Actions of Compact Groups on Spheres and on Generalized Quadrangles

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Abstract

The actions of sufficiently high-dimensional compact connected groups on spheres and on two types of compact Tits buildings are classified explicitly. The result for spheres may be summarized as follows: every effective continuous action of a compact connected group whose dimension exceeds $1 + \dim \mathrm{SO}_{n-2}\mathbb{R}$ on an n-sphere is linear, i.e. it is equivalent to the natural action of a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$. Under similar hypotheses, we study actions on finite-dimensional compact generalized quadrangles whose point rows have dimension either 1 or 4. We find that every effective action of a sufficiently high-dimensional compact group is equivalent to an action on a Moufang quadrangle, i.e. on a coset geometry associated to a BN-pair in a simple Lie group. Both for spheres and for generalized quadrangles, the classification arises from an explicit description of the actions.

One main source for this thesis is the investigation of compact projective planes and, recently, other compact generalized polygons by Salzmann and his school. They developed the specific hypothesis of a sufficiently large group dimension, which here is applied to generalized quadrangles for the first time. Compactness of the group is a strong additional assumption which allows us to introduce the sophisticated theory of actions of compact groups on (cohomology) manifolds further into topological incidence geometry. Conversely, the results about spheres, which lie completely within the scope of the classical theory, are rendered possible by Salzmann's specific question. When combined with a thorough exploitation of the classification of compact Lie groups, it essentially reduces the problem to the consideration of a small number of series of groups. To obtain the results about generalized quadrangles, we first show transitivity of the action and then use, and partly re-prove, recent classification results.

Zusammenfassung

Alle Wirkungen kompakter zusammenhängender Gruppen von genügend großer Dimension auf Sphären und auf zwei Arten von verallgemeinerten Vierecken werden im einzelnen beschrieben. Für Sphären läßt sich das Ergebnis wie folgt zusammenfassen: Jede treue stetige Wirkung einer kompakten zusammenhängenden Gruppe, deren Dimension $1 + \dim \mathrm{SO}_{n-2}\mathbb{R}$ übersteigt, auf einer n-Sphäre ist linear, also äquivalent zur natürlichen Wirkung einer Untergruppe von $\mathrm{SO}_{n+1}\mathbb{R}$. Unter ähnlichen Voraussetzungen untersuchen wir Wirkungen auf endlichdimensionalen kompakten verallgemeinerten Vierecken, deren Punktreihen Dimension 1 oder 4 haben. Hier zeigen wir, daß jede treue Wirkung einer kompakten Gruppe von genügend großer Dimension äquivalent ist zu einer Wirkung auf einem Moufang-Viereck, also auf einer Nebenklassengeometrie einer einfachen Lie-Gruppe, die durch ein BN-Paar beschrieben wird.

Die vorliegende Arbeit steht in der Tradition der Untersuchung kompakter projektiver Ebenen und neuerdings anderer kompakter verallgemeinerter Polygone durch Salzmann und seine Schule. Der dabei entstandene Leitgedanke, nur die Wirkung einer Gruppe von genügend großer Dimension vorauszusetzen, wird in dieser Arbeit erstmals für verallgemeinerte Vierecke durchgeführt. Wir setzen zusätzlich voraus, daß die Gruppe kompakt ist, um die hochentwickelte Theorie der Wirkungen kompakter Gruppen auf (Kohomologie-) Mannigfaltigkeiten für die topologische Inzidenzgeometrie weiter zu erschließen. Umgekehrt ermöglicht erst die spezifische Salzmannsche Fragestellung die Ergebnisse über Sphären, die ja dem Bereich der klassischen Theorie angehören. Indem die Klassifikation der kompakten Lie-Gruppen konsequent ausgenutzt wird, läßt sich das Problem auf die Behandlung weniger Serien von Gruppen zurückführen. Bei verallgemeinerten Vierecken zeigt man dagegen zuerst die Transitivität der Wirkung und benutzt dann die bestehende (teilweise hier neu bewiesene) Klassifikation.

Contents

List of Tables **vii**

Preface **ix**

Deutschsprachige Kurzfassung **xv**

1 Foundations **1**
1.1 Locally compact groups 2
1.2 Cohomology manifolds 4
1.3 Generalized spheres 12
1.4 Compact quadrangles 14

2 Actions of compact groups **23**
2.1 General theory 23
2.2 Actions on cohomology manifolds 36
2.3 Consequences of a theorem of Mann's 38
2.4 Calculating homology 41
2.5 Large subalgebras of compact Lie algebras 49

3 Spheres **63**
3.1 Very large orbits 63
3.2 Complex unitary groups 73
3.3 A homogeneity property of torus groups 83
3.4 Other particular groups 85
3.5 Local type of compact Lie transformation groups 91
3.6 Reconstruction of actions 102

4 Compact $(1,m)$-quadrangles **115**
4.1 Miscellaneous tools for compact polygons 115
4.2 Line-homogeneous quadrangles 119
4.3 Local type of compact Lie transformation groups 130
4.4 Reconstruction of actions 145

5 Compact $(4,m)$-quadrangles **151**
5.1 Compact Lie groups 152
5.2 Compact non-Lie groups 161
5.3 Characterization theorems 182

Bibliography **187**

Notation **201**

Index **203**

List of Tables

2.1 The simple compact Lie algebras and their subalgebras of maximal dimension 50

2.2 Structure of semi-simple compact Lie groups acting on spaces of small dimension 60

3.1 The semi-simple Lie algebras of rank at most 5 78

3.2 The compact Lie algebras for effective actions on $\mathbb{S}_8$ . . 94

3.3 Actions of large compact connected groups on n-spheres 114

5.1 Large compact Lie algebras of given rank 153

Preface

To study an object is to study its symmetry – this thought is the heart of Felix Klein's influential Erlangen Programme. In the present thesis, it is applied to topological spheres and to two classes of topological Tits buildings. In each setting, the actions of sufficiently high-dimensional compact connected groups are determined. The results are most easily formulated for spheres. The group $\mathrm{SO}_{n+1}\mathbb{R}$ acts naturally on the n-dimensional sphere $\mathbb{S}_n$. If a compact connected group whose dimension is greater than that of $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{n-2}\mathbb{R}$ acts continuously and effectively on $\mathbb{S}_n$ then the action is equivalent to the natural action of a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$. The proof combines the theory of Lie groups with algebraic topology.

The Tits buildings which we study are generalized quadrangles. These geometries consist of points and lines. Their characteristic property is that for every non-incident point-line pair, there is a unique incident pair such that the four elements form a chain. We assume that this chain depends continuously on its ends with respect to some finite-dimensional compact topology, and that the points which lie on any one line form sets of dimension either 1 or 4. Each effective action of a sufficiently high-dimensional compact connected group on such a generalized quadrangle is equivalent to an action on a Moufang quadrangle, i.e. on one which is associated to a BN-pair in a simple Lie group; the group, the quadrangle, and the action are described explicitly. In particular, we obtain sharp upper bounds for the group dimensions.

The interest in generalized quadrangles is the reason for working in the category of continuous actions. In the absence of differentiability assumptions, we are referred to the theory of topological transformation groups which flourished in the context of the solution of Hilbert's

Fifth Problem by Montgomery and Zippin. Their work, as well as contributions by Borel, Bredon, and others, is essential.

The second main source of inspiration for this thesis, and in fact its original motivation, is the work of Salzmann on compact projective planes, which are the smallest non-trivial compact buildings. Salzmann and his school developed the use of the group dimension as the principal measure of symmetry, and many of their methods are relevant in our context. More recently, the investigation of general topological buildings was begun by Burns, Spatzier, and Thorbergsson in differential geometry, and by Forst, Grundhöfer, Joswig, Knarr, Kramer, Schroth, the Stroppels, and Van Maldeghem in Salzmann's tradition.

A more detailed description of the results will now be given in connection with an overview of the five chapters. In the first, we provide some background material. Section 1.2 introduces cohomology manifolds. Taking only sheaf-theoretic (co-)homology for granted, we collect the further concepts needed for the definition of these spaces, and for their characterization among general (1.2.9) and among metrizable spaces (1.2.14). We improve Löwen's Theorem (1.2.11) on euclidean neighbourhood retracts (ENRs) with contractible point complements by observing that compactness is obtained for free. Section 1.4 gives an overview of the theory of compact generalized quadrangles. The points and lines of a generalized quadrangle form the vertices of a bipartite graph, with incidence as adjacency. The diameter of this graph is 4, its cycles are of length at least 8, and every vertex has at least 3 neighbours. We choose this point of view as our definition of a generalized quadrangle. It captures the geometric intuition described above while providing certain notational advantages and stressing the inherent duality, i.e. the fact that points and lines can be interchanged.

Chapter 2 treats actions of compact groups on general spaces. We work out some material by Montgomery and Zippin about the correspondence between a point and its stabilizer. Then we use work by Bredon to deduce an observation which will be crucial: if an effective action of a compact group on a locally homogeneous cohomology manifold has an orbit of codimension at most 2 then the group is a Lie group, and the space is a genuine manifold (2.2.2). In Section 2.4, we develop a method to calculate the homology of a space on which a compact connected Lie group acts in such a way that all orbits are equivalent and that some closed normal subgroup has sphere orbits. The most important contribution of this chapter is Section 2.5. The results (2.5.5) can be rephrased as follows: to each compact Lie algebra, we associate the smallest possible codimension of a subalgebra which does not contain any non-trivial ideal. This invariant is very well-behaved under products. In fact, it is additive unless both factors contain the ideal $\mathfrak{o}_3\mathbb{R}$ with odd multiplicity, in which case its value for the product is one less than the sum of the values for the two factors. The invariant is known for simple compact algebras, so that it can recursively be calculated for arbitrary compact algebras. Its relevance consists in the fact that it can be interpreted as the minimum dimension of a principal orbit in an effective action on a (cohomology) manifold. Given the dimension of the manifold, we thus obtain a finite list of the Lie algebras of possible compact Lie transformation groups (Table 2.2).

Not every Lie algebra in this list will belong to an action on every manifold of the right dimension. The tool which allows to exclude some algebras and to reconstruct the action for others is Mostert's Theorem (3.1.2). This result, which stands at the beginning of Chapter 3, asserts that if a compact connected Lie group acts on a sphere with an orbit of codimension 1 then there are at most three types of orbits, and the action is determined uniquely by three corresponding stabilizers. When the Lie algebra of the group is known, we will often conclude that Mostert's Theorem applies by using the methods of Section 2.5. They also provide information about the maximal orbit type, which can be used to prove that all stabilizers are connected (3.1.4), and to determine the action explicitly under the hypothesis that the group is semi-simple (3.1.6). In addition to these general results, the procedure which has just been sketched is the key to the treatment of concrete Lie algebras which are not semi-simple (e.g. 3.4.5). The 'local' result (3.5.1, 3.5.4) is that the Lie algebra of a compact Lie group which acts effectively on $\mathbb{S}_n$ is embedded into $\mathfrak{o}_{n+1}\mathbb{R}$ if $n \leq 8$ or if the dimension of the group is greater than that of $\mathbb{T}^3 \times \mathrm{SO}_{n-4}\mathbb{R}$. Under stronger hypotheses, we obtain the global description of the action which was stated at the beginning of this preface (3.6.11, Table 3.3).

Chapter 4 enters the realm of topological incidence geometry. We first deal with compact quadrangles whose point rows (i.e. sets of points on single lines) are one-dimensional, the so-called compact $(1, m)$-quadrangles. In this notation, the number m is the dimension of the line pencils, i.e. of the sets of lines through single points. All natural numbers occur for m. This phenomenon constitutes the additional challenge of the topic when compared to the theory of compact projective planes. Point rows and line pencils of finite-dimensional compact connected quadrangles are cohomology manifolds and homotopy equivalent to spheres, so that the results of previous chapters apply. The compact Moufang quadrangles with parameters $(1, m)$ are the real orthogonal quadrangles $Q(m + 3, \mathbb{R})$. They can be described by a bilinear form of Witt index 2 in $\mathbb{R}^{m+4}$. Their points and lines are the one- and two-dimensional vector subspaces which are totally isotropic. Any maximal compact connected subgroup of $\mathrm{Aut}\, Q(m+3, \mathbb{R})$ is isomorphic to $G_m := P(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{m+2}\mathbb{R})$.

The first two sections give new proofs of the classification of line-homogeneous quadrangles with one-dimensional point rows (4.2.15), and of the fact that a locally compact group which acts effectively on a finite-dimensional compact polygon is itself finite-dimensional (4.1.6). This also leads to upper bounds for the dimensions of compact and, in particular, of compact abelian groups (4.1.7). In Sections 4.3 and 4.4, we study compact $(1, m)$-quadrangles in the spirit of the third chapter (4.4.2). Transformation group theory shows that a compact connected group which acts effectively and whose dimension exceeds that of $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{m-1}\mathbb{R}$ is a Lie group. Using methods similar to those of Chapter 3, we show that the Lie algebra of a compact Lie group which acts effectively and whose dimension is greater than that of $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_5\mathbb{R} \times \mathrm{SO}_{m-2}\mathbb{R}$ admits an embedding into $\mathbb{R} \times \mathfrak{o}_{m+2}\mathbb{R}$. If the group is connected and its dimension exceeds that of $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{m+1}\mathbb{R}$ then we determine not only the group, but also the quadrangle. In fact, the action is equivalent to the action of either G_m or $(G_m)' \cong \mathrm{SO}_{m+2}\mathbb{R}$ on $Q(m + 3, \mathbb{R})$ or, if $m = 1$, on the dual quadrangle.

Chapter 5 treats compact $(4, m)$-quadrangles, i.e. those whose point rows are four-dimensional. Here, the Moufang quadrangles are described by sesquilinear forms on vector spaces over Hamilton's quaternion skew field $\mathbb{H}$. Only the numbers $m \in \{1, 3, 5, 7, 11, 15, 19, \dots\}$ occur as second parameters of compact Moufang $(4, m)$-quadrangles. There is reason to conjecture that the same holds for non-Moufang quadrangles.

In order to study compact quadrangles with four-dimensional point rows, we need to develop new methods once more. To deal with the actions of compact Lie groups, we observe that there is a sharp upper bound on the rank of a group which fixes a line pointwise (5.1.3), and that a bound on the rank yields a bound on the group dimension (5.1.1). In a compact connected non-Lie group, the commutator subgroup is a semi-simple Lie group, and one expects that its dimension and that of the centre balance each other. This idea is made precise with the help of a classification of point orbits under the action of a compact connected abelian group (5.2.4). Combining these results with work by Grundhöfer, Knarr, and Kramer, we find that any effective action of a sufficiently high-dimensional compact connected group on a compact $(4, m)$-quadrangle, where m is as described above, is equivalent to the action of a Lie group on a Moufang quadrangle (5.3.1, 5.3.3).

I would like to express my gratitude to some of those whose generous help has made this thesis possible. During its preparation, I have worked in the groups of Prof. Theo Grundhöfer, Prof. Hermann Hähl, and Prof. Karl-Hermann Neeb, whose hospitality, encouragement, and support I appreciate. Prof. Karl H. Hofmann, Prof. Rainer Löwen, and Dr. Linus Kramer have given me further important advice. I was granted scholarships by the State of Bavaria and by the Evangelisches Studienwerk Villigst (Lutheran Foundation for Advanced Studies). My warmest thanks go to my supervisor, Dr. Markus Stroppel. He has suggested the topic which I felt was very rewarding. He has patiently set me on my way, and he has diligently discussed the progressing work. Before all, it is to him that I owe the introduction to the specific interplay of group theory, topology, and geometry which constitutes the beauty of this area of mathematics.

Deutschsprachige Kurzfassung

Das geeignete Maß für die Symmetrie eines Objekts ist die größte mögliche Dimension einer treu darauf wirkenden Gruppe – diese Maxime ist als das Salzmann-Programm für topologische Geometrien bekannt. In der vorliegenden Arbeit wird sie auf topologische Sphären und auf zwei Klassen topologischer Tits-Gebäude angewandt. Hauptergebnis ist jeweils die Beschreibung aller Wirkungen kompakter zusammenhängender Gruppen von hinreichend großer Dimension. So wirkt etwa auf der n-dimensionalen Sphäre $\mathbb{S}_n$ die Gruppe $\mathrm{SO}_{n+1}\mathbb{R}$ in natürlicher Weise. Wirkt nun eine kompakte zusammenhängende Gruppe, deren Dimension größer als die von $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{n-2}\mathbb{R}$ ist, treu und stetig auf $\mathbb{S}_n$, so ist ihre Wirkung äquivalent zur natürlichen Wirkung einer Untergruppe von $\mathrm{SO}_{n+1}\mathbb{R}$. Der Beweis dieses Satzes verbindet die Theorie der Lie-Gruppen mit algebraischer Topologie.

Die untersuchten Tits-Gebäude sind verallgemeinerte Vierecke. Diese Inzidenzgeometrien bestehen aus Punkten und Geraden. Ihre kennzeichnende Eigenschaft ist es, daß es zu jedem nicht-inzidenten Punkt-Geraden-Paar genau ein inzidentes Paar so gibt, daß die vier Elemente einen Streckenzug bilden. Man kann also von einem Punkt auf eine vorbeilaufende Gerade eindeutig ein „inzidenzgeometrisches Lot“ fällen. Lotgerade und Lotfußpunkt sollen dabei von den beiden gegebenen Elementen stetig abhängen, und zwar bezüglich einer kompakten Topologie von positiver endlicher Dimension. Eine erste Klasseneinteilung gewinnt man über die sogenannten topologischen Parameter, zwei natürliche Zahlen, die die Dimensionen der Punktreihen von Geraden und der Geradenbüschel von Punkten angeben. Man spricht kurz von kompakten (m, m')-Vierecken. Wir betrachten kompakte Vierecke, deren Punktreihen die Dimension 1 oder 4 haben. Jede treue Wirkung einer kompakten zusammenhängenden Gruppe von genügend großer Dimension auf einem solchen Viereck ist äquivalent zu einer Wirkung auf einem Moufang-Viereck, also auf einem Viereck, das mit Hilfe eines BN-Paares als Nebenklassengeometrie einer einfachen Lie-Gruppe entsteht. Da die kompakten zusammenhängenden Moufang-Vierecke von Grundhöfer und Knarr [48] klassifiziert wurden, sind Gruppe, Viereck und Wirkung somit explizit bekannt. Insbesondere finden wir scharfe obere Schranken an die Gruppendimension.

Eine genauere Beschreibung des Inhalts der Arbeit soll nun in Verbindung mit einer Übersicht über ihre fünf Kapitel gegeben werden. Das erste Kapitel stellt einige Grundlagen der weiteren Arbeit zusammen. Zunächst wird über Höhepunkte aus der Theorie der lokal kompakten Gruppen referiert. Wir zitieren den Satz über die Approximation durch Lie-Gruppen, van Kampens Struktursatz für kompakte zusammenhängende Gruppen und den Satz von Mal'cev und Iwasawa.

Der folgende Abschnitt über Kohomologiemannigfaltigkeiten ist der eigentliche Beitrag dieses Kapitels. Die wesentliche Eigenschaft dieser Räume ist es, daß ihre Homologie relativ zu Punktkomplementen die gleiche wie in Mannigfaltigkeiten ist. Sie treten in der Theorie der stetigen Wirkungen kompakter Gruppen in natürlicher Weise auf. Zum Beispiel gibt es dort Situationen, in denen man eine Zerlegung eines Raums als Produkt gewinnt. Ist der ursprüngliche Raum eine Kohomologiemannigfaltigkeit, so sind es die Faktoren wieder. Sie müssen aber auch dann keine echten Mannigfaltigkeiten sein, wenn dies auf den ursprünglichen Raum zutrifft. Ähnliches gilt für Fixpunktmengen. Ein anderer Zusammenhang, in dem Kohomologiemannigfaltigkeiten auftreten, ist die topologische Charakterisierung von Mannigfaltigkeiten. Aus der Sicht des Topologen sind Kohomologiemannigfaltigkeiten relativ einfache Objekte. Zum Beispiel benötigt man für ihre Definition keine Homöomorphismen, und deren Konstruktion ist im allgemeinen schwierig. So ist es nicht erstaunlich, daß man von kompakten zusammenhängenden Polygonen endlicher Dimension zwar zeigen konnte, daß ihre Punktreihen und Geradenbüschel Kohomologiemannigfaltigkeiten sind, was sich dann auf Punkt- und Geradenraum überträgt, daß aber die Vermutung noch immer offen ist, diese Räume müßten stets Mannigfaltigkeiten sein. Dabei sind die genannten Räume in einem Polygon sogar lokal homogen, es gibt also zu je zwei Punkten einen

Homöomorphismus von einer Umgebung des ersten auf eine Umgebung des zweiten, der den ersten Punkt auf den zweiten abbildet.

Leider ist die Definition einer Kohomologiemannigfaltigkeit recht technisch, wenn diese Räume auch durch die Neuauflage von Bredons Monographie [16] wesentlich zugänglicher geworden sind. Vor allem benötigt man besonders leistungsfähige Homologie- und Kohomologietheorien, die Borel-Moore-Homologie und die Garben-Kohomologie. Wir nehmen diese beiden Theorien als gegeben hin, geben aber alle weiteren Definitionen, etwa die der kohomologischen Dimension. Danach stellen wir aus der Literatur mehrere alternative Eigenschaften zusammen, die die Kohomologiemannigfaltigkeiten unter allgemeinen und unter metrisierbaren Räumen kennzeichnen (Satz 1.2.9 und Korollar 1.2.14). Wir gewinnen daraus eine Verallgemeinerung des Satzes von Löwen. Es stellt sich nämlich heraus, daß dort Kompaktheit nicht vorausgesetzt werden muß. In der folgenden Fassung ist dies berücksichtigt. Die allgemeinste mir bekannte Formulierung ist in 1.2.11 zu finden.

Satz (vgl. Löwen [81, 6.2]). *Sei X ein euklidischer Umgebungsretrakt (ENR), in dem jedes Punktkomplement kontrahierbar ist. Dann ist X eine kompakte Kohomologiemannigfaltigkeit und homotopieäquivalent zu einer Sphäre gleicher Dimension.*

Für die topologischen Räume, die in diesem Satz auftreten, führen wir den Begriff einer *verallgemeinerten Sphäre* ein, wenn sie zusätzlich lokal homogen sind. (Tatsächlich ist unsere Definition noch etwas stärker. Wir möchten nämlich nicht auf die allgemeine Poincaré-Vermutung zurückgreifen, um zu zeigen, daß eine verallgemeinerte Sphäre eine gewöhnliche Sphäre ist, wenn sie eine Mannigfaltigkeit ist.) Unter den Begriff der verallgemeinerten Sphäre fallen sowohl gewöhnliche Sphären als auch Punktreihen und Geradenbüschel kompakter zusammenhängender Polygone endlicher Dimension. Diese Räume simultan zu behandeln ist der Grund für die Definition.

Die Punkte und Geraden eines verallgemeinerten Vierecks bilden die Ecken eines bipartiten Graphen Q, dessen Nachbarschaftsrelation durch die Inzidenz gegeben ist. Der Graph hat die folgenden drei Eigenschaften:

(i) Der Durchmesser von Q ist 4.

(ii) Die kleinste Länge eines Zykels in Q ist 8.

(iii) Jede Ecke hat wenigstens 3 Nachbarn.

Wir benutzen diese Eigenschaften als Definition eines verallgemeinerten Vierecks. Sie geben die inzidenzgeometrische Intuition, also die Existenz eines eindeutig bestimmten Lots, treu wieder. Zugleich führen sie zu einer vorteilhaften Notation. Wenn nämlich $Q' \subseteq Q$ als Fixpunktmenge oder als ein geometrisches Erzeugnis auftritt, so lassen sich der Punktraum von Q, die Geradenbüschel usw. einfach als Schnittmengen schreiben. Außerdem betont diese Definition das Dualitätsprinzip, das besagt, daß die Rolle von Punkten und Geraden in einem verallgemeinerten Viereck austauschbar ist. Von dieser Wahl der Definition abgesehen, folgt unsere Darstellung weitgehend der Arbeit von Grundhöfer und Knarr [48].

Das zweite Kapitel behandelt die Theorie kompakter Transformationsgruppen. Der erste Abschnitt lehnt sich zunächst an die Monographie von Montgomery und Zippin [93] an, gibt jedoch viele selbständige Beweise und führt mit Satz 2.1.15 über die Vorlage hinaus. Im zweiten Abschnitt benutzen wir eine Arbeit von Bredon [10], um den folgenden Satz zu beweisen, der für alles weitere von zentraler Bedeutung ist:

Satz 2.2.2. *Eine kompakte Gruppe G wirke treu auf einer zusammenhängenden metrisierbaren Kohomologiemannigfaltigkeit X über einem Hauptidealring R. Außerdem sei X lokal homogen, und eine Bahn maximaler Überdeckungsdimension habe die Kodimension höchstens 2. Dann ist G eine Lie-Gruppe und X eine Mannigfaltigkeit.*

Hier ist die Kodimension die Differenz zwischen der kohomologischen Dimension von X und der Überdeckungsdimension der Bahn. Aus dem Satz folgt, daß die Überdeckungsdimension von X endlich ist, so daß sie mit der kohomologischen Dimension zusammenfällt.

Die erste der beiden Folgerungen des Satzes ist wichtiger als die zweite. Für Wirkungen kompakter Gruppen, die keine Lie-Gruppen sind, hat man nur wenige Werkzeuge, unter denen sich die untere Schranke an die Kodimension einer Bahn als ein sehr wirkungsvolles erweist. Dagegen wurde schon bemerkt, daß Kohomologiemannigfaltigkeiten in der Theorie der stetigen Wirkungen kompakter Lie-Gruppen die natürlichen Räume sind. Dennoch wurde in der älteren

Literatur gelegentlich auf diese Allgemeinheit verzichtet, und es ist bequem, daß der Satz es erlaubt, Ergebnisse über Wirkungen kompakter Lie-Gruppen auf Mannigfaltigkeiten wie etwa das von Mostert [94] unverändert zu zitieren. Als Beispiel für ein Resultat, wie es auch für Wirkungen auf Mannigfaltigkeiten schöner nicht sein könnte, geben wir einen Satz von Montgomery und Yang 2.2.3 wieder, der uns noch gute Dienste leisten wird. Er zeigt beispielsweise, daß es für treue Wirkungen stets eine Bahn gibt, auf der allein die Wirkung schon treu ist. Man kann sich also zunächst auf treue transitive Wirkungen zurückziehen.

Die kombinatorische Behandlung der Lie-Algebra $\mathfrak{g}$ einer kompakten Lie-Gruppe, die treu und transitiv auf einem Raum der Dimension n wirkt, ist ein Grundpfeiler der vorliegenden Arbeit. Sie wird in Abschnitt 2.3 vorbereitet und in Abschnitt 2.5 vollends bereitgestellt. Als kompakte Lie-Algebra ist $\mathfrak{g}$ das direkte Produkt seines Zentrums mit einer halbeinfachen Algebra, also mit einem Produkt von einfachen kompakten Lie-Algebren. In einer Arbeit von Mann [83] wurde eine obere Schranke für die Dimension von $\mathfrak{g}$ in Abhängigkeit von der Dimension des Zentrums und der Anzahl und Dimension der einfachen Faktoren aufgestellt. Man kann etwa sagen, daß $\mathfrak{g}$ nur dann große Dimension haben kann, wenn $\mathfrak{g}$ einen großen einfachen Faktor hat und wenn das Zentrum und die Gesamtzahl der einfachen Faktoren klein sind. Das Ergebnis von Mann ist ein Korollar der folgenden Aussage:

Definition und Satz. *Für eine kompakte Lie-Algebra $\mathfrak{g}$ sei $\mu(\mathfrak{g})$ die kleinste mögliche Dimension eines Raumes, auf dem eine Gruppe mit Lie-Algebra $\mathfrak{g}$ treu und transitiv wirkt. (Äquivalent ist es, die Zahl $\mu(\mathfrak{g})$ als die kleinste mögliche Kodimension einer Unteralgebra von $\mathfrak{g}$, die keinen direkten Faktor enthält, zu definieren.)*

Sei $\mathfrak{h}$ eine Unteralgebra von $\mathfrak{g}$, die keinen direkten Faktor enthält und deren Kodimension $\mu(\mathfrak{g})$ ist. Dann wird $\mathfrak{h}$ von Satz 2.5.5 beschrieben; im wesentlichen ist $\mathfrak{h}$ ein Produkt von Unteralgebren der Faktoren von $\mathfrak{g}$.

Seien $\mathfrak{g}_1$ und $\mathfrak{g}_2$ kompakte Lie-Algebren. Falls die einfache Lie-Algebra $\mathfrak{o}_3\mathbb{R}$ sowohl als Faktor von $\mathfrak{g}_1$ als auch als Faktor von $\mathfrak{g}_2$ mit ungerader Vielfachheit auftritt, dann gilt

$$\mu(\mathfrak{g}_1 \times \mathfrak{g}_2) = \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2) - 1.$$

Andernfalls gilt

$$\mu(\mathfrak{g}_1 \times \mathfrak{g}_2) = \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2).$$

Die einfachen kompakten Lie-Algebren wurden durch Arbeiten von Killing und Weyl klassifiziert. Sie treten in vier unendlichen Serien auf, zu denen noch fünf sogenannte Ausnahmealgebren hinzukommen. Für alle einfachen kompakten Algebren $\mathfrak{g}$ ist $\mu(\mathfrak{g})$ in Tabelle 2.1 auf Seite 50 zu finden. Für eine abelsche kompakte Algebra $\mathfrak{a}$ ist $\mu(\mathfrak{a}) = \dim \mathfrak{a}$, da jede Unteralgebra ein direktes Komplement hat. Damit kann $\mu(\mathfrak{g})$ für jede kompakte Algebra rekursiv bestimmt werden. Insbesondere gilt für kompakte Algebren $\mathfrak{g}$, die nicht zu der eindimensionalen Algebra $\mathbb{R}$ isomorph sind, stets $\mu(\mathfrak{g}) \geq 2$. Daher gilt

$$\mu(\mathfrak{g}_1 \times \mathfrak{g}_2) > \max\{\mu(\mathfrak{g}_1), \mu(\mathfrak{g}_2)\},$$

so daß es es für jedes $n \in \mathbb{N}$ nur endlich viele kompakte Algebren $\mathfrak{g}$ mit $\mu(\mathfrak{g}) \leq n$ gibt. Für $n \leq 9$ sind sie in der Tabelle 2.2 auf Seite 60 zusammengestellt, die man für größere n leicht beliebig verlängern kann.

Wir haben vorgegriffen und Abschnitt 2.4 übersprungen, der einige Techniken zur Berechnung von Homologie- und Kohomologiegruppen bereitstellt, die sämtlich die Wirkung einer Gruppe ausnutzen, um einen gegebenen Raum als Totalraum eines lokal trivialen Faserbündels zu erkennen. Er wird damit für die Maschinerie der Spektralsequenzen zugänglich, von der wir aber nur einfache Konsequenzen brauchen, nämlich die exakten Gysin-Sequenzen zu Faserbündeln, deren Fasern Sphären sind. Das Hauptergebnis lautet in leicht vereinfachter Formulierung:

Satz 2.4.7. *Eine kompakte zusammenhängende Lie-Gruppe G wirke so auf einem vollständig regulären Raum X, daß alle Standgruppen zu $H \leq G$ konjugiert sind. Dann besagt ein klassisches Ergebnis, daß die Quotientenabbildung $X \to X/G$ auf den Bahnenraum die Projektion eines (lokal trivialen) Faserbündels mit Faser G/H ist. Sei N ein abgeschlossener Normalteiler von G. Dann faktorisiert $X \to X/G$ als*

$$X \longrightarrow X/N \longrightarrow X/G,$$

wobei beide Abbildungen Projektionen in Faserbündeln sind. Die Faser des linken ist $N/(H \cap N)$, die des rechten G/HN.

Wenn wir diesen Satz anwenden, wird $N/(H \cap N)$ eine Sphäre sein, und der Satz wird zeigen, daß X/N eine Mannigfaltigkeit von höchstens der gleichen Dimension ist. Die Gysin-Sequenz liefert dann das Ergebnis

$$H_*(X; \mathbb{Z}/2) \cong H_*\left(\frac{N}{H \cap N} \times \frac{X}{N};\ \ \mathbb{Z}/2\right).$$

Das dritte Kapitel behandelt nun die Wirkungen großer kompakter Lie-Gruppen auf verallgemeinerten Sphären, wie sie ja in kompakten Polygonen auftreten. Die Ergebnisse sind aber auch für gewöhnliche Sphären neu, und Anwendungen in verschiedenen Gebieten der Geometrie sind denkbar. Das Kapitel beginnt mit einem Abschnitt, der über die Klassifikation homogener Kohomologiesphären berichtet und dann einen zweiten Grundpfeiler meiner Arbeit bereitstellt, nämlich die systematische Ausnutzung eines Satzes von Mostert [94], der später von ihm in einer gemeinsamen Arbeit mit Hofmann [58] verbessert wurde. Dieser Satz behandelt Wirkungen einer kompakten zusammenhängenden Lie-Gruppe auf einer Mannigfaltigkeit X, bei denen eine Bahn Kodimension 1 hat. Ist X kompakt und einfach zusammenhängend, so ist der Bahnenraum ein Intervall. Alle Bahnen zu inneren Punkten sind äquivalent und haben die Kodimension 1. Unter geeigneten Voraussetzungen an die Homologie von X, etwa wenn X eine Sphäre ist, sind die Bahnen zu den beiden Endpunkten des Bahnenraums von kleinerer Dimension, und es besteht ein Zusammenhang zwischen ihrer Homologie und der von X. Es gibt ein Paar von natürlichen Abbildungen einer maximalen Bahn auf die beiden kleineren. Der doppelte Abbildungszylinder zu diesem Paar ist äquivariant zu X homöomorph. Das bedeutet, daß die Wirkung aus der Kenntnis von drei Standgruppen eindeutig rekonstruiert werden kann. Das ist die für diese Arbeit zentrale Rekonstruktionsmethode. Oft wird sie auch verwendet, um die Wirkung einer kompakten Gruppe mit gegebener Lie-Algebra auf X auszuschließen.

Erst im Zusammenhang mit den oben beschriebenen kombinatorischen Methoden entfaltet Mosterts Satz seine volle Wirkung. Betrachten wir eine kompakte zusammenhängende Lie-Gruppe G, die treu auf einer verallgemeinerten n-Sphäre S wirkt. Dann gibt es eine Bahn x^G, auf der G treu wirkt. Für die Lie-Algebra $\mathfrak{g}$ von G gilt $\mu(\mathfrak{g}) \leq \dim x^G$. Dadurch gewinnen wir eine endliche Liste kompakter Lie-Algebren, die die möglichen Isomorphietypen von $\mathfrak{g}$ umfaßt.

(Man beachte, daß der Isomorphietyp von $\mathfrak{g}$ dem lokalen Isomorphietyp von G entspricht.) Unser erstes Ziel ist es, zu zeigen, daß die tatsächlich auftretenden Lie-Algebren – oder zumindest diejenigen, die eine Mindestdimension erreichen – eine Einbettung in die Lie-Algebra $\mathfrak{o}_{n+1}\mathbb{R}$ zulassen. (Diese Algebra gehört zu der Gruppe $\mathrm{SO}_{n+1}\mathbb{R}$, die in natürlicher Weise auf der n-Sphäre $\mathbb{S}_n$ wirkt.) Die Liste wird noch einige Isomorphietypen enthalten, die auszuschließen sind. Es zeigt sich nun, daß für diese Algebren $\mathfrak{g}$ die Beziehung $\mu(\mathfrak{g}) = n - 1$ gilt, so daß die Bahn x^G höchstens die Kodimension 1 haben kann und die Lie-Algebra der Standgruppe G_x gut bekannt ist. Diese Isomorphietypen lassen sich dann ausschließen, indem der Rekonstruktionsversuch zu einem Widerspruch führt. Danach möchten wir von den größten unter den möglichen Isomorphietypen zeigen, daß die Gruppe nicht nur lokal, sondern auch global und mitsamt ihrer Wirkung auf S eindeutig festgelegt ist. Wieder stellt sich heraus, daß die Kombinatorik von $\mathfrak{g}$ dazu führt, daß Mosterts Satz erfolgreich angewandt werden kann.

Die entsprechenden Argumente lassen sich in großer Allgemeinheit durchführen. Man gewinnt das folgende Resultat:

Sätze 3.1.4 und 3.1.6. *Eine kompakte zusammenhängende Lie-Gruppe G wirke fast treu auf einer verallgemeinerten n-Sphäre S, und die Wirkung sei nicht transitiv. Für die Lie-Algebra $\mathfrak{g}$ von G gelte $\mu(\mathfrak{g}) = n - 1$, so daß Mosterts Satz Anwendung findet.*

Dann ist jede Standgruppe zusammenhängend.

Die Wirkung sei sogar treu, und die Standgruppe eines Punktes mit maximaler Bahndimension sei in der Kommutatorgruppe G' enthalten. (Das ist etwa erfüllt, wenn G halbeinfach ist.) Dann tritt einer der beiden folgenden Fälle ein:

(i) *Die Wirkung ist äquivalent zur Einhängung einer transitiven Wirkung von G auf $\mathbb{S}_{n-1}$. Entweder gilt $G \cong \mathrm{SO}_n\mathbb{R}$, oder es gilt $n = 7$, und G ist isomorph zur Ausnahmegruppe G_2.*

(ii) *Die Wirkung ist äquivalent zum „join“ von zwei transitiven Wirkungen auf Sphären positiver Dimension. Es gibt also kompakte zusammenhängende Lie-Gruppen H_0 und H_1, die transitiv auf Sphären $\mathbb{S}_{n_0}$ und $\mathbb{S}_{n_1}$ wirken, und zwar so, daß die Wirkung von G auf S zur Wirkung von $H_0 \times H_1$ auf $\mathbb{S}_{n_0} * \mathbb{S}_{n_1}$ äquivalent ist. Es gilt dann $n = n_0 + n_1 + 1$. Jede der beiden Gruppen H_j*

ist isomorph zu $\mathrm{SO}_{n_j+1}\mathbb{R}$ *oder zu* G_2*, und im zweiten Fall gilt* $n_j = 6$.

Insbesondere ist die Wirkung von G *auf* S *zur natürlichen Wirkung einer Untergruppe von* $\mathrm{SO}_{n+1}\mathbb{R}$ *auf* $\mathbb{S}_n$ *äquivalent.*

Mit diesen Methoden erhalten wir drei Sätze über Sphären beliebiger Dimension, die die Ergebnisse von Richardson über höchstens vierdimensionale Sphären [111] fortsetzen. Um die auftretenden Binomialkoeffizienten einzuordnen, beachte man, daß $\binom{n}{2}$ die Dimension der Gruppe $\mathrm{SO}_n\mathbb{R}$ ist.

Sätze 3.5.1 und 3.5.4. *Eine kompakte Lie-Gruppe mit Lie-Algebra* $\mathfrak{g}$ *wirke treu auf einer verallgemeinerten* n*-Sphäre. Es gelte* $n \leq 8$ *oder* $\dim \mathfrak{g} > \binom{n-4}{2} + 3$*. Dann läßt* $\mathfrak{g}$ *eine Einbettung in* $\mathfrak{o}_{n+1}\mathbb{R}$ *zu.*

Das oben skizzierte Beweisverfahren liefert explizite Listen der möglichen Lie-Algebren.

Satz 3.6.11. *Eine kompakte zusammenhängende Lie-Gruppe* G *wirke treu auf einer verallgemeinerten* n*-Sphäre* S*. Es gelte* $\dim G > \binom{n-2}{2}+1$*. Dann ist die Wirkung von* G *auf* S *äquivalent zur natürlichen Wirkung einer Untergruppe von* $\mathrm{SO}_{n+1}\mathbb{R}$ *auf* $\mathbb{S}_n$*. Eine vollständige Liste der Gruppen und ihrer Wirkungen ist in Tabelle 3.3 auf Seite 114 zu finden.*

Man beachte, daß es eine Familie von treuen Wirkungen der Gruppe $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$ auf $\mathbb{S}_5$ gibt, die differenzierbar, aber nicht linear sind, und auch eine treue stetige Wirkung, die nicht differenzierbar ist.

Kapitel 4 führt ein ähnliches Programm für kompakte $(1,m)$-Vierecke durch. Ziel ist der Vergleich mit den sogenannten reell-orthogonalen Vierecken $Q(m+3,\mathbb{R})$, die als Absolutgeometrien von Polaritäten des reellen projektiven Raums mit Witt-Index 2 entstehen und die klassischen Referenzobjekte darstellen. Sie sind gerade die kompakten Moufang-Vierecke mit Parametern $(1,m)$, und sie treten in jeder Dimension auf. Das letztere Phänomen ist die zusätzliche Herausforderung, die die Theorie der kompakten Vierecke von ihrem Vorbild unterscheidet, der Theorie der kompakten Dreiecke bzw. projektiven Ebenen, wie sie von Salzmann und seiner Schule entwickelt worden ist.

Der erste Abschnitt stellt einige Werkzeuge von allgemeiner Bedeutung bereit. Insbesondere enthält er einen neuen, kurzen Beweis des

Satzes von Stroppel und Stroppel [129], daß eine lokal kompakte Gruppe, die treu auf einem kompakten Polygon von endlicher Dimension wirkt, ebenfalls von endlicher Dimension sein muß. Der Beweis läßt sich leicht auf andere topologische Inzidenzgeometrien übertragen. Er liefert auch erste Oberschranken für die Dimension wirkender kompakter bzw. kompakter abelscher Gruppen. Diese Schranken sind zwar selten scharf, aber sie sind brauchbar und für abelsche Gruppen auch nicht mehr weit zu verbessern.

In einem zweiten Abschnitt wird gezeigt, daß jede geradentransitive Wirkung einer kompakten zusammenhängenden Gruppe auf einem kompakten $(1, m)$-Viereck äquivalent zu einer Wirkung auf $Q(m+3, \mathbb{R})$ ist, wodurch auch die Gruppe bekannt ist. Dieser Satz ist auch in der Klassifikation der fahnenhomogenen kompakten Polygone durch Grundhöfer, Knarr und Kramer [50] implizit enthalten, wird dort aber mit anderen Methoden bewiesen.

Ziel der beiden übrigen Abschnitte ist eine Behandlung kompakter Gruppen auf kompakten $(1, m)$-Vierecken in Analogie zu den Ergebnissen für Sphären. Da die Methoden den dort beschriebenen ähneln, beschränken wir uns auf die Wiedergabe des Hauptresultats.

Satz 4.4.2 (Charakterisierung von $Q(m+3, \mathbb{R})$). *Sei G eine kompakte zusammenhängende Gruppe, die treu auf einem kompakten $(1, m)$-Viereck $Q = P \mathbin{\dot\cup} L$ wirkt, und sei $d := \dim G$.*

(a) *Gilt $d > \binom{m-1}{2} + 1$, so ist G eine Lie-Gruppe und der Punktraum P eine topologische Mannigfaltigkeit.*

(b) *Gilt $d > \binom{m-1}{2} + 4$, so ist jedes Geradenbüschel homöomorph zu $\mathbb{S}_m$. Dies gilt schon für $d > \binom{m-1}{2} + 2$, wenn eine der Aussagen $m = 7$ oder $m \geq 9$ gilt.*

(c) *Nun sei G eine Lie-Gruppe, und es gelte wenigstens eine der drei Bedingungen*

$$m \leq 4, \quad (m = 5 \text{ und } G \not\cong \mathbb{T}^5), \quad d > \binom{m-2}{2} + 11.$$

Dann läßt sich die Lie-Algebra von G in $\mathbb{R} \times \mathfrak{o}_{m+2}\mathbb{R}$ einbetten.

(d) *Falls schließlich d größer als $\binom{m+1}{2} + 1$ ist (für $m = 2$ muß d sogar größer als 5 sein), dann ist Q zum reell-orthogonalen Viereck*

$Q(m+3,\mathbb{R})$ isomorph (für $m=1$ bis auf Dualität), und die Wirkung von G auf Q ist zur Wirkung einer der beiden Gruppen $\mathrm{SO}_{m+2}\mathbb{R}$ oder $P(\mathrm{SO}_2\mathbb{R}\times\mathrm{SO}_{m+2}\mathbb{R})$ auf $Q(m+3,\mathbb{R})$ äquivalent.

Die Betrachtung kompakter $(4,m)$-Vierecke im letzten Kapitel erfordert noch einmal neue Methoden. Der Schlüssel für die Behandlung kompakter Lie-Gruppen ist die Beobachtung, daß die Dimension einer kompakten Lie-Gruppe vom Rang r, von wenigen Ausnahmen abgesehen, kleiner als die Dimension der Gruppe $\mathrm{U}_r\mathbb{H}$ ist, verbunden mit dem folgenden Ergebnis.

Satz 5.1.3. *Die elementar-abelsche Gruppe $G=(\mathbb{Z}/p)^r$ wirke treu auf einem kompakten (m,m')-Viereck und halte dabei eine offene Teilmenge einer Punktreihe punktweise fest. Dann hält G ein gewöhnliches Viereck fest. Ist $p=2$, so gilt $r\leq\frac{m'-1}{m}+1$. Für $p>2$ wird in der folgenden Tabelle eine obere Schranke für r angegeben.*

	m ungerade	*m gerade*
m' ungerade	$r\leq\dfrac{m'-1}{m+1}$	$r\leq\dfrac{m'-1}{m}$
m' gerade	$r\leq\dfrac{m'-2}{m+1}+1$	$r\leq\dfrac{m'-2}{m}+1$

Sind p und m' ungerade, so ist $\mathrm{Fix}\,G$ ein kompaktes (m,m'_0)-Unterviereck, dessen zweiter Parameter die Ungleichung $m'_0\leq m'-rm$ erfüllt.

Kompakte zusammenhängende Gruppen von endlicher Dimension, die keine Lie-Gruppen sind, sind das fast direkte Produkt einer kompakten zusammenhängenden abelschen Gruppe mit einer halbeinfachen kompakten Lie-Gruppe. Die Dimensionen dieser beiden Faktoren begrenzen sich gegenseitig. Um das einzusehen, werden einige Ergebnisse über Bahnen kompakter Gruppen entwickelt, die in eine Klassifikation der Bahnen kompakter zusammenhängender abelscher Gruppen in Satz 5.2.4 münden. Es stellt sich heraus, daß die Dimension einer kompakten Gruppe, die keine Lie-Gruppe ist, kleiner ist als die Dimensionen derjenigen Lie-Gruppen, die im ersten Abschnitt erfolgreich behandelt wurden.

Die klassischen Referenzobjekte sind hier Untergeometrien von projektiven Räumen über dem Schiefkörper $\mathbb{H}$ der Quaternionen.

Satz 5.3.1 (Charakterisierung von $H(n+1, \mathbb{H})$). *Sei G eine kompakte zusammenhängende Gruppe, die treu auf einem kompakten $(4, 4n-5)$-Viereck $Q = P \cup L$ wirkt. Die Dimension von G erfülle die folgenden Voraussetzungen:*

$$\begin{aligned} n < 4 \quad &: \quad \dim G > \binom{2n+1}{2} + 6 \\ n = 4 \quad &: \quad \dim G > \binom{2n+1}{2} + 9 = 45 \\ n > 4 \quad &: \quad \dim G > \binom{2n+1}{2} + 5. \end{aligned}$$

Dann gilt

$$G \cong \frac{\mathrm{U}_2\mathbb{H} \times \mathrm{U}_n\mathbb{H}}{\langle(-1,-1)\rangle} \quad \textit{oder} \quad G \cong \frac{\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H} \times \mathrm{U}_n\mathbb{H}}{\langle(-1,-1,-1)\rangle} \quad .$$

Die Wirkung von G auf Q ist zu der natürlichen Wirkung dieser Gruppe auf dem quaternional-hermiteschen Viereck $H(n+1, \mathbb{H})$ äquivalent.

Für $n = 2$ ist dieses Ergebnis scharf: Es gibt eine fast treue Wirkung der Gruppe $\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H} \times \mathrm{U}_2\mathbb{H}$ auf einem nicht-klassischen kompakten $(4, 3)$-Viereck.

Kompakte $(4, m)$-Vierecke, deren zweiter Parameter nicht von der Gestalt $4n-5$ ist, treten vermutlich nur für $m = 1$ und $m = 5$ auf. Der erste Fall ist nach dem Dualitätsprinzip schon in Satz 4.4.2 behandelt worden. Das Ergebnis für kompakte $(4, 5)$-Vierecke erreicht die gleiche Qualität; es ist in Satz 5.3.3 zu finden.

Chapter 1

Foundations

In this chapter, we collect a number of results which we will employ in the sequel. There are sections on locally compact groups, on cohomology manifolds, on a class of topological spaces called generalized spheres, and on compact generalized quadrangles. The first and fourth section are mere reproductions of known and easily accessible statements. They are repeated here to fix notation, and for ease of reference. Generalized spheres are introduced in order to have an axiomatic description which covers genuine spheres and, at the same time, point rows and line pencils of compact connected quadrangles. The overview of the theory of cohomology manifolds may be convenient as a kind of 'user's guide', although these spaces have become more accessible by the new edition of Bredon's monograph [16] on which our exposition rests. This section also contains a small generalization of Löwen's Theorem on metric ANRs with contractible point complements, see Remark 1.2.11.

There are at least two large areas of mathematics which we do not describe in spite of their importance for our work. Some algebraic topology will be needed, including the definition and basic properties of singular homology and cohomology, the definition of homotopy groups, and the exact homotopy sequence associated with a transitive action of a locally compact group. This exact sequence can be found in Chapter 9 of the monograph by Salzmann et al. [115, 96.12], which generally provides a lot of enjoyable background material. This remark applies also to the theory of compact Lie groups, the other important theory which we will use but not describe. In particular, we will need some classification and structure theory of these groups, and the representation theory of their Lie algebras. Further sources of reference in this field are Hofmann and Morris [57] and Tits [136].

1.1 Locally compact groups

There are different notions of the dimension of a topological space which is not a manifold. The reader is referred to the survey by Fedorchuk [41] or to Section 92 of Salzmann et al. [115]. The three most prominent dimension functions are covering dimension and small and large inductive dimension. These agree on separable metric spaces (cf. [115, 92.6 and 92.7]) and on coset spaces of locally compact groups with respect to closed subgroups (Pasynkov [99], cf. [115, 93.7]). As these are the classes of spaces in which we are mainly interested, we need not worry about the more subtle differences of the three dimension functions. For the sake of conciseness, the symbol dim will denote covering dimension. The reason for this choice is that covering dimension fits best with cohomological dimension, cf. Remark 1.2.6.

The foundations of the theory of locally compact groups, as developed in the first chapter of Hofmann's and Morris's book [57], will usually be taken for granted. To give the reader an idea of what we mean by foundations, we mention the existence of small open normal subgroups in totally disconnected compact groups [57, 1.34]. In this section, we will reproduce a few highlights of the theory, equally important but less elementary, which will be used at some point. The first of these is still comparatively easy.

1.1.1 Theorem (Open Mapping Theorem). *Let G be a locally compact σ-compact group acting continuously and transitively on a locally compact (Hausdorff) space X. Then for every point $x \in X$, the evaluation map*

$$g \longmapsto x^g : G \longrightarrow X$$

is open. In particular, any surjective continuous homomorphism of G onto a locally compact group is open.

Proof. See Freudenthal [46]; cf. Hohti [60] and Hewitt and Ross [54, 5.29]. □

The main tool for locally compact non-Lie groups is the fact that they can be approximated by Lie groups. This is exploited in Stroppel's paper [132]. For ease of reference, we repeat statement and proof of [132, Theorem 2.1].

1.1.2 Theorem (Approximation Theorem). *Let G be a locally compact group such that the quotient G/G^1 of G by its identity component G^1 is compact.*

(a) *For every neighbourhood U of 1 in G there exists a compact normal subgroup N of G such that $N \subseteq U$ and G/N admits local analytic coordinates that render the group operations analytic.*

(b) *If, moreover, $\dim G < \infty$, then there exists a neighbourhood V of 1 such that every subgroup $H \subseteq V$ satisfies $\dim H = 0$. That is, there is a totally disconnected compact normal subgroup N such that G/N is a Lie group with $\dim G = \dim G/N$.*

Proof. Montgomery and Zippin [93, Chapter IV], Gluškov [47, Theorem 9], see also Kaplansky [71, II.10, Theorem 18]. For compact groups, the theorem is due to Pontryagin [102], cf. van Kampen [70, Theorem 6]. □

The significance of the Approximation Theorem lies in the fact that it often allows to deduce results about locally compact groups from statements about Lie groups. Prominent examples are the following theorems.

1.1.3 Theorem (Structure of compact connected groups). *Let G be a compact connected group. Then there is a family $(S_j)_{j \in J}$ of simply connected almost simple compact Lie groups and there is a surjective homomorphism*

$$\eta : Z(G)^1 \times \prod_{j \in J} S_j \longrightarrow G$$

whose kernel is totally disconnected and has trivial intersection with $Z(G)^1 \times 1$. Moreover, the homomorphism η is unique up to an automorphism of its domain.

Proof. This is due to van Kampen [69]. For the Lie group case, he refers to Cartan [25]. A recent proof can be found in Hofmann's and Morris's book [57, 9.24]. □

1.1.4 Theorem (Mal'cev and Iwasawa). *Let G be a locally compact group such that G/G^1 is compact. Then every compact subgroup is contained in a maximal compact subgroup K, and all maximal compact subgroups are conjugate. There is a finite family of homomorphic embeddings*

$$\rho_1, \dots, \rho_n : \mathbb{R} \longrightarrow G$$

such that the map

$$\begin{array}{ccc} \mathbb{R}^n \times K & \longrightarrow & G \\ (t_1, \dots, t_n, k) & \longmapsto & \rho_1(t_1) \cdots \rho_n(t_n) \cdot k \end{array}$$

is a homeomorphism.

Proof. This has been proved for connected groups by Iwasawa [66, Theorem 13], and for Lie groups by Mal'cev [82]; cf. Hochschild [56, 3.1] and Hofmann and Terp [59]. □

1.2 Cohomology manifolds

In the theory of continuous actions of compact Lie groups, the appropriate class of topological spaces is the class of cohomology manifolds. These spaces appear as fixed point sets and as direct factors of (cohomology or genuine) manifolds, in particular as slices. Moreover, they share many of the nice properties of genuine manifolds, such as all kinds of Poincaré duality. Unfortunately, the definition is quite technical. The main difficulty is the need for specially tailored homology and cohomology theories.

Coefficients will usually be constant, and will be taken from a principal ideal domain R. We use Borel–Moore homology with compact supports, denoted by $\bar{H}^c_*(-)$, and sheaf cohomology with closed supports, denoted by $\bar{H}^*(-)$. Supports are chosen as to obtain the closest possible analogy with singular (co-)homology, see Remark 1.2.2 below. For the definition of sheaf-theoretic dimension, we shall also need sheaf cohomology with compact supports, which will be denoted by $\bar{H}^*_c(-)$. All spaces will be locally compact, which is necessary for the definition of Borel–Moore homology. Moreover, local compactness implies that compact subsets form a paracompactifying family of supports.

Our main source is Bredon's book on sheaf theory [16]. However, we will always give definitions which could also be used with the familiar singular (co-)homology. When necessary, we will supply the arguments which show that our definitions agree with Bredon's. Shorter treatments than ours are given by Löwen [81, Sections 3 and 4], which is the seminal paper for the application of sheaf-theoretic cohomology to topological incidence geometries, and by Kramer [74, Section 6.3].

1.2.1 Definition. A space X is *singular homology locally connected* (*HLC*) if for each point $x \in X$ and neighbourhood U of x, there is a neighbourhood $V \subseteq U$ of x such that the inclusion of V into U induces a trivial homomorphism

$$\tilde{H}_*(V;\mathbb{Z}) \xrightarrow{0} \tilde{H}_*(U;\mathbb{Z}),$$

where $\tilde{H}_*(-)$ denotes reduced singular homology.

Similarly, a space X is *cohomology locally connected over R in every degree* (clc_R^∞) if, given a point $x \in X$, a neighbourhood U of x and a degree $i \in \mathbb{N}_0$, there is a neighbourhood $V \subseteq U$ of x such that the inclusion of V into U induces a trivial homomorphism

$$\tilde{\bar{H}}^i(U;R) \xrightarrow{0} \tilde{\bar{H}}^i(V;R).$$

1.2.2 Remark. The condition clc_R^∞ is the main local connectivity property in sheaf-theoretic algebraic topology. It coincides with the analogous condition for Borel–Moore homology: a space is clc_R^∞ if and only if it is locally connected in every degree with respect to Borel–Moore homology over R (hlc_R^∞), see [16, V.12.4 and V.12.10]. Applying the Universal Coefficient Theorem [16, II.15.3] to compact neighbourhoods and using an elementary fact about exact sequences [16, II.17.3], we find that a locally compact space which is $clc_{\mathbb{Z}}^\infty$ is clc_R^∞ for any principal ideal domain R.

The significance of the *HLC* condition is that it implies the coincidence of the sheaf-theoretic with the singular theory. More precisely, if (X, A) is a pair of paracompact *HLC* spaces then there is a natural isomorphism $\bar{H}^*(X, A; R) \cong H^*(X, A; R)$, see [16, III.2.1]. Similarly, if (X, A) is a pair of locally compact *HLC* spaces then there is a natural isomorphism $\bar{H}^c_*(X, A; R) \cong H_*(X, A; R)$ by [16, V.13.6], cf. [16, V.1.19]. An *HLC* space is $clc_{\mathbb{Z}}^\infty$ by [16, p. 195], but the converse is not true, as examples show [16, II.17.12f.]. Since these examples are non-metric, it is conceivable (but could not be verified) that there is a closer connection between the two conditions in a restricted class of topological spaces, for example, in the class of completely metrizable spaces.

The reason for this guess is the following: in degree 0, the *clc* condition means local connectedness [16, p. 126], which, for complete metric spaces, implies local pathwise connectedness (Mazurkiewicz–Moore–Menger Theorem, see Kuratowski [79, §45, II.1], cf. Arkhangel'skiĭ and Fedorchuk [2, 8.3]), and this is just the *HLC* condition in degree 0. Also note that a second countable locally compact space is completely metrizable (see Querenburg [103, 10.16 and 13.16]).

1.2.3 Definition. The *homology sheaf* $\mathcal{H}(X;R)$ of a locally compact space X is the sheaf generated by the presheaf

$$U \longmapsto \bar{H}^c_*(X, X \setminus U; R).$$

Its stalk at $x \in X$, the so-called *local homology group* at x, is denoted by $\mathcal{H}_*(X;R)_x$ and satisfies

$$\mathcal{H}_*(X;R)_x = \bar{H}^c_*(X, X \setminus \{x\}; R).$$

1.2.4 Remark. Bredon [16, p. 293] defines $\mathcal{H}(X;R)$ to be generated by the presheaf $U \mapsto \bar{H}^{cld}_*(U;R)$, where the superscript denotes closed supports. If U is an open relatively compact subset of X then [16, V.5.10] yields that $\bar{H}^c_*(X, X \setminus U; R)$ is naturally isomorphic to $\bar{H}^{cld}_*(U;R)$. Hence the two definitions agree. Moreover

$$\bar{H}^c_*(X, X \setminus \{x\}; R) \cong \bar{H}^{cld}_*(X, X \setminus \{x\}; R) \cong \mathcal{H}_*(X;R)_x,$$

where the first isomorphism is given by [16, V.5.9], the second by [16, V.5.11].

Note, by the way, that $\bar{H}^{cld}_*(X;R) \cong \bar{H}^c_*(X \cup \{\infty\}, \{\infty\}; R)$, where $X \cup \{\infty\}$ is the one-point compactification of X, by [16, V.5.10]. Similarly, we obtain $\bar{H}^*_c(X;R) \cong \bar{H}^*(X \cup \{\infty\}, \{\infty\}; R)$ from [16, II.12.3].

A discussion of the homology sheaf for singular homology can be found in Salzmann et al. [115, Section 54].

1.2.5 Definition. The *sheaf-theoretic dimension* of a locally compact space X is defined as

$$\dim_R X := -1 + \min\{n \in \mathbb{N}_0 \mid \bar{H}^n_c(U;R) = 0 \text{ for each open } U \subseteq X\}.$$

(If the set on the right-hand side is empty then $\dim_R X := \infty$.)

1.2.6 Remark. A comprehensive treatment of sheaf-theoretic dimension can be found in Section II.16 of [16]. The definition which we have given comes from [16, II.16.14]. If $n > \dim_R X$ then $\bar{H}^n_c(A;R)$ vanishes for every locally closed subset A of X by [16, II.10.1, II.16.3 and II.16.6]. Moreover, if A is compact then $\bar{H}^*_c(A;R) = \bar{H}^*(A;R)$. In particular, if $\dim_R X$ is finite then the condition clc^∞_R is equivalent to clc_R, i.e. we can choose the smaller neighbourhood V independently of the degree i in Definition 1.2.1. Another important fact is that $\dim_R X$ is dominated by covering dimension and by small and large inductive dimension [16, II.16.34 and II.16.38f.]. Moreover $\dim_R X \leq \dim_{\mathbb{Z}} X$ by [16, II.16.15], and $\dim_{\mathbb{Z}} X$ agrees with the covering dimension of X if the latter is finite and X is paracompact, but need not agree if the covering dimension is infinite (see [16, p. 122] or Deo and Singh [30]; cf. Löwen [81, Section 4]).

1.2.7 Definition. An *n-dimensional homology manifold* over R (an *n-hm_R*) is a locally compact space X which satisfies the following properties:

(i) $\dim_R X = n$

(ii) For all $x \in X$ and $i \in \mathbb{N}_0$, $\mathcal{H}_i(X;R)_x \cong \begin{cases} R & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$

(iii) *Local orientability:* the homology sheaf $\mathcal{H}_*(X;R)$ is locally constant.

An *n-dimensional cohomology manifold* over R (*n-cm_R*) is an *n-hm_R* which is clc^∞_R (cf. [16, V.16.8]).

1.2.8 Lemma. *Let X be an n-hm_R, choose a point $x \in X$, and let A be a connected relatively compact subset of X which contains x and over which $\mathcal{H}_*(X;R)$ is constant. Suppose that A is either open or closed in X. Then the canonical map $\bar{H}^c_n(X, X \setminus A; R) \to \mathcal{H}_n(X;R)_x$ is an isomorphism.*

Proof. Löwen [81, 3.1] gives a proof for open A, using homology with closed supports. In view of [16, V.5.9], this carries over to compact supports when A is relatively compact. The proof for compact A follows the same lines but is slightly easier. Indeed, Poincaré duality [16, p. 330] yields a natural isomorphism $\bar{H}^c_n(X, X \setminus A; R) \cong$

$\bar{H}^0(A;\mathcal{H}_n(X;R))$, and $\bar{H}^0(A;\mathcal{H}_n(X;R))$ is naturally isomorphic to $\Gamma(\mathcal{H}_n(X;R)|A)$, the group of sections of the restriction of $\mathcal{H}_n(X;R)$ to A, by [16, p. 39]. Since $\mathcal{H}_n(X;R)|A$ is a constant sheaf, the canonical map $\Gamma(\mathcal{H}_n(X;R)|A) \to \mathcal{H}_n(X;R)_x$ is also an isomorphism. □

1.2.9 Theorem (Characterization of cohomology manifolds). *Let X be a locally compact connected clc_R^∞ space. Then the following are equivalent:*

(i) *X is an n-cm_R.*

(ii) *$\dim_R X < \infty$, and for all $x \in X$ and $i \in \mathbb{N}_0$,*

$$\mathcal{H}_i(X;R)_x \cong \begin{cases} R & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

(iii) *$\dim_R X = n < \infty$, and $\mathcal{H}_i(X;R)$ is locally constant for each i.*

If X is second countable then a fourth equivalent condition is

(iv) *$\dim_R X = n < \infty$, and the stalks $\mathcal{H}_i(X;R)_x$ are finitely generated and mutually isomorphic (i.e. independent of x) for each i.*

If R is a field or $R \cong \mathbb{Z}$ (but X need not be second countable) then a fifth equivalent condition is

(v) *$\dim_R X < \infty$, and for all $x \in X$ and $i \in \mathbb{N}_0$,*

$$\bar{H}^i(X, X \setminus \{x\}; R) \cong \begin{cases} R & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

Proof. See Bredon [16, V.16.3, V.16.8, V.16.9 and V.16.14]. □

1.2.10 Corollary. *Let X be a locally compact clc_R^∞ space with $\dim_R X = n < \infty$ such that every point $x \in X$ has acyclic complement, i.e.*

$$\forall\, x \in X : \tilde{\bar{H}}^c_*(X \setminus \{x\}; R) = 0.$$

Then X is an n-cm_R, and $\bar{H}^c_*(X;R) \cong \bar{H}^c_*(\mathbb{S}_n;R)$.

Proof. Choose $x \in X$. The long exact homology sequence of the pair $(X, X \setminus \{x\})$ (see [16, p. 305]) shows that the map $\tilde{\bar{H}}^c_*(X;R) \to \mathcal{H}_*(X;R)_x$ induced by inclusion is an isomorphism. This implies that $\mathcal{H}_*(X;R)$ is a constant sheaf (cf. Salzmann et al. [115, 54.4]). Therefore, the space X is an n-cm_R by part (iii) of the preceding theorem, which entails that

$$\tilde{\bar{H}}^c_*(X;R) \cong \mathcal{H}_*(X;R)_x \cong \tilde{\bar{H}}^c_*(\mathbb{S}_n;R).$$

□

1.2.11 Remark (Löwen's Theorem). This corollary, to which Löwen alludes at the beginning of [81, Section 6], allows to drop the compactness hypothesis from Löwen's Theorem [81, 6.2] on metric absolute neighbourhood retracts (ANRs) with contractible point complements. Without supposing compactness, Kramer's generalized version [74, 6.3.6] reads as follows: let X be a locally compact ANR of finite covering dimension n in which every point has acyclic complement, and suppose that there are two points whose complements are simply connected. Then X is an n-$cm_{\mathbb{Z}}$, and homotopy equivalent to $\mathbb{S}_n$. Except for the use of Corollary 1.2.10, Kramer's proof goes through unchanged.

If an n-cm_R X with $\bar{H}^c_*(X;R) \cong \bar{H}^c_*(\mathbb{S}_n;R)$ is second countable then X is compact after all. Indeed, the space X is connected by [16, V.5.14], and the Universal Coefficient Theorem [16, V.12.8] shows that $\bar{H}^n(X;R) \cong R$. Compactness follows from [16, p. 414, no. 26].

1.2.12 Proposition (Change of rings). *If X is an n-$cm_{\mathbb{Z}}$ then X is an n-cm_R for every principal ideal domain R.*

Proof. Since X is locally connected, it is the topological sum of its connected components, which can therefore be treated separately. We have seen in Remarks 1.2.2 and 1.2.6 that X is clc^∞_R, and that $\dim_R X < \infty$. By the preceding theorem, the proposition follows from the universal coefficient sequence [16, (13) on p. 294] which connects the stalks of $\mathcal{H}_*(X;\mathbb{Z})$ with those of $\mathcal{H}_*(X;R)$, because change of rings is valid for the Borel–Moore homology of X (cf. [16, V.15.1]). □

For implications in the opposite direction, see Raymond [106].

1.2.13 Theorem. *Let R be a countable principal ideal domain (respectively, a field) and X a locally compact metrizable space with $\dim_R X < \infty$. Then X is clc_R^∞ if and only if for all $x \in X$ and $i \in \mathbb{N}_0$, the stalk $\mathcal{H}_i(X;R)_x$ is countable (respectively, of countable dimension over R).*

Proof. See Mitchell [86] (cf. also Harlap [52, Theorem 8]) and, for the parenthetical case, Bredon [16, V.16.13], and note that clc_R and clc_R^∞ are equivalent for finite-dimensional spaces. ☐

Note that if R is a field then it is in fact unnecessary to suppose that X is metrizable. The first axiom of countability is sufficient. This implies that the following corollary could be slightly stronger if R is a field.

1.2.14 Corollary (Characterization in the metrizable case). *Let R be a countable principal ideal domain (respectively, a field) and X a locally compact connected metrizable space. Then the following are equivalent:*

(i) *X is an n-cm_R.*

(ii) *$\dim_R X < \infty$, and for all $x \in X$ and $i \in \mathbb{N}_0$,*

$$\mathcal{H}_i(X;R)_x \cong \begin{cases} R & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

(iii) *$\dim_R X = n < \infty$, all stalks $\mathcal{H}_i(X;R)_x$ are countable (respectively, of countable dimension over R), and $\mathcal{H}_i(X;R)$ is locally constant for each i.*

(iv) *$\dim_R X = n < \infty$, and the stalks $\mathcal{H}_i(X;R)_x$ are finitely generated and mutually isomorphic (i.e. independent of x) for each i.*

Proof. This follows directly from the two preceding theorems. When (iv) is supposed, note that each point in a locally compact metrizable space has an open neighbourhood which is second countable. Such a neighbourhood is an n-cm_R by the two theorems, and being an n-cm_R is a local property. ☐

1.2.15 Remark (Alternative definitions). The notion of a (co-)homology manifold goes back to the early days of topology. Wilder's book [143] gives a substantial exposition. In the theory of group actions, Smith [122] has introduced spaces which are closely related to cohomology manifolds, as was recognized by Conner and Floyd [26]. A culmination of this theory is represented by the seminar report of Borel et al. [8]. Note that Borel's definition of a cohomology manifold is equivalent to ours, as was shown by Raymond [106] (cf. Lemma 1.2.8). A proof of this fact for the case that R is a Dedekind ring was already given by Borel and Moore [9, 7.12]; cf. also Harlap [52, Theorem 11].

Under the name of generalized manifolds, homology manifolds that are euclidean neighbourhood retracts (ENRs) play an important role in geometric topology. (A topological space is an ENR if and only if it is separable, metrizable, of finite dimension, and locally contractible, see Hurewicz and Wallman [64, V.3] and Kuratowski [79, §49, VII.6], cf. Dugundji [35].) Cannon [22] has formulated the influential conjecture that among generalized manifolds, manifolds should be characterized by a small number of easily recognizable properties. Important progress was made by Cannon et al. [24], Edwards [40], Quinn [104], and others. Overviews are given by Lacher [80] and by Repovš ([108], [109], and [110]), and there is a monograph by Daverman [27]. For recent developments, see Daverman and Repovš [28] and Bryant et al. [20].

The definition of a cohomology manifold could be less technical if one was willing to restrict oneself to a nice class $\mathcal{C}$ of topological spaces. For example, local singular homology alone characterizes cohomology manifolds among locally compact finite-dimensional HLC metrizable spaces and, in particular, among ENRs. This approach would be really satisfactory if the class $\mathcal{C}$ would satisfy the following property: if a locally closed subset X of a cohomology manifold of class $\mathcal{C}$ is a cohomology manifold, then X belongs to $\mathcal{C}$. In addition, it might help to suppose that X is either a direct factor or the set of fixed points under a group action, and for certain classes $\mathcal{C}$ such as those mentioned above, a solution is easy in the former case.

Nevertheless, I have not succeeded in finding a class $\mathcal{C}$ with this property. Arguably, if a simpler notion of a generalized manifold was possible in the theory of group actions, even at the expense of generality, then Borel and others would not have built up the impressive technology of which we have had a glimpse.

1.3 Generalized spheres

We introduce a class of topological spaces which contains all spheres. Spaces from this class will occur as point rows and line pencils in compact quadrangles, which will be defined in the next section.

We recall a few definitions: a topological space X is called *locally homogeneous* if for every pair (x, y) of points of X there is a homeomorphism from some open neighbourhood of x onto an open neighbourhood of y which maps x to y. Secondly, a topological space X is called *pseudo-isotopically contractible to $x \in X$ relative to x* if there exists a homotopy $F : X \times [0,1] \to X$ such that $F(\cdot, 0) = \mathrm{id}_X$, both $F(\cdot, 1)$ and $F(x, \cdot)$ are constant maps to x, and for every $t \in]0,1[$, the map $F(\cdot, t)$ is a homeomorphism of X onto itself. This implies that X is locally contractible at x, i.e. every neighbourhood U of x contains a neighbourhood V of x such that the inclusion $V \hookrightarrow U$ is homotopic to a constant map. Indeed, a homotopy $F : X \times [0,1] \to X$ with the properties described above maps the compact set $\{x\} \times [0,1]$ to $\{x\}$, whence for any U, there is a neighbourhood V such that $F(V \times [0,1]) \subseteq U$, so that the claim follows.

1.3.1 Definition. A *generalized n-sphere* is a locally homogeneous n-dimensional ENR in which every point complement is non-empty and pseudo-isotopically contractible to one of its points relative to that point.

Note that $\mathbb{S}_n$ is indeed a generalized n-sphere.

1.3.2 Lemma. *Every generalized n-sphere is a compact n-$\mathrm{cm}_{\mathbb{Z}}$ and homotopy equivalent to $\mathbb{S}_n$.*

Proof. This follows immediately from Löwen's Theorem in the version which we have stated in Remark 1.2.11. □

In fact, we shall only need the following four properties of generalized spheres.

1.3.3 Lemma (Principal orbits). *Let G be a compact Lie group acting effectively on a generalized n-sphere S. Then G acts effectively on every single principal orbit.*

Proof. In view of the preceding lemma, this is the Montgomery–Yang Theorem 2.2.3. □

1.3.4 Lemma (Orbits of full dimension). *If $n > 0$ then every action of a compact Lie group G on a generalized n-sphere S which has an orbit of full dimension n is transitive.*

Proof. Let x^G be an n-dimensional orbit in S. Then x^G is a closed subset of S, and S is locally compact, separable metric, locally homogeneous, and locally contractible. Seidel [119] shows that x^G contains a non-empty open subset U of S. (This also follows from Bredon [16, V.16.18]). We infer that $x^G = U^G$ is open. Since it is also closed and S is connected, we conclude that $x^G = S$. □

1.3.5 Lemma (Orbits of small codimension). *Let G be a compact group acting effectively on a generalized n-sphere S, and suppose that some orbit has codimension at most 2. Then $S \approx \mathbb{S}_n$, and G is a Lie group.*

Proof. By Theorem 2.2.2 and Lemma 1.3.2, the group G is a Lie group, and S is a topological manifold. As in the proof given by Salzmann et al. [115, Theorem 52.3], we can infer from a theorem of Brown's [19] that S is homeomorphic to $\mathbb{S}_n$; see also Harrold [53]. □

1.3.6 Lemma (Smith's rank restriction). *Let G be a compact Lie group acting almost effectively on a generalized n-sphere S. Then* $\operatorname{rk} G \leq \lfloor \frac{n+1}{2} \rfloor = \operatorname{rk} \mathrm{SO}_{n+1}\mathbb{R}$.

Proof. This has been proved by Smith [123, no. 4], cf. Borel et al. [8, V.2.6]. Alternatively, it follows easily from [8, XIII.2.3]. □

1.3.7 Remark. In the definition of a generalized n-sphere, we require that every point complement is pseudo-isotopically contractible to one of its points relative to that point. This hypothesis may seem quite technical. However, in the applications which we have in mind, that is, in the theory of topological incidence geometries, the condition is verified naturally when one shows that the spaces under consideration are locally contractible, with contractible point complements. Alternatively, we could weaken the definition of a generalized sphere, requiring such a space to be a locally homogeneous ENR which satisfies the hypotheses of Löwen's Theorem as stated in Remark 1.2.11.

Then most of what we have done so far would go through. The only proof to fail would be that of the first conclusion of Lemma 1.3.5, that is, a generalized n-sphere – in the alternative sense – which is a topological manifold would not necessarily be an ordinary sphere. This probably does not do much harm since the methods which we will use for non-transitive actions in Chapter 3, including Lemma 1.3.6, work equally well for homotopy spheres, and the classification of effective and transitive actions on spheres can be used since a homogeneous homotopy sphere is a sphere by Theorem 3.1.1. The validity of some other statements such as Richardson's [111, 1.2] remains to be checked. Alternatively, one could use the generalized Poincaré conjecture which is known to hold if $n \neq 3$: a manifold which is a homotopy sphere (and whose dimension is different from 3) is an ordinary sphere, see Freedman [45] and Newman [96] and compare Kramer [74, 6.5.3]. In dimension 3, a simply connected manifold which admits a non-trivial action of a compact connected group is known to be a 3-sphere (Raymond [107, p. 52], cf. Orlik and Raymond [98, p. 298]). However, we have no need to invoke these deep results. We may as well stick to our original Definition 1.3.1.

1.4 Compact quadrangles

This section gives an overview of the theory of generalized quadrangles, as far as we will need it. Although all relevant terms are defined, the exposition is not intended to be an introduction for the beginner. For this purpose, we recommend the paper by Grundhöfer and Knarr [48]. Most of the facts which we reproduce here are taken from there, unless a reference is given.

Let us at least mention a few important pieces of work in this area. Generalized quadrangles, which are one kind of Tits buildings, have been introduced by Tits in [135] when he developed the theory of buildings (see Tits [137], Brown [18], or Ronan [113]). Monographs which concentrate on their algebraic and combinatorial theory have been written by Payne and Thas [100] and by Van Maldeghem [140]. Topological generalized quadrangles have first been investigated by Forst [44], and later by Grundhöfer and Knarr [48], Grundhöfer and Van Maldeghem [51], Knarr [73], and many others. Schroth has found a connection, in small dimensions, with topological circle planes, cf.

[118]. Kramer [74] has clarified the algebraic topology of compact connected quadrangles. Classification under hypotheses of transitive group actions have been accomplished by Grundhöfer, Knarr and Kramer ([49] and [50]), and by Kramer [75]. (The appendix of [49] is another source for Kramer's results on the algebraic topology of generalized quadrangles.) The study of non-transitive groups has been begun by Stroppel and Stroppel ([127] and [128]).

Before we state the definition of a generalized quadrangle, recall that the *girth* of a graph is defined as the length of a shortest cycle.

1.4.1 Definition. A *generalized quadrangle* is a triple $(Q, *, \tau)$, where $(Q, *)$ is a graph with vertex set Q and adjacency relation $*$, and $\tau : Q \to \{\text{'point'}, \text{'line'}\}$ is a map, subject to the following conditions:

(i) The graph $(Q, *)$ is bipartite, of diameter 4, of girth 8, and every vertex has at least 3 neighbours.

(ii) Setting $P := \tau^{-1}(\{\text{'point'}\})$ and $L := \tau^{-1}(\{\text{'line'}\})$, we obtain the given partition of $Q = P \mathbin{\dot\cup} L$.

The elements of P are called *points*, those of L *lines*. The set $F := \{(p, l) \in P \times L \mid p * l\}$ is called the set of *flags*. Adjacent vertices are also called *incident*. (Note that in a bipartite graph, adjacency is an anti-reflexive relation.)

Let d denote the path metric on the graph $(Q, *)$. For $k \in \mathbb{N}_0$ and $x \in Q$, define

$$\begin{aligned} D_k &:= \{(y, z) \in Q \times Q \mid d(y, z) = k\} \\ D_k(x) &:= \{y \in Q \mid d(x, y) = k\}. \end{aligned}$$

In particular, the *line pencil* of a point $p \in P$ is $L_p := D_1(p)$, the *point row* of a line $l \in L$ is $P_l := D_1(l)$, and $F = D_1 \cap (P \times L)$.

We will usually suppress the adjacency relation $*$ and the type map τ from notation. The type decomposition of the vertex set will be indicated by formulations like 'let $Q = P \mathbin{\dot\cup} L$ be a generalized quadrangle'. When no ambiguity arises, the notations of the preceding definition such as d, D_k and F will tacitly be used. This rarely causes trouble because one usually works with only one quadrangle (and with its substructures) at a time. A subset of the vertex set is always thought of as equipped with the restricted adjacency relation and type map, i.e. it will be regarded as a full subgraph.

Note that the roles of points and lines are completely symmetric, and can be interchanged. Formally, one gets a new generalized quadrangle by using the type map

$$\begin{array}{rcl} \tau' : Q & \longrightarrow & \{\text{`point'}, \text{`line'}\} \\ x & \longmapsto & \begin{cases} \text{`point'} & \text{if } \tau(x) = \text{`line'} \\ \text{`line'} & \text{if } \tau(x) = \text{`point'} \end{cases} \end{array}$$

instead of τ. The new generalized quadrangle is called the *dual quadrangle* of Q. Similarly, to *dualize* a statement about generalized quadrangles means to interchange points and lines in this statement. The dual of a generally valid assertion is again generally valid. Therefore, one often states only one of a pair of dual theorems.

The girth condition implies that whenever x and y are vertices of a generalized quadrangle whose distance satisfies $d(x, y) \leq 3$, the two are connected by exactly one path of length $d(x, y)$. One should imagine points and lines as such, and adjacency as incidence, i.e. a point p is adjacent to a line l if p 'lies on' l or, equivalently, if l 'runs through' p. Then any two different points p and p' are joined by at most one line, which is called $p \vee p'$ if it exists, and any two different lines l and l' meet in at most one point $l \wedge l'$. The characteristic property of a generalized quadrangle, however, is the following.

> Whenever (p, l) is a non-incident point-line pair, there is a unique line $\lambda(p, l)$ through p which meets l, and there is a unique point $\pi(p, l)$ on l which is joined to p.

Together with the mild assumptions that not every point lies on every line and that each vertex has at least three neighbours, this property actually characterizes generalized quadrangles. Thus we obtain an alternative definition which has the virtue of being more geometrically intuitive. We will comment on this point shortly.

1.4.2 Definition. Let $Q = P \mathbin{\dot\cup} L$ be a generalized quadrangle. A subset $S \subseteq Q$ is called *geometrically closed* if, for any $x, y \in S$ with $d(x, y) \leq 3$, the vertices of the unique shortest path from x to y belong to S. The *geometric closure* $\langle S \rangle$ of a subset $S \subseteq Q$ is the smallest subset Q which contains S and is geometrically closed. It is also called the subset which is *generated* by S.

The geometric closure of a subset $S \subseteq Q$ certainly contains all vertices on paths of length at most 3 which join two vertices of S, and also all vertices on 'short' paths between these new vertices. It is easy to see that every vertex of $\langle S \rangle$ occurs in this way after a finite number of steps. Indeed, the set which is constructed by countable repetition of this joining process is geometrically closed.

Many authors define a generalized quadrangle as a triple (P, L, F) which satisfies the alternative axioms described above. In the context of topological generalized quadrangles, this might be regarded as the established notation. The corresponding definition is hardly longer and gives better geometric insight. Our definition, which has been known for an equally long time, gives notational advantages in connection with the fixed subgeometry of a group action and, in particular, with the concept of geometric closure. The flag set, line pencils, point rows, etc. of $\operatorname{Fix} G$ or of $\langle S \rangle$ can simply be written as intersections. Thus we combine the notation from graph theory with the intuition from incidence geometry.

An isomorphism of generalized quadrangles is a graph isomorphism which maps points to points and lines to lines. When a group G acts on a generalized quadrangle $Q = P \mathbin{\dot\cup} L$, it is understood that G acts by automorphisms. A generalized quadrangle is called a *Moufang quadrangle* if it admits the action of a group G such that the following condition is satisfied.

> For every chain $a * b * c * d$ of pairwise different vertices, the subgroup of G which fixes $D_1(a) \cup D_1(b) \cup D_1(c)$ elementwise acts transitively on $D_1(d) \setminus \{c\}$.

Tits and Weiss work on an explicit classification of Moufang quadrangles (see [138] and [139]). Under the topological hypothesis which we define now, such a classification has been obtained by Grundhöfer and Knarr [48, 5.2].

1.4.3 Definition. A *compact quadrangle* is a generalized quadrangle $Q = P \mathbin{\dot\cup} L$ such that Q is a compact (Hausdorff) space, both P and L are closed in Q, and D_1 is closed in $Q \times Q$.

In a compact quadrangle, the vertices on a path of fixed length less than 4 depend continuously on the end vertices of the path. In other words, joining points and meeting lines are continuous operations, and the map

$$\begin{array}{rcl} (P \times L) \setminus F & \longrightarrow & F \\ (p, l) & \longmapsto & (\pi(p, l), \lambda(p, l)) \end{array}$$

is continuous.

Suppose that x and y are 'opposite' vertices of Q, i.e. that their distance satisfies $d(x, y) = 4$. (Note that x and y are either both points or both lines.) There is a bijection from $D_1(x)$ onto $D_1(y)$ which maps $x' \in D_1(x)$ to the unique $y' \in D_1(y)$ which satisfies $d(x', y') = 2$. Such a bijection is called a *perspectivity*. A concatenation of perspectivities is called a *projectivity*. It is not hard to see that the group of all self-projectivities of $D_1(x)$ acts doubly transitive. If Q is a compact quadrangle then every projectivity is a homeomorphism. In particular, any two point rows (respectively, line pencils) are homeomorphic, and each single one is doubly homogeneous. By a similar geometric construction, one finds, for every pair of adjacent vertices (x, y), a bijection

$$D_4(x) \cong (D_1(x) \setminus \{y\}) \times (D_1(y) \setminus \{x\})^2, \tag{1.1}$$

which is a homeomorphism in the case of a compact quadrangle. In this case $D_4(x)$ is an open subset of Q. More generally, when Q is a compact quadrangle and $0 \leq k \leq 3$, the subset $D_{\leq k} := \bigcup_{j \leq k} D_j \subseteq Q \times Q$ is compact. This entails that point rows and line pencils of compact quadrangles are compact.

A subset $R \subseteq Q = P \mathbin{\dot\cup} L$ of a compact quadrangle is called a *grid* if it is geometrically closed and contains an ordinary quadrangle, and if for every $p \in P \cap R$, the intersection $L_p \cap R$ consists of at most two elements. In fact, this implies that $|L_p \cap R| = 2$ whenever $p \in P \cap R$. Let l_1, l_2, l_1', l_2' be the lines of an ordinary quadrangle in R, with $d(l_1, l_1') = 4$. Then every line of R meets either l_1 or l_2. Thus $L \cap R$ can be written as the disjoint union

$$\begin{array}{rcl} L \cap R & = & (D_2(l_1) \cap R) \cup (D_2(l_2) \cap R) \\ & = & \big(\lambda(P_{l_1} \times \{l_1'\}) \cap R\big) \cup \big(\lambda(P_{l_2} \times \{l_2'\}) \cap R\big) \end{array}$$

of two relatively closed subsets. The two continuous maps

$$p \longmapsto \lambda(p, l_i') : P_{l_i} \cap R \longrightarrow D_2(l_i) \cap R$$

are homeomorphisms since their inverses $l \mapsto l \wedge l_i$ are continuous. Similarly, there is a homeomorphism

$$\begin{array}{rcl}(D_2(l_1) \cap R) \times (D_2(l_2) \cap R) & \longrightarrow & P \cap R \\ (k_1, k_2) & \longmapsto & k_1 \wedge k_2,\end{array}$$

because this map has local inverses of the type $p \mapsto (\lambda(p, l_1), \lambda(p, l_2))$. We record that

$$(P_{l_1} \cap R) \times (P_{l_2} \cap R) \approx (D_2(l_1) \cap R) \times (D_2(l_2) \cap R) \approx P \cap R.$$

Now suppose that $R = \langle S \rangle$ for some subset $S \subseteq Q$. Then S cannot be a connected subset of L. Suppose that S consists of points. An investigation of the generating process shows that every line of R runs through a point of S, i.e. $L \cap R = D_1(S) \cap R$. Suppose that S is closed in P and hence compact. Then $D_1(S)$ is compact, and so are the two sets

$$P_{l_i} \cap R \approx D_2(l_i) \cap R = \lambda(P_{l_i} \times \{l_i'\}) \cap D_1(S).$$

In the literature, a subset of a generalized quadrangle is sometimes called a *weak subquadrangle* if it is a subquadrangle, a grid, or a dual grid. In order to emphasize that a subquadrangle is not a weak subquadrangle, one can call it *thick*.

If $Q = P \cup L$ is a compact quadrangle then the spaces P, L, P_l, and L_p are either all connected or all totally disconnected. In the first case, the quadrangle Q is called a *compact connected quadrangle*.

Compactness of Q implies that P_l and L_p are metrizable separable spaces, so that the usual dimension functions agree on them. Their dimensions $m := \dim P_l$ and $m' := \dim L_p$ are called the *topological parameters* of Q. By the local product formula 1.1 and the product inequality for (small inductive or covering) dimension (see Salzmann et al. [115, 92.10]), the numbers m and m' are finite if and only if the dimension of P (or equivalently, that of L) is finite. For the sake of conciseness, we will use the following terminology:

1.4.4 Definition. A *compact (m, m')-quadrangle* is a compact quadrangle $Q = P \cup L$ with $\dim P_l = m$ and $\dim L_p = m'$ for all $l \in L$ and $p \in P$, where m and m' are positive integers.

1.4.5 Theorem. *Let $Q = P \cup L$ be a compact (m, m')-quadrangle. Then every point row P_l and every line pencil L_p is a generalized sphere in the sense of Section 1.3.*

Proof. Grundhöfer and Knarr [48, 3.1, 4.1, and 4.2], and Hurewicz and Wallman [64, V.3] □

The local product formula 1.1 implies that the point space P and the line space L are ENR cohomology manifolds over $\mathbb{Z}$, and that $\dim P = 2m + m'$ and $\dim L = 2m' + m$ (cf. Bredon [16, V.16.11]).

1.4.6 Theorem (Knarr and Kramer). *Let Q be a compact (m, m')-quadrangle with $m, m' > 1$. Then either $m = m' \in \{2, 4\}$, or $m + m'$ is odd.*

Proof. This was proved by Knarr under the hypothesis that point rows and line pencils are genuine manifolds [73], and by Kramer in the general case [74, 3.3.6]. □

Stolz has recently proved a result in differential geometry [125] which implies further restrictions on m and m'. The state of the art seems to be the following: the pair $(4, 4)$ is impossible (Kramer and Van Maldeghem [78, 2.7]). Suppose that $m < m'$, and set

$$k := |\{s \in \mathbb{Z}_{>0} \mid s < m,\ s \leq m' - m, \text{ and } s \equiv 0, 1, 2, 4 \pmod 8\}|.$$

Then $m + m' + 1 \in 2^k\mathbb{Z}$ (Markert [85], see also Kramer [75, 7.20] and [76]).

Kramer [74] has computed many homotopy groups and cohomology rings of compact (m, m')-quadrangles. His results are reproduced by Grundhöfer, Knarr and Kramer [49, Appendix].

Examples for compact (m, m')-quadrangles are obtained in a number of ways. We sketch the construction of the so-called *classical quadrangles* which are subgeometries of projective spaces. Let $\mathbb{F}$ be a locally compact connected skew field, i.e. let $\mathbb{F}$ be isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and choose a non-degenerate sesquilinear form β of Witt index 2 on the right vector space $\mathbb{F}^{n+1}$. (Recall that the Witt index of a sesquilinear form β is the dimension of a maximal totally isotropic subspace, i.e. of a maximal subspace on which β vanishes identically. By Witt's Theorem, cf. Scharlau [117, Ch. 7 §9], all such subspaces are conjugate under semi-linear automorphisms of $\mathbb{F}^{n+1}$ which preserve β.)

Suppose first that β is hermitian with respect to the standard anti-automorphism $x \mapsto \overline{x}$ of $\mathbb{F}$. Up to a change of basis, such a form can be written as

$$\begin{array}{rcl} \beta\colon \mathbb{F}^{n+1} \times \mathbb{F}^{n+1} & \longrightarrow & \mathbb{F} \\ (x,y) & \longmapsto & -\overline{x_0}y_0 - \overline{x_1}y_1 + \overline{x_2}y_2 + \cdots + \overline{x_n}y_n. \end{array}$$

Let P (respectively L) be the set of one-dimensional (respectively, two-dimensional) totally isotropic subspaces of $\mathbb{F}^{n+1}$, and take inclusion as adjacency relation. Then $Q = P \cup L$ is a compact $(d, d(n-2)-1)$-quadrangle, where $d := \dim_{\mathbb{R}} \mathbb{F}$. These quadrangles are called the real orthogonal quadrangles $Q(n,\mathbb{R})$ and the hermitian quadrangles $H(n,\mathbb{C})$ and $H(n,\mathbb{H})$.

Apart from these infinite series, a few more examples arise from the same procedure: if β is a symplectic form on $\mathbb{F}^4$ with $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ then the construction yields a compact (d,d)-quadrangle, the symplectic quadrangle $W(\mathbb{F})$. A symmetric bilinear form on $\mathbb{C}^5$ leads to a compact $(2,2)$-quadrangle, the complex orthogonal quadrangle $Q(4,\mathbb{C})$, and an anti-hermitian form on $\mathbb{H}^{n+1}$ (where $n \in \{3,4\}$) gives rise to a compact $(4, 4n-11)$-quadrangle, the anti-unitary quadrangle $H^\alpha(n,\mathbb{H})$.

Note that some pairs of parameters appear twice, possibly in the reverse order. Whenever this is the case, the corresponding quadrangles are dual to each other. Together with a coset geometry of the real simple Lie group $E_{6(-14)}$, the classical quadrangles, i.e. those described in this paragraph, and their duals form the complete list of compact connected Moufang quadrangles.

Let us mention the other important construction principles. Using work of Ferus, Karcher, and Münzner [42], Thorbergsson [134] has constructed a compact connected quadrangle out of every real representation of a real Clifford algebra (see Husemoller [65] for their representation theory). These examples include the compact connected Moufang quadrangles, and they cover all parameters (m, m') known to date.

There are two ways of obtaining further examples, which may be quite inhomogeneous. Joswig [68, 2.23] has shown that a certain construction which is due to Tits (see [68, 1.37], cf. Dembowski [29, p. 304]) leads to compact $(1,m)$-quadrangles. Finally, Forst [44] and Schroth [118] have developed an intimate connection between locally compact circle planes and compact quadrangles whose topological parameters are $(1,1)$ or $(2,2)$.

The theory of generalized quadrangles forms part of the theory of buildings or, more precisely, of generalized polygons. These are precisely the spherical buildings of rank 2, i.e. those spherical buildings which consist of points and lines only when they are considered as incidence structures. A *generalized n-gon* can be defined as a triple $(V, *, \tau)$ where $(V, *)$ is a bipartite graph of diameter n and girth $2n$ in which every vertex has at least 3 neighbours, and $\tau : V \to \{\text{'point'}, \text{'line'}\}$ is a type map which is compatible with the partition of V. Thus generalized 4-gons are just generalized quadrangles. Similarly, generalized 3-gons are just projective planes. Most of the notions and results described in this section carry over to generalized polygons. The general reference is Kramer's thesis [74]; for projective planes, a wealth of results is collected in the monograph by Salzmann et al. [115]. The Knarr–Kramer Theorem 1.4.6 was only stated for generalized quadrangles above. Its complete version says that if $P \cup L$ is a compact connected n-gon of finite dimension then $n \in \{3,4,6\}$. If $n = 3$ then $m = m' \in \{1,2,4,8\}$, if $n = 4$ then m and m' are as described above, and if $n = 6$ then $m = m' \in \{1,2,4\}$. For projective planes, the statement is due to Löwen [81]; examples for each possible pair of topological parameters are provided by the so-called classical projective planes, i.e. those over $\mathbb{R}$, over $\mathbb{C}$, over $\mathbb{H}$, and over Cayley's octonion algebra $\mathbb{O}$.

Chapter 2

Actions of compact groups

2.1 General theory

When a compact group acts on a Hausdorff space, there is a nice correspondence between a point and its stabilizer, and between an orbit and the kernel of the restricted action on that orbit. This correspondence can be expressed by the Hausdorff topology on the set of all closed subgroups, but we may content ourselves with direct statements in terms of the group topology, which are in fact stronger. We first give the elementary results 2.1.1 and 2.1.3. Then we quote a theorem about Lie groups which entails a particularly close relation between stabilizers of neighbouring points in actions of compact Lie groups (2.1.5f.). After two results about finiteness of dimension (2.1.7f.), we use approximation by Lie groups to prove Theorem 2.1.15 which describes how the orbit dimension and the identity components of stabilizers and kernels vary in the case of arbitrary compact groups. The counterexample in Remark 2.1.16 shows that no stronger statement is possible in general.

The most important results of this section, namely Theorem 2.1.5 and its corollary, Theorem 2.1.7, and most of Theorem 2.1.15 together with its corollaries, are essentially due to Montgomery and Zippin [93, Chapter VI], and so is some of the preparatory material. As they point out in their preface, they do not consider their presentation to be "a complete and detailed exposition". Therefore, it may not be superfluous to give new proofs at some points. This also leads to statements which are a little stronger.

After Montgomery's and Zippin's classic [93], a number of comprehensive treatises on actions of compact groups have been written. We mention the books by Borel et al. [8], Bredon [14], Hsiang [62], tom Dieck [31], and Allday and Puppe [1]. The generality of the setting varies. In particular, tom Dieck is almost exclusively concerned with differentiable actions of Lie groups on manifolds.

2.1.1 Lemma (Close points have close stabilizers). *Let G be a compact group acting on a Hausdorff space X, choose a point $x \in X$, and let U be a neighbourhood of the stabilizer G_x of x. Then x has a neighbourhood V such that U contains the stabilizers of all points in V:*

$$\exists\, V \in \mathcal{U}(x)\ \forall\, y \in V : G_y \subseteq U$$

Proof. We may assume that U is open so that its complement $G \setminus U$ is compact. Then x is not contained in the compact set $x^{G\setminus U}$, and hence there are disjoint neighbourhoods V_1 of x and W of $x^{G\setminus U}$. Since $x^{G\setminus U}$ is a continuous image of $\{x\} \times (G \setminus U)$, there is a neighbourhood V_2 of x such that

$$V_2^{G\setminus U} = \{y^g | y \in V_2, g \in G \setminus U\} \subseteq W.$$

Let $V := V_1 \cap V_2$. □

There is an analogous result for kernels of the actions on close orbits. In order to deduce this from Lemma 2.1.1, we need a lemma about uniform spaces. For the sake of simplicity, it will be formulated as a statement about topological groups.

2.1.2 Lemma. *Let G be a Hausdorff group, let $\mathcal{K}$ be a set of compact subsets of G, and let U be a neighbourhood of $\bigcap \mathcal{K}$ in G. Then there is a neighbourhood V of the identity in G and a finite subset $\{K_1, \ldots, K_n\} \subseteq \mathcal{K}$ such that*

$$\bigcap_{i=1}^{n} K_i V \subseteq U.$$

Proof. By compactness, there is a finite subset $\{K_1, \dots, K_n\} \subseteq \mathcal{K}$ such that $\bigcap_{i=1}^n K_i \subseteq U$. We may suppose that U is an open subset of G. Let $C := K_1 \times \cdots \times K_n \subseteq G^n$, and set

$$D := \{(x, x, \dots, x) \in G^n \mid x \in G \setminus U\}.$$

Then C is compact, the subset D is closed, and $C \cap D = \emptyset$, whence there is a neighbourhood V of 1 in G such that the neighbourhood $K_1V \times \cdots \times K_nV$ of C does not meet D. This implies that $\bigcap_{i=1}^n K_iV$ is contained in U. $\square$

2.1.3 Lemma (Close orbits have close kernels). *Let G be a compact group acting on a Hausdorff space X, choose a point $x \in X$, and let U be a neighbourhood of the kernel $G_{[x^G]}$ of the action on the orbit x^G. Then x has a neighbourhood V such that U contains all kernels which correspond to orbits of points in V:*

$$\exists\, V \in \mathcal{U}(x)\; \forall\, y \in V : G_{[y^G]} \subseteq U$$

Proof. Applying Lemma 2.1.2 to $\mathcal{K} := \{{G_x}^g | g \in G\}$, we find a neighbourhood W of $1 \in G$ and a finite subset $F \subseteq G$ such that

$$\bigcap_{g \in F} {G_x}^g W \subseteq U.$$

Set $W' := \bigcap_{g\in F} gWg^{-1}$. Then Lemma 2.1.1 yields a neighbourhood V of $x \in X$ such that $G_y \subseteq G_xW'$ holds for each $y \in V$. This entails

$$G_{[y^G]} \subseteq \bigcap_{g\in F} {G_y}^g \subseteq \bigcap_{g\in F} (G_xW')^g \subseteq \bigcap_{g\in F} {G_x}^g W \subseteq U.$$

$\square$

2.1.4 Remark. For the sake of completeness and comparison with Montgomery and Zippin [93], we note that analogues of Lemmas 2.1.1 and 2.1.3 hold for the identity components of stabilizers and kernels. This is due to the following topological fact: if H is a closed subgroup of a compact group G and U is a neighbourhood of the identity component H^1, then there is a neighbourhood V of H such that for every subgroup K of G which is contained in V, the identity component K^1 lies within U.

Indeed, since connected components and quasi-components coincide in the compact Hausdorff space H, there is a subset L of H which is both open and closed in H, which lies within U, and which contains 1. Let V_1 and V_2 be disjoint neighbourhoods of the compact sets L and $H \setminus L$, respectively, and set $V := (U \cap V_1) \cup V_2$. Then V is a neighbourhood of H which has the desired property.

Stronger results hold if we suppose that G is a compact Lie group (Corollary 2.1.6) or if we restrict our attention to identity components (Theorem 2.1.15).

2.1.5 Theorem (Close subgroups of Lie groups). *Let G be a Lie group, let K be a compact subgroup of G, and let U be a neighbourhood of the identity element in G. Then K has a neighbourhood V such that every subgroup H of G contained in V is conjugate to a subgroup of K by an element g of U:*

$$\exists\, V \in \mathcal{U}(K)\; \forall\, H \leq G : H \subseteq V \Rightarrow\; \exists\, g \in U : H^g \leq K$$

Proof. This is a minor generalization of a result due to Montgomery and Zippin [93, Section 5.3]. They prove that

$$\exists\, W \in \mathcal{U}(K)\; \forall\, H \leq G : H \text{ compact, } H \subseteq W \Rightarrow\; \exists\, g \in U : H^g \leq K$$

If we choose a neighbourhood V of K whose closure is compact and contained in W then every subgroup contained in V has compact closure, and this closure is contained in W and hence is the conjugate of a subgroup of K by an element of U. For alternative approaches, see Corollary II.5.6 and the following Remark in Bredon's book [14]. $\square$

Recall that a *principal stabilizer* of some action of a compact Lie group is a stabilizer of minimal dimension which, subject to this first condition, has the smallest possible number of components. The corresponding orbit is called a *principal orbit*.

2.1.6 Corollary (Stabilizers in compact Lie groups). *Let G be a compact Lie group acting on a Hausdorff space X, and let U be a neighbourhood of the identity element in G. Then every point $x \in X$ has a neighbourhood V such that all stabilizers of points in V are conjugate to subgroups of the stabilizer G_x of x by elements of U:*

$$\exists\, V \in \mathcal{U}(x)\; \forall\, y \in V\; \exists\, g \in U : {G_y}^g \leq G_x$$

In particular, principal orbits exist, the points on principal orbits form an open subset, and the kernel of the action of G on a principal orbit fixes a neighbourhood of that orbit pointwise.

Proof. Suppose $x \in X$, and let W be a neighbourhood of G_x such that every subgroup of G contained in W is conjugate to a subgroup of G_x by an element of U. By Lemma 2.1.1, there is a neighbourhood V of x such that all stabilizers of points in V are contained in W.

The existence of principal orbits follows because, in a Lie group, every descending chain of compact subgroups becomes stationary. If the point x lies on a principal orbit then the stabilizers of points in V belong to a single conjugacy class. □

We collect some applications of the Approximation Theorem 1.1.2 to actions of non-Lie groups. The first result concerns transitive actions of locally compact groups.

2.1.7 Theorem (Transitive actions on finite-dimensional spaces). *Let G be a locally compact group acting effectively and transitively on a locally compact space X. Suppose that G/G^1 is compact, and that $n := \dim X$ is finite. Then the dimension of G is finite. If G is compact and connected then $\dim G \le \binom{n+1}{2}$.*

Proof. The Open Mapping Theorem 1.1.1 shows that the action of G on X is equivalent to the action on a coset space. Hence the finiteness of $\dim G$ is Theorem 6.2.2 of Montgomery and Zippin [93]. Their standing hypothesis that G is separable metric is not essential to their proof.

If G is compact, then the upper bound on $\dim G$ is stated as Corollary 2 on page 243 of Montgomery and Zippin [93]. We indicate a proof due to Mann [84]. Choose a point $x \in X$. Since G is of finite dimension, Theorem 1.1.2 provides us with a totally disconnected compact normal subgroup N of G such that G/N is a Lie group. Since G/N acts transitively on $X/N \approx G/(G_xN)$, this space is a manifold. We use Nagami's Dimension Formula

$$\dim G = \dim N + \dim G/N \tag{2.1}$$

(see Salzmann et al. [115, 93.7] and note that the formula holds whenever N is a closed subgroup of a locally compact group G). The formula shows that

$$\dim G_x N = \dim \frac{G_x N}{N} = \dim \frac{G_x}{G_x \cap N} = \dim G_x,$$

which implies $\dim X/N = \dim X$. By the corresponding result for Lie groups (Montgomery and Zippin [93, page 243]), we have $\dim G/N \leq \binom{n+1}{2}$. □

2.1.8 Corollary (Comparing codimensions). *Let G be a locally compact group such that G/G^1 is compact, and let H_1 and H_2 be closed subgroups of G such that $H_1 \leq H_2$. Then*

$$\dim G/H_1 \geq \dim G/H_2,$$

and if $\dim G/H_1 = \dim G/H_2 < \infty$ then ${H_1}^1 = {H_2}^1$.

Proof. If $\dim G/H_1$ is infinite, there is nothing to prove. Otherwise, the preceding theorem shows that the kernel of the action of G on G/H_1 is of finite codimension, and so is its identity component

$$K := \left(\bigcap_{g \in G} {H_1}^g \right)^1$$

by the Dimension Formula 2.1. The subgroup H_1 is a stabilizer of the natural action of G on $\frac{G/K}{H_1/K}$, whence

$$G/H_1 \approx \frac{G/K}{H_1/K} \quad .$$

The same statement holds for H_2 in the place of H_1. Using this and the Dimension Formula, we find

$$\begin{aligned} \dim G/H_1 &= \dim \frac{G/K}{H_1/K} \\ &= \dim G/K - \dim H_1/K \\ &\geq \dim G/K - \dim H_2/K \\ &= \dim \frac{G/K}{H_2/K} \\ &= \dim G/H_2. \end{aligned}$$

Equality holds if and only if $\dim H_1/K = \dim H_2/K$. It follows from the Dimension Formula that this implies

$$H_1{}^1/K = (H_1/K)^1 = (H_2/K)^1 = H_2{}^1/K$$

(cf. Salzmann et al. [115, 93.12]), where the outer equalities come from the fact that connectedness is an extension property. We conclude that $H_1{}^1 = H_2{}^1$. □

There is no complete analogue of Corollary 2.1.6 for compact non-Lie groups: stabilizers of close points are no longer related in such a nice way. Nevertheless, there is a "connected version" of Corollary 2.1.6 for orbits of finite dimension. This will be made precise in Theorem 2.1.15 which we prepare by some results of a more general nature.

2.1.9 Lemma (Connected component and intersection). *If Y is a topological space and $\mathcal{Z}$ is a collection of subsets of Y then for every point $y \in Y$,*

$$\left(\bigcap_{Z\in\mathcal{Z}} Z^y\right)^y = \left(\bigcap \mathcal{Z}\right)^y$$

where Z^y denotes the connected component of Z which contains y.

Proof. The forward inclusion is obvious. To see the reverse inclusion, first note that for every $Z \in \mathcal{Z}$, the connected component Z^y contains $(\bigcap \mathcal{Z})^y$. □

2.1.10 Proposition (Identity component and homomorphic image). *Let $\varphi : G \to H$ be an open surjective homomorphism of locally compact groups. Then*

$$\overline{\varphi(G^1)} = H^1.$$

Proof. The left-hand side is contained in the right-hand side since $\varphi(G^1)$ is connected and H^1 is closed. To see the reverse inclusion, it suffices to prove that $H_0 := H/\overline{\varphi(G^1)}$ is totally disconnected. To achieve this, we will show that H_0 contains small open subgroups.

Let U be a neighbourhood of the identity element in H_0, and let $\overline{\varphi} : G/G_1 \to H_0$ be the natural surjection induced by φ. As G/G^1 is a totally disconnected locally compact group, the preimage $\overline{\varphi}^{\,-1}(U)$ contains an open subgroup K. Since $\overline{\varphi}$ is an open map, the image $\overline{\varphi}(K)$ is an open subgroup of H_0, and it is contained in U. □

To illustrate the subtlety of the situation, we draw attention to the following example. Choose a prime number p, and let

$$\begin{aligned}
\mathbb{Z}_p &= \left\{ (k_n + p^n\mathbb{Z})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \frac{\mathbb{Z}}{p^n\mathbb{Z}} \;\middle|\; \forall n \in \mathbb{N} : k_{n+1} - k_n \in p^n\mathbb{Z} \right\} \\
\mathbb{T}_p &= \left\{ (x_n + p^n\mathbb{Z})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \frac{\mathbb{R}}{p^n\mathbb{Z}} \;\middle|\; \forall n \in \mathbb{N} : x_{n+1} - x_n \in p^n\mathbb{Z} \right\}
\end{aligned}$$

be the group of p-adic integers and the p-adic solenoid (cf. Hofmann and Morris [57, 1.28]). Consider the continuous open surjective homomorphism

$$\begin{aligned}
\varphi : \qquad\qquad \mathbb{R} \times \mathbb{Z}_p &\longrightarrow \mathbb{T}_p \\
\big(x, (k_n + p^n\mathbb{Z})_{n\in\mathbb{N}}\big) &\longmapsto (x + k_n + p^n\mathbb{Z})_{n\in\mathbb{N}}.
\end{aligned}$$

Then the image of the identity component $\mathbb{R} \times \{0\}$ is a proper dense subgroup of $\mathbb{T}_p$. In fact, the homomorphism φ is a covering map since its kernel is discrete, so that $\varphi(\mathbb{R} \times \{0\})$ is the path component of the identity element.

2.1.11 Proposition (Identity component and complex product). *Let G be a locally compact σ-compact group, and let H and N be closed subgroups of G. Suppose that N is a normal subgroup, and that H or N is compact. Then*

$$H^1N^1 = (HN)^1.$$

Proof. We form the semi-direct product $H \ltimes N$, where H acts on N by conjugation, and we consider the surjective homomorphism

$$\begin{aligned}
\mu : H \ltimes N &\longrightarrow HN \\
(h, n) &\longmapsto hn.
\end{aligned}$$

As either H or N is compact, their product HN is a closed subgroup of G. Now HN is σ-compact, whence the Open Mapping Theorem 1.1.1 shows that we can apply Proposition 2.1.10 to μ. Since H^1N^1 is closed as well (and for the same reason), this yields

$$H^1N^1 = \overline{H^1N^1} = \overline{\mu(H^1 \ltimes N^1)} = \overline{\mu\left((H \ltimes N)^1\right)} = (HN)^1.$$

$\square$

2.1.12 Remark (Alternative proof). If $\dim HN/N < \infty$ then there is a shorter if less elementary proof of the last proposition. Use that $(H^1 \cap N^1)^1 = (H \cap N)^1$, and observe

$$\dim \tfrac{H^1N^1}{N^1} = \dim \tfrac{H^1}{H^1 \cap N^1} = \dim \tfrac{H}{H \cap N} = \dim \tfrac{HN}{N} = \dim \tfrac{(HN)^1}{N^1}.$$

2.1.13 Proposition (Close connected subgroups of locally compact groups). *Let G be a locally compact group such that G/G^1 is compact, let K be a compact subgroup of G whose codimension $\dim G/K$ is finite, and let U be a neighbourhood of the identity element in G. Then K has a neighbourhood V such that every connected subgroup H of G contained in V is conjugate to a subgroup of K by an element g of U:*

$$\exists\, V \in \mathcal{U}(K)\ \forall\, H \leq G : H \text{ conn.}, H \subseteq V \Rightarrow\ \exists\, g \in U : H^g \leq K$$

Proof. Theorem 2.1.7 on transitive actions shows that the kernel $L := \bigcap_{g \in G} K^g$ of the action of G on the finite-dimensional space G/K has finite codimension. By part (b) of the Approximation Theorem 1.1.2, the group G/L does not have small connected subgroups. Thus we find a neighbourhood W_1 of L such that L contains the identity components of all subgroups of G which lie within W_1. Let N be a compact normal subgroup of G such that N is contained in W_1 and G/N is a Lie group. Then the identity component N^1 of N is contained in L and hence in K.

Let $\mathrm{pr} : G \to G/N$ denote the canonical projection. By Theorem 2.1.5, there is a neighbourhood W_2 of $\mathrm{pr}(K)$ in G/N such that every subgroup of G/N which is contained in W_2 is conjugate to a subgroup of $\mathrm{pr}(K)$ by an element of $\mathrm{pr}(U)$. Set $V := \mathrm{pr}^{-1}(W_2)$, and let H be a connected subgroup of G which lies within V. Then $\mathrm{pr}(H)$ is a subgroup of G/N which is contained in W_2. Therefore, we find an element $g \in U$ such that $\mathrm{pr}(H^g) = \mathrm{pr}(H)^{\mathrm{pr}(g)} \leq \mathrm{pr}(K)$. This entails

$$H^g \leq \mathrm{pr}^{-1}(\mathrm{pr}(H^g)) \leq \mathrm{pr}^{-1}(\mathrm{pr}(K)) = KN.$$

Using Proposition 2.1.11, we infer that

$$H^g = (H^g)^1 \leq (KN)^1 = K^1 N^1 = K^1 \leq K.$$

□

2.1.14 Remark. As follows from the statements about small subgroups made in the Approximation Theorem 1.1.2, neither the finite codimension of K nor the connectedness of H is dispensable if G is not a Lie group.

2.1.15 Theorem (Stabilizers in compact non-Lie groups). *Let G be a compact group acting on a Hausdorff space X, let U be a neighbourhood of the identity element in G, and choose $x \in X$. Suppose that the orbit x^G has finite dimension, and let $G_{[x^G]}$ denote the kernel of the action of G on x^G. Then*

$$\exists\, V \in \mathcal{U}(x)\; \forall\, y \in V\; \exists\, g \in U : ({G_y}^g)^1 \leq G_x.$$

In particular, the relations $\dim y^G \geq \dim x^G$ and $(G_{[y^G]})^1 \leq G_{[x^G]}$ hold for every $y \in V$. Moreover, if $\dim y^G = \dim x^G$ then $({G_y}^g)^1 = (G_x)^1$ and $(G_{[y^G]})^1 = (G_{[x^G]})^1$.

Proof. As the dimension of $G/G_x \approx x^G$ is finite, Proposition 2.1.13 yields a neighbourhood W of G_x such that every connected subgroup of G which is contained in W is conjugate to a subgroup of G_x by an element g of U. Lemma 2.1.1 provides a neighbourhood V of x in X such that for every point $y \in V$, the stabilizer G_y is contained in W, so that there is an element $g \in U$ such that $({G_y}^g)^1 \leq G_x$. The Dimension Formula 2.1 and Corollary 2.1.8 yield

$$\dim y^G = \dim \frac{G}{{G_y}^g} = \dim \frac{G}{({G_y}^g)^1} \geq \dim \frac{G}{G_x} = \dim x^G.$$

Lemma 2.1.9 allows to conclude that

$$\begin{array}{rcccc}
\left(G_{[y^G]}\right)^1 & = & \left(\bigcap_{h\in G} G_y{}^h\right)^1 & = & \left(\bigcap_{h\in G} (G_y{}^g)^h\right)^1 \\
& = & \left(\bigcap_{h\in G}\left((G_y{}^g)^h\right)^1\right)^1 & = & \left(\bigcap_{h\in G}\left((G_y{}^g)^1\right)^h\right)^1 \\
& \le & \left(\bigcap_{h\in G}\left((G_x)^1\right)^h\right)^1 & = & \left(\bigcap_{h\in G}(G_x{}^h)^1\right)^1 \\
& = & \left(\bigcap_{h\in G} G_x{}^h\right)^1 & = & \left(G_{[x^G]}\right)^1.
\end{array}$$

If $\dim y^G = \dim x^G$ then Corollary 2.1.8 implies that $(G_y{}^g)^1 = (G_x)^1$, whence also $(G_{[y^G]})^1 = (G_{[x^G]})^1$. □

2.1.16 Remark (A counterexample). The full stabilizer can "jump up" when G is not a compact Lie group, and even its identity component can do the same when the orbit dimension is infinite. To see this, let G be a locally compact group in which there is a descending sequence

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \dots$$

of non-trivial closed subgroups whose intersection is trivial. Let X be the quotient space of $\tilde{X} := [0,1] \times G$ obtained by identifying $\{1\} \times G$ to G/H_1 and, for every $n \ge 2$,

$$\left[\frac{1}{n}, \frac{1}{n-1}\right[\times G \quad \text{to} \quad \left[\frac{1}{n}, \frac{1}{n-1}\right[\times G/H_n$$

in the obvious way. For example, one can obtain a compactification of an infinite tree of valency 3 from the group of 2-adic integers in this way. The space X is a Hausdorff space, and since it contains $\{0\}\times G$ as a closed subspace, it is compact if and only if G is. Similarly, the space X is connected if and only if G/H_1 is. Moreover, the natural action of G on $\tilde{X}$ induces an action of G on X by the following commutative diagram:[1]

$$\begin{array}{ccc} \tilde{X} \times G & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ X \times G & \longrightarrow & X \end{array}$$

This action is continuous since the left-hand side vertical arrow is a quotient map (see Bredon [15, I.13.19]). Now the action of G on $\{0\} \times G$ is free, but all stabilizers of points outside $\{0\} \times G$ are non-trivial.

A similar counterexample is described by Montgomery and Zippin in [92, p. 786].

There is some more fruit to be harvested from Theorem 2.1.15.

2.1.17 Corollary (Orbits of maximal dimension). *Let G be a compact group acting on a Hausdorff space X. Suppose that $k := \max\{\dim x^G | x \in X\}$ is a finite number, and let $Y \subseteq X$ be the set of points on k-dimensional orbits. Then Y is an open subset of X. Moreover, every point $x \in Y$ has a neighbourhood V such that the identity components of all kernels of actions on orbits which meet V coincide, and the identity components of all stabilizers of points in V are conjugate.*

In particular, the effective quotient $G|_{V^G}$ of G with respect to the action on the open set V^G is finite-dimensional. If G is connected then $\dim G|_{V^G} \leq \binom{k+1}{2}$.

Proof. Suppose that $x \in Y$, let $U := G$, and choose a neighbourhood V of x as in Theorem 2.1.15. Then $V \subseteq Y$. Moreover, for all $y \in V$, we have

$$\left(G_{[y^G]}\right)^1 = \left(G_{[x^G]}\right)^1 \quad \text{and} \quad \exists\, g \in G : ({G_y}^g)^1 = (G_x)^1.$$

The last claims follow from Theorem 2.1.7 on transitive actions. $\square$

2.1.18 Corollary (Uniform orbit dimension). *Let G be a compact group acting effectively on a connected Hausdorff space X, and*

[1] Commutative diagrams have been set by the "diagrams" TEX package written by Paul Taylor, Queen Mary and Westfield College, London.

suppose that all orbits have the same finite dimension k. Then the identity components of all stabilizers are conjugate, and the action of G on every single orbit is almost effective. If G is connected then the dimension of G is at most $\binom{k+1}{2}$.

Proof. Choose a point $x \in X$, and set

$$Y := \{y \in X \mid \exists\, g \in G : ({G_y}^g)^1 = (G_x)^1\}.$$

Then Y is an open subset of X by Corollary 2.1.17. If z is a point on the topological boundary of Y then the same Corollary shows that $z \in Y$. Thus Y is closed as well, and $Y = X$ by connectedness. Hence

$$\forall\, y \in X\ \exists\, g \in G : ({G_y}^g)^1 = (G_x)^1.$$

As above, we infer from Lemma 2.1.9 that the identity component of the kernel of the action on x^G satisfies

$$\left(G_{[x^G]}\right)^1 = \left(\bigcap_{g \in G} ({G_x}^g)^1\right)^1.$$

As this is the same for every orbit and G acts effectively, we conclude that $(G_{[x^G]})^1 = 1$, so that the action of G on x^G is almost effective, and the Dimension Formula 2.1 yields

$$\dim G = \dim \frac{G}{G_{[x^G]}}.$$

If G is connected then Theorem 2.1.7 on transitive actions shows that $\dim G \leq \binom{k+1}{2}$. □

2.1.19 Remark. We point out that subsets of a compact group which are conjugate by small elements are also close in the sense of the Hausdorff topology. To be precise, let G be a compact group, and let V be a neighbourhood of $1 \in G$. Then there is a neighbourhood U of $1 \in G$ such that for all elements $g \in U$ and for all subsets $S \subseteq G$, the relations

$$S^g \subseteq SV \quad \text{and} \quad S \subseteq S^g V$$

hold. Namely, choose U as a symmetric neighbourhood of $1 \in G$ such that $G \times U$ is mapped into V by the commutator map

$$\begin{array}{rcl} G \times G & \longrightarrow & G \\ (s, g) & \longmapsto & s^{-1}g^{-1}sg. \end{array}$$

Then for all $g \in U$ and $s \in G$, we have

$$s^g \in sV \quad \text{and} \quad s = (s^g)^{g^{-1}} \in s^g V,$$

the latter because $g^{-1} \in U$.

2.2 Actions on cohomology manifolds

2.2.1 Theorem (Orbits of low codimension in cohomology manifolds, I). *Let G be a compact group acting on a metrizable cohomology n-manifold X over some principal ideal domain R. Suppose that $k \in \mathbb{Z}_{\geq 0}$ is the highest covering dimension of any orbit. If $k \geq n-2$ then the set Y of points on k-dimensional orbits is a manifold.*

Note that Y is an open subset of X by Corollary 2.1.17.

Proof. Let x be a point of Y. We infer from Corollary 2.1.17 that the action of G on some invariant neighbourhood V of the orbit x^G is effectively finite-dimensional. Theorem 4a of Bredon [10] yields the existence of a closed subset C of Y containing x and of a k-cell K in $G/G_{[V]}$ such that the natural map $C \times K \to C^K$ is a homeomorphism onto a neighbourhood of x. We can find a relatively open subset U of C and an open k-ball B contained in K such that U^B is open in Y. Being an open subset of a cohomology manifold, the space $U^B \approx U \times B$ is itself a cohomology manifold of dimension n over R. Bredon [16, V.16.11] shows that the direct factor U is a cohomology manifold over R of dimension $n - k \leq 2$. Moreover, we can choose U to be relatively compact, whence it is second countable. This implies by [16, V.16.32, cf. V.16.8] that U is in fact a manifold. Therefore, the open neighbourhood U^B of x is a manifold. As this holds for every $x \in Y$ and Y, being metrizable, is paracompact, the space Y is a manifold as well. $\square$

Bredon was aware of the relevance of his results from [10] for cohomology manifolds. According to his remark in [12, p. 165], the main results of the earlier paper [10] just carry over.

2.2.2 Theorem (Orbits of low codimension in cohomology manifolds, II). *Let G be a compact group acting effectively on a metrizable cohomology n-manifold X over some principal ideal domain R. Suppose that X is connected and locally homogeneous, and that the highest covering dimension of any orbit is some number $k \geq n-2$. Then X is a manifold, and G is a Lie group.*

Under similar hypotheses, a proof that G is a Lie group was given by Raymond [105].

Proof. Let Y be the set of points whose orbits have maximal dimension. Then Y is open by Corollary 2.1.17, and it is a manifold by Theorem 2.2.1. Local homogeneity forces the whole space X to be a manifold. Therefore, Theorems 10 and 11 of Bredon [10] show that G is a Lie group. □

2.2.3 Theorem (Montgomery and Yang [91]: Principal orbits in cohomology manifolds). *Let G be a compact Lie group acting effectively on a connected cohomology manifold X over $\mathbb{Z}$. Then the subset $Y \subseteq X$ formed by the points on principal orbits is open and dense in X, and Y/G is connected. Therefore, all principal stabilizers are conjugate, and the action of G on every single principal orbit is effective. If G is connected then so is Y.*

Proof. The subset Y is open by Corollary 2.1.6, and Montgomery and Yang [91, Lemma 2] have shown that it is dense (cf. Borel et al. [8, IX, Lemma 3.2]). They have also proved that the image Y/G of Y under the projection of X onto the orbit space X/G is connected ([91, Lemma 4], cf. [8, IX, Lemma 3.4]). If y is a point of Y, then Corollary 2.1.6 provides a neighbourhood V of y such that all stabilizers of points in V belong to a single conjugacy class. The image V^G/G of V under the projection of X onto X/G is open, and it consists of orbits of one type. As Y/G is connected, we infer that all stabilizers G_y of points y of Y are conjugate in G. If y^G is a principal orbit, then the kernel $G_{[y^G]} = \bigcap_{g\in G} {G_y}^g$ fixes every point of Y. Hence this kernel is trivial. Finally, if G and Y/G are connected then the same holds for Y.

(This is elementary and has been noted in [91, Lemma 4], cf. [8, IX, Lemma 3.4].) $\square$

Bredon [14, IV.3.1] gives a proof of the last theorem in the easier case of a smooth action on a manifold.

2.3 Consequences of a theorem of Mann's

2.3.1 Theorem (Mann [83]). *Every compact connected Lie group G is covered by some direct product*

$$\mathbb{T}^q \times S_1 \times \cdots \times S_n,$$

where every S_i is a compact connected simply connected Lie group, either almost simple or $\mathrm{Spin}_4\mathbb{R}$, *and $S_i \cong \mathrm{Spin}_3\mathbb{R}$ occurs at most once. This slightly unusual 'normal form' permits the following statement: if G acts almost effectively on a connected cohomology manifold X over $\mathbb{Z}$ then*

$$q + \sum_{i=1}^{n} \min\left\{ s \in \mathbb{N} \,\middle|\, \dim S_i \leq \binom{s+1}{2} \right\} \leq \max\{\dim x^G | x \in X\}.$$

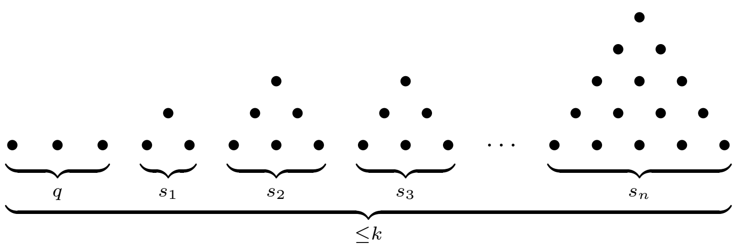
This result is easier to visualize than one might expect. We set

$$k := \max\{\dim x^G | x \in X\}$$

and, for each $i \in \{1, \ldots, n\}$,

$$s_i := \min\left\{ s \in \mathbb{N} \,\middle|\, \dim S_i \leq \binom{s+1}{2} \right\}.$$

Then the dimension of G is bounded by the number of dots in the following diagram:



Proof. The action of G induced on a single principal orbit is effective by the Montgomery–Yang Theorem 2.2.3. Therefore, the statement is just Mann's result [83, Theorem 1]. $\square$

2.3.2 Remark. When this result is applied to topological incidence geometries, the deep Montgomery–Yang Theorem 2.2.3 can usually be replaced by an argument involving Corollary 2.1.6 about stabilizers in compact Lie groups.

Section 2.5 will be devoted to a proof of a refinement of Mann's Theorem, see Corollary 2.5.6. Our proof will be largely independent, but we will use Table 2.1 on page 50 which has been taken from Mann's paper [83]. Moreover, Mann's original proof is considerably shorter than our Section 2.5.

Mann's result develops its full strength in connection with other theorems, as we will see in subsequent Chapters. It is also the cornerstone of Mann's two papers [83] and [84]. In the spirit of [84, Section 3], we draw some immediate consequences.

The following preparatory lemma begins to use the classification of simple compact Lie algebras. See Table 2.1 on page 50, which also contains their subalgebras of maximal dimension.

2.3.3 Lemma. *Let $\mathfrak{g}$ be a simple compact Lie algebra, and suppose that $\mathfrak{g}$ is the Lie algebra of a (necessarily compact) Lie group which acts almost effectively on a connected cohomology n-manifold over a principal ideal domain. Suppose that $\dim \mathfrak{g} \geq \binom{n-1}{2}$, and that $\mathfrak{g}$ is not isomorphic to a real orthogonal algebra. Then the action is transitive, and either $n = 4$ and $\mathfrak{g} \cong \mathfrak{a}_2$, or $n = 6$ and $\mathfrak{g} \cong \mathfrak{g}_2$, or $n = 8$ and $\mathfrak{g} \cong \mathfrak{a}_4$, or $n = 8$ and $\mathfrak{g} \cong \mathfrak{c}_3$.*

Proof. Assume first that $\mathfrak{g} \cong \mathfrak{a}_r$ and $r \notin \{1, 3\}$. (The algebras $\mathfrak{a}_1$ and $\mathfrak{a}_3$ are isomorphic to $\mathfrak{o}_3\mathbb{R}$ and $\mathfrak{o}_6\mathbb{R}$, respectively.) By Table 2.1 on page 50, we have $n \geq 2r$. If $r \geq 5$ then

$$\binom{n-1}{2} \geq \binom{2r-1}{2} = 2r^2 - 3r + 1 > r^2 + 2r = \dim \mathfrak{g}.$$

Suppose $\mathfrak{g} \cong \mathfrak{a}_2$. Then $\dim \mathfrak{g} = 8 < \binom{n-1}{2}$ if $n \geq 5$. Taking Table 2.1 into account again, we find that $n = 4$ and that some orbit has dimension 4. Invariance of domain (Bredon [16, V.16.19]) yields that the action is transitive. Similarly, if $\mathfrak{g} \cong \mathfrak{a}_4$ then $n = 8$, and the action is transitive.

Assume, then, that $\mathfrak{g} \cong \mathfrak{c}_r$ and $r \geq 3$. (The algebra $\mathfrak{c}_2$ is isomorphic to $\mathfrak{o}_5\mathbb{R}$.) Table 2.1 yields $n \geq 4(r-1)$. If $r \geq 4$ then

$$\binom{n-1}{2} \geq \binom{4r-5}{2} > \binom{2r+1}{2} = \dim \mathfrak{g}.$$

If $\mathfrak{g} \cong \mathfrak{c}_3$ then $n \geq 8$. Now $\dim \mathfrak{g} = 21 \geq \binom{n-1}{2}$ forces $n = 8$, and the action must be transitive.

Finally, assume that $\mathfrak{g}$ is an exceptional simple compact Lie algebra. If $\mathfrak{g} \cong \mathfrak{g}_2$ then Table 2.1 and the dimension hypothesis entail that $n = 6$, and the action is transitive. If $\mathfrak{g} \cong \mathfrak{f}_4$ then $n \geq 16$ by Table 2.1, whence $\binom{n-1}{2} \geq 105 > \dim \mathfrak{g}$. The other exceptional algebras are excluded in the same way. □

When the dimension hypothesis is weakened, further exceptions arise, the next of which is $\mathfrak{g} \cong \mathfrak{a}_5$ and $n = 10$ with a transitive action. However, these exceptions are always finite in number as long as the lower bound on $\dim \mathfrak{g}$ is of the form $\binom{n-const}{2} \pm const$. By the same method of proof, one can work out arbitrarily strong forms of the lemma.

2.3.4 Theorem. *Let G be a compact connected group acting almost effectively and transitively on a Hausdorff space X of finite dimension $n \geq 4$, and suppose that $\dim G \geq \binom{n-1}{2} + 4$. (For $n = 4$, the weaker hypothesis that $\dim G \geq 6$ is sufficient.) Then G is covered by one of the following groups:*

- $\mathrm{Spin}_{n+1}\mathbb{R}$;
- $Z \times \mathrm{Spin}_n\mathbb{R}$, *where Z is a compact connected abelian group of dimension 1;*
- $\mathrm{Spin}_n\mathbb{R}$;
- $\mathrm{SU}_3\mathbb{C}$, *and $n = 4$;*
- G_2, *and $n = 6$.*

Proof. (Cf. Mann [84, Theorem 6].) The dimension of G is finite by Theorem 2.1.7. Let N be a totally disconnected closed normal subgroup of G such that G/N is a Lie group. As in the proof of Theorem 2.1.7, we find that X/N is a homogeneous space of G/N with $\dim X/N = \dim X$. Mann's Theorem 2.3.1, applied to the almost effective action of G/N on X/N, yields that the centre of G/N (and hence that of G) is at most one-dimensional, and the commutator subgroup of G/N (and hence that of G) is almost simple or covered by $\mathrm{Spin}_4\mathbb{R}$.

By van Kampen's Theorem 1.1.3, the compact group G is covered by the direct product $Z(G)^1 \times G'$. Lemma 2.3.3 shows the possible Lie algebras $\mathfrak{l}(G')$ of G'. The case $\mathfrak{l}(G') \cong \mathfrak{c}_3$ is excluded by the hypothesis on $\dim G$, and so is $\mathfrak{l}(G') \cong \mathfrak{a}_4$ if $Z(G)^1$ is trivial. The cases $\mathfrak{l}(G') \cong \mathfrak{a}_2$ and $\mathfrak{l}(G') \cong \mathfrak{g}_2$ of Lemma 2.3.3 lead to the last two cases in the present theorem.

Suppose that $Z(G)^1 \neq 1$, and choose a point $x \in X$. As the action is almost effective, the stabilizer G_x does not contain $Z(G)^1$. Therefore $X/Z(G) \approx G/(Z(G)G_x)$ is a homogeneous space of G' with $\dim X/Z(G) = n - 1$. The Lie algebra $\mathfrak{l}(G')$ cannot be isomorphic to one of the non-orthogonal algebras from Lemma 2.3.3, because Table 2.1 on page 50 shows that the corresponding groups can only act trivially on a space of dimension $n - 1$.

Whether $Z(G)^1$ is trivial or not, we may now suppose that $\mathfrak{l}(G')$ is isomorphic to a real orthogonal algebra, i.e. the commutator group G' is covered by some group $\mathrm{Spin}_k\mathbb{R}$. As above, we apply Mann's Theorem 2.3.1 to the action of G/N on X/N, and we find that $k \in \{n, n+1\}$, with $Z(G)^1 = 1$ if $k = n + 1$. □

2.4 Calculating homology

We collect some techniques for calculating homology and cohomology groups, all of which exploit group actions. The general aim is to see that some topological space is the total space of a locally trivial fibre bundle. It is then accessible to the machinery of spectral sequences. However, we will only have to apply special consequences of this such as the following.

2.4.1 Lemma. *Let $E \to B$ be a q-sphere bundle (i.e., a fibre bundle with fibre $\mathbb{S}_q$), where $q > 0$. Suppose that $H_i(B; \mathbb{Z}/2) = 0$ for $i > q$.*

Then

$$H_i(E;\mathbb{Z}/2) \cong \begin{cases} H_i(B;\mathbb{Z}/2) & \text{if } i \leq q-1 \\ H_0(B;\mathbb{Z}/2) \oplus H_q(B;\mathbb{Z}/2) & \text{if } i = q \\ H_{i-q}(B;\mathbb{Z}/2) & \text{if } i \geq q+1. \end{cases}$$

The same result holds if homology is replaced by cohomology.

Proof. This is immediate from the exact Gysin sequences (see Spanier [124, 5.7.11]; cf. [124, 5.7.18]):

$$H_{i+1}(B) \longrightarrow H_{i-q}(B) \longrightarrow H_i(E) \longrightarrow H_i(B) \longrightarrow H_{i-q-1}(B)$$

$$H^{i-q-1}(B) \longrightarrow H^i(B) \longrightarrow H^i(E) \longrightarrow H^{i-q}(B) \longrightarrow H^{i+1}(B)$$

□

2.4.2 Corollary. *We can rephrase the result as follows:*

$$\begin{array}{lclcl} H_*(E;\mathbb{Z}/2) & \cong & H_*(B;\mathbb{Z}/2) \otimes H_*(\mathbb{S}_q;\mathbb{Z}/2) & \cong & H_*(B \times \mathbb{S}_q;\mathbb{Z}/2) \\ H^*(E;\mathbb{Z}/2) & \cong & H^*(B;\mathbb{Z}/2) \otimes H^*(\mathbb{S}_q;\mathbb{Z}/2) & \cong & H^*(B \times \mathbb{S}_q;\mathbb{Z}/2) \end{array}$$

Proof. In both lines, the second isomorphism comes from the Künneth Theorem, see Bredon [15, VI.3.2]. □

As an analogue of the Gysin sequences for sphere bundles, there are the Wang sequences for fibre bundles whose base space is a (simply connected homology) sphere, see Spanier [124, 8.5.6, 9.3.2, and 9.5.1].

Let us now describe some situations in which fibre bundles arise.

2.4.3 Lemma. *Let G be a compact Lie group. Suppose that G acts freely on a completely regular space X, and that G acts on a Hausdorff space Y. Let $X \times_G Y$ denote the orbit space of the action of G on $X \times Y$. Then*

$$\begin{array}{ccc} X \times_G Y & \longrightarrow & X/G \\ (x,y)^G & \longmapsto & x^G \end{array}$$

is a fibre bundle with fibre Y and structure group $G/G_{[Y]}$, where $G_{[Y]}$ denotes the kernel of the action on Y.

Proof. The orbit projection $X \to X/G$ is a principal G-bundle (Bredon [14, II.5.8]), and $X \times_G Y \to X/G$ is the associated Y-bundle, see [14, II.2.4]. The kernel $G_{[Y]}$ is factored out since the action of the structure group on the fibre should be effective. $\square$

A typical application is the following:

2.4.4 Proposition. *Let G_1 and G_2 be topological groups, and let $H_1 \le G_1$ and $H_2 \le G_2$ be closed subgroups. Let $N \le G_1 \times G_2$ be a subgroup which contains and normalizes $H_1 \times H_2$, and suppose that $N/(H_1 \times H_2)$ is a compact Lie group and that $N \cap (1 \times G_2) = 1 \times H_2$. For $i \in \{1,2\}$, let $\mathrm{pr}_i : G_1 \times G_2 \to G_i$ be the projection. Then*

$$\begin{array}{rccc} p: & \dfrac{G_1 \times G_2}{N} & \longrightarrow & \dfrac{G_1}{\mathrm{pr}_1 N} \\ & N(g_1, g_2) & \longmapsto & (\mathrm{pr}_1 N) g_1 \end{array}$$

is a fibre bundle with fibre G_2/H_2 whose structure group is the compact Lie group $(\mathrm{pr}_2 N)/H_2$.

Proof. Set $K := N/(H_1 \times H_2)$. We will describe actions of K on G_1/H_1 and on G_2/H_2 such that

$$\frac{G_1 \times G_2}{N} \approx \frac{G_1}{H_1} \times_K \frac{G_2}{H_2}.$$

As the spaces G_1/H_1 and G_2/H_2 are completely regular (see Hewitt and Ross [54, 8.14(a)]), this will allow us to apply the preceding lemma.

A typical element of K is of the form $H_1 n_1 \times H_2 n_2$ with $(n_1, n_2) \in N \le G_1 \times G_2$. Note that n_1 normalizes H_1, and that n_2 normalizes H_2. Let $i \in \{1,2\}$. A continuous right action of K on G_i/H_i is given by

$$\begin{array}{ccc} \dfrac{G_i}{H_i} \times K & \longrightarrow & \dfrac{G_i}{H_i} \\ (H_i g_i, H_1 n_1 \times H_2 n_2) & \longmapsto & {n_i}^{-1} H_i g_i = H_i {n_i}^{-1} g_i. \end{array}$$

Continuity can be inferred from the following commutative diagram in which the vertical arrows on the left-hand side are products of canonical projections with identity maps. Therefore, they are open and continuous, and we can apply the universal property of topological quotient maps (Dugundji [36, VI.3.1]) to conclude that the bottom horizontal map is continuous.

$$\begin{tikzcd}
G_i \times N \arrow[r] \arrow[d] & G_i \arrow[d] \\
\frac{G_i}{H_i} \times N \arrow[r] \arrow[d] & \frac{G_i}{H_i} \arrow[d] \\
\frac{G_i}{H_i} \times K \arrow[r] & \frac{G_i}{H_i}
\end{tikzcd}$$

In the action of K on G_1/H_1, the stabilizer of an arbitrary point is

$$\frac{N \cap (H_1 \times G_2)}{H_1 \times H_2}.$$

Now $H_1 \times H_2 \leq N$ implies that

$$N \cap (H_1 \times G_2) = (N \cap (1 \times G_2))(H_1 \times 1).$$

The hypothesis $N \cap (1 \times G_2) = 1 \times H_2$ entails that the action of K on G_1/H_1 is free.

Similarly, every stabilizer of the action of K on G_2/H_2 is the normal subgroup

$$\frac{N \cap (G_1 \times H_2)}{H_1 \times H_2} = K \cap \frac{G_1 \times H_2}{H_1 \times H_2}$$

of K. We claim that the effective quotient of K, i.e. the quotient of K by this stabilizer, is isomorphic to $(\mathrm{pr}_2 N)/H_2$. To see this, consider the following commutative diagram in which all maps are the natural ones.

$$\begin{tikzcd}
K \arrow[d] & N \arrow[l] \arrow[r] \arrow[d] & \mathrm{pr}_2 N \arrow[d] \\
\frac{K}{K \cap \frac{G_1 \times H_2}{H_1 \times H_2}} & \frac{N}{N \cap (G_1 \times H_2)} \arrow[l] \arrow[r] & \frac{\mathrm{pr}_2 N}{H_2}
\end{tikzcd}$$

The two maps in the bottom row are algebraic isomorphisms. Applying the universal property of topological quotient maps, we find that both are continuous, and that the left map is even a homeomorphism. Now all three groups in the bottom row have Hausdorff topologies, and the bottom left group is compact. Hence the bottom right map is a homeomorphism as well, which proves our claim. In particular, the group $(\mathrm{pr}_2 N)/H_2$ is a compact Lie group.

The map

$$\begin{array}{ccc} \dfrac{G_1 \times G_2}{N} & \longrightarrow & \dfrac{G_1}{H_1} \times_K \dfrac{G_2}{H_2} \\ N(g_1, g_2) & \longmapsto & (H_1 g_1, H_2 g_2)^K \end{array}$$

is a well-defined surjection. It is also injective, and we infer from the following commutative diagram that it is continuous and open, hence a homeomorphism.

$$\begin{tikzcd} G_1 \times G_2 \arrow[r] \arrow[d] & \dfrac{G_1}{H_1} \times \dfrac{G_2}{H_2} \arrow[d] \\ \dfrac{G_1 \times G_2}{N} \arrow[r] & \dfrac{G_1}{H_1} \times_K \dfrac{G_2}{H_2} \end{tikzcd}$$

Similarly,

$$\begin{array}{ccc} \dfrac{G_1}{\mathrm{pr}_1 N} & \longrightarrow & \dfrac{G_1/H_1}{K} \\ (\mathrm{pr}_1 N) g_1 & \longmapsto & (H_1 g_1)^K \end{array}$$

is a homeomorphism. Since also the diagram

$$\begin{tikzcd} \dfrac{G_1 \times G_2}{N} \arrow[r] \arrow[d, "p"'] & \dfrac{G_1}{H_1} \times_K \dfrac{G_2}{H_2} \arrow[d] \\ \dfrac{G_1}{\mathrm{pr}_1 N} \arrow[r] & \dfrac{G_1/H_1}{K} \end{tikzcd}$$

commutes, Lemma 2.4.3 shows that p is the projection in a fibre bundle whose fibre and structure group are as stated. $\square$

2.4.5 Remark. When G_2 is a compact Lie group, it is much easier to prove that

$$p : \frac{G_1 \times G_2}{N} \longrightarrow \frac{G_1}{\mathrm{pr}_1 N}$$

is a fibre bundle with fibre G_2/H_2. The following lemma shows that all stabilizers of the natural action of $1 \times G_2$ on $(G_1 \times G_2)/N$ are conjugate to $1 \times H_2$. Hence the orbit projection of this action is a fibre bundle (see Bredon [14, II.5.8]). The statement follows since the orbit space is homeomorphic to $G_1/(\mathrm{pr}_1 N)$.

Nevertheless, we did not want to suppress the more general proposition.

2.4.6 Lemma. *Let G be a group acting on a set X, and suppose that all stabilizers are conjugate to $H \leq G$. Let K be a subgroup of G.*

(a) *If $G = N_G(H)\, C_G(K)\, K$ then all stabilizers of the induced action of K on X are conjugate to $H \cap K$ in K.*

(b) *If K is a normal subgroup of G then all stabilizers of the action of G on the orbit space X/K are conjugate to HK.*

Proof. Choose $x \in X$. Then the stabilizer of x is $G_x = H^g$ for some $g \in G$. Write $g = nck$, with $n \in N_G(H)$, $c \in C_G(K)$, and $k \in K$. Then

$$\begin{array}{ccccccc} K_x & = & G_x \cap K & = & H^g \cap K & & \\ & = & H^{nck} \cap K & = & H^{ck} \cap K & & \\ & = & H^{ck} \cap K^{ck} & = & (H \cap K)^{ck} & = & (H \cap K)^k \end{array}$$

is conjugate to $H \cap K$ by the element k of K.

If K is a normal subgroup of G then G acts on the orbit space X/K, and an element $h \in G$ stabilizes the orbit x^K if and only if $x^h \in x^K$, which is equivalent to

$$h \in G_x K = H^g K = H^g K^g = (HK)^g.$$

□

2.4.7 Proposition. *Let G be a compact connected Lie group acting on a completely regular space X, and suppose that all stabilizers are* *conjugate to $H \leq G$. Let N be a closed normal subgroup of G. Then the orbit map $X \to X/N$ is the projection in a fibre bundle with fibre $N/(H \cap N)$ and structure group $N_N(H \cap N)/(H \cap N)$. Similarly, the orbit map $X/N \to X/G$ of the action of G on X/N is the projection in a fibre bundle with fibre G/HN and structure group $N_G(HN)/HN$.*

Proof. In order to use the preceding lemma, it suffices to prove that $G = C_G(N)\,N$. The identity component N^1 is the almost direct product of its intersections with $Z(G)^1$ and G', whence there is a closed normal subgroup K of G such that G is the almost direct product of K and N^1 (cf. Corollary 3.3.4 and Knapp [72, IV.4.25]). In particular, the complement K centralizes N^1. As N^1 is open in N, the intersection $K \cap N^1$ is a discrete open subgroup of $K \cap N$. Hence $K \cap N$ is a discrete normal subgroup of G and therefore central. The modular law shows that $N = N^1(K \cap N)$. This implies that K centralizes N, which proves our claim.

Hence for all $x \in X$, the stabilizer N_x is conjugate to $H \cap N$ in N. Therefore, the statement about the orbit map $X \to X/N$ can be found in Bredon's book [14, Theorem II.5.8]. The same theorem also yields the statement about $X/N \to X/G$ if we can show that the orbit space X/N is completely regular. But this follows from the fact that given a continuous function $f : X \to [0, 1]$ with $f(x_0) = 1$ and $f(x) = 0$ for all points x which are not contained in some neighbourhood U of x_0, we obtain a function

$$x^N \longmapsto \int_N f(x^g)d\mu_N(g) : X/N \longrightarrow [0, 1]$$

which is strictly positive in ${x_0}^N$ and vanishes outside U^N/N. □

In the particular case of a transitive action, we obtain the following corollary.

2.4.8 Corollary. *Let G be a compact connected Lie group, let H and N be closed subgroups of G, and suppose that N is normal in G. Then the natural map $G/H \to G/HN$ is the projection in a fibre bundle with fibre $N/(H \cap N)$ and structure group $N_N(H \cap N)/(H \cap N)$.* □

One often represents this locally trivial fibre bundle by the following diagram.

$$\begin{tikzcd}
\frac{N}{H \cap N} \arrow[r] & \frac{G}{H} \arrow[d] \\
 & \frac{G}{HN}
\end{tikzcd}$$

In the preceding proposition, we have thus started with a fibre bundle whose fibre was itself the total space of a bundle

$$\begin{tikzcd}
\frac{N}{H \cap N} \arrow[r] & \frac{G}{H} \arrow[r] \arrow[d] & X \arrow[d] \\
 & \frac{G}{HN} & \frac{X}{G}
\end{tikzcd}$$

and we have transformed it into a fibre bundle whose base space is the total space of a bundle.

$$\begin{tikzcd}
\frac{N}{H \cap N} \arrow[r] & X \arrow[d] \\
\frac{G}{HN} \arrow[r] & \frac{X}{N} \arrow[d] \\
 & \frac{X}{G}
\end{tikzcd}$$

This is an advantage because the fibres of these two bundles are simpler spaces. When we will use Proposition 2.4.7, the space $N/(H \cap N)$ will be a sphere, and X/N will be a manifold of sufficiently small dimension, so that we can apply Lemma 2.4.1.

2.5 Large subalgebras of compact Lie algebras

When a compact connected Lie group G acts effectively on a connected cohomology manifold over $\mathbb{Z}$, every principal orbit is an effective homogeneous space of G by the Montgomery–Yang Theorem 2.2.3. Therefore, it is useful to know the minimal dimension of an almost effective homogeneous space G/H if the Lie algebra of G is given. This problem is solved in the present section. Moreover, we obtain a detailed description of the Lie algebras of those H for which the minimal dimension is attained.

2.5.1 Theorem. *For each simple compact Lie algebra $\mathfrak{g}$, the isomorphism type of a proper subalgebra $\mathfrak{h}_M$ of maximal dimension is as described in Table 2.1.*

With the exception of $\mathfrak{b}_3 \hookrightarrow \mathfrak{d}_4$, the inclusion of $\mathfrak{h}_M$ in $\mathfrak{g}$ is unique up to conjugation under inner automorphisms. The inclusion $\mathfrak{b}_3 \hookrightarrow \mathfrak{d}_4$ is unique up to conjugation under the full automorphism group of $\mathfrak{d}_4$.

Concatenation of an embedding $\mathfrak{b}_3 \hookrightarrow \mathfrak{d}_4$ with the triality automorphism of $\mathfrak{d}_4$ yields a second embedding which is not conjugate to the first one under any inner automorphism, see Salzmann et al. [115, 17.16].

Proof. The table is taken from Mann [83, Section 4], who has adapted it from Dynkin's work ([37] and [38]) on simple complex Lie algebras (see also Borel and de Siebenthal [7] and Seitz, [120] and [121]). If $\operatorname{rk} \mathfrak{h}_M = \operatorname{rk} \mathfrak{g}$ then uniqueness of the inclusion follows from Wolf [144, Theorem 8.10.8].

Every embedding of $\mathfrak{b}_{r-1} = \mathfrak{o}_{2r-1}\mathbb{R}$ into $\mathfrak{d}_r = \mathfrak{o}_{2r}\mathbb{R}$ induces the structure of a $\mathfrak{b}_{r-1}$-module on $\mathbb{R}^{2r}$. Such a module is the direct sum of simple submodules. (We shall freely use representation theory of semi-simple Lie algebras as outlined in Tits [136], cf. also Salzmann et al. [115, Chapter 95]. As an additional piece of notation, we follow Onishchik and Vinberg [97] in writing $R(\lambda)$ for a simple complex module of highest weight λ.) Returning to our embedding, we first note that for $2 \leq i \leq r-2$ the fundamental weight λ_i of $\mathfrak{b}_{r-1}$ is of real type. Hence the corresponding simple real module $R(\lambda_i)^{(\mathbb{R})}$ of $R(\lambda_i)$ satisfies

$$\dim_{\mathbb{R}} R(\lambda_i)^{(\mathbb{R})} = \dim_{\mathbb{C}} R(\lambda_i) =: \dim \lambda_i .$$

Table 2.1: The simple compact Lie algebras $\mathfrak{g}$ and their subalgebras $\mathfrak{h}_M$ of maximal dimension according to Mann [83, Section 4]

$\mathfrak{g}$	$\dim \mathfrak{g}$	$\dim \mathfrak{g} - \dim \mathfrak{h}_M$	$\mathfrak{h}_M$
$\mathfrak{a}_r$ $(r \neq 3)$	$r(r+2)$	$2r$	$\mathbb{R} \times \mathfrak{a}_{r-1}$
$\mathfrak{b}_r$ $(r \geq 3)$	$r(2r+1)$	$2r$	$\mathfrak{d}_r$
$\mathfrak{c}_r$ $(r \geq 2)$	$r(2r+1)$	$4(r-1)$	$\mathfrak{a}_1 \times \mathfrak{c}_{r-1}$
$\mathfrak{d}_r$ $(r \geq 3)$	$r(2r-1)$	$2r-1$	$\mathfrak{b}_{r-1}$
$\mathfrak{e}_6$	78	26	$\mathfrak{f}_4$
$\mathfrak{e}_7$	133	54	$\mathbb{R} \times \mathfrak{e}_6$
$\mathfrak{e}_8$	248	112	$\mathfrak{a}_1 \times \mathfrak{e}_7$
$\mathfrak{f}_4$	52	16	$\mathfrak{b}_4$
$\mathfrak{g}_2$	14	6	$\mathfrak{a}_2$

A table of dimensions of many irreducible representations is given in [97, pp. 300–305]. We find that $\dim \lambda_i = \binom{2r-1}{i} \geq \binom{2r-1}{2} > 2r$ (note that $r \geq 4$). This implies that the coefficient of λ_i in the highest weight of a simple summand of $\mathbb{R}^{2r}$ is 0. Similarly, the coefficient of λ_1 is at most 1 since $2\lambda_1$ is of real type, and $\dim 2\lambda_1 = \frac{2r+1}{2r-1}\binom{2r-1}{2} > 2r$ for $r \geq 3$. If $r = 3$ then λ_{r-1} is of quaternionic type, whence

$$\dim_{\mathbb{R}} R(\lambda_{r-1})^{(\mathbb{R})} = 2 \dim_{\mathbb{C}} R(\lambda_{r-1}) = 2^r > 2r.$$

If $r \geq 5$ then no matter of what type λ_{r-1} is, we have

$$\dim_{\mathbb{R}} R(\lambda_{r-1})^{(\mathbb{R})} \geq \dim_{\mathbb{C}} R(\lambda_{r-1}) = 2^{r-1} > 2r.$$

Hence if $r \neq 4$ then every non-trivial simple $\mathfrak{b}_{r-1}$-module of dimension at most $2r$ has highest weight λ_1 and dimension $2r-1$. The $\mathfrak{b}_{r-1}$-module $\mathbb{R}^{2r}$ therefore decomposes as

$$\mathbb{R}^{2r} \cong R(0)^{(\mathbb{R})} \oplus R(\lambda_1)^{(\mathbb{R})}.$$

We conclude that all $2r$-dimensional representations of $\mathfrak{b}_{r-1}$ are equivalent. This means that whenever ι_1 and ι_2 are embeddings of $\mathfrak{b}_{r-1}$

into $\mathfrak{d}_r$, there is an element $A \in \mathrm{GL}_{2r}\mathbb{R}$ such that $\iota_2(X) = A^{-1}\iota_1(X)A$ holds for all $X \in \mathfrak{b}_{r-1}$. A well-known argument involving the Cartan decomposition of $\mathfrak{gl}_{2r}\mathbb{R}$ shows that A can be chosen from $\mathrm{O}_{2r}\mathbb{R}$. (Details can be found in Warner [141, 1.1.3.7] and will be given for the case $(\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{e}_6 + i\mathfrak{e}_6$ at the end of this proof; cf. Kramer [75, 4.6].) Since the standard embedding of $\mathfrak{b}_{r-1}$ into $\mathfrak{d}_r$ is centralized by an element of $\mathrm{O}_{2r}\mathbb{R} \setminus \mathrm{SO}_{2r}\mathbb{R}$, the same holds for every embedding, so that we can even choose A from $\mathrm{SO}_{2r}\mathbb{R}$. Hence $\iota_1(\mathfrak{b}_{r-1})$ and $\iota_2(\mathfrak{b}_{r-1})$ are indeed conjugate under an inner automorphism of $\mathfrak{d}_r$.

The case $r = 4$ is special due to the triality automorphism of $\mathfrak{d}_4$. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two subalgebras of $\mathfrak{d}_4 = \mathfrak{o}_8\mathbb{R}$ with $\mathfrak{h}_1 \cong \mathfrak{h}_2 \cong \mathfrak{b}_3 = \mathfrak{o}_7\mathbb{R}$, and let H_1 and H_2 be closed connected subgroups of $\mathrm{Spin}_8\mathbb{R}$ with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Salzmann et al. [115, 17.16] show that some automorphism of $\mathrm{Spin}_8\mathbb{R}$ maps H_1 to H_2. Derivation yields an automorphism of $\mathfrak{o}_8\mathbb{R}$ which maps $\mathfrak{h}_1$ to $\mathfrak{h}_2$.

The second case in which a subalgebra of maximal dimension is not of full rank is the inclusion of $\mathfrak{f}_4$ into $\mathfrak{e}_6$. To show its uniqueness up to conjugation, let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be subalgebras of $\mathfrak{e}_6$ which are isomorphic to $\mathfrak{f}_4$. Seitz [121, 15.1] shows that their complexifications are conjugate under an inner automorphism of $\mathfrak{e}_6{}^{\mathbb{C}}$. In other words, there is an element $g \in \mathrm{Int}\,\mathfrak{e}_6{}^{\mathbb{C}}$ such that $g(\mathfrak{h}_1{}^{\mathbb{C}}) = \mathfrak{h}_2{}^{\mathbb{C}}$. The image $g(\mathfrak{h}_1)$ is a compact real form of $\mathfrak{h}_2{}^{\mathbb{C}}$, hence it is conjugate to $\mathfrak{h}_2$ under an inner automorphism (see Knapp [72, 6.20]). Therefore, we may assume that $g(\mathfrak{h}_1) = \mathfrak{h}_2$. Now $(\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{e}_6 + i\mathfrak{e}_6$ is a Cartan decomposition of $\mathfrak{e}_6{}^{\mathbb{C}}$, regarded as a real Lie algebra, since the symmetric real bilinear form $\beta : (\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}} \times (\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}} \to \mathbb{R}$ with

$$\begin{aligned} &\beta(X + iY, X' + iY') \\ &\quad= \; -\mathrm{Tr}\big(\mathrm{ad}_{(\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}}}(X + iY)\ \mathrm{ad}_{(\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}}}(X' - iY')\big) \\ &\quad= \; -2\,\mathrm{Tr}\big((\mathrm{ad}_{\mathfrak{e}_6} X)(\mathrm{ad}_{\mathfrak{e}_6} X') + (\mathrm{ad}_{\mathfrak{e}_6} Y)(\mathrm{ad}_{\mathfrak{e}_6} Y')\big) \end{aligned}$$

(where $X, Y, X', Y' \in \mathfrak{e}_6$) is positive definite [72, 6.13f.]. By the corresponding Cartan decomposition of $\mathrm{Int}\,\mathfrak{e}_6{}^{\mathbb{C}} = \mathrm{Int}(\mathfrak{e}_6{}^{\mathbb{C}})^{\mathbb{R}}$ (see [72, p. 57 and 6.31]), we can write $g = \exp(\mathrm{ad}\,X)\exp(\mathrm{ad}\,iY)$ for some elements $X, Y \in \mathfrak{e}_6$. Then $\mathfrak{e}_6$ contains

$$\exp(-\mathrm{ad}\,X)\mathfrak{h}_2 = \exp(\mathrm{ad}\,iY)\mathfrak{h}_1 = \cosh(\mathrm{ad}\,iY)\mathfrak{h}_1 + \sinh(\mathrm{ad}\,iY)\mathfrak{h}_1.$$

In the last sum, the first summand lies in $\mathfrak{e}_6$, the second in $i\mathfrak{e}_6$. Hence the second summand vanishes. The endomorphism $\mathrm{ad}\,iY$ of $\mathfrak{e}_6{}^{\mathbb{C}}$ is symmetric with respect to β, hence it is diagonalizable with real eigenvalues. This entails that $(\operatorname{ad} iY)\mathfrak{h}_1$ vanishes as well, from which we infer that $\exp(-\operatorname{ad} X)\mathfrak{h}_2 = \mathfrak{h}_1$. In other words, the subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate under an inner automorphism of $\mathfrak{e}_6$. $\square$

2.5.2 Proposition. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} := {\mathfrak{a}_1}^n$ which does not contain any non-zero ideal of $\mathfrak{g}$. Then $\dim \mathfrak{h} \leq \left\lfloor \frac{3n}{2} \right\rfloor$.*

Suppose that $\dim \mathfrak{h} = \left\lfloor \frac{3n}{2} \right\rfloor$. Let $m := \left\lfloor \frac{n}{2} \right\rfloor$. Then there are automorphisms $\varphi_1, \ldots, \varphi_m$ of $\mathfrak{a}_1$ such that, after rearranging the factors of $\mathfrak{g}$, the following holds:

(a) *If n is even then $\mathfrak{h}$ is the direct product of the graphs*

$$\Gamma_{\varphi_i} := \{(x, \varphi_i(x)) \mid x \in \mathfrak{a}_1\}$$

of the φ_i, i.e.

$$\mathfrak{h} = \Gamma_{\varphi_1} \times \ldots \times \Gamma_{\varphi_m}.$$

(b) *If n is odd then there is a one-dimensional subalgebra $\mathfrak{z}$ of $\mathfrak{a}_1$ such that*

$$\mathfrak{h} = \mathfrak{z} \times \Gamma_{\varphi_1} \times \ldots \times \Gamma_{\varphi_m}.$$

Proof. We use induction on n, the case $n = 0$ being trivial. For $n = 1$, note that every proper non-zero subalgebra of $\mathfrak{a}_1 = \mathfrak{o}_3\mathbb{R}$ is one-dimensional.

As a vector space, the compact Lie algebra $\mathfrak{h}$ is the direct sum of its centre $Z(\mathfrak{h})$ and its commutator algebra $\mathfrak{h}'$. Both are ideals. In such a case, one writes $\mathfrak{h} = Z(\mathfrak{h}) \oplus \mathfrak{h}'$.[2] For $1 \leq i \leq n$, let $\operatorname{pr}_i : {\mathfrak{a}_1}^n \to \mathfrak{a}_1$ be the projection onto the i-th factor. Then $\operatorname{pr}_i(Z(\mathfrak{h}))$ is an abelian subalgebra of $\mathfrak{a}_1$, whence $\dim \operatorname{pr}_i(Z(\mathfrak{h})) \leq 1$. If $\mathfrak{h}' = \{0\}$ then this implies that $\dim \mathfrak{h} \leq n$, since

$$\mathfrak{h} \leq \operatorname{pr}_1(\mathfrak{h}) \times \ldots \times \operatorname{pr}_n(\mathfrak{h}).$$

Now if $n \geq 2$ then $n < \left\lfloor \frac{3n}{2} \right\rfloor$.

[2] Note that this is isomorphic to $Z(\mathfrak{h}) \times \mathfrak{h}'$. Hence $Z(\mathfrak{h}) \oplus \mathfrak{h}'$ denotes the 'internal direct product' of $Z(\mathfrak{h})$ and $\mathfrak{h}'$, a term which is not common. We may occasionally speak of a direct sum, although this must not be understood in the sense of category theory.

Therefore, we may assume that the commutator algebra $\mathfrak{h}'$ is non-trivial. Note that $\mathfrak{h}'$ is a semi-simple Lie algebra. If $\mathfrak{k}$ is a simple ideal of $\mathfrak{h}'$ then all images $\mathrm{pr}_i(\mathfrak{k})$ are either isomorphic to $\mathfrak{k}$ or trivial, and the latter is not the case for all i. Hence $\mathfrak{k} \cong \mathfrak{a}_1$, and $\mathfrak{h}'$ is a power of $\mathfrak{a}_1$. We fix a simple ideal $\mathfrak{k}$ of $\mathfrak{h}'$. Then we may assume that $\mathrm{pr}_n(\mathfrak{k}) = \mathfrak{a}_1$. If all other images $\mathrm{pr}_i(\mathfrak{k})$, where $1 \leq i \leq n-1$, are trivial, then $\mathfrak{k} = \{0\}^{n-1} \times \mathfrak{a}_1$, which contradicts the hypothesis that $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$. Therefore, we may also assume that $\mathrm{pr}_{n-1}(\mathfrak{k}) = \mathfrak{a}_1$.

Compact Lie algebras are characterized by the fact that they admit an invariant positive definite bilinear form. Obviously, the orthogonal complement of an ideal with respect to such a form is again an ideal. Hence we can write $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp$, and $\mathfrak{k}^\perp$ is contained in the centralizer $Z_\mathfrak{g}(\mathfrak{k})$ of $\mathfrak{k}$ in $\mathfrak{g}$. As in any direct product, we have that

$$Z_\mathfrak{g}(\mathfrak{k}) = Z_{\mathfrak{a}_1}(\mathrm{pr}_1(\mathfrak{k})) \times \ldots \times Z_{\mathfrak{a}_1}(\mathrm{pr}_n(\mathfrak{k})).$$

Hence $\mathfrak{k}^\perp \leq {\mathfrak{a}_1}^{n-2} \times \{0\}^2$. In particular, induction applies to $\mathfrak{k}^\perp$, whence

$$\dim \mathfrak{h} = \dim \mathfrak{k} + \dim \mathfrak{k}^\perp \leq 3 + \left\lfloor \frac{3(n-2)}{2} \right\rfloor = \left\lfloor \frac{3n}{2} \right\rfloor .$$

Suppose that equality holds. By induction, there are automorphisms $\varphi_1, \ldots, \varphi_{m-1}$ of $\mathfrak{a}_1$ such that, after rearranging the first $n-2$ factors of $\mathfrak{g}$, we obtain

$$\mathfrak{k}^\perp = \mathfrak{z}^e \times \Gamma_{\varphi_1} \times \ldots \times \Gamma_{\varphi_{m-1}} \times \{0\}^2,$$

where $\mathfrak{z}$ is a one-dimensional subalgebra of $\mathfrak{a}_1$, the exponent e is 0 if n is even, and it is 1 if n is odd. As $\mathfrak{k} \leq Z_\mathfrak{g}(\mathfrak{k}^\perp)$, this shows that $\mathrm{pr}_i(\mathfrak{k}) = \{0\}$ for $i \leq n-2$. Using the automorphism $\varphi_m := \mathrm{pr}_n \circ (\mathrm{pr}_{n-1}|_\mathfrak{k})^{-1}$ of $\mathfrak{a}_1$, we infer that $\mathfrak{k} = \{0\}^{n-2} \times \Gamma_{\varphi_m}$. Hence

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp = \mathfrak{z}^e \times \Gamma_{\varphi_1} \times \ldots \times \Gamma_{\varphi_m}.$$

□

2.5.3 Proposition. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} := {\mathfrak{a}_2}^n$ which does not contain any non-zero ideal of $\mathfrak{g}$. Then $\dim \mathfrak{h} \leq 4n$.*

If $\dim \mathfrak{h} = 4n$ *then there exist a non-negative integer* $m \leq \frac{n}{2}$*, four-dimensional subalgebras* $\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-2m}$ *of* $\mathfrak{a}_2$*, and automorphisms* $\varphi_1, \ldots, \varphi_m$ *of* $\mathfrak{a}_2$ *such that, after rearranging the factors of* $\mathfrak{g}$*,*

$$\mathfrak{h} = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_{n-2m} \times \Gamma_{\varphi_1} \times \ldots \times \Gamma_{\varphi_m}.$$

Note that every four-dimensional subalgebra of $\mathfrak{a}_2 = \mathfrak{su}_3\mathbb{C}$ *is conjugate to* $\mathfrak{u}_2\mathbb{C}$ *in its standard embedding.*

Proof. We use induction on n. The case $n = 0$ is trivial. Any proper subalgebra of $\mathfrak{a}_2$ whose dimension is maximal is conjugate to $\mathfrak{u}_2\mathbb{C}$. This is part of Theorem 2.5.1. Nevertheless, we sketch a proof: the algebra ${\mathfrak{a}_1}^2$ cannot be embedded into $\mathfrak{a}_2$ since $\mathfrak{sl}_2\mathbb{C}^2$ does not admit an effective three-dimensional complex representation. As the rank of a subalgebra of $\mathfrak{a}_2$ is at most 2, this yields the isomorphism type of a subalgebra of maximal dimension. For uniqueness of the embedding see Lemma 3.2.2 below. Thus the statement is valid for $n = 1$.

For $1 \leq i \leq n$, let $\mathrm{pr}_i : {\mathfrak{a}_2}^n \to \mathfrak{a}_2$ be the projection onto the i-th factor. If $\mathfrak{k}$ is a simple ideal of $\mathfrak{h}$ then $\mathrm{pr}_i|_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{a}_2$ is an embedding for some i. Therefore $\mathfrak{k}$ is isomorphic to either $\mathfrak{a}_1$ or $\mathfrak{a}_2$.

Suppose first that no simple ideal of $\mathfrak{h}$ is isomorphic to $\mathfrak{a}_2$. For all i, the image $\mathfrak{h}_i := \mathrm{pr}_i(\mathfrak{h}) \leq \mathfrak{a}_2$ is isomorphic to an ideal of $\mathfrak{h}$ which complements $\mathfrak{h} \cap \ker \mathrm{pr}_i$. Hence $\dim \mathfrak{h}_i \leq 4$. Since $\mathfrak{h} \leq \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_n$, we have indeed that $\dim \mathfrak{h} \leq 4n$. If $\dim \mathfrak{h} = 4n$ then equality $\mathfrak{h} = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_n$ holds, and every $\mathfrak{h}_i$ is a four-dimensional subalgebra of $\mathfrak{a}_2$.

Assume now that $\mathfrak{h}$ contains an ideal $\mathfrak{k}$ with $\mathfrak{k} \cong \mathfrak{a}_2$, and write $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp$. We may assume that $\mathrm{pr}_n(\mathfrak{k}) = \mathfrak{a}_2$. Since $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$, we may further assume that $\mathrm{pr}_{n-1}(\mathfrak{k}) = \mathfrak{a}_2$. Hence the centralizer $Z_{\mathfrak{g}}(\mathfrak{k})$ of $\mathfrak{k}$ in $\mathfrak{g}$ is contained in ${\mathfrak{a}_2}^{n-2} \times \{0\}^2$. This centralizer contains the complement $\mathfrak{k}^\perp$ of $\mathfrak{k}$ in $\mathfrak{h}$. Hence induction applies to $\mathfrak{k}^\perp$, so that $\dim \mathfrak{h} = \dim \mathfrak{k} + \dim \mathfrak{k}^\perp \leq 4n$.

Suppose that $\dim \mathfrak{h} = 4n$, so that $\dim \mathfrak{k}^\perp = 4(n-2)$. By induction, there exist a non-negative integer $m \leq \frac{n-2}{2}$, four-dimensional subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-2-2m}$ of $\mathfrak{a}_2$, and automorphisms $\varphi_1, \ldots, \varphi_m$ of $\mathfrak{a}_2$ such that

$$\mathfrak{k}^\perp = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_{n-2(m+1)} \times \Gamma_{\varphi_1} \times \ldots \times \Gamma_{\varphi_m} \times \{0\}^2.$$

Since none of the $\mathfrak{h}_i$ is central in the simple algebra $\mathfrak{a}_2$, this implies that $\mathrm{pr}_i(\mathfrak{k}) = 0$ for all $i \leq n-2$. Letting $\varphi_{m+1} := \mathrm{pr}_n \circ (\mathrm{pr}_{n-1}|_{\mathfrak{k}})^{-1}$, we infer that $\mathfrak{k} = \{0\}^2 \times \Gamma_{\varphi_{m+1}}$. Hence $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ is indeed of the form which has been described in the statement. $\square$

The situation is easier for semi-simple compact algebras $\mathfrak{g}$ which do not contain $\mathfrak{a}_1$ or $\mathfrak{a}_2$ as ideals. If $\mathfrak{h} \le \mathfrak{g}$ is a subalgebra which has maximal dimension among the subalgebras not containing a non-trivial ideal of $\mathfrak{g}$ then $\mathfrak{h}$ is the direct sum of its intersections with the simple ideals of $\mathfrak{g}$. In other words, diagonals do no longer appear as ideals of $\mathfrak{h}$.

2.5.4 Proposition. *Let $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ be simple compact Lie algebras. For each i, suppose that $\dim \mathfrak{s}_i \ge 10$, and let t_i be the maximal dimension of a proper subalgebra of $\mathfrak{s}_i$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} := \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n$ which does not contain any non-zero ideal of $\mathfrak{g}$. Then $\dim \mathfrak{h} \le t_1 + \ldots + t_n$.*

If $\dim \mathfrak{h} = t_1 + \ldots + t_n$ then there are subalgebras $\mathfrak{h}_i \le \mathfrak{s}_i$ such that $\dim \mathfrak{h}_i = t_i$, and $\mathfrak{h} = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_n$.

The subalgebras $\mathfrak{h}_i$ are described by Theorem 2.5.1.

Proof. We use induction on n, the cases $n = 0$ and $n = 1$ being trivial. Let $\mathrm{pr}_i : \mathfrak{g} \to \mathfrak{s}_i$ be the projection onto the i-th factor. We may assume that $\dim \mathfrak{s}_n = \max\{\dim \mathfrak{s}_i \mid 1 \le i \le n\}$. In contrast to the notation of the previous proof, let $\mathfrak{k} := \mathfrak{h} \cap \ker \mathrm{pr}_n$, and write $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{k}^\perp$. Then

$$\mathrm{pr}_n|_{\mathfrak{k}^\perp} : \mathfrak{k}^\perp \longrightarrow \mathrm{pr}_n(\mathfrak{h})$$

is an isomorphism.

Suppose first that $\mathrm{pr}_n(\mathfrak{h}) = \mathfrak{s}_n$. As $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$, the image $\mathrm{pr}_i(\mathfrak{k}^\perp)$ is non-trivial for some $i \le n-1$. We may therefore assume that $\mathrm{pr}_{n-1}(\mathfrak{k}^\perp) \ne \{0\}$. Since $\mathfrak{k}^\perp \cong \mathfrak{s}_n$ is simple and $\dim \mathfrak{s}_{n-1} \le \dim \mathfrak{s}_n$, this entails that

$$\mathrm{pr}_{n-1}|_{\mathfrak{k}^\perp} : \mathfrak{k}^\perp \longrightarrow \mathfrak{s}_{n-1}$$

is an isomorphism as well. In particular, this shows that $t_{n-1} = t_n$. We also conclude that

$$\mathfrak{k} \le Z_{\mathfrak{g}}(\mathfrak{k}^\perp) \le \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_{n-2} \times \{0\}^2.$$

By induction $\dim \mathfrak{k} \le t_1 + \ldots + t_{n-2}$. Since $\dim \mathfrak{s}_n \ge 10$, we infer from Table 2.1 on page 50 that $\dim \mathfrak{k}^\perp = \dim \mathfrak{s}_n < 2t_n = t_{n-1} + t_n$. Therefore

$$\dim \mathfrak{h} = \dim \mathfrak{k} + \dim \mathfrak{k}^\perp < t_1 + \ldots + t_n.$$

Now suppose that $\mathfrak{h}_n := \mathrm{pr}_n(\mathfrak{h})$ is a proper subalgebra of $\mathfrak{s}_n$. Then $\dim \mathfrak{k}^\perp = \dim \mathfrak{h}_n \le t_n$. Moreover,

$$\mathfrak{k} = \mathfrak{h} \cap (\mathfrak{s}_1 \times \ldots \times \mathfrak{s}_{n-1} \times \{0\})$$

satisfies $\dim \mathfrak{k} \le t_1 + \ldots + t_{n-1}$ by induction. Hence $\dim \mathfrak{h} \le t_1 + \ldots + t_n$. Suppose that equality holds. Then $\dim \mathfrak{k} = t_1 + \ldots + t_{n-1}$, and $\dim \mathfrak{h}_n = t_n$. By induction, there are subalgebras $\mathfrak{h}_1 \le \mathfrak{s}_1, \ldots, \mathfrak{h}_{n-1} \le \mathfrak{s}_{n-1}$ such that $\dim \mathfrak{h}_i = t_i$, and

$$\mathfrak{k} = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_{n-1} \times \{0\}.$$

For all $i \le n-1$, the subalgebra $\mathfrak{h}_i + Z_{\mathfrak{s}_i}(\mathfrak{h}_i) \le \mathfrak{s}_i$ contains, and normalizes, the subalgebra $\mathfrak{h}_i$. Since $\mathfrak{s}_i$ is simple and $\mathfrak{h}_i$ is maximal among the proper subalgebras of $\mathfrak{s}_i$, it follows that $Z_{\mathfrak{s}_i}(\mathfrak{h}_i) \le \mathfrak{h}_i$. Together with $\mathfrak{k}^\perp \le Z_{\mathfrak{g}}(\mathfrak{k})$, this implies that $\mathrm{pr}_i(\mathfrak{k}^\perp) \le Z_{\mathfrak{s}_i}(\mathrm{pr}_i(\mathfrak{k})) = Z_{\mathfrak{s}_i}(\mathfrak{h}_i) \le \mathfrak{h}_i$. Hence

$$\mathfrak{h} \le \mathrm{pr}_1(\mathfrak{h}) \times \ldots \times \mathrm{pr}_n(\mathfrak{h}) = \mathfrak{h}_1 \times \ldots \times \mathfrak{h}_n,$$

and equality follows since the dimensions agree. □

2.5.5 Theorem. *Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ be simple compact Lie algebras. For each i, suppose that $\dim \mathfrak{s}_i \ge 10$, and let t_i be the maximal dimension of a proper subalgebra of $\mathfrak{s}_i$. Let $\mathfrak{h}$ be a subalgebra of*

$$\mathfrak{g} := \mathbb{R}^k \times {\mathfrak{a}_1}^l \times {\mathfrak{a}_2}^m \times \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n$$

which does not contain any non-zero ideal of $\mathfrak{g}$. Then

$$\dim \mathfrak{h} \le \left\lfloor \frac{3l}{2} \right\rfloor + 4m + t_1 + \ldots + t_n.$$

If $\dim \mathfrak{h} = \left\lfloor \frac{3l}{2} \right\rfloor + 4m + t_1 + \ldots + t_n$ then there exist a subalgebra $\mathfrak{h}_1 \le {\mathfrak{a}_1}^l$ with $\dim \mathfrak{h}_1 = \left\lfloor \frac{3l}{2} \right\rfloor$, a subalgebra $\mathfrak{h}_2 \le {\mathfrak{a}_2}^m$ with $\dim \mathfrak{h}_2 = 4m$, and a subalgebra $\mathfrak{h}_3 \le \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n$ with $\dim \mathfrak{h}_3 = t_1 + \ldots + t_n$ such that $\mathfrak{h}$ is the graph of some homomorphism of $\mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$ into $\mathbb{R}^k$.

The subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$, and $\mathfrak{h}_3$ are described in Propositions 2.5.2, 2.5.3, and 2.5.4, respectively.

Proof. We shall use the projections of $\mathfrak{g}$ onto some of its direct factors:

$$\begin{aligned}
\mathrm{pr}_{\mathfrak{z}} : \mathfrak{g} &\longrightarrow \mathbb{R}^k \\
\mathrm{pr}_1 : \mathfrak{g} &\longrightarrow {\mathfrak{a}_1}^l \\
\mathrm{pr}_2 : \mathfrak{g} &\longrightarrow {\mathfrak{a}_2}^m \\
\mathrm{pr}_3 : \mathfrak{g} &\longrightarrow \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n \\
\mathrm{pr}_{12} : \mathfrak{g} &\longrightarrow {\mathfrak{a}_1}^l \times {\mathfrak{a}_2}^m \\
\mathrm{pr}' : \mathfrak{g} &\longrightarrow {\mathfrak{a}_1}^l \times {\mathfrak{a}_2}^m \times \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n
\end{aligned}$$

Thus $\mathrm{pr}_{12} = \langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle$, $\mathrm{pr}' = \langle \mathrm{pr}_{12}, \mathrm{pr}_3 \rangle$, and $\mathrm{id}_{\mathfrak{g}} = \langle \mathrm{pr}_{\mathfrak{z}}, \mathrm{pr}' \rangle$. Define

$$\mathfrak{k}_i := \mathfrak{h} \cap \ker \mathrm{pr}_i \quad \text{and} \quad \mathfrak{h}_i := \mathrm{pr}_i(\mathfrak{h})$$

for $i \in \{1, 2, 3\}$, and write $\mathfrak{h} = \mathfrak{k}_i \oplus {\mathfrak{k}_i}^\perp$.

Suppose first that $k = n = 0$. We claim that $\mathfrak{h}_2$ does not contain an ideal of ${\mathfrak{a}_2}^m$. If this is false then we may assume that $\mathfrak{h}_2$ contains $\{0\}^{m-1} \times \mathfrak{a}_2$. Let $\mathfrak{j}$ be the inverse image of this ideal under the restriction of pr_2 to ${\mathfrak{k}_2}^\perp$. As this restriction is an isomorphism onto $\mathfrak{h}_2$, we find that $\mathfrak{j} \cong \mathfrak{a}_2$, and $\mathfrak{j}$ is an ideal of ${\mathfrak{k}_2}^\perp$ and hence of $\mathfrak{h}$. Now $\mathrm{pr}_1(\mathfrak{j})$ is trivial since $\mathfrak{a}_2$ does not embed into ${\mathfrak{a}_1}^l$. Hence $\mathfrak{j} = \{0\}^{l+m-1} \times \mathfrak{a}_2$ which contradicts the hypothesis that $\mathfrak{h}$ does not contain any non-zero ideal of $\mathfrak{g}$. Thus our claim is proved.

We infer from Proposition 2.5.3 that $\dim {\mathfrak{k}_2}^\perp = \dim \mathfrak{h}_2 \le 4m$. Moreover, Proposition 2.5.2 shows that $\dim \mathfrak{k}_2 \le \left\lfloor \frac{3l}{2} \right\rfloor$. Hence $\dim \mathfrak{h} \le \left\lfloor \frac{3l}{2} \right\rfloor + 4m$. If equality holds then $\dim \mathfrak{k}_2 = \left\lfloor \frac{3l}{2} \right\rfloor$ and $\dim {\mathfrak{k}_2}^\perp = 4m$. Using Proposition 2.5.2 again, we infer that

$$Z_{{\mathfrak{a}_1}^l \times \{0\}^m}(\mathfrak{k}_2) \le \mathfrak{k}_2.$$

This implies that $\mathrm{pr}_1({\mathfrak{k}_2}^\perp) \times \{0\}^m \le \mathfrak{k}_2$, whence $\mathfrak{h}_1 \times \{0\}^m = \mathfrak{k}_2$, and $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_2$ since $\mathfrak{h}$ is contained in the right-hand side, and the dimensions agree.

Suppose now that $k = 0$, but $n > 0$. We proceed exactly as in the first part of this proof. The image $\mathfrak{h}_3$ of $\mathfrak{h}$ under pr_3 cannot contain an ideal of $\mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n$ since no $\mathfrak{s}_i$ embeds into ${\mathfrak{a}_1}^l \times {\mathfrak{a}_2}^m$. Hence $\dim {\mathfrak{k}_3}^\perp = \dim \mathfrak{h}_3 \le t_1 + \ldots + t_n$ by Proposition 2.5.4, so that the upper bound on $\dim \mathfrak{h}$ follows with the help of the first part of this proof. If $\dim \mathfrak{h}$ equals this bound then $\dim \mathfrak{k}_3 = \left\lfloor \frac{3l}{2} \right\rfloor + 4m$, whence

$\mathfrak{k}_3 = \mathrm{pr}_1(\mathfrak{k}_3) \times \mathrm{pr}_2(\mathfrak{k}_3) \times \{0\}^n$. Moreover, Propositions 2.5.2 and 2.5.3 imply that

$$Z_{\mathfrak{a}_1{}^l \times \mathfrak{a}_2{}^m \times \{0\}^n}(\mathfrak{k}_3) \leq \mathfrak{k}_3,$$

whence $\mathrm{pr}_{12}(\mathfrak{k}_3{}^\perp) \times \{0\}^n \leq \mathfrak{k}_3$. As above, this implies that $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$.

Finally, suppose that $k > 0$. Then the restriction of pr' to $\mathfrak{h}$ is an embedding since $\mathfrak{h} \cap \ker \mathrm{pr}'$ is trivial, being an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. If $\mathfrak{j}$ is a simple ideal of $\mathfrak{h}$ then $\mathrm{pr}_{\mathfrak{z}}(\mathfrak{j})$ is trivial. As before, this shows that $\mathrm{pr}'(\mathfrak{h})$ cannot contain an ideal of $\mathfrak{a}_1{}^l \times \mathfrak{a}_2{}^m \times \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_n$. Hence $\dim \mathfrak{h} = \dim \mathrm{pr}'(\mathfrak{h}) \leq \lfloor \frac{3l}{2} \rfloor + 4m + t_1 + \ldots + t_n$. If equality holds then $\mathrm{pr}'(\mathfrak{h}) = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$, and the dimensions of the $\mathfrak{h}_i$ are as claimed. To end the proof, note that $\mathfrak{h}$ is the graph of the homomorphism

$$\mathrm{pr}_{\mathfrak{z}} \circ (\mathrm{pr}'|_{\mathfrak{h}})^{-1} : \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3 \longrightarrow \mathbb{R}^k.$$

□

2.5.6 Corollary (Mann's Theorem 2.3.1, revisited). *Consider the compact connected Lie group*

$$G = \mathbb{T}^q \times S_1 \times \cdots \times S_n,$$

where every S_i is a compact connected simply connected Lie group, either almost simple or $\mathrm{Spin}_4\mathbb{R}$, and $S_i \cong \mathrm{Spin}_3\mathbb{R}$ occurs at most once. Let s_i be the minimal codimension of a proper subgroup of S_i if S_i is almost simple, and set $s_i := 3$ if $S_i \cong \mathrm{Spin}_4\mathbb{R}$. Suppose that G acts almost effectively on a connected cohomology manifold X over $\mathbb{Z}$. Then

$$q + \sum_{i=1}^{n} s_i \leq \max\{\dim x^G | x \in X\}.$$

Recall that every compact connected Lie group is covered by a unique group of the kind which occurred in the statement. Also note that the numbers s_i can be found in Table 2.1 on page 50. For small values of $\sum s_i$, the Lie algebras $\mathfrak{l}(G)$ are listed in Table 2.2 on page 60.

This corollary is stronger than Mann's Theorem 2.3.1. To see this, we apply Theorem 2.1.7 to the almost effective action of S_i on its quotient by a proper subgroup of maximal dimension (respectively, to the natural action on $\mathbb{S}_3$ if $S_i \cong \mathrm{Spin}_4\mathbb{R}$). We find that $\dim S_i \leq \binom{s_i+1}{2}$. Hence Theorem 2.3.1 follows from the corollary.

Proof. Choose a principal orbit x^G. By the Montgomery–Yang Theorem 2.2.3, the action of G on x^G is almost effective. Hence the Lie algebra $\mathfrak{l}(G_x)$ does not contain a non-trivial ideal of the Lie algebra $\mathfrak{l}(G)$. Theorem 2.5.5 yields that

$$q + \sum_{i=1}^{n} s_i \leq \dim G - \dim G_x = \dim x^G.$$

The assertion follows. □

2.5.7 Lemma. *Let $\mathfrak{g}$ be a compact Lie algebra, and let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be subalgebras of $\mathfrak{g}$ with $\mathfrak{h}_1 \leq \mathfrak{h}_2$. Suppose that $\dim \mathfrak{h}_1$ is as large as possible under the condition that $\mathfrak{h}_1$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then $\mathfrak{g}$ contains an ideal $\mathfrak{j}$ such that $\mathfrak{h}_2 = \mathfrak{h}_1 + \mathfrak{j}$. If $\mathfrak{j}$ is chosen as large as possible then $\mathfrak{j}$ is abelian or $\mathfrak{h}_1 \cap \mathfrak{j} \neq \{0\}$.*

Proof. Let $\mathfrak{j}$ be the largest ideal of $\mathfrak{g}$ which is contained in $\mathfrak{h}_2$, and let $\mathfrak{j}^\perp$ be the complement of $\mathfrak{j}$ in $\mathfrak{g}$ with respect to an invariant positive definite bilinear form on $\mathfrak{g}$. Then $\mathfrak{h}_2 = (\mathfrak{h}_2 \cap \mathfrak{j}^\perp) \oplus \mathfrak{j}$. Set $\mathfrak{k} := (\mathfrak{h}_1 + \mathfrak{j}) \cap \mathfrak{j}^\perp$, so that $\mathfrak{k}$ is the image of $\mathfrak{h}_1$ under the projection onto $\mathfrak{j}^\perp$ with kernel $\mathfrak{j}$. Then $\mathfrak{k} \oplus (\mathfrak{h}_1 \cap \mathfrak{j})$ is a subalgebra of $\mathfrak{g}$ whose dimension is equal to $\dim \mathfrak{h}_1$. Since $\mathfrak{k} \leq \mathfrak{h}_2 \cap \mathfrak{j}^\perp$ and the right-hand side does not contain a non-trivial ideal of $\mathfrak{g}$, we infer that $\mathfrak{k} \oplus (\mathfrak{h}_1 \cap \mathfrak{j})$ is a subalgebra of $\mathfrak{g}$ which does not contain a non-trivial ideal, and it is in fact maximal among these subalgebras. If $\mathfrak{j}$ is not abelian then this shows that $\mathfrak{h}_1 \cap \mathfrak{j} \neq \{0\}$. Moreover $\mathfrak{k}$ is maximal among the subalgebras of $\mathfrak{j}^\perp$ which do not contain a non-trivial ideal. This implies that $\mathfrak{k} = \mathfrak{h}_2 \cap \mathfrak{j}^\perp$, whence

$$\mathfrak{h}_2 = (\mathfrak{h}_2 \cap \mathfrak{j}^\perp) + \mathfrak{j} = ((\mathfrak{h}_1 + \mathfrak{j}) \cap \mathfrak{j}^\perp) + \mathfrak{j} = \mathfrak{h}_1 + \mathfrak{j}.$$

□

2.5.8 Remark. Note that this proof does not depend on the classification in Theorem 2.5.5. However, this classification allows an important further observation under the additional hypothesis that $\mathfrak{h}_1$ is

Table 2.2: Structure of semi-simple compact Lie groups acting on spaces of small dimension, according to Corollary 2.5.6

$\sum_{i=1}^n s_i$	n	$(s_1, \dots, s_n)$	$\mathfrak{s}_1 \times \dots \times \mathfrak{s}_n$
2	1	(2)	$\mathfrak{o}_3\mathbb{R}$
3	1	(3)	$\mathfrak{o}_4\mathbb{R}$
4	1	(4)	$\mathfrak{o}_5\mathbb{R}$, $\mathfrak{su}_3\mathbb{C}$
5	1	(5)	$\mathfrak{o}_6\mathbb{R}$
	2	$(2,3)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
6	1	(6)	$\mathfrak{o}_7\mathbb{R}$, $\mathfrak{g}_2$
	2	$(2,4)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$, $\mathfrak{o}_3\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$
		$(3,3)$	$\mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
7	1	(7)	$\mathfrak{o}_8\mathbb{R}$
	2	$(2,5)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_6\mathbb{R}$
		$(3,4)$	$\mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$, $\mathfrak{o}_4\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$
8	1	(8)	$\mathfrak{o}_9\mathbb{R}$, $\mathfrak{su}_5\mathbb{C}$, $\mathfrak{u}_3\mathbb{H}$
	2	$(2,6)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_7\mathbb{R}$, $\mathfrak{o}_3\mathbb{R} \times \mathfrak{g}_2$
		$(3,5)$	$\mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_6\mathbb{R}$
		$(4,4)$	$\mathfrak{o}_5\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$, $\mathfrak{su}_3\mathbb{C} \times \mathfrak{o}_5\mathbb{R}$, $\mathfrak{su}_3\mathbb{C} \times \mathfrak{su}_3\mathbb{C}$
	3	$(2,3,3)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
9	1	(9)	$\mathfrak{o}_{10}\mathbb{R}$
	2	$(2,7)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_8\mathbb{R}$
		$(3,6)$	$\mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_7\mathbb{R}$, $\mathfrak{o}_4\mathbb{R} \times \mathfrak{g}_2$
		$(4,5)$	$\mathfrak{o}_5\mathbb{R} \times \mathfrak{o}_6\mathbb{R}$, $\mathfrak{su}_3\mathbb{C} \times \mathfrak{o}_6\mathbb{R}$
	3	$(2,3,4)$	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$, $\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$

Let $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ be compact Lie algebras, either simple or $\mathfrak{o}_4\mathbb{R}$, of which at most one is isomorphic to $\mathfrak{o}_3\mathbb{R}$. For each $i \in \{1, \dots, n\}$, let s_i be the minimal codimension of a proper subalgebra of $\mathfrak{s}_i$ if $\mathfrak{s}_i$ is simple, and set $s_i := 3$ if $\mathfrak{s}_i \cong \mathfrak{o}_4\mathbb{R}$. For the smallest values of $\sum s_i$, the table shows $\mathfrak{s}_1 \times \cdots \times \mathfrak{s}_n$ up to isomorphism. In each line, the algebras are ordered by descending dimension.

Every effective homogeneous space of a compact Lie group whose Lie algebra is $\mathbb{R}^q \times \mathfrak{s}_1 \times \mathfrak{s}_2 \times \cdots \times \mathfrak{s}_n$ has dimension at least $q + \sum s_i$.

contained in the commutator algebra $\mathfrak{g}'$ of $\mathfrak{g}$. Let $\mathfrak{k}$ be a simple ideal of $\mathfrak{h}_1$. Then either $\mathfrak{k}$ is contained in an ideal of $\mathfrak{g}'$, or there is an ideal of $\mathfrak{g}'$ which is isomorphic to $\mathfrak{a}_1 \times \mathfrak{a}_1$ or to $\mathfrak{a}_2 \times \mathfrak{a}_2$ and which contains $\mathfrak{k}$ as a diagonal subalgebra. If $\mathfrak{k}$ is not contained in $\mathfrak{j}^\perp$ then this implies that $\mathfrak{k} + \mathfrak{j}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{k} + \mathfrak{j}$ is contained in $\mathfrak{h}_2$, we must have $\mathfrak{k} \leq \mathfrak{j}$. Moreover, the centre of $\mathfrak{h}_1$ can be written as a direct product of subalgebras of ideals of $\mathfrak{g}'$. We conclude that

$$\mathfrak{h}_1 = (\mathfrak{h}_1 \cap \mathfrak{j}^\perp) \oplus (\mathfrak{h}_1 \cap \mathfrak{j})$$

if $\mathfrak{h}_1 \leq \mathfrak{g}'$. Note that this holds for an arbitrary complement $\mathfrak{j}^\perp$ of $\mathfrak{j}$.

It does not suffice to suppose that $\mathfrak{h}_1$ is maximal among the subalgebras which do not contain a non-trivial ideal of $\mathfrak{g}$. To see this, let $\varphi : \mathfrak{g}_2 \hookrightarrow \mathfrak{d}_4$ be an embedding. Let $\mathfrak{k}_1$ be a proper subalgebra of $\mathfrak{d}_4$ which properly contains $\operatorname{im} \varphi$. Then $\mathfrak{k}_1 \cong \mathfrak{b}_3$, as one finds by checking all compact algebras of rank at most 4. Set $\mathfrak{g} := \mathfrak{g}_2 \times \mathfrak{d}_4$, and let $\mathfrak{h}_1 \leq \mathfrak{g}$ be the graph of φ. Suppose that $\mathfrak{h}_2$ is a subalgebra of $\mathfrak{g}$ which sits above $\mathfrak{h}_1$ but does not contain a non-trivial ideal of $\mathfrak{g}$. Set $\mathfrak{k}_2 := \mathfrak{h}_2 \cap (\mathfrak{g}_2 \times \{0\})$. Then $\mathfrak{h}_2 = \mathfrak{k}_2 \oplus {\mathfrak{k}_2}^\perp$, and ${\mathfrak{k}_2}^\perp$ is the graph of a homomorphism from some subalgebra of $\mathfrak{d}_4$ into $\mathfrak{g}_2$. Taking all this together, one can show that $\mathfrak{h}_2 = \mathfrak{h}_1$. But $\mathfrak{h}_1 < \mathfrak{g}_2 \times \mathfrak{k}_1$, so that we have indeed established a counterexample.

2.5.9 Remark. If a compact Lie group G acts almost effectively and transitively on a Hausdorff space, and if H is a stabilizer of this action, then H does not contain a normal subgroup of G whose dimension is positive. In this situation, Theorem 2.5.5 yields an upper bound on $\dim H$. If this upper bound is attained then the same theorem describes the connected component H^1, since this is the exponential image of the Lie algebra of H. Moreover, the subgroup H normalizes its connected component. In this context, note that if H is a subgroup of a direct product $G_1 \times G_2$ then

$$\mathrm{pr}_i(N_{G_1 \times G_2}(H)) \leq N_{G_i}(\mathrm{pr}_i(H)),$$

where pr_i is one of the two projections. To see that this may be a proper inclusion, let H be the graph of an embedding of $\mathbb{Z}/3$ into the symmetric group S_3. However, if $H_1 \leq G_1$ and $H_2 \leq G_2$ then

$$N_{G_1 \times G_2}(H_1 \times H_2) = N_{G_1}(H_1) \times N_{G_2}(H_2).$$

Chapter 3

Spheres

As we want to apply the results of this chapter to the theory of compact connected generalized quadrangles, we shall work with generalized spheres. These have been introduced in Section 1.3. We point out that the results of this chapter are of interest also for ordinary spheres.

3.1 Very large orbits

This section reports on the classification of transitive actions of compact connected groups on (cohomology) spheres, and it contains general results about actions in which an orbit has codimension 1.

3.1.1 Theorem (Homogeneous cohomology spheres). *Let G be a compact connected Lie group, and let H be a closed subgroup of G. Suppose that there is an isomorphism $H^*(G/H;\mathbb{Z}) \cong H^*(\mathbb{S}_n;\mathbb{Z})$ of graded groups for some $n \in \mathbb{N}$, and that the action of G on G/H is effective. Then either $G \cong \mathrm{SO}_3\mathbb{R}$, and H is an icosahedral subgroup of G, or $G/H \approx \mathbb{S}_n$. In the first case, the homogeneous space G/H is called the* Poincaré homology 3-sphere.

If $G/H \approx \mathbb{S}_n$ then the action of G on G/H is equivalent to the natural action of a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$ on the unit sphere in $\mathbb{R}^{n+1}$. Suppose that $n \geq 2$. Then there is an almost simple closed normal subgroup N of G which acts transitively on G/H. Let $\mathbb{F}$ denote the centralizer of N in $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{n+1})$. Then $\mathbb{F}$ is a skew field, and G is the *almost direct product of N with a compact connected subgroup of $\mathbb{F}^\times$. Explicitly, the following cases are possible:*

(a) $\mathbb{F} \cong \mathbb{R}$, *and* $G \cong \mathrm{SO}_{n+1}\mathbb{R}$.

(b) $\mathbb{F} \cong \mathbb{C}$, *and* $G \cong \mathrm{SU}_k\mathbb{C}$ *or* $G \cong \mathrm{U}_k\mathbb{C}$, *where* $n = 2k-1$.

(c) $\mathbb{F} \cong \mathbb{H}$, *and* $G \cong \mathrm{U}_k\mathbb{H}$, $G \cong \mathrm{U}_1\mathbb{C} \cdot \mathrm{U}_k\mathbb{H}$, *or* $G \cong \mathrm{U}_1\mathbb{H} \cdot \mathrm{U}_k\mathbb{H}$, *where* $n = 4k-1$.

(d) $\mathbb{F} \cong \mathbb{R}$, *and* G *and* n *are as follows:* $G \cong \mathrm{G}_2$ *and* $n = 6$, *or* $G \cong \mathrm{Spin}_7\mathbb{R}$ *and* $n = 7$, *or* $G \cong \mathrm{Spin}_9\mathbb{R}$ *and* $n = 15$.

In each case, the action of N is unique up to linear equivalence.

Note that $H^*(G/H;\mathbb{Z}) \cong H^*(\mathbb{S}_n;\mathbb{Z})$ is a consequence of $H_*(G/H;\mathbb{Z}) \cong H_*(\mathbb{S}_n;\mathbb{Z})$, as can be seen from the Universal Coefficient Theorem (cf. Bredon [15, V.7.2]). The converse also holds (see Spanier [124, 5.5.12] and note that the homology of a compact manifold is finitely generated, cf. [15, E.5]).

Proof. The first part is due to Bredon [11]. The structure of a compact group which acts transitively on a sphere was found by Montgomery and Samelson [89]. Borel ([5] and [6]) has given the explicit list of almost simple groups. Linearity over $\mathbb{R}$ has been proved by Poncet [101]. By Schur's Lemma (see Salzmann et al. [115, 95.4] for the appropriate version), the centralizer $\mathbb{F}$ is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Representation theory allows to decide the isomorphism type of $\mathbb{F}$; for a convenient reference, see the list in Salzmann et al. [115, 95.10]. □

3.1.2 Theorem (Mostert [94]). *Let G be a compact group acting on a generalized n-sphere S (where $n \geq 2$), and suppose that some orbit x^G has codimension 1. Then $S \approx \mathbb{S}_n$, and G is a Lie group. There are closed subgroups H_0 and H_1 of G such that the following conditions are satisfied.*

(i) *The stabilizer G_x is contained in both H_0 and H_1.*

(ii) *The two quotient spaces H_j/G_x are integral homology spheres of positive dimension.*

(iii) *The action of G on S is equivalent to the natural action of G on the double mapping cylinder of the two quotient maps*

$$\frac{G}{H_0} \longleftarrow \frac{G}{G_x} \longrightarrow \frac{G}{H_1} \ ,$$

which is defined as the quotient space

$$\frac{\frac{G}{H_0} \cup ([0,1] \times \frac{G}{G_x}) \cup \frac{G}{H_1}}{H_0 g \sim (0, G_x g), (1, G_x g) \sim H_1 g} \ .$$

In particular, all orbits of codimension 1 are equivalent, and they are exactly the principal orbits. There are exactly two non-principal orbits ${y_0}^G$ and ${y_1}^G$. The points $y_0, y_1 \in S$ can be chosen such that their stabilizers are H_0 and H_1. The orbit space S/G is a compact interval with end points ${y_0}^G$ and ${y_1}^G$. There are homotopy equivalences

$$x^G \simeq S \setminus ({y_0}^G \cup {y_1}^G), \quad {y_0}^G \simeq S \setminus {y_1}^G, \quad \textit{and} \quad {y_1}^G \simeq S \setminus {y_0}^G.$$

The principal orbits are orientable. For any abelian group A, there are the following isomorphisms, where $1 \le i \le n-2$.

$$\begin{array}{rcl} H_i(x^G; A) & \cong & H_i({y_0}^G; A) \oplus H_i({y_1}^G; A) \\ H^i(x^G; A) & \cong & H^i({y_0}^G; A) \oplus H^i({y_1}^G; A) \end{array}$$

For the concept of a double mapping cylinder, see tom Dieck, Kamps, and Puppe [32, 1.29].

Proof. By Lemma 1.3.5, the existence of an orbit of codimension 1 entails that $S \approx \mathbb{S}_n$, and that G is a Lie group. The orbit space S/G is simply connected by Montgomery and Yang [90, Corollary 2]. Hence parts (i) and (iii) are given by Mostert [94, Theorem 4], cf. Richardson [111, 1.1]. According to Hofmann and Mostert [58, Footnote 2], the spaces H_j/G_x are integral homology spheres. The homotopy equivalences are an immediate consequence of the description of S as a double mapping cylinder.

All this is valid under the hypothesis that S is a simply connected compact manifold. The fact that the two non-principal orbits are of smaller dimension and the statements about homology and cohomology groups depend more closely on S being a sphere. In degrees

$1 \leq i \leq n-2$, the latter statements follow from the Mayer–Vietoris sequences [15, V.8.3] for the triple

$$\left(S,\ \ S \setminus {y_0}^G,\ \ S \setminus {y_1}^G\right).$$

This also yields a short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}/2 \longrightarrow & H_{n-1}(x^G; \mathbb{Z}/2) \\ & \longrightarrow H_{n-1}({y_0}^G; \mathbb{Z}/2) \oplus H_{n-1}({y_1}^G; \mathbb{Z}/2) \longrightarrow 0 \end{aligned}$$

from which we infer with the help of the dimension criterion given by Bredon [15, VI.7.12] that $H_{n-1}({y_j}^G; \mathbb{Z}/2) = 0$. As $\dim {y_j}^G \leq n-1$, this shows that $\dim {y_j}^G < n-1$, whence $H_{n-1}({y_j}^G; A) = 0$ for arbitrary coefficients A. The above short exact sequence, with $\mathbb{Z}/2$ replaced by $\mathbb{Z}$, yields that $H_{n-1}(x^G; \mathbb{Z}) \cong \mathbb{Z}$. In other words, the principal orbit x^G is orientable. □

Mostert's Theorem is a tool of general importance in the theory of compact Lie groups acting on manifolds or, in view of Theorem 2.2.2, on locally homogeneous metrizable cohomology manifolds. For actions on spheres, it develops remarkable additional strength when we employ the fact that a sphere does neither admit a non-trivial covering nor a product decomposition. This approach leads to the next results.

3.1.3 Lemma. *Let G be a compact connected group acting on a generalized n-sphere S (where $n \geq 2$). Suppose that some orbit x^G has codimension 1, and that the stabilizer G_x is not connected. Choose $y_0, y_1 \in S$ as in Mostert's Theorem 3.1.2. Then for $j = 0$ or for $j = 1$, the intersection $(G_{y_j})^1 \cap G_x$ is not connected. In particular, the quotient G_{y_j}/G_x is homeomorphic either to $\mathbb{S}_1$ or to the Poincaré homology 3-sphere.*

Proof. Since $\dim (G_{y_j})^1 = \dim G_{y_j}$ and G_{y_j}/G_x is connected, the action of $(G_{y_j})^1$ on G_{y_j}/G_x is transitive, so that $G_{y_j} = G_x (G_{y_j})^1$. Moreover, the intersection $(G_{y_j})^1 \cap G_x$ is a stabilizer of this action. If it is not connected then G_{y_j}/G_x is not simply connected (cf. Salzmann et al. [115, 94.4]), and it is an integral homology sphere by Mostert's Theorem 3.1.2. Therefore, the last claim follows from the main statement by means of the Classification Theorem 3.1.1.

Suppose that both $(G_{y_0})^1 \cap G_x$ and $(G_{y_1})^1 \cap G_x$ are connected, so that they are both equal to $(G_x)^1 =: H$. Set $\tilde{S} := [0,1] \times G/H$. Since H is normal in G_x, the quotient group G_x/H acts on $\tilde{S}$ by

$$\begin{array}{rcl} \tilde{S} \times G_x/H & \longrightarrow & \tilde{S} \\ ((t, Hg), Hk) & \longmapsto & (t, Hk^{-1}g). \end{array}$$

As in the proof of Proposition 2.4.4, we infer from the commutative diagram

$$\begin{tikzcd} ([0,1] \times G) \times G_x \arrow[r] \arrow[d] & {[0,1] \times G} \arrow[d] \\ \tilde{S} \times G_x \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\ \tilde{S} \times \dfrac{G_x}{H} \arrow[r] & \tilde{S} \end{tikzcd}$$

that the action is continuous. Moreover, it is free. Define an equivalence relation $\sim$ on $\tilde{S}$ by

$$(j, Hg_1) \sim (j, Hg_2) :\Longleftrightarrow g_1 {g_2}^{-1} \in (G_{y_j})^1 \text{ for } j \in \{0, 1\},$$

and let $S' := \tilde{S}/\!\sim$. Then S' is naturally isomorphic to the double mapping cylinder of the two quotient maps

$$\frac{G}{(G_{y_0})^1} \longleftarrow \frac{G}{H} \longrightarrow \frac{G}{(G_{y_1})^1} \quad .$$

Being a subgroup of G_{y_j}, the group G_x normalizes $(G_{y_j})^1$, whence the action of G_x/H on $\tilde{S}$ is compatible with $\sim$. Therefore, this group also acts on S'. This action is still continuous, and the stabilizer of the $\sim$-class of (j, Hg) is

$$\begin{aligned} &\{Hk \in G_x/H | Hk^{-1}g \sim Hg\} \\ &\quad = \{Hk \in G_x/H | k^{-1}gg^{-1} \in (G_{y_j})^1\} = \frac{(G_{y_j})^1 \cap G_x}{H} = 1. \end{aligned}$$

In other words, the action of G_x/H on S' is still free. Now the orbit space of S' with respect to this action is homeomorphic to the double mapping cylinder of

$$\frac{G}{G_{y_0}} \longleftarrow \frac{G}{G_x} \longrightarrow \frac{G}{G_{y_1}} \quad ,$$

hence it is homeomorphic to $\mathbb{S}_n$ by Mostert's Theorem 3.1.2. But the orbit projection is a covering map since the action is free, and there is no proper connected covering space of $\mathbb{S}_n$ for $n \geq 2$, see Bredon [15, III.8.1]. This contradiction shows that at least one of the spaces $(G_{y_0})^1 \cap G_x$ and $(G_{y_1})^1 \cap G_x$ is not connected. □

We will often use Theorem 2.5.5 to prove that the codimension of a principal orbit is at most 1. The hypotheses of Theorem 2.5.5 are therefore reflected in those of the following theorem.

3.1.4 Theorem (Groups which must act with a codimension 1 orbit, I). *Let G be a compact connected Lie group acting almost effectively and non-transitively on a generalized n-sphere S. Suppose that the codimension of every subgroup of G which does not contain a non-trivial connected normal subgroup is at least $n-1$. Then each stabilizer is connected.*

Proof. The statement is trivial for $n = 1$, so that we assume $n \geq 2$. Choose a principal orbit x^G. By Lemma 1.3.3, the restricted action of G on x^G is almost effective. Therefore, the stabilizer G_x does not contain a non-trivial connected normal subgroup, whence $\dim x^G = \dim G/G_x \geq n-1$, and equality must hold because the action is not transitive, see Lemma 1.3.4. Hence we can apply Mostert's Theorem 3.1.2. In particular, there are exactly two non-principal orbits ${y_0}^G$ and ${y_1}^G$.

Suppose that G_x is not connected. By Lemma 3.1.3, we may assume that $(G_{y_0})^1 \cap G_x$ is not connected. Let N be the largest closed connected normal subgroup of G which is contained in G_{y_0}. Then N is not trivial because $\dim G_{y_0} > \dim G_x$, and we infer from Lemma 2.5.7 that $(G_{y_0})^1 = (G_x)^1 N$. Since $(G_{y_0})^1$ acts transitively on G_{y_0}/G_x, the action of N on this space is transitive as well. It is also almost effective since $G_x \cap N$ is a stabilizer, and every connected normal subgroup of N is normal in G. By Lemma 3.1.3, the action of N on G_{y_0}/G_x is equivalent to the action of $\mathrm{SO}_3\mathbb{R}$ or $\mathrm{Spin}_3\mathbb{R}$ on the Poincaré homology sphere or to an action of $\mathbb{T}$ on the circle. The Poincaré homology sphere is excluded since Lemma 2.5.7 would yield the contradiction that $G_x \cap N$ is not totally disconnected. Hence $N \cong \mathbb{T}$, and being normal in G, the subgroup N is indeed contained in $Z(G)$. We may suppose that the intersection $Z(G) \cap G_x$ is trivial. (Otherwise, it can be factored out by Lemma 1.3.3.) But then

$$(G_{y_0})^1 \cap G_x = \big((G_x)^1 N\big) \cap G_x = (G_x)^1 (N \cap G_x) = (G_x)^1$$

is connected, a contradiction.

This shows that the principal stabilizer G_x is connected. We infer from Mostert's Theorem 3.1.2 that the two non-principal stabilizers are connected as well. □

3.1.5 Remark. In Lemma 3.1.3 and Theorem 3.1.4, the generalized n-sphere S can be replaced by any connected simply connected locally homogeneous metrizable cohomology n-manifold X over $\mathbb{Z}$ whose $\mathbb{Z}/2$-homology in degree $n-1$ vanishes. Indeed, if a compact group acts on X with an orbit of codimension one then X is a manifold by Theorem 2.2.2, and one can conclude that X does not admit a proper connected covering space (cf. Bredon [15, III.8.1]). Mostert's Theorem applies, and the hypothesis on the homology of X entails that the two non-principal orbits are of smaller dimension. An example for such a space X is the point space of a compact connected (m, m')-polygon with $m \geq 2$.

If, in addition, the topological parameters m and m' are equal then the point space is not homeomorphic to a product of two topological spaces, as was proved by Kramer [74, 3.3.8]. If the space X has this additional property then it can take the place of the generalized n-sphere S in the following theorem as well. Note that the conclusion of the theorem contains the statement that X is a sphere. Therefore, the hypotheses of the theorem cannot be satisfied for other spaces.

3.1.6 Theorem (Groups which must act with a codimension 1 orbit, II). *Let G be a compact connected Lie group which acts non-transitively on a generalized n-sphere S, where $n \geq 2$. Suppose that the action is effective, that some principal stabilizer is contained in the commutator subgroup G' of G, and that the codimension of every subgroup of G which does not contain a non-trivial connected normal subgroup is at least $n-1$. Then one of the following statements holds.*

(i) *The action is equivalent to the suspension of a transitive action of G on $\mathbb{S}_{n-1}$, and either $G \cong \mathrm{SO}_n\mathbb{R}$, or $G \cong \mathrm{G}_2$ and $n = 7$.*

(ii) *The action is equivalent to the join of two transitive actions on spheres of positive dimension. This means that there are compact connected Lie groups H_0 and H_1 which act transitively on $\mathbb{S}_{n_0}$ and $\mathbb{S}_{n_1}$, respectively, such that the action of G on S is equivalent to the action of $H_0 \times H_1$ on the join $\mathbb{S}_{n_0} * \mathbb{S}_{n_1}$. Thus $n = n_0 + n_1 + 1$. Each H_j satisfies either $H_j \cong \mathrm{SO}_{n_j+1}\mathbb{R}$, or $H_j \cong \mathrm{G}_2$ and $n_j = 6$.*

In particular, the action of G on S is equivalent to the natural action of a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$ on $\mathbb{S}_n$.

It is conceivable that this theorem also holds without the hypothesis that some principal stabilizer is contained in the commutator group. There are, however, additional difficulties. For example, one has to exclude an effective action of a Lie group with algebra $\mathbb{R} \times \mathfrak{e}_7$ on $\mathbb{S}_{56}$. (There is no such linear action, cf. Tits [136].) Therefore, we prefer to treat such actions concretely when they arise, even if this should imply some repetition of arguments.

Proof. Let $\tilde{G}$ be the universal compact covering group of G. As before, the hypotheses imply that we are in the situation of Mostert's Theorem 3.1.2. Choose a principal orbit x^G, and let ${y_0}^G$ and ${y_1}^G$ be the two non-principal orbits, where $\tilde{G}_x \leq \tilde{G}_{y_0} \cap \tilde{G}_{y_1}$. By the Classification Theorem 3.1.1 and Theorem 3.1.4, the quotient $\tilde{G}_{y_j}/\tilde{G}_x$ is homeomorphic to $\mathbb{S}_{n_j}$ for some $n_j \in \mathbb{N}$. For $j \in \{0, 1\}$, let $\tilde{H}_j$ be the largest closed connected normal subgroup which is contained in $\tilde{G}_{y_j}$. Then Lemma 2.5.7 and Theorem 3.1.4 show that $\tilde{G}_{y_j} = \tilde{G}_x \tilde{H}_j$.

Choose a closed connected normal subgroup N of $\tilde{G}$ which complements $\tilde{H}_0\tilde{H}_1$ (cf. Corollary 3.3.4). Set $H := (\tilde{H}_0 \cap \tilde{H}_1)^1$, and choose closed connected normal complements N_0 and N_1 of H in $\tilde{H}_0$ and in $\tilde{H}_1$, respectively. Then $\tilde{H}_0\tilde{H}_1$ is the internal direct product of N_0, N_1, and H, whence $\tilde{G}$ is the internal direct product of its four subgroups N, N_0, N_1, and H.

As all principal stabilizers are conjugate, the commutator group G' contains G_x. This is in fact just a statement about the respective Lie algebras since G_x is connected by Theorem 3.1.4. Thus the inclusion $\tilde{G}_x \leq \tilde{G}'$ also holds. Remark 2.5.8 shows that

$$\tilde{G}_x = (\tilde{G}_x \cap NN_{1-j})(\tilde{G}_x \cap \tilde{H}_j).$$

Under the given hypotheses, this decomposition is equivalent to the statement that every almost simple closed connected normal subgroup of $\tilde{G}_x$ is contained in either NN_{1-j} or $\tilde{H}_j$, which is in fact how it was proved. This implies that every almost simple closed connected normal subgroup of $\tilde{G}_x$ is contained in one of the four direct factors N, N_0, N_1, and H. As $Z(\tilde{G}_x)^1$ is the direct product of its intersections with these four direct factors by Theorem 2.5.5, we conclude that

$$\tilde{G}_x = (\tilde{G}_x \cap N)(\tilde{G}_x \cap N_0)(\tilde{G}_x \cap N_1)(\tilde{G}_x \cap H).$$

We obtain the following product decompositions of the various quotient spaces:

$$\begin{array}{ccccc}
x^G & \approx & \dfrac{\tilde{G}}{\tilde{G}_x} & \approx & \dfrac{N}{\tilde{G}_x \cap N} \times \dfrac{N_0}{\tilde{G}_x \cap N_0} \times \dfrac{N_1}{\tilde{G}_x \cap N_1} \times \dfrac{H}{\tilde{G}_x \cap H} \\
{y_0}^G & \approx & \dfrac{\tilde{G}}{\tilde{G}_{y_0}} & \approx & \dfrac{N}{\tilde{G}_x \cap N} \times \dfrac{N_1}{\tilde{G}_x \cap N_1} \\
{y_1}^G & \approx & \dfrac{\tilde{G}}{\tilde{G}_{y_1}} & \approx & \dfrac{N}{\tilde{G}_x \cap N} \times \dfrac{N_0}{\tilde{G}_x \cap N_0} \\
\mathbb{S}_{n_0} & \approx & \dfrac{\tilde{G}_{y_0}}{\tilde{G}_x} & \approx & \dfrac{N_0}{\tilde{G}_x \cap N_0} \times \dfrac{H}{\tilde{G}_x \cap H} \\
\mathbb{S}_{n_1} & \approx & \dfrac{\tilde{G}_{y_1}}{\tilde{G}_x} & \approx & \dfrac{N_1}{\tilde{G}_x \cap N_1} \times \dfrac{H}{\tilde{G}_x \cap H}
\end{array}$$

In particular, each of the three orbits contains the quotient $N/(\tilde{G}_x \cap N)$ as a direct factor. Therefore, the double mapping cylinder of

$$\frac{\tilde{G}}{\tilde{G}_{y_0}} \longleftarrow \frac{\tilde{G}}{\tilde{G}_x} \longrightarrow \frac{\tilde{G}}{\tilde{G}_{y_1}}$$

is homeomorphic to the product of $N/(\tilde{G}_x \cap N)$ with the double mapping cylinder of

$$\frac{N_1}{\tilde{G}_x \cap N_1} \longleftarrow \frac{N_0 N_1 H}{\tilde{G}_x \cap N_0 N_1 H} \longrightarrow \frac{N_0}{\tilde{G}_x \cap N_0} .$$

This shows that $N = 1$, since $S \approx \mathbb{S}_n$ does not have any non-trivial manifold as a direct factor, as one sees from the Künneth Theorem with coefficients $\mathbb{Z}/2$ (see Bredon [15, VI.3.2]). Similarly, the indecomposability of spheres entails that either $N_0 = N_1 = 1$ or $H = 1$.

In the first case, the two non-principal orbits are fixed points, and the reconstruction of the action as an action on a double mapping cylinder described in Mostert's Theorem 3.1.2 shows that the action of G on S is equivalent to the suspension of the action of G on $x^G \approx \mathbb{S}_{n-1}$. Together with Theorem 2.5.5 and the Classification Theorem 3.1.1, the fact that G_x is of maximal dimension among the subgroups of G which do not contain a non-trivial connected normal subgroup implies the classification of G contained in the statement. Moreover, the action of G on $\mathbb{S}_n$ is equivalent to a linear action by the Classification Theorem 3.1.1, and the suspension of a linear action is again linear. This is because the suspension of the natural action of a subgroup of $\mathrm{SO}_n\mathbb{R}$ on $\mathbb{S}_{n-1}$ is equivalent to the corresponding action on $\mathbb{S}_n$ which is obtained from an embedding of $\mathrm{SO}_n\mathbb{R}$ into $\mathrm{SO}_{n+1}\mathbb{R}$.

Suppose that $H = 1$. Then

$$ {y_{1-j}}^G \approx \frac{N_j}{\tilde{G}_x \cap N_j} \approx \mathbb{S}_{n_j}, $$

and the reconstruction of the action yields that the action of $\tilde{G}$ on S is equivalent to the join of the two actions of the groups $\tilde{G}/\tilde{G}_{y_{1-j}} \cong \tilde{H}_j = N_j$ on the spheres $\mathbb{S}_{n_j}$. The kernel of this action is the internal direct product of the kernels which belong to the actions of N_j on $\mathbb{S}_{n_j}$. When it is factored out, we obtain a decomposition of G into an internal direct product of two subgroups H_0 and H_1. Since for each $j \in \{0, 1\}$, the subgroup $G_x \cap H_j$ of H_j is of maximal dimension among the subgroups of H_j which do not contain a non-trivial connected normal subgroup, Theorem 2.5.5 and the Classification Theorem 3.1.1 lead to the possible cases for the H_j which the statement describes. Finally, the groups H_j act linearly on the spheres $\mathbb{S}_{n_j}$, and the join of two linear actions is equivalent to a linear action. Indeed, the two embeddings $H_j \hookrightarrow \mathrm{SO}_{n_j+1}\mathbb{R}$ give rise to an embedding $H_0 \times H_1 \hookrightarrow \mathrm{SO}_{n_0+n_1+2}\mathbb{R}$, and the action on $\mathbb{S}_n$ which we obtain in this way is equivalent to the join action, because $\mathbb{S}_n \subseteq \mathbb{R}^{n_0+1} \times \mathbb{R}^{n_1+1}$ can be written as

$$ \mathbb{S}_n = \left\{ \left(x \cos \tfrac{\pi}{2} t, y \sin \tfrac{\pi}{2} t\right) \;\middle|\; x \in \mathbb{S}_{n_0},\ y \in \mathbb{S}_{n_1},\ t \in [0,1] \right\} \approx \mathbb{S}_{n_0} * \mathbb{S}_{n_1}. $$

□

We conclude this section with a result of a more special nature.

3.1.7 Lemma (Montgomery and Samelson [89, Lemma 1]). *Let G be a compact connected Lie group acting effectively and transitively on a Hausdorff space X of dimension n, and suppose that the Lie algebra of G is isomorphic to $\mathfrak{o}_{n+1}\mathbb{R}$. Then the action of G on X is equivalent either to the natural action of $\mathrm{SO}_{n+1}\mathbb{R}$ on $\mathbb{S}_n$ or to that of $\mathrm{PSO}_{n+1}\mathbb{R}$ on $P_n\mathbb{R}$. In particular, G is covered by $\mathrm{SO}_{n+1}\mathbb{R}$.* □

3.2 Complex unitary groups

3.2.1 Lemma. *If $k \geq 4$ then all closed connected subgroups of $\mathrm{SU}_k\mathbb{C}$ which are locally isomorphic to $\mathrm{SU}_{k-1}\mathbb{C}$ are conjugate.*

The hypothesis that $k \geq 4$ is necessary because there are natural inclusions of the groups $\mathrm{SO}_3\mathbb{R}$ and $\mathrm{SU}_2\mathbb{C}$ into $\mathrm{SU}_3\mathbb{C}$, and the two groups are locally isomorphic but not isomorphic.

Proof. Let G be a closed connected subgroup of $\mathrm{SU}_k\mathbb{C}$ which is locally isomorphic to $\mathrm{SU}_{k-1}\mathbb{C}$, and let $\mathfrak{g} \leq \mathfrak{su}_k\mathbb{C}$ be the Lie algebra of G, so that $\mathfrak{g} \cong \mathfrak{su}_{k-1}\mathbb{C}$. The natural action of $\mathfrak{su}_k\mathbb{C}$ on $\mathbb{C}^k$ gives rise to an action of $\mathfrak{g}$ on $\mathbb{C}^k$ which in turn induces an action of the complexification $\mathfrak{g}_\mathbb{C} \cong \mathfrak{sl}_{k-1}\mathbb{C}$. Since this is a (semi-)simple Lie algebra, the module $\mathbb{C}^k$ is a direct sum of simple submodules by Weyl's Theorem (see Humphreys [63, 6.3]). We claim that there are two of these and that their dimensions are $k-1$ and 1. It suffices to show that every complex vector space of dimension at most k on which $\mathfrak{sl}_{k-1}\mathbb{C}$ acts non-trivially and irreducibly is in fact $(k-1)$-dimensional.

An irreducible complex $\mathfrak{sl}_{k-1}\mathbb{C}$-module of finite dimension is characterized by its highest weight $\lambda = m_1\lambda_1 + \ldots m_l\lambda_l$, where $l = k-2$ is the rank of $\mathfrak{sl}_{k-1}\mathbb{C}$ and the m_i are non-negative integers [63, 21.2]. Its dimension is given by Weyl's formula, which in the special case of $\mathfrak{sl}_{k-1}\mathbb{C}$ reads

$$\dim\lambda = \prod_{1\leq a\leq b\leq l}\left(1 + \frac{\sum_{i=a}^{b} m_i}{b+1-a}\right),$$

see Dynkin [37, p. 358]. It is an increasing function of each m_i if the others are fixed. (This phenomenon occurs for all simple Lie algebras, cf. Salzmann et al. [115, 95.7].) Therefore, we shall only need the following special cases which can also be found in the book by Onishchik and Vinberg [97, p. 300].

$$\begin{aligned} \dim \lambda_i &= \binom{l+1}{i} \\ \dim m\lambda_1 = \dim m\lambda_l &= \binom{l+m}{l} \\ \dim(\lambda_1 + \lambda_l) &= l(l+2) \end{aligned}$$

Thus $\dim \lambda_1 = \dim \lambda_l = k-1$, and if $2 \le i \le k-3$ (which only occurs for $k \ge 5$) then $\dim \lambda_i = \binom{k-1}{i} \ge \binom{k-1}{2} > k$. Similarly, if $m \ge 2$ then $\dim m\lambda_1 = \dim m\lambda_l = \binom{l+m}{l} \ge \binom{l+2}{l} = \binom{k}{2} > k$. Finally $\dim(\lambda_1 + \lambda_l) = k(k-2) > k$. Hence the highest weights of irreducible complex $\mathfrak{sl}_{k-1}\mathbb{C}$-modules whose dimension is at most k are precisely λ_1, λ_l, and 0, where the latter corresponds to the trivial one-dimensional module.

We have seen that $\mathfrak{g}$ has two invariant subspaces of respective dimensions $k-1$ and 1. These subspaces are orthogonal to each other since $\mathfrak{g}$ is contained in $\mathfrak{su}_k\mathbb{C}$. For the linear Lie group $\mathrm{SU}_k\mathbb{C}$, both exp and log can be written as power series. Therefore, the group G and its Lie algebra $\mathfrak{g}$ leave the same subspaces invariant. Hence there is an element $g \in \mathrm{SU}_k\mathbb{C}$ such that the conjugate G^g is contained in

$$H := \left\{ \left(\begin{array}{c|c} \det A^{-1} & \\ \hline & A \end{array} \right) \middle| A \in \mathrm{U}_{k-1}\mathbb{C} \right\}.$$

Since G^g is almost simple, it is in fact contained in the commutator group

$$H' = \left\{ \left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \middle| A \in \mathrm{SU}_{k-1}\mathbb{C} \right\}.$$

Now H' is connected, and $\dim G^g = \dim H'$, whence $G^g = H'$. □

3.2.2 Lemma. *Up to conjugacy, there is a unique closed subgroup G of $\mathrm{SU}_k\mathbb{C}$ (where $k \ge 3$) such that $\dim G = \dim \mathrm{U}_{k-1}\mathbb{C}$.*

In particular, every transitive action of $\mathrm{SU}_k\mathbb{C}$ on a Hausdorff space of dimension $2k-2$ is equivalent to the action of that group on $P_{k-1}\mathbb{C}$.

The lemma fails for $k = 2$ since $\mathrm{SU}_2\mathbb{C}$ contains closed one-dimensional subgroups which are not connected.

Proof. Let $G \leq \mathrm{SU}_k\mathbb{C}$ be a closed subgroup whose dimension equals $\dim \mathrm{U}_{k-1}\mathbb{C} = (k-1)^2$. Our first claim is that G is locally isomorphic to $\mathrm{U}_{k-1}\mathbb{C}$. In other words, we claim that the Lie algebra $\mathfrak{g} \leq \mathfrak{su}_k\mathbb{C}$ of G is isomorphic to $\mathbb{R} \times \mathfrak{a}_{k-2}$. If $k \neq 4$ then this follows from Table 2.1 on page 50. If $k = 4$ then $\dim \mathfrak{g} = 9$ and $\operatorname{rk} \mathfrak{g} \leq 3$, whence $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{a}_2$ or $\mathfrak{g} \cong {\mathfrak{a}_1}^3$. Suppose the latter. Then the natural action of $\mathfrak{su}_4\mathbb{C}$ on $\mathbb{C}^4$ induces an effective action of the complexification of $\mathfrak{g}$, which is $\mathfrak{g}_\mathbb{C} = (\mathfrak{sl}_2\mathbb{C})^3$. Since this Lie algebra is semi-simple, the module $\mathbb{C}^4$ decomposes as a direct sum of simple submodules by Weyl's Theorem (see Humphreys [63, 6.3]). Moreover, every simple module is a tensor product of three simple modules of the factors of $(\mathfrak{sl}_2\mathbb{C})^3$, see Samelson [116, 3.4 Thm. E p. 104]. Since the dimension of an effective $\mathfrak{sl}_2\mathbb{C}$-module is at least 2, it is easy to see that $\mathbb{C}^4$ cannot be an effective module of $(\mathfrak{sl}_2\mathbb{C})^3$. Hence in fact $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{a}_2$.

Let K be the commutator subgroup of the connected component of G. Then K is closed, connected, and locally isomorphic to $\mathrm{SU}_{k-1}\mathbb{C}$. If $k \geq 4$ then the previous lemma shows that

$$K^g = \left\{ \left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \middle| \, A \in \mathrm{SU}_{k-1}\mathbb{C} \right\}$$

for some $g \in \mathrm{SU}_k\mathbb{C}$. We want to show that the same holds for $k = 3$. As in the previous proof, it suffices to show that the action of K on $\mathbb{C}^3$ is not irreducible. Suppose the opposite. Then by Schur's Lemma, the centralizer C of this action in $\mathrm{End}(\mathbb{C}^3)$ is a skew field, and its dimension over the centre $Z(\mathrm{End}(\mathbb{C}^3))$ of $\mathrm{End}(\mathbb{C}^3)$ is finite. Since $Z(\mathrm{End}(\mathbb{C}^3)) \cong \mathbb{C}$ is algebraically closed, this implies that $C = Z(\mathrm{End}(\mathbb{C}^3))$. Now $Z(\mathrm{End}(\mathbb{C}^3)) \cap \mathrm{SU}_3\mathbb{C}$ is finite, contradicting the fact that this intersection contains the centre of G which is one-dimensional.

Since K is a (topologically) characteristic subgroup of G, the group G is contained in the normalizer N of K in $\mathrm{SU}_k\mathbb{C}$. The two K-invariant subspaces of $\mathbb{C}^k$ are also invariant under N, whence

$$N^g = \left\{ \left(\begin{array}{c|c} \det A^{-1} & \\ \hline & A \end{array} \right) \middle| \, A \in \mathrm{U}_{k-1}\mathbb{C} \right\}.$$

Since G is a full-dimensional subgroup of the connected Lie group N, we have in fact $G = N$.

Finally, all $(2k-2)$-dimensional homogeneous spaces of $\mathrm{SU}_k\mathbb{C}$ are equivalent since all possible stabilizers are conjugate, and $P_{k-1}\mathbb{C}$ is such a homogeneous space.

For $k \neq 4$, note that the lemma also follows from Theorem 2.5.1. $\square$

3.2.3 Lemma. *Let $G := \mathrm{SU}_k\mathbb{C}$, where $k \notin \{1, 2, 4\}$, act almost effectively on a generalized $2k$-sphere S. Then every principal orbit has codimension 1.*

In particular $S \approx \mathbb{S}_{2k}$.

For the values of k which have been excluded, there are counterexamples to this lemma. To see this, note that $\mathrm{SU}_4\mathbb{C} \cong \mathrm{Spin}_6\mathbb{R}$.

Proof. Let m be the dimension of a principal orbit. Then m is positive because the action is not trivial. Table 2.1 on page 50 shows that $m \geq 2k - 2$, whence $S \approx \mathbb{S}_{2k}$ by Lemma 1.3.5. The action cannot be transitive by Theorem 3.1.1, so that Lemma 1.3.4 implies that $m \leq 2k - 1$.

Suppose that $m = 2k - 2$. By Lemma 3.2.2 every principal orbit is homeomorphic to $P_{k-1}\mathbb{C}$. Let $X \subseteq S$ be the set of points on principal orbits, and set $Y := S \setminus X$. Richardson [111, 1.3] has proved that Y is non-empty. Hence [111, 1.2] shows that the orbit space S/G is homeomorphic to a two-dimensional disc $\mathbb{D}_2$. Moreover, the image X/G of X under the natural continuous surjection $S \to S/G$ is the interior of S/G, and Y/G is its boundary. Finally, Richardson's result also shows that the dimension of every orbit in Y is strictly smaller than m. Hence Y consists of fixed points by Table 2.1. Thus $Y = \mathrm{Fix}\, G$ is compact, and Y/G is the image of Y under a continuous bijection, whence $Y \approx Y/G \approx \mathbb{S}_1$. Its complement X is homeomorphic to $\mathring{\mathbb{D}}_2 \times P_{k-1}\mathbb{C}$ by [111, 1.6]. Hence X is homotopy equivalent to $P_{k-1}\mathbb{C}$. By Alexander duality (see Dold [34, VIII.8.17]), we have $H_2(X; \mathbb{Z}) \cong H^{2k-3}(Y; \mathbb{Z})$. The right-hand side is trivial, whereas the left-hand side is isomorphic to $H_2(P_{k-1}\mathbb{C}; \mathbb{Z}) \cong \mathbb{Z}$, see [34, V.4.9]. This contradiction shows that $m = 2k - 1$. $\square$

3.2.4 Corollary. *Every almost effective action of $\mathrm{SU}_k\mathbb{C}$, where $k \notin \{1, 2, 4\}$, on a generalized $(2k - 1)$-sphere S is transitive.*

Proof. Suppose, to the contrary, that $\mathrm{SU}_k\mathbb{C}$ acts non-transitively on S. Then Table 2.1 on page 50 and Lemma 1.3.5 show that $S \approx \mathbb{S}_{2k-1}$. The action of $\mathrm{SU}_k\mathbb{C}$ on $\mathbb{S}_{2k}$ which is obtained as the suspension of the action on S contradicts the preceding lemma. $\square$

3.2.5 Corollary. *If the group* $\mathrm{SU}_k\mathbb{C}$*, where* $k \notin \{1,2,4\}$*, acts almost effectively but not transitively on a generalized* n*-sphere* S *then* $n \geq 2k$*.*

Proof. If $n < 2k$ then $n = 2k-1$ by Table 2.1 on page 50 and Lemma 1.3.4. But this contradicts the preceding corollary. □

The following results prepare the actual reconstruction of an action of $\mathrm{SU}_k\mathbb{C}$ on $\mathbb{S}_{2k}$, at least for some small values of k. It is conceivable that the next lemma holds for arbitrary $k \geq 4$. A proof might use Dynkin's work [37]. If this works out then the following proofs go through for all $k \geq 4$.

3.2.6 Lemma. *If* $k \in \{4,5,6,7\}$ *then all closed connected subgroups* G *of* $\mathrm{SU}_k\mathbb{C}$ *with* $\dim G = \dim \mathrm{SU}_{k-1}\mathbb{C}$ *are conjugate.*

In particular, every transitive action of $\mathrm{SU}_k\mathbb{C}$ *(*k *as above) on a simply connected Hausdorff space of dimension* $2k-1$ *is equivalent to the usual action of this group on* $\mathbb{S}_{2k-1}$*.*

Proof. By Lemma 3.2.1, it suffices to show that the Lie algebra $\mathfrak{g} \leq \mathfrak{su}_k\mathbb{C}$ of G is isomorphic to $\mathfrak{su}_{k-1}\mathbb{C}$. Now $\dim \mathfrak{g} = \dim \mathfrak{su}_{k-1}\mathbb{C} = k(k-2)$, and $\operatorname{rk} \mathfrak{g} \leq \operatorname{rk} \mathfrak{su}_k\mathbb{C} = k-1$. Let $c := \dim \mathfrak{z}(\mathfrak{g})$. Note that $\mathfrak{g}$ is the direct product of its centre $\mathfrak{z}(\mathfrak{g})$ and its commutator algebra $\mathfrak{g}'$, which is semi-simple. We have $\dim \mathfrak{g}' = k(k-2) - c$ and $\operatorname{rk} \mathfrak{g}' = \operatorname{rk} \mathfrak{g} - c \leq k-1-c$. A glance at the classification of simple compact Lie algebras in Table 2.1 on page 50 shows that $\dim \mathfrak{g}' \equiv \operatorname{rk} \mathfrak{g}'$ (mod 2). On the other hand, the integers $k(k-2)$ and $k-1$ have opposite parities. Therefore in fact $\operatorname{rk} \mathfrak{g}' \leq k-2-c$, which is equivalent to $\operatorname{rk} \mathfrak{g} \leq k-2$.

For each of the four values of k, we will use the classification of compact Lie algebras to obtain a short list of possible isomorphism types of $\mathfrak{g}$. Representation theory will then allow us to exclude all cases except $\mathfrak{g} \cong \mathfrak{a}_{k-2} = \mathfrak{su}_{k-1}\mathbb{C}$.

If $k = 4$ then $\dim \mathfrak{g} = 8$ and $\operatorname{rk} \mathfrak{g} \leq 2$. This alone implies that $\mathfrak{g} \cong \mathfrak{a}_2$.

Suppose that $k = 5$, so that $\dim \mathfrak{g} = 15$ and $\operatorname{rk} \mathfrak{g} \leq 3$. If $\mathfrak{g}$ is not isomorphic to $\mathfrak{a}_3$ then $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{g}_2$. (To see such statements about compact algebras of small rank, one can use the list of semi-simple compact algebras of rank at most 5 given by Table 3.1 on page 78.) So suppose that $\mathfrak{g}_2$ embeds into $\mathfrak{su}_5\mathbb{C}$. Then the complexification $(\mathfrak{g}_2)_\mathbb{C}$ acts effectively

Table 3.1: The semi-simple Lie algebras of rank at most 5

	1	2	3	4	5
3	$\mathfrak{a}_1$				
4					
5					
6		${\mathfrak{a}_1}^2$			
7					
8		$\mathfrak{a}_2$			
9			${\mathfrak{a}_1}^3$		
10		$\mathfrak{x}_2$			
11			$\mathfrak{a}_2\times\mathfrak{a}_1$		
12				${\mathfrak{a}_1}^4$	
13			$\mathfrak{x}_2\times\mathfrak{a}_1$		
14		$\mathfrak{g}_2$		$\mathfrak{a}_2\times{\mathfrak{a}_1}^2$	
15			$\mathfrak{d}_3$		${\mathfrak{a}_1}^5$
16				$\mathfrak{x}_2\times{\mathfrak{a}_1}^2$, ${\mathfrak{a}_2}^2$	
17			$\mathfrak{g}_2\times\mathfrak{a}_1$		$\mathfrak{a}_2\times{\mathfrak{a}_1}^3$
18				$\mathfrak{d}_3\times\mathfrak{a}_1$, $\mathfrak{x}_2\times\mathfrak{a}_2$	
19					$\mathfrak{x}_2\times{\mathfrak{a}_1}^3$, ${\mathfrak{a}_2}^2\times\mathfrak{a}_1$
20				$\mathfrak{g}_2\times{\mathfrak{a}_1}^2$, ${\mathfrak{x}_2}^2$	
21			$\mathfrak{x}_3$		$\mathfrak{d}_3\times{\mathfrak{a}_1}^2$, $\mathfrak{x}_2\times\mathfrak{a}_2\times\mathfrak{a}_1$
22				$\mathfrak{g}_2\times\mathfrak{a}_2$	
23					$\mathfrak{d}_3\times\mathfrak{a}_2$, $\mathfrak{g}_2\times{\mathfrak{a}_1}^3$, ${\mathfrak{x}_2}^2\times\mathfrak{a}_1$
24				$\mathfrak{a}_4$, $\mathfrak{x}_3\times\mathfrak{a}_1$, $\mathfrak{g}_2\times\mathfrak{x}_2$	
25					$\mathfrak{d}_3\times\mathfrak{x}_2$, $\mathfrak{g}_2\times\mathfrak{a}_2\times\mathfrak{a}_1$
26					
27					$\mathfrak{a}_4\times\mathfrak{a}_1$, $\mathfrak{x}_3\times{\mathfrak{a}_1}^2$, $\mathfrak{g}_2\times\mathfrak{x}_2\times\mathfrak{a}_1$
28				$\mathfrak{d}_4$, ${\mathfrak{g}_2}^2$	
29					$\mathfrak{x}_3\times\mathfrak{a}_2$, $\mathfrak{d}_3\times\mathfrak{g}_2$
30					
31					$\mathfrak{d}_4\times\mathfrak{a}_1$, $\mathfrak{x}_3\times\mathfrak{x}_2$, ${\mathfrak{g}_2}^2\times\mathfrak{a}_1$
.					
35					$\mathfrak{a}_5$, $\mathfrak{x}_3\times\mathfrak{g}_2$
36				$\mathfrak{x}_4$	
.					
39					$\mathfrak{x}_4\times\mathfrak{a}_1$
.					
45					$\mathfrak{d}_5$
.					
52				$\mathfrak{f}_4$	
.					
55					$\mathfrak{x}_5$, $\mathfrak{f}_4\times\mathfrak{a}_1$

The symbol $\mathfrak{x}_r$ stands for either of the two Lie algebras $\mathfrak{b}_r$ or $\mathfrak{c}_r$, which have equal rank and dimension.

on $\mathbb{C}^5$. As we have stated in the proofs of Lemmas 3.2.1 and 3.2.2, the module $\mathbb{C}^5$ is a direct sum of simple submodules, and these are characterized by their highest weights. The smallest dimension of a non-trivial complex module corresponds to a fundamental weight. The dimensions of these fundamental representations have been listed by Tits [136] and by Onishchik and Vinberg [97, pp. 299–305]. We find that the dimension of a non-trivial $(\mathfrak{g}_2)_\mathbb{C}$-module is at least 7. Thus we have obtained a contradiction.

Assume that $k = 6$. Then $\dim \mathfrak{g} = 24$ and $\operatorname{rk} \mathfrak{g} \leq 4$. This leads to the possibilities

$$\mathfrak{a}_4, \mathfrak{a}_1 \times \mathfrak{b}_3, \mathfrak{a}_1 \times \mathfrak{c}_3, \text{ and } \mathfrak{b}_2 \times \mathfrak{g}_2$$

for the isomorphism type of $\mathfrak{g}$. Now $\mathfrak{b}_3$ and $\mathfrak{g}_2$ cannot be included in $\mathfrak{su}_6\mathbb{C}$ since for both of them, the minimal dimension of an effective module over their complexification is 7. For $\mathfrak{c}_3$, this number is 6. However, if $\mathfrak{a}_1 \times \mathfrak{c}_3$ were included in $\mathfrak{su}_6\mathbb{C}$ then $\mathbb{C}^6$ would be an effective module over the complexification $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sp}_6\mathbb{C}$. Every irreducible module over a direct product of simple complex Lie algebras is a tensor product of modules over the simple factors. Hence $\mathbb{C}^6$ cannot be effective.

Suppose that $k = 7$, so that $\dim \mathfrak{g} = 35$ and $\operatorname{rk} \mathfrak{g} \leq 5$. Then $\mathfrak{g}$ is isomorphic to one of

$$\mathfrak{a}_5, \mathfrak{g}_2 \times \mathfrak{b}_3, \text{ or } \mathfrak{g}_2 \times \mathfrak{c}_3.$$

The above argument yields that the minimal dimension of an effective module is 14 for $(\mathfrak{g}_2 \times \mathfrak{b}_3)_\mathbb{C}$, and 13 for $(\mathfrak{g}_2 \times \mathfrak{c}_3)_\mathbb{C}$. Hence $\mathfrak{g} \cong \mathfrak{a}_5$.

Finally, consider a transitive action of $\mathrm{SU}_k\mathbb{C}$, where $k \in \{4, 5, 6, 7\}$, on a simply connected Hausdorff space whose dimension is $2k - 1$. Let G be a stabilizer. Then G is connected (see Salzmann et al. [115, 94.4]). Therefore, it is conjugate to a stabilizer of the usual action of $\mathrm{SU}_k\mathbb{C}$ on $\mathbb{S}_{2k-1}$, and the two actions are equivalent. □

3.2.7 Lemma. *Suppose $k \in \{3, 5, 6, 7\}$, and let $G := \mathrm{SU}_k\mathbb{C}$ act almost effectively on a generalized $2k$-sphere S. Then the connected component of some principal stabilizer G_x is*

$$(G_x)^1 = \left\{ \left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \middle| \, A \in \mathrm{SU}_{k-1}\mathbb{C} \right\}.$$

Proof. If $k \in \{5,6,7\}$ then this follows from the preceding lemma, together with Lemma 3.2.3. So suppose that $k = 3$. Then Lemma 3.2.3 still applies, yielding $\dim G_x = 3$ for every principal stabilizer G_x. Since $\operatorname{rk} G_x \le \operatorname{rk} G = 2$, this implies that G_x is locally isomorphic to to $\mathrm{SU}_2\mathbb{C}$. This group can act on $\mathbb{C}^3$ in exactly two inequivalent effective ways, namely reducibly and irreducibly, see Bröcker and tom Dieck [17, II.5.3]. If the action of $(G_x)^1$ on $\mathbb{C}^3$ is reducible then we see as in the proof of Lemma 3.2.1 that $(G_x)^1$ is conjugate to the group appearing in the statement of the present lemma. In this case, we may even assume equality.

Suppose that $(G_x)^1$ acts irreducibly on $\mathbb{C}^3$. Then there is an element $g \in \mathrm{GL}_3\mathbb{C}$ such that $(G_x)^1 = (\mathrm{SO}_3\mathbb{R})^g$. Using the Cartan decomposition of $\mathrm{GL}_3\mathbb{C}$, one finds that g can be chosen in $\mathrm{U}_3\mathbb{C}$, and in fact in $\mathrm{SU}_3\mathbb{C}$ (cf. Warner [141, 1.1.3.7] or Kramer [75, 4.6f.]). Let $X \subseteq S$ be the set of points on principal orbits, and set $Y := S \setminus X$. Mostert's Theorem 3.1.2 yields that Y consists of exactly two orbits ${y_0}^G$ and ${y_1}^G$. Each stabilizer G_{y_j} contains a conjugate of G_x, and $\dim G_{y_j} > \dim G_x$. If $G_{y_j} \neq G$ then Table 2.1 on page 50 shows that $\dim G_{y_j} = 4$. Lemma 3.2.2 entails that G_{y_j} acts reducibly on $\mathbb{C}^3$. This contradicts the irreducibility of the action of $(G_x)^1$. We infer that $G_{y_j} = G$. Mostert's Theorem 3.1.2 also states that G_{y_j}/G_x is an integral homology sphere. Thus $x^G \approx \mathbb{S}_5$ by Theorem 3.1.1. Moreover, we have an exact sequence

$$\pi_2(x^G) \longrightarrow \pi_1(G_x) \longrightarrow \pi_1(G),$$

see Salzmann et al [115, 96.12]. Now $\pi_2(x^G) = 0$ (Rotman [114, 11.31]), and $\pi_1(G) = 0$ (Hilgert and Neeb [55, 11.4]), so that $\pi_1(G_x) = 0$ as well. But this contradicts $\pi_1(G_x) = \pi_1(\mathrm{SO}_3\mathbb{R}) \cong \mathbb{Z}/2$. $\square$

3.2.8 Lemma. *Let $G := \mathrm{SU}_k\mathbb{C}$ act almost effectively on a generalized $2k$-sphere S. Suppose that $k \ge 3$ and that the connected component of some principal stabilizer G_x is*

$$(G_x)^1 = \left\{ \left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \middle| A \in \mathrm{SU}_{k-1}\mathbb{C} \right\}.$$

Then the action of G on S is equivalent to the suspension of the transitive action on $\mathbb{S}_{2k-1}$.

Proof. We first show that every proper subgroup H of G which contains $(G_x)^1$ is contained in the normalizer

$$N_G((G_x)^1) = \left\{ \left(\begin{array}{c|c} \det A^{-1} & \\ \hline & A \end{array} \right) \middle| A \in \mathrm{U}_{k-1}\mathbb{C} \right\}.$$

Note that $\dim H < \dim G$ since G is connected. If $\dim H = \dim G_x$ then $(G_x)^1$ is normal in H since it is the connected component. Suppose that $\dim H > \dim G_x$. We claim that H is locally isomorphic to $\mathbb{T} \times \mathrm{SU}_{k-1}\mathbb{C}$. If $k \neq 4$ then this can be inferred from Table 2.1 on page 50. If $k = 4$ then $\operatorname{rk} H \leq \operatorname{rk} G = 3$, and $\dim H \leq 10$ by Table 2.1. Since also $\dim H \geq 9$, the Lie algebra $\mathfrak{h}$ of H must be isomorphic to one of

$$\mathfrak{b}_2, {\mathfrak{a}_1}^3, \text{ or } \mathbb{R} \times \mathfrak{a}_2.$$

Moreover, the Lie algebra of G_x, which is isomorphic to $\mathfrak{a}_2$, is embedded in $\mathfrak{h}$. Therefore, Table 2.1 excludes $\mathfrak{b}_2$, and ${\mathfrak{a}_1}^3$ is impossible since the image of $\mathfrak{a}_2$ under each projection onto a simple factor would be trivial. Having established the local isomorphism type of H, we see that $(G_x)^1$ is the commutator subgroup of the connected component of H, whence it is indeed normal in H. Also note that if $\dim H > \dim G_x$ then $H = N_G((G_x)^1)$.

The hypotheses imply that the codimension of x^G in S is 1. Hence Mostert's Theorem 3.1.2 applies. Its results will go into many arguments in the remainder of this proof, and we use the notation which was introduced there. Since each of the two non-principal stabilizers G_{y_j} satisfies $\dim G_{y_j} > \dim G_x$, the first part of the present proof shows that each G_{y_j} equals either $N_G((G_x)^1)$ or G. Therefore, each non-principal orbit ${y_j}^G$ is either a fixed point or homeomorphic to $P_{k-1}\mathbb{C}$. Moreover,

$$H^{2k-2}({y_0}^G; \mathbb{Z}) \oplus H^{2k-2}({y_1}^G; \mathbb{Z}) \cong H^{2k-2}(x^G; \mathbb{Z}) \cong H_1(x^G; \mathbb{Z}),$$

where the last isomorphism is Poincaré duality (Dold [34, VIII.8.1]; note that x^G is orientable). The Hurewicz Theorem (see Bredon [15, IV.3.4]) shows that

$$H_1(x^G; \mathbb{Z}) \cong \frac{\pi_1(x^G)}{\pi_1(x^G)'} \quad .$$

The exact homotopy sequence

$$1 = \pi_1(G) \longrightarrow \pi_1(x^G) \longrightarrow \pi_0(G_x) \longrightarrow \pi_0(G) = 1$$

(see Salzmann et al. [115, 96.12]) entails that $\pi_1(x^G) \cong G_x/(G_x)^1$, and this is a finite group, whence so is $H_1(x^G;\mathbb{Z})$. Since $H^{2k-2}(P_{k-1}\mathbb{C};\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, see [15, VI.10.2], we conclude that both non-principal orbits are fixed points. Hence G/G_x is an integral homology sphere, and indeed a sphere by Theorem 3.1.1. This implies that G_x is connected. Now that we know the stabilizers of the action of G on S, we infer from Mostert's Theorem 3.1.2 that this action is unique up to equivalence. It is therefore equivalent to the action described in the statement of the present lemma. □

3.2.9 Corollary. *Every almost effective action of* $\mathrm{SU}_k\mathbb{C}$ *on a generalized* $2k$*-sphere* S*, where* $k \in \{3, 5, 6, 7\}$*, is equivalent to the suspension of the transitive action on* $\mathbb{S}_{2k-1}$*.*

Proof. This is an immediate consequence of the two last lemmas. □

3.2.10 Proposition. *Let* G *be a compact connected group acting effectively on a generalized* $2k$*-sphere* S*, where* $k \in \{3, 5, 6, 7\}$*, and suppose that some normal subgroup* N *of* G *is locally isomorphic to* $\mathrm{SU}_k\mathbb{C}$*. Then* G *is isomorphic to* $\mathrm{U}_k\mathbb{C}$ *or to* $\mathrm{SU}_k\mathbb{C}$*, and its action on* S *is equivalent to the suspension of the transitive action on* $\mathbb{S}_{2k-1}$*.*

Proof. If $\dim G = \dim \mathrm{SU}_k\mathbb{C}$ then $G = N$ is locally isomorphic to $\mathrm{SU}_k\mathbb{C}$. Hence the latter group acts almost effectively, and the result follows immediately from the preceding corollary.

Assume that $\dim G > \dim \mathrm{SU}_k\mathbb{C}$. Together with Lemma 1.3.5, the classification of transitive actions on spheres in Theorem 3.1.1 shows that G cannot act transitively on S. Choose $x \in S$ such that x^N is a principal N-orbit. Then $\dim x^G = 2k-1 = \dim x^N$ by Lemmas 1.3.4 and 3.2.3, and the group G is a Lie group by Lemma 1.3.5. Therefore, the action of N on the connected manifold x^G is transitive. In other words, each principal N-orbit is invariant under G. This implies that the two fixed points of N are also fixed by G. Corollary 3.2.9 shows that $x^N \approx \mathbb{S}_{2k-1}$, and the action of G on $x^G = x^N$ is effective by Lemma 1.3.3. Hence Theorem 3.1.1 shows that this action is equivalent to the usual action of $\mathrm{U}_k\mathbb{C}$.

As above, we infer from Mostert's Theorem 3.1.2 that the action of G on S is determined up to equivalence by its stabilizers. In particular, it is equivalent to the action of $\mathrm{U}_k\mathbb{C}$ on $\mathbb{S}_{2k}$ which has been described in the statement. $\square$

3.3 A homogeneity property of torus groups

This brief section is devoted to a convenient property of torus groups: any two closed connected subgroups of equal dimension are conjugate by a continuous group automorphism. We give an elementary proof of this fact. The concluding remark contains a short alternative proof based on Pontryagin duality.

3.3.1 Lemma. *For all $(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, there exists $A \in \mathrm{GL}_2\mathbb{Z}$ such that*

$$(a,b)A = (0, \gcd(a,b)).$$

Proof. Let $r := \gcd(a,b)$, and choose $s,t \in \mathbb{Z}$ such that $r = as + bt$. Then

$$A := \begin{pmatrix} \frac{b}{r} & s \\ -\frac{a}{r} & t \end{pmatrix}$$

is an element of $\mathrm{GL}_2\mathbb{Z}$ with the required property. $\square$

3.3.2 Lemma. *All closed connected one-dimensional subgroups of $\mathbb{T}^n$ are conjugate under $\operatorname{Aut}\mathbb{T}^n$.*

Proof. Let H be a closed connected subgroup of $\mathbb{T}^n$ with $\dim H = 1$. There is a closed connected one-dimensional subgroup $\tilde{H}$ of $\mathbb{R}^n$ whose image under the canonical projection $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ is H. As H is closed, we find $a_1, \ldots, a_n \in \mathbb{Z}$ such that $\tilde{H} = \mathbb{R}(a_1, \ldots, a_n)$. We may even choose these integers in such a way that $\gcd(a_1, \ldots, a_n) = 1$. After rearranging, we may also assume that possible zeroes appear at the beginning of $(a_1, \ldots, a_n)$. By induction, the preceding lemma yields an element $A \in \mathrm{GL}_n\mathbb{Z}$ such that $(a_1, \ldots, a_n)A = (0, \ldots, 0, 1)$. The automorphism $x \mapsto xA$ of $\mathbb{R}^n$ induces an automorphism of $\mathbb{T}^n$ which maps $H = \mathrm{pr}(\tilde{H})$ to $\mathrm{pr}(\mathbb{R}(0, \ldots, 0, 1))$. $\square$

3.3.3 Proposition. *Let H and K be closed connected subgroups of $\mathbb{T}^n$ such that $\dim H = \dim K$. Then H and K are conjugate under $\operatorname{Aut} \mathbb{T}^n$.*

Proof. We may assume that $\dim H \neq 0$. The preceding lemma yields an automorphism φ of $\mathbb{T}^n$ such that H^φ contains $\mathbb{T} \times 1^{n-1}$. Let $H_1 \leq \mathbb{T}^{n-1}$ be the image of H^φ under the projection of $\mathbb{T}^n$ onto $\mathbb{T}^{n-1}$ with kernel $\mathbb{T} \times 1^{n-1}$. Then H_1 is closed and connected, and $\dim H_1 = \dim H - 1$. Moreover,

$$H_1 = \{(t_2, \dots, t_n) \in \mathbb{T}^{n-1} | (1, t_2, \dots, t_n) \in H^\varphi\},$$

whence $H^\varphi = \mathbb{T} \times H_1$. Similarly, there is an automorphism ψ of $\mathbb{T}^n$ and a closed connected subgroup K_1 of $\mathbb{T}^{n-1}$ such that $K^\psi = \mathbb{T} \times K_1$. By induction, there is an automorphism of $\mathbb{T}^{n-1}$ which maps H_1 to K_1. This extends to an automorphism of $\mathbb{T}^n$ which maps $\mathbb{T} \times H_1$ to $\mathbb{T} \times K_1$. □

3.3.4 Corollary. *Every closed connected subgroup of $\mathbb{T}^n$ is complemented.*

Proof. As $\mathbb{T}^k \times 1^{n-k}$ has the complement $1^k \times \mathbb{T}^{n-k}$ in $\mathbb{T}^n$, every closed connected k-dimensional subgroup of $\mathbb{T}^n$ has a complement. □

3.3.5 Remark. Conversely, this corollary immediately implies the preceding proposition.

We give a short independent proof of the corollary which is based on the Pontryagin Duality Theorem. This Theorem states that the functor $\operatorname{Hom}(-, \mathbb{T})$ is a self-duality of the category of locally compact abelian groups. For a proof, see Roeder [112] or Hewitt and Ross [54, 24.8]. The corollary can be rephrased as saying that every monomorphism between torus groups splits. Equivalently, one can prove the dual statement. As the Pontryagin dual of $\mathbb{T}$ is $\mathbb{Z}$ (see [54, 23.27]), the dual of a torus group is a free abelian group, whence the dual statement is that every epimorphism between free abelian groups splits. But this is obvious.

Note that we have only needed a few special consequences of Pontryagin duality, namely, the facts that $\mathbb{T}^n$ and $\mathbb{Z}^n$ correspond, that monomorphisms correspond to epimorphisms, and that this correspondence is natural in the suitable sense.

3.4 Other particular groups

3.4.1 Lemma. *For $n \geq 7$, the group $\mathrm{G}_2 \times \mathrm{Spin}_{n-5}\mathbb{R}$ cannot act almost effectively on a generalized n-sphere.*

Proof. An action of this group cannot be transitive by Theorem 3.1.1. Lemma 1.3.4 shows that the codimension of every stabilizer is at most $n-1$. On the other hand, Theorem 2.5.5 shows that every closed subgroup which does not contain any normal subgroup of positive dimension has codimension at least n. Therefore, every stabilizer contains a normal subgroup of positive dimension, and Lemma 1.3.3 shows that the action cannot be almost effective. □

3.4.2 Lemma. *If the group $G := \mathrm{U}_3\mathbb{H}$ acts almost effectively on a generalized n-sphere S then $n \geq 11$.*

Note that $\mathbb{S}_{11}$ is the unit sphere in $\mathbb{H}^3$, so that it does admit an effective transitive action of $\mathrm{U}_3\mathbb{H}$.

Proof. Suppose, to the contrary, that $n \leq 10$. By Theorem 3.1.1, the action of G on S is not transitive. Lemma 1.3.4 shows that the dimension of every stabilizer G_x is at least 12. Since there is no compact Lie group of dimension 12 and rank at most 3, we find that $\dim G_x \geq 13$. If G_x is a proper subgroup of G then we infer from Theorem 2.5.1 that the connected component of G_x is conjugate to

$$\left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array} \right) \middle| \, A \in \mathrm{U}_1\mathbb{H}, B \in \mathrm{U}_2\mathbb{H} \right\}.$$

Now G_x normalizes its connected component and hence leaves the same subspaces of $\mathbb{H}^3$ invariant. This implies that G_x is indeed connected. We conclude that every orbit in S is either a fixed point or homeomorphic to $P_2\mathbb{H}$. Moreover, Lemma 1.3.5 shows that $S \approx \mathbb{S}_n$ unless G acts trivially.

If $n = 8$ then the action of G on S must be transitive. But this is impossible since $\mathbb{S}_8 \not\approx P_2\mathbb{H}$.

In the case $n = 9$, we use Mostert's Theorem 3.1.2. Every principal orbit is homeomorphic to $P_2\mathbb{H}$. Being of smaller dimension, each non-principal orbit is a fixed point. This yields the contradiction $0 = H_4(P_2\mathbb{H}; \mathbb{Z}) \cong \mathbb{Z}$, see Dold [34, V.4.9]. Alternatively, we could have excluded this case by quoting Theorem 3.1.6.

Finally, suppose that $n = 10$. Let $X \subseteq S$ be the set of points whose G-orbits are homeomorphic to $P_2\mathbb{H}$, and let $Y \subseteq S$ be the set of points fixed under the action of G, so that $S = X \mathbin{\dot\cup} Y$. Richardson [111, 1.2 and 1.3] proves that the orbit space S/G is a compact 2-disc, and Y/G is its boundary. The restriction of the orbit map to the compact set Y is a continuous bijection onto Y/G, whence $Y \approx \mathbb{S}_1$. Using also [111, 1.6], we find that $X \approx \mathbb{R}^2 \times P_2\mathbb{H}$. Alexander duality (see Dold [34, VIII.8.17]) yields a contradiction:

$$0 = H^5(Y;\mathbb{Z}) \cong H_4(X;\mathbb{Z}) \cong H_4(P_2\mathbb{H};\mathbb{Z}) \cong \mathbb{Z}.$$

□

3.4.3 Lemma. *The following groups cannot act almost effectively on a generalized n-sphere if n is in the given range.*

(a) $n \geq 6: \ \mathbb{T}^2 \times \mathrm{Spin}_{n-2}\mathbb{R}$

(b) $n \geq 7: \ \mathrm{SU}_3\mathbb{C} \times \mathrm{Spin}_{n-4}\mathbb{R}$

(c) $n \geq 8: \ \mathbb{T} \times \mathrm{Spin}_4\mathbb{R} \times \mathrm{Spin}_{n-4}\mathbb{R}$

Note that the hypotheses on n are sharp, as is shown by easy counterexamples.

Proof. Suppose that one of these groups does act almost effectively on a generalized n-sphere, where n is as described in the statement. Theorem 3.1.1 yields that the action is not transitive. Theorems 2.5.5 and 3.1.4 show that all stabilizers are connected. Every principal stabilizer is contained in the commutator subgroup. For the groups in (a) and in (c), this follows because a principal stabilizer is semi-simple by Theorem 2.5.5, and the group in (b) is itself semi-simple and hence equals its commutator group. Theorem 3.1.6 yields a contradiction. Note that for even n, the groups in (a) and in (c) are already excluded by Smith's rank restriction given in Lemma 1.3.6. □

3.4.4 Lemma. *Let K be a finite cyclic subgroup of $\mathrm{SU}_2\mathbb{C}$, and set $k := |K|$. Then*

$$H_i(\mathrm{SU}_2\mathbb{C}/K;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0,3\} \\ \mathbb{Z}/k & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $X := \mathrm{SU}_2\mathbb{C}/K$, and suppress coefficients in homology and cohomology groups. Since X is connected, we have that $H_0(X) \cong \mathbb{Z}$. Next, we find that $H_1(X) \cong \pi_1(X) \cong \mathbb{Z}/k$ by the Hurewicz Theorem (see Bredon [15, IV.3.4]) and the exact homotopy sequence (see Salzmann et al. [115, 96.12]). As $\dim X = 3$ is odd, the Euler characteristic $\chi(X)$ vanishes (see Dold [34, VIII.8.7]). Since $H_1(X)$ is finite, this is only possible if $H_3(X) \cong \mathbb{Z}$, i.e. if X is orientable, cf. [15, VI.7.12]. Poincaré duality [34, VIII.8.1] yields that $H_2(X) \cong H^1(X) = 0$, where the last part follows from the Universal Coefficient Theorem [15, V.7.2]. □

The space $\mathrm{SU}_2\mathbb{C}/K$ is known as the lens space $L(|K|, 1)$, cf. Bredon [15, p. 363].

3.4.5 Lemma. *Let G be a compact connected Lie group which acts effectively on a generalized 5-sphere S. Suppose that the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}$. Then the action of G is equivalent to the natural action of*

$$\left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array} \right) \in \mathrm{SO}_6\mathbb{R} \;\middle|\; A \in \mathrm{SO}_2\mathbb{R}, B \in \mathrm{U}_2\mathbb{C} \leq \mathrm{SO}_4\mathbb{R} \right\}$$

on $\mathbb{S}_5$.

Proof. The group G is covered by $\tilde{G} := \mathbb{T}^2 \times \mathrm{SU}_2\mathbb{C}$. The covering map, which induces an almost effective action of $\tilde{G}$ on S, can be chosen such that $\mathbb{T}^2 \times 1$ acts effectively. Let

$$\mathrm{pr}_1 : \tilde{G} \longrightarrow \mathbb{T}^2 \quad \text{and} \quad \mathrm{pr}_2 : \tilde{G} \longrightarrow \mathrm{SU}_2\mathbb{C}$$

be the projections.

The Classification Theorem 3.1.1 shows that the action of G on S is not transitive. Choose a principal orbit x^G. Lemma 1.3.3 and Theorem 2.5.5 imply that $\dim \tilde{G}_x = 1$, and that the action of $\mathbb{T}^2 \times 1$ on x^G is effective. Hence $\mathbb{T}^2 \times 1$ intersects $\tilde{G}_x$ trivially, the restriction of pr_2 to $\tilde{G}_x$ is an embedding, and there is a homomorphism φ from $H := \mathrm{pr}_2(\tilde{G}_x)$ into $\mathbb{T}^2$ whose graph is $\tilde{G}_x$. Lemma 1.3.5 shows that $S \approx \mathbb{S}_5$. Theorem 3.1.4 yields that $\tilde{G}_x$ is connected. Moreover, we infer from Theorem 3.1.6 that φ is non-trivial. In other words, the subgroup $S_1 := \varphi(H) = \mathrm{pr}_1(\tilde{G}_x)$ of $\mathbb{T}^2$ is isomorphic to $\mathbb{T}$. Note that $\tilde{G}_x$ is contained in $S_1 \times \mathrm{SU}_2\mathbb{C}$.

Proposition 3.3.3 yields that $S_1 \leq \mathbb{T}^2$ has a closed connected complement $S_2 \leq \mathbb{T}^2$. It follows that $\tilde{G} = (S_2 \times \mathrm{SU}_2\mathbb{C})\tilde{G}_x$. Hence $S_2 \times \mathrm{SU}_2\mathbb{C}$ acts transitively on x^G. In this action, the stabilizer of x is

$$(S_2 \times \mathrm{SU}_2\mathbb{C}) \cap \tilde{G}_x = (1 \times \mathrm{SU}_2\mathbb{C}) \cap \tilde{G}_x = 1 \times K$$

for the finite cyclic subgroup $K := \ker \varphi \leq H \cong \mathbb{T}$ of $\mathrm{SU}_2\mathbb{C}$. This entails that

$$x^G \approx \mathbb{S}_1 \times \frac{\mathrm{SU}_2\mathbb{C}}{K}.$$

Set $k := |K|$.

By Mostert's Theorem 3.1.2, there are exactly two non-principal orbits ${y_0}^G$ and ${y_1}^G$, and the points y_0 and y_1 can be chosen in such a way that $\tilde{G}_x \leq \tilde{G}_{y_0} \cap \tilde{G}_{y_1}$. Then the two spaces $\tilde{G}_{y_j}/\tilde{G}_x$ are integral homology spheres of positive dimension. Theorem 3.1.1 shows that the dimension of $\tilde{G}_{y_j}/\tilde{G}_x$ is either 1 or 3. In the latter case, the subgroup $1^2 \times \mathrm{SU}_2\mathbb{C}$ of $\tilde{G}$ must be contained in $\tilde{G}_{y_j}$, and we infer that $\tilde{G}_{y_j}/\tilde{G}_x \approx \mathbb{S}_3$ and $\tilde{G}_{y_j} = S_1 \times \mathrm{SU}_2\mathbb{C}$, so that $k = 1$ and ${y_j}^G \approx \mathbb{S}_1$.

Suppose that $\dim(\tilde{G}_{y_j}/\tilde{G}_x) = 1$. It follows from Theorem 2.5.5 that some closed connected normal subgroup T_j of $\tilde{G}$ is contained in $\tilde{G}_{y_j}$. This T_j must be a one-dimensional subgroup of $Z(\tilde{G})$. The action of T_j on $\tilde{G}_{y_j}/\tilde{G}_x \approx \mathbb{S}_1$ is not trivial and hence transitive, so that the general Frattini argument shows that $\tilde{G}_{y_j} = T_j\tilde{G}_x$. If $T_j = S_1$ then ${y_j}^G \approx \mathbb{S}_1 \times \mathbb{S}_2$. Otherwise, the intersection $T_j \cap S_1$ is finite, with k_j elements, say. Then $\mathrm{pr}_1(\tilde{G}_{y_j}) = \mathbb{T}^2$, whence $\tilde{G} = (1 \times \mathrm{SU}_2\mathbb{C})\tilde{G}_{y_j}$, so that ${y_j}^G$ is a homogeneous space of $\mathrm{SU}_2\mathbb{C}$. The stabilizer of y_j in $\mathrm{SU}_2\mathbb{C}$ is $\varphi^{-1}(T_j) \leq H \cong \mathbb{T}$, which is is a cyclic subgroup of order k_jk.

We calculate the $\mathbb{Z}$-homology of the possible orbits with the help of Lemma 3.4.4 and of the Künneth Theorem (see Bredon [15, VI.1.6]):

j	$H_i(x^G;\mathbb{Z})$	$H_i(\mathbb{S}_1;\mathbb{Z})$	$H_i(\mathbb{S}_1 \times \mathbb{S}_2;\mathbb{Z})$	$H_i\left(\frac{\mathrm{SU}_2\mathbb{C}}{\varphi^{-1}(T_j)};\mathbb{Z}\right)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/k \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/k_jk$
2	$\mathbb{Z}/k$	0	$\mathbb{Z}$	0
3	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$
4	$\mathbb{Z}$	0	0	0

Mostert's Theorem 3.1.2 shows that

$$H_i(x^G;\mathbb{Z}) \cong H_i({y_0}^G;\mathbb{Z}) \oplus H_i({y_1}^G;\mathbb{Z}) \text{ if } 1 \le i \le 3.$$

We infer that $k = 1$, and the non-principal stabilizers are $\tilde{G}_{y_0} = S_1 \times \mathrm{SU}_2\mathbb{C}$ and $\tilde{G}_{y_1} = S_2\tilde{G}_x$. Of course, this only holds up to the choice of the complement S_2, i.e. up to an automorphism of $\tilde{G}$ which fixes $S_1 \times \mathrm{SU}_2\mathbb{C}$, and up to exchanging y_0 and y_1. We infer that the group which acts effectively is $G \cong \mathrm{SO}_2\mathbb{R} \times \mathrm{U}_2\mathbb{C}$. Theorem 3.1.2 allows us to reconstruct the action uniquely. □

3.4.6 Lemma. *For $n \ge 8$, the group $\mathbb{T}^2 \times \mathrm{Spin}_3\mathbb{R} \times \mathrm{Spin}_{n-4}\mathbb{R}$ cannot act almost effectively on a generalized n-sphere S.*

The lemma does not hold for $n = 7$, as is shown by the linear actions of the groups $(\mathrm{SO}_2\mathbb{R})^2 \times \mathrm{SO}_4\mathbb{R} \le \mathrm{SO}_8\mathbb{R}$ and $(\mathrm{U}_2\mathbb{C})^2 \le \mathrm{U}_4\mathbb{C}$ on $\mathbb{S}_7$.

Proof. If n is even then the lemma follows from Smith's rank restriction 1.3.6. In particular, we may assume that $n \ge 9$.

By the Classification Theorem 3.1.1, there is no transitive action of the group $\mathbb{T}^2 \times \mathrm{Spin}_3\mathbb{R} \times \mathrm{Spin}_{n-4}\mathbb{R}$ on S. The Lie algebra of this group is isomorphic to $\mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{n-4}\mathbb{R}$. Under an isomorphism, the Lie algebra of a principal stabilizer is mapped onto the graph of some non-trivial morphism from $\mathbb{R} \times \mathfrak{o}_{n-5}\mathbb{R} \le \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{n-4}\mathbb{R}$ into $\mathbb{R}^2$, see Lemma 1.3.3, Theorem 2.5.5, and Theorem 3.1.6. By Lemma 3.1.7, we can factor out a part of the kernel to obtain an almost effective action of the group

$$G := \mathbb{T} \times \mathbb{T} \times \mathrm{SU}_2\mathbb{C} \times \mathrm{SO}_{n-4}\mathbb{R}$$

on S such that the action of $Z(G)$ is effective. Let G_x be a principal stabilizer. Theorem 3.1.4 shows that G_x is connected. Using Corollary 3.3.4, we infer from the above description of $\mathfrak{l}(G_x)$ that

$$G_x = 1 \times H \times \mathrm{SO}_{n-5}\mathbb{R}$$

holds without loss of generality, where $H \cong \mathbb{T}$ is a subgroup of $\mathbb{T} \times \mathrm{SU}_2\mathbb{C}$ which is not contained in $1 \times \mathrm{SU}_2\mathbb{C}$ and has trivial intersection with $\mathbb{T} \times 1$. Define $K \le \mathrm{SU}_2\mathbb{C}$ by $1 \times K := H \cap (1 \times \mathrm{SU}_2\mathbb{C})$. Then $1 \times K$ is a discrete subgroup of H, and hence it is a finite cyclic group.

As $\mathbb{T}\times\mathrm{SU}_2\mathbb{C}$ is the complex product of its subgroups H and $1\times\mathrm{SU}_2\mathbb{C}$, we find that $\mathrm{SU}_2\mathbb{C}$ acts transitively on the coset space $(\mathbb{T}\times\mathrm{SU}_2\mathbb{C})/H$, whence this coset space is homeomorphic to $\mathrm{SU}_2\mathbb{C}/K$. We conclude that the principal orbit x^G satisfies

$$x^G \approx \mathbb{S}_1 \times \frac{\mathrm{SU}_2\mathbb{C}}{K} \times \mathbb{S}_{n-5}.$$

The situation is described by Mostert's Theorem 3.1.2. There are exactly two non-principal orbits ${y_0}^G$ and ${y_1}^G$. We may assume that $G_x \leq G_{y_j}$. Then the coset spaces G_{y_j}/G_x are integral homology spheres. For $j\in\{0,1\}$, let N_j be the largest closed connected subgroup of G which is contained in G_{y_j}. Lemma 2.5.7 shows that $G_{y_j} = G_xN_j$, whence

$$\frac{G_{y_j}}{G_x} \approx \frac{N_j}{G_x\cap N_j} \quad .$$

If N_j contains $1^3\times\mathrm{SO}_{n-4}\mathbb{R}$ then $\mathbb{S}_{n-5}$ is a direct factor of $N_j/(G_x\cap N_j)$, whence $N_j = 1^3\times\mathrm{SO}_{n-4}\mathbb{R}$, and

$${y_j}^G \approx \mathbb{S}_1 \times \frac{\mathrm{SU}_2\mathbb{C}}{K} \quad .$$

Similarly, if N_j contains $1^2\times\mathrm{SU}_2\mathbb{C}\times 1$ then $N_j = 1\times\mathbb{T}\times\mathrm{SU}_2\mathbb{C}\times 1$, and

$${y_j}^G \approx \mathbb{S}_1 \times \mathbb{S}_{n-5}.$$

If neither of these two cases holds then N_j is contained in the centre of G. Then $G_x\cap N_j = 1$, so that $N_j\cong\mathbb{T}$ and $\dim {y_j}^G = n-2$. Define $K_j \leq \mathrm{SU}_2\mathbb{C}$ by

$$\begin{aligned} 1^2\times K_j\times 1 \quad &:= \quad \big(N_j(1\times H\times 1)\big)\cap(1^2\times\mathrm{SU}_2\mathbb{C}\times 1) \\ &\leq \quad \big((\mathbb{T}^2\times 1^2)(1\times H\times 1)\big)\cap(1^2\times\mathrm{SU}_2\mathbb{C}\times 1) \\ &\cong \quad \mathbb{T}. \end{aligned}$$

Then K_j contains K. If $N_j = 1\times\mathbb{T}\times 1^2$ then $K_j\cong\mathbb{T}$, and

$${y_j}^G \approx \mathbb{S}_1\times\mathbb{S}_2\times\mathbb{S}_{n-5}.$$

Otherwise,

$$N_j(1\times H\times 1)(1^2\times\mathrm{SU}_2\mathbb{C}\times 1) = \mathbb{T}^2\times\mathrm{SU}_2\mathbb{C}\times 1.$$

Hence $\mathrm{SU}_2\mathbb{C} \times \mathrm{SO}_{n-4}\mathbb{R}$ acts transitively on ${y_j}^G$, and $K_j \times \mathrm{SO}_{n-5}\mathbb{R}$ is the stabilizer of y_j in this action. In this case, the group K_j is discrete, and in fact finite cyclic. Moreover,

$$ {y_j}^G \approx \frac{\mathrm{SU}_2\mathbb{C}}{K_j} \times \mathbb{S}_{n-5}. $$

Since $S \approx \mathbb{S}_n$ is not homeomorphic to a direct product, it is not possible that both non-principal orbits contain the direct factor $\mathbb{S}_1$, or that both contain the direct factor $\mathbb{S}_{n-5}$. We conclude that the relations

$$ \begin{array}{rcl} x^G & \approx & \mathbb{S}_1 \times \dfrac{\mathrm{SU}_2\mathbb{C}}{K} \times \mathbb{S}_{n-5} \\ {y_0}^G & \approx & \mathbb{S}_1 \times \dfrac{\mathrm{SU}_2\mathbb{C}}{K} \\ {y_1}^G & \approx & \dfrac{\mathrm{SU}_2\mathbb{C}}{K_1} \times \mathbb{S}_{n-5} \end{array} $$

hold, up to exchange of y_0 and y_1. Mostert's Theorem 3.1.2 shows that

$$ H_{n-3}(x^G;\mathbb{Z}) \cong H_{n-3}({y_0}^G;\mathbb{Z}) \oplus H_{n-3}({y_1}^G;\mathbb{Z}). $$

Using Lemma 3.4.4 and the Künneth Theorem (Bredon [15, VI.1.6]) to calculate the homology groups, we find that

$$ \begin{array}{rcl} H_{n-3}(x^G;\mathbb{Z}) & \cong & \mathbb{Z}/|K| \\ H_{n-3}({y_0}^G;\mathbb{Z}) & = & 0 \\ H_{n-3}({y_1}^G;\mathbb{Z}) & = & 0. \end{array} $$

This contradiction completes the proof. □

3.5 Local type of compact Lie transformation groups

3.5.1 Theorem. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G which acts almost effectively on a generalized n-sphere S, where $n \leq 8$. Then $\mathfrak{g}$ is embedded into $\mathfrak{o}_{n+1}\mathbb{R}$.*

We prove the theorem by determining the possible local isomorphism types of G. Therefore, the proof also yields lists of all subalgebras of $\mathfrak{o}_{n+1}\mathbb{R}$, which we write out explicitly for $4 \leq n \leq 8$. These lists are complete up to isomorphism. Note that the Lie algebras of transitive subgroups of $\mathrm{SO}_{n+1}\mathbb{R}$ are omitted; they can be inferred from Theorem 3.1.1. Also note that isomorphic subalgebras need not be conjugate under the adjoint action of $\mathrm{SO}_{n+1}\mathbb{R}$.

The reason for the hypothesis $n \leq 8$ is that we cannot easily exclude almost effective actions of some smaller groups on $\mathbb{S}_9$. The largest of these groups is $\mathbb{T}^3 \times \mathrm{Spin}_5\mathbb{R}$. We will comment on this problem in more detail later.

Nevertheless, this upper bound is satisfactory for the application of our results to compact generalized triangles (i.e. compact projective planes) of finite positive dimension. Their point rows and line pencils are generalized spheres of equal dimension, and this dimension divides 8.

Proof. We may assume throughout that G is connected and nontrivial.

If G acts transitively on S then $S \approx \mathbb{S}_n$ by Lemma 1.3.5. The effective quotient $G^1|_S$ of the connected component of G is equivalent to a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$ by Theorem 3.1.1, whence the claim follows.

From now on, we therefore assume that the action is not transitive. Let $x^G \subseteq S$ be a principal orbit. Then $\dim x^G \leq n-1$ by Lemma 1.3.4, and Lemma 1.3.3 shows that we can apply Corollary 2.5.6 to the almost effective action of G on the manifold x^G.

If $n \leq 4$ then it suffices to observe that $\dim G \leq \binom{n}{2}$ by Theorem 2.1.7. Together with Smith's rank restriction 1.3.6, this allows to deduce the statement easily. For example, if $n = 4$ then $\dim G \leq 6$ and $\operatorname{rk} G \leq 2$. Hence $\mathfrak{g}$ is isomorphic to one of

$$\mathfrak{o}_4\mathbb{R}, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R}, \quad \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R}^2, \quad \text{or } \mathbb{R},$$

all of which are indeed embedded into $\mathfrak{o}_5\mathbb{R}$.

For greater n, we make detailed use of Corollary 2.5.6. For convenience, the possible semi-simple parts of $\mathfrak{g}$ are collected in Table 2.2 on page 60. If $n = 5$ then we infer from this table and from Lemma 1.3.6 that the possible isomorphism types of $\mathfrak{g}$ are

$$\mathfrak{o}_5\mathbb{R}, \quad \mathbb{R} \times \mathfrak{o}_4\mathbb{R}, \quad \mathfrak{o}_4\mathbb{R}, \quad \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R}, \quad \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R}^3, \quad \mathbb{R}^2, \quad \text{and } \mathbb{R}.$$

Note that $\mathfrak{su}_3\mathbb{C}$ has been omitted since Corollary 3.2.4 shows that $\mathrm{SU}_3\mathbb{C}$ can only act transitively on $\mathbb{S}_5$. The algebras contained in the list are isomorphic to subalgebras of $\mathfrak{o}_6\mathbb{R}$. This is obvious in all cases except, perhaps, for $\mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}$. But this is the Lie algebra of the group $\mathrm{U}_1\mathbb{C} \times \mathrm{U}_2\mathbb{C}$ which is indeed embedded into $\mathrm{SO}_6\mathbb{R}$.

For $n = 6$, we find, in the same way, that $\mathfrak{g}$ is isomorphic to one of the following subalgebras of $\mathfrak{o}_7\mathbb{R}$:

$$\begin{array}{llllll}
\mathfrak{o}_6\mathbb{R}, & \mathbb{R} \times \mathfrak{o}_5\mathbb{R}, & \mathfrak{o}_5\mathbb{R}, & \mathbb{R} \times \mathfrak{su}_3\mathbb{C}, & \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}, & \\
 & \mathfrak{su}_3\mathbb{C}, & \mathbb{R} \times \mathfrak{o}_4\mathbb{R}, & \mathfrak{o}_4\mathbb{R}, & \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}, & \\
 & \mathbb{R} \times \mathfrak{o}_3\mathbb{R}, & \mathfrak{o}_3\mathbb{R}, & \mathbb{R}^3, & \mathbb{R}^2, & \mathbb{R}.
\end{array}$$

They are ordered by descending dimension.

For $n = 7$, these methods do not exclude the Lie algebras $\mathbb{R}^2 \times \mathfrak{o}_5\mathbb{R}$ and $\mathfrak{o}_3\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$. However, Lemma 3.4.3 shows that these algebras do not occur. Hence we are left with

$$\begin{array}{llllll}
\mathfrak{o}_7\mathbb{R}, & \mathbb{R} \times \mathfrak{o}_6\mathbb{R}, & \mathfrak{o}_6\mathbb{R}, & \mathfrak{g}_2, & \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_5\mathbb{R}, \\
 & \mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_4\mathbb{R}, & \mathbb{R} \times \mathfrak{o}_5\mathbb{R}, & \mathfrak{o}_5\mathbb{R}, & \mathbb{R}^2 \times \mathfrak{su}_3\mathbb{C}, \\
 & \mathbb{R} \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}, & \mathbb{R} \times \mathfrak{su}_3\mathbb{C}, & \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}, & \mathfrak{su}_3\mathbb{C}, \\
 & \mathbb{R}^2 \times \mathfrak{o}_4\mathbb{R}, & \mathbb{R} \times \mathfrak{o}_4\mathbb{R}, & \mathfrak{o}_4\mathbb{R}, & \mathbb{R}^3 \times \mathfrak{o}_3\mathbb{R}, \\
 & \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}, & \mathbb{R} \times \mathfrak{o}_3\mathbb{R}, & \mathbb{R}^4, & \mathfrak{o}_3\mathbb{R}, \\
 & \mathbb{R}^3, & \mathbb{R}^2, & \mathbb{R}. &
\end{array}$$

Again, we have ordered these algebras by descending dimension. All of them are embedded into $\mathfrak{o}_8\mathbb{R}$. As above, this is easy to see if one keeps in mind that $\mathbb{R} \times \mathfrak{o}_3\mathbb{R}$ is the Lie algebra of $\mathrm{U}_2\mathbb{C}$; also note that $\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$ is the Lie algebra of $\mathrm{U}_3\mathbb{C}$.

In the case $n = 8$, the only algebra which is not excluded by Corollary 2.5.6 and Lemma 1.3.6 is $\mathfrak{o}_4\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$. But this algebra is ruled out by Lemma 3.4.3. Table 3.2 on page 94 shows the remaining Lie algebras, all of which are isomorphic to subalgebras of $\mathfrak{o}_9\mathbb{R}$. □

A similar statement is possible for larger spheres if the group dimension is sufficiently high. With the help of Corollary 2.5.6, a large dimension of $\mathfrak{g}$ entails, in most cases, that $\mathfrak{g}$ contains a large orthogonal algebra as an ideal. We shall deal with this situation first, and then treat the exceptions, which only arise for relatively small n.

The hypothesis $\dim \mathfrak{g} > \binom{n-4}{2} + 3$ which occurs in Theorem 3.5.4 is not really sharp. In fact, I do not know any example of a compact

Table 3.2: The Lie algebras of compact Lie groups which act effectively on $\mathbb{S}_8$, ordered by descending dimension

dim	
36	$\mathfrak{o}_9\mathbb{R}$
.	
28	$\mathfrak{o}_8\mathbb{R}$
.	
22	$\mathbb{R} \times \mathfrak{o}_7\mathbb{R}$
21	$\mathfrak{o}_7\mathbb{R}$
.	
18	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_6\mathbb{R}$
17	
16	$\mathbb{R} \times \mathfrak{o}_6\mathbb{R}, \quad \mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$
15	$\mathfrak{o}_6\mathbb{R}, \quad \mathbb{R} \times \mathfrak{g}_2$
14	$\mathfrak{g}_2, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$
13	$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$
12	$\mathbb{R}^2 \times \mathfrak{o}_5\mathbb{R}, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{su}_3\mathbb{C}, \quad \mathfrak{o}_4\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
11	$\mathbb{R} \times \mathfrak{o}_5\mathbb{R}, \quad \mathfrak{o}_3\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$
10	$\mathfrak{o}_5\mathbb{R}, \quad \mathbb{R}^2 \times \mathfrak{su}_3\mathbb{C}, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
9	$\mathbb{R} \times \mathfrak{su}_3\mathbb{C}, \quad \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
8	$\mathfrak{su}_3\mathbb{C}, \quad \mathbb{R}^2 \times \mathfrak{o}_4\mathbb{R}$
7	$\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$
6	$\mathfrak{o}_4\mathbb{R}, \quad \mathbb{R}^3 \times \mathfrak{o}_3\mathbb{R}$
5	$\mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}$
4	$\mathbb{R} \times \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R}^4$
3	$\mathfrak{o}_3\mathbb{R}, \quad \mathbb{R}^3$
2	$\mathbb{R}^2$
1	$\mathbb{R}$

Note that the list is complete up to isomorphism, but not up to conjugacy of subalgebras of $\mathfrak{o}_9\mathbb{R}$. Also note that the only effective and transitive action of a compact connected group on $\mathbb{S}_8$ is the natural action of $\mathrm{SO}_9\mathbb{R}$ by Theorem 3.1.1.

connected Lie group which acts continuously on $\mathbb{S}_n$ but is not locally isomorphic to a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$. However, our methods do not allow us to exclude an almost effective action of $\mathbb{T}^3 \times \mathrm{Spin}_{n-4}\mathbb{R}$ on $\mathbb{S}_n$ if n is odd. (For even n, the rank of this group is too large by Lemma 1.3.6.) The problem is that Theorem 2.5.5 allows a principal orbit of $\mathbb{T}^3 \times \mathrm{Spin}_{n-4}\mathbb{R}$ to have codimension 2. Our treatment of the codimension 2 case rests on the methods developed by Richardson [111, 1.2f.], and these work for semi-simple groups only. Nevertheless, the bound on the group dimension might be lowered considerably for actions on even-dimensional spheres, as is shown by Proposition 3.5.2. However, there are groups to exclude where the codimension of a principal orbit, according to Theorem 2.5.5, could be greater than 2. An example for this situation is the problem whether $\mathrm{SU}_3\mathbb{C} \times \mathrm{U}_3\mathbb{H}$ can act on $\mathbb{S}_{15}$, or on $\mathbb{S}_{16}$. If the codimension of a principal orbit is 3 or greater then the situation is desperate. It is not even known whether the orbit space is a manifold with boundary, and this information would only be the starting point for notorious hard topological problems.

3.5.2 Proposition. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G which acts almost effectively, but not transitively, on a generalized n-sphere S, where $n \geq 9$. Suppose that $\mathfrak{g} \cong \mathfrak{h} \times \mathfrak{o}_m\mathbb{R}$ for some $m \geq n-6$, and that $\mathfrak{g} \not\cong \mathbb{R}^3 \times \mathfrak{o}_{n-4}\mathbb{R}$. If $m = n-6$ then suppose that $\dim \mathfrak{h} > 10$, and that $n \geq 10$.*

Then $\mathfrak{h}$ admits an embedding into $\mathfrak{o}_{n+1-m}\mathbb{R}$. In particular, the Lie algebra $\mathfrak{g}$ is embedded into $\mathfrak{o}_{n+1}\mathbb{R}$.

The case $m = 3$ is excluded for technical reasons because the proof rests on Corollary 2.5.6.

For large n, this proposition yields information about Lie groups whose dimension is greater than $\binom{n-6}{2} + 10$. The lower bound is due to the problem whether $G = \mathbb{T} \times (\mathrm{Spin}_3\mathbb{R})^3 \times \mathrm{Spin}_{n-6}\mathbb{R}$ can act effectively on $\mathbb{S}_n$. This group is accessible to Mostert's Theorem 3.1.2 and to the machinery derived from it, but we have not worked out the details. Note that for even n, the group G is excluded by its rank.

Proof. As in the proof of Theorem 3.5.1, we first note that every principal orbit $x^G \subseteq S$ satisfies $\dim x^G \leq n-1$ by Lemma 1.3.4, and that the action of G on x^G is almost effective by Lemma 1.3.3. We shall apply Corollary 2.5.6 to this action. As the subalgebra $\mathfrak{l}(G_x)$ of $\mathfrak{g}$

does not contain a non-trivial ideal and has codimension at most $n-1$, we infer from Corollary 2.5.6 that $\mathfrak{h}$ has a subalgebra which does not contain a non-trivial ideal and has codimension at most $n-m$. Therefore, a list of possible isomorphism types of $\mathfrak{h}$ can conveniently be derived from Table 2.2 on page 60 for each m.

Theorem 2.1.7 implies that $m \le n$. If $m = n$ then $\mathfrak{g} \cong \mathfrak{o}_n\mathbb{R}$, and if $m = n-1$ then either $\mathfrak{g} = \mathbb{R} \times \mathfrak{o}_{n-1}\mathbb{R}$ or $\mathfrak{g} = \mathfrak{o}_{n-1}\mathbb{R}$. If $m = n-2$ then $\mathfrak{h}$ is either trivial or isomorphic to one of $\mathfrak{o}_3\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}$. The case $\mathfrak{h} \cong \mathbb{R}^2$ is excluded by Lemma 3.4.3.

The possible isomorphism types of $\mathfrak{h}$ for $m = n-3$ are

$$\mathfrak{o}_4\mathbb{R}, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R}, \quad \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R}^3, \quad \mathbb{R}^2, \quad \mathbb{R}, \quad \text{and } 0.$$

The case $\mathfrak{h} \cong \mathbb{R}^3$ is ruled out by Lemma 1.3.6 since

$$\operatorname{rk}(\mathbb{R}^3 \times \mathfrak{o}_{n-3}\mathbb{R}) = 3 + \left\lfloor \frac{n-3}{2} \right\rfloor = \left\lfloor \frac{n+3}{2} \right\rfloor > \operatorname{rk} \mathfrak{o}_{n+1}\mathbb{R}.$$

Suppose that $m = n-4$. Lemma 1.3.6 shows that $\operatorname{rk}\mathfrak{h} \le 3$. Using Corollary 2.5.6, we find that we only have to exclude the case that $\mathfrak{h}$ is isomorphic to one of the algebras

$$\mathfrak{su}_3\mathbb{C}, \quad \mathbb{R} \times \mathfrak{o}_4\mathbb{R}, \quad \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}, \quad \text{or } \mathbb{R}^3.$$

This is done by Lemma 3.4.3, Theorems 2.5.5 and 3.1.6, Lemma 3.4.6, and the hypothesis, respectively.

For $m = n-5$, we find again that $\operatorname{rk}\mathfrak{h} \le 3$. The isomorphism types to exclude are $(\mathfrak{o}_3\mathbb{R})^3$ and $\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$. To achieve this, we use Theorems 2.5.5 and 3.1.6.

If $m = n-6$ then $\dim \mathfrak{h} > 10$ by hypothesis, and $\operatorname{rk}\mathfrak{h} \le 4$. After the use of Corollary 2.5.6, we only have to show that $\mathfrak{h}$ is not isomorphic to one of the algebras

$$\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_5\mathbb{R}, \quad \mathfrak{o}_3\mathbb{R} \times \mathfrak{su}_3\mathbb{C}, \quad (\mathfrak{o}_3\mathbb{R})^4, \quad \mathbb{R} \times \mathfrak{o}_6\mathbb{R}, \quad \text{or } \mathbb{R}^2 \times \mathfrak{o}_5\mathbb{R}.$$

All of these algebras are excluded by Theorems 2.5.5 and 3.1.6. □

3.5.3 Proposition. *Let G be a compact connected group acting effectively and transitively on a generalized n-sphere S, and suppose that $n \ge 9$ and that $\dim G > \binom{n-4}{2} + 3$. Then one of the following statements holds:*

(a) $G \cong \mathrm{SO}_{n+1}\mathbb{R}$;

(b) $n = 2m - 1 \leq 15$, *and* $G \cong \mathrm{SU}_m\mathbb{C}$ *or* $G \cong \mathrm{U}_m\mathbb{C}$.

For each group, the action is unique up to equivalence.

Proof. We apply the classification of effective and transitive actions of compact groups on spheres given in Theorem 3.1.1. In all cases, the group G is a Lie group, and the action is equivalent to a linear action. If $n \geq 9$ then apart from one exceptional action of the group $\mathrm{Spin}_9\mathbb{R}$, whose dimension is too small, on $\mathbb{S}_{15}$, all actions fall into one of three series which we now treat separately.

In the first case, we have the natural action of $\mathrm{SO}_{n+1}\mathbb{R}$ on $\mathbb{S}_n$. The dimension of this group is $\binom{n+1}{2}$, so that it satisfies the hypothesis of the present proposition.

In the second case, the group is isomorphic to either $\mathrm{U}_m\mathbb{C}$ or $\mathrm{SU}_m\mathbb{C}$, and it acts on $\mathbb{S}_{2m-1}$ in the familiar way. Then

$$m^2 \geq \dim G > \binom{n-4}{2} + 3 = \binom{2m-5}{2} + 3.$$

Straightforward calculation leads to $(m-5)(m-6) < 12$, whence $m \leq 8$.

Finally, the group can be a subgroup of $\mathrm{SO}_{4m}\mathbb{R}$ consisting of $\mathrm{U}_m\mathbb{H}$ (in its natural representation) and of a second factor which centralizes the first. This second factor embeds into $\mathrm{U}_1\mathbb{H}$. Hence

$$\binom{2m+1}{2} + 3 \geq \dim G > \binom{n-4}{2} + 3 = \binom{4m-5}{2} + 3,$$

which entails that $m \leq 2$, whence $n \leq 7$. □

3.5.4 Theorem. *Let G be a compact connected Lie group acting effectively on a generalized n-sphere S, and suppose that $n \geq 9$ and that $\dim G > \binom{n-4}{2} + 3$. Then the Lie algebra of G is embedded into $\mathfrak{o}_{n+1}\mathbb{R}$.*

The transitive actions are described in Proposition 3.5.3. If G acts non-transitively then either the Lie algebra $\mathfrak{g}$ of G has an ideal isomorphic to $\mathfrak{o}_m\mathbb{R}$ for some $m \geq n-6$, and it is described in Proposition 3.5.2, or one of the following cases occurs, all of which are possible:

(a) $n = 9$, *and $\mathfrak{g}$ is isomorphic to $\mathfrak{o}_3\mathbb{R} \times \mathfrak{g}_2$, to $\mathbb{R} \times \mathfrak{g}_2$, or to $\mathfrak{g}_2$.*

(b) $n = 10$, *and* $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{su}_5\mathbb{C}$ *or* $\mathfrak{g} \cong \mathfrak{su}_5\mathbb{C}$.

(c) $n = 11$, *and* $\mathfrak{g} \cong \mathbb{R}^2 \times \mathfrak{su}_5\mathbb{C}$ *or* $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{su}_5\mathbb{C}$.

(d) $n = 12$, *and* $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{su}_6\mathbb{C}$ *or* $\mathfrak{g} \cong \mathfrak{su}_6\mathbb{C}$.

(e) $n = 14$, *and* $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{su}_7\mathbb{C}$ *or* $\mathfrak{g} \cong \mathfrak{su}_7\mathbb{C}$.

Theorem 3.1.6 describes the action if $n = 9$ and $\mathfrak{g} \cong \mathfrak{o}_3\mathbb{R} \times \mathfrak{g}_2$. If $n = 2m$ and $\mathfrak{g}$ contains $\mathfrak{su}_m\mathbb{C}$ as an ideal then the action is described in Proposition 3.2.10.

Proof. Suppose that G does not act transitively. As above, we apply Corollary 2.5.6 to the effective action of G on a principal orbit, whose dimension is at most $n - 1$. We treat the case $n = 9$ first. Then we give a precise meaning to the intuitive notion that $\mathfrak{g}$ must contain a large simple ideal. After that, we will treat the dimensions from 10 to 14 one by one, and finally we will prove the theorem for $n \geq 15$.

Suppose that $n = 9$. Then $\dim \mathfrak{g} > 13$, and Lemma 1.3.6 shows that $\operatorname{rk} \mathfrak{g} \leq 5$. If some ideal of $\mathfrak{g}$ is isomorphic to $\mathfrak{o}_m\mathbb{R}$ with $m \geq 4$ then we are in the situation of Proposition 3.5.2. Suppose that this is not the case. Table 2.2 on page 60 yields that $\mathfrak{g}$ is isomorphic to one of the Lie algebras

$$\mathfrak{su}_5\mathbb{C}, \quad \mathfrak{u}_3\mathbb{H}, \quad \mathfrak{o}_3\mathbb{R} \times \mathfrak{g}_2, \quad \mathfrak{su}_3\mathbb{C} \times \mathfrak{su}_3\mathbb{C}, \quad \mathbb{R}^2 \times \mathfrak{g}_2, \quad \mathbb{R} \times \mathfrak{g}_2, \quad \text{and } \mathfrak{g}_2.$$

Corollary 3.2.4 asserts that $\mathrm{SU}_5\mathbb{C}$ can only act transitively. The algebra $\mathfrak{u}_3\mathbb{H}$ is excluded by Lemma 3.4.2. Theorems 2.5.5 and 3.1.6 rule out $\mathfrak{su}_3\mathbb{C} \times \mathfrak{su}_3\mathbb{C}$ and $\mathbb{R}^2 \times \mathfrak{g}_2$. The remaining three algebras occur as exceptions in point (a) of the statement. They admit an embedding into $\mathfrak{o}_{10}\mathbb{R}$ since $\mathfrak{g}_2$ can be embedded into $\mathfrak{o}_7\mathbb{R}$.

We use Corollary 2.5.6 to deduce a general statement about the existence of a large simple ideal. Let $q := \dim \mathfrak{z}(\mathfrak{g})$, and write $\mathfrak{g}' \cong \mathfrak{s}_1 \times \ldots \times \mathfrak{s}_k$ with compact Lie algebras $\mathfrak{s}_i$ which are either simple or $\mathfrak{o}_4\mathbb{R}$, and of which at most one is isomorphic to $\mathfrak{o}_3\mathbb{R}$. For the sake of definiteness, assume that $\dim \mathfrak{s}_1 \leq \cdots \leq \dim \mathfrak{s}_k$. If $\mathfrak{s}_i$ is simple then define s_i to be the minimal codimension of a proper subalgebra of $\mathfrak{s}_i$. Note that the number s_i can be found in Table 2.1 on page 50. Theorem 2.1.7 entails that $\dim \mathfrak{s}_i \leq \binom{s_i+1}{2}$. If $\mathfrak{s}_i \cong \mathfrak{o}_4\mathbb{R}$ then set

$s_i := 3$. Note that $\dim \mathfrak{s}_i$ satisfies the same inequality. Corollary 2.5.6 yields that

$$q + s_1 + \cdots + s_k \leq n - 1.$$

We find that

$$\begin{aligned} \dim \mathfrak{g} &\leq q + \sum_{i=1}^{k} \frac{(s_i+1)s_i}{2} \leq q + \frac{s_k+1}{2} \sum_{i=1}^{k} s_i \\ &\leq \frac{s_k+1}{2}\left(q + \sum_{i=1}^{k} s_i\right) \leq \frac{s_k+1}{2}(n-1). \end{aligned}$$

Note that the second and third inequalities are sharp if all s_i are equal and if $q = 0$. This means that we have lost as little information as possible. We rephrase the result as

$$s_k \geq \frac{2\dim \mathfrak{g}}{n-1} - 1 \geq \frac{2}{n-1}\left(\binom{n-4}{2} + 4\right) - 1,$$

which yields the inequality

$$s_k \geq n - 9 + \frac{20}{n-1} \quad .$$

Set $\mathfrak{h} := \mathbb{R}^q \times \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_{k-1}$, so that $\mathfrak{g} \cong \mathfrak{h} \times \mathfrak{s}_k$. There is a subalgebra of $\mathfrak{h}$ which does not contain any non-trivial ideal and whose codimension is $s' := q + s_1 + \cdots + s_{k-1}$. Note that $s' \leq n - 1 - s_k$. Theorem 2.1.7 shows that

$$\dim \mathfrak{h} \leq \binom{s'+1}{2} \leq \binom{n-s_k}{2}.$$

We conclude that

$$\dim \mathfrak{g} \leq \binom{n-s_k}{2} + \dim \mathfrak{s}_k \leq \binom{n-s_k}{2} + \binom{s_k+1}{2}. \tag{3.1}$$

Suppose that $10 \leq n \leq 14$. Then $n - 7 \leq s_k \leq n - 1$, and

$$\dim G > \binom{n-4}{2} + 3 > \binom{n-6}{2} + 10.$$

Suppose that $\mathfrak{s}_k \cong \mathfrak{o}_m\mathbb{R}$ for some $m \in \mathbb{N}$. Then $m = s_k+1$. Hence $m \geq n-6$, so that the theorem follows from Proposition 3.5.2. Therefore, we may assume that $\mathfrak{s}_k$ is not a real orthogonal algebra.

If $n = 10$ then we even have that $s_k \geq 4$, and $\dim \mathfrak{g} > 18$. The possible isomorphism types of $\mathfrak{s}_k$ are

$$\mathfrak{su}_5\mathbb{C}, \quad \mathfrak{u}_3\mathbb{H}, \quad \mathfrak{g}_2, \quad \text{and } \mathfrak{su}_3\mathbb{C}.$$

If $\mathfrak{h} \cong \mathfrak{su}_5\mathbb{C}$ then $s_k = 8$, whence $\dim \mathfrak{h} \leq 1$. This leads to the first two cases of point (b). The Lie algebra $\mathfrak{u}_3\mathbb{H}$ is excluded by Lemma 3.4.2. If $\mathfrak{s}_k \cong \mathfrak{g}_2$ then $s' \leq 3$. As $\dim \mathfrak{h} > 4$, we find that $\mathfrak{h} \cong \mathfrak{o}_4\mathbb{R}$. Hence this case is covered by Proposition 3.5.2. Finally, if $\mathfrak{s}_k \cong \mathfrak{su}_3\mathbb{C}$ then $s' \leq 5$ and $\dim \mathfrak{h} > 10$. The possible isomorphism types of $\mathfrak{h}$ can be inferred from Table 2.2 on page 60. None of them satisfies the assumption that $\dim \mathfrak{s}_i \leq \dim \mathfrak{s}_k$ for all i.

Suppose that $n = 11$. Then $s_k \geq 4$ and $\dim \mathfrak{g} > 24$. The ideal $\mathfrak{s}_k$ is isomorphic to one of the algebras

$$\mathfrak{su}_6\mathbb{C}, \quad \mathfrak{su}_5\mathbb{C}, \quad \mathfrak{u}_3\mathbb{H}, \quad \mathfrak{g}_2, \quad \text{and } \mathfrak{su}_3\mathbb{C}.$$

Corollary 3.2.4 shows that $\mathrm{SU}_6\mathbb{C}$ can only act transitively. If $\mathfrak{s}_k \cong \mathfrak{su}_5\mathbb{C}$ then $s' \leq 2$ and $\dim \mathfrak{h} > 0$. This leads to point (c) of the statement. If $\mathfrak{s}_k \cong \mathfrak{u}_3\mathbb{H}$ then $s' \leq 2$, and if $\mathfrak{s}_k \cong \mathfrak{g}_2$ then $s' \leq 4$. Both cases are excluded by the dimension formula 3.1. Finally, if $\mathfrak{s}_k \cong \mathfrak{su}_3\mathbb{C}$ then $s' \leq 6$ and $\dim \mathfrak{h} > 16$. We obtain a contradiction by the argument used for $n = 10$.

If $n = 12$ then $s_k \geq 5$ and $\dim \mathfrak{g} > 31$. The possibilities for $\mathfrak{s}_k$ are

$$\mathfrak{su}_6\mathbb{C}, \quad \mathfrak{su}_5\mathbb{C}, \quad \mathfrak{u}_3\mathbb{H}, \quad \text{and } \mathfrak{g}_2.$$

The first of these algebras leads to point (d), and the others are excluded by the dimension formula. The cases $n = 13$ and $n = 14$ are treated in exactly the same way.

Now assume that $n \geq 15$. If $6 \leq s_k \leq n-7$ then the dimension formula 3.1 implies that

$$\dim \mathfrak{g} \leq \binom{n-s_k}{2} + \binom{s_k+1}{2} \leq \binom{n-6}{2} + \binom{7}{2} \leq \binom{n-4}{2} + 3.$$

Hence s_k cannot lie within this range. As $s_k \geq n-8$, we infer that $s_k \geq n-6$. This implies that $s' \leq 5$, so that $\dim \mathfrak{h} \leq 15$, and we conclude that

$$\dim \mathfrak{s}_k > \binom{n-4}{2} - 12.$$

If $\mathfrak{s}_k \cong \mathfrak{o}_m\mathbb{R}$ for some $m \in \mathbb{N}$ then $m \geq n-5$ since $\binom{n-4}{2} - 12 \geq \binom{n-6}{2}$, and the result follows from Proposition 3.5.2.

Suppose that $\mathfrak{s}_k \cong \mathfrak{su}_m\mathbb{C}$ for some $m \in \mathbb{N}$. Then $s_k = 2m-2$ by Table 2.1 on page 50, and

$$m^2 - 1 = \dim \mathfrak{s}_k > \binom{n-4}{2} - 12.$$

On the other hand, Corollary 3.2.5 shows that $n \geq 2m$. Hence

$$\begin{aligned} m^2 - 1 &> \binom{2m-4}{2} - 12 = 2m^2 - 9m - 2 \\ 21 &> m^2 - 9m + 20 = (m-4)(m-5) \\ m &\leq 9. \end{aligned}$$

Moreover, if $n \geq 15$ then the lower bound for $\dim \mathfrak{s}_k$ entails that $m \geq 7$.

If $m = 7$ then the same inequality shows that $n = 15$. Hence $s' \leq 2$, so that we obtain the contradiction $\dim \mathfrak{g} \leq 51$. Similarly, if $m = 8$ then $n \leq 16$, and equality holds because $n \geq 2m$. We infer that $s' \leq 1$ and $\dim \mathfrak{g} \leq 64$. Finally, if $m = 9$ then $n \leq 18$. Equality holds, and $\dim \mathfrak{g} \leq 81$.

Suppose, then, that $\mathfrak{s}_k \cong \mathfrak{u}_m\mathbb{H}$ for some $m \in \mathbb{N}$. Then

$$m(2m+1) = \dim \mathfrak{s}_k > \binom{n-4}{2} - 12.$$

In particular, this relation shows that $m \geq 5$. Table 2.1 on page 50 yields that $4(m-1) = s_k \leq n-1$. Combining this with the first inequality, we find that

$$24 > (2m-5)(3m-8).$$

which contradicts $m \geq 5$.

If $\mathfrak{s}_k \cong \mathfrak{f}_4$ then we obtain the contradictory inequalities $52 = \dim \mathfrak{s}_k > \binom{n-4}{2} - 12$ and $16 = s_k \leq n-1$. The exceptional Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$ are ruled out in the same way. $\square$

3.6 Reconstruction of actions

In this section, we prove that every effective action of a compact connected Lie group whose dimension is strictly greater than $\binom{n-2}{2} + 1$ on a generalized n-sphere is equivalent to the natural action of a subgroup of $\mathrm{SO}_{n+1}\mathbb{R}$ on $\mathbb{S}_n$. In the course of the proof, we describe all such actions explicitly.

The proof is based on the local classification of sufficiently large Lie groups acting effectively on generalized spheres, which was obtained in Section 3.5. Most of the methods used here have occurred in previous sections.

An effective action of a compact connected Lie group G whose dimension is equal to $\binom{n-2}{2}+1$ on a generalized n-sphere S may be more complicated. It is still true that S is homeomorphic to an ordinary sphere (Lemma 1.3.3, Theorem 2.1.7, and Lemma 1.3.5). However, there is a family of non-linear differentiable actions of $\mathbb{T} \times \mathrm{SO}_3\mathbb{R}$ on $\mathbb{S}_5$ (see W.-C. and W.-Y. Hsiang [61], cf. Bredon [13]). For another example, let $I \le \mathrm{SO}_3\mathbb{R}$ be the icosahedral subgroup, so that $\mathrm{SO}_3\mathbb{R}/I$ is the Poincaré homology 3-sphere, cf. Bredon [14, I.8] or Bredon [15, VI.8.10]. There is a natural effective action of $\mathbb{T} \times \mathrm{SO}_3\mathbb{R}$ on the join $\mathbb{S}_1 * \mathrm{SO}_3\mathbb{R}/I$. This space is the double suspension of $\mathrm{SO}_3\mathbb{R}/I$. By a famous theorem proved by Cannon [23] and by Edwards [39], it is homeomorphic to $\mathbb{S}_5$. A principal stabilizer of the join action is conjugate to $1 \times I$. This shows that the action is not equivalent to a linear action, or to one of the actions from the non-linear differentiable family mentioned above. Therefore, it is not even differentiable (see the classification by Straume [126], who also describes the examples of this paragraph, or by Asoh [4]). Generalizing the last example, we find non-differentiable effective actions of $\mathrm{SO}_{n-3}\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$ on $\mathbb{S}_n \approx \mathbb{S}_{n-4} * \mathrm{SO}_3\mathbb{R}/I$ for all $n \ge 5$. However, if $n > 5$ then the dimension of this group is slightly smaller relative to $\dim \mathrm{SO}_{n+1}\mathbb{R}$.

The action of a sufficiently high-dimensional group forces a generalized sphere to be a sphere (Lemma 3.6.1). Therefore, generalized spheres of dimension $n \in \{3, 4\}$ have been treated completely by Richardson [111]. For $n = 1$, see Salzmann et al. [115, 96.29]. The case $n = 2$ is treated in Lemma 3.6.2 below. Therefore, we will often suppose that $n \ge 5$. However, if such a hypothesis occurs in a statement, this statement may still hold for smaller n if the proof is suitably modified.

Let $\mathfrak{g}$ be the Lie algebra of a compact connected Lie group which acts effectively on a generalized n-sphere, and suppose that $\dim \mathfrak{g} > \binom{n-2}{2} + 1$. Then we see from the tables of Section 3.5 (proof of Theorem 3.5.1; Proposition 3.5.2 and Theorem 3.5.4) that one of the following cases arises:

(a) The action is transitive and therefore linear by Theorem 3.1.1.

(b) The Lie algebra $\mathfrak{g}$ is isomorphic to one of $\mathfrak{o}_n\mathbb{R}$, $\mathbb{R}\times\mathfrak{o}_{n-1}\mathbb{R}$, $\mathfrak{o}_{n-1}\mathbb{R}$, or $\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{n-2}\mathbb{R}$.

(c) $n = 5$, and $\mathfrak{g} \cong \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}$.

(d) $n = 6$, and $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{su}_3\mathbb{C}$ or $\mathfrak{g} \cong \mathfrak{su}_3\mathbb{C}$. In this case, the action is described by Proposition 3.2.10; in particular, it is linear.

(e) $n = 7$, and $\mathfrak{g} \cong \mathfrak{g}_2$ or $\mathfrak{g} \cong (\mathfrak{o}_3\mathbb{R})^4$.

We will reconstruct the action for each isomorphism type of $\mathfrak{g}$, starting with the series and treating the three remaining sporadic cases at the end.

As a preparatory step, we note that the generalized spheres which we will encounter are in fact spheres.

3.6.1 Lemma. *Let G be a compact connected Lie group acting effectively on a generalized n-sphere S, and suppose that* $\dim G > \binom{n-2}{2}$. *Then $S \approx \mathbb{S}_n$.*

Proof. The action of G on any principal orbit x^G is effective by Lemma 1.3.3. This entails that $\dim x^G \geq n-2$ (Theorem 2.1.7), whence $S \approx \mathbb{S}_n$ (Lemma 1.3.5). □

3.6.2 Lemma. *Every effective action of a non-trivial compact connected group G on $\mathbb{S}_2$ is equivalent to the usual action of either* $\mathrm{SO}_3\mathbb{R}$ *or* $\mathbb{T}$.

Proof. A result due to Bredon [10, Theorem 10] implies that the group G is a Lie group. Suppose first that the action of G is transitive. Then one can use the Classification Theorem 3.1.1 or the following direct argument. Theorem 2.1.7 shows that $\dim G \leq 3$, whence G is either abelian or covered by $\mathrm{Spin}_3\mathbb{R}$. Every homogeneous space of a compact connected abelian Lie group is homeomorphic to a power of $\mathbb{S}_1$. Hence G is covered by $\mathrm{Spin}_3\mathbb{R}$. Each stabilizer of the induced action of $\mathrm{Spin}_3\mathbb{R}$ is one-dimensional, and it is connected (Salzmann et al. [115, 94.4]), so that it is (the image of) a one-parameter subgroup. All these are conjugate, whence $G \cong \mathrm{SO}_3\mathbb{R}$, and the action is equivalent to the linear action of G on $\mathbb{S}_2$.

Suppose now that the action of G is not transitive. Then every non-trivial orbit of G is homeomorphic to a circle. Mostert's Theorem 3.1.2 implies that both non-principal orbits are fixed points, and that the action is equivalent to the usual action of the circle group. □

Let us return to spheres of arbitrary dimension. If $\mathfrak{g}$ is isomorphic to one of $\mathfrak{o}_n\mathbb{R}$, $\mathbb{R} \times \mathfrak{o}_{n-1}\mathbb{R}$, or $\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{n-2}\mathbb{R}$ then the codimension of a principal orbit is at most 1 by Theorem 2.5.5, so that we can apply Mostert's Theorem 3.1.2. These are the cases that will be treated first.

3.6.3 Proposition. *Let G be a compact connected Lie group which acts effectively, but not transitively, on $\mathbb{S}_n$, where $n \geq 5$. Suppose that the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{o}_k\mathbb{R} \times \mathfrak{o}_{n+1-k}\mathbb{R}$ for some $k \in \{1, 2, \dots, n\}$. Then either the action of G is equivalent to the natural action of*

$$\left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array} \right) \in \mathrm{SO}_{n+1}\mathbb{R} \;\middle|\; A \in \mathrm{SO}_k\mathbb{R}, B \in \mathrm{SO}_{n+1-k}\mathbb{R} \right\},$$

or $n = 5$, $k = 3$, and the action of G is equivalent to the natural action of

$$\left\{ \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & A \end{array} \right) \in \mathrm{SO}_6\mathbb{R} \;\middle|\; A \in \mathrm{SO}_4\mathbb{R} \right\}.$$

For $n = 4$, the group $\mathrm{U}_2\mathbb{C}$ is also of the type treated here. With this exception, the group is contained in $\mathrm{SO}_{n+1}\mathbb{R}$ as the stabilizer of a subspace of $\mathbb{R}^{n+1}$ also if $n < 5$.

Proof. Suppose that $\mathfrak{g}$ is not isomorphic to $\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_3\mathbb{R}$. Lemma 1.3.3 and Theorem 2.5.5 yield that the Lie algebra $\mathfrak{h}$ of a principal stabilizer is isomorphic to $\mathfrak{o}_{k-1}\mathbb{R} \times \mathfrak{o}_{n-k}\mathbb{R}$. In particular, the codimension of $\mathfrak{h}$ in $\mathfrak{g}$ is $n - 1$, and either $\mathfrak{g}$ or $\mathfrak{h}$ is semi-simple, whence $\mathfrak{h}$ is contained in the commutator subalgebra of $\mathfrak{g}$. Therefore, Theorem 3.1.6 applies, and the proposition follows immediately.

Suppose now that there is an isomorphism $\varphi : \mathfrak{g} \to \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_3\mathbb{R}$, so that $n = 5$. As the action of G on $\mathbb{S}_5$ is not transitive, the codimension of a principal stabilizer $\mathfrak{h}$ is at most 4, and it is at least 3 by Proposition 2.5.2. Codimension 3 will be treated in Proposition 3.6.9 below. Suppose that $\dim \mathfrak{h} = 2$, so that $\mathfrak{h} \cong \mathbb{R}^2$. Then $\varphi(\mathfrak{h})$ is the direct product of two one-dimensional subalgebras of the two direct factors $\mathfrak{o}_3\mathbb{R}$ since neither of these factors contains a two-dimensional subalgebra. Fixing embeddings of $\mathbb{R}$ into $\mathfrak{o}_3\mathbb{R}$, we write $\varphi(\mathfrak{h}) = \mathbb{R} \times \mathbb{R}$. Mostert's Theorem 3.1.2 shows that there are exactly two non-principal orbits ${y_0}^G$ and ${y_1}^G$, and the points y_j can be chosen such that the Lie algebras of their stabilizers properly contain $\mathfrak{h}$, which belongs to the stabilizer of some point $x \in \mathbb{S}_5$. Since the two coset spaces G_{y_j}/G_x are integral homology spheres, the isomorphism φ maps each of the two Lie algebras $\mathfrak{l}(G_{y_j})$ onto either $\mathfrak{o}_3\mathbb{R} \times \mathbb{R}$ or $\mathbb{R} \times \mathfrak{o}_3\mathbb{R}$. We infer from Lemma 3.1.3 that G_x is connected, whence so are all stabilizers. In particular, there is an isomorphism $\psi : G \to \mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$, and ψ maps G_{y_j} to $(\mathrm{SO}_3\mathbb{R} \times 1) \cdot \psi(G_x)$ or to $(1 \times \mathrm{SO}_3\mathbb{R}) \cdot \psi(G_x)$. Recall from Theorem 3.1.2 that the action can be reconstructed uniquely from the three stabilizers G_x, G_{y_0}, and G_{y_1}. If G_{y_0} and G_{y_1} are equal than this reconstruction yields a homeomorphism of $\mathbb{S}_5$ onto $\mathbb{S}_3 \times \mathbb{S}_2$. This contradiction shows that $G_{y_0} \neq G_{y_1}$, whence the action is equivalent to the linear action which has been described in the statement of the proposition. □

The preceding proposition treats three of the four series of isomorphism types of $\mathfrak{g}$ which have been described in point (b) of the introduction to this section. It remains to deal with $\mathfrak{g} \cong \mathfrak{o}_{n-1}\mathbb{R}$. In this case, the codimension of a principal orbit will usually be 2. Since the non-principal orbits will then be fixed points, the treatment will not be more difficult. However, we need some preparation to deal with the possibility that the codimension of a principal orbit is 1. This will in fact occur for $n = 5$ and for $n = 8$.

Let us first note an elementary fact about the behaviour of normalizers under group epimorphisms.

3.6.4 Lemma. *Let $\varphi : G \to H$ be a surjective group homomorphism, and let K be a subgroup of H. Then*

$$N_G(\varphi^{-1}(K)) = \varphi^{-1}(N_H(K)).$$

Proof. If $g \in G$ then $K^{\varphi(g)} = \varphi(\varphi^{-1}(K))^{\varphi(g)} = \varphi(\varphi^{-1}(K)^g)$. This shows that the left-hand side is contained in the right-hand side. Since $\ker\varphi \leq \varphi^{-1}(K)^g$, we also infer that $\varphi^{-1}(K^{\varphi(g)}) = \varphi^{-1}(K)^g$, which implies the reverse inclusion in the statement of the lemma. $\square$

3.6.5 Remark. The lemma breaks down if φ is not surjective. For example, let φ be the inclusion map from $G := \langle(1\,2\,3)\rangle$ into $H := S_3$, and let $K := \langle(1\,2)\rangle$.

While the lemma implies that $\varphi(N_G(K)) = N_H(\varphi(K))$ if $\ker\varphi \leq K \leq G$, this statement does not hold if $\ker\varphi$ is not contained in K. For example, let φ map $G := A_4$ onto $H := \mathbb{Z}/3$, and let $K := \langle(1\,2)(3\,4)\rangle$.

3.6.6 Remark. As a first step, we will study subgroups of $\mathrm{SO}_8\mathbb{R}$, in particular those which are isomorphic to G_2, $\mathrm{Spin}_7\mathbb{R}$, or $\mathrm{SO}_7\mathbb{R}$, together with their normalizers. Representation theory shows that any two isomorphic subgroups of this kind are conjugate in $\mathrm{O}_8\mathbb{R}$ (see the list by Salzmann et al. [115, 95.10] and note that equivalent representations are actually conjugate under $\mathrm{O}_8\mathbb{R}$, see, for instance, Kramer [75, 4.6]). We fix inclusions of $\mathrm{SO}_7\mathbb{R}$ and of $\mathrm{Spin}_7\mathbb{R}$ into $\mathrm{SO}_8\mathbb{R}$. By the preceding observation, the action of $\mathrm{Spin}_7\mathbb{R}$ on $\mathbb{S}_7$ which is induced by the chosen inclusion is transitive (cf. Theorem 3.1.1 or [115, 17.14]).

By the (general) Frattini argument, this transitivity implies that $\mathrm{SO}_8\mathbb{R} = \mathrm{SO}_7\mathbb{R} \cdot \mathrm{Spin}_7\mathbb{R}$ as a complex product. Since the subgroup $\mathrm{SO}_7\mathbb{R}$ is centralized by an involution $s \in \mathrm{O}_8\mathbb{R} \setminus \mathrm{SO}_8\mathbb{R}$ (for which there are two choices), the inclusion of this subgroup is even unique up to conjugation in $\mathrm{SO}_8\mathbb{R}$. Therefore, any two copies of $\mathrm{SO}_7\mathbb{R}$ are conjugate under $\mathrm{Spin}_7\mathbb{R}$. Moreover, any subgroup of $\mathrm{SO}_8\mathbb{R}$ which is isomorphic to $\mathrm{Spin}_7\mathbb{R}$ is conjugate under $\mathrm{SO}_7\mathbb{R}$ to either $\mathrm{Spin}_7\mathbb{R}$ or $(\mathrm{Spin}_7\mathbb{R})^s$.

Let us show that $\mathrm{SO}_7\mathbb{R} \cap \mathrm{Spin}_7\mathbb{R} \cong \mathrm{G}_2$. Indeed, since $\mathrm{SO}_7\mathbb{R}$ is a point stabilizer of the action of $\mathrm{SO}_8\mathbb{R}$ on $\mathbb{S}_7$, this intersection is a point stabilizer in $\mathrm{Spin}_7\mathbb{R}$. We infer from the long exact homotopy sequence [115, 96.12] that it is connected and simply connected. By the dimension formula, its dimension is 14. Its rank is at most that of $\mathrm{Spin}_7\mathbb{R}$, which is 3. The classification of compact connected Lie groups yields that the intersection is isomorphic to G_2, cf. Table 3.1 on page 78. A more geometric proof of this fact is given in [115, 17.15].

We fix an embedding of G_2 into $\mathrm{SO}_8\mathbb{R}$ by writing $\mathrm{G}_2 = \mathrm{SO}_7\mathbb{R} \cap \mathrm{Spin}_7\mathbb{R}$. Note that the above involution $s \in \mathrm{O}_8\mathbb{R}\setminus\mathrm{SO}_8\mathbb{R}$ centralizes G_2, which shows that any two isomorphic copies of this group are even conjugate in $\mathrm{SO}_8\mathbb{R}$.

The centre Z of $\mathrm{Spin}_7\mathbb{R}$ coincides with the centre of $\mathrm{SO}_8\mathbb{R}$. Indeed, let z be the central involution of $\mathrm{Spin}_7\mathbb{R}$. Then the subspace of $\mathbb{R}^8$ which is fixed by z is invariant under $\mathrm{Spin}_7\mathbb{R}$. As the action of $\mathrm{Spin}_7\mathbb{R}$ on $\mathbb{R}^8$ is irreducible, this implies that $\mathrm{Fix}\, z = \{0\}$, whence $z = -1$, the central involution of $\mathrm{SO}_8\mathbb{R}$.

We claim that the normalizer of G_2 in $\mathrm{SO}_8\mathbb{R}$ is $Z \cdot \mathrm{G}_2$. This normalizer leaves the unique G_2-invariant one-dimensional subspace of $\mathbb{R}^8$ invariant. Hence it is contained in $Z\cdot\mathrm{SO}_7\mathbb{R}$, and its identity component is contained in $\mathrm{SO}_7\mathbb{R}$. Let N be the normalizer of $\mathrm{G}_2 = \mathrm{SO}_7\mathbb{R}\cap\mathrm{Spin}_7\mathbb{R}$ in $\mathrm{Spin}_7\mathbb{R}$. Then the identity component N^1, being a subgroup of both $\mathrm{SO}_7\mathbb{R}$ and $\mathrm{Spin}_7\mathbb{R}$, is equal to G_2. Now N contains Z and N^1, whence $Z \cdot \mathrm{G}_2 \le N$. On the other hand, the group $(Z \cdot \mathrm{SO}_7\mathbb{R}) \cap \mathrm{Spin}_7\mathbb{R}$ acts on $(Z \cdot \mathrm{SO}_7\mathbb{R})/\mathrm{SO}_7\mathbb{R}$, and G_2 is a stabilizer of this action, whence the index of G_2 in $(Z \cdot \mathrm{SO}_7\mathbb{R}) \cap \mathrm{Spin}_7\mathbb{R}$ is at most 2. We infer that $N = Z \cdot \mathrm{G}_2$. Now let $p : \mathrm{Spin}_7\mathbb{R} \to \mathrm{SO}_7\mathbb{R}$ be the natural projection. Then the image of G_2 under p is again G_2 because this group is simple. Therefore $p^{-1}(\mathrm{G}_2) = Z \cdot \mathrm{G}_2$. Since this inverse image is self-normalizing in $\mathrm{Spin}_7\mathbb{R}$, Lemma 3.6.4 shows that G_2 is self-normalizing in $\mathrm{SO}_7\mathbb{R}$. Therefore the index of G_2 in its normalizer in $Z \cdot \mathrm{SO}_7\mathbb{R}$ is at most 2. Hence $Z \cdot \mathrm{G}_2$ is the normalizer of G_2 in $Z \cdot \mathrm{SO}_7\mathbb{R}$ and also in $\mathrm{SO}_8\mathbb{R}$.

By a similar argument, we will also show that $\mathrm{Spin}_7\mathbb{R}$ is a self-normalizing subgroup of $\mathrm{SO}_8\mathbb{R}$. Let $p_1 : \mathrm{Spin}_8\mathbb{R} \to \mathrm{SO}_8\mathbb{R}$ be a projection. Then the inverse image of $\mathrm{SO}_7\mathbb{R}$ under p_1 is isomorphic to $\mathrm{Spin}_7\mathbb{R}$ because the quotient space $\mathrm{Spin}_8\mathbb{R}/{p_1}^{-1}(\mathrm{SO}_7\mathbb{R})$ is homeomorphic to $\mathbb{S}_7$. Choose a projection $p_2 : \mathrm{Spin}_8\mathbb{R} \to \mathrm{SO}_8\mathbb{R}$ whose kernel is not contained in ${p_1}^{-1}(\mathrm{SO}_7\mathbb{R})$ (cf. [115, 17.13]). Then the restriction of p_2 to ${p_1}^{-1}(\mathrm{SO}_7\mathbb{R})$ is an isomorphism, whence we may assume that $p_2({p_1}^{-1}(\mathrm{SO}_7\mathbb{R})) = \mathrm{Spin}_7\mathbb{R}$. The inverse image ${p_2}^{-1}(\mathrm{Spin}_7\mathbb{R})$ has two connected components, whence $p_1({p_2}^{-1}(\mathrm{Spin}_7\mathbb{R})) = Z \cdot \mathrm{SO}_7\mathbb{R}$. Since this subgroup is self-normalizing, Lemma 3.6.4 implies that the same holds for

$$p_2({p_1}^{-1}(Z \cdot \mathrm{SO}_7\mathbb{R})) = \mathrm{Spin}_7\mathbb{R}.$$

Also note that $\mathrm{SO}_7\mathbb{R}$, $\mathrm{Spin}_7\mathbb{R}$, and $(\mathrm{Spin}_7\mathbb{R})^s$ are the only subgroups of $\mathrm{SO}_8\mathbb{R}$ which contain G_2 and are locally isomorphic to $\mathrm{SO}_7\mathbb{R}$. To see this, let $G \le \mathrm{SO}_8\mathbb{R}$ be such a subgroup. If $G \cong \mathrm{SO}_7\mathbb{R}$ then it follows immediately from the decomposition $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ of $\mathbb{R}^8$

under the induced action of G_2 that $G = \mathrm{SO}_7\mathbb{R}$. So suppose that $G \cong \mathrm{Spin}_7\mathbb{R}$, and choose $g \in \mathrm{SO}_7\mathbb{R}$ such that G^g equals either $\mathrm{Spin}_7\mathbb{R}$ or $(\mathrm{Spin}_7\mathbb{R})^s$. Then ${\mathrm{G}_2}^g$ is contained in either $\mathrm{SO}_7\mathbb{R} \cap \mathrm{Spin}_7\mathbb{R}$ or $\mathrm{SO}_7\mathbb{R} \cap (\mathrm{Spin}_7\mathbb{R})^s$. But both of these intersections equal G_2, whence g normalizes G_2. Therefore g lies in $Z \cdot \mathrm{G}_2$, which is contained in both $\mathrm{Spin}_7\mathbb{R}$ and $(\mathrm{Spin}_7\mathbb{R})^s$, so that g normalizes these two subgroups, and G equals one of them.

There are no further subgroups of $\mathrm{SO}_8\mathbb{R}$ which sit above G_2. Indeed, the rank of such an intermediate subgroup would be at most $\operatorname{rk} \mathrm{SO}_8\mathbb{R} = 4$. The subgroup could not be simple, since Theorem 2.5.1 shows that $\mathfrak{o}_6\mathbb{R}$ and $\mathfrak{u}_3\mathbb{H}$ do not contain $\mathfrak{g}_2$, while $\mathfrak{su}_5\mathbb{C}$ is not contained in $\mathfrak{o}_8\mathbb{R}$. If $\mathfrak{g}_2$ is contained in a non-simple subalgebra of $\mathfrak{o}_8\mathbb{R}$ then we use the projections of this subalgebra onto its centre and onto its simple ideals to find that such a subalgebra must contain $\mathfrak{g}_2$ as an ideal. But we have seen that $\mathfrak{g}_2$ is self-normalizing.

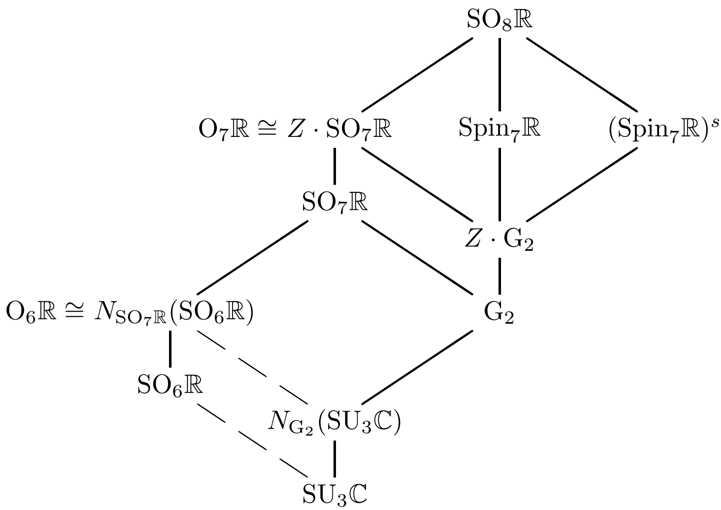
Finally, we record the homeomorphism types of some of the occurring quotient spaces. The sphere $\mathbb{S}_7$ arises as $\mathrm{SO}_8\mathbb{R}/\mathrm{SO}_7\mathbb{R}$ and as $\mathrm{Spin}_7\mathbb{R}/\mathrm{G}_2$. The natural transitive action of $\mathrm{SO}_8\mathbb{R}$ on $P_7\mathbb{R}$ induces a transitive action of $\mathrm{Spin}_7\mathbb{R}$ whose stabilizer is $(Z \cdot \mathrm{SO}_7\mathbb{R}) \cap \mathrm{Spin}_7\mathbb{R} = Z \cdot \mathrm{G}_2$. Factoring out the centre Z, we also obtain a transitive action of $\mathrm{SO}_7\mathbb{R}$ with stabilizer G_2. The group $\mathrm{SO}_7\mathbb{R}$ also acts transitively on $\mathrm{SO}_8\mathbb{R}/\mathrm{Spin}_7\mathbb{R}$, and again the stabilizer is G_2. Hence

$$P_7\mathbb{R} \approx \frac{\mathrm{SO}_8\mathbb{R}}{Z \cdot \mathrm{SO}_7\mathbb{R}} \approx \frac{\mathrm{Spin}_7\mathbb{R}}{Z \cdot \mathrm{G}_2} \approx \frac{\mathrm{SO}_7\mathbb{R}}{\mathrm{G}_2} \approx \frac{\mathrm{SO}_8\mathbb{R}}{\mathrm{Spin}_7\mathbb{R}}.$$

3.6.7 Remark. Later in this section, we will need similar results for $\mathrm{SO}_7\mathbb{R}$ and its subgroups $\mathrm{SU}_3\mathbb{C}$, G_2, and $\mathrm{SO}_6\mathbb{R}$. It is convenient to record these facts now. These subgroups are unique up to conjugation in $\mathrm{SO}_7\mathbb{R}$ by [115, 95.10] and [75, 4.7]. In particular, the induced action of G_2 on $\mathbb{S}_6$ is transitive by Theorem 3.1.1. Therefore, the Frattini argument shows that $\mathrm{SO}_7\mathbb{R} = \mathrm{SO}_6\mathbb{R} \cdot \mathrm{G}_2$. Moreover, the intersection $\mathrm{SO}_6\mathbb{R} \cap \mathrm{G}_2$ is a stabilizer of the action of G_2 on $\mathbb{S}_6$. Hence it is a connected, simply connected group of dimension 8 and rank at most 2. We infer that $\mathrm{SO}_6\mathbb{R} \cap \mathrm{G}_2 = \mathrm{SU}_3\mathbb{C}$. (Alternatively, see [115, 11.34].)

The normalizer of $\mathrm{SU}_3\mathbb{C}$ in G_2 is $N_{\mathrm{SO}_7\mathbb{R}}(\mathrm{SO}_6\mathbb{R}) \cap \mathrm{G}_2$, where $\mathrm{O}_6\mathbb{R} \cong N_{\mathrm{SO}_7\mathbb{R}}(\mathrm{SO}_6\mathbb{R}) =: N$. Indeed, it is contained in N since it leaves the decomposition $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^6$ of $\mathbb{R}^7$ under the induced action of $\mathrm{SU}_3\mathbb{C}$ invariant. On the other hand, the subgroup $\mathrm{SU}_3\mathbb{C} = \mathrm{SO}_6\mathbb{R} \cap \mathrm{G}_2$ has

Figure 3.1: Subgroups of $\mathrm{SO}_8\mathbb{R}$



This is a part of the subgroup lattice of $\mathrm{SO}_8\mathbb{R}$. Thick unbroken lines indicate that no intermediate groups are missing. In fact, the figure contains the complete interval from G_2 to $\mathrm{SO}_8\mathbb{R}$, as well as that from $\mathrm{SU}_3\mathbb{C}$ to G_2. In particular, the normalizer of G_2 in $\mathrm{SO}_8\mathbb{R}$ is $Z \cdot \mathrm{G}_2$. For details, see the discussion in 3.6.6 and 3.6.7.

index at most 2 in $N \cap \mathrm{G}_2$, whence $\mathrm{SU}_3\mathbb{C}$ is normalized by this intersection. Note that this normalizer properly contains $\mathrm{SU}_3\mathbb{C}$ since it is a stabilizer of the transitive action of G_2 on $P_6\mathbb{R}$ which is induced by the transitive action of $\mathrm{SO}_7\mathbb{R}$. Also note that the normalizer of $\mathrm{SU}_3\mathbb{C}$ in $\mathrm{SO}_6\mathbb{R}$ contains $\mathrm{U}_3\mathbb{C}$. (In fact, equality holds, since this normalizer also normalizes the centralizer of $\mathrm{SU}_3\mathbb{C}$ in $\mathrm{End}(\mathbb{R}^6)$, which is $\mathbb{C}$. But the normalizer of $\mathbb{C}$ in $\mathrm{GL}_6\mathbb{R}$ is $\mathrm{GL}_3\mathbb{C}$ extended by complex conjugation, and complex conjugation is represented by an element of $\mathrm{O}_6\mathbb{R} \setminus \mathrm{SO}_6\mathbb{R}$. Hence the normalizer of $\mathrm{SU}_3\mathbb{C}$ in $\mathrm{SO}_6\mathbb{R}$ consists of $\mathbb{C}$-linear elements.) We will content ourselves with the description of the subgroup lattice of $\mathrm{SO}_7\mathbb{R}$ which we have achieved now, even if it is less complete than that which we have obtained for $\mathrm{SO}_8\mathbb{R}$.

Since $\mathfrak{su}_3\mathbb{C}$ is a maximal subalgebra of $\mathfrak{g}_2$ by Theorem 2.5.1, there are no further subgroups between $\mathrm{SU}_3\mathbb{C}$ and G_2. In particular, the spaces $\mathbb{S}_6$ and $P_6\mathbb{R}$ are the only six-dimensional homogeneous spaces of G_2.

3.6.8 Lemma. *Let X be an n-dimensional homogeneous space of $G := \mathrm{Spin}_n\mathbb{R}$. Then one of the following statements holds:*

(a) *$n = 3$, and $X \approx \mathbb{S}_3$ or $X \approx P_3\mathbb{R}$.*

(b) *$n = 4$, and X is homeomorphic to ${\mathbb{S}_2}^2$, ${\mathbb{S}_2}^2/\pm$, $\mathbb{S}_2 \times P_2\mathbb{R}$, or $P_2\mathbb{R}^2$.*

(c) *$n = 6$ (so that $G \cong \mathrm{SU}_4\mathbb{C}$), and $X \approx P_3\mathbb{C}$.*

(d) *$n = 7$, and $X \approx \mathbb{S}_7$ or $X \approx P_7\mathbb{R}$.*

In each case, the action of G on X is unique up to equivalence.

Here ${\mathbb{S}_2}^2/\pm$ denotes the orbit space of the action

$$((x, y), \varepsilon) \longmapsto (x\varepsilon, y\varepsilon) : {\mathbb{S}_2}^2 \times \{1, -1\} \longrightarrow {\mathbb{S}_2}^2.$$

Proof. Let H be a stabilizer of the action of G on X. If $n = 3$ then $H \leq Z(G) \cong \mathbb{Z}/2$, whence the claim follows.

Suppose $n = 4$. Then $\dim H = 2$. Since $\mathfrak{o}_3\mathbb{R}$ does not contain any 2-dimensional subalgebra, the connected component of H is conjugate to $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$. Up to conjugation, this implies that

$$H^1 \leq H \leq N_{\mathrm{SO}_3\mathbb{R}\times\mathrm{SO}_3\mathbb{R}}(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_2\mathbb{R}) = \mathrm{O}_2\mathbb{R} \times \mathrm{O}_2\mathbb{R} =: N.$$

Corresponding to the five subgroups of $N/H^1 \cong (\mathbb{Z}/2)^2$, there are five possible actions, two of which are equivalent.

The case $n = 5$ does not occur since H would have dimension 5 and rank at most 2, which is impossible.

If $n = 6$ then $G \cong \mathrm{SU}_4\mathbb{C}$, and the claim follows from Lemma 3.2.2.

Suppose that $n \geq 7$. The action of H on $\mathbb{S}_{n-1}$ which is induced by the natural action of G is almost effective. We can apply Mann's main result [83, Theorem 2] since $\dim H = \binom{n-1}{2} - 1$ falls into the gap between $\binom{n-2}{2} + \binom{3}{2}$ and $\binom{n-1}{2}$. We find that $n = 7$ and $H^1 \cong \mathrm{G}_2$. Our claim now follows from the discussion in 3.6.6. $\square$

3.6.9 Proposition. *Let G be a compact connected Lie group G which acts effectively, but not transitively, on $\mathbb{S}_n$, where $n \geq 5$. Suppose that the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{o}_{n-1}\mathbb{R}$. Then the action of G is equivalent to the natural action of one of the following subgroups of $\mathrm{SO}_{n+1}\mathbb{R}$ on $\mathbb{S}_n$:*

(a) $$\left\{ \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & A \end{array} \right) \in \mathrm{SO}_{n+1}\mathbb{R} \;\middle|\; A \in \mathrm{SO}_{n-1}\mathbb{R} \right\}$$

(b) $$\left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array} \right) \in \mathrm{SO}_6\mathbb{R} \;\middle|\; A \in \mathrm{SO}_3\mathbb{R}, B \in \mathrm{SO}_3\mathbb{R} \right\} \quad \textit{(so that } n = 5\textit{)}$$

(c) $$\left\{ \left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \in \mathrm{SO}_9\mathbb{R} \;\middle|\; A \in \mathrm{Spin}_7\mathbb{R} \leq \mathrm{SO}_8\mathbb{R} \right\} \quad \textit{(so that } n = 8\textit{)}$$

Up to equivalence, there are exactly three almost effective actions of $\mathrm{Spin}_3\mathbb{R}$ on $\mathbb{S}_4$ (Richardson [111]). The group $\mathbb{T}$ can act in several different ways on $\mathbb{S}_3$, see [111] and Jacoby [67].

Proof. By Lemma 1.3.3, the action of G on each single principal orbit x^G is effective. Therefore, Theorem 2.5.5 shows that the dimension of a principal orbit x^G is at least $n-2$. Let X be the union of all principal orbits, and set $Y := \mathbb{S}_n \setminus X$.

Suppose first that $\dim x^G = n - 2$. As G is semi-simple, Richardson [111, 1.2 and 1.3] has shown that $\mathbb{S}_n/G$ is a two-disc with interior $X/G \approx \mathbb{R}^2$ and boundary $Y/G \approx \mathbb{S}_1$. Moreover, the set X is equivariantly homeomorphic to $\mathbb{R}^2 \times x^G$ by [111, 1.6], and Y consists of orbits whose dimension is strictly smaller than $\dim x^G$. The Lie algebra of a principal stabilizer is a maximal subalgebra of $\mathfrak{g}$. (If $n \geq 6$ then this follows from Theorem 2.5.1; if $n = 5$ then it is not hard to see that a diagonal subalgebra of $\mathfrak{o}_4\mathbb{R} \cong (\mathfrak{o}_3\mathbb{R})^2$ is maximal.) Therefore, every non-principal orbit is a fixed point. These orbits form the subset Y of $\mathbb{S}_n$. As the fixed point set Y is compact, we find that $Y \approx Y/G \approx \mathbb{S}_1$. In particular, this implies that

$$\begin{aligned} H_{n-3}(x^G; \mathbb{Z}/2) &\cong H_{n-3}(\mathbb{R}^2 \times x^G; \mathbb{Z}/2) \\ &\cong H_{n-3}(X; \mathbb{Z}/2) \cong H^2(Y; \mathbb{Z}/2) = 0 \end{aligned}$$

by Alexander duality (see Dold [34, VIII.8.17]). Using Lemma 3.1.7, we conclude that $G \cong \mathrm{SO}_{n-1}\mathbb{R}$ and $x^G \approx \mathbb{S}_{n-2}$. The fact that Y consists of fixed points also implies that the projection of $\mathbb{S}_n$ onto its orbit space admits a continuous section. To see this, recall that X is equivariantly homeomorphic to $X/G \times x^G$. Choose a continuous section $X/G \to X$, and extend this section to a section $\mathbb{S}_n/G \to \mathbb{S}_n$. The resulting map is continuous because every neighbourhood of a fixed point in $\mathbb{S}_n$ contains a G-invariant neighbourhood. By the general reconstruction principle formulated by Richardson [111, 1.4], the action of G on $\mathbb{S}_n$ is equivalent to the natural action which was described in point (a) of the proposition.

Since G does not act transitively on $\mathbb{S}_n$, the dimension of a principal orbit is at most $n - 1$. Suppose that equality holds, so that Mostert's Theorem 3.1.2 applies. Lemma 3.6.8 shows that there is only a small number of possibilities for the action of G on x^G. In particular $n \in \{5, 7, 8\}$. Suppose that $n = 5$. We have seen in the proof of Proposition 3.6.3 that every action of a group G with Lie algebra $\mathfrak{o}_4\mathbb{R} \cong (\mathfrak{o}_3\mathbb{R})^2$ on $\mathbb{S}_5$ whose principal orbits have codimension 1 is equivalent to the natural action of $(\mathrm{SO}_3\mathbb{R})^2$ which we have described in (b). If $n = 7$ then the action of G on x^G is equivalent to the natural action of $\mathrm{PSU}_4\mathbb{C}$ on $P_3\mathbb{C}$. By Theorem 2.5.1, the Lie algebra of a principal stabilizer of this action, which is isomorphic to $\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$, is a maximal subalgebra of the Lie algebra of G. Therefore, the two non-principal orbits ${y_0}^G$ and ${y_1}^G$ are fixed points. Theorem 3.1.2 leads to the contradiction

$$\mathbb{Z} \cong H_2(x^G; \mathbb{Z}) \cong H_2({y_0}^G; \mathbb{Z}) \oplus H_2({y_1}^G; \mathbb{Z}) = 0.$$

Finally, suppose that $n = 8$. Then the discussion in 3.6.6 implies that the two non-principal orbits are again fixed points. Mostert's

Theorem 3.1.2 yields that $x^G \approx \mathbb{S}_7$, and $G \cong \mathrm{Spin}_7\mathbb{R}$. By the usual reconstruction argument, the action is determined uniquely by its three orbit types, whence it is equivalent to the action described in (c). □

All that is left to do is to deal with the 'sporadic' isomorphism types of $\mathfrak{g}$, i.e. those described in points (c), (d), and (e) of the introduction to this section. If $n = 5$ and $\mathfrak{g} \cong \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}$ then the action of G is equivalent to the linear action of $\mathrm{SO}_2\mathbb{R} \times \mathrm{U}_2\mathbb{C}$ by Lemma 3.4.5. The case $n = 7$ and $\mathfrak{g} \cong (\mathfrak{o}_3\mathbb{R})^4$ is contained in Proposition 3.6.3.

3.6.10 Proposition. *Every non-trivial action of* G_2 *on* $\mathbb{S}_7$ *is equivalent to the natural action induced by the inclusion of* G_2 *into* $\mathrm{SO}_8\mathbb{R}$.

Proof. Since an action of G_2 on $\mathbb{S}_7$ is not transitive by Theorem 3.1.1, Theorem 2.5.1 shows that the dimension of a principal orbit is 6. Moreover, the two non-principal orbits are fixed points. Together with the discussion in 3.6.7, Alexander duality yields that every principal orbit is homeomorphic to $\mathbb{S}_6$. From this information, the action can uniquely be reconstructed. □

The material of this section, together with the classification of transitive actions of compact connected Lie groups on spheres in Theorem 3.1.1, amounts to the proof of the following final result.

3.6.11 Theorem. *Let G be a compact connected Lie group acting effectively on a generalized n-sphere S (in the sense of Definition 1.3.1). Suppose that* $\dim G > \binom{n-2}{2} + 1$. *Then $S \approx \mathbb{S}_n$, and the action of G is equivalent to the natural action of a subgroup of* $\mathrm{SO}_{n+1}\mathbb{R}$. *A complete list of groups and actions is given in Table 3.3 on page 114.* □

3.6.12 Remark. If a compact connected group acts effectively on an ordinary sphere $\mathbb{S}_n$ and satisfies $\dim G > \binom{n-3}{2} + 1$ then G is a Lie group, see Mann [84, Theorem 7].

A little more is known for the smallest spheres. Let G be a compact connected Lie group acting effectively on $\mathbb{S}_n$. If $\dim G = 1$ and $n = 3$ then Jacoby [67] has proved that the action of G is equivalent to the action of a subgroup of $\mathrm{SO}_4\mathbb{R}$. If $\dim G = 2$ and $n = 4$ then Richardson [111] shows that all possible actions are equivalent (and, in particular, linear). We can either drop the hypothesis that G is a Lie group or allow G to act on a generalized sphere. The second generalization is possible by Theorem 2.2.3 and Lemma 1.3.5, while the first follows again from [84, Theorem 7].

Table 3.3: Actions of large compact connected groups on n-spheres

n	Codimension of principal orbits		
	0	1	2
arbitrary	$\mathrm{SO}_{n+1}\mathbb{R}$	$\mathrm{SO}_n\mathbb{R}$, $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{n-1}\mathbb{R}$, $\mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_{n-2}\mathbb{R}$	$\mathrm{SO}_{n-1}\mathbb{R}$
Additional possibilities occur for small n only:			
3	$\mathrm{U}_2\mathbb{C}$, $\mathrm{SU}_2\mathbb{C}$	—	—
4	—	$\mathrm{U}_2\mathbb{C}$, $\mathrm{SU}_2\mathbb{C}$, $\mathrm{SO}_3\mathbb{R}$	—
5	$\mathrm{U}_3\mathbb{C}$, $\mathrm{SU}_3\mathbb{C}$	$\mathrm{SO}_2\mathbb{R} \times \mathrm{U}_2\mathbb{C}$	—
6	G_2	$\mathrm{U}_3\mathbb{C}$, $\mathrm{SU}_3\mathbb{C}$	—
7	$\mathrm{Spin}_7\mathbb{R}$, $\mathrm{U}_4\mathbb{C}$, $\mathrm{SU}_4\mathbb{C}$, $\mathrm{U}_1\mathbb{H} \cdot \mathrm{U}_2\mathbb{H}$	G_2, $\mathrm{SO}_4\mathbb{R} \times \mathrm{SO}_4\mathbb{R}$	—
8	—	$\mathrm{Spin}_7\mathbb{R}$	—
9	$\mathrm{U}_5\mathbb{C}$, $\mathrm{SU}_5\mathbb{C}$	—	—

The table shows all compact connected groups which act effectively on $\mathbb{S}_n$ and whose dimension is strictly greater than $\binom{n-2}{2} + 1$. Up to equivalence, each entry corresponds to a unique action.

Note that the table also applies to actions of compact connected Lie groups on generalized n-spheres.

Chapter 4

Compact $(1, m)$-quadrangles

4.1 Miscellaneous tools for compact polygons

The first section contains some facts about locally compact σ-compact groups which act effectively on finite-dimensional compact generalized polygons. We give a new and simple proof that the dimension of such a group is finite, together with some not too rough bounds for the dimensions of compact and of compact abelian groups. Then we quote two theorems from Smith Theory about actions of finite abelian groups on Čech (co-)homology spheres. The last two results are specifically concerned with compact $(1, m)$-quadrangles.

4.1.1 Lemma. *Let G be a group acting on a generalized polygon $P \,\dot\cup\, L$, and suppose that the stabilizer G_p of every point $p \in P$ acts transitively on the line pencil L_p of that point. Then G acts transitively on L.*

Proof. We show that the orbit of any line $l \in L$ contains every other line $k \in L$. If k intersects l then the two lines are already conjugate under the point stabilizer $G_{k \wedge l}$. Otherwise, there is a finite sequence $l_0, l_1, \ldots, l_n$ of lines such that $l_0 = l$, the line l_i meets l_{i-1} for each $i \in \{1, \ldots, n\}$, and $l_n = k$. Then $k \in l_{n-1}^G = l_{n-2}^G = \ldots = l^G$. $\square$

4.1.2 Lemma. *Let G be a locally compact σ-compact group acting effectively on a locally compact (Hausdorff) space X which has a countable basis. Then the topology of G has a countable basis.*

Proof. As G is σ-compact, it is enough to find a countable basis for some open subgroup of G. Since G/G^1 is totally disconnected, we can choose an open subgroup G_0 of G such that G_0/G^1 is compact. The Mal'cev–Iwasawa Theorem 1.1.4, applied to G_0, shows that it suffices to find a countable basis for a maximal compact subgroup K of G_0. Since K acts on X as a topological transformation group, we can introduce the compact-open topology on K. This is coarser than the original topology, cf. Dugundji [36, XII.2.4]. Since it also is a Hausdorff topology [36, XII.1.3], and since the original topology of K is compact, the two topologies actually coincide. This entails that K has a countable basis, see [36, XII.5.2].

For the reader's convenience, we reproduce the details: let $\mathcal{B}$ be a countable basis of the topology of X. We may assume that all elements of $\mathcal{B}$ have compact closure. Then the sets $\{k \in K \mid \overline{U}^k \subseteq V\}$ form a countable subbasis for the compact-open topology on K as U and V run through $\mathcal{B}$. Indeed, suppose that $g \in K$ maps a compact subset $C \subseteq X$ into an open subset $W \subseteq X$. For every point $x \in C$, choose $V_x \in \mathcal{B}$ such that $x^g \in V_x \subseteq W$, and choose $U_x \in \mathcal{B}$ such that $x \in U_x$ and g maps $\overline{U_x}$ into V_x. By compactness of C, there is a finite subset $F \subseteq C$ such that the family $(U_x)_{x \in F}$ covers C, so that

$$g \in \bigcap_{x \in F} \left\{ k \in K \;\middle|\; \left(\overline{U_x}\right)^k \subseteq V_x \right\} \subseteq \{k \in K \mid C^k \subseteq W\}.$$

Hence K has a countable basis. □

4.1.3 Corollary. *Let G be a locally compact σ-compact group acting effectively on a compact generalized polygon $P \cup L$. Then the topology of G has a countable basis.*

Proof. Both P and L have countable bases, see Grundhöfer, Knarr, and Kramer [49, 1.5]. □

On the group of continuous automorphisms of a compact generalized polygon, the compact-open topology is a locally compact group topology. This was proved by Burns and Spatzier [21, 2.1].

4.1.4 Proposition. *Let G be a compact Lie group acting effectively on a non-discrete Hausdorff polygon. Then G acts effectively on every single principal orbit.*

Proof. The kernel of the action of G on a principal orbit fixes a neighbourhood of this orbit by Corollary 2.1.6. Every automorphism which fixes an open set elementwise is trivial. (For quadrangles, see Stroppel and Stroppel [128, 2.2]. Their proof of the general statement [129, 2.3] follows the same lines.) □

4.1.5 Theorem. *Let G be a compact group acting almost effectively on a finite-dimensional compact generalized polygon $P \mathbin{\dot\cup} L$. Then the dimension of G is finite, and G acts almost effectively on every point orbit of maximal dimension.*

Proof. Let p^G be a point orbit of maximal dimension. Then $(G_{[p^G]})^1$ fixes a neighbourhood of the orbit p^G by Corollary 2.1.17. As we have stated in the previous proof, every automorphism which fixes an open set is trivial. Hence $(G_{[p^G]})^1 = 1$, i.e. the action of G on p^G is almost effective. Theorem 2.1.7 yields that $\dim G = \dim G/G_{[p^G]}$ is finite. □

4.1.6 Corollary (Stroppel and Stroppel [127], [129]). *If G is a locally compact group acting effectively on a compact generalized polygon $P \mathbin{\dot\cup} L$ of finite dimension then also the dimension of G is finite.*

Proof. The Mal'cev–Iwasawa Theorem 1.1.4, applied to the connected component of G, shows that it suffices to prove finiteness of the dimension of a maximal compact connected subgroup of G. But this is just the preceding theorem. □

4.1.7 Corollary. *Let G be a compact group acting almost effectively on a finite-dimensional compact connected generalized polygon $P \mathbin{\dot\cup} L$. Set $k := \dim P$. Then $\dim G \leq \binom{k+1}{2}$. If G is not a Lie group then $\dim G \leq \binom{k-2}{2}$, and if the identity component G^1 is not a Lie group then $\dim G \leq \binom{k-3}{2} + 1$. If G is a torus group then $\dim G \leq k-1$, and if G is an abelian non-Lie group then $\dim G \leq k-3$.*

In Propositions 4.3.6 and 5.2.3, we will give slightly stronger results for actions of torus groups on quadrangles.

Proof. The last theorem shows that the identity component G^1 acts almost effectively on every point orbit p^{G^1} whose dimension is maximal. Therefore $\dim G \leq \binom{k+1}{2}$ by Theorem 2.1.7. If G is not a Lie group then Theorem 2.2.2 shows that $\dim p^{G^1} \leq k-3$, whence $\dim G \leq \binom{k-2}{2}$, again by Theorem 2.1.7. If the identity component G^1 is not a Lie group then $\dim G \leq \binom{k-3}{2}+1$, as was proved by Mann [84, Theorem 6].

Suppose that G is abelian. Then $G_p = G_{[p^G]}$, whence $\dim G = \dim p^G$. If G is not a Lie group then the claim follows immediately from Theorem 2.2.2. If G is a Lie group and $\dim G = k$ then G acts transitively on P. This implies that $\pi_1(P) \cong \mathbb{Z}^k$, contradicting the list given by Grundhöfer, Knarr, and Kramer [49, Appendix]. □

4.1.8 Lemma. *Let G be a compact group acting effectively on the real line $\mathbb{R}$. Then G fixes a point and is of order at most 2.*

Proof. Every element $g \in G$ induces a homeomorphism of $\mathbb{R}$ onto itself, which either preserves or reverses the order of $\mathbb{R}$. Hence $x^G = \{\min x^G, \max x^G\}$ for every $x \in \mathbb{R}$. In particular, the action of every non-trivial element of G is order-reversing, whence $|G| \leq 2$, and G fixes a point. □

A more sophisticated tool is Smith's theory of finite abelian transformation groups. We recall two important results.

4.1.9 Theorem (Floyd [43, 5.2], cf. Salzmann et al. [115, 55.24]). *Let G be a finite abelian p-group, where p is prime, and let X be a locally compact finite-dimensional Hausdorff space which shares the* mod p *Čech homology of an m-sphere. Then the fixed point set of any action of G on X has the* mod p *Čech homology of an n-sphere for some $n \in \{-1, 0, 1, \ldots, m\}$. If p is odd then $m-n$ is even.* □

4.1.10 Theorem (Smith [123, no. 6], cf. Salzmann et al. [115, 55.27]). *Let the finite abelian group G act on a locally compact finite-dimensional Hausdorff space X, leaving a closed subset Y invariant and acting freely on $X \setminus Y$. If both X and Y have the same integral Čech cohomology as spheres of respective dimensions m and n, where $n \leq m-2$, then G is cyclic.* □

4.1.11 Proposition. *Let G be a compact Lie group acting effectively on a compact $(1, m)$-quadrangle $Q = P \cup L$, and suppose that G fixes an open subset U of the line pencil of some point $p \in P$ elementwise. Then G has at most two elements and fixes an ordinary quadrangle.*

Proof. Let n be an odd prime, and assume that G contains a nontrivial finite abelian n-group A. If $l \in U$ then Lemma 4.1.8 shows that A fixes every point on l. If $q \in P_l$ is such a point then Theorem 4.1.9 yields that A fixes at least a second line k through q, and k is also fixed pointwise. Now U and P_k generate Q geometrically (cf. Stroppel and Stroppel [128, 1.5]). Hence A acts trivially on Q, which contradicts effectiveness. We infer that G must be totally disconnected, and in fact a finite 2-group. Again by Theorem 4.1.9, the group G fixes an ordinary quadrangle. If l is any line through p fixed by G then the action of G on $P_l \setminus \{p\} \approx \mathbb{R}$ is therefore effective. Lemma 4.1.8 yields $|G| \leq 2$. □

4.1.12 Corollary. *Let G be a compact connected Lie group acting effectively on a compact $(1, m)$-quadrangle $Q = P \cup L$, suppose $m \geq 2$, and assume that G fixes a point $p \in P$. If the action of G on the line pencil L_p is transitive then it is also effective.*

Proof. Let l be any line through p. Transitivity yields an exact sequence

$$1 = \pi_1(L_p) \longrightarrow \pi_0(G_l) \longrightarrow \pi_0(G) = 1$$

which shows that the stabilizer G_l of l is connected. Hence G_l acts trivially on the point row P_l. The kernel of the action of G on L_p is contained in G_l. Hence it fixes P_l pointwise, and it also fixes an ordinary quadrangle by the preceding proposition. Therefore, this kernel is trivial. □

4.2 Line-homogeneous quadrangles

We show that every line-homogeneous compact $(1, m)$-quadrangle is isomorphic to a real orthogonal quadrangle (up to duality if $m = 1$). For these quadrangles, line-homogeneity therefore implies the Moufang property. Theorems 4.2.3 and 4.2.15 provide this result, together with a list of all line-transitive compact connected groups.

We point out that the result is due to Kramer [74, 5.2.7] if $m = 1$, and that our proof is similar to his. For $m > 1$, Grundhöfer, Knarr, and Kramer [50] have classified the flag-homogeneous quadrangles. In fact, their proof only uses line-homogeneity. However, our proof differs considerably from theirs. Its flavour is more geometric, due to the observation that point orbits are ovoids.

The case $m = 1$ will be dealt with first, since it is not accessible to most of the techniques used for larger line pencils.

4.2.1 Proposition. *Let G be a locally compact connected group acting effectively on a compact $(1, 1)$-quadrangle $Q = P \cup L$. Then G is a Lie group.*

Moreover, if $\pi_1(L)$ is finite and the action of G on L is transitive then every maximal compact subgroup of G acts transitively on L.

Proof. The point space P is a connected manifold, see Grundhöfer and Knarr [48, 4.5]. By Montgomery [88] or Corollary 4.1.6, the dimension of G is finite. Hence G is a Lie group, see Bredon [10, Theorem 8]. For transitivity of maximal compact subgroups, see Montgomery [87, Corollary 3] or Salzmann et al. [115, 96.19]. □

4.2.2 Remark. Up to duality, the fundamental groups of P and L are $\pi_1(P) \cong \mathbb{Z}$ and $\pi_1(L) \cong \mathbb{Z}/2$ (see Kramer [74, 3.4.11] or Grundhöfer, Knarr, and Kramer [49, 4_1]). Hence the last hypothesis of the preceding proposition simply stipulates that G be transitive on the element of $\{P, L\}$ whose fundamental group is $\mathbb{Z}/2$.

4.2.3 Theorem. *Let G be a compact connected group acting effectively and line-transitively on a compact $(1, 1)$-quadrangle $Q = P \cup L$. Then one of the following statements holds:*

(i) *$G \cong \mathrm{SO}_3\mathbb{R}$, and $Q \cong Q(4, \mathbb{R})$;*

(ii) *$G \cong \mathbb{T} \times \mathrm{SO}_3\mathbb{R}$, and $Q \cong Q(4, \mathbb{R})$ or $Q \cong W(\mathbb{R})$.*

Proof. Since the dimension of G is at least 3, Theorem 4.4.1 below yields that the quadrangle Q is isomorphic to the real orthogonal quadrangle $Q(4, \mathbb{R})$ or to its dual, the symplectic quadrangle $W(\mathbb{R})$. Moreover, the group G is embedded into a maximal compact connected automorphism group of these quadrangles, which is isomorphic to $\mathbb{T} \times \mathrm{SO}_3\mathbb{R}$. If $G \cong \mathrm{SO}_3\mathbb{R}$ then the exact sequence

$$\pi_1(G) \longrightarrow \pi_1(L) \longrightarrow \pi_0(G_l)$$

shows that $\pi_1(L)$ is finite, which excludes $W(\mathbb{R})$. (See also Kramer [74, 5.2.7].) $\square$

For the remainder of this section, we shall suppose that $m \geq 2$. By $Q = P \cup L$ we will always denote a compact $(1,m)$-quadrangle with point space P and line space L. Then $\pi_1(P) \cong \mathbb{Z}$ and $\pi_1(L) = 1$ (see Kramer [74, 3.4.11] or Grundhöfer, Knarr, and Kramer [49, 4_2]). Recall from Section 1.4 that $\dim P = m + 2$ and $\dim L = 2m + 1$.

4.2.4 Lemma. *Let G be a locally compact σ-compact group acting effectively and line-transitively on Q. Then the following statements hold:*

(a) *The group G is a Lie group.*

(b) *The connected component G^1 of G acts transitively on L.*

(c) *Every line stabilizer in G^1 is connected.*

(d) *The action of every maximal compact subgroup of G^1 on L is transitive.*

Proof. Corollary 4.1.3 shows that G has a countable basis. Therefore, Szenthe's Theorem (see Salzmann et al. [115, 96.14]) yields that G is a Lie group. The connected component G^1 acts transitively on L by [115, 96.11]. Since the sequence

$$1 = \pi_1(L) \longrightarrow \pi_0((G^1)_l) \longrightarrow \pi_0(G^1) = 1$$

is exact for every line $l \in L$, all line stabilizers in G^1 are connected. As above, we infer transitivity of maximal compact subgroups of G^1 from Montgomery [87, Corollary 3] or from [115, 96.19]. $\square$

4.2.5 Lemma. *Let G be a compact group acting on Q. Then the action of G on L is transitive if and only if for every point $p \in P$, the stabilizer G_p acts transitively on the line pencil L_p.*

Proof. Salzmann et al. [115, 96.11] show that we may assume that G is connected. We may also assume that the action of G on Q is effective. Then G is a Lie group by Lemma 4.2.4.

If G_p acts transitively on L_p for every $p \in P$ then G acts transitively on L by Lemma 4.1.1. Conversely, suppose that the action of G on L

is transitive. Assume that there is a point $p \in P$ such that G_p does not act transitively on L_p. For an arbitrary line l through p, we have

$$\begin{aligned}
\dim p^G &= \dim G - \dim G_p \\
&= \dim G - \dim G_{p,l} - \dim l^{G_p} \\
&\geq \dim G - \dim G_l - \dim l^{G_p} \\
&= \dim L - \dim l^{G_p} \\
&\geq \dim L - (\dim L_p - 1) \\
&= \dim P.
\end{aligned}$$

Equality must hold in each step. In particular, each line $l \in L_p$ has an orbit l^{G_p} of codimension 1 in L_p, which contradicts Mostert's Theorem 3.1.2. □

4.2.6 Proposition. *Let G be a compact connected group acting effectively and line-transitively on Q. Then G has a closed connected subgroup H with the following properties:*

(a) *The subgroup H is transitive on L but not on P.*

(b) *Either $G = H$, or G is the almost direct product of H with a one-dimensional torus group.*

4.2.7 Remark. Proposition 4.2.11 below will show that the group H is semi-simple (actually, almost simple if $m > 2$). Hence the centre of G is at most one-dimensional.

Proof of 4.2.6. The group G is a Lie group by Lemma 4.2.4. Its structure is described by van Kampen's Theorem 1.1.3. Note that $Z(G)^1$, being a compact connected abelian Lie group, is simply a torus group. We may assume that the action of G on P is transitive. Choose a point $p \in P$. Then the exact sequence

$$\pi_1(G) \longrightarrow \pi_1(P) \longrightarrow \pi_0(G_p)$$

shows that $\pi_1(G)$ is infinite, since $\pi_1(P) \cong \mathbb{Z}$ and $\pi_0(G_p)$ is finite. This implies that the connected component $Z(G)^1$ of the centre of G is a non-trivial torus group. The same argument shows that the commutator group G' cannot act transitively on P. Hence G_pG' is a proper subgroup of G. Let C be a closed connected subgroup of $Z(G)$

which has codimension 1 and contains $\big((G_pG')\cap Z(G)\big)^1$, and let $H := (G_p)^1 G'C$. Then H is a closed connected subgroup of G, and G is the almost direct product of H with any closed connected one-dimensional subgroup of $Z(G)$ which is not contained in C. The action of H on P is not transitive because $\dim H < \dim G$ and $\dim H_p = \dim G_p$. If $q \in P$ is an arbitrary point then $(G_q)^1$ is conjugate to $(G_p)^1$ in G since G acts transitively on P. Hence $(G_q)^1$ is contained in H. Lemma 4.2.5 shows that the action of G_q on L_q is transitive. This carries over to the action of $(G_q)^1$, see Salzmann et al. [115, 96.11]. Hence H_q acts transitively on L_q for every point $q \in P$. Therefore, the action of H on L is transitive. $\square$

Transitivity of the action of H on L also follows since the exact homotopy sequence for the action of G on L implies that $(G_l)^1$ cannot be contained in H for any line $l \in L$.

Recall that an *ovoid* O in a generalized quadrangle is a set of points which meets (the point row of) every line in exactly one point. If the quadrangle is compact and O is closed then each point complement in O is homeomorphic to $(P_l \setminus \{p\}) \times (L_p \setminus \{l\})$ for every flag (p, l), and O is the one-point compactification of any point complement. If, furthermore, the quadrangle has finite positive dimension then O is a generalized sphere in the sense of Section 1.3. In particular, if O is a manifold then it is homeomorphic to a sphere by Brown's result [19], see the proof of Lemma 1.3.5. For more information, see Kramer and Van Maldeghem [78].

4.2.8 Lemma. *Let G be a locally compact connected group acting on Q. Suppose that every point stabilizer G_p acts transitively on the corresponding line pencil L_p. Then every point orbit is either an ovoid or open. If G is compact, this means that either every point orbit is an ovoid, or G is transitive on P and hence on the flag space.*

Proof. Lemma 4.1.1 shows that the action of G on L is transitive. Choose an arbitrary point $p \in P$. For every $l \in L$, there is an element $g \in G$ such that $l^g \in L_p$, whence $p^{g^{-1}} \in P_l$. Hence the orbit p^G of p meets every line. Now suppose that p^G is not an ovoid. Then some line $l \in L$ is met by p^G in at least two different points q and q^g. Since the stabilizer of q^g is transitive on the line pencil L_{q^g}, we may as well assume that g fixes l. Hence G_l acts non-trivially on P_l. Since G_l is connected by Lemma 4.2.4, this implies that $\dim G_{l,q} < \dim G_l$. We infer that

$$\begin{aligned}
\dim p^G &= \dim q^G \\
&= \dim G - \dim G_q \\
&= \dim G - (\dim G_{l,q} + \dim l^{G_q}) \\
&> \dim G - \dim G_l - \dim l^{G_q} \\
&= \dim L - \dim L_q \\
&= m + 1,
\end{aligned}$$

whence the orbit p^G is open (see Salzmann et al. [115, 96.11]). □

4.2.9 Remark. We sketch an alternative arrangement of the material on compact $(1,m)$-quadrangles $Q = P \cup L$ with $m \geq 2$ which we have obtained so far. Let G be a compact connected (Lie) group which acts almost effectively and line-transitively on Q. The first claim is that the commutator subgroup G' acts transitively on L.

Suppose that this claim fails, and choose $l \in L$. Then $G_l G'$ is a proper subgroup of G. Replacing G by a covering group, we may assume that G' is simply connected, and that G is the internal direct product of G' with the torus group $Z(G)^1$. By Corollary 3.3.4, there is a non-trivial closed connected subgroup T of $Z(G)$ such that G is the internal direct product of T with $G_l G'$. Hence $L = l^G$ is homeomorphic to the product of T with the orbit of l under G', which contradicts simple connectedness of L. (A similar proof is given by Grundhöfer, Knarr, and Kramer [50, 1.3].)

We claim that every point stabilizer G_p acts transitively on the corresponding line pencil L_p, and that every line stabilizer $(G')_l$ in the commutator subgroup acts trivially on the corresponding point row P_l.

Indeed, choose a point $p \in P$ and a line l through p. We may assume that G is semi-simple. Then the fundamental group $\pi_1(G)$ is finite. As we have seen, the homotopy sequence shows that the action of G on P is not transitive. We obtain the sequence

$$\begin{aligned}
\dim l^{G_p} &= \dim G_p - G_{p,l} \geq \dim G_p - \dim G_l \\
&= \dim G - \dim p^G - \dim G_l \geq \dim L - (\dim P - 1) = \dim L_p.
\end{aligned}$$

in which equality must hold throughout. Hence the action of G_p on L_p is transitive, and the dimensions of the groups $G_{p,l}$ and G_l are equal.

Since G_l is connected, this implies that the groups are equal. Thus G_l fixes the point $p \in P_l$, whose choice was arbitrary.

The last part of the second claim reduces the length of the proof that all point orbits under G' are ovoids.

We shall make extensive use of Stroppel's reconstruction method, see [130], [131], [133, Lemma 4] and also Wich's comprehensive treatment [142]. For us, the following facts are most relevant: let G be a locally compact σ-compact group acting on a compact generalized polygon $P' \mathbin{\dot\cup} L'$, and suppose that for every point $p \in P'$, the stabilizer G_p acts transitively on the line pencil L'_p. Then the action of G on L' is transitive. Choose a line $l \in L'$. Then P'_l contains a set R of representatives for the point orbits. If $p, q \in R$ and $p \neq q$ then $G_p \neq G_q$. The map

$$\begin{array}{rcl} G/G_l & \longrightarrow & L' \\ G_l g & \longmapsto & l^g \end{array}$$

is a homeomorphism, and

$$\begin{array}{rcl} \bigcup_{p \in R} G/G_p & \longrightarrow & P' \\ G_p g & \longmapsto & p^g \end{array}$$

is a bijection. Observe that p^g is incident with l^h if and only if $G_p g \cap G_l h \neq \emptyset$. The triple

$$(G, \{G_p | p \in R\}, G_l)$$

is called a *sketch* of $P' \mathbin{\dot\cup} L'$, and the polygon is called *sketched.*

4.2.10 Lemma. *Let G be a compact connected group acting line-transitively on a compact $(1, m)$-quadrangle $Q = P \mathbin{\dot\cup} L$ with $m \geq 2$. If the action of G is also transitive on the point space then it is even transitive on the space of flags, so that Q is sketched by $(G, \{G_p\}, G_l)$ for any flag (p, l). Otherwise, for every line $l \in L$, the quadrangle Q is sketched by*

$$(G, \{G_p | p \in P_l\}, G_l).$$

Proof. Lemma 4.2.5 shows that every point stabilizer G_p acts transitively on the line pencil L_p. If G acts point-transitively then this implies that G acts transitively on the space of flags. If the action of G on the point set P is not transitive then every point orbit is an ovoid. In particular, every point row P_l is a set of representatives for the action of G on P. □

4.2.11 Proposition. *Let G be a compact connected group acting effectively on a compact $(1, m)$-quadrangle $Q = P \cup L$, where $m \geq 2$. Suppose that G is transitive on L but not on P. Then all point orbits are equivalent. For any flag (p, l), one of the following statements holds:*

(i) $(G, G_p, G_l) \cong (\mathrm{SO}_{m+2}\mathbb{R}, \mathrm{SO}_{m+1}\mathbb{R}, \mathrm{SO}_m\mathbb{R})$;

(ii) $(G, G_p, G_l) \cong (\mathrm{G}_2, \mathrm{SU}_3\mathbb{C}, \mathrm{SU}_2\mathbb{C})$, *and* $m = 5$;

(iii) $(G, G_p, G_l) \cong (\mathrm{Spin}_7\mathbb{R}, \mathrm{G}_2, \mathrm{SU}_3\mathbb{C})$, *and* $m = 6$.

Moreover, G_p is contained in G as the stabilizer of an effective and transitive action on $\mathbb{S}_{m+1}$, and G_l is contained in G_p as the stabilizer of an effective and transitive action on $\mathbb{S}_m$.

Proof. We have shown that G is a Lie group (Lemma 4.2.4), that all point stabilizers act transitively on the corresponding line pencils (Lemma 4.2.5), and that all point orbits are ovoids (Lemma 4.2.8). In particular, all occurring spaces are manifolds, and every line stabilizer G_l acts trivially on the corresponding point row P_l. Exactness of the sequence

$$1 = \pi_1(p^G) \longrightarrow \pi_0(G_p) \longrightarrow \pi_0(G) = 1$$

yields that all point stabilizers are connected. This implies that they act effectively on the corresponding line pencils (Corollary 4.1.12). By Corollary 2.1.6, it also entails that all point stabilizers are conjugate because they are all of the same dimension. Hence all point orbits are principal.

Choose an orbit p^G. By Proposition 4.1.4, the action of G on $p^G \approx \mathbb{S}_{m+1}$ is effective and, of course, transitive. Moreover, the point stabilizer G_p acts effectively and transitively on $L_p \approx \mathbb{S}_m$. We employ the classification of effective and transitive actions of compact connected groups on spheres (see Theorem 3.1.1).

The first possibility is that G is a special orthogonal group. Then $G \cong \mathrm{SO}_{m+2}\mathbb{R}$, and the point stabilizer $G_p \cong \mathrm{SO}_{m+1}\mathbb{R}$ can indeed act transitively on $\mathbb{S}_m$. The stabilizer of l is $G_l = G_{p,l} \cong \mathrm{SO}_m\mathbb{R}$.

Assume that G is embedded in a unitary group $\mathrm{U}_{r+1}\mathbb{C}$, and that $m = 2r$. Then the image of G in $\mathrm{U}_{r+1}\mathbb{C}$ contains $\mathrm{SU}_{r+1}\mathbb{C}$. The stabilizer G_p lies between $\mathrm{SU}_r\mathbb{C}$ and $\mathrm{U}_r\mathbb{C}$. This contradicts the transitive action of G_p on $L_p \cong \mathbb{S}_{2r}$.

Analogously, it is impossible that the group G is a transitive subgroup of $\mathrm{U}_r\mathbb{H} \cdot \mathrm{U}_1\mathbb{H}$ and $m = 4r - 2$, since $\mathrm{U}_{r-1}\mathbb{H} \cdot \mathrm{U}_1\mathbb{H}$ does not act transitively and effectively on $\mathbb{S}_{4r-2}$.

If $m = 5$ then G can be isomorphic to the exceptional simple group G_2. The point stabilizer G_p has dimension 8 and rank at most 2; the exact sequence

$$1 = \pi_2(p^G) \longrightarrow \pi_1(G_p) \longrightarrow \pi_1(G) = 1$$

shows that G_p is simply connected. Hence $G_p \cong \mathrm{SU}_3\mathbb{C}$. This group can indeed act transitively on $\mathbb{S}_5$, and the stabilizer of this action is $G_l \cong \mathrm{SU}_2\mathbb{C}$. (Note that this argument, as well as the following, has already appeared in Remarks 3.6.6 and 3.6.7.)

If $G \cong \mathrm{Spin}_7\mathbb{R}$ and $m = 6$ then the stabilizer G_p has dimension 14 and rank at most 3, whence $G_p \cong \mathrm{G}_2$. We have seen that the stabilizer of the transitive action of G_2 on $\mathbb{S}_6$ is $G_l \cong \mathrm{SU}_3\mathbb{C}$.

The last possibility, namely $G \cong \mathrm{Spin}_9\mathbb{R}$ and $m = 14$, is excluded, since the stabilizer G_p would be $\mathrm{Spin}_7\mathbb{R}$, and this group does not act transitively on $\mathbb{S}_{14}$. □

For a treatment of the three last ('exceptional') transitive actions on spheres see Salzmann et al. [115, 11.30–34, 17.15, and 18.13].

4.2.12 Remark. Note that all three cases of the preceding proposition actually occur for orthogonal quadrangles $Q(m + 3, \mathbb{R})$. Indeed, any maximal semi-simple compact automorphism group of this quadrangle is isomorphic to $\mathrm{SO}_{m+2}\mathbb{R}$. This group contains G_2 if $m = 5$ and $\mathrm{Spin}_7\mathbb{R}$ if $m = 6$ because the transitive actions of these two groups on the corresponding spheres are equivalent to linear actions.

4.2.13 Proposition. *If the exceptional group* G_2 *acts effectively on a compact* $(1, 5)$*-quadrangle* Q *then the action is precisely the one described in the preceding proposition, i.e. it is not transitive on the* *point space, and every point stabilizer acts transitively on the line pencil.*

Proof. The exact homotopy sequence shows that the action on P is not transitive. Hence the dimension of every point orbit is at most 6, which is the minimal dimension of any homogeneous space of G_2. No point is fixed since G_2 cannot act effectively on the five-dimensional line pencil. Therefore, every point stabilizer G_p is an eight-dimensional subgroup of G_2. It follows that its connected component $(G_p)^1$ is covered by $\mathrm{SU}_3\mathbb{C}$, whence the latter group acts almost effectively on the line pencil L_p. This action is transitive by Corollary 3.2.4. $\square$

4.2.14 Lemma. *Let G be a compact connected group acting effectively on a compact $(1, m)$-quadrangle $Q = P \cup L$ with $m \geq 2$. Suppose that G is transitive on L but not on P. Corresponding to the action of G on a point orbit $p^G \approx \mathbb{S}_{m+1}$, there is an effective orthogonal action of G on $\mathbb{R}^{m+2}$. Suppose that p_1, p_2 and p_3 are collinear points of Q, and that v_1, v_2 and v_3 are points of the unit sphere $\mathbb{S}_{m+1} \subseteq \mathbb{R}^{m+2}$ such that $G_{p_i} = G_{v_i}$ for $i = 1, 2, 3$. Then v_1, v_2 and v_3 are linearly dependent.*

Proof. Linearity of the action of G on p^G follows from Theorem 3.1.1. Let $l \in L$ be the line through p_1, p_2 and p_3. The stabilizer G_l fixes every point on l. In other words, every stabilizer of a point on l contains G_l. Hence G_l also fixes v_1, v_2 and v_3. If G is not isomorphic to G_2 then the statement follows from the fact that the fixed space of G_l has dimension 2. However, the following argument applies to all possible groups G.

Let V be the subspace of $\mathbb{R}^{m+2}$ which is spanned by v_1, v_2 and v_3, and assume that the dimension of V is 3. Let w be the image of v_3 under the reflection of V in the plane spanned by v_1 and v_2. We have seen in Proposition 4.2.11 that each stabilizer G_{v_i} acts transitively on $v_i^\perp \cap \mathbb{S}_{m+1}$ and, in particular, on the set of planes through v_i. Therefore, for $i \in \{1, 2\}$, there are elements $g_i \in G_{v_i}$ which satisfy $v_3^{g_i} = w$. Since

$$\begin{array}{ccc} \mathbb{S}_{m+1} & \longrightarrow & p_3^G \\ v_3^g & \longmapsto & p_3^g \qquad (g \in G) \end{array}$$

is a homeomorphism, we find that $p_3^{g_1} = p_3^{g_2} =: q$. Now since g_1 fixes p_1, the points p_1 and q are collinear, and so are p_2 and q. But neither

g_1 nor g_2 are elements of G_l since this group fixes V pointwise. Hence p_1, p_2 and q form a triangle in the generalized quadrangle Q, which is a contradiction. $\square$

4.2.15 Theorem. *Let G be a locally compact σ-compact group acting effectively and line-transitively on a compact $(1, m)$-quadrangle $Q = P \cup L$, where $m \geq 2$. Then G is a Lie group, and Q is isomorphic to the real orthogonal quadrangle $Q(m+3, \mathbb{R})$.*

Let K be the commutator subgroup of a maximal compact connected subgroup M of G. Then the following assertions hold.

(i) *Either $M = K$, or M is an almost direct product of K with a one-dimensional torus group.*

(ii) *$K \cong \mathrm{SO}_{m+2}\mathbb{R}$, or $m = 5$ and $K \cong \mathrm{G}_2$, or $m = 6$ and $K \cong \mathrm{Spin}_7\mathbb{R}$.*

(iii) *The action of K on Q is equivalent to the action of K on the real orthogonal quadrangle $Q(m+3, \mathbb{R})$ obtained from the embeddings*

$$K \hookrightarrow \mathrm{SO}_{m+2}\mathbb{R} \hookrightarrow \mathrm{P}(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{\mathrm{m+2}}\mathbb{R}).$$

In particular, the action of K_p on L_p is transitive for every point p, and all point orbits are ovoids.

Proof. Lemma 4.2.4 (d) shows that every maximal compact connected subgroup M of G acts transitively on L. By Proposition 4.2.6, we can choose a closed connected subgroup H of M such that H acts transitively on L but not on P, and such that M either is equal to H or is the almost direct product of H with a one-dimensional torus group. Proposition 4.2.11 yields that the isomorphism type of H is indeed as we have claimed. In particular, the group H is semi-simple, whence it is contained in the commutator subgroup $K := M'$. As the quotient M/H is abelian, we infer that $H = K$. Moreover, we obtain an action of K on $Q(m+3, \mathbb{R}) =: P' \cup L'$ by the embeddings $K \hookrightarrow \mathrm{SO}_{m+2}\mathbb{R} \hookrightarrow \mathrm{P}(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{\mathrm{m+2}}\mathbb{R})$. Representation theory shows that the first of these is unique up to conjugation in $\mathrm{O}_{m+2}\mathbb{R}$ (see, for instance, Kramer [75, 4.B and 4.C]). The image of the second embedding is the commutator subgroup.

Choose a line $l \in L$. Then

$$(K, \{K_p | p \in P_l\}, K_l)$$

is a sketch of Q by Lemma 4.2.10. The choice of the embedding of K into $\mathrm{SO}_{m+2}\mathbb{R}$ yields an action of K on $\mathbb{R}^{m+2}$. By Lemma 4.2.14, the one-dimensional fixed spaces of the stabilizers K_p of all points p on l are contained in a two-dimensional subspace V of $\mathbb{R}^{m+2}$. Choose an orthonormal basis $\{x, y\}$ of V, and let

$$l' := \mathbb{R}(1, 0, x) + \mathbb{R}(0, 1, y) \in L'.$$

This gives a sketch

$$(K, \{K_{p'} | p' \in P'_{l'}\}, K_{l'})$$

of $Q(m+3, \mathbb{R})$. We have $K_l \leq K_{l'}$, and even $K_l = K_{l'}$ since the two Lie groups are isomorphic. Moreover, for every point $p \in P_l$, there is a point $p' \in P'_{l'}$ such that $K_p = K_{p'}$. Therefore, the sketch of Q embeds into that of $Q(m+3, \mathbb{R})$, yielding an embedding of generalized quadrangles $Q \hookrightarrow Q(m+3, \mathbb{R})$. The restriction of this embedding to L is in fact a homeomorphism onto L'. Hence P goes onto P'. As the topologies are determined by the line spaces, we have an isomorphism of topological quadrangles. Since this comes from the two sketches, it is also an isomorphism of K-spaces. $\square$

The proof that the action is equivalent to an action on a real orthogonal quadrangle does not depend on the full strength of the classification of compact connected groups acting transitively on spheres (Theorem 3.1.1). We have only needed the fact that every such action is equivalent to a linear action, and an argument which shows the transitivity of stabilizers.

4.3 Local type of compact Lie transformation groups

Let G be a compact Lie group which acts effectively on a compact $(1,m)$-quadrangle. In an investigation similar to that of Section 3.5, we show that if m is sufficiently small, or if the dimension of G is sufficiently large, then the Lie algebra of G embeds into the Lie algebra of the maximal compact automorphism group of the real orthogonal quadrangle $Q(m+3,\mathbb{R})$, which is $\mathbb{R} \times \mathfrak{o}_{m+2}\mathbb{R}$.

Together with Theorem 4.2.15, the following lemma will always allow us to assume that the principal point stabilizers do not act transitively on line pencils.

4.3.1 Lemma. *Let G be a compact connected Lie group acting on a compact (m, m')-quadrangle $Q = P \mathbin{\dot\cup} L$. Choose a point $x \in P$ on a principal orbit, and suppose that the stabilizer G_x acts transitively on the line pencil L_x. Then G_p acts transitively on L_p for each $p \in P$. In particular, the quadrangle is sketched by G in the sense of Section 4.2.*

Proof. The result holds if p is a point of the orbit x^G. By Theorem 2.2.3, all principal stabilizers are conjugate, and the points on principal orbits form a dense subset of P. Therefore, Corollary 2.1.6 implies that every point stabilizer G_p contains the stabilizer of some point x^g. Since G_{x^g} fixes x^g and p and acts transitively on L_{x^g}, the points x^g and p are either equal or opposite, whence G_{x^g} also acts transitively on L_p. A fortiori, the action of G_p on L_p is transitive. □

Unless explicitly stated otherwise, homology and cohomology will always be taken over $\mathbb{Z}/2$.

The next result is the application of Mostert's Theorem [94] (cf. Theorem 3.1.2) to compact $(1, m)$-quadrangles. The reconstruction of the action is described in a varied formulation.

4.3.2 Proposition. *Let G be a compact connected Lie group acting on a compact $(1, m)$-quadrangle $Q = P \mathbin{\dot\cup} L$, and suppose that the codimension of some point orbit is 1. Then the orbit space P/G is homeomorphic to a circle $\mathbb{S}_1$ or to a compact interval $[0, 1]$.*

In the first case, all point orbits are equivalent and simply connected, so that all point stabilizers are connected and mutually conjugate. The orbit map $P \to P/G$ is the projection in a fibre bundle with fibre p^G and structure group $N_G(G_p)/N_G(G_p)^1$, where $p \in P$ is arbitrary.

In the second case, there are exactly two non-principal point orbits ${y_0}^G$ and ${y_1}^G$. Setting $Y := {y_0}^G \cup {y_1}^G$ and $X := P \setminus Y$ and choosing $x \in X$, we obtain the following relations between the cohomology *groups of* x^G *and* Y*: if* $m > 1$ *then*

$$\begin{aligned} H^1(Y) &\cong H^1(x^G) \oplus \mathbb{Z}/2 \\ H^j(Y) &\cong H^j(x^G) \text{ if } 2 \le j \le m-1 \end{aligned}$$

$$\begin{pmatrix} \dim_{\mathbb{Z}/2} H^m(Y^G) \\ \dim_{\mathbb{Z}/2} H^{m+1}(Y^G) \end{pmatrix} \in \left\{ \begin{pmatrix} -1+\dim_{\mathbb{Z}/2} H^m(x^G) \\ 0 \end{pmatrix}, \begin{pmatrix} \dim_{\mathbb{Z}/2} H^m(x^G) \\ 1 \end{pmatrix} \right\}.$$

If $m = 1$ *then*

$$\begin{pmatrix} \dim_{\mathbb{Z}/2} H^1(Y^G) \\ \dim_{\mathbb{Z}/2} H^2(Y^G) \end{pmatrix} \in \left\{ \begin{pmatrix} \dim_{\mathbb{Z}/2} H^1(x^G) \\ 0 \end{pmatrix}, \begin{pmatrix} 1+\dim_{\mathbb{Z}/2} H^1(x^G) \\ 1 \end{pmatrix} \right\}.$$

The points x*,* y_0 *and* y_1 *can be chosen in such a way that the following assertions hold: the principal stabilizer* G_x *is contained in both* G_{y_0} *and* G_{y_1}*; the two spaces* G_{y_i}/G_x *are integral homology spheres; and the action of* G *on* P *is equivalent to the natural action of* G *on the quotient space obtained from* $[0,1]\times G/G_x$ *by identifying, for* $i \in \{0,1\}$*, the points* $(i, G_x g)$ *and* $(i, G_x h)$ *whenever* $gh^{-1} \in G_{y_i}$*.*

Note that a fibre bundle with trivial structure group is a trivial bundle, i.e. homeomorphic to the product of its fibre with its base space.

Proof. By Theorem 2.2.2, the point space P is a topological manifold. Since P and hence P/G are compact, most of the proposition is contained in Mostert's work [94]. If P is a locally trivial fibre bundle over $P/G \approx \mathbb{S}_1$ then there is an exact homotopy sequence (cf. Bredon [15, VII.6.7 and VII.6.12])

$$\pi_2(P/G) \longrightarrow \pi_1(p^G) \longrightarrow \pi_1(P) \longrightarrow \pi_1(P/G) \longrightarrow \pi_0(p^G)$$

which proves that p^G is simply connected since $\pi_1(P) \cong \mathbb{Z}$ (see Grundhöfer, Knarr, and Kramer [49, Appendix]), and $\pi_0(p^G)$ is a singleton. (Note that the homotopy sequence excludes the case $m = 1$ and $\pi_1(P) = \mathbb{Z}/2$.) Salzmann et al. [115, 94.4] show that G_p is connected.

Suppose that the orbit space is a compact interval. Let $\mathrm{pr} : P \to [0,1]$ be the composition $P \to P/G \approx [0,1]$. Set $A := \mathrm{pr}^{-1}([0,1[)$ and $B := \mathrm{pr}^{-1}(]0,1])$. From the description of P as a quotient space of $[0,1] \times G/G_x$, one obtains the following homotopy equivalences:

$$A \simeq {y_0}^G, \quad A \cap B \simeq x^G, \quad B \simeq {y_1}^G.$$

The statements about the cohomology of x^G and $Y = {y_0}^G \cup {y_1}^G$ follow from the Mayer–Vietoris cohomology sequence [15, V.8.3] for (P, A, B) and from the description of $H^*(P)$ which is given in [49, Appendix]. $\square$

4.3.3 Remark. Using Leray–Serre spectral sequences, one can show under certain additional assumptions that if the orbit space is a circle then the $\mathbb{Z}/2$-cohomology of each fibre is isomorphic to that of $\mathbb{S}_{m+1}$. However, we avoid this additional technique.

4.3.4 Lemma. *Let $\mathfrak{g}$ be a simple compact Lie algebra, and suppose that $\mathfrak{o}_n\mathbb{R} \cong \mathfrak{h} < \mathfrak{g}$ for some $n \geq 2$, and that $d := \dim \mathfrak{g} - \dim \mathfrak{o}_{n+1}\mathbb{R} \leq 6$. Then either $\mathfrak{g} \cong \mathfrak{o}_{n+1}\mathbb{R}$, or $d \geq 2$ and the triple $(\mathfrak{g}, \mathfrak{o}_n\mathbb{R}, d)$ is one of the following, up to isomorphism of $\mathfrak{g}$:*

$$\begin{array}{cccc} (\mathfrak{su}_3\mathbb{C}, \mathfrak{o}_2\mathbb{R}, 5) & (\mathfrak{su}_3\mathbb{C}, \mathfrak{o}_3\mathbb{R}, 2) & (\mathfrak{o}_5\mathbb{R}, \mathfrak{o}_3\mathbb{R}, 4) & (\mathfrak{g}_2, \mathfrak{o}_4\mathbb{R}, 4) \\ (\mathfrak{o}_6\mathbb{R}, \mathfrak{o}_4\mathbb{R}, 5) & (\mathfrak{o}_7\mathbb{R}, \mathfrak{o}_5\mathbb{R}, 6) & (\mathfrak{u}_3\mathbb{H}, \mathfrak{o}_5\mathbb{R}, 6) & (\mathfrak{su}_5\mathbb{C}, \mathfrak{o}_6\mathbb{R}, 3) \end{array}$$

To see that all these triples actually occur, note the exceptional isomorphisms $\mathfrak{o}_5\mathbb{R} \cong \mathfrak{u}_2\mathbb{H}$ and $\mathfrak{o}_6\mathbb{R} \cong \mathfrak{su}_4\mathbb{C}$; the algebra $\mathfrak{g}_2$ contains $\mathfrak{o}_4\mathbb{R} \cong \mathfrak{su}_2\mathbb{C} \times \mathfrak{su}_2\mathbb{C}$ as a maximal subalgebra of maximal rank (see Borel and de Siebenthal [7] or Grundhöfer, Knarr, and Kramer [49]). Also note that different embeddings are possible in some cases. For example, $\mathfrak{o}_3\mathbb{R}$ is embedded in three essentially different ways into $\mathfrak{o}_5\mathbb{R}$.

4.3.5 Remark. One can weaken the hypothesis $\dim \mathfrak{g} \leq \dim \mathfrak{o}_{n+1}\mathbb{R} + 6$ if a stronger statement is needed. If we suppose that $\dim \mathfrak{g} \leq \dim \mathfrak{o}_{n+1}\mathbb{R} + 7$ then the triples $(\mathfrak{o}_8\mathbb{R}, \mathfrak{o}_6\mathbb{R}, 7)$ and $(\mathfrak{f}_4, \mathfrak{o}_9\mathbb{R}, 7)$ occur as further exceptions, since $\mathfrak{o}_9\mathbb{R} = \mathfrak{b}_4$ is indeed contained in the exceptional algebra $\mathfrak{f}_4$ as a maximal subalgebra of maximal rank. Representation theory shows that, on the other hand, the algebra $\mathfrak{o}_7\mathbb{R}$ is not embedded into $\mathfrak{su}_6\mathbb{C}$, and $\mathfrak{o}_{16}\mathbb{R}$ does not fit into $\mathfrak{su}_{12}\mathbb{C}$.

Proof of 4.3.4. We investigate the possible isomorphism types of $\mathfrak{g}$ given by the classification of simple compact Lie algebras.

If $\mathfrak{g} \cong \mathfrak{su}_{r+1}\mathbb{C}$ for some $r \in \mathbb{N}$ then the codimension of $\mathfrak{h}$ in $\mathfrak{g}$ is bounded above by the hypothesis, and it is bounded below by the fact that the minimal codimension of any subalgebra of $\mathfrak{su}_{r+1}\mathbb{C}$ is $2r$ if $r \neq 3$ (see Table 2.1 on page 50). We may suppose that $r \neq 3$ because $\mathfrak{su}_4\mathbb{C}$ is isomorphic to $\mathfrak{o}_6\mathbb{R}$. Explicitly, we have

$$2r \leq \dim \mathfrak{g} - \dim \mathfrak{h} = r(r+2) - \binom{n}{2} \leq \binom{n+1}{2} + 6 - \binom{n}{2},$$

which leads to the inequalities

$$\binom{n}{2} \leq r^2 \quad \text{and} \quad (r+1)^2 \leq \binom{n+1}{2} + 7.$$

Eliminating r, we obtain

$$\sqrt{\binom{n}{2}} + 1 \leq r+1 \leq \sqrt{\binom{n+1}{2} + 7},$$

from which we deduce the following sequence of equivalent inequalities:

$$\begin{aligned}
\binom{n}{2} + 2\cdot\sqrt{\binom{n}{2}} + 1 \quad &\leq \quad \binom{n+1}{2} + 7 \\
2\cdot\sqrt{\binom{n}{2}} \quad &\leq \quad n+6 \\
2n(n-1) \quad &\leq \quad n^2 + 12n + 36 \\
n^2 - 14n \quad &\leq \quad 36 \\
(n-7)^2 \quad &\leq \quad 85
\end{aligned}$$

As n is a natural number, this implies $n - 7 \leq 9$ and

$$n \leq 16.$$

Since the two squares r^2 and $(r+1)^2$ lie between $\binom{n}{2}$ and $\binom{n+1}{2} + 7$, the number n cannot be one of $\{5, 7, 8, 10, 11, 12, 13, 14, 15, 16\}$. The algebra $\mathfrak{o}_9\mathbb{R}$ cannot be embedded into $\mathfrak{su}_7\mathbb{C}$ because it would map onto a subalgebra of minimal codimension, and these are all isomorphic to $\mathfrak{su}_6\mathbb{C} \times \mathbb{R}$ by Table 2.1 on page 50. The other values $\{2, 3, 4, 6\}$ of n lead to some of the exceptions listed in the statement of the lemma; note that $n = 4$ corresponds to $r = 3$.

Some more exceptions arise for $\mathfrak{g} \cong \mathfrak{o}_m\mathbb{R}$. However, if $n \geq 6$ and $m \geq n+2$ then $\dim \mathfrak{o}_m\mathbb{R} \geq \dim \mathfrak{o}_{n+2}\mathbb{R} > \dim \mathfrak{o}_{n+1}\mathbb{R} + 6$.

The case $\mathfrak{g} \cong \mathfrak{u}_r\mathbb{H}$ yields only one additional exception. If $r \geq 4$ then $\binom{n}{2} < \dim \mathfrak{u}_r\mathbb{H} = \binom{2r+1}{2} \leq \binom{n+1}{2} + 6$ entails $n = 2r$, contradicting the fact that the minimal codimension of any subalgebra of $\mathfrak{u}_r\mathbb{H}$ is $4(r-1)$.

The exceptional algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ and $\mathfrak{f}_4$ do not occur because they do not contain any sufficiently large subalgebra. However, the exceptional algebra $\mathfrak{g}_2$ contains $\mathfrak{o}_4\mathbb{R}$ as a subalgebra of maximal rank. □

The following proposition partly refines Corollary 4.1.7 for the specific case of compact $(1, m)$-quadrangles. Quadrangles with general parameters will be treated in Proposition 5.2.3.

4.3.6 Proposition. *Let G be a compact abelian group acting almost effectively on a compact $(1, m)$-quadrangle $Q = P \mathbin{\dot\cup} L$. Then $\dim G \leq m + 1$. If G is not a Lie group then $\dim G \leq m - 1$. If $m \geq 3$ then $\dim G \leq m$.*

Proof. In view of Corollary 4.1.7, it remains to assume that $m \geq 3$ and that $G = \mathbb{T}^{m+1}$, and to obtain a contradiction. Since the action of G on a principal orbit x^G is almost effective by Proposition 4.1.4, we have $x^G \approx {\mathbb{S}_1}^{m+1}$, so that we can apply Proposition 4.3.2. The orbit space P/G is a compact interval since x^G is not simply connected. Each of the two non-principal orbits ${y_i}^G$ is homeomorphic to ${\mathbb{S}_1}^m$ or to ${\mathbb{S}_1}^{m+1}$. By an induction argument based on the Künneth Theorem, we find that $\dim_{\mathbb{Z}/2} H^1(x^G) = m+1$ and $\dim_{\mathbb{Z}/2} H^1({y_i}^G) \in \{m, m+1\}$. This contradicts $H^1({y_0}^G) \oplus H^1({y_1}^G) \cong H^1(x^G) \oplus \mathbb{Z}/2$. □

4.3.7 Lemma. *Let G be a compact Lie group acting transitively on the point space P of a compact $(1, m)$-quadrangle. Choose $p \in P$. Then*

$$1 \leq \dim Z(G^1) \leq 1 + \dim Z\big((G_p)^1\big).$$

Proof. The compact connected Lie group G^1 has a finite covering group whose fundamental group is free abelian of rank $\dim Z(G^1)$, see Salzmann et al. [115, 94.31]. Hence $\pi_1(G)$ is a finitely generated abelian group whose free rank is $\dim Z(G^1)$. The analogous statement holds for $\pi_1(G_p)$. Since $\pi_1(P) \cong \mathbb{Z}$ (cf. Grundhöfer, Knarr, and Kramer [49, 4_2]) and $\pi_0(G_p)$ is finite, the statement of the lemma follows from the exact homotopy sequence [115, 96.12]

$$\pi_1(G_p) \longrightarrow \pi_1(G) \longrightarrow \pi_1(P) \longrightarrow \pi_0(G_p).$$

□

4.3.8 Proposition. *Let G be a compact Lie group with Lie algebra $\mathfrak{g} = \mathfrak{o}_{j+1}\mathbb{R} \times \mathfrak{o}_{k+1}\mathbb{R}$, where $j \geq 2$ and $k \geq 3$. Then G cannot act almost effectively on a compact $(1, j+k-1)$-quadrangle.*

Proof. Suppose that G does act almost effectively on a compact $(1, j+k-1)$-quadrangle $Q = P \cup L$. The preceding lemma shows that the action of G on P is not transitive. Therefore, the dimension of any point orbit is at most $j+k$. The action of G on a principal point orbit x^G is almost effective by Proposition 4.1.4. Set $H := G_x$. By Theorem 2.5.5 and Lemma 3.1.7, we may assume that $G = \mathrm{SO}_{j+1}\mathbb{R} \times \mathrm{SO}_{k+1}\mathbb{R}$ and that $H^1 = \mathrm{SO}_j\mathbb{R} \times \mathrm{SO}_k\mathbb{R}$. In particular, Proposition 4.3.2 applies.

If P/G is a circle then H is connected. It is no loss of generality to assume that $j \leq k$. We apply Proposition 2.4.7 to the normal subgroup $N := 1 \times \mathrm{SO}_{k+1}\mathbb{R}$ of G. It shows that P is a $\mathbb{S}_k$-bundle over P/N, and that this base space is itself a $\mathbb{S}_j$-bundle over $\mathbb{S}_1$. Lemma 2.4.1 shows that $H^*(P/N) \cong H^*(\mathbb{S}_1 \times \mathbb{S}_j)$. The cohomology Gysin sequence (see Spanier [124, 5.7.11])

$$H^{k+1}(P) \longrightarrow H^1(P/N) \longrightarrow H^{k+2}(P/N)$$

yields a contradiction because the three consecutive groups are 0 (see Grundhöfer, Knarr, and Kramer [49, 4_2]), $\mathbb{Z}/2$, and 0.

Suppose that P/G is a compact interval. Since

$$H^1 \leq H \leq N_G(H^1) = \mathrm{O}_j\mathbb{R} \times \mathrm{O}_k\mathbb{R},$$

there are five distinct possibilities for H. They will be excluded, one by one. Suppose that H is connected, so that $x^G \approx \mathbb{S}_j \times \mathbb{S}_k$. Then each of the two non-principal orbits ${y_i}^G$ is homeomorphic to one of the spaces

$$P_j\mathbb{R} \times \mathbb{S}_k,\ \ \mathbb{S}_j \times P_k\mathbb{R},\ \ \frac{\mathbb{S}_j \times \mathbb{S}_k}{\pm},\ \ \mathbb{S}_j,\ \ \mathbb{S}_k,$$

where we have used the fact that G_{y_i}/H is an integral homology sphere. Using Proposition 2.4.7, we infer from Lemma 2.4.1 that

$$H^*\left(\frac{\mathbb{S}_j \times \mathbb{S}_k}{\pm}\right) \cong H^*(P_j\mathbb{R} \times \mathbb{S}_k),$$

where we still assume that $j \leq k$. Proposition 4.3.2 yields that

$$H^{j+k-1}(Y) = 0 \quad \text{and} \quad H^{j+k}(Y) \cong \mathbb{Z}/2,$$

which is impossible.

If $H = \mathrm{O}_j\mathbb{R} \times \mathrm{SO}_k\mathbb{R}$, so that $x^G \approx P_j\mathbb{R} \times \mathbb{S}_k$, then ${y_i}^G \approx P_j\mathbb{R} \times P_k\mathbb{R}$ or ${y_i}^G \approx P_j\mathbb{R}$. But this is not compatible with $H^2(Y) \cong H^2(x^G) \cong \mathbb{Z}/2$. If $H = \mathrm{SO}_j\mathbb{R} \times \mathrm{O}_k\mathbb{R}$ then the fact that $H^2(Y) \cong H^2(x^G)$ entails that $j = 2$ and that ${y_0}^G \approx {y_1}^G \approx P_k\mathbb{R}$. Reconstruction of the action leads to the contradiction $P \approx \mathbb{S}_3 \times P_k\mathbb{R}$.

Suppose that H/H^1 is the diagonal subgroup of $(\mathrm{O}_j\mathbb{R} \times \mathrm{O}_k\mathbb{R})/H^1$. Then $x^G \approx (\mathbb{S}_j \times \mathbb{S}_k)/\pm$, and ${y_i}^G$ is homeomorphic to $P_j\mathbb{R} \times P_k\mathbb{R}$, $P_j\mathbb{R}$, or $P_k\mathbb{R}$. We assume that $j \leq k$, whence $H^*(x^G) \cong H^*(P_j\mathbb{R} \times \mathbb{S}_k)$. Then $\mathbb{Z}/2 \cong H^2(x^G) \cong H^2(Y)$, which is impossible.

Finally H cannot be equal to $\mathrm{O}_j\mathbb{R} \times \mathrm{O}_k\mathbb{R}$ since G_{y_i}/H is an integral homology sphere. □

4.3.9 Proposition. *Let G be a compact Lie group with Lie algebra $\mathfrak{g} = \mathbb{R}^3 \times \mathfrak{o}_{m-1}\mathbb{R}$, where $m \geq 5$. Then G cannot act almost effectively on a compact $(1, m)$-quadrangle.*

Proof. Suppose, to the contrary, that G acts almost effectively on a compact $(1, m)$-quadrangle $Q = P \cup L$. It is no loss of generality to assume that the group G is connected. Choose $x \in P$ such that x^G is a principal orbit, and set $H := G_x$. Then $\dim H \geq \dim G - \dim P = \binom{m-2}{2} - 1$. Suppose first that equality holds. Then the action of G on P is transitive. Lemma 4.3.7 shows that $\dim Z(H^1) \geq 2$. The projection of $\mathfrak{g}$ onto $\mathfrak{g}/Z(\mathfrak{g}) \cong \mathfrak{o}_{m-1}\mathbb{R}$ induces an embedding of the Lie algebra $\mathfrak{h}$ of H into $\mathfrak{o}_{m-1}\mathbb{R}$. Lemma 3.6.8 implies that $m = 5$ and $\mathfrak{h} \cong \mathbb{R}^2$. We claim that the intersection of H with the commutator subgroup G' is discrete. Otherwise, the image of H under the projection of G onto G/G' is at most one-dimensional, whence there is a connected subgroup $K \leq G$ with $G' \leq K$ and $\dim Z(K) = 1$ such that $H^1 \leq K$. The map $\pi_1(H) \to \pi_1(G)$ induced by inclusion factors through $\pi_1(K)$. In particular, the free rank of its image is at most 1, so that the exact homotopy sequence used in the proof of Lemma 4.3.7 yields a contradiction. Having proved our claim, we apply Proposition 2.4.7 to the normal subgroup G' of G. Since all stabilizers of the action of G' on P are discrete, Proposition 4.3.2 shows that they are all trivial, and that $G' \cong \mathrm{SU}_2\mathbb{C}^2$. Let N be a normal subgroup of G' such that

$N \cong \mathrm{SU}_2\mathbb{C}$. Then P is an $\mathbb{S}_3$-bundle over P/N, which is itself a $\mathbb{S}_3$-bundle over $\mathbb{S}_1$. By Lemma 2.4.1, we have

$$(\dim_{\mathbb{Z}/2} H^j(P/N))_{j\in\mathbb{N}} = (1, 1, 0, 1, 1, 0, 0, \dots).$$

Since $H^4(P) = 0$, see Grundhöfer, Knarr, and Kramer [49, 4_2], the cohomology Gysin sequence (cf. Spanier [124, 5.7.11])

$$H^4(P) \longrightarrow H^1(P/N) \longrightarrow H^5(P/N)$$

yields a contradiction.

Thus the action of G on P is not transitive. By Theorem 2.5.5, the dimension of the principal stabilizer H is $\binom{m-2}{2}$. Together with Lemma 3.1.7, the same theorem shows that, without loss of generality, $G = \mathbb{T}^3 \times \mathrm{SO}_{m-1}\mathbb{R}$ and $H^1 = 1^3 \times \mathrm{SO}_{m-2}\mathbb{R}$. We apply Proposition 4.3.2. Since the principal orbit x^G is not simply connected, the orbit space P/G is a compact interval. We may assume that $H \cap Z(G)$ is trivial. Then there are three essentially different possibilities for H.

Firstly, the principal stabilizer H can be connected. Then $x^G \approx {\mathbb{S}_1}^3 \times \mathbb{S}_{m-2}$. Since the spaces G_{y_i}/G_x are integral homology spheres, the possible homeomorphism types of the non-principal orbits ${y_i}^G$ are

$${\mathbb{S}_1}^3 \times \mathbb{S}_{m-2},\ \ {\mathbb{S}_1}^2 \times \frac{\mathbb{S}_1 \times \mathbb{S}_{m-2}}{\pm},\ \ {\mathbb{S}_1}^3 \times P_{m-2}\mathbb{R},\ \ {\mathbb{S}_1}^2 \times \mathbb{S}_{m-2},\ \ {\mathbb{S}_1}^3.$$

The first two of these spaces have the same cohomology. We find that $\dim_{\mathbb{Z}/2} H^2({y_i}^G) \in \{1, 3, 7\}$, which contradicts the fact that $H^2(Y) \cong H^2(x^G) \cong (\mathbb{Z}/2)^3$.

Suppose that $H = 1^3 \times \mathrm{O}_{m-2}\mathbb{R} \le G'$. Then $x^G \approx {\mathbb{S}_1}^3 \times P_{m-2}\mathbb{R}$, and ${y_i}^G \approx x^G$ or ${y_i}^G \approx {\mathbb{S}_1}^2 \times P_{m-2}\mathbb{R}$. Hence $\dim_{\mathbb{Z}/2} H^1(x^G) = 4$, and $\dim_{\mathbb{Z}/2} H^1({y_i}^G) \in \{3, 4\}$, which is impossible by Proposition 4.3.2.

Finally, suppose that H is the graph of a non-trivial morphism from $\mathrm{O}_{m-2}\mathbb{R}$ to $\mathbb{T}^3$. Then $x^G \approx {\mathbb{S}_1}^2 \times (\mathbb{S}_1 \times \mathbb{S}_{m-2})/\pm$, and the possible homeomorphism types of ${y_i}^G$ are

$$\begin{gathered} {\mathbb{S}_1}^2 \times \frac{\mathbb{S}_1 \times \mathbb{S}_{m-2}}{\pm}, \\ {\mathbb{S}_1}^3 \times P_{m-2}\mathbb{R},\ \ \mathbb{S}_1 \times \frac{\mathbb{S}_1 \times \mathbb{S}_{m-2}}{\pm},\ \ {\mathbb{S}_1}^2 \times P_{m-2}\mathbb{R},\ \ {\mathbb{S}_1}^3. \end{gathered}$$

We find that $\dim_{\mathbb{Z}/2} H^2(x^G) = 3$, and $\dim_{\mathbb{Z}/2} H^2({y_i}^G) \in \{1, 3, 4, 7\}$, which contradicts the fact that $H^2(Y) \cong H^2(x^G)$. □

4.3.10 Theorem. *Let G be a compact Lie group which acts almost effectively on a compact $(1,m)$-quadrangle $Q = P \cup L$, and suppose that $m \leq 5$. Then the Lie algebra of G is isomorphic to a subalgebra of $\mathbb{R} \times \mathfrak{o}_{m+2}\mathbb{R}$ unless $m = 5$ and $G^1 \cong \mathbb{T}^5$.*

No action of $\mathbb{T}^5$ on a compact $(1,5)$-quadrangle is known. Unfortunately, our methods do not allow to exclude such an action.

Proof. Since the action of a point stabilizer on the corresponding line pencil is almost effective by Corollary 4.1.12, the Lie algebras of point stabilizers are described by the results of Section 3.5. We assume that the group G is connected. Moreover, we may assume that the principal point stabilizers do not act transitively on line pencils, see Lemma 4.3.1 and Theorem 4.2.15. For each parameter m, we will obtain a reasonably short list of possible Lie algebras, and some of these will have to be excluded.

Set $\mathfrak{g} := L(G)$, and let $\mathfrak{h} \leq \mathfrak{g}$ be the Lie algebra of a principal point stabilizer. Then

$$\dim \mathfrak{g} - \dim \mathfrak{h} \leq \dim P = m + 2,$$

and equality holds if and only if G acts point-transitively.

If $m = 1$ then $\dim G_p \leq 1$, whence $\dim G \leq 4$. Moreover, if G is abelian then $\dim G \leq 2$ by Proposition 4.3.6. The claim follows.

Suppose that $m = 2$. As a principal point stabilizer does not act transitively on the line pencil, we find that $\dim \mathfrak{h} \leq 1$ and $\dim \mathfrak{g} \leq 5$. Moreover, $\dim G \leq 3$ if G is abelian, again by Proposition 4.3.6. This completes the proof for $m = 2$.

If $m = 3$ then $\dim \mathfrak{h} \leq 3$, whence $\dim \mathfrak{g} \leq 8$. If $\dim \mathfrak{g} = 8$ then G acts transitively on P, and $\dim \mathfrak{h} = 3$, whence $\mathfrak{h} \cong \mathfrak{o}_3\mathbb{R}$. Lemma 4.3.7 yields that $\dim Z(G) = 1$, which is impossible. Hence $\dim \mathfrak{g} \leq 7$, and $\operatorname{rk} \mathfrak{g} \leq 3$ by Proposition 4.3.6. The non-trivial compact Lie algebras with these two properties are

$$\mathbb{R} \times \mathfrak{o}_4\mathbb{R}, \quad \mathfrak{o}_4\mathbb{R}, \quad \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R} \times \mathfrak{o}_3\mathbb{R}, \quad \mathfrak{o}_3\mathbb{R}, \quad \mathbb{R}^3, \quad \mathbb{R}^2, \quad \text{and } \mathbb{R},$$

and these are indeed embedded into $\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$.

Suppose that $m = 4$ and $\dim \mathfrak{h} \leq 6$. From now on, it will be advantageous to treat the possible isomorphism types of $\mathfrak{h}$ one by one.

They can be found in the proof of Theorem 3.5.1. Suppose that $\mathfrak{h} \cong \mathfrak{o}_4\mathbb{R}$, so that $\dim \mathfrak{g} \leq 12$. With the help of Theorem 2.5.5, we find that $\mathfrak{g}$ is isomorphic to $\mathfrak{o}_3\mathbb{R}^4$, or to $\mathbb{R}^j \times \mathfrak{o}_5\mathbb{R}$ for some $j \leq 2$. Lemma 4.3.7 excludes $\mathfrak{o}_3\mathbb{R}^4$ and $\mathbb{R}^2 \times \mathfrak{o}_5\mathbb{R}$. If $\mathfrak{h} \cong \mathbb{R} \times \mathfrak{o}_3\mathbb{R}$ then the dimension of $\mathfrak{g}$ is at most 10, and the possible isomorphism types of $\mathfrak{g}$ are $\mathfrak{o}_5\mathbb{R}$, $\mathbb{R}^j \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$ for $j \leq 1$, and $\mathbb{R}^j \times \mathfrak{su}_3\mathbb{C}$ for $j \leq 2$. Among these, the algebras containing $\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_4\mathbb{R}$ are excluded by Proposition 4.3.8, and $\mathfrak{o}_5\mathbb{R}$ is now impossible by Lemma 4.3.7. If $\mathfrak{h} \cong \mathfrak{o}_3\mathbb{R}$ then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{su}_3\mathbb{C}$ for $j \leq 1$, or $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}$, where Lemma 4.3.7 shows that $j \leq 2$. If $\mathfrak{h} \cong \mathbb{R}^2$ then Lemma 4.3.7 excludes $\mathfrak{g} \cong \mathfrak{su}_3\mathbb{C}$. Hence $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}$, where $j \leq 2$ since $\dim \mathfrak{g} \leq 8$. If $\mathfrak{h} \cong \mathbb{R}$ then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}$ with $j \leq 1$, or $\mathfrak{g} \leq \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R}$, where $j \leq 3$ by Lemma 4.3.7. Finally, suppose that $\mathfrak{h} = 0$. If $\mathfrak{g}$ is not abelian then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R}$ with $j \leq 2$. For abelian $\mathfrak{g}$, Proposition 4.3.6 shows that $\dim \mathfrak{g} \leq 4$. In all cases, we have found that $\mathfrak{g}$ is embedded into $\mathbb{R} \times \mathfrak{o}_6\mathbb{R}$.

Suppose that $m = 5$, and that the principal point stabilizers do not act transitively on line pencils, whence $\dim \mathfrak{h} \leq 10$. If $\mathfrak{h} \cong \mathfrak{o}_5\mathbb{R}$ then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_6\mathbb{R}$, where $j \leq 1$ by Lemma 4.3.7. If $\mathfrak{h} \cong \mathbb{R} \times \mathfrak{o}_4\mathbb{R}$ then either $\mathfrak{g} \cong \mathfrak{o}_4\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$, which contradicts Lemma 4.3.7, or $\mathfrak{g}$ must contain $\mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_5\mathbb{R}$, which is impossible by Proposition 4.3.8. If $\mathfrak{h} \cong \mathfrak{o}_4\mathbb{R}$ then the same proposition shows that $\mathfrak{g}$ does not contain $\mathfrak{o}_4\mathbb{R}^2$, so that we must have $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_5\mathbb{R}$ with $j \leq 2$.

Suppose that $\mathfrak{h} \cong \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}$. Since $\mathfrak{h}$ is not embedded into $\mathfrak{o}_5\mathbb{R}$, we must have $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{su}_3\mathbb{C}$ with $j \leq 1$. It suffices to exclude an almost effective action of $G = \mathrm{SU}_2\mathbb{C} \times \mathrm{SU}_3\mathbb{C}$. By Lemma 4.3.7, such an action cannot be point-transitive, whence Theorem 2.5.5 shows that a principal stabilizer H satisfies $H^1 = \mathbb{T} \times \mathrm{U}_2\mathbb{C}$ for suitable embeddings $\mathbb{T} \hookrightarrow \mathrm{SU}_2\mathbb{C}$ and $\mathrm{U}_2\mathbb{C} \hookrightarrow \mathrm{SU}_3\mathbb{C}$. (All such embeddings are conjugate.) We apply Proposition 4.3.2. Suppose first that P/G is a compact interval. Using the fact that the two spaces G_{y_i}/G_x are integral homology spheres, we find that $x^G \approx \mathbb{S}_2 \times P_2\mathbb{C}$, and that ${y_i}^G \approx P_2\mathbb{C}$ or ${y_i}^G \approx P_2\mathbb{R} \times P_2\mathbb{C}$. Now Proposition 4.3.2 yields that $H^5(Y) = 0$ and $H^6(Y) \cong \mathbb{Z}/2$. This is clearly impossible. Therefore, the orbit space P/G is a circle, and all orbits are equivalent and simply connected, whence H is connected. We will apply Proposition 2.4.7 to the normal subgroup $N := 1 \times \mathrm{SU}_3\mathbb{C}$ of G. As $H \cap N = 1 \times \mathrm{U}_2\mathbb{C}$ is a self-normalizing subgroup of N, the point space P is a trivial fibre bundle over P/N with fibre $N/(H \cap N) \approx P_2\mathbb{C}$. The base space P/N is a compact three-manifold. The Künneth Theorem (cf. Bredon [15, VI.3.2]) yields

$$\begin{aligned}\dim_{\mathbb{Z}/2} H^5(P) &= \dim_{\mathbb{Z}/2}(H^*(P_2\mathbb{C}) \otimes H^*(P/N))_5 \\ &\geq \dim_{\mathbb{Z}/2} H^2(P_2\mathbb{C}) \otimes H^3(P/N) \\ &= 1,\end{aligned}$$

which contradicts the fact that $H^5(P) = 0$, see Grundhöfer, Knarr, and Kramer [49, 4_2].

Suppose that $\mathfrak{h} \cong \mathbb{R} \times \mathfrak{o}_3\mathbb{R}$. Then $\mathfrak{g}$ is isomorphic to one of

$$\mathbb{R}^j \times \mathfrak{o}_5\mathbb{R}\ (j \leq 1),\quad \mathbb{R}^j \times \mathfrak{su}_3\mathbb{C}\ (j \leq 2),\quad \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R}^3\ (j \leq 2).$$

Among these, only $\mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}^3$ is not embedded into $\mathbb{R} \times \mathfrak{o}_7\mathbb{R}$. Suppose, then, that $\mathfrak{g} \cong \mathbb{R}^2 \times \mathfrak{o}_3\mathbb{R}^3$. Then G acts transitively on P. We claim that the identity component of any point stabilizer G_p is not contained in the derived subgroup G'. Otherwise, the map $\pi_1(G_p) \to \pi_1(G)$ induced by inclusion factors through $\pi_1(G')$. In particular, its image is finite since G' is semi-simple. The exact homotopy sequence used in the proof of Lemma 4.3.7 yields a contradiction. Using Proposition 2.4.7, we find that $K := G'$ acts on P in such a way that all orbits are equivalent, and their codimension is 1. Hence they are simply connected, so that a typical point stabilizer H in K is connected. Therefore, we may assume that $K = \mathrm{SU}_2\mathbb{C} \times \mathrm{SO}_4\mathbb{R}$ and $H = 1 \times \mathrm{SO}_3\mathbb{R}$. We set $N := 1 \times \mathrm{SO}_4\mathbb{R}$ and apply Proposition 2.4.7 again. It shows that P is a locally trivial fibre bundle with fibre $\mathbb{S}_3$ over P/N, and that this base space is itself a $\mathbb{S}_3$-bundle over $\mathbb{S}_1$. Lemma 2.4.1 shows that $H^*(P/N) \cong H^*(\mathbb{S}_1 \times \mathbb{S}_3)$. Since $H^4(P) = 0$ by [49, 4_2], the cohomology Gysin sequence (Spanier [124, 5.7.11])

$$H^4(P) \longrightarrow H^1(P/N) \longrightarrow H^5(P/N)$$

yields a contradiction.

If $\mathfrak{h} \cong \mathfrak{o}_3\mathbb{R}$ then $\mathfrak{g}$ is isomorphic to one of

$$\mathbb{R}^j \times \mathfrak{o}_3\mathbb{R}^3\ (j \leq 1),\quad \mathbb{R}^j \times \mathfrak{su}_3\mathbb{C}\ (j \leq 1),\quad \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}\ (j \leq 2),$$

where $\mathbb{R}^3 \times \mathfrak{o}_4\mathbb{R}$ has been excluded by Proposition 4.3.9. If $\mathfrak{h} \cong \mathbb{R}^3$ then Theorem 2.5.5 shows that $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{o}_3\mathbb{R}^3$ or $\mathfrak{g} \cong \mathfrak{o}_3\mathbb{R}^3$. If $\mathfrak{h} \cong \mathbb{R}^2$ then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{su}_3\mathbb{C}$ with $j \leq 1$ or $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}$ with $j \leq 2$.

Suppose that $\mathfrak{h} \cong \mathbb{R}$. Then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}$ with $j \leq 2$ or $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R}$ with $j \leq 4$. Assume that $\mathfrak{g} \cong \mathbb{R}^4 \times \mathfrak{o}_3\mathbb{R}$. Then every principal orbit x^G

satisfies $x^G \approx {\mathbb{S}_1}^3 \times M$, where M is a three-dimensional homogeneous space of $\mathbb{T}\times\mathrm{SU}_2\mathbb{C}$. One shows that the possible homeomorphism types of M and cohomology modules of x^G are as follows:

M	$(\dim_{\mathbb{Z}/2} H^j(x^G))_{0\le j\le 6}$						
$\frac{\mathrm{SU}_2\mathbb{C}}{\mathbb{Z}/k}$, k odd	(1,	3,	3,	2,	3,	3,	1)
$\frac{\mathrm{SU}_2\mathbb{C}}{\mathbb{Z}/k}$, k even; $\mathbb{S}_1\times\mathbb{S}_2$; $\frac{\mathbb{S}_1\times\mathbb{S}_2}{\pm}$	(1,	4,	7,	8,	7,	4,	1)
$\mathbb{S}_1 \times P_2\mathbb{R}$	(1,	5,	11,	14,	11,	5,	1)

In particular, the principal orbit x^G is not simply connected, whence Proposition 4.3.2 shows that P/G is a compact interval. The Lie algebras of the two non-principal stabilizers are isomorphic to $\mathbb{R}$, $\mathbb{R}^2$, $\mathfrak{o}_3\mathbb{R}$, or $\mathbb{R}\times\mathfrak{o}_3\mathbb{R}$. In addition to the possible cohomology modules of x^G, the cohomology of a non-principal orbit ${y_i}^G$ can be as follows:

$(\dim_{\mathbb{Z}/2} H^j({y_i}^G))_{0\le j\le 6}$						
(1,	2,	1,	1,	2,	1,	0)
(1,	3,	4,	4,	3,	1,	0)
(1,	4,	7,	7,	4,	1,	0)
(1,	4,	6,	4,	1,	0,	0)
(1,	3,	3,	1,	0,	0,	0)

Investigation of the H^1- and H^2-columns yields a contradiction to Proposition 4.3.2.

Finally, suppose that $\mathfrak{h} = 0$. Then $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_4\mathbb{R}$ with $j \le 1$, $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R}$ with $j \le 3$, or $\mathfrak{g} \cong \mathbb{R}^j$, where $j \le 5$ by Proposition 4.3.6. This completes the proof for the case $m = 5$. □

4.3.11 Theorem. *Let G be a compact Lie group which acts almost effectively on a compact $(1,m)$-quadrangle $Q = P \,\dot\cup\, L$, and suppose that $\dim G > \binom{m-2}{2} + 11$. Then the Lie algebra of G is isomorphic to a subalgebra of $\mathbb{R}\times\mathfrak{o}_{m+2}\mathbb{R}$.*

Proof. We follow the same strategy as in the last proof. Assume, as we may, that the group G is connected. Let $H = G_x$ be a principal point stabilizer. By Theorem 4.2.15 and Lemma 4.3.1, we may assume that the action of H on L_x is not transitive. Moreover, this action is almost effective by Corollary 4.1.12. By the preceding theorem, we may assume that $m \geq 6$. Since

$$\dim H > \binom{m-2}{2} + 11 - (m+2) = \binom{m-3}{2} + 6,$$

we infer from the proof of Theorem 3.5.1 and from Theorem 3.5.4 that the Lie algebra $\mathfrak{h}$ of H is isomorphic to one of those in the following table.

$\mathfrak{o}_m\mathbb{R}$, $\mathbb{R}\times\mathfrak{o}_{m-1}\mathbb{R}$, $\mathfrak{o}_{m-1}\mathbb{R}$	$(m \geq 6)$
$\mathfrak{o}_3\mathbb{R}\times\mathfrak{o}_{m-2}\mathbb{R}$	$(m \geq 7)$
$\mathbb{R}\times\mathfrak{o}_{m-2}\mathbb{R}$	$(m \geq 9)$
$\mathfrak{o}_{m-2}\mathbb{R}$	$(m \geq 10)$
$\mathfrak{g}_2$	$(m = 7)$

We will work our way through this list, using Lemma 4.3.4 as an essential tool.

Suppose that $\mathfrak{h} \cong \mathfrak{o}_m\mathbb{R}$. By Lemma 4.3.7, the action of G' on P is not transitive, whence the derived algebra $\mathfrak{g}'$ satisfies

$$\dim \mathfrak{g}' \leq \dim \mathfrak{h} + m + 1 = \binom{m+1}{2} + 1.$$

In particular, this shows that $\dim \mathfrak{h} > \frac{1}{2}\dim \mathfrak{g}'$. Now $\mathfrak{h}$ is simple. Using the projections of $\mathfrak{g}$ onto its simple ideals, we find that $\mathfrak{h}$ must be included in one of these, say $\mathfrak{k}$. This inclusion is proper since the action of G on x^G is almost effective. Therefore, Lemma 4.3.4 implies that $\mathfrak{k} \cong \mathfrak{o}_{m+1}\mathbb{R}$, and $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_{m+1}\mathbb{R}$, where $j \leq 1$ by Lemma 4.3.7.

If $\mathfrak{h} \cong \mathbb{R}\times\mathfrak{o}_{m-1}\mathbb{R}$ then $\dim \mathfrak{g}' \leq \binom{m}{2} + 3 < 2\dim \mathfrak{o}_{m-1}\mathbb{R}$. As above, we find that $\mathfrak{h}' \cong \mathfrak{o}_{m-1}\mathbb{R}$ is properly contained in some simple ideal $\mathfrak{k}$ of $\mathfrak{g}$. By Lemma 4.3.4, we find that $\mathfrak{k} \cong \mathfrak{o}_m\mathbb{R}$, or $m = 7$ and $\mathfrak{k} \cong \mathfrak{su}_5\mathbb{C}$. In the first case, it follows from Theorem 2.5.5 that $\mathfrak{g} \cong \mathbb{R}\times\mathfrak{o}_3\mathbb{R}\times\mathfrak{o}_m\mathbb{R}$ or $\mathfrak{g} \cong \mathfrak{o}_3\mathbb{R}\times\mathfrak{o}_m\mathbb{R}$. Both isomorphism types are excluded by Proposition 4.3.8. It now suffices to exclude the case $m = 7$ and $G = \mathrm{SU}_5\mathbb{C}$. Since G does not act transitively on P, Theorem 2.5.5 entails that all stabilizers of non-fixed points are conjugate to $\mathrm{U}_4\mathbb{C}$, which is a self-normalizing subgroup of $\mathrm{SU}_5\mathbb{C}$. We apply Proposition 4.3.2. Since there is no subgroup of $\mathrm{SU}_5\mathbb{C}$ whose quotient by $\mathrm{U}_4\mathbb{C}$ is an integral homology sphere, the orbit space P/G is a circle. As $\mathrm{U}_4\mathbb{C}$ is a self-normalizing connected subgroup of G, the fibre bundle $P \to P/G$ is trivial, so that $P \approx \mathbb{S}_1 \times P_4\mathbb{C}$. But the cohomology modules of these spaces do not agree.

Suppose that $\mathfrak{h} \cong \mathfrak{o}_{m-1}\mathbb{R}$. Then $\dim \mathfrak{g}' \le \binom{m}{2}+2$, and $\mathfrak{h}$ is properly contained in some simple ideal $\mathfrak{k}$ of $\mathfrak{g}$, which must be isomorphic to $\mathfrak{o}_m\mathbb{R}$. The Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_m\mathbb{R}$, where $j \le 2$ by Lemma 4.3.7.

If $\mathfrak{h} \cong \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{m-2}\mathbb{R}$ then $m \ge 7$. Suppose first that $m \ge 8$. We find that $\dim \mathfrak{g}' \le \binom{m-1}{2} + 6 < 2\dim \mathfrak{o}_{m-2}\mathbb{R}$. There must be a simple ideal $\mathfrak{k}$ of $\mathfrak{g}$ which properly contains $\mathfrak{h}$. We apply Lemma 4.3.4. By Theorem 2.5.5, it is impossible that $m = 8$ and $\mathfrak{k} \cong \mathfrak{su}_5\mathbb{C}$. Hence $\mathfrak{k} \cong \mathfrak{o}_{m-1}\mathbb{R}$. Then Theorem 2.5.5 shows that $\mathfrak{g}$ contains $\mathfrak{o}_4\mathbb{R}\times\mathfrak{o}_{m-1}\mathbb{R}$, which contradicts Proposition 4.3.8. There are more possibilities for $m = 7$. If $\mathfrak{o}_5\mathbb{R}$ is not contained in an ideal of $\mathfrak{g}$, there must be at least two simple ideals into which $\mathfrak{o}_5\mathbb{R}$ projects non-trivially. This entails $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_5\mathbb{R}^2$ with $j \le 1$. But then the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ cannot contain $\mathfrak{o}_3\mathbb{R}$, a contradiction. Suppose that $\mathfrak{o}_5\mathbb{R}$ is properly contained in a simple ideal $\mathfrak{k}$ of $\mathfrak{g}$. The above argument excludes $\mathfrak{k} \cong \mathfrak{o}_6\mathbb{R}$, whence $\mathfrak{k} \cong \mathfrak{o}_7\mathbb{R}$ or $\mathfrak{k} \cong \mathfrak{u}_3\mathbb{H}$. In the first case, there is an embedding of $\mathfrak{g}$ into $\mathbb{R} \times \mathfrak{o}_7\mathbb{R}$. In the second case, the group $\mathrm{U}_3\mathbb{H}$ acts on P, and all stabilizers of non-fixed points must be conjugate to the self-normalizing subgroup $\mathrm{U}_1\mathbb{H}\times\mathrm{U}_2\mathbb{H}$ of $\mathrm{U}_3\mathbb{H}$ by Theorem 2.5.1. As above, we infer from Proposition 4.3.2 that $P \approx \mathbb{S}_1 \times P_2\mathbb{H}$, which contradicts cohomology.

If $\mathfrak{h} \cong \mathbb{R} \times \mathfrak{o}_{m-2}\mathbb{R}$ then $m \ge 9$ and

$$\dim \mathfrak{g}' \le \binom{m-1}{2} + 4 < 2\dim \mathfrak{o}_{m-2}\mathbb{R}.$$

Hence $\mathfrak{h}' \cong \mathfrak{o}_{m-2}\mathbb{R}$ is properly contained in a simple ideal $\mathfrak{k}$ of $\mathfrak{g}$, and $\mathfrak{k} \cong \mathfrak{o}_{m-1}\mathbb{R}$. This leads to $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{m-1}\mathbb{R}$ with $j \le 2$. By Theorem 2.5.5 and Lemma 3.1.7, the group which we must exclude is $G = \mathbb{T}^2 \times \mathrm{SU}_2\mathbb{C} \times \mathrm{SO}_{m-1}\mathbb{R}$. To achieve this, we will apply Proposition 2.4.7 twice. The action of G on P is transitive. Using Proposition 3.3.3, we may assume that the identity component of the stabilizer H is contained in $N_1 := 1 \times \mathbb{T} \times \mathrm{SU}_2\mathbb{C} \times \mathrm{SO}_{m-1}\mathbb{R}$. Then Proposition 4.3.2 implies that $H \cap N_1$ is connected. Let $N_2 := 1^3 \times \mathrm{SO}_{m-1}\mathbb{R}$. Then $H \cap N_2 = 1^3 \times \mathrm{SO}_{m-2}\mathbb{R}$. Hence P is a $\mathbb{S}_{m-2}$-bundle over the four-dimensional manifold $P/N_2 \approx G/HN_2$. Therefore, Lemma 2.4.1 leads to a contradiction.

Suppose that $\mathfrak{h} \cong \mathfrak{o}_{m-2}\mathbb{R}$. Then $\dim \mathfrak{g}' \le \binom{m-1}{2} + 3$. By the familiar arguments, we find that $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{m-1}\mathbb{R}$ with $j \le 1$ or $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_{m-1}\mathbb{R}$, where $j \le 2$ by Proposition 4.3.9.

Finally, suppose that $m = 7$ and that $\mathfrak{h} \cong \mathfrak{g}_2$. Then $\dim \mathfrak{g}' \le 22$, whence $\mathfrak{h}$ is contained in some simple ideal of $\mathfrak{g}$. Theorem 2.5.1 shows that this ideal is isomorphic to $\mathfrak{o}_7\mathbb{R}$. Hence $\mathfrak{g} \cong \mathbb{R}^j \times \mathfrak{o}_7\mathbb{R}$ with $j \le 1$ by Lemma 4.3.7. □

4.4 Reconstruction of actions

We show that every compact $(1,m)$-quadrangle on which a compact group of sufficiently large dimension acts is isomorphic to a real orthogonal quadrangle (up to duality if $m = 1$).

As in Section 4.2, we deal with the case $m = 1$ separately.

4.4.1 Theorem. *Let G be a compact connected group acting effectively on a compact $(1,1)$-quadrangle $Q = P \mathbin{\dot\cup} L$, and suppose $\dim G > 2$. Then Q is isomorphic to the real orthogonal quadrangle $Q(4,\mathbb{R})$ or to its dual, the real symplectic quadrangle $W(\mathbb{R})$, and the action of G on Q is equivalent to the natural action of either $\mathrm{SO}_3\mathbb{R}$ or $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$.*

Proof. Proposition 4.2.1 shows that G is a Lie group. By Proposition 4.1.4, the action of G on every principal point orbit p^G is effective. Assume that G is abelian. Then G acts freely on p^G, and $\dim G = \dim p^G \le 3$. Hence $\dim G = 3$ by hypothesis, and G acts transitively on P, which yields the exact sequence

$$1 = \pi_1(G_p) \longrightarrow \pi_1(G) \longrightarrow \pi_1(P).$$

By passing to the dual of Q, if necessary, we may assume $\pi_1(P) \cong \mathbb{Z}$ and $\pi_1(L) \cong \mathbb{Z}/2$, see Kramer [74, 3.4.11] or Grundhöfer, Knarr, and Kramer [49, 4_1]. But $G \cong \mathbb{T}^3$, whence $\pi_1(G) \cong \mathbb{Z}^3$. This contradiction shows that G is not abelian. Therefore, some subgroup H of G is covered by $\mathrm{Spin}_3\mathbb{R}$.

The result now follows from [74, 5.2.4]. We give a more detailed and slightly different proof.

The action of H on P cannot be transitive, because otherwise we would obtain an exact sequence

$$\pi_1(H) \longrightarrow \pi_1(P) \longrightarrow \pi_0(H_p)$$

whose outer terms are finite, while $\pi_1(P) \cong \mathbb{Z}$. Therefore, the dimension of every point stabilizer H_p is at least 1, whence H_p contains the centre of H. Fixing every point, the centre of H must be trivial, whence $H \cong \mathrm{SO}_3\mathbb{R}$. Moreover, the stabilizer H_p of an arbitrary point $p \in P$ acts almost effectively on L_p by Proposition 4.1.11. Since $L_p \approx \mathbb{S}_1$, the action of H_p on L_p is transitive. Lemma 4.1.1 shows that H acts transitively on L. The stabilizer H_l of a line $l \in L$ is discrete, and the exact sequence

$$1 = \pi_1(H_l) \longrightarrow \pi_1(H) \longrightarrow \pi_1(L) \longrightarrow \pi_0(H_l) \longrightarrow \pi_0(H) = 1$$

shows that H_l is connected, whence it is trivial.

The action of $H \cong \mathrm{SO}_3\mathbb{R}$ on L is thus sharply transitive. Therefore, every point stabilizer H_p is homeomorphic to L_p. Hence H_p is a one-parameter subgroup of H. One could now use the result of Dienst [33] to find that Q is isomorphic to $Q(4,\mathbb{R}) =: P' \mathbin{\dot\cup} L'$, cf. [49, 3.4]. We present an independent proof. Choose a line $l \in L$. Exactly as in the proofs of Lemma 4.2.14 and Theorem 4.2.15, we find a line $l' \in L'$ with $H_{l'} = H_l$, and such that for all points $p \in P_l$ there is a point $p' \in P'_{l'}$ with $H_p = H_{p'}$. Therefore, the sketch $(H, \{H_p | p \in P_l\}, H_l)$ of Q embeds into the sketch $(H, \{H_{p'} | p' \in P'_{l'}\}, H_{l'})$ of $Q(4,\mathbb{R})$. This yields an embedding ι of Q into $Q(4,\mathbb{R})$ which maps L homeomorphically onto L'. Hence ι is an H-equivariant isomorphism of topological quadrangles. The group G contains H and embeds into the maximal compact connected group of automorphisms of $Q(4,\mathbb{R})$, which is $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$. □

The next theorem treats the remaining parameters. It also gathers much of the information which we have obtained so far.

4.4.2 Theorem (Characterization of $Q(m+3,\mathbb{R})$). *Let G be a compact connected group acting effectively on a compact $(1,m)$-quadrangle $Q = P \mathbin{\dot\cup} L$, and let $d := \dim G$.*

(a) *If $d > \binom{m-1}{2} + 1$ then G is a Lie group and the point space P is a manifold.*

(b) *If $d > \binom{m-1}{2} + 4$ then every line pencil is homeomorphic to $\mathbb{S}_m$. The same conclusion already holds for $d > \binom{m-1}{2} + 2$ if $m = 7$ or $m \geq 9$.*

(c) *Suppose that G is a Lie group, and that at least one of the following three conditions holds:*

$$m \leq 4, \quad (m = 5 \text{ and } G \ncong \mathbb{T}^5), \quad d > \binom{m-2}{2} + 11.$$

Then the Lie algebra of G embeds into $\mathbb{R} \times \mathfrak{o}_{m+2}\mathbb{R}$.

(d) *If $d > \binom{m+1}{2} + 1$ ($d > 5$ if $m = 2$) then Q is isomorphic to the real orthogonal quadrangle $Q(m+3, \mathbb{R})$ (up to duality if $m = 1$), and the action of G on Q is equivalent to the action of either $\mathrm{SO}_{m+2}\mathbb{R}$ or $P(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{m+2}\mathbb{R})$ on $Q(m+3, \mathbb{R})$.*

Proof. If G is not a Lie group then $d \leq \binom{m-1}{2} + 1$ by Corollary 4.1.7. Suppose, then, that G is a Lie group, but that P is not a manifold. This implies that L_p is not a manifold for any point $p \in P$. Choose a point $p \in P$, and let $l \in L_p$ be a line through p whose orbit under $(G_p)^1$ is principal. By Corollary 2.1.6 and Proposition 4.1.11, the action of $(G_p)^1$ on the orbit $l^{(G_p)^1}$ is almost effective. Theorem 2.2.2 implies $\dim l^{(G_p)^1} \leq m - 3$ and $\dim p^G \leq m - 1$. Thus $\dim G_p \leq \binom{m-2}{2}$ by Theorem 2.1.7, and

$$\dim G = \dim G_p + \dim p^G \leq \binom{m-1}{2} + 1.$$

Suppose that the line pencils are not homeomorphic to $\mathbb{S}_m$. Then they are not manifolds. For arbitrary $p \in P$, we find as before that $\dim G_p \leq \binom{m-2}{2}$, whence $\dim G \leq \dim G_p + \dim P \leq \binom{m-1}{2} + 4$. Assume that $m \geq 7$, and that $\dim G > \binom{m-1}{2} + 2$. Then G is a Lie group. Moreover, we find that $\dim G_p > \binom{m-2}{2} - 2$. We have seen that $(G_p)^1$ acts almost effectively on the orbit $l^{(G_p)^1}$ of some line $l \in L_p$, and that $\dim l^{(G_p)^1} \leq m - 3$. In fact, Theorem 2.1.7 shows that $\dim l^{(G_p)^1} = m - 3$. Theorem 2.3.4 yields that $(G_p)^1$ is covered by $\mathrm{Spin}_{m-2}\mathbb{R}$. Now choose p in such a way that p^G is a principal orbit. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h} \leq \mathfrak{g}$ be the Lie algebra of G_p, so that $\mathfrak{h} \cong \mathfrak{o}_{m-2}\mathbb{R}$. In particular, $\dim \mathfrak{h} > \frac{1}{2} \dim \mathfrak{g}$, and $\mathfrak{h}$ is simple, whence there must be a simple ideal $\mathfrak{k}$ of $\mathfrak{g}$ which contains $\mathfrak{h}$. Moreover, Proposition 4.1.4 implies that $\mathfrak{h}$ is properly contained in $\mathfrak{k}$. Suppose that $m \neq 8$. Then Lemma 4.3.4 shows that $\mathfrak{k} \cong \mathfrak{o}_{m-1}\mathbb{R}$. Proposition 4.3.9 entails that $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{m-1}\mathbb{R}$ or $\mathfrak{g} \cong \mathfrak{o}_3\mathbb{R} \times \mathfrak{o}_{m-1}\mathbb{R}$.

It suffices to obtain a contradiction in the latter case. By Lemma 3.1.7, the group G is covered by $\tilde{G} = \mathrm{SU}_2\mathbb{C} \times \mathrm{SO}_{m-1}\mathbb{R}$, and we may assume that $(\tilde{G}_p)^1 = 1 \times \mathrm{SO}_{m-2}\mathbb{R}$. In particular, Proposition 4.3.2 applies. If the orbit space $P/\tilde{G}$ is a compact interval then this proposition implies that there is a point orbit of smaller dimension, which is impossible in our case since all point stabilizers are locally isomorphic. Hence $P/\tilde{G}$ is a circle, and $\tilde{G}_p$ is connected. When applied to the normal subgroup $N = 1 \times \mathrm{SO}_{m-1}\mathbb{R}$ of $\tilde{G}$, Proposition 2.4.7 shows that P is a $\mathbb{S}_{m-2}$-bundle over P/N, and P/N is a $\mathbb{S}_3$-bundle over $\mathbb{S}_1$. Lemma 2.4.1 yields that

$$H^*(P; \mathbb{Z}/2) \cong H^*(\mathbb{S}_1 \times \mathbb{S}_3 \times \mathbb{S}_{m-2}; \mathbb{Z}/2),$$

which is a contradiction to Grundhöfer, Knarr and Kramer [49, 4_2].

This proves parts (a) and (b) of the theorem. Part (c) is taken from Theorems 4.3.10 and 4.3.11.

To prove the last part, first note that G is a Lie group. We may assume that $m > 1$. Let G_p be a principal point stabilizer. Then $\dim G_p \geq \dim G - \dim P \geq \binom{m}{2}$. By Theorem 4.2.15 and Lemma 4.3.1, it suffices to prove that G_p acts transitively on L_p. If this is not the case then we use essentially the same arguments as above to infer that $\dim G_p = \binom{m}{2}$, that $(G_p)^1$ is in fact covered by $\mathrm{Spin}_m\mathbb{R}$, and that the Lie algebra of G is isomorphic to $\mathbb{R}^2 \times \mathfrak{o}_{m+1}\mathbb{R}$. If $m > 2$ then this contradicts Lemma 4.3.7. □

4.4.3 Remark. In part (a), it should be noted that it is more or less a coincidence that we have obtained the same upper bounds for actions of compact connected non-Lie groups and for actions of Lie groups on quadrangles whose point spaces are not manifolds.

If $m = 8$ then it is conceivable that part (b) cannot be improved. For the group $\mathrm{SO}_2\mathbb{R} \times \mathrm{SU}_5\mathbb{C}$ acts on $Q(11, \mathbb{R})$ in such a way that all point stabilizers are conjugate to $\mathrm{SU}_4\mathbb{C} \cong \mathrm{Spin}_6\mathbb{R}$. This group could act on a line pencil in such a way that all orbits are at most five-dimensional. (Of course, this is not the classical situation.) However, it might be possible to prove the second half of part (b) for $m \leq 6$.

For part (d), note that the usual action of $\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_{m+1}\mathbb{R}$ on $Q(m+3, \mathbb{R})$ is neither transitive on the points nor on the lines. Hence there is not much hope of reconstruction. If a compact connected group G of dimension 5 acts effectively on a compact $(1, 2)$-quadrangle then one can show that $G \cong (\mathrm{SO}_2\mathbb{R} \times \mathrm{U}_2\mathbb{C})/\langle(-1,-1)\rangle$, that G acts transitively on the point set, and that each line stabilizer acts transitively on the point row of the corresponding line. Thus in principle, reconstruction is possible, although it has not been obtained yet. Finally, note that we do not need the full force of Theorem 4.2.15 if we exploit the large dimension of G.

4.4.4 Remark. The group $\mathrm{SO}_m\mathbb{R}$ can act effectively on a non-classical compact $(1, m)$-quadrangle Q, where m is arbitrary. Indeed, the quadrangle Q can be chosen as a translation quadrangle which comes from the Tits construction, see Joswig [68, 1.37 and 2.23]. Let h be a hyperplane in real projective $(m+2)$-space $\mathrm{PG}_{m+2}\mathbb{R}$. Choose a closed ovoid O in h whose global stabilizer G in $(\mathrm{PGL}_{m+3}\mathbb{R})_h$ does not act transitively on O and contains a subgroup K which is isomorphic to $\mathrm{SO}_m\mathbb{R}$. Let Q be the compact translation quadrangle associated to O by the Tits construction. Then h is a point of Q, and the points of O form the line pencil of h in Q. Let H be the stabilizer of h in $\mathrm{Aut}\, Q$. It is immediate from the construction that G acts effectively on Q as a group of automorphisms which fix h. This yields an injective homomorphism $\varphi : G \to H$ which is compatible with the action. Every element of H is induced by a projective collineation, so that φ is an isomorphism of topological groups, see [68, 2.25]. Hence $K \cong \mathrm{SO}_m\mathbb{R}$ acts effectively on Q and fixes a point. As the action of H on the line pencil of h is not transitive, the quadrangle Q is not isomorphic to $Q(m + 3, \mathbb{R})$.

Since Q is a translation quadrangle, it is equipped with the action of a particularly large non-compact group. This suggests that other non-classical examples with parameters $(1, m)$ might admit the action of a compact group which is even larger than $\mathrm{SO}_m\mathbb{R}$.

Chapter 5

Compact $(4, m)$-quadrangles

As our last topic, we treat actions of compact groups on compact quadrangles whose topological parameters are larger than 1. The general results which we obtain are most fruitful for compact $(4, m)$-quadrangles. For all m for which such quadrangles are known to exist, we obtain sharp upper bounds for the group dimension. These upper bounds are the dimensions of the maximal compact automorphism groups of Moufang quadrangles. (If Stolz's result [125] carries over to quadrangles as expected then the parameters $(4, m)$ treated here cover all possible values.) If the group dimension is close to its upper bound, we show that the group is a Lie group, and that it acts transitively on the set of points, lines, or even flags. The flag-transitive actions have been determined by Grundhöfer, Knarr, and Kramer ([49] and [50]).Kramer [75] has described the homogeneous spaces whose integral cohomology coincides with $H^*(P; \mathbb{Z})$, where P is the point space of some compact (m, m')-quadrangle with $m, m' \geq 3$. He has also determined the point-transitive actions of compact connected groups on compact quadrangles for several series of topological parameters (see also Kramer [74]). These explicit classifications allow us to reconstruct the action of a compact group whose dimension is close to the maximal possible value. It turns out that the corresponding quadrangles are exactly the compact Moufang $(4, m)$-quadrangles.

5.1 Compact Lie groups

The theory of actions of elementary abelian groups, due to Smith [122] and others, yields rank restrictions for certain subgroups of compact Lie groups which act on compact quadrangles. This approach works particularly well if one of the topological parameters equals 1 or 4. For compact $(1, m)$-quadrangles, the techniques of Sections 4.3 and 4.4 have turned out to be even more powerful. In this section, we draw the consequences for compact $(4, m)$-quadrangles.

5.1.1 Proposition. *Let $\mathfrak{g}$ be a compact Lie algebra of rank at most r. If $\dim \mathfrak{g} > \binom{2r+1}{2}$ then either $\operatorname{rk} \mathfrak{g} = r$, and $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2$, to $\mathfrak{f}_4$, or to $\mathfrak{e}_7$, or $8 \le r \le 11$ and some ideal of $\mathfrak{g}$ is isomorphic to $\mathfrak{e}_8$.*

If $\dim \mathfrak{g} = \binom{2r+1}{2}$ then $\operatorname{rk} \mathfrak{g} = r$, and $\mathfrak{g}$ is isomorphic to $\mathfrak{b}_r$, to $\mathfrak{c}_r$, or to one of the following algebras:

$$\mathfrak{a}_1 \times \mathfrak{f}_4, \quad \mathfrak{e}_6, \quad \mathfrak{a}_1 \times \mathfrak{e}_7, \quad \mathbb{R}^2 \times \mathfrak{a}_1 \times \mathfrak{e}_8, \quad \mathfrak{f}_4 \times \mathfrak{e}_8.$$

Only the first part of this proposition will actually be used. The second part demonstrates that the isomorphism type of a Lie algebra can be determined if its dimension is large in relation to its rank. This is also illustrated by Table 5.1 on the next page. The regular pattern which evolves in the table shows that one can treat a wider range of Lie algebra dimensions if necessary.

Proof. We exploit the classification of compact Lie algebras. The list of simple compact Lie algebras can be found in Table 2.1 on page 50.

In the first part of the proof, assume that no ideal of $\mathfrak{g}$ is isomorphic to $\mathfrak{e}_8$. If $\mathfrak{g}$ is simple then inspection of the list yields that $\dim \mathfrak{g} < \dim \mathfrak{b}_r$ except for the cases described in the proposition. Obviously $\dim \mathbb{R} < \dim \mathfrak{b}_1$. In the remaining cases, let $\mathfrak{h}$ and $\mathfrak{k}$ be non-trivial complementary ideals of $\mathfrak{g}$, so that

$$\mathfrak{g} \cong \mathfrak{h} \times \mathfrak{k}.$$

Set $s := \operatorname{rk} \mathfrak{h}$ and $t := \operatorname{rk} \mathfrak{k}$. Then $s + t \le r$. If $\dim \mathfrak{h} \le \dim \mathfrak{b}_s$ and $\dim \mathfrak{k} \le \dim \mathfrak{b}_t$ then the middle inequality in the line

$$\dim \mathfrak{g} \le \binom{2s+1}{2} + \binom{2t+1}{2} < \binom{2(s+t)+1}{2} \le \dim \mathfrak{b}_r$$

Table 5.1: Large compact Lie algebras of given rank

r	$\dim \mathfrak{a}_r$	$\dim \mathfrak{b}_r$	$\dim \mathfrak{d}_r$	compact Lie algebras of rank at most r and of dimension at least $\binom{2r}{2}$, ordered by descending dimension; here $\mathfrak{x}$ stands for several appropriate small algebras
1	3			$\mathfrak{a}_1,\ \mathbb{R}$
2	8	10		$\mathfrak{g}_2,\ \mathfrak{b}_2,\ \mathfrak{a}_2,\ \mathfrak{a}_1 \times \mathfrak{a}_1$
3	15	21		$\mathfrak{b}_3,\ \mathfrak{c}_3,\ \mathfrak{a}_1 \times \mathfrak{g}_2,\ \mathfrak{a}_3,\ \mathbb{R} \times \mathfrak{g}_2$
4	24	36	28	$\mathfrak{f}_4,\ \mathfrak{b}_4,\ \mathfrak{c}_4,\ \mathfrak{d}_4,\ \mathfrak{g}_2 \times \mathfrak{g}_2$
5	35	55	45	$\mathfrak{b}_5,\ \mathfrak{c}_5,\ \mathfrak{a}_1 \times \mathfrak{f}_4,\ \mathbb{R} \times \mathfrak{f}_4,\ \mathfrak{f}_4,\ \mathfrak{d}_5$
6	48	78	66	$\mathfrak{b}_6,\ \mathfrak{c}_6,\ \mathfrak{e}_6,\ \mathfrak{d}_6,\ \mathfrak{g}_2 \times \mathfrak{f}_4$
7	63	105	91	$\mathfrak{e}_7,\ \mathfrak{b}_7,\ \mathfrak{c}_7,\ \mathfrak{d}_7$
8	80	136	120	$\mathfrak{e}_8,\ \mathfrak{b}_8,\ \mathfrak{c}_8,\ \mathfrak{a}_1 \times \mathfrak{e}_7,\ \mathbb{R} \times \mathfrak{e}_7,\ \mathfrak{e}_7,\ \mathfrak{d}_8$
9	99	171	153	$\mathfrak{a}_1 \times \mathfrak{e}_8,\ \mathbb{R} \times \mathfrak{e}_8,\ \mathfrak{e}_8,\ \mathfrak{b}_9,\ \mathfrak{c}_9,\ \mathfrak{d}_9$
10	120	210	190	$\mathfrak{x} \times \mathfrak{e}_8,\ \mathfrak{e}_8,\ \mathfrak{b}_{10},\ \mathfrak{c}_{10},\ \mathfrak{d}_{10}$
11	143	253	231	$\mathfrak{x} \times \mathfrak{e}_8,\ \mathfrak{b}_{11},\ \mathfrak{c}_{11},\ \mathfrak{x} \times \mathfrak{e}_8,\ \mathfrak{e}_8,\ \mathfrak{d}_{11}$
12	168	300	276	$\mathfrak{b}_{12},\ \mathfrak{c}_{12},\ \mathfrak{x} \times \mathfrak{e}_8,\ \mathfrak{d}_{12}$
13	195	351	325	$\mathfrak{b}_{13},\ \mathfrak{c}_{13},\ \mathfrak{d}_{13}$
14	224	406	378	$\mathfrak{b}_{14},\ \mathfrak{c}_{14},\ \mathfrak{d}_{14}$
15	255	465	435	$\mathfrak{b}_{15},\ \mathfrak{c}_{15},\ \mathfrak{d}_{15}$
16	288	528	496	$\mathfrak{b}_{16},\ \mathfrak{c}_{16},\ \mathfrak{d}_{16},\ \mathfrak{e}_8 \times \mathfrak{e}_8$
≥ 17	$r(r+2)$	$\binom{2r+1}{2}$	$\binom{2r}{2}$	$\mathfrak{b}_r,\ \mathfrak{c}_r,\ \mathfrak{d}_r$

can be checked directly. If one of the ideals, say $\mathfrak{k}$, has greater dimension then by induction on r, it is isomorphic to $\mathfrak{g}_2$, to $\mathfrak{f}_4$, or to $\mathfrak{e}_7$. If $\dim\mathfrak{h} \le \dim\mathfrak{b}_s$ then direct calculation shows that $\dim\mathfrak{g} \le \dim\mathfrak{b}_r$. This inequality is strict unless $\mathrm{rk}\,\mathfrak{g} = r$, the ideal $\mathfrak{h}$ is isomorphic to $\mathfrak{a}_1 \cong \mathfrak{b}_1$, and $\mathfrak{k}$ is isomorphic to $\mathfrak{f}_4$ or to $\mathfrak{e}_7$. If also $\dim\mathfrak{h} > \dim\mathfrak{b}_t$ then the following table, which contains the dimensions of the possible products $\mathfrak{h}\times\mathfrak{k}$, shows that $\dim\mathfrak{g} < \dim\mathfrak{b}_r$.

	$\mathfrak{g}_2$	$\mathfrak{f}_4$	$\mathfrak{e}_7$
$\mathfrak{g}_2$	28	66	147
$\mathfrak{f}_4$		104	185
$\mathfrak{e}_7$			266

Suppose that $\mathfrak{g}$ contains ideals which are isomorphic to $\mathfrak{e}_8$. We use induction on the number n of such ideals. Suppose that $n = 1$, so that

$$\mathfrak{g} \cong \mathfrak{h} \times \mathfrak{e}_8,$$

where $\mathfrak{h}$ does not contain $\mathfrak{e}_8$ as an ideal. If $\dim\mathfrak{h} > \dim\mathfrak{b}_{r-8}$ then $\mathfrak{h}$ is isomorphic to $\mathfrak{g}_2$, to $\mathfrak{f}_4$, or to $\mathfrak{e}_7$, and we find that

$$\begin{aligned} \dim(\mathfrak{g}_2\times\mathfrak{e}_8) &> \dim\mathfrak{b}_{10} \\ \dim(\mathfrak{f}_4\times\mathfrak{e}_8) &= \dim\mathfrak{b}_{12} \\ \dim(\mathfrak{e}_7\times\mathfrak{e}_8) &< \dim\mathfrak{b}_{15}. \end{aligned}$$

Suppose that $\dim\mathfrak{h} \le \dim\mathfrak{b}_{r-8}$. Then calculation shows that $\dim\mathfrak{g} < \dim\mathfrak{b}_r$ if $r > 11$. If $\dim\mathfrak{g} = \dim\mathfrak{b}_r$ then $r = 11$. Hence $\mathrm{rk}\,\mathfrak{h} \le 3$ and $\dim\mathfrak{h} = \dim\mathfrak{b}_{11} - \dim\mathfrak{e}_8 = 5$, which implies that $\mathfrak{h} \cong \mathbb{R}^2 \times \mathfrak{a}_1$.

Suppose that $n \ge 2$, so that

$$\mathfrak{g} \cong \mathfrak{h} \times \mathfrak{e}_8 \times \mathfrak{e}_8.$$

Observe that $\dim{\mathfrak{e}_8}^2 = 496 = \dim\mathfrak{d}_{16}$. Applying induction to the Lie algebra $\mathfrak{h}\times\mathfrak{d}_{16}$ whose dimension equals that of $\mathfrak{g}$, we find that $\dim\mathfrak{g} < \dim\mathfrak{b}_r$. □

5.1.2 Theorem (Kramer and Van Maldeghem [78, 4.1]). *Let Q be a compact (m,m')-quadrangle, and let Q' be a proper closed connected subquadrangle of Q whose topological parameters are (k,k'). If $k = m$ then $k + k' \le m'$.* □

Note that Q' has the property that it contains the full point row of each of its lines. Such a subquadrangle is called 'full'.

The following result is a generalization of Proposition 4.1.11.

5.1.3 Proposition (Elementary abelian groups fixing all points on a line). *Let $Q = P \cup L$ be a compact (m, m')-quadrangle on which $G = (\mathbb{Z}/p)^r$ acts effectively, where p is a prime. Suppose that G fixes an open subset U of some point row P_l elementwise. Then G fixes an ordinary quadrangle. If $p = 2$ then $r \leq \frac{m'-1}{m} + 1$. If $p > 2$ then an upper bound for r is given in the following table.*

	m odd	m even
m' odd	$r \leq \dfrac{m'-1}{m+1}$	$r \leq \dfrac{m'-1}{m}$
m' even	$r \leq \dfrac{m'-2}{m+1} + 1$	$r \leq \dfrac{m'-2}{m} + 1$

If $p > 2$ and m' is odd then $\operatorname{Fix} G$ *is an (m, m'_0)-subquadrangle of Q whose second parameter m'_0 satisfies $m'_0 \leq m' - rm$.*

Proof. By the Smith–Floyd Theorem 4.1.9, the fixed point set of the action of G on a generalized sphere has the mod p Čech homology of a sphere. In particular, the fixed point set cannot consist of a single point. This shows that whenever G fixes a flag (x, l), it fixes a second point on l and a second line through x. Continuing the argument, we find that G fixes an ordinary quadrangle.

Choose a point $x \in U$. For any subgroup $H \leq G$, we use the cohomological version of the Smith–Floyd Theorem 4.1.9 (cf. Bredon [14, III.7.11]) to define $m'(H)$ as the integer between -1 and m' such that $L_x \cap \operatorname{Fix} H$ is a mod p Čech cohomology $m'(H)$-sphere. Then Borel has proved in [8, XIII.2.3] that

$$m' - m'(G) = \sum_{[G:H]=p} \big(m'(H) - m'(G)\big).$$

(Using Čech and sheaf cohomology simultaneously is justified since both agree by Bredon [16, III.4.12].) As G fixes U as well as an ordinary quadrangle in Q, the action of G on L_x is effective. Therefore, this action is not trivial unless $G = 1$. It follows from [8, V.2.6] that $m'(G) < m'$. Hence if $G \neq 1$ then there is a subgroup $H_1 < G$ of index p such that $m'(G) < m'(H_1)$. We continue by induction: if $H_i \neq 1$ then there is a subgroup $H_{i+1} < H_i$ whose index is p and which satisfies $m'(H_i) < m'(H_{i+1})$. Thus we obtain a strictly decreasing chain

$$G > H_1 > H_2 > \cdots > H_r = 1$$

of subgroups such that the chain

$$\operatorname{Fix} G \subset \operatorname{Fix} H_1 \subset \operatorname{Fix} H_2 \subset \cdots \subset \operatorname{Fix} H_r = Q$$

is strictly increasing. All these fixed subgeometries contain an ordinary quadrangle, the set U, and all except possibly $\operatorname{Fix} G$ contain at least three lines through the point x. Therefore, each $\operatorname{Fix} H_i$ is a compact (m, m'_i)-quadrangle for $m'_i := m'(H_i) > 0$; the first parameter is always m because $\dim U = m$. By Theorem 5.1.2, this implies that $m \leq m'_{i+1} - m'_i$, for all i. If $p = 2$ then we conclude that

$$(r-1)m \leq m'_r - m'_1 = m' - m'_1 \leq m' - 1,$$

whence $r \leq \frac{m'-1}{m} + 1$.

For $p > 2$, this argument can be refined. The Smith–Floyd Theorem [14, III.7.11] shows that $m' - m'_i$ and $m' - m'(G)$ are even numbers. This is useful in several ways: if m' is odd then $\operatorname{Fix} G$ is a compact (m, m'_0)-quadrangle for $m'_0 := m'(G) > 0$, and the above argument shows that $m'_0 \leq m'_1 - m \leq m' - rm$. If m' is even then $m'_1 \geq 2$. Finally, if m is odd then $m + 1 \leq m'_{i+1} - m'_i$. These facts combine to yield the upper bounds on r which are contained in the table. □

5.1.4 Remark. Note from the proof that there is a maximal subgroup H_1 of G such that $\operatorname{Fix} H_1$ is an (m, m'_1)-subquadrangle of Q whose second parameter satisfies $m'_1 \leq m' - (r-1)m$. If $p > 2$ and m is odd then $m'_1 \leq m' - (r-1)(m+1)$.

Smith [123, no. 4] describes a slightly longer but more elementary way to construct the chain $G > H_1 > \cdots > H_r = 1$ of subgroups.

5.1.5 Lemma. *A compact Lie group with Lie algebra $\mathfrak{e}_6$ cannot act effectively on a compact $(4, 19)$-quadrangle.*

Proof. Let G be a simply connected compact Lie group with $\mathfrak{l}(G) \cong \mathfrak{e}_6$, and suppose that G acts almost effectively on a compact $(4, 19)$-quadrangle $Q = P \mathbin{\dot\cup} L$. Then $\dim G = 78$ and $\dim P = 27$. We infer from Table 3.1 on page 78 that there is no compact Lie group of rank at most 6 and dimension $51 = \dim G - \dim P$. Therefore, the action of G on P cannot be transitive. By Theorem 2.5.1, every non-trivial point orbit has codimension 1. According to Remark 3.1.5, Mostert's Theorem entails that there are exactly two non-principal point orbits, and that their dimensions are not maximal. Hence these two orbits are fixed points, and every principal orbit p^G is an integral homology sphere. This contradicts the classification of homogeneous cohomology spheres in Theorem 3.1.1. Thus the proof of the lemma is complete. □

5.1.6 Lemma. *Let G be a compact connected Lie group acting effectively on a compact $(4, 4n-5)$-quadrangle $Q = P \mathbin{\dot\cup} L$, where $n \in \{6, 7\}$. Suppose that there is a closed connected subgroup $H \leq G$ which fixes all points on some line $l \in L$ and satisfies $\mathfrak{l}(H) \cong \mathfrak{f}_4$.*

If $n = 6$ then $\dim G \leq 72$, and H is a normal subgroup of G.

If $n = 7$ then $\dim G \leq 88$.

Proof. Suppose first that H is a normal subgroup of G. Then G is the almost direct product of H with its centralizer $C := C_G(H)$. Fix an embedding $(\mathbb{Z}/3)^4 \hookrightarrow H$. (Here 3 could be replaced by any odd prime.) Proposition 5.1.3 shows that $Q' := \mathrm{Fix}(\mathbb{Z}/3)^4$ is a compact connected subquadrangle of Q whose second parameter is at most $4n - 21$. Now Q' is invariant under C, so that

$$\dim l^C \leq \dim(L \cap Q') \leq 4 + 2(4n - 21) = 8n - 38.$$

Let $C_{[P_l]}$ denote the kernel of the action of C_l on P_l. Then the dimension of $C_l/C_{[P_l]}$ is at most 10, as can be seen from Theorem 2.1.7 and the Montgomery–Yang Theorem 2.2.3. Finally, observe that $C_{[P_l]}$ is the centralizer of H in $G_{[P_l]}$. Since $\operatorname{rk} G_{[P_l]} \leq n - 2$ by Proposition 5.1.3, this implies that $\dim C_{[P_l]} = 0$ if $n = 6$, and $\dim C_{[P_l]} \leq 3$ if $n = 7$. We conclude that

$$\begin{aligned} \dim G &= \dim H + \dim l^C + \dim \frac{C_l}{C_{[P_l]}} + \dim C_{[P_l]} \\ &\leq \begin{cases} 72 & \text{if } n = 6 \\ 83 & \text{if } n = 7. \end{cases} \end{aligned}$$

Suppose that H is not normal in G. As a preliminary step, we establish an upper bound for $\dim G$. Since $\dim(G_l/G_{[P_l]}) \le 10$ and $\operatorname{rk} G_{[P_l]} \le n-2$, we infer from Proposition 5.1.1 that

$$\begin{aligned}\dim G &= \dim l^G + \dim \frac{G_l}{G_{[P_l]}} + \dim G_{[P_l]} \\ &\le \begin{cases} 104 & \text{if } n = 6 \\ 115 & \text{if } n = 7. \end{cases}\end{aligned}$$

Under the projections of $\mathfrak{l}(G)$ onto its simple ideals, the image of the simple subalgebra $\mathfrak{l}(H) \cong \mathfrak{f}_4$ is either trivial or again isomorphic to $\mathfrak{f}_4$. Representation theory shows that $\mathfrak{f}_4$ cannot be embedded into a classical simple Lie algebra of dimension at most 115, see Tits [136]. As $\dim P \le 31$, Theorem 2.5.5 shows that the Lie algebra $\mathfrak{l}(G)$ cannot contain ${\mathfrak{f}_4}^2$. We conclude that $\mathfrak{l}(H)$ must be contained in an ideal $\mathfrak{j}$ of $\mathfrak{l}(G)$ which is isomorphic to $\mathfrak{e}_6$. By the preceding lemma, this implies that $n = 7$. We infer from Kramer's classification [75, 3.15] that the action of G on P cannot be transitive. Let $\mathfrak{k}$ be the ideal of $\mathfrak{l}(G)$ which complements $\mathfrak{j}$. Since G acts effectively on each principal point orbit p^G by Proposition 4.1.4 and $\dim p^G \le 30$, we infer from Theorem 2.5.5 that $\mathfrak{k}$ has a subalgebra whose codimension is at most 4 and which does not contain a non-trivial ideal of $\mathfrak{k}$. Then the same theorem entails that $\dim \mathfrak{k} \le 10$, whence $\dim G \le 88$. □

5.1.7 Remark. We could have avoided Kramer's result [75, 3.15] at this point. Admitting a point-transitive action in the last part of the proof, we would only have found that $\dim G \le 93$ if $n = 7$. Nevertheless, this weaker result would be sufficient for our purposes in this chapter. By this modification, the present section would become independent of Kramer's deep work, and so is the next section. The situation will be different in the final section of this chapter when Kramer's classification will be crucial.

5.1.8 Theorem. *Let G be a compact Lie group acting effectively on a compact $(4, 4n-5)$-quadrangle $Q = P \cup L$. Then $\dim G \le d_{\max}$, where*

$$d_{\max} := \begin{cases} \binom{2n+1}{2} + 10 & \text{if } n \ne 4 \\ \binom{2n+1}{2} + 14 = 50 & \text{if } n = 4. \end{cases}$$

If $\dim G = d_{\max}$ *then* G *acts flag-transitively on* Q*. If* $\dim G > d_{\max} - 4$ *then each line stabilizer* G_l *acts transitively on the point row* P_l*. The same conclusion holds for* $\dim G = d_{\max} - 4$ *unless* G *acts transitively on* L*.*

Transitivity of the action of G_l on P_l for all $l \in L$ means that G acts transitively on P, and that the quadrangle Q is 'sketched' by the group G in the sense of Stroppel [130].

If $n \neq 4$ then the upper bound $d_{\max}$ on $\dim G$ is attained for the quaternion hermitian quadrangles $H(n+1, \mathbb{H})$. For $n = 4$, a sharp upper bound on the group dimension will be obtained in Theorem 5.3.1.

Proof. Choose an arbitrary line $l \in L$. Then

$$\begin{aligned} \dim G &= \dim l^G + \dim G_l \\ &= \dim l^G + \dim \frac{G_l}{G_{[P_l]}} + \dim G_{[P_l]}. \end{aligned}$$

The dimension of l^G is bounded above by $\dim L = 4 + 2(4n-5) = 8n - 6$. The group $H := G_l/G_{[P_l]}$ acts effectively on the generalized 4-sphere P_l. By Theorem 2.1.7, its dimension is at most 10 if the action is transitive. Using also Lemma 1.3.4 and the Montgomery–Yang Theorem 2.2.3, we find that $\dim H \le 6$ if the action is non-transitive. Proposition 5.1.3 shows that

$$\operatorname{rk} G_{[P_l]} \le \left\lfloor \frac{4n-6}{4} \right\rfloor = n - 2.$$

This will allow us to apply Proposition 5.1.1.

First, we show that the theorem holds if $\dim G_{[P_l]} \le \binom{2n-3}{2}$. Indeed, this condition implies that

$$\dim G \le 8n - 6 + 10 + \binom{2n-3}{2} = \binom{2n+1}{2} + 10.$$

Equality can only hold if we have $\dim l^G = \dim L$ and $\dim H = 10$ for every line $l \in L$. Hence G acts transitively on L, and every line stabilizer acts transitively on the corresponding point row, so that the action of G on the space of flags is transitive. If $\dim G > \binom{2n+1}{2} + 6$ then we can still conclude that $\dim H = 10$, whence every line stabilizer

G_l acts transitively on P_l. If $\dim G = \binom{2n+1}{2} + 6$ and G_l does not act transitively on P_l then $\dim H \leq 6$, and G must act transitively on L.

We treat the exceptions which occur in Proposition 5.1.1. If the Lie algebra $\mathfrak{l}(G_{[P_l]})$ is isomorphic to $\mathfrak{f}_4$ then either $\dim G \leq \binom{2n-3}{2}$ or $n = 6$. In the latter case, Lemma 5.1.6 shows that $\dim G \leq 72$. Suppose that $\mathfrak{l}(G_{[P_l]}) \cong \mathfrak{e}_7$. Proposition 5.1.3 entails that $G_{[P_l]}$ acts effectively on L_p whenever p is a point on l. Theorem 2.5.1 implies that $4n - 5 = \dim L_p \geq 54$, whence $n \geq 15$. Similarly, if $\mathfrak{l}(G_{[P_l]})$ has got an ideal which is isomorphic to $\mathfrak{e}_8$ then $4n - 5 \geq 112$, whence $n \geq 30$. In both cases, we infer that $\dim G_{[P_l]} \leq \binom{2n-3}{2}$. Hence by Proposition 5.1.1, we may assume that $\dim G_{[P_l]} \leq \binom{2n-3}{2}$ unless $n = 4$ and $\mathfrak{l}(G_{[P_l]}) \cong \mathfrak{g}_2$. In this case, we find that

$$\dim G \leq 8n - 6 + 14 + \binom{2n-3}{2} = \binom{2n+1}{2} + 14 = 50.$$

As above, we find that each line stabilizer G_l acts transitively on the corresponding point row P_l if $\dim G > 46$, and also if $\dim G = 46$ unless G acts transitively on L. $\square$

5.1.9 Theorem. *Let G be a compact Lie group acting effectively on a compact $(4, 5)$-quadrangle $Q = P \cup L$. Then $\dim G \leq 27$. If $\dim G > 23$ then G acts flag-transitively on Q. If $\dim G = 23$ then one of the following statements holds.*

(i) *The group G acts transitively both on P and on L.*

(ii) *For each point $p \in P$, the stabilizer G_p acts transitively on the line pencil L_p.*

(iii) *For each line $l \in L$, the stabilizer G_l acts transitively on the point row P_l.*

In Theorem 5.3.3, we will obtain a sharp upper bound on the group dimension. It will be crucial that a group must act flag-transitively even if its dimension is a little lower than the upper bound obtained here.

Proof. We first prove that a point stabilizer G_p whose dimension is at least 11 acts transitively on the line pencil L_p. Let $p \in P$ be a point with $\dim G_p \geq 11$, and suppose that the action of G_p on

L_p is not transitive. The dual of Proposition 5.1.3 implies that the kernel $G_{[L_p]}$ of this action has dimension at most 3. Therefore, the dimension of the effective group $G_p/G_{[L_p]}$ is at least 8. We infer from Theorem 3.5.1 that the Lie algebra of this group is isomorphic to $\mathfrak{o}_5\mathbb{R}$. Hence G_p has a closed connected normal subgroup H with $\mathfrak{l}(H) \cong \mathfrak{o}_5\mathbb{R}$. By Theorem 3.6.11, the action of $H/H_{[L_p]}$ on L_p is equivalent to the suspension of the transitive action of $\mathrm{SO}_5\mathbb{R}$ on $\mathbb{S}_4$. Therefore, there is a line l through p which is fixed by H. The action of H on the point row P_l is trivial since H fixes the point $p \in P_l$. This contradicts the fact that $\operatorname{rk} G_{[P_l]} \leq 1$ by Proposition 5.1.3.

Now we repeat some arguments from the proof of Theorem 5.1.8. For any line $l \in L$, we find that

$$\dim G = \dim l^G + \dim \frac{G_l}{G_{[P_l]}} + \dim G_{[P_l]} \leq 14 + 10 + 3 = 27.$$

If the dimension of $G_l/G_{[P_l]}$ is greater than 6 then the action of G_l on the point row P_l is transitive. This follows for all lines if $\dim G > 23$, and also if $\dim G = 23$ and G does not act transitively on L. If it holds for all lines then G acts transitively on P by the dual of Lemma 4.1.1. If $\dim G > 23$ then for every point $p \in P$, we find that $\dim G_p \geq \dim G - \dim P \geq 11$, whence the action of G_p on L_p is transitive. The same conclusion holds if $\dim G = 23$ and the action of G on P is not transitive. $\square$

5.2 Compact non-Lie groups

The idea for the treatment of compact connected non-Lie groups is that the dimensions of the centre and of the commutator group should balance each other. The centre acts almost effectively on each point orbit of maximal dimension. We prove some geometric properties of orbits which culminate in a classification of orbit types in Proposition 5.2.4. The commutator group is a Lie group, and it is therefore accessible to the techniques of the previous section. If a compact non-Lie group acts effectively on a compact $(4, m)$-quadrangle then our main results show that its dimension must just be smaller than the dimensions for which Lie groups were successfully treated in the first section.

5.2.1 Lemma. *Let A be a non-empty locally closed subset of a cohomology manifold X over a principal ideal domain R. Suppose that A*

is locally homogeneous as a topological space. Then $\dim_R A = \dim_R X$ *if and only if* A *is open in* X*.*

Proof. Suppose that $\dim_R A = \dim_R X$. Bredon [16, V.16.18] shows that A contains an open subset U of X. Now U is a cohomology manifold over R, hence so is A, by local homogeneity. Invariance of domain [16, V.16.19] yields that A is open in X. Conversely, the result [16, V.16.18] also shows that an open subset of X has full dimension. $\square$

5.2.2 Lemma. *Let* $Q = P \cup L$ *be a compact* (m, m')*-quadrangle, and let* $A \subseteq P$ *be a compact subset which is locally homogeneous as a topological space. (For example, the subset* A *could be a group orbit.) Suppose that no two points of* A *are collinear, and that* $\dim A \geq m + m'$*. Then* A *is an ovoid. If, in particular, the subset* A *is a homogeneous space of some compact group then* $A \approx \mathbb{S}_{m+m'}$*.*

Proof. Assume that A is not an ovoid, and choose a line l whose point row does not meet A. Then

$$p \longmapsto \lambda(p, l) : A \longrightarrow D_2(l)$$

is an embedding of A onto some compact subset $A' \subseteq D_2(l)$. Moreover,

$$l' \longmapsto l' \wedge l : D_2(l) \longrightarrow P_l$$

is the projection in a locally trivial fibre bundle, which shows that $D_2(l)$ is a cohomology $(m + m')$-manifold over $\mathbb{Z}$. Therefore, the preceding lemma shows that A' is an open subset of $D_2(l)$. Now A' is compact and $D_2(l)$ is connected, whence $A' = D_2(l)$. This contradicts the fact that $D_2(l)$ is not compact.

Kramer and Van Maldeghem [78, 3.1f.] have shown that every compact ovoid in Q is a cohomology $(m + m')$-manifold over $\mathbb{Z}$ and homotopy equivalent to $\mathbb{S}_{m+m'}$. Suppose that A is a homogeneous space of some compact group. Since P is metrizable, Theorems 2.2.2 and 3.1.1 show that $A \approx \mathbb{S}_{m+m'}$. $\square$

5.2.3 Proposition. *Let* Z *be a compact abelian group acting effectively on a compact* (m, m')*-quadrangle* $Q = P \cup L$*. Then*

$$\dim Z \leq \begin{cases} \dim P - 1 & \textit{if } \dim P \leq 4 \\ \dim P - 2 & \textit{if } \dim P > 4 \end{cases}$$

Proof. Corollary 4.1.7 yields that $\dim Z \leq \dim P - 1$, and that $\dim Z \leq \dim P - 3$ if Z is not a Lie group. We may suppose that Z is connected. Therefore, it suffices to assume that $Z \cong \mathbb{T}^{\dim P-1}$, and to deduce that $\dim P \leq 4$. By Proposition 4.1.4, the codimension of a principal point orbit is 1. Hence we can apply Mostert's Theorem [94]. The orbit space P/G is a compact interval. If $m = 1$ then this follows from Proposition 4.3.2 because there are no simply connected orbits, and if $m > 1$ then it holds since P/G is compact and simply connected (see Grundhöfer, Knarr, and Kramer [49, Appendix] and Montgomery and Yang [90, Corollary 2]). Every principal point orbit x^Z is homeomorphic to ${\mathbb{S}_1}^{\dim P-1}$. There are exactly two non-principal point orbits ${y_0}^Z$ and ${y_1}^Z$, each of which is homeomorphic to either ${\mathbb{S}_1}^{\dim P-2}$ or ${\mathbb{S}_1}^{\dim P-1}$. Using the homotopy equivalences

$$x^Z \simeq P \setminus ({y_0}^Z \cup {y_1}^Z), \quad {y_0}^Z \simeq P \setminus {y_1}^Z, \quad \text{and} \quad {y_1}^Z \simeq P \setminus {y_0}^Z$$

and the Mayer–Vietoris sequence of the triple $(P, P \setminus {y_0}^Z, P \setminus {y_0}^Z)$, see Bredon [15, V.8.3], we obtain the exact sequence

$$H^1(P; \mathbb{Z}/2) \longrightarrow H^1({y_0}^Z; \mathbb{Z}/2) \oplus H^1({y_1}^Z; \mathbb{Z}/2) \longrightarrow H^1(x^Z; \mathbb{Z}/2).$$

Now $\dim_{\mathbb{Z}/2} H^1(P; \mathbb{Z}/2) \leq 1$ (see [49, Appendix] and use the Universal Coefficient Theorem, Spanier [124, 5.5.10]). The Künneth Theorem (see [15, VI.3.2]) shows that $\dim_{\mathbb{Z}/2} H^1({\mathbb{S}_1}^n; \mathbb{Z}/2) = n$. Hence the exact sequence shows that

$$\begin{aligned} 2(\dim P - 2) &\leq \dim_{\mathbb{Z}/2}\big(H^1({y_0}^Z; \mathbb{Z}/2) \oplus H^1({y_1}^Z; \mathbb{Z}/2)\big) \\ &\leq \dim_{\mathbb{Z}/2} H^1(P; \mathbb{Z}/2) + \dim_{\mathbb{Z}/2} H^1(x^Z; \mathbb{Z}/2) \\ &\leq \dim P, \end{aligned}$$

whence $\dim P \leq 4$. □

The following proposition refines the description of orbit types due to Stroppel and Stroppel [128, 3.2 and 3.8].

5.2.4 Proposition. *Let Z be a compact connected abelian group acting on a compact (m, m')-quadrangle $Q = P \cup L$. Choose a non-trivial point orbit $p^Z \subseteq P$.*

If the orbit p^Z contains collinear points then the following (mutually exclusive) cases are possible.

(i) *There is a unique line $l \in L \cap \operatorname{Fix} Z$ such that $p^Z \subseteq P_l$, and*

$$\dim p^Z \leq \max\left\{m-3, \frac{m+1}{2}\right\}.$$

(ii) *There is a unique point $q \in P$ such that $p^Z \subseteq D_2(q)$. Then $q \in \operatorname{Fix} Z$, and*

$$\dim p^Z \leq \max\left\{m-3, \left\lfloor\frac{m+1}{2}\right\rfloor\right\} + \max\left\{m'-3, \left\lfloor\frac{m'+1}{2}\right\rfloor\right\}.$$

(iii) *The orbit p^Z generates a grid, i.e. $\langle p^Z\rangle$ contains an ordinary quadrangle, and for arbitrary $q \in P \cap \langle p^Z\rangle$, the intersection $L_q \cap \langle p^Z\rangle$ consists of exactly two lines, say l_1 and l_2. In this case,*

$$\dim p^Z \leq \dim(P_{l_1} \cap \langle p^Z\rangle) + \dim(P_{l_2} \cap \langle p^Z\rangle),$$

and $\dim p^Z \leq 2m-1$ unless $m=1$.

(iv) *The orbit p^Z generates a subquadrangle of Q. If (k,k') denotes the pair of topological parameters of the closure $\overline{\langle p^Z\rangle}$ then*

$$\dim p^Z \leq \begin{cases} 2k+k'-1 & \text{if } 2k+k' \leq 4 \\ 2k+k'-2 & \text{if } 2k+k' > 4 \end{cases}$$

If no two points of p^Z are collinear then $\dim p^Z \leq m+m'-1$, and for every line l through p, exactly one of the following cases arises.

(v) *The subset $p^Z \cup \{l\} \subseteq Q$ generates a grid, and*

$$\dim p^Z \leq \max\{m-1, 1\}.$$

(vi) *There is a point $q \in P_l$ such that $p^Z \subseteq D_2(q)$, and*

$$\dim p^Z \leq \max\{m'-1, 1\}.$$

(vii) *The subset $p^Z \cup \{l\} \subseteq Q$ generates a subquadrangle of Q. If (k,k') denotes the pair of topological parameters of the closure $\overline{\langle p^Z \cup \{l\}\rangle}$ then*

$$\dim p^Z \leq k+k'-1.$$

Proof. Suppose first that there is a line l which joins two points of p^Z. We may assume that l goes through p. Note that l is fixed by the stabilizer Z_p. If $p^Z \subseteq P_l$ then $l \in \operatorname{Fix} Z$. Consider the action of Z on P_l. If $\dim p^Z > m-3$ then $Z/Z_{[P_l]}$ is a Lie group by Theorem 2.2.2, and Smith's rank restriction 1.3.6 shows that $\dim p^Z \leq \frac{m+1}{2}$.

Suppose that p^Z is not contained in P_l, and that there is a point $q \in P$ such that $p^Z \subseteq \{q\} \cup D_2(q)$. Then $\{q\} = \pi((p^Z \setminus P_l) \times \{l\})$. This shows that q is determined uniquely by p^Z. In particular, it is a fixed point of Z. Hence L_q is invariant under Z. As above, we find that the dimension of $l^Z \leq L_q$ is at most $\max\{m'-3, \frac{m'+1}{2}\}$. Since also $\dim p^{Z_l} \leq \max\{m-3, \frac{m+1}{2}\}$, we obtain the upper bound for

$$\dim p^Z = \dim Z - \dim Z_p = \dim Z - \dim Z_{p,l} = \dim l^Z + \dim p^{Z_l}$$

which has been given in the statement.

Suppose that neither of these two cases arises. Then $R := \langle p^Z \rangle$ contains an ordinary quadrangle. If R is a subquadrangle of Q then the upper bounds for $\dim p^Z$ are given by Proposition 5.2.3. Otherwise, either $|L_q \cap R| \leq 2$ for each $q \in P$ or, dually, $|P_l \cap R| \leq 2$ for each $l \in L$. In other words, the subgeometry R is either a grid or a dual grid. Recall from Section 1.4 that the point space of a dual grid can be written as a disjoint union of two relatively closed non-empty subsets each of which consists of pairwise opposite points. Therefore, a dual grid cannot be generated by a connected set of points. Hence R is a grid. Let l_1 and l_2 be the two lines of R which run through p, and recall that

$$P \cap R \approx (P_{l_1} \cap R) \times (P_{l_2} \cap R),$$

where both direct factors are compact. The product inequality for (small inductive or covering) dimension (see Salzmann et al. [115, 92.10]) implies that

$$\dim p^Z \leq \dim(P_{l_1} \cap R) + \dim(P_{l_2} \cap R).$$

In particular, this shows that $\dim p^Z \leq 2m$. Suppose that equality holds, and recall that $P_{l_i} \cap R$ is homeomorphic to $D_2(l_i) \cap R$, which is an orbit under Z. Since both spaces $P_{l_i} \cap R$ are compact, Lemma 5.2.1 shows that they are open in the respective point rows P_{l_i}. As the point rows P_{l_1} and P_{l_2} are connected, they must be contained in R. Hence

$P \cap R \approx P_{l_1} \times P_{l_2}$ is a cohomology manifold over Z, and it contains p^Z as a subset of full dimension. Using Lemma 5.2.1 once more, we find that the action of Z on R is transitive. By Theorem 2.2.2, this implies that $R \approx {\mathbb{S}_1}^{2m}$, and we conclude that $m = 1$.

In the second part of the proof, suppose that no two points of the orbit p^Z are collinear, and let $l \in L_p$ be an arbitrary line through p. Suppose first that $R' := \langle p^Z \cup \{l\} \rangle$ is a subquadrangle of Q, and let (k, k') be the topological parameters of its closure $\overline{R'}$. Since p^Z cannot be homeomorphic to $\mathbb{S}_{k+k'}$, Lemma 5.2.2 implies that $\dim p^Z \leq k + k' - 1$. Suppose that R' is not a subquadrangle of Q. Then the continuous map

$$\begin{array}{rcl} \varphi : p^Z \setminus \{p\} & \longrightarrow & P_l \\ p^z & \longmapsto & \pi(p^z, l) \end{array}$$

is either injective or constant. In the first case, the set R' is a grid. Then $p^Z \approx R' \cap D_2(l)$, and there is an embedding

$$l' \longmapsto l' \wedge l : R' \cap D_2(l) \longrightarrow P_l.$$

We infer from Lemma 5.2.1 and Theorem 2.2.2 that $\dim p^Z \leq \max\{m-1, 1\}$. If the image of φ consists of a single point $q \in P_l$ then the map

$$p^z \longmapsto p^z \vee q : p^Z \longrightarrow L_q$$

is an embedding, and we find that $\dim p^Z \leq \max\{m' - 1, 1\}$. $\square$

5.2.5 Lemma. *Let G be a compact group which acts effectively on a finite-dimensional compact generalized polygon $P \cup L$. Set*

$$d := \max\{\dim x^G \mid x \in P\},$$

and choose a point $p \in P$ such that $\dim Z(G)_p = 0$. Then the following inequalities hold:

$$\begin{array}{rcl} 2\dim Z(G) + \dim p^{\overline{G'}} & \leq & 2d \\ \dim Z(G) + \dim p^{\overline{G'}} & \leq & \dfrac{3}{2}d \end{array}$$

Note that $d \leq \dim P - 3$ if G is a non-Lie group, or if G is contained in a compact non-Lie Group which also acts effectively (Theorem 2.2.2).

The existence of a point $p \in P$ with $\dim Z(G)_p = 0$ follows from Theorem 4.1.5, which also shows that the dimension of G is finite. If G is connected then the commutator group G' is automatically closed by van Kampen's Theorem 1.1.3; see Hofmann and Morris [57, E 6.6] for disconnected groups.

Proof. The structure theory of compact groups yields that $Z(G)G'$ contains G^1, and that the intersection $Z(G) \cap \overline{G'}$ is totally disconnected (Hofmann and Morris [57, 9.23]). In particular, this shows that

$$\begin{aligned} \dim G - \dim Z(G) &= \dim Z(G)\overline{G'} - \dim Z(G) \\ &= \dim \frac{Z(G)\overline{G'}}{Z(G)} \\ &= \dim \frac{\overline{G'}}{Z(G) \cap \overline{G'}} \\ &= \dim \overline{G'}. \end{aligned}$$

Applying this to G_p and observing that $\overline{(G_p)'} \leq (\overline{G'})_p$, we find that

$$\begin{aligned} \dim Z(G_p) &= \dim G_p - \dim \overline{(G_p)'} \\ &\geq \dim G_p - \dim (\overline{G'})_p \\ &= \dim G - \dim p^G - \dim \overline{G'} + \dim p^{\overline{G'}} \\ &\geq \dim Z(G) - d + \dim p^{\overline{G'}}. \end{aligned}$$

As $Z(G) \cap Z(G_p) = Z(G)_p$ is totally disconnected, the abelian subgroup $Z(G)Z(G_p)$ of G has dimension $\dim Z(G) + \dim Z(G_p)$, and it acts almost freely on some point orbit by Theorem 4.1.5. Hence

$$\dim Z(G) + \dim Z(G_p) \leq d.$$

Combining these inequalities, we find that

$$\dim Z(G) - d + \dim p^{\overline{G'}} \leq \dim Z(G_p) \leq d - \dim Z(G),$$

which implies that

$$2 \dim Z(G) + \dim p^{\overline{G'}} \leq 2d.$$

Adding the inequality $\dim p^{\overline{G'}} \leq d$ and dividing by 2, we obtain that

$$\dim Z(G) + \dim p^{\overline{G'}} \leq \frac{3}{2}d.$$

$\square$

5.2.6 Theorem (Borel et al. [8, IX.2.2]). *Let G be a compact connected Lie group acting non-trivially on a cohomology n-manifold X over $\mathbb{Z}$. Then the dimension of any orbit is at most $n - 1 - \dim \operatorname{Fix} G$.* $\square$

In fact, the result which we have quoted is stronger. Let $X_{\leq t}$ be the union over all orbits of dimension at most t, where t is smaller than the maximal orbit dimension. Then the dimension of an arbitrary orbit is at most $n - 1 + t - \dim X_{\leq t}$.

5.2.7 Theorem. *Let G be a compact non-Lie group acting effectively on a compact $(4, 4n - 5)$-quadrangle $Q = P \cup L$. Then $\dim G \leq d_{\max} - 5$, where*

$$d_{\max} := \begin{cases} \binom{2n+1}{2} + 10 & \text{if } n \neq 4 \\ \binom{2n+1}{2} + 14 = 50 & \text{if } n = 4. \end{cases}$$

Note that the bound for $\dim G$ given here fits nicely with the bounds of Theorem 5.1.8.

Proof. The dimension of G is finite by Theorem 4.1.5. Let Z be the identity component of the centre of G^1, and let S be the commutator group of G^1. Van Kampen's Theorem 1.1.3 shows that S is a Lie group, and that G^1 is the almost direct product of Z and S. We will first prove that $\dim S \leq d_{\max} - 7$. This implies the theorem if $\dim Z \leq 2$. Choose a line l such that l^S is a principal orbit under S. Theorem 2.2.2 shows that $\dim l^S \leq \dim L - 3 = 8n - 9$. Combining this theorem with the dual of Lemma 4.3.1, we find that S_l does not act transitively on P_l. By Theorem 2.1.7 and the Montgomery–Yang Theorem 2.2.3, this implies that $\dim(S_l/S_{[P_l]}) \leq 6$. Hence

$$\begin{aligned} \dim S &= \dim l^S + \dim \frac{S_l}{S_{[P_l]}} + \dim S_{[P_l]} \\ &\leq 8n - 9 + 6 + \dim S_{[P_l]} \\ &= 8n - 3 + \dim S_{[P_l]}. \end{aligned}$$

Proposition 5.1.3 yields that $\operatorname{rk} S_{[P_l]} \leq n-2$. If $\dim S_{[P_l]} \leq \binom{2n-3}{2}$ then

$$\dim S \leq 8n - 3 + \binom{2n-3}{2} = \binom{2n+1}{2} + 3.$$

Suppose that $\dim S_{[P_l]} > \binom{2n-3}{2}$. According to Proposition 5.1.1, there are only finitely many possibilities for the isomorphism type of $\mathfrak{l}(S_{[P_l]})$. If $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{g}_2$ then $n = 4$, and $\dim S \leq 8n - 3 + 14 = 43$. If $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{f}_4$ then $n = 6$, and $\dim S \leq 72$ by Lemma 5.1.6. Suppose that $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{e}_7$. Then we have seen in the proof of Theorem 5.1.8 that $n \geq 15$. Similarly, if some ideal of $\mathfrak{l}(S_{[P_l]})$ is isomorphic to $\mathfrak{e}_8$ then $\operatorname{rk} S_{[P_l]} \leq 11$ and $n \geq 30$. In both cases, we find a contradiction to the assumption that $\dim S_{[P_l]} > \binom{2n-3}{2}$. Thus we have shown that $\dim S \leq d_{\max} - 7$, which proves the theorem in the case that $\dim Z \leq 2$.

In the general case, we first infer from Corollary 2.1.17 that there is an open subset U of P such that the Z-orbits of the points in U have maximal dimension, and for all points $p \in U$, Theorem 4.1.5 shows that $(Z_p)^1$ is trivial. By the Montgomery–Yang Theorem 2.2.3, the set of points on principal S-orbits is dense in P. Therefore, we may choose a point $p \in P$ such that $(Z_p)^1 = 1$ and p^S is a principal orbit under S. In particular, this implies that $\dim Z = \dim p^Z$. The remainder of the proof consists of two parts, depending on whether or not the orbit p^Z contains collinear points.

Part A Suppose that p^Z contains collinear points.

Step A.1 Choice of $l \in L_p$.

Among the lines which join p to other points of p^Z, choose l such that $\dim(P_l \cap \overline{\langle p^Z \rangle})$ is as large as possible. Let $r := \operatorname{rk} S_{[P_l]}$, and fix an embedding $(\mathbb{Z}/3)^r \hookrightarrow S_{[P_l]}$. (Here 3 could be replaced by any odd prime.) Let $Q' := \operatorname{Fix}(\mathbb{Z}/3)^r$. Then Q' is invariant under Z. Proposition 5.1.3 shows that $r \leq n-2$, and that Q' is a compact connected subquadrangle of Q whose second parameter is at most $4(n-r)-5$.

Step A.2 Claim: $\dim Z + \dim \frac{S_p}{S_{[P_l]}} \leq 4(n-r)+1$.

The stabilizer S_p fixes $\langle p^Z \rangle$, whence $l \in \operatorname{Fix} S_p$, and S_p acts on P_l. In view of Lemma 1.3.5 and the Montgomery–Yang Theorem 2.2.3, Richardson [111] has shown that the action of $(S_p)^1$ on P_l is equivalent to a linear action on $\mathbb{S}_4$ unless the dimension of $S_p/S_{[P_l]}$ is at most 1.

The classification of compact orbits in Proposition 5.2.4 yields that S_p fixes at least three points on l. We infer that $\dim(S_p/S_{[P_l]}) \leq 3$. If $n=2$ then $r=0$, and $\dim G = \dim Z + \dim p^S + \dim S_p$. Lemma 5.2.5 shows that $\dim Z + \dim p^S \leq 12$, which implies the theorem in this case. For the remainder of this part of the proof, we may therefore assume that $n \geq 3$.

We consider the orbit types described by Proposition 5.2.4. If p^Z is contained in a fixed line of Z then $\dim Z \leq 2$. If all points of p^Z are collinear to a fixed point of Z then

$$\dim Z \leq \max\{4(n-r)-6,\ 2(n-r)\}.$$

Suppose that $\langle p^Z \rangle$ is a grid, and set $k := \dim(P_l \cap \overline{\langle p^Z \rangle})$. As we have chosen l such that k is as large as possible, the inequality $\dim Z \leq \min\{7, 2k\}$ holds. Now the dimension of a principal S_p-orbit in P_l is at most $3-k$ by Theorem 5.2.6, so that Theorem 2.1.7 and the Montgomery–Yang Theorem 2.2.3 yield that

$$\dim \frac{S_p}{S_{[P_l]}} \leq \binom{4-k}{2}.$$

In particular, if $\langle p^Z \rangle$ is a grid then $\dim Z + \dim(S_p/S_{[P_l]}) \leq 7$. Suppose that $\langle p^Z \rangle$ is a subquadrangle of Q', and let (k, k') be the pair of topological parameters of $\overline{\langle p^Z \rangle}$. Then $k' \leq 4(n-r)-5$. If $k=1$ then $\dim Z \leq 4(n-r)-5$. If $k=2$ then $\dim Z \leq 4(n-r)-3$ and $\dim(S_p/S_{[P_l]}) \leq 1$. If $k \geq 3$ then $\dim Z \leq 4(n-r)+1$ and $\dim(S_p/S_{[P_l]}) = 0$. We conclude that the inequality

$$\dim Z + \dim \frac{S_p}{S_{[P_l]}} \leq 4(n-r)+1$$

holds no matter of what type the orbit p^Z is. Moreover, the codimension of p^S in P is at least 3 by Theorem 2.2.2, whence $\dim p^S \leq 4n$. Combining these facts, we obtain that

$$\begin{aligned} \dim G &= \dim Z + \dim p^S + \dim \frac{S_p}{S_{[P_l]}} + \dim S_{[P_l]} \\ &\leq 8n - 4r + 1 + \dim S_{[P_l]} \end{aligned}$$

Step A.3 Suppose that $\dim S_{[P_l]} \leq \binom{2r+1}{2}$.

In this case, the relation

$$\dim G \leq 8n - 4r + 1 + \binom{2r+1}{2}$$

holds, and we claim that

$$8n - 4r + 1 + \binom{2r+1}{2} \leq \binom{2n+1}{2} + 5.$$

Indeed, straightforward calculation shows that this inequality is equivalent to

$$\left(r - \frac{3}{4}\right)^2 \leq \left(n - \frac{7}{4}\right)^2 - \frac{1}{2} \ ,$$

which is equivalent to

$$\frac{1}{2} \leq \left(n + r - \frac{5}{2}\right)\left(n - r - 1\right).$$

This holds since $n \geq 3$ and $n - r \geq 2$.

Step A.4 Suppose that $\dim S_{[P_l]} > \binom{2r+1}{2}$.

There are only finitely many isomorphism types of the Lie algebra $\mathfrak{l}(S_{[P_l]})$, which are described by Proposition 5.1.1. Suppose first that $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{g}_2$. Then $n \geq 4$, and

$$\dim G \leq 8n - 8 + 1 + 14 = 8n + 7 \leq \binom{2n+1}{2} + 5.$$

Suppose that $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{f}_4$. If $n = 6$ then $\dim S \leq 72$ by Lemma 5.1.6, and $\dim Z \leq \dim Z + \dim(S_p/S_{[P_l]}) \leq 9$, whence $\dim G \leq 81$. Otherwise $n > 6$, and

$$\dim G \leq 8n - 16 + 1 + 52 = 8n + 37 \leq \binom{2n+1}{2} + 5.$$

If $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{e}_7$ then $n \geq 15$, as we have seen in the proof of Theorem 5.1.8. Therefore,

$$\dim G \leq 8n - 28 + 1 + 133 = 8n + 106 \leq \binom{2n+1}{2} + 5.$$

Similarly, if some ideal of $\mathfrak{l}(S_{[P_l]})$ is isomorphic to $\mathfrak{e}_8$ then $n \geq 30$ and $8 \leq r \leq 11$, whence $\dim \mathfrak{l}(S_{[P_l]}) \leq 21 + 248 = 269$, and

$$\dim G \leq 8n - 32 + 1 + 269 = 8n + 238 \leq \binom{2n+1}{2} + 5.$$

This completes the proof of the theorem in the case that the orbit p^Z contains collinear points.

Part B Suppose that no two points of p^Z are collinear.

Step B.1 Choice of $l \in L_p$.

Choose $l \in L_p$ such that $\dim l^{S_p}$ is as small as possible. The action of S on L is not transitive (Theorem 2.2.2). As p^S is a principal S-orbit, Lemmas 4.1.1 and 4.3.1 show that the action of S_p on L_p is not transitive. By Lemma 1.3.4 and Mostert's Theorem 3.1.2, the codimension of the orbit of l under S_p in L_p is at least 2, i.e. $\dim l^{S_p} \leq 4n - 7$. As above, let $r := \operatorname{rk} S_{[P_l]}$, note that $r \leq n - 2$, and fix an embedding $(\mathbb{Z}/3)^r \hookrightarrow S_{[P_l]}$. Then $Q' := \operatorname{Fix}(\mathbb{Z}/3)^r$ is invariant under Z, and it is a subquadrangle of Q whose second parameter is at most $4(n - r) - 5$.

Step B.2 Claim: $\dim Z + \dim \frac{S_{p,l}}{S_{[P_l]}} \leq 4(n - r) - 2$.

The case $\dim Z \leq 2$ has been treated at the beginning of this proof. Suppose that $\dim Z \geq 3$, and set

$$H := \frac{S_{p,l}}{S_{[P_l]}} \quad .$$

To prove our claim, first note that $\dim Z \leq 4(n - r) - 2$ by Proposition 5.2.4, and $\dim H \leq 6$ by Theorem 2.1.7 and the Montgomery–Yang Theorem 2.2.3. Hence our claim holds if $\dim Z \leq 4(n - r) - 8$.

We consider the possibilities for $R := \langle p^Z \cup \{l\} \rangle$ which have been described in Proposition 5.2.4. Suppose that R is a grid. Then $\dim Z \leq 3$, and $k := \dim(P_l \cap \overline{R})$ is strictly positive. As above, we infer from Theorem 5.2.6 that $\dim H \leq \binom{4-k}{2} \leq 3$. Hence $\dim Z + \dim H \leq 6$.

Suppose that there is a point $q \in P_l$ such that $p^Z \subseteq D_2(q)$. Then $\dim Z \leq 4(n - r) - 6$ by Proposition 5.2.4, whence $n - r \geq 3$ because $\dim Z \geq 3$. Moreover, we may suppose that $\dim Z \geq 4(n - r) - 7$. Choose a point $q' \in P_l \setminus \{p, q\}$. We will show that $S_{p,l,q'}/S_{[P_l]}$ is totally disconnected. Suppose that this is not the case. Then there is an embedding of $\mathbb{Z}/5$ into $S_{p,l,q'}$ whose image is not contained in $S_{[P_l]}$. (Again, we could replace 5 by any odd prime; we avoid 3 since it has been used before.) Now $\mathrm{Fix}(\mathbb{Z}/5)$ contains p^Z, l, q, q', and an ordinary quadrangle, the latter by the Smith–Floyd Theorem 4.1.9. Hence $\mathrm{Fix}(\mathbb{Z}/5)$ is a compact subquadrangle of Q', and it is connected since it contains the connected set p^Z and hence cannot be totally disconnected. Let (k, k') be the pair of topological parameters of $\mathrm{Fix}(\mathbb{Z}/5)$. By the Smith–Floyd Theorem 4.1.9, the difference $\dim(L_p \cap Q') - k'$ is even, whence k' is odd. Moreover, Lemma 5.2.1 entails that $k' > \dim Z$. We infer that $k' = 4(n-r) - 5$. Applying the dual of Theorem 5.1.2 to the inclusion of $\mathrm{Fix}(\mathbb{Z}/5)$ into Q', we find that $k = 4$. In particular, the group $\mathbb{Z}/5$ fixes every point on l. This contradiction shows that $S_{p,l,q'}/S_{[P_l]}$ is indeed totally disconnected. We conclude that $\dim H = \dim (q')^{S_{p,l}} \leq 3$.

Finally, suppose that the set R generated by $p^Z \cup \{l\}$ is a subquadrangle of Q'. Let (k, k') be the pair of topological parameters of the closure $\overline{R}$. Then $\dim H \leq \binom{4-k}{2}$, and $k' \leq 4(n-r) - 5$. If $k \geq 3$ then $\dim H = 0$. If $k = 2$ then $\dim H \leq 1$ and $\dim Z \leq 4(n-r) - 4$. If $k = 1$ then $\dim H \leq 3$ and $\dim Z \leq 4(n-r) - 5$. This completes the proof of our claim that $\dim Z + \dim H \leq 4(n-r) - 2$.

As before, the dimension of p^S is at most $4n$, whence

$$\begin{aligned} \dim G &= \dim Z + \dim p^S + \dim l^{S_p} + \dim H + \dim S_{[P_l]} \\ &\leq 12n - 4r - 9 + \dim S_{[P_l]}. \end{aligned}$$

Step B.3 Suppose that $\dim S_{[P_l]} \leq \binom{2r+1}{2}$.

The theorem is implied by the inequality

$$12n - 4r - 9 + \binom{2r+1}{2} \leq \binom{2n+1}{2} + 5.$$

In turn, this inequality is equivalent to

$$0 \leq \left(n + r - \frac{7}{2}\right)\left(n - r - 2\right),$$

which holds whenever $r \leq n-2$ unless $r = 0$ and $n = 3$.

Step B.4 Suppose that $r = 0$ and $n = 3$.

Then $\dim S_{[P_l]} = 0$, whence $\dim S_{p,l} = \dim H$. Above, we have seen that $\dim Z \leq 10$ and $\dim S_{p,l} \leq 6$. Moreover, if $\dim Z \geq 5$ then $\dim S_{p,l} \leq 3$. The dimension of S_p satisfies

$$\dim S_p = \dim l^{S_p} + \dim S_{p,l} \leq 5 + 6 = 11.$$

Suppose that equality holds. Then $\dim S_{p,l} = 6$, whence $\dim Z \leq 4$. We will show that $\dim Z \leq 3$. Set $K := (S_p)^1$. As K_l acts almost effectively on the generalized 4-sphere P_l and fixes p, its Lie algebra satisfies $\mathfrak{l}(K_l) \cong \mathfrak{o}_4\mathbb{R}$. The kernel $K_{[L_p]}$ of the action of K on L_p has rank at most 1 by the dual of Proposition 5.1.3, and it is a normal subgroup of K and hence of K_l. This shows that $\dim K_{[L_p]} \in \{0, 3\}$, whence $\dim(K/K_{[L_p]}) \in \{11, 8\}$. Since L_p is a generalized 7-sphere, we infer from Theorem 3.5.1 that $\mathfrak{l}(K/K_{[L_p]})$ is isomorphic to one of the following Lie algebras:

$$\mathbb{R} \times \mathfrak{o}_5\mathbb{R}, \quad \mathfrak{su}_3\mathbb{C}, \quad \mathbb{R}^2 \times \mathfrak{o}_4\mathbb{R}.$$

The case $\mathfrak{l}(K/K_{[L_p]}) \cong \mathfrak{su}_3\mathbb{C}$ will lead to a contradiction to our choice of l. Namely, all orbits of K in L_p have dimension at least $\dim l^K = \dim l^{S_p} = 5$. Richardson [111, 1.3] shows that there must be an orbit of dimension at least 6, and there cannot be an orbit of dimension 7 because K does not act transitively on L_p. Hence Mostert's Theorem 3.1.2 applies. The dimension of a principal stabilizer in $K/K_{[L_p]}$ is 2, and the dimension of a non-principal stabilizer must be 3, which is impossible. We conclude that $\mathfrak{l}(K/K_{[L_p]}) \not\cong \mathfrak{su}_3\mathbb{C}$. In particular, the centre of K has positive dimension. Let N be the normalizer of K_l in K. Then $Z(K) \leq N$, and K_l is contained in the commutator subgroup of K, so that $\dim N > \dim K_l$. Hence the orbit $l^N \subseteq L_p$ has positive dimension, and $l^N \subseteq \operatorname{Fix} K_l$. Now suppose that $\dim Z = 4$, and consider $\langle p^Z \cup \{l\}\rangle \subseteq \operatorname{Fix} K_l$. As $\dim K_l = 6$, this set cannot be a subquadrangle of Q. By Proposition 5.2.4, there is a point $q \in P_l$ such that $p^Z \subseteq D_2(q)$. Choose a point $q' \in P_l \setminus \{p, q\}$, and let $Q'' := \operatorname{Fix} K_{l,q'}$. Since Q'' contains a non-trivial connected subset of L_p, the orbit p^Z, and the three points p, q, and q' on l, it is a connected subquadrangle of Q. Its second parameter k' satisfies $k' > \dim Z$. For any line $l' \in L_p \setminus Q''$, Theorem 5.2.6 yields that $\dim(l')^{K_{l,q'}} \leq 6 - k' \leq 1$. The subgeometry $\operatorname{Fix} K_{l,q',l'}$ is a subquadrangle whose second parameter is at least 6, and the same argument shows that the identity component $(K_{l,q',l'})^1$ acts trivially on L_p. By the dual of Theorem 5.1.2, this implies that $\dim K_{l,q',l'} = 0$, which yields the contradiction

$$6 = \dim K_l = \dim(q')^{K_l} + \dim(l')^{K_{l,q'}} \leq 3 + 1 = 4.$$

Hence if $\dim S_p = 11$ then $\dim Z \leq 3$.

Finally, Lemma 5.2.5 shows that $\dim Z + \dim p^S \leq 18$. We have seen that $\dim p^S \leq 12$ and that $\dim S_p \leq 11$. If $\dim Z = 4$ then $\dim S_p \leq 10$. If $\dim Z \geq 5$ then $\dim S_{p,l} \leq 3$, whence $\dim S_p = \dim l^{S_p} + \dim S_{p,l} \leq 8$. Putting all this into the formula

$$\dim G = \dim Z + \dim p^S + \dim S_p,$$

we find that $\dim G \leq 26$. This proves the theorem if $r = 0$ and $n = 3$, so that we are done with the case that $\dim S_{[P_l]} \leq \binom{2r+1}{2}$.

Step B.5 Suppose that $\dim S_{[P_l]} > \binom{2r+1}{2}$.

We proceed exactly as in the first part. The possible isomorphism types of $\mathfrak{l}(S_{[P_l]})$, whose number is finite, are described by Proposition 5.1.1. Suppose that $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{g}_2$. If $n = 4$ then

$$\dim G \leq 12n - 4r - 9 + \dim S_{[P_l]} = 12n - 3 = 45$$

as was stated in the theorem. Otherwise $n \geq 5$, and

$$\dim G \leq 12n - 3 \leq \binom{2n+1}{2} + 5.$$

Suppose that $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{f}_4$. If $n = 6$ then $\dim Z \leq \dim Z + \dim H \leq 4(n-r) - 2 = 6$, whence $\dim G \leq 78$ by Lemma 5.1.6. If $n = 7$ then the same lemma yields that $\dim G \leq 98$ since $\dim Z \leq 10$. Otherwise $n \geq 8$, and

$$\dim G \leq 12n - 16 - 9 + 52 = 12n + 27 \leq \binom{2n+1}{2} + 5.$$

If $\mathfrak{l}(S_{[P_l]}) \cong \mathfrak{e}_7$ then $n \geq 15$, and

$$\dim G \leq 12n - 28 - 9 + 133 = 12n + 96 \leq \binom{2n+1}{2} + 5.$$

Finally, if some ideal of $\mathfrak{l}(S_{[P_l]})$ is isomorphic to $\mathfrak{e}_8$ then $n \geq 30$ and $8 \leq r \leq 11$, whence $\dim \mathfrak{l}(S_{[P_l]}) \leq 21 + 248 = 269$, and

$$\dim G \leq 12n - 32 - 9 + 269 = 12n + 228 \leq \binom{2n+1}{2} + 5.$$

This finishes the proof of the theorem in the case that no two points of the orbit p^Z are collinear. Thus we have completed the proof. $\square$

One might expect that the upper bound for the group dimension grows tighter if the dimension of the centre is larger. For the parameters $(4, 3)$, this is a part of the following refinement of Theorem 5.2.7.

5.2.8 Theorem. *Let G be a compact non-Lie group acting effectively on a compact $(4, 3)$-quadrangle $Q = P \cup L$. Then $\dim G \leq 14$, and $\dim Z(G^1) \leq 7$.*

If $\dim Z(G^1) \geq 4$ then $\dim G \leq 13$.
If $\dim Z(G^1) = 6$ then $\dim G \leq 12$.
If $\dim Z(G^1) = 7$ then $\dim G \leq 10$.

Proof. Those parts of this proof which are close to the proof of Theorem 5.2.7 will be given with little detail.

The identity component G^1 of G is the almost direct product of $Z := Z(G^1)^1$ and the commutator subgroup $S := (G^1)'$, which is a semi-simple Lie group. As there is a line $l \in L$ such that $(Z_l)^1 = 1$, Theorem 2.2.2 shows that $\dim Z \leq 7$. If $\dim Z \leq 1$ then the theorem follows from the fact that $\dim S \leq 13$.

Suppose that $\dim Z \geq 2$, and choose a point $p \in P$ such that $(Z_p)^1 = 1$ and p^S is a principal orbit under S. Then $\dim p^Z = \dim Z$, and $\dim p^S \leq 8$. Lemma 5.2.5 yields that $\dim Z + \dim p^S \leq 12$.

Suppose first that p^Z contains collinear points. Among the lines which join p to other points of p^Z, choose l such that the dimension of $P_l \cap \overline{\langle p^Z \rangle}$ is as large as possible. Proposition 5.1.3 shows that $\dim S_{[P_l]} = 0$. Considering the almost effective action of S_p on P_l, we infer that $\dim S_p \leq 3$, whence $\dim S = \dim p^S + \dim S_p \leq 11$. Therefore, we may assume that $\dim Z \geq 4$. Then p^Z cannot be contained in P_l.

Suppose that all points of p^Z are collinear to $q \in P \cap \operatorname{Fix} Z$. Then $\dim Z = \dim l^Z + \dim p^{Z_l}$. Combining Theorem 2.2.2 and Smith's rank restriction 1.3.6 as in the proof of Proposition 5.2.4, we find that both summands are at most 2. Hence $\dim Z = 4$, and $\dim p^{Z_l} = 2$. Since $p^{Z_l} \subseteq P_l \cap \operatorname{Fix} S_p$, this implies that $\dim S_p \leq 1$, whence $\dim S \leq 9$ and $\dim G \leq 13$.

Assume that $\langle p^Z \rangle$ is a grid, and set $k := \dim(P_l \cap \overline{\langle p^Z \rangle})$. Then $\dim Z \leq \min\{2k, 7\}$, and $\dim S_p \leq \binom{4-k}{2}$. If $\dim Z \geq 5$ then $\dim S_p =$

0, and $\dim G \le 12$. Otherwise $\dim Z = 4$, and $\dim S_p \le 1$, so that $\dim G \le 13$.

Suppose that $\langle p^Z \rangle$ is a subquadrangle of Q, and let (k, k') denote the topological parameters of its closure. Then $\dim Z \ge 4$ implies that $k \ge 2$. If $k = 2$ then $k' \le 2$ by the dual of Theorem 5.1.2, whence $\dim Z = 4$, and $\dim S_p \le 1$. This leads to $\dim G \le 13$. Finally, if $k \ge 3$ then $\dim S_p = 0$, and $\dim G \le 12$.

Suppose now that no two points of p^Z are collinear, and let l be a line through p such that $\dim l^{S_p} \le 1$. Then $\dim S_{[P_l]} = 0$. Suppose that $\langle p^Z \cup \{l\} \rangle$ is a grid. Then $\dim Z \le 3$, and $\operatorname{Fix} S_{p,l}$ contains the homeomorphic image $P_l \cap \langle p^Z \cup \{l\} \rangle$ of p^Z, whence $\dim S_{p,l} \le 1$. This implies that $\dim S_p = \dim l^{S_p} + \dim S_{p,l} \le 2$, so that $\dim G \le 13$.

Suppose that there is a point q on l such that $p^Z \subseteq D_2(q)$. Then $\dim Z = 2$. The action of $(S_p)^1$ on L_p is either trivial, or Lemma 1.3.5 shows that $L_p \approx \mathbb{S}_3$, and the induced effective action on L_p is linear (see Richardson [111, Theorem A]). This implies that the group $H := (S_{p,l})^1$ fixes l and at least one more line through p, whence it fixes an ordinary quadrangle. Suppose that $\dim H \ge 4$. Then the action of $H/H_{[P_l]}$ on P_l is equivalent to the suspension of the transitive action of either $\mathrm{SO}_4\mathbb{R}$ or $\mathrm{U}_2\mathbb{C}$ on $\mathbb{S}_3$ by [111, Theorem B]. Choose $q' \in P_l \setminus \{p, q\}$. Then $\operatorname{Fix} H_{q'}$ is a compact subquadrangle of Q whose topological parameters (k, k') are either $(1, 3)$ or $(4, 3)$. (See Lemma 5.2.1 for k', and then the dual of Theorem 5.1.2 for k.) In the second case, we find that $\dim H_{q'} = 0$, which is impossible if $\dim H \ge 4$. Hence $\dim(P_l \cap \operatorname{Fix} H_{q'}) = 1$. This implies that $H/H_{[P_l]} \not\cong \mathrm{U}_2\mathbb{C}$. But for every $q'' \in P_l \setminus \operatorname{Fix} H_{q'}$, the stabilizer $H_{q',q''}$ is trivial, which shows that $H/H_{[P_l]} \not\cong \mathrm{SO}_4\mathbb{R}$. This contradiction implies that $\dim H \le 3$. We conclude that $\dim S \le 12$, whence $\dim G \le 14$.

Finally, suppose that $\langle p^Z \cup \{l\} \rangle$ is a subquadrangle of Q, and let (k, k') be the topological parameters of its closure Q'. Then $\dim Z \le k + k' - 1 \le 6$. If $k \ge 3$ then $\dim S_{p,l} = 0$, whence $\dim S_p \le 1$ and $\dim G \le 13$. If $k = 2$ then $\dim S_{p,l} \le 1$, and $k' \le 2$ by the dual of Theorem 5.1.2. Hence $\dim Z \le 3$, so that $\dim G \le 13$. Suppose that $k = 1$. Then $\dim S_{p,l} \le 3$, and $\dim Z \le 3$. As $\dim S \le 12$, the theorem follows if $\dim Z = 2$. Suppose that $\dim Z = 3$. Then $k' = 3$. Choose $q \in P_l \setminus Q'$. Then $\dim S_{p,l,q} = 0$, and $\dim q^{S_{p,l}} \le 2$. Hence $\dim S_p \le 3$, and $\dim G \le 14$.

If $\dim Z = 6$ then we have seen that $\dim G \le 13$. Hence $\dim S = \dim G - \dim Z \le 7$. As S is semi-simple, this implies that $\dim S \le 6$, whence $\dim G \leq 12$. Similarly, the case $\dim Z = 7$ occurs only if $\langle p^Z \rangle$ is a grid or subquadrangle of Q, and in both cases, it leads to the inequality $\dim G \leq 12$. Therefore, we find that $\dim S \leq 5$, whence in fact $\dim S \leq 3$. Thus $\dim G \leq 10$. □

Note that the last argument works for small dimensions only. Indeed, the dimensions of semi-simple compact Lie groups cover all integers greater than 7.

The following lemma will not be needed elsewhere, it is given for its own sake.

5.2.9 Lemma. *Let G be a compact non-Lie group acting effectively on a compact $(4,5)$-quadrangle $Q = P \cup L$. Suppose that $\dim G > 18$, and let Z be the identity component of the centre of G^1. If $p^Z \subseteq P$ is an orbit of maximal dimension then p^Z does not contain a pair of collinear points.*

Proof. The dimension of G is finite by Theorem 4.1.5. Hence the commutator subgroup $S := (G^1)'$ of G^1 is a semi-simple Lie group by van Kampen's Theorem 1.1.3, and G^1 is the almost direct product of Z and S. Let $p^Z \subseteq P$ be an orbit of maximal dimension, and suppose that it contains collinear points. Then we will show that $\dim G \leq 18$.

Using Theorem 4.1.5 again, we find that $\dim Z_p = 0$. Theorem 2.2.2 implies that $\dim p^S \leq 10$, and Lemma 5.2.5 shows that $\dim Z + \dim p^S \leq 15$. Among the lines which join p to other points of p^Z, choose l such that the dimension of $P_l \cap \overline{\langle p^Z \rangle}$ is as high as possible. Note that l is fixed by the stabilizer S_p, whence there is an action of S_p on P_l. Set

$$H := \left(\frac{S_p}{S_{[P_l]}} \right)^1 .$$

If the dimension of H is greater than 1 then the Montgomery–Yang Theorem 2.2.3 shows that some orbit of S_p in P_l has dimension at least 2, whence $P_l \approx \mathbb{S}_4$ by Lemma 1.3.5. Thus Richardson's classification [111] shows that the action of H on P_l is equivalent to a linear action if the group dimension is greater than 1. Since the description of orbit types in Proposition 5.2.4 shows that l meets $\langle p^Z \rangle$ in at least 3 points, we conclude that

$$\dim H \leq 3.$$

Proposition 5.1.3 shows that the rank of $S_{[P_l]}$ is at most 1. If this rank vanishes then the equality

$$\dim G = \dim Z + \dim p^S + \dim H + \dim S_{[P_l]}$$

implies that $\dim G \leq 18$. Hence we may assume that the rank of $S_{[P_l]}$ is 1. Then the dimension of this group is at most 3. Fix an embedding of $\mathbb{Z}/3$ into $S_{[P_l]}$. (Here 3 could be replaced by an arbitrary odd prime.) Proposition 5.1.3 yields that $Q' := \mathrm{Fix}(\mathbb{Z}/3)$ is a subquadrangle of Q whose parameters are $(4, 1)$. It suffices to show that either H is trivial, or

$$\dim Z + \dim H \leq 5.$$

To achieve this, we go through the classification of orbit types in Proposition 5.2.4. As $\dim H \leq 3$, we may assume that $\dim Z \geq 3$. Then p^Z cannot be contained in the point row P_l. Suppose that there is a fixed point q of Z which is collinear to all points of the orbit p^Z. Then $\dim Z = 3$. Considering the action of Z on the line pencil L_q, we find that

$$\dim Z = \dim l^Z + \dim p^{Z_l},$$

whence $\dim p^{Z_l} = 2$. This implies that the dimension of H is at most 1.

Suppose that the orbit p^Z generates a grid, and let k be the dimension of $P_l \cap \overline{\langle p^Z \rangle}$. The choice of l implies that $\dim Z \leq 2k$. If $k \geq 3$ then the Smith–Floyd Theorem 4.1.9 shows that H is trivial. If $k = 2$ then $\dim H \leq 1$, and if $k = 1$ then $\dim H \leq 3$. In both cases, the result follows since $\dim Z + \dim H$ is at most 5.

Finally, suppose that the orbit p^Z generates a subquadrangle of Q', and let (k, k') be the topological parameters of its closure. Then $k' = 1$. For each possible value of k, we apply the dual of Proposition 5.2.3 to obtain an upper bound for the dimension of Z. We find the following implications:

$$\begin{aligned}
k \geq 3 \quad &\Longrightarrow \quad \dim Z \leq 4, \quad \dim H = 0 \\
k = 2 \quad &\Longrightarrow \quad \dim Z \leq 3, \quad \dim H \leq 1 \\
k = 1 \quad &\Longrightarrow \quad \dim Z \leq 2, \quad \dim H \leq 3
\end{aligned}$$

We conclude that $\dim Z + \dim H \leq 5$, which completes the proof. □

5.2.10 Theorem. *Let G be a compact non-Lie group acting effectively on a compact $(4, 5)$-quadrangle $Q = P \cup L$. Then*

$$\dim G \leq 21.$$

Proof. By van Kampen's Theorem 1.1.3, the identity component G^1 of G is the almost direct product of the identity component Z of its centre with its commutator subgroup $S := (G^1)'$, which is a semi-simple Lie group since the dimension of G is finite by Theorem 4.1.5. We first show that the dimension of S is at most 20. Indeed, if l^S is a principal line orbit then we combine Theorem 2.2.2 with the duals of Lemmas 4.1.1 and 4.3.1 to find that the action of S_l on P_l is not transitive. Hence the dimension of $S_l/S_{[P_l]}$ is at most 6. Theorem 2.2.2 also shows that the dimension of l^S is at most 11, and Proposition 5.1.3 yields that $S_{[P_l]}$ has rank at most 1 and hence dimension at most 3. Putting these upper bounds together, we find that

$$\dim S \leq \dim l^S + \dim \frac{S_l}{S_{[P_l]}} + \dim S_{[P_l]} \leq 11 + 6 + 3 = 20.$$

In particular, we may assume that $\dim Z \geq 2$.

The points of Q whose orbits under Z have maximal dimension form an open subset of P by Corollary 2.1.17. On the other hand, the Montgomery–Yang Theorem 2.2.3 shows that the points on principal S-orbits form a dense subset of P. As the action of S on L is not transitive by Theorem 2.2.2, Lemmas 4.1.1 and 4.3.1 show that a principal point stabilizer does not act transitively on the corresponding line pencil. Therefore, we can choose a point $p \in P$ such that p^Z is of the highest possible dimension and the action of S_p on L_p is not transitive. Theorem 4.1.5 implies that $\dim Z_p = 0$. Theorem 2.2.2 yields that $\dim p^S \leq 10$, and Lemma 5.2.5 shows that $\dim Z + \dim p^S \leq 15$.

Suppose first that the orbit p^Z contains collinear points. We could use Lemma 5.2.9. However, there is a short argument which shows that $\dim G \leq 21$. Let l be a line which joins p to another point of p^Z. Then l is fixed by the stabilizer S_p. The classification of orbit types in Proposition 5.2.4 yields that l meets $\langle p^Z \rangle$ in at least 3 points. Together with Lemma 1.3.5 and the Montgomery–Yang Theorem 2.2.3, Richardson's classification [111] of compact groups which act on $\mathbb{S}_4$ implies that the dimension of $S_p/S_{[P_l]}$ is at most 3. As before, we infer from Proposition 5.1.3 that $S_{[P_l]}$ has rank at most 1 and hence dimension at most 3. We conclude that

$$\dim G = \dim Z + \dim p^S + \dim \frac{S_p}{S_{[P_l]}} + \dim S_{[P_l]} \leq 15 + 3 + 3 = 21.$$

We may now assume that no two points of the orbit p^Z are collinear. Since the action of S_p on the generalized 5-sphere L_p is not transitive, Mostert's Theorem 3.1.2 allows us to choose a line l through p whose orbit under S_p is at most 3-dimensional. Then

$$\dim G = \underbrace{\dim Z + \underbrace{\dim p^S + \dim l^{S_p}}_{\leq 13}}_{\leq 18} + \dim \frac{S_{p,l}}{S_{[P_l]}} + \dim S_{[P_l]},$$

and the rank of $S_{[P_l]}$ is at most 1. Suppose that it is 0. If $\langle p^Z \cup \{l\}\rangle$ is a grid or a subquadrangle of Q then $S_{p,l}$ fixes a non-trivial connected subset of the point row P_l. Hence $\dim S_{p,l} \leq 3$, and $\dim G \leq 21$. Suppose that there is a point q on l such that p^Z is contained in $D_2(q)$. Proposition 5.2.4 shows that $\dim Z \leq 4$. If $\dim Z \leq 2$ then $\dim G \leq 21$ because $\dim S_{p,l} \leq 6$, so assume that $\dim Z \geq 3$. Choose a point $q' \in P_l \setminus \{p, q\}$. We claim that $S_{p,l,q'}$ is totally disconnected. If this fails then we may fix an embedding of $\mathbb{Z}/3$ into $S_{p,l,q'}$. (As usual, the number 3 could be replaced by any odd prime.) The Smith–Floyd Theorem 4.1.9 entails that $\mathrm{Fix}(\mathbb{Z}/3)$ is a subquadrangle of Q. Let (k, k') be the pair of its topological parameters. When we apply Lemma 5.2.1 to the compact subset $p^Z \vee q$ of the connected set $L_q \cap \mathrm{Fix}(\mathbb{Z}/3)$, we find that $k' \geq 4$. Now the Smith–Floyd Theorem 4.1.9 shows that $k' = 5$. The dual of Theorem 5.1.2 yields that $\mathbb{Z}/3$ fixes the whole quadrangle Q, which contradicts effectiveness of the action. This shows that $\dim S_{p,l,q'} = 0$ as claimed. Hence

$$\dim S_{p,l} = \dim (q')^{S_{p,l}} \leq 3,$$

and we conclude that $\dim G \leq 20$.

Suppose that the rank of $S_{[P_l]}$ is 1. Fix an embedding of $\mathbb{Z}/3$ into $S_{[P_l]}$, and let $Q' := \mathrm{Fix}(\mathbb{Z}/3)$. Proposition 5.1.3 shows that Q' is a subquadrangle of Q whose topological parameters are $(4, 1)$. As $\dim Z \geq 2$, there cannot be a point on l to which all points of p^Z are collinear. Therefore, the set R generated by $p^Z \cup \{l\}$ is either a grid or a subquadrangle. Let k be the dimension of $P_l \cap \overline{R}$. Proposition 5.2.4 implies that $\dim Z \le k$. If $k = 2$ then the dimension of $S_{p,l}/S_{[P_l]}$ is at most 1, and by the Smith–Floyd Theorem 4.1.9, it vanishes if $k \ge 3$. Hence the inequality

$$\dim Z + \dim \frac{S_{p,l}}{S_{[P_l]}} \le 4$$

holds, and it implies that $\dim G \le 20$. □

5.3 Characterization theorems

We combine our results with those due to Grundhöfer, Knarr, and Kramer [50] and Kramer [75].

5.3.1 Theorem (Characterization of $H(n+1, \mathbb{H})$). *Let G be a compact connected group acting effectively on a compact $(4, 4n-5)$-quadrangle $Q = P \cup L$. Suppose that the dimension of G satisfies the following hypothesis.*

$$\begin{aligned} n < 4 \quad &: \quad \dim G > \binom{2n+1}{2} + 6 \\ n = 4 \quad &: \quad \dim G > \binom{2n+1}{2} + 9 = 45 \\ n > 4 \quad &: \quad \dim G > \binom{2n+1}{2} + 5. \end{aligned}$$

Then

$$G \cong \frac{\mathrm{U}_2\mathbb{H} \times \mathrm{U}_n\mathbb{H}}{\langle(-1,-1)\rangle} \quad \text{or} \quad G \cong \frac{\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H} \times \mathrm{U}_n\mathbb{H}}{\langle(-1,-1,-1)\rangle} \quad .$$

The action of G on Q is equivalent to the natural action of this group on the quaternion hermitian quadrangle $H(n+1, \mathbb{H})$.

The dimensions of these groups are

$$\begin{aligned} \dim \frac{\mathrm{U}_2\mathbb{H} \times \mathrm{U}_n\mathbb{H}}{\langle(-1,-1)\rangle} \quad &= \quad \binom{2n+1}{2} + 10 \quad \text{and} \\ \dim \frac{\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H} \times \mathrm{U}_n\mathbb{H}}{\langle(-1,-1,-1)\rangle} \quad &= \quad \binom{2n+1}{2} + 6. \end{aligned}$$

Hence the dimension hypothesis excludes the second group if $n \le 4$.

Proof. Theorem 5.2.7 shows that G is a Lie group, so that we can apply Theorem 5.1.8. The large parameters have been treated by Kramer [75]. Indeed, if $n \geq 5$ then G acts transitively on P or on L by Theorem 5.1.8. Then [75, 7.18] shows that $Q \cong H(n+1, \mathbb{H})$. The action of G is equivalent to the natural action of a subgroup of $(\mathrm{U}_2\mathbb{H} \times \mathrm{U}_n\mathbb{H})/\langle(-1,-1)\rangle$, which is determined by its dimension.

For $n \leq 4$, we apply the classification due to Grundhöfer, Knarr, and Kramer [50]. A group of dimension $\binom{2n+1}{2} + 6$ need not act flag-transitively on Q, as is shown by the action of $\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H} \times \mathrm{U}_n\mathbb{H}$ on $H(n+1, \mathbb{H})$. Indeed, if $n = 2$ then this group acts on a non-classical example, see Remark 5.3.2 below. For this reason and because of the irritation caused by $\mathfrak{g}_2$ in the proofs of Theorems 5.1.8 and 5.2.7, we need stronger hypotheses for the three smallest values of n.

Suppose that $n \in \{2, 3\}$. Theorem 5.1.8 shows that the action of G on P is transitive. Now P has the integral cohomology of $\mathbb{S}_4 \times \mathbb{S}_{4n-1}$, and homogeneous spaces with this property have been classified by Kramer [75, 3.15], so that we need only go through his tables. If $n = 2$ and $16 < \dim G \leq 20$ then we find that $\dim G = 20$. Similarly, if $n = 3$ and $27 < \dim G \leq 31$ then $\dim G = 31$. In both cases, we infer from Theorem 5.1.8 that the action of G on the flag space is transitive. Thus the result follows from Grundhöfer, Knarr, and Kramer [50].

Suppose, then, that $n = 4$. Theorem 5.1.8 yields that G acts transitively either on P, whose integral cohomology is that of $\mathbb{S}_4 \times \mathbb{S}_{15}$, or on L, whose integral cohomology is that of $\mathbb{S}_{11} \times \mathbb{S}_{15}$. Using Kramer's classification and the fact that $45 < \dim G \leq 50$, we find that the Lie algebra of G is isomorphic to $\mathfrak{b}_2 \times \mathfrak{b}_4$ or to $\mathfrak{c}_2 \times \mathfrak{c}_4$. The first case, in which the point space P is the product of two homogeneous spheres, will be excluded first. Suppose that $\mathfrak{l}(G) \cong \mathfrak{b}_2 \times \mathfrak{b}_4$. Kramer's classification shows that G cannot act transitively on L. Choose a principal line orbit l^G. Then

$$\dim \frac{G_l}{G_{[P_l]}} + \dim G_{[P_l]} = 46 - \dim l^G > 46 - \dim L = 20.$$

This implies that $\dim G_l/G_{[P_l]} = 10$, and $\mathfrak{l}(G_{[P_l]}) \cong \mathfrak{g}_2$. Therefore, the image of $\mathfrak{l}(G_l)$ in $\mathfrak{b}_2 \times \mathfrak{b}_4$ under the isomorphism $\mathfrak{l}(G) \cong \mathfrak{b}_2 \times \mathfrak{b}_4$ has an ideal which can be identified with $\mathfrak{g}_2$, and whose complement $\mathfrak{h}$ is of dimension 10. Using the projections onto the two factors, we find that the ideal $\mathfrak{g}_2$ is contained in $\{0\} \times \mathfrak{b}_4$. Representation theory shows that there is a unique embedding $\mathfrak{g}_2 \hookrightarrow \mathfrak{b}_4$ (see Salzmann et al. [115, 95.10]). The centralizer $\mathfrak{z}$ of $\mathfrak{g}_2$ in $\mathfrak{b}_4$ preserves the isotypic decomposition of $\mathbb{R}^9$ as a module over $\mathfrak{g}_2$, which implies that $\mathfrak{z}$ is one-dimensional. Under the projection of $\mathfrak{b}_2 \times \mathfrak{b}_4$ onto $\mathfrak{b}_4$, the complement $\mathfrak{h}$ of $\mathfrak{g}_2$ in $\mathfrak{l}(G_l)$ is mapped into $\mathfrak{z}$. Hence the intersection of $\mathfrak{h}$ with $\mathfrak{b}_2 \times \{0\}$ has dimension at least 9. Theorem 2.5.1 yields that $\mathfrak{h} = \mathfrak{b}_2 \times \{0\}$. Hence $\mathfrak{l}(G_l)$ contains a non-trivial ideal of $\mathfrak{l}(G)$. This means that G_l contains a non-trivial connected normal subgroup of G, which contradicts the fact that G acts effectively on l^G by Proposition 4.1.4. Thus we have found that $\mathfrak{l}(G) \cong \mathfrak{c}_2 \times \mathfrak{c}_4$. As $\mathfrak{g}_2$ cannot be embedded into either $\mathfrak{c}_2$ or $\mathfrak{c}_4$ (cf. Tits [136]), it cannot be embedded into $\mathfrak{l}(G)$. We conclude that $\dim G_{[P_l]} \leq 10$ holds for every line $l \in L$. This entails that G acts flag-transitively on Q, and the result follows again from Grundhöfer, Knarr, and Kramer [50]. $\square$

5.3.2 Remark. For compact $(4, 3)$-quadrangles, the assertion that $\dim G \leq 16$ if Q is not a classical quadrangle is sharp. Indeed, the group

$$G := \frac{\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H} \times \mathrm{U}_2\mathbb{H}}{\langle(-1, -1, -1,)\rangle}$$

acts effectively and line-transitively on the non-classical compact $(4, 3)$-quadrangle $Q := \mathit{FKM}(4, 8, 0)$. This is one of the examples due to Ferus, Karcher, and Münzner [42] and Thorbergsson [134]. The action of G on $Q^{\mathrm{dual}} \cong \mathit{FKM}(3, 8)$ is described by Kramer in [76], and in [77] he shows that the quadrangle is not classical.

Incidentally, Theorem 5.3.1 shows that G is a maximal compact connected subgroup of $\operatorname{Aut} Q$.

By duality, actions of sufficiently large compact groups on compact $(4, 1)$-quadrangles are covered by Theorem 4.4.2. It remains to treat quadrangles with five-dimensional line pencils.

5.3.3 Theorem (Characterization of $H^\alpha(4, \mathbb{H})$). *Let G be a compact connected group acting effectively on a compact $(4, 5)$-quadrangle $Q = P \cup L$. If $\dim G > 21$ then G is a Lie group. If $\dim G > 22$ then the action of G on Q is equivalent to the natural action of either $\mathrm{U}_5\mathbb{C}/\langle -1\rangle$ or $\mathrm{SU}_5\mathbb{C}$ on the anti-unitary quadrangle $H^\alpha(4, \mathbb{H})$.*

Proof. If $\dim G > 21$ then G is a Lie group by Theorem 5.2.10. Suppose that $\dim G > 22$. Then Theorem 5.1.9 shows that G acts transitively on either P or L. Kramer's result [75, 3.15] yields that $\dim G > 23$. Using Theorem 5.1.9 again, we find that G acts flag-transitively, and the present theorem follows from the classification by Grundhöfer, Knarr, and Kramer [50]. □

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Notation

$\cong$	isomorphism, e.g. of topological groups
$\approx$	homeomorphism of topological spaces
$\simeq$	homotopy equivalence of topological spaces
$\sim$	equivalence relation
G^1	identity component of the topological group G
$H_i(\cdot)$, $H^i(\cdot)$	singular (co-)homology
x^g	result of the (right) action of the group element g on x; if x belongs to the same group then usually $x^g := g^{-1}xg$
x^G	orbit of x under the group G
G_x	stabilizer of x in G
$G_{[X]}$	pointwise stabilizer of the set X in G, i.e. $\bigcap_{x \in X} G_x$
$P \mathbin{\dot\cup} L$	incidence geometry consisting of points P and lines L; formally, this is considered as a bipartite graph
$F \subseteq P \times L$	set of flags (incident point-line pairs) of $P \mathbin{\dot\cup} L$
P_l	point row of a line l, i.e. $\{p \in P \mid (p, l) \in F\}$
L_p	line pencil of a point p, i.e. $\{l \in L \mid (p, l) \in F\}$
$\pi(p, l)$, $\lambda(p, l)$	geometric operations in a generalized quadrangle (see p. 16)

Index

ANR, 9

building, 14, 22

Cartan decomposition, 51, 80
chain
    descending, 27, 156
change of rings, 9
clc, **5**, 7, 10
Clifford algebra, *see* representation
closed
    geometrically, **16**
closure
    geometric, **16**
cohomology
    Čech, 118
    of quadrangles, 20
    sheaf, 4
        and Čech, 155
        and singular, 5
cohomology manifold, **7**, 12, 20, 66, 161
    actions, 36–39, 58, 69, 168
contractible
    locally, 11, **12**
    pseudo-isotopically, **12**
Corollary
    characterization of metrizable cohomology manifolds, 10
    codimensions of comparable subgroups, 28
    finite dimension of automorphism group, 117
    orbits of maximal dimension, 34
    refinement of Mann's Theorem, 58
    stabilizers in compact Lie groups, 26
    uniform orbit dimension, 34

Dedekind ring, 11
$\dim_R$, *see* dimension, sheaf theoretic
dimension, 2
    sheaf theoretic, **6**
domain invariance, 39, 162
duality
    in (homology) manifolds, 7
    in (homology) manifolds, 4, 76, 86, 87, 112, 113
    in generalized quadrangles, **16**
    Pontryagin, 84

ENR, 11–13, 20
Euler characteristic, 87

fibre bundle, 41–43, 47, 48, 131, 132, 162
Frattini argument, 88, 106, 108
fundamental group, 20, 120, 124, 135

generalized manifold, 11
generate
- geometrically, **16**

girth, **15**
grid, **18**, 164

HLC, **5**, 11
homogeneous
- locally, **12**, 13, 37, 66, 69, 161, 162

homology
- Čech, 118
- Borel–Moore, 4
  - and singular, 5
- finitely generated, 64
- local, 6

homology manifold, **7**
homology sheaf, **6**, 7
- singular, 6

isomorphism
- exceptional, 133

isotropic
- totally, **20**

Lemma
- close orbits have close kernels, 25
- close points have close stabilizers, 24
- connected component and intersection, 29
- effective action on principal orbit, 12
- Montgomery and Samelson, 73
- orbits of full dimension, 13
- orbits of small codimension, 13
- Smith's rank restriction, 13

lens space, 87
line pencil, **15**, 18, 20, 92
locally connected
- pathwise, 6
- w.r.t. Borel–Moore homology, 5
- w.r.t. sheaf cohomology, *see clc*
- w.r.t. singular homology, *see HLC*

metric space, 10, 36, 37, 66, 69
- complete, 6
- separable, 2, 11, 19

n-gon, *see* polygon
neighbourhood retract
- absolute, *see* ANR
- euclidean, *see* ENR

non-linear, 102

opposite, **18**
orbit
- principal, 12, **26**, 37, 64, 114, 117, 131

orientable, 65
- locally, **7**

ovoid, **123**, 149, 162

p-adic integers, 30, 33
parameters
- topological, **19**, 20–22, 69, 151

perspectivity, **18**
plane
  circle, 21
  projective, 22, 92
Poincaré conjecture, 14
point row, **15**, 18, 20, 92
polygon
  generalized, **22**, 69, 115–117, 125
projectivity, **18**
Proposition
  change of rings in cohomology manifolds, 9
  close connected subgroups of locally compact groups, 31
  elementary abelian groups fixing all points on a line, 155
  identity component and complex product, 30
  identity component and homomorphic image, 29

quadrangle
  anti-unitary, **21**, 184
  classical, **20**
  compact, **17**
  compact (m, m')-, **19**
  connected, **19**
  dual, **16**
  examples, 20–22
  Ferus–Karcher–Münzner, 21, 184
  flag-homogeneous, 120, 151, 158, 160
  generalized, **15**
  hermitian, **21**, 159, 182
  line-homogeneous, 120, 125, 129, 151, 158, 160, 184
  Moufang, **17**, 21, 119, 151
  non-classical, 21, 149, 184
  orthogonal, **21**, 120, 127, 129, 145, 146
  point-homogeneous, 129, 151, 158, 160
  symplectic, 21, 120, 145
  thick, **19**
  translation, 149
  weak, **19**

representation
  of a Clifford algebra, 21
  of a Lie algebra, 2, 49, 79

sequence
  exact homotopy, 135
  exact homology, 9
  exact homotopy, 1, 80, 82, 87, 106, 119, 121–124, 126–128, 132, 137, 141, 145, 146
  Gysin, 42, 136, 138, 141
  Mayer–Vietoris, 66, 133, 163
  spectral, 41, 133
  universal coefficient, 9
  Wang, 42
sketch, **125**, 131, 146, 159
smooth action, 38
solenoid, 30
sphere
  generalized, **12**, 20
  Poincaré homology, **63**, 102

stabilizer
- principal, **26**, 131

subquadrangle
- full, **155**

suspension, 70, 80, 82
- double, 102

Theorem
- approximation by Lie groups, 3
- characterization of $H(n+1,\mathbb{H})$, 182
- characterization of $H^\alpha(4,\mathbb{H})$, 184
- characterization of $Q(m+3,\mathbb{R})$, 146
- characterization of cohomology manifolds, 8
- close subgroups of compact Lie groups, 26
- groups which must act with a codimension 1 orbit, 68, 69
- homogeneous cohomology spheres, 63
- Hurewicz, 81, 87
- Knarr and Kramer, 20
- Kramer and Van Maldeghem: full subquadrangles, 154
- Löwen, 9
- Mal'cev and Iwasawa, 4
- Mann, 38
- Mazurkiewicz, Moore, and Menger, 6
- Montgomery and Yang: principal orbits in cohomology manifolds, 37
- Mostert, 64, 131
- Open Mapping Theorem, 2
- orbits of low codimension in cohomology manifolds, 36, 37
- Smith, 118
- Smith and Floyd, 118
- stabilizers in compact non-Lie groups, 32
- structure of compact connected groups, 3
- Szenthe, 121
- Tits and Weiss, 17
- transitive actions on finite-dimensional spaces, 27
- Universal Coefficient, 5, 9, 64, 87, 163
- Weyl, 73, 75

Tits construction, 21, 149

topology
- compact-open, 116
- Hausdorff, 23, 35

torus, 83–84, 117, 122, 129, 135, 162

tree, 33

triality automorphism, 49, 51

triangle
- generalized, *see* plane, projective

two-disc, 76, 111

uniform space, 24

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