differential equations and in a second step the constraint forces are calculated algebraically. For ideal applied forces both steps can be executed successively while contact forces require simultaneous execution.

Holonomic systems with proportional or proportional – differential forces result in *ordinary* multibody systems. The equations of motion follow from the Newton–Euler equations, applying d'Alembert's principle.

The equations of motion of holonomic systems are found according to d'Alembert's principle by premultiplication of (28) with $\bar{J}^T$ as

$$M(y,t)\ddot{y} + k(y,\dot{y},t) = q(y,\dot{y},t) \quad . \tag{30}$$

Here, the number of equations is reduced from $6p$ to f, the $f \times f$–inertia matrix $M(y,t)$ is completely symmetrized, $M(y,t) = \bar{J}^T \; \overline{M} \; \bar{J} > 0$, and the constraint forces and torques are eliminated. The remaining $f \times 1$–vector k describes the generalized coriolis forces and the $f \times 1$–vector q includes the generalized applied forces.

Nonholonomic systems with proportional–integral forces produce *general* multibody systems. The equations of motion are obtained from the Newton–Euler equations (29) where the proportional–integral forces (24) and Jourdain's principle has to be regarded. However, the equations of motion are not sufficient, they have to be completed by the nonholonomic constraint equation (12). Thus, the complete equations read as

$$\begin{aligned} &M(y,z,t)\dot{z} + k(y,z,t) = q(y,z,w,t) \quad , \\ &\dot{y} = \dot{y}(y,z,t) \quad , \quad \dot{w} = \dot{w}(y,z,t) \quad . \end{aligned} \tag{31}$$

Now, the number of equations is reduced from $6p$ to g and the $g \times g$–symmetric inertia matrix $M(y,t) = \bar{L}^T \; \overline{M} \; \bar{L} > 0$ appears. Further, k and q are $g \times 1$–vectors of generalized coriolis and applied forces. The equations (31) are in the literature also denoted as Kane's equations.

The equations of motion presented may be automatically generated by the formalism NEWEUL described in the Multibody Systems Handbook[20], too.

NEWEUL is a software package for the dynamic analysis of mechanical systems with the multibody system method. It comprises the computation of the symbolic equations of motion. NEWEUL has been successfully applied in industrial and academic research institutions since 1979. The major fields of application are vehicle dynamics, dynamics of machinery, robot dynamics, biomechanics, satellite dynamics and dynamics of mechanisms. The input data for NEWEUL have to be entered in input files prepared with prompts and comments.

The resulting equations of motion may be linear, partially linearized or nonlinear symbolic differential equations. Constant parameters can be included in numerical form. Nonlinear coupling elements in kinematically linear models are also permitted.

For the output format of the equations of motion several options are possible. A FORTRAN compatible output allows the equations to be included in commercial software packages for dynamic analysis and simulation such as, for instance, ACSL. Another output format allows the processing of the equations with the formula manipulation program MAPLE.

The software module NEWSIM included in the NEWEUL package allows the simulation of motion by numerical integration of the symbolic equations of motion provided. It automatically generates a problem specific simulation program. The user simply has to add the specification of force laws, system parameter values, and initial conditions. The simulation results are stored in ASCII data files that can be visualized with arbitrary graphic packages.

The simulation results may contain the time history of the state variables, the kinematical data of observation points, data for animation, the time history of the reaction forces, and user−defined output data.

The software package NEWEUL is written in FORTRAN 77 and can be implemented on any workstation or mainframe with a FORTRAN 77 compiler. NEWEUL uses its own formula manipulator.

NONLINEARLY CONTROLLED NONHOLONOMIC VEHICLE

The state equations of a four−wheeled vehicle, Fig. 1, are generated based on the following assumptions.

1. The vehicle is considered as one rigid body. Front axle and rear axle including the wheels are considered as rigid and massless.
2. Steering follows from rotation of the whole front axle.
3. The wheels roll on a rough plane. Sliding or loss of contact are excluded.
4. The drive force acts always on the center of gravity C of the vehicle.

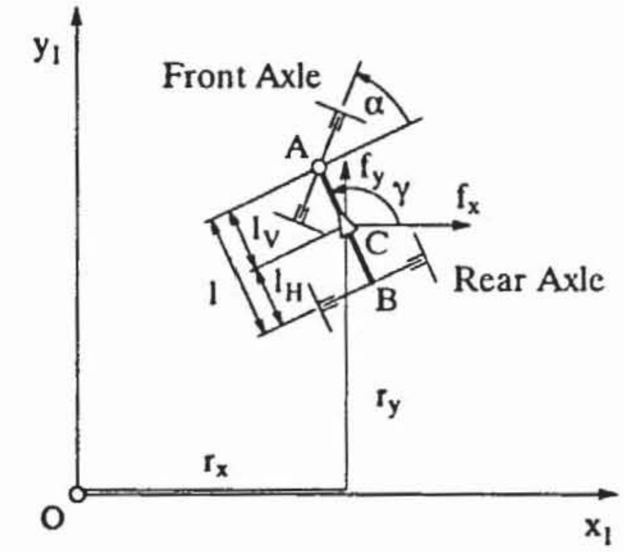

Fig. 1: Vehicle model

The rough plane the vehicle is rolling on is described by fixed axes x_I, y_I. The vehicle position within the plane is defined by coordinates r_x, r_y of the center of gravity C and angle γ due to the vehicle longitudinal axes. Thus, the vehicle has three positional degrees of freedom. However, the velocity degrees of freedom of the vehicle are constrained by the rolling condition. The instantaneous velocities v_A, v_B of the points A and B are vanishing in the directions of the axles leading to the implicit contraints

$$v_A \equiv (\dot{r}_x \cos\gamma + \dot{r}_y \sin\gamma)\tan\alpha - l\dot{\gamma} = 0 \ , \tag{32}$$

$$v_B \equiv -\dot{r}_x \sin\gamma + \dot{r}_y \cos\gamma - l_H\dot{\gamma} = 0 \ . \tag{33}$$

The constraints (32) and (33) are linear, nonholonomic and rheonomic, if the steering angle $\alpha(t)$ is any function of time. The constraints may be expressed explicitly, too,

$$\left.\begin{aligned} \dot{r}_x &= v_H[\cos\gamma - (l_H/l)\tan\alpha\sin\gamma] \ , \\ \dot{r}_y &= v_H[\sin\gamma + (l_H/l)\tan\alpha\cos\gamma] \ , \\ \gamma &= v_H(1/l)\tan\alpha \ , \end{aligned}\right\} \tag{34}$$

where the velocity of the rear axle,

$$v_H = \dot{r}_x \cos\gamma + \dot{r}_y \sin\gamma \quad , \tag{35}$$

was introduced as abbreviation. Thus, the vehicle has one degree of freedom with respect to the velocity, described definitely by the generalized velocity v_H.

The free control variables of the vehicle are at first given by the steering angle $\alpha(t)$ and the coordinates $f_x(t)$, $f_y(t)$ of the driving force. Then, supplying the vehicle with a velocity dependent steering control, which is following the law

$$\alpha = \arctan(v_H/v_L) \quad , \tag{36}$$

leads to a loss of one free control variable, and the vehicle motion remains controlled only by the driving forces $f_x(t)$, $f_y(t)$. Here, v_L is a reference velocity.

Together with (36), constraint (32) reads now

$$v_A \equiv (\dot{r}_x \cos\gamma + r_y \sin\gamma)^2 - v_L l\dot{\gamma} = 0 \quad . \tag{37}$$

Hence, the vehicle is governed by a nonlinear, nonholonomic and skleronomic constraint, and a nonlinear, nonholonomic system is given. The explicit form of the constraints are changed correspondingly,

$$\left.\begin{aligned} \dot{r}_x &= v_H \cos\gamma - (v_H^2 l_H / v_L l) \sin\gamma] \quad , \\ \dot{r}_y &= v_H \sin\gamma + (v_H^2 l_H / v_L l) \cos\gamma] \quad , \\ \dot{\gamma} &= v_H^2 / v_L l \quad , \end{aligned}\right\} \quad . \tag{38}$$

The number of degrees of freedom with respect to the position and the velocity will not be changed by the steering control.

The equation of motion will be generated by Jourdain's principle, see e. g. Schiehlen[3]. According to that, for the plain motion of a rigid body one gets

$$(m\ddot{r}_x - f_x)\delta'\dot{r}_x + (m\ddot{r}_y - f_y)\delta'\dot{r}_y + I\ddot{\gamma}\delta'\dot{\gamma} = 0 \quad , \tag{39}$$

where m is the vehicle mass and I the moment of inertia with respect to C. One gets the accelerations directly by differentiating (38), while Jourdain's variations $\delta'\dot{r}_x$, $\delta'\dot{r}_y$, $\delta'\dot{\gamma}$ have to be derived from (34) and (36), since $\delta' t = 0$. After some lenghty calculations, from (39) one gets the equation of motion

$$\begin{aligned} &\left[m + 2(I + m l_H^2) \frac{v_H^2}{v_L^2 l^2} \right] \dot{v}_H \\ &= f_x \left(\cos\gamma - \frac{v_H l_H}{v_L l} \sin\gamma \right) + f_y \left(\sin\gamma + \frac{v_H l_H}{v_L l} \cos\gamma \right) \end{aligned} \tag{40}$$

The vehicle state is completely defined by the coupled nonlinear system of differential equations (38) and (40). Using the dimensionless quantities

$$x = \frac{r_x}{l} \quad , \quad y = \frac{r_y}{l} \quad , \quad v = \frac{v_H}{v_L} \quad , \quad u = \frac{lf}{m v_L^2} \quad , \quad \tau = \frac{v_L t}{l} \quad , \tag{41}$$

one gets dimensionless state equations

$$
\begin{aligned}
x' &= v\cos\gamma - v^2\varkappa\sin\gamma \quad , \\
y' &= v\sin\gamma + v^2\varkappa\cos\gamma \quad , \\
\gamma' &= v^2 \quad , \\
[1 + 2(i^2 + \varkappa^2)v^2]v' &= u_x(\cos\gamma - v\varkappa\sin\gamma) + u_y(\sin\gamma + v\varkappa\cos\gamma) \quad ,
\end{aligned} \tag{42}
$$

where the abbreviations $\varkappa = l_H/l$ and $i^2 = I/ml^2$ have been introduced.

DYNAMICAL ANALYSIS OF MOTION

Stationary and instationary motion of the vehicle will be considered, see also Schiehlen[21]. A further reduction is obtained for motion under gravity only. The resulting conservative system, or system with symmetry, respectively, has an energy integral omitting one differential equation.

Stationary Motion

Without drive,

$$u_x = u_y = 0 \quad , \tag{43}$$

the vehicle performs a stationary motion. Directly from (42.3) and (42.4) one gets

$$v = v_0 = const \quad , \tag{44}$$

$$\gamma = \gamma_0 + v_0^2\tau \quad . \tag{45}$$

In addition, from (42.1) and (42.2) follow by integration

$$x = x_m + \varkappa\cos\gamma + (1/v_0)\sin\gamma \, , \tag{46}$$

$$y = y_m + \varkappa\sin\gamma - (1/v_0)\cos\gamma \, . \tag{47}$$

Thus, the center of gravity is moving on a circular path,

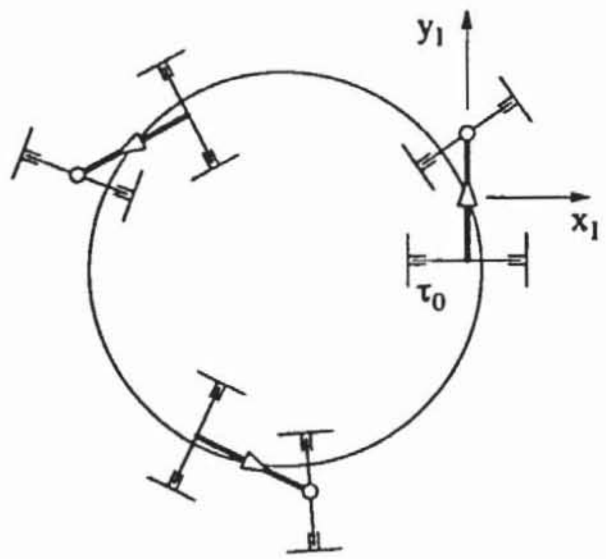

Fig. 2: Stationary circular motion

$$(x - x_m)^2 + (y - y_m)^2 = \varkappa^2 + (1/v_0)^2 = a^2 \quad . \tag{48}$$

Due to the steering law (36), the vehicle can't perform a stationary straight motion. The radius a of the circular path is decreasing with increasing initial velocity v_0. The circle center has the coordinates

$$
\left.
\begin{aligned}
x_m &= x_0 + \varkappa\cos\gamma_0 - (1/v_0)\sin\gamma_0 \quad , \\
y_m &= y_0 - \varkappa\sin\gamma_0 + (1/v_0)\cos\gamma_0 \quad ,
\end{aligned}
\right\} \tag{49}
$$

defined by the initial conditions x_0, y_0 and γ_0 at time τ_0.

Fig. 2 shows the stationary motion of the circular motion with $v_0 = 0{,}8$. The radius is $a = 1{,}346$, and the period time reads $T = 2\pi/v_0^2 = 9{,}817$.

Instationary Motion

The vehicle with drive performs instationary motions. Here, the most important cases of rear axle drive and of running down an inclined plane will be treated.

The rear axle drive is characterized by a force acting always in the direction of the longitudinal vehicle axes:

$$u_x = u\cos\gamma \ , \ u_y = u\sin\gamma \ , \tag{50}$$

where u represents the value of the driving force. Then it follows from (42.4) the equation

$$\left[1 + 2(i^2 + \varkappa^2)v^2\right]v' = u(t) \tag{51}$$

with one free control variable $u(t)$. Equation (42.1) to (42.3) remain unchanged.

The differential equation (51) might be integrated directly. The result is a cubic equation for the velocity,

$$\frac{2}{3}(i^2 + \varkappa^2)v^3 + v - \left[\frac{2}{3}(i^2 + \varkappa^2)v_0^3 + v_0 + \int_{\tau_0}^{\tau} u d\sigma\right] = 0 \ . \tag{52}$$

This cubic equation can be solved elementarily, but this isn't very profitable, because in a general case equations (42.1) to (42.3) can be integrated only numerically.

Fig. 3 shows the accelerated vehicle with rear axle drive starting from a rest. One recognizes the parabolic curve due to small velocities near the origin, while the circular path with decreasing radius appears distinctly in the case of high volocities. In Fig. 4 appears a peak in the trajectory for the decelerated motion. The peak refers to a change in direction of the vehicle with a momentary rest position.

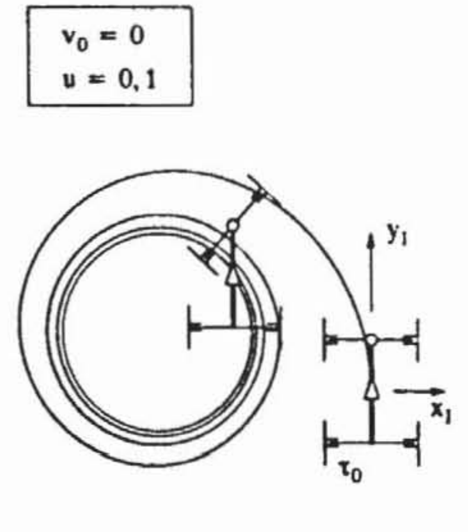

Fig. 3: Constant acceleration starting at a rest position

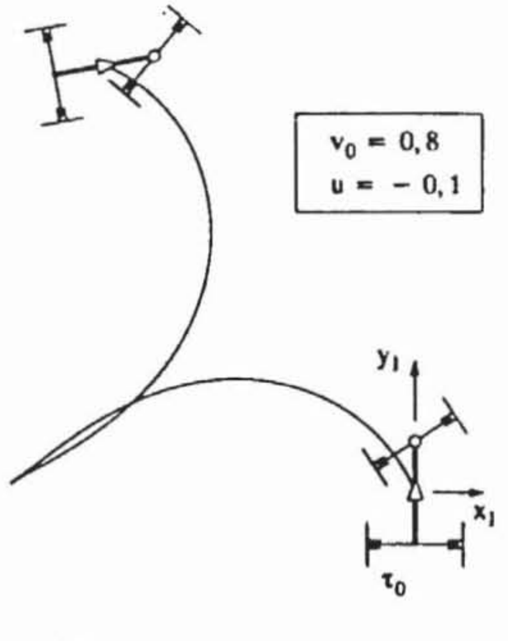

Fig. 4: Constant deceleration from circular motion

Destined motions of the vehicle can be achieved by using a Bang−Bang control of the rear axle drive, i. e., by a single control variable. The control laws for the displacement maneuver and the turning maneuver are put together in Tab. 1. The control laws were determined by parameter studies, but they may be determined graphically, as well. Fig. 5 shows the corresponding trajectories. The displacement of the vehicle can be realized only by circular curves. The turn maneuver can be achieved by a common change in direction. With the change in direction there appears the characteristic trajectory peak, again.

Table 1: Control laws

Displacement maneuver		Turning maneuver	
Time interval	Control variable	Time interval	Control variable
$0 = \tau_0 < \tau <$ 2,661	$u = +1$	$0 = \tau_0 < \tau <$ 1,500	$u = +1$
2,661 $< \tau <$ 5,322	$u = -1$	1,500 $< \tau <$ 4,500	$u = -1$
5,322 $< \tau <$ 7,983	$u = +1$	4,500 $< \tau <$ 6,000	$u = +1$
7,983 $< \tau <$ 10,644	$u = -1$		

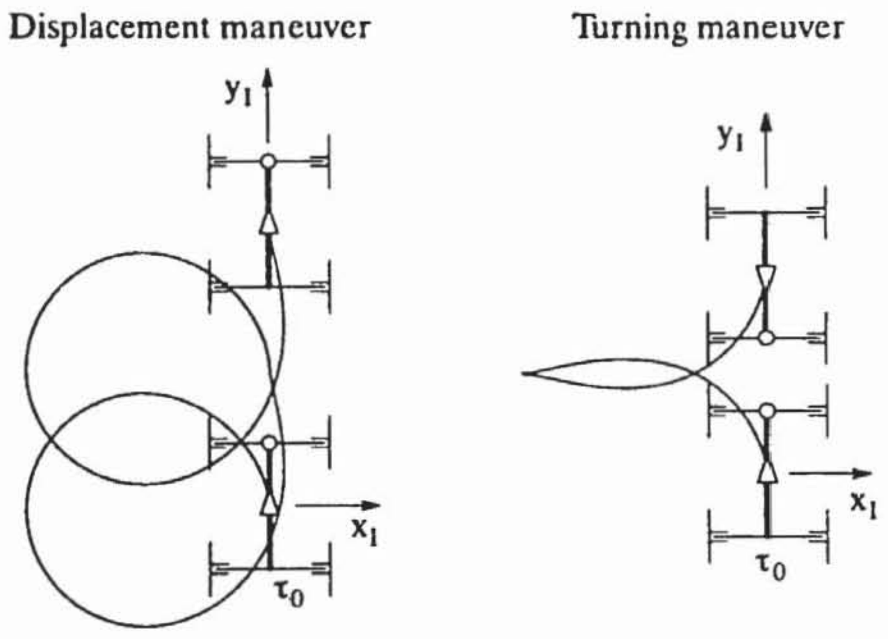

Fig. 5: Bang−Bang control

Motion Under Gravity

The inclined plane is rotated relative to the horizontal with respect to the x−axes by the angle δ. Then, the applied drive force is always directed parallel to the y−axes,

$$u_x = 0 \quad , \quad u_y = h = gl(1/v_L)^2 \sin\delta \quad , \tag{53}$$

where g denotes the gravity. Then, from (42.4) follows the equation

$$\left[1 + 2(i^2 + \varkappa^2)v^2\right]v' = h(\sin\gamma + v\varkappa\cos\gamma) \ . \tag{54}$$

The state equations (42.3) and (54) now are coupled and can't solved separately any longer. However, the number of state equations might be diminished by the energy integral. Based on vanishing initial conditions, $y_0 = v_0 = 0$, one finds from (42.2) and (54) the integral

$$(i^2 + \varkappa^2)v^4 + v^2 - 2hy = 0 \ . \tag{55}$$

There remain the differential equations (42.1) to (42.3) as state equations. Thus, the numerical integration of the general solution is simplified.

On the inclined plane the vehicle running downwards performs a change in direction as shown in Fig. 6. Thereby, the center of gravity reaches the starting height in a rest position. Then, the vehicle starts again to run down the inclined plane, while a mean displacement slanting to the inclined plane takes place. With increasing loss of height the velocity increases, too, and the steering angle grows.

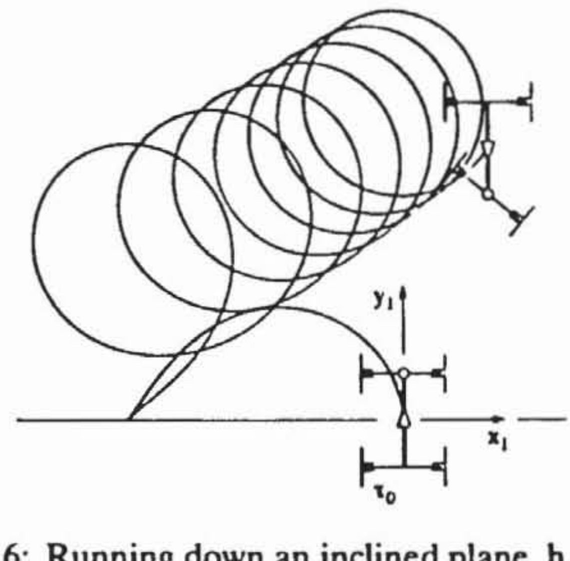

Fig. 6: Running down an inclined plane, h = 0,5

Fig. 7 shows an interesting special case. The vehicle starts slanting to the plane from a rest and reaches after changing the direction a horizontal, stationary rest, again. The vehicle's motion is self-restrained.

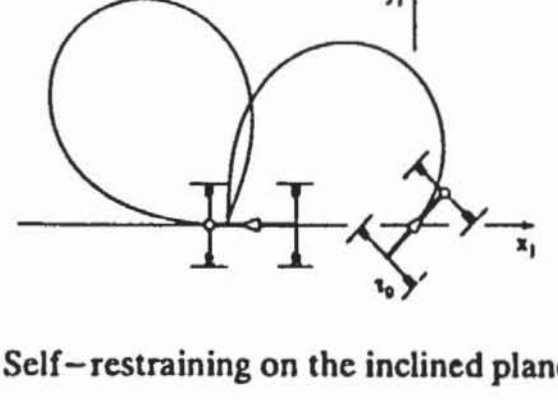

Fig. 7: Self-restraining on the inclined plane, h = 0.5

CONCLUSION

Nonholonomic constraints of mechanical systems result in an additional reduction of the dimension of the dynamical system under consideration. If, in addition, the applied forces follow from a potential, a dynamical system with symmetry is given which means a further reduction of the Riemannian space representing the system's motion. It is shown that a nonlinearily controlled nonholonomic vehicle features full controllability operating on a rigid surface even with one control input.

REFERENCES

1. Neimark, J. I.; Futaev, N. A. (1972). *Dynamics of Nonholonomic Systems*, Translations of Mathematical Monographs, Vol. 33, American Mathematical Society, Providence.
2. Bahar, L. Y. (1993). *On a nonholonomic problem proposed by Greenwood,* Int. J. Nonlinear Mechanics 28, pp. 169–186.
3. Schiehlen, W. (1986). *Technische Dynamik*, Stuttgart.
4. Kane, T. R. (1978). *Nonholonomic Multibody Systems Containing Gyrostats*, in Dynamics of Multibody Systems (Ed. Magnus, K.), pp. 97–107, Springer–Verlag, Berlin.
5. Wittenburg, J. (1977). *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart.
6. Marsden, J. E.; Montgomery, R. and Ratiu, T. (1990). *Reduction, Symmetry and Phases in Mechanics*. Memoirs of the American Mathematical Society.
7. Krishnaprasad, P. S.; Yang, R. and Dayawansa, W. P. (1991). *Control Problems an Principal Bundles and Nonholonomic Mechanics*, in Proc. 30th IEEE Conference Decision and Control, pp. 1133–1138.
8. Yang, R. (1992). *Nonholonomic Geometry, Mechanics and Control*, Ph. D. Theses, University of Maryland, College Park.
9. Roberson, R. E. and Schwertassek, R. (1988). *Dynamics of Multibody Systems*, Springer–Verlag, Berlin.
10. Nikravesh, P. E. (1988). *Computer–aided Analysis of Mechanical Systems*, Prentice–Hall, New Jersey.
11. Haug, E. J. (1989). *Computer–aided Kinematics and Dynamics of Mechanical Systems*, Allyn and Bacon, Boston.
12. Shabana, A. (1989). *Dynamics of Multibody Systems*, Wiley, New York.
13. Magnus, K. (ed.) (1978). *Dynamics of Multibody Systems*. Springer–Verlag, Berlin.
14. Slibar, A.; Springer, H. (eds.) (1978). *Dynamics of Vehicles on Roads and Railway Tracks*, Swets and Zeitlinger.
15. Haug, E. J. (ed.) (1984). *Computer–aided Analysis and Optimization of Mechanical System Dynamics*, Springer–Verlag, Berlin.
16. Kortüm, W.; Schiehlen, W. (1985). *General purpose vehicle system dynamics*

software based on multibody formalisms, Vehicle System Dynamics 14, pp. 229–263.
17. Bianchi, G.; Schiehlen, W. (eds.) (1986). *Dynamics of Multibody Systems*, Springer–Verlag, Berlin.
18. Kortüm, W.; Sharp, R. S. (eds.) (1993). *Multibody Computer Codes in Vehicle System Dynamics*, Swets and Zeitlinger, Amsterdam
19. Haug, E. J. (ed.) (1993). *Concurrent Engineering: Tools and Technologies for Mechanical System Design*, Springer–Verlag, Berlin.
20. Schiehlen, W. (ed.) (1990). *Multibody System Handbook*, Springer–Verlag, Berlin.
21. Schiehlen, W. (1977). *Bewegungsverhalten eines nichtlinearen, nichtholonomen Systems*, in VII. Int. Konferenz über nichtlineare Schwingungen, pp. 331–340, Band II,2, Akademie–Verlag, Berlin.