A rate-independent model for phase transformations in shape-memory alloys

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Introduction

In this thesis we consider so-called rate-independent systems and prove the existence of time evolution for such systems by using the energetic formulation, which was for the first time systematically used for treating of rate-independent systems by A. Mielke and F. Theil [MT99]. The framework of this approach is purely energetic and avoids the derivatives in space. These properties make it possible to apply this formulation to a wide class of rate-independent models without taking care of smoothness of solutions, which can not always be expected. The existence results in [MT99, MT04] were based on the linear structure of the process space. In particular, the reflexivity of the process space was crucial for the proofs. Such assumptions are often not satisfied in mechanical models of rate-independent systems. For example, in many models the process space is given as L^1-space, which is non-reflexive. These restrictions motivate to find suitable generalisation of the previous existence results for the energetic approach. In this paper we present a possible generalisation, which allows us to completely abandon the linear structure of the state space. In the last chapter we give a simple model for phase transformation processes in solids and use the obtained general existence results for studying the existence of solutions.

In first chapter we treat general rate-independent systems. Such systems are typically driven by an external loading on a time scale much slower than any internal time scale (like viscous relaxation times) but still much faster than the time needed to find the thermodynamical equilibrium. Typical phenomena involve dry friction, elasto-plasticity, certain hysteresis models for shape-memory alloys and quasistatic delamination or fracture. The main feature is the rate-independence of the system response, which means that a loading with twice (or half) the speed will lead to a response with exactly twice (or half) the speed. We refer to [BS96, KP89, Vis94, MM93] for approaches to these phenomena involving either differential inclusions or abstract hysteresis operators. The energetic method is different since we avoid time derivatives and use energy principles instead.

As it is well-known from dry friction, such systems will not necessarily relax into a complete equilibrium, since friction forces do not tend to 0 for vanishing velocities. One way to explain this phenomenon on a purely energetic basis is via so-called “wiggly energies”, where the macroscopic energy functional has a super-imposed fluctuating part with many local minimisers. Only after reaching a certain activation energy it is possible to leave these local minima and generate macroscopic changes, cf. [ACJ96, Men02]. Here we use a different approach which involves a dissipation distance which locally behaves homogeneous of degree 1, in contrast to viscous dissipation which is homogeneous of degree 2. This approach was introduced in [MT99, MT04, MTL02, GMH02] for models for shape-memory alloys and is now generalised to many other rate-independent systems. See [Mie03a] for a general setup for rate-independent material models in the framework of "standard generalised materials".

As basis for our considerations in the first chapter we take the following continuum mechanical model. Let $\Omega \subset \mathbb{R}^d$ be the undeformed body and $t \in [0,T]$ the slow process time. The deformation or displacement $\varphi(t) : \Omega \to \mathbb{R}^d$ is considered to lie in the space $\mathcal{F}$ of admissible deformations containing suitable Dirichlet boundary conditions. The internal variable $z(t) : \Omega \to Z \subset \mathbb{R}^m$ describes the internal state which may involve plastic deformations, hardening variables, magnetisation or phase indicators. The elastic (Gibb's) stored energy is given via

$$\mathcal{E}(t,\varphi,z) = \int_\Omega W(x, \mathrm{D}\varphi(x), z(x))\,\mathrm{d}x - \langle \ell(t), \varphi\rangle,$$

where $\langle \ell(t), \varphi\rangle = \int_\Omega f_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x + \int_{\partial\Omega} g_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x$ denotes the external loading depending on the process time t.

Changes of the internal variables are associated with dissipation of energy which is given constitutively via a dissipation potential $\Delta : \Omega \times \mathrm{T}Z \to [0,\infty]$, i.e., an internal process $Z : [t_0,t_1] \times \Omega \to Z$ dissipates the energy

$$\mathrm{Diss}(z,[t_0,t_1]) = \int_{t_0}^{t_1}\int_\Omega \Delta(x, z(t,x), \dot z(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

Rate-independence is obtained via homogeneity: $\Delta(x,z,\alpha v) = \alpha\Delta(x,z,v)$ for $\alpha \geq 0$. We associate with Δ a global dissipation distance $\mathcal{D}$ on the set of all internal states:

$$\mathcal{D}(z_0,z_1) = \inf\left\{\, \mathrm{Diss}(z,[0,1]) \mid z \in \mathrm{C}^1([0,1]\times\Omega, Z),\ z(0)=z_0,\ z(1)=z_1 \,\right\}.$$

In the setting of smooth continuum mechanics the evolution equations associated with such a process are given through the theory of standard generalised materials (cf. [Mie03a] and the references therein). They are the elastic equilibrium and the force balance for the internal variables:

$$\left.\begin{aligned} -\operatorname{div}\tfrac{\partial W}{\partial F}(x, \mathrm{D}_x\varphi(t,x), z(t,x)) &= f_{\mathrm{ext}}(t,x) \\ 0 \in \partial^{\mathrm{sub}}_{\dot z}\Delta(x, z(t,x), \dot z(t,x)) + \tfrac{\partial W}{\partial z}(x, \mathrm{D}_x\varphi(t,x), z(t,x)) & \end{aligned}\right\} \text{ in } \Omega,$$

where boundary conditions need to be added and ∂^{sub} denotes the subdifferential of a convex function. Using the functionals this system can be written in an abstract form as

$$\mathrm{D}_\varphi\mathcal{E}(t,\varphi(t),z(t)) = 0, \quad 0 \in \partial^{\mathrm{sub}}_{z_2}\widetilde{\mathcal{D}}(z(t),\cdot)[\dot z(t)] + \mathrm{D}_z\mathcal{E}(t,\varphi(t),z(t)), \tag{0.0.1}$$

which has the form of the doubly nonlinear problems studied in [CV90].

It was realised in [MT99, MTL02, Mie03a] that this problem can be rewritten in a derivative-free, energetic form which does not require solutions to be smooth in time or space. Hence, it is much more adequate for many mechanical systems. Moreover, the energetic formulation allows for the usage of powerful tools of the modern theory of the calculus of variations, such as lower semi-continuity, quasi- and poly-convexity and nonsmooth techniques. A pair $(\varphi,z) : [0,T] \to \mathcal{F}\times\mathcal{Z}$ is called a solution of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{D}$ if (S) and (E) hold:

(S) Stability: *For all* $t \in [0,T]$ *and all* $(\widetilde{\varphi}, \widetilde{z}) \in \mathcal{F} \times \mathcal{Z}$ *we have*
$\mathcal{E}(t, \varphi(t), z(t)) \leq \mathcal{E}(t, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z(t), \widetilde{z})$.

(E) Energy equality: *For all* $t \in [0,T]$ *we have*
$\mathcal{E}(t, \varphi(t), z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0,t]) = \mathcal{E}(0, \varphi(0), z(0)) - \int_0^t \langle \dot{\ell}(\tau), \varphi(\tau) \rangle \, \mathrm{d}\tau$.

At this point it is suitable to say that as far as the author knows the very special version of the energetic formulation was first used in the paper by G. A. Francfort and J.-J. Marigo about the Griffith model of crack propagation [FM93]. But instead of the natural and general condition **(E)** the authors used very special condition which is not fulfilled in most situations.

The following functionals $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{D}}$ provide us the first simple nontrivial application of the abstract theory. Let

$$\mathcal{E}(t, z) = \int_\Omega \tfrac{a(x)}{2} |\mathrm{D}_x z(x)|^2 - g_{\mathrm{ext}}(t,x) z(x) \, \mathrm{d}x \quad \text{on } \mathcal{Z} = \mathrm{H}^1_0(\Omega)$$

and $\widetilde{\mathcal{D}}(z_0, z_1) = \int_\Omega \kappa |z_1(x) - z_0(x)| \, \mathrm{d}x$ with $\kappa > 0$. Then, $\Delta(x, z, \dot{z}) = \kappa |\dot{z}|$ and (0.0.1) reduces to the partial differential inclusion

$$0 \in \kappa \, \mathrm{Sign}(\dot{z}(t,x)) - \mathrm{div}\Big(a(x) \mathrm{D}_x z(t,x)\Big) - g_{\mathrm{ext}}(t,x), \tag{0.0.2}$$

where Sign denotes the set-valued signum function. Our general theory using (S) & (E) will provide a generalised solution to this problem which satisfies $z \in \mathrm{BV}([0,T], \mathrm{L}^1(\Omega)) \cap \mathrm{L}^\infty([0,T], \mathrm{H}^1_0(\Omega))$ whenever $g_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathrm{H}^{-1}(\Omega))$, see Theorem 1.4.6. However, using the uniform convexity of $\mathcal{E}(t, \cdot)$ this result can be considerably improved; the theory in [MT04, Sect.7] provides uniqueness and $z \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathrm{H}^1_0(\Omega))$.

Under the assumptions that the sets $\mathcal{F}$ and $\mathcal{Z}$ are closed, convex subspaces of a suitable Banach space and that $\widetilde{\mathcal{D}}(z_0, z_1) = \mathbf{\Delta}(z_1 - z_0)$, an existence theory was developed in the above-mentioned work and certain refinements were added in [MR03, Efe03, KMR03].

In the first chapter of this thesis we consider an abstract framework, which was developed in [MM03] and which allows us to construct solutions to (S) & (E) without relying on any underlying linear structure in $\mathcal{Y} = \mathcal{F} \times \mathcal{Z}$. This abstract framework helps us to extend the previous existence results for the rate-independent problems, cf. [MT04], to the more general class of such systems. In particular, it was shown how the abstract theory lays the basis for the treatment of the delamination problem in [KMR03]. Moreover, it was shown in [MM03] that the model of brittle fracture introduced in [FM93] and developed further in [FM98, DMT02, Cha03, DMFT04] can be formulated as a special case of the abstract theory. It was shown that the conditions posed there are equivalent to conditions (S) & (E) which gives the theory a clearer mechanical interpretation. Furthermore, it seems that the abstract theory provides the opportunity to study genuinely nonlinear mechanical models such as elasto-plasticity with finite strains, see [OR99, CHM02, Mie02, Mie03a, LMD03, Mie03b].

In the second chapter a short overview of the theory of functions of bounded variation is provided. Such functions play an important role in several classical problems of the calculus of variation, for instance in the theory of graphs with minimal area. At present, this class of functions is heavily used to study problems, whose solutions develop discontinuities along hypersurfaces. Typical examples come from image recognition and fracture mechanics. Of course, the complete survey of the theory can not be provided in this thesis. It is also not the main topic of this thesis. Nevertheless this chapter provides all results which are needed in order to introduce and to study our phase transformation model. In this chapter we omit mostly all proofs. All missing proofs can be found in the modern book about the theory of function of bounded variation [AFP00] written by L. Ambrosio, N. Fusco and D. Pallara. The interested reader can find further details of the theory in the following books: [Fed69, Giu84, Maz85, VH85, Zie89].

In the last chapter we use the results of the second chapter for introducing of a simple model for phase transition. The modelling of phase transition processes plays an important role in the material science. Especially in the context of shape-memory alloys such modelling has been subjected to intensive theoretical and experimental research in the last years. It is surely related to the importance of smart materials in the aerospace and civil engineering. There exist yet some applications to human medicine. Such smart materials are characterised by an existence of different possible atomic grids (phases) and by a strong dependence of elastic properties on the actual structure of atomic grid. The grid with higher symmetry (mostly cubic) is referred as austenite phase while the lower-symmetrical grids (smart materials may have more than one lower-symmetrical grid) are called martensite phases. Under an external mechanical loading a smart material passes through an elastic deformation, but by attainment of a certain activation stress the phase transformation occurs. At this moment the energy, which is needed for the phase transformation, is partially dissipated to heat and partially stored in the new phase interface. Practical experiments show that the phase transformation processes can be considered, except very fast time scales, as rate-independent. This fact leads to the opportunity to treat the time evolution of phase transformation as a rate-independent process and to apply the abstract existence theory.

There exist some previous works, which try to apply the energetic approach to the modelling of phase transformation in solids. We can mention the papers by A. Mielke, F. Theil & V.I. Levitas [MTL02] and A. Mielke & T. Roubìček [MR03]. In these papers the authors consider a mesoscopic level model for phase transformation. Accordingly it is assumed that the phase state at every material point is given as a mixture of a pure crystallographic phases. The main aim of the mentioned papers was the modelling of microstructure evolution in shape-memory alloys. The research direction was strongly motivated by practical experiments, where the formation of very fine laminates was observed. In order to allow the formation of microstructure the energy stored in the phase interface was completely neglected.

In the model for phase transformation, which is presented in this thesis, we assume that the phase state at every material point is given by one pure crystallographic phase. It means that this model can be considered as a microscopic one. We assume also that one part of the stored energy is saved in the phase interfaces. This assumption is realised through an interface energy term of total stored energy. This term is introduced as an integral over the phase interface of some suitable interface density function. Surely, the interface energy term forbids the formation of microstructure, but at the same time this additional term allows us to model nucleation effects, which were also observed in experiments.

To be more specific, we provided a rough overview of ingredients of the model. Let, as in general setup of rate-independent processes, $\Omega \subset \mathbb{R}^d$ be the undeformed body and $t \in [0,T]$ the slow process time. The deformation is again denoted by φ and lies in the space $\mathcal{F}$ of admissible deformations. We denote the set of possible crystallographic phases by $Z \subset \mathbb{R}^m$. Thus the phase state can be prescribed by an internal variable $z(t) : \Omega \to Z \subset \mathbb{R}^m$. The stored energy is given via

$$\mathcal{E}(t,\varphi,z) = \int_\Omega W(x, \mathrm{D}\varphi(x), z(x))\,\mathrm{d}x + \int_{\text{phase interfaces}} \psi(z^+(a), z^-(a))\,\mathrm{d}a \;\; - \langle \ell(t), \varphi\rangle.$$

Here $\ell(t)$ denotes again the external loading depending on the process time t, the function ψ is a density of energy stored in the phase interfaces and $z^+(a)$, $z^-(a)$ denote the phase states on both sides of the phase interface. We also assume that the energy, which is dissipated by change from internal phase state z_1 to the internal state z_2 is given as $\int_\Omega D(z_1(x), z_2(x))\,\mathrm{d}x$. Furthermore we call this value the dissipation distance. Using the theory developed in the first chapter we are able to show the existence of solution for the evolution problem in the (S) & (E) formulation.

At the end of the introduction it is suitable to mention that the modelling of smart materials is a topic of many papers and the models, which are mentioned here, cover this area only very partially. The interested reader can find a good overview of this huge area in [Rou04]. In particular, the author considers also atomic and macroscopic models for evolution in shape-memory alloys.

Chapter 1

General existence theory for rate-independent systems

In this chapter we consider a rate-independent system whose evolution runs in evolution space $\mathcal{F} \times \mathcal{Z}$. In typical mechanical applications the set $\mathcal{F}$ is given as a set of admissible deformation and the set $\mathcal{Z}$ is a set of admissible internal states, i.e. phase state, magnetisation, etc. We do not require the linear structure on the evolution space. In fact, our application to memory-shape alloys in Chapter 3 forbids explicitly the linear structure in $\mathcal{Z}$. Like in the introduction $\mathcal{E} : [0, T] \times \mathcal{F} \times \mathcal{Z} \to \mathbb{R}$ and $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ denote the stored energy and the dissipation distance of the system. We present existence results in two different situations, which are called in the following "convex" and "non-convex" cases. We speak about the "convex" case if the functional $\mathcal{E}_{t,z} := \mathcal{E}(t, \cdot, z)$ has a unique minimiser for any (t, z), i.e. the deformation φ can be considered as a function of the internal state z. In this case the abstract framework introduced in [MM03] can be immediately applied in order to obtain the suitable existence results. For this we have to consider a pair (φ, z) as a single variable $y \in \mathcal{Y} := \mathcal{F} \times \mathcal{Z}$ and the dissipation distance as a function on $\mathcal{Y} \times \mathcal{Y}$. Then the convexity assumption allows us to verify one of central assumptions in [MM03], cf. condition (A4), to establish the existence of a solution for the evolution problem (S) & (E). Since the above method does not work in the "non-convex" case we avoid it in the "convex" case in favour of obtaining important a priori estimates, which coincide in both cases. The notation "convex" (resp. "non-convex") is motivated by the elasticity theory, where φ is a commonly accepted notation for elastic deformation. It is well-known that the convexity in the deformation gradient implies in the elasticity theory the uniqueness of the elastic equilibrium.

The existence proofs are based on the commonly used time-incremental approach which leads to the following minimisation problems.

(IP) *Given a pair* $(\varphi_0, z_0) \in \mathcal{F} \times \mathcal{Z}$ *and a partition of a time interval* $0 = t_0 < \ldots < t_N = T$ *find* $(\varphi_1, z_1), \ldots, (\varphi_N, z_N)$ *such that for any* k
$$\mathcal{E}(t_k, \varphi_k, z_k) = \inf \{ \mathcal{E}(t_k, \varphi, z) + \mathcal{D}(z_{k-1}, z) \mid (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \}.$$

We equip the spaces $\mathcal{F}$ and $\mathcal{Z}$ with Hausdorff topologies $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{Z}}$ such that the functions $\mathcal{E} : [0, T] \times \mathcal{F} \times \mathcal{Z} \to [\mathcal{E}_{\min}, \infty]$ and $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ are s-lower semicontinuous, where "s-" stands for "sequentially". Moreover, we assume that the reachable sets

$\mathcal{R}(t) = \{ (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \mid \mathcal{E}(t, \varphi, z) + \mathcal{D}(z_0, z) \le \mathcal{E}(t, \varphi_0, z_0) + C_{\mathcal{E}} t + 1 \}$ are s-compact. From this we can deduce existence of solutions for (IP).

Solutions to the time-continuous problem (S) & (E) are obtained as limits of incremental solutions for a sequence of nested partitions $P(n) = \{0{=}t_0^{(n)}{<}t_1^{(n)}{<}\cdots{<}t_{N(n)}^{(n)}{=}T\} \subset P(n{+}1)$ whose fineness $\phi(P(n)) = \max\left\{ t_j^{(n)} - t_{j-1}^{(n)} \mid j = 1, \ldots, N(n) \right\}$ tends to 0. Under the assumption

$$|\partial_t \mathcal{E}(t, \varphi, z)| \le C_{\mathcal{E}}$$

all these solutions (Φ^n, Z^n) satisfy the a priori bound

$$\mathrm{Diss}_{\mathcal{D}}(Z^n, [0,T]) = \textstyle\sum_{j=1}^{N(n)} \mathcal{D}(Z^n(t_{j-1}^{(n)}), Z^n(t_j^{(n)})) \le \mathcal{E}(0, \varphi_0, z_0) - \mathcal{E}_{\min} + C_{\mathcal{E}} T.$$

Here (Φ^n, Z^n) denotes the piecewise constant interpolant with $(\Phi^n, Z^n)(t) = (z_j^{(n)}, \varphi_j^{(n)})$ for $t \in [t_j^{(n)}, t_{j+1}^{(n)})$. Using the dissipation bound and a generalised, abstract version of Helly's selection principle (see Section 1.3) we extract a subsequence (Φ^{n_k}, Z^{n_k}) such that $Z^{n_k}(t) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z^\infty(t)$. For this we need an additional compatibility between the topology $\mathcal{T}_{\mathcal{Z}}$ and the dissipation distance $\mathcal{D}$, namely that $\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0$ implies $z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z$.

In the "convex" case the convergence in z-parts of incremental solutions leads immediately to the convergence in φ-parts. Hence, we obtain $\Phi^{n_k}(t) \xrightarrow{\mathcal{T}_{\mathcal{F}}} \Phi^\infty(t)$. Using s-continuity of $\partial_t \mathcal{E}(t, \cdot, \cdot)$ and assuming stability (S) for (Φ^∞, Z^∞) it is then straightforward to deduce the energy equality (E) for the limit (Φ^∞, Z^∞). In the "non-convex" case this argumentation does not work due to the impossibility to control the oscillation in the φ-part. In this situation we use the argument introduced first in [DMFT04] in order to select Φ^∞ pointwise in such a way that the energy inequality holds. Using the results of the set-valued analysis we show that this selection can be realised by a measurable function.

The major task is to show that (Φ^∞, Z^∞) satisfies (S). For this we use the set of stable states, shortly called the stable set:

$$\mathcal{S}_{[0,T]} = \cup_{t \in [0,T]} (t, \mathcal{S}(t))$$

with

$$\mathcal{S}(t) = \{ (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \mid \mathcal{E}(t, \varphi, z) \le \mathcal{E}(t, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z, \widetilde{z}) \text{ for all } (\widetilde{\varphi}, \widetilde{z}) \in \mathcal{F} \times \mathcal{Z} \}.$$

From the incremental problem we obtain $(t_k^{(n)}, \varphi_k^{(n)}, z_k^{(n)}) \in \mathcal{S}_{[0,T]}$. Hence s-closedness of the stable set is sufficient to conclude $(t, \Phi^\infty(t), Z^\infty(t)) \in \mathcal{S}_{[0,T]}$ for all t, which is exactly the condition (S).

In Theorem 1.4.3 we summarise the main existence result in the "convex" case and provide afterwards a typical application to the Banach space setting. In Section 1.5 we discuss abstract conditions on $\mathcal{E}$ and $\mathcal{D}$ which guarantee the s-closedness of $\mathcal{S}_{[0,T]}$. In Section 1.6 we prove the main existence result in the "non-convex" case.

1.1 Abstract setup of the problem

We start with topological Hausdorff spaces $(\mathcal{F}, \mathcal{T}_{\mathcal{F}})$, $(\mathcal{Z}, \mathcal{T}_{\mathcal{Z}})$. We will write $\varphi_k \xrightarrow{\mathcal{T}_{\mathcal{F}}} \varphi$ and $z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z$ to denote the convergence in these spaces respectively. In typical mechanical applications the space $(\mathcal{F}, \mathcal{T}_\mathcal{F})$ prescribes the set of admissible elastic deformation and it is given as the subspace of $\mathrm{W}^{1,p}$ equipped with weak $\mathrm{W}^{1,p}$-topology. The space $(\mathcal{Z}, \mathcal{T}_\mathcal{Z})$ can by considered as a space of internal variables which describes non-elastic parameters such as phase state, fracture, magnetisation. For our aims it will be sufficient to consider sequential closedness, compactness and continuity. We will indicate this fact by writing s-closedness, s-compactness and s-continuity.

The first ingredient of the energetic formulation is the *dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$ satisfying $\mathcal{D}(z, z) = 0$ and the triangle inequality:

$$\mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3) \quad \text{for all } z_1, z_2, z_3 \in \mathcal{Z}. \tag{A1}$$

We enforce neither strict positivity (i.e., $\mathcal{D}(z_1, z_2) = 0$ for $z_1 \ne z_2$ is allowed) nor symmetry (i.e., we allow for $\mathcal{D}(z_0, z_1) \ne \mathcal{D}(z_1, z_0)$ as is needed in Chapter 3). Other typical applications which require the non-symmetry of $\mathcal{D}$ are delamination problem and brittle fracture. We call $\mathcal{D}(z_0, z_1)$ the dissipation distance from z_0 to z_1.

One major point of the theory is the interplay between the topology $\mathcal{T}_\mathcal{Z}$ and the dissipation distance. To have a typical nontrivial application in mind, one may consider $\mathcal{Z} = \left\{ z \in \mathrm{L}^1(\Omega, \mathbb{R}^k) \mid \|z\|_{\mathrm{L}^\infty} \le 1 \right\}$ equipped with the weak L^1 topology and the dissipation distance $\mathcal{D}(y_1, y_2) = \|z_1 - z_2\|_{\mathrm{L}^1}$.

For a given curve $z : [0, T] \to \mathcal{Z}$ we define the total dissipation on $[s, t]$ via

$$\mathrm{Diss}_\mathcal{D}(z; [s, t]) = \sup \left\{ \textstyle\sum_1^N \mathcal{D}(z(\tau_{j-1}), z(\tau_j)) \mid N{\in}\mathbb{N}, s{=}\tau_0{<}\tau_1{<}\cdots{<}\tau_N{=}t \right\}. \tag{1.1.1}$$

Further we define the following set of functions:

$$\mathrm{BV}_\mathcal{D}([0, T], \mathcal{Z}) := \{ u : [0, T] \to \mathcal{Z} \mid \mathrm{Diss}_\mathcal{D}(u; [0, T]) < \infty \}.$$

Note that the functions are defined everywhere and changing it at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\mathrm{Diss}_\mathcal{D}(z; [r, t]) = \mathrm{Diss}_\mathcal{D}(z; [r, s]) + \mathrm{Diss}_\mathcal{D}(z; [s, t]) \quad \text{for all } r < s < t.$$

The second ingredient is the energy-storage functional $\mathcal{E} : [0, T] \times \mathcal{F} \times \mathcal{Z} \to [\mathcal{E}_{\min}, \infty]$, which is assumed to be bounded from below by a fixed constant $\mathcal{E}_{\min}$. Here $t \in [0, T]$ plays the rôle of a (very slow) process time which changes the underlying system via loading conditions. We assume that for all (φ, z) with $\mathcal{E}(0, \varphi, z) < \infty$ the function $t \mapsto \mathcal{E}(t, \varphi, z)$ is Lipschitz continuous, i.e.,

$$\partial_t\mathcal{E}(\cdot, \varphi, z) : [0, T] \to \mathbb{R} \text{ is measurable and } |\partial_t\mathcal{E}(t, \varphi, z)| \le C_\mathcal{E}. \tag{A2}$$

Condition (A2) can be also replaced with weaker conditions, which are sufficient for proving suitable existence results. Since the above condition is satisfied for a suitable large class of external loadings, we avoid the weak formulations and use the strong condition (A2) instead. Possible weaker assumptions can be found in [FM04].

Definition 1.1.1. A curve $(\varphi, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ is called a **solution** of the rate-independent model $(\mathcal{D}, \mathcal{E})$, if **global stability (S)** and **energy equality (E)** hold:

> ***(S)*** *For all $t \in [0, T]$ and all $(\widehat{\varphi}, \widehat{z}) \in \mathcal{F} \times \mathcal{Z}$ we have*
> $$\mathcal{E}(t, \varphi(t), z(t)) \leq \mathcal{E}(t, \widehat{\varphi}, \widehat{z}) + \mathcal{D}(z(t), \widehat{z}).$$
> ***(E)*** *For all $t \in [0, T]$ we have*
> $$\mathcal{E}(t, \varphi(t), z(t)) + \mathrm{Diss}_{\mathcal{D}}(z; [0, t]) = \mathcal{E}(0, \varphi(0), z(0)) + \int_0^t \partial_t \mathcal{E}(\tau, \varphi(\tau), z(\tau))\, \mathrm{d}\tau.$$

The definition of solutions of (S) & (E) is such that it implies the two natural requirements for evolutionary problems, namely that *restrictions* and *concatenations* of solutions remain solutions. To be more precise, for any solution $(\varphi, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ and any subinterval $[s, t] \subset [0, T]$, the restriction $(\varphi, z)|_{[s,t]}$ solves (S) & (E) with initial datum $(\varphi(s), z(s))$. Moreover, if $(\varphi_1, z_1) : [0, t] \to \mathcal{F} \times \mathcal{Z}$ and $(\varphi_2, z_2) : [t, T] \to \mathcal{F} \times \mathcal{Z}$ solve (S) & (E) on the respective intervals and if $(\varphi_1, z_1)(t) = (\varphi_2, z_2)(t)$, then the concatenation $(\varphi, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ solves (S) & (E) on the interval $[0, T]$ as well.

In the previous works on rate-independent systems condition (E) was often substituted by the following weakened version (see [MT04, MTL02]):

> ***(E)$_{\mathbf{weak}}$*** *For all $t \in [0, T]$ we have*
> $$\mathcal{E}(t, \varphi(t), z(t)) + \mathrm{Diss}_{\mathcal{D}}(z; [0, t]) \leq \mathcal{E}(0, \varphi(0), z(0)) + \int_0^t \partial_t \mathcal{E}(s, \varphi(s), z(s))\, \mathrm{d}s.$$

Condition (E)$_{\text{weak}}$ enables *concatenations* of solutions, but doesn't guarantee that *restrictions* remain solutions. Although condition (E)$_{\text{weak}}$ seems to be weaker than the condition (E), it is shown in [MT04] that (S) and (E)$_{\text{weak}}$ together imply that (E) holds (see Proposition 1.4.2 for a simple proof).

Rate-independence manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that a pair (φ, z) is a solution for $(\mathcal{D}, \mathcal{E})$ if and only if the reparametrised curve $(\widetilde{\varphi}, \widehat{z}) : t \mapsto (\varphi, z)(\alpha(t))$, where $\dot{\alpha} > 0$, is a solution for $(\mathcal{D}, \widetilde{\mathcal{E}})$ with $\widetilde{\mathcal{E}}(t, \varphi, z) = \mathcal{E}(\alpha(t), \varphi, z)$. In particular, the stability (S) is a static concept and the energy balance (E) is rate-independent, since the dissipation defined via (1.1.1) is scale invariant like the length of a curve.

The major importance of the energetic formulation is that neither the given functionals $\mathcal{D}$ and $\mathcal{E}(t, \cdot)$ nor the solutions $(\varphi, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ need to be differentiable. In particular, applications to continuum mechanics often have low smoothness. Of course, under additional smoothness assumptions on $\mathcal{D}$ and $\mathcal{E}$ the weak energetic form (S) & (E) can be replaced by local formulations in the form of differential inclusions like (0.0.1) ([CV90, Vis01]) or in the form of variational inequalities. See [MT04] for a discussion of the implications between these different formulations.

1.2 Incremental solutions and a priori bounds

Existence of solutions of (S) & (E) is shown via time-incremental minimisation problems. In this section we give the incremental formulation of (S) & (E) problem and show that under suitable assumptions this incremental formulation has a solution. Moreover we prove a priori bound estimates for incremental problem. These estimates help us later to select a subsequence of incremental solutions which converges to a solution of the original (S) & (E) problem.

Furthermore we assume that the functionals $\mathcal{E}(t,\cdot,\cdot) : \mathcal{F} \times \mathcal{Z} \to [\mathcal{E}_{\min}, \infty]$ and $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$ are s-lower semicontinuous. In the standard case $\mathcal{F}$ and $\mathcal{Z}$ are reflexive Banach spaces (like the Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)$ with $p \in]1,\infty[$) equipped with weak topology $\mathfrak{T}$ such that closed and bounded sets are compact. Then, s-lower semicontinuity of $\mathcal{E}$ and $\mathcal{D}$ in $(\mathcal{F},\mathfrak{T}_{\mathcal{F}}) \times (\mathcal{Z},\mathfrak{T}_{\mathcal{Z}})$ and $(\mathcal{Z},\mathfrak{T}_{\mathcal{Z}})$ respectively is the same as the classical weak sequential lower semicontinuity in the calculus of variations [Dac89].

We reformulate the stability condition (S) by defining the stable sets

$$\begin{aligned}
&\mathcal{S}(t) := \{\, (\varphi,z) \in \mathcal{F} \times \mathcal{Z} \mid \mathcal{E}(t,\varphi,z) \le \mathcal{E}(t,\widehat{\varphi},\widehat{z}) + \mathcal{D}(z,\widehat{z}) \text{ for all } (\widehat{\varphi},\widehat{z}) \in \mathcal{F} \times \mathcal{Z} \,\}, \\
&\mathcal{S}_{[0,T]} := \{\, (t,\varphi,z) \in [0,T] \times \mathcal{F} \times \mathcal{Z} \mid (\varphi,z) \in \mathcal{S}(t) \,\} = \cup_{t\in[0,T]}(t,\mathcal{S}(t)).
\end{aligned}$$

In this notation condition (S) means, that $(\varphi(t),z(t))$ lies in $\mathcal{S}(t)$ for every $t \in [0,T]$. The properties of the stable sets turn out to be crucial for deriving existence results.

For the time discretisation we choose a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $[0,T]$ and seek for a (φ_k,z_k) which approximates the solution (φ,z) of (S) & (E) at t_k, i.e., $(\varphi_k,z_k) \approx (\varphi,z)(t_k)$. Our energetic approach has the major advantage that the pairs (φ_k,z_k) can be found incrementally via minimisation problems. Since the methods of the calculus of variations are especially suited for applications in material modelling this will allow for a rich field of applications.

In our general setting the incremental problem associated with a given partition of time interval takes the following form:

(IP) *For $(\varphi_0,z_0) \in \mathcal{S}(0) \subset \mathcal{F} \times \mathcal{Z}$ find $(\varphi_1,z_1),\ldots,(\varphi_N,z_N) \in \mathcal{F} \times \mathcal{Z}$ such that*

$$(\varphi_k,z_k) \in \arg\min \{\, \mathcal{E}(t_k,\varphi,z) + \mathcal{D}(z_{k-1},z) \mid (\varphi,z) \in \mathcal{F} \times \mathcal{Z} \,\} \qquad \textit{for } k = 1,\ldots,N. \tag{1.2.1}$$

Here "arg min" denotes the set of all minimisers. The following result shows that (IP) is intrinsically linked to (S) & (E). Without any smallness assumptions on the time steps, the solutions of (IP) satisfy properties which are closely related to (S) & (E).

Theorem 1.2.1. *Let (A1) and (A2) hold. Any solution of the incremental problem (1.2.1) satisfies the following properties:*

(i) (φ_k,z_k) *is stable for time* t_k*, i.e.,* $(\varphi_k,z_k) \in \mathcal{S}(t_k)$*;*

(ii) $\int_{[t_{k-1},t_k]} \partial_s\mathcal{E}(s,\varphi_k,z_k)\,\mathrm{d}s \le \mathcal{E}(t_k,\varphi_k,z_k) - \mathcal{E}(t_{k-1},\varphi_{k-1},z_{k-1}) + \mathcal{D}(z_{k-1},z_k)$
$\le \int_{[t_{k-1},t_k]} \partial_s\mathcal{E}(s,\varphi_{k-1},z_{k-1})\,\mathrm{d}s$ *for* $k = 1,\ldots,N$*;*

(iii) $\mathcal{E}(t_j,\varphi_j,z_j) + \sum_{k=1}^{j} \mathcal{D}(z_{k-1},z_k) \le \mathcal{E}(0,\varphi_0,z_0) + C_{\mathcal{E}}t_j$*.*

Proof. (i) The stability follows from minimisation properties of the solutions and the triangle inequality. For all $(\widehat{\varphi},\widehat{z}) \in \mathcal{F} \times \mathcal{Z}$ we have

$$\begin{aligned}
\mathcal{E}(t_k,\widehat{\varphi},\widehat{z}) + \mathcal{D}(z_k,\widehat{z}) &= \mathcal{E}(t_k,\widehat{\varphi},\widehat{z}) + \mathcal{D}(z_{k-1},\widehat{z}) + \mathcal{D}(z_k,\widehat{z}) - \mathcal{D}(z_{k-1},\widehat{z}) \\
&\ge \mathcal{E}(t_k,\varphi_k,z_k) + \mathcal{D}(z_{k-1},z_k) + \mathcal{D}(z_k,\widehat{z}) - \mathcal{D}(z_{k-1},\widehat{z}) \ge \mathcal{E}(t_k,\varphi_k,z_k).
\end{aligned}$$

(ii) The first estimate is deduced from $(\varphi_{k-1}, z_{k-1}) \in \mathcal{S}(t_{k-1})$ as follows:

$$\begin{aligned}
&\mathcal{E}(t_k, \varphi_k, z_k) + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, \varphi_{k-1}, z_{k-1}) \\
&\quad = \quad \mathcal{E}(t_{k-1}, \varphi_k, z_k) + \textstyle\int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, \varphi_k, z_k)\,\mathrm{d}s + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, \varphi_{k-1}, z_{k-1}) \\
&\quad \geq \quad \textstyle\int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, \varphi_k, z_k)\,\mathrm{d}s.
\end{aligned}$$

Since $(\varphi_k, z_k) \in \arg\min \{ \mathcal{E}(t_k, \varphi, z) + \mathcal{D}(z_{k-1}, z) \mid (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \}$ the second estimate follows via

$$\begin{aligned}
&\mathcal{E}(t_k, \varphi_k, z_k) - \mathcal{E}(t_{k-1}, \varphi_{k-1}, z_{k-1}) + \mathcal{D}(z_{k-1}, z_k) \leq \\
&\mathcal{E}(t_k, \varphi_{k-1}, z_{k-1}) - \mathcal{E}(t_{k-1}, \varphi_{k-1}, z_{k-1}) + \mathcal{D}(z_{k-1}, z_{k-1}) = \textstyle\int_{[t_{k-1},t_k]} \partial_s \mathcal{E}(s, \varphi_{k-1}, z_{k-1})\,\mathrm{d}s.
\end{aligned} \tag{1.2.2}$$

(iii) This estimate is obtained by adding up the second estimate in (ii) for $k = 1, \ldots, j$. ∎

For each incremental solution $(\varphi_k, z_k)_{k=1,\ldots,N}$ of (IP) we define two piecewise constant functions with values in $\mathcal{F} \times \mathcal{Z}$ which take the values (φ_k, z_k) at t_k and are constant in-between: (Φ^P, Z^P) is continuous from the right and $(\widehat{\Phi}^P, \widehat{Z}^P)$ is continuous from the left. Summing up the estimates (ii) in Theorem 1.2.1 over $k = j, \ldots, m$ we find the following two-sided energy estimate:

Corollary 1.2.2. *Let (A1) and (A2) hold and let P be any partition of $[0, T]$. Then, any solution $(y_k)_{k=0,\ldots,N}$ of (IP) satisfies, for $0 \leq j < m \leq N$, the two-sided energy inequality*

$$\begin{aligned}
&\mathcal{E}(t_j, \Phi^P(t_j), Z^P(t_j)) + \textstyle\int_{t_j}^{t_m} \partial_s \mathcal{E}(s, \widehat{\Phi}^P(s), \widehat{Z}^P(s))\,\mathrm{d}s \\
&\quad \leq \quad \mathcal{E}(t_m, \Phi^P(t_m), Z^P(t_m)) + \mathrm{Diss}_{\mathcal{D}}(Z^P, [t_j, t_m]) \\
&\quad \leq \quad \mathcal{E}(t_j, \Phi^P(t_j), Y^P(t_j)) + \textstyle\int_{t_j}^{t_m} \partial_s \mathcal{E}(s, \Phi^P(s), Z^P(s))\,\mathrm{d}s.
\end{aligned}$$

So far, we have not yet proved the existence of solutions to (IP). However, the above theorem already indicates that we can use induction arguments to provide compactness and hence existence results. We define first the reachable sets

$$\begin{aligned}
&\mathcal{R}_{[0,T]} \quad := \{ (t, \varphi, z) \in [0, T] \times \mathcal{F} \times \mathcal{Z} \mid \mathcal{E}(t, \varphi, z) + \mathcal{D}(z_0, z) \leq \mathcal{E}(0, \varphi_0, z_0) + C_{\mathcal{E}} t + 1 \} \\
&\text{and } \mathcal{R}(t) := \left\{ (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \mid (t, \varphi, z) \in \mathcal{R}_{[0,T]} \right\}.
\end{aligned} \tag{1.2.3}$$

With (A2) we conclude $\mathcal{R}(s) \leq \mathcal{R}(t)$ for $s < t$. As a consequence we have $\mathcal{R}_{[0,T]} \subset [0, T] \times \mathcal{R}(T)$. The following two assumptions will ensure the existence of a solution to (IP).

The set $\mathcal{R}(T)$ is s-compact in $(\mathcal{F}, \mathcal{T}_{\mathcal{F}}) \times (\mathcal{Z}, \mathcal{T}_{\mathcal{Z}})$. (A3)

For all $t \in [0, T]$ and all $(\widehat{\varphi}, \widehat{z}) \in \mathcal{R}(T)$ the mapping $(\varphi, z) \mapsto \mathcal{E}(t, \varphi, z) + \mathcal{D}(\widehat{z}, z)$ is s-lower semi-continuous on $\mathcal{R}(t) \subset \mathcal{F} \times \mathcal{Z}$. (A4)

Using (A4) it is easy to see that $\mathcal{R}_{[0,T]}$ is a closed subset of $[0, T] \times \mathcal{F} \times \mathcal{Z}$. Hence, together with (A3) we conclude that each $\mathcal{R}(t)$ and $\mathcal{R}_{[0,T]}$ are s-compact.

Theorem 1.2.3. *Let (A1), (A2), (A3), and (A4) hold. Then, (IP) has a solution.*

Proof. The proof works by induction over $k = 1, \ldots, N$, since y_0 is given.

In step k the value (φ_{k-1}, z_{k-1}) is given and we have to find

$$(\varphi_k, z_k) \in \arg\min\{\mathcal{E}(t_k, y) + \mathcal{D}(z_{k-1}, y)\}.$$

Since (φ_{k-1}, z_{k-1}) was the minimiser in the previous step we have $(\varphi_{k-1}, z_{k-1}) \in \mathcal{R}(t_{k-1})$. In fact, by Theorem 1.2.1 (iii) we have

$$\mathcal{E}(t_{k-1}, \varphi_{k-1}, z_{k-1}) + \mathcal{D}(y_0, z_{k-1}) \leq \mathcal{E}(0, y_0) + C_{\mathcal{E}} t_{k-1}.$$

Let $(\varphi^l, z^l)_{l\in\mathbb{N}}$ be an infimising sequence for $\mathcal{E}(t_k, \cdot, \cdot) + \mathcal{D}(z_{k-1}, \cdot)$ with

$$\mathcal{E}(t_k, \varphi^l, z^l) + \mathcal{D}(z_{k-1}, z^l) \leq \inf_{(\varphi,z)\in\mathcal{F}\times\mathcal{Z}} \{\mathcal{E}(t_k, \varphi, z) + \mathcal{D}(z_{k-1}, z)\} + 1$$

for all $l \in \mathbb{N}$. Following estimate (1.2.2) in Theorem 1.2.1 we obtain $(\varphi^l, z^l) \in \mathcal{R}(t_k)$ for all $l \in \mathbb{N}$. Using (A3) and (A4) we conclude the existence of $(\varphi^*, z^*) \in \mathcal{R}(t_k)$ and a subsequence (φ^{l_m}, z^{l_m}) with $\varphi^{l_m} \xrightarrow{\mathcal{T}_{\mathcal{F}}} \varphi^*$ and $z^{l_m} \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z^*$. Moreover,

$$\begin{aligned}
\mathcal{E}(t_k, \varphi^*, z^*) + \mathcal{D}(z_0, z^*) &\leq \liminf_{m\to\infty} \mathcal{E}(t_k, \varphi^{l_m}, z^{l_m}) + \mathcal{D}(z_{k-1}, z^{l_m}) \\
&= \inf_{(\varphi,z)\in\mathcal{F}\times\mathcal{Z}} \{\mathcal{E}(t_k, \varphi, z) + \mathcal{D}(z_{k-1}, z)\}.
\end{aligned}$$

Hence, we let $(\varphi_k, z_k) = (\varphi^*, z^*)$ and the induction step is completed. ∎

In order to proof the stability of one point $(\varphi(t), z(t))$ at time t we need to test with an arbitrary pair (φ, z). If conditions (A3) and (A4) hold on the whole set $\mathcal{F} \times \mathcal{Z}$, we can reduce the test set using the following result.

Theorem 1.2.4. *Assume that conditions (A3) and (A4) hold on the whole set $\mathcal{F} \times \mathcal{Z}$. Then the following holds:*

> $(\varphi^*, z^*) \notin \mathcal{S}(t)$ *if and only if there exists* $(\varphi_s, z_s) \in \mathcal{S}(t)$ *such that*
> $\mathcal{E}(t, \varphi^*, z^*) > \mathcal{E}(t, \varphi_s, z_s) + \mathcal{D}(z^*, z_s)$.

Proof. Let $(\varphi^*, z^*) \notin \mathcal{S}(t)$. Using (A3), (A4) and the classical result of direct methods we conclude that the set $M := \arg\min\{\mathcal{E}(t, \varphi, z) + \mathcal{D}(z^*, z) \mid (\varphi, z) \in \mathcal{F} \times \mathcal{Z}\}$ is not empty. For $(\varphi_s, z_s) \in M$ and any pair $(\widehat{\varphi}, \widehat{z})$ we have:

$$\begin{aligned}
\mathcal{E}(t, \widehat{\varphi}, \widehat{z}) + \mathcal{D}(z_s, \widehat{z}) &= \mathcal{E}(t, \widehat{\varphi}, \widehat{z}) + \mathcal{D}(z^*, \widehat{z}) + \mathcal{D}(z_s, \widehat{z}) - \mathcal{D}(z^*, \widehat{z}) \\
&\geq \mathcal{E}(t, \varphi_s, z_s) + \mathcal{D}(z^*, z_s) + \mathcal{D}(z_s, \widehat{z}) - \mathcal{D}(z^*, \widehat{z}) \geq \mathcal{E}(t, \varphi_s, z_s).
\end{aligned}$$

Hence, one direction is proved. The other direction follows immediately from the definition of stable sets. ∎

Remark 1.2.5. Assume that $\mathcal{F}$ and $\mathcal{Z}$ are subsets of some Banach spaces and all stable sets $S(t)$ lie in some set $B_{\mathcal{F}} \times B_{\mathcal{Z}}$. We can consider the reduced evolution problem by replacing $\mathcal{F}$ with $\mathcal{F} \cap B_{\mathcal{F}}$ and $\mathcal{Z}$ with $\mathcal{Z} \cap B_{\mathcal{Z}}$. Theorem 1.2.4 implies that any solution of the reduced problem is also a solution of the original non-reduced problem.

1.3 Selection result in the spirit of Helly's selection principle

In this section we present one possible generalisation of Helly's selection principles, which allows us to develop the existence theory for rate-independent systems in abstract Hausdorff-spaces. More precisely, we apply the results of this section to incremental approximations (Φ_n, Z_n), whose z-parts satisfy a priori estimates of bounded variation in time. Theorem 1.3.3 implies then that we can select a subsequence $(\Phi_{n_k}, Z_{n_k})_{k\in\mathbb{N}}$ such that the z-parts converge pointwise to some function Z^∞. Theorem 1.3.3 and Theorem 1.3.4 was first formulated and proved in [MM03].

Before we formulate and prove the main result in this section, we recall the Helly's selection principle and give an extension of this principle to the case of Banach-space valued functions.

The classical selection theorem of Helly states that a bounded sequence of monotone functions on the real line always has a subsequence which converges pointwise everywhere.

Theorem 1.3.1. ***(Helly's selection principle)*** *Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval $[a,b]$. We assume that there exist positive numbers C and M such that for any $n \in \mathbb{N}$ holds:*

$$\mathrm{V}_a^b(Z_n) \le C, \quad \sup_{x\in[a,b]} |Z_n(x)| \le M, \tag{1.3.1}$$

where $\mathrm{V}_a^b(Z_n)$ is the total variation of Z_n on $[a,b]$. Then the sequence $(Z_n)_{n\in\mathbb{N}}$ contains a subsequence $(Z_{n_k})_{k\in\mathbb{N}}$ which converges for every $x \in [a,b]$ to some function Z. Moreover, the limit function Z satisfies also condition (1.3.1)

The proof of this theorem can be found in many analysis books (see [KF75], Ch. 10, Sect. 36, Thms 4 and 5).

In the book [BP86] the following generalisation of Helly's selection principle to the case of reflexive Banach spaces was presented.

Theorem 1.3.2. *Let $(X, \|\cdot\|)$ be a separable, reflexive Banach space with separable dual X^*. Assume that for the sequence of functions $Z_n : [0,T] \to X$ there exists a constant C such that*

$$\|Z_n\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\|\cdot\|}(Z_n; [0,T]) \le C.$$

Then, there exist a function $Z \in \mathrm{BV}_{\|\cdot\|}([0,T], X)$ and a subsequence $(Z_{n_k})_{k\in\mathbb{N}}$ such that for all $t \in [0,T]$ we have $Z_{n_k}(t) \rightharpoonup Z(t)$ in X (weak convergence).

This result was used in [MT04] to obtain the existence theory for rate-independent systems in the case of reflexive Banach spaces. The proof of the above theorem can be found in [BP86] (Ch. 1, Thm. 3.5) and is heavily based on the linear structure of Banach spaces and reflexivity. Since we are interested in the general existence results for rate-independent systems, we have to avoid the settings of vector spaces. We give now the extension of Helly's selection principle, which works in the case of Hausdorff-space $(\mathcal{Z}, \mathcal{T}_\mathcal{Z})$ and which is adequate for our theory.

Theorem 1.3.3. *Let $\mathcal{Z}_{[0,T]} = \cup_{[0,T]}(t, \mathcal{Z}(t))$ be a s-compact subset of $[0,T] \times \mathcal{Z}$ and $\mathcal{V}^\mathcal{Z}_{[0,T]} = \cup_{[0,T]}(t, \mathcal{V}^\mathcal{Z}(t)) \subset \mathcal{Z}_{[0,T]}$. Assume that $\mathcal{D}$, $\mathcal{T}_\mathcal{Z}$, $\mathcal{Z}_{[0,T]}$, and $\mathcal{V}^\mathcal{Z}_{[0,T]}$ satisfy the following two compatibility conditions:*

(V1) For all $t_1, t_2 \in [0,T]$ the functional $\mathcal{D}(\cdot,\cdot) : \mathcal{Z}(t_1) \times \mathcal{Z}(t_2) \to [0,\infty]$ is s-lower semi-continuous.

(V2) If $(t_k, z_k) \in \mathcal{V}^{\mathcal{Z}}_{[0,T]}$ with $t_k \to t$ and $\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0$, then $z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z$.

Consider a sequence of functions $Z_n : [0,T] \to \mathcal{Z}$ such that there exists a constant $C > 0$ such that $\mathrm{Diss}_{\mathcal{D}}(Z_n; [0,T]) \le C$ for all $n \in \mathbb{N}$. Moreover, for all $t \in [0,T]$ we have

$$Z_n(t) \in \mathcal{Z}(t) \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \mathrm{acc}_{\mathcal{T}_{\mathcal{Z}}}(Z_k(t))_{k\in\mathbb{N}} \subset \mathcal{V}^{\mathcal{Z}}(t),$$

where $\mathrm{acc}_{\mathcal{T}_{\mathcal{Z}}}(z_k)_{k\in\mathbb{N}}$ denotes the set of all possible accumulation points, i.e., $\mathcal{T}_{\mathcal{Z}}$-limits of subsequences.

Then, there exist a subsequence $(Z_{n_k})_{k\in\mathbb{N}}$ and functions $V_\infty \in \mathrm{BV}([0,T],\mathbb{R})$, $Z^\infty \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z})$ such that the following holds:

(a) $V_{n_k}(t) := \mathrm{Diss}_{\mathcal{D}}(Z_{n_k}, [0,t]) \to V_\infty(t)$ for all $t \in [0,T]$,

(b) $Z_{n_k}(t) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z^\infty(t) \in \mathcal{V}^{\mathcal{Z}}(t)$ for all $t \in [0,T]$,

(c) $\mathrm{Diss}_{\mathcal{D}}(Z^\infty, [t_0, t_1]) \le V_\infty(t_1) - V_\infty(t_0)$ for all $0 \le t_0 < t_1 \le T$.

Proof. The functions $\mathrm{V}_n : [0,T] \to [0,C];\ t \mapsto \mathrm{Diss}_{\mathcal{D}}(Z_n, [0,t])$ are nondecreasing. The scalar version of Helly's selection principle guarantees the existence of a function $\mathrm{V}_\infty : [0,T] \to [0,C]$ and of a subsequence $(n_l)_{l\in\mathbb{N}}$ with $\mathrm{V}_{n_l}(t) \to \mathrm{V}_\infty(t)$ for all $t \in [0,T]$. Thus, we have proved (a).

Since V_∞ is monotone and bounded, the set J of all its discontinuity points is at most countable. We choose a countable set M with the following properties:

$$J \subset M, \qquad M \text{ is dense in } [0,T], \qquad 0 \in M.$$

Using $Z_n(t) \in \mathcal{Z}(t)$ and the s-compactness of $\mathcal{Z}(t)$ we select, by the aid of Cantor's diagonal process, a subsequence (n_k) of the sequence (n_l) such that $Z_{n_k}(t)$ converges in $(\mathcal{Z}, \mathcal{T}_{\mathcal{Z}})$ for all $t \in M$. The limit of the sequence $(Z_{n_k}(t))$ is denoted by $Z^\infty(t)$, such that $Z^\infty : M \to \mathcal{Z}$ is defined.

We now show that this subsequence also converges for $t \in [0,T]\backslash M$, which provides the extension of Z^∞ to the whole interval. Fix an arbitrary $t \in [0,T]\backslash M$. The s-compactness of $\mathcal{Z}(t)$ guarantees an accumulation point $Z^\infty(t) \in \mathcal{V}^{\mathcal{Z}}(t)$, i.e., $Z_{\widetilde{n}_m}(t) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z^\infty(t)$ for a subsequence $(Z_{\widetilde{n}_m})$ of (Z_{n_k}). It remains to show that this accumulation point is unique. For this we use (V1) and (V2).

Take any sequence $(t_i)_{i\in\mathbb{N}}$ such that $t_i \in M$ and $t_i \to t$. Then, if $t_i < t$, (V1) implies

$$\begin{aligned} \mathcal{D}(Z^\infty(t_i), Z^\infty(t)) &\le \liminf_{m\to\infty} \mathcal{D}(Z_{\widetilde{n}_m}(t_i), Z_{\widetilde{n}_m}(t)) \le \liminf_{m\to\infty} \mathrm{Diss}(Z_{\widetilde{n}_m}; [t_i, t]) \\ &= \liminf_{m\to\infty} \left[\mathrm{V}_{\widetilde{n}_m}(t) - \mathrm{V}_{\widetilde{n}_m}(t_i)\right] = \mathrm{V}_\infty(t) - \mathrm{V}_\infty(t_i). \end{aligned}$$

Similarly, if $t < t_i$ we obtain $\mathcal{D}(Z^\infty(t), Z^\infty(t_i)) \le \mathrm{V}_\infty(t_i) - \mathrm{V}_\infty(t)$ and together with the continuity of V_∞ at the time t we conclude

$$\min\{\mathcal{D}(Z^\infty(t_i), Z^\infty(t)), \mathcal{D}(Z^\infty(t), Z^\infty(t_i))\} \le |\mathrm{V}_\infty(t) - \mathrm{V}_\infty(t_i)| \to 0 \text{ for } i \to \infty.$$

Now we employ (V2) which implies $Z^\infty(t_i) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z^\infty(t)$. Since $(\mathcal{Z}, \mathcal{T})$ is a Hausdorff space, the limit of a converging sequence is unique, and we conclude that $(Z_{n_k})_{k\in\mathbb{N}}$ has exactly one accumulation point. Thus, we have proved (b).

For assertion (c) we consider any discretisation $t_0 = \theta_0 < \theta_1 < \ldots < \theta_N = t_1$ of the segment $[t_0, t_1]$. Using $Z_{n_k}(\theta_i) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z^\infty(\theta_i)$ for $i = 0, 1, \ldots, N$ and (V1) we obtain

$$\begin{aligned}&\textstyle\sum_{j=1}^N \mathcal{D}(Z^\infty(\theta_{j-1}), Z^\infty(\theta_j)) \leq \liminf_{k\to\infty} \sum_{j=1}^N \mathcal{D}(Z_{n_k}(\theta_{j-1}), Z_{n_k}(\theta_j))\\ &\leq \liminf_{m\to\infty} \mathrm{Diss}(Z_{n_k}; [t_0, t_1]) = \liminf_{m\to\infty} \big[\mathrm{V}_{n_k}(t_1) - \mathrm{V}_{n_k}(t_0)\big] = \mathrm{V}_\infty(t_1) - \mathrm{V}_\infty(t_0).\end{aligned}$$

Taking the supremum on the left-hand side gives the desired estimate (c). ∎

We now collect a few results on functions $Z \in \mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Z})$ which will be useful later on.

Theorem 1.3.4. *Let $\mathcal{V}^{\mathcal{Z}}_{[0,T]}$ be a s-compact subset of $[0,T] \times \mathcal{Z}$ and assume that $\mathcal{D}$ satisfies (V1) and (V2). Furthermore, assume that $Z \in \mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Z})$ satisfies $(t, Z(t)) \in \mathcal{V}^{\mathcal{Z}}_{[0,T]}$ for all $t \in [0,T]$.*

(a) Then, $t \mapsto Z(t)$ is s-continuous (w.r.t. $\mathcal{T}_{\mathcal{Z}}$) at all continuity points of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Z, [0,t])$.

(b) For all $t \in [0,T]$ the $\mathcal{T}$-limits from the right $Z_+(t) = \lim_{\tau\searrow t} Z(\tau)$ and from the left $Z_-(t) = \lim_{\tau\nearrow t} Z(\tau)$ are well defined. Moreover, $\lim_{\tau\nearrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [\tau, t]) = \mathcal{D}(Z_-(t), Z(t))$ and $\lim_{\tau\searrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [t, \tau]) = \mathcal{D}(Z(t), Z_+(t))$.

(c) If $P(n) = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots t_{N_n-1}^{(n)} < t_{N_n}^{(n)} = T\}$ defines a sequence of partitions of the interval $[0,T]$ such that the fineness $\phi(P(n)) = \max_{1\leq k\leq N_n} t_k^{(n)} - t_{k-1}^{(n)}$ tends to 0, then the piecewise constant interpolants $Z^{(n)}$ with $Z^{(n)}(t) = Z(t_k^{(n)})$ for $t \in (t_{k-1}^{(n)}, t_k^{(n)}]$ lie in $\mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Z})$, and for almost all $t \in [0,T]$ we have $Z^{(n)}(t) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z(t)$. In fact, the convergence holds for all t except in the (at most countable) set of jump points of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Z, [0,t])$.

Proof. Let t be a continuity point of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Z, [0,t])$. Then

$$\lim_{\tau\nearrow t} \mathcal{D}(Z(\tau), Z(t)) \leq \lim_{\tau\nearrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [\tau, t]) = \mathrm{Diss}_{\mathcal{D}}(Z, [0,t]) - \lim_{\tau\nearrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [0,\tau]) = 0.$$

Using (V2) we obtain $\lim_{\tau\nearrow t} Z(\tau) = Z(t)$. Similarly we can show that $\lim_{\tau\searrow t} Z(\tau) = Z(t)$. Thus, we have proven (a).

Fix an arbitrary t in $[0,T]$ and consider a monotone increasing sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow t$. All points $(t_n, Z(t_n))$ lie in the set $\mathcal{V}^{\mathcal{Z}}_{[0,T]}$ and therefore we can select a subsequence $(n_k)_{k\in\mathbb{N}}$ with $Z(t_{n_k}) \to z_*$. We let $Z_-(t) := z_*$ and have to show that $Z_-(t) = \lim_{\tau\nearrow t} Z(\tau)$: The function $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Z, [0,t])$ is nondecreasing and bounded. This fact implies the existence of the limit $v_-(t) := \lim_{s\nearrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [0,s])$. Let $\varepsilon > 0$ be arbitrary. There exists a number $s_0 < t$ such that for all $s \in (s_0, t)$ holds

$$v_-(t) - \varepsilon \leq \mathrm{Diss}_{\mathcal{D}}(Z, [0,s]) \leq v_-(t)$$

and therefore $\mathcal{D}(Z(s), Z(s_1)) \leq \varepsilon$ for all s, s_1 with $s_0 < s < s_1 < t$. Using (V1) and $t_{n_k} \to t$ we obtain

$$\mathcal{D}(Z(s), Z_-(t)) \leq \liminf_{k\to\infty} \mathcal{D}(Z(s), Z(t_{n_k})) \leq \varepsilon \tag{1.3.2}$$

which in conjunction with (V2) implies that $Z_-(t) = \lim_{\tau\nearrow t} Z(\tau)$.

On the one hand, the estimate $\mathcal{D}(Z(s), Z(t)) \leq \mathrm{Diss}_{\mathcal{D}}(Z, [s,t])$ and the additivity of the dissipation give $\mathrm{Diss}_{\mathcal{D}}(Z, [0,s]) + \mathcal{D}(Z(s), Z(t)) \leq \mathrm{Diss}_{\mathcal{D}}(Z, [0,t])$. Taking the limit $s \nearrow t$, using $Z(s) \to Z_-(t)$ and (V1), we obtain

$$v_-(t) + \mathcal{D}(Z_-(t), Z(t)) \leq \mathrm{Diss}_{\mathcal{D}}(Z, [0,t]). \tag{1.3.3}$$

On the other hand, for each partition $P = \{0 = t_0 < t_1 < \ldots t_{N-1} < t_N = t\}$ of the interval $[0,t]$ with $t_{N-1} > s_0$ we obtain by using the triangle inequality and (1.3.2)

$$\begin{aligned} \textstyle\sum_{j=1}^{N} \mathcal{D}(Z(t_{j-1}), Z(t_j)) &\leq \mathrm{Diss}_{\mathcal{D}}(Z, [0, t_{N-1}]) + \mathcal{D}(Z(t_{N-1}), Z_-(t)) + \mathcal{D}(Z_-(t), Z(t)) \\ &\leq v_-(t) + \varepsilon + \mathcal{D}(Z_-(t), Z(t)). \end{aligned} \tag{1.3.4}$$

Taking the supremum over all partitions and using that $\varepsilon > 0$ is arbitrary we infer with (1.3.3) that $\mathrm{Diss}_{\mathcal{D}}(Z, [0,t]) = \mathcal{D}(Z_-(t), Z(t)) + v_-(t)$. From this we find

$$\lim_{\tau \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [\tau, t]) = \mathcal{D}(Z_-(t), Z(t)) + v_-(t) - \lim_{\tau \nearrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [0, \tau]) = \mathcal{D}(Z_-(t), Z(t)).$$

Likewise we can show the existence of $Z_+(t) = \lim_{\tau \searrow t} Z(\tau)$ and $\lim_{\tau \searrow t} \mathrm{Diss}_{\mathcal{D}}(Z, [t, \tau]) = \mathcal{D}(Z(t), Z_+(t))$. This proves (b).

Since Z is s-continuous at all points t except in the at most countable set of jump points of $t \mapsto \mathrm{Diss}_{\mathcal{D}}(Z, [0,t])$, part (c) follows immediately. ■

1.4 Existence result in the convex case

In this section we develop an existence theory for the initial value problem, i.e. we find a solution of the (S) & (E) problem which additionally satisfies $(\varphi, z)(0) = (\varphi_0, z_0)$. In general, the uniqueness of solution can not be expected without imposing further conditions like smoothness and uniform convexity of $\mathcal{E}(t, \cdot)$ and $\mathcal{D}$, see [MT04].

We establish the existence of solutions for the time-continuous problem (S) & (E) by extracting a suitable subsequence of approximate solutions obtained from incremental problems and by showing that the limit of this suitable subsequence is a solution. In order to select a suitable subsequence we use the general selection result from Section 1.3. We recall that Theorem 1.3.3 allows us to select a subsequence of approximate solutions whose z-parts converge to some function Z^∞. Unfortunately, we can not apply Theorem 1.3.3 to the φ-parts of approximate solutions. In fact, the proof of Theorem 1.3.3 is based on the possibility to control the convergence by the dissipation distance $\mathcal{D}$. But the dissipation distance $\mathcal{D}$ does not depend on φ. Hence, we are unable to control the oscillation of approximate solutions in φ-parts. In this section we make the following assumption

$$\begin{array}{l} \text{For any } z \in \mathcal{Z} \text{ and } t \in [0,T] \text{ the functional} \\ \mathcal{E}_{t,z} : \mathcal{F} \to \mathbb{R}\ ;\ \varphi \mapsto \mathcal{E}(t, \varphi, z) \text{ has a unique minimiser} \end{array} \tag{CC}$$

and show that this assumption implies the pointwise convergence of φ-parts. Later we show in Section 1.6 how to remove this restriction.

We first give a rough overview of the proof which illuminates the structure and the assumptions needed.

Theorem 1.2.3 and Theorem 1.2.1 show that $\mathcal{S}(t)$ is not empty for each $t \in [0,T]$. For all incremental solutions the points (t_k, φ_k, z_k) lie in the set

$$\mathcal{V}_{[0,T]} := \mathcal{R}_{[0,T]} \cap \mathcal{S}_{[0,T]} = \cup_{t\in[0,T]}(t, \mathcal{V}(t)) \text{ where } \mathcal{V}(t) = \mathcal{R}(t) \cap \mathcal{S}(t).$$

To construct approximate solutions we choose a sequence $(P(n))_{n\in\mathbb{N}}$ of discretisations whose fineness $\phi(P(n)) = \max\left\{ t_j^{(n)} - t_{j-1}^{(n)} \mid j = 1, \ldots, N_n \right\}$ tends to 0. Moreover, we assume that the sequence is nested, i.e., $P(n) \subset P(n{+}1)$. We write shortly $(\Phi^n, Z^n) = (\Phi^{P(n)}, Z^{P(n)})$ (resp. $(\widehat{\Phi}^n, \widehat{Z}^n) = (\widehat{\Phi}^{P(n)}, \widehat{Z}^{P(n)})$) for the right (resp. left) continuous, piecewise constant interpolant associated with the partitions $P(n)$.

The dissipation bound (iii) of Theorem 1.2.1 provides an a priori bound in $\mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Z})$ for the z-parts of the interpolants:

$$\mathrm{Diss}_{\mathcal{D}}(Z^n, [0,T]) \leq \mathcal{E}(0, \varphi_0, z_0) - \mathcal{E}_{\min} + C_{\mathcal{E}}T.$$

Using the abstract version of Helly's selection principle allows us to extract a subsequence $(n_l)_{l\in\mathbb{N}}$ such that for all $t \in [0,T]$ the sequence $Z^{n_l}(t) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} Z^\infty(t)$ with

$$\mathrm{Diss}_{\mathcal{D}}(Z^\infty, [t_0, t_1]) \leq \lim_{l\to\infty} \mathrm{Diss}_{\mathcal{D}}(Z^{n_l}, [t_0, t_1]).$$

Moreover, the condition (CC) implies that $\Phi n_l(t) \xrightarrow{\mathcal{T}_{\mathcal{F}}} \Phi^\infty(t)$ for all $t \in [0,T]$, if we further assume that

$$\mathcal{V}_{[0,T]} \text{ is s-compact.} \tag{A5}$$

Proposition 1.4.1. *Let $(t_n, \varphi_n, z_n)_{n\in\mathbb{N}} \subset \mathcal{V}_{[0,T]}$ with $t_n \to t^*$ and $z_n \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z$. If (CC) and (A5) hold, then*

$$\varphi_n \xrightarrow{\mathcal{T}_{\mathcal{F}}} \varphi^* := M_{t^*,z^*},$$

where M_{t^*,z^*} is a unique minimiser of the functional $\mathcal{E}_{t^*,z^*}$.

Proof. Since the set $\mathcal{V}_{[0,T]}$ is compact, there exists a subsequence $(t_{n_k}, \varphi_{n_k}, z_{n_k})_{k\in\mathbb{N}}$, which converges to $(\widehat{t}, \widehat{\varphi}, \widehat{z}) \in \mathcal{S}_{[0,T]}$. It follows immediately that $\widehat{t} = t^*$ and $\widehat{z} = z^*$. Using condition (CC) we conclude that the set $\left\{ (t, \varphi, z) \in \mathcal{S}_{[0,T]} \mid t = t^* \text{ and } z = t^* \right\}$ is a one-element-set $\{M_{t^*,z^*}\}$ defined by

$$\{M_{t^*,z^*}\} = \left\{ (t^*, \varphi^*, z^*) \mid \varphi^* \text{ is a unique minimizer of the functional } \mathcal{E}_{t^*,z^*} \right\}.$$

Hence, the whole sequence $(t_n, \varphi_n, z_n)_{n\in\mathbb{N}}$ converges to (t^*, φ^*, z^*). ∎

In order to use the selection result of Section 1.3 we need to enforce two additional conditions.

$$\mathcal{D} : \Pi_{\mathcal{Z}}\mathcal{R}(T) \times \Pi_{\mathcal{Z}}\mathcal{R}(T) \to [0, \infty] \text{ is s-lower semicontinuous (see (V1)),} \tag{A6}$$

$$\text{If } (t_k, z_k) \in \Pi_{\mathcal{Z}}\mathcal{V}_{[0,T]} \text{ with } t_k \to t \text{ and } \min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0, \text{ then } z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z \text{ (see (V2))} \tag{A7}$$

where $\Pi_{\mathcal{Z}}$ is a projection on $\mathcal{Z}$, i.e. $\Pi_{\mathcal{Z}} : \mathcal{F} \times \mathcal{Z} \to \mathcal{Z}\ ;\ (\varphi, z) \mapsto z$.

Now we need to show that (Φ^∞, Z^∞) is a solution of (S) & (E). Stability is obtained via the stability of the incremental solutions at the discretisation points which become dense in the limit of $n \to \infty$. It turns out that condition (A5) is a sufficient condition for a such limit argument. This condition (A5) is certainly the most restrictive assumption and it will be considered in the next section in more detail.

Using Corollary 1.2.2 it is easy to give conditions which guarantee that $(\Phi,^\infty, Z^\infty)$ satisfies $(\mathrm{E})_{\mathrm{weak}}$. From this we obtain energy equality (E) via using the stability (S). The following theorem relies on the additional assumption:

$$\begin{aligned}&\text{It holds for almost every } t \in [0,T] :\\ &\text{If } (\varphi, z) \in \mathcal{V}(t) \text{ and } (\varphi_k, z_k) \in \mathcal{V}(t_k) \text{ with } t_k \to t, \varphi_k \xrightarrow{\mathcal{T}_\mathcal{F}} \varphi, z_k \xrightarrow{\mathcal{T}_\mathcal{Z}} z,\\ &\text{then } \partial_t \mathcal{E}(t, \varphi_k, z_k) \to \partial_t \mathcal{E}(t, \varphi, z)\end{aligned} \tag{A8}$$

Surely, in the convex case we do not need to enforce $\varphi_k \xrightarrow{\mathcal{T}_\mathcal{F}} \varphi$ like above. In fact, this immediately follows from Proposition 1.4.1. Since we will use condition (A8) in the "non-convex" case we prefer to give a general formulation.

Proposition 1.4.2. *Let $(\varphi, z)(t) \in \mathcal{V}(t) \subset \mathcal{S}(t)$ for $t \in [0,T]$. Additionally, we assume that $z \in \mathrm{BV}_\mathcal{D}([0,T], \mathcal{Z})$. If (A1), (A2), (A5), (A8) and (CC) hold and if z is continuous for all t except on a set, which is at most countable, then for all $0 \le r < s \le T$ we have the opposite energy inequality*

$$\mathcal{E}(s, \varphi(s), z(s)) + \mathrm{Diss}_\mathcal{D}(z, [r,s]) \ge \mathcal{E}(r, \varphi(r), z(r)) + \textstyle\int_r^s \partial_t \mathcal{E}(t, \varphi(\tau), z(\tau))\, \mathrm{d}\tau. \tag{1.4.1}$$

Proof. Proposition 1.4.1 provides that φ is continuous for all t except on a set which is at most countable. In fact, the set of discontinuity points of the function φ lies in the set of discontinuity points of the function z.

We consider the equidistant partition $P(n)$ with $t_k = r + k(s-r)/n$ of the segment $[r,s]$. Moreover, we set $(\varphi_k, z_k) = (\varphi, z)(t_k)$ and $(\widehat{\Phi}^n, \widehat{Z}^n)$ for the piecewise constant interpolant which is continuous from the left. As in Corollary 1.2.2 (see also the proof of part (ii) of Theorem 1.2.1), where only the stability was used, we obtain the lower estimate

$$\mathcal{E}(s, \varphi(s), z(s)) + \mathrm{Diss}_\mathcal{D}(\widehat{Z}^n, [r,s]) \ge \mathcal{E}(r, \varphi(r), z(r)) + \textstyle\int_r^s \partial_t \mathcal{E}(t, \widehat{\Phi}^n(t), \widehat{Z}^n(t))\, \mathrm{d}t.$$

Here the left-hand side is a lower bound for the left-hand side in (1.4.1). The convergence of the right-hand side to $\mathcal{E}(r, \varphi(r), z(r)) + \int_r^s \partial_t \mathcal{E}(t, \varphi(\tau), z(\tau))\, \mathrm{d}\tau$ follows by Lebesgue's majorated convergence theorem. In fact, since (φ, z) is continuous for all t except on an at most countable set, $(\widehat{\Phi}^n, \widehat{Z}^n)(t)$ converges to $(\varphi, z)(t)$ for almost all $t \in [0,T]$. Condition (A8) implies now the pointwise convergence under integral. This proves the result. ■

Now we are ready to turn the above construction into a rigorous existence proof.

Theorem 1.4.3. *Let the conditions (A1)–(A8) and (CC) be satisfied. Assume additionally that*

$$\mathcal{E} : \mathcal{R}_{[0,T]} \to [\mathcal{E}_{\min}, \infty] \text{ is s-lower semicontinuous.} \tag{A9}$$

Then for each $(\varphi_0, z_0) \in \mathcal{S}(0)$ there is at least one solution (φ, z) of (S) & (E) with $z \in \mathrm{BV}_\mathcal{D}([0,T], \mathcal{Z})$ and $(\varphi, z)(0) = (\varphi_0, z_0)$.

Moreover, for the above incremental approximations there exists a subsequence $(\Phi^{n_k}, Z^{n_k})_{k\in\mathbb{N}}$ with the following convergence properties for $k \to \infty$:

(i) $Z^{n_k} \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z})$.

(ii) *For all* $t \in [0,T]$ *we have* $(\Phi^{n_k}, Z^{n_k})(t) \to (\varphi, z)(t)$ *in* $\mathcal{F} \times \mathcal{Z}$.

(iii) *For* $0 \leq r < s \leq T$ *we have* $\mathrm{Diss}_{\mathcal{D}}(Z^{n_k},[r,s]) \to \mathrm{Diss}_{\mathcal{D}}(z,[r,s])$.

(iv) *For all* $t \in [0,T]$ *we have* $\mathcal{E}(t,\Phi^{n_k}(t), Z^{n_k}(t)) \to \mathcal{E}(t,\varphi(t),z(t))$.

Proof. Theorem 1.2.3 provides the existence of a solution for the incremental problem (1.2.1) for any partition. We take a sequence of hierarchical partitions $P(n) = \{0 = t_0^n, t_1^n, \ldots, t_{N_n}^n = T\}$ which is nested, i.e., $P(n) \subset P(n{+}1)$, and whose fineness tends to 0, i.e., $\phi(P(n)) = \max\left\{\, t_j^n - t_{j-1}^n \mid j = 1,\ldots,N_n \,\right\} \to 0$. For each partition $P(n)$ we have an incremental solution $(\varphi_k^n, z_k^n)_{k=0,\ldots,N_n}$ and we define two piecewise constant functions (Φ^n, Z^n) (continuous from the right) and $(\widehat{\Phi}^n, \widehat{Z}^n)$ (continuous from the left).

Using $\mathcal{R}(r) \subset \mathcal{R}(s)$ for $r < s$ we conclude that $(\Phi^n, Z^n)(t) \in \mathcal{R}(t)$ for all t and n. To apply our selection result in Theorem 1.3.3 on Z^n we have to show that the accumulation points of each sequence $(Z^n(t))_{n\in\mathbb{N}}$ lie in $\Pi_{\mathcal{Z}}\mathcal{V}(t) = \Pi_{\mathcal{Z}}(\mathcal{S}(t) \cap \mathcal{R}(t))$. We fix t and assume $Z^{n_m}(t) \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z$, then we know that $Z^{n_m}(t) = z_k^{n_m}$ with $t \in [t_k^{n_m}, t_{k+1}^{n_m})$. Since $\mathcal{V}_{[0,T]}$ is s-compact, $(t_k^{n_m}, z_k^{n_m}) \in \Pi_{\mathcal{Z}}\mathcal{V}_{[0,T]}$, $t_k^{n_m} \to t$ and $z_k^{n_m} \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z$, we conclude $z \in \Pi_{\mathcal{Z}}\mathcal{V}(t)$ as desired.

Thus, the selection principle is applicable and we obtain a subsequence $(Z^{n_k})_{k\in\mathbb{N}}$ which converges for all t and its limit Z^∞ satisfies $Z^\infty(t) \in \Pi_{\mathcal{Z}}\mathcal{V}(t) \subset \Pi_{\mathcal{Z}}\mathcal{S}(t)$ and

$$\mathrm{Diss}_{\mathcal{D}}(Z^\infty,[r,s]) \leq \lim_{k\to\infty} \mathrm{Diss}_{\mathcal{D}}(Z^{n_k},[r,s]) =: \mathrm{V}_\infty(s) - \mathrm{V}_\infty(r) \tag{1.4.2}$$

for $0 \leq r < s \leq T$. Using Proposition 1.4.1 we obtain immediately that

$$\Phi^{n_k}(t) \xrightarrow{\mathcal{T}_{\mathcal{F}}} \Phi^\infty(t)$$

for all $t \in [0,T]$. Here, $\Phi^\infty(t)$ is a unique minimiser of the functional $\mathcal{E}_{t,Z^\infty(t)}$.

In order to show that one desired solution (φ, z) is this particular (Φ^∞, Z^∞) we have to prove that the stability condition (S) and the energy equality (E) hold. The stability condition (S) follows from construction of (Φ^∞, Z^∞). In fact, for all $t \in [0,T]$

$$Z^\infty(t) \in \Pi_{\mathcal{Z}}\mathcal{V}(t) \quad \text{and} \quad \Phi^\infty(t) \text{ is a unique minimiser of } \mathcal{E}_{t,Z^\infty(t)}.$$

Hence, $(\Phi^\infty, Z^\infty) \in \mathcal{V}(t)$.

To prove the energy equality (E) together with the convergence results stated in (iii) and (iv) we introduce the real-valued functions E_k, V_k, W_k and $\widehat{W}_k$ via

$$\begin{aligned} E_k(t) &:= \mathcal{E}(t, \Phi^{n_k}, Z^{n_k}(t)), & \mathrm{V}_k(t) &:= \mathrm{Diss}_{\mathcal{D}}(Z^{n_k},[0,t]),\\ W_k(t) &:= \textstyle\int_0^t \partial_t\mathcal{E}(s,\Phi^{n_k}(s),Z^{n_k}(s))\,\mathrm{d}s, & \widehat{W}_k(t) &:= \textstyle\int_0^t \partial_t\mathcal{E}(s,\widehat{\Phi}^{n_k}(s),\widehat{Z}^{n_k}(s))\,\mathrm{d}s. \end{aligned}$$

Using Collorary 1.2.2 and (A2) we obtain for all t and all k the two-sided energy estimate

$$\widehat{W}_k(t) - C_{\mathcal{E}}\phi_k \leq E_k(t) + \mathrm{V}_k(t) - \mathcal{E}(0,\varphi_0,z_0) \leq W_k(t) + 2C_{\mathcal{E}}\phi_k,$$

where $\phi_k = \phi(P(n_k))$ denotes the fineness of the partitions. For grid points $t \in P(n_k)$ the estimate holds without the corrections $\pm 2C_{\mathcal{E}}\phi_k$. For general points we use (A2) (i.e., $|\partial_t\mathcal{E}| \leq C_{\mathcal{E}}$) and the fact that (Φ^{n_k}, Z^{n_k}) is piecewise constant.

In the limit $k \to \infty$ the left-hand and the right-hand side converge to the same limit $W_\infty(t) = \int_0^t \partial_t \mathcal{E}(s, \Phi^\infty(s), Z^\infty(s))\,\mathrm{d}s$ by Propositions 1.3.4(a), 1.4.1 and (A8). Using $\mathrm{V}_k(t) \to \mathrm{V}_\infty(t)$ we conclude that the limit $E_\infty(t) := \lim_{k\to\infty} E_k(t)$ exists. Moreover, by (A9) and (1.4.2) we have

$$\mathcal{E}(t, \Phi^\infty, Z^\infty(t)) + \mathrm{Diss}_{\mathcal{D}}(Z^\infty, [0,t]) \le E_\infty(t) + \mathrm{V}_\infty(t) = \mathcal{E}(0, y_0) + W_\infty(t),$$

which is $(\mathrm{E})_{\mathrm{weak}}$. Together with the opposite inequality derived in Proposition 1.4.2 we obtain (E).

In particular, this means that $\mathcal{E}(t, \Phi^\infty(t), Z^\infty(t)) + \mathrm{Diss}_{\mathcal{D}}(Z^\infty, [0,t]) = E_\infty(t) + \mathrm{V}_\infty(t)$ in addition to $\mathcal{E}(t, \Phi^\infty(t), Z^\infty(t)) \le E_\infty(t)$ and $\mathrm{Diss}_{\mathcal{D}}(Z^\infty, [0,t]) \le \mathrm{V}_\infty(t)$. This implies equality in both cases and (iii) and (iv) are established. ■

Remark 1.4.4. Assumption (A4) follows immediately from (A9) and (A6).

Remark 1.4.5. If $\mathcal{E}(t, \cdot, \cdot) : \mathcal{R}(t) \to [\mathcal{E}_{\min}, \infty]$ is s-lower semicontinuous for all $t \in [0,T]$, then assumption (A2) implies that (A9) also holds.

Our solution concept is such that solutions are well-defined for all $t \in [0,T]$ in contrast to definitions for almost every $t \in [0,T]$. In particular, both, the left-hand limit $(\varphi_-, z_-)(t)$ and the right-hand limit $(\varphi_+, z_+)(t)$, may differ from $(\varphi, z)(t)$. However, if (φ, z) is a solution of (S) & (E), then also (φ_-, z_-) and (φ_+, z_+) are solutions (with a possible change of initial value in the latter case).

Moreover, the energy equality and stability imply that at jump points the following identities hold:

$$\begin{aligned}
\mathcal{E}(t, \varphi_-(t), z_-(t)) &= \mathcal{E}(t, \varphi(t), z(t)) + \mathcal{D}(z_-(t), z(t)), \\
\mathcal{E}(t, \varphi(t), z(t)) &= \mathcal{E}(t, \varphi_+(t), z_+(t)) + \mathcal{D}(z(t), z_+(t)), \\
\mathcal{D}(z_-(t), z_+(t)) &= \mathcal{D}(z_-(t), z(t)) + \mathcal{D}(z(t), z_+(t)).
\end{aligned} \tag{1.4.3}$$

Note that all three points $(\varphi, z)(t)$, $(\varphi_-, y_-)(t)$ and $(\varphi_+, y_+)(t)$ lie in the stable set $\mathcal{S}(t)$.

We formulate now a special version of Theorem 1.4.3, which is based on Banach spaces and which is easy to apply to several models in continuum mechanics.

Theorem 1.4.6. *Let Y_1 and Y be Banach spaces. Suppose that Y_1 is compactly embedded in Y and that $\{\, y \in Y_1 \mid \|y\|_{Y_1} \le 1 \,\}$ is closed in Y. The dissipation distance $\mathcal{D} : Y \times Y \to \mathbb{R}$ is the Y-norm, i.e., $\mathcal{D}(y_1, y_2) = \|y_1 - y_2\|_Y$. Furthermore, the functional $\mathcal{E} : [0,T] \times Y \to [\mathcal{E}_{\min}, \infty]$ has the following properties:*

(a) $\mathcal{E}$ is s-lower semicontinuous on $[0,T] \times Y$ (with respect to the norm topology of Y).

(b) For some real numbers $c_1 > 0$, C_2 and $\alpha > 1$ we have

$$\mathcal{E}(t, y) \ge c_1 \|y\|_{Y_1}^\alpha - C_2 \quad (\textit{i.e., } \mathcal{E}(t,y) = \infty \textit{ for } y \in Y \setminus Y_1). \tag{1.4.4}$$

(c) The map $\partial_t \mathcal{E}(t, \cdot) : Y_1 \to \mathbb{R}$ is s-continuous with respect to the norm topology Y.

(d) There exists C_3 such that $|\partial_t \mathcal{E}(t, y)| \le C_3(1 + \|y\|_{Y_1})$ for all $t \in [0,T]$ and $y \in Y_1$.

Then, for each $y_0 \in \mathcal{S}(0)$ there exists at least one solution $y \in \mathrm{BV}_{\mathcal{D}}([0,T], Y) \cap B([0,T], Y_1)$ of (S) & (E) with $y(0) = y_0$ and all the conclusions of Theorem 1.4.3 also hold.

Here $B([0,T], Y_1)$ denotes the set of mappings y such that $t \mapsto \|y(t)\|_{Y_1}$ is bounded.

Remark 1.4.7. Before giving the proof of this result, we show that it provides a generalised solution to (0.0.2), i.e., to the partial differential inclusion $0 \in \kappa(x)\,\mathrm{Sign}(\dot{z}(t,x)) - \mathrm{div}(a(x)\mathrm{D}_x z(t,x)) - g_{\mathrm{ext}}(t,x)$ with $z(t,\cdot)|_{\partial\Omega} = 0$. To this end, take $Y = \mathrm{L}^1(\Omega), Y_1 = \mathrm{H}^1_0(\Omega)$ and $\mathcal{D}$ and $\mathcal{E}$ as defined for (0.0.2). Since $\mathcal{E}$ is quadratic, the assumptions (a) and (b) hold with $\alpha = 2$. Moreover, with $g_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathrm{H}^{-1}(\Omega))$ we obtain $|\partial_t \mathcal{E}(t,z)| = |\langle \partial_t g_{\mathrm{ext}}(t), z\rangle| \le C\|z\|_{\mathrm{H}^1}$.

Proof of Theorem 1.4.6: Since we want to use Theorem 1.4.3, one putative problem is the absence of an estimate similar to (A2). The main idea of our proof is to solve the problem (S) & (E) on the set $B_R^{Y_1}(0) := \{\, y \in Y_1 \mid \|y\|_{Y_1} \le R \,\}$ equipped with the Y-topology and to show that the constructed solution does not depend on R. In the sequel, these problems are called restricted problems.

The proof is done in three steps:
Step 1: Show that the problem (S) & (E) on the set $B_R^{Y_1}(0)$ equipped with the Y-topology has a solution y_R for all $R \ge \|y_0\|_{Y_1}$.
Step 2: Give a number r_{st} such that all solutions of the restricted problems with $R \ge \|y_0\|_{Y_1}$ as well as possible solutions of the problem on Y lie in $B_{r_{\mathrm{st}}}^{Y_1}(0)$.
Step 3: Give a number R_{dist} such that for all y, which are stable on the set $B_{r_{\mathrm{st}}}^{Y_1}(0)$, the inequality $\mathcal{E}(t,y) \le \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y})$ holds for all $\widehat{y} \in Y \setminus B_{R_{\mathrm{dist}}}^{Y_1}(0)$.

If these three steps are completed, it easy to see that each solution obtained for any $R > \max\{r_{\mathrm{st}}, R_{\mathrm{dist}}\}$ remains a solution for any $\widehat{R} > R$. Hence, each such solution is a solution of the full problem. We now work out Steps 1 to 3.

Step 1: Let $R > \|y_0\|_{Y_1}$. We define the space $\mathfrak{Y}$ as $B_R^{Y_1}(0)$ and equip it with the Y-topology. From the compact embedding of Y_1 in Y and the Y-closedness of the Y_1-balls in Y it follows that $\mathfrak{Y}$ is a compact, topological Hausdorff space.

We need to verify all the assumptions of Theorem 1.4.3.

Each stable point $y \in \mathfrak{Y}$ for the problem (S) & (E) on Y is also stable for the restricted problem on $\mathfrak{Y}$. Hence, y_0 is stable at the time $t = 0$ for the restricted problem. Using (a) and (b) we infer that conditions (A8) and (A9) hold on $\mathfrak{Y}$. Since $\mathcal{D}(y_1,y_2)$ is equal to $\|y_1 - y_2\|_Y$, conditions (A1), (A6) and (A7) follow immediately. Using (d) we obtain that the assumption (A2) holds on $\mathfrak{Y}$ with $C_{\mathcal{E}} = C_3(1{+}R)$.

The map $y \mapsto \mathcal{E}(t,y) + \mathcal{D}(y_0,y)$ is s-lower semicontinuous. Hence, the set $\mathcal{R}(T)$ of the restricted problem is s-closed in $\mathfrak{Y}$. Since $\mathfrak{Y}$ is compact, condition (A3) holds.

Using Theorem 1.5.1 from below we obtain condition (A5) from (A3) and the Y-continuity of the dissipation distance. Thus, Step 1 is proved.

Step 2: We give an a priori bound for $\|y\|_{Y_1}$ for all solutions y of the whole problem or of the restricted problems. If y is a solution, then $y(t)$ is stable for all $t \in [0,T]$. Using $y \in \mathcal{S}(t)$, (1.4.4) and assumption (d), we get the following estimate

$$c_1\|y\|_{Y_1}^{\alpha} - C_2 \le \mathcal{E}(t,y) \le \mathcal{E}(t,y_0) + \mathcal{D}(y,y_0) \le \mathcal{E}(0,y_0) + t\,[C_3(1{+}\|y_0\|_{Y_1})] + \|y - y_0\|_Y. \tag{1.4.5}$$

Since Y_1 is continuously embedded in Y there exists a $K > 0$ such that $\|y\|_Y \le K\|y\|_{Y_1}$ for all $y \in Y_1$. Hence, using (1.4.5) we obtain

$$c_1\|y\|_{Y_1}^{\alpha} - C_2 - K\|y\|_{Y_1} \le \mathcal{E}(0,y_0) + t\,[C_3(1{+}\|y_0\|_{Y_1})] + K\|y_0\|_{Y_1}. \tag{1.4.6}$$

The last estimate implies that there exists a number $r_{\rm st} > 0$ such that

$$\|y\|_{Y_1} \le r_{\rm st} \quad \text{ for all } y \in \cup_{t\in[0,T]}\mathcal{S}(t).$$

Therefore, all solutions of the whole problem or of the restricted problems lie in $B^{Y_1}_{r_{\rm st}}(0)$. This proves Step 2.

Step 3: Suppose that y is stable on $B^{Y_1}_{r_{\rm st}}(0)$ at a time $t \in [0,T]$ and that there exists $\widehat{y} \in Y_1$ such that

$$\mathcal{E}(t,y) > \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}). \tag{1.4.7}$$

Our aim is to give an a priori upper bound for the Y_1-norm of $\widehat{y}$. Using the stability of the point y on $B^{Y_1}_{r_{\rm st}}(0)$ and assumption (d), we obtain the following estimate:

$$\begin{aligned}\mathcal{E}(t,y) \le \mathcal{E}(t,y_0) + \mathcal{D}(y,y_0) &\le \mathcal{E}(0,y_0) + t\,[C_3(1{+}\|y_0\|_{Y_1})] + \mathcal{D}(y,y_0)\\ &\le \mathcal{E}(0,y_0) + T\,[C_3(1{+}r_{\rm st})] + 2r_{\rm st}.\end{aligned}$$

Combining this estimate, (1.4.7) and (1.4.4) we obtain

$$\mathcal{E}(0,y_0) + T\,[C_3(1{+}r_{\rm st})] + 2r_{\rm st} \ge \mathcal{E}(t,y) > \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y}) \ge c_1\|\widehat{y}\|^{\alpha}_{Y_1} - C_2. \tag{1.4.8}$$

The last estimate allows us to give an a priori upper bound $R_{\rm dist}$ for the norm of $\widehat{y}$, i.e., for all $\widehat{y}$ with $\|\widehat{y}\| > R_{\rm dist}$ the estimate

$$\mathcal{E}(t,y) \le \mathcal{E}(t,\widehat{y}) + \mathcal{D}(y,\widehat{y})$$

holds. Thus, Step 3 is proved. ∎

1.5 Closedness of the stable set

The major assumption of our existence result in Theorem 1.4.3 is the s-compactness of $\mathcal{V}_{[0,T]}$ stated in (A5). Since $\mathcal{V}_{[0,T]} = \mathcal{S}_{[0,T]} \cap \mathcal{R}_{[0,T]}$ is a subset of the s-compact set $\mathcal{R}_{[0,T]}$, it suffices to show that $\mathcal{S}_{[0,T]}$ (or just $\mathcal{S}_{[0,T]} \cap \mathcal{R}_{[0,T]}$) is s-closed.

Before deriving abstract results in this direction we give two simple nontrivial applications of the theorem and thus highlight that the choice of the topology $\mathcal{T}$ is crucial. For both examples let $\mathcal{Y} = \mathrm{L}^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open and bounded, and choose the dissipation distance $\mathcal{D}(y_0,y_1) = \|y_1{-}y_0\|_{\mathcal{Y}} = \int_\Omega |y_1(x){-}y_0(x)|\,\mathrm{d}x$.

For the first example consider

$$\mathcal{E}_1(t,y) = \int_\Omega a(x)|y(x)|^\alpha {-} g(t,x)y(x)\,\mathrm{d}x,$$

where $a(x) \ge a_0 > 0$, $\alpha > 1$, and $g \in \mathrm{C}^1([0,T],\mathrm{L}^\infty(\Omega))$. Since $\mathcal{E}_1(t,\cdot)$ is convex and lower semi-continuous, the set $\mathcal{R}(T)$ is a closed convex set which lies in the intersection of an L^1-ball and an L^α-ball. Hence, taking $\mathcal{T}$ to be the weak topology on $\mathcal{Y}$, we obtain that the s-compactness condition (A3) holds. Note that $\mathcal{R}(T)$ is not compact in the norm topology of $\mathrm{L}^1(\Omega)$. The stable sets for $\mathcal{E}_1$ are given by

$$\mathcal{S}_1(t) = \left\{ y \in \mathrm{L}^1(\Omega) \mid |y(x)|^{\beta-2}y(x) \in [\tfrac{g(t,x)-c_{\mathcal{D}}}{a(x)\alpha}, \tfrac{g(t,x)+c_{\mathcal{D}}}{a(x)\alpha}] \text{ for a.a. } x \in \Omega \right\},$$

which shows that they are s-closed with respect to $\mathcal{T}$, since they are convex and closed in the norm topology. Hence, with $\mathcal{T}$ as the weak topology in $\mathcal{Y} = \mathrm{L}^1(\Omega)$ all conditions of Theorem 1.4.3 can be satisfied.

For the second example consider the nonconvex energy functional

$$\mathcal{E}_2(t,y) = \int_\Omega \tfrac{1}{2}|\mathrm{D}y(x)|^2 + f(t,x,y(x))\,\mathrm{d}x \quad \text{for } y \in \mathrm{H}^1(\Omega) \quad \text{and } +\infty \text{ else},$$

where $f : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $\partial_t f$ are continuous and bounded. Now, $\mathcal{R}(T)$ is already compact in the norm topology of $Y = \mathrm{L}^1(\Omega)$, since it is closed and contained in a Y_1-ball, where $Y_1 = H^1(\Omega)$ is compactly embedded in Y. With these properties, it can be shown that if we select $\mathcal{T}$ as the strong topology of $\mathrm{L}^1(\Omega)$ then all conditions of Theorem 1.4.6 are satisfied.

As a first abstract result we show that s-continuity of $\mathcal{D}$ leads to s-continuity of $\mathcal{E}$ on the stable set. Thus s-closedness of the stable set follows.

Theorem 1.5.1. *Let (A2) hold. Assume that $\mathcal{E}$ is s-lower semicontinuous on $[0,T] \times \mathcal{F} \times \mathcal{Z}$ and that $\mathcal{D}$ is s-continuous on $\mathcal{Z} \times \mathcal{Z}$. Then, $\mathcal{E} : \mathcal{S}_{[0,T]} \to [\mathcal{E}_{\min}, \infty)$ is s-continuous as well and the set $\mathcal{S}_{[0,T]}$ is s-closed.*

Proof. For $(s,\varphi_s,z_s),(t,\varphi_t,y_t) \in \mathcal{S}_{[0,T]}$ we have by stability

$$-C_{\mathcal{E}}|t-s| - \mathcal{D}(z_s,z_t) \le \mathcal{E}(t,\varphi_t,y_t) - \mathcal{E}(s,\varphi_s,y_s) \le C_{\mathcal{E}}|t-s| + \mathcal{D}(z_t,z_s).$$

This estimate together with the s-continuity of $\mathcal{D}$ implies the s-continuity of $\mathcal{E}$.

Now, consider a sequence $(t_k,\varphi_k,z_k)_{k\in\mathbb{N}}$ in $\mathcal{S}_{[0,T]}$ with $t_k \to t^*$, $z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z^*$ and $\varphi_k \xrightarrow{\mathcal{T}_{\mathcal{F}}} \varphi^*$. It remains to show that $(\varphi^*,z^*) \in \mathcal{S}(t^*)$. For an arbitrary $(\varphi,z) \in \mathcal{F} \times \mathcal{Z}$ we have $\mathcal{E}(t_k,\varphi_k,z_k) \le \mathcal{E}(t_k,\varphi,z) + \mathcal{D}(z_k,z)$ for all $k \in \mathbb{N}$. Taking the limit $k \to \infty$ the s-continuities yield $\mathcal{E}(t^*,\varphi^*,z^*) \le \mathcal{E}(t^*,\varphi,z) + \mathcal{D}(z^*,z)$. Since $(\varphi,z) \in \mathcal{F} \times \mathcal{Z}$ is arbitrary, it follows that $y^* \in \mathcal{S}(t^*)$. ■

Remark 1.5.2. By using Theorem 1.5.1 we can also verify condition (A8). For this we need the following additional assumption on $\partial_t\mathcal{E}$:

> If $(t_k,\varphi_k,z_k)_{k\in\mathbb{N}} \subset \mathcal{S}_{[0,T]}$ with $t_k \to t, \varphi_k \xrightarrow{\mathcal{T}_{\mathcal{F}}} \varphi, z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z,$ then there exist positive numbers $k_0 \in \mathbb{N}, h_0 > 0$ and a function $\omega : \mathbb{R} \to \mathbb{R}_+$ with $\lim_{t\searrow 0}\omega(t) = 0$ such that for $k \ge k_0$ and $h \in (0,h_0)$ it holds:
> $$\Big|\tfrac{1}{h}\left[\mathcal{E}(t \pm h,\varphi_k,z_k) - \mathcal{E}(t,\varphi_k,z_k)\right] \mp \partial_t E(t,\varphi_k,z_k)\Big| \le \omega(h) \tag{1.5.1}$$

For details see [FM04].

The next result is a strengthened version of the previous theorem.

Theorem 1.5.3. *Let condition (A2) hold. Assume that for each sequence $(t_k,y_k)_{k\in\mathbb{N}}$ with $(t_k,\varphi_k,z_k) \in \mathcal{S}_{[0,T]}$, $t_k \to t^*$, $z_k \xrightarrow{\mathcal{T}_{\mathcal{Z}}} z^*$ and $\varphi_k \xrightarrow{\mathcal{T}_{\mathcal{F}}} \varphi^*$ the following condition holds:*

$$\forall(\varphi,z) \in \mathcal{F} \times \mathcal{Z} : \liminf_{k\to\infty}\left[\mathcal{E}(t_k,\varphi_k,z_k) - \mathcal{D}(z_k,z)\right] \ge \mathcal{E}(t^*,\varphi^*,z^*) - \mathcal{D}(z^*,z). \tag{1.5.2}$$

Then, the set $\mathcal{S}_{[0,T]}$ is s-closed.

Proof. Let $(\varphi, z) \in \mathcal{F} \times \mathcal{Z}$ be arbitrary. We have to show that $\mathcal{E}(t^*, \varphi^*, z^*) \leq \mathcal{E}(t^*, \varphi, z) + \mathcal{D}(z^*, z)$. Since $(t_k, \varphi_k, y_k) \in \mathcal{S}_{[0,T]}$ we have the following estimates

$$\begin{aligned}
\mathcal{E}(t^*, \varphi^*, y^*) &= \mathcal{E}(t^*, \varphi^*, y^*) - \mathcal{E}(t_k, \varphi_k, z_k) + \mathcal{E}(t_k, \varphi_k, z_k) \\
&\leq \mathcal{E}(t^*, \varphi^*, z^*) - \mathcal{E}(t_k, \varphi_k, z_k) + \mathcal{E}(t_k, \varphi, z) + \mathcal{D}(z_k, z) \\
&= \mathcal{E}(t^*, \varphi, z) + \mathcal{D}(z^*, z) + (\mathcal{E}(t_k, \varphi, z) - \mathcal{E}(t^*, \varphi, z)) \\
&- [\mathcal{E}(t_k, \varphi_k, z_k) - \mathcal{D}(z_k, z) - \mathcal{E}(t^*, \varphi^*, z^*) + \mathcal{D}(z^*, z)].
\end{aligned}$$

Using (A2) (i.e., $|\partial_t \mathcal{E}| \leq C_{\mathcal{E}}$) and condition (1.5.2) we obtain the desired result by taking the limit $k \to \infty$. ∎

Further results, which guarantee the s-clossdness of the stable set, can be found in [DMFT04, MM03]. For example, in [MM03] the following result was proved.

Theorem 1.5.4. *Let (A1), (A2), (A6), (A3), and (A9) hold and assume that there exists an s-closed set $\mathcal{M}_{[0,T]}$ with $\mathcal{S}_{[0,T]} \subset \mathcal{M}_{[0,T]} \subset [0,T]\times\mathcal{Y}$ such that $\mathcal{E} : \mathcal{M}_{[0,T]} \to [\mathcal{E}_{\min}, \infty]$ is s-continuous. Moreover, assume that $\mathcal{D}$ satisfies the following condition:*

$$\begin{array}{l}
\textit{For all } (t, \widehat{y}), (t_k, y_k) \in \mathcal{S}_{[0,T]} \textit{ with } (t_k, y_k) \xrightarrow{\mathcal{T}} (t, y) \\
\textit{there exists } \widehat{y}_k \in \mathcal{M}(t_k) \textit{ such that} \\
\widehat{y}_k \xrightarrow{\mathcal{T}} \widehat{y} \quad \textit{and} \quad \liminf_{k\to\infty} \mathcal{D}(y_k, \widehat{y}_k) \leq \mathcal{D}(y, \widehat{y}).
\end{array} \tag{1.5.3}$$

Then, the set $\mathcal{S}_{[0,T]}$ is s-closed.

The assumption on $\mathcal{D}$ in Theorem 1.5.4 is weaker than the continuity assumed in Theorem 1.5.1. It make possible to apply this result in situations, where $\mathcal{D}$ can take $+\infty$ as a value. Theorem 1.5.4 was especially used to treat the delamination problem, see [KMR03].

1.6 Existence result in the non-convex case

Theorem 1.6.1. *Let the conditions (A1)–(A9) be satisfied. Assume additionally that*

$$\textit{the family } \Big\{ \partial_t \mathcal{E}(\cdot, \Phi, Z) \mid (\Phi, Z) \in \mathcal{R}(T) \Big\} \subset \mathrm{C}^0([0,T]) \textit{ is equicontinuous.} \tag{A10}$$

Then for each $(\varphi_0, z_0) \in \mathcal{S}(0)$ there exists a function $t \to Z^\infty(t) \in \mathcal{Z}$ and a set-valued function $t \rightsquigarrow F(t) \subset \mathcal{F}$ with non-empty closed images such that for any selection Φ^∞ of F, i.e, $\Phi^\infty(t) \in F(t)$ for every $t \in [0,T]$, a pair (Φ^∞, Z^∞) is a solution of (S) & (E) with $Z^\infty \in \mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Z})$ and $(\Phi^\infty, Z^\infty)(0) = (\varphi_0, z_0)$.

Moreover, for the above incremental approximations there exists a subsequence $(\Phi^n, Z^n)_{n\in\mathbb{N}}$ with the following convergence properties for $n \to \infty$:
(i) $Z^n \in \mathrm{BV}_{\mathcal{D}}([0,T], \mathcal{Z})$.
(ii) *For all $t \in [0,T]$ we have $Z^n(t) \to Z^\infty(t)$ in $\mathcal{Z}$.*
(iii) *For $0 \leq r < s \leq T$ we have $\mathrm{Diss}_{\mathcal{D}}(Z^n, [r,s]) \to \mathrm{Diss}_{\mathcal{D}}(Z^\infty, [r,s])$.*
(iv) *For all $t \in [0,T]$ we have $\mathcal{E}(t, \Phi^n(t), Z^n(t)) \to \mathcal{E}(t, \Phi^\infty(t), Z^\infty(t))$.*

Proof. Our proof is essentially the same as in the "convex" case. We also take a sequence of hierarchical partitions $P(n)$ and consider the piecewise constant approximated solutions (Φ^n, Z^n). Furthermore we use the following notation:

$$\theta_n : [0,T] \to \mathbb{R}; t \mapsto \partial_t \mathcal{E}(t, \Phi^n(t), Z^n(t)).$$

Without loss of generality we assume that

$$\begin{gathered}\text{the sequence } (Z^n) \text{ converges pointwise in } (\mathcal{F}, \mathcal{T}_{\mathcal{F}}) \text{ to } Z^\infty \\ \text{and} \\ \text{the sequence } (\theta_n) \text{ weakly}^* \text{ converges to } \theta^* \text{ in } \mathrm{L}^\infty([0,T],\mathbb{R})\end{gathered}$$

with $Z^\infty \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z})$ and $\theta^* \in \mathrm{L}^\infty([0,T],\mathbb{R})$. In fact, the former follows as in the proof of Theorem 1.4.3. The latter follows from the boundedness of θ_n in $\mathrm{L}^\infty([0,T],\mathbb{R})$. We notice also that, as in Theorem 1.4.3, for all $t \in [0,T]$ the following holds:

$$\begin{aligned}&V^\infty(t) := \lim_{n\to\infty} \mathrm{Diss}_{\mathcal{D}}(Z^n, [0,t]) \\ &\mathrm{Diss}_{\mathcal{D}}(Z^\infty, [0,t]) \le V^\infty(t).\end{aligned} \tag{1.6.1}$$

Let a function $\Theta : [0,T] \to \mathbb{R}$ be given by

$$\Theta(t) := \limsup_{n\to\infty} \theta_n(t).$$

Using Fatou's Lemma we obtain immediately

$$\textstyle\int_0^t \Theta(s)\,\mathrm{d}s \ge \int_0^t \theta^*(s)\,\mathrm{d}s \text{ for a.e. } t \in [0,T]. \tag{1.6.2}$$

We take the set-valued function F as follows

$$F(t) = \{\, \Phi \mid \Phi \in \mathrm{Limsup}_{n\to\infty}\{\Phi^n(t)\} \text{ and } \partial_t\mathcal{E}(t, \Phi, Z^\infty(t)) = \Theta(t) \,\}\,.$$

Here, $\mathrm{Limsup}_{n\to\infty}\{\Phi^n(t)\}$ is the set of the cluster points of the sequence $(\Phi^n(t))_{n\in\mathbb{N}}$, see also Definition A.5.1. We have to prove that $F(s)$ is a non-empty closed set. By definition of Θ there exists a subsequence $(n_k^s)_{k\in\mathbb{N}}$ such that $\theta_{n_k^s}(s) \to \Theta(s)$. By construction of approximated solutions there exists a sequence $(t_{n_k^s})_{k\in\mathbb{N}} \subset [0,T]$ with $t_{n_k^s} \to s$ such that for any $k \in N$

$$\begin{aligned}&Z^{n_k^s}(s) = Z^{n_k^s}(t_{n_k^s}) \text{ and } \Phi^{n_k^s}(s) = \Phi^{n_k^s}(t_{n_k^s}), \\ &(\Phi^{n_k^s}(t_{n_k^s}), Z^{n_k^s}(t_{n_k^s})) \in \mathcal{V}(t_{n_k^s}).\end{aligned}$$

Using condition (A5) we obtain that there exists $\Phi \in \mathcal{F}$ such that $(s, \Phi, Z^\infty(s))$ is a cluster point of the sequence $(t_{n_k^s}, \Phi^{n_k^s}(t_{n_k^s}), Z^{n_k^s}(t_{n_k^s}))$ in $\mathcal{V}_{[0,T]}$. Condition (A8) implies

$$\partial_t\mathcal{E}(s, \Phi, Z^\infty(s)) = \Theta(s)$$

Hence, the set $F(s)$ is non-empty. The same arguments show also that

$$(\Phi, Z^\infty(s)) \in \mathcal{V}(s) \tag{1.6.3}$$

for any $\Phi \in F(s)$. Since the set of cluster points is closed, this observation implies by using (A8) the closedness of $F(s)$.

Consider an arbitrary selection Φ^∞ of the set-valued map F. The conclusion (1.6.3) shows that a pair (Φ^∞, Z^∞) satisfies the stability condition **(S)**. We have to verify the energy balance **(E)** for (Φ^∞, Z^∞).

Let $t \in [0,T]$ be fix. Using s-lower semicontinuity of $\mathcal{E}$ and (1.6.1) we obtain:

$$\begin{aligned}&\mathcal{E}(t,\Phi^\infty(t),Z^\infty(t)) + \mathrm{Diss}(Z^\infty,[0,t]) - \mathcal{E}(0,\varphi_0,z_0)\\&\quad\le \limsup_{n\to\infty}\,[\mathcal{E}(t,\Phi^n(t),Z^n(t)) + \mathrm{Diss}(Z^n,[0,t])] - \mathcal{E}(0,\varphi_0,z_0)\\&\quad\le \limsup_{n\to\infty} \int_0^t \partial_t\mathcal{E}(s,\Phi^n(s),Z^n(s))\,\mathrm{d}s = \int_0^t \theta^*(s)\,\mathrm{d}s\end{aligned} \tag{1.6.4}$$

We show now that

$$\mathcal{E}(t,\Phi^\infty(t),Z^\infty(t)) + \mathrm{Diss}(Z^\infty,[0,t]) - \mathcal{E}(0,\varphi_0,z_0) \ge \int_0^t \Theta(s)\,\mathrm{d}s. \tag{1.6.5}$$

Unfortunately we can not guarantee, that the set of the discontinuity points of function Φ^∞ is at most countable. Hence, we can not apply Theorem 1.4.2 directly. Instead of that we use a trick based on the possibility to approximate the Lebesgue integral with suitable Riemann sums (the idea was suggested in [DMFT04, FM04]).

Let $\varepsilon > 0$. Using (A10) we find $\delta > 0$ such that

$$|\partial_t\mathcal{E}(t_1,\Phi^\infty(t_2),Z^\infty(t_2)) - \partial_t\mathcal{E}(t_2,\Phi^\infty(t_2),Z^\infty(t_2))| \le \varepsilon \tag{1.6.6}$$

holds for all $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| < \delta$. Using Theorem A.3.1 we select a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_N = t$ such that

$$\left|\int_0^t \partial_t\mathcal{E}(s,\Phi^\infty(s),Z^\infty(s))\,\mathrm{d}s - \sum_{i=1}^N \partial_t\mathcal{E}(\tau_i,\Phi^\infty(\tau_i),Z^\infty(\tau_i))(\tau_i - \tau_{i-1})\right| < \varepsilon. \tag{1.6.7}$$

Without loss of generality we assume that the fineness of partition $(\tau_i)_{i=0,\ldots,N}$ is smaller than δ . In fact, if we can divide the interval [0,T] into L equidistant subintervals and apply Theorem A.3.1 to each subinterval, then we obtain a partition of $[0,T]$ with fineness smaller than $1/L$. As in Theorem 1.4.2 the stability provides

$$\begin{aligned}&\mathcal{E}(t,\Phi^\infty(t),Z^\infty(t)) + \mathrm{Diss}(Z^\infty,[0,t]) - \mathcal{E}(0,\varphi_0,z_0)\\&\quad\ge \sum_{i=1}^N [\mathcal{E}(\tau_i,\Phi^\infty(\tau_i),Z^\infty(\tau_i)) + \mathcal{D}(Z^\infty(\tau_{i-1}),Z^\infty(\tau_i)) - \mathcal{E}(\tau_{i-1},\Phi^\infty(\tau_{i-1}),Z^\infty(\tau_{i-1}))]\\&\quad\ge \sum_{i=1}^N [\mathcal{E}(\tau_i,\Phi^\infty(\tau_i),Z^\infty(\tau_i)) - \mathcal{E}(\tau_{i-1},\Phi^\infty(\tau_i),Z^\infty(\tau_i))]\\&\quad= \sum_{i=1}^N \int_{\tau_{i-1}}^{\tau_i} \partial_t\mathcal{E}(s,\Phi^\infty(\tau_i),Z^\infty(\tau_i))\,\mathrm{d}s.\end{aligned} \tag{1.6.8}$$

Combining (1.6.6), (1.6.7) and (1.6.8) we conclude

$$\begin{aligned}&\mathcal{E}(t,\Phi^\infty(t),Z^\infty(t)) + \mathrm{Diss}(Z^\infty,[0,t]) - \mathcal{E}(0,\varphi_0,z_0)\\&\quad\ge \int_0^t \partial_t\mathcal{E}(s,\Phi^\infty(s),Z^\infty(s))\,\mathrm{d}s - \varepsilon - \varepsilon T = \int_0^t \Theta(s)\,\mathrm{d}s - (1+T)\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is an arbitrary number, we obtain the desired estimate (1.6.5). The estimates (1.6.2), (1.6.4) and (1.6.5) provides

$$\begin{aligned}\int_0^t \Theta(s)\,\mathrm{d}s &\le \mathcal{E}(t,\Phi^\infty(t),Z^\infty(t)) + \mathrm{Diss}(Z^\infty,[0,t]) - \mathcal{E}(0,\varphi_0,z_0)\\&\le \limsup_{n\to\infty}\,[\mathcal{E}(t,\Phi^n(t),Z^n(t)) + \mathrm{Diss}(Z^n,[0,t])] - \mathcal{E}(0,\varphi_0,z_0)\\&\le \limsup_{n\to\infty} \int_0^t \partial_t\mathcal{E}(s,\Phi^n(s),Z^n(s))\,\mathrm{d}s = \int_0^t \theta^*(s)\,\mathrm{d}s \le \int_0^t \Theta(s)\,\mathrm{d}s.\end{aligned}$$

The last estimate and $\Theta(s) = \partial_t\mathcal{E}(t, \Phi(s)^\infty(s), Z^\infty(s))$ imply that the energy balance holds. Moreover, we obtain

$$\theta^*(t) = \Theta(t) \text{ for a.e. } t \in [0,T]$$

and by using s-lower semicontinuity of $\mathcal{E}$ and (1.6.1) we conclude that

$$\begin{aligned} \mathrm{Diss}_{\mathcal{D}}(Z^n, [0,t]) &\to \mathrm{Diss}_{\mathcal{D}}(Z^\infty, [0,t]), \\ \mathcal{E}(t, \Phi^n(t), Z^n(t)) &\to \mathcal{E}(t, \Phi^\infty(t), Z^\infty(t)). \end{aligned}$$

■

Corollary 1.6.2. *Assume that $(\mathcal{F}, \mathcal{T}_{\mathcal{F}})$ is a separable metric space and all assumptions of the previous theorem hold. Then the set-valued function F in Theorem 1.6.1 can be chosen (by taking a further subsequence) as measurable function (in the sense of Definition A.5.3).*

Proof. We use the notations of the previous theorem. We know

$$\begin{aligned} \theta_n = \textstyle\int_0^t \partial_t\mathcal{E}(s, Z^n, \Phi^n) &\overset{*}{\rightharpoonup} \theta^* \ \text{ in } \mathrm{L}^\infty([0,T],\mathbb{R}), \\ \Theta(t) = \theta^*(t) & \quad \text{for a.e. } t \in [0,T] \end{aligned}$$

with $\Theta(t) := \limsup_{n\to\infty} \theta_n(t)$. By using Theorem A.2.2 we obtain

$$\theta_n \to \theta^* \text{ in } \mathrm{L}^p([0,T],\mathbb{R}), \quad p < \infty.$$

Hence, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\theta_{n_k}(t) \to \Theta(t) = \theta^*(t)$ for almost every $t \in [0,T]$.

Consider the sequence (Φ^{n_k}, Z^{n_k}) of approximated solutions and define

$$\widehat{F} = \mathrm{Limsup}_{k\to\infty}\{\Phi^{n_k}\} \quad \text{(the set of cluster points, see Def. A.5.1)}$$

Theorem A.5.4 implies that $\widehat{F}$ is a measurable set-valued function with non-empty closed images. Let now $t \in [0,T]$ such that $\Theta(t) = \lim_{k\to\infty} \partial_t\mathcal{E}(t, \Phi^{n_k}(t), Z^{n_k}(t))$. Using (A8) and $Z^{n_k}(t) \to Z^\infty(t)$ we obtain for every $\Phi \in \widehat{F}(t)$

$$\Theta(t) = \lim_{k\to\infty} \partial_t\mathcal{E}(t, \Phi^{n_k}(t), Z^{n_k}(t)) = \partial_t\mathcal{E}(t, \Phi, Z^\infty(t)).$$

As in Theorem 1.6.1 we conclude that any selection Φ^∞ of $\widehat{F}$ is a solution of (S) & (E) and all conclusions of Theorem 1.6.1 hold. ■

Using Theorem A.5.6 we obtain immediately the following theorem:

Theorem 1.6.3. *Assume that $(\mathcal{F}, \mathcal{T}_{\mathcal{F}})$ is a separable metric space and all assumptions of the previous theorem hold. Then for each $(\varphi_0, z_0) \in \mathcal{S}(0)$ there exists at least one solution (Φ^∞, Z^∞) of (S) & (E) with $Z^\infty \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z})$, Φ^∞ measurable and $(\Phi^\infty, Z^\infty)(0) = (\varphi_0, z_0)$.*

Moreover, for the above incremental approximations there exists a subsequence $(\Phi^n, Z^n)_{n\in\mathbb{N}}$ with the following convergence properties for $n \to \infty$:

(i) $Z^n \in \mathrm{BV}_{\mathcal{D}}([0,T],\mathcal{Z})$.

(ii) *For all* $t \in [0,T]$ *we have* $Z^n(t) \to Z^\infty(t)$ *in* $\mathcal{Z}$.

(iii) *For* $0 \le r < s \le T$ *we have* $\mathrm{Diss}_{\mathcal{D}}(Z^n,[r,s]) \to \mathrm{Diss}_{\mathcal{D}}(Z^\infty,[r,s])$.

(iv) *For all* $t \in [0,T]$ *we have* $\mathcal{E}(t,\Phi^n(t),Z^n(t)) \to \mathcal{E}(t,\Phi^\infty(t),Z^\infty(t))$.

Proof. In fact, Theorem A.5.6 provides a sequence $(\Phi_l^\infty, Z_l^\infty)_{l\in\mathbb{N}}$ of measurable solutions with

$$\overline{\cup_{l\in\mathbb{N}}\Phi_l^\infty(t)} = \widehat{F}(t)$$

for all $t \in [0,T]$. Here $\widehat{F}$ is a set-valued function defined in the proof of Corollary 1.6.2. ∎

Chapter 2

Functions of bounded variation

In this chapter we formulate some results of $\mathrm{BV}(\Omega)$-theory and the related theory of sets of finite perimeter. We use these results in the next chapter to introduce and to investigate our phase transition model. There exist many classical books about $\mathrm{BV}(\Omega)$-theory (see [Alt99, Fed69, Giu84, Maz85, VH85, Zie89]). In this chapter we follow mainly the excellent monograph [AFP00] on $\mathrm{BV}(\Omega)$-spaces, which also treats general variational problems in BV. All missing proofs in this section can be found in [AFP00], Chapters 3,4,5.

In the first section of this chapter we give a short introduction to BV spaces. We define the space BV as a functions with a measure distributional derivative. We introduce also two new convergence concepts, namely weak* convergence and strict convergence and formulate compactness of embedding in L^1. The second section contains as well finer results on the structure of the distributional derivative as properties of sets of finite perimeter. Finally, in the last section we introduce the space of SBV-functions and give sufficient conditions for lower semicontinuity of integral functionals in SBV with respect to the weak* convergence.

2.1 The space BV and sets of finite perimeter

Throughout this chapter we denote by Ω an open set in $\mathbb{R}^N$.

Definition 2.1.1. A function $u \in \mathrm{L}^1(\Omega, \mathbb{R})$ whose partial derivatives in the sense of distributions are finite Radon measures in Ω is called a function of bounded variation. The vector space of all functions of bounded variation in Ω is denoted by $\mathrm{BV}(\Omega, \mathbb{R})$. Thus $u \in \mathrm{BV}(\Omega, \mathbb{R})$ if and only if there exist some $\mathbb{R}^N$-valued Radon measures $\mathrm{D}u = (\mathrm{D}_1 u, \dots, \mathrm{D}_N u)$ in Ω such that

$$\int_\Omega u \tfrac{\partial \varphi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega \varphi\,\mathrm{d}\mathrm{D}_i u, \qquad i = 1, \dots, N.$$

hold for all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega, \mathbb{R})$.

The last N formulae can be summarised in a single one by writing:

$$\int_\Omega u \operatorname{div} \varphi\,\mathrm{d}x = -\sum_{i=1}^N \int_\Omega \varphi_i\,\mathrm{d}\mathrm{D}_i u \quad \text{for all } \varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega, \mathbb{R}^N).$$

The same notation can be used for functions $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$. In this case $\mathrm{D}u$ is an $m \times N$ matrix of measures $\mathrm{D}_i u^j$ in Ω satisfying

$$\sum_{j=1}^m \int_\Omega u^j \operatorname{div} \varphi^j \, \mathrm{d}x = -\sum_{j=1}^m \sum_{i=1}^N \int_\Omega \varphi_i^j \, \mathrm{d}\mathrm{D}_i u^j \quad \text{for all } \varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega, \mathbb{R}^{m\times N}).$$

Definition 2.1.2. (Variation) Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. The variation $V(u, \Omega)$ of u in Ω is defined by

$$V(u,\Omega) := \sup \left\{ \sum_{j=1}^m \int_\Omega u^j \operatorname{div} \varphi^j \, \mathrm{d}x \mid \varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega, \mathbb{R}^{m\times N}) \text{ with } \|\varphi(x)\|_\infty \leq 1 \text{ for all } x \in \Omega \right\}.$$

The following proposition provides the most important property of variation.

Proposition 2.1.3. *The mapping $u \to V(u, \Omega) \in [0, \infty]$ is lower semicontinuous in the space $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$*

Proof. Consider the sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ which converges to the function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$-topology. For any function $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\Omega, \mathbb{R}^{m\times N})$ with $\|\varphi\|_\infty \leq 1$ we obtain

$$\liminf_{k\to\infty} V(u_k, \Omega) \geq \liminf_{k\to\infty} \sum_{\alpha=1}^m \int_\Omega u_k^\alpha \operatorname{div} \varphi^\alpha \, \mathrm{d}x = \sum_{\alpha=1}^m \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \, \mathrm{d}x.$$

The conclusion follows by taking a supremum over all functions φ with $\|\varphi\|_\infty \leq 1$ in the above estimate. ■

The next theorem builds a connection between the concept of variation and the class of functions of bounded variation.

Theorem 2.1.4. *Let $u \in \mathrm{L}^1(\Omega, \mathbb{R}^m)$. Then, u belongs to $\mathrm{BV}(\Omega, \mathbb{R}^m)$ if and only if $V(u, \Omega) < \infty$. In addition, $V(u, \Omega)$ coincides with $|\mathrm{D}u|(\Omega)$ for any $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ and $u \mapsto |\mathrm{D}u|(\Omega)$ is lower semicontinuous in $\mathrm{BV}(\Omega, \mathbb{R}^m)$ with respect to the $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ topology.*

For a proof see [AFP00], Proposition 3.6.

Remark 2.1.5. The above theorem shows that $\mathrm{BV}(\Omega, \mathbb{R}^m)$, endowed with the norm $\|u\|_{\mathrm{BV}} := \int_\Omega |u| \, \mathrm{d}x + |\mathrm{D}u|(\Omega) = \int_\Omega |u| \, \mathrm{d}x + V(u, \Omega)$, is a Banach space. It provides that the BV space coincides with the set of functions $u \in \mathrm{L}^1(\Omega, \mathbb{R}^m)$, who have the finite variation. Using the equivalence of norms in $\mathbb{R}^{m\times N}$ we can replace the $\|\cdot\|_\infty$-norm in Definition 2.1.2 by an arbitrary norm $\|\cdot\|$ in $\mathbb{R}^{m\times N}$ and obtain a corresponding variation $V_{\|\cdot\|}(u, \Omega)$. This new variation leads to the equivalent norm $\mathrm{BV}(\Omega, \mathbb{R}^m)$, defined by $\|u\|_{\mathrm{BV}} := \int_\Omega |u| \, \mathrm{d}x + V_{\|\cdot\|}(u, \Omega)$. Moreover, the new variation $V_{\|\cdot\|}(u, \Omega)$ is also lower semicontinuous in $\mathrm{BV}(\Omega, \mathbb{R}^m)$ with respect to the $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ topology.

Unfortunately, the norm-topology is not very useful. The following simple example shows that the norm-topology is too strong for our theory.

Example 2.1.6. Consider the functions $u_k = \chi_{(1/2+1/k,1)}$, $k = 1, \ldots$. The sequence (u_k) converges in $L^1(0, 1)$ to the function $u = \chi_{(1/2,1)}$ as $k \to \infty$. But for any k the inequality

$$\|u - u_k\|_{\mathrm{BV}((0,1),\mathbb{R})} \geq 2$$

holds. Since $\|\chi_{(\alpha,1)} - \chi_{(\beta,1)}\|_{\mathrm{BV}} \geq 2$ for $\alpha, \beta \in (0, 1)$ with $\alpha \neq \beta$ we see that the space BV is not not separable.

In order to bypass this problem we introduce weak* convergence and strict convergence in $\mathrm{BV}(\Omega)$. The former is useful for its compactness properties, which play a crucial role in our theory. The latter is useful to prove several identities in BV by smoothing arguments, but it not really needed for our considerations. Both convergences are much weaker than the norm convergence.

Definition 2.1.7. (Weak* convergence) Let $u, u_k \in \mathrm{BV}(\Omega, \mathbb{R}^m)$. We say that $(u_k)_{k\in\mathbb{N}}$ converges weakly* to u in $\mathrm{BV}(\Omega, \mathbb{R}^m)$ if $(u_k)_{k\in\mathbb{N}}$ converges to u in $\mathrm{L}^1(\Omega, \mathbb{R}^m)$ and $(\mathrm{D}u_k)_{k\in\mathbb{N}}$ converges weakly* to $\mathrm{D}u$ in Ω, i.e.

$$\lim_{k\to\infty} \int_\Omega \varphi \,\mathrm{dD}u_k = \int_\Omega \varphi \,\mathrm{dD}u,$$

for all $\varphi \in \mathrm{C}_0(\Omega, \mathbb{R})$.

Definition 2.1.8. (Strict convergence) Let $u, u_k \in \mathrm{BV}(\Omega, \mathbb{R}^m)$. We say that $(u_k)_{k\in\mathbb{N}}$ converges strictly to u in $\mathrm{BV}(\Omega, \mathbb{R}^m)$ if $(u_k)_{k\in\mathbb{N}}$ converges to u in $\mathrm{L}^1(\Omega, \mathbb{R}^m)$ and the variations $|\mathrm{D}u_k|(\Omega)$ converge to $|\mathrm{D}u|(\Omega)$ as $k \to \infty$.

The following theorem gives us an equivalent definition of weak* convergence, which is simpler to verify in many applications.

Theorem 2.1.9. *Let $(u_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega, \mathbb{R}^m)$. Then $(u_k)_{k\in\mathbb{N}}$ weakly* converges to u in $\mathrm{BV}(\Omega, \mathbb{R}^m)$ if and only if $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{BV}(\Omega, \mathbb{R}^m)$ and converges to u in $\mathrm{L}^1(\Omega, \mathbb{R}^m)$.*

For a proof see [AFP00], Proposition 3.13.

By Theorem 2.1.9 strict convergence implies weak* convergence. The opposite implication is not true in general. In fact, the sequence $(\sin(kx)/k)_{k\in\mathbb{N}}$ weakly* converges to 0 in $\mathrm{BV}(0, 2\pi)$, but $|\mathrm{D}u_k|((0, 2\pi)) = 4$ for any k.

Before we study deeper the concept of weak* convergence, we show that the space $\mathrm{BV}(\Omega, \mathbb{R}^m)$ can be considered as the dual of a separable Banach space. Later we show that, if the domain Ω is sufficiently regular, the weak*-convergence in the usual sense coincides with weak*-convergence in the sense of Definition 2.1.7.

Theorem 2.1.10. ***(BV as dual space)*** *Let L be a following subspace of the separable Banach space $X := [C_0(\Omega)]^{N+1}$*

$$L := \left\{ f := (f_0, f_1, \dots, f_N) \in X \mid \hat{f} = (f_1, \dots, f_N) \in [\mathrm{C}_c^\infty(\Omega)]^N, f_0 = \operatorname{div} \hat{f} \right\}.$$

The space $\mathrm{BV}(\Omega, \mathbb{R})$ is isomorphic to the dual of the factor space $X/\overline{L}$. Here $\overline{L}$ is the closure of L in the space X.

Proof. The Riesz representation theorem says that the space X^*, the dual of space X, is the space of finite $\mathbb{R}^{N+1}$-valued Radon measures under the paring

$$\langle f, \mu\rangle := \sum_{l=0}^N \int_\Omega f_l \,\mathrm{d}\mu_l.$$

Moreover, the total variation $|\mu|(\Omega)$ is the dual norm.

For our proof we use that $(X/\overline{L})^* = (\overline{L})^\perp$ with $(\overline{L})^\perp := \left\{ \ell \in X^* \mid \langle \varphi, \ell\rangle = 0 \text{ for all } \varphi \in \overline{L} \right\}$ (for details of proof see Theorem 4.9 in [Rud75]).

We consider now the mapping $T : \mathrm{BV}(\Omega, \mathbb{R}) \to X^*; u \mapsto Tu = (u\mathcal{L}^N, \mathrm{D}_1 u, \ldots, \mathrm{D}_N u)$ and notice furthermore for convenience $u\mathcal{L}^N$ as $\mathrm{D}_0 u$. It is obvious that the mapping T is linear. Using the definition of the BV-norm we obtain

$$\begin{aligned}\|Tu\|_{X^*} &= |Tu|(\Omega) = \sup\Big\{ \textstyle\sum_{i=0}^N \int_\Omega \varphi_i \,\mathrm{d}\mathrm{D}_i u \mid \varphi \in [\mathrm{C_c}(\Omega)]^{N+1}, \|\varphi\|_\infty \le 1 \Big\} \\ &= \sup\Big\{ \textstyle\int_\Omega |u| \,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \hat\varphi_i \,\mathrm{d}\mathrm{D}_i u \mid \hat\varphi \in [\mathrm{C_c^\infty}(\Omega)]^{N}, \|\hat\varphi\|_\infty \le 1 \Big\} = \|u\|_{\mathrm{BV}(\Omega,\mathbb{R})}.\end{aligned}$$

The last estimate shows that the image of T is isomorphic to $\mathrm{BV}(\Omega, \mathbb{R})$.

We have to prove that the image of T is isomorphic to the dual of the factor space $X/\overline{L}$. We proof first that $\mathrm{Image}(T) \subset (X/\overline{L})^*$. The equality

$$\begin{aligned}\langle \varphi, Tu\rangle &= \textstyle\sum_{i=0}^N \int_\Omega \varphi_i \,\mathrm{d}\mathrm{D}_i u = \int_\Omega \varphi_0 u \,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \varphi_i \,\mathrm{d}\mathrm{D}_i u \\ &= \textstyle\int_\Omega \varphi_0 u \,\mathrm{d}x - \sum_{i=1}^N \frac{\partial}{\partial x_i}\varphi_i u \,\mathrm{d}x = \int_\Omega (\varphi_0 - \operatorname{div}\hat\varphi) u \,\mathrm{d}x = 0\end{aligned}$$

holds for any $\varphi \in L$. Hence, $\overline{L}$ is a subset of the set $\ker(Tu)$. This implies that the image of T lies in the dual of the factor space $X/\overline{L}$.

In order to prove $(X/\overline{L})^* \subset \mathrm{Image}(T)$ we have to show that any $\mu \in X^*$ whose kernel contains L is equal to Tu for some $u \in \mathrm{BV}(\Omega, \mathbb{R})$.

Let $\mu := (\mu_0, \mu_1, \ldots, \mu_N) \in X^*$ be such that the kernel μ contains L. This means that

$$\int_\Omega \operatorname{div}\hat\varphi \,\mathrm{d}\mu_0 = -\sum_{i=1}^N \int_\Omega \hat\varphi_i \,\mathrm{d}\mu_i \tag{2.1.1}$$

holds for any $\hat\varphi \in [\mathrm{C_c^\infty}(\Omega)]^N$. The last equality implies that μ_i, $i = 1, \ldots, N$ are partial distributional derivatives of the finite Radon measure μ_0. Consider a sequence of mollifiers $(\rho_{1/h})_{h\in\mathbb{N}}$ and define for any $h \in \mathbb{N}$

$$\mu_0^h = \mu_0 * \rho_{1/h} := \int_\Omega \rho_{1/h}(x-y) \,\mathrm{d}\mu_0(y).$$

Using properties of convolution of functions with Radon measures (see [AFP00], Th. 2.2) we conclude that $\mu_0^h \in C^\infty(\Omega_{1/h})$ for all $h \in \mathbb{N}$ with $\Omega_{1/h} := \big\{ x \in \mathbb{R}^N \mid \operatorname{dist}(x, \Omega) < 1/h \big\}$. Moreover, the measures μ_0^h locally weakly* converge in Ω to μ_0 as $h \to \infty$. Since μ_0 is a finite Radon measure we obtain for any $h \in \mathbb{N}$

$$\begin{aligned}\int_\Omega |\mu_0^h|(x) \,\mathrm{d}x &= \int_\Omega \Big|\int_\Omega \rho_{1/h}(x-y) \,\mathrm{d}\mu_0(y)\Big| \,\mathrm{d}x \le \int_\Omega\int_\Omega \rho_{1/h}(x-y) \,\mathrm{d}|\mu_0|(y) \,\mathrm{d}x \\ &= \int_\Omega\int_\Omega \rho_{1/h}(x-y) \,\mathrm{d}x \,\mathrm{d}|\mu_0|(y) \le |\mu_0|(\Omega).\end{aligned} \tag{2.1.2}$$

Hence, the sequence $(\mu_0^h)_{h\in\mathbb{N}}$ is uniform bounded in $\mathrm{L}^1(\Omega, \mathbb{R})$. We calculate now the partial derivatives of the functions $(\mu_0^h)_{h\in\mathbb{N}}$. Using (2.1.1) we obtain for any $i = 1, \ldots, N$

$$\begin{aligned}\mathrm{D}_i \mu_0^h(x) &= \tfrac{\partial}{\partial x_i} \int_\Omega \rho_{1/h}(x-y) \,\mathrm{d}\mu_0(y) = \int_\Omega \tfrac{\partial}{\partial x_i}\rho_{1/h}(x-y) \,\mathrm{d}\mu_0(y) \\ &= -\int_\Omega \tfrac{\partial}{\partial y_i}\rho_{1/h}(x-y) \,\mathrm{d}\mu_0(y) = \int_\Omega \rho_{1/h}(x-y) \,\mathrm{d}\mu_i(y) = \mu_i * \rho_{1/h}.\end{aligned}$$

Using the last equality we conclude as in (2.1.2) that the inequalities

$$\int_\Omega \big|\mathrm{D}_i \mu_0^h(x)\big| \,\mathrm{d}x \le |\mu_i|(\Omega), \qquad i = 1, \ldots, N$$

hold for any $h \in \mathbb{N}$. Thus the sequence $(\mu_0^h)_{h\in\mathbb{N}}$ is uniform bounded in $\mathrm{BV}(\Omega, \mathbb{R})$. Theorem 2.1.12 implies that there exists a subsequence $\big(\mu_0^{h(k)}\big)_{k\in\mathbb{N}}$ and a function $u \in$

$\mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R})$ such that $\big(\mu_0^{h(k)}\big)_{k\in\mathbb{N}}$ weakly* converges to u. By construction the measures $\mu_0^{h(k)}$ locally weakly* converge in Ω to μ_0 as $k \to \infty$. Hence, $\mu_0 = u\mathcal{L}^N$. Since μ_i, $i = 0, \dots, N$ are finite Radon measures, it follows that $u \in \mathrm{BV}(\Omega, \mathbb{R})$. Moreover, using typical argumention of the theory of distributions (the density function of a regular distribution is unique) we conclude that the function u is unique in L^1-sense. The proof is complete.

■

Remark 2.1.11. We prove Theorem 2.1.10 for the space $\mathrm{BV}(\Omega, \mathbb{R})$. In the vector-valued case we obtain the result by arguing on the single components of the involved BV-functions.

The following compactness theorem for BV functions shows the usefulness of the concept of weak* convergence. This result plays an important role in many variational problems. In particular, we use this result in order to obtain the existence theory for our simple model of phase transitions.

Theorem 2.1.12. ***(Compactness in*** BV***)*** *Consider* $(u_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega, \mathbb{R}^m)$ *such that for any open set* $A \subset\subset \Omega$ *the inequality*

$$\sup\Big\{ \int_A |u_k| \,\mathrm{d}x + |\mathrm{D}u_k|(A) \;\Big|\; k \in \mathbb{N} \Big\} < \infty$$

holds. Then, there exists a subsequence $(u_{k_l})_{l\in\mathbb{N}}$*, which converges in* $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *to a function* $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$*. If* Ω *is a bounded Lipschitz domain and the sequence is bounded in* $\mathrm{BV}(\Omega, \mathbb{R}^m)$ *we can say that* $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ *and that the subsequence weakly* converges to* u*.*

For a proof see [AFP00], Theorem 3.23.

The last result is the reformulation of the well-known fact in the theory of Sobolev spaces, namely the compact embedding of the space $\mathrm{W}^{1,1}$ into the space L^1. Many other results of the theory of Sobolev spaces can also be reformulated for the BV. For example, the embedding theorem and the Poincaré inequality can be proved for BV-functions.

Theorem 2.1.13. ***(Embedding theorem)*** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain. Then the embedding* $\mathrm{BV}(\Omega, \mathbb{R}) \hookrightarrow \mathrm{L}^{1^*}(\Omega)$ *is continuous and the embeddings* $\mathrm{BV}(\Omega, \mathbb{R}) \hookrightarrow \mathrm{L}^p(\Omega)$ *are compact for* $1 \le p < 1^*$*, with* $1^* = \infty$ *if* $N = 1$ *and* $1^* = N/(N-1)$ *otherwise.*

For a proof see [AFP00], Corollary 3.49.

Theorem 2.1.14. ***(Poincaré inequality)*** *If* Ω *is a bounded connected Lipschitz domain, then there exists a real number* C *such that for any* $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ *the inequality*

$$\int_\Omega |u - u_\Omega| \,\mathrm{d}x \le C |\mathrm{D}u|(\Omega),$$

with $u_\Omega := \fint_\Omega u(x) \,\mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega u(x) \,\mathrm{d}x$ *holds.*

For a proof see [AFP00], Theorem 3.44.

The next Theorem gives the condition for the coinciding of weak*-convergence in the sense of Definition 2.1.7 and the weak*-convergence in the sense of duality and legitimates in this way the used notation in Definition 2.1.7.

Theorem 2.1.15. *If Ω a bounded Lipschitz domain, then the weak*-convergence in the sense of duality coincides with weak*-convergence in the sense of Definition 2.1.7.*

Proof. First we show that the weak*-convergence in the sense of Definition 2.1.7 is stronger than the weak*-convergence in the sense of duality. For convenience we notice the weak*-convergence in the sense of Definition 2.1.7 as weak*-def-convergence.

Consider a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega,\mathbb{R}^m)$ which weakly*-def converges to $u \in \mathrm{BV}(\Omega,\mathbb{R}^m)$. Then per definition $(u_k, \mathrm{D}_1 u_k, \ldots, \mathrm{D}_N u_k)$ weakly* converges to $(u, \mathrm{D}_1 u, \ldots, \mathrm{D}_N u)$ in the space $[\mathcal{M}(\Omega)]^{N+1} := ([C_0(\Omega)]^{N+1})^*$ of finite $\mathbb{R}^{N+1}$-valued Radon measures. All u_k's and u lie in $([C_0(\Omega)]^{N+1}/\,\overline{L}\,)^*$, where $\overline{L}$ is the closed subspace of the space $[C_0(\Omega)]^{N+1}$ defined in Theorem 2.1.10. Overall we conclude that

$$(u_k, \mathrm{D}_1 u_k, \ldots, \mathrm{D}_N u_k) \text{ weakly}^* \text{ converges to } (u, \mathrm{D}_1 u, \ldots, \mathrm{D}_N u) \text{ in } ([C_0(\Omega)]^{N+1}/\,\overline{L}\,)^*.$$

Now we show that for bounded Lipschitz domains the weak*-convergence in the sense of duality implies weak*-def-convergence. Consider a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega,\mathbb{R}^m)$ which weakly* converges to $u \in \mathrm{BV}(\Omega,\mathbb{R}^m)$ and assume that the sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega,\mathbb{R}^m)$ does not weakly*-def converge to $u \in \mathrm{BV}(\Omega,\mathbb{R}^m)$. The weak*-convergence implies that the sequence $(\|u_k\|_{\mathrm{BV}(\Omega,\mathbb{R})})_{k\in\mathbb{N}}$ is bounded. By using Theorem 2.1.12 this fact allows us to select a subsequence $(\|u_{k_l}\|_{\mathrm{BV}(\Omega,\mathbb{R}^m)})_{l\in\mathbb{N}}$ which weakly*-def converges to some $\hat{u}$. Our assumption provides $\hat{u} \neq u$. Since weak*-def convergence is stronger than weak* convergence, we get a contradiction. The proof is complete. ■

Since all functions in our transition model take only countably many values, we need the knowledge about the particular class of BV functions, namely the characteristic functions of the sets of finite perimeter. Such functions are the simplest examples of functions of bounded variation, which are not lying in the Sobolev space $\mathrm{W}^{1,1}$. Since many fine results of BV theory are obtained by studying their properties, these functions and corresponded sets of finite perimeter play an important role from the theoretical point of view. We give now only a definition of the sets of finite perimeter and formulate finer results later.

Definition 2.1.16. Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For any open set $\Omega \subset \mathbb{R}^N$ the perimeter of E in Ω, denoted by $P(E,\Omega)$, is the variation of χ_E in Ω, i.e.

$$P(E,\Omega) := \sup\left\{ \textstyle\int_E \operatorname{div}\varphi \,\mathrm{d}x \mid \varphi \in \mathrm{C}^1_\mathrm{c}(\Omega,\mathbb{R}^N) \text{ with } \|\varphi(x)\|_\infty \le 1 \text{ for all } x \in \Omega \right\}.$$

We say that E is a set of finite perimeter in Ω, if $P(E,\Omega) < \infty$.

The sets of finite perimeter are closely related to the theory of BV. In fact, if $|E \cap \Omega|$ is finite, then $\chi_E \in \mathrm{L}^1(\Omega,\mathbb{R})$ and Theorem 2.1.4 implies that E is a finite perimeter set, if and only if χ_E lies in $\mathrm{BV}(\Omega,\mathbb{R})$.

Example 2.1.17. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $E \subset \mathbb{R}^N$ be a bounded set with C^2 boundary. It obvious that $\int_\Omega \chi_E \,\mathrm{d}x = |E \cap \Omega| < \infty$. Moreover, by using the Gauss-Green theorem we obtain that for any $\varphi \in \mathrm{C}^1_\mathrm{c}(\Omega,\mathbb{R}^N)$ the estimate

$$\textstyle\int_\Omega \chi_E \operatorname{div}\varphi \,\mathrm{d}x = \int_E \operatorname{div}\varphi \,\mathrm{d}x = -\int_{\partial E} \langle \nu_E, \varphi \rangle \,\mathrm{d}\mathcal{H}^{N-1}$$

holds. Here, ν_E is the inward unit normal to ∂E. Since $|\nu_E| \equiv 1$ we conclude that

$$P(E,\Omega) \leq \mathcal{H}^{N-1}(\partial E \cap \Omega).$$

Since E has C^2 boundary, ν_E is a C^1 vector valued function. Thus, there exists a function $\hat{\nu} \in \mathrm{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ with $\hat{\nu}(x) = \nu(x)$ for all $x \in \partial E$ such that $|\hat{\nu}(x)| \leq 1$ for all $x \in \mathbb{R}^N$. If $\eta \in \mathrm{C}_\mathrm{c}^\infty(\Omega, \mathbb{R})$ with $|\eta| \leq 1$, then we have

$$\int_E \operatorname{div}(\eta\hat{\nu})\,\mathrm{d}x = \int_{\partial E} \eta\,\mathrm{d}\mathcal{H}^{N-1}.$$

Hence, $P(E,\Omega) = \mathcal{H}^{N-1}(\partial E \cap \Omega)$.

By applying Theorem 2.1.14 to characteristic functions χ_E of sets of finite perimeter we obtain the following interesting theorem.

Theorem 2.1.18. ***(Isoperimetric inequality)*** *Let $N > 1$ be an integer. For any set E of finite perimeter in $\mathbb{R}^N$ either E or $\mathbb{R}^N \setminus E$ has a finite Lebesgue measure and there exists a real constant γ_N such that*

$$\min\left\{|E|, |\mathbb{R}^N \setminus E|\right\} \leq \gamma_N \left(P(E, \mathbb{R}^N)\right)^{N/(N-1)}.$$

For a proof see [AFP00], Theorem 3.46.

In our phase transition model the isoperimetric inequality will be responsible for the phenomenon of finite nucleation size.

2.2 Fine properties of sets of finite perimeter and BV functions

In this section we give some deeper results of the BV theory and the theory of the sets of finite perimeter. These results allow us to formulate the Gauss-Green formula for sets of finite perimeter and to investigate the approximate continuity and differentiability properties of BV functions. Finally, we introduce the decomposition of distributional derivatives in three parts: absolutely continuous part, jump part and Cantor part.

In Example 2.1.17 we have seen that the class of sets of finite perimeter includes the sets E with smooth C^2-boundary and it holds $P(E,\Omega) = \mathcal{H}^{N-1}(\partial E \cap \Omega)$. The following example shows that for non-smooth sets of finite perimeter the latter can not be true. The situation can be even more complex. There exists a set of finite perimeter whose topological boundary has a strictly positive Lebesgue measure.

Example 2.2.1. Let x_i be a sequence of all rational points in $\mathbb{R}^2$. Consider the sequence of open balls B_i with centre x_i and radius 2^{-i} and define the set E as the union of balls B_i. It can be easily seen that E has a finite perimeter and $|E| < \infty$. However, the set E is dense in $\mathbb{R}^2$. Thus $|\partial E| = \infty$ which implies that $\mathcal{H}^1 = \infty$.

The above example motivates to introduce a new concept of boundary, which adopts the role of topological boundary of smooth sets.

Definition 2.2.2. (Reduced boundary) Let E be a Lebesgue measurable subset of R^N and Ω be the largest open set such that E is locally of finite perimeter in Ω. The point x belongs to the reduced boundary $\mathcal{F}E$, if $x \in \operatorname{supp}|\mathrm{D}\chi_E| \cap \Omega$ and the limit

$$\nu_E(x) := \lim_{\rho \searrow 0} \frac{\mathrm{D}\chi_E(B_\rho(x))}{|\mathrm{D}\chi_E|(B_\rho(x))}$$

exists and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : \mathcal{F}E \to \mathbb{S}^{N-1}$ is called the generalised inner normal to E.

The following theorem characterises the reduced boundary and provides the essential properties.

Theorem 2.2.3. ***(De Giorgi)*** *Let E be an $\mathcal{L}^N$-measurable set of $\mathbb{R}^N$. Then the reduced boundary $\mathcal{F}E$ is countably $(N-1)$-rectifiable and $|\mathrm{D}\chi_E| = \mathcal{H}^{N-1} \llcorner \mathcal{F}E$, i.e. $|\mathrm{D}\chi_E|(F) = \mathcal{H}^{N-1}(F \cap \mathcal{F}E)$ (see Definition A.7.1). In addition, for any $x_0 \in \mathcal{F}E$ the following properties hold:*

(a) the sets $(E - x_0)/\rho$ locally converge in measure in $\mathbb{R}^N$ as $\rho \searrow 0$ to the halfspace H orthogonal to $\nu_E(x_0)$ and containing $\nu_E(x_0)$;

(b) $\mathrm{Tan}^{N-1}(\mathcal{H}^{N-1} \llcorner \mathcal{F}E, x_0) = \mathcal{H}^{N-1} \llcorner \nu_E^\perp(x_0)$ (see Definition A.7.7) and

$$\lim_{\rho \searrow 0} \frac{\mathcal{H}^{N-1}(\mathcal{F}E \cap B_\rho(x_0))}{\omega_{N-1}\rho^{N-1}} = 1.$$

Here, ω_{N-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$.

For a proof see [AFP00], Theorem 3.59.

Definition 2.2.4. (Points of density t and essential boundary) For every $t \in [0, 1]$ and every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$ we denote by E^t the set

$$\left\{ x \in \mathbb{R}^N \mid \lim_{\rho \searrow 0} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = t \right\}$$

of all points where E has density t. The essential boundary $\partial^* E$ of E is the set of all points, where the density is neither 1 nor 0.

The sets E^1 and E^0 could be considered as the measure theoretic interior and exterior of E, and this motivate the definition of essential boundary. The following theorem gives the description of density sets.

Theorem 2.2.5. ***(Federer)*** *Let E be a set of finite perimeter in Ω. Then*

$$\mathcal{F}E \cap \Omega \subset E^{1/2} \subset \partial^* E \text{ and } \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \mathcal{F}E \cup E^1)) = 0.$$

In particular, E has density either 1 or 1/2 or 0 at $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E \cap \Omega$ belongs to $\mathcal{F}E$.

For a proof see [AFP00], Proposition 3.61.

Now we can formulate the generalised Gauss-Green formula for a set of finite perimeter E, which follows from Definitions 2.1.16, 2.2.2 and Theorem 2.2.3:

$$\int_E \operatorname{div}\varphi \,\mathrm{d}x = -\int_\Omega \langle \nu_E, \varphi\rangle \,\mathrm{d}|\mathrm{D}\chi_E| = -\int_{\mathcal{F}E} \langle \nu_E, \varphi\rangle \,\mathrm{d}\mathcal{H}^{N-1}.$$

Using Theorem 2.2.5 we can replace $\mathcal{F}E$ in the last integral with $\partial^* E$.

Now we introduce the concept of approximate discontinuity points. After that, we extract the particular class of discontinuity points, namely so-called jump approximate discontinuity points. These correspond to an approximate jump discontinuity between two values a along a direction ν. We also define the sets of approximate differential points. All these sets are used for the introduction of the decomposition of the distributional derivative and allow us to prescribe the subspace of the space BV, which is adequate to our application to the phase transition.

Definition 2.2.6. (Approximate limit) Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. We say that u has an approximate limit at $x \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \searrow 0} \fint_{B_\rho(x)} |u(y) - z| \,\mathrm{d}y = 0. \tag{2.2.1}$$

The set S_u of points where this properties does not hold is called the approximate discontinuity set. For any $x \in \Omega \setminus S_u$ the vector z, uniquely determined by (2.2.1), is called the approximate limit of u and denoted by $\tilde{u}(x)$.

In order to define the set of approximate jump points we use the following convenient notation

$$B^+_\rho(x,\nu) := \{\, y \in B_\rho(x) \mid \langle y - x, \nu\rangle > 0 \,\},$$
$$B^-_\rho(x,\nu) := \{\, y \in B_\rho(x) \mid \langle y - x, \nu\rangle < 0 \,\}.$$

Definition 2.2.7. (Approximate jump points) Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and $x \in \Omega$. The point x is called an approximation jump point of u if there exist $a, b \in \mathbb{R}^m$ with $a \neq b$ and $\nu \in \mathbb{S}^{N-1}$ such that

$$\lim_{\rho \searrow 0} \fint_{B^+_\rho(x,\nu)} |u(y) - a| \,\mathrm{d}y = 0 \qquad \text{and} \qquad \lim_{\rho \searrow 0} \fint_{B^-_\rho(x,\nu)} |u(y) - b| \,\mathrm{d}y = 0. \tag{2.2.2}$$

The triplet (a, b, ν), uniquely determined by (2.2.2) up to permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. We denote the set of all approximate jump points of u by J_u.

It can be proved that the set J_u is the Borel subset of S_u and there exist Borel functions $(u^+(x), u^-(x), \nu_u(x)) : J_u \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{N-1}$ such that (2.2.2) holds at any $x \in J_u$. For details see Proposition 3.69 in [AFP00].

Definition 2.2.8. (Approximate differentiability) Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and let $x \in \Omega \setminus S_u$. The function u is approximately differentiable at x if there exists an $(m \times N)$-matrix L such that

$$\lim_{\rho \searrow 0} \fint_{B_\rho(x)} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{\rho} \,\mathrm{d}y = 0. \tag{2.2.3}$$

The matrix L, uniquely determined by (2.2.3), is called the approximate differential of u at the point x and denoted by $\nabla u(x)$. The set of approximate differentiability points is denoted by $\mathcal{D}_u$.

We use now introduced concepts and formulate results, which enables us to understand the structure of functions of bounded variation.

Theorem 2.2.9. *Let $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$. Then $|\mathrm{D}u| \geq |u^+ - u^-|\mathcal{H}^{N-1} \llcorner J_u$ and for any Borel set $B \subset \Omega$ the following two implications hold:*

$$\mathcal{H}^{N-1}(B) = 0 \Longrightarrow |\mathrm{D}u|(B) = 0;$$
$$\mathcal{H}^{N-1}(B) < \infty \text{ and } B \cap S_u = \emptyset \Longrightarrow |\mathrm{D}u|(B) = 0.$$

For a proof see [AFP00], Lemma 3.76.

Theorem 2.2.10. ***(Federer-Vol'pert)*** *For any $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ the discontinuity set S_u is countably $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. In addition, $\mathrm{D}u \llcorner J_u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner J_u$ and for $\mathcal{H}^{N-1}$ – a.e. $x \in J_u$ holds:*

$$\begin{aligned}\mathrm{Tan}^{N-1}(J_u, x) &= \nu_u(x)^{\perp}, \\ \mathrm{Tan}^{N-1}(|\mathrm{D}u| \llcorner J_u, x) &= |u^+(x) - u^-(x)|\mathcal{H}^{N-1} \llcorner \nu_u^{\perp}.\end{aligned}$$

For a proof see [AFP00], Proposition 3.78.

Theorem 2.2.11. ***(Calderón-Zygmund)*** *Any function $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ is approximately differentiable at $\mathcal{L}^N$-almost every point of Ω. In addition, the approximate differential ∇u is the density of the absolutely continuous part of $\mathrm{D}u$ with respect to $\mathcal{L}^N$.*

For a proof see [AFP00], Proposition 3.83.

The last part of this section is devoted to the decomposition of distributional derivatives of functions of bounded variation. This decomposition allows us to introduce in the next section a particular class of BV functions, which is suitable for the modelling of phase distribution in our application.

According to the Radon-Nikodým Theorem the distributional derivative $\mathrm{D}u$ can be split into the absolutely continuous part $\mathrm{D}^a u$ with respect to $\mathcal{L}^N$ and the singular part $\mathrm{D}^s u$ with respect to $\mathcal{L}^N$. Moreover, we can split the singular part $\mathrm{D}^s u$ into the jump part $\mathrm{D}^j u$ and the Cantor part $\mathrm{D}^c u$. This splitting is well-known from the theory of BV functions of one variable. The famous Cantor-Vitali function provides an example of a function, whose distributional derivative has no absolutely continuous part and no jump part. The following definition gives the formal description of the jump part and the Cantor part.

Definition 2.2.12. (Jump and Cantor parts) For any $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ the measures

$$\mathrm{D}^j u := \mathrm{D}^s u \llcorner J_u \text{ and } \mathrm{D}^c := \mathrm{D}^s \llcorner (\Omega \setminus S_u)$$

are called the jump part of the derivative and the Cantor part of the derivative.

Using the Federer-Vol'pert Theorem and Theorem 2.2.9 we obtain immediately the following decomposition of $\mathrm{D}u$:

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u = \mathrm{D}^a u + \mathrm{D}^j u + D^c u.$$

2.3 The space SBV and semicontinuity in BV

Since all functions in our transition model take only countably many values, the space BV is too general for our purpose. In this section we consider an adequate restriction of BV, namely the space of so-called piecewise constant BV functions. We introduce this space as a subspace of the space SBV of special BV functions and consider related compactness results. Finally, we give a result, which provides sufficient conditions for lower semicontinuity of integral functionals in SBV.

Definition 2.3.1. A function $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ is called a special function with bounded variation, if the Cantor part of its derivate $\mathrm{D}^c u$ is zero. We denote the space of such functions as $\mathrm{SBV}(\Omega, \mathbb{R}^m)$.

Remark 2.3.2. It can be easily proved that the space $\mathrm{SBV}(\Omega, \mathbb{R}^m)$ is a closed subspace of $\mathrm{BV}(\Omega, \mathbb{R}^m)$.

In view of applications to variational problems, it is important to have closure and compactness properties of the space $\mathrm{SBV}(\Omega, \mathbb{R}^m)$ with respect to weak* convergence in $\mathrm{BV}(\Omega, \mathbb{R}^m)$. The two following theorems give sufficient conditions for such properties.

Theorem 2.3.3. ***(Closure of*** **SBV*)*** *Let $\varphi : [0, \infty) \to [0, \infty], \theta : (0, \infty) \to (0, \infty]$ be lower semicontinuous increasing functions and assume that*

$$\lim_{t\to\infty} \varphi(t)/t = \infty \quad \text{and} \quad \lim_{t\to 0} \theta(t)/t = \infty.$$

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and let $(u_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}(\Omega, \mathbb{R}^m)$ be such that

$$\sup_k \left\{ \int_\Omega \varphi(|\nabla u_k|)\,\mathrm{d}x + \int_{J_{u_k}} \theta(|u_k^+ - u_k^-|)\,\mathrm{d}\mathcal{H}^{n-1} \right\} < \infty. \tag{2.3.1}$$

If (u_k) weakly converges to u in $\mathrm{BV}(\Omega, \mathbb{R}^m)$, then $u \in \mathrm{SBV}(\Omega, \mathbb{R}^m)$, the approximate gradients ∇u_k weakly converge to ∇u in $\mathrm{L}^1(\Omega, \mathbb{R}^{m\times N})$ and $\mathrm{D}^j u_h$ weakly* converge to $\mathrm{D}^j u$ in Ω. If the function φ is convex then*

$$\int_\Omega \varphi(|\nabla u|)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega \varphi(|\nabla u_k|)\,\mathrm{d}x$$

additionally holds. If the function θ is concave then

$$\int_{J_u} \theta(|u^+ - u^-|)\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{k\to\infty} \int_{J_{u_k}} \theta(|u_k^+ - u_k^-|)\,\mathrm{d}\mathcal{H}^{n-1}$$

additionally holds.

For a proof see [AFP00], Theorem 4.7.

Theorem 2.3.4. ***(Compactness of*** SBV) *Let φ, θ, Ω be as in Theorem 2.3.3. Let a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}(\Omega, \mathbb{R}^m)$ be satisfying (2.3.1) and assume, in addition, that $\|u\|_\infty$ is uniformly bounded in k. Then, there exists a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ weakly* converging to $u \in \mathrm{SBV}(\Omega, \mathbb{R}^m)$ in $\mathrm{BV}(\Omega, \mathbb{R}^m)$.*

For a proof see [AFP00], Theorem 4.8.

Now we present the properties of partitions of a domain Ω in the sets of finite perimeter. Such partitions are called Caccioppoli partitions. By using such partitions we would be able to define the piecewise constant functions, i.e. SBV functions whose level sets generate a Caccioppoli partition.

Definition 2.3.5. (Partition) Let $\Omega \subset \mathbb{R}^N$. We call a family $(E)_{i\in I}$ of sets a partition of Ω, if

$$\bigcup_{i\in I} E_i = \Omega \qquad \text{and} \qquad E_i \cap E_j = \emptyset \text{ for } i \neq j.$$

Definition 2.3.6. (Caccioppoli partitions) Let $\Omega \subset \mathbb{R}^N$ be an open set and $I \subset \mathbb{N}$. The partition $\{E_i\}_{i\in I}$ of Ω is a Caccioppoli partition if $\sum_{i\in I} P(E_i, \Omega) < \infty$. A Caccioppoli partition is called ordered if $|E_i| \geq |E_j|$ for any $i \leq j$.

The next theorem describes the structure of countable Caccioppoli partitions.

Theorem 2.3.7. ***(Local structure of Caccioppoli partitions)*** *Let $\{E_i\}_{i\in\mathbb{N}}$ be a Caccioppoli partition of Ω. Then*

$$\bigcup_{i\in I} (E_i)^1 \cup \bigcup_{i,j\in I, i\neq j} \mathcal{F}E_i \cap \mathcal{F}E_j$$

contains $\mathcal{H}^{N-1}$-almost all of Ω. Here the sets $(E_i)^1$ are the measure theoretic interiors of the sets E_i, see Definition 2.2.4.

For a proof see [AFP00], Theorem 4.17.

Definition 2.3.8. (Piecewise constant functions) The function $u : \Omega \to \mathbb{R}^m$ is called piecewise constant in Ω if there exist a Caccioppoli partition $\{E_i\}_{i\in I}$ of Ω and a map $t : I \to \mathbb{R}^m$ such that

$$u = \sum_{i\in I} t_i \chi_{E_i}.$$

Using the compactness result for $\mathrm{SBV}(\Omega, \mathbb{R}^m)$ we can obtain the following compactness result for piecewise constant functions.

Theorem 2.3.9. ***(Compactness of piecewise constant functions)*** *Let Ω be a bounded open set with Lipschitz boundary. Let $(u_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}(\Omega, \mathbb{R}^m)$ be a sequence of piecewise constant functions such that $(\|u_k\|_\infty + \mathcal{H}^{N-1}(S_{u_k}))_{k\in\mathbb{N}}$ is bounded. Then, there exists a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ converging in L^1 to a piecewise constant function u.*

For a proof see [AFP00], Theorem 4.25.

In Chapter 1 we use the direct method of calculus of variation in order to construct a solution of the evolution problem (S) & (E). The usage of such methods requires that the minimised functional I has s-compact sublevels $\{\, u \mid I(u) < \alpha \,\}$. Such conditions are mainly guaranteed by enforcing the sequential lower semicontinuity of the minimised functional. In the last part of this section we present the result, which provides sufficient conditions for the sequential lower semicontinuity of functionals defined on the set of piecewise constant functions with respect to the weak* topology. Finally, we give examples of sequential lower semicontinuous functionals, which are suitable for our phase transformation model discussed below.

Definition 2.3.10. (Joint convexity) Let $K \subset \mathbb{R}^m$ be compact and $\phi : K \times K \times \mathbb{R}^N \to [0, \infty]$. The function ϕ is called jointly convex if there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset \mathrm{C}(K, \mathbb{R}^N)$ such that the estimate

$$\phi(i, j, p) = \sup_{k\in\mathbb{N}} \langle g_k(i) - g_k(j), p\rangle$$

holds for any $(i, j, p) \in K \times K \times \mathbb{R}^N$.

It follows immediately from the definition that the necessery conditions for the joint convexity of ϕ are lower semicontinuity (as supremum of lower semicontinuous functions) and

$$\begin{cases} \phi(i, i, p) = 0 \text{ for all } i \in K, p \in \mathbb{R}^N, \\ \phi(i, k, p) \le \phi(i, j, p) + \phi(j, k, p), \\ \phi(i, j, \cdot) \text{ convex and positively 1-homogeneous for all } i, j \in K. \end{cases}$$

Example 2.3.11. Let $K \subset \mathbb{R}^m$ be compact, let $\theta : K \times K \to [0, \infty)$ be continuous and $\psi : \mathbb{R}^N \to [0, \infty]$ be lower semicontinuous. Assume additionally that θ is a positive, symmetric function satisfying the triangle inequality and ψ is even, positively 1-homogeneous and convex. The function

$$\phi(i, j, p) := \theta(i, j)\psi(p)$$

is jointly convex.

In fact, according to Theorem A.2.1 there exists a sequence $(\gamma_l)_{l\in\mathbb{N}} \subset \mathbb{R}^N$ such that for any $p \in \mathbb{R}^N$ holds $\psi(p) = \sup\langle\gamma_k, p\rangle$. We take a countable dense sequence $(c_h)_{h\in\mathbb{N}} \subset K$ and define

$$g_{h,l} : K \to \mathbb{R}^N \; ; \; c \mapsto \theta(c, c_h)\gamma_l.$$

For any $(i, j, p) \in K \times K \times \mathbb{R}^N$ we obtain $\phi(i, j, p) = \sup_{h,l\in\mathbb{N}}\langle g_{h,l}(i) - g_{h,l}(j), p\rangle$.

If K is a finite set and ϕ is real valued the joint convexity can be restated by the following theorem

Theorem 2.3.12. *Let $K \subset \mathbb{R}^m$ be a finite set and $\phi : K \times K \times \mathbb{R}^N \to [0, \infty)$. Then, ϕ is jointly convex if and only if $\phi(i, j, p) = \phi(j, i, -p)$ and*

$$\sum_{n=0}^{r} \phi(i_n, j_n, p_n) \ge \phi(i, j, p)$$

for all $i, j \in K$, $p \in \mathbb{R}^N$ and all $i_k, j_k \in K$, $p_k \in \mathbb{R}^N$ with $k = 1, \ldots, r$ such that

$$\sum_{k=1}^{r} (\delta_{i_k} - \delta_{j_k})p_k = (\delta_i - \delta_j)p \quad \in [\mathcal{M}(\mathbb{R}^m)]^N. \tag{2.3.2}$$

Here $\delta_c \in \mathcal{M}(\mathbb{R}^m)$, $c \in \mathbb{R}^m$ denotes the Dirac measure in c, i.e. $\delta_c(F) = 1$ if $c \in F$ and $\delta_c(F) = 0$ otherwise.

The condition (2.3.2) can be considered as a generalised subadditivity condition. For a proof see [AFP00], Proposition 5.19.

The next theorem provides sufficient conditions for lower semicontinuity of integral functionals on $\mathrm{SBV}(\Omega, \mathbb{R}^m)$.

Theorem 2.3.13. *Let $K \subset \mathbb{R}^m$ be compact and $\phi : K \times K \times R^N \to [0, \infty]$ be a jointly convex function. Assume additionally that there exists a constant $c > 0$ such that the estimate*

$$\phi(i, j, p) \geq c|p|$$

holds for any (i, j, p) with $i \neq j$. Let $(u_k)_{k \in \mathbb{N}} \subset \mathrm{SBV}(\Omega, \mathbb{R}^m)$ be a sequence converging to u in $\mathrm{L}^1(\Omega, \mathbb{R}^m)$, such that $(\nabla u_k)_{k \in \mathbb{N}}$ is equiintegrable and, for any $k \in \mathbb{N}$, $u_k(x) \in K$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. Then $u \in \mathrm{SBV}(\Omega, \mathbb{R}^m)$ with $u(x) \in K$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and

$$\int_{J_u} \phi(u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{N-1} \leq \liminf_{k \to \infty} \int_{J_{u_k}} \phi(u_k^+, u_k^-, \nu_{u_k}) \, \mathrm{d}\mathcal{H}^{N-1}.$$

For a proof see [AFP00], Theorem 5.22.

In our phase transition model the set K is often finite and the function ϕ has the form:

$$\phi(i, j, p) = \theta(i, j)|p|.$$

Here θ is a positive, symmetric function satisfying the triangle inequality. In this case the lower semicontinuity of the functional $u \to \int_{J_u} \phi(u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{N-1}$ can be also proved by defining a new variation as mentioned in Remark 2.1.5. It provides the possibility to consider intergral functionals on the space of piecewise constant functions without disscusion of finer properties of $\mathrm{BV}(\Omega, \mathbb{R})$-functions (another idea of proof can be found in [Bal90]).

Theorem 2.3.14. *Let $K \subset \mathbb{R}^m$ be a finite set and $\phi : K \times K \times \mathbb{R}^N \to [0, \infty)$ such that*

$$\phi(i, j, p) = \theta(i, j)|p|,$$

where $\theta : K \times K \to [0, \infty)$ is a positive, symmetric function satisfying the triangle inequality. Then, there exists a norm $\|\cdot\|_{m^* \times N}$ on $\mathbb{R}^{m \times N}$ such that for any piecewise constant function $u \in \mathrm{SBV}(\Omega, \mathbb{R}^m)$ with values in K holds:

$$\int_{J_u} \phi(u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{N-1} = \tilde{V}(u, \Omega)$$

with

$$\tilde{V}(u, \Omega) := \sup \left\{ \sum_{j=1}^m \int_\Omega u^j \operatorname{div} \varphi^j \, \mathrm{d}x \mid \varphi \in \mathrm{C}_c^\infty(\Omega, \mathbb{R}^{m \times N}), \|\varphi(x)\|_{m^* \times N} \leq 1 \text{ for all } x \in \Omega \right\}.$$

Proof. Without loss of generality we can assume that the set K is given as a set of unit vectors $\{e_1, \ldots, e_m\}$ in $\mathbb{R}^m$. Theorem A.1.4 implies that there exists a norm $\|\cdot\|_m$ in $\mathbb{R}^m$ with

$$\|e_i - e_j\|_m = \theta(e_i, e_j).$$

We denote with $\|\cdot\|_N$ the Euclidean norm $\mathbb{R}^N$ and define the new variation by

$$\tilde{V}(u, \Omega) := \sup \left\{ \sum_{j=1}^m \int_\Omega u^j \operatorname{div} \varphi^j \, \mathrm{d}x \mid \varphi \in \mathrm{C}_c^\infty(\Omega, \mathbb{R}^{m \times N}), \|\varphi\|_{m^* \times N} \leq 1 \right\}.$$

Here, the norm $\|\cdot\|_{m^* \times N}$ in $\mathbb{R}^{m \times N}$ is defined as in Definition A.1.1. The proof is now done in three steps.

Step 1: $\big(\tilde{V}(u,\Omega) \le \int_{J_u} \phi(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}\big)$
Proposition 2.1.3 implies that $\tilde{V}$ is lower semicontinuous. Moreover, if $E_1, E_2, \dots, E_m$ is a Caccioppoli partition of Ω and if a piecewise constant function $u \in \mathrm{SBV}(\Omega,\mathbb{R}^m)$ is given by

$$u : \Omega \to \mathbb{R}^m \ ; \ x \mapsto \big\{ \ e_i \quad \text{for } x \in E_i,$$

then $\int_{J_u} \phi(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{N-1} = \tilde{V}(u,\Omega)$. In fact, for any function $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\Omega,\mathbb{R}^{m\times N})$ with $\|\varphi\|_{m^*\times N} \le 1$ holds

$$\begin{aligned}
\sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \,\mathrm{d}x &= -\int \varphi : \,\mathrm{d}\mathrm{D}u = -\int_{J_u} \varphi : (u^+ - u^-) \otimes \nu_u \,\mathrm{d}\mathcal{H}^{N-1} \\
&= \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{m} \int_{\mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega} (e_i - e_j)^{\mathrm{T}} \varphi \nu_u \,\mathrm{d}\mathcal{H}^{N-1} \le \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{m} \int_{\mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega} \|e_i - e_j\|_m \|\varphi\nu_u\|_{m^*} \,\mathrm{d}\mathcal{H}^{N-1} \\
&\le \sum_{\substack{i,j=1\\ i<j}}^{m} \int_{\mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega} \|e_i - e_j\|_m \|\varphi\|_{m^*\times N} \|\nu_u\|_N \,\mathrm{d}\mathcal{H}^{N-1} \le \sum_{\substack{i,j=1\\ i<j}}^{m} \int_{\mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega} \|e_i - e_j\|_m \,\mathrm{d}\mathcal{H}^{N-1} \\
&= \int_{J_u} \phi(u^+,u^-,\nu_u) \,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned}$$

Hence, $\tilde{V}(u,\Omega) \le \int_{J_u} \phi(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}$.

Step 2: $\big($**Special case** $\tilde{V}(u,\Omega) \ge \int_{J_u} \phi(u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}\big)$
Consider a particular piecewise function $\hat{u}$ given by

$$\hat{u} : \Omega \to \mathbb{R}^m \ ; \ x \mapsto \begin{cases} e_i & \text{for } x \in \Omega_1, \\ e_j & \text{for } x \in \Omega_2. \end{cases}$$

Here, Ω_1, Ω_2 is a Caccioppoli partition of Ω. Since $\hat{u}$ is a piecewise constant function, it holds $\mathcal{H}^{N-1}(J_{\hat{u}}) < \infty$. Additionally, we assume that there exists a C^1-function $g : \mathbb{R}^{N-1} \to \mathbb{R}^N \ ; \ y \mapsto (y, \hat{g}(y))$ such that $J_{\hat{u}} \subset g(R^{N-1})$ and $g : g^{-1}(J_u) \to \mathbb{R}^N$ is a Lipschitz function.

Using our assumption on $J_{\hat{u}}$ we conclude that there exists a continuous function $\nu_{\hat{u}} : g(\mathbb{R}^{N-1}) \to \mathbb{R}^N$ such that for each $y \in J_{\hat{u}}$ the vector $\nu_{\hat{u}}(y)$ is the normal unit vector to $J_{\hat{u}}$ in y which is directed into Ω_1. Starting from the function g we define the extension function g_{ext} as

$$g_{\text{ext}}; \mathbb{R}^N \to \mathbb{R}^N \ ; \ (x_1,\dots,x_N) \mapsto (x_1,\dots,x_{N-1}, \hat{g}(x_1,\dots,x_{N-1}) + x_N).$$

Obviously the function g_{ext} is a homeomorphism of $\mathbb{R}^N$ on $\mathbb{R}^N$. Using the function g_{ext} we can extend $\nu_{\hat{u}}$ continuously on the whole $\mathbb{R}^N$. For this we consider the continuous function $\nu_u \circ g_{\text{ext}} : \big\{\, x \in \mathbb{R}^N \mid x_N = 0 \,\big\} \to \mathbb{R}^N$ and extend continuously this function with constant values in e_N direction, i.e.

$$\eta : \mathbb{R}^N \to \mathbb{R}^N \ ; \ (x_1,\dots,x_N) \mapsto \nu_u \circ g_{\text{ext}}(x_1,\dots,x_{N-1},0).$$

Using properties of the functions g_{ext} and η we conclude immediately that the function $\eta \circ (g_{\text{ext}})^{-1} : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous extension of the function $\nu_{\hat{u}} : g(\mathbb{R}^{N-1}) \to \mathbb{R}^N$. For simplicity we denote this extension again by $\nu_{\hat{u}}$.

Let now $\varepsilon > 0$. Our assumptions on $J_{\hat{u}}$ implies that there exists a compact set $K \subset \mathbb{R}^{N-1}$ such that $\mathcal{H}^{N-1}(J_{\hat{u}} \setminus g(K)) < \varepsilon$. By using a suitable cut-off function we can construct a continuous function $\gamma \in \mathrm{C_c}(\Omega, \mathbb{R}^N)$ with $\sup_{x\in\Omega} \|\gamma(x)\|_N \leq 1$, such that the estimate

$$\gamma(x) = \nu_{\hat{u}}(x).$$

holds for any $x \in g(K)$. Let $b \in \mathrm{C_c}(\Omega, \mathbb{R}^m)$ with $\|b(x)\|_{m^*} \leq 1$ for any $x \in \Omega$. Proposition A.1.3 implies that for a function $\varphi := -b \otimes \gamma$ holds $\|\varphi\|_{m^* \times N} \leq 1$. Consider a sequence of mollifiers $(\rho_{1/h})_{h\in\mathbb{N}}$ and define $\varphi_h := \varphi * \rho_{1/h}$. Then $\|\varphi_h\|_{m^*\times N} \leq 1$ holds. Using uniform convergence φ_h to φ and testing with φ_h we obtain

$$\begin{aligned}\tilde{V}(\hat{u}, \Omega) &\geq \lim_{h\to\infty} \int_\Omega \sum_{j=1}^m \hat{u}^j \operatorname{div} \varphi_h^j \,\mathrm{d}x = \lim_{h\to\infty} -\int_{J_{\hat{u}}} \varphi_h : (e_i - e_j) \otimes \nu_{\hat{u}} \,\mathrm{d}\mathcal{H}^{N-1} \\ &\geq \int_{J_{\hat{u}}} \langle e_i - e_j, b(x)\rangle \,\mathrm{d}\mathcal{H}^{N-1} - 2\int_{J_{\hat{u}}\setminus g(K)} |\langle e_i - e_j, b(x)\rangle| \,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Since $\|e_i - e_j\|_m = \sup_{\|b\|_{m^*}\leq 1} \langle e_i - e_j, b\rangle$ and ε is arbitrary small, we conclude

$$\tilde{V}(\hat{u}, \Omega) \geq \int_{J_{\hat{u}}} \theta(e_i, e_j) \,\mathrm{d}\mathcal{H}^{N-1} = \int_{J_{\hat{u}}} \phi(\hat{u}^+, \hat{u}^-, \nu_{\hat{u}}) \,\mathrm{d}\mathcal{H}^{N-1}.$$

Step 3: $\big($**General case** $\tilde{V}(u, \Omega) \geq \int_{J_u} \phi(u^+, u^-, \nu_u) \,\mathrm{d}\mathcal{H}^{N-1}\big)$
Let now $u \in \mathrm{SBV}(\Omega, \mathbb{R}^m)$ be an arbitrary piecewise constant function with values in $\{e_1, \ldots, e_m\}$. From Definition 2.3.6 follows immediately that $\mathcal{H}^{N-1}(J_u) < \infty$. Moreover, by the Federer-Vol'pert Theorem 2.2.10 the set J_u is countably $\mathcal{H}^{N-1}$-rectifiable. Theorem 2.3.7 implies that $J_u = \cup_{i,j=1}^m \mathcal{F}E_i \cap \mathcal{F}E_j$. Here (E_i) is a Caccioppoli partition, which consists of the level sets of the function u, i.e. $E_i = \{\, x \in \Omega \mid u(x)u = e_i \,\}$. Hence, the sets $\mathcal{F}E_i \cap \mathcal{F}E_j$ are measurable and countably $\mathcal{H}^{N-1}$-rectifiable. It is obvious that the sets $\mathcal{F}E_i \cap \mathcal{F}E_j$ with $1 \leq i < j \leq m$ are disjoint.

Using Theorem A.7.6 we find, for all $1 \leq i < j \leq m$ with $i < j$, that there exist countably many Lipschitz $(N-1)$-graphs $\Gamma_l^{ij} \subset \mathbb{R}^N$, $l \in \mathbb{N}$ such that

$$\mathcal{H}^{N-1}\Big((\mathcal{F}E_i \cap \mathcal{F}E_j) \setminus \bigcup_{l\in\mathbb{N}} \Gamma_l^{ij}\Big) = 0$$

and the Lipschitz constants of the corresponding functions are less the 1.

In next step we use the so-called Whitney extension theorem (see Theorem A.7.8) in order to replace the Lipschitz graphs with C^1-graphs. In fact, for i, j, l with $1 \leq i < j \leq m$ and $l \in \mathbb{N}$ there exist an $(N-1)$-plane pi_l^{ij} in $\mathbb{R}^N$ and a Lipschitz function $f_l^{ij} : \pi_l^{ij} \to \big(\pi_l^{ij}\big)^\perp$ such that

$$\Gamma_l^{ij} := \Big\{\, x \in \mathbb{R}^N \mid f_l^{ij}(\pi_l^{ij} x) = \big(\pi_l^{ij}\big)^\perp x \,\Big\}.$$

According to the collorary of the Whitney extension theorem A.7.9 there exist C^1-functions $g_l^{ij} : \pi_l^{ij} \to \big(\pi_l^{ij}\big)^\perp$ such that for all $1 \leq i < j \leq m$ and $l \in \mathbb{N}$

$$\mathcal{L}^{N-1}(\{\, x \in \pi_l^{ij} \mid g_l(x) \neq f_l(x) \,\}) < \frac{\varepsilon}{2^{l+N} m^2}$$

holds. Furthermore, we denote the graphs of the functions g_l^{ij} by $\widetilde{\Gamma}_l^{ij}$. Using Theorem A.7.3 and the properties of functions f_l^{ij} we conclude that

$$\mathcal{H}^{N-1}(\Gamma_l^{ij} \setminus \widetilde{\Gamma}_l^{ij}) < \frac{\varepsilon}{2^{l+1}m^2}.$$

It follows immediately that the estimate

$$\mathcal{H}^{N-1}\Big((\mathcal{F}E_i \cap \mathcal{F}E_j) \setminus \bigcup_{l\in\mathbb{N}} \widetilde{\Gamma}_l^{ij}\Big) < \frac{\varepsilon}{2m^2}.$$

holds for all i, j with $1 \leq i < j \leq m$. The last estimation implies that there exists a number $N(\varepsilon) \in \mathbb{N}$ such that

$$\mathcal{H}^{N-1}\Big((\mathcal{F}E_i \cap \mathcal{F}E_j) \setminus \bigcup_{l=1}^{N(\varepsilon)} \widetilde{\Gamma}_l^{ij}\Big) < \frac{\varepsilon}{m^2}.$$

Hence, there exist disjoint compact sets K_l^{ij}, with $1 \leq i < j \leq m$ and $l = 1, \dots, N(\varepsilon)$, such that $K_l^{ij} \subset \big(\Gamma_l^{ij} \cap \widetilde{\Gamma}_l^{ij}\big) \cap \big(\mathcal{F}E_i \cap \mathcal{F}E_j\big)$ and

$$\mathcal{H}^{N-1}\Big((\mathcal{F}E_i \cap \mathcal{F}E_j) \setminus \bigcup_{l=1}^{N(\varepsilon)} K_l^{ij}\Big) < \frac{2\varepsilon}{m^2}.$$

Summarily, we have

$$\mathcal{H}^{N-1}\Big(J_u \setminus \bigcup_{\substack{i,j=1\\ i<j}}^{m} \bigcup_{l=1}^{N(\varepsilon)} K_l^{ij}\Big) < \varepsilon.$$

The last estimate means that we approximate J_u with a finite number of disjoint compact sets, which are given as subsets $(N-1)$-graphs of C^1-functions. Now the conclusion follows as before. In fact, using the compactness of the sets K_l^{ij} with $1 \leq i < j \leq m$ and $l = 1, \dots, N(\varepsilon)$ we find disjoint open sets Ω_l^{ij} such that $K_l^{ij} \subset \Omega_l^{ij} \subset \Omega$. We construct $\varphi_l^{ij} \in \mathrm{C_c}(\Omega_l^{ij}, \mathbb{R}^m)$ like in the special case and define $\varphi \in \mathrm{C_c}(\Omega, \mathbb{R}^m)$ by

$$\varphi(x) = \begin{cases} \varphi_l^{ij} & x \in \Omega_l^{ij}, \\ 0 & \text{else.} \end{cases}$$

Then the proof follows like Step 2 by testing with φ.

∎

Chapter 3

Phase transition model

In this chapter we formulate a simple phase transition model for materials, whose elastic properties depend on an actual crystallographic phase state. Such classes of materials, so-called smart materials, are treated in particular in the context of modelling of shape-memory alloys and are a subject of theoretical and experimental research since the invention of the first material NiTi, whose elastic strongly depends on an actual grid structure [BW65].

Our model is a microscopic one. It means that, in each microscopic point $x \in \Omega$, an elastic material is free to choose one of m crystallographic phases Z and that the elastic energy density W depends on a grid structure and is then given by $W_j(\mathrm{D}\varphi)$, $j = 1, \ldots, m$. We assume that the phase transformation process is rate-independent and write the evolution problem in the (S) & (E) formulation.

The idea to use energetic formulation in order to study the phase transformation process was applied previously in [MTL02, GMH02]. In these papers the authors consider mesoscopic level model, where the phase state in each material point x is given as a mixture of pure phases. More specifically, the phase state is modelled through an internal variable $z(x) \in \mathbb{R}^m$, whose j-component $z^{(j)}(x)$ prescribes the portion of the j-th phase, i.e $z(x) \in Z = \left\{ z \in [0,1]^m \subset \mathbb{R}^m \mid \sum_1^m z^{(j)} = 1 \right\}$. Hence, the set of admissible internal states is given as $\mathcal{Z} = \mathrm{L}^1(\Omega, Z) \subset \mathrm{L}^1(\Omega, \mathbb{R}^m)$. The material properties are given via a mixture function $W : \mathbb{R}^{d\times d} \times Z \to [0, \infty]$, for details see [MTL02, GMH02]. It was shown, that the dissipation has the form $D(z_0, z_1) = \int_\Omega d(z_1(x){-}z_0(x))\,\mathrm{d}x$ with $d(v) = \max\{\, \sigma_l \cdot v \mid l = 1, \ldots, L \,\} \geq C_d|v|$, where $\sigma_m \in \mathbb{R}^m$ are thermodynamically conjugated threshold values. In difference to the model which is presented here, the model in [MTL02] neglects in favour of laminate modelling the energy stored in the phase interface. At present, the existence results for the mesoscopic model is not proved. However, the case with only two phases $m = 2$ has been treated in [MTL02] under the additional assumption that the elastic behaviour is linear and both phases have the same elastic tensor. Using a careful analysis of H-measures the authors were able to prove the existence of solutions in this case. Unfortunately, it was shown in [Pre03] that the used methods can not be generalised to the case $m > 2$. Moreover, an example of non-closed stable set $\mathcal{S}_{[0,T]}$ (cf. Chapter 1), those closedness is crucial for the general existence theory in Chapter 1, was constructed.

In a microscopic model there are no phase mixtures allowed, i.e., we assume $z \in Z := \{e_1, e_2, \ldots, e_m\} \subset \mathbb{R}^m$, where e_j is the j-th unit vector. Thus, the functions $z \in \mathcal{Z}$ are like characteristic functions which indicate exactly one phase at each material point. The dissipation is assumed as above, but now the elastic energy contains an additional term measuring the energy, which is stored in the surface area of the interfaces between the different regions, and is given by

$$\mathcal{E}(t,\varphi,z) = \int_\Omega W(\mathrm{D}\varphi,z)\,\mathrm{d}x + \int_{J_z} \phi(z^+,z^-)\,\mathrm{d}\mathcal{H}^{d-1} - \langle \ell_{\mathrm{ext}}(t),\varphi\rangle,$$

with z^+, z^-, J_z as in Chapter 2.

The main aim of this chapter is to prove the existence results for phase transformation processes. In Theorems 3.2.2 and 3.3.1 we provide conditions, which are sufficient for the existence of a solution for the evolution problem. Additionally, we consider an one-dimensional example, which demonstrates the main features of our model, namely hysteretic behaviour and finite size nucleation. The last one is the consequence of an interplay between the interface energy term and the dissipation energy and can be expected due to the isoperimetric inequality (see. Theorem 2.1.18).

3.1 Mathematical setup of the model

We consider an elastic body $\Omega \subset \mathbb{R}^d$ given as a bounded Lipschitz domain. The set Ω prescribes the reference configuration of an elastic deformable body. The elastic deformation is denoted by $\varphi : \Omega \to \mathbb{R}^d$, and $F = \mathrm{D}\varphi$ refers to the strain tensor. We assume that the set of possible solid phases is at most countable and we denote this set as Z. For convenience we assume that the set Z is a bounded subset of some Euclidean space $\mathbb{R}^m$. In the case of a finite number m of crystallographic phases we identify simply the phase j with the unit vector e_j in $\mathbb{R}^m$. In this case the set Z takes also the form $Z := \{e_1, e_2, \ldots, e_m\} \subset \mathbb{R}^m$, where e_j is the j-th unit vector. Thus, the internal variable $z : \Omega \to Z$ prescribes on the microscopic level the phase state at every point of the body. We assume that the internal variable z is a piecewise constant function in the sense of Definition 2.3.6. This means that the level sets

$$E_c := \{\, x \in \Omega \mid z(x) = c \,\}$$

induce the Caccioppoli partition of Ω. We denote by $\mathcal{Z}$ the class of all piecewise constant functions with values in Z. Using the definitions and the theorems of the previous chapter the set $\mathcal{Z}$ can be also defined as:

$$\mathcal{Z} = \left\{\, z \in \mathrm{BV}(\Omega,\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega,Z) \mid \mathrm{D}z \text{ is concentrated on } J_z \text{ and } \mathcal{H}^{d-1}(J_z) < \infty \,\right\}.$$

The elastic material properties are given by the stored energy density $W_c(F)$, which depends on the phase state $c \in Z$. For a given internal variable $z \in \mathcal{Z}$ and $\varphi \in \mathrm{W}^{1,p}(\Omega)$ the elastic bulk energy takes the form

$$\mathcal{E}_{\mathrm{elast}}(\varphi,z) = \int_\Omega \sum_{c\in Z} W_c(x,\mathrm{D}\varphi(x))\chi_{E_c}\,\mathrm{d}x = \int_\Omega W(x,\mathrm{D}\varphi(x),z(x))\,\mathrm{d}x.$$

The boundary conditions on the deformation φ are given through the set of admissible deformations $\mathcal{F}$. In this paper we consider only the case of time-independent Dirichlet boundary conditions. This means that the set $\mathcal{F}$ has the following form

$$\mathcal{F} = \left\{\, \varphi \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^d) \mid (\varphi - \varphi_{\mathrm{Dir}})|_{\Gamma_{\mathrm{Dir}}} = 0 \,\right\},$$

where $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ has a positive surface measure and $\varphi_{\mathrm{Dir}} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)$ is fixed. We understand this condition in the sense of the Sobolev trace theorem. It means that

$$Tr(\varphi) \equiv Tr(\varphi_{\mathrm{Dir}}) \text{ in } \mathrm{L}^p(\partial\Omega, \mathbb{R}^d),$$

where $Tr : W^{1,p}(\Omega, \mathbb{R}^d) \to \mathrm{L}^p(\partial\Omega, \mathbb{R}^d)$ is the unique trace operator (see [Alt99], A. 6.6). This implies that the set $\mathcal{F}$ is weakly closed with respect to the weak $\mathrm{W}^{1,p}$-topology. Throughout this chapter we assume that $p > 1$.

In our model we want that the whole stored energy has a term $\mathcal{E}_{\mathrm{inter}}$ which depends on the interfaces between regions with different phase states. We assume that this term has the following form

$$\mathcal{E}_{\mathrm{inter}}(z) := \sum_{c_1 \in Z} \sum_{c_2 \neq c_1} \int_{\mathcal{F}E_{c_1} \cap \mathcal{F}E_{c_2} \cap \Omega} \psi(c_1, c_2, \nu_{E_{c_1}})\, \mathrm{d}\mathcal{H}^{d-1}$$

for some function

$$\psi : Z \times Z \times \mathbb{S}^{d-1} \to [0, \infty]$$

with $\psi(c, c, \nu) = 0$ for any $c \in Z$ and $\nu \in \mathbb{S}^{d-1}$. The set $\mathcal{F}E_c$ is the reduced boundary of the set E_c in the sense of Definition 2.2.2. The function ψ can be interpreted as the interface energy density and it can be easily extended on the whole space $Z \times Z \times \mathbb{R}^d$ by setting

$$\psi(c_1, c_2, \nu) = \begin{cases} |\nu|\psi(c_1, c_2, \frac{\nu}{|\nu|}) & \text{if } \nu \neq 0, \\ 0 & \text{if } \nu = 0. \end{cases}$$

Moreover, by setting $\phi(c_1, c_2, \nu) = \psi(c_1, c_2, \nu) + \psi(c_2, c_1, -\nu)$ and using Theorems 2.2.10 and 2.3.7 we obtain

$$\mathcal{E}_{\mathrm{inter}}(z) = \int_{J_z} \phi(z^+, z^-, \nu_z)\, \mathrm{d}\mathcal{H}^{d-1}.$$

Finally, the stored energy $\mathcal{E}$ takes the form

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(x, \mathrm{D}\varphi, z) + \int_{J_z} \phi(z^+, z^-, \nu_z)\, \mathrm{d}\mathcal{H}^{d-1} - \langle \ell_{\mathrm{ext}}(t), \varphi \rangle.$$

The last term on the right side arises from the potential of external forces. We assume that the external loading ℓ_{ext} is smooth enough, i.e. $\ell_{\mathrm{ext}} \in \mathrm{Lip}([0, T], W^{1,p}(\Omega, \mathbb{R}^d)^*)$.

The next ingredient of our mathematical model is the dissipation energy. We assume that the dissipation energy is given by

$$\mathcal{D}(z_1, z_2) = \int_\Omega D(z_1, z_2)\, \mathrm{d}x$$

for some function

$$D : Z \times Z \to [0, \infty],$$

with $D(c, c) = 0$ for any $c \in Z$ and D satisfies the triangle inequality, i.e

$$D(c_1, c_3) \leq D(c_1, c_2) + D(c_2, c_3) \tag{3.1.1}$$

for any $c_1, c_2, c_3 \in Z$. The triangle inequality for D is a natural assumption in our model. In fact, this means that the energy, which is needed for the direct phase change c_1-c_3, is not greater than the energy, which is required for the phase change c_1-c_3 via a temporary state c_2. Additionally, we assume that there exists a constant $\gamma_1 > 0$ such that for any $c_1, c_2 \in Z$ holds

$$D(c_1, c_2) \geq \gamma_1 |c_1 - c_2|.$$

We recall that the dissipation along a path $z : [0,T] \to \mathcal{Z}$ over the interval $[t_0, t_1]$ is defined as

$$\mathrm{Diss}_{\mathcal{D}}(z; [s,t]) = \sup \Big\{ \textstyle\sum_1^N \mathcal{D}(z(\tau_{j-1}), z(\tau_j)) \mid N{\in}\mathbb{N}, s{=}\tau_0{<}\tau_1{<}\cdots{<}\tau_N{=}t \Big\}$$

and a pair $(\varphi, z) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ is called a solution of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{D}$ if (S) and (E) hold:

(S) Stability: *For all $t \in [0,T]$ and all $(\widetilde{\varphi}, \widetilde{z}) \in \mathcal{F} \times \mathcal{Z}$ we have*
$\mathcal{E}(t, \varphi(t), z(t)) \leq \mathcal{E}(t, \widetilde{\varphi}, \widetilde{z}) + \mathcal{D}(z(t), \widetilde{z})$.

(E) Energy equality: *For all $t \in [0,T]$ we have*
$\mathcal{E}(t, \varphi(t), z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0,t]) = \mathcal{E}(0, \varphi(0), z(0)) - \int_0^t \langle \dot{\ell}_{\mathrm{ext}}(\tau), \varphi(\tau) \rangle \,\mathrm{d}\tau$.

For this definition we implicitly require that $\tau \to \langle \dot{\ell}_{\mathrm{ext}}(\tau), \varphi(\tau) \rangle$ is measurable. Moreover, condition **(E)** implies that the dissipation $\mathrm{Diss}_{\mathcal{D}}(z, [0,t])$ is finite. This leads immediately to $z \in \mathrm{BV}([0,T], \mathrm{L}^1(\Omega, \mathbb{R}^m))$.

3.2 Convex case

Now we want to apply the general theory developed in Chapter 1, Section 1.4 in order to prove the existence of solutions for our phase transition model. For this we have to equip the sets $\mathcal{F}$ and $\mathcal{Z}$ with suitable topologies. In our case we take the weak topology in $\mathcal{F} \subset \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)$ and the strong L^1-topology in $\mathcal{Z} \subset \mathrm{BV}(\Omega, \mathbb{R}^m)$.

We recall that the stability condition **(S)** can be reformulated by defining the stable sets

$$\begin{aligned}
\mathcal{S}(t) &:= \{ (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \mid \mathcal{E}(t, \varphi, z) \leq \mathcal{E}(t, \widehat{\varphi}, \widehat{z}) + \mathcal{D}(\varphi, z, \widehat{\varphi}, \widehat{z}) \text{ for all } (\widehat{\varphi}, \widehat{z}) \in \mathcal{F} \times \mathcal{Z} \}, \\
\mathcal{S}_{[0,T]} &:= \{ (t, \varphi, z) \in [0,T] \times \mathcal{F} \times \mathcal{Z} \mid (\varphi, z) \in \mathcal{S}(t) \} = \cup_{t \in [0,T]} (t, \mathcal{S}(t)).
\end{aligned}$$

In this notation condition **(S)** means that $(\varphi(t), z(t))$ lies in $\mathcal{S}(t)$ for every $t \in [0,T]$. In order to proof the stability of one point $(\varphi(t), z(t))$ at time t, we need to test with an arbitrary pair (φ, z). If we have some information about $\mathcal{S}(t)$, we can reduce the test set using the following result, which states that is suffices to prove stability by testing with stable states only.

Theorem 3.2.1. *Assume that $\mathcal{E}(t, \cdot, \cdot) + \mathcal{D}(y, \cdot)$ is s-lower semicontinuous and coercive. Then the following holds:*

$(\varphi^*, z^*) \notin \mathcal{S}(t)$ if and only if there exists $(\varphi_s, z_s) \in \mathcal{S}(t)$ such that
$\mathcal{E}(t, \varphi^*, z^*) > \mathcal{E}(t, \varphi_s, z_s) + \mathcal{D}(z^*, z_s)$.

Proof. Let $y^* \notin \mathcal{S}(t)$. Using Theorem 2.1.12 and the classical result of direct methods (see Th. A.6.1) we conclude that the set

$$M := \arg\min \{ \mathcal{E}(t,\varphi,z) + \mathcal{D}(z^*,z) \mid (\varphi,z) \in \mathcal{F} \times \mathcal{Z} \}$$

is not empty. For $(\varphi_s, z_s) \in M$ and any pair $(\widehat{\varphi}, \widehat{z})$

$$\begin{aligned} \mathcal{E}(t,\widehat{\varphi},\widehat{z}) + \mathcal{D}(z_s,\widehat{z}) &= \mathcal{E}(t,\widehat{\varphi},\widehat{z}) + \mathcal{D}(z^*,\widehat{z}) + \mathcal{D}(z_s,\widehat{z}) - \mathcal{D}(z^*,\widehat{z}) \\ &\geq \mathcal{E}(t,\varphi_s,z_s) + \mathcal{D}(z^*,z_s) + \mathcal{D}(z_s,\widehat{z}) - \mathcal{D}(z^*,\widehat{z}) \geq \mathcal{E}(t,\varphi_s,z_s). \end{aligned}$$

holds. Hence, the "$\Longrightarrow$" direction is proved. The "$\Longleftarrow$" direction follows immediately from the definition of stable sets. ∎

Theorem 3.2.2. *Let $Z \subset \mathbb{R}^m$ be compact and $D : Z \times Z \to [0,\infty)$ be a function satisfying (3.1.1) such that the estimate*

$$\gamma_1 |c_1 - c_2| \leq D(c_1,c_2) \leq \gamma_2 |c_1 - c_2| \tag{3.2.1}$$

holds for some $\gamma_1, \gamma_2 > 0$. Let $\phi : Z \times Z \times \mathbb{R}^d \to [0,\infty]$ be a jointly convex function. Assume that there exists $\gamma > 0$ such that for all $c_1, c_2 \in Z$ with $c_1 \neq c_2$ and $\nu \in \mathbb{R}^d$ holds:

$$\phi(c_1,c_2,\nu) \geq \gamma|\nu|.$$

Assume that the dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty)$ is defined by

$$\mathcal{D} : (z_1,z_2) \mapsto \int_\Omega D(z_1(x), z_2(x))\,\mathrm{d}x$$

and the energy functional $\mathcal{E} : [0,T] \times \mathcal{F} \times \mathcal{Z} :\to \mathbb{R}$ is given by

$$\mathcal{E}(t,\varphi,z) = \textstyle\int_\Omega W(x,\mathrm{D}\varphi,z)\,\mathrm{d}x + \int_{J_z} \phi(z^+,z^-,\nu_z)\,\mathrm{d}\mathcal{H}^{d-1} - \langle \ell_{ext}(t), \varphi\rangle.$$

Additionally, assume that the following conditions hold:

(a) The functional $(\varphi,z) \to \int_\Omega W(x,\mathrm{D}\varphi,z)$ is s-lower semicontinuous on $\mathcal{F} \times \mathcal{Z}$ (with respect to the $\mathfrak{T}_\mathcal{F} \times \mathfrak{T}_\mathcal{Z}$-topology).

(b) For some real numbers $C_1 > 0$, $C_2 > 0$ we have for all $x \in \Omega$, $A \in \mathbb{R}^{d\times d}$, $z \in \mathbb{R}^m$

$$W(x,A,z) \geq C_1|A|^p - C_2.$$

(c) For every $z \in \mathcal{Z} \cap \mathrm{BV}(\Omega,\mathbb{R}^m)$ and $t \in [0,T]$ the functional $\mathcal{E}_{t,z} : \mathcal{F} \to \mathbb{R}; \varphi \mapsto \mathcal{E}(t,\varphi,z)$ has a unique minimiser.

Then, for every pair $(\varphi_0,z_0) \in \mathcal{S}(0)$ there exists at least one solution (φ,z) with $\varphi \in \mathrm{L}^\infty([0,T],\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d))$, $z \in \mathrm{BV}([0,T],\mathrm{L}^1(\Omega)) \cap \mathrm{B}([0,T],\mathrm{BV}(\Omega))$ of (S)&(E) with $(\varphi,z)(0) = (\varphi_0,z_0)$ and all the conclusions of Theorem 1.4.3 also hold.

Proof. In the proof we denote with $B_R^X(0)$ the norm-closed ball with radius R and centre 0 in some Banach space X.

Estimate (3.2.1) implies that the dissipation distance $\mathcal{D}$ is continuous with respect to the L^1-topology. Using Theorem 2.3.13 and condition (a) we obtain that the functional

$\mathcal{E}$ is s-lower semicontinuous on $[0,T]\times\mathcal{F}\times\mathcal{Z}$. For an arbitrary triplet $(t,\varphi,z)\in\mathcal{S}_{[0,T]}$ we have by stability

$$\begin{aligned}\mathcal{E}(t,\varphi,z) &\leq \mathcal{E}(t,\varphi_0,z_0)+\mathcal{D}(z,z_0)\\ &\leq \mathcal{E}(0,\varphi_0,z_0)+TC_{\ell_{\text{ext}}}\|\varphi_0\|_{W^{1,p}(\Omega,\mathbb{R}^d)}+\gamma_2\operatorname{diam}(Z)\mathrm{Vol}(\Omega).\end{aligned}$$

Combining this estimate with condition (b) we obtain

$$C_1\|\mathrm{D}\varphi\|^p_{L^p}-C_2\mathrm{Vol}(\Omega)+\gamma\int_{\mathrm{J_z}}\mathrm{d}\mathcal{H}^{\mathrm{d}-1}-\mathrm{C}_{\ell_{\text{ext}}}\|\varphi\|_{\mathrm{W^{1,p}(\Omega,\mathbb{R}^d)}}\leq\mathrm{B}_{\mathcal{E}}$$

with $B_{\mathcal{E}}=\mathcal{E}(0,\varphi_0,z_0)+TC_{\ell_{\text{ext}}}\|\varphi_0\|_{W^{1,p}(\Omega,\mathbb{R}^d)}+\gamma_2\operatorname{diam}(Z)\mathrm{Vol}(\Omega)$. Using the Poincaré inequality and $p>1$ we conclude that there exists $R_{\text{st}}>0$ such that

$$\mathcal{S}_{[0,T]}\subset[0,T]\times B^{\mathrm{W}^{1,p}}_{R_{\text{st}}}(0)\times B^{\mathrm{BV}(\Omega,\mathbb{R}^m)}_{R_{\text{st}}}(0). \tag{3.2.2}$$

We restrict now our spaces $(\mathcal{F},\mathfrak{T}_{\mathcal{F}})$ and $(\mathcal{Z},\mathfrak{T}_{\mathcal{Z}})$ to the spaces

$$(\mathcal{F}_{\text{res}},\mathfrak{T}_{\mathcal{F}}):=(\mathcal{F}\cap B^{\mathrm{W}^{1,p}}_{R_{\text{st}}}(0),\mathfrak{T}_{\mathcal{F}})\text{ and }(\mathcal{Z}_{\text{res}},\mathfrak{T}_{\mathcal{Z}}):=(\mathcal{Z}\cap B^{\mathrm{BV}(\Omega,\mathbb{R}^m)}_{R_{\text{st}}}(0),\mathfrak{T}_{\mathcal{Z}}).$$

For these restricted spaces we can consider the associated (S) & (E) problem. Furthermore, we call this problem the restricted (S) & (E) problem. Theorems 2.3.9 and 2.1.4 imply that the space $(\mathcal{Z}_{\text{res}},\mathfrak{T}_{\mathcal{Z}})$ is a compact topological space. The set $\mathcal{F}_{\text{res}}$ is a bounded, weakly closed subset of the reflexive Banach space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)$. Using Alaoglu's Theorem we conclude that the space $(\mathcal{F}_{\text{res}},\mathfrak{T}_{\mathcal{F}})$ is also a compact topological space. Using Theorem 3.2.1 we obtain that the stable set $\mathcal{S}_{[0,T]}$ of the restricted (S) & (E) problem is the same as in the original problem. This means that a solution of the restricted (S) & (E) problem is also a solution of the original (S) & (E) problem. We prove now that the restricted (S) & (E) problem has a solution.

We set $C_{\mathcal{E}}:=C_{\ell_{\text{ext}}}R_{\text{st}}$. It is obvious that condition (A2) in Theorem 1.4.3 holds. Theorem 1.5.1 implies the closedness of the stable set $\mathcal{S}_{[0,T]}$ with respect to $\mathfrak{T}_{\mathcal{F}}\times\mathfrak{T}_{\mathcal{Z}}$. Using the last observation we obtain immediately that the assumption (A5) in Theorem 1.4.3 holds. All other assumptions of the existence result in the convex case (Th. 1.4.3) can also be easily verified. Hence, there exists a solution of the restricted (S) & (E) problem, which is on the other hand a solution of the original problem. Moreover, Theorem 1.4.3 implies that our solution $(\varphi^\infty,z^\infty)$ is obtained as a limit of piecewise constant incremental approximations $(\varphi^N,z^N)_{N\in\mathbb{N}}$, i.e.

$$\begin{aligned}\varphi^N(t)&\rightharpoonup\varphi^\infty(t)\ \text{ in }\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)\text{ and}\\ z^N(t)&\to z^\infty(t)\ \text{ in }\mathrm{L}^1(\Omega,\mathbb{R}^m)\end{aligned}$$

for all $t\in[0,T]$. Hence, φ^∞ is weakly measurable. Using Pettis Theorem A.4.2, separability of the reflexive Banach space $\mathrm{W}^{1,p}(\Omega,R^d)$ and (3.2.2) we follow immediately that

$$\varphi^\infty\in\mathrm{L}^\infty([0,T],\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)).$$

■

Remark 3.2.3. The convergence in z coincides with weak*-convergence in the sense of Definition 2.1.7. In fact, we have, for all $t \in [0,T]$, $Z^N(t) \to z^\infty(t)$ in $B_{R_{\rm st}}^{\mathrm{BV}(\Omega,\mathbb{R}^m)}(0)$ with respect to L^1-topology. Hence, Theorem 2.1.9 implies the weak*-convergence in $\mathrm{BV}(\Omega,\mathbb{R}^m)$.

Combining the fact that $\mathrm{BV}(\Omega,\mathbb{R}^m)$ is a dual space for some separable Banach space X (see Theorem 2.1.10), and proof techniques for Pettis Theorem, we obtain the following result.

Theorem 3.2.4. *The solution (φ,z) of (S) & (E) obtained in Theorem 3.2.2 satisfies:*
(a) z is weakly measurable in $\mathrm{BV}(\Omega,\mathbb{R}^m)$, i.e. $t \mapsto \langle f, z(t)\rangle$ is measurable for any $f \in X$.*
(b) $\|z(\cdot)\|_{\mathrm{BV}(\Omega,\mathbb{R}^m)} \in \mathrm{L}^\infty([0,T],\mathbb{R})$.

Proof. (a) In the following we use the notation of the previous theorem. The constructed in Theorem 3.2.2 solution (φ,z) was obtained as a limit of the incremental solution (φ^N, z^N) with

$$z^N(t) \to z^\infty(t) \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^m) \text{ with respect to } \mathrm{L}^1\text{-topology.}$$

Moreover, for all $t \in [0,T]$ and all $N \in \mathbb{N}$ it holds that

$$\|z^N(t)\|_{\mathrm{BV}(\Omega)} \le \mathbb{R}_{\rm st},$$
$$\|z(t)\|_{\mathrm{BV}(\Omega)} \le \mathbb{R}_{\rm st},$$

with positive constant $\mathbb{R}_{\rm st}$ as in Theorem 3.2.2. Using Theorems 2.1.9 and 2.1.15 we conclude that for all $t \in [0,T]$

$$z^{N_k}(t) \xrightarrow{*} z(t) \in \mathrm{BV}(\Omega,\mathbb{R}^m).$$

Since all functions z^N, $N \in \mathbb{N}$ are finite-valued, conclusion (a) follows.
(b) Since X is separable, there exists a countable set $M = \{\, f_i \in X \mid \|f_i\|_X \le 1,\ i \in \mathbb{N} \,\}$, such that $B_1^X(0) = \overline{M}$. Let $a \in \mathbb{R}$. We consider the sets $A = \left\{\, t \in [0,T] \mid \|z(t)\|_{\mathrm{BV}(\Omega)} \le a \,\right\}$ and $A_f = \{\, t \in [0,T] \mid\ |\langle z(t), f\rangle| \le a \,\}$, where $f \in X$. It follows immediately that

$$A = \bigcap\nolimits_{\|f\|_X \le 1} A_f.$$

It is also clear that $\bigcap_{\|f\|_X \le 1} A_f \subset \bigcap_{i\in\mathbb{N}} A_{f_i}$. For any $f \notin M$ with $\|f\| \le 1$ there exists a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ such that $f_{i_k} \to f$. For $t \in \bigcap_{i\in\mathbb{N}} A_{f_i}$ holds:

$$\langle z(t), f\rangle = \lim_{k\to\infty} \langle z(t), f_{i_k}\rangle \le a.$$

Hence, t lies in A_x. Using $A = \cap_{i\in\mathbb{N}} A_{f_i}$ we conclude now that A is measurable. This proves (b). ∎

Remark 3.2.5. Since the simple running front $t \mapsto \chi_{[0,t]}$ has a non-separable range in $\mathrm{BV}(\Omega)$ we can not expect $z \in \mathrm{L}^\infty([0,T],\mathrm{BV}(\Omega,\mathbb{R}^m))$ in general.

Remark 3.2.6. The statements on measurability obtained in Theorems 3.2.2 and 3.2.4 are based only on the separability, reflexivity or on the opportunity to consider the related space as a dual space of some separable Banach space. This means that under similar assumptions we can also obtain the same statements in the abstract general case.

Before formulating the existence result for phase transformation in the non-convex case we give a simple example for phase transition in an one-dimensional bar made of a material with two different crystallographic phases.

Example 3.2.7. Consider an one-dimensional bar with length l, which is mounted at point 0. Assume that the material of the bar can take two different crystallographic phases and denote these phases by 0 and 1. Assume that the storage energy and the dissipation are given by

$$\begin{aligned}\mathcal{E}(t,\varphi,z) &:= \int_0^l (\frac{1}{2}(\varphi')^2 + az - b\varphi' z)\,\mathrm{d}x + \sigma\#J + \int_0^l f_{\mathrm{ext}}(t,x)\varphi\,\mathrm{d}x - \lambda(t)\varphi(l),\\ \mathcal{D}(z_0,z_1) &= \int_0^l \delta|z_0 - z_1|\,\mathrm{d}x.\end{aligned}$$

Here a, b and δ are some real numbers and $\#J$ is equal to the number of phase changes. The parameter σ is the price of one phase change. The functions f_{ext} and λ can be interpreted as an external force and an external contact force respectively. The existence result guarantees the existence of (S) & (E)-solutions for this problem. We show now that this simple model has hysteretic behaviour and that the phenomenon of finite nucleation can be observed as well. We make our computation for special values of parameters, which we fix later.

The stability condition **(S)** provides that for fixed time t and fixed phase distribution z the functional $\mathcal{E}(t,\cdot,z)$ satisfies the Euler-Lagrange equation

$$(\varphi' - bz)' = f_{\mathrm{ext}}, \tag{3.2.3}$$

$$\varphi(0) = 0, \tag{3.2.4}$$

$$(\varphi' - bz)(l) = \lambda(t). \tag{3.2.5}$$

Using (3.2.3) and (3.2.5) we obtain the unique solution $\varphi = \Phi(z)$ from

$$\varphi(0) = 0 \text{ and } \varphi' - bz = F(t,x) := \lambda(t) - \textstyle\int_x^l f_{\mathrm{ext}}(t,\xi)\,\mathrm{d}\xi. \tag{3.2.6}$$

By partial integration we calculate

$$\begin{aligned}\mathcal{E}(t,\Phi(z),z) &= \textstyle\int_0^l (\frac{1}{2}(\varphi')^2 + az - b\varphi' z)\,\mathrm{d}x + \sigma\#J + \int_0^l f_{\mathrm{ext}}(t,x)\varphi\,\mathrm{d}x - \lambda(t)\varphi(l)\\ &= \textstyle\int_0^l (\frac{1}{2}(\varphi')^2 + az - b\varphi' z)\,\mathrm{d}x + \sigma\#J + \int_0^l F(t,x)\varphi' + F(t,x)\varphi|_0^l - \lambda(t)\varphi(l)\\ &= \textstyle\int_0^l (\frac{1}{2}(\varphi')^2 + az - b\varphi' z)\,\mathrm{d}x + \sigma\#J - \int_0^l (\varphi' - bz)\varphi'\,\mathrm{d}x\\ &= \textstyle\int_0^l (-\frac{1}{2}(\varphi')^2 + az)\,\mathrm{d}x + \sigma\#J = \int_0^l (-\frac{1}{2}(bz + F(t,x))^2 + az)\,\mathrm{d}x + \sigma\#J\end{aligned}$$

Since z takes the values in $\{0,1\}$ we rewrite the last estimate as follows

$$\mathcal{E}(t,\Phi(z),z) = \int_0^l \left\{ \left(a - \frac{1}{2}b^2 - bF(t,x) \right) z - \frac{1}{2}F(t,x)^2 \right\} \mathrm{d}x + \sigma\#J.$$

Since the term on the right side does not depend on the deformation φ, the last formula allows us to make easy numerical experiments. For a special choice of parameters

$f_{\text{ext}}(t,\xi) := 2(\xi - \frac{l}{2})$, $b := 1$, $a := \frac{1}{2} + \left(\frac{l}{2}\right)^2$ we obtain the following simple expression for $\mathcal{E}(t, \Phi(z), z)$

$$\mathcal{E}(t, \Phi(z), z) = \int_0^l z \left[\left(x - \frac{l}{2} \right)^2 - \lambda(t) \right] \mathrm{d}x + \frac{1}{2} \int_0^l F(t,x)^2 \, \mathrm{d}x.$$

The second term on the right side does depend on z. Hence, this term does not influence the solution of the (S) & (E)-problem. The first term shows that new nucleus can appear only in the middle of the bar. The following pictures show phase transformation process of a bar with length $l = 5$ under the external contact force $\lambda(t) := \max\{t, 16 - t\}$. We assume that at the time $t := 0$ the whole bar was in phase 0. The price of one phase change σ is taking equal to 0.05. Figure 3.1 presents the contribution of phases in the bar in dependence on the external loading λ. The points in phase 0 are represented in black, the points in phase 1 are given in grey.

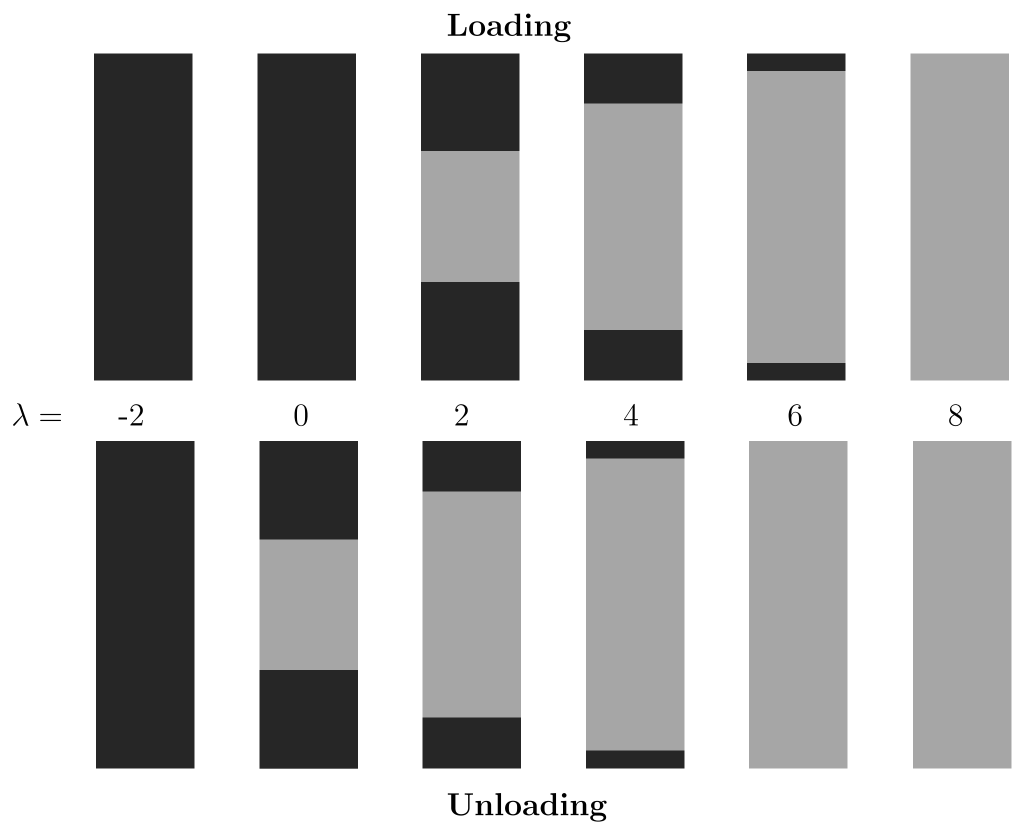

Figure 3.1: Phase distribution in dependence on time

Figure 3.2 presents the same process as a diagram of the length of 1-phase in dependence on the external contact loading $\lambda(t)$. Here the lower (grey) path corresponds to the loading phase. The upper (black) path corresponds to the unloading phase. We see a typical hysteretic loop. The jumps match the appearance and disappearance of new phase boundaries, i.e. the appearance and disappearance of a nucleus of finite size.

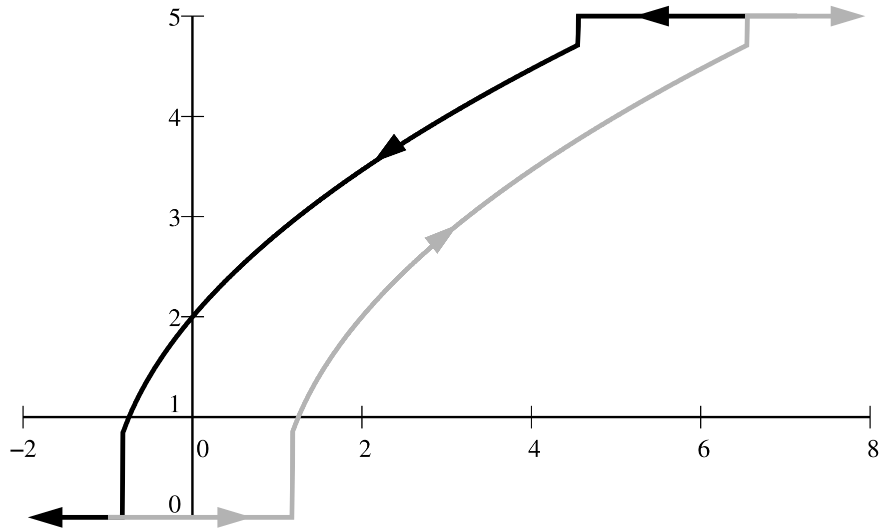

Figure 3.2: Length of 1-phase in dependence on the contact loading

3.3 Non-convex case

Condition (c) of Theorem 3.2.2 is really strong and unsatisfying in many cases. This condition is obviously not fulfilled for the case of finite elasticity. In this section we use the abstract existence theory in the non-convex case in order to obtain an existence result for our model in the non-convex case, which allows us to consider polyconvex energy densities.

Theorem 3.3.1. *Let Z, D, ϕ, $\mathcal{D}$ and $\mathcal{E}$ be given as in Theorem 3.2.2. Additionally, assume that the following conditions hold:*

(a) The functional $(\varphi, z) \to \int_\Omega W(x, \mathrm{D}\varphi, z)$ is s-lower semicontinuous on $\mathcal{F} \times \mathcal{Z}$ (with respect to the $\mathfrak{T}_{\mathcal{F}} \times \mathfrak{T}_{\mathcal{Z}}$-topology).

(b) For some real numbers $C_1 > 0$, $C_2 > 0$ we have for all $x \in \Omega$, $A \in \mathbb{R}^{d\times d}$, $z \in \mathbb{R}^m$

$$W(x, A, e_j) \geq C_1 |A|^p - C_2, \quad j = 1, \dots, m.$$

(c) $\ell_{\mathrm{ext}} \in \mathrm{C}^1([0,T], W^{1,p}(\Omega, \mathbb{R}^d)^*)$. Then, for every pair $(\varphi_0, z_0) \in \mathcal{S}(0)$ there exists at least one solution (φ, z) with $\varphi \in \mathrm{L}^\infty([0,T], \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d))$, $z \in \mathrm{BV}([0,T], \mathrm{L}^1(\Omega)) \cap \mathrm{B}([0,T], \mathrm{BV}(\Omega))$ of (S) & (E) with $(\varphi, z)(0) = (\varphi_0, z_0)$ and all the conclusions of Theorems 1.6.3 and 3.2.4 hold.

Proof. The proof follows the proof of Theorem 3.2.2. We conclude as in the convex case that the energy functional $\mathcal{E}$ is s-lower semicontinuous and that the dissipated energy $\mathcal{D}$ is continuous. As in Theorem 3.2.2 there exists the positive number R_{st} such that

$$S_{[0,T]} \subset [0,T] \times B_{R_{\mathrm{st}}}^{\mathrm{W}^{1,p}}(0) \times B_{R_{\mathrm{st}}}^{\mathrm{BV}(\Omega,\mathbb{R}^m)}(0).$$

We consider again the restricted (S) & (E) problem. Theorem 3.2.1 implies that solutions of the restricted problem are solutions of the original problem. Moreover, the inverse state holds also. Like before the restricted spaces are compact topological spaces and the conditions (A1)–(A9) hold with $C_{\mathcal{E}} := C_{\ell_{\text{ext}}} R_{\text{st}}$. Using $\dot{\ell}_{\text{ext}} \in \mathrm{C}^0([0,T], \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d)^*)$, $\|\varphi\|_{\mathrm{W}^{1,p}} \leq R_{\text{st}}$ and the estimate

$$\partial_t \mathcal{E}(t_1, \varphi) - \partial_t \mathcal{E}(t_2, \varphi) = \langle \dot{\ell}_{\text{ext}}(t_1) - \dot{\ell}_{\text{ext}}(t_2), \varphi \rangle \leq \|\dot{\ell}_{\text{ext}}(t_1) - \dot{\ell}_{\text{ext}}(t_2)\|_{\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)^*} \|\varphi\|_{\mathrm{W}^{1,p}(\Omega,\mathbb{R}^d)}$$

we obtain also the condition (A10).

■

Remark 3.3.2. If ℓ_{ext} is piecewise C^1, then the (S) & (E) problem has also at least one solution. In fact, consider a partition $0 = t_0 < t_1 \ldots < t_P = T$ such that for all $j > 1$

$$\ell_{\text{ext}} \in \mathrm{C}^1([t_{j-1}, t_j], W^{1,p}(\Omega, \mathbb{R}^d)^*).$$

Using the previous theorem we obtain a solution (φ, z) on the interval $[t_0, t_1]$ with a start value (φ_0, z_0). Using $(\varphi(t_1), z(t_1))$ as start value we obtain a solution on the interval $[t_1, t_2]$ and so on. By concatenation of the obtained solutions we get a solution on the whole interval $[0, T]$.

The following result shows that there exist energy densities W, which satisfy the assumptions (a) and (b) of Theorem 3.3.1

Theorem 3.3.3. *Let $W : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$. Assume that there exist positive constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $q < p$ such that for all $x \in \Omega$, $F \in \mathbb{R}^{d\times d}$, $z, \bar{z} \in \mathbb{R}^m$ the following hold:*
(i) $\forall z \in \mathcal{Z}$: $\varphi \mapsto \int_\Omega W(x, \mathrm{D}\varphi, z)$ is s-lower semicontinuous on $\mathcal{F}$,
(ii) $|W(x, F, z) - W(x, F, \bar{z})| \leq C_3(1 + |F|^q)|z - \bar{z}|$.
Then, if $(\varphi_n, z_n) \to (\varphi, z)$ in $\mathcal{F} \times \mathcal{Z}$ with respect to $\mathfrak{T}_{\mathcal{F}} \times \mathfrak{T}_{\mathcal{Z}}$, it follows that

$$\int_\Omega W(x, \mathrm{D}\varphi, z)\,\mathrm{d}x \leq \liminf_{n\in\mathbb{N}} \int_\Omega W(x, \mathrm{D}\varphi_n, z_n)\,\mathrm{d}x.$$

Proof. Since $\varphi_n \rightharpoonup \varphi$ in $\mathrm{W}^{1,p}$, there exists $R > 0$ such that

$$\|\varphi\|_{\mathrm{W}^{1,p}} \leq R \text{ and } \|\varphi_n\|_{\mathrm{W}^{1,p}} \leq R \text{ for all } n.$$

Since Z is compact, the uniform boundedness of the sequence $(z_n(x))_{n\in\mathbb{N}}$ follows. Moreover, the L^1-convergence of $(z_n(x))_{n\in\mathbb{N}}$ to z implies immediately the L^s-convergence of $(z_n(x))_{n\in\mathbb{N}}$ to z for any $s > 1$. Using the Hölder inequality we obtain immediately

$$\lim_{n\to\infty} \int_\Omega |\mathrm{D}\varphi_n|^q |z - z_n|\,\mathrm{d}x \leq \lim_{n\to\infty} \left[\left(\int_\Omega |\mathrm{D}\varphi_n|^p\,\mathrm{d}x \right)^{q/p} \left(\int_\Omega |z - z_n|^{p/(p-q)} \right)^{1-q/p} \right] = 0. \tag{3.3.1}$$

It holds

$$\begin{aligned} &\textstyle\int_\Omega W(x, \mathrm{D}\varphi, z) - W(x, \mathrm{D}\varphi_n, z_n)\,\mathrm{d}x \\ &\qquad = \textstyle\int_\Omega W(x, \mathrm{D}\varphi, z) - W(x, \mathrm{D}\varphi_n, z)\,\mathrm{d}x + \int_\Omega W(x, \mathrm{D}\varphi_n, z) - W(x, \mathrm{D}\varphi_n, z_n)\,\mathrm{d}x \\ &\qquad \leq \textstyle\int_\Omega W(x, \mathrm{D}\varphi, z) - W(x, \mathrm{D}\varphi_n, z)\,\mathrm{d}x + \int_\Omega C(1 + |\mathrm{D}\varphi_n|^q)|z - z_n|\,\mathrm{d}x. \end{aligned}$$

Taking the lim inf on both sides in the above estimate and using (3.3.1) and condition (ii) we obtain the desired conclusion. ■

Remark 3.3.4. If W is convex in the deformation gradient, i.e. $W(x,\cdot,z)$ is convex, then condition (i) is fulfilled, see Theorem A.6.2. Moreover, if $W(x,\cdot,z)$ is strict convex, then the deformation φ is uniquely defined by z. This means that we have the "convex case".

Remark 3.3.5. The condition (i) can be guaranteed in "non-convex" case e.g. by polyconvexity of the elastic energy density W and by weak convergence of all involved minors of the defomation gradient. This shows, the right choice of p in Theorem 3.3.1 depends not only on the growth behaviour of the elastic energy densitity but also on the deep structure of it. Further details can be found in Theorems A.6.4 and A.6.6.

Now we give an example of a typical energy functional, which satisfies the assumption of Theorem 3.3.1.

Example 3.3.6. Let $\Omega \subset \mathbb{R}^3$ and Z be given as a set of unit vectors in $\mathbb{R}^m$, i.e. $Z := \{e_1, \dots, e_m\}$. Let $\rho_0 : Z \to \mathbb{R}$, $\rho_1 : Z \to \mathbb{R}$ and $\phi : Z \times Z \to [0,\infty)$ such that there exists $\gamma > 0$ and for all $c_1, c_2, c_3 \in Z$ hold

$$\begin{aligned} &\phi(c_1,c_1) = 0, && \phi(c_1,c_2) \geq \gamma, \\ &\phi(c_1,c_2) = \phi(c_2,c_1), && \phi(c_1,c_3) \leq \phi(c_1,c_2) + \phi(c_2,c_3). \end{aligned}$$

Additionally, let $\psi : \mathbb{R}^N \to [0,\infty]$ be a lower semicontinuous function. If ψ is even, positively 1-homogeneous and convex, then the energy function $\mathcal{E}$ defined by

$$\begin{aligned} \mathcal{E}(t,\varphi,z) = &\int_\Omega \left[\det{}^2(\mathrm{D}\varphi) + \frac{1}{\det{}^2(\mathrm{D}\varphi)} + |\mathrm{D}\varphi|^4 + \rho_1(z)|\mathrm{D}\varphi|^2 + \rho_0(z) + \langle \ell(t), \varphi\rangle\right] \mathrm{d}x \\ &+ \int_{J_z} \phi(z^+, z^-)\psi(\nu_z)\,\mathrm{d}\mathcal{H}^{d-1} \end{aligned}$$

satisfies all assumptions of Theorem 3.3.1 with $p = 4$. In fact, the energy density function W is polyconvex. Remark A.6.7 implies that all conditions of Theorem A.6.2 are fulfilled. Notice that the triangle inequality for the function ϕ can be interpreted as a reasonable physical restriction. In fact, this condition forbids the building of infinitesimal layers in the phase c_2 between interfaces of c_1 and c_3 phases. The function ψ allows us to model crystals with preferred directions of interface building. The simplest case $\psi(\nu) := |\nu|$ corresponds to crystals without preferred directions of interface building.

Appendix A

Mixed theoretical results

A.1 Norms in $\mathbb{R}^m$

Definition A.1.1. If $\|\cdot\|_m$ is an arbitrary norm in $\mathbb{R}^m$, then we denote with $\|\cdot\|_{m^*}$ a norm in $\mathbb{R}^m$ defined by:

$$\|x\|_{m^*} = \sup_{\|y\|_m=1} \langle x, y\rangle.$$

If $\|\cdot\|_m$ and $\|\cdot\|_N$ are arbitrary norms in $\mathbb{R}^m$ and in $\mathbb{R}^N$ respectively, then we denote by $\|\cdot\|_{m\times N}$ a norm in $\mathbb{R}^{m\times N}$ defined by:

$$\|A\|_{m\times N} = \sup_{\|x\|_N=1} \|Ax\|_m.$$

Remark A.1.2. The norm $\|\cdot\|_{m^*}$ is so-called operator norm associated with $\|\cdot\|_m$. It is well known that the operator norm associated with $\|\cdot\|_{m^*}$ is the original norm $\|\cdot\|_m$.

Proposition A.1.3. *Let $\|\cdot\|_m$, $\|\cdot\|_N$ be arbitrary norms in $\mathbb{R}^m$ and in $\mathbb{R}^N$ respectively. For any $b \in \mathbb{R}^m$ and $n \in \mathbb{R}^N$ holds:*

$$\|b \otimes n\|_{m\times n} = \|b\|_m \|n\|_{N^*}.$$

Proof. Using the definition of the $m \times N$-norm we obtain immediately that

$$\|b \otimes n\|_{m\times N} = \sup_{\|y\|_N=1} \|b \otimes n\, y\|_m = \sup_{\|y\|_N=1} \|b\|_m \langle n, y\rangle = \|b\|_m \|n\|_{N^*}.$$

■

Theorem A.1.4. *Let $\{e_1, e_2, \ldots, e_m\}$ be a canonical basis in $\mathbb{R}^m$. Let k_{ij} with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$ be arbitrary positive real numbers which satisfy the following conditions*

(a) $k_{ii} = 0$ *for all* $i = 1, 2, \ldots, m$,

(b) ***(Symmetry)*** $k_{ij} = k_{ji} > 0$ *for all* $i \neq j$,

(c) ***(Triangle inequality)*** $k_{ij} + k_{jl} \geq k_{il}$ *for all* $\{i, j, l\}$.

Then there exists a norm $\|\cdot\|_m$ in $\mathbb{R}^m$, such that for any i and j holds:

$$\|e_i - e_j\|_m = k_{ij}.$$

Proof. We denote the vectors $(e_i - e_j)$ by e_{ij}. Since the vector $\gamma = \{1, 1, \ldots, 1\} \in \mathbb{R}^m$ is orthogonal to all vectors e_{ij}, the vectors e_{ij} lie in the $m-1$ dimensional subspace V given by

$$\{\, x \in \mathbb{R}^m \mid x_1 + x_2 + \ldots + x_m = 1 \,\}.$$

For convenience we define $\widetilde{e}_{ij} := e_{ij}/k_{ij}$, $i \neq j$. We consider the convex hull C of the points $\widetilde{e}_{ij}$.

We have to show that for any i, j the point $\widetilde{e}_{ij}$ is a boundary point of C. In fact, if all points $\widetilde{e}_{ij}$ are boundary points of C, then we can take the convex hull $C \cup \{\gamma\}$ as a closed one-ball $\overline{B_1(0)}$ in the $\|\cdot\|_m$-norm. By using the Minkowsky-functional we define immediately the norm $\|\cdot\|_m$ on the whole $\mathbb{R}^m$.

We prove now, that the point $\widetilde{e}_{12}$ lies on the boundary of the set C. For other points the conclusion can be obtained in the same way.

We consider now the hyperplane P in V which is spanned on the points $\widetilde{e}_{12}, \widetilde{e}_{13}, \ldots, \widetilde{e}_{1n}$. We show that all points $\{\widetilde{e}_{ij}\} \subset V$ lie on the same side of the hyperplane P as a point 0. For this, we introduce new coordinates in V with basis vectors $\hat{e}_1, \ldots, \hat{e}_{m-1}$ given by

$$\hat{e}_{k-1} = \widetilde{e}_{1k}, \quad k \in \{2, \ldots, m\}.$$

It follows immediately from the definition that the vectors $\hat{e}_1, \ldots, \hat{e}_{m-1}$ are linearly independent. Hence, the new coordinates are well-defined.

In new coordinates the hyperplane P is given by $l(\hat{x}) = 1$ with

$$l(\hat{x}) := \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{m-1}.$$

Now we write the points $\widetilde{e}_{ij}$ in new coordinates in order to show that these points lie on the same side of P. Furthermore, we consider all possible situations step by step:

Case ($i = 1$): In this case the new coordinates of the point $\widetilde{e}_{ij}$ have the form

$$(0, \ldots, 0, \underset{j-1\text{pos}}{1}, 0, \ldots, 0).$$

Since the point lies on the hyperplane P, it holds $l(\hat{e}_{ij}) = 1$.

Case ($j = 1$): In this case the new coordinates $\hat{e}_{ij}$ of the point $\widetilde{e}_{ij}$ have the form

$$(0, \ldots, 0, \underset{i-1\text{pos}}{-1}, 0, \ldots, 0).$$

It follows $l(\hat{e}_{ij}) = -1 < 1$.

Case ($i \neq 1$ and $j \neq 1$): The new coordinates $\hat{e}_{ij}$ of the point $\widetilde{e}_{ij}$ can be calculated as follows:

$$\widetilde{e}_{ij} = \frac{1}{k_{ij}}(e_i - e_j) = \frac{1}{k_{ij}}(e_i - e_1 + e_1 - e_j) = \frac{1}{k_{ij}}(k_{1j}\widetilde{e}_{1j} - k_{1i}\widetilde{e}_{1i}) = \frac{k_{1j}}{k_{ij}}\hat{e}_{j-1} - \frac{k_{1i}}{k_{ij}}\hat{e}_{i-1}.$$

Using $k_{1j} \leq k_{1i} + k_{ij}$ we obtain

$$l(\hat{e}_{ij}) \leq 1.$$

Since $l(0) = 0 < 1$, we conclude that all points $\widetilde{e}_{ij}$ lie on the same side of P as point 0. The point $\widetilde{e}_{12}$ has the new coordinates $(1, 0, \ldots, 0)$ and lies on the hyperplane P. The set $l(\hat{x}) \leq 1$ is convex. Hence, the set C lies on one side of the hyperplane P. The previous conclusion implies that the point $\widetilde{e}_{12}$ lies on the boundary of the set C. In fact, if the point $\widetilde{e}_{12}$ is an internal point of C, then there exists a small open ball $B_r(\widetilde{e}_{12}) \subset C$. Since $\widetilde{e}_{12}$ lies on the hyperplane P, there exist two points $x_1, x_2 \in B_r(\widetilde{e}_{12}) \subset C$ which lie on the different sides of P. Since the set C lies on one side of the hyperplane P, we get a contradiction. ∎

Remark A.1.5. It is obviously that the previous theorem gives also necessary conditions for the existence of such kind of norms. It is clear that the uniqueness is not given. In fact, we constructed the norm which satisfies the condition of the theorem and which takes the greatest possible values in the subspace $\{\, x \in \mathbb{R}^m \mid x_1 + x_2 + \ldots + x_m = 1 \,\}$.

A.2 Analysis results

Theorem A.2.1. *Let X be a separable Banach space and $f : X^* \to \mathbb{R} \cup \{+\infty\}$ be convex. Then*

(i) f is weakly* lower semicontinuous if and only if f is sequentially weakly* lower semicontinuous.

(ii) f is weakly* lower semicontinuous if and only if there exist sequences $(x_k) \subset X$, $(a_k) \subset \mathbb{R}$ such that for any $\ell \in X^*$ holds:

$$f(l) = \sup_{k\in\mathbb{N}} (l(x_k) + a_k).$$

Moreover, f is positively 1-homogeneous if and only if we can take $a_k \equiv 0$.

For a proof see [AFP00], Proposition 2.31.

Theorem A.2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be an uniformly bounded sequence in $\mathrm{L}^\infty([0,T],\mathbb{R})$ such that*

$$f_n \overset{*}{\rightharpoonup} f^\infty := \limsup f_n \text{ in } \mathrm{L}^\infty([0,T],\mathbb{R}).$$

Then, $f_n \to f$ in $\mathrm{L}^p([0,T],\mathbb{R})$ for any $p \in [1,\infty)$.

Proof. Let $g_k := \sup_{n\geq k} f_k$. It follows immediately

$$g_k - f_k \geq 0 \quad \text{and} \quad g_k \searrow f^\infty \text{ in } L^\infty([0,T],\mathbb{R}).$$

The above estimate implies that $g_k \to f^\infty$ in $L^1([0,T],\mathbb{R})$. Moreover, we obtain

$$\lim_{k\to\infty} \int_0^T |g_k - f_k| \,\mathrm{d}x = \lim_{k\to\infty} \left[\int_0^T g_k \,\mathrm{d}x - \int_0^T f_k \,\mathrm{d}x \right] = 0.$$

Hence, $(g_k - f_k) \to 0$ in $L^1([0,T],\mathbb{R})$. Using the previous estimates and $\|f - f_k\|_{\mathrm{L}^1} \leq \|f - g_k\|_{\mathrm{L}^1} + \|g_k - f_k\|_{\mathrm{L}^1}$ we conclude that $f_n \to f^\infty$ in $L^1([0,T],\mathbb{R})$. Since $(f_n)_{n\in\mathbb{N}}$ is an uniformly bounded sequence in $\mathrm{L}^\infty([0,T],\mathbb{R})$ the conclusion follows. ∎

A.3 Approximation of Lebesgue integral with Riemann sums

The following theorem gives an useful approximation of Lebesgue integrals of L^∞-functions with help of suitable Riemann sums.

Theorem A.3.1. *Let $f \in \mathrm{L}^\infty([0,T],\mathbb{R})$. For every $\varepsilon > 0$ there exists the partition $0 = \tau_0 < t_1 < \ldots \tau_N = T$ of the interval $[0,T]$ such that*

$$\left| \int_{[0,T]} f - \sum_{i=1}^{N} f(\tau_i)(\tau_i - \tau_{i-1}) \right| < \varepsilon.$$

Proof. The theorem holds obviously for $\|f\|_\infty = 0$. We assume $\|f\|_\infty > 0$ and set $K := \max\{\|f\|_\infty, f(T)\}$.

Since the Lebesgue-integral is absolutely continuous, there exists $\delta > 0$, with $\delta \in (\frac{\varepsilon}{K}, 0)$, such that for any $M \subset [0,T]$ with $\mathcal{L}^1(M) < \delta$ holds:

$$\textstyle\int_M |f| < \varepsilon. \tag{A.3.1}$$

Using the Luzin Theorem we find the set $M \subset [0,T]$ and a continuous function $g : [0,T] \to [-K,K]$ such that $M = \{\, x \mid f(x) \neq g(x) \,\}$ with $\mathcal{L}^1(M) < \delta$. The function g, as bounded and continuous function, is Riemann-integrable. This implies that there exists a partition $0 = s_0 < s_1 < \ldots < s_L = T$ with $s_j - s_{j-1} < \frac{\varepsilon}{K}$ such that for any $\eta_1, \ldots, \eta_L$ with $\eta_j \in [s_{j-1}, s_j]$ holds

$$\left| \textstyle\int_{[0,T]} g - \sum_{j=1}^{L} g(\eta_j)(s_j - s_{j-1}) \right| < \varepsilon. \tag{A.3.2}$$

We divide the indices of the subintervals $[s_{j-1}, s_j]$ in two classes by defining the following two sets

$$\begin{aligned} I_1 &= \{\, j \in \{1, \ldots, L\} \mid |[s_{j-1}, s_j] \setminus M| > 0 \,\} \\ I_2 &= \{\, j \in \{1, \ldots, L\} \mid |[s_{j-1}, s_j] \setminus M| = 0 \,\}, \end{aligned}$$

i.e. the set I_1 consists of all indices j, whose associated subintervals $[s_{j-1}, s_j]$ are not almost everywhere covered by M. From the definition of the set I_2 follows immediately

$$\sum_{j \in I_2} s_j - s_{j-1} < \delta. \tag{A.3.3}$$

Using $\mathcal{L}^1(M) < \delta$ we select the numbers $\hat{s}_j$'s with $j \in I_1$ such that

$$\begin{aligned} &\hat{s}_j \notin M \text{ for } j \in I_1, \\ &s_{j-1} < \hat{s}_j \leq s_j \text{ for all } j \in I_1, \\ &\textstyle\sum_{j \in I_1} s_j - \hat{s}_j < \delta. \end{aligned} \tag{A.3.4}$$

For $j \in I_2$ we set $\hat{s}_j := s_j$. We renumerate the numbers s_j, $j \in I_1$ and $\hat{s}_j$, $j \in I_1$ as t_i with $j = 1, \ldots, N$ and as τ_i with $j = 1, \ldots, N$ respectively. Additionally, we set $\tau_0 := 0$.

By (A.3.1), (A.3.2) and (A.3.3) we obtain

$$\begin{aligned}
&\left|\int_{[0,T]} f - \sum_{i=1}^N f(\tau_i)(\tau_i - \tau_{i-1})\right| \le \left|\int_{[0,1]} f - g\right| + \left|\int_{[0,T]} g - \sum_{j=1}^L g(\hat{s}_j)(s_j - s_{j-1})\right| \\
&\quad + \left|\sum_{j\in I_2} g(s_j)(s_j - s_{j-1})\right| + \left|\sum_{j\in I_1} g(\hat{s}_j)(s_j - s_{j-1}) - \sum_{i=1}^N f(\tau_i)(\tau_i - \tau_{i-1})\right| \\
&\quad < 2\varepsilon + \varepsilon + K\mu\left(\cup_{j\in I_2}[s_{j-1}, s_j]\right) + K\delta + \left|\sum_{j=1}^N g(\tau_j)(t_j - t_{j-1}) - \sum_{i=1}^N f(\tau_i)(\tau_i - \tau_{i-1})\right| \\
&\quad < 2\varepsilon + \varepsilon + K\mu\left(\cup_{j\in I_2}[s_{j-1}, s_j]\right) + K\delta + 2K\delta < 7\varepsilon.
\end{aligned}$$

If τ_N is equal to T, then we are ready. In the other case we set $\tau_{N+1} = T$. Using (A.3.3), (A.3.4) and the last estimate we conclude

$$\left|\int_{[0,T]} f - \sum_{i=1}^{N+1} f(\tau_i)(\tau_i - \tau_{i-1})\right| \le 7\varepsilon + |f(T)(\tau_{N+1} - \tau_N)| \le 9\varepsilon.$$

The proof is complete. ■

Remark: By a simple cut-off argument the previous theorem can be proved for $f \in \mathrm{L}^1$.

A.4 The weak and strong measurability. Pettis' Theorem

In this section X denotes a Banach space and $(S, \mathcal{B}, \mu)$ denotes a measure space.

Definition A.4.1. Let u be a function defined on S with values in X.

1. The function u is called weakly $\mathcal{B}$-measurable if for any $\ell \in Y'$ the numerical function $s \mapsto \langle u(s), \ell\rangle$ is $\mathcal{B}$-measurable.

2. The function u is called finite-valued if there exists a finite number of disjoint $\mathcal{B}$-measurable sets B_i with $\mu(B_i) < \infty$ and corresponding numbers $c_i \neq 0$ such that

$$u(s) = \begin{cases} c_i & \text{if } x \in B_i, \\ 0 & \text{if } x \in S \setminus \bigcup_i B_i. \end{cases}$$

3. The function u is called strongly $\mathcal{B}$-measurable if the exists a sequence of finite-valued functions which are strongly convergent to u μ-a.e. on S.

4. The function u is called separable-valued if its range $\{\, u(s) \mid s \in S \,\}$ is separable. It is μ-almost separably valued, if there exists a $\mathcal{B}$-measurable set N of μ-measure zero such that $\{\, u(s) \mid s \in S \setminus N \,\}$ is separable.

Theorem A.4.2. ***(Pettis)*** *Let u be a function defined on S with values in X. The function u is strongly $\mathcal{B}$-measurable if and only if it is weakly $\mathcal{B}$-measurable and m-almost separably valued.*

For a proof see [Yos71], Ch. 5, Sec. 4, page 131.

A.5 Set-valued functions

In this section we collect theorems and definitions, which are used in order to obtain the regularity statement in the existence result for rate-independent systems in the non-convex case. We follow the monograph "Set-Valued Analysis" by J.P. Aubin and H. Frankowska [AF90].

Definition A.5.1. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of metric space (X, d). The set

$$\operatorname{Limsup}_{n\to\infty} K_n := \Big\{ x \in X \mid \liminf_{n\to\infty} d(x, K_n) = 0 \Big\}$$

is called the upper limit of the sequence K_n and the set

$$\operatorname{Liminf}_{n\to\infty} K_n := \Big\{ x \in X \mid \lim_{n\to\infty} d(x, K_n) = 0 \Big\}$$

is called the lower limit of the sequence K_n.

For our theory we are especially interesting in $\operatorname{Limsup}_{n\to\infty} K_n$. The sets introduced in Definition A.5.1 can be defined in an alternative way by the following proposition.

Proposition A.5.2. *If $(K_n)_{n\in\mathbb{N}}$ is a sequence of subsets of a metric space, then $\operatorname{Liminf}_{n\to\infty} K_n$ is the set of limits of sequences $x_n \in K_n$ and $\operatorname{Limsup}_{n\to\infty} K_n$ is the set of cluster points of sequences $x_n \in K_n$.*

For a proof see [AF90], Ch. 1, Prop. 1.1.2, page 18.

Now we give the definition of measurable set-valued maps, which is closely related to the definition of measurably single-valued functions.

Definition A.5.3. Consider a measurable space $(\Omega, \mathcal{A})$, a complete separable metric space X and a set-valued map $F : \Omega \rightsquigarrow X$ with closed images.

The map F is called measurable if the inverse image of each open set is a measurable set, i.e. for every open subset $\mathcal{O} \subset X$ holds

$$F^{-1}(\mathcal{O}) := \{ \omega \in \Omega \mid F(\omega)\mathcal{O} \neq \emptyset \} \in \mathcal{A}.$$

The following theorem gives us the opportunity to construct new measurable set-valued mappings from a given sequence of measurable set-valued functions.

Theorem A.5.4. ***(Lower and upper limits)*** *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space, X a complete separable metric space and $F_n : \Omega \rightsquigarrow X$, $n \geq 1$ set-valued maps with closed images. Then the maps*

$$\Omega \ni \omega \rightsquigarrow \operatorname{Liminf}_{n\to\infty} F_n(\omega) \quad \text{and} \quad \Omega \ni \omega \rightsquigarrow \operatorname{Limsup}_{n\to\infty} F_n(\omega)$$

are measurable. Consequently, if for every $\omega \in \Omega$, the images $F_n(\omega)$ converge to a subset $F(\omega)$, the set-valued map F (called the pointwise limit) is measurable.

For a proof see [AF90], Ch. 8, Th. 8.2.5, page 312.

In our theory we work on single-valued functions. In order to obtain a single-valued measurable function from a set-valued map we need some kind of selection. The next definition introduces the concept of measurable selection, which provides the needed selection operation. The following theorem supplies then an existence result for measurable selections, which is convenient in the framework of our theory of rate-independent systems.

Definition A.5.5. **(Measurable selection)** Let $(\Omega, \mathcal{A})$ be a measurable space and X be a complete separable metric space. Consider a set-valued map $F : \Omega \rightsquigarrow X$. A measurable function $f : \Omega \to X$ satisfying

$$\forall \omega \in \Omega, \quad f(\omega) \in F(\omega)$$

is called a measurable selection of F.

Theorem A.5.6. ***(Characterisation theorem)*** *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space, X a complete separable metric space and $F : \Omega \rightsquigarrow X$ a set-valued map with non-empty closed images. Then the following properties are equivalent:*

(i) *F is measurable.*

(ii) *The graph of F belongs to $\mathcal{A} \times \mathcal{B}(X)$.*

(iii) *$F^{-1}(C) \in \mathcal{A}$ for every closed set $C \subset X$.*

(iv) *$F^{-1}(B) \in \mathcal{A}$ for every Borel set $B \subset X$.*

(v) *For all $x \in X$ the map $d(x, F(\cdot))$ is measurable.*

(vi) *There exists a sequence of measurable selections $(f_n)_{n\in\mathbb{N}}$ of F such that $\forall \omega \in \Omega$, $F(\omega) = \overline{\cup_{n\in\mathbb{N}} f_n(\omega)}$.*

For a proof see [AF90], Ch. 8, Th. 8.1.4, page 310.

Theorem A.5.7. ***(Filippov)*** *Consider a complete σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, complete separable metric spaces X, Y and a measurable set-valued map $F : \Omega \rightsquigarrow X$ with closed nonempty images. Let $g : \Omega \times X \to Y$ be a Carathéodory map. Then for every measurable map $h : \Omega \to Y$ satisfying*

$$h(\omega) \in g(\omega, F(\omega)) \text{ for almost all } \omega \in \Omega$$

there exists a measurable selection $f(\omega) \in F(\omega)$ such that

$$h(\omega) = g(\omega, f(\omega)) \text{ for almost all } \omega \in \Omega.$$

For a proof see [AF90], Ch. 8, Th. 8.2.10, page 316.

A.6 Variational methods

Theorem A.6.1. *Suppose that X is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset X$ be a weakly closed subset X. Assume that the following conditions hold:*

(a) ***(Coercivity)*** *$E(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$*

(b) ***(Weak lower semi-continuity)*** *for any $u \in M$, any sequence $(u_m)_{m\in N}$ in M such that $u \rightharpoonup u$ weakly in X there holds:*

$$E(u) \leq \liminf_{m\to\infty} E(u_m).$$

Then E is bounded from below on M and attains its infimum in M.

For a proof see [Str90], Ch. 1, Th. 1.2, page 4.

Theorem A.6.2. *Let Ω be a domain in $\mathbb{R}^n$, and assume that $W : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory function satisfying the conditions*

(a) $W(x, u, A) \geq h(x)$ *for almost every x, u, A, where $h \in \mathrm{L}^1(\Omega)$*

(b) $W(x, u, \cdot)$ *is convex in A for almost every x, u.*

If $u_k, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that

$$u_k \to u \text{ in } \mathrm{L}^1(\Omega') \quad \text{and} \quad \mathrm{D}u_k \rightharpoonup \mathrm{D}u \text{ weakly in } \mathrm{L}^1(\Omega')$$

for all $\Omega' \subset\subset \Omega$, then

$$\int_\Omega W(x, u, \mathrm{D}u)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega W(x, u_k, \mathrm{D}u_k)\,\mathrm{d}x.$$

For a proof see [Str90], Ch. 1, Th. 1.6, page 8.

Furthermore, we denote the set of all $s \times s$ minors of some matrix $A \in \mathbb{R}^{nm}$ as $\mathrm{adj}_s A$. The number of all possible minors of some matrix $A \in \mathbb{R}^{nm}$ is denoted by $\tau(n, m)$. For convenience we use the following notation: $n \wedge m := \min\{n, m\}$.

Definition A.6.3. A function $f : \mathbb{R}^{nm} \to \mathbb{R} \cup \{\infty\}$ is called polyconvex if there exists a convex function $g : R^{\tau(n,m)} \to \mathbb{R} \cup \{\infty\}$, such that

$$f(A) = g(T(A)),$$

where $T : \mathbb{R}^{nm} \to \mathbb{R}^{\tau(n,m)}$ is defined by

$$T(A) = (A, \mathrm{adj}_2, \ldots, \mathrm{adj}_{n\wedge m} A).$$

The following theorem shows that minors are weakly continuous functions.

Theorem A.6.4. *Let $\Omega \in \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, and let*

$$u_k \rightharpoonup u \text{ in } \mathrm{W}^{1,p}(\Omega, \mathbb{R}^m)$$

(if $p = +\infty$, $u_k \overset{*}{\rightharpoonup} u$). It holds:

(i) *If $m = n = 2$ and $p \geq 2$, then* $\det \mathrm{D}u_k \rightharpoonup \det \mathrm{D}u$ *in* $\mathcal{D}'(\Omega)$.

(ii) *If $m = n = 3$ and $p \geq 2$, then* $\mathrm{adj}_2 \mathrm{D}u_k \rightharpoonup \mathrm{adj}_2 \mathrm{D}u$ *in* $\mathcal{D}'(\Omega)^9$.

(iii) *If $m = n = 3$ and $p \geq 3$, then* $\det \mathrm{D}u_k \rightharpoonup \det \mathrm{D}u$ *in* $\mathcal{D}'(\Omega)$.

(iv) *If $m = n$ and $p \geq n$, then* $\det \mathrm{D}u_k \rightharpoonup \det \mathrm{D}u$ *in* $\mathcal{D}'(\Omega)$.

(v) *If $m, n \geq 2, 2 \leq s \leq n \wedge m$ and $p \geq s$, then*

$$\mathrm{adj}_s \mathrm{D}u_k \rightharpoonup \mathrm{adj}_s \mathrm{D}u \text{ in } \mathcal{D}'(\Omega)^{\sigma(s)},$$

where $\sigma(s) = \binom{m}{s}\binom{n}{s}$.

(vi) *If $m, n \geq 2, 2 \leq s \leq n \wedge m$ and*

$$\mathrm{adj}_{s-1} \mathrm{D}u_k \rightharpoonup \mathrm{adj}_{s-1} \mathrm{D}u \text{ in } \mathrm{L}^r(\Omega)^{\sigma(s-1)}$$

with $r > 1$ and $1/p + 1/r \leq 1$, then

$$\mathrm{adj}_s \mathrm{D}u_k \rightharpoonup \mathrm{adj}_s \mathrm{D}u \text{ in } \mathcal{D}'(\Omega)^{\sigma(s)}.$$

For a proof see [Dac89], Ch. 4, Th. 2.6, page 172.

Remark A.6.5. If $m = n$ and $p > n$, then

$$\operatorname{adj}_s \mathrm{D}u_k \rightharpoonup \operatorname{adj}_s \mathrm{D}u \text{ in } \mathrm{L}^{p/s}(\Omega)^{\sigma(s)},$$

for every $1 \leq s \leq n$. In fact, the sequence $(\operatorname{adj}_s \mathrm{D}u_k)_{k\in\mathbb{N}}$ is obviously bounded in $\mathrm{L}^{p/s}(\Omega)^{\sigma(s)}$. Hence, there exists a subsequence $(\operatorname{adj}_s \mathrm{D}u_{k_l})_{l\in\mathbb{N}}$, which weakly converges in $\mathrm{L}^{p/s}(\Omega)^{\sigma(s)}$. The conclusion follows by uniqueness of the distributional limit.

Theorem A.6.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\infty\}$. Let $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^{\tau(n,m)} \to \mathbb{R} \cup \{\infty\}$ be a Carathéodory function which is such that*

$$\begin{cases} g(x, u, \cdot) \text{ is convex for every } u \in \mathbb{R}^m \text{ and almost every } x \in \Omega \\ W(x, u, A) \equiv g(x, u, T(A)) \geq \alpha(x) + \sum_{s=1}^{n\wedge m} \beta_s |\operatorname{adj}_s A|_s^p \end{cases}$$

where $\alpha \in \mathrm{L}^1(\Omega), \beta_s > 0$ and $p_1 := p \geq 2$, $p_s \geq \frac{p_1}{p_1-1}$ if $2 \leq s < n \wedge m$ and $p_{n\wedge m} > 1$. Let

$$\inf \left\{ I(u) = \int_\Omega W(x, u(x), \mathrm{D}u(x))\, \mathrm{d}x \mid u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^m) \right\}. \tag{P}$$

Assume that there exists $\tilde{u} \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^m)$ such that $I(\tilde{u}) < \infty$.

Then (P) admits at least one solution.

For a proof see [Dac89], Ch. 4, Th. 2.10, page 182.

Remark A.6.7. The complex lower bound for W is used only to obtain weak continuity in minors by applying Theorem A.6.4. If $m = n$ and $p > n$ then by using Remark A.6.5 we can replace the complex lower bound for W with the following simple coercivity condition

$$W(x, u, A) \equiv g(x, u, T(A)) \geq \alpha(x) + \beta_1 |A|^p,$$

where $\alpha \in \mathrm{L}^1(\Omega)$ and $\beta_1 > 0$.

A.7 Geometric measure theory

Definition A.7.1. (Restriction)
Let μ be a positive, real or vector measure on the measure space $(X, \mathcal{A})$. If $E \in \mathcal{A}$ we set $\mu \llcorner E(F) = \mu(E \cap F)$ for every $F \in \mathcal{A}$

Definition A.7.2. (Hausdorff measures)
Let $k \in [0, \infty)$ and $E \subset \mathbb{R}^N$. The k-dimensional Hausdorff measure of E is given by

$$\mathcal{H}^k(E) := \lim_{\delta \searrow 0} \mathcal{H}^k_\delta(E)$$

where, for $0 < \delta \leq \infty$, $\mathcal{H}^k_\delta(E)$ is defined by

$$\mathcal{H}^k_\delta(E) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{i\in I} [\operatorname{diam}(E_i)]^k \mid \operatorname{diam}(E_i) < \delta, E \subset \bigcup_{i\in I} E_i \right\}$$

for finite or countable covers $\{E_i\}_{i\in I}$, with the convention $\operatorname{diam}(\emptyset) = 0$. Here $\omega_k = \pi^{k/2}/\Gamma(1 + k/2)$.

Theorem A.7.3. ***(Properties of Hausdorff measures)*** *The measure $\mathcal{H}^k$ in $\mathbb{R}^N$ enjoys the following properties:*

(i) *The measures $\mathcal{H}^k$ are outer measure in $\mathbb{R}^N$ and, in particular, σ-additive on $\mathcal{B}(\mathbb{R}^N)$;*

(ii) *$\mathcal{H}^k$ has the following behaviour with respect to translation and homotheties:*

$$\mathcal{H}^k(E+z) = \mathcal{H}^k(E) \quad \forall z \in \mathbb{R}^N, \quad \mathcal{H}^k(\lambda E) = \lambda^k \mathcal{H}^k(E) \quad \forall \lambda > 0$$

for any $E \subset \mathbb{R}^N$, and is identically zero if $k > N$;

(iii) *if $k > k' \geq 0$, then*

$$\mathcal{H}^k(E) > 0 \Longrightarrow \mathcal{H}^{k'}(E) = \infty;$$

(iv) *if $f : \mathbb{R}^N \to \mathbb{R}^m$ is a Lipschitz function, then*

$$\mathcal{H}^k(f(E)) \leq [\mathrm{Lip(f)}]^{\mathrm{k}} \mathcal{H}^{\mathrm{k}}(\mathrm{E}) \quad \forall \mathrm{E} \subset \mathbb{R}^{\mathrm{N}}.$$

For a proof see [AFP00], Proposition 2.49.

Theorem A.7.4. *For any Borel set $B \subset \mathbb{R}^N$ and any $\delta \in (0, \infty]$ there holds*

$$\mathcal{L}^N(B) = \mathcal{H}^N_\delta(B) = \mathcal{H}^N(B).$$

For a proof see [AFP00], Proposition 2.53.

Definition A.7.5. (Rectifiable sets)
Let $E \subset \mathbb{R}^N$ be a $\mathcal{H}^k$-measurable set. The set E is countably k-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^N$ such that

$$E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k).$$

The set E is countably $\mathcal{H}^k$-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^N$ such that

$$\mathcal{H}^k\big(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k)\big) = 0.$$

Finally, the set E is $\mathcal{H}^k$-rectifiable if E is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(E) < \infty$.

Theorem A.7.6. *Any $\mathcal{H}^k$-measurable set E is countably $\mathcal{H}^k$-rectifiable if and only if there exist countably many Lipschitz k-graphs $\Gamma_i \subset \mathbb{R}^N$ such that*

$$\mathcal{H}^k\big(E \setminus \bigcup_{i=0}^{\infty} \Gamma_i\big) = 0.$$

Moreover, given any $\varepsilon > 0$, the graphs Γ_i can be chosen in such a way that their Lipschitz constants are less than ε.

For a proof see [AFP00], Proposition 2.76.

Definition A.7.7. (Approximate tangent space to a measures)
Let μ be an R^m-valued Radon measure in an open set $\Omega \subset \mathbb{R}^N$ and $x \in \Omega$. We say that μ has approximate tangent space $\pi \in G_k$ with multiplicity $\theta \in \mathbb{R}^m$ at x, and write

$$\mathrm{Tan}^k(\mu, x) = \theta \mathcal{H}^k \llcorner \pi,$$

if $\rho^{-k}\mu_{x,\rho}$ locally weakly* converges to $\theta\mathcal{H}^k \llcorner \pi$ as $\rho \searrow 0$. Here $\mu_{x,\rho}$ are rescaled measures defined by

$$\mu_{x,\rho}(B) := \mu(x + \rho B).$$

Theorem A.7.8. ***(Whitney extension theorem)*** *Suppose that Y is a normed vector space, k is a nonnegative integer, A is a closed subset of R^N, and to each $a \in A$ corresponds a polynomial function*

$$P_a : \mathbb{R}^N \to Y \text{ with degree } P_a \leq k.$$

Whenever $C \subset A$ and $\delta > 0$ let $\rho(C, \delta)$ be the supremum of the set of all numbers

$$\|\mathrm{D}^i P_a(b) - \mathrm{D}^i P_b(b)\|_Y |a - b|^{i-k}(k - i)!$$

corresponding to $i = 0, \ldots, k$ and $a, b \in C$ with $0 < |a - b| \leq \delta$. If $\rho(C, \delta) \to 0$ as $\delta \searrow 0$ for each compact subset C of A, then there exists a map $g : \mathbb{R}^N \to Y$ of class k such that $\mathrm{D}^i g(a) = \mathrm{D}^i P_a(a)$ for $i = 0, \ldots, k$ and $a \in A$.

For a proof see [Fed69], Theorem 3.1.14.

Theorem A.7.9. *If $A \subset \mathbb{R}^N$, $f : A \to \mathbb{R}^m$ and*

$$\text{ap limsup}_{x \to a} \frac{|f(x) - f(a)|}{|x - a|} < \infty$$

for $\mathcal{L}^N$-almost all x in A, then for each $\varepsilon > 0$ there exists a C^1-map $g : \mathbb{R}^N \to \mathbb{R}^m$ such that

$$\mathcal{L}^N(A \setminus \{\, x \mid f(x) = g(x) \,\}) < \varepsilon.$$

Here, $\text{ap limsup}\, x \to a \frac{|f(x)-f(a)|}{|x-a|}$ *is defined as the greatest lower bound of the set of all numbers t such that the set $\{\, z \mid f(z) > t \,\}$ has a density 0 at the point a.*

For a proof see [Fed69], Theorem 3.1.16.

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Deutsche Zusammenfassung

In dieser Arbeit wird eine spezielle Klasse der mechanischen Systeme betrachtet, nämlich die ratenunabhängigen Systeme. Solche Systeme werden von externen Kräften angetrieben, die eine viel langsamere Zeitskala haben als die interne System-Zeitskala. Man spricht also von quasi-statischen Systemen. Als typische Beispiele können an dieser Stelle die Trockenreibung, Bruchbildung und -ausbreitung genannt werden. Das Ziel dieser Dissertation ist die Entwicklung eines mathematischen Apparates, der die mathematisch exakte Existenzaussagen für ein breites Spektrum der ratenunabhängigen Prozesse erlaubt. Im letzten Kapitel wird die entwickelte Theorie exemplarisch auf ein einfaches Modell für Phasen-Transformationen in Festkörpern angewendet, um die Existenz des Evolutionsprozesses nachzuweisen.

Im ersten Teil dieser Arbeit werden ganz allgemein mechanische Systeme untersucht, die so genannte Ratenunabhängigkeit aufweisen. Diese Eigenschaft besagt, dass die Reskalierung der Zeit zu den reskalierten Lösungen führt. Mathematisch lässt sich dieser Sachverhalt wie folgt ausdrücken:

> *Seien $\ell : [0,T] \to X$ (Kraftraum) die antreibende Kraft des Systems und die Funktion $\alpha : [0,\hat{T}] \to [0,T]$ eine monotone Reparametrisierung des Zeitintervalls. Die Funktion $y : [0,T] \to \mathcal{Y}$ (Zustandsraum) beschreibt genau dann eine mögliche evolutionäre Entwicklung des Systems, wenn $y \circ \alpha$ eine mögliche evolutionäre Entwicklung des Systems mit der treibenden Kraft $\ell \circ \alpha$ ist.*

In der Literatur [BS96, KP89, Vis94, MM93] wurden bereits die Theorie der Differential-Inklusionen und die Theorie der Hysterese-Operatoren erfolgreich angewendet um solche Probleme zu erforschen. Die letztere Methode wurde extra für das Studium der Probleme mit Hysterese entwickelt, die absolut typisch für ratenunabhängige Systeme ist. Leider erfordern beide Methoden zusätzliche Eigenschaften (Banachstruktur, Konvexität, Skalarwertigkeit), die im Allgemeinen nicht gegeben sind. In dieser Arbeit folgen wir einem anderen in [MT99] entwickelten Zugang, den wir jetzt kurz beschreiben.

Jedes ratenunabhängige System besitzt zwei wichtige Kenngrößen, nämlich die im System gespeicherte Energie $\mathcal{E} : [0,T] \times \mathcal{Y} \to \mathbb{R}$ und die Dissipationsdistanz $\mathcal{D} : \mathcal{Y} \times \mathcal{Y} \to [0,\infty]$. Letztere Funktion gibt die Energie an, die bei dem Übergang aus einem Zustand in einen anderen verbraucht wird. Mit Hilfe der Dissipationsdistanz lässt sich die längs eines Evolutionsprocesses $y : [0,T] \to \mathcal{Y}$ verbrauchte Energie wie folgt berechnen:

$$\mathrm{Diss}_{\mathcal{D}}(z;[s,t]) = \sup \left\{ \sum_{1}^{N} \mathcal{D}(z(\tau_{j-1}), z(\tau_j)) \mid N \in \mathbb{N}, s = \tau_0 < \tau_1 < \cdots < \tau_N = t \right\}.$$

Der in [MT99] vorgeschlagene Zugang basiert nun darauf, die Entwicklung des Systems y durch zwei folgende Bedingungen zu beschreiben:

(S) *Für alle* $t \in [0,T]$ *and alle* $\widehat{y} \in \mathcal{Y}$ *soll gelten*

$$\mathcal{E}(t, y(t)) \leq \mathcal{E}(t, \widehat{y}) + \mathcal{D}(y(t), \widehat{y}).$$

(E) *Für alle* $t \in [0,T]$ *soll gelten*

$$\mathcal{E}(t, y(t)) + \mathrm{Diss}_{\mathcal{D}}(y; [0,t]) = \mathcal{E}(0, y(0)) + \int_0^t \partial_t \mathcal{E}(\tau, y(\tau))\, \mathrm{d}\tau.$$

Die erste Bedingung besagt, dass zu jedem Zeitpunkt t der Zustand $y(t)$ stabil ist, d.h. ein Sprung in einen anderen Zustand $\widehat{y}$ energetisch ungünstig ist. Die zweite Bedingung postuliert die Energiebilanz.

In [MT99] wurde die Existenz der Lösungen für das (S) & (E)-Problem in einem Spezialfall gezeigt. Es wurde angenommen, dass $\mathcal{Y}$ ein reflexiver Banachraum ist und die Dissipationsdistanz $\mathcal{D}$ eine endliche konvexe Funktion ist. In dieser Arbeit folgen wir der in [MM03] entwickelten Verallgemeinerung, die Annahmen über lineare Struktur des Zustandsraumes und folglich auch Konvexitätsannahmen für $\mathcal{D}$ völlig meidet. Dies erlaubt, unsere Theorie auf eine breite Klasse der ratenunabhängigen Prozesse anzuwenden. Zum Beispiel kann damit die Bruchausbreitung im Griffith-Modell untersucht werden. An dieser Stelle sei bemerkt, dass eine sehr spezielle Version der (S) & (E)-Formulierung zum ersten Mal gerade bei den Untersuchungen der Bruchausbreitung angewendet wurde, siehe [FM93].

In vielen Anwendungen will man die Evolution eines Material-Stückes beschreiben, das sowohl eine elastische Verformung als auch die Veränderung des inneren Zustandes (Magnetisierung, Riss-Bildung, Phasen-Übergänge) erlebt. Folglich lässt sich der Zustandsraum $\mathcal{Y}$ als Produkt zweier Räume $\mathcal{F} \times \mathcal{Z}$ schreiben. Dabei beschreibt der Raum $\mathcal{F}$ die Menge der möglichen elastischen Deformationen und der Raum $\mathcal{Z}$ beschreibt die Menge der möglichen inneren Zustände. Diese Aufspaltung hängt damit zusammen, dass die Dissipationsdistanz in vielen Fällen nur von den inneren Zuständen abhängt. Dies macht zwei Fälle möglich. Im ersten Fall (konvexer Fall) lässt sich die Deformation als Funktion des inneren Zustandes schreiben in dem zweitem (nicht-konvexer Fall) nicht. Im ersten Kapitel dieser Arbeit werden die Bedingungen für die Existenz der Lösungen der (S) & (E) Formulierung angegeben. Dabei sind $\mathcal{F}$ und $\mathcal{Z}$ beliebige topologische Hausdorff-Räume. Die Beweisidee basiert auf der Konstruktion einer Folge der stückweise konstanten Approximierenden, die folgendes inkrementelles Problem lösen:

(IP) *Sei* $(\varphi_0, z_0) \in \mathcal{F} \times \mathcal{Z}$ *und eine Zerlegung* $0 = t_0 < \ldots < t_N = T$ *gegeben.*
Gesucht $(\varphi_1, z_1), \ldots, (\varphi_N, z_N)$*, so dass für jedes* $k = 0, \ldots, N$

$$\mathcal{E}(t_k, \varphi_k, z_k) = \inf \{ \mathcal{E}(t_k, \varphi, z) + \mathcal{D}(z_{k-1}, z) \mid (\varphi, z) \in \mathcal{F} \times \mathcal{Z} \} \qquad \textit{gilt.}$$

Im weiteren Schritt verwenden wir eine verallgemeinerte Version des Hellyschen Satzes, um die Konvergenz der Approximierenden zu zeigen. Dabei wird die Tatsache verwendet, dass die Dissipation entlang der konstruierten Approximierenden einer a priori Abschätzung nach oben genügt. Schließlich wird gezeigt, dass für die so konstruierten Grenzwerte die (S) & (E)-Formulierung gilt. Im nicht-konvexen Fall muss dieses Verfahren leicht verfeinert werden, da die Kontrolle über die Dissipationen keine Kontrolle über die Deformationen liefert. Dazu wird wie in [DMFT04, FM04] der Deformationsanteil einer Lösung separat so konstruiert, dass die Energiebilanz gilt. Zusätzlich verwenden wir die Theorie der mengenwertigen Abbildungen, um die Messbarkeit des Deformationsanteiles zu garantieren.

Im zweiten Kapitel geben wir eine kurze Einführung in die Theorie der Funktionen mit beschränkter Variation. Die in diesem Kapitel gesammelten Aussagen sind allesamt mehr oder minder bekannt. Allerdings sind diese nur einem engen Kreis der Spezialisten bekannt und in der Fachliteratur zu sehr verstreut. Die Beweise der Sätze 2.1.10, 2.1.15, 2.3.14 konnte der Autor in ihm bekannten Standardwerken über die $\mathrm{BV}(\Omega, \mathbb{R})$-Theorie nicht finden.

Im letzten Kapitel wird schließlich ein einfaches Modell für Phasen-Transformationen in Festkörpern vorgestellt, das auf ein ratenunabhängiges System führt (aus Experimenten weiß man, dass die Phasentransformationen z.B. in Gedächtnis-Legierungen fast ratenunabhängig sind), für welches schließlich mit Hilfe der im ersten Kapitel entwickelten Methoden die Existenz eines Evolutionprozesses zeigen lässt. Dazu betrachten wir einen Körper Ω aus einem Material mit einer endlichen Anzahl der möglichen Phasen-Zustände Z. Die im System gespeicherte Energie wird in der Form

$$\mathcal{E}(t, \varphi, z) = \int_\Omega W(x, \mathrm{D}\varphi(x), z(x))\,\mathrm{d}x + \int_{\text{phase interface}} \psi(z^+(a), z^-(a))\,\mathrm{d}a \;- \langle \ell(t), \varphi \rangle$$

angenommen. Hier bezeichnet $\ell(t)$ wieder die externe zeitabhängige Kraft, die Funktion ψ beschreibt die Dichte der in den Phasengrenzen gespeicherten Energie. Die Funktionen $z^+(a)$, $z^-(a)$ stehen für die Phasen-Zustände auf beiden Seiten des Phasen-Übergangs. Weiterhin nehmen wir an, dass die Dissipationsdistanz folgende Form hat

$$\mathcal{D}(z_1, z_2) := \int_\Omega D(z_1(x), z_2(x))\,\mathrm{d}x.$$

Hierbei beschreiben die Funktionen z_1 und z_2 die Phasen-Verteilung im Körper. So konstruiertes Modell gehört zur der Klasse der mikroskopischen Modelle, da es angenommen wird, dass in jedem Körperpunkt reiner Phasenzustand herrscht. Der zweite Term in der gespeicherten Energie beschreibt dabei die Energie, die durch die Entstehung der Phasengrenzen gespeichert wird. Diese Energie unterbindet zwar die experimentell beobachtete Ausbildung der Mikrostrukturen im Material, liefert aber eine Regularisierung der gespeicherten Energie, welche die für unsere Theorie essentielle Existenz der Minimierer garantiert. Die Zunahme solcher Terme wird aber auch durch die Physik gefordert, denn in Experimenten wurde der Effekt der Keimbildung bei Gedächtnis-Legierungen beobachtet. Dieser Effekt kann aber nur dadurch erklärt werden, dass die Ausbildung der neuen Phasengrenzen doch eine zusätzliche Energie benötigt. Insgesamt erscheint das vorgestellte Modell recht interessant zu sein, da es einerseits sehr einfach und im Rahmen der entwickelten Theorie mathematisch behandelbar ist, andererseits aber auch einige der beobachteten Effekte erklärt. Die Frage nach dem „richtigen“ Modell bleibt weiterhin nicht endgültig geklärt. Es ist allerdings auch zu bezweifeln, dass ein einziges Modell alle auftretende Effekte auf allen Längenskalen abdecken kann. Der interessierte Leser kann sich mit Hilfe von [Rou04] eine breite Übersicht über mögliche Modelle für Gedächtnis-Legierungen verschaffen. In der obigen Arbeit werden z.B. Beispiele der mesoskopischen Modelle genannt, die ebenfalls auf ratenunabhängige Systeme führen. Allerdings sind zur Zeit nur Existenzaussagen für den Fall zweier Phasen verfügbar (s. [MTL02]).

Lebenslauf

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