On Unipotent Specht Modules of Finite General Linear Groups

Von der Fakultät Mathematik und Physik der Universität Stuttgart zur Erlangung der Würde eines Doktors der Naturwissenschaften (Dr. rer. nat.) genehmigte Abhandlung

Vorgelegt von

Marco Brandt

aus Böblingen

Hauptberichter:	Prof. Dr. R. Dipper
Mitberichter:	Prof. G. D. James
	Prof. Dr. W. Kimmerle
Tag der mündlichen Prüfung:	12. Februar 2004

Institut für Algebra und Zahlentheorie der Universität Stuttgart

2004

Contents

Introduction

Many outstanding problems in representation theory can be solved with a proper understanding of the irreducible unipotent modules for the finite general linear group $GL_n(q)$ [5, 6].
In [11], Gordon James investigated these irreducible unipotent modules: For each partition λ of n there is a Specht module S^λ for $GL_n(q)$, defined over a field F in terms of the intersection of the kernels of certain homomorphisms. If F is a field of characteristic zero, then S^λ is irreducible and $\{S^\lambda \mid \lambda$ is a partition of $n\}$ is a complete set of pairwise non-isomorphic irreducible unipotent modules for $GL_n(q)$. If the characteristic of F is coprime to q, then, in general, S^λ has a unique top composition factor D^λ and the D^λ's are the irreducible unipotent modules for $GL_n(q)$.
For each Specht module S^λ, a generating element e_λ is known but, in general, no explicit basis for S^λ as a vector space over F has been found. In [7], Richard Dipper and Gordon James make significant progress towards the construction of a basis of S^λ for a two part partition λ. My thesis is based on this paper and further develops and improves the techniques introduced there.
Chapter 1 sets the scene and gives an overview of the fundamental definitions and propositions in the area of compositions, partitions, λ-tableaux and Gaussian polynomials. Furthermore, we introduce λ-flags and a manageable notation Ξ_λ for these chains of vector subspaces. We conclude the chapter by delivering a short insight into the theory of finite groups with a BN-pair.
In chapter 2 we define M^λ as vector space over F with basis Ξ_λ. The canonical operation of $GL_n(q)$ on Ξ_λ turns M^λ into an $FGL_n(q)$-module. As we can assign a λ-tableau to each λ-flag and we have a total ordering on the set of λ-tableaux, we define, for an element $v = \sum_{X \in \Xi_\lambda} c_X X \in M^\lambda$, $last(v)$ as the last λ-tableau which can be assigned to a λ-flag X occurring in this sum with nonzero coefficient c_X.
Motivated by the fact that the unipotent Specht module S^λ is a submodule of M^λ, we carefully examine M^λ and the operation of $GL_n(q)$ on M^λ. Next we define S^λ as the intersection of the kernels of certain homomorphisms and present our main tool for understanding the structure of S^λ, namely the branching theorem.
Sinéad Lyle proves in her thesis [14] that, for every element $v \in S^\lambda$, $last(v)$ is a standard λ-tableau. This leads us to the definition of a standard basis of S^λ, i.e. a basis $\mathfrak{B}^\lambda = \{b_i \mid i \in \mathfrak{I}\}$ of S^λ, which is defined independently of the concrete choice of the field F, together with a set of polynomials $\{p_\mathfrak{t}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ such that $p_\mathfrak{t}(q) = |\{b \in \mathfrak{B}^\lambda \mid last(b) = \mathfrak{t}\}|$ and $p_\mathfrak{t}(1) = 1$ holds for every $\mathfrak{t} \in Std(\lambda)$.
Finding a standard basis of S^λ for a two part partition $\lambda = (n-m, m)$ is the goal

of chapter 3. We start with the introduction of a short notation of the $(n-m,m)$-flags. Then we define, for every $\mathfrak{t} \in Std((n-m,m))$, a subset $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ of $\Xi_{(n-m,m)}$ and set $p_{\mathfrak{t}}(q) := |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)|$. By a recursive approach we develop an algorithm to calculate $p_{\mathfrak{t}}(q)$ and prove that all $p_{\mathfrak{t}}(q)$ are polynomials over q. The main theorem of this chapter is the existence of a set $\mathfrak{B}^{(n-m,m)} = \{b_L \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}$ of linearly independent vectors in $S^{(n-m,m)}$. The proof of this theorem is constructive and lists, for every $\mathfrak{t} \in Std(\lambda)$ and every $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$, the operations necessary to obtain the element b_L from the generator $e_{(n-m,m)}$ of $S^{(n-m,m)}$. Unfortunately we can't prove that $\mathfrak{B}^{(n-m,m)}$ is a generating system of $S^{(n-m,m)}$ and therefore it remains only a conjecture. But with the help of GAP [9], the idea from [7] to divide $Std((n-m,m))$ in some special intervals and the branching theorem, we collect a lot of evidence for this conjecture. We formulate two further conjectures and finally prove that $\mathfrak{B}^{(n-m,m)}$ is a standard basis of $S^{(n-m,m)}$ with corresponding polynomials $\{p_{\mathfrak{t}}(q) \mid \mathfrak{t} \in Std((n-m,m))\}$, if $1 \leq m \leq 11$.
In chapter 4 we deal with the Specht module $S^{(2,2,2)}$. The third part in the partition $(2,2,2)$ significantly complicates the task. But again the branching theorem turns out to be very powerful and helps us to construct a standard basis of $S^{(2,2,2)}$.

Acknowledgments

Many people have supported, encouraged and helped me during the time I spent working on this thesis. I wish to express my gratitude to all of them.

First of all, I would like to thank my supervisor Prof. Dr. Richard Dipper. He has been a great source of motivation and I am grateful to him for having introduced me to the fascinating research area of representation theory of the finite general linear group and for guiding my research work that led to this thesis.

Furthermore, I would like to thank my co-supervisors Prof. Gordon James and Prof. Dr. Wolfgang Kimmerle for reading this thesis.

Many thanks to my colleagues and friends at the "Abteilung für Darstellungstheorie" and the "Fachschaft Mathematik" who have made me feel very comfortable at the University of Stuttgart.

I would also like to thank Vanessa Miemietz for proof-reading this thesis.

For financial support I am grateful to the Deutsche Forschungsgemeinschaft (DFG) and my grandparents.

Finally, I would like to thank my parents for their encouragement and their invaluable support over the last years which allowed me to fully concentrate on my research and thus significantly contributed to the successful completion of this thesis.

Chapter 1

Basics

1.1 The setting

Throughout this thesis n is a natural number, p a prime, q a power of p and F a field whose characteristic is coprime to p and which contains a primitive p^{th} root of unity. $GF(q)$ denotes the finite field of q elements, $GF(q)^*$ its multiplicative group and $GL_n(q)$ the group of invertible $n \times n$ matrices over $GF(q)$. The monoid of $a \times b$ matrices over $GF(q)$ is referred to as $\mathfrak{M}_{a,b}(q)$.
Let X be a set. Then we denote by $\mathfrak{S}_X$ the group of permutations on X. Moreover $\mathfrak{S}_n := \mathfrak{S}_{\{1,2,\ldots,n\}}$ is the symmetric group on n numbers.
We embed $\mathfrak{S}_n$ into $GL_n(q)$ by assigning to a permutation π the appropriate permutation matrix $P = (p_{ij}) \in GL_n(q)$, where

$$p_{ij} := \begin{cases} 1 & \text{for } j = i\pi \\ 0 & \text{otherwise.} \end{cases}$$

A permutation of the form (i, j) is called transposition and a permutation of the form $(i, i+1)$ is a basic transposition.
If $\underline{a}$ and $\underline{b}$ are vectors over $GF(q)$ of the same length l, we have the canonical scalar product

$$\langle \underline{a}, \underline{b} \rangle := \sum_{i=1}^{l} a_i b_i \in GF(q).$$

Furthermore, we fix, once and for all, a non-trivial group homomorphism

$$\theta : (GF(q), +) \to F^*.$$

Thus, θ is a linear F-character of the group $(GF(q),+)$.

1.2 Compositions and Partitions

In this section we introduce the fundamental definitions of compositions and partitions. Thereby we follow [12] and [15].

1.2.1 Definition:

1.) $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a composition of n, if $\lambda_1, \lambda_2, \lambda_3, \ldots$ are non-negative integers with

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i = n.$$

The non-zero λ_i are called the parts of λ. The last part is denoted by λ_h (h standing for "height").

2.) A partition of n is a composition λ of n for which

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots.$$

In the notation of compositions we often suppress the zeros at the end.
For example

$$(2, 0, 3, 1, 4, 0, 0, \ldots) = (2, 0, 3, 1, 4).$$

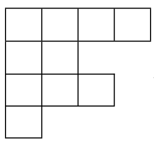
In partitions the sequence of entries is unique, because they must decrease. Therefore we can indicate repeated parts by a superscript.
For example

$$(4, 2, 2, 2, 1, 0, 0, \ldots) = (4, 2^3, 1).$$

1.2.2 Definition: If λ is a composition of n, then the diagram $[\lambda]$ is the set $\{(i, j) \mid i, j \in \mathbb{Z}, 1 \leq i, 1 \leq j \leq \lambda_i\}$. If $(i, j) \in [\lambda]$, then (i, j) is called a node of $[\lambda]$. The k^{th} row (respectively, column) of a diagram consists of those nodes whose first (respectively, second) coordinate is k.

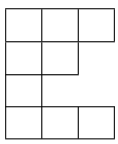
We shall draw diagrams as in the following example.

1.2.3 Example:

For $\lambda = (4, 2, 3, 1)$ we have $[\lambda] =$  .

1.2.4 Definition: Suppose λ is a composition. If λ'_j ($j \geq 1$) equals the number of nodes in column j of $[\lambda]$, then $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \ldots)$ is obviously a partition. It is called the conjugate partition of λ.

1.2.5 Example:

For $\lambda = (3, 2, 1, 3)$ with $[\lambda] =$  we get $\lambda' = (4, 3, 2)$.

We define a partial order on the set of compositions of n.

1.2.6 Definition: If λ and μ are compositions of n, we write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \mu'_i \text{ for all } j \geq 1$$

and $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ but $\lambda \neq \mu$.

1.2.7 Example:

1.) We have the following order on the partitions of 3:

$$\ydiagram{3} \trianglerighteq \ydiagram{2,1} \trianglerighteq \ydiagram{1,1,1}.$$

2.) It is possible to have $\lambda \trianglelefteq \mu$ and $\mu \trianglelefteq \lambda$ without $\lambda = \mu$:

$$\ydiagram{2,1} \trianglerighteq \ydiagram{1,2} \trianglerighteq \ydiagram{2,1}.$$

3.) $\trianglerighteq$ is only a partial order. For example, we can't order the elements

$$\ydiagram{4,1,1} \text{ and } \ydiagram{3,3}.$$

1.2.8 Definition: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition of n and $(i, j) \in [\lambda]$. Then the (i, j)-hook of $[\lambda]$ consists of the (i, j)-node along with the $\lambda_i - j$ nodes to the right of it and the $\lambda'_j - i$ nodes below it.
The length of the (i, j)-hook is

$$h_{ij} = \lambda_i + \lambda'_j + 1 - i - j.$$

If we replace the (i, j)-node of $[\lambda]$ by the number h_{ij} for each node, we obtain the hook graph.

1.2.9 Example: Let $\lambda = (4, 2, 1)$. Then

$$\begin{ytableau} 6 & 4 & 2 & 1 \\ 3 & 1 \\ 1 \end{ytableau}$$

is the hook graph of λ.

1.3 λ-tableaux

We continue with the introduction of λ-tableaux and related definitions. Our main reference is [12].

1.3.1 Definition: Let λ be a composition of n. A λ-tableau is one of the $n!$ arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integers $1, 2, 3, \ldots, n$, allowing no repeats.

1.3.2 Example: Let $\lambda = (1, 2, 4, 1)$.
Then

$$\mathfrak{t}_1 = \begin{ytableau} 1 \\ 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 \end{ytableau} \text{ and } \mathfrak{t}_2 = \begin{ytableau} 1 \\ 2 & 5 \\ 3 & 6 & 7 & 8 \\ 4 \end{ytableau}$$

are λ-tableaux.

The symmetric group $\mathfrak{S}_n$ acts on the set of λ-tableaux by permuting the integers $1, 2, 3, \dots, n$. For example

$$\mathfrak{t}_1(3,5,6,7,8,4) = \mathfrak{t}_2.$$

1.3.3 Definition: If λ is a composition and $\mathfrak{t}$ a λ-tableau, then

- the row i of $\mathfrak{t}$ is the set $\{x \in \mathbb{N} \mid$ there exists a node in row i of $[\lambda]$ which, in $\mathfrak{t}$, is replaced with $x\}$ and
- the column j of $\mathfrak{t}$ is the set $\{x \in \mathbb{N} \mid$ there exists a node in column j of $[\lambda]$ which, in $\mathfrak{t}$, is replaced with $x\}$.

1.3.4 Example: In example 1.3.2 the set $\{3,5\}$ is column 2 of $\mathfrak{t}_1$ and the set $\{3,6,7,8\}$ is row 3 of $\mathfrak{t}_2$.

1.3.5 Definition: Suppose λ is a composition of n and $\mathfrak{t}$ a λ-tableau. We define functions

$$\begin{aligned} row_{\mathfrak{t}} : \{1,2,\dots,n\} &\to \{1,2,\dots\} \\ b &\mapsto i \end{aligned}$$

if b belongs to row i of $\mathfrak{t}$ and

$$\begin{aligned} col_{\mathfrak{t}} : \{1,2,\dots,n\} &\to \{1,2,\dots\ n\} \\ b &\mapsto j \end{aligned}$$

if b belongs to column j of $\mathfrak{t}$.

1.3.6 Definition: Let λ be a composition and $\mathfrak{t}$ a λ-tableau. Then $\mathfrak{t}$ is

1.) row-standard, if the numbers increase along the rows of $\mathfrak{t}$. We write $\mathcal{T}_{rs}(\lambda)$ for the set of row-standard λ-tableaux.

2.) column-standard, if λ is a partition and the numbers increase down the columns of $\mathfrak{t}$.

3.) standard, if $\mathfrak{t}$ is row-standard and column-standard. We denote the set of standard λ-tableaux by $Std(\lambda)$.

1.3.7 Definition: Let λ be a composition of n. The λ-tableau, where the nodes are replaced with the numbers $1, 2, \dots, n$ in order along

1.) the rows is called the initial tableau and is denoted by $\mathfrak{t}^\lambda$.

2.) the columns is denoted by $\mathfrak{t}_\lambda$.

1.3.8 Example: Let $\lambda = (1,2,4,1)$ and $\mathfrak{t}_1, \mathfrak{t}_2$ as in example 1.3.2. Then

$$\mathfrak{t}^\lambda = \mathfrak{t}_1 \text{ and } \mathfrak{t}_\lambda = \mathfrak{t}_2.$$

1.3.9 Definition: If $\mathfrak{t} \in \mathcal{T}_{rs}(\lambda)$ and $\pi \in \mathfrak{S}_n$ then the numbers in the rows of $\mathfrak{t}\pi$ may not be in ascending order. But we can rearrange those numbers row by row to get an element of $\mathcal{T}_{rs}(\lambda)$ and denote this element by

$$\mathfrak{t} \circ \pi.$$

With "$\circ$" we obtain an operation of $\mathfrak{S}_n$ on the set $\mathcal{T}_{rs}(\lambda)$.

1.3.10 Example: Let $\lambda = (1, 2, 4, 1)$ and $\mathfrak{t}_1$ as in example 1.3.2. Then

$$\mathfrak{t}_1 \circ (2, 3) = \mathfrak{t}_1.$$

We totally order $\mathcal{T}_{rs}(\lambda)$ by the following definition.

1.3.11 Definition: Let λ be a composition and $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}_{rs}(\lambda)$. Then $\mathfrak{t}_1 < \mathfrak{t}_2$ if and only if for some i

1.) $row_{\mathfrak{t}_1}(j) = row_{\mathfrak{t}_2}(j)$ for $j > i$ and

2.) $row_{\mathfrak{t}_1}(i) < row_{\mathfrak{t}_2}(i)$.

1.3.12 Example: We get the following order on the standard $(3, 2)$-tableaux.

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}.$$

1.3.13 Remark: If λ is a partition, then for every $\mathfrak{t} \in Std(\lambda)$

$$\mathfrak{t}_\lambda \leq \mathfrak{t} \leq \mathfrak{t}^\lambda.$$

1.3.14 Definition: If λ is a composition and $\mathfrak{t}$ a λ-tableau, then the column stabilizer $C_\mathfrak{t}$ of $\mathfrak{t}$ is the subgroup of $\mathfrak{S}_n$, keeping the columns of $\mathfrak{t}$ fixed setwise, i.e.

$$C_\mathfrak{t} = \{\pi \in \mathfrak{S}_n \mid \text{for all } i, i \text{ and } i\pi \text{ belong to the same column of } \mathfrak{t}\}.$$

1.3.15 Example: The column stabilizer of the tableau $\mathfrak{t}_1$ in example 1.3.2 is

$$C_{\mathfrak{t}_1} = \mathfrak{S}_{\{1,2,4,8\}} \times \mathfrak{S}_{\{3,5\}} \times \mathfrak{S}_{\{6\}} \times \mathfrak{S}_{\{7\}}.$$

1.4 Gaussian polynomials

Later we will need Gaussian polynomials. In this section we define these polynomials and list some of their basic properties. Most of the following and a lot more can be found in [1], [7] and [11].

1.4.1 Definition:

1.) For $r \in \mathbb{N}_0$ we define

$$[r] := \sum_{i=0}^{r-1} q^i = 1 + q + q^2 + \ldots + q^{r-1}.$$

In particular $[0] = 0, [1] = 1$ and $[r] = \frac{q^r-1}{q-1}$.
Furthermore,

$$[r]! := \prod_{i=1}^{r} [i] = [1] \cdot [2] \cdot [3] \cdot \ldots \cdot [r].$$

2.) Let $r, s \in \mathbb{N}_0$. Then we set

$$\begin{bmatrix} r \\ s \end{bmatrix} := \begin{cases} \frac{[r]!}{[s]![r-s]!} & \text{if } r \geq s \\ 0 & \text{otherwise.} \end{cases}$$

$\begin{bmatrix} r \\ s \end{bmatrix}$ is a polynomial in q, known as a Gaussian polynomial.

3.) For $a, b \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we denote by $r_{a,b}(q, k)$ the number of $a \times b$ matrices of rank k with entries in $GF(q)$.

1.4.2 Example:

$$\begin{aligned} \begin{bmatrix} 4 \\ 2 \end{bmatrix} &= \frac{[4]!}{[2]![2]!} = \frac{[4][3]}{[2][1]} = \frac{(q^3 + q^2 + q + 1)(q^2 + q + 1)}{q + 1} \\ &= (q^2 + 1)(q^2 + q + 1) = q^4 + q^3 + 2q^2 + q + 1. \end{aligned}$$

A well-known fact is the following proposition.

1.4.3 Proposition: *Let $k \in \mathbb{N}_0$. Then we have*

$$|GL_k(q)| = \prod_{i=0}^{k-1} (q^k - q^i).$$

Proof: See, for example, 6.2 Hilfssatz in [10]. ∎

The Gaussian polynomials have a nice property.

1.4.4 Proposition: *Let $r, s \in \mathbb{N}_0$. Then $\begin{bmatrix} r \\ s \end{bmatrix}$ counts the number of s-dimensional subspaces of an r-dimensional vector space $GF(q)^r$ over $GF(q)$.*

Proof: See Theorem 13.1. in [1]. ∎

The next proposition contains a series of facts about Gaussian polynomials.

1.4.5 Proposition:
1.) If $1 \leq s \leq r$, then

$$(q^r - 1)(q^r - q)\dots(q^r - q^{s-1}) = \begin{bmatrix} r \\ s \end{bmatrix} |GL_s(q)|. \tag{1.1}$$

2.) If $1 \leq s_1 \leq s_2 \leq r$, then

$$\prod_{i=s_1}^{s_2-1} (q^r - q^i) \begin{bmatrix} r \\ s_1 \end{bmatrix} |GL_{s_1}(q)| = \begin{bmatrix} r \\ s_2 \end{bmatrix} |GL_{s_2}(q)|. \tag{1.2}$$

3.) Let $a, b \in \mathbb{N}$ and $k \leq \min\{a, b\} \in \mathbb{N}_0$. Then we have

$$r_{a,b}(q, k) = \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix} |GL_k(q)| \tag{1.3}$$

and

$$\sum_{j=0}^{\min\{a,b\}} \begin{bmatrix} a \\ j \end{bmatrix} \begin{bmatrix} b \\ j \end{bmatrix} |GL_j(q)| = q^{ab}. \tag{1.4}$$

4.) For $0 \leq l \leq m \leq n$

$$\begin{bmatrix} n-l \\ m-l \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix}. \tag{1.5}$$

Proof:

1.)

$$\begin{aligned} \frac{(q^r - 1)(q^r - q)\dots(q^r - q^{s-1})}{(q^s - 1)(q^s - q)\dots(q^s - q^{s-1})} &= \frac{(q^r - 1)(q^{r-1} - 1)\dots(q^{r-s+1} - 1)}{(q^s - 1)(q^{s-1} - 1)\dots(q - 1)} \\ &= \frac{[r][r-1]\dots[r-s+1]}{[s][s-1]\dots[1]} = \begin{bmatrix} r \\ s \end{bmatrix}. \end{aligned}$$

2.)

$$\prod_{i=s_1}^{s_2-1} (q^r - q^i) \begin{bmatrix} r \\ s_1 \end{bmatrix} |GL_{s_1}(q)| \overset{(1.1)}{=} \prod_{i=0}^{s_2-1} (q^r - q^i) \overset{(1.1)}{=} \begin{bmatrix} r \\ s_2 \end{bmatrix} |GL_{s_2}(q)|.$$

3.) See 2.9 Proposition und 2.10 Corollary in [7].

4.) Follows directly from the definition of the Gaussian polynomials.

$$\begin{aligned} \begin{bmatrix} n-l \\ m-l \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} &= \frac{[n-l]}{[n-l-m+l][m-l]} \frac{[n]}{[n-l][l]} = \frac{[n]}{[n-m][m-l][l]} \\ &= \frac{[n]}{[n-m][m]} \frac{[m]}{[m-l][l]} = \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix}. \end{aligned}$$

■

1.4.6 Definition: Let $a, b, k \in \mathbb{N}$ with $1 \leq k \leq \min\{a, b\}$ and $\mathfrak{B} = (v_1, \ldots, v_k)$ a k-tuple of linearly independent vectors of the vector space $GF(q)^b$. Then we denote by $\mathfrak{M}_{a,b,k}(q, \mathfrak{B})$ the set of matrices $M \in \mathfrak{M}_{a,b}(q)$ of rank k with the following property: There exist natural numbers $1 \leq a_1 < a_2 < \ldots < a_k \leq a$ such that for $1 \leq i \leq k$

1.) the a_i^{th} row of M equals v_i and

2.) the span of the first $a_i - 1$ rows of M is equal to the span of the vectors $v_1, v_2, \ldots, v_{i-1}$.

We define

$$r_{a,b,k}(q, \mathfrak{B}) := |\mathfrak{M}_{a,b,k}(q, \mathfrak{B})|.$$

1.4.7 Example: Let $\mathfrak{B} = ((0, 1, 0), (1, 0, 0))$. Then

$$\mathfrak{M}_{3,3,2}(q, \mathfrak{B}) = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ r & s & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & t & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \;\middle|\; r, s, t \in GF(q) \right\}.$$

1.4.8 Proposition: *Suppose that $a, b, k \in \mathbb{N}$ with $1 \leq k \leq \min\{a, b\}$ and $\mathfrak{B} = (v_1, \ldots, v_k)$ is a k-tuple of linearly independent vectors of the vector space $GF(q)^b$. Then*

$$r_{a,b,k}(q, \mathfrak{B}) = \begin{bmatrix} a \\ k \end{bmatrix}.$$

Proof: Let $\mathfrak{M}(a, k, q)$ denote the set of matrices $M = (m_{ij}) \in \mathfrak{M}_{a,k}(q)$ with the following property: There exist natural numbers $1 \leq a_1 < a_2 < \ldots < a_k \leq a$ such that, for $1 \leq i \leq k$,

$$m_{a_i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad m_{j,i} = 0 \text{ if } j < a_i.$$

For example, with the data of example 1.4.7,

$$\mathfrak{M}(a, k, q) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ r & s \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; r, s, t \in GF(q) \right\}.$$

Furthermore, we get the element B of $\mathfrak{M}_{k,b}(q)$ if we consecutively insert the vectors $v_1, v_2, \ldots, v_k$ into the rows of a matrix.
Then the following map

$$\begin{aligned} \mathfrak{M}(a, k, q) &\longrightarrow \mathfrak{M}_{a,b,k}(q, \mathfrak{B}) \\ M &\longmapsto MB \end{aligned}$$

is a bijection between the sets $\mathfrak{M}(a, k, q)$ and $\mathfrak{M}_{a,b,k}(q, \mathfrak{B})$.
On the other hand, we have a bijection between $\mathfrak{M}(a, k, q)$ and the set $\mathfrak{V}(a, k, q)$ of k-dimensional subspaces of the vector space $GF(q)^a$: The column span of an element $M \in \mathfrak{M}(a, k, q)$ is an element of $\mathfrak{V}(a, k, q)$ and we get every element of $\mathfrak{V}(a, k, q)$ exactly once since the columns of $M \in \mathfrak{M}(a, k, q)$ are in a kind of

"normal form".
Thus

$$r_{a,b,k}(q,\mathfrak{B}) = |\mathfrak{M}_{a,b,k}(q,\mathfrak{B})| = |\mathfrak{M}(a,k,q)| = |\mathfrak{V}(a,k,q)| \overset{1.4.4}{=} \begin{bmatrix} a \\ k \end{bmatrix}.$$

■

1.5 λ-flags

To define the Specht module S^λ it is necessary to know what a λ-flag is. According to [11] we have the following definition.

1.5.1 Definition: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a composition of n. Then a set of subspaces $V_0, V_1, V_2, \ldots, V_h$ of the vector space $V = GF(q)^n$ with the properties

$$V = V_0 \geq V_1 \geq \ldots \geq V_{h-1} \geq V_h = 0 \quad \text{and} \quad dim(V_{i-1}/V_i) = \lambda_i \ (1 \leq i \leq h)$$

is called a λ-flag. The set of λ-flags is denoted by $\mathcal{F}(\lambda)$.

Now we want to find a manageable notation for such a λ-flag.

1.5.2 Definition: Suppose, that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is a composition of n and $\{V_0, V_1, V_2, \ldots, V_h\}$ a λ-flag. Obviously we can find a basis $\{v_1, v_2, \ldots, v_n\}$ of V, such that the last λ_h vectors form a basis of V_{h-1}, the last $\lambda_{h-1} + \lambda_h$ vectors a basis of V_{h-2}, the last $\lambda_{h-2} + \lambda_{h-1} + \lambda_h$ vectors a basis of V_{h-3} and so on. Such a basis is called adapted to the flag.

Following Gordon James and Richard Dipper we write the λ-flag as an $n \times n$ matrix (v_{ij}) over $GF(q)$ by consecutively inserting the basis vectors $\{v_1, v_2, \ldots, v_n\}$ into the rows. To mark the different subspaces we draw lines after the first λ_1 rows, the first $\lambda_1 + \lambda_2$ rows, ... and finally the first $\sum_{i=1}^{h-1} \lambda_i$ rows.

1.5.3 Example: Let $\lambda = (1,2,1)$ and $\{e_1, e_2, e_3, e_4\}$ the natural basis of the vector space $V = GF(q)^4$ over $GF(q)$. The set $\{V_0, V_1, V_2, V_3\}$ with

$$V_0 = V, V_1 = \langle e_1, e_3, e_4 \rangle, V_2 = \langle e_3 \rangle, V_3 = 0$$

forms a λ-flag. In our new notation we write

$$\begin{array}{|cccc|} \hline 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array}\,.$$

This notation is obviously not unique since in general the choice of the basis $\{v_1, v_2, \ldots, v_n\}$ is not unique. But if we require some properties to hold for the basis we get a unique notation. For this purpose we generalize the set $\Xi_{m,n}$ of [7].

1.5.4 Definition: Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ be a composition of n. Then we denote by Ξ_λ the set of $n \times n$ matrices

$$\begin{array}{|cccc|l}
\hline
x^{(1)}_{1,1} & x^{(1)}_{1,2} & \cdots & x^{(1)}_{1,n} & \\
x^{(1)}_{2,1} & x^{(1)}_{2,2} & \cdots & x^{(1)}_{2,n} & \\
\vdots & \vdots & & \vdots & 1^{st} \text{ row segment} \\
x^{(1)}_{\lambda_1,1} & x^{(1)}_{\lambda_1,2} & \cdots & x^{(1)}_{\lambda_1,n} & \\
\hline
x^{(2)}_{1,1} & x^{(2)}_{1,2} & \cdots & x^{(2)}_{1,n} & \\
\vdots & \vdots & & \vdots & 2^{nd} \text{ row segment} \\
x^{(2)}_{\lambda_2,1} & x^{(2)}_{\lambda_2,2} & \cdots & x^{(2)}_{\lambda_2,n} & \\
\hline
\vdots & \vdots & & \vdots & \\
\hline
x^{(h)}_{1,1} & x^{(h)}_{1,2} & \cdots & x^{(h)}_{1,n} & \\
\vdots & \vdots & & \vdots & h^{th} \text{ row segment} \\
x^{(h)}_{\lambda_h,1} & x^{(h)}_{\lambda_h,2} & \cdots & x^{(h)}_{\lambda_h,n} & \\
\hline
\end{array} \tag{1.6}$$

over $GF(q)$ with the property that there exist integers $1 \leq j_1^{(k)} < j_2^{(k)} < \dots < j_{\lambda_k}^{(k)} \leq n$ $(1 \leq k \leq h)$ such that for all $1 \leq k \leq h$ and all $1 \leq i \leq \lambda_k$

- $x^{(k)}_{i,j_i^{(k)}} = 1$ and $x^{(k)}_{i,j} = 0$ if $j > j_i^{(k)}$ (to the right of entry $x^{(k)}_{i,j_i^{(k)}} = 1$ are only zeros) and
- $x^{(s)}_{r,j_i^{(k)}} = 0$ if either $s = k$ and $r \neq i$ or $s < k$ (in the column of entry $x^{(k)}_{i,j_i^{(k)}} = 1$ are the only other nonzero entries in row segments to numbers bigger than k).

We call $x^{(k)}_{i,j_i^{(k)}} = 1$ the last one in its row.

1.5.5 Remark: This implies in particular that all the $j_i^{(k)}$ for $1 \leq k \leq h$ and $1 \leq i \leq \lambda_k$ are pairwise different and hence the ones $x^{(k)}_{i,j_i^{(k)}} = 1$ are all in pairwise different columns. Moreover we have

$$\{j_i^{(k)} \mid 1 \leq k \leq h, 1 \leq i \leq \lambda_k\} = \{1, 2, \dots n\}$$

and therefore every element of Ξ_λ is invertible.

1.5.6 Proposition: *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ be a composition of n and $\{V_0, V_1, V_2, \dots, V_h\}$ a λ-flag. Then there exists exactly one adapted basis $\{v_1, v_2, \dots, v_n\}$ of the vector space $V = GF(q)^n$ with the property that if we consecutively insert the basis vectors $v_1, v_2, \dots, v_n$ into the rows of an $n \times n$ matrix, we get an element of Ξ_λ.*

Proof: Existence of the basis:
We start with an arbitrary adapted basis $\{v_1, v_2, \dots, v_n\}$ of V satisfying the first property of the proposition and consecutively insert the vectors $v_1, v_2, \dots, v_n$ into the rows of an $n \times n$ matrix whose entries are indexed as in (1.6). Beginning with the last row we execute the following actions on every row:

- Divide the row $(x_{i,1}^{(k)}, x_{i,2}^{(k)}, \ldots, x_{i,n}^{(k)})$ by the last nonzero element. The position of this element defines $j_i^{(k)}$.
- Add multiples of the new row to the other rows in the first k row segments, such that the entries in column $j_i^{(k)}$ of these rows are zero.

The first action gives us $x_{i,j_i^{(k)}}^{(k)} = 1$ and $x_{i,j}^{(k)} = 0$ if $j > j_i^{(k)}$. With the second action we have achieved the condition $x_{r,j_i^{(k)}}^{(s)} = 0$ if either $s = k$ and $r \neq i$ or $s < k$. Finally we permute the rows in every row segment to achieve $1 \leq j_1^{(i)} < j_2^{(i)} < ... < j_{\lambda_i}^{(i)} \leq n$. Since the described actions don't affect the first property, we have proved the existence.

<u>Uniqueness of the basis:</u>
Again we consecutively insert the vectors $v_1, v_2, \ldots, v_n$ into the rows of an $n \times n$ matrix. If the last $\lambda_{i+1} + \ldots + \lambda_h$ rows should form a basis of V_i $(0 \leq i \leq h-1)$ we can only perform the following actions on the rows of the matrix:

1.) multiply a row by a scalar,

2.) permute the rows in one row segment and

3.) add a multiple of one row to another row in the same or a higher row segment.

Starting with a basis that has all required properties we can't get another one since:

- action 1.) destroys the property $x_{i,j_i^{(k)}}^{(k)} = 1$,
- action 2.) destroys the property $1 \leq j_1^{(i)} < j_2^{(i)} < ... < j_{\lambda_i}^{(i)} \leq n$ and
- action 3.) destroys the property $x_{r,j_i^{(k)}}^{(s)} = 0$ if either $s = k$ and $r \neq i$ or $s < k$.

$\blacksquare$

1.5.7 Corollary: *For each composition λ we have a bijection between the set $\mathcal{F}(\lambda)$ of λ-flags and the set Ξ_λ.*

1.5.8 Definition: From now on we will write the λ-flags as elements of Ξ_λ, the corresponding element of Ξ_λ is called the normal form of a λ-flag.

In the future we will omit the first row segment since it is uniquely determined by the other ones.

1.5.9 Example: What is the first row segment of the following λ-flag?

$$\begin{array}{|cccc|}\hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline\end{array}.$$

The full notation of the λ-flag is a quadratic matrix. Hence $\lambda = (2,1,1)$. The only possible completion to an element of $\Xi_{(2,1,1)}$ is

$$\begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array}\,.$$

1.6 Finite groups with a BN-pair

Following [4] we want to introduce the concept of finite groups with a BN-pair.

1.6.1 Definition: A finite group with a BN-pair is a finite group G containing a pair of subgroups B and N satisfying the following axioms:

1.) $G = \langle B, N\rangle$

2.) $B \cap N \trianglelefteq N$

3.) Let $W = N/(B \cap N)$, and for each $w \in W$ choose a coset representative $\dot{w} \in N$. Then W is generated by a set $S = \{s_1, s_2, ..., s_k\}$ of involutions such that
$$\dot{s}_i B\dot{w} \subseteq B\dot{w}B \cup B\dot{s}_i wB$$
and
$$\dot{s}_i B\dot{s}_i \neq B$$
for each $w \in W$ and each $s_i \in S$.

We will denote $B \cap N$ by T and call the group $W = N/T$ the Weyl group associated with the BN-pair. The subgroup B is called a Borel subgroup of G.

1.6.2 Proposition: (Bruhat Decomposition) *Let G be a finite group with a BN-pair. Then*
$$G = \dot{\bigcup_{w\in W}} BwB. \tag{1.7}$$

Proof: cf. Theorem 65.4 in [4]. ∎

Let $k \in \mathbb{N}$. We define the following subgroups of $GL_k(q)$:

- $B := \{(a_{ij}) \in GL_k(q) \mid a_{ij} = 0 \text{ if } i > j\}$, the group of upper triangular matrices,
- $N :=$ group of monomial matrices in $GL_k(q)$,
- $T :=$ group of diagonal matrices in $GL_k(q)$,

where a monomial matrix is an element $g \in GL_k(q)$ such that each row and each column of g contains exactly one nonzero entry. Clearly $T = B \cap N$.

Then we get the following proposition.

1.6.3 Proposition: *Let $k \in \mathbb{N}$. The subgroups B and N defined above form a BN-pair in $GL_k(q)$, whose Weyl group W is isomorpic to the symmetric group $\mathfrak{S}_k$.*

Proof: cf. Theorem 65.10 in [4]. ■

1.6.4 Proposition: *Let $k \in \mathbb{N}$ and $A \in GL_k(q)$. Then there exist lower triangular matrices B_1 and B_2 and a permutation matrix P such that*

$$A = B_1 P B_2.$$

Proof: Let $A \in GL_k(q)$. Then the transposed matrix A^T is an element of $GL_k(q)$ as well. Proposition 1.6.2 guarantees that we find upper triangular matrices $\hat{B}_1$ and $\hat{B}_2$ and a permutation matrix $\hat{P}$ such that $A^T = \hat{B}_1 \hat{P} \hat{B}_2$. Therefore

$$A = \hat{B}_2^T \hat{P}^T \hat{B}_1^T$$

and the proposition follows directly with the lower triangular matrices $\hat{B}_2^T$ and $\hat{B}_1^T$ and the permutation matrix $\hat{P}^T$. ■

Chapter 2

Facts about Specht modules S^λ

2.1 The permutation module M^λ

All results of this section are based on [7]. But in most cases they are a little more general than the corresponding lemmas and propositions in [7].
Let λ be a composition of n and V the vector space $GF(q)^n$. Then the action of $GL_n(q)$ on V induces a permutation of the λ-flags. The advantage of writing a λ-flag as an element X of Ξ_λ is the following: The operation of $g \in GL_n(q)$ on the flag X is simply by matrix multiplication by g because the rows of the matrix Xg form a basis of the vector subspaces of the new flag in the usual way. Unfortunately Xg is not automatically an element of Ξ_λ. To bring it into normal form we must in general carry out some of the row operations described in the proof of proposition 1.5.6. The resulting matrix is denoted by $X \circ g \in \Xi_\lambda$.

2.1.1 Example: Let $\lambda = (2,2)$, $X = \begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline\end{array} \in \Xi_{(2,2)}$ and $g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Then

$$Xg = \begin{array}{|cccc|}\hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline\end{array} \quad \text{and} \quad X \circ g = \begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline\end{array}.$$

2.1.2 Definition: M^λ is defined as the vector space over F with basis Ξ_λ. The action "$\circ$" of $GL_n(q)$ turns M^λ into an $FGL_n(q)$-module.

2.1.3 Definition: If λ is a composition and $X \in \Xi_\lambda$ then $tab(X)$ denotes the row standard λ-tableau, whose i^{th} $(1 \leq i \leq h)$ row has the entries

$$j_1^{(i)} < j_2^{(i)} < \ldots < j_{\lambda_i}^{(i)}$$

(cf. definition 1.5.4). It is called the tableau which belongs to X.

2.1.4 Example:

$$tab\left(\begin{array}{|cccc|}\hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}.$$

2.1.5 Definition: Suppose that $v \in M^\lambda$, i.e. there exist elements $c_X \in F$ such that

$$v = \sum_{X \in \Xi_\lambda} c_X X.$$

1.) For each $\mathfrak{t} \in \mathcal{T}_{rs}(\lambda)$ let

$$v(\mathfrak{t}) := \sum_{X \in \Xi_\lambda | tab(X) = \mathfrak{t}} c_X X.$$

2.) If $v \neq 0$ then $last(v)$ denotes the last tableau $\mathfrak{t} \in \mathcal{T}_{rs}(\lambda)$ (with respect to the total order of definition 1.3.11) such that $v(\mathfrak{t}) \neq 0$.

3.) For $v \neq 0$ we define $top(v) := v(last(v))$.

2.1.6 Example: Suppose $\lambda = (2,1,1), v = \sum_{a \in GF(q)} \begin{array}{|cccc|}\hline 0&0&1&0\\ \hline a&0&0&1\\ \hline\end{array} - \sum_{a \in GF(q)} \begin{array}{|cccc|}\hline 0&0&0&1\\ \hline a&0&1&0\\ \hline\end{array}$ and $\mathfrak{t} = \begin{array}{|c|c|}\hline 1&2\\ \hline 4\\ \cline{1-1} 3\\ \cline{1-1}\end{array}$. Then

$$v(\mathfrak{t}) = -\sum_{a \in GF(q)} \begin{array}{|cccc|}\hline 0&0&0&1\\ \hline a&0&1&0\\ \hline\end{array},\ last(v) = \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1} 4\\ \cline{1-1}\end{array} \text{ and } top(v) = \sum_{a \in GF(q)} \begin{array}{|cccc|}\hline 0&0&1&0\\ \hline a&0&0&1\\ \hline\end{array}.$$

2.1.7 Proposition: *Suppose that $X \in \Xi_\lambda$ and $\mathfrak{t} = tab(X)$. If $\pi = (k, k+1) \in \mathfrak{S}_n$ is a basic transposition then $tab(X \circ \pi) \in \{tab(X), tab(X) \circ \pi\}$.*

Proof: The operation of π on X permutes the columns k and $k+1$ of X. We have three cases:

Case 1 $row_\mathfrak{t}(k) = row_\mathfrak{t}(k+1)$	Case 2 $row_\mathfrak{t}(k) < row_\mathfrak{t}(k+1)$	Case 3 $row_\mathfrak{t}(k) > row_\mathfrak{t}(k+1)$
$\begin{array}{\|cccccc\|}\hline &&\vdots&\vdots&&\\ &&0&0&&\\ \hline &&\vdots&\vdots&&\\ &&0&0&&\\ \dots&?&1&0&0&\dots\\ \dots&?&0&1&0&\dots\\ &&0&0&&\\ &&\vdots&\vdots&&\\ \hline &&?&?&&\\ &&\vdots&\vdots&&\\ \hline\end{array}$	$\begin{array}{\|cccccc\|}\hline &&\vdots&\vdots&&\\ &&0&0&&\\ \hline &&\vdots&\vdots&&\\ &&0&0&&\\ \dots&?&1&0&0&\dots\\ &&0&0&&\\ &&\vdots&\vdots&&\\ \hline &&\vdots&\vdots&&\\ &&?&0&&\\ \dots&?&a&1&0&\dots\\ &&?&0&&\\ &&\vdots&\vdots&&\\ \hline &&?&?&&\\ &&\vdots&\vdots&&\\ \hline\end{array}$	$\begin{array}{\|cccccc\|}\hline &&\vdots&\vdots&&\\ &&0&0&&\\ \hline &&\vdots&\vdots&&\\ &&0&0&&\\ \dots&?&0&1&0&\dots\\ &&0&0&&\\ &&\vdots&\vdots&&\\ \hline &&\vdots&\vdots&&\\ &&0&?&&\\ \dots&?&1&0&0&\dots\\ &&0&?&&\\ &&\vdots&\vdots&&\\ \hline &&?&?&&\\ &&\vdots&\vdots&&\\ \hline\end{array}$

The table illustrates for each case the element X (mainly column k and $k+1$). Hereby the question marks are arbitrary elements of $GF(q)$.
We easily conclude:

- Case 1: $tab(X \circ \pi) = \mathfrak{t}$ with $\mathfrak{t} = \mathfrak{t} \circ \pi$.
- Case 2: Note that in this case we may have a nonzero $a \in GF(q)$ immediately to the left of the last one in the row in which the last one occurs in column $k+1$. After switching columns k and $k+1$ this entry has to be cleared to obtain an element of Ξ_λ.
    - i) $a = 0 : tab(X \circ \pi) = \mathfrak{t} \circ \pi$ with $\mathfrak{t} > \mathfrak{t} \circ \pi$.
    - ii) $a \neq 0 : tab(X \circ \pi) = \mathfrak{t}$ with $\mathfrak{t} > \mathfrak{t} \circ \pi$.
- Case 3: $tab(X \circ \pi) = \mathfrak{t} \circ \pi$ with $\mathfrak{t} < \mathfrak{t} \circ \pi$.

■

2.1.8 Lemma: *Suppose that λ is a composition of n with at most 3 parts, $\pi = (k, k+1) \in \mathfrak{S}_n$ is a basic transposition and $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}_{rs}(\lambda)$. If $\mathfrak{t}_1 < \mathfrak{t}_2 < \mathfrak{t}_2 \circ \pi$ then $\mathfrak{t}_1 \circ \pi < \mathfrak{t}_2 \circ \pi$.*

Proof: We have:

- $\mathfrak{t}_1 < \mathfrak{t}_2 \Rightarrow \exists i_1$ such that

$$row_{\mathfrak{t}_1}(j) = row_{\mathfrak{t}_2}(j) \text{ for } j > i_1 \tag{2.1}$$
$$row_{\mathfrak{t}_1}(i_1) < row_{\mathfrak{t}_2}(i_1). \tag{2.2}$$

- $\mathfrak{t}_2 < \mathfrak{t}_2 \circ \pi \Rightarrow$

$$row_{\mathfrak{t}_2}(j) = row_{\mathfrak{t}_2 \circ \pi}(j) \text{ for } j \notin \{k, k+1\} \tag{2.3}$$
$$row_{\mathfrak{t}_2}(k+1) < row_{\mathfrak{t}_2 \circ \pi}(k+1) \tag{2.4}$$
$$row_{\mathfrak{t}_2}(k) > row_{\mathfrak{t}_2 \circ \pi}(k). \tag{2.5}$$

We want to prove $\mathfrak{t}_1 \circ \pi < \mathfrak{t}_2 \circ \pi$, that means there exists an i_2 such that

$$row_{\mathfrak{t}_1 \circ \pi}(j) = row_{\mathfrak{t}_2 \circ \pi}(j) \text{ for } j > i_2 \tag{2.6}$$
$$row_{\mathfrak{t}_1 \circ \pi}(i_2) < row_{\mathfrak{t}_2 \circ \pi}(i_2). \tag{2.7}$$

We have the following cases:

- $i_1 < k$, that means

$$row_{\mathfrak{t}_1}(k) = row_{\mathfrak{t}_2}(k) \tag{2.8}$$
$$row_{\mathfrak{t}_1}(k+1) = row_{\mathfrak{t}_2}(k+1). \tag{2.9}$$

  The operation of π exchanges at most k and $k+1$ in $\mathfrak{t}_1$ and $\mathfrak{t}_2$. Therefore $i_2 := i_1$ satisfies (2.6) and (2.7) because of (2.1) and (2.2).

- $i_1 = k$, that means

$$row_{\mathfrak{t}_1}(k) < row_{\mathfrak{t}_2}(k) \tag{2.10}$$
$$row_{\mathfrak{t}_1}(k+1) = row_{\mathfrak{t}_2}(k+1). \tag{2.11}$$

  In this case we set $i_2 := k+1$. Then (2.6) follows from (2.1) and (2.7) from (2.10) and (2.11).

- $i_1 = k+1$. Therefore

$$row_{\mathfrak{t}_1}(k+1) \overset{(2.2)}{<} row_{\mathfrak{t}_2}(k+1) \overset{(2.4)}{<} row_{\mathfrak{t}_2\circ\pi}(k+1).$$

  As λ has at most 3 parts and the operation of π exchanges at most k and $k+1$,

$$row_{\mathfrak{t}_1}(k+1) = 1 \tag{2.12}$$
$$row_{\mathfrak{t}_2}(k+1) = row_{\mathfrak{t}_2\circ\pi}(k) = 2 \tag{2.13}$$
$$row_{\mathfrak{t}_2\circ\pi}(k+1) = row_{\mathfrak{t}_2}(k) = 3. \tag{2.14}$$

  If $row_{\mathfrak{t}_1}(k) \in \{1,2\}$ we set $i_2 := k+1$ and if $row_{\mathfrak{t}_1}(k) = 3$ we set $i_2 := k$. (2.6) and (2.7) then follows from (2.12), (2.13), (2.14) and (2.1).

- $i_1 > k+1$. If we set $i_2 := i_1$, (2.6) and (2.7) are direct consequences of (2.1) and (2.2).

■

Indeed, we must require λ to have at most 3 parts, as we can see in the following example.

2.1.9 Example: Let $\lambda = (4,3,2,1)$, $\pi = (9,10)$. If we define

$$\mathfrak{t}_1 := \begin{array}{|c|c|c|c|}\hline 1&2&3&10\\\hline 4&5&6\\\cline{1-3} 7&8\\\cline{1-2} 9\\\cline{1-1}\end{array} \quad\text{and } \mathfrak{t}_2 := \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&10\\\cline{1-3} 7&9\\\cline{1-2} 8\\\cline{1-1}\end{array}$$

then $\mathfrak{t}_1 < \mathfrak{t}_2 < \mathfrak{t}_2\circ\pi$, but $\mathfrak{t}_1\circ\pi > \mathfrak{t}_2\circ\pi$.

In the next propositions we examine for $v \in M^\lambda$ and some special matrices $g \in GL_n(q)$ the relationship between $last(v)$ and $last(v\circ g)$.

2.1.10 Proposition: *Suppose that λ is a composition with at most 3 parts, $\mathfrak{t} \in \mathcal{T}_{rs}(\lambda)$ and $\pi \in \mathfrak{S}_n$ is a basic transposition such that $\mathfrak{t} < \mathfrak{t}\circ\pi$. If $0 \neq v \in M^\lambda$ and $last(v) = \mathfrak{t}$ then $last(v\circ\pi) = \mathfrak{t}\circ\pi$ and $top(v\circ\pi) = top(v)\circ\pi$.*

Proof: We may write

$$v = \sum_{tab(X)<\mathfrak{t}} c_X X + \sum_{tab(X)=\mathfrak{t}} d_X X,$$

where $c_X, d_X \in F$ and $d_X \neq 0$ for some X with $tab(X) = \mathfrak{t}$. There are two cases for the λ-flags X in this sum:

- $tab(X) = \mathfrak{t}$.
  In this case we can deduce from $\mathfrak{t} < \mathfrak{t}\circ\pi$ that $tab(X\circ\pi) = tab(X)\circ\pi = \mathfrak{t}\circ\pi$ (cf. case 3 of the proof of proposition 2.1.7)

- $tab(X) < \mathfrak{t}$.
  Let $\tilde{\mathfrak{t}} := tab(X)$. Then $tab(X\circ\pi) \in \{\tilde{\mathfrak{t}}, \tilde{\mathfrak{t}}\circ\pi\}$ by proposition 2.1.7. Since $\tilde{\mathfrak{t}} < \mathfrak{t} < \mathfrak{t}\circ\pi$ we can conclude, using lemma 2.1.8, that $\tilde{\mathfrak{t}}\circ\pi < \mathfrak{t}\circ\pi$. Altogether we get $tab(X\circ\pi) < \mathfrak{t}\circ\pi$.

We obtain

$$v\circ\pi = \sum_{tab(X\circ\pi)<\mathfrak{t}\circ\pi} c_X(X\circ\pi) + \sum_{tab(X\circ\pi)=\mathfrak{t}\circ\pi} d_X(X\circ\pi).$$

Therefore $last(v\circ\pi) = \mathfrak{t}\circ\pi$ and $top(v\circ\pi) = top(v)\circ\pi$. ■

2.1.11 Proposition: *Suppose that λ is a composition, $0 \neq v \in M^\lambda$ and $last(v) = \mathfrak{t}$. If $g \in GL_n(q)$ is a lower triangular matrix then $last(v\circ g) = \mathfrak{t}$ and $top(v\circ g) = top(v)\circ g$.*

Proof: We may write

$$v = \sum_{tab(X)<\mathfrak{t}} c_X X + \sum_{tab(X)=\mathfrak{t}} d_X X,$$

where $c_X, d_X \in F$ and $d_X \neq 0$ for some X with $tab(X) = \mathfrak{t}$.
We have $tab(X\circ g) = tab(X)$ for every λ-flag X because g is lower triangular.
We obtain

$$v\circ g = \sum_{tab(X)<\mathfrak{t}} c_X(X\circ g) + \sum_{tab(X)=\mathfrak{t}} d_X(X\circ g).$$

Therefore $last(v\circ g) = \mathfrak{t}$ and $top(v\circ g) = top(v)\circ g$. ■

2.1.12 Proposition: *Suppose that λ is a composition with at most 2 parts, $0 \neq v \in M^\lambda$ and $last(v) = \mathfrak{t}$. If there are integers $x, l \in \mathbb{N}$ such that*

$$row_{\mathfrak{t}}(x) = row_{\mathfrak{t}}(x+1) = \ldots = row_{\mathfrak{t}}(x+l)$$

and $\pi \in \mathfrak{S}_n$ is a permutation such that $j\pi = \pi$ for all $j \notin \{x, x+1, \ldots, x+l\}$ then $last(v\circ\pi) = \mathfrak{t}$ and $top(v\circ\pi) = top(v)\circ\pi$.

Proof: We may write

$$v = \sum_{tab(X)<\mathfrak{t}} c_X X + \sum_{tab(X)=\mathfrak{t}} d_X X,$$

where $c_X, d_X \in F$ and $d_X \neq 0$ for some X with $tab(X) = \mathfrak{t}$.
It is enough to prove the proposition for a basic transposition $\pi = (k, k+1) \in \mathfrak{S}_n$ since we can write π as product of basic transpositions. The restriction $j\pi = \pi$ for all $j \notin \{x, x+1, \ldots, x+l\}$ is equivalent to $x \leq k < x+l$. There are two cases for the λ-flags X in the above sum:

- $tab(X) = \mathfrak{t}$.
  We are in case 1 of the proof of proposition 2.1.7 since $row_{\mathfrak{t}}(k) = row_{\mathfrak{t}}(k+1)$. Therefore $tab(X \circ \pi) = \mathfrak{t}$.

- $tab(X) < \mathfrak{t} \Rightarrow$ there exists an i such that

$$row_{tab(X)}(j) = row_{\mathfrak{t}}(j) \text{ for } j > i$$
$$row_{tab(X)}(i) < row_{\mathfrak{t}}(i).$$

  We want to show that $tab(X \circ \pi) < \mathfrak{t}$. This is immediately clear if we don't have $tab(X \circ \pi) > tab(X)$. Therefore we only need to look at case 3 of the proof of proposition 2.1.7, i.e. $row_{tab(X)}(k) > row_{tab(X)}(k+1)$. Together with the facts that $row_{\mathfrak{t}}(k) = row_{\mathfrak{t}}(k+1)$ and we only deal with 2 rows since λ has at most two parts we have two possibilities:

1.)

	$\mathfrak{t}$	$\mathfrak{t} \circ \pi$	$tab(X)$	$tab(X \circ \pi)$
k in row	1	1	2	1
$k+1$ in row	1	1	1	2

Then $tab(X) < \mathfrak{t} \Rightarrow i > k+1 \Rightarrow tab(X \circ \pi) < \mathfrak{t}$.

2.)

	$\mathfrak{t}$	$\mathfrak{t} \circ \pi$	$tab(X)$	$tab(X \circ \pi)$
k in row	2	2	2	1
$k+1$ in row	2	2	1	2

This means

$$row_{tab(X \circ \pi))}(k) < row_{\mathfrak{t}}(k)$$
$$row_{tab(X \circ \pi)}(k+1) = row_{\mathfrak{t}}(k+1).$$

Since $tab(X) < \mathfrak{t}$ and the operation of π exchanges at most the integers k and $k+1$ we get $tab(X \circ \pi) < \mathfrak{t}$.

We obtain

$$v \circ \pi = \sum_{tab(X \circ \pi) < \mathfrak{t}} c_X (X \circ \pi) + \sum_{tab(X) = \mathfrak{t}} d_X (X \circ \pi).$$

Therefore $last(v \circ \pi) = \mathfrak{t}$ and $top(v \circ \pi) = top(v) \circ \pi$. ∎

We can't omit the requirement that λ has at most 2 parts. The following example shows that, for compositions with more parts, the proposition is wrong in general.

2.1.13 Example: Let $\lambda = (4, 3, 1)$. We set

$$g := (7,8) \in \mathfrak{S}_8 \text{ and } v := \begin{array}{|cccccccc|} \hline 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ \hline 0&0&0&0&1&0&0&0 \\ \hline \end{array} + \begin{array}{|cccccccc|} \hline 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ \hline 0&0&0&0&0&0&1&0 \\ \hline \end{array} \in M^\lambda.$$

Then $last(v) = \begin{array}{|c|c|c|c|} \hline 1&2&3&6 \\ \hline 4&7&8 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$, $x = 7$ and $l = 1$ satisfy the assumptions of proposition 2.1.12.

We get

$$v \circ g = \begin{array}{|cccccccc|}\hline 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ \hline 0&0&0&0&1&0&0&0\\ \hline\end{array} + \begin{array}{|cccccccc|}\hline 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ \hline 0&0&0&0&0&0&0&1\\ \hline\end{array}.$$

But this means $last(v \circ g) = \begin{array}{|c|c|c|c|}\hline 1&2&3&7\\ \hline 4&5&6\\ \cline{1-3} 8\\ \cline{1-1}\end{array} \neq last(v)$.

2.2 The Specht module S^λ

According to [7] and [11] we define the unipotent Specht module S^λ as a kernel intersection.

2.2.1 Definition: If λ is any composition of n, then S^λ is defined by

$$S^\lambda := \bigcap_{\mu \rhd \lambda} \{ker\ \theta \mid \theta \in Hom_{FGL_n(q)}(M^\lambda, M^\mu)\}.$$

If λ is a partition we can specify the homomorphisms in the kernel intersection more precisely.

2.2.2 Definition: Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is a partition, $1 \leq d \leq h-1$ and $0 \leq i \leq \lambda_d$. We set

$$\mu := (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}, \lambda_d + \lambda_{d+1} - i, i, \lambda_{d+2}, \ldots, \lambda_h).$$

Then the $FGL_n(q)$-homomorphism

$$\psi_{d,i} : M^\lambda \to M^\mu$$

sends a λ-flag

$$V = V_0 \geq V_1 \geq \ldots \geq V_{d-1} \geq V_d \geq V_{d+1} \geq \ldots \geq V_h = 0$$

to the sum of all μ-flags

$$V = V_0 \geq V_1 \geq \ldots \geq V_{d-1} \geq W_d \geq V_{d+1} \geq \ldots \geq V_h = 0$$

with the property that $W_d \leq V_d$.

2.2.3 Proposition: *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition. Then the Specht module S^λ is given by*

$$S^\lambda = \bigcap_{d=2}^{h} \bigcap_{i=0}^{\lambda_d - 1} ker\ \psi_{d-1,i}.$$

Proof: cf. [11]. ■

Our work relies heavily on the following theorem of Sinéad Lyle.

2.2.4 Theorem: *Suppose that* $0 \neq v \in S^\lambda$. *Then* $last(v)$ *is a standard* λ*-tableau.*

Proof: cf. [14]. ∎

Therefore the next definition makes sense.

2.2.5 Definition: A basis $\mathfrak{B}^\lambda = \{b_i \mid i \in \mathfrak{I}\}$ of the $FGL_n(q)$-module S^λ together with a set of polynomials $\{p_\mathfrak{t}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ is called a standard basis if

1.) it is defined independently of the concrete choice of the field F,

2.) the elements $\{top(b_i) \mid i \in \mathfrak{I}\}$ of M^λ are linearly independent and

3.) $p_\mathfrak{t}(q) = |\{b \in \mathfrak{B}^\lambda \mid last(b) = \mathfrak{t}\}|$ and $p_\mathfrak{t}(1) = 1$ holds for every $\mathfrak{t} \in Std(\lambda)$.

The polynomials $\{p_\mathfrak{t}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ are called the corresponding polynomials of the standard basis $\mathfrak{B}^\lambda$.

2.2.6 Remark: The second postulation in the definition 2.2.5 of a standard basis of S^λ (linear independence of $\{top(b_i) \mid i \in \mathfrak{I}\}$) provides that the polynomials $p_\mathfrak{t}(q)$ are well defined.
Otherwise, suppose that $e_1, e_2, \ldots, e_q$ is a part of a basis $\mathfrak{B}^\lambda$ coming from $\mathfrak{t}_1$ and $f_1, f_2, \ldots, f_q$ is another part of $\mathfrak{B}^\lambda$ coming from $\mathfrak{t}_2$ (giving $p_{\mathfrak{t}_1} = p_{\mathfrak{t}_2} = q$). Then we could equally well use $e_1, e_1 + f_1, e_2 + f_1, \ldots, e_q + f_1, f_2, f_3, \ldots, f_q$ (giving $p_{\mathfrak{t}_1} = 1$ and $p_{\mathfrak{t}_2} = 2q - 1$).

According to [8] we can calculate the dimension of the Specht module S^λ explicitly.

2.2.7 Proposition: *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ *be a partition of* n *and* h_{ij} *the hook length for the* (i,j)*-node in* $[\lambda]$. *Then we have*

$$dim(S^\lambda) = q^k \frac{[n]!}{\prod\limits_{(i,j)\in[\lambda]} [h_{ij}]}, \tag{2.15}$$

where $k := \sum\limits_{l=1}^{h} (l-1)\lambda_l$.

Proof: cf. [8] ∎

2.3 The branching theorem

In this section we follow completely [11] and introduce an important tool to understand the structure of the Specht module S^λ, namely the branching theorem.

2.3.1 Definition: We denote by I_k ($k > 0$) the identity matrix in $GL_k(q)$.

2.3.2 Definition: Suppose that $1 \leq r \leq n$.

1.) Let $\Gamma(r) := \{(i,j) \mid 1 \le j < i \le n, j \le r\}$.

2.) We denote by $G(\Gamma(r))$ the set of matrices $A = (a_{ij}) \in GL_n(q)$ with the following property:

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \text{ and } (i,j) \notin \Gamma(r). \end{cases}$$

3.) Let

$$E_r := \frac{1}{q^{|\Gamma(r)|}} \sum_{A=(a_{ij}) \in G(\Gamma(r))} \prod_{k=1}^{r-1} \theta(a_{k+1,k}) A.$$

E_r is an element of $FGL_n(q)$.

4.) We define three subsets of $GL_n(q)$ by

$$G_r(q) := \left\{ \begin{pmatrix} I_{n-r} & 0 \\ 0 & A \end{pmatrix} \;\middle|\; A \in GL_r(q) \right\},$$

$$G_r^*(q) := \left\{ \begin{pmatrix} I_{n-r} & 0 \\ A & B \end{pmatrix} \;\middle|\; A \in \mathfrak{M}_{r,n-r}(q), B \in GL_r(q) \right\} \text{ and}$$

$$H_r^*(q) := \left\{ \begin{pmatrix} I_{n-r} & 0 \\ A & B \end{pmatrix} \;\middle|\; A \in \mathfrak{M}_{r,n-r}(q), B = (b_{ij}) \in GL_r(q), b_{1j} = \delta_{1j} \right\}.$$

2.3.3 Example: If $n = 4$, then

$$E_1 = \frac{1}{q^3} \sum_{\underline{a} \in GF(q)^3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{pmatrix}, E_2 = \frac{1}{q^5} \sum_{\underline{a} \in GF(q)^5} \theta(a_1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_4 & 1 & 0 \\ a_3 & a_5 & 0 & 1 \end{pmatrix},$$

$$E_3 = \frac{1}{q^6} \sum_{\underline{a} \in GF(q)^6} \theta(a_1)\theta(a_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_4 & 1 & 0 \\ a_3 & a_5 & a_6 & 1 \end{pmatrix} \text{ and}$$

$$E_4 = \frac{1}{q^6} \sum_{\underline{a} \in GF(q)^6} \theta(a_1)\theta(a_4)\theta(a_6) \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_4 & 1 & 0 \\ a_3 & a_5 & a_6 & 1 \end{pmatrix}.$$

2.3.4 Definition: Assume that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is a partition of n and $1 \le r \le h$. We denote by $\mathfrak{R}_r^*$ the set of subsets R of $\{1, 2, \ldots, h\}$ which have the properties that

1.) $|R| = r$ and

2.) if $b \in R$ and $\lambda_b = \lambda_{b+1}$, then $b + 1 \in R$.

For distinct elements R_1, R_2 of $\mathfrak{R}_r^*$, we write $R_1 < R_2$ if the largest element of $R_1 \setminus R_2$ is less than the largest element of $R_2 \setminus R_1$.
For each $R \in \mathfrak{R}_r^*$, we define the partition λ_R of $n-r$ by

$$\lambda_R = (\lambda_1 - \varepsilon_1, \lambda_2 - \varepsilon_2, \dots, \lambda_h - \varepsilon_h),$$

where

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \in R \\ 0 & \text{if } i \notin R. \end{cases}$$

2.3.5 Theorem: (Branching theorem)
If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ is a partition of n, then as $FG_{n-1}^*(q)$-modules

$$S^\lambda = S^\lambda E_1 \oplus \sum_{r=2}^{h} S^\lambda E_r FG_{n-1}^*(q)$$

and $S^\lambda E_i$ $(1 \le i \le h)$ has a series of $FG_{n-i}(q)$-submodules

$$S^\lambda E_i = S_k > S_{k-1} > \dots > S_1 > S_0 = 0,$$

such that for each j with $1 \le j \le k$, S_j/S_{j-1} is $FG_{n-i}(q)$-isomorphic to $S^{\lambda_{R_j}}$ (here $R_1 < R_2 < \dots < R_k$ are the elements of $\mathfrak{R}_i^*$ ordered as in definition 2.3.4).

Proof: cf. [11] ■

Furthermore, we have the following dimension formulas.

2.3.6 Proposition: *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ be a partition of n and $1 \le r \le h$. Then*

$$dim(S^\lambda E_r) = \sum_{R \in \mathfrak{R}_r^*} dim(S^{\lambda_R})$$

and

$$dim(S^\lambda E_r FG_{n-1}^*(q)) = (q^{n-1} - 1)(q^{n-2} - 1) \dots (q^{n-r+1} - 1) dim(S^\lambda E_r).$$

For the dimension of the Specht module we obtain

$$dim(S^\lambda) = \sum_{r=1}^{h} (q^{n-1} - 1)(q^{n-2} - 1) \dots (q^{n-r+1} - 1) \sum_{R \in \mathfrak{R}_r^*} dim(S^{\lambda_R}).$$

Proof: cf. [11] ■

Chapter 3

The Specht modules $S^{(n-m,m)}$

In this chapter m is an arbitrary but fixed integer with $1 \leq m \leq \frac{1}{2}n$. Therefore $(n-m,m)$ is a partition. From now on we set $\lambda := (n-m,m)$ unless stated otherwise.

3.0.7 Goal: Find a standard basis of $S^{(n-m,m)}$.

3.1 The permutation module $M^{(n-m,m)}$

We recall that the permutation module M^λ is defined as a vector space over F with basis $\mathcal{F}(\lambda)$ or equivalently Ξ_λ. Therefore the dimension of M^λ equals the number of m-dimensional subspaces of the n-dimensional vector space $GF(q)^n$, which is $\begin{bmatrix} n \\ m \end{bmatrix}$ according to proposition 1.4.4.
To simplify the notation of elements of Ξ_λ and M^λ we introduce some conventions.

3.1.1 Convention: We don't write the first row segment.

3.1.2 Example:

$$\begin{array}{|cccccc|} \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} := \begin{array}{|cccccc|} \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}.$$

3.1.3 Convention: Zeros are written as dots.

3.1.4 Example:

$$\begin{array}{|cccccc|} \hline \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline \end{array} := \begin{array}{|cccccc|} \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}.$$

3.1.5 Convention: Stars indicate the summation over $GF(q)$ (as illustrated in the following example).

3.1.6 Example:

$$\begin{array}{|cccccc|}\hline \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & 1 & \cdot & \cdot \\ * & \cdot & * & \cdot & \cdot & 1 \\ \hline\end{array} := \sum_{a,b,c\in GF(q)} \begin{array}{|cccccc|}\hline \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & 1 & \cdot & \cdot \\ b & \cdot & c & \cdot & \cdot & 1 \\ \hline\end{array}.$$

3.1.7 Convention: We omit the columns $j_1^{(2)}, j_2^{(2)}, \ldots, j_m^{(2)}$ with the ones (cf. definition 1.5.4) and mark the area which was formerly to the right of the ones with one line. By definition, this area has only zero entries. Hence the dots are unnecessary and we don't write them.

3.1.8 Example: If $x_1, x_2, x_3, x_4 \in GF(q)$ then

$$\begin{array}{|ccc|}\hline \cdot & & \\ * & \cdot & \\ * & * & \cdot \\ \hline\end{array} := \sum_{a,b,c\in GF(q)} \begin{array}{|cccccc|}\hline \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & 1 & \cdot & \cdot \\ b & \cdot & c & \cdot & \cdot & 1 \\ \hline\end{array} \text{ and}$$

$$\begin{array}{|cccc|}\hline & & & \\ x_1 & x_2 & & \\ x_3 & x_4 & & \\ \hline\end{array} := \begin{array}{|ccccccc|}\hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x_1 & x_2 & 1 & \cdot & \cdot & \cdot \\ \cdot & x_3 & x_4 & \cdot & 1 & \cdot & \cdot \\ \hline\end{array}.$$

We don't get in trouble with this short notation since we can reconstruct the element of Ξ_λ easily: We just insert the m columns with the ones according to the heights of the defining line.

3.1.9 Definition:

1.) Let $X \in \Xi_\lambda$. After applying the conventions 3.1.1 and 3.1.7 the notation of X is an $m \times (n-m)$ array A with a defining line starting at the top left vertex of A and ending at the bottom right vertex (parts of this line overlap with the boundary of A). We denote this object with $short(X)$ and call it the short notation of X.

2.) Let $v \in M^\lambda$. If we can write v with conventions 3.1.1, 3.1.5 and 3.1.7 as a single $m \times (n-m)$ array A (again with a defining line; but now the entries can be stars as well as elements of $GF(q)$) we denote this object with $short(v)$ and call it the short notation of v.

3.1.10 Example: If we write $\begin{array}{|ccc|}\hline \cdot & & \\ * & \cdot & \\ * & * & \cdot \\ \hline\end{array}$ then we mean the linear combination of flags

$$\sum_{a,b,c\in GF(q)} \begin{array}{|cccccc|}\hline \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & 1 & \cdot & \cdot \\ b & \cdot & c & \cdot & \cdot & 1 \\ \hline\end{array}.$$

But $short(\begin{array}{|ccc|}\hline \cdot & & \\ * & \cdot & \\ * & * & \cdot \\ \hline\end{array})$ is simply a 3×3 array with zero and star entries together with a defining line.

3.2 Basic properties of $S^{(n-m,m)}$

Following [7] we get definition 3.2.1 and proposition 3.2.2. But first we recall from definition 1.3.7 that

$$\mathfrak{t}_\lambda = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \dots & 2m-1 & 2m+1 & \dots & n \\ \hline 2 & 4 & 6 & \dots & 2m & \multicolumn{3}{c}{} \\ \cline{1-5}\end{array}.$$

3.2.1 Definition: Let S be the set of all matrices $X = (x_{ij}) \in \Xi_\lambda$ such that $tab(X) = \mathfrak{t}_\lambda$ and

$$x_{n-m+1,1} = x_{n-m+2,3} = x_{n-m+3,5} = \ldots = x_{n,2m-1} = 0.$$

Define e_λ to be the following element of M^λ.

$$\begin{aligned} e_\lambda &:= \sum_{X \in S, \pi \in C_{\mathfrak{t}_\lambda}} sign(\pi)(X \circ \pi) \\ &= \sum_{X \in S} X \circ ((1-(1,2))(1-(3,4))\dots(1-(2m-1,2m))). \end{aligned}$$

3.2.2 Proposition: *e_λ belongs to S^λ.*

Proof: Let $A = (a_{ij}) \in GL_n(q)$ be the matrix which agrees with the identity matrix, except that

$$a_{12} = a_{34} = a_{56} = \ldots = a_{2m-1,2m} = -1.$$

Then $e_\lambda = u \circ A$, where u is the generator for S^λ described in 11.17(v) of [11]. ■

3.2.3 Example:

1.) If $\lambda = (3,1)$ then

$$e_\lambda = \begin{array}{|c|c|}\hline \cdot & \quad \\ \hline\end{array} - \begin{array}{|c|}\hline \quad\quad \\ \hline\end{array}, last(e_\lambda) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \text{ and } top(e_\lambda) = \begin{array}{|c|c|}\hline \cdot & \quad \\ \hline\end{array}.$$

2.) If $\lambda = (3,2)$ then

$$e_\lambda = \begin{array}{|cc|}\hline \cdot & \\ * & \cdot \\ \hline\end{array} - \begin{array}{|cc|}\hline & \\ * & \cdot \\ \hline\end{array} - \begin{array}{|c|c|}\hline \cdot & \\ * & \\ \hline\end{array} + \begin{array}{|cc|}\hline & \\ * & \\ \hline\end{array},$$

$$last(e_\lambda) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \text{ and } top(e_\lambda) = \begin{array}{|cc|}\hline \cdot & \\ * & \cdot \\ \hline\end{array}.$$

3.) If $\lambda = (3,3)$ then

$$e_\lambda = \begin{array}{|ccc|}\hline \cdot & & \\ * & \cdot & \\ * & * & \cdot \\ \hline\end{array} \circ ((1-(1,2))(1-(3,4))(1-(5,6))),$$

$$last(e_\lambda) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array} \text{ and } top(e_\lambda) = \begin{array}{|ccc|}\hline \cdot & & \\ * & \cdot & \\ * & * & \cdot \\ \hline\end{array}.$$

3.2.4 Lemma: *Let S be the same set as in definition 3.2.1. Then*

$$last(e_\lambda) = \mathfrak{t}_\lambda \text{ and } top(e_\lambda) = \sum_{X\in S} X.$$

Proof: Let π be a nontrivial permutation in $C_{\mathfrak{t}_\lambda} \leq \mathfrak{S}_n$ and $X \in S$. Then there exists a maximal integer $1 \leq k \leq n$ such that $k\pi \neq k$. As $C_{\mathfrak{t}_\lambda}$ is generated by the basic transpositions $(1,2), (3,4), \ldots$ and $(2m-1, 2m)$ we know that

- k is an even number,
- $row_{tab(X\circ\pi)}(k) = 1$ and $row_{\mathfrak{t}_\lambda}(k) = 2$,
- $row_{tab(X\circ\pi)}(i) = row_{\mathfrak{t}_\lambda}(i)$ for $i > k$.

Hence $tab(X\circ\pi) < \mathfrak{t}_\lambda$. Since $tab(X) = \mathfrak{t}_\lambda$ for all $X \in S$ we know that $last(e_\lambda) = \mathfrak{t}_\lambda$ and $top(e_\lambda) = \sum_{X\in S} X$. ■

3.2.5 Definition: Suppose that $\mathfrak{t} \in Std(\lambda)$ and row 2 of $\mathfrak{t}$ consists of the integers $j_1 < j_2 < \ldots < j_m$. Then, for $1 \leq i \leq m$, we set

$$\begin{aligned} \pi_{\mathfrak{t}}^{(i)} &:= (2i, 2i+1)(2i+1, 2i+2)\ldots(j_i - 1, j_i) \in \mathfrak{S}_n, \\ \pi_{\mathfrak{t}} &:= \pi_{\mathfrak{t}}^{(m)}\pi_{\mathfrak{t}}^{(m-1)}\ldots\pi_{\mathfrak{t}}^{(2)}\pi_{\mathfrak{t}}^{(1)}, \\ v_{\mathfrak{t}} &:= e_\lambda \circ \pi_{\mathfrak{t}}. \end{aligned}$$

3.2.6 Proposition: *Let $\mathfrak{t} \in Std(\lambda)$. Then*

$$last(v_{\mathfrak{t}}) = \mathfrak{t} \text{ and } top(v_{\mathfrak{t}}) = top(e_\lambda) \circ \pi_{\mathfrak{t}}.$$

Proof: By construction $\pi_{\mathfrak{t}}$ has the following properties:

- Since row 2 of $\mathfrak{t}_\lambda$ is $\{2, 4, 6, \ldots, 2m\}$ and the operation of $\pi_{\mathfrak{t}}^{(i)}$ provides that in this row $2i$ is replaced with j_i $(1 \leq i \leq m)$ we have

$$\mathfrak{t}_\lambda \circ \pi_{\mathfrak{t}} = \mathfrak{t}. \tag{3.1}$$

- $\pi_{\mathfrak{t}}$ is a product of basic transpositions. If we write this product as

$$\pi_{\mathfrak{t}} = \pi_1\pi_2\ldots\pi_k$$

  we get for $1 \leq i \leq k$ and $\pi_0 := ()$

$$\mathfrak{t}_\lambda \circ (\pi_1\pi_2\ldots\pi_{i-1}) < \mathfrak{t}_\lambda \circ (\pi_1\pi_2\ldots\pi_{i-1}\pi_i) \tag{3.2}$$

  because each of the basic transpositions π_i shifts a certain integer x from row 1 to row 2 and $x-1$ from row 2 to row 1.
  Since $\mathfrak{t}_\lambda$ is a standard λ-tableau, induction using the same arguments yields

$$\mathfrak{t}_\lambda \circ (\pi_1\pi_2\ldots\pi_{i-1}\pi_i) \in Std(\lambda). \tag{3.3}$$

With (3.1), (3.2) and (3.3) the proposition follows directly from proposition 2.1.10. ■

This means that we have found for each tableau $\mathfrak{t} \in Std(\lambda)$ an element $v_{\mathfrak{t}} \in S^{\lambda}$ with $last(v_{\mathfrak{t}}) = \mathfrak{t}$.

3.2.7 Example:

1.) $\lambda = (3,1)$

$\mathfrak{t}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$
$top(v_{\mathfrak{t}})$	$\begin{array}{ccc}\cdot & & \end{array}$	$\begin{array}{ccc}\cdot & \cdot & \end{array}$	$\begin{array}{ccc}\cdot & \cdot & \cdot\end{array}$

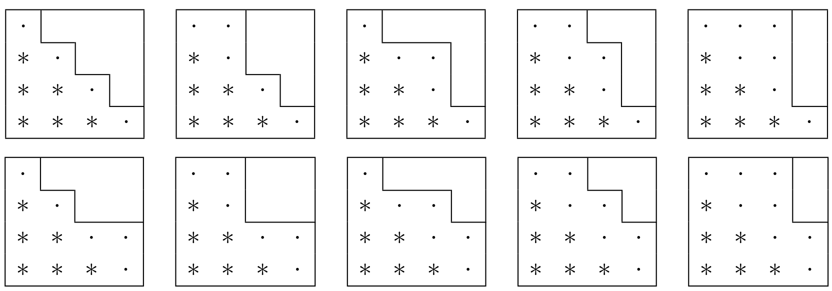
2.) $\lambda = (3,2)$

$\mathfrak{t}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$
$top(v_{\mathfrak{t}})$	$\begin{array}{ccc}\cdot & & \\ * & \cdot & \end{array}$	$\begin{array}{ccc}\cdot & \cdot & \\ * & \cdot & \end{array}$	$\begin{array}{ccc}\cdot & & \\ * & \cdot & \cdot\end{array}$	$\begin{array}{ccc}\cdot & \cdot & \\ * & \cdot & \cdot\end{array}$	$\begin{array}{ccc}\cdot & \cdot & \cdot \\ * & \cdot & \cdot\end{array}$

3.) $\lambda = (3,3)$

$\mathfrak{t}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}$
$top(v_{\mathfrak{t}})$	$\begin{array}{ccc}\cdot & & \\ * & \cdot & \\ * & * & \cdot\end{array}$	$\begin{array}{ccc}\cdot & \cdot & \\ * & \cdot & \\ * & * & \cdot\end{array}$	$\begin{array}{ccc}\cdot & & \\ * & \cdot & \cdot \\ * & * & \cdot\end{array}$	$\begin{array}{ccc}\cdot & \cdot & \\ * & \cdot & \cdot \\ * & * & \cdot\end{array}$	$\begin{array}{ccc}\cdot & \cdot & \cdot \\ * & \cdot & \cdot \\ * & * & \cdot\end{array}$

4.) $\lambda = (4,4)$. Summary of the different $top(v_{\mathfrak{t}})$ for $\mathfrak{t} \in Std(\lambda)$:



$$\begin{array}{cccc} \cdot & & & \\ * & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{array} \quad \begin{array}{cccc} \cdot & \cdot & & \\ * & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{array} \quad \begin{array}{cccc} \cdot & \cdot & \cdot & \\ * & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{array} \quad \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & \cdot \end{array} .$$

This example suggests that $short(top(v_{\mathfrak{t}}))$ for $\mathfrak{t} \in Std(\lambda)$ is an $m \times (n-m)$ array that consists only of dot entries, star entries and empty entries together with a defining line. The next lemma characterizes this array more precisely.

3.2.8 Lemma: *An $m \times (n-m)$ array $A = (a_{ij})$ that consists of dot entries, star entries and empty entries and that has a defining line from the top left vertex to the bottom right vertex that separates the dot and star entries from the empty entries equals $short(top(v_{\mathfrak{t}}))$ for a tableau $\mathfrak{t} \in Std(\lambda)$ if and only if*

1.) a_{ij} is a star if and only if $1 \leq j < i \leq m$,

2.) a_{ij} is a dot for $1 \leq j = i \leq m$ and

3.) if we start at the top left vertex of A then the defining line runs on its way to the bottom right vertex of A only to the right and to the bottom.

Proof: " $\Rightarrow$ " : Let $\mathfrak{t} \in Std(\lambda)$ and $A = (a_{ij}) = short(top(v_{\mathfrak{t}}))$. The first property follows directly from the construction of $v_{\mathfrak{t}}$ and proposition 3.2.6.
How do we get the defining line? The defining line starts at the top left vertex of A. If we count from 1 to n then it runs one line segment to the right if the actual number is in the first row of $\mathfrak{t}$ and one line segment to the bottom if the actual number is in the second row of $\mathfrak{t}$. After $n-m$ line segments to the right and m line segments to the bottom the defining line ends at the right bottom vertex of A. This is the third property.
$\mathfrak{t} \in Std(\lambda)$ ensures that the defining line runs further to the right than to the bottom and therefore the second property holds.
" $\Leftarrow$ " : Let $A = (a_{ij})$ be an array for which all the required properties hold. The third property says that the defining line starting from the left top vertex runs only to the right and to the bottom. Therefore we can construct a λ-tableau $\mathfrak{t}$ if we follow the defining line from the top left vertex of A to the bottom right vertex and insert for each line segment successively one of the numbers from 1 to n into the nodes of $[\lambda]$: For each horizontal line segment we insert the actual number into the leftmost free node of the first row of $[\lambda]$ and for each vertical line segment into the leftmost free node of the second row of $[\lambda]$.
The resulting λ-tableau $\mathfrak{t}$ is standard because of the second property. The first property provides that $A = short(top(v_{\mathfrak{t}}))$. ∎

3.3 The polynomials $p_{\mathfrak{t}}(q)$

In this section we will define for each $\mathfrak{t} \in Std(\lambda)$ a polynomial $p_{\mathfrak{t}}(q)$ in q with $p_{\mathfrak{t}}(1) = 1$. Later we will construct, for all $\mathfrak{t} \in Std(\lambda)$, $p_{\mathfrak{t}}(q)$ linearly independent

elements $\{b_i^{(\mathfrak{t})} \mid i \in \mathfrak{I}^{(\mathfrak{t})}\}$ of S^λ with $last(b_i^{(\mathfrak{t})}) = \mathfrak{t}$.
Our goal is to show that

$$\{b_i^{(\mathfrak{t})} \mid \mathfrak{t} \in Std(\lambda), i \in \mathfrak{I}^{(\mathfrak{t})}\}$$

is a standard basis of S^λ with corresponding polynomials $\{p_{\mathfrak{t}}(q) \mid \mathfrak{t} \in Std(\lambda)\}$.

3.3.1 Definition: Let $\mathfrak{t} \in Std(\lambda)$. Then we denote by $\delta(\mathfrak{t})$ the tuple $(d, \underline{h}, \underline{b}, \underline{r})$, which we get in the following way:

- The vector $\underline{b}$ consists of the lengths of sequences of consecutive numbers in the first row of $\mathfrak{t}$.
- The vector $\underline{h}$ consists of the lengths of sequences of consecutive numbers in the second row of $\mathfrak{t}$.
- d is the length of the vector $\underline{h}$ minus one. There are two cases:

  1.) $\lambda = (m, m)$:
  $\underline{b}$ and $\underline{h}$ have the same length since the numbers $1, 2, 3, \dots 2m$ alternately form sequences in the first and the second row of $\mathfrak{t}$ starting with the sequence containing the integer 1 in the first row and ending with the sequence containing the integer $2m$ in the second row.

  2.) $\lambda = (n - m, m)$ with $n - m > m$:
  The length of $\underline{b}$ equals the length of $\underline{h}$ if the integer $2m$ is in the second row of $\mathfrak{t}$ and one plus the length of $\underline{h}$ if the integer $2m$ is in the first row of $\mathfrak{t}$.

  We index the entries of $\underline{b}$ and $\underline{h}$ in the following way:

  $$\underline{h} = (h_0, \dots, h_d) \in \mathbb{N}^{d+1},$$
  $$\underline{b} = (b_1, \dots, b_{d+1}) \in \mathbb{N}^{d+1} \text{ or } \underline{b} = (b_1, \dots, b_{d+1}, b_{d+2}) \in \mathbb{N}^{d+2}.$$

- $\underline{r} = (r_1, \dots r_d)$ is given by

  $$r_j := \sum_{i=1}^{j} b_i - \sum_{i=0}^{j-1} h_i \text{ for } 1 \leq j \leq d. \tag{3.4}$$

  $\mathfrak{t} \in Std(\lambda)$ guarantees that $\underline{r} \in \mathbb{N}_0^d$.

3.3.2 Example:

1.) Let $\lambda = (5, 4)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 9 \\ \hline 3 & 5 & 7 & 8 & \\ \cline{1-4}\end{array}$. Then the sequences of the first and second row of $\mathfrak{t}$ are $\{1, 2\}, \{4\}, \{6\}, \{9\}$ and $\{3\}, \{5\}, \{7, 8\}$ respectively. Therefore

$$d = 2, \underline{h} = (1, 1, 2), \underline{b} = (2, 1, 1, 1), \underline{r} = (1, 1).$$

2.) Let $\lambda = (6,6)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&5&8&10\\\hline 4&6&7&9&11&12\\\hline\end{array}$. Then the sequences of the first and second row of $\mathfrak{t}$ are $\{1,2,3\},\{5\},\{8\},\{10\}$ and $\{4\},\{6,7\},\{9\},\{11,12\}$ respectively. Therefore

$$d = 3, \underline{h} = (1,2,1,2), \underline{b} = (3,1,1,1), \underline{r} = (2,1,1).$$

We can also obtain the tuple $\delta(\mathfrak{t})$ directly from $short(top(v_{\mathfrak{t}}))$:

3.3.3 Lemma: *Let $\mathfrak{t} \in Std(\lambda)$. If we extend the line segments of the defining line in $short(top(v_{\mathfrak{t}}))$ to the left and to the bottom then there exists an $l \geq 0$ such that $top(v_{\mathfrak{t}})$ is of the form*

$$\begin{array}{|c|c|c|c|c|c|}\hline D_1 & & & & & \\\cline{1-2} X_{11} & D_2 & & & & \\\cline{1-2} X_{21} & X_{22} & \ddots & & & \\\cline{1-2}\cline{4-4} \dots & \dots & & D_l & & \\\cline{1-2}\cline{4-5} X_{l1} & X_{l2} & \dots & X_{ll} & D_{l+1} & \\\hline\end{array} \quad \textit{or} \quad \begin{array}{|c|c|c|c|c|c|}\hline D_1 & & & & & \\\cline{1-2} X_{11} & D_2 & & & & \\\cline{1-2} X_{21} & X_{22} & \ddots & & & \\\cline{1-2}\cline{4-4} \dots & \dots & & D_l & & \\\cline{1-2}\cline{4-5} X_{l1} & X_{l2} & \dots & X_{ll} & D_{l+1} & \\\hline\end{array},$$

where the blocks D_i and X_{ij} consist of dots and stars.
Now $\delta(\mathfrak{t}) = (d, \underline{h}, \underline{b}, \underline{r})$ is given by:

- *$d = l$,*
- *b_i is the width of the block D_i $(1 \leq i \leq d+1)$ and b_{d+2} the number of zero columns to the right of D_{d+1},*
- *h_i is the height of the block D_{i+1} $(0 \leq i \leq d)$ and*
- *r_i is the number of dots in the first row of X_{ii} $(1 \leq i \leq d)$.*

Before we prove the lemma we illustrate it with an example.

3.3.4 Example: Let $\lambda = (5,4)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|}\hline 1&2&4&6&9\\\hline 3&5&7&8\\\cline{1-4}\end{array}$.
If we extend the segments of the defining line to clarify what the blocks are then we get

$$top(v_{\mathfrak{t}}) = \begin{array}{|cc|c|c|c|}\hline \cdot & \cdot & & & \\\cline{1-3} * & \cdot & \cdot & & \\\cline{3-4} * & * & \cdot & \cdot & \\ * & * & * & \cdot & \\\hline\end{array}.$$

This means

$$D_1 = \boxed{\cdot \quad \cdot}, D_2 = \boxed{\cdot}, D_3 = \begin{array}{|c|}\hline \cdot \\ \cdot \\\hline\end{array}, X_{11} = \boxed{* \quad \cdot}, X_{21} = \begin{array}{|cc|}\hline * & * \\ * & * \\\hline\end{array} \text{ and } X_{22} = \begin{array}{|c|}\hline \cdot \\ * \\\hline\end{array}$$

and therefore as in example 3.3.2

$$d = 2, \underline{h} = (1,1,2), \underline{b} = (2,1,1,1), \underline{r} = (1,1).$$

Proof: (of lemma 3.3.3). Starting at the top left vertex of $short(top(v_{\mathfrak{t}}))$ the defining line runs one entry to the right for each number in the first row and one entry to the bottom for each number in the second row. Therefore we get the correct values for $d, \underline{b}$ and $\underline{h}$.
$r_j = \sum_{i=1}^{j} b_i - \sum_{i=0}^{j-1} h_i$ is a measure how much further the defining line runs to the right than to the bottom after the j^{th} vertical line segment. Since the stars form a triangle beneath the diagonal we get the same result if we count the dots in the first row of X_{jj}. ∎

For the moment we will assume that $n = 2m$, i.e. $\lambda = (m, m)$.

3.3.5 Definition: Suppose that $\lambda = (m, m)$ and $\mathfrak{t} \in Std(\lambda)$. We set

$$\mathfrak{M}_{\mathfrak{t}}(q) := \left\{ \begin{array}{|c|c|c|c|c|} \hline D_1 & & & & \\ \hline X_{11} & D_2 & & & \\ \hline X_{21} & X_{22} & \ddots & & \\ \hline \dots & \dots & & D_d & \\ \hline X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} \\ \hline \end{array} \;\middle|\; \begin{array}{l} D_i \in \mathfrak{M}_{h_{i-1},b_i}(q) \ (1 \leq i \leq d+1) \text{ and} \\ X_{ij} \in \mathfrak{M}_{h_i,b_j}(q) \ (1 \leq j \leq i \leq d) \text{ with } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t}) \end{array} \right\} \subseteq \Xi_{(m,m)}$$

and

$$\mathfrak{M}_{\mathfrak{t}}^{rk}(q) := \left\{ M = \begin{array}{|c|c|c|c|c|} \hline D_1 & & & & \\ \hline X_{11} & D_2 & & & \\ \hline X_{21} & X_{22} & \ddots & & \\ \hline \dots & \dots & & D_d & \\ \hline X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} \\ \hline \end{array} \in \mathfrak{M}_{\mathfrak{t}}(q) \;\middle|\; \begin{array}{l} rank(xmat(M,i)) \leq r_i \\ \text{for } 1 \leq i \leq d \text{ with } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t}) \end{array} \right\} \subseteq \Xi_{(m,m)},$$

where $xmat(M, i)$ $(1 \leq i \leq d)$ denotes the matrix which consists of the blocks $\{X_{kl} \mid 1 \leq l \leq i \leq k \leq d\}$.

3.3.6 Example: In the above notation we get

$$xmat(M,1) = \begin{pmatrix} X_{11} \\ X_{21} \\ \dots \\ X_{d1} \end{pmatrix}, xmat(M,2) = \begin{pmatrix} X_{21} & X_{22} \\ \dots & \dots \\ X_{d1} & X_{d2} \end{pmatrix} \text{ and}$$

$$xmat(M,d) = \begin{pmatrix} X_{d1} & X_{d2} & \dots & X_{dd} \end{pmatrix}$$

3.3.7 Definition: Let $\lambda = (m, m)$ and $\mathfrak{t} \in Std(\lambda)$. Then we set

$$p_{\mathfrak{t}}(q) := |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)|.$$

In the next section we will see that $p_{\mathfrak{t}}(q)$ is a polynomial in q.

We obtain subsets of $\mathfrak{M}_{\mathfrak{t}}(q)$ and $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ in definition 3.3.5 if we assume that the blocks D_i $(1 \le i \le d+1)$ have only zero entries.

3.3.8 Definition: Suppose that $\lambda = (m, m)$ and $\mathfrak{t} \in Std(\lambda)$. We define

$$\hat{\mathfrak{M}}_{\mathfrak{t}}(q) := \left\{ \begin{array}{|c|c|ccc|} \hline 0 & & & & \\ \cline{1-2} X_{11} & 0 & & & \\ \cline{1-2} X_{21} & X_{22} & \ddots & & \\ \cline{1-2} \cline{4-4} \dots & \dots & & \multicolumn{1}{|c|}{0} & \\ \hline X_{d1} & X_{d2} & \dots & \multicolumn{1}{|c|}{X_{dd}} & 0 \\ \hline \end{array} \;\middle|\; X_{ij} \in \mathfrak{M}_{h_i, b_j}(q) \text{ for } 1 \le j \le i \le d \right.$$

$$\left. \text{with } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t}) \right\} \subseteq \mathfrak{M}_{\mathfrak{t}}(q)$$

and

$$\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q) := \hat{\mathfrak{M}}_{\mathfrak{t}}(q) \cap \mathfrak{M}_{\mathfrak{t}}^{rk}(q).$$

Since the only restriction on the matrices D_i $(1 \le i \le d+1)$ in the definition of the elements of the set $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ is

$$D_i \in \mathfrak{M}_{h_{i-1}, b_i}(q)$$

we immediately have the following lemma.

3.3.9 Lemma: *Let $\lambda = (m, m)$ and $\mathfrak{t} \in Std(\lambda)$. Then*

$$p_{\mathfrak{t}}(q) = \begin{cases} q^k |\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| & \text{if } d \neq 0 \\ q^{h_0 b_1} & \text{if } d = 0, \end{cases}$$

where $k := \sum_{i=1}^{d+1} h_{i-1} b_i$.

3.3.10 Example:

1.) Let $\lambda = (4, 4)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 & 8 \\ \hline \end{array}$.

With extended line segments $top(v_{\mathfrak{t}}) = \begin{array}{|cc|c|c|} \hline \cdot & \cdot & & \\ \cline{3-3} * & \cdot & \cdot & \\ \cline{4-4} * & * & \cdot & \cdot \\ * & * & * & \cdot \\ \hline \end{array}$.

Then

$$d = 2, \underline{h} = (1, 1, 2), \underline{b} = (2, 1, 1), \underline{r} = (1, 1),$$
$$p_{\mathfrak{t}}(q) = |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| = q^5 |\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)|,$$

$$\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q) = \left\{ \begin{array}{|cc|c|c|} \hline \cdot & \cdot & & \\ \cline{1-3} a_1 & a_2 & \cdot & \\ \cline{1-4} a_3 & a_4 & a_5 & \cdot \\ a_6 & a_7 & a_8 & \cdot \\ \hline \end{array} \;\middle|\; \underline{a} \in GF(q)^8, rank \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_6 & a_7 \end{pmatrix} \le 1, \right.$$

$$\left. rank \begin{pmatrix} a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \end{pmatrix} \le 1 \right\}.$$

In the next section we will derive a formula to calculate $|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)|$. For this example we will get

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = q^5 + 2q^4 - q^3 - q^2.$$

Therefore

$$p_{\mathfrak{t}}(q) = q^{10} + 2q^9 - q^8 - q^7.$$

2.) Let $\lambda = (6,6)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 8 & 10 \\ \hline 4 & 6 & 7 & 9 & 11 & 12 \\ \hline\end{array}$.

With extended line segments $top(v_{\mathfrak{t}}) = \begin{array}{|ccc|c|c|c|}\hline \cdot & \cdot & \cdot & & & \\ * & \cdot & \cdot & \cdot & & \\ * & * & \cdot & \cdot & & \\ * & * & * & \cdot & \cdot & \\ * & * & * & * & \cdot & \cdot \\ * & * & * & * & * & \cdot \\ \hline\end{array}$.

We get

$$d = 3, \underline{h} = (1,2,1,2), \underline{b} = (3,1,1,1), \underline{r} = (2,1,1),$$
$$p_{\mathfrak{t}}(q) = |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| = q^8 |\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)|,$$

$$\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q) = \left\{ \begin{array}{|ccc|c|c|c|}\hline \cdot & \cdot & \cdot & & & \\ a_1 & a_2 & a_3 & \cdot & & \\ a_4 & a_5 & a_6 & \cdot & & \\ a_7 & a_8 & a_9 & a_{10} & \cdot & \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdot \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & \cdot \\ \hline\end{array} \;\middle|\; \underline{a} \in GF(q)^{20}, rank \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{11} & a_{12} & a_{13} \\ a_{16} & a_{17} & a_{18} \end{pmatrix} \le 2, \right.$$
$$\left. rank \begin{pmatrix} a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{16} & a_{17} & a_{18} & a_{19} \end{pmatrix} \le 1 \text{ and } rank \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{pmatrix} \le 1 \right\}.$$

We will see that

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = q^{12} + 3q^{11} + 2q^{10} - 2q^9 - 4q^8 - q^7 + 2q^6.$$

Thus

$$p_{\mathfrak{t}}(q) = q^{20} + 3q^{19} + 2q^{18} - 2q^{17} - 4q^{16} - q^{15} + 2q^{14}.$$

From now on λ is again an arbitrary partition $(m-n, n)$ of n. We want to generalize $\mathfrak{M}_{\mathfrak{t}}(q)$, $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ and $p_{\mathfrak{t}}(q)$ and define them for the more general $\mathfrak{t} \in Std(\lambda)$.

3.3.11 Definition: Let $\mathfrak{t} \in Std(\lambda)$. We set

$$\mathfrak{M}_{\mathfrak{t}}(q) := \left\{ \begin{array}{|c|c|c|c|c|c|}\hline D_1 & & & & & \\ X_{11} & D_2 & & & & \\ X_{21} & X_{22} & \ddots & & & \\ \dots & \dots & & D_d & & \\ X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} & \underbrace{\qquad}_{b_{d+2}} \\ \hline\end{array} \;\middle|\; D_i \in \mathfrak{M}_{h_{i-1},b_i}(q)\ (1 \le i \le d+1) \text{ and} \right.$$
$$\left. X_{ij} \in \mathfrak{M}_{h_i,b_j}(q)\ (1 \le j \le i \le d) \text{ with } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t}) \right\} \subseteq \Xi_{(n-m,m)}.$$

The number b_{d+2} might be zero, especially if $\lambda = (m, m)$. Then we get the same definition of $\mathfrak{M}_{\mathfrak{t}}(q)$ as in definition 3.3.5.

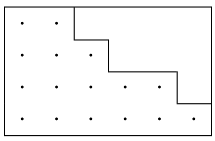
3.3.12 Definition: Let $X \in \Xi_{(n-m,m)}$. Then X is of the form

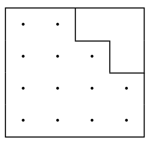
$$X = \begin{array}{|c|c|ccc|c|}\hline D_1 & & & & & \\ \cline{1-2} X_{11} & D_2 & & & & \\ \cline{1-2} X_{21} & X_{22} & \ddots & & & \\ \cline{1-2}\cline{4-4} \dots & \dots & & \multicolumn{1}{|c|}{D_d} & & \\ \cline{1-2}\cline{4-5} X_{d1} & X_{d2} & \dots & \multicolumn{1}{|c|}{X_{dd}} & \multicolumn{1}{c|}{D_{d+1}} & \\ \hline \end{array}\,\underbrace{}_{b_{d+2}}.$$

If we write down only the first m columns of $short(X)$ we get an array

$$A = \begin{array}{|c|c|c|c|}\hline D_1 & & & \\ \cline{1-2} X_{11} & D_2 & & \\ \cline{1-2} X_{21} & X_{22} & \ddots & \\ \cline{1-2}\cline{4-4} \dots & \dots & & \tilde{D}_i \\ \cline{1-2}\cline{4-4} X_{i1} & X_{i2} & \dots & \tilde{X}_{ii} \\ \cline{1-2}\cline{4-4} \dots & \dots & \dots & \dots \\ \cline{1-2}\cline{4-4} X_{d1} & X_{d2} & \dots & \tilde{X}_{di} \\ \hline \end{array},$$

where $1 \leq i \leq d+1$ and $\tilde{D}_i, \tilde{X}_{ii}, \dots, \tilde{X}_{di}$ consist of the first x columns of $D_i, X_{ii}, \dots, X_{di}$ respectively for an $x \in \mathbb{N}$. Now there exists an element $Y \in \Xi_{(m,m)}$ such that $A = short(Y)$ and we denote this element Y by $quad(X)$.

3.3.13 Example: Let $X =$  $\in \Xi_{(6,4)}$. Then

$quad(X) =$  $\in \Xi_{(4,4)}$.

3.3.14 Lemma: *Let $\mathfrak{t} \in Std(\lambda)$. Then, for every element $X \in \mathfrak{M}_{\mathfrak{t}}(q)$, $tab(quad(X))$ is the same standard (m,m)-tableau. We will denote this tableau by $quad(\mathfrak{t})$.*

Proof: For $X \in \mathfrak{M}_{\mathfrak{t}}(q)$ the tableau $tab(quad(X))$ is uniquely determined by the defining line in $short(X)$. But the defining line is equal for all elements of $\mathfrak{M}_{\mathfrak{t}}(q)$. The fact that $quad(\mathfrak{t})$ is standard follows directly from the fact that $\mathfrak{t}$ is standard. ∎

3.3.15 Example: Let $\lambda = (6,4)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&4&6&7&9\\ \hline 3&5&8&10 \\ \cline{1-4}\end{array} \in Std(\lambda)$. Then example 3.3.13 gives us an example for an element $X \in \mathfrak{M}_\mathfrak{t}(q)$ and $quad(X)$. We get

$$quad(\mathfrak{t}) = \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&5&7&8\\ \hline\end{array} \in Std((4,4)).$$

3.3.16 Definition: Let $\mathfrak{t} \in Std(\lambda)$. We set

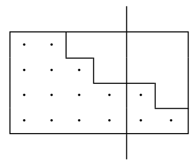
$$\mathfrak{M}^{rk}_\mathfrak{t}(q) := \left\{X \in \mathfrak{M}_\mathfrak{t}(q) \mid quad(X) \in \mathfrak{M}^{rk}_{quad(\mathfrak{t})}(q)\right\} \subseteq \Xi_{(n-m,m)}$$

and

$$p_\mathfrak{t}(q) := |\mathfrak{M}^{rk}_\mathfrak{t}(q)|.$$

For the special case $\lambda = (m,m)$ this definition is consistent with definition 3.3.5 since for $\mathfrak{t} \in Std((m,m))$ and $X \in \mathfrak{M}_\mathfrak{t}(q)$ we have $quad(X) = X$ and $quad(\mathfrak{t}) = \mathfrak{t}$.

3.3.17 Definition: Let $\mathfrak{t} \in Std(\lambda)$. Then for every element $X \in \mathfrak{M}_\mathfrak{t}(q)$ the number of entries below the defining line in the columns $\{m+1, m+2, \dots, n-m\}$ of $short(X)$ is the same. We will denote this number by $numb(\mathfrak{t})$.

3.3.18 Example: Let $\lambda = (6,4)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&4&6&7&9\\ \hline 3&5&8&10 \\ \cline{1-4}\end{array}$. Then 

is an example for an element of $\mathfrak{M}_\mathfrak{t}(q)$. Therefore $numb(\mathfrak{t}) = 3$.

If we divide the array $short(X)$ for $X \in \mathfrak{M}^{rk}_\mathfrak{t}(q)$ into the two parts $short(quad(X))$ and the remaining part to the right, then the parts must meet two independent restrictions to guarantee $X \in \mathfrak{M}^{rk}_\mathfrak{t}(q)$:

- $quad(X)$ must be an element of $\mathfrak{M}^{rk}_{quad(\mathfrak{t})}(q)$ and
- the entries below the defining line in the right part of $short(X)$ must be elements of $GF(q)$.

Since $p_\mathfrak{t}(q) = |\mathfrak{M}^{rk}_\mathfrak{t}(q)|$ and $p_{quad(\mathfrak{t})}(q) = |\mathfrak{M}^{rk}_{quad(t)}(q)|$ we immediately get the following lemma.

3.3.19 Lemma: *Let* $\mathfrak{t} \in Std(\lambda)$. *Then*

$$p_\mathfrak{t}(q) := q^{numb(\mathfrak{t})} \cdot p_{quad(\mathfrak{t})}(q).$$

3.3.20 Example: Let $\lambda = (6,4)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&4&6&7&9\\ \hline 3&5&8&10 \\ \cline{1-4}\end{array}$. In the last examples we have calculated $quad(\mathfrak{t}) = \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&5&7&8\\ \hline\end{array}$ and $numb(\mathfrak{t}) = 3$.
Together with example 3.3.10 we finally get

$$p_\mathfrak{t}(q) = q^3 \cdot (q^{10} + 2q^9 - q^8 - q^7) = q^{13} + 2q^{12} - q^{11} - q^{10}.$$

3.3.21 Corollary: *Let $\mathfrak{t} \in Std(\lambda)$. Then $p_{\mathfrak{t}}(1) = 1$.*

Proof: By lemma 3.3.9 and lemma 3.3.19 there exists an $l \in \mathbb{N}$ such that

$$p_{\mathfrak{t}}(q) = q^l \cdot |\hat{\mathfrak{M}}^{rk}_{quad(t)}(q)|.$$

But formally speaking $|\hat{\mathfrak{M}}^{rk}_{quad(t)}(1)| = 1$ and the corollary follows. ■

3.4 Calculation of the polynomials $p_{\mathfrak{t}}(q)$

Suppose that $\mathfrak{t} \in Std(\lambda)$. Then lemma 3.3.9 and lemma 3.3.19 ensure that we can calculate $p_{\mathfrak{t}}(q)$ if we know a formula for $|\hat{\mathfrak{M}}^{rk}_{quad(\mathfrak{t})}(q)|$. Therefore we assume in this section that $n = 2m$ and $\lambda = (m, m)$. To calculate the number of elements in the set $\hat{\mathfrak{M}}^{rk}_{quad(\mathfrak{t})}(q)$ we choose a recursive approach.

3.4.1 Definition: Let M be a matrix over $GF(q)$ with rank $k \in N$. Then we get a k-tuple $(v_1, v_2, \ldots, v_k)$ of linearly independent vectors in the following way: We look at the rows of M from top to bottom. v_1 is the first non-zero row, v_2 the next row that is not linearly dependent of v_1, v_3 the next row that is not linearly dependent of v_1 and v_2 and so on. We denote this k-tuple with $tup(M)$.

3.4.2 Lemma: *Let $h, b \in \mathbb{N}, 0 \leq r_Y \leq r_X \leq \min\{h, b\}$ and Y a matrix over $GF(q)$ with width b and rank r_Y. Then we have*

$$p^{(1)}_{h,b,r_Y,r_X}(q) = q^{r_Y(h-r_X+r_Y)} \begin{bmatrix} h \\ r_X - r_Y \end{bmatrix} \prod_{i=r_Y}^{r_X-1} (q^b - q^i) \tag{3.5}$$

possibilities to expand Y by a matrix $X \in \mathfrak{M}_{h,b}(q)$ such that the matrix $\left(\begin{array}{|c|}\hline X \\ \hline Y \\ \hline\end{array}\right)$ has rank r_X.

Proof: Since the matrix Y has rank r_Y and the new matrix $\left(\begin{array}{|c|}\hline X \\ \hline Y \\ \hline\end{array}\right)$ shall have rank r_X, we need $r_X - r_Y$ linearly independent rows in X that are linearly independent from the rows of Y.
Therefore

$$p^{(1)}_{h,b,r_Y,r_X}(q) = p_1(q)p_2(q)p_3(q),$$

where

- $p_1(q)$ counts the possibilities to construct an $(r_X - r_Y)$-tuple $\mathfrak{B} = (v_1, \ldots, v_{r_X-r_Y})$ of linearly independent vectors that are linearly independent of the rows of Y,
- $p_2(q)$ counts the possibilities to construct a matrix X with $tup(X) = \mathfrak{B}$ and

- $p_3(q)$ counts the possibilities to add multiples of the r_Y linearly independent rows of Y to the remaining $h-(r_X-r_Y)$ rows of X that are not fixed by $\mathfrak{B}$.

We immediately get

$$p_1(q)=\prod_{i=r_Y}^{r_X-1}(q^b-q^i) \text{ and } p_3(q)=q^{r_Y(h-r_X+r_Y)}.$$

By proposition 1.4.8 we have

$$p_2(q)=r_{h,b,r_X-r_Y}(q,\mathfrak{B})=\begin{bmatrix} h \\ r_X-r_Y \end{bmatrix}$$

and the lemma follows. ∎

3.4.3 Corollary: *Alternatively we obtain*

$$\begin{aligned} p^{(1)}_{h,b,r_Y,r_X}(q) &= q^{r_Y h}\begin{bmatrix} h \\ r_X-r_Y \end{bmatrix}\begin{bmatrix} b-r_Y \\ r_X-r_Y \end{bmatrix}|GL_{r_X-r_Y}(q)| \\ &= q^{r_Y(h-r_X+r_Y)}\frac{\begin{bmatrix} b \\ r_X \end{bmatrix}\begin{bmatrix} h \\ r_X-r_Y \end{bmatrix}}{\begin{bmatrix} b \\ r_Y \end{bmatrix}}\frac{|GL_{r_X}(q)|}{|GL_{r_Y}(q)|}. \end{aligned} \tag{3.6}$$

Proof: Using lemma 3.4.2 we get

$$\begin{aligned} p^{(1)}_{h,b,r_Y,r_X}(q) &= q^{r_Y(h-r_X+r_Y)}\begin{bmatrix} h \\ r_X-r_Y \end{bmatrix}\prod_{i=r_Y}^{r_X-1}(q^b-q^i) \\ &= q^{r_Y(h-r_X+r_Y)}\begin{bmatrix} h \\ r_X-r_Y \end{bmatrix}q^{r_Y(r_X-1-r_Y+1)}\prod_{i=0}^{r_X-r_Y-1}(q^{b-r_Y}-q^i) \\ &\overset{(1.1)}{=} q^{r_Y h}\begin{bmatrix} h \\ r_X-r_Y \end{bmatrix}\begin{bmatrix} b-r_Y \\ r_X-r_Y \end{bmatrix}|GL_{r_X-r_Y}(q)| \end{aligned}$$

The second equality follows directly from lemma 3.4.2 and (1.2). ∎

3.4.4 Lemma: *Let $h,b\in\mathbb{N}$ and $r_A,r_B,r_Y,r_X\in\mathbb{N}_0$ with the following properties*

- $r_Y\le r_A\le r_X$ *and* $r_Y\le r_B\le r_X$,
- $r_A+r_B-r_Y\le r_X\le\min\{h,b\}$.

Furthermore, suppose that Y is a matrix over $GF(q)$ with rank r_Y, A is a matrix over $GF(q)$ with h rows such that the matrix $\left(\begin{array}{|c|}\hline A \\ \hline Y \\ \hline\end{array}\right)$ has rank r_A and B is a *matrix over $GF(q)$ with b columns such that the matrix $\left(\begin{array}{|c|c|}\hline Y & B\\\hline\end{array}\right)$ has rank r_B. If we set $k := (h - r_X + r_B)(r_B - r_Y) + b(r_A - r_Y)$ then we have*

$$p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q) = q^k \begin{bmatrix} h - (r_A - r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \prod_{i=r_B-r_Y}^{r_X-r_A-1} (q^b - q^i) \tag{3.7}$$

possibilities to replace the zero matrix in the matrix $\left(\begin{array}{|c|c|}\hline A & 0\\\hline Y & B\\\hline\end{array}\right)$ by a matrix $X \in \mathfrak{M}_{h,b}(q)$ such that $\left(\begin{array}{|c|c|}\hline A & X\\\hline Y & B\\\hline\end{array}\right)$ has rank r_X.

Proof: Since Y has rank r_Y and the matrix $\left(\begin{array}{|c|}\hline A\\\hline Y\\\hline\end{array}\right)$ has rank r_A we find $r_A - r_Y$ linearly independent rows in A that are linearly independent of the rows in Y. We can extend these rows in X by arbitrary elements of $GF(q)$ in

$$p_1(q) = q^{b(r_A - r_Y)}$$

possible ways.
We need $r_X - r_A - r_B + r_Y$ new linearly independent rows $v_1, v_2, \ldots, v_{r_X - r_A - r_B + r_Y}$ in X, that are not linearly dependent of the $r_B - r_Y$ rows in B which ensure that $\left(\begin{array}{|c|c|}\hline Y & B\\\hline\end{array}\right)$ has rank r_B. To do this, we have

$$p_2(q) = \prod_{i=r_B-r_Y}^{r_X-r_A-1} (q^b - q^i)$$

possibilities.
$r_A - r_Y$ rows of X are already fixed. The remaining $h - (r_A - r_Y)$ rows form a matrix $\hat{X}$. We can fill this matrix with linear combinations of the vectors in the tuple $\mathfrak{B} = (v_1, v_2, \ldots, v_{r_X - r_A - r_B + r_Y})$ such that $tup(\hat{X}) = \mathfrak{B}$. By proposition 1.4.8 we have

$$p_3(q) = r_{h-(r_A-r_Y),b,r_X-r_A-r_B+r_Y}(q, \mathfrak{B}) = \begin{bmatrix} h - (r_A - r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix}$$

possibilities to find such a matrix $\hat{X}$.
Finally we can add linear combinations of rows of B to the rows of X. But we may take only the $r_B - r_Y$ rows of B that ensure that $\left(\begin{array}{|c|c|}\hline Y & B\\\hline\end{array}\right)$ has rank r_B. And we must not add them to the rows that ensure that $\left(\begin{array}{|c|c|}\hline A & X\\\hline Y & B\\\hline\end{array}\right)$ has rank r_X because those are already fixed. This leaves

$$p_4(q) = q^{(h-(r_X-r_B))(r_B-r_Y)}$$

possibilities.
At last

$$p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q) = p_1(q)p_2(q)p_3(q)p_4(q)$$

and the lemma follows. ■

3.4.5 Corollary: *If we set*

$$l := b(r_A - r_Y) + h(r_B - r_Y) - (r_A - r_Y)(r_B - r_Y)$$

we alternatively obtain

$$\begin{aligned} p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q) = \ & q^l \begin{bmatrix} h-(r_A-r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \begin{bmatrix} b-(r_B-r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \cdot \\ & \cdot |GL_{r_X-r_A-r_B+r_Y}(q)| \\ = \ & q^l \frac{\begin{bmatrix} b \\ r_X - r_A \end{bmatrix} \begin{bmatrix} h \\ r_X - r_B \end{bmatrix} \begin{bmatrix} r_X - r_A \\ r_B - r_Y \end{bmatrix} \begin{bmatrix} r_X - r_B \\ r_A - r_Y \end{bmatrix}}{\begin{bmatrix} b \\ r_B - r_Y \end{bmatrix} \begin{bmatrix} h \\ r_A - r_Y \end{bmatrix}} \cdot \\ & \cdot |GL_{r_X-r_A-r_B+r_Y}(q)|. \end{aligned} \tag{3.8}$$

The expected symmetry

$$p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q) = p^{(2)}_{b,h,r_Y,r_B,r_A,r_X}(q)$$

becomes evident in both formulas.

Proof: We set

$$\begin{aligned} k_1 &:= (r_B - r_X)(r_B - r_Y) + h(r_B - r_Y) + b(r_A - r_Y), \\ k_2 &:= (r_B - r_Y)(r_X - r_A - r_B + r_Y) \end{aligned}$$

and obtain by lemma 3.4.4

$$\begin{aligned} p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q) = \ & q^{k_1} \begin{bmatrix} h-(r_A-r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \prod_{i=r_B-r_Y}^{r_X-r_A-1} (q^b - q^i) \\ = \ & q^{k_1} \begin{bmatrix} h-(r_A-r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} q^{k_2} \prod_{i=0}^{r_X-r_A-r_B+r_Y-1} (q^{b-r_B+r_Y} - q^i) \\ \overset{(1.1)}{=} \ & q^l \begin{bmatrix} h-(r_A-r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \begin{bmatrix} b-(r_B-r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \cdot \\ & \cdot |GL_{r_X-r_A-r_B+r_Y}(q)| \\ \overset{(1.5)}{=} \ & q^l \frac{\begin{bmatrix} b \\ r_X - r_A \end{bmatrix} \begin{bmatrix} h \\ r_X - r_B \end{bmatrix} \begin{bmatrix} r_X - r_A \\ r_B - r_Y \end{bmatrix} \begin{bmatrix} r_X - r_B \\ r_A - r_Y \end{bmatrix}}{\begin{bmatrix} b \\ r_B - r_Y \end{bmatrix} \begin{bmatrix} h \\ r_A - r_Y \end{bmatrix}} \cdot \\ & \cdot |GL_{r_X-r_A-r_B+r_Y}(q)|. \end{aligned}$$

■

3.4.6 Definition: Let $\mathfrak{t} \in Std(\lambda)$, $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$ and $L \in \hat{\mathfrak{M}}^{rk}_{\mathfrak{t}}(q)$, that means L is of the form

$$L = \begin{array}{|c|c|ccc|}
\hline
0 & & & & \\
\cline{1-2}
X_{11} & 0 & & & \\
\cline{1-2}
X_{21} & X_{22} & \ddots & & \\
\cline{1-2} \cline{4-4}
\dots & \dots & & \multicolumn{1}{|c|}{0} & \\
\hline
X_{d1} & X_{d2} & \dots & \multicolumn{1}{|c|}{X_{dd}} & 0 \\
\hline
\end{array}.$$

1.) For $1 \leq j \leq i \leq d$ we denote by $mat(X_{ij})$ the matrix

$$\begin{pmatrix} X_{i1} & X_{i2} & \dots & X_{ij} \\ X_{i+1,1} & X_{i+1,2} & \dots & X_{i+1,j} \\ \vdots & \vdots & & \vdots \\ X_{d1} & X_{d2} & \dots & X_{dj} \end{pmatrix}.$$

2.) We can assign a tuple

$$(r_{ij})_L = (r_{11}, r_{21}, r_{22}, r_{31}, r_{32}, r_{33}, \dots, r_{d1}, r_{d2}, \dots, r_{dd})$$

to L by setting $r_{ij} := rank(mat(X_{ij}))$ for $1 \leq j \leq i \leq d$.

3.) We define $\mathfrak{I}^{rk}_{\mathfrak{t}}$ to be the set of all tuples

$$(r_{ij}) = (r_{11}, r_{21}, r_{22}, r_{31}, r_{32}, r_{33}, \dots, r_{d1}, r_{d2}, \dots, r_{dd})$$

with the following properties

- $r_{ij} \in \mathbb{N}$ for $1 \leq j \leq i \leq d$
- $r_{ii} \leq r_i$ for $1 \leq i \leq d$,
- $r_{i+1,j} \leq r_{ij}$ for $1 \leq j \leq i \leq d-1$,
- $r_{i,j-1} \leq r_{ij}$ for $2 \leq j \leq i \leq d$ and
- $r_{i,j-1} + r_{i+1,j} - r_{i+1,j-1} \leq r_{ij}$ for $2 \leq j \leq i \leq d-1$.

An easy consequence of this definition is the following lemma.

3.4.7 Lemma: *Let $\mathfrak{t} \in Std(\lambda)$. Then*

$$\mathfrak{I}^{rk}_{\mathfrak{t}} = \{(r_{ij})_L \mid L \in \hat{\mathfrak{M}}^{rk}_{\mathfrak{t}}(q)\}.$$

Now we are ready to calculate $|\hat{\mathfrak{M}}^{rk}_{\mathfrak{t}}(q)|$ for an arbitrary $\mathfrak{t} \in Std(\lambda)$. We explain the algorithm in the next example.

3.4.8 Example: Let $\mathfrak{t} \in Std(\lambda)$ and $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$, where

1.) $d = 1$. Then

$$\hat{\mathfrak{M}}^{rk}_{\mathfrak{t}}(q) = \left\{ \begin{array}{|c|c|} \hline 0 & \\ \hline X_{11} & 0 \\ \hline \end{array} \;\middle|\; X_{11} \in \mathfrak{M}_{h_1,b_1}(q), rank(X_{11}) \leq r_1 \right\}$$

and by proposition 1.4.5 we obtain

$$|\hat{\mathfrak{M}}^{rk}_{\mathfrak{t}}(q)| = \sum_{r_{11}=0}^{r_1} r_{h_1,b_1}(q, r_{11}) = \sum_{(r_{ij}) \in \mathfrak{I}^{rk}_{\mathfrak{t}}} \begin{bmatrix} b_1 \\ r_{11} \end{bmatrix} \begin{bmatrix} h_1 \\ r_{11} \end{bmatrix} |GL_{r_{11}}(q)|. \tag{3.9}$$

2.) $d = 2$. Then

$$\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q) = \left\{ \begin{array}{|c|c|c|} \hline 0 & & \\ \hline X_{11} & 0 & \\ \hline X_{21} & X_{22} & 0 \\ \hline \end{array} \;\middle|\; X_{ij} \in \mathfrak{M}_{h_i,b_j}(q) \text{ for } 1 \le j \le i \le 2 \text{ and} \right.$$

$$\left. rank(mat(X_{11})) \le r_1, rank(mat(X_{22})) \le r_2, \right\}.$$

If we use the case "$d = 1$", lemma 3.4.2 and (3.6) twice we get

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = \sum_{(r_{ij}) \in \mathfrak{I}_{\mathfrak{t}}^{rk}} q^l \begin{bmatrix} b_1 \\ r_{11} \end{bmatrix} \begin{bmatrix} h_1 \\ r_{11} - r_{21} \end{bmatrix} \begin{bmatrix} h_2 \\ r_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ r_{22} - r_{21} \end{bmatrix} \frac{|GL_{r_{22}}(q)||GL_{r_{11}}(q)|}{|GL_{r_{21}}(q)|},$$

where $l := r_{21}(h_1 + b_2 + 2r_{21} - r_{22} - r_{11})$.
With this formula we can calculate $|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)|$ for the first example in 3.3.10. We had

$$d = 2, \underline{h} = (1,1,2), \underline{b} = (2,1,1), \underline{r} = (1,1).$$

Therefore we obtain

$$\mathfrak{I}_{\mathfrak{t}}^{rk} = \{(0,0,0), (1,0,0), (0,0,1), (1,0,1), (1,1,1)\}$$

and

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = 1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}(q-1) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}(q-1) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}(q-1)^2 +$$

$$+ q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}(q-1) = q^5 + 2q^4 - q^3 - q^2.$$

3.) $d = 3$. Then

$$\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q) = \left\{ \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline X_{11} & 0 & & \\ \hline X_{21} & X_{22} & 0 & \\ \hline X_{31} & X_{32} & X_{33} & 0 \\ \hline \end{array} \;\middle|\; X_{ij} \in \mathfrak{M}_{h_i,b_j}(q) \text{ for } 1 \le j \le i \le 3 \text{ and} \right.$$

$$\left. rank(mat(X_{ii})) \le r_i \text{ for } 1 \le i \le 3 \right\}.$$

If we use the case "$d = 2$", lemma 3.4.2 together with (3.6) twice and lemma 3.4.4 together with (3.8) we obtain

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = \sum_{(r_{ij}) \in \mathfrak{I}_{\mathfrak{t}}^{rk}} q^l \begin{bmatrix} b_1 \\ r_{11} \end{bmatrix} \begin{bmatrix} h_3 \\ r_{33} \end{bmatrix} \begin{bmatrix} b_2 \\ r_{22} - r_{21} \end{bmatrix} \begin{bmatrix} h_2 \\ r_{22} - r_{32} \end{bmatrix} \begin{bmatrix} b_3 \\ r_{33} - r_{32} \end{bmatrix} \begin{bmatrix} h_1 \\ r_{11} - r_{21} \end{bmatrix}$$

$$\begin{bmatrix} r_{22} - r_{21} \\ r_{32} - r_{31} \end{bmatrix} \begin{bmatrix} r_{22} - r_{32} \\ r_{21} - r_{31} \end{bmatrix} \frac{|GL_{r_{11}}(q)||GL_{r_{33}}(q)||GL_{r_{22}-r_{32}-r_{21}+r_{31}}(q)|}{|GL_{r_{31}}(q)|},$$

where $l := r_{31}^2 - r_{32}r_{21} + r_{32}(b_3 + h_2 - r_{33} + r_{32}) + r_{21}(h_1 + b_2 - r_{11} + r_{21})$.
With this formula we can calculate

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = q^{12} + 3q^{11} + 2q^{10} - 2q^9 - 4q^8 - q^7 + 2q^6$$

for the second example in 3.3.10.

If we continue with this recursive algorithm, we finally obtain a general formula for $|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)|$. The result is our next theorem.

3.4.9 Theorem: *Let $\mathfrak{t} \in Std(\lambda)$ and $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$. Then we have*

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = \sum_{(r_{ij}) \in \mathfrak{I}_{\mathfrak{t}}^{rk}} r^{(1)}_{\mathfrak{t},(r_{ij})}(q) r^{(2)}_{\mathfrak{t},(r_{ij})}(q) r^{(3)}_{\mathfrak{t},(r_{ij})}(q),$$

where

$$\begin{aligned} r^{(1)}_{\mathfrak{t},(r_{ij})}(q) :=& \begin{bmatrix} b_1 \\ r_{11} \end{bmatrix} \begin{bmatrix} b_d \\ r_{dd} \end{bmatrix} \prod_{2 \leq j \leq i \leq d-1} \begin{bmatrix} r_{ij} - r_{i,j-1} \\ r_{i+1,j} - r_{i+1,j-1} \end{bmatrix} \begin{bmatrix} r_{ij} - r_{i+1,j} \\ r_{i,j-1} - r_{i+1,j-1} \end{bmatrix} \\ & \prod_{1 \leq k \leq d-1} \begin{bmatrix} h_k \\ r_{kk} - r_{k+1,k} \end{bmatrix} \begin{bmatrix} b_{k+1} \\ r_{k+1,k+1} - r_{k+1,k} \end{bmatrix}, \\ r^{(2)}_{\mathfrak{t},(r_{ij})}(q) :=& \frac{|GL_{r_{11}}(q)||GL_{r_{dd}}(q)|}{|GL_{r_{d1}}(q)|} \prod_{2 \leq j \leq i \leq d-1} |GL_{r_{ij} - r_{i,j-1} - r_{i+1,j} + r_{i+1,j-1}}(q)| \text{ and} \\ r^{(3)}_{\mathfrak{t},(r_{ij})}(q) :=& q^l, \end{aligned}$$

where

$$l := \begin{cases} 0 & \text{if } d = 1 \\ r_{21}(h_1 + b_2 + 2r_{21} - r_{22} - r_{11}) & \text{if } d = 2 \\ r_{d1}^2 + r_{21}^2 + r_{d,d-1}^2 - r_{11}r_{21} - r_{dd}r_{d,d-1} + & \\ + \sum_{i=2}^{d} r_{i,i-1}(b_i + h_{i-1}) - \sum_{1 \leq j < i \leq d-1} r_{ij}r_{i+1,j+1} + & \\ + \sum_{2 \leq j < i \leq d-1} r_{ij}(r_{i,j-1} + r_{i+1,j}) - \sum_{2 \leq j < i-1 \leq d-2} r_{ij}^2 & \text{if } d > 2. \end{cases}$$

3.4.10 Corollary: *If $\mathfrak{t} \in Std(\lambda)$ then $p_{\mathfrak{t}}(q)$ is a polynomial in q.*

Proof: Follows directly from the recursive algorithm and the fact that $p^{(1)}_{h,b,r_Y,r_X}(q)$ in (3.5) and $p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q)$ in (3.7) are polynomials in q. ∎

3.5 The idempotents e_L

Our goal is to construct for every $\mathfrak{t} \in Std(\lambda)$ and every $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ an element b_L of S^λ such that

- $last(b_L) = \mathfrak{t}$,
- $\{b_L \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}$ is part of a basis of S^λ and
- the elements $\{top(b_L) \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}$ of M^λ are linearly independent.

The main tool to prove the linear independence of the elements b_L and $top(b_L)$ are the idempotents e_L introduced in [7] which we want to generalize for our purpose.

3.5.1 Definition: Let $\mathfrak{t} \in Std(\lambda)$ and M an arbitrary element of $\mathfrak{M}_\mathfrak{t}(q)$. Then $\mathfrak{I}_\mathfrak{t} \subseteq \{1, 2, \ldots n\} \times \{1, 2, \ldots n\}$ indexes the entries of M which appear below the defining line in $short(M)$. Obviously this is well defined.

3.5.2 Example: Suppose that $\lambda = (2,2)$ and $\mathfrak{t} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$. Then

$$\mathfrak{M}_\mathfrak{t}(q) = \left\{ \begin{array}{|cc|}\hline a & \\ b & c \\ \hline \end{array} \;\middle|\; a, b, c \in GF(q) \right\} = \left\{ \begin{array}{|cccc|}\hline 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \hline a & 1 & \cdot & \cdot \\ b & \cdot & c & 1 \\ \hline \end{array} \;\middle|\; a, b, c \in GF(q) \right\}$$

and therefore

$$\mathfrak{I}_\mathfrak{t} = \{(3,1), (4,1), (4,3)\}.$$

We can calculate $|\mathfrak{I}_\mathfrak{t}|$ easily.

$$|\mathfrak{I}_\mathfrak{t}| = \sum_{i=0}^{d} \sum_{j=1}^{i+1} h_i b_j,$$

where $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$.
For $\mathfrak{t} \in Std(\lambda)$ we introduce an addition $\diamond$ on $\mathfrak{M}_\mathfrak{t}(q)$ by pointwise adding the entries below the defining line.

3.5.3 Example: Let $a_1, a_2, b_1, b_2, c_1, c_2 \in GF(q)$. Then

$$\begin{array}{|cc|}\hline a_1 & \\ b_1 & c_1 \\ \hline \end{array} \diamond \begin{array}{|cc|}\hline a_2 & \\ b_2 & c_2 \\ \hline \end{array} = \begin{array}{|cc|}\hline a_1 + a_2 & \\ b_1 + b_2 & c_1 + c_2 \\ \hline \end{array}.$$

Recall that this is only shorthand notation for

$$\begin{array}{|cccc|}\hline a_1 & 1 & & \\ b_1 & & c_1 & 1 \\ \hline \end{array} \diamond \begin{array}{|cccc|}\hline a_2 & 1 & & \\ b_2 & & c_2 & 1 \\ \hline \end{array} = \begin{array}{|cccc|}\hline a_1 + a_2 & 1 & & \\ b_1 + b_2 & & c_1 + c_2 & 1 \\ \hline \end{array}.$$

Then $(\mathfrak{M}_\mathfrak{t}(q), \diamond)$ is an abelian group of order $q^{|\mathfrak{I}_\mathfrak{t}|}$. Therefore we find $q^{|\mathfrak{I}_\mathfrak{t}|}$ linear irreducible F-characters of $\mathfrak{M}_\mathfrak{t}(q)$. Such a character χ is a group homomorphism from $\mathfrak{M}_\mathfrak{t}(q)$ to the multiplicative group F^*. In particular

$$\chi(M \diamond N) = \chi(M)\chi(N) \text{ for } M, N \in \mathfrak{M}_\mathfrak{t}(q).$$

We regard the set X of F-linear characters of $\mathfrak{M}_\mathfrak{t}(q)$ as a vector space over $GF(q)$ by setting

$$(\chi_1 + \chi_2)(M) = \chi_1(M)\chi_2(M)$$
$$\alpha\chi(M) = \chi(\alpha M)$$

for all $M \in \mathfrak{M}_\mathfrak{t}(q)$ and $\alpha \in GF(q)$.
For $(i,j) \in \mathfrak{I}_\mathfrak{t}$ we denote by ε_{ij} the (i,j) coordinate function from $\mathfrak{M}_\mathfrak{t}(q)$ to $GF(q)$. Then

$$\{\theta\varepsilon_{ij} \mid (i,j) \in \mathfrak{I}_\mathfrak{t}\}$$

is a basis of the $GF(q)$-vector space X.
Thus, if $\chi \in X$ then

$$\chi = \sum_{(i,j)\in\mathfrak{I}_\mathfrak{t}} l_{ij}(\theta\varepsilon_{ij})$$

for uniquely determined elements l_{ij} of $GF(q)$.
Vice versa, given a matrix $L = (l_{ij}) \in \mathfrak{M}_\mathfrak{t}(q)$, we let

$$\chi_L := \sum_{(i,j)\in\mathfrak{I}_\mathfrak{t}} l_{ij}(\theta\varepsilon_{ij}),$$

so that

$$X = \{\chi_L \mid L \in \mathfrak{M}_\mathfrak{t}(q)\}$$

and for $M = (m_{ij}) \in \mathfrak{M}_\mathfrak{t}(q)$, we have

$$\chi_L(M) = \prod_{(i,j)\in\mathfrak{I}_\mathfrak{t}} \theta(l_{ij}m_{ij}).$$

Now let $\mathfrak{V}_\mathfrak{t}(q)$ be the group algebra of $\mathfrak{M}_\mathfrak{t}(q)$ over F. Thus, $\mathfrak{V}_\mathfrak{t}(q)$ is a $q^{|\mathfrak{I}_\mathfrak{t}|}$-dimensional vector subspace of M^λ, with natural basis

$$\{M \mid M \in \mathfrak{M}_\mathfrak{t}(q)\}.$$

3.5.4 Definition: Suppose that $\mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_\mathfrak{t}(q)$. We define

$$e_L := \frac{1}{q^{|\mathfrak{I}_\mathfrak{t}|}} \sum_{M\in\mathfrak{M}_\mathfrak{t}(q)} \chi_L(-M)M.$$

Then e_L is the idempotent in $\mathfrak{V}_\mathfrak{t}(q)$ affording the linear character χ_L. In fact,

$$\{e_L \mid L \in \mathfrak{M}_\mathfrak{t}(q)\}$$

is a complete set of primitive orthogonal idempotents in $F\mathfrak{M}_\mathfrak{t}(q)$, and so

$$\mathfrak{V}_\mathfrak{t}(q) = \bigoplus_{L\in\mathfrak{M}_\mathfrak{t}(q)} Fe_L$$

is the decomposition of the regular module of $F\mathfrak{M}_\mathfrak{t}(q)$ into pairwise non-isomorphic irreducible $F\mathfrak{M}_\mathfrak{t}(q)$-modules.

3.6 Construction of the elements b_L

Let $\mathfrak{t} \in Std(\lambda)$ and $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$ and $M \in \mathfrak{M}_{\mathfrak{t}}(q)$. Then M is of the form

$$M = \left[\begin{array}{cccccc} D_1 & & & & & \\ X_{11} & D_2 & & & & \\ X_{21} & X_{22} & \ddots & & & \\ \dots & \dots & & D_d & & \\ X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} & \underbrace{\qquad}_{b_{d+2}} \end{array}\right],$$

where $D_i \in \mathfrak{M}_{h_{i-1},b_i}(q)$ for $1 \leq i \leq d+1$ and $X_{ij} \in \mathfrak{M}_{h_i,b_j}(q)$ for $1 \leq j \leq i \leq d$. For

- $A_i \in GL_{b_i}$ $(1 \leq i \leq d+1)$,
- $B_i \in GL_{h_{i-1}}$ $(1 \leq i \leq d+1)$,
- $T_i \in \mathfrak{M}_{h_{i-1},b_i}$ $(1 \leq i \leq d+1)$ and
- $T_{ij} \in \mathfrak{M}_{h_i,b_j}$ $(1 \leq j < i \leq d)$

we want to analyse the operation $\circ$ of the element

$$g = \begin{pmatrix} A_1 & & & & & & & & & \\ B_1T_1 & B_1 & & & & & & & & \\ & & A_2 & & & & & & & \\ B_2T_{11} & & B_2T_2 & B_2 & & & & & & \\ & & & & A_3 & & & & & \\ B_3T_{21} & & B_3T_{22} & & B_3T_3 & B_3 & & & & \\ \cdots & & \cdots & & \cdots & \ddots & \ddots & & & \\ & & & & & & & A_{d+1} & & \\ B_{d+1}T_{d1} & & B_{d+1}T_{d2} & & B_{d+1}T_{d3} & & \cdots & B_{d+1}T_{d+1} & B_{d+1} & \\ & & & & & & & & & I_{b_{d+2}} \end{pmatrix} \tag{3.10}$$

of $GL_n(q)$ on the flag M. To do this, it is more convenient not to omit the columns with the ones in the notation of M.

$$M = \left[\begin{array}{ccccccccc} D_1 & I_{h_0} & & & & & & & \\ X_{11} & & D_2 & I_{h_1} & & & & & \\ X_{21} & & X_{22} & & \ddots & & & & \\ \dots & & \dots & & & D_d & I_{h_{d-1}} & & \\ X_{d1} & & X_{d2} & & \dots & X_{dd} & & D_{d+1} \quad I_{h_d} & \underbrace{\qquad}_{b_{d+2}} \end{array}\right].$$

Then we get

$$Mg = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\tilde{D}_1 & B_1 & & & & & & & & \\
\tilde{X}_{11} & & \tilde{D}_2 & B_2 & & & & & & \\
\tilde{X}_{21} & & \tilde{X}_{22} & & \ddots & & & & & \\
\dots & & \dots & & & \tilde{D}_d & B_d & & & \\
\tilde{X}_{d1} & & \tilde{X}_{d2} & & \dots & \tilde{X}_{dd} & & \tilde{D}_{d+1} & B_{d+1} & \\
\hline
\multicolumn{9}{c}{} & \underbrace{\qquad}_{b_{d+2}}
\end{array},$$

where

- $\tilde{D}_i = D_i A_i + B_i T_i$ $(1 \le i \le d+1)$ and
- $\tilde{X}_{ij} = X_{ij} A_j + B_{i+1} T_{ij}$ $(1 \le j < i \le d)$.

If we multiply the first row of blocks by B_1^{-1} on the left, the second block by B_2^{-1} on the left, ... and the $(d+1)^{th}$ row of blocks by B_{d+1}^{-1} on the left we get an element of Ξ_λ.
Therefore

$$M \circ g = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\hat{D}_1 & I_{h_0} & & & & & & & & \\
\hat{X}_{11} & & \hat{D}_2 & I_{h_1} & & & & & & \\
\hat{X}_{21} & & \hat{X}_{22} & & \ddots & & & & & \\
\dots & & \dots & & & \hat{D}_d & I_{h_{d-1}} & & & \\
\hat{X}_{d1} & & \hat{X}_{d2} & & \dots & \hat{X}_{dd} & & \hat{D}_{d+1} & I_{h_d} & \\
\hline
\multicolumn{9}{c}{} & \underbrace{\qquad}_{b_{d+2}}
\end{array} \in \mathfrak{M}_{\mathfrak{t}}(q),$$

where

- $\hat{D}_i = B_i^{-1} D_i A_i + T_i$ $(1 \le i \le d+1)$ and
- $\hat{X}_{ij} = B_{i+1}^{-1} X_{ij} A_j + T_{ij}$ $(1 \le j < i \le d)$.

Again in our short notation

$$M \circ g = \begin{array}{|c|c|c|c|c|c|}
\hline
\hat{D}_1 & & & & & \\
\hat{X}_{11} & \hat{D}_2 & & & & \\
\hat{X}_{21} & \hat{X}_{22} & \ddots & & & \\
\dots & \dots & & \hat{D}_d & & \\
\hat{X}_{d1} & \hat{X}_{d2} & \dots & \hat{X}_{dd} & \hat{D}_{d+1} & \\
\hline
\multicolumn{5}{c}{} & \underbrace{\qquad}_{b_{d+2}}
\end{array} \in \mathfrak{M}_{\mathfrak{t}}(q).$$

3.6.1 Example: In the above notation

$$
\begin{array}{|c|c|c|}
\hline
D_1 & & \\
\hline
X_{11} & D_2 & \\
\hline
X_{21} & X_{22} & D_3 \\
\hline
\end{array}
\circ
\left(
\begin{array}{cccccc}
A_1 & & & & & \\
B_1T_1 & B_1 & & & & \\
 & & A_2 & & & \\
B_2T_{11} & & B_2T_2 & B_2 & & \\
 & & & & A_3 & \\
B_3T_{21} & & B_3T_{22} & & B_3T_3 & B_3
\end{array}
\right)
$$

$$
= \begin{array}{|c|c|c|}
\hline
B_1^{-1}D_1A_1 + T_1 & & \\
\hline
B_2^{-1}X_{11}A_1 + T_{11} & B_2^{-1}D_2A_2 + T_2 & \\
\hline
B_3^{-1}X_{21}A_1 + T_{21} & B_3^{-1}X_{22}A_2 + T_{22} & B_3^{-1}D_3A_3 + T_3 \\
\hline
\end{array}\,.
$$

This means, that by an appropriate operation of an element $g \in GL_n(q)$ we can

- add a matrix T to a block D_i or X_{ij} of M,
- multiply the blocks of a column of blocks of M simultaneously on the right by an invertible matrix A and
- multiply the blocks of a row of blocks of M simultaneously on the left by an invertible matrix B.

The operation of a matrix $g \in GL_n(q)$ as in (3.10) has further nice properties as we can see in the following two propositions which, for some special cases, can be found in [7].

3.6.2 Proposition: *Let* $\mathfrak{t} \in Std(\lambda)$, $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$, $v \in M^\lambda$ *with* $last(v) = \mathfrak{t}$ *and* $g \in GL_n(q)$ *as in (3.10). Then*

$$last(v \circ g) = \mathfrak{t} \text{ and } top(v \circ g) = top(v) \circ g.$$

Proof: By proposition 1.6.4 we find lower triangular matrices $Q_i^{(1)}, Q_i^{(2)}, R_i^{(1)}$ and $R_i^{(2)}$ and permutation matrices P_i and S_i such that we can write the matrices A_i and B_i on the diagonal of g in the following form.

$$A_i = Q_i^{(1)} P_i Q_i^{(2)} \text{ and } B_i = R_i^{(1)} S_i R_i^{(2)} \text{ for } 1 \leq i \leq d+1.$$

Then

$$g = \hat{Q}_1^{(1)} \hat{P}_1 \hat{Q}_1^{(2)} \hat{R}_1^{(1)} \hat{S}_1 \hat{R}_1^{(2)} \hat{Q}_2^{(1)} \hat{P}_2 \hat{Q}_2^{(2)} \hat{R}_2^{(1)} \hat{S}_2 \hat{R}_2^{(2)} \dots \hat{Q}_{d+1}^{(1)} \hat{P}_{d+1} \hat{Q}_{d+1}^{(2)} \hat{R}_{d+1}^{(1)} \hat{S}_{d+1} \hat{R}_{d+1}^{(2)},$$

where, for $1 \leq i \leq d+1$, we obtain

- $\hat{Q}_i^{(1)}$ (resp. $\hat{P}_i$) from g by replacing A_i with $Q_i^{(1)}$ (resp. P_i), the other blocks on the diagonal with identity matrices and the blocks under the diagonal with zero matrices,
- $\hat{Q}_i^{(2)}$ from g by replacing A_i with $Q_i^{(2)}$, the other blocks on the diagonal with identity matrices and the blocks under the diagonal which are not in the same column of blocks as A_i with zero matrices and

- $\hat{R}_i^{(1)}$ (resp. $\hat{S}_i$, $\hat{R}_i^{(2)}$) from g by replacing B_i with $R_i^{(1)}$ (resp. S_i, $R_i^{(2)}$), the other blocks on the diagonal with identity matrices and the blocks under the diagonal with zero matrices.

The proposition now follows directly from proposition 2.1.11 for the multiplication by the lower triangular matrices $\hat{Q}_i^{(1)}, \hat{Q}_i^{(2)}, \hat{R}_i^{(1)}$ and $\hat{R}_i^{(2)}$ and proposition 2.1.12 for the multiplication by the permutation matrices $\hat{P}_i$ and $\hat{S}_i$. ■

3.6.3 Proposition: *Let* $\mathfrak{t} \in Std(\lambda), L \in \mathfrak{M}_{\mathfrak{t}}(q), (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$ *and*

$$g := \begin{pmatrix} \boxed{A_1} & & & & & & & \\ & \boxed{B_1} & & & & & & \\ & & \boxed{A_2} & & & & & \\ & & & \boxed{B_2} & & & & \\ & & & & \ddots & & & \\ & & & & & \boxed{A_{d+1}} & & \\ & & & & & & \boxed{B_{d+1}} & \\ & & & & & & & \boxed{I_{b_{d+2}}} \end{pmatrix} \in GL_n(q), \tag{3.11}$$

where $A_i \in GL_{b_i}$ $(1 \leq i \leq d+1)$ *and* $B_i \in GL_{h_{i-1}}$ $(1 \leq i \leq d+1)$.
Then we have

$$e_L \circ g = e_{L \circ (g^{-1})^T}. \tag{3.12}$$

Proof: We have the following equivalences:

$$\begin{aligned} & e_L \circ g = e_{L \circ (g^{-1})^T} \\ \Longleftrightarrow & \frac{1}{q^{|\mathfrak{I}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}(q)} \chi_L(-M)(M \circ g) = \frac{1}{q^{|\mathfrak{I}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}(q)} \chi_{L \circ (g^{-1})^T}(-M)M \\ \Longleftrightarrow & \sum_{N \in \mathfrak{M}_{\mathfrak{t}}(q)} \chi_L(-N \circ g^{-1})N = \sum_{M \in \mathfrak{M}_{\mathfrak{t}}(q)} \chi_{L \circ (g^{-1})^T}(-M)M \\ \Longleftrightarrow & \chi_L(-M \circ g^{-1}) = \chi_{L \circ (g^{-1})^T}(-M) \text{ for all } M \in \mathfrak{M}_{\mathfrak{t}}(q). \end{aligned} \tag{3.13}$$

First we want to show (3.13) for the special case

$$g := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \boxed{A_i} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \tag{3.14}$$

for an arbitrary but fixed $1 \leq i \leq d+1$.
Let $M = (m_{ij}) \in \mathfrak{M}_{\mathfrak{t}}(q), L = (l_{ij}), M \circ g^{-1} = (\tilde{m}_{ij})$ and $L \circ (g^{-1})^T = (\tilde{l}_{ij})$.

Then

$$\begin{aligned}
&\chi_L(-M\circ g^{-1}) = \chi_{L\circ (g^{-1})^T}(-M)\\
\iff& \prod_{(i,j)\in\mathfrak{I}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) = \prod_{(i,j)\in\mathfrak{I}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij})\\
\iff& \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) \prod_{(i,j)\in\mathfrak{I}^{(2)}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) = \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij}) \prod_{(i,j)\in\mathfrak{I}^{(2)}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij}),
\end{aligned}$$

where $\mathfrak{I}^{(1)}_{\mathfrak{t}} \subseteq \mathfrak{I}_{\mathfrak{t}}$ indexes the entries of M which appear in the i^{th} column of blocks below the defining line in $short(M)$ and

$$\mathfrak{I}^{(2)}_{\mathfrak{t}} := \mathfrak{I}_{\mathfrak{t}} \backslash \mathfrak{I}^{(1)}_{\mathfrak{t}}.$$

Hence

$$\begin{aligned}
&\prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) \prod_{(i,j)\in\mathfrak{I}^{(2)}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) = \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij}) \prod_{(i,j)\in\mathfrak{I}^{(2)}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij})\\
\iff& \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) \prod_{(i,j)\in\mathfrak{I}^{(2)}_{\mathfrak{t}}} \theta(-l_{ij}m_{ij}) = \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij}) \prod_{(i,j)\in\mathfrak{I}^{(2)}_{\mathfrak{t}}} \theta(-l_{ij}m_{ij})\\
\iff& \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-l_{ij}\tilde{m}_{ij}) = \prod_{(i,j)\in\mathfrak{I}^{(1)}_{\mathfrak{t}}} \theta(-\tilde{l}_{ij}m_{ij}).
\end{aligned}$$

So we have reduced the proof of the special case (3.14) to the task to prove

$$\prod_{i=1}^{h}\prod_{j=1}^{b} \theta(-l_{ij}\tilde{m}_{ij}) = \prod_{i=1}^{h}\prod_{j=1}^{b} \theta(-\tilde{l}_{ij}m_{ij})$$

for integers $h,b \in \mathbb{N}, L=(l_{ij}) \in \mathfrak{M}_{h,b}(q), M=(m_{ij}) \in \mathfrak{M}_{h,b}(q), A^{-1}=(a_{ij}) \in GL_b(q), MA^{-1}=(\tilde{m}_{ij})$ and $L(A^{-1})^T=(\tilde{l}_{ij})$.
We obtain

$$\tilde{m}_{ij} = \sum_{k=1}^{b} m_{ik}a_{kj} \text{ and } \tilde{l}_{ij} = \sum_{k=1}^{b} l_{ik}a_{jk}.$$

Therefore

$$\begin{aligned}
&\prod_{i=1}^{h}\prod_{j=1}^{b} \theta(-l_{ij}\tilde{m}_{ij}) = \prod_{i=1}^{h}\prod_{j=1}^{b} \theta(-\sum_{k=1}^{b} l_{ij}m_{ik}a_{kj}) = \prod_{i=1}^{h}\prod_{j=1}^{b}\prod_{k=1}^{b} \theta(-l_{ij}m_{ik}a_{kj})\\
=&\prod_{i=1}^{h}\prod_{j=1}^{b}\prod_{k=1}^{b} \theta(-l_{ik}m_{ij}a_{jk}) = \prod_{i=1}^{h}\prod_{j=1}^{b} \theta(-\sum_{k=1}^{b} l_{ik}m_{ij}a_{jk}) = \prod_{i=1}^{h}\prod_{j=1}^{b} \theta(-\tilde{l}_{ij}m_{ij})
\end{aligned}$$

and hence the proposition holds for the special case (3.14).

The proof of (3.13) for the special case

$$g := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \boxed{B_i} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \tag{3.15}$$

for an arbitrary but fixed $1 \leq i \leq d+1$ is quite similar: We reduce the problem to the i^{th} row of blocks and recall that the operation $\circ$ of g^{-1} is on this row the multiplication by B_i on the left.
Now the proposition follows inductively from the two special cases (3.14) and (3.15) since

$$\begin{aligned} e_L \circ (g_1 g_2) &= (e_L \circ g_1) \circ g_2 = e_{L \circ (g_1^{-1})^T} \circ g_2 = e_{L \circ (g_1^{-1})^T \circ (g_2^{-1})^T} \\ &= e_{L \circ ((g_1^{-1})^T (g_2^{-1})^T)} = e_{L \circ ((g_2^{-1} g_1^{-1})^T)} = e_{L \circ ((g_1 g_2)^{-1})^T}. \end{aligned}$$

■

3.6.4 Definition: Let $k, h, b \in \mathbb{N}$, $x, y \in \mathbb{N}$ with $x + h - 1 \leq k, y + b - 1 \leq k$ and $M \in \mathfrak{M}_{h,b}(q)$. Then $E(k, x, y, M) \in \mathfrak{M}_{k,k}(q)$ is defined as the matrix that differs from the identity matrix $I_k \in GL_k(q)$ only in the block M starting at position (x, y).

3.6.5 Example: Let $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{M}_{2,2}(q)$. Then

$$E(4,1,1,M) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E(4,3,3,M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix},$$

$$E(4,3,1,M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \text{ and } E(4,3,2,M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 1 \end{pmatrix}.$$

3.6.6 Definition: Let $\mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_{\mathfrak{t}}(q)$. This means that L is of the form

$$L = \begin{array}{|c|c|c|c|c|c|} \hline D_1 & & & & & \\ X_{11} & D_2 & & & & \\ X_{21} & X_{22} & \ddots & & & \\ \dots & \dots & & D_d & & \\ X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} & \underbrace{}_{b_{d+2}} \\ \hline \end{array}.$$

1.) We recall from definition 3.3.5 that, for $1 \leq i \leq d$, $xmat(L,i)$ denotes the matrix which consists of the blocks $\{X_{kl} \mid 1 \leq l \leq i \leq k \leq d\}$.

2.) If $1 \leq i \leq d$ and $r \in \mathbb{N}$ is not greater than the width of $xmat(L,i)$, then $xmat(L,i,r)$ is defined as the matrix that consists of the last r columns of $xmat(L,i)$.

3.) The matrix
$$\begin{pmatrix} X_{i-1,1} & \dots & X_{i-1,i-1} & D_i \end{pmatrix}$$
is called $dmat(L,i)$ $(2 \leq i \leq d+1)$. It consists of the blocks which are to the left of the defining line in the i^{th} row of blocks of $short(L)$.

4.) If $2 \leq i \leq d+1$ and $r \in \mathbb{N}$ is not greater than the width of $dmat(L,i)$, then $dmat(L,i,r)$ denotes the last r columns of $dmat(L,i)$.

To prove the next theorem we need the following technical lemma.

3.6.7 Lemma: *Let $b, h_1, h_2 \in \mathbb{N}$, $b_1, b_2 \in \mathbb{N}_0$ with $b_1 + b_2 = b$, $M_1 \in \mathfrak{M}_{h_1,b}(q)$ and $M_2 \in \mathfrak{M}_{h_2,b}(q)$ with $rank(M_2) \leq b_2$. Then there exist matrices $A \in GL_b(q)$ and $B \in GL_{h_1}(q)$ such that*

- *BM_1A is an upper triangular matrix, i.e. the entries below the diagonal are zero (in general BM_1A is not a quadratic matrix) and*
- *all entries in the first b_1 columns of M_2A are zero.*

Proof: By multiplication by a matrix $A \in GL_b(q)$ on the right we can carry out arbitrary column operations on the matrix M_2. Since $rank(M_2) \leq b_2$ we find a matrix $A_1 \in GL_b(q)$ such that the first $b - b_2 = b_1$ columns of M_2A_1 have only zero entries. We split the matrix M_1A_1 into two parts, namely the first b_1 columns M_{11} and the last b_2 columns M_{12}.

$$\begin{array}{cc} & b \\ \begin{array}{c} h_1 \\ h_2 \end{array} & \left(\begin{array}{|c|} \hline M_1 \\ \hline M_2 \\ \hline \end{array}\right) \end{array} A_1 = \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \left(\begin{array}{|c|c|} \hline M_{11} & M_{12} \\ \hline 0 & ? \\ \hline \end{array}\right) \end{array}$$

By multiplication by a matrix $B \in GL_{h_1}$ on the left and a matrix $A \in GL_{b_1}(q)$ on the right we can carry out arbitrary row and column operations on M_{11}. Therefore we find a matrix $A_2 \in GL_{b_1}(q)$ and a matrix $B_2 \in GL_{h_1}(q)$ such that the nondiagonal entries of $B_2M_{11}A_2$ are zero. Then there are two cases:

1.) $b_1 \geq h_1$: $B_2M_1A_1E(b,1,1,A_2)$ is an upper triangular matrix. The lemma follows directly with the matrices
$$A := A_1E(b,1,1,A_2) \text{ and } B := B_2.$$

2.) $b_1 < h_1$: We split the matrix $B_2M_1A_1E(b,1,1,A_2)$ into 4 parts

$$B_2M_1A_1E(b,1,1,A_2) = \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \left(\begin{array}{|c|c|} \hline M_{11} & M_{12} \\ \hline 0 & M_{22} \\ \hline \end{array}\right) \end{array} \begin{array}{c} b_1 \\ h_1 - b_1 \end{array}$$

where the nondiagonal entries of M_{11} are zero.
As before we find matrices $A_3 \in GL_{b_2}(q)$ and $B_3 \in GL_{h_1-b_1}(q)$ such that the nondiagonal entries of $B_3 M_{22} A_3$ are zero. The lemma follows directly with the matrices

$$\begin{aligned} A &:= A_1 E(b,1,1,A_2) E(b, b_1+1, b_1+1, A_3) \\ &\text{and } B := E(h_1, b_1+1, b_1+1, B_3) B_2. \end{aligned}$$

■

3.6.8 Theorem: *Let $\mathfrak{t} \in Std(\lambda)$. Then, for every $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$, there exists an element $b_L \in S^\lambda$ such that $last(b_L) = \mathfrak{t}$ and $top(b_L) = e_L$.*

Before we prove the theorem we demonstrate the difficulty of constructing such an element b_L.

3.6.9 Example: Let $n := 4, \lambda := (2,2), \mathfrak{t} := \boxed{\begin{array}{c|c} 1 & 2 \\ \hline 3 & 4 \end{array}} \in Std(\lambda), q := 2$ and

1.) $L := \boxed{\begin{array}{cc} \cdot & 1 \\ \cdot & \cdot \end{array}} \in \mathfrak{M}_{\mathfrak{t}}^{rk}(2)$.
In this case it is easy to construct an element b_L.

$$b_L := \frac{1}{2^4} \sum_{\underline{y} \in GF(2)^3} \theta(-y_2)(e_{(2,2)} \circ (2,3)) \circ \begin{pmatrix} 1 & & & \\ & 1 & & \\ y_1 & y_2 & 1 & \\ & y_3 & & 1 \end{pmatrix}$$

since

$$last(b_L) = \mathfrak{t}$$

and

$$\begin{aligned} top(b_L) &= \frac{1}{2^4} \sum_{\underline{y} \in GF(2)^3} \theta(-y_2) \boxed{\begin{array}{cc} \cdot & \cdot \\ * & \cdot \end{array}} \circ \begin{pmatrix} 1 & & & \\ & 1 & & \\ y_1 & y_2 & 1 & \\ & y_3 & & 1 \end{pmatrix} \\ &= \frac{1}{2^4} \sum_{\underline{y} \in GF(2)^3} \theta(-y_2) \boxed{\begin{array}{cc} y_1 & y_2 \\ * & y_3 \end{array}} = e_L. \end{aligned}$$

2.) $L := \boxed{\begin{array}{cc} \cdot & \cdot \\ 1 & \cdot \end{array}} \in \mathfrak{M}_{\mathfrak{t}}^{rk}(2)$.
This case is much more difficult because the summation "$*$" in

$$top(e_{(2,2)} \circ (2,3))$$

is just at the position where the "1" is located in L. The proof of theorem 3.6.8 shows the solution of this problem.

Proof: (of theorem 3.6.8)
First we prove the theorem for the special case $\lambda := (m, m)$:
For this purpose let $\mathfrak{t} \in Std(\lambda)$, $\delta(\mathfrak{t}) = (d, \underline{h}, \underline{b}, \underline{r})$ and

$$L = \begin{array}{|c|c|c|c|c|}
\hline
D_1 & & & & \\
\cline{1-2}
X_{11} & D_2 & & & \\
\cline{1-2}
X_{21} & X_{22} & \ddots & & \\
\cline{1-2} \cline{4-4}
\dots & \dots & & D_d & \\
\cline{1-2} \cline{4-5}
X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} \\
\hline
\end{array} \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q).$$

We will construct an element $b_L \in S^\lambda$ with $last(b_L) = \mathfrak{t}$ and $top(b_L) = e_L$. Our algorithm has two parts: In the first part we show the existence of several matrices with special properties. In the second part we use these matrices to operate on e_λ to get an element b_L.

<u>First part:</u> We set $L_1 := L$ and $r_0 := 0$ and successively execute for $i = 1, 2, 3, \dots d$ the following step i:
We define $E_i := dmat(L_i, i, b_i + r_{i-1})$ and $Y_i := xmat(L_i, i, b_i + r_{i-1})$.
Using lemma 3.6.7 and the fact that $rank(xmat(L_i, i)) \leq r_i$, we find matrices $A_i \in GL_{b_i + r_{i-1}}(q)$ and $B_i \in GL_{h_{i-1}}(q)$ such that

- $B_i E_i A_i$ is an upper triangular matrix and
- all but the last r_i columns of $Y_i A_i$ have only zero entries.

Now we get L_{i+1} from L_i by replacing the blocks E_i and Y_i in L_i with $B_i E_i A_i$ and $Y_i A_i$ and proceed with the next step.
We complete the first part with step $d+1$ which is similar to the previous steps except that we don't have a matrix Y_{d+1}. We define $E_{d+1} := dmat(L_{d+1}, d+1, b_{d+1} + r_d)$ and find matrices $A_{d+1} \in GL_{b_{d+1}+r_d}(q)$ and $B_{d+1} \in GL_{h_d}(q)$ such that $B_{d+1} E_{d+1} A_{d+1}$ is an upper triangular matrix. Finally we replace the block E_{d+1} in L_{d+1} with $B_{d+1} E_{d+1} A_{d+1}$ and get L_{d+2}.
By construction the block $B_i E_i A_i$ $(1 \leq i \leq d+1)$ has its uppermost leftmost entry at position

$$\left(\sum_{j=0}^{i-2} h_j + 1, \sum_{j=1}^{i} b_j - (b_i + r_{i-1}) + 1 \right) \tag{3.16}$$

and its lowest rightmost entry at position

$$\left(\sum_{j=0}^{i-1} h_j, \sum_{j=1}^{i} b_j \right) \tag{3.17}$$

in $short(L_{i+1})$.
And the block $Y_i A_i$ $(1 \leq i \leq d)$ has its uppermost leftmost entry at position

$$\left(\sum_{j=0}^{i-1} h_j + 1, \sum_{j=1}^{i} b_j - (b_i + r_{i-1}) + 1 \right) \tag{3.18}$$

and its lowest rightmost entry at position

$$\left(m, \sum_{j=1}^{i} b_j\right) \tag{3.19}$$

in $short(L_{i+1})$.
Furthermore, $short(L_{d+2})$ is an upper triangular matrix because

- the uppermost leftmost entry of the matrix $B_iE_iA_i$ lies on the diagonal of $short(L_{d+2})$ since

$$\begin{aligned}\sum_{j=1}^{i} b_j - (b_i + r_{i-1}) + 1 &\overset{(3.4)}{=} \sum_{j=1}^{i} b_j - b_i - (\sum_{j=1}^{i-1} b_j + \sum_{j=0}^{i-2} h_j) + 1 \\ &= \sum_{j=0}^{i-2} h_j + 1.\end{aligned} \tag{3.20}$$

- the matrix Y_iA_i below $B_iE_iA_i$ may have nonzero entries only in the last r_i columns, but exactly those columns are again part of Y_{i+1} in the next step. Indeed, if we subtract the position of the last column of Y_iA_i in $short(L_{i+1})$ from the position of the first column of Y_{i+1} we get

$$\sum_{j=1}^{i} b_j - (\sum_{j=1}^{i+1} b_j - (b_{i+1} + r_i) + 1) = r_i - 1$$

  which means that Y_iA_i and Y_{i+1} overlap each other in r_i columns.

We want to illustrate the first part by an example.
Let $\lambda := (7,7), \mathfrak{t} := \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 7 & 8 & 12 \\ \hline 5 & 6 & 9 & 10 & 11 & 13 & 14 \\ \hline \end{array} \in Std(\lambda)$ and $q := 2$.
Then

$$\underline{h} = (2,3,2), \underline{b} = (4,2,1), \underline{r} = (2,1), d = 2.$$

Furthermore, we choose

$$L := \begin{array}{|cccc|cc|c|}\hline 1 & 1 & 1 & \cdot & & & \\ 1 & 1 & \cdot & 1 & & & \\ \hline 1 & \cdot & \cdot & 1 & \cdot & 1 & \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \hline 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \hline \end{array} \in \mathfrak{M}_{\mathfrak{t}}^{rk}(2).$$

<u>Step 1:</u>

$$L_1 := L, E_1 := \begin{pmatrix} 1 & 1 & 1 & \cdot \\ 1 & 1 & \cdot & 1 \end{pmatrix}, Y_1 := \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot \end{pmatrix}.$$

We find

$$A_1 = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ 1 & 1 & 1 & 1 \\ \cdot & 1 & \cdot & 1 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}$$

such that

$$B_1E_1A_1 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, Y_1A_1 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \text{ and } L_2 = \begin{array}{|cccc|cc|c|} \hline 1 & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & 1 & \cdot & & & \\ \cline{1-6} \cdot & \cdot & \cdot & 1 & \cdot & 1 & \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \hline \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\ \hline \end{array}.$$

Step 2:

$$E_2 := \begin{pmatrix} \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, Y_2 := \begin{pmatrix} 1 & \cdot & 1 & 1 \\ 1 & \cdot & 1 & 1 \end{pmatrix}.$$

We find

$$A_2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & 1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

such that

$$B_2E_2A_2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, Y_2A_2 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \text{ and } L_3 = \begin{array}{|cccc|cc|c|} \hline 1 & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & 1 & \cdot & & & \\ \cline{1-6} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \hline \end{array}.$$

Step 3:

$$E_3 := \begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix}.$$

We find

$$A_3 = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

such that

$$B_3E_3A_3 = \begin{pmatrix} 1 & 1 \\ \cdot & 1 \end{pmatrix} \text{ and } L_4 = \begin{array}{|cccc|cc|c|} \hline 1 & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & 1 & \cdot & & & \\ \cline{1-6} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline \end{array}.$$

Second part: We start with several definitions:

- Using the permutations $\pi_{\mathfrak{t}}^{(1)}, \dots, \pi_{\mathfrak{t}}^{(m)}$ from definition 3.2.5, for $1 \leq i \leq d+1$, we set
$$\pi_i := \pi_{\mathfrak{t}}^{(m-k_i)} \pi_{\mathfrak{t}}^{(m-k_i-1)} \dots \pi_{\mathfrak{t}}^{(m-k_i-h_{i-1}+1)},$$
where $k_i := \sum_{j=i}^{d} h_j$.
This means for example that
$$\pi_{d+1} = \pi_{\mathfrak{t}}^{(m)} \pi_{\mathfrak{t}}^{(m-1)} \dots \pi_{\mathfrak{t}}^{(m-h_d+1)} \text{ and } \pi_1 = \pi_{\mathfrak{t}}^{(h_0)} \pi_{\mathfrak{t}}^{(h_0-1)} \dots \pi_{\mathfrak{t}}^{(1)}$$
and by construction we get
$$\pi_{d+1}\pi_d \dots \pi_2 \pi_1 = \pi_{\mathfrak{t}}. \tag{3.21}$$
In our example of the first part
$$\begin{aligned} \pi_1 &= \pi_{\mathfrak{t}}^{(2)} \pi_{\mathfrak{t}}^{(1)} = (4,5)(5,6)(2,3)(3,4)(4,5), \\ \pi_2 &= \pi_{\mathfrak{t}}^{(5)} \pi_{\mathfrak{t}}^{(4)} \pi_{\mathfrak{t}}^{(3)} = (10,11)(8,9)(9,10)(6,7)(7,8)(8,9), \\ \pi_3 &= \pi_{\mathfrak{t}}^{(7)} \pi_{\mathfrak{t}}^{(6)} = (12,13). \end{aligned}$$

- We define for $1 \leq i \leq d+1$
$$\hat{A}_i := E(n, k_i^{(A)}, k_i^{(A)}, A_i^T) \text{ and } \hat{B}_i := E(n, k_i^{(B)}, k_i^{(B)}, (B_i^{-1})^T),$$
where
$$\begin{aligned} k_i^{(A)} &:= \left(\sum_{j=1}^{i-1} b_j \right) + \left(\sum_{j=0}^{i-2} h_j \right) - r_{i-1} + 1 \text{ and} \\ k_i^{(B)} &:= \left(\sum_{j=1}^{i} b_j \right) + \left(\sum_{j=0}^{i-2} h_j \right) + 1. \end{aligned}$$
For example
$$\begin{aligned} \hat{A}_1 &= E(n, 1, 1, A_1^T), \\ \hat{A}_{d+1} &= E(n, n - b_{d+1} - h_d - r_d + 1, n - b_{d+1} - h_d - r_d + 1, A_{d+1}^T), \\ \hat{B}_1 &= E(n, b_1 + 1, b_1 + 1, (B_1^{-1})^T) \text{ and} \\ \hat{B}_{d+1} &= E(n, n - h_d + 1, n - h_d + 1, (B_{d+1}^{-1})^T). \end{aligned}$$
In our example of the first part
$$\begin{gathered} \hat{A}_1 := E(14, 1, 1, A_1^T), \hat{A}_2 := E(14, 5, 5, A_2^T), \hat{A}_3 := E(14, 11, 11, A_3^T), \\ \hat{B}_1 := E(14, 5, 5, (B_1^{-1})^T), \hat{B}_2 := E(14, 9, 9, (B_2^{-1})^T), \\ \hat{B}_3 := E(14, 13, 13, (B_3^{-1})^T). \end{gathered}$$

- For $a, b \in \mathbb{N}$ we denote by $\mathfrak{M}_{a,b}^{up}(q) \subseteq \mathfrak{M}_{a,b}(q)$ the set of $a \times b$ matrices whose entries below and on the diagonal are zero.

- In the set $\mathfrak{I}_{\mathfrak{t}_\lambda}$ the tupels (k,l) which fulfill $2(k-m)-1=l$ index exactly the diagonal entries of $short(M)$ for an arbitrary $M \in \mathfrak{M}_{\mathfrak{t}_\lambda}(q)$.
  We define $N_{d+2} = (n_{kl}) \in \mathfrak{M}_{\mathfrak{t}_\lambda}(q)$ by

$$n_{kl} := \begin{cases} 0 & \text{if } 2(k-m)-1 \neq l, (k,l) \in \mathfrak{I}_{\mathfrak{t}_\lambda} \\ l_{kl} & \text{if } 2(k-m)-1 = l, (k,l) \in \mathfrak{I}_{\mathfrak{t}_\lambda}, \end{cases}$$

  where $L_{d+2} = (l_{ij})$.
  After operating with π_{d+1} on N_{d+2} we get a block in $short(N_{d+2} \circ \pi_{d+1})$ which has its uppermost leftmost entry at position $(m-h_d+1, m-h_d+1)$ and its lowest rightmost entry at position (m,m). If we replace this block with $B_{d+1}E_{d+1}A_{d+1}$ we get the short notation of $N_{d+1}^{(1)} \in \mathfrak{M}_{\mathfrak{t}_\lambda \circ \pi_{d+1}}(q)$ and if we replace this block with E_{d+1} we get the short notation of $N_{d+1}^{(2)} \in \mathfrak{M}_{\mathfrak{t}_\lambda \circ \pi_{d+1}}(q)$.
  For $d \geq i \geq 1$ we define the elements $N_i^{(1)}$ and $N_i^{(2)}$ recursively in the following way:
  After operating with π_i on $N_{i+1}^{(2)}$ we get in $short(N_{i+1}^{(2)} \circ \pi_i)$ a block $P_i^{(0)}$ which has its uppermost leftmost entry at position

$$\left(m - \sum_{j=i-1}^{d} h_j + 1, m - \sum_{j=i-1}^{d} h_j + 1\right)$$

  and its lowest rightmost entry at position

$$\left(m, m - \sum_{j=i+1}^{d+1} b_j\right).$$

  First we look at the subblock $P_i^{(1)}$ of this block which has its uppermost leftmost entry at position

$$\left(m - \sum_{j=i-1}^{d} h_j + 1, m - \sum_{j=i-1}^{d} h_j + 1\right) \tag{3.22}$$

  and its lowest rightmost entry at position

$$\left(m - \sum_{j=i}^{d} h_j, m - \sum_{j=i+1}^{d+1} b_j\right). \tag{3.23}$$

Fact 1 *The diagonal entries of $P_i^{(1)}$ in $short(N_{i+1}^{(2)} \circ \pi_i)$ are identical to the diagonal entries of $B_iE_iA_i$.*

Proof: The positions in (3.16) and (3.22) are equal and also the positions in (3.17) and (3.23) since $\sum_{j=0}^{d} h_j = m$ and $\sum_{j=1}^{d+1} b_j = m$ and (3.20) holds.

Therefore $B_iE_iA_i$ is in $short(L_{d+2})$ at the same position as $P_i^{(1)}$ in $short(N_{i+1}^{(2)} \circ \pi_i)$. ∎

If we replace in $short(N_{i+1}^{(2)} \circ \pi_i)$ the block $P_i^{(1)}$ with $B_iE_iA_i$ we get the short notation of $N_i^{(1)} \in \mathfrak{M}_{\mathfrak{t}_\lambda \circ (\pi_{d+1}\pi_d \ldots \pi_i)}(q)$.
Now we look at the subblock $P_i^{(2)}$ of $P_i^{(0)}$ which remains if we omit $P_i^{(1)}$. $P_i^{(2)}$ has its uppermost leftmost entry at position

$$\left(m - \sum_{j=i}^{d} h_j + 1, m - \sum_{j=i-1}^{d} h_j + 1\right) \tag{3.24}$$

and its lowest rightmost entry at position

$$\left(m, m - \sum_{j=i+1}^{d+1} b_j\right). \tag{3.25}$$

Fact 2 *The block $P_i^{(2)}$ in $N_i^{(1)}$ equals exactly Y_iA_i.*

Proof: The position of $P_i^{(2)}$ in $short(N_i^{(1)})$ is the same as the position of Y_iA_i in $short(L_{i+1})$ (cf. (3.18), (3.19), (3.24), (3.25) and (3.20)). ∎

If we replace in $short(N_{i+1}^{(2)} \circ \pi_i)$ the block $P_i^{(1)}$ with E_i and the block $P_i^{(2)}$ with Y_i we get the short notation of $N_i^{(2)} \in \mathfrak{M}_{\mathfrak{t}_\lambda \circ (\pi_{d+1}\pi_d \ldots \pi_i)}(q)$.
Finally we get the element $N_1^{(2)} \in \mathfrak{M}_{\mathfrak{t}_\lambda \circ (\pi_{d+1}\pi_d \ldots \pi_1)}(q) \overset{(3.1) \text{ and } (3.21)}{=} \mathfrak{M}_{\mathfrak{t}}(q)$.

Fact 3 *We have*

$$N_1^{(2)} = L. \tag{3.26}$$

Proof: By the same arguments applied to the positions of the E_i and Y_i as before. ∎

In our example of the first part

$$N_4 = \begin{array}{|ccccccc|}\hline 1 & & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & 1 & & & & \\ \cdot & \cdot & \cdot & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline\end{array}, N_3^{(1)} = \begin{array}{|ccccccc|}\hline 1 & & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & 1 & & & & \\ \cdot & \cdot & \cdot & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline\end{array}, N_3^{(2)} = \begin{array}{|ccccccc|}\hline 1 & & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & 1 & & & & \\ \cdot & \cdot & \cdot & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \hline\end{array},$$

$$N_2^{(1)} = \begin{array}{|ccccccc|}\hline 1 & & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \hline\end{array} \text{ and } N_2^{(2)} = \begin{array}{|ccccccc|}\hline 1 & & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\ \hline\end{array},$$

$$N_1^{(1)} = \begin{array}{|cccc|cc|c|} \hline 1 & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & 1 & \cdot & & & \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & 1 & \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \hline \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\ \hline \end{array} \text{ and } N_1^{(2)} = \begin{array}{|cccc|cc|c|} \hline 1 & 1 & 1 & \cdot & & & \\ 1 & 1 & \cdot & 1 & & & \\ \hline 1 & \cdot & \cdot & 1 & \cdot & 1 & \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\ \hline 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \hline \end{array} = L.$$

- We set

$$v_{d+2}^{(2)} := \frac{1}{q^k} \sum_{\underline{y} \in GF(q)^m} \left(\prod_{j=1}^{m} \theta(-l_{jj} y_j) \right) e_\lambda \circ \begin{pmatrix} 1 & & & & & & & \\ y_1 & 1 & & & & & & \\ & & 1 & & & & & \\ & & y_2 & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & & 1 & \\ & & & & & & y_m & 1 \end{pmatrix},$$

where $(l_{ij}) := short(L_{d+2})$ and $k := \sum_{j=1}^{m} j = \frac{m(m+1)}{2}$.
Finally, for $d+1 \geq i \geq 1$, we define recursively

$$v_i^{(1)} := \frac{1}{|\mathfrak{M}_i(q)|} \sum_{M=(m_{rs}) \in \mathfrak{M}_i(q)} \left(\prod_{r=1}^{h_{i-1}} \prod_{s=1}^{b_i + r_{i-1}} \theta(-m_{rs} n_{rs}^{(i)}) \right) v_{i+1}^{(2)} \circ (\pi_i E(n, k_i^{(B)}, k_i^{(A)}, M)) \text{ and}$$

$$v_i^{(2)} := v_i^{(1)} \circ (\hat{A}_i \hat{B}_i),$$

where $\mathfrak{M}_i(q) := \mathfrak{M}^{up}_{h_{i-1}, b_i + r_{i-1}}(q)$ and $(n_{rs}^{(i)}) := B_i E_i A_i$.

Fact 4

$$top(v_{d+2}^{(2)}) = e_{N_{d+2}}.$$

Proof: Follows directly with proposition 3.6.2. ■

Fact 5 *For $d+1 \leq i \leq 1$*

$$last(v_i^{(1)}) = \mathfrak{t}_\lambda \circ (\pi_{d+1} \pi_d \ldots \pi_i) \text{ and } top(v_i^{(1)}) = e_{N_i^{(1)}}.$$

Proof: Follows directly with propositions 2.1.10 and 2.1.11, fact 1 and fact 2. ■

Fact 6 *For $d+1 \leq i \leq 1$*

$$last(v_i^{(2)}) = \mathfrak{t}_\lambda \circ (\pi_{d+1} \pi_d \ldots \pi_i) \text{ and } top(v_i^{(2)}) = e_{N_i^{(2)}}.$$

Proof: Follows directly with propositions 3.6.2 and 3.6.3. ■

This means

$$top(v_1^{(2)}) = e_{N_1^{(2)}} \overset{(3.26)}{=} e_L \text{ and}$$

$$last(v_1^{(2)}) = \mathfrak{t}_\lambda \circ (\pi_{d+1} \pi_d \ldots \pi_1) \overset{(3.21)}{=} \mathfrak{t}_\lambda \circ \pi_{\mathfrak{t}} \overset{(3.1)}{=} \mathfrak{t}$$

and therefore, for the special case $\lambda = (m,m)$, $\mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$, we have proved the existence of an element b_L with the required properties, namely $b_L := v_1^{(2)}$.
We illustrate this with our example of the first part:

$$top(v_4^{(2)}) = \frac{1}{2^{28}} \sum_{\underline{y} \in GF(2)^7} \prod_{i \in \mathfrak{J}_4^{(2)}} \theta(-y_i) \begin{array}{|ccccccc|} \hline y_1 & & & & & & \\ * & y_2 & & & & & \\ * & * & y_3 & & & & \\ * & * & * & y_4 & & & \\ * & * & * & * & y_5 & & \\ * & * & * & * & * & y_6 & \\ * & * & * & * & * & * & y_7 \\ \hline \end{array} = e_{N_4},$$

where $\mathfrak{J}_4^{(2)} := \{1,3,4,5,6,7\}$.

$$top(v_3^{(1)}) = \frac{1}{2^{29}} \sum_{\underline{y} \in GF(2)^8} \prod_{i \in \mathfrak{J}_3^{(1)}} \theta(-y_i) \begin{array}{|ccccccc|} \hline y_1 & & & & & & \\ * & y_2 & & & & & \\ * & * & y_3 & & & & \\ * & * & * & y_4 & & & \\ * & * & * & * & y_5 & & \\ * & * & * & * & * & y_6 & y_8 \\ * & * & * & * & * & * & y_7 \\ \hline \end{array} = e_{N_3^{(1)}},$$

where $\mathfrak{J}_3^{(1)} := \{1,3,4,5,6,7,8\}$.

$$top(v_3^{(2)}) = \frac{1}{2^{29}} \sum_{\underline{y} \in GF(2)^9} \prod_{i \in \mathfrak{J}_3^{(2)}} \theta(-y_i) \begin{array}{|ccccccc|} \hline y_1 & & & & & & \\ * & y_2 & & & & & \\ * & * & y_3 & & & & \\ * & * & * & y_4 & & & \\ * & * & * & * & y_5 & & \\ * & * & * & * & * & y_6 & y_7 \\ * & * & * & * & * & y_8 & y_9 \\ \hline \end{array} = e_{N_3^{(2)}},$$

where $\mathfrak{J}_3^{(2)} := \{1,3,4,5,6,7,8\}$.

$$top(v_2^{(1)}) = \frac{1}{2^{35}} \sum_{\underline{y} \in GF(2)^{15}} \prod_{i \in \mathfrak{J}_2^{(1)}} \theta(-y_i) \begin{array}{|ccccccc|} \hline y_1 & & & & & & \\ * & y_2 & & & & & \\ * & * & y_3 & y_{10} & y_{11} & y_{12} & \\ * & * & * & y_4 & y_{13} & y_{14} & \\ * & * & * & * & y_5 & y_{15} & \\ * & * & * & * & * & y_6 & y_7 \\ * & * & * & * & * & y_8 & y_9 \\ \hline \end{array} = e_{N_2^{(1)}},$$

where $\mathfrak{J}_2^{(1)} := \{1,3,4,5,6,7,8\}$.

$$top(v_2^{(2)}) = \frac{1}{2^{35}} \sum_{\underline{y} \in GF(2)^{24}} \prod_{i \in \mathfrak{J}_2^{(2)}} \theta(-y_i) \begin{array}{|ccccccc|} \hline y_1 & & & & & & \\ * & y_2 & & & & & \\ * & * & y_3 & y_4 & y_5 & y_6 & \\ * & * & y_7 & y_8 & y_9 & y_{10} & \\ * & * & y_{11} & y_{12} & y_{13} & y_{14} & \\ * & * & y_{15} & y_{16} & y_{17} & y_{18} & y_{19} \\ * & * & y_{20} & y_{21} & y_{22} & y_{23} & y_{24} \\ \hline \end{array} = e_{N_2^{(2)}},$$

where $\mathfrak{I}_2^{(2)} := \{1, 4, 6, 7, 13, 15, 17, 18, 19, 20, 22, 23\}$.

$$top(v_1^{(1)}) = \frac{1}{2^{40}} \sum_{\underline{y} \in GF(2)^{29}} \prod_{i \in \mathfrak{I}_1^{(1)}} \theta(-y_i) \begin{array}{|cccc|cc|c|}
\hline
y_1 & y_{25} & y_{26} & y_{27} & & & \\
* & y_2 & y_{28} & y_{29} & & & \\
\hline
* & * & y_3 & y_4 & y_5 & y_6 & \\
* & * & y_7 & y_8 & y_9 & y_{10} & \\
* & * & y_{11} & y_{12} & y_{13} & y_{14} & \\
\hline
* & * & y_{15} & y_{16} & y_{17} & y_{18} & y_{19} \\
* & * & y_{20} & y_{21} & y_{22} & y_{23} & y_{24} \\
\hline
\end{array} = e_{N_1^{(1)}},$$

where $\mathfrak{I}_1^{(1)} := \{1, 4, 6, 7, 13, 15, 17, 18, 19, 20, 22, 23, 28\}$.

$$top(v_1^{(2)}) = \frac{1}{2^{40}} \sum_{\underline{y} \in GF(2)^{40}} \prod_{i \in \mathfrak{I}_1^{(2)}} \theta(-y_i) \begin{array}{|cccc|cc|c|}
\hline
y_1 & y_2 & y_3 & y_4 & & & \\
y_5 & y_6 & y_7 & y_8 & & & \\
\hline
y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} & \\
y_{15} & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} & \\
y_{21} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & \\
\hline
y_{27} & y_{28} & y_{29} & y_{30} & y_{31} & y_{32} & y_{33} \\
y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} & y_{40} \\
\hline
\end{array} = e_{N_1^{(2)}},$$

where

$$\mathfrak{I}_1^{(2)} := \{1, 2, 3, 5, 6, 8, 9, 12, 14, 15, 16, 17, 25, 27, \\ 28, 29, 31, 32, 33, 34, 35, 36, 38, 39\}.$$

Now let $\lambda = (n - m, m)$ be an arbitrary two-part partition of n, $\mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$.
The proof of this case follows easily from the special case $\lambda = (m, m)$. We illustrate it with an example; the general case is similar.
Let $n := 17, \lambda := (10, 7), \mathfrak{t} := \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 7 & 8 & 12 & 13 & 15 & 17 \\ \hline 5 & 6 & 9 & 10 & 11 & 14 & 16 \\ \cline{1-7}\end{array} \in Std(\lambda), q := 2$ and

$$L := \begin{array}{|cccc|cc|cc|cc|}
\hline
1 & 1 & 1 & \cdot & & & & & & \\
1 & 1 & \cdot & 1 & & & & & & \\
\hline
1 & \cdot & \cdot & 1 & \cdot & 1 & & & & \\
1 & 1 & 1 & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & & & & \\
\hline
1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \\
\hline
\end{array} \in \mathfrak{M}_{\mathfrak{t}}^{rk}(2).$$

The previous example of the special case ensures that we can generate an element v_1 of S^λ with

$$top(v_1) = \frac{1}{2^{40}} \sum_{\underline{y} \in GF(2)^{40}} \prod_{i \in \mathfrak{I}_1} \theta(-y_i) \begin{array}{|cccc|cc|c|ccc|}
\hline
y_1 & y_2 & y_3 & y_4 & & & & & & \\
y_5 & y_6 & y_7 & y_8 & & & & & & \\
\hline
y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} & & & & \\
y_{15} & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} & & & & \\
y_{21} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & & & & \\
\hline
y_{27} & y_{28} & y_{29} & y_{30} & y_{31} & y_{32} & y_{33} & & & \\
y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} & y_{40} & & & \\
\hline
\end{array},$$

where

$$\mathfrak{I}_1 := \{1, 2, 3, 5, 6, 8, 9, 12, 14, 15, 16, 17, 25, 27, \\ 28, 29, 31, 32, 33, 34, 35, 36, 38, 39\}.$$

For this purpose we just sum and operate in the same way as in the special case. The only difference is that we don't operate with the elements $g \in GL_{14}(2)$ but with the elements $E(17,1,1,g) \in GL_{17}(2)$.
We set

$$v_2 := \frac{1}{2^3} \sum_{\underline{y} \in GF(2)^3} (\theta(-y_1)\theta(-y_3))v_1 \circ ((14,15)(15,16)(13,14)E(17,13,13,M(\underline{y}))),$$

where

$$M(\underline{y}) := \begin{pmatrix} 1 & & & \\ y_1 & 1 & & \\ & & 1 & \\ y_2 & & y_3 & 1 \end{pmatrix}.$$

By construction

$$last(v_2) = \mathfrak{t}$$

and

$$top(v_2) = \frac{1}{2^{43}} \sum_{\underline{y} \in GF(2)^{43}} \prod_{i \in \mathfrak{I}_2} \theta(-y_i) \begin{array}{|cccc|cc|cc|c|c|} \hline y_1 & y_2 & y_3 & y_4 & & & & & & \\ y_5 & y_6 & y_7 & y_8 & & & & & & \\ \hline y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} & & & & \\ y_{15} & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} & & & & \\ y_{21} & y_{22} & y_{23} & y_{24} & y_{25} & y_{26} & & & & \\ \hline y_{27} & y_{28} & y_{29} & y_{30} & y_{31} & y_{32} & y_{33} & y_{41} & & \\ \hline y_{34} & y_{35} & y_{36} & y_{37} & y_{38} & y_{39} & y_{40} & y_{42} & y_{43} & \\ \hline \end{array} = e_L,$$

where

$$\mathfrak{I}_2 := \{1,2,3,5,6,8,9,12,14,15,16,17,25,27,$$
$$28,29,31,32,33,34,35,36,38,39,41,43\}.$$

Thus we have found an element b_L with the required properties, namely $b_L := v_2$. This example suggests the steps for the general case:

1.) Use the special case $\lambda = (m,m)$.

2.) Bring the remaining columns with the ones in the correct place by operating with basic transpositions.

3.) Finally operate with matrices of the form (3.10) to get an element b_L with $top(b_L) = e_L$. Hereby we don't get in trouble as in the second case of example 3.6.9 since we don't have summations "$*$" in the remaining columns.

■

3.6.10 Definition: According to theorem 3.6.8 we find for every $\mathfrak{t} \in Std(\lambda)$ and every $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ an element $b_L \in S^\lambda$ such that $last(b_L) = \mathfrak{t}$ and $top(b_L) = e_L$. We fix, for every L, such an element b_L and define

$$\mathfrak{B}^\lambda := \{b_L \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}.$$

3.6.11 Corollary: *The subsets $\{top(b_L) \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}$ of M^λ and $\mathfrak{B}^\lambda$ of S^λ are both linearly independent.*

Proof: Follows directly from theorem 3.6.8, the linear independence of the elements e_L and the fact that elements $v_1, v_2 \in S^\lambda$ with $last(v_1) \neq last(v_2)$ are linearly independent. ∎

3.7 Is $\mathfrak{B}^\lambda$ a basis of S^λ?

In this section we want to give some evidence for the following conjecture.

3.7.1 Conjecture: $\mathfrak{B}^\lambda$ is a standard basis of S^λ with corresponding polynomials $\{p_{\mathfrak{t}}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ or equivalently

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda). \tag{3.27}$$

A first approach to prove this conjecture is to show the equality (3.27). Proposition 2.2.7 provides a formula to calculate the dimension S^λ. But for our special partition $\lambda = (n-m, m)$ an easier formula holds.

3.7.2 Proposition:

$$dim(S^\lambda) = \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}.$$

Proof: cf. [11] ∎

From corollary 3.6.11 we immediately obtain the following proposition.

3.7.3 Proposition:

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) \leq dim(S^\lambda).$$

Proof:

$$dim(S^\lambda) \overset{3.6.11}{\geq} |\mathfrak{B}^\lambda| = \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| = \sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q).$$

∎

I have written a program in GAP ([9]) that computes the polynomials $p_{\mathfrak{t}}(q)$ for $\mathfrak{t} \in Std(\lambda), \lambda = (m, m)$ and $1 \leq m \leq 11$. The output proves the following theorem.

3.7.4 Theorem: *Suppose that $1 \leq m \leq 11$ and $\lambda = (m, m)$. Then*

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda)$$

and therefore $\mathfrak{B}^\lambda$ is a standard basis of S^λ.

In appendix B we give an overview over the polynomials $p_{\mathfrak{t}}(q)$ for

$$\mathfrak{t} \in Std((2,2)) \cup Std((3,3)) \cup Std((4,4)) \cup Std((5,5)).$$

The next approach to shed further light on conjecture 3.7.1 is to divide $Std(\lambda)$ in intervals. All ideas and proofs for this approach can be found in [7]. We will translate them into our notation.

3.7.5 Definition: Let $\lambda = (m,m)$. Then, for $1 \leq k \leq m$, we define

$$\mathcal{J}_k(\lambda) := \{\mathfrak{t} \in Std(\lambda) \mid h_d = k \text{ for } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})\} \subseteq Std(\lambda).$$

3.7.6 Example: Let $\lambda = (3,3)$. Then

$$\mathcal{J}_1(\lambda) = \left\{ \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \right\}, \ \mathcal{J}_2(\lambda) = \left\{ \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \right\} \text{ and}$$
$$\mathcal{J}_3(\lambda) = \left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \right\}.$$

3.7.7 Lemma: *If $\lambda = (m,m)$ then*

$$Std(\lambda) = \mathcal{J}_1(\lambda) \,\dot{\cup}\, \mathcal{J}_2(\lambda) \,\dot{\cup}\, \mathcal{J}_3(\lambda) \,\dot{\cup} \ldots \dot{\cup}\, \mathcal{J}_m(\lambda)$$

and, for $1 \leq k \leq m$, the set $\mathcal{J}_k(\lambda)$ contains all standard λ-tableaux $\mathfrak{t}$ with the following property:

$$row_{\mathfrak{t}}(2m - i) = \begin{cases} 1 & \text{if } i = k \\ 2 & \text{if } 0 \leq i < k. \end{cases}$$

Proof: Clear. ■

3.7.8 Proposition: *We have*

$$dim(S^{(n-m,m)}) = \sum_{k=1}^{m} q^{mk+k-m} dim(S^{(m-1,m-k)}) \begin{bmatrix} n-2m+k \\ k \end{bmatrix}$$

and in particular

$$dim(S^{(m,m)}) = \sum_{k=1}^{m} q^{mk+k-m} dim(S^{(m-1,m-k)}).$$

Proof: See 4.11 Proposition in [7]. ■

3.7.9 Conjecture: If $\lambda = (m,m)$ and $1 \leq k \leq m$ then

$$\sum_{\mathfrak{t} \in \mathcal{J}_k(\lambda)} p_{\mathfrak{t}}(q) = q^{mk+k-m} dim(S^{(m-1,m-k)}).$$

As in [7] we can prove this conjecture for the special cases $k = m-1$ and $k = m$ and obtain an inductive approach for $k = 1$. These results are the content of the following three propositions.

3.7.10 Proposition: *Let $\lambda = (m, m)$. Then*

$$\sum_{\mathfrak{t}\in\mathcal{J}_m(\lambda)} p_{\mathfrak{t}}(q) = q^{m^2} dim(S^{(m-1,0)}) = q^{m^2}.$$

Proof: By lemma 3.7.7

$$\mathfrak{t} := \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & m \\ \hline m+1 & m+2 & \dots & 2m \\ \hline\end{array}$$

is the only element of $\mathcal{J}_m(\lambda)$. We obtain

$$\delta(\mathfrak{t}) = (0, (m), (m), ())$$

and therefore, by lemma 3.3.9,

$$p_{\mathfrak{t}}(q) = q^{m^2}.$$

Proposition 2.2.7 provides $dim(S^{(m-1,0)}) = 1$ and the proposition follows. ∎

3.7.11 Proposition: *Let $\lambda = (m, m)$. Then*

$$\sum_{\mathfrak{t}\in\mathcal{J}_{m-1}(\lambda)} p_{\mathfrak{t}}(q) = q^{m^2-m-1} dim(S^{(m-1,1)}).$$

Proof: By lemma 3.7.7

$$\mathcal{J}_{m-1}(\lambda) = \{\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_{m-1}\},$$

where, for $1 \leq i \leq m-1$, $\mathfrak{t}_i$ is the uniquely defined standard λ-tableau with second row $\{i+1, m+2, m+3, \dots, 2m\}$.
We obtain

$$\delta(\mathfrak{t}_i) = (1, (1, m-1), (i, m-i), (i-1))$$

and by lemma 3.3.9 and (3.9)

$$p_{\mathfrak{t}_i}(q) = q^{i+(m-1)(m-i)} \sum_{r_{11}=0}^{i-1} \begin{bmatrix} i \\ r_{11} \end{bmatrix} \begin{bmatrix} m-1 \\ r_{11} \end{bmatrix} |GL_{r_{11}}(q)|.$$

Finally

$$\begin{aligned}\sum_{\mathfrak{t}\in\mathcal{J}_{m-1}(\lambda)} p_{\mathfrak{t}}(q) &= \sum_{i=1}^{m-1} q^{i+(m-1)(m-i)} \sum_{r_{11}=0}^{i-1} \begin{bmatrix} i \\ r_{11} \end{bmatrix} \begin{bmatrix} m-1 \\ r_{11} \end{bmatrix} |GL_{r_{11}}(q)| \\ &\overset{[7]}{=} q^{m^2-m-1} dim(S^{(m-1,1)}).\end{aligned}$$

The last equality is proved in [7]. ∎

3.7.12 Proposition: *Suppose that $\lambda = (m, m)$ and*

$$\sum_{\mathfrak{t} \in Std((m-1,m-1))} p_{\mathfrak{t}}(q) = dim(S^{(m-1,m-1)}).$$

Then

$$\sum_{\mathfrak{t} \in \mathcal{J}_1(\lambda)} p_{\mathfrak{t}}(q) = q\ dim(S^{(m-1,m-1)}).$$

Proof: Let $\mathfrak{t} \in \mathcal{J}_1(\lambda)$. Then, by lemma 3.7.7, $\mathfrak{t}$ is of the form

$$\mathfrak{t} = \begin{array}{|c|c|} \hline \hat{\mathfrak{t}} & \begin{array}{c} 2m-1 \\ \hline 2m \end{array} \\ \hline \end{array},$$

where $\hat{\mathfrak{t}}$ is an element of $Std((m-1, m-1))$.
Furthermore, by lemma 3.2.8 and lemma 3.3.3, every element $M \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ is a flag

$$M = \begin{array}{|ccc|c|} \hline & & & \\ & A & & \\ & & & \\ \hline 0 & \dots & 0 & a \\ \hline \end{array}, \tag{3.28}$$

where $A \in \mathfrak{M}_{\hat{\mathfrak{t}}}^{rk}(q)$ and $a \in GF(q)$.
Therefore

$$\sum_{\mathfrak{t} \in \mathcal{J}_1(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| \leq q \sum_{\hat{\mathfrak{t}} \in Std((m-1,m-1))} |\mathfrak{M}_{\hat{\mathfrak{t}}}^{rk}(q)| \tag{3.29}$$

Vice versa let $\hat{\mathfrak{t}} \in Std((m-1, m-1))$.
If we extend $\hat{\mathfrak{t}}$ by the column $\begin{array}{|c|} \hline 2m-1 \\ \hline 2m \\ \hline \end{array}$, we get an element of $\mathcal{J}_1(\lambda)$ and denote it by $\mathfrak{t}$. For every $a \in GF(q)$ we can extend an element $A \in \mathfrak{M}_{\hat{\mathfrak{t}}}^{rk}(q)$ as in (3.28) to an element $M \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$. Together with (3.29) we conclude

$$\sum_{\mathfrak{t} \in \mathcal{J}_1(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| = q \sum_{\hat{\mathfrak{t}} \in Std((m-1,m-1))} |\mathfrak{M}_{\hat{\mathfrak{t}}}^{rk}(q)|.$$

This means

$$\begin{aligned} \sum_{\mathfrak{t} \in \mathcal{J}_1(\lambda)} p_{\mathfrak{t}}(q) = \sum_{\mathfrak{t} \in \mathcal{J}_1(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| &= q \sum_{\hat{\mathfrak{t}} \in Std((m-1,m-1))} |\mathfrak{M}_{\hat{\mathfrak{t}}}^{rk}(q)| \\ &= q \sum_{\hat{\mathfrak{t}} \in Std((m-1,m-1))} p_{\hat{\mathfrak{t}}}(q) = q\ dim(S^{(m-1,m-1)}). \end{aligned}$$

■

3.7.13 Proposition: *Suppose that $\lambda = (n - m, m)$. If conjecture 3.7.9 holds then*

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda)$$

and therefore $\mathfrak{B}^\lambda$ is a standard basis of S^λ.

Proof: We have

$$\begin{aligned}
\sum_{\mathfrak{t}\in Std(\lambda)} p_{\mathfrak{t}}(q) &\overset{3.3.19}{=} \sum_{\mathfrak{t}\in Std(\lambda)} q^{numb(\mathfrak{t})} \cdot p_{quad(\mathfrak{t})}(q) \\
&\overset{3.7.7}{=} \sum_{k=1}^{m} \sum_{\mathfrak{t}\in Std(\lambda)|quad(\mathfrak{t})\in\mathcal{J}_k((m,m))} q^{numb(\mathfrak{t})} \cdot p_{quad(\mathfrak{t})}(q) \qquad (3.30)\\
&= \sum_{k=1}^{m} \sum_{\hat{\mathfrak{t}}\in\mathcal{J}_k((m,m))} p_{\hat{\mathfrak{t}}}(q) \sum_{\mathfrak{t}\in Std(\lambda)|quad(\mathfrak{t})=\hat{\mathfrak{t}}} q^{numb(\mathfrak{t})}.
\end{aligned}$$

$\sum_{\mathfrak{t}\in Std(\lambda)|quad(\mathfrak{t})=\hat{\mathfrak{t}}} q^{numb(\mathfrak{t})}$ counts the number of possibilities to extend the first $2m-k$ columns of an element $\hat{M} \in \mathfrak{M}_{\hat{\mathfrak{t}}}^{rk}(q)$ by $n-2m+k$ further columns to an element of $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ for a $\mathfrak{t} \in Std(\lambda)$ with $quad(\mathfrak{t}) = \hat{\mathfrak{t}}$. By lemma 3.2.8 and the definition of $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$, every possible extension by $n-2m+k$ columns is of the form

$$\begin{array}{cc}
\left(\begin{array}{|c|}\hline 0 \\ \hline A \\ \hline\end{array}\right) & \begin{array}{c} m-k \\ k \end{array} \\
n-2m+k &
\end{array},$$

where $A \in \mathfrak{M}_{k,n-2m+k}$ is the second row segment of an element of $\Xi_{(n-2m,k)}$. Since every element of $\Xi_{(n-2m,k)}$ is uniquely determined by its second row segment, we have

$$\sum_{\mathfrak{t}\in Std(\lambda)|quad(\mathfrak{t})=\hat{\mathfrak{t}}} q^{numb(\mathfrak{t})} = |\Xi_{(n-2m,k)}| = \begin{bmatrix} n-2m+k \\ k \end{bmatrix}.$$

Together with (3.30) we obtain

$$\begin{aligned}
\sum_{\mathfrak{t}\in Std(\lambda)} p_{\mathfrak{t}}(q) &= \sum_{k=1}^{m} \sum_{\hat{\mathfrak{t}}\in\mathcal{J}_k((m,m))} p_{\hat{\mathfrak{t}}}(q) \begin{bmatrix} n-2m+k \\ k \end{bmatrix} \\
&\overset{3.7.9}{=} \sum_{k=1}^{m} q^{mk+k-m} dim(S^{(m-1,m-k)}) \begin{bmatrix} n-2m+k \\ k \end{bmatrix} \\
&\overset{3.7.8}{=} dim(S^{\lambda}).
\end{aligned}$$

■

Using the polynomials $p_{\mathfrak{t}}(q)$ computed by the previously mentioned GAP program we can verify that conjecture 3.7.9 holds for $1 \leq m \leq 11$. Therefore we immediately obtain the following theorem.

3.7.14 Theorem: *If $1 \leq m \leq 11$ and $\lambda = (n-m,m)$ then $\mathfrak{B}^{\lambda}$ is a standard basis of S^{λ}.*

The third approach to prove conjecture 3.7.1 is the branching theorem 2.3.5. There are two cases for our special partition $\lambda = (n-m,m)$:

1.) $\lambda = (m,m)$. Then $R_1^* = \{\{2\}\}, R_2^* = \{\{1,2\}\}$ and

$$S^\lambda = S^\lambda E_1 \oplus S^\lambda E_2 FG_{n-1}^*(q) \text{ as } FG_{n-1}^*(q)\text{-modules,}$$

where

$$S^\lambda E_1 \cong S^{(m,m-1)} \text{ as } FG_{n-1}(q)\text{-modules,}$$
$$S^\lambda E_2 \cong S^{(m-1,m-1)} \text{ as } FG_{n-2}(q)\text{-modules.}$$

For the dimensions we obtain

$$dim(S^\lambda E_1) = dim(S^{(m,m-1)}), \tag{3.31}$$
$$dim(S^\lambda E_2) = dim(S^{(m-1,m-1)}) \text{ and} \tag{3.32}$$
$$dim(S^\lambda) = dim(S^{(m,m-1)}) + (q^{2m-1}-1)dim(S^{(m-1,m-1)}). \tag{3.33}$$

2.) $\lambda = (n-m,m)$ with $n-m > m$. Then $R_1^* = \{\{1\},\{2\}\}, R_2^* = \{\{1,2\}\}$ and

$$S^\lambda = S^\lambda E_1 \oplus S^\lambda E_2 FG_{n-1}^*(q) \text{ as } FG_{n-1}^*(q)\text{-modules,}$$

where

$$S^\lambda E_1 = S_2 > S_1 > S_0 = 0,$$
$$S^\lambda E_2 \cong S^{(n-m-1,m-1)} \text{ as } FG_{n-2}(q)\text{-modules}$$

with

$$S_2/S_1 \cong S^{\lambda_{\{2\}}} = S^{(n-m,m-1)} \text{ as } FG_{n-1}(q)\text{-modules,}$$
$$S_1 \cong S^{\lambda_{\{1\}}} = S^{(n-m-1,m)} \text{ as } FG_{n-1}(q)\text{-modules.}$$

For the dimensions we obtain

$$dim(S^\lambda E_1) = dim(S^{(n-m-1,m)}) + dim(S^{(n-m,m-1)}),$$

$$dim(S^\lambda E_2) = dim(S^{(n-m-1,m-1)}) \text{ and} \tag{3.34}$$

$$\begin{aligned} dim(S^\lambda) =& dim(S^{(n-m-1,m)}) + dim(S^{(n-m,m-1)}) + \\ &+ (q^{n-1}-1)dim(S^{(n-m-1,m-1)}). \end{aligned} \tag{3.35}$$

3.7.15 Definition: Let $\lambda = (n-m,m)$ and $\mathfrak{t} \in Std(\lambda)$. Then we set

$$\begin{aligned} \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) := \{L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q) \mid\ & x_{i,1}^{(2)} = 0 \text{ for } 1 \le i \le m, \text{ where} \\ & L \text{ is written as in (1.6)}\} \text{ and} \\ \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q) = \{L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q) \mid\ & x_{i,1}^{(2)} = -\delta_{i,1} \text{ for } 1 \le i \le m \text{ and } x_{1,i}^{(2)} = \delta_{i-1,1} \\ & \text{for } 2 \le i \le n, \text{ where } L \text{ is written as in (1.6)}\}. \end{aligned}$$

3.7.16 Proposition: *Let* $\lambda = (n-m, m), \mathfrak{t} \in Std(\lambda)$ *and* $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$. *Then*

$$top(b_L \circ E_1) = \begin{cases} top(b_L) & \text{if } L \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) \\ 0 & \text{if } L \notin \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q). \end{cases} \tag{3.36}$$

If $\mathfrak{B}^{(m,m-1)}$ *is a standard basis of* $S^{(m,m-1)}$ *then*

$$\{b_L \circ E_1 \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q), \mathfrak{t} \in Std((m,m))\}$$

is a basis of $S^{(m,m)}E_1$ *and*

$$\sum_{\mathfrak{t} \in Std((m,m))} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| = dim(S^{(m,m-1)}). \tag{3.37}$$

Proof: We start with a small example. Let $\lambda = (3,2)$ and $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$.
Then

$$\mathfrak{M}_{\mathfrak{t}}^{rk}(q) = \left\{ \begin{array}{|cc|c|}\hline y_1 & y_2 & \\ y_3 & y_4 & \\ \hline\end{array} \;\middle|\; \underline{y} \in GF(q)^4 \right\}.$$

Let $\underline{l} \in GF(q)^4$ and therefore $L = \begin{array}{|cc|c|}\hline l_1 & l_2 & \\ l_3 & l_4 & \\ \hline\end{array}$ a fixed element of $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$.
Then we obtain

$$top(b_L) = \frac{1}{q^4} \sum_{\underline{y} \in GF(q)^4} \theta(-\langle \underline{y}, \underline{l} \rangle) \begin{array}{|cc|c|}\hline y_1 & y_2 & \\ y_3 & y_4 & \\ \hline\end{array}$$

and by proposition 2.1.11

$$\begin{aligned} top(b_L \circ E_1) &= top(b_L) \circ E_1 \\ &= \frac{1}{q^4} \sum_{\underline{y} \in GF(q)^4} \theta(-\langle \underline{y}, \underline{l} \rangle) \begin{array}{|cc|c|}\hline y_1 & y_2 & \\ y_3 & y_4 & \\ \hline\end{array} \circ \frac{1}{q^4} \sum_{\underline{a} \in GF(q)^4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ a_3 & 0 & 0 & 1 & 0 \\ a_4 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{q^8} \sum_{\underline{a}, \underline{y} \in GF(q)^4} \theta(-\langle \underline{y}, \underline{l} \rangle) \begin{array}{|cc|c|}\hline y_1 + a_1 y_2 + a_2 & y_2 & \\ y_3 + a_1 y_4 + a_3 & y_4 & \\ \hline\end{array}. \end{aligned}$$

What is the coefficient $c(b_1, b_2, b_3, b_4)$ of a fixed element $\begin{array}{|cc|c|}\hline b_1 & b_2 & \\ b_3 & b_4 & \\ \hline\end{array}$ of $\mathfrak{M}_{\mathfrak{t}}(q)$?
We obtain the following equalities.

$$\begin{aligned} y_2 &= b_2, \\ y_4 &= b_4, \\ y_1 + a_1 y_2 + a_2 = b_1 &\Rightarrow y_1 = b_1 - a_1 b_2 - a_2, \\ y_3 + a_1 y_4 + a_3 = b_3 &\Rightarrow y_3 = b_3 - a_1 b_4 - a_3. \end{aligned}$$

Therefore

$$\begin{aligned} &c(b_1, b_2, b_3, b_4) = \\ &\quad = \frac{1}{q^8} \sum_{\underline{a} \in GF(q)^4} \theta(-(b_1 - a_1 b_2 - a_2) l_1 - b_2 l_2 - (b_3 - a_1 b_4 - a_3) l_3 - b_4 l_4). \end{aligned}$$

We have two cases:

1.) $L \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)$, i.e. $l_1 = l_3 = 0$.
Then

$$c(b_1, b_2, b_3, b_4) = \frac{1}{q^8} \sum_{\underline{a} \in GF(q)^4} \theta(-b_2 l_2 - b_4 l_4) = \frac{1}{q^4} \theta(-b_2 l_2 - b_4 l_4)$$

and

$$top(b_L \circ E_1) = \frac{1}{q^4} \sum_{\underline{b} \in GF(q)^4} \theta(-\langle \underline{b}, \underline{l} \rangle) \boxed{\begin{array}{cc|c} b_1 & b_2 & \\ b_3 & b_4 & \end{array}} = top(b_L).$$

2.) $L \notin \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)$, i.e. $l_1 \neq 0$ or $l_3 \neq 0$.
We obtain

$$c(b_1, b_2, b_3, b_4) = \begin{cases} \frac{1}{q^8} \sum\limits_{a_1, a_3, a_4, x \in GF(q)} \theta(x) = 0 & \text{if } l_1 \neq 0 \\ \frac{1}{q^8} \sum\limits_{a_1, a_2, a_4, x \in GF(q)} \theta(x) = 0 & \text{if } l_3 \neq 0 \end{cases}$$

and therefore

$$top(b_L \circ E_1) = 0.$$

The general case is much more complex to write down but the proof is the same.
Thus (3.36) holds.
We define a map

$$\begin{aligned} \varphi : \bigcup_{\mathfrak{t} \in Std((m,m))} \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) &\to \bigcup_{\mathfrak{t} \in Std((m,m-1))} \mathfrak{M}_{\mathfrak{t}}^{rk}(q) \\ M &\mapsto \varphi(M) \end{aligned}$$

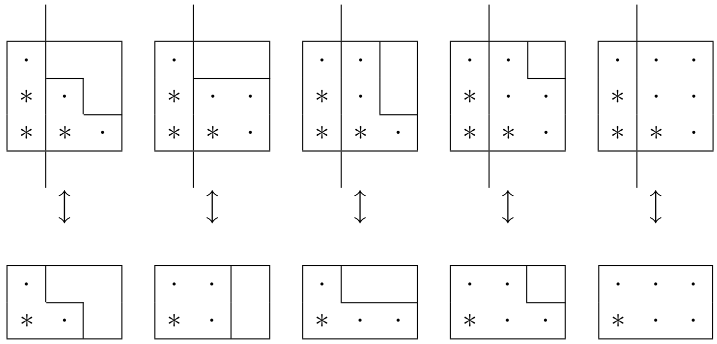
by the following algorithm:

1.) Remove the first column of $short(M)$ (it consists only of zeros),

2.) reflect the new object at the bisector that starts at the bottom left vertex and

3.) interpret the result $\varphi(M)$ as an element of $\bigcup\limits_{\mathfrak{t} \in Std((m,m-1))} \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$.

We illustrate this by an example ($m = 3$) and write down the corresponding $top(v_{\mathfrak{t}})$.



If we use lemma 3.2.8 and the fact that the rank conditions are invariant under our special reflection then we see that φ is well defined and bijective.
Hence

$$\sum_{\mathfrak{t}\in Std((m,m))} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| = \sum_{\mathfrak{t}\in Std((m,m-1))} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)|.$$

By assumption $\mathfrak{B}^{(m,m-1)}$ is a standard basis of $S^{(m,m-1)}$ and therefore

$$\sum_{\mathfrak{t}\in Std((m,m-1))} p_{\mathfrak{t}}(q) = dim(S^{m,m-1}).$$

Altogether

$$\begin{aligned}\sum_{\mathfrak{t}\in Std((m,m))} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| &= \sum_{\mathfrak{t}\in Std((m,m-1))} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| \\ &= \sum_{\mathfrak{t}\in Std((m,m-1))} p_{\mathfrak{t}}(q) = dim(S^{(m,m-1)})\end{aligned}$$

and (3.37) holds.
We examine the set

$$S := \{b_L \circ E_1 \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q), \mathfrak{t} \in Std((m,m))\}.$$

We have

- $S \subseteq S^{(m,m)}E_1$,
- $|S| = dim(S^{(m,m-1)})$ and
- the elements of S are linear independent since

$$top(b_L \circ E_1) = top(b_L) = e_L.$$

By (3.31), the dimension of $S^{(m,m)}E_1$ is equal to $dim(S^{(m,m-1)})$ and therefore we immediately obtain that S is a basis of $S^{(m,m)}E_1$. ■

3.7.17 Proposition: *Let $\lambda = (n-m, m), \mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$. Then*

$$top(b_L \circ E_2) = \begin{cases} top(b_L) & \text{if } L \in \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q) \\ 0 & \text{if } L \notin \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q). \end{cases} \tag{3.38}$$

If $\mathfrak{B}^{(n-m-1,m-1)}$ is a standard basis of $S^{(n-m-1,m-1)}$ then

$$\{b_L \circ E_2 \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q), \mathfrak{t} \in Std(\lambda)\}$$

is a basis of $S^{(n-m,m)}E_2$ and

$$\sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)| = dim(S^{(n-m-1,m-1)}). \tag{3.39}$$

Proof: Again we will illustrate the proof in special cases to simplify the notation. The proof of the general case is the same.
Let $\mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$. Since E_2 differs from the identity matrix only in the first two columns we must examine two cases:

1.) There exists an $3 \leq i \leq n$ such that $x_{1i}^{(2)} \neq 0$ (we write L as in (1.6)). As in the proof of proposition 3.7.16 we choose

$$\lambda := (3,2), \mathfrak{t} := \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \text{ and } L := \begin{array}{|cc|c|} \hline l_1 & l_2 & \\ l_3 & l_4 & \\ \hline \end{array} \text{ for } \underline{l} \in GF(q)^4 \text{ fixed.}$$

Then

$$\begin{aligned} top(b_L \circ E_2) &= top(b_L) \circ E_2 \\ &= \frac{1}{q^4} \sum_{\underline{y} \in GF(q)^4} \theta(-\langle \underline{y}, \underline{l} \rangle) \begin{array}{|cc|c|} \hline y_1 & y_2 & \\ y_3 & y_4 & \\ \hline \end{array} \\ &\circ \frac{1}{q^7} \sum_{\underline{a} \in GF(q)^7} \theta(a_1) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ a_2 & a_5 & 1 & 0 & 0 \\ a_3 & a_6 & 0 & 1 & 0 \\ a_4 & a_7 & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{q^{11}} \sum_{\underline{a}, \underline{y}} \theta(a_1 - \langle \underline{y}, \underline{l} \rangle) \begin{array}{|cc|c|} \hline y_1 + a_1 y_2 + a_2 & y_2 + a_5 & \\ y_3 + a_1 y_4 + a_3 & y_4 + a_6 & \\ \hline \end{array}. \end{aligned}$$

What is the coefficient $c(b_1, b_2, b_3, b_4)$ of a fixed element $\begin{array}{|cc|c|} \hline b_1 & b_2 & \\ b_3 & b_4 & \\ \hline \end{array}$ of $\mathfrak{M}_{\mathfrak{t}}(q)$? We obtain the following equalities.

$$\begin{aligned} y_2 &= b_2 - a_5, \\ y_4 &= b_4 - a_6, \\ y_1 &= b_1 - a_1(b_2 - a_5) - a_2, \\ y_3 &= b_3 - a_1(b_4 - a_6) - a_3. \end{aligned}$$

Therefore

$$c(b_1,b_2,b_3,b_4) = \frac{1}{q^{11}} \sum_{\underline{a}\in GF(q)^7} \theta(a_1 - (b_1 - a_1(b_2-a_5) - a_2)l_1 - (b_2-a_5)l_2 \\ - (b_3 - a_1(b_4-a_6) - a_3)l_3 - (b_4-a_6)l_4).$$

There is no suitable choice for l_1, l_2, l_3, l_4 to obtain an argument for θ which is independent of a_1. Thus

$$c(b_1,b_2,b_3,b_4) = \frac{1}{q^{11}} \sum_{x,a_2,\ldots,a_7\in GF(q)} \theta(x) = 0$$

and therefore

$$top(b_L \circ E_2) = 0.$$

2.) $x_{1i}^{(2)} = 0$ for all $3 \leq i \leq n$ (we write L as in (1.6)).
Since $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ for an $\mathfrak{t} \in Std(\lambda)$ we automatically get $x_{12}^{(2)} = 1$ and $x_{i1}^{(2)} = 0$ for $2 \leq i \leq m$. We choose

$$\lambda := (3,2), \mathfrak{t} := \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \text{ and } L := \boxed{\begin{array}{cc} l_1 & \\ 0 & l_2 \end{array}} \text{ for } \underline{l} \in GF(q)^2 \text{ fixed.}$$

Then

$$\begin{aligned}
top(b_L \circ E_2) &= top(b_L) \circ E_2 \\
&= \frac{1}{q^3} \sum_{\underline{y}\in GF(q)^4} \theta(-\langle \underline{y}, \underline{l}\rangle) \boxed{\begin{array}{cc} y_1 & \\ y_3 & y_2 \end{array}} \\
&\quad \circ \frac{1}{q^7} \sum_{\underline{a}\in GF(q)^7} \theta(a_1) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ a_2 & a_5 & 1 & 0 & 0 \\ a_3 & a_6 & 0 & 1 & 0 \\ a_4 & a_7 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{q^{10}} \sum_{\underline{a},\underline{y}} \theta(a_1 - \langle \underline{y}, \underline{l}\rangle) \boxed{\begin{array}{cc} y_1 + a_1 & \\ x & y_2 \end{array}},
\end{aligned}$$

where $x := y_3 + a_2 y_2 + a_3 - (y_1 + a_1)(y_2 a_5 + a_6)$.
What is the coefficient $c(b_1,b_2,b_3,b_4)$ of a fixed element $\boxed{\begin{array}{cc} b_1 & \\ b_3 & b_2 \end{array}}$ of $\mathfrak{M}_{\mathfrak{t}}(q)$?
We obtain the equalities

$$y_1 = b_1 - a_1 \text{ and } y_2 = b_2,$$

therefore

$$c(b_1,b_2,b_3) = \frac{1}{q^{10}} \sum_{\underline{a}\in GF(q)^7} \theta(a_1 - (b_1 - a_1)l_1 - b_2 l_2).$$

We see immediately

$$top(b_L \circ E_2) = \begin{cases} top(b_L) & \text{if } l_1 = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Altogether we have established (3.38).
We define a map

$$\varphi : \bigcup_{\mathfrak{t}\in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q) \rightarrow \bigcup_{\mathfrak{t}\in Std((n-m-1,m-1))} \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$$
$$M \mapsto \varphi(M)$$

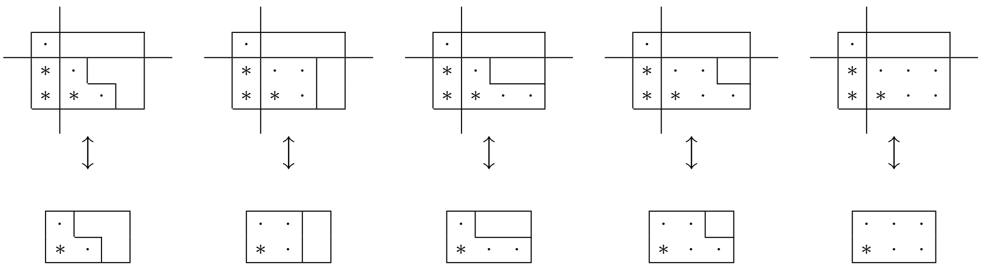
by the following algorithm:

1.) Remove the first column $(-1,0,0,\ldots,0)$ and then the first row $(0,0,\ldots,0)$ of $short(M)$ and

2.) interpret the resulting object $\varphi(M)$ as an element of

$$\bigcup_{\mathfrak{t}\in Std((n-m-1,m-1))} \mathfrak{M}_{\mathfrak{t}}^{rk}(q).$$

We illustrate this by an example ($n = 7, m = 3$) and write down the corresponding $top(v_{\mathfrak{t}})$.



If we use lemma 3.2.8 then we see that φ is well defined and bijective.
Hence

$$\sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)| = \sum_{\mathfrak{t}\in Std((n-m-1,m-1))} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)|.$$

By assumption $\mathfrak{B}^{(n-m-1,m-1)}$ is a standard basis of $S^{(n-m-1,m-1)}$ and therefore

$$\sum_{\mathfrak{t}\in Std((n-m-1,m-1))} p_{\mathfrak{t}}(q) = dim(S^{n-m-1,m-1}).$$

Altogether

$$\begin{aligned}\sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)| &= \sum_{\mathfrak{t}\in Std((n-m-1,m-1))} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| \\ &= \sum_{\mathfrak{t}\in Std((n-m-1,m-1))} p_{\mathfrak{t}}(q) = dim(S^{(n-m-1,m-1)})\end{aligned}$$

and (3.39) holds.
We examine the set

$$S := \{b_L \circ E_2 \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q), \mathfrak{t} \in Std(\lambda)\}.$$

We have

- $S \subseteq S^\lambda E_2$,
- $|S| = dim(S^{(n-m-1,m-1)})$ and
- the elements of S are linearly independent since

$$top(b_L \circ E_2) = top(b_L) = e_L.$$

By (3.32) and (3.34) the dimension of $S^\lambda E_2$ is equal to $dim(S^{(n-m-1,m-1)})$ and therefore we immediately get that S is a basis of $S^\lambda E_2$. ■

3.7.18 Proposition: *Suppose that* $\lambda = (n-m,m)$. *Then we have*

$$\sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| + (q^{n-1}-1) \sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)| \leq \sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)|.$$

Proof: We define a map

$$\varphi : \bigcup_{\mathfrak{t}\in Std(\lambda)} (\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) \cup \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)) \to \mathfrak{P}(\{L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q) \mid \mathfrak{t} \in Std(\lambda)\})$$

$$M \mapsto \begin{cases} \{M\} & \text{if } M \in \bigcup_{\mathfrak{t}\in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) \\ S(M) & \text{if } M \in \bigcup_{\mathfrak{t}\in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q), \end{cases}$$

where $\mathfrak{P}(N)$ denotes the power set of the set N and $S(M)$ is defined by the following algorithm

- In $short(M)$ we can replace $m_{11} = -1$ with an arbitrary element $x_0 \in GF(q)^*$ and get again an element of $\mathfrak{M}_{\mathfrak{t}}^{rk}(q)$. We denote by $S_0(M)$ the set of these elements and set $pos_1^{(0)} := 1$ and $pos_2^{(0)} := 1$ (the position of x_0). For $i = 1, 2, \ldots, n-2$ we obtain $S_i(M)$ recursively:

  $$S_i(M) := \bigcup_{L\in S_{i-1}(M)} T(L),$$

  where $T(L) = \{L(x_i) \mid x_i \in GF(q)\} \subseteq \bigcup_{\mathfrak{t}\in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ and the definition of $L(x_i)$ depends on the position $(pos_1^{(i-1)}+1, pos_2^{(i-1)}+1)$ in $short(L)$.

  Case 1 The position $(pos_1^{(i-1)}+1, pos_2^{(i-1)}+1)$ lies in the area to the top and to the right of the defining line.
  Then we obtain $L(x_i)$ from $short(L)$ if we interchange the rows $pos_1^{(i-1)}$ and $pos_1^{(i-1)}+1$ and then add the new row $pos_1^{(i-1)}+1$ multiplied by x_i to the new row $pos_1^{(i-1)}$.
  $L(x_i)$ is an element of $\bigcup_{\mathfrak{t}\in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ because the above row operations don't change the rank.
  We define $pos_1^{(i)} := pos_1^{(i-1)}+1$ and $pos_2^{(i)} := pos_2^{(i-1)}$ (the new position of x_{i-1}).

Case 2 The position $(pos_1^{(i-1)}+1, pos_2^{(i-1)}+1)$ lies in the area to the bottom and to the left of the defining line.
Then we obtain $L(x_i)$ from $short(L)$ if we write x_i at position $(pos_1^{(i-1)}, pos_2^{(i-1)}+1)$ and change the defining line from $\lfloor x_i$ to $\overline{x_i}\rceil$. $L(x_i)$ is an element of $\bigcup_{\mathfrak{t}\in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ since we get a new degree of freedom at the position of x_i by moving the defining line.
We define $pos_1^{(i)} := pos_1^{(i-1)}$ and $pos_2^{(i)} := pos_2^{(i-1)}+1$ (the position of x_i).

- By looking at the first column and the position of the defining line, we see that
$$S_i(M)\cap S_j(M)=\emptyset \quad (0\le i\ne j\le n-2) \tag{3.40}$$
and set
$$S(M) := \bigcup_{i=0}^{n-2} S_i(M).$$

We give an example for this algorithm. Let

$$\lambda := (4,3), \mathfrak{t} := \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & 7 & \\ \cline{1-3}\end{array} \text{ and } M := \begin{array}{|cccc|}\hline -1 & & & \\ 0 & a & b & \\ 0 & c & d & e \\ \hline\end{array} \in \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q).$$

Then

$$S_0(M) = \left\{ \begin{array}{|cccc|}\hline x_0 & & & \\ 0 & a & b & \\ 0 & c & d & e \\ \hline\end{array} \;\middle|\; x_0\in GF(q)^* \right\},$$

$$S_1(M) = \left\{ \begin{array}{|cccc|}\hline x_0 & x_1 & & \\ 0 & a & b & \\ 0 & c & d & e \\ \hline\end{array} \;\middle|\; x_0\in GF(q)^*, x_1\in GF(q) \right\},$$

$$S_2(M) = \left\{ \begin{array}{|cccc|}\hline x_0 & x_1 & x_2 & \\ 0 & a & b & \\ 0 & c & d & e \\ \hline\end{array} \;\middle|\; x_0\in GF(q)^*, x_1, x_2\in GF(q) \right\},$$

$$S_3(M) = \left\{ \begin{array}{c|cccc|}\cline{2-5} x_3 & 0 & a & b & \\ & x_0 & x_1 & x_2 & \\ & 0 & c & d & e \\ \cline{2-5}\end{array} \;\middle|\; x_0\in GF(q)^*, x_1, x_2, x_3\in GF(q) \right\},$$

where the x_3 in front of the flag means: Add to this row the row starting with x_0 multiplied by x_3. Furthermore,

$$S_4(M) = \left\{ \begin{array}{c|cccc|}\cline{2-5} x_3 & 0 & a & b & \\ & x_0 & x_1 & x_2 & x_4 \\ & 0 & c & d & e \\ \cline{2-5}\end{array} \;\middle|\; x_0\in GF(q)^*, x_1, x_2, x_3, x_4\in GF(q) \right\},$$

$$S_5(M) = \left\{ \begin{array}{c|cccc|}\cline{2-5} x_3 & 0 & a & b & \\ x_5 & 0 & c & d & e \\ & x_0 & x_1 & x_2 & x_4 \\ \cline{2-5}\end{array} \;\middle|\; x_0\in GF(q)^*, x_1, x_2, x_3, x_4, x_5\in GF(q) \right\}.$$

With (3.40) we obtain

$$|S(M)| = \sum_{i=0}^{n-2} (q-1)q^i = q^{n-1} - 1.$$

We have

$$S(M_1) \cap S(M_2) = \emptyset \text{ for } M_1 \neq M_2 \in \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)$$

since we can easily find an "inverse" algorithm that reconstructs M from every element of the set $S(M)$ (look at the last row r with a nonzero first entry, remove the last entry of r before the defining line if possible, otherwise interchange r with the row $\hat{r}$ above it and add a multiple of r to $\hat{r}$ such that the first entry of $\hat{r}$ is zero and so on).
Thus we get

$$\varphi(M_1) \cap \varphi(M_2) = \emptyset \text{ for } M_1 \neq M_2 \in \bigcup_{\mathfrak{t} \in Std(\lambda)} (\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) \cup \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)) \tag{3.41}$$

because the first column of an element $M \in \bigcup_{\mathfrak{t} \in Std(\lambda)} \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)$ is zero. Altogether

$$\begin{aligned}
\sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| &\geq |\bigcup_{\mathfrak{t} \in Std(\lambda)} \bigcup_{M \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) \cup \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)} \varphi(M)| \\
&\overset{(3.41)}{=} \sum_{\mathfrak{t} \in Std(\lambda)} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q) \cup \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)} |\varphi(M)| \\
&= \sum_{\mathfrak{t} \in Std(\lambda)} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)} 1 + \sum_{\mathfrak{t} \in Std(\lambda)} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)} (q^{n-1} - 1) \\
&= \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| + (q^{n-1} - 1) \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)|.
\end{aligned}$$

■

3.7.19 Conjecture: If $\lambda = (n-m, m)$ with $n - m > m$ then

$$\sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| = dim(S^{(n-m-1,m)}) + dim(S^{(n-m,m-1)}).$$

3.7.20 Proposition: *Let $\lambda = (n-m, m)$. If conjecture 3.7.19 holds then*

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^{\lambda})$$

and $\mathfrak{B}^{\lambda}$ is a standard basis of S^{λ}.

Proof: We prove the proposition by induction on n. By Theorem 3.7.14 the statement holds for small n. Now we suppose that the proposition is proved for all two-part partitions λ with $|\lambda| \leq n-1$. For $\lambda = (n-m, m)$ we have two cases:

Case 1 $n = 2m$.
By the induction hypothesis we can use the formulas (3.37) and (3.39) and obtain

$$\begin{aligned} dim(S^\lambda) &\overset{(3.33)}{=} dim(S^{(m,m-1)}) + (q^{2m-1} - 1) dim(S^{(m-1,m-1)}) \\ &\overset{(3.37),(3.39)}{=} \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| + (q^{2m-1} - 1) \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)| \\ &\overset{3.7.18}{\leq} \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| = \sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q). \end{aligned}$$

Case 2 $n > 2m$.
By the induction hypothesis we can use the formula (3.39) and obtain together with conjecure 3.7.19

$$\begin{aligned} dim(S^\lambda) &\overset{(3.35)}{=} dim(S^{(n-m-1,m)}) + dim(S^{(n-m,m-1)}) + \\ &\quad + (q^{n-1} - 1) dim(S^{(n-m-1,m-1)}) \\ &\overset{3.7.19,(3.39)}{=} \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,0}(q)| + (q^{2m-1} - 1) \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk,1}(q)| \\ &\overset{3.7.18}{\leq} \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)| = \sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q). \end{aligned}$$

Combined with proposition 3.7.3 we conclude in both cases

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda)$$

and the statement follows. ∎

Chapter 4

The Specht module $S^{(2,2,2)}$

In this chapter we examine the Specht module $S^{(2,2,2)}$. The three parts of the partition $(2,2,2)$ confront us with completely new problems that don't appear in the case of Specht modules $S^{(n-m,m)}$. But the goal is the same.

4.0.21 Goal: Find a standard basis of $S^{(2,2,2)}$.

Unfortunately we will only find a weaker notion of basis of $S^{(2,2,2)}$.

4.0.22 Definition: A basis $\mathfrak{B}^\lambda = \{b_i \mid i \in \mathfrak{I}\}$ of the $FGL_n(q)$-module S^λ together with a set of polynomials $\{p_\mathfrak{t}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ is called a basis with corresponding polynomials if

1.) it is defined independently of the concrete choice of the field F and

2.) $p_\mathfrak{t}(q) = |\{b \in \mathfrak{B}^\lambda \mid last(b) = \mathfrak{t}\}|$ and $p_\mathfrak{t}(1) = 1$ holds for every $\mathfrak{t} \in Std(\lambda)$.

Since the only partition that appears in this chapter is $(2,2,2)$, we from now on set $\lambda := (2,2,2)$ to simplify the notation.

4.1 Basic definitions and properties

4.1.1 Definition: There are exactly 5 standard λ-tableaux. We denote them by

$$\mathfrak{t}_1 := \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} = \mathfrak{t}_\lambda, \mathfrak{t}_2 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}, \mathfrak{t}_3 := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array},$$

$$\mathfrak{t}_4 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \text{ and } \mathfrak{t}_5 := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} = \mathfrak{t}^\lambda.$$

As in the last chapter we use three conventions to simplify the notation of the elements of Ξ_λ and M^λ (cf. the conventions 3.1.1, 3.1.3 and 3.1.5).

4.1.2 Convention: We don't write the first row segment.

4.1.3 Convention: A zero is written

- as dot if we can exchange the zero with an arbitrary element of $GF(q)$ and in either case obtain an element of Ξ_λ and
- as empty place otherwise.

4.1.4 Convention: Stars indicate the summation over $GF(q)$.

Following an idea of Gordon James and Richard Dipper we get the next definition.

4.1.5 Definition: We define the following element of M^λ.

$$\begin{aligned} e_\lambda := & \sum_{\underline{x}\in GF(q)^3,\ \pi\in C_{\mathfrak{t}_\lambda}} sign(\pi) \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ x_1 & & & \cdot & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ x_2 & x_3 & & \cdot & \cdot & 1 \\ \hline\end{array} \circ \pi \\ = & \sum_{\pi\in\mathfrak{S}_{\{1,2,3\}}\times\mathfrak{S}_{\{4,5,6\}}} sign(\pi) \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & \cdot & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline\end{array} \circ \pi. \end{aligned}$$

4.1.6 Proposition: *e_λ belongs to S^λ.*

Proof: We have

$$\begin{aligned} e_\lambda = {} & + \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & \cdot & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline\end{array} - \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & 1 & & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline\end{array} - \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & \cdot & & 1 \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & 1 & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & 1 & & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & 1 & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & & \cdot & 1 \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & 1 & & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & 1 & & \\ \hline\end{array} \\ & - \begin{array}{|cccccc|}\hline 1 & & & & & \\ & * & & \cdot & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline\end{array} + \begin{array}{|cccccc|}\hline 1 & & & & & \\ & * & & 1 & & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline\end{array} + \begin{array}{|cccccc|}\hline 1 & & & & & \\ & * & & \cdot & & 1 \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & 1 & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline 1 & & & & & \\ & * & & 1 & & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & 1 & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline 1 & & & & & \\ & * & & & \cdot & 1 \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & 1 & & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline 1 & & & & & \\ & * & & & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & 1 & & \\ \hline\end{array} \\ & - \begin{array}{|cccccc|}\hline \cdot & & 1 & & & \\ * & & & \cdot & 1 & \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & \cdot & 1 \\ \hline\end{array} + \begin{array}{|cccccc|}\hline \cdot & & 1 & & & \\ * & & & 1 & & \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & \cdot & 1 \\ \hline\end{array} + \begin{array}{|cccccc|}\hline \cdot & & 1 & & & \\ * & & & \cdot & & 1 \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & 1 & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline \cdot & & 1 & & & \\ * & & & 1 & & \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & 1 & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline \cdot & & 1 & & & \\ * & & & & \cdot & 1 \\ \hline \cdot & 1 & & & & \\ * & & * & 1 & & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline \cdot & & 1 & & & \\ * & & & & 1 & \\ \hline \cdot & 1 & & & & \\ * & & * & 1 & & \\ \hline\end{array} \\ & + \begin{array}{|cccccc|}\hline 1 & & & & & \\ & & * & \cdot & 1 & \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & \cdot & 1 \\ \hline\end{array} - \begin{array}{|cccccc|}\hline 1 & & & & & \\ & & * & 1 & & \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & \cdot & 1 \\ \hline\end{array} - \begin{array}{|cccccc|}\hline 1 & & & & & \\ & & * & \cdot & & 1 \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & 1 & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline 1 & & & & & \\ & & * & 1 & & \\ \hline \cdot & 1 & & & & \\ * & & * & \cdot & 1 & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline 1 & & & & & \\ & & * & & \cdot & 1 \\ \hline \cdot & 1 & & & & \\ * & & * & 1 & & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline 1 & & & & & \\ & & * & & 1 & \\ \hline \cdot & 1 & & & & \\ * & & * & 1 & & \\ \hline\end{array} \\ & + \begin{array}{|cccccc|}\hline & \cdot & 1 & & & \\ & * & & \cdot & 1 & \\ \hline 1 & & & & & \\ & * & * & \cdot & \cdot & 1 \\ \hline\end{array} - \begin{array}{|cccccc|}\hline & \cdot & 1 & & & \\ & * & & 1 & & \\ \hline 1 & & & & & \\ & * & * & \cdot & \cdot & 1 \\ \hline\end{array} - \begin{array}{|cccccc|}\hline & \cdot & 1 & & & \\ & * & & \cdot & & 1 \\ \hline 1 & & & & & \\ & * & * & \cdot & 1 & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline & \cdot & 1 & & & \\ & * & & 1 & & \\ \hline 1 & & & & & \\ & * & * & \cdot & 1 & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline & \cdot & 1 & & & \\ & * & & & \cdot & 1 \\ \hline 1 & & & & & \\ & * & * & 1 & & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline & \cdot & 1 & & & \\ & * & & & 1 & \\ \hline 1 & & & & & \\ & * & * & 1 & & \\ \hline\end{array} \\ & - \begin{array}{|cccccc|}\hline & 1 & & & & \\ & & * & \cdot & 1 & \\ \hline 1 & & & & & \\ & * & * & \cdot & \cdot & 1 \\ \hline\end{array} + \begin{array}{|cccccc|}\hline & 1 & & & & \\ & & * & 1 & & \\ \hline 1 & & & & & \\ & * & * & \cdot & \cdot & 1 \\ \hline\end{array} + \begin{array}{|cccccc|}\hline & 1 & & & & \\ & & * & \cdot & & 1 \\ \hline 1 & & & & & \\ & * & * & \cdot & 1 & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline & 1 & & & & \\ & & * & 1 & & \\ \hline 1 & & & & & \\ & * & * & \cdot & 1 & \\ \hline\end{array} - \begin{array}{|cccccc|}\hline & 1 & & & & \\ & & * & & \cdot & 1 \\ \hline 1 & & & & & \\ & * & * & 1 & & \\ \hline\end{array} + \begin{array}{|cccccc|}\hline & 1 & & & & \\ & & * & & 1 & \\ \hline 1 & & & & & \\ & * & * & 1 & & \\ \hline\end{array}. \end{aligned}$$

By proposition 2.2.3 e_λ belongs to S^λ if

$$e_\lambda \in ker\ \psi_{1,0} \cap ker\ \psi_{1,1} \cap ker\ \psi_{2,0} \cap ker\ \psi_{2,1}.$$

But

$$\psi_{1,0}(e_\lambda) = 0 \text{ and } \psi_{1,1}(e_\lambda) = 0$$

because under the maps $\psi_{1,0}$ and $\psi_{1,1}$ the elements in row 1 and 3, 2 and 5, 4 and 6 in the above notation of e_λ eliminate each other column by column.
And

$$\psi_{2,0}(e_\lambda) = 0 \text{ and } \psi_{2,1}(e_\lambda) = 0$$

because under the maps $\psi_{2,0}$ and $\psi_{2,1}$ the elements in row 1 and 2, 3 and 4, 5 and 6 in the above notation of e_λ eliminate each other column by column. ■

4.1.7 Definition: Let $\mathfrak{t} \in Std(\lambda)$. In analogy to definition 3.2.5, we define the permutation $\pi_\mathfrak{t} \in \mathfrak{S}_6$ by the following equality

$$\mathfrak{t}_\lambda \circ \pi_\mathfrak{t} = \mathfrak{t}$$

and set

$$v_\mathfrak{t} := e_\lambda \circ \pi_\mathfrak{t}.$$

By easy calculation we obtain

$$\begin{aligned}
\pi_{\mathfrak{t}_1} = (), last(v_{\mathfrak{t}_1}) = \mathfrak{t}_1, top(v_{\mathfrak{t}_1}) &= \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & & \cdot & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline \end{array}, \\
\pi_{\mathfrak{t}_2} = (3,4), last(v_{\mathfrak{t}_2}) = \mathfrak{t}_2, top(v_{\mathfrak{t}_2}) &= \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & \cdot & & 1 & \\ \hline \cdot & \cdot & \cdot & 1 & & \\ * & * & \cdot & & \cdot & 1 \\ \hline \end{array}, \\
\pi_{\mathfrak{t}_3} = (2,3,4), last(v_{\mathfrak{t}_3}) = \mathfrak{t}_3, top(v_{\mathfrak{t}_3}) &= \begin{array}{|cccccc|}\hline \cdot & \cdot & 1 & & & \\ * & \cdot & & & 1 & \\ \hline \cdot & \cdot & \cdot & 1 & & \\ * & \cdot & * & & \cdot & 1 \\ \hline \end{array}, \\
\pi_{\mathfrak{t}_4} = (3,5,4), last(v_{\mathfrak{t}_4}) = \mathfrak{t}_4, top(v_{\mathfrak{t}_4}) &= \begin{array}{|cccccc|}\hline \cdot & 1 & & & & \\ * & & \cdot & 1 & & \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \\ * & * & \cdot & \cdot & & 1 \\ \hline \end{array}, \\
\pi_{\mathfrak{t}_5} = (2,3,5,4), last(v_{\mathfrak{t}_5}) = \mathfrak{t}_5, top(v_{\mathfrak{t}_5}) &= \begin{array}{|cccccc|}\hline \cdot & \cdot & 1 & & & \\ * & \cdot & & 1 & & \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \\ * & \cdot & * & \cdot & & 1 \\ \hline \end{array}.
\end{aligned} \tag{4.1}$$

4.1.8 Definition: We define

$$\begin{aligned} pot : Std(\lambda) &\to \mathbb{N} \\ \mathfrak{t} &\mapsto \begin{cases} 9 & \text{if } \mathfrak{t} = \mathfrak{t}_1 \\ 10 & \text{if } \mathfrak{t} = \mathfrak{t}_2 \\ 11 & \text{if } \mathfrak{t} = \mathfrak{t}_3 \\ 11 & \text{if } \mathfrak{t} = \mathfrak{t}_4 \\ 12 & \text{if } \mathfrak{t} = \mathfrak{t}_5 \end{cases} \end{aligned}$$

and for $\mathfrak{t} \in Std(\lambda)$

$$\begin{aligned} \psi_\mathfrak{t} : GF(q)^{pot(\mathfrak{t})} &\to \Xi_\lambda \\ \underline{y} &\mapsto \psi_\mathfrak{t}(\underline{y}), \end{aligned}$$

where we get $\psi_\mathfrak{t}(\underline{y})$ if we replace the dots and stars along the columns in our notation of $top(v_\mathfrak{t})$ in (4.1) with the entries of $\underline{y}$.
Furthermore, for $\mathfrak{t} \in Std(\lambda)$, we set

$$\mathfrak{M}_\mathfrak{t}(q) := \left\{ \psi_\mathfrak{t}(\underline{y}) \mid \underline{y} \in GF(q)^{pot(\mathfrak{t})} \right\}.$$

Then we get a map

$$\begin{aligned}\phi_{\mathfrak{t}} : \mathfrak{M}_{\mathfrak{t}}(q) &\to GF(q)^{pot(\mathfrak{t})}\\ L &\mapsto \psi_{\mathfrak{t}}^{-1}(L).\end{aligned}$$

4.1.9 Example: For $\underline{y} \in GF(q)^{11}$ we have

$$\psi_{\mathfrak{t}_4}(\underline{y}) = \begin{array}{|ccccc|}\hline y_1 & 1 & & & \\ y_2 & & y_7 & 1 & \\ \hline y_3 & y_5 & y_8 & y_{10} & 1 \\ y_4 & y_6 & y_9 & y_{11} & \quad 1 \\ \hline\end{array} \text{ and } \phi_{\mathfrak{t}_4}(\psi_{\mathfrak{t}_4}(\underline{y})) = \underline{y}.$$

4.1.10 Definition: Suppose that $\mathfrak{t} \in Std(\lambda)$ and $L \in \mathfrak{M}_{\mathfrak{t}}(q)$. Then we obtain, similar to the two-part partitions, an idempotent

$$e_L := \frac{1}{q^{pot(\mathfrak{t})}} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}(q)} \theta(-\langle \phi_{\mathfrak{t}}(L), \phi_{\mathfrak{t}}(M) \rangle) M$$

of the group algebra of $\mathfrak{M}_{\mathfrak{t}}(q)$ over F.

4.2 Using the branching theorem

The branching theorem 2.3.5 and proposition 2.3.6 provide the following information.
As $FG_5^*(q)$-modules

$$S^\lambda = S^\lambda E_1 \oplus S^\lambda E_2 FG_5^*(q) \oplus S^\lambda E_3 FG_5^*(q), \tag{4.2}$$

where

$$S^\lambda E_1 \cong S^{(2,2,1)} \text{ as } FG_5(q)\text{-modules}, \tag{4.3}$$
$$S^\lambda E_2 \cong S^{(2,1,1)} \text{ as } FG_4(q)\text{-modules and} \tag{4.4}$$
$$S^\lambda E_3 \cong S^{(1,1,1)} \text{ as } FG_3(q)\text{-modules}. \tag{4.5}$$

For the dimension we get

$$dim(S^\lambda E_2 FG_5^*(q)) = (q^5 - 1) dim(S^\lambda E_2) \tag{4.6}$$
$$dim(S^\lambda E_3 FG_5^*(q)) = (q^5 - 1)(q^4 - 1) dim(S^\lambda E_3) \tag{4.7}$$

and

$$\begin{aligned}dim(S^\lambda) =& dim(S^{(2,2,1)}) + (q^5 - 1) dim(S^{(2,1,1)}) + \\ &+ (q^5 - 1)(q^4 - 1) dim(S^{(1,1,1)}).\end{aligned}$$

With the dimension formula (2.15) for Specht modules we can compute

$$dim(S^\lambda) = q^6 \frac{[6]!}{[4][3][3][2][2]} = q^{12} + q^{10} + q^9 + q^8 + q^6, \tag{4.8}$$

$$dim(S^{(2,2,1)}) = q^4 \frac{[5]!}{[4][3][2]} = q^8 + q^7 + q^6 + q^5 + q^4, \tag{4.9}$$

$$dim(S^{(2,1,1)}) = q^3 \frac{[4]!}{[4][2]} = q^5 + q^4 + q^3 \text{ and} \tag{4.10}$$

$$dim(S^{(1,1,1)}) = q^3 \frac{[3]!}{[3][2]} = q^3. \tag{4.11}$$

First we examine the summand $S^\lambda E_1$ of the direct sum (4.2).

4.2.1 Definition: We define the sets

$$\mathfrak{M}^0_{\mathfrak{t}_1}(q) := \left\{ \begin{array}{|cccccc|} \hline 0 & 1 & & & & \\ 0 & & & y_7 & 1 & \\ \hline 0 & y_5 & 1 & & & \\ 0 & 0 & & y_8 & y_9 & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\},$$

$$\mathfrak{M}^0_{\mathfrak{t}_2}(q) := \left\{ \begin{array}{|cccccc|} \hline 0 & 1 & & & & \\ 0 & & y_7 & & 1 & \\ \hline 0 & y_5 & y_8 & 1 & & \\ 0 & 0 & y_9 & & y_{10} & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\},$$

$$\mathfrak{M}^0_{\mathfrak{t}_3}(q) := \left\{ \begin{array}{|cccccc|} \hline 0 & y_5 & 1 & & & \\ 0 & y_6 & & & 1 & \\ \hline 0 & y_7 & y_9 & 1 & & \\ 0 & y_8 & 0 & & y_{11} & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\},$$

$$\mathfrak{M}^0_{\mathfrak{t}_4}(q) := \left\{ \begin{array}{|cccccc|} \hline 0 & 1 & & & & \\ 0 & & y_7 & 1 & & \\ \hline 0 & y_5 & y_8 & y_{10} & 1 & \\ 0 & y_6 & y_9 & y_{11} & & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\},$$

$$\mathfrak{M}^0_{\mathfrak{t}_5}(q) := \left\{ \begin{array}{|cccccc|} \hline 0 & y_5 & 1 & & & \\ 0 & y_6 & & 1 & & \\ \hline 0 & y_7 & y_9 & y_{11} & 1 & \\ 0 & y_8 & y_{10} & y_{12} & & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\}.$$

4.2.2 Proposition: *There exists a basis*

$$\mathfrak{B}_1 = \{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}}(q), \mathfrak{t} \in Std(\lambda)\}$$

of $S^\lambda E_1$, where the elements b_L have the properties

- *$last(b_L) = \mathfrak{t}$ and*
- *$top(b_L) = e_L$ if $\mathfrak{t} \neq \mathfrak{t}_3$ or if $\mathfrak{t} = \mathfrak{t}_3$ and $l_8 = 0$, where $\underline{l} = \phi_{\mathfrak{t}_3}(L)$.*

Proof: We construct the elements

- $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_1}(q)\}$. Let $L \in \mathfrak{M}^0_{\mathfrak{t}_1}(q)$. We set

$$b_L := \frac{1}{q^9} \sum_{\substack{\underline{y} \in GF(q)^9, \\ y_2 = y_4 = y_6 = 0}} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \underline{y}\rangle) v_{\mathfrak{t}_1} \circ \begin{pmatrix} 1 & & & & & \\ y_1 & 1 & & & & \\ y_3 & y_5 & 1 & & & \\ & & & 1 & & \\ & & & y_7 & 1 & \\ & & & y_8 & y_9 & 1 \end{pmatrix}. \tag{4.12}$$

Then, by proposition 2.1.11 and (4.1), we have

$$last(b_L) = \mathfrak{t}_1$$

and

$$\begin{aligned} top(b_L) &= \frac{1}{q^9} \sum_{\substack{\underline{y} \in GF(q)^9, \\ y_2 = y_4 = y_6 = 0}} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \underline{y}\rangle) \begin{array}{|cccccc|} y_1 & 1 & & & & \\ * & & & y_7 & 1 & \\ \hline y_3 & y_5 & 1 & & & \\ * & * & & y_8 & y_9 & 1 \\ \hline \end{array} \\ &= \frac{1}{q^9} \sum_{M \in \mathfrak{M}_{\mathfrak{t}_1}(q)} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \phi_{\mathfrak{t}_1}(M)\rangle) M = e_L. \end{aligned}$$

- $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_2}(q)\}$. Let $L \in \mathfrak{M}^0_{\mathfrak{t}_2}(q)$. We set

$$b_L := \frac{1}{q^{10}} \sum_{\substack{\underline{y} \in GF(q)^{10}, \\ y_2 = y_4 = y_6 = 0}} \theta(-\langle \phi_{\mathfrak{t}_2}(L), \underline{y}\rangle) v_{\mathfrak{t}_2} \circ \begin{pmatrix} 1 & & & & & \\ y_1 & 1 & & & & \\ & & 1 & & & \\ y_3 & y_5 & y_8 & 1 & & \\ & & y_7 & & 1 & \\ & & y_9 & & y_{10} & 1 \end{pmatrix}. \tag{4.13}$$

Then, by proposition 2.1.11 and (4.1), we have

$$last(b_L) = \mathfrak{t}_2$$

and

$$\begin{aligned} top(b_L) &= \frac{1}{q^{10}} \sum_{\substack{\underline{y} \in GF(q)^{10}, \\ y_2 = y_4 = y_6 = 0}} \theta(-\langle \phi_{\mathfrak{t}_2}(L), \underline{y}\rangle) \begin{array}{|cccccc|} y_1 & 1 & & & & \\ * & & y_7 & & 1 & \\ \hline y_3 & y_5 & y_8 & 1 & & \\ * & * & y_9 & & y_{10} & 1 \\ \hline \end{array} \\ &= \frac{1}{q^{10}} \sum_{M \in \mathfrak{M}_{\mathfrak{t}_2}(q)} \theta(-\langle \phi_{\mathfrak{t}_2}(L), \phi_{\mathfrak{t}_2}(M)\rangle) M = e_L. \end{aligned}$$

- $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_3}(q)\}$. Let $L \in \mathfrak{M}^0_{\mathfrak{t}_3}(q)$. We set

$$b_L := \frac{1}{q^{11}} \sum_{\substack{\underline{y} \in GF(q)^{11}, \\ y_2 = y_4 = y_{10} = 0}} \theta(-\langle \phi_{\mathfrak{t}_3}(L), \underline{y}\rangle) v_{\mathfrak{t}_3} \circ \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ y_1 & y_5 & 1 & & & \\ y_3 & y_7 & y_9 & 1 & & \\ & y_6 & & & 1 & \\ & y_8 & & & y_{11} & 1 \end{pmatrix}.$$

By proposition 2.1.11 and (4.1), we then have

$$last(b_L) = \mathfrak{t}_3$$

and

$$\begin{aligned} top(b_L) &= \frac{1}{q^{11}} \sum_{\substack{x \in GF(q), \underline{y} \in GF(q)^{11}, \\ y_2=y_4=y_{10}=0}} \theta(-\langle \phi_{\mathfrak{t}_3}(L), \underline{y}\rangle) \begin{array}{|cccccc|} y_1 & y_5 & 1 & & & \\ * & y_6 & & & 1 & \\ \hline y_3 & y_7 & y_9 & 1 & & \\ * & y_8 + y_5 x & x & & y_{11} & 1 \\ \hline \end{array} \\ &= \frac{1}{q^{11}} \sum_{\underline{y} \in GF(q)^{11}} \theta(-\langle \underline{l}, \underline{y}\rangle + y_5 y_{10} l_8) \psi_{\mathfrak{t}_3}(\underline{y}), \end{aligned} \tag{4.14}$$

where $\underline{l} = \phi_{\mathfrak{t}_3}(L)$.
This means: If $l_8 = 0$ then $top(b_L) = e_L$.
Nevertheless the elements $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_3}(q)\}$ are linearly independent: If we write the coefficients

$$\frac{1}{q^{11}} \theta(-\langle \underline{l}, \underline{y}\rangle + y_5 y_{10} l_8)$$

of the flags $\psi_{\mathfrak{t}_3}(\underline{y}) \in \mathfrak{M}_{\mathfrak{t}_3}(q)$ in the sum in (4.14) into a matrix A where the columns are parametrized by the elements $\underline{y} \in GF(q)^{11}$ and the rows by the elements $L \in \mathfrak{M}^0_{\mathfrak{t}_3}(q)$ and if we write the elements

$$\frac{1}{q^{11}} \theta(-\langle \underline{l}, \underline{y}\rangle)$$

into a matrix B in the same way, then we get the columns of A by permuting the columns of B: Column $\underline{y}$ of A is column $\underline{\hat{y}}$ of B, where

$$\hat{y}_i := \begin{cases} y_i & \text{if } i \neq 8 \\ y_5 y_{10} - y_8 & \text{if } i = 8. \end{cases}$$

B has full rank because the idempotents e_L are linearly independent. Thus A has full rank and therefore the elements $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_3}(q)\}$ are linearly independent.

- $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_4}(q)\}$. Let $\mathfrak{L} := \left\{ \begin{array}{|cccccc|} 0 & 1 & & & & \\ 0 & & l_7 & 1 & & \\ \hline 0 & l_5 & l_8 & l_{10} & 1 & \\ 0 & 0 & l_9 & l_{11} & & 1 \\ \hline \end{array} \;\middle|\; l_i \in GF(q) \right\}$ and $L \in \mathfrak{L}$.

  We set

$$b_L := \frac{1}{q^{11}} \sum_{\substack{\underline{y} \in GF(q)^{11}, \\ y_2=y_4=y_6=0}} \theta(-\langle \phi_{\mathfrak{t}_4}(L), \underline{y}\rangle) v_{\mathfrak{t}_4} \circ \begin{pmatrix} 1 & & & & & \\ y_1 & 1 & & & & \\ & & 1 & & & \\ & & y_7 & 1 & & \\ y_3 & y_5 & y_8 & y_{10} & 1 & \\ & & y_9 & y_{11} & & 1 \end{pmatrix}. \tag{4.15}$$

Then, by proposition 2.1.11 and (4.1), we have

$$last(b_L) = \mathfrak{t}_4$$

and

$$
\begin{aligned}
top(b_L) &= \frac{1}{q^{11}} \sum_{\substack{\underline{y}\in GF(q)^{11},\\ y_2=y_4=y_6=0}} \theta(-\langle \phi_{\mathfrak{t}_4}(L), \underline{y}\rangle) \begin{array}{|cccccc|} y_1 & 1 & & & & \\ * & & y_7 & 1 & & \\ \hline y_3 & y_5 & y_8 & y_{10} & 1 & \\ * & * & y_9 & y_{11} & & 1 \\ \hline \end{array} \\
&= \frac{1}{q^{11}} \sum_{M\in\mathfrak{M}_{\mathfrak{t}_4}(q)} \theta(-\langle \phi_{\mathfrak{t}_4}(L), \phi_{\mathfrak{t}_4}(M)\rangle) M = e_L.
\end{aligned}
$$

Let $B \in GL_2(q)$ and $g := E(6,5,5,(B^T)^{-1})$. By proposition 1.6.4

$$g = B_1 P B_2$$

for some lower triangular matrices B_1, B_2 and $P \in \left\{ I_6, E(6,5,5,\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$. Then, by proposition 2.1.11 and the fact that in the tableau $\mathfrak{t}_4$ the numbers 5 and 6 are already in the last row, we conclude

$$last(b_L \circ g) = \mathfrak{t}_4 \text{ and } top(b_L \circ g) = top(b_L) \circ g.$$

Furthermore, we can prove as in proposition 3.6.3 that

$$top(b_L \circ g) = top(b_L) \circ g = e_L \circ g = e_{L\circ(g^{-1})^T}$$

holds.

But if $L = \begin{array}{|cccccc|} \hline 0 & 1 & & & & \\ 0 & & l_7 & 1 & & \\ \hline 0 & l_5 & l_8 & l_{10} & 1 & \\ 0 & 0 & l_9 & l_{11} & & 1 \\ \hline \end{array}$ then $L \circ (g^{-1})^T = \begin{array}{|cccccc|} \hline 0 & 1 & & & & \\ 0 & & l_7 & 1 & & \\ \hline 0 & l'_5 & l'_8 & l'_{10} & 1 & \\ 0 & l'_6 & l'_9 & l'_{11} & & 1 \\ \hline \end{array}$, where

$$\begin{pmatrix} l'_5 & l'_8 & l'_{10} \\ l'_6 & l'_9 & l'_{11} \end{pmatrix} = B^{-1} \begin{pmatrix} l_5 & l_8 & l_{10} \\ 0 & l_9 & l_{11} \end{pmatrix}.$$

This means that, for every $L' \in \mathfrak{M}^0_{\mathfrak{t}_4}(q)$, we find appropriate elements $g \in GL_6(q)$ and $L \in \mathfrak{L}$ such that $b_{L'} := b_L \circ g$ has the properties

$$last(b_{L'}) = \mathfrak{t}_4 \text{ and } top(b_{L'}) = e_{L'}.$$

- $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_5}(q)\}$. Let $\mathfrak{L} := \left\{ \begin{array}{|cccccc|} \hline 0 & l_5 & 1 & & & \\ 0 & l_6 & & 1 & & \\ \hline 0 & l_7 & l_9 & l_{11} & 1 & \\ 0 & 0 & 0 & l_{12} & & 1 \\ \hline \end{array} \;\middle|\; l_i \in GF(q) \right\}$ and $L \in \mathfrak{L}$.

  We set

$$b_L := \frac{1}{q^{12}} \sum_{\substack{\underline{y}\in GF(q)^{12},\\ y_2=y_4=y_{10}=0}} \theta(-\langle \phi_{\mathfrak{t}_5}(L), \underline{y}\rangle) v_{\mathfrak{t}_5} \circ \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ y_1 & y_5 & 1 & & & \\ & y_6 & & 1 & & \\ y_3 & y_7 & y_9 & y_{11} & 1 & \\ & y_8 & & y_{12} & & 1 \end{pmatrix}.$$

Then, by proposition 2.1.11 and (4.1), we have

$$last(b_L) = \mathfrak{t}_5$$

and

$$\begin{aligned} top(b_L) &= \frac{1}{q^{12}} \sum_{\substack{x \in GF(q), \underline{y} \in GF(q)^{12}, \\ y_2 = y_4 = y_{10} = 0}} \theta(-\langle \phi_{\mathfrak{t}_5}(L), \underline{y} \rangle) \begin{array}{|cccccc|} \hline y_1 & y_5 & 1 & & & \\ * & y_6 & & 1 & & \\ \hline y_3 & y_7 & y_9 & y_{11} & 1 & \\ * & y_8 + y_5 x & x & y_{12} & & 1 \\ \hline \end{array} \\ &= \frac{1}{q^{12}} \sum_{M \in \mathfrak{M}_{\mathfrak{t}_5}(q)} \theta(-\langle \phi_{\mathfrak{t}_5}(L), \phi_{\mathfrak{t}_5}(M) \rangle) M = e_L. \end{aligned}$$

We define $\mathfrak{L}' := \left\{ \begin{array}{|cccccc|} \hline 0 & l_5 & 1 & & & \\ 0 & l_6 & & 1 & & \\ \hline 0 & l_7 & l_9 & l_{11} & 1 & \\ 0 & 0 & l_{10} & l_{12} & & 1 \\ \hline \end{array} \;\middle|\; l_i \in GF(q) \right\}$.

Let $A \in GL_2(q)$ and $g := E(6, 3, 3, (A^T)^{-1})$.
Then, by proposition 1.6.4, proposition 2.1.11 and the fact that $\mathfrak{t}_5 = \mathfrak{t}^\lambda$, we conclude

$$last(b_L \circ g) = \mathfrak{t}_5 \text{ and } top(b_L \circ g) = top(b_L) \circ g.$$

Furthermore, we can prove as in proposition 3.6.3 that

$$top(b_L \circ g) = top(b_L) \circ g = e_L \circ g = e_{L \circ (g^{-1})^T}$$

holds.

But if $L = \begin{array}{|cccccc|} \hline 0 & l_5 & 1 & & & \\ 0 & l_6 & & 1 & & \\ \hline 0 & l_7 & l_9 & l_{11} & 1 & \\ 0 & 0 & 0 & l_{12} & & 1 \\ \hline \end{array}$ then $L \circ (g^{-1})^T = \begin{array}{|cccccc|} \hline 0 & l_5' & 1 & & & \\ 0 & l_6' & & 1 & & \\ \hline 0 & l_7 & l_9' & l_{11}' & 1 & \\ 0 & 0 & l_{10}' & l_{12}' & & 1 \\ \hline \end{array}$, where

$$\begin{pmatrix} l_5' \\ l_6' \end{pmatrix} = A^{-1} \begin{pmatrix} l_5 \\ l_6 \end{pmatrix} \text{ and } \begin{pmatrix} l_9' & l_{11}' \\ l_{10}' & l_{12}' \end{pmatrix} = \begin{pmatrix} l_9 & l_{11} \\ 0 & l_{12} \end{pmatrix} A.$$

This means that, for every $L' \in \mathfrak{L}'$, we find appropriate elements $g \in GL_6(q)$ and $L \in \mathfrak{L}$ such that $b_{L'} := b_L \circ g$ has the properties

$$last(b_{L'}) = \mathfrak{t}_5 \text{ and } top(b_{L'}) = e_{L'}.$$

With $B \in GL_2(q)$ and $g := E(6, 5, 5, (B^T)^{-1})$ we finally prove as in the last step of the construction of the elements $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_4}(q)\}$ that, for every $L'' \in \mathfrak{M}^0_{\mathfrak{t}_5}(q)$, we find appropriate elements $g' \in GL_6(q)$ and $L' \in \mathfrak{L}'$ such that $b_{L''} := b_L' \circ g'$ has the properties

$$last(b_{L''}) = \mathfrak{t}_5 \text{ and } top(b_{L''}) = e_{L''}.$$

This means, for every $L \in \mathfrak{M}^0_{\mathfrak{t}}(q)$ with $\mathfrak{t} \in Std(\lambda)$, we can construct and fix an element b_L. We set $\mathfrak{B}_1 := \{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}}(q), \mathfrak{t} \in Std(\lambda)\}$.
The elements $b_L \in \mathfrak{B}_1$ are linearly independent since

- the elements of the set $\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}}(q)\}$ are linearly independent for every $\mathfrak{t} \in Std(\lambda)$ and
- two elements $b_1, b_2 \in S^\lambda$ are linearly independent if $last(b_1) \neq last(b_2)$.

Furthermore, we have for all elements $b_L \in \mathfrak{B}_1$

$$top(b_L \circ E_1) = top(b_L).$$

This is proved as in the proof of proposition 3.7.16 since the first column of every element L in the set $\{L \in \mathfrak{M}^0_{\mathfrak{t}}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ consists only of zeros and the anomaly of the elements in (4.14) lies in the second column.
Thus $\mathfrak{B}_1$ is a set of linearly independent vectors of $S^\lambda E_1$ and even a basis because

$$|\mathfrak{B}_1| = \sum_{\mathfrak{t} \in Std(\lambda)} |\mathfrak{M}^0_{\mathfrak{t}}(q)| = q^8 + q^7 + q^6 + q^5 + q^4 \overset{(4.3),(4.9)}{=} dim(S^\lambda E_1).$$

$\blacksquare$

Next we analyse $S^\lambda E_2$.

4.2.3 Definition: We define the sets

$$\mathfrak{M}^1_{\mathfrak{t}_1}(q) := \left\{ \begin{array}{|cccccc|} \hline -1 & 1 & & & & \\ 0 & & & y_7 & 1 & \\ \hline 0 & 0 & 1 & & & \\ 0 & 0 & & y_8 & y_9 & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\},$$

$$\mathfrak{M}^1_{\mathfrak{t}_2}(q) := \left\{ \begin{array}{|cccccc|} \hline -1 & 1 & & & & \\ 0 & & y_7 & & 1 & \\ \hline 0 & 0 & y_8 & 1 & & \\ 0 & 0 & y_9 & & y_{10} & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\}, \mathfrak{M}^1_{\mathfrak{t}_3}(q) := \emptyset,$$

$$\mathfrak{M}^1_{\mathfrak{t}_4}(q) := \left\{ \begin{array}{|cccccc|} \hline -1 & 1 & & & & \\ 0 & & y_7 & 1 & & \\ \hline 0 & 0 & y_8 & y_{10} & 1 & \\ 0 & 0 & y_9 & y_{11} & & 1 \\ \hline \end{array} \;\middle|\; y_i \in GF(q) \right\} \text{ and } \mathfrak{M}^1_{\mathfrak{t}_5}(q) := \emptyset.$$

4.2.4 Proposition: *There exists a basis*

$$\mathfrak{B}_2 = \{b_L \mid L \in \mathfrak{M}^1_{\mathfrak{t}}(q), \mathfrak{t} \in Std(\lambda)\}$$

of $S^\lambda E_2$, where the elements b_L have the properties

$$last(b_L) = \mathfrak{t} \quad and \quad top(b_L) = e_L.$$

Proof: We construct the elements $\{b_L \mid L \in \mathfrak{M}^1_{\mathfrak{t}}(q), \mathfrak{t} \in \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_4\}\}$ as in (4.12), (4.13) and (4.15) and immediately get

$$last(b_L) = \mathfrak{t} \quad \text{and} \quad top(b_L) = e_L.$$

We set $\mathfrak{B}_2 := \{b_L \mid L \in \mathfrak{M}^1_{\mathfrak{t}}(q), \mathfrak{t} \in Std(\lambda)\}$.
Furthermore, we have

$$top(b_L \circ E_2) = top(b_L) = e_L$$

for all elements $b_L \in \mathfrak{B}_2$. This is proved as in the proof of proposition 3.7.17 since the first two columns of every element L in the set $\{L \in \mathfrak{M}^1_{\mathfrak{t}}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ are $\begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $\mathfrak{B}_2$ is a set of linearly independent vectors of $S^\lambda E_2$ and even a basis because

$$|\mathfrak{B}_2| = \sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}^1_{\mathfrak{t}}(q)| = q^5 + q^4 + q^3 \overset{(4.4),(4.10)}{=} dim(S^\lambda E_2).$$

■

4.2.5 Definition: We define the sets

$$\mathfrak{M}^2_{\mathfrak{t}_1}(q) := \left\{ \boxed{\begin{array}{llll} -1 & 1 & & \\ 0 & & y_7 & 1 \\ \hline 0 & -1 & 1 & \\ 0 & 0 & y_8 & y_9 \;\; 1 \end{array}} \;\middle|\; y_i \in GF(q) \right\},$$

$$\mathfrak{M}^2_{\mathfrak{t}_2}(q) := \emptyset, \mathfrak{M}^2_{\mathfrak{t}_3}(q) := \emptyset, \mathfrak{M}^2_{\mathfrak{t}_4}(q) := \emptyset \text{ and } \mathfrak{M}^2_{\mathfrak{t}_5}(q) := \emptyset.$$

Then we get the following proposition for $S^\lambda E_3$.

4.2.6 Proposition: *There exists a basis*

$$\mathfrak{B}_3 = \{b_L \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q)\}$$

of $S^\lambda E_3$, where the elements b_L have the properties

$$last(b_L) = \mathfrak{t}_1 \quad and \quad top(b_L) = e_L.$$

Proof: We construct the elements $\{b_L \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q)\}$ as in (4.12) and immediately get

$$last(b_L) = \mathfrak{t}_1 \quad \text{and} \quad top(b_L) = e_L.$$

We set $\mathfrak{B}_3 := \{b_L \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q)\}$.
Furthermore, we have for all elements $b_L \in \mathfrak{B}_2$

$$\begin{aligned} top(b_L \circ E_3) &= top(b_L) \circ E_3 \\ &= \frac{1}{q^9} \sum_{\underline{y}\in GF(q)^9} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \underline{y}\rangle)\psi_{\mathfrak{t}_1}(\underline{y}) \circ \\ &\quad \frac{1}{q^{12}} \sum_{\underline{a}\in GF(q)^{12}} \theta(a_1 + a_6) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 & 0 \\ a_2 & a_6 & 1 & 0 & 0 & 0 \\ a_3 & a_7 & a_{10} & 1 & 0 & 0 \\ a_4 & a_8 & a_{11} & 0 & 1 & 0 \\ a_5 & a_9 & a_{12} & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{q^{21}} \sum_{\substack{\underline{y}\in GF(q)^9, \\ \underline{a}\in GF(q)^{12}}} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \underline{y}\rangle + a_1 + a_6) \boxed{\begin{array}{llll} y_1 + a_1 & 1 & & \\ x_1(\underline{y}, \underline{a}) & & y_7 & 1 \\ \hline x_2(\underline{y}, \underline{a}) & y_5 + a_6 & 1 & \\ x_3(\underline{y}, \underline{a}) & x_4(\underline{y}, \underline{a}) & y_8 & y_9 \;\; 1 \end{array}}, \end{aligned}$$

where the $x_i(\underline{y},\underline{a})$ are polynomial functions in $y_1, y_2, \ldots, y_9$ and $a_1, a_2, \ldots a_{12}$.

What is the coefficient $c(\underline{b})$ of a fixed element $\begin{array}{|cccccc|} b_1 & 1 & & & & \\ b_2 & & & b_7 & 1 & \\ \hline b_3 & b_5 & 1 & & & \\ b_4 & b_6 & & b_8 & b_9 & 1 \\ \hline \end{array} \in \mathfrak{M}_{\mathfrak{t}_1}(q)$ in this sum?

We obtain the following equalities.

$$\begin{aligned} y_1 + a_1 = b_1 &\Rightarrow y_1 = b_1 - a_1, \\ y_5 + a_6 = b_5 &\Rightarrow y_5 = b_5 - a_6, \\ y_7 = b_7, y_8 &= b_8 \text{ and } y_9 = b_9 \end{aligned}$$

Therefore

$$\begin{aligned} c(\underline{b}) &= \frac{1}{q^{21}} \sum_{\underline{a} \in GF(q)^{12}} \theta(y_1 + y_5 - y_7 b_7 - y_8 b_8 - y_9 b_9 + a_1 + a_6) \\ &= \frac{1}{q^{21}} \sum_{\underline{a} \in GF(q)^{12}} \theta(b_1 - a_1 + b_5 - a_6 - y_7 b_7 - y_8 b_8 - y_9 b_9 + a_1 + a_6) \\ &= \frac{1}{q^9} \theta(b_1 + b_5 - y_7 b_7 - y_8 b_8 - y_9 b_9) = \frac{1}{q^9} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \underline{b} \rangle) \end{aligned}$$

and

$$top(b_L \circ E_3) = \frac{1}{q^9} \sum_{M \in \mathfrak{M}_{\mathfrak{t}_1}(q)} \theta(-\langle \phi_{\mathfrak{t}_1}(L), \phi_{\mathfrak{t}_1}(M) \rangle) M = top(b_L) = e_L.$$

Thus $\mathfrak{B}_3$ is a set of linearly independent vectors of $S^\lambda E_3$ and even a basis because

$$|\mathfrak{B}_3| = |\mathfrak{M}^2_{\mathfrak{t}_1}(q)| = q^3 \overset{(4.5),(4.11)}{=} dim(S^\lambda E_3).$$

■

If we recall definition 3.6.4 then we get a complete set of

- right coset representatives $\mathfrak{R}_1$ of $H_4^*(q)$ in $G_4^*(q)$ by

$$\mathfrak{R}_1 = \mathfrak{R}_1^{(1)} \cup \mathfrak{R}_1^{(2)} \cup \mathfrak{R}_1^{(3)} \cup \mathfrak{R}_1^{(4)},$$

where

$$\begin{aligned} \mathfrak{R}_1^{(1)} &:= \{E(6,3,3,(r_0)) \mid r_0 \in GF(q)^*\}, \\ \mathfrak{R}_1^{(2)} &:= \{E(6,3,3,(r_0))(3,4)E(6,4,3,(r_1)) \mid r_0 \in GF(q)^*, r_1 \in GF(q)\}, \\ \mathfrak{R}_1^{(3)} &:= \{E(6,3,3,(r_0))(3,4)(4,5)E(6,5,3,(r_1,r_2)) \mid r_0 \in GF(q)^*, \\ &\qquad r_1, r_2 \in GF(q)\} \text{ and} \\ \mathfrak{R}_1^{(4)} &:= \{E(6,3,3,(r_0))(3,4)(4,5)(5,6)E(6,6,3,(r_1,r_2,r_3)) \mid r_0 \in GF(q)^*, \\ &\qquad r_1, r_2, r_3 \in GF(q)\}. \end{aligned} \tag{4.16}$$

- right coset representatives $\mathfrak{R}_2$ of $H_5^*(q)$ in $G_5^*(q)$ by

$$\mathfrak{R}_2 = \mathfrak{R}_2^{(1)} \cup \mathfrak{R}_2^{(2)} \cup \mathfrak{R}_2^{(3)} \cup \mathfrak{R}_2^{(4)} \cup \mathfrak{R}_2^{(5)},$$

where

$$\begin{aligned}
\mathfrak{R}_2^{(1)} &:= \{E(6,2,2,(r_0)) \mid r_0 \in GF(q)^*\},\\
\mathfrak{R}_2^{(2)} &:= \{E(6,2,2,(r_0))(2,3)E(6,3,2,(r_1)) \mid r_0 \in GF(q)^*, r_1 \in GF(q)\},\\
\mathfrak{R}_2^{(3)} &:= \{E(6,2,2,(r_0))(2,3)(3,4)E(6,4,2,(r_1,r_2)) \mid \\
&\qquad r_0 \in GF(q)^*, r_1, r_2 \in GF(q)\},\\
\mathfrak{R}_2^{(4)} &:= \{E(6,2,2,(r_0))(2,3)(3,4)(4,5)E(6,5,2,(r_1,r_2,r_3)) \mid \\
&\qquad r_0 \in GF(q)^*, r_1, r_2, r_3 \in GF(q)\} \text{ and}\\
\mathfrak{R}_2^{(5)} &:= \{E(6,2,2,(r_0))(2,3)(3,4)(4,5)(5,6)E(6,6,2,(r_1,r_2,r_3,r_4)) \mid \\
&\qquad r_0 \in GF(q)^*, r_1, r_2, r_3, r_4 \in GF(q)\}.
\end{aligned} \tag{4.17}$$

The next two propositions give us a basis of the summands $S^\lambda E_2 FG_5^*(q)$ and $S^\lambda E_3 FG_5^*(q)$ in the direct sum (4.2).

4.2.7 Proposition: *The set*

$$\mathfrak{B}_2^{mult} := \{b_L \circ g \mid b_L \in \mathfrak{B}_2, g \in \mathfrak{R}_2\}$$

is a basis of $S^\lambda E_2 FG_5^*(q)$.

Proof: We have the basis $\mathfrak{B}_2$ of the $FG_4(q)$-module $S^\lambda E_2$ and search a basis of the $FG_5^*(q)$-module $S^\lambda E_2 FG_5^*(q)$. Furthermore,

$$G_4(q) \leq G_4^*(q) \leq H_5^*(q) \leq G_5^*(q).$$

By [11] the multiplication of $S^\lambda E_2$ by $FG_5^*(q)$ is the extension of the operation to $FH_5^*(q)$ and then induction to $FG_5^*(q)$.
Since $\mathfrak{R}_2$ is a set of right coset representatives of $H_5^*(q)$ in $G_5^*(q)$ we get by

$$\mathfrak{B}_2^{mult} := \{b_L \circ g \mid b_L \in \mathfrak{B}_2, g \in \mathfrak{R}_2\}$$

a basis of $S^\lambda E_2 FG_5^*(q)$. ∎

4.2.8 Proposition: *The set*

$$\mathfrak{B}_3^{mult} := \{b_L \circ (gh) \mid b_L \in \mathfrak{B}_3, g \in \mathfrak{R}_1, h \in \mathfrak{R}_2\}$$

is a basis of $S^\lambda E_3 FG_5^*(q)$.

Proof: We have the basis $\mathfrak{B}_3$ of the $FG_3(q)$-module $S^\lambda E_3$ and search a basis of the $FG_5^*(q)$-module $S^\lambda E_3 FG_5^*(q)$. Furthermore,

$$G_3(q) \leq G_3^*(q) \leq H_4^*(q) \leq G_4^*(q) \leq H_5^*(q) \leq G_5^*(q).$$

By [11] the multiplication of $S^\lambda E_3$ by $FG_5^*(q)$ is the extension of the operation to $FH_4^*(q)$, then induction to $FG_4^*(q)$, again extension of the operation to $FH_5^*(q)$ and finally induction to $FG_5^*(q)$.
Since $\mathfrak{R}_1$ is a set of right coset representatives of $H_4^*(q)$ in $G_4^*(q)$ and $\mathfrak{R}_2$ a set of right coset representatives of $H_5^*(q)$ in $G_5^*(q)$ we get by

$$\mathfrak{B}_3^{mult} := \{b_L \circ (gh) \mid b_L \in \mathfrak{B}_3, g \in \mathfrak{R}_1, h \in \mathfrak{R}_2\}$$

a basis of $S^\lambda E_3 FG_5^*(q)$. ■

4.2.9 Corollary: $\mathfrak{B}^\lambda := \mathfrak{B}_1 \cup \mathfrak{B}_2^{mult} \cup \mathfrak{B}_3^{mult}$ *is a basis of* S^λ.

Proof: Follows directly from (4.2). ■

Finally we will calculate, for every $\mathfrak{t} \in Std(\lambda)$, the number of elements in the set

$$\{b \in \mathfrak{B}^\lambda \mid last(b) = \mathfrak{t}\}.$$

If we have in mind that all matrices $E(6, i, j, M)$ in (4.16) and (4.17) are lower triangular matrices and therefore, for all $v \in S^\lambda$, $last(v \circ E(6, i, j, M)) = last(v)$ holds, then we obtain

- for the elements b of $\mathfrak{B}_1$

b element of	$last(b)$
$\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_1}(q)\}$	$\mathfrak{t}_1$
$\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_2}(q)\}$	$\mathfrak{t}_2$
$\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_3}(q)\}$	$\mathfrak{t}_3$
$\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_4}(q)\}$	$\mathfrak{t}_4$
$\{b_L \mid L \in \mathfrak{M}^0_{\mathfrak{t}_5}(q)\}$	$\mathfrak{t}_5$

- for the elements b of $\mathfrak{B}_2^{mult}$

b element of	$last(b)$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_1}(q), g \in \mathfrak{R}_2^{(1)}\}$	$\mathfrak{t}_1$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_1}(q), g \in \mathfrak{R}_2^{(2)}\}$	$\mathfrak{t}_1$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_1}(q), g \in \mathfrak{R}_2^{(3)}\}$	$\mathfrak{t}_2$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_1}(q), g \in \mathfrak{R}_2^{(4)}\}$	$\mathfrak{t}_4$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_1}(q), g \in \mathfrak{R}_2^{(5)}\}$	$\mathfrak{t}_4$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_2}(q), g \in \mathfrak{R}_2^{(1)}\}$	$\mathfrak{t}_2$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_2}(q), g \in \mathfrak{R}_2^{(2)}\}$	$\mathfrak{t}_3$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_2}(q), g \in \mathfrak{R}_2^{(3)}\}$	$\mathfrak{t}_3$

$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_2}(q), g \in \mathfrak{R}^{(4)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_2}(q), g \in \mathfrak{R}^{(5)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_4}(q), g \in \mathfrak{R}^{(1)}_2\}$	$\mathfrak{t}_4$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_4}(q), g \in \mathfrak{R}^{(2)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_4}(q), g \in \mathfrak{R}^{(3)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_4}(q), g \in \mathfrak{R}^{(4)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ g \mid L \in \mathfrak{M}^1_{\mathfrak{t}_4}(q), g \in \mathfrak{R}^{(5)}_2\}$	$\mathfrak{t}_5$

- for the elements b of $\mathfrak{B}^{mult}_3$

b element of	$last(b)$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(1)}_1, h \in \mathfrak{R}^{(1)}_2\}$	$\mathfrak{t}_1$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(1)}_1, h \in \mathfrak{R}^{(2)}_2\}$	$\mathfrak{t}_1$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(1)}_1, h \in \mathfrak{R}^{(3)}_2\}$	$\mathfrak{t}_2$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(1)}_1, h \in \mathfrak{R}^{(4)}_2\}$	$\mathfrak{t}_4$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(1)}_1, h \in \mathfrak{R}^{(5)}_2\}$	$\mathfrak{t}_4$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(2)}_1, h \in \mathfrak{R}^{(1)}_2\}$	$\mathfrak{t}_2$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(2)}_1, h \in \mathfrak{R}^{(2)}_2\}$	$\mathfrak{t}_3$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(2)}_1, h \in \mathfrak{R}^{(3)}_2\}$	$\mathfrak{t}_3$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(2)}_1, h \in \mathfrak{R}^{(4)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(2)}_1, h \in \mathfrak{R}^{(5)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(3)}_1, h \in \mathfrak{R}^{(1)}_2\}$	$\mathfrak{t}_4$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(3)}_1, h \in \mathfrak{R}^{(2)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(3)}_1, h \in \mathfrak{R}^{(3)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(3)}_1, h \in \mathfrak{R}^{(4)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(3)}_1, h \in \mathfrak{R}^{(5)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(4)}_1, h \in \mathfrak{R}^{(1)}_2\}$	$\mathfrak{t}_4$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(4)}_1, h \in \mathfrak{R}^{(2)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(4)}_1, h \in \mathfrak{R}^{(3)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(4)}_1, h \in \mathfrak{R}^{(4)}_2\}$	$\mathfrak{t}_5$
$\{b_L \circ (gh) \mid L \in \mathfrak{M}^2_{\mathfrak{t}_1}(q), g \in \mathfrak{R}^{(4)}_1, h \in \mathfrak{R}^{(5)}_2\}$	$\mathfrak{t}_5$

The cardinalities of the sets in the last three tables are polynomials in q. We summarize them in the next table.

	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{B}_1$	q^4	q^5	q^6	q^7	q^8
$\mathfrak{B}^{mult}_2$	$(q-1)q^3$ $(q-1)q^4$	$(q-1)q^5$		$(q-1)q^6$ $(q-1)q^7$	
		$(q-1)q^4$	$(q-1)q^5$ $(q-1)q^6$		$(q-1)q^7$ $(q-1)q^8$
				$(q-1)q^5$	$(q-1)q^6$ $(q-1)q^7$

					$(q-1)q^8$ $(q-1)q^9$
$\mathfrak{B}_3^{mult}$	$(q-1)^2q^3$ $(q-1)^2q^4$	$(q-1)^2q^5$		$(q-1)^2q^6$ $(q-1)^2q^7$	
		$(q-1)^2q^4$	$(q-1)^2q^5$ $(q-1)^2q^6$		$(q-1)^2q^7$ $(q-1)^2q^8$
				$(q-1)^2q^5$	$(q-1)^2q^6$ $(q-1)^2q^7$ $(q-1)^2q^8$ $(q-1)^2q^9$
				$(q-1)^2q^6$	$(q-1)^2q^7$ $(q-1)^2q^8$ $(q-1)^2q^9$ $(q-1)^2q^{10}$
$\sum$	q^6	q^7	q^8	$q^9+q^8-q^7$	$q^{12}+q^{10}-q^8$

Altogether we have proved the following theorem.

4.2.10 Theorem: $\mathfrak{B}^\lambda$ *is a basis of* S^λ *with corresponding polynomials*

$$p_{\mathfrak{t}_1}(q) = q^6, p_{\mathfrak{t}_2}(q) = q^7, p_{\mathfrak{t}_3}(q) = q^8,$$
$$p_{\mathfrak{t}_4}(q) = q^9 + q^8 - q^7 \text{ and } p_{\mathfrak{t}_5}(q) = q^{12} + q^{10} - q^8.$$

$\mathfrak{B}^\lambda$ is not a standard basis of S^λ. But we have the following bold conjecture.

4.2.11 Conjecture: There is a standard basis of S^λ with corresponding polynomials

$$p_{\mathfrak{t}_1}(q) = q^6, p_{\mathfrak{t}_2}(q) = q^7, p_{\mathfrak{t}_3}(q) = q^9 + q^8 - q^7,$$
$$p_{\mathfrak{t}_4}(q) = q^{10} \text{ and } p_{\mathfrak{t}_5}(q) = q^{12}.$$

Chapter 5

German summary

Viele ungelöste Probleme in der Darstellungstheorie können mit einem geeigneten Verständnis der irreduziblen unipotenten Moduln der endlichen generellen linearen Gruppe $GL_n(q)$ gelöst werden [5, 6].
Gordon James gibt in [11] einen Überblick über diese irreduziblen unipotenten Moduln: Für jede Partition λ von n gibt es einen Spechtmodul S^λ für $GL_n(q)$, der über einem Körper F als Schnitt von Kernen gewisser Homomorphismen definiert ist. Ist F ein Körper der Charakteristik null, dann ist S^λ irreduzibel und $\{S^\lambda \mid \lambda \text{ ist eine Partition von } n\}$ ist eine vollständige Menge von paarweise nicht isomorphen irreduziblen unipotenten $FGL_n(q)$-Moduln. Ist hingegen die Charakteristik von F koprim zu q, dann besitzt S^λ im allgemeinen einen eindeutigen oberen Kompositionsfaktor D^λ und die D^λ sind die irreduziblen unipotenten $FGL_n(q)$-Moduln.
Für jeden Spechtmodul S^λ ist ein erzeugendes Element e_λ bekannt, aber im allgemeinen wurde noch keine explizite Basis von S^λ als F-Vektorraum gefunden. In [7] machen Richard Dipper and Gordon James einen großen Schritt in Richtung zu einer Basis von S^λ, falls λ eine Partition mit zwei Teilen ist. Meine Arbeit basiert auf [7] und entwickelt die dort eingeführten Techniken weiter.

Kapitel 1

Das erste Kapitel dient dazu, die grundlegenden Begriffe und Hilfsmittel einzuführen. Dabei starten wir mit der Beschreibung der Ausgangslage: In der ganzen Arbeit ist n eine natürliche Zahl, p eine Primzahl und q eine Potenz von p. Weiter soll F ein Körper sein, dessen Charakteristik koprim zu p ist und der eine primitive p. Einheitswurzel enthält. Zudem wählen wir ein für alle Mal einen nichttrivialen linearen F-Charakter θ der Gruppe $(GF(q), +)$.
In den folgenden Unterkapiteln geben wir dann einen Überblick über die für uns wichtigen Definitionen und Sätze auf dem Gebiet der Kompositionen, Partitionen, λ-Tableaux und Gauß-Polynome. Desweiteren führen wir λ-Fahnen und mit der Menge Ξ_λ eine handhabbare Schreibweise dieser Untervektorraumketten ein und schließen letztendlich das Kapitel mit einem kurzen Einblick in die endlichen Gruppen mit BN-Paar, für die die sogenannte Bruhatzerlegung existiert.

Kapitel 2

Nun nähern wir uns langsam den Spechtmoduln. Für eine Komposition λ von n führen wir den $FGL_n(q)$-Modul M^λ ein, der als F-Basis die λ-Fahnen besitzt, auf denen die generelle lineare Gruppe $GL_n(q)$ in kanonischer Form operiert. Diese Operation schreiben wir mit $\circ$.
Die Tatsache, daß der Spechtmodul S^λ ein Untermodul von M^λ ist, motiviert uns, M^λ genauer zu untersuchen. Dabei ist uns behilflich, daß wir jeder λ-Fahne X ein λ-Tableau $tab(X)$ zuordnen können. Da wir auf den λ-Tableaux eine totale Ordnung haben, macht es Sinn, für ein Element $v \in M^\lambda$ mit $last(v)$ das größte λ-Tableau zu bezeichnen, das in der Darstellung

$$v = \sum_{X \in \Xi_\lambda} c_X X \quad (c_X \in F)$$

einer Fahne X mit $c_X \neq 0$ zugeordnet werden kann. $top(v)$ steht dann für die Teilsumme $\sum\limits_{X | tab(X) = last(v)} c_X X$.
Nach der Definition von $last(v)$ und $top(v)$ stellt sich natürlich sofort die Frage, ob wir bei der Operation mit einem Element $g \in GL_n(q)$ auf v irgendwelche Aussagen über $last(v \circ g)$ und $top(v \circ g)$ treffen können. Für spezielle Elemente g geben die folgenden drei Sätze darüber Auskunft.

Satz 1 (2.1.10) *Sei λ eine Komposition mit maximal 3 Teilen, $\mathfrak{t}$ ein zeilenstandard λ-Tableau und $\pi \in \mathfrak{S}_n$ eine Fundamentaltransposition mit $\mathfrak{t} < \mathfrak{t} \circ \pi$. Wenn $0 \neq v \in M^\lambda$ und $last(v) = \mathfrak{t}$ ist, dann haben wir*

$$last(v \circ \pi) = \mathfrak{t} \circ \pi \text{ und } top(v \circ \pi) = top(v) \circ \pi.$$

Satz 2 (2.1.11) *Sei λ eine Komposition, $0 \neq v \in M^\lambda$ und $last(v) = \mathfrak{t}$. Ist $g \in GL_n(q)$ eine untere Dreiecksmatrix, dann gilt*

$$last(v \circ g) = \mathfrak{t} \text{ und } top(v \circ g) = top(v) \circ g.$$

Satz 3 (2.1.12) *Sei λ eine Komposition mit maximal 2 Teilen, $0 \neq v \in M^\lambda$ und $last(v) = \mathfrak{t}$. Wenn natürliche Zahlen x und l existieren, so daß*

$$row_\mathfrak{t}(x) = row_\mathfrak{t}(x+1) = \ldots = row_\mathfrak{t}(x+l)$$

gilt und $\pi \in \mathfrak{S}_n$ eine Permutation ist, so daß $j\pi = \pi$ für alle $j \notin \{x, x+1, \ldots, x+l\}$ gilt, dann haben wir

$$last(v \circ \pi) = \mathfrak{t} \text{ und } top(v \circ \pi) = top(v) \circ \pi.$$

Den nächsten Abschnitt bildet die Definition des Spechtmoduls S^λ als Schnitt von Kernen gewisser $FGL_n(q)-$Homomorphismen. Sinéad Lyle zeigt in ihrer Dissertation eine interessante Eigenschaft der Elemente von S^λ.

Theorem 1 (2.2.4) *Sei $0 \neq v \in S^\lambda$. Dann ist $last(v)$ ein standard λ-Tableau.*

Dies führt uns zur Definition der Standardbasis des Spechtmoduls S^λ.

Definition 1 (2.2.5) *Eine Basis* $\mathfrak{B}^\lambda = \{b_i \mid i \in \mathfrak{I}\}$ *des* $FGL_n(q)$*-Moduls* S^λ *zusammen mit einer Menge von Polynomen* $\{p_\mathfrak{t}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ *heißt Standardbasis, wenn*

1.) sie unabhängig von der konkreten Wahl des Körpers F *definiert ist,*

2.) die Elemente $\{b_i \mid i \in \mathfrak{I}\}$ *von* M^λ *linear unabhängig sind und*

3.) $p_\mathfrak{t}(q) = |\{b \in \mathfrak{B}^\lambda \mid last(b) = \mathfrak{t}\}|$ *und* $p_\mathfrak{t}(1) = 1$ *für jedes* $\mathfrak{t} \in Std(\lambda)$ *gilt.*

Die Polynome $\{p_\mathfrak{t}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ *bezeichnen wir als die zur Standardbasis zugehörigen Polynome.*

Abgerundet wird das Kapitel mit unserem wichtigsten Hilfsmittel, die Struktur des Spechtmoduls zu verstehen, nämlich dem Branching Theorem, das uns in vielen Beweisen nützlich sein wird.

Theorem 2 (2.3.5) *Für eine Partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ *von* n *gilt*

$$S^\lambda = S^\lambda E_1 \oplus \sum_{r=2}^{h} S^\lambda E_r FG^*_{n-1}(q),$$

wobei wir alle Moduln als $FG^*_{n-1}(q)$*-Moduln betrachten.*
Zudem besitzt $S^\lambda E_i$ $(1 \leq i \leq h)$ *eine Kette von* $FG_{n-i}(q)$*-Untermoduln*

$$S^\lambda E_i = S_k > S_{k-1} > \ldots > S_1 > S_0 = 0,$$

so daß S_j/S_{j-1} *für jedes* j *mit* $1 \leq j \leq k$ $FG_{n-i}(q)$*-isomorph zu* $S^{\lambda_{R_j}}$ *ist (hierbei sind die Elemente* $R_1 < R_2 < \ldots < R_k$ *von* $\mathfrak{R}_i^*$ *geordnet wie in Definition 2.3.4).*

Kapitel 3

In diesem Kapitel widmen wir uns der Untersuchung des Spechtmoduls S^λ für eine beliebige aber feste 2-Teile-Partition $\lambda = (n-m, m)$. Dabei verfolgen wir das Ziel, eine Standardbasis von S^λ zu finden. Leider werden wir unsere Ergebnisse nicht vollständig beweisen können.
Nach Einführung mehrerer Konventionen, die Schreibweise der λ-Fahnen zu vereinfachen (wir schreiben ab sofort λ-Fahnen als $m \times (n-m)$ - Felder mit einer Grenzlinie, die von der oberen linken Ecke bis zur unteren rechten Ecke verläuft), identifizieren wir in Satz 3.2.2 mit

$$e_\lambda := \sum_{X \in S} X \circ ((1-(1,2))(1-(3,4))\ldots(1-(2m-1,2m)))$$

ein Element von S^λ, wobei S die Menge aller Matrizen $X = (x_{ij}) \in \Xi_\lambda$ bezeichnet, so daß $tab(X)$ das kleinste λ-Tableau und

$$x_{n-m+1,1} = x_{n-m+2,3} = x_{n-m+3,5} = \ldots = x_{n,2m-1} = 0.$$

ist.
Ausgehend von e_λ konstruieren wir in Definition 3.2.5 Permutationen $\pi_{\mathfrak{t}} \in \mathfrak{S}_n$ und Elemente $v_{\mathfrak{t}}$ des Spechtmoduls S^λ, von denen wir in Satz 3.2.6 zeigen können, daß

$$last(v_{\mathfrak{t}}) = \mathfrak{t} \text{ und } top(v_{\mathfrak{t}}) = top(e_\lambda) \circ \pi_{\mathfrak{t}}.$$

gilt. Das heißt, wir haben für jedes standard λ-Tableau $\mathfrak{t}$ ein Element $v_{\mathfrak{t}}$ mit $last(v_{\mathfrak{t}}) = \mathfrak{t}$ gefunden. Da die Struktur von $top(v_{\mathfrak{t}})$ für spätere Definitionen wichtig werden wird, untersuchen wir diese in Lemma 3.2.8.
Der nächste Abschnitt dient dazu, für jedes standard λ-Tableau $\mathfrak{t}$ ein Polynom $p_{\mathfrak{t}}(q)$ einzuführen. Wie die Namensgebung schon andeutet, sollen die Polynome $p_{\mathfrak{t}}(q)$ später die zu einer Standardbasis zugehörigen Polynome sein. Um $p_{\mathfrak{t}}(q)$ zu definieren, benötigen wir einige Ausgangsdaten, die wir in dem Tupel

$$\delta(\mathfrak{t}) := (d, \underline{h}, \underline{b}, \underline{r})$$

zusammenfassen. Diese Daten können wir sowohl an dem λ-Tableau $\mathfrak{t}$ (siehe Definition 3.3.1) als auch an $top(v_{\mathfrak{t}})$ ablesen (siehe Lemma 3.3.3).
Sie sind so gewählt, daß wir mit der Menge

$$\mathfrak{M}_{\mathfrak{t}}(q) := \left\{ \begin{array}{|c|c|c|c|c|c|} \hline D_1 & & & & & \\ \hline X_{11} & D_2 & & & & \\ \hline X_{21} & X_{22} & \ddots & & & \\ \hline \dots & \dots & & D_d & & \\ \hline X_{d1} & X_{d2} & \dots & X_{dd} & D_{d+1} & \underbrace{}_{b_{d+2}} \\ \hline \end{array} \;\middle|\; D_i \in \mathfrak{M}_{h_{i-1},b_i}(q)\ (1 \leq i \leq d+1) \text{ und} \right.$$

$$\left. X_{ij} \in \mathfrak{M}_{h_i,b_j}(q)\ (1 \leq j \leq i \leq d) \text{ mit } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t}) \right\} \subseteq \Xi_{(n-m,m)}.$$

genau die Elemente X von $\Xi_{(n-m,m)}$ erhalten, für die $last(X) = \mathfrak{t}$ gilt.
Ohne hier auf die genaue Definition von $\mathfrak{M}^{rk}_{\mathfrak{t}}(q)$ einzugehen ($\mathfrak{M}^{rk}_{\mathfrak{t}}(q)$ ist eine Teilmenge von $\mathfrak{M}_{\mathfrak{t}}(q)$, bei der noch einige Rangbedingungen erfüllt sein müssen, bei denen der Vektor $\underline{r}$ aus $\delta(\mathfrak{t})$ Bedeutung erlangt; siehe 3.3.5 und 3.3.16) definieren wir

$$p_{\mathfrak{t}}(q) := |\mathfrak{M}^{rk}_{\mathfrak{t}}(q)|.$$

Mit den Lemmata 3.3.9 und 3.3.19 sehen wir, daß $p_{\mathfrak{t}}(q)$ eine leicht aus $\mathfrak{t}$ zu bestimmende q-Potenz mal der Mächtigkeit der Menge

$$\hat{\mathfrak{M}}^{rk}_{\hat{\mathfrak{t}}}(q) := \left\{ \begin{array}{|c|c|c|c|c|} \hline 0 & & & & \\ \hline X_{11} & 0 & & & \\ \hline X_{21} & X_{22} & \ddots & & \\ \hline \dots & \dots & & 0 & \\ \hline X_{d1} & X_{d2} & \dots & X_{dd} & 0 \\ \hline \end{array} \;\middle|\; X_{ij} \in \mathfrak{M}_{h_i,b_j}(q) \text{ für } 1 \leq j \leq i \leq d, \right.$$

$$\left. rank(xmat(M,i)) \leq r_i \text{ für } 1 \leq i \leq d \text{ mit } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\hat{\mathfrak{t}}) \right\} \subseteq \mathfrak{M}_{\hat{\mathfrak{t}}}(q)$$

ist, wobei $\hat{\mathfrak{t}} := quad(\mathfrak{t})$ ist (siehe 3.3.12 und 3.3.14) und $xmat(M,i)$ $(1 \le i \le d)$ diejenige Matrix bezeichnet, die aus den Blöcken $\{X_{kl} \mid 1 \le l \le i \le k \le d\}$ besteht.
Um eben diese Mächtigkeit von $\hat{\mathfrak{M}}^{rk}_{\hat{\mathfrak{t}}}(q)$ zu ermitteln, wählen wir einen rekursiven Ansatz, der auf den beiden nächsten Lemmata basiert.

Lemma 1 (3.4.2) *Seien $h, b \in \mathbb{N}, 0 \le r_Y \le r_X \le \min\{h, b\}$ und Y eine Matrix über $GF(q)$ mit Breite b und Rang r_Y. Dann gibt es*

$$p^{(1)}_{h,b,r_Y,r_X}(q) = q^{r_Y(h-r_X+r_Y)} \begin{bmatrix} h \\ r_X - r_Y \end{bmatrix} \prod_{i=r_Y}^{r_X-1} (q^b - q^i)$$

Möglichkeiten, Y um eine Matrix $X \in \mathfrak{M}_{h,b}(q)$ zu erweitern, so daß die resultierende Matrix $\left(\begin{array}{|c|}\hline X \\ \hline Y \\ \hline\end{array}\right)$ Rang r_X besitzt.

Lemma 2 (3.4.4) *Seien $h, b \in \mathbb{N}$ und $r_A, r_B, r_Y, r_X \in \mathbb{N}_0$ mit*

- $r_Y \le r_A \le r_X$ *und* $r_Y \le r_B \le r_X$,
- $r_A + r_B - r_Y \le r_X \le \min\{h, b\}$.

Weiter sei Y eine Matrix über $GF(q)$ mit Rang r_Y, A eine Matrix über $GF(q)$ mit h Zeilen, so daß die Matrix $\left(\begin{array}{|c|}\hline A \\ \hline Y \\ \hline\end{array}\right)$ Rang r_A besitzt, und B eine Matrix über $GF(q)$ mit b Spalten, so daß die Matrix $\left(\begin{array}{|c|c|}\hline Y & B \\ \hline\end{array}\right)$ Rang r_B besitzt.
Wenn wir nun $k := (h - r_X + r_B)(r_B - r_Y) + b(r_A - r_Y)$ setzen, dann haben wir

$$p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q) = q^k \begin{bmatrix} h - (r_A - r_Y) \\ r_X - r_A - r_B + r_Y \end{bmatrix} \prod_{i=r_B-r_Y}^{r_X-r_A-1} (q^b - q^i)$$

Möglichkeiten, die Nullmatrix in der Matrix $\left(\begin{array}{|c|c|}\hline A & 0 \\ \hline Y & B \\ \hline\end{array}\right)$ durch eine Matrix $X \in \mathfrak{M}_{h,b}(q)$ zu ersetzen, so daß die Matrix $\left(\begin{array}{|c|c|}\hline A & X \\ \hline Y & B \\ \hline\end{array}\right)$ Rang r_X besitzt.

Definieren wir jetzt $\mathfrak{I}^{rk}_{\mathfrak{t}}$ als die Menge aller Tupel

$$(r_{ij}) = (r_{11}, r_{21}, r_{22}, r_{31}, r_{32}, r_{33}, \ldots, r_{d1}, r_{d2}, \ldots, r_{dd}),$$

die die Eigenschaften

- $r_{ij} \in \mathbb{N}$ für $1 \le j \le i \le d$
- $r_{ii} \le r_i$ für $1 \le i \le d$,
- $r_{i+1,j} \le r_{ij}$ für $1 \le j \le i \le d-1$,
- $r_{i,j-1} \le r_{ij}$ für $2 \le j \le i \le d$ und

- $r_{i,j-1} + r_{i+1,j} - r_{i+1,j-1} \leq r_{ij}$ für $2 \leq j \leq i \leq d-1$

erfüllen, dann erhalten wir als Ergebnis der Rekursion das folgende Theorem.

Theorem 3 (3.4.9) *Sei* $\mathfrak{t} \in Std(\lambda)$ *und* $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$. *Dann gilt*

$$|\hat{\mathfrak{M}}_{\mathfrak{t}}^{rk}(q)| = \sum_{(r_{ij}) \in \mathfrak{I}_{\mathfrak{t}}^{rk}} r^{(1)}_{\mathfrak{t},(r_{ij})}(q) r^{(2)}_{\mathfrak{t},(r_{ij})}(q) r^{(3)}_{\mathfrak{t},(r_{ij})}(q),$$

wobei

$$\begin{aligned}
r^{(1)}_{\mathfrak{t},(r_{ij})}(q) :=& \begin{bmatrix} b_1 \\ r_{11} \end{bmatrix} \begin{bmatrix} b_d \\ r_{dd} \end{bmatrix} \prod_{2 \leq j \leq i \leq d-1} \begin{bmatrix} r_{ij} - r_{i,j-1} \\ r_{i+1,j} - r_{i+1,j-1} \end{bmatrix} \begin{bmatrix} r_{ij} - r_{i+1,j} \\ r_{i,j-1} - r_{i+1,j-1} \end{bmatrix} \\
& \prod_{1 \leq k \leq d-1} \begin{bmatrix} h_k \\ r_{kk} - r_{k+1,k} \end{bmatrix} \begin{bmatrix} b_{k+1} \\ r_{k+1,k+1} - r_{k+1,k} \end{bmatrix}, \\
r^{(2)}_{\mathfrak{t},(r_{ij})}(q) :=& \frac{|GL_{r_{11}}(q)||GL_{r_{dd}}(q)|}{|GL_{r_{d1}}(q)|} \prod_{2 \leq j \leq i \leq d-1} |GL_{r_{ij} - r_{i,j-1} - r_{i+1,j} + r_{i+1,j-1}}(q)| \text{ und} \\
r^{(3)}_{\mathfrak{t},(r_{ij})}(q) :=& q^l \text{ ist}
\end{aligned}$$

mit

$$l := \begin{cases} 0 & \text{falls } d = 1 \\ r_{21}(h_1 + b_2 + 2r_{21} - r_{22} - r_{11}) & \text{falls } d = 2 \\ r_{d1}^2 + r_{21}^2 + r_{d,d-1}^2 - r_{11}r_{21} - r_{dd}r_{d,d-1} + & \\ + \sum_{i=2}^{d} r_{i,i-1}(b_i + h_{i-1}) - \sum_{1 \leq j < i \leq d-1} r_{ij} r_{i+1,j+1} + & \\ + \sum_{2 \leq j < i \leq d-1} r_{ij}(r_{i,j-1} + r_{i+1,j}) - \sum_{2 \leq j < i-1 \leq d-2} r_{ij}^2 & \text{falls } d > 2. \end{cases}$$

Damit sind die Polynome $p_{\mathfrak{t}}(q)$ für jedes $\mathfrak{t} \in Std(\lambda)$ vollständig bekannt. Der nächste Schritt in Richtung Standardbasis wird sein, dazu eine linear unabhängige Teilmenge $\mathfrak{B}^\lambda$ von S^λ zu finden, so daß in dieser für jedes $\mathfrak{t} \in Std(\lambda)$ genau $p_{\mathfrak{t}}(q)$ Elemente v_i mit $last(v_i) = \mathfrak{t}$ liegen. Dazu benötigen wir allerdings noch einige Hilfsmittel. Das erste werden die Idempotente e_L sein.
Indizieren wir für ein standard λ-Tableau $\mathfrak{t}$ mit $\mathfrak{I}_{\mathfrak{t}}$ die Einträge eines Elements $M \in \mathfrak{M}_{\mathfrak{t}}(q)$ unterhalb der Grenzlinie (siehe Definition 3.5.1), dann wird $\mathfrak{M}_{\mathfrak{t}}(q)$ mit punktweiser Addition $\diamond$ unterhalb der Grenzlinie (siehe Beispiel 3.5.3) zu einer abelschen Gruppe der Ordnung $q^{|\mathfrak{I}_{\mathfrak{t}}|}$.
Bezeichnen wir für $(i,j) \in \mathfrak{I}_{\mathfrak{t}}$ mit ε_{ij} die (i,j)-Koordinatenfunktion von $\mathfrak{M}_{\mathfrak{t}}(q)$ nach $GF(q)$, so erhalten wir für jedes $L = (l_{ij}) \in \mathfrak{M}_{\mathfrak{t}}(q)$ mit

$$\chi_L := \sum_{(i,j) \in \mathfrak{I}_{\mathfrak{t}}} l_{ij}(\theta \varepsilon_{ij}),$$

einen linearen Charakter von $(\mathfrak{M}_{\mathfrak{t}}(q), \diamond)$, der ein $M = (m_{ij}) \in \mathfrak{M}_{\mathfrak{t}}(q)$ auf

$$\chi_L(M) = \prod_{(i,j) \in \mathfrak{I}_{\mathfrak{t}}} \theta(l_{ij} m_{ij})$$

abbildet. Für jedes $L \in \mathfrak{M}_{\mathfrak{t}}(q)$ bekommen wir dann mittels

$$e_L := \frac{1}{q^{|\mathfrak{I}_{\mathfrak{t}}|}} \sum_{M \in \mathfrak{M}_{\mathfrak{t}}(q)} \chi_L(-M)M$$

ein Idempotent der Gruppenalgebra $\mathfrak{V}_{\mathfrak{t}}(q)$ von $\mathfrak{M}_{\mathfrak{t}}(q)$ über F, das zum linearen Charakter χ_L gehört.
Das nächstes Hilfsmittel auf dem Weg zur linear unabhängigen Teilmenge $\mathfrak{B}^\lambda$ werden die beiden folgenden Sätze sein, die die Operation spezieller Elemente $g \in GL_n(q)$ genauer untersuchen und damit bei der Konstruktion der Elemente von $\mathfrak{B}^\lambda$ behilflich sind.

Satz 4 (3.6.2) *Sei $\mathfrak{t} \in Std(\lambda)$, $(d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$, $v \in M^\lambda$ mit $last(v) = \mathfrak{t}$ und $g \in GL_n(q)$ wie in (3.10). Dann gilt*

$$last(v \circ g) = \mathfrak{t} \text{ und } top(v \circ g) = top(v) \circ g.$$

Satz 5 (3.6.3) *Sei $\mathfrak{t} \in Std(\lambda), L \in \mathfrak{M}_{\mathfrak{t}}(q), (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})$ und $g \in GL_n(q)$ wie in (3.11). Dann haben wir*

$$e_L \circ g = e_{L \circ (g^{-1})^T}.$$

Nun sind wir bereit, das folgende Theorem zu beweisen.

Theorem 4 (3.6.8) *Sei $\mathfrak{t} \in Std(\lambda)$. Dann existiert für jedes $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ ein Element $b_L \in S^\lambda$, so daß $last(b_L) = \mathfrak{t}$ und $top(b_L) = e_L$ ist.*

Der Beweis ist sehr technisch, dafür aber konstruktiv. Wir geben nämlich für jedes $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ eine genaue Liste von Operationen auf e_λ an, die ein Element $b_L \in S^\lambda$ mit den geforderten Eigenschaften erzeugen.
Für jedes $L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q)$ fixieren wir nun ein derart konstruiertes Element b_L (die Konstruktion ist bei weitem nicht eindeutig) und setzen

$$\mathfrak{B}^\lambda := \{b_L \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}.$$

Zusammen mit der Tatsache, daß zwei Elemente v_1 und v_2 von S^λ linear unabhängig sind, falls $last(v_1) \neq last(v_2)$ gilt, und der linearen Unabhängigkeit der Idempotente e_L folgern wir das folgende Korollar.

Korollar 1 (3.6.11) *Die Teilmengen $\{top(b_L) \mid L \in \mathfrak{M}_{\mathfrak{t}}^{rk}(q), \mathfrak{t} \in Std(\lambda)\}$ von M^λ und $\mathfrak{B}^\lambda$ von S^λ sind beide linear unabhängig.*

Es stellt sich nun sofort die Frage, ob $\mathfrak{B}^\lambda$ auch ein Erzeugendensystem und damit eine Basis von S^λ ist. Wäre dies nämlich der Fall, würde $\mathfrak{B}^\lambda$ zusammen mit den Polynomen $p_{\mathfrak{t}}(q) = |\mathfrak{M}_{\mathfrak{t}}^{rk}(q)|$ alle Voraussetzungen für eine Standardbasis erfüllen und wir hätten unser Ziel erreicht. Leider bleibt dies nur eine Vermutung, die wir allerdings im Folgenden mit einigen Fakten untermauern werden.

Vermutung 1 (3.7.1) *$\mathfrak{B}^\lambda$ ist eine Standardbasis von S^λ mit zugehörigen Polynomen $\{p_{\mathfrak{t}}(q) \mid \mathfrak{t} \in Std(\lambda)\}$ beziehungsweise, äquivalent dazu, es gilt*

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda). \tag{5.1}$$

Da mittels

$$dim(S^\lambda) = \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}$$

die Dimension des Spechtmoduls S^λ bekannt ist, bietet es sich an, mit Hilfe eines Computerprogramms die Gleichheit (5.1) in Spezialfällen nachzuprüfen. Dies habe ich unter Verwendung von GAP [9] getan. Die Berechnungen liefern uns das folgende Theorem.

Theorem 5 (3.7.4) *Sei* $1 \leq m \leq 11$ *und* $\lambda = (m, m)$. *Dann ist*

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda)$$

und damit $\mathfrak{B}^\lambda$ *eine Standardbasis von* S^λ.

Eine andere Vorgehensweise, sich der Vermutung 1 zu nähern, ist die Idee aus [7], die standard (m, m)-Tableaux in mehrere Klassen zu unterteilen. Dies geschieht dadurch, daß wir für $1 \leq k \leq m$

$$\mathcal{J}_k(\lambda) := \{\mathfrak{t} \in Std(\lambda) \mid h_d = k \text{ für } (d, \underline{h}, \underline{b}, \underline{r}) = \delta(\mathfrak{t})\} \subseteq Std(\lambda)$$

setzen und führt zu einer weiteren Vermutung.

Vermutung 2 (3.7.9) *Wenn* $\lambda = (m, m)$ *und* $1 \leq k \leq m$ *ist, dann gilt*

$$\sum_{\mathfrak{t} \in \mathcal{J}_k(\lambda)} p_{\mathfrak{t}}(q) = q^{mk+k-m} dim(S^{(m-1,m-k)}).$$

Können wir diese Vermutung beweisen, dann haben wir sofort unser Ziel erreicht, wie der folgende Satz zeigt.

Satz 6 (3.7.13) *Sei* $\lambda = (n - m, m)$. *Wenn Vermutung 2 wahr ist, dann gilt*

$$\sum_{\mathfrak{t} \in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda)$$

und damit ist $\mathfrak{B}^\lambda$ *eine Standardbasis von* S^λ.

Leider können wir Vermutung 2 wie in [7] nur in den Spezialfällen $k = m-1$ und $k = m$ beweisen (siehe Satz 3.7.10 und Satz 3.7.11) und einen induktiven Ansatz für $k = 1$ zeigen (siehe Satz 3.7.12).
Für $1 \leq m \leq 11$ bestätigt uns das vorhin erwähnte GAP-Programm die Richtigkeit der Vermutung 2, so daß wir mit Satz 6 sofort das nächste Theorem folgern können.

Theorem 6 (3.7.14) *Wenn* $1 \leq m \leq 11$ *und* $\lambda = (n - m, m)$ *ist, dann ist* $\mathfrak{B}^\lambda$ *eine Standardbasis von* S^λ.

Den letzten Ansatz, Vermutung 1 zu beweisen, liefert das Branching Theorem, welches uns eine direkte Zerlegung von S^λ als $FG^*_{n-1}(q)$-Modul liefert:

$$S^\lambda = S^\lambda E_1 \oplus S^\lambda E_2 FG^*_{n-1}(q).$$

Die Isomorphien

$$\begin{aligned} S^\lambda E_1 &\cong S^{(m,m-1)} \text{ als } FG_{n-1}(q)\text{-Moduln und} \\ S^\lambda E_2 &\cong S^{(m-1,m-1)} \text{ als } FG_{n-2}(q)\text{-Moduln} \end{aligned}$$

für $n-m=m$ beziehungsweise

$$\begin{aligned} S^\lambda E_1 &= S_2 > S_1 > S_0 = 0 \text{ mit} \\ S_2/S_1 &\cong S^{(n-m,m-1)} \text{ als } FG_{n-1}(q)\text{-Moduln,} \\ S_1 &\cong S^{(n-m-1,m)} \text{ als } FG_{n-1}(q)\text{-Moduln und} \\ S^\lambda E_2 &\cong S^{(n-m-1,m-1)} \text{ als } FG_{n-2}(q)\text{-Moduln} \end{aligned}$$

für $n-m>m$ sind der optimale Ansatz für einen induktiven Beweis.
Wir führen in Definition 3.7.15 die Teilmengen $\mathfrak{M}^{rk,0}_{\mathfrak{t}}(q)$ und $\mathfrak{M}^{rk,1}_{\mathfrak{t}}(q)$ ein und beweisen in den nachfolgenden Sätzen 3.7.16, 3.7.17 und 3.7.18 fast alle Bausteine, die wir für die Induktion benötigen. Ein Mosaikstein fehlt uns allerdings, den wir in der folgenden Vermutung formulieren.

Vermutung 3 (3.7.19) *Für $\lambda = (n-m,m)$ mit $n-m>m$ gilt*

$$\sum_{\mathfrak{t}\in Std(\lambda)} |\mathfrak{M}^{rk,0}_{\mathfrak{t}}(q)| = dim(S^{(n-m-1,m)}) + dim(S^{(n-m,m-1)}).$$

Unter Annahme der Richtigkeit dieser Vermutung funktioniert der Induktionsbeweis, und wir erhalten abschließend in diesem Kapitel den folgenden Satz.

Satz 7 (3.7.20) *Sei $\lambda = (n-m,m)$. Falls Vermutung 3 wahr ist, dann gilt*

$$\sum_{\mathfrak{t}\in Std(\lambda)} p_{\mathfrak{t}}(q) = dim(S^\lambda)$$

und $\mathfrak{B}^\lambda$ ist eine Standardbasis von S^λ.

Kapitel 4

In Kapitel 4 untersuchen wir den Spechtmodul S^λ für die 3-Teile-Partition $\lambda := (2,2,2)$. Auch für diesen Spechtmodul begeben wir uns auf die Suche nach einer Standardbasis. Leider finden wir nur eine schwächere Form von Basis von S^λ, die wir in Definition 4.0.22 einführen und Basis mit zugehörigen Polynomen nennen. Aber zuerst einmal gibt es genau 5 standard λ-Tableaux:

$$\mathfrak{t}_1 := \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}, \mathfrak{t}_2 := \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array}, \mathfrak{t}_3 := \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline\end{array}, \mathfrak{t}_4 := \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array} \text{ und } \mathfrak{t}_5 := \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}.$$

Desweiteren identifizieren wir in Satz 4.1.6 mit

$$e_\lambda := \sum_{\pi \in \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5,6\}}} sign(\pi) \begin{array}{|cccccc|} \hline \cdot & 1 & & & & \\ * & & & \cdot & 1 & \\ \hline \cdot & \cdot & 1 & & & \\ * & * & & \cdot & \cdot & 1 \\ \hline \end{array} \circ \pi$$

ein Element von S^λ.
Das entscheidende Hilfsmittel auf dem Weg zur Standardbasis ist das Branching Theorem. Es liefert uns eine direkte Zerlegung von S^λ als $FG_5^*(q)$-Modul.

$$S^\lambda = S^\lambda E_1 \oplus S^\lambda E_2 FG_5^*(q) \oplus S^\lambda E_3 FG_5^*(q),$$

wobei

$$\begin{aligned} S^\lambda E_1 &\cong S^{(2,2,1)} \text{ als } FG_5(q)\text{-Moduln,} \\ S^\lambda E_2 &\cong S^{(2,1,1)} \text{ als } FG_4(q)\text{-Moduln und} \\ S^\lambda E_3 &\cong S^{(1,1,1)} \text{ als } FG_3(q)\text{-Moduln ist.} \end{aligned}$$

In den darauffolgenden Sätzen konstruieren wir der Reihe nach Basen $\mathfrak{B}_1$ von $S^\lambda E_1$, $\mathfrak{B}_2$ von $S^\lambda E_2$, $\mathfrak{B}_3$ von $S^\lambda E_3$, $\mathfrak{B}_2^{mult}$ von $S^\lambda E_2 FG_5^*(q)$ und schließlich $\mathfrak{B}_3^{mult}$ von $S^\lambda E_3 FG_5^*(q)$.
Insgesamt erhalten wir das folgende Korollar.

Korollar 2 $\mathfrak{B}^\lambda := \mathfrak{B}_1 \cup \mathfrak{B}_2^{mult} \cup \mathfrak{B}_3^{mult}$ *ist eine Basis von* S^λ.

Bleibt nur noch, für jedes $\mathfrak{t} \in Std(\lambda)$ die Anzahl der Elemente in der Menge

$$\{b \in \mathfrak{B}^\lambda \mid last(b) = \mathfrak{t}\}$$

zu untersuchen. Dabei stellt sich heraus, daß die konstruierte Basis $\mathfrak{B}^\lambda$ eine Basis mit zugehörigen Polynomen ist.

Theorem 7 $\mathfrak{B}^\lambda$ *ist eine Basis von* S^λ *mit zugehörigen Polynomen*

$$\begin{gathered} p_{\mathfrak{t}_1}(q) = q^6, p_{\mathfrak{t}_2}(q) = q^7, p_{\mathfrak{t}_3}(q) = q^8, \\ p_{\mathfrak{t}_4}(q) = q^9 + q^8 - q^7 \ \textit{und}\ p_{\mathfrak{t}_5}(q) = q^{12} + q^{10} - q^8. \end{gathered}$$

$\mathfrak{B}^\lambda$ ist keine Standardbasis von S^λ. Wir stellen aber noch die folgende gewagte Vermutung auf.

Vermutung 4 *Es gibt eine Standardbasis von* S^λ *mit zugehörigen Polynomen*

$$\begin{gathered} p_{\mathfrak{t}_1}(q) = q^6, p_{\mathfrak{t}_2}(q) = q^7, p_{\mathfrak{t}_3}(q) = q^9 + q^8 - q^7, \\ p_{\mathfrak{t}_4}(q) = q^{10} \ \textit{und}\ p_{\mathfrak{t}_5}(q) = q^{12}. \end{gathered}$$

Appendix A

Notation

Symbol	Page	Description
b_L	63, 84, 89, 90	(basis) element of S^λ
$\mathfrak{B}^\lambda$	63, 93	linearly independent set (basis) of S^λ
$\mathfrak{B}_1$	84	part of a basis of $S^{(2,2,2)}$
$\mathfrak{B}_2$	89	part of a basis of $S^{(2,2,2)}$
$\mathfrak{B}_2^{mult}$	92	part of a basis of $S^{(2,2,2)}$
$\mathfrak{B}_3$	90	part of a basis of $S^{(2,2,2)}$
$\mathfrak{B}_3^{mult}$	92	part of a basis of $S^{(2,2,2)}$
$C_{\mathfrak{t}}$	5	column stabilizer of $\mathfrak{t}$
χ_L	45	character of $\mathfrak{M}_{\mathfrak{t}}(q)$
$col_{\mathfrak{t}}$	4	column function
$\delta(\mathfrak{t})$	30	tuple $(d, \underline{h}, \underline{b}, \underline{r})$
$dmat(L, i)$	52	submatrix of L
$dmat(L, i, r)$	52	submatrix of L
e_λ	26, 81	element of S^λ
e_L	45, 83	idempotent
E_r	22	element of $FGL_n(q)$
$E(k, x, y, M)$	51	element of $\mathfrak{M}_{k,k}(q)$
F	1	field, $p \nmid char(F)$, contains a p^{th} root of unity
$\mathcal{F}(\lambda)$	9	set of λ-flags
$\Gamma(r)$	22	index set
$G(\Gamma(r))$	22	subset of $GL_n(q)$
$GF(q)$	1	finite field of q elements
$GF(q)^*$	1	multiplicative group of $GF(q)$
$GL_n(q)$	1	group of invertible $n \times n$ matrices over $GF(q)$
$G_r(q)$	22	subset of $GL_n(q)$
$G_r^*(q)$	22	subset of $GL_n(q)$
h_{ij}	3	(i, j)-hook
$H_r^*(q)$	22	subset of $GL_n(q)$
I_k	21	identity matrix in $GL_k(q)$
$\mathfrak{I}_{\mathfrak{t}}$	44	index set
$\mathfrak{I}_{\mathfrak{t}}^{rk}$	41	index set
$\mathcal{J}_k(\lambda)$	65	subset of $Std(\lambda)$

Symbol	Page	Description
λ_R	23	partition derived from λ
$last$	15	function from M^λ to $\mathcal{T}_{rs}(\lambda)$
M^λ	14	vector space over F with basis Ξ_λ
$\mathfrak{M}_{a,b}(q)$	1	monoid of $a \times b$ matrices over $GF(q)$
$\mathfrak{M}_{a,b,k}(q, \mathfrak{B})$	8	subset of $\mathfrak{M}_{a,b}(q)$
$\mathfrak{M}_\mathfrak{t}(q)$	32, 34, 82	subset of Ξ_λ
$\mathfrak{M}_\mathfrak{t}^{rk}(q)$	32, 36	subset of $\Xi_{(n-m,m)}$
$\mathfrak{M}_\mathfrak{t}^{0}(q)$	84	subset of $\Xi_{(2,2,2)}$
$\mathfrak{M}_\mathfrak{t}^{1}(q)$	89	subset of $\Xi_{(2,2,2)}$
$\mathfrak{M}_\mathfrak{t}^{2}(q)$	90	subset of $\Xi_{(2,2,2)}$
$\mathfrak{M}_\mathfrak{t}^{rk,0}(q)$	69	subset of $\Xi_{(n-m,m)}$
$\mathfrak{M}_\mathfrak{t}^{rk,1}(q)$	69	subset of $\Xi_{(n-m,m)}$
$\hat{\mathfrak{M}}_\mathfrak{t}(q)$	33	subset of $\Xi_{(n-m,m)}$
$\hat{\mathfrak{M}}_\mathfrak{t}^{rk}(q)$	33	subset of $\Xi_{(n-m,m)}$
$mat(X_{ij})$	41	matrix over $GF(q)$
$numb$	36	function from $Std(\lambda)$ to $\mathbb{N}_0$
$p^{(1)}_{h,b,r_Y,r_X}(q)$	37	polynomial in q
$p^{(2)}_{h,b,r_Y,r_A,r_B,r_X}(q)$	39	polynomial in q
$p_\mathfrak{t}(q)$	32, 36, 95	polynomial in q
$\phi_\mathfrak{t}$	83	function from $\mathfrak{M}_\mathfrak{t}(q)$ to $GF(q)^{pot(\mathfrak{t})}$
$\pi_\mathfrak{t}$	27, 82	element of $\mathfrak{S}_n$
$\pi_\mathfrak{t}^{(i)}$	27	element of $\mathfrak{S}_n$
pot	82	function from $Std((2,2,2))$ to $\mathbb{N}$
$\psi_{d,i}$	20	$FGL_n(q)$-homomorphism
$\psi_\mathfrak{t}$	82	function from $GF(q)^{pot(\mathfrak{t})}$ to $\mathfrak{M}_\mathfrak{t}(q)$
$quad(X)$	35	element of $\Xi_{(m,m)}$
$quad(\mathfrak{t})$	35	element of $Std((m,m))$
$r_{a,b}(q,k)$	6	$\lvert\{A \in \mathfrak{M}_{a,b}(q) \mid rank(A) = k\}\rvert$
$r_{a,b,k}(q,\mathfrak{B})$	8	$\lvert\mathfrak{M}_{a,b,k}(q,\mathfrak{B})\rvert$
$(r_{ij})_L$	41	tuple of elements of $\mathbb{N}_0$
$row_\mathfrak{t}$	4	row function
$\mathfrak{R}_1$	91	set of right coset representatives
$\mathfrak{R}_1^{(i)}$	91	set of right coset representatives
$\mathfrak{R}_2$	92	set of right coset representatives
$\mathfrak{R}_2^{(i)}$	92	set of right coset representatives
$\mathfrak{R}_r^*$	22	set of subsets of $\{1, 2, \ldots, h\}$
$\mathfrak{S}_n$	1	symmtric group on n numbers
$\mathfrak{S}_X$	1	group of permutations on X
$short(v)$	25	short notation of v
$short(X)$	25	short notation of X
$Std(\lambda)$	4	set of standard λ-tableaux
S^λ	20	Specht module
$\mathfrak{t}_\lambda$	4	λ-tableau
$\mathfrak{t}^\lambda$	4	initial λ-tableau

Symbol	**Page**	**Description**
$tab(X)$	14	tableau which belongs to X
θ	1	linear F-character of the group $(GF(q),+)$
top	15	function from M^λ to M^λ
$\mathcal{T}_{rs}(\lambda)$	4	set of row-standard λ-tableaux
$tup(M)$	37	tuple of linearly independent vectors
$v_\mathfrak{t}$	27, 82	element of S^λ
Ξ_λ	10	notation for the λ-flags
$xmat(L,i,r)$	52	submatrix of L
$xmat(M,i)$	32	submatrix of M
$\langle \underline{a}, \underline{b} \rangle$	1	canonical scalar product in $GF(q)^l$
$[r]$	6	polynomial in q
$[r]!$	6	polynomial in q
$\begin{bmatrix} r \\ s \end{bmatrix}$	6	Gaussian polynomial

Appendix B

Some polynomials $p_{\mathfrak{t}}(q)$

Table 1 The polynomials $p_{\mathfrak{t}}(q)$ for $\mathfrak{t} \in Std((2,2))$

$\mathfrak{t}$	$p_{\mathfrak{t}}(q)$ q^2	q^3	q^4
2 4	1		
3 4			1
$dim(S^{(2,2)})$	1		1

To simplify the notation of the tableau $\mathfrak{t}$ we write down only its second row. In the table we list the coefficients of the polynomials $p_{\mathfrak{t}}(q)$ and of the polynomial $dim(S^{(2,2)})$.

Table 2 The polynomials $p_{\mathfrak{t}}(q)$ for $\mathfrak{t} \in Std((3,3))$

$\mathfrak{t}$	q^3	q^4	q^5	$p_{\mathfrak{t}}(q)$ q^6	q^7	q^8	q^9
2 4 6	1						
3 4 6			1				
2 5 6			1				
3 5 6			-1	1	1		
4 5 6							1
$dim(S^{(3,3)})$	1		1	1	1		1

Table 3 The polynomials $p_{\mathfrak{t}}(q)$ for $\mathfrak{t} \in Std((4,4))$

$\mathfrak{t}$	q^4	q^5	q^6	q^7	q^8	q^9	$p_{\mathfrak{t}}(q)$ q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	q^{16}
2 4 6 8	1												
3 4 6 8			1										
2 5 6 8			1										
3 5 6 8			-1	1	1								
4 5 6 8							1						
2 4 7 8			1										
3 4 7 8					1								
2 5 7 8			-1	1	1								
3 5 7 8				-1	-1	2	1						
4 5 7 8						-1		1	1				
2 6 7 8							1						
3 6 7 8						-1		1	1				
4 6 7 8						1	-1	-1		1	1		
5 6 7 8													1
$dim(S^{(4,4)})$	1		1	1	2	1	2	1	2	1	1		1

Table 4 The polynomials $p_{\mathfrak{t}}(q)$ for $\mathfrak{t} \in Std((5,5))$

$\mathfrak{t}$	$p_{\mathfrak{t}}(q)$																				
	q^{5}	q^{6}	q^{7}	q^{8}	q^{9}	q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	q^{16}	q^{17}	q^{18}	q^{19}	q^{20}	q^{21}	q^{22}	q^{23}	q^{24}	q^{25}
2 4 6 8 10	1																				
3 4 6 8 10			1																		
2 5 6 8 10			1																		
3 5 6 8 10			-1	1	1																
4 5 6 8 10							1														
2 4 7 8 10			1																		
3 4 7 8 10					1																
2 5 7 8 10			-1	1	1																
3 5 7 8 10				-1	-1	2	1														
4 5 7 8 10						-1		1	1												
2 6 7 8 10							1														
3 6 7 8 10						-1		1	1												
4 6 7 8 10						1	-1	-1		1	1										
5 6 7 8 10													1								
2 4 6 9 10			1																		
3 4 6 9 10					1																
2 5 6 9 10					1																
3 5 6 9 10					-1	1	1														
4 5 6 9 10									1												
2 4 7 9 10			-1	1	1																
3 4 7 9 10					-1	1	1														
2 5 7 9 10				-1	-1	2	1														
3 5 7 9 10						-3		3	1												
4 5 7 9 10							-1	-1		2	1										
2 6 7 9 10						-1		1	1												
3 6 7 9 10						1	-2	-2	1	2	1										
4 6 7 9 10							1		-2	-2	1	2	1								
5 6 7 9 10											-1			1	1						
2 4 8 9 10							1														
3 4 8 9 10									1												
2 5 8 9 10						-1		1	1												
3 5 8 9 10							-1	-1		2	1										
4 5 8 9 10									-1	-1	1	1	1								
2 6 8 9 10						1	-1	-1		1	1										
3 6 8 9 10							1		-2	-2	1	2	1								
4 6 8 9 10								1	1	-1	-3	-1	1	2	1						
5 6 8 9 10										1		-1	-1	-1	1	1	1				
2 7 8 9 10													1								
3 7 8 9 10											-1			1	1						
4 7 8 9 10										1		-1	-1	-1	1	1	1				
5 7 8 9 10										-1	1	1			-2			1	1		
6 7 8 9 10																					1
$dim(S^{(5,5)})$	1		1	1	2	2	3	2	4	3	4	3	4	2	3	2	2	1	1		1

Bibliography

[1] G. E. Andrews, The Theory of Partitions, Encyclopedia of Math. Appl., Vol. 2, Addison-Wesley (1976)

[2] R. W. Carter, Finite Groups of Lie Type, Wiley, New York (1985)

[3] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience Publishers, New York (1962)

[4] C. W. Curtis and I. Reiner, Methods of Representation Theory - with Applications to Finite Groups and Orders (Volume II), Wiley Classics Library Edition, New York (1994)

[5] R. Dipper and G. James, Identification of the Irreducible Modular Representations of $GL_n(q)$, Journal of Algebra **104** (1986), 266-288

[6] R. Dipper and G. James, The q-Schur Algebra, Proc. London Math. Soc. **59** (1989), 23-50

[7] R. Dipper and G. James, On Specht Modules for General Linear Groups, to appear

[8] P. Fong and B. Srinivasan, The Blocks of Finite General Linear and Unitary Groups, Invent. math. **69**, 109-153 (1982)

[9] The GAP Group, GAP – Groups, Algorithms and Programming, Version 4.3 (2002), http://www.gap-system.org

[10] B. Huppert, Endliche Gruppen I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin (1967)

[11] G. D. James, Representations of General Linear Groups, London Mathematical Society, Lecture Note Series **94** (1984)

[12] G. D. James, The Representation Theory of the Symmetric Groups, Springer Verlag, Lecture Notes in Mathematics **682** (1978)

[13] S. Lang, Algebra, Addison-Wesley Publishing Company (1993)

[14] S. Lyle, Ph.D. Thesis, Imperial College, London (2002)

[15] I. G. MacDonald, Symmetric Functions and Hall Polynomials, Second Edition, Oxford mathematical monographs (1995)